NE 155/255, Fall 2019
Simplified Scattering, Diffusion Equation
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Scattering

Double differential scattering can be simplified. Let's start by recalling

$$\Sigma_s(\vec{r}, E', \hat{\Omega}' \to E, \hat{\Omega}) = \Sigma_s(\vec{r}, E') f_s(E', \hat{\Omega}' \to E, \hat{\Omega}) .$$

We can represent the fractional probability as a product of two independent fractional probabilities (which is technically an assumption, but is well-supported by reality):

$$f_s(E', \hat{\Omega}' \to E, \hat{\Omega}) = f_{sE}(E' \to E) f_{s\Omega}(\hat{\Omega}' \to \hat{\Omega}) .$$

In the **monoenergetic** case, $f_{sE}(E \to E') = 1$ and with **isotropic scattering**, $f_{s\Omega}(\hat{\Omega}' \to \hat{\Omega}) = \dfrac{1}{4\pi}$, which would combine to give $\Sigma_s(\vec{r}, E', \hat{\Omega}' \to E, \hat{\Omega}) = \dfrac{\Sigma_s(\vec{r})}{4\pi}$.

If **scattering is not isotropic**, what do we do? That is, when we can't simplify so easily, we at least need to represent it in a way we can numerically manage. There are several approaches; here is one of them.

We expand the scattering cross section in Legendre Polynomials, which are a sequence of orthogonal polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1 \right)^n \right] ,$$

as follows

$$\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl} P_l(\mu_0) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl} P_l(\hat{\Omega}') P_l(\hat{\Omega}) .$$

Note that we used the Legendre Addition Thoerem, which uses spherical harmonics. For now, just note that there are identities and manipulations that allow this to be true.

Legendre polynomials are solutions to Legendre's differential equations,

$$\frac{d}{dx}\big[(1-x^2)\frac{d}{dx}P_n(x)\big] + n(n+1)P_n(x) = 0\,,$$

and are orthogonal on $[-1,1]$:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m} = \quad \frac{2}{2n+1}\text{if n = m,} \quad 0 \text{ if n} \neq \text{m.}$$

$$\therefore P_0(\mu) = 1$$

$$P_1(\mu) = \mu\,.$$

We can see the first several polynomials in Figure 1.

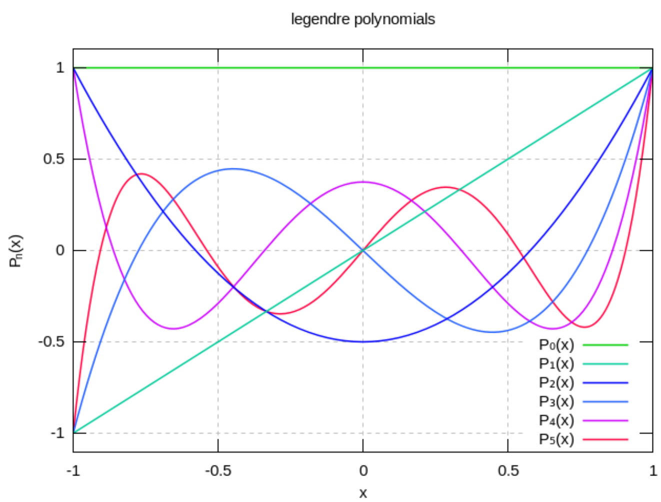

Figure 1: Legendre polynomials.

We get the scattering expansion coefficients from the orthogonality of Legendre Polynomials. Any function can be expanded on the interval $-1 \leq x \leq 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} f_n P_n(x)$$

To find f_n, multiply by P_l and integrate over $(-1,1)$:

$$\int_{-1}^{1} dx\, f(x)P_l(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} f_n \int_{-1}^{1} dx\, P_n(x)P_l(x)$$
$$\therefore f_n = \int_{-1}^{1} dx\, f(x)P_l(x)$$
$$\text{and by analogy}\Sigma_{sl} = 2\pi \int_{-1}^{1} d\mu_0\, P_l(\mu_0)\Sigma_s(\mu_0)\,.$$

