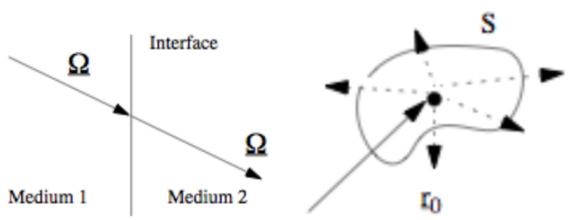
NE 155/255, Fall 2019
Transport Equation Conditions and Simplifications
September 18, 2019

Additional Conditions

- Finiteness condition: $0 \leq \psi(\vec{r}, E, \hat{\Omega}, t) \leq \infty$
- Interface condition: $\psi_1(\vec{r}, E, \hat{\Omega}, t) = \psi_2(\vec{r}, E, \hat{\Omega}, t) \qquad \forall \vec{r} \in S_i$, all energies, and all $\hat{\Omega}$
- Source condition: $Q(\vec{r_0}, E, \hat{\Omega}, t) = Q_0(E, \hat{\Omega}, t)\delta(\vec{r} - \vec{r_0})$



Simplified Forms

Time Independence

We assume that losses and sources are balanced, and thus there is no rate of change with time:

$$\frac{\partial \psi}{\partial t} = 0$$

We can then remove the time dependence from all terms in our equation. As noted, for reactors we often

- solve a steady-state form of the equation,
- perform depletion calculations that characterize material evolution (using the Batemann equations, which we're skipping for now)
- solve a new steady-state calculation with updated material specifications.

One Speed

Assume all particles are at the same speed, $\vec{v} = v_0 \cdot \hat{\Omega}$. Then

$$n(\vec{r}, v, \hat{\Omega}, t) = n(\vec{r}, \hat{\Omega}, t)\delta(v - v_0)$$
$$\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) = \Sigma_s(E, \hat{\Omega}' \to \hat{\Omega})\delta(E' - E)$$

Now we can remove E integration and E dependence:

$$\frac{1}{v}\frac{\partial\psi(\vec{r}, \hat{\Omega}, t)}{\partial t} + \hat{\Omega} \cdot \nabla\psi(\vec{r}, \hat{\Omega}, t) + \Sigma_t\psi(\vec{r}, \hat{\Omega}, t) =$$
$$\int_{4\pi} d\hat{\Omega}'\Sigma_s(\hat{\Omega}' \to \hat{\Omega})\psi(\vec{r}, \hat{\Omega}', t) + \frac{\nu\Sigma_f}{4\pi}\int_{4\pi} d\hat{\Omega}'\psi(\vec{r}, \hat{\Omega}', t) + S(\vec{r}, \hat{\Omega}, t)$$

Isotropic Source

It is often the case that an external source is (or we can approximate it as) isotropic:

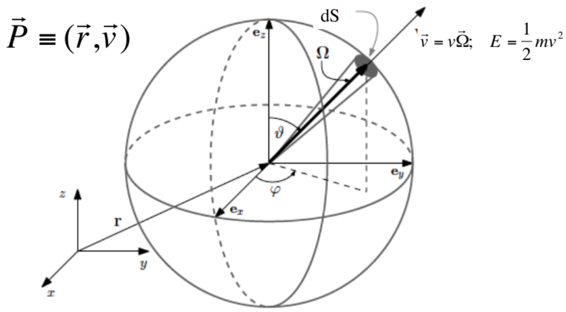
$$S(\vec{r}, E, \hat{\Omega}, t) = \frac{1}{4\pi}S(\vec{r}, E, t)$$

One Dimensional and Rotational (Azimuthal) Symmetry

We also often simplify by only worrying about fewer dimensions. Here we'll look at just one: we'll get rid of x and y.

