NE 155/255, Fall 2019
Transport Equation and Boundary Conditions
September 16, 2019

Transport Equation (cont'd.)

In order to build the transport equation, we sum the terms we derived previously with appropriate signs for loss and gain, to the overall rate of change. Letting $\triangle\beta$ approach a differential element and canceling it, we obtain

$$\begin{aligned}\frac{\partial n}{\partial t} = &-\left[\frac{\partial(\dot{x}n)}{\partial x} + \frac{\partial(\dot{y}n)}{\partial y} + \frac{\partial(\dot{z}n)}{\partial z} + \frac{\partial(\dot{E}n)}{\partial E} + \frac{\partial(\dot{\theta}n)}{\partial \theta} + \frac{\partial(\dot{\varphi}n)}{\partial \varphi}\right] \\ &- v\Sigma_a(\vec{r},E)n \\ &+ \int_0^\infty \int_{4\pi} v'\Sigma_s(\vec{r},E'\to E,\hat{\Omega}'\to\hat{\Omega})n(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE' \\ &- \int_0^\infty \int_{4\pi} v\Sigma_s(\vec{r},E\to E',\hat{\Omega}\to\hat{\Omega}')n(\vec{r},E,\hat{\Omega},t)d\hat{\Omega}'dE' \\ &+ S(\vec{r},E,\hat{\Omega},t),\end{aligned} \tag{1}$$

where $n = n(\vec{r},E,\hat{\Omega},t)$. Since particles travel in a straight line between collisions, $\dot{\theta} = \dot{\varphi} = 0$. Furthermore, $\dot{E} = 0$ because particles stream with no change in energy. Finally, performing the outscattering integral:

$$\begin{aligned}&\frac{1}{v}\frac{\partial \psi}{\partial t}(\vec{r},E,\hat{\Omega},t) + \hat{\Omega}\cdot\nabla\psi(\vec{r},E,\hat{\Omega},t) + \Sigma_t(\vec{r},E)\psi(\vec{r},E,\hat{\Omega},t) \\ &\quad = \int_0^\infty \int_{4\pi} \Sigma_s(\vec{r},E'\to E,\hat{\Omega}'\to\hat{\Omega})\psi(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE' + S(\vec{r},E,\hat{\Omega},t).\end{aligned} \tag{2}$$

We can easily generalize this equation to include nuclear fission. To do that, we must revisit our treatment of Σ_a; there are two main processes responsible for the absorption of particles in the system: *radiative capture* and *nuclear fission* (note that we're ignoring (n,xn) reactions for the time being). Now, we define

$$\Sigma_\gamma(\vec{r}, E)d\beta = \text{probability of capture}$$

and

$$\Sigma_f(\vec{r}, E)d\beta = \text{probability of a fission event,}$$

such that

$$\Sigma_a(\vec{r}, E) = \Sigma_\gamma(\vec{r}, E) + \Sigma_f(\vec{r}, E)\,.$$

While a captured neutron is simply removed from the system, a neutron with energy E that induces a fission event causes the target nucleus to split into two smaller daughter nuclei, and

$$\nu(E) = \text{the mean number of fission neutrons that are released.}$$

Of this number, $\nu(E)[1 - M(E)]$ are *prompt* fission neutrons (being emitted within 10^{-15} seconds of the fission event). These fission neutrons are emitted isotropically, with an energy distribution given by the fission spectrum $\chi_p(E)$. Also, $\nu(E)M(E)$ *delayed* fission neutrons (being released roughly 0.1 to 60 seconds after the fission event) are created; a delayed neutron is produced when a radioactive daughter nucleus undergoes a radioactive decay process in which a neutron is emitted.

Assuming (for simplicity) that the number of delayed neutrons emitted by fission is very small [$M(E) << 1$], we can neglect the delayed neutron terms and rewrite the transport equation as

$$
\begin{aligned}
&\underbrace{\frac{1}{v}\frac{\partial \psi}{\partial t}(\vec{r},E,\hat{\Omega},t)}_{\text{time rate of change}} + \underbrace{\hat{\Omega}\cdot\nabla\psi(\vec{r},E,\hat{\Omega},t)}_{\text{streaming loss rate}} + \underbrace{\Sigma_t(\vec{r},E)\psi(\vec{r},E,\hat{\Omega},t)}_{\text{total interaction loss rate}} \\
&\qquad = \underbrace{\int_0^\infty \int_{4\pi} \Sigma_s(\vec{r},E'\to E,\hat{\Omega}'\to\hat{\Omega})\psi(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE'}_{\text{in scattering source rate}} \\
&\qquad\quad + \underbrace{\frac{\chi_p(E)}{4\pi}\int_0^\infty \int_{4\pi} \nu(E')\Sigma_f(\vec{r},E')\psi(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE'}_{\text{fission source rate}} \\
&\qquad\quad + \underbrace{S(\vec{r},E,\hat{\Omega},t)}_{\text{external source rate}}.
\end{aligned} \tag{3}
$$

Boundary and Initial Conditions

These equations require both spatial and temporal boundary conditions. *Assuming that the physical system of interest is nonreentrant* (convex) and characterized by a volume V, it is sufficient to specify the flux of particles at all points of the bounding surface of the system in the incoming directions. This implies

$$\psi(\vec{r}_s,E,\hat{\Omega},t) = \psi_b(\vec{r}_s,E,\hat{\Omega},t)\,, \quad \mathbf{n}\cdot\hat{\Omega} < 0,$$

where ψ_b is a specified function at the boundary, $\vec{r}_s$ is a point on the surface, and **n** is the unit outward normal vector at this point. In the time variable, we assume the range of interest $0 \leq t < \infty$ and specify the initial condition at $t = 0$, such that

$$\psi(\vec{r},E,\hat{\Omega},0) = \psi_0(\vec{r},E,\hat{\Omega}),$$

where ψ_0 is a specified function.

A few other boundary conditions that we frequently use in nuclear engineering:

- mirror reflecting: $\psi(\vec{r},E,\hat{\Omega},t) = \psi(\vec{r},E,\hat{\Omega}',t) \quad \forall\vec{r} \in$ surface S, where $(\hat{e})\cdot\hat{\Omega} < 0$

- isotropic reflecting: $\psi(\vec{r}, E, \hat{\Omega}, t) = \dfrac{\phi(\vec{r}, E, t)}{4\pi}$

- vacuum: $\psi(\vec{r}, E, \hat{\Omega}, t) = 0 \quad \forall \vec{r} \in S$, where S is a surface, and $(\hat{e}) \cdot \hat{\Omega} < 0$

  $J^{-}(\vec{r}, E, t) = \int_{(\hat{e})\cdot\hat{\Omega}<0} d\hat{\Omega}\, |\hat{(e)} \cdot \hat{\Omega}| \psi(\vec{r}, E, \hat{\Omega}, t) = 0$

- periodic: $\psi(\vec{r}, E, \hat{\Omega}, t) = \psi(\vec{r} + d, E, \hat{\Omega}, t)$

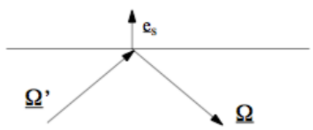

FIGURE XII.7. Mirror reflection

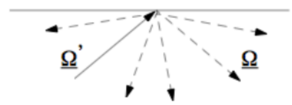

FIGURE XII.8. Isotropic reflective boundary condition