Best mean-square approximation polynomial
Chebyshev
KZ

$$\texttt{BMSchebyshev2[n\_]} := \sqrt{2/\texttt{Pi}}\ \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}$$

$$\texttt{aCoefficient[fun\_, n\_]} := \int_{-1}^{1} \texttt{fun} * \sqrt{1-x^2} * \texttt{BMSchebyshev2[n]}\, dx$$

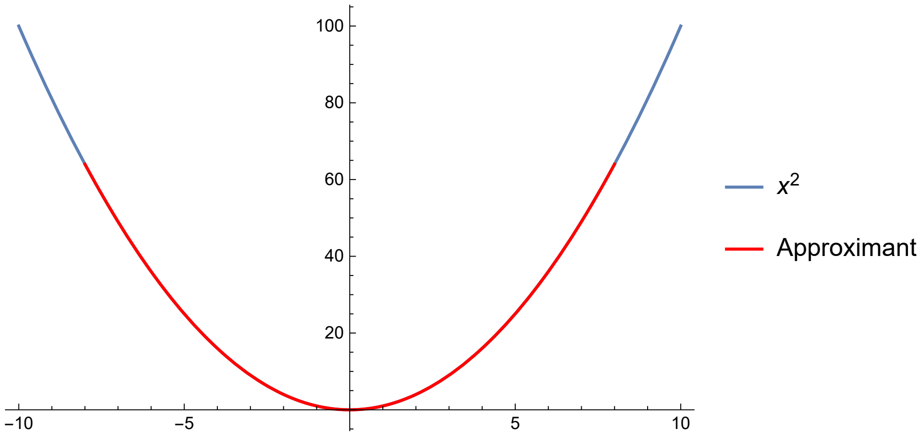
```
approximant[fun_, n_] :=
 Sum[N@BMSchebyshev2[k] * aCoefficient[fun, k], {k, 0, n}] // FullSimplify
```

Test 1

```
polynomial = approximant[x^2, 2]
```

$0. + 1.\ x^2$

```
Show[{Plot[x^2, {x, -10, 10}, PlotLegends → {"x^2"}],
  Plot[polynomial /. x → k, {k, -8, 8}, PlotStyle → Red, PlotLegends → {"Approximant"}]}]
```

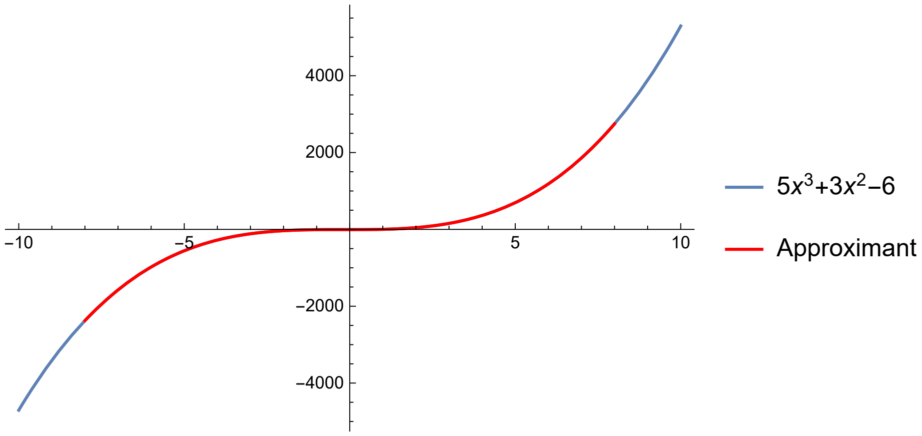


Test 2

```
polynomial2 = approximant[5 x^3 + 3 x^2 - 6, 3]
```

$-6. + x^2\ (3. + 5.\ x)$

```
Show[{Plot[5 x^3 + 3 x^2 - 6, {x, -10, 10}, PlotLegends → {"5x^3+3x^2-6"}], Plot[
   polynomial2 /. x → k, {k, -8, 8}, PlotStyle → Red, PlotLegends → {"Approximant"}]}]
```

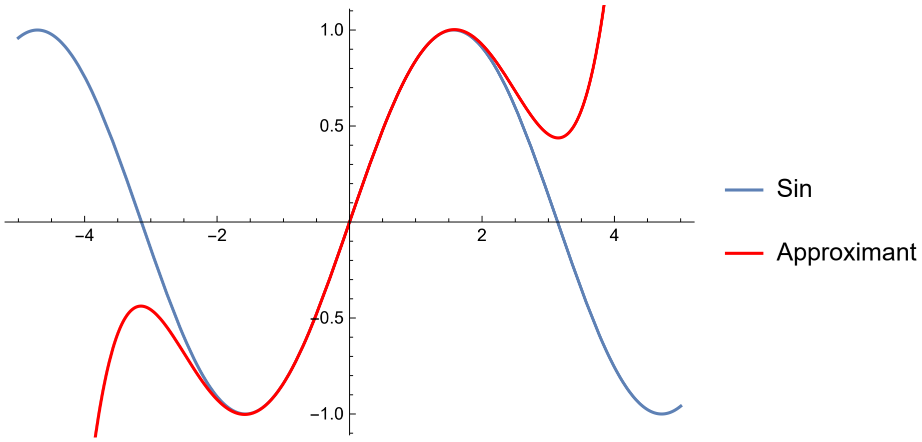


Test 3

```
polynomial3 = approximant[Sin[x], 5]
```

$0.999988\,x - 0.166545\,x^3 + 0.00804032\,x^5$

```
Show[{Plot[Sin[x], {x, -5, 5}, PlotLegends → {"Sin"}], Plot[polynomial3 /. x → k,
   {k, -5, 5}, PlotStyle → Red, PlotLegends → {"Approximant"}]}]
```



Recursion

```
BMSCheb[0] = Sqrt[2/Pi];

BMSCheb[1] = 2 x Sqrt[2/Pi];

BMSCheb[n_] := (2 x BMSCheb[n - 1] - BMSCheb[n - 2]) Sqrt[2/Pi]

aCoefficient2[fun_, n_] := ∫_{-1}^{1} fun * Sqrt[1 - x^2] * BMSCheb[n] dx

approximant2[fun_, n_] :=
 Sum[N@BMSCheb[k] * aCoefficient2[fun, k], {k, 0, n}] // FullSimplify

pol2 = approximant2[5 x^3 + 3 x^2 - 6, 3]
```

$-5.72746 + x\,(1.9368 + (1.90986 + 0.99978\,x)\,x)$

```
Show[{Plot[5 x^3 + 3 x^2 - 6, {x, -8, 8}, PlotLegends → {"Sin"}],
  Plot[pol2 /. x → k, {k, -5, 5}, PlotStyle → Red, PlotLegends → {"Approximant"}]}]
```