Faculty Development Program on
NETWORK SCIENCE:
FOUNDATION OF SOCIAL NETWORK ANALYSIS

Centrality in Networks

Module 2

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Outline

1. Node Centrality
2. Classification of Node Centrality Measures
3. Degree based Centrality Measures
4. Flow based Centrality Measures
5. Centrality Measures for Directed Networks

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- In terms of flow of information
    - Passes thru?
    - Reachability?
- Importance of a node depends on context
    - Spread of information, brokerage, opportunities
- Used in sociology to study notion of *power / influence / control*

Formal Definition of Centrality

- Function $C : V \longrightarrow \mathcal{R}^+$ that induces a total order on V
- Higher the value of centrality, more important is the node in the network
- A node v_i is more central than v_j iff $C(v_i) > C(v_j)$

Applications of Centrality

- Information cascading in the network
- Preventing spread of malicious messages
- Detecting the most influential person in network
- Heavily used junctions in a transportation network
- Ranking of web pages
- Planning for a facility location in a city
- Spreading awareness about government policies
- Viral marketing to increase brand awareness

No consensus on Centrality till date!!!

"There is certainly no unanimity on exactly what centrality is or on its conceptual foundations, and there is little agreement on the proper procedure for its measurement."

Freeman(1979)

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Will learn about different centrality measures in the following slides!!!

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2. **Classification of Node Centrality Measures**
3. Degree based Centrality Measures
4. Flow based Centrality Measures
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Classification of Node Centrality Measures

- Degree based measures
  - Consider topology of neighborhood
- Flow based measures
  - Consider shortest path between nodes
- Measures for directed graphs
  - Consider direction of edges

Outline

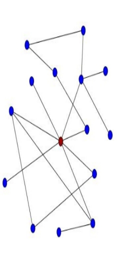
Degree based Centrality Measures

1. Degree centrality
2. Eigenvector centrality

Degree Centrality (DC)

Higher the degree, higher is the importance of node

1. Local measure
2. Considers one hop connections
3. Captures direct influence on neighbors
4. Quantifies favors from neighbors



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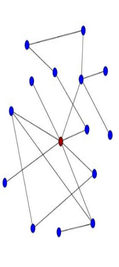

Applications

- **Identifying influential actors in social networks**
- **Finding top trading companies in economic networks**
- **Hubs in computer networks**
- **Identifying super spreaders in epidemics**

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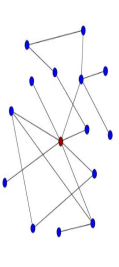

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Each link is equally important in an undirected graph

Formal Definition of Degree Centrality (DC)

Degree Centrality - based on count of neighbours

$\mathcal{DC}(v_i) = d_i$ **where** d_i **is the degree of** v_i

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Normalized Degree Centrality - useful for comparing central nodes of different sized networks

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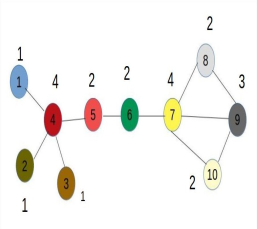
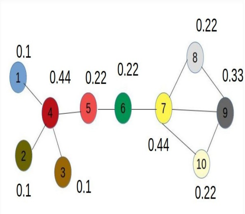
Degree Centrality



Normalized Degree Centrality



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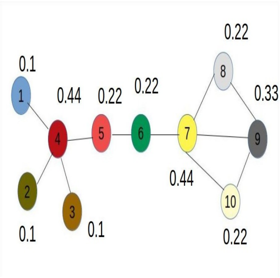

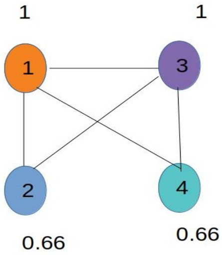

Limitations of DC

1. **Does not consider topology of entire network**
2. **Higher degree nodes may not transmit information to the entire network**
3. **High centrality nodes may not receive information from distant nodes in the network**

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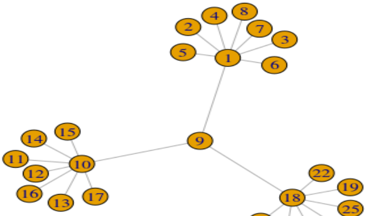
Which is the most central node??



