NE 155/255, Fall 2019
Assumptions and Terms of the Transport Equation
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Assumptions

1. Particles are point objects ($\lambda = h/(mv)$) is small compared to the atomic diameter): its state is fully described by its location, velocity vector, and a given time. This ignores rotation and quantum effects.
2. Neutral particles travel in straight lines between collisions.
3. Particle-particle collisions are negligible (makes TE linear).
4. Material properties are isotropic (generally valid unless velocities are very low).
5. Material composition is time-independent (generally valid over short time scales).
6. Quantities are expected values: fluctuations about the mean for very low densities are not accounted for.

The Transport Equation

We consider a six-dimensional volume (as a six-dimensional cube) fixed in space, of dimensions $\triangle x$, $\triangle y$, $\triangle z$, $\triangle E$, $\triangle \theta$, $\triangle \varphi$. Then, the number of particles within this volume at time t is

$$n(\vec{r}, E, \hat{\Omega}, t)\triangle x \triangle y \triangle z \triangle E \triangle \theta \triangle \varphi = n(\vec{r}, E, \hat{\Omega}, t)\triangle \beta,$$

where all arguments of N are "average" arguments in the increment of six-dimensional phase space $\triangle \beta$. The number of particles in this cube changes with time:

$$\triangle \beta \frac{\partial}{\partial t} n(\vec{r}, E, \hat{\Omega}, t) = \begin{array}{l}\text{time rate of change of the number of} \\ \text{particles in the six-dimensional cube } \triangle \beta.\end{array}$$

This time rate of change is due to five separate processes. One is the rate of streaming of particles out of the volume through the boundaries. The other processes occur within the six-dimensional "cube": the rate of absorption; the rate of scattering from E, $\hat{\Omega}$ to all other energies and directions, known as outscattering; the rate of scattering into E, $\hat{\Omega}$ from all other energies and directions, known as inscattering; and the rate of production of particles due to an internal source.

Now, let us consider the surfaces of the cube perpendicular to the x-axis. For the net rate of particles leaving the cube through these two surfaces, we have

$$(\text{streaming})_x = \dot{x}n(\vec{r}, E, \hat{\Omega}, t) \mid_x^{x+\triangle x} \triangle y \triangle z \triangle E \triangle \theta \triangle \varphi,$$

where $\dot{x}$ is the x component of the particle velocity, and $\triangle y \triangle z \triangle E \triangle \theta \triangle \varphi$ is the surface area. Letting $\triangle x$ go to the differential dx, we rewrite

$$(\text{streaming})_x = \triangle \beta \frac{\partial}{\partial x} \left[\dot{x}n(\vec{r}, E, \hat{\Omega}, t) \right].$$

Using the same procedure for the flow from the cube in the other five "directions", we obtain

$$\begin{aligned} \text{streaming} = \Bigg[&\frac{\partial}{\partial x}(\dot{x}n) + \frac{\partial}{\partial y}(\dot{y}n) + \frac{\partial}{\partial z}(\dot{z}n) \\ &+ \frac{\partial}{\partial E}(\dot{E}n) + \frac{\partial}{\partial \theta}(\dot{\theta}n) + \frac{\partial}{\partial \varphi}(\dot{\varphi}n) \Bigg] \triangle \beta, \end{aligned}$$

where $n = n(\vec{r}, E, \hat{\Omega}, t)$.

The rate of absorption within the cube is the product of the number of particles in the cube and the probability of absorption per particle per unit of time. This probability is given by the product of the absorption cross section and the particle speed v. That is,

$$\text{absorption} = v\Sigma_a(\vec{r}, E)n(\vec{r}, E, \hat{\Omega}, t)\triangle \beta.$$

Using similar arguments and the fact that we need to sum the scattering from (to) E, $\hat{\Omega}$ to (from) all other energies and directions E', $\hat{\Omega}'$, we find

$$\text{outscattering} = \triangle\beta \int_0^\infty \int_{4\pi} v\Sigma_s(\vec{r}, E \to E', \hat{\Omega} \to \hat{\Omega}')n(\vec{r}, E, \hat{\Omega}, t)d\hat{\Omega}'dE',$$
$$\text{inscattering} = \triangle\beta \int_0^\infty \int_{4\pi} v'\Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega})n(\vec{r}, E', \hat{\Omega}', t)d\hat{\Omega}'dE',$$

where $\Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega})$ is the macroscopic differential scattering cross section. Since the distribution function in the integrand of the outscattering term is independent of the integration variables, we can rewrite outscattering as $\triangle\beta v\Sigma_s(\vec{r}, E)n(\vec{r}, E, \hat{\Omega}, t)$. Finally, we need to consider the internal source of particles. We quantify this source by introducing the function $S(\vec{r}, E, \hat{\Omega}, t)$ such that the rate of introduction of particles into the cube is given by

$$\text{source} = S(\vec{r}, E, \hat{\Omega}, t)\triangle\beta.$$