We first assume that **things only vary in the z dimension**, so $\vec{r} \to z$, e.g. $\psi(\vec{r}, E, \hat{\Omega}, t) \to \psi(z, E, \hat{\Omega}, t)$

We next assume (which is often true) that our systems is **azimuthally symmetric**, which means that the scattering is symmetric about the azimuthal direction (φ). Therefore, scattering only depends on the angle of the scattering cosine, $\mu_0 = \hat{\Omega}' \cdot \hat{\Omega}$, the angle between where it was heading and where it is heading. We can use this assumption to simplify two things.



$$d\hat{\Omega} = \sin(\theta)d\theta d\varphi = d\mu d\varphi$$
$$\mu = \cos(\theta), \text{ so } d\mu = \sin(\theta)d\theta$$
$$\Omega_z = \cos(\theta) = \mu$$

Recall our new definition $\mu_0 = \hat{\Omega}' \cdot \hat{\Omega}$.

$$\int_{4\pi} d\hat{\Omega} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta)d\theta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu = 4\pi$$
$$\psi(z,\hat{\Omega},E,t)d\hat{\Omega} = \psi(z,\varphi,\mu,E,t)d\varphi d\mu = \psi(z,\mu,E,t)d\varphi d\mu$$

With azimuthal symmetry, we can rewrite the scattering cross section since it is no longer a function of φ.

$$\Sigma_s(z,E' \to E,\hat{\Omega}' \to \hat{\Omega}) \to \Sigma_s(z,E' \to E,\hat{\Omega}' \cdot \hat{\Omega}) = \Sigma_s(z,E' \to E,\mu_0)\ .$$

Further, we can execute the angular integration over φ since nothing depends on it anymore:

$$\int_{4\pi} d\hat{\Omega}\, \psi(z,\hat{\Omega},E,t) = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu\, \psi(z,\hat{\Omega},E,t) = 2\pi \int_{-1}^{1} d\mu\, \psi(z,\mu,E,t)\ .$$

The next simplification is in the streaming term:

$$\hat{\Omega} \cdot \nabla\psi(\vec{r},\hat{\Omega},E,t) \to \Omega_z \frac{\partial\psi(z,\mu,E,t)}{\partial z} = \mu\frac{\partial\psi(z,\mu,E,t)}{\partial z}$$

We can combine all of this into a 1-D equation, using $\mu_0 = \hat{\Omega}' \cdot \hat{\Omega}$:

$$
\begin{aligned}
\frac{1}{v}\frac{\partial\psi}{\partial t}(z,E,\hat{\Omega},t) &+ \Omega_z\frac{\partial\psi(z,\hat{\Omega},E,t)}{\partial z} + \Sigma_t(z)\psi(z,\hat{\Omega},E,t) \\
&= \frac{\chi(E)}{4\pi}\int_0^\infty dE' \int_{4\pi} d\hat{\Omega}'\, \nu(E')\Sigma_f(z,E',t)\psi(z,\hat{\Omega}',E',t) \\
&+ \int_0^\infty dE' \int_{4\pi} d\hat{\Omega}'\, \Sigma_s(z,E'\to E,\mu_0)\psi(z,\hat{\Omega}',E',t) + S(z,E,\hat{\Omega},t)\ ,
\end{aligned}
$$

which we can then integrate over the φ component of angle:

$$
\begin{aligned}
\frac{1}{v}\frac{\partial\psi}{\partial t}(z,E,\mu,t) &+ \mu\frac{\partial\psi(z,\mu,E,t)}{\partial z} + \Sigma_t(z)\psi(z,\mu,E,t) \\
&= 2\pi\frac{\chi(E)}{4\pi}\int_0^\infty dE'\, \nu(E')\Sigma_f(z,E') \int_{-1}^1 d\mu'\, \psi(z,\mu',E',t) \\
&+ 2\pi\int_0^\infty dE' \int_{-1}^1 d\mu'\, \Sigma_s(z,E'\to E,\mu_0)\psi(z,\mu',E',t) + S(z,E,\mu,t)
\end{aligned}
$$

Note that the fission term becomes

$$
\frac{\chi(E)}{2}\int_0^\infty dE'\, \nu(E')\Sigma_f(z,E')\underbrace{\phi(z,E',t)}_{\text{scalar flux}}\ .
$$

Combination

If we combine one-speed, time-independent, isotropic source, one-dimensional, and azimuthally-symmetric, we get

$$
\begin{aligned}
\mu\frac{\partial\psi(z,\mu)}{\partial z} &+ \Sigma_t(z)\psi(z,\mu) = \\
&\frac{\nu\Sigma_f(z)}{2}\phi(z) + 2\pi\int_{-1}^1 d\mu'\, \Sigma_s(z,\mu_0)\psi(z,\mu') + \frac{S(z)}{2}\ .
\end{aligned}
$$