Eigenvector Centrality (EVC)

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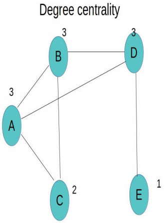
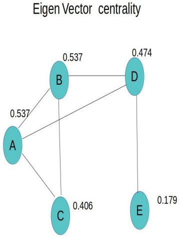
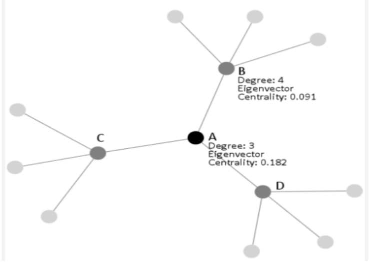




More examples ..



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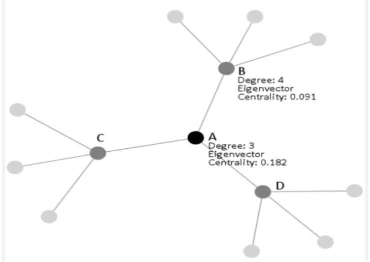

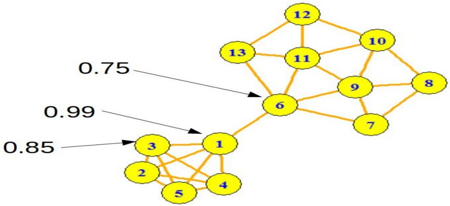

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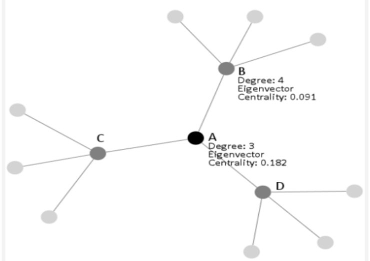
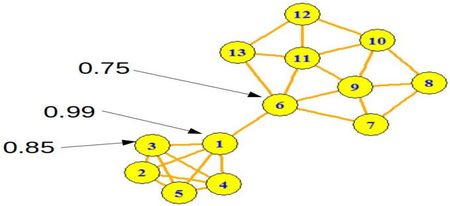
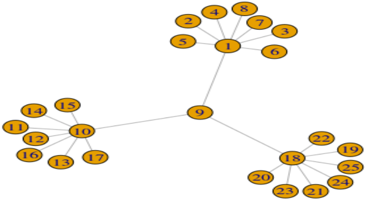




High degree does not imply high EVC

Effect of neighbours's importance on the node



EVC of the selected nodes

NODE	1	5	9	10	11	18	19
EVC	**1**	0.31	**0.94**	**1**	0.31	**1**	0.31
DC	**8**	1	3	**8**	3	**8**	1
NDC	**0.33**	0.042	0.13	**0.33**	0.042	**0.33**	0.042

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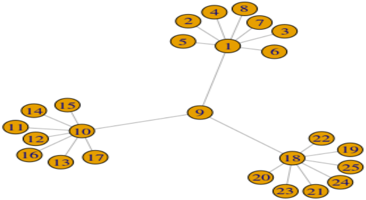

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Affiliation with important nodes increases the importance of the node itself

Applications of EVC

- A famous individualś cook
- A twitter account followed by someone with huge following
- An entrepreneur who knows Sundar Pichai
- A researcher who is coauthoring with prolific researcher

A node with high EVC has more connections as well as more important connections

About Eigenvectors and Eigenvalues

Origin in Physics to solve motion related problems

Applied in differential equation, quantum mechanics ...

Eigenvectors are used for understanding linear transformations

Eigenvectors are the axes (directions) along which a linear transformation is simplified by stretching/compressing/flipping

Eigenvalues are the factors by which this compression occurs i.e. change in length of the eigenvector from the original length

A number is the eigenvalue for a square matrix $\mathcal{A}$ if and only if there exists a nonzero vector X such that $\mathcal{A}X = \lambda X$ where X is vector, λ is a scalar value (number)

Here λ is the eigenvalue and X is the eigenvector.

Formal definition of EVC

Given a network with N nodes and adjacency matrix $\mathcal{A}$, Eigenvector Centrality ($\mathcal{E}(v_i)$) of node v_i is:

$$\mathcal{E}(v_i) = \frac{1}{\lambda} \sum_{j=1}^{N} \mathcal{A}_{ij}\mathcal{E}(v_j)$$

where λ is an eigenvalue and $\mathcal{E}$ is an eigenvector.

