Probabilistic Primality Testing and Analysis of Probabilistic AKS

Emilie Ma

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Introduction

Abstract

This paper aims to analyze the Fermat, the Euler (Solovay-Strassen), and the Miller-Rabin primality tests, three well-known probabilistic algorithms based on Fermat's little theorem. The Agarwal-Kayal-Saxena test, the first polynomial time deterministic primality test developed, is also discussed, as well as a proposal for a new probabilistic adaptation. This probabilistic AKS was found to deliver significant running time decreases, at the expense of eliminating determinism and passing a considerable amount of pseudoprimes.

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- In layman's terms: looked at a variety of tests for checking if a number is prime, and analyzed their performance with regards to speed and accuracy

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- Well-known primality tests include the Fermat, Euler (Solovay-Strassen), Miller-Rabin, and Agarwal-Kayal-Saxena (AKS) tests

Probabilistic vs Deterministic

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- Why wouldn't we always use deterministic tests?
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    - High performance demands where 100% accuracy can be sacrificed
- Research goal: how can we take the best of both worlds of probabilistic and deterministic tests?

Background Theory

Preliminary Notes

- Assumes a basic knowledge of modular congruences
- All $\log n$ shown are $\log_2 n$ unless otherwise marked

Agarwal-Kayal-Saxena Primality Test

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- Based on the theorem that for any a where a is coprime to n and where $n \geq 2$, the following holds within the polynomial ring $\mathbb{Z}[x]$:

$$(X + a)^n \equiv X^n + a \pmod{n}$$

Agarwal-Kayal-Saxena Primality Test

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    - Check $n \leq r$; if so, n is prime.
    - Compute all $(X + a)^n \equiv X^n + a \pmod{n}$ for $1 \leq a \leq \lfloor \sqrt{\phi(r)} \log n \rfloor$; if n passes all these tests, it is prime, otherwise it is composite.

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- Probabilistic AKS: this research project's proposal for a new variant of AKS
    - Instead of checking all a $1 \leq a \leq \lfloor \sqrt{\phi(r)} \log n \rfloor$, choose k random a values from that range to check
    - Similar idea as the random choice of a in the Fermat, Euler, and Miller-Rabin tests

Probabilistic AKS Primality Test

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  - Instead of having to check $\lfloor \sqrt{\phi(r)} \log n \rfloor$ calculations $(O(\sqrt{(\log^3 n)} \times \log n) = O(\log^{5/2} n))$, k equations can be checked, leading to a worst-case complexity of:

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Analysis Methods

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- To consistently analyze the four primality tests (AKS was not tested for pseudoprimes as it is proven correct and would have taken too long), integers in the range $10^5 \leq x \leq 10^6$ were randomly chosen
    - For the Fermat, Euler, and Miller-Rabin tests, 10^4 integers were chosen
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- Sympy's `isprime` function was used as a reference to test for primality

Results

Probabilistic primality tests - accuracy

Figure 1: The effect of increasing the number of base trials k on pseudoprimes passed

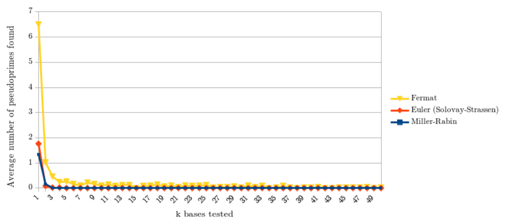
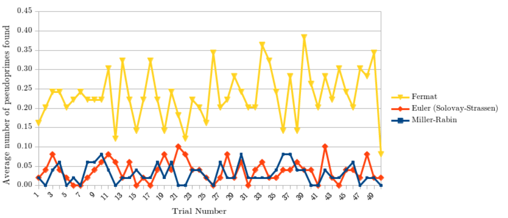


Figure 2: Average pseudoprimes passed across all $1 \leq k \leq 50$ values per trial



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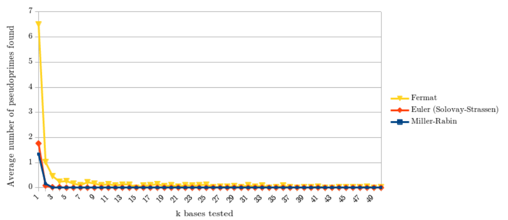

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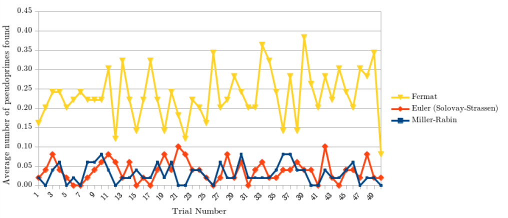


- Fermat consistently passed more pseudoprimes
- Euler and Miller-Rabin appear to perform quite similarly but on average Euler will pass more pseudoprimes

