NE 155/255
Equations and Vector Norms
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1 Equations, continued

Integro-Differential Equations

...are equations that involve both integrals and derivatives of a function.

The general first-order, linear (only with respect to the term involving the derivative) integro-differential equation is of the form

$$\frac{d}{dx}u(x) + \int_{x_0}^{x} f(t, u(t))dt = g(x, u(x)) \,, \qquad u(x_0) = u_0 \,, \qquad x_0 \geq 0 \,.$$

This is the equation type we will likely deal with the most.

Nuclear Engineering Example:

One-dimensional in space, one-dimensional in angle, time-independent, monoenergetic neutron transport equation:

$$\mu\frac{\partial\psi(x,\mu)}{\partial x} + \Sigma_t\psi(x,\mu) = \frac{\Sigma_s}{2}\int_{-1}^{1} d\mu'\psi(x,\mu') + S(x,\mu)$$

where the angular neutron flux is a function of one spatial variable (x) and one angular variable ($\mu = \cos(\theta)$).

Integral Equations

...are equations in which an unknown function appears under an integral sign.

Integral equations are classified according to three different dichotomies, creating eight different kinds:

1. Limits of integration

   (a) both fixed: Fredholm equation

   $$f(x) = \int_a^b K(x,t)\,\varphi(t)\,dt$$

   (b) one variable: Volterra equation

   $$f(x) = \int_a^x K(x,t)\,\varphi(t)\,dt$$

2. Placement of unknown function

   (a) only inside integral: first kind (both above examples)

   (b) both inside and outside integral: second kind

   $$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\,\varphi(t)\,dt$$

3. Nature of known function, f

   (a) identically zero: homogeneous

   (b) not identically zero: inhomogeneous

Both Fredholm and Volterra equations are linear integral equations, due to the linear behaviour of $\phi(x)$ under the integral. A nonlinear Volterra integral equation has the general form

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\,F(x,t,\varphi(t))\,dt$$

where F is a known function.

The integral form of the Neutron Transport Equation is

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \ \exp\left[-\int_0^{\rho'} d\rho''\,\Sigma_t(\vec{r}-\rho''\hat{\Omega},E)\right] q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E)$$

where q contains fixed, inscattering, and fission sources. That is, $q = f(\psi)$.

2 Vector Norms

Definition A *Vector Norm* on $\mathbb{R}^n$ is a function $||\cdot||$ mapping $\mathbb{R}^n \to \mathbb{R}$ with the following properties:

1. $||\vec{x}|| \geq 0$ for all $\vec{x} \in \mathbb{R}^n$
2. $||\vec{x}|| = 0$ iff $\vec{x} = 0$
3. $||\alpha\vec{x}|| = |\alpha|||\vec{x}||$ for all $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ (scalar multiplication)
4. $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ (triangle inequality)

Common Norms

- The l_1 norm is given by
$$||\vec{x}||_1 = \sum_{i=1}^{n} |x_i|$$

- The *Max norm, Sup norm*, or l_∞ norm, is given by
$$||\vec{x}||_\infty = \max_{1\leq i\leq n} |x_i|$$

- The *Euclidean Norm*, or l_2 norm, is given by
$$||\vec{x}||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

  Note: this norm represents the usual notion of distance (Pythagorean theorem)

Example Consider $\vec{x} = (1, -3, 5)^T$. Then

i. $||\vec{x}||_1 = 9$

ii. $||\vec{x}||_\infty = 5$

iii. $||\vec{x}||_2 = \sqrt{1+9+25} = \sqrt{35}$

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Cauchy-Schwarz and Triangle Inequality

It is easy to see that all the properties of a vector norm are satisfied for l_1 and l_∞, but we need to show that the triangle inequality holds for l_2. We will need the following theorem:

Theorem 1. *(Cauchy-Schwarz in $\mathbb{R}^n$)*
For each $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\vec{x}^T\vec{y} = \sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2} = ||\vec{x}||_2||\vec{y}||_2$$

Proof. Consider the following quadratic polynomial in $z \in \mathbb{R}$:

$$0 \leq (x_1 z + y_1)^2 + ... + (x_n z + y_n)^2 = \left(\sum_{i=1}^{n} x_i^2 \right) z^2 + 2 \left(\sum_{i=1}^{n} x_i y_i \right) z + \left(\sum_{i=1}^{n} y_i^2 \right).$$

Since it is nonnegative, it has at most one real root for z. Hence, its discriminant is less than or equal to zero; that is,

$$\left(\sum_{i=1}^{n} x_i y_i \right)^2 - \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right) \leq 0.$$

□

<u>Proof of the Triangle Inequality for l_2</u>:

Proof. Using Cauchy-Schwarz, for each $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$||\vec{x} + \vec{y}||_2^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \leq ||\vec{x}||_2^2 + 2||\vec{x}||_2 ||\vec{y}||_2 + ||\vec{y}||_2^2.$$

Taking the square root of both sides, we obtain

$$||\vec{x} + \vec{y}||_2 \leq ||\vec{x}||_2 + ||\vec{y}||_2.$$

□