In matrix notation $\mathcal{E}' = \mathcal{A}\mathcal{E}$

Involves repetitive matrix computation for better estimates of Eigenvector $\mathcal{E}$ as:

$$\mathcal{E}^t = (\mathcal{A})^t \mathcal{E}^0$$

where $\mathcal{E}^0$ is the initial Eigenvector.

EVC continued..

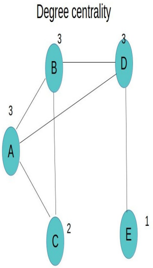
- **Eigenvector centrality is $\mathcal{E}^t = (\mathcal{A})^t \mathcal{E}^0$**
- **Expensive computations of repeated matrix operations**
- **Fastest known matrix multiplication algorithm is super-quadratic**

- Known result: With increasing t, vector $\mathcal{E}^t$ converges to dominant eigenvector of $\mathcal{A}^t$.
- Use *Power iteration method* to compute $\mathcal{E}^t$

Power iteration Method: Dominant Eigenvector

1. $t \leftarrow 0$
2. $\mathcal{E}^0 \leftarrow 1$ //All nodes are of equal importance
3. Repeat
   1. $t \leftarrow t + 1$ //next iteration
   2. $\mathcal{E}^t \leftarrow \mathcal{A}\mathcal{E}^{t-1}$ //Eigenvector estimate
   3. $i \leftarrow \arg\max_j \{\mathcal{E}^t[j]\}$ // maximum value index
   4. $\lambda = \frac{\mathcal{E}^t[i]}{\mathcal{E}^{t-1}[i]}$ //Eigen Value estimate
   5. $\mathcal{E}^t = \frac{1}{\mathcal{E}^t[i]}\mathcal{E}^t$//scale vector
4. until $||\mathcal{E}^t - \mathcal{E}^{t-1} \leq \epsilon||$
5. $\mathcal{E} = \frac{1}{||\mathcal{E}^t||}\mathcal{E}^t$ // Normalize Eigenvector

Example showing computations for EVC ($\mathcal{E}$)



$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathcal{E}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathcal{E}^1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0.66 \\ 1 \\ 0.33 \end{bmatrix}$$

(scaling by $\lambda = 3$)

continuing...

$$\mathcal{E}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0.66 \\ 1 \\ 0.33 \end{bmatrix} = \begin{bmatrix} 2.66 \\ 2.66 \\ 2 \\ 2.33 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0.75 \\ 0.86 \\ 0.38 \end{bmatrix}$$

(scaling by $\lambda = 2.66$)

$\vdots$

$$\mathcal{E}^5 = \begin{bmatrix} 2.62 \\ 2.62 \\ 2 \\ 2.34 \\ 0.87 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0.76 \\ 0.89 \\ 0.33 \end{bmatrix} \rightarrow \begin{bmatrix} 0.537 \\ 0.537 \\ 0.40 \\ 0.47 \\ 0.18 \end{bmatrix}$$

(scaling by $\lambda = 2.62$)
at converging point

Normalizing Eigenvector
NET EVC

ONLINE Calculator for EVC

Use this link for online computation of EVC
http://comnuan.com/cmnn01002/

Limitations of EVC

- Works best for undirected connected networks

- Directed network has asymmetric adjacency matrix
  - Which of the two leading eigenvectors (left/right)?

- Nodes with no incoming links have centrality value 0
- Nodes with 0 EVC do not contribute to the importance of the neighbouring nodes

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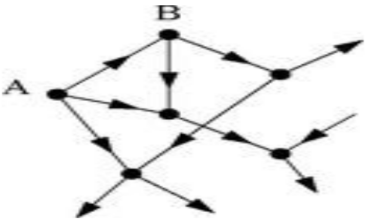

Figure: Eigenvector centrality in directed network.

Fig. Source: Ref. [1]

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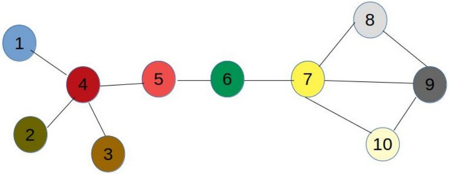
3. Degree based Centrality Measures
4. **Flow based Centrality Measures**
5. Centrality Measures for Directed Networks

Flow based Centrality Measures

Consider shortest paths on which a node lies

1. Consider all shortest paths w.r.t all the nodes
2. Consider the topological structure of the entire network
3. Capture influence of all neighbours which are connected directly or indirectly



Flow based Centrality

1. Eccentricity centrality
2. Closeness Centrality
3. Decay Centrality
4. Betweeness Centrality
5. Spanning tree Centrality

Eccentricity Centrality

Recall that eccentricity $e(v_i)$ is the maximum distance of node from any other node in the network

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Lesser the eccentricity of a node, more central it is

$e(v_i) = \mathbf{max}(D(v_i, v_j))\ \forall j$

Eccentricity Centrality of node v_i is

$\mathcal{EC}(v_i) = \frac{1}{e(v_i)}$

- **Nodes in center region of** G have low eccentricity, i.e. $e(v_i) \approx r(G)$
- **Nodes in periphery** typically have high eccentricity

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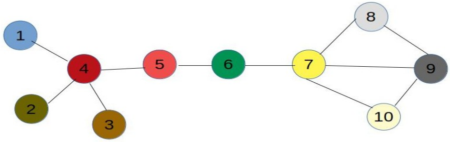
- **Nodes in center region of** G have low eccentricity, i.e. $e(v_i) \approx r(G)$
- **Nodes in periphery** typically have high eccentricity

Higher the eccentricity, lower is the rate of diffusion in information spreading

Useful in applications where requirement is to minimize the maximum distance to any node E.g. Location of a hospital

Computation of Eccentricity Centrality



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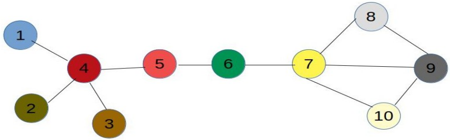
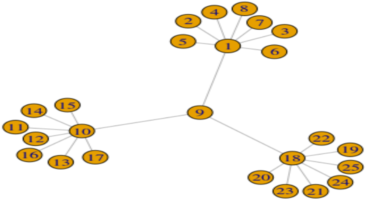


SHORTEST DISTANCE

Vertex	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10
V1	0	2	2	1	2	3	4	5	5	5
V2	2	0	2	1	2	3	4	5	5	5
V3	2	2	0	1	2	3	4	5	5	5
V4	1	1	1	0	1	2	3	4	4	4
V5	2	2	2	1	0	1	2	3	3	3
V6	3	3	3	2	1	0	1	2	2	2
V7	4	4	4	3	2	1	0	1	1	1
V8	5	5	5	4	3	2	1	0	1	0
V9	5	5	5	4	3	2	1	1	0	1
V10	5	5	5	4	3	2	1	2	1	2

MAX DIST	EC
5	0.20
5	0.20
5	0.20
4	0.25
3	**0.33**
3	**0.33**
4	0.25
5	0.20
5	0.20
5	0.20

Effect of the maximum distance on EC



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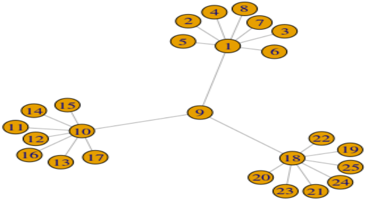

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Smaller the distance from the farthest node, higher is the importance of the node itself

Closeness Centrality

Extends degree centrality by looking at neighborhoods of all sizes

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Captures closeness with rest of the nodes in the network

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Extends degree centrality by looking at neighborhoods of all sizes

Captures closeness with rest of the nodes in the network

Far-*ness* is the sum of distances of v_i from other nodes
Closeness is the inverse of far-*ness* and is computed as:

$$\mathcal{CC}(v_i) = \frac{1}{\Sigma_{\forall j} D(v_i, v_j)}$$

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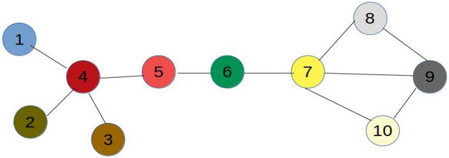
Useful for efficient information transmission

Applications of Closeness Centrality

1. Fast access to information
2. Planning for immunization strategy for controlling spread of infectious disease
3. Adoption of new technology by spreading its benefits
4. Deciding for location of a facility e.g. shopping complex

Computations for Closeness Centrality



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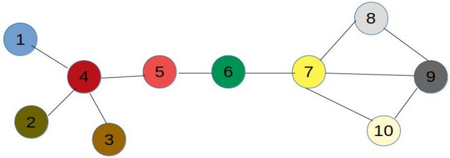

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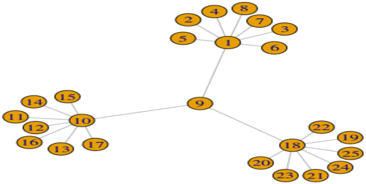
V1	0	2	2	1	2	3	4	5	5	5
V2	2	0	2	1	2	3	4	5	5	5
V3	2	2	0	1	2	3	4	5	5	5
V4	1	1	1	0	1	2	3	4	4	4
V5	2	2	2	1	0	1	2	3	3	3
V6	3	3	3	2	1	0	1	2	2	2
V7	4	4	4	3	2	1	0	1	1	1
V8	5	5	5	4	3	2	1	0	1	0
V9	5	5	5	4	3	2	1	1	0	1
V10	5	5	5	4	3	2	1	2	1	2

SUM DIST	CC
29	0.034
29	0.034
29	0.034
21	0.048
19	0.053
19	0.053
21	0.048
26	0.038
27	0.037
30	0.033

Effect of reachability from all nodes



Closeness Centrality of few selected nodes

NODE	1	5	9	10	11	18	19
CC	0.018	0.012	**0.022**	0.018	0.012	0.018	0.012
DC	**8**	1	3	**8**	3	**8**	1
EVC	**1**	0.31	0.94	**1**	0.31	**1**	0.31

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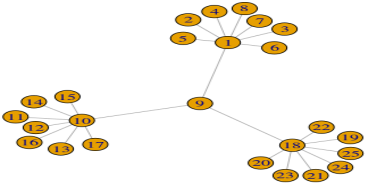

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DC	**8**	1	3	**8**	3	**8**	1
EVC	**1**	0.31	0.94	**1**	0.31	**1**	0.31

Smaller the total distance from all nodes, higher is the reachability of the node

Decay Centrality

Extension of closeness centrality

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Information traveling along paths decays over time

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Accounts for closeness, but with decayed weightage to the distant nodes

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Decay Centrality of node v_i is
$\mathcal{DEC}^{\delta}(v_i) = \Sigma_{h \leq n-1} \delta^h n_i^h$
where h: number of hops, δ: decay parameter, n_i^h: number of nodes at distance h from v_i

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Decay parameter δ controls the importance of nodes at distance h from v_i
As $\delta \to 1$, decay centrality measures the size of the component in which the node lies
As $\delta \to 0$, decay centrality becomes degree centrality

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As $\delta \rightarrow 0$, decay centrality becomes degree centrality

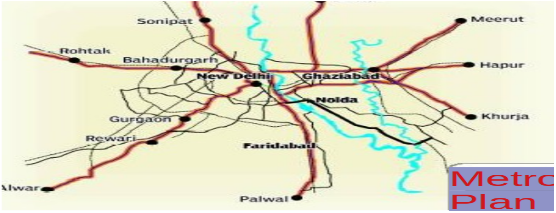
1. Identifying source for funds distribution
2. Monitoring the impact of innovation from the origin to nearby regions

Betweenness Centrality

Considers how many shortest paths passes thru the node

Useful in capturing bridging connections

Connectivity route to NCR Towns under RRTS,Source: Times of India



Captures the ability of the node to control the spread of information flowing through the network.

High betweenness centrality means high control on information

Betweenness using Toy Network

- X and V lies between no two other vertices
- Y lies between 3 pairs of vertices (X,Z), (X,U) and (X,V)
- Z lies between 4 pairs of vertices (X,U),(X,V),(Y,U),(Y,V)

Hence C betweeness is high!!

Formal Definition of Betweenness centrality

Let η_{jk} be the number of shortest paths between nodes j and k

Let $\eta_{jk}(v_i)$ be the number of shortest paths between nodes j and k passing thru v_i.