Probabilistic primality tests - runtime

Figure 3: The effect of increasing the number of base trials k on pseudoprimes passed

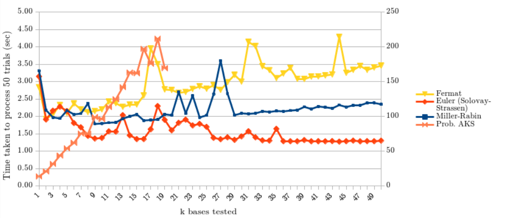
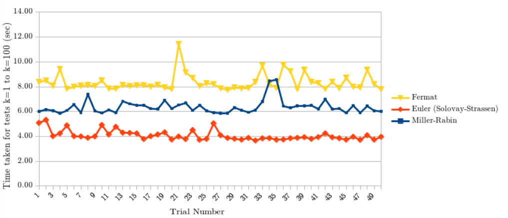


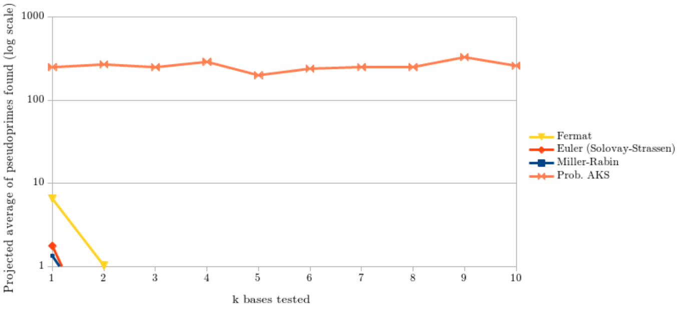
Figure 4: Average time elapsed across all $1 \leq k \leq 50$ values per trial



- A possible explanation for more efficient runtime of Euler and Miller-Rabin is the speed at which numbers are discarded as composite
  - The congruences used in the Euler and Miller-Rabin tests may do so more quickly - though this has not been theoretically verified yet

Projected pseudoprimes vs. k value

Figure 5: Projected average number of pseudoprimes passed by probabilistic AKS versus other tests



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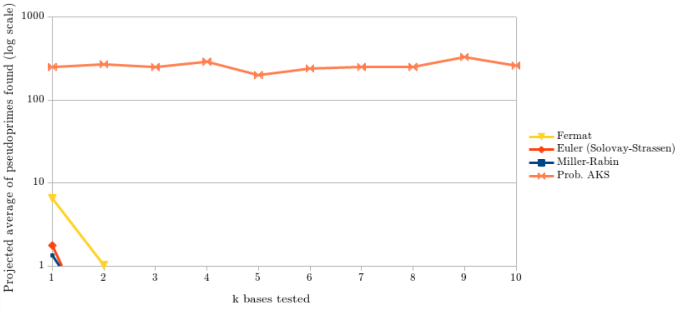


- Significant increase in number of projected pseudoprimes passed by probabilistic AKS compared to the existing tests (200 vs. <10)
  - Projection method: multiply pseudoprimes found by 10^2
- Due to the relatively very low k values tested for probabilistic AKS

Running time of probabilistic AKS

Figure 3: The effect of increasing the number of base trials k on pseudoprimes passed

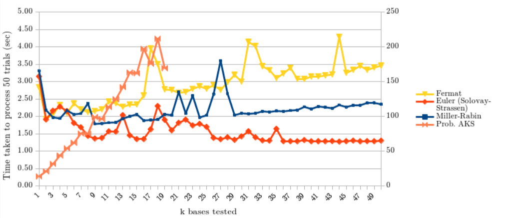
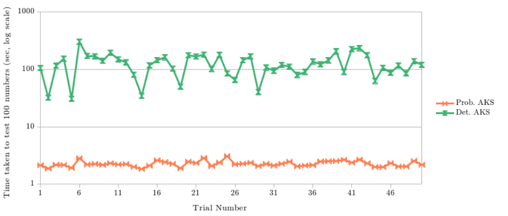


Figure 6: Elapsed running time for probabilistic versus deterministic AKS



- Probabilistic AKS is considerably slower than the existing primality tests, but also significantly faster than deterministic AKS
    - Discrepancies in runtime from probabilistic AKS and existing primality tests can be attributed to time complexity

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  - Potentially misleading `isprime` Sympy function used; also relies on probabilistic methods

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  - Precomputing an array of deterministically verified (with AKS) values to check primality against
  - Adjusting the bounds of r in probabilistic AKS to further improve the theoretical runtime complexity

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- Deterministic AKS: no pseudoprimes passed, and very slow to run $\rightarrow$ good for applications where speed is irrelevant
- Probabilistic AKS: high numbers of pseudoprimes passed at low k, much faster than deterministic AKS $\rightarrow$ may provide a suitable alternative to deterministic AKS for more practical applications given high enough k values

Acknowledgements

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