$$\int \frac{2}{e^x + e^{-x}}\,dx = \int \frac{2x\frac{1}{u}}{u+\frac{1}{u}}\,du = 2\int \frac{1}{u^2+1}\,du$$

$u = e^x$

$du = e^x dx$

$dx = \frac{du}{e^x} = \frac{du}{u}$

Calculus

Final Exam

December 23rd, 2009

Time: 100 mins.
Score: Every problem is worth 6 points.

Part A - Right Or Wrong: Classify the statements as either (A) true; or (B) wrong; or (C) true in some cases, wrong in others. Carefully justify your choice!

1. Let $f(x)$ be a continuous function with $0 \leq f(x) \leq 1$.

   C (a) $\lim_{x\to\infty} f(x)$ exists.

   A (b) $\lim_{x\to\infty} f(x)/x$ exists.

   revise → C (c) $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} f'(x)$. $\frac{f'(x)}{1}$

   B (d) The integral

   $$< \int_1^\infty \frac{f(x)}{x}dx \leq \int_1^\infty \frac{1}{x}dx$$

   is convergent.

Part B - Problems: For the remaining problems, argue carefully, work neatly and show all your work!

2. (a) Evaluate the following limit

   $$\lim_{x\to 0+} \frac{1}{x}\int_0^x (\cos(t))^{1/t^2}\,dt.$$

   (b) Find all continuous functions $g(x)$ and all c such that

   $$\int_c^x g(t)dt = x^2 + x - 6.$$

3. (a) Show that

$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) = \operatorname{sech}^2(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}(x) = -\tanh(x)\operatorname{sech}(x).$$

(b) Let $n, m \geq 0$ be integers. Show that

$$\int \tanh^n(x)\operatorname{sech}^m(x)\mathrm{d}x$$

can be solved if either $n, m \geq 1$ and n is odd or $m \geq 2$ and m is even.

(c) Evaluate

$$\int \operatorname{sech}(x)\mathrm{d}x.$$

4. Let $p \in \mathbb{R}$. Consider

$$\int_1^\infty \frac{\ln x}{x^p}\mathrm{d}x.$$

(a) Without evaluating the integral, show that it is divergent for $p < 1$.

(b) Show that the integral is convergent for $p > 1$. Also, find its value.

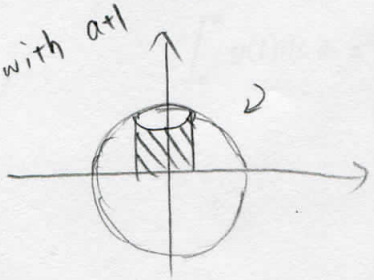
(c) Is the integral convergent or divergent for $p = 1$?

5. Consider a sphere of radius 2 with center at the origin of the coordinate systems. A cylindrical hole with radius 1 is drilled out of the sphere with axis of the cylinder the z-axis.

(a) Plot the cross-sections perpendicular to the x-axis and the z-axis.

(b) Choose one direction from part (a) and compute the volume of the solid with the method of cross-sections.

(c) Verify your result from part (b) by computing the volume with the method of cylindrical shells.



Good Luck and Merry Xmas!