Then, fraction of paths thru v_i is

$$\gamma_{jk}(v_i) = \frac{\eta_{jk}(v_i)}{\eta_{jk}}$$

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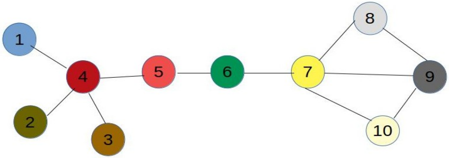
Betweenness Centrality $\mathcal{BC}(v_i)$ is:

$$\sum_{j \neq i} \sum_{k \neq i, k > j} \gamma_{jk}(v_i)$$

$$= \sum_{j \neq i} \sum_{k \neq i, k > j} \frac{\eta_{jk}(v_i)}{\eta_{jk}}$$

Computation of Betweenness Centrality



Number of shortest path

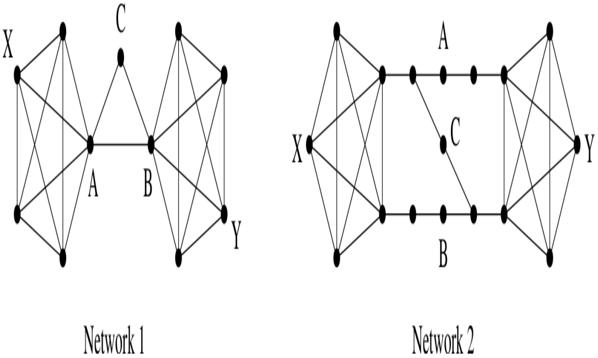
Vertex	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10
V1	0	1	1	1	1	1	1	1	1	1
V2	1	0	1	1	1	1	1	1	1	1
V3	1	1	0	1	1	1	1	1	1	1
V4	1	1	1	0	1	1	1	1	1	1
V5	1	1	1	1	0	1	1	1	1	1
V6	1	1	1	1	1	0	1	1	1	1
V7	1	1	1	1	1	1	0	1	1	1
V8	1	1	1	1	1	1	1	0	1	2
V9	1	1	1	1	1	1	1	1	0	1
V10	1	1	1	1	1	1	1	2	1	0

BC
0
0
0
21
20
20
18.5
0
0.5
0

Applications of Betweeness Centrality

1. Control over the information flowing in the network
2. Positioning of server and scheduling its maintenance activities in the communication network
3. Detecting dominant pathways in chemical network for computational diagnostics
4. Positioning of junction in the transportation network

Identify nodes with High Betweenness!!



Vertices A and B will have high betweenness as per geodesic paths but vertex C will not!!

Ref [1]

Variations of Betweenness Centrality

Flow Betweeness

- Spreading of news/rumor/message need not to be through shortest (geodesic) paths but wander around more randomly
- Need to include non-geodesic paths in addition to geodesic paths to maximize the spread

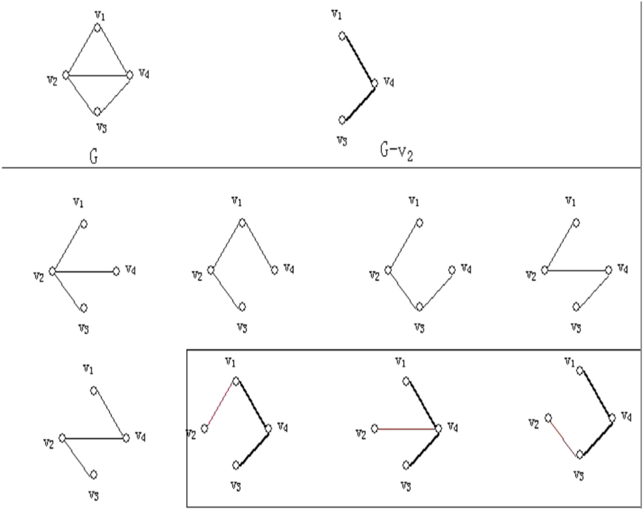
Random Walk Betweeness

- Delivery of a message that originated at some source follows random path to reach the destination
- Need to include random walks originating at source and ending up at destination

Spanning Tree centrality

- Recall that spanning trees model the reach of a network
- Spanning tree centrality (STC) of a vertex measures the role of the vertex in keeping network connected.
- Used for identifying vulnerable nodes.
- Assigns importance to a node on the basis of number of times the node appears as a cut vertex

Role of Cut Vertex V_2



Total 8 spanning tree out which vertex V_2 acts as cut vertex in 5