Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of $f$ and closely spaced level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function $f$ such that $\mathbf{F} = \nabla f$. In this situation $f$ is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}
\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\
&= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\
&= \mathbf{F}(x, y, z)
\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

## 16.1 EXERCISES

**1–10** Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

**1.** $\mathbf{F}(x, y) = 0.3\,\mathbf{i} - 0.4\,\mathbf{j}$

**2.** $\mathbf{F}(x, y) = \frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$

**3.** $\mathbf{F}(x, y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

**4.** $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

**5.** $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

**6.** $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

**7.** $\mathbf{F}(x, y, z) = \mathbf{i}$

**8.** $\mathbf{F}(x, y, z) = z\,\mathbf{i}$

**9.** $\mathbf{F}(x, y, z) = -y\,\mathbf{i}$

**10.** $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

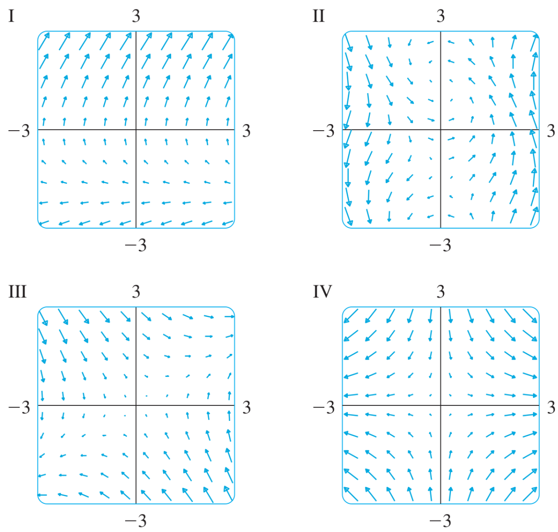
**11–14** Match the vector fields F with the plots labeled I–IV. Give reasons for your choices.

**11.** $\mathbf{F}(x, y) = \langle x, -y \rangle$

**12.** $\mathbf{F}(x, y) = \langle y, x - y \rangle$

**13.** $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

**14.** $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$



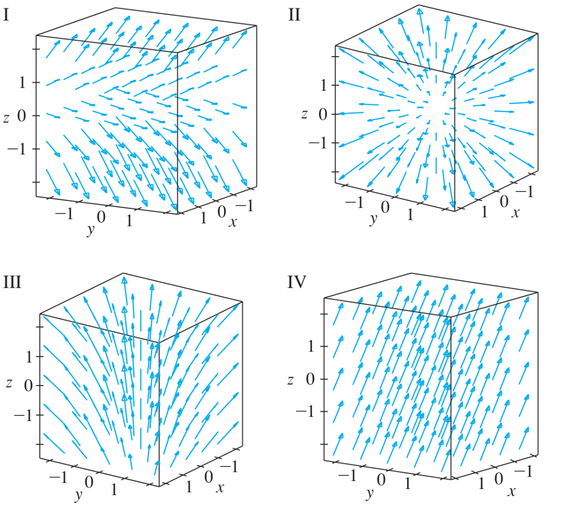
**15–18** Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

**15.** $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

**16.** $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

**17.** $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

**18.** $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$




**19.** If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points $(x, y)$ such that $\mathbf{F}(x, y) = \mathbf{0}$.

**20.** Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

**21–24** Find the gradient vector field of $f$.

**21.** $f(x, y) = y \sin(xy)$

**22.** $f(s, t) = \sqrt{2s + 3t}$

**23.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

**24.** $f(x, y, z) = x^2 y e^{y/z}$

**25–26** Find the gradient vector field $\nabla f$ of $f$ and sketch it.

**25.** $f(x, y) = \frac{1}{2}(x - y)^2$

**26.** $f(x, y) = \frac{1}{2}(x^2 - y^2)$

**27–28** Plot the gradient vector field of $f$ together with a contour map of $f$. Explain how they are related to each other.

**27.** $f(x, y) = \ln(1 + x^2 + 2y^2)$

**28.** $f(x, y) = \cos x - 2 \sin y$

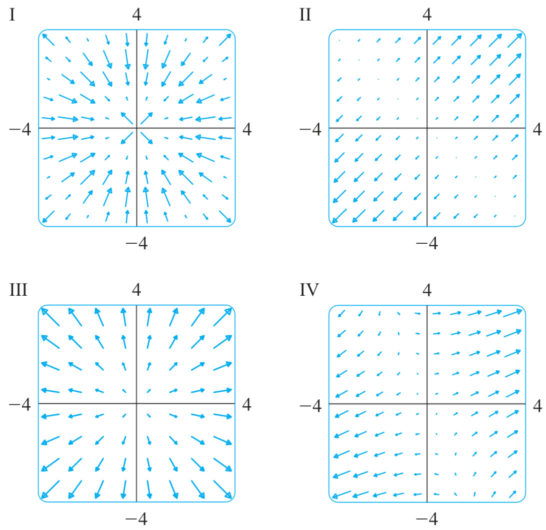
**29–32** Match the functions $f$ with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

**29.** $f(x, y) = x^2 + y^2$

**30.** $f(x, y) = x(x + y)$

**31.** $f(x, y) = (x + y)^2$

**32.** $f(x, y) = \sin\sqrt{x^2 + y^2}$




**33.** A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

**34.** At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

**35.** The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

**36.** (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.