All of that was to say, we can actually deal with scattering when it's *not* isotropic by expressing the scattering cross section as an expansion in Legendre polynomials. In our simplified transport equation, this looks like:

$$\mu\frac{\partial\psi(z,\mu)}{\partial z} + \Sigma_t(z)\psi(z,\mu) = \\ \frac{\nu\Sigma_f(z)}{2}\phi(z) + \sum_{l=0}^{\infty}\frac{2l+1}{2}\Sigma_{sl}(z)\int_{-1}^{1} d\mu'\, P_l(\mu_0)\psi(z,\mu') + \frac{S(z)}{2}\,.$$

In reality, we truncate the expansion at some point. Some common truncations:

$$l = 0 \text{ is isotropic; (note } P_0(\hat{\Omega}) = 1)$$
$$\Sigma_s(\hat{\Omega}'\cdot\hat{\Omega}) \cong \frac{1}{4\pi}\Sigma_{s0}$$
$$l = 1 \text{ is linearly isotropic; (note } P_1(\hat{\Omega}) = \hat{\Omega})$$
$$\Sigma_s(\hat{\Omega}'\cdot\hat{\Omega}) \cong \frac{1}{4\pi}\big(\Sigma_{s0} + 3\hat{\Omega}'\hat{\Omega}\Sigma_{s1}\big)$$

In describing representations of the TE, people commonly describe the scattering by saying the "P_n" expansion of scattering.

Diffusion Equation

The diffusion approximation is a widely used simplification that reduces the computational complexity of the transport equation.

The approximation is that the **angular dependence of the flux is unimportant**, so the direction component of the transport equation can be discarded. Physically this means that neutrons move against their concentration gradient like just heat diffuses through a conductor.

The information in this section is derived from Duderstadt and Hamilton's *Nuclear Reactor Analysis* and neglects fission for simplicity.

The first step in applying this approximation is to integrate the angular dependence out of the transport equation, resulting in the neutron continuity equation:

$$\nabla \cdot J(\vec{r}, E) + \Sigma(\vec{r}, E)\phi(\vec{r}, E) = \int dE' \, \Sigma_s(\vec{r}, E' \to E)\phi(\vec{r}, E') + Q_{ex}(\vec{r}, E) \; , \tag{1}$$

where the following definitions have been used:

- $J(\vec{r}, E) = \int d\hat{\Omega} \, \hat{\Omega}\psi(\vec{r}, \hat{\Omega}, E)$ is the neutron current
- $\phi(\vec{r}, E) = \int d\hat{\Omega} \, \psi(\vec{r}, \hat{\Omega}, E)$ is the scalar flux, and
- $Q_{ex}(\vec{r}, E) = \int d\hat{\Omega} \, q_{ex}(\vec{r}, \hat{\Omega}, E)$ is the external source.

Unfortunately, this simplifying approximation added another unknown, J, which leaves one equation with two unknowns.

In an attempt to eliminate one of these unknowns, Equation (1) is multiplied by $\hat{\Omega}$ and integrated over angle again to obtain the first angular moment:

$$\begin{aligned} \nabla \cdot \int d\hat{\Omega} \, \hat{\Omega}\hat{\Omega}\psi(\vec{r}, \hat{\Omega}, E) + \Sigma(\vec{r}, E)J(\vec{r}, E) = & \\ \int dE' \, \Sigma_{s1}(\vec{r}, E' \to E)J(\vec{r}, E') + \int d\hat{\Omega} \int d\hat{\Omega} \, \hat{\Omega} q_{ex} \; , & \end{aligned} \tag{2}$$

where $\Sigma_{s1} = \int d\hat{\Omega} \, \hat{\Omega}\Sigma_s$. The first angular moment form of the equation cannot be solved either because the streaming (first) term is still unknown.

To make Equation (2) solvable, the original assumption is modified to assert that the angular flux is weakly, in fact **linearly, dependent on angle rather than independent of angle**.

To implement this assumption the angular flux is expanded in angle and only the first two terms are retained:

$$\psi(\vec{r}, \hat{\Omega}, E) \cong \frac{1}{4\pi}\phi(\vec{r}, E) + \frac{3}{4\pi}J(\vec{r}, E) \cdot \hat{\Omega} \, . \tag{3}$$

The truncated angular flux is then inserted into the streaming term in Equation (2), giving

$$\nabla \cdot \int d\hat{\Omega} \, \hat{\Omega}\hat{\Omega}\psi(\vec{r}, \hat{\Omega}, E) \cong \frac{1}{3}\nabla\phi(\vec{r}, E) \, . \tag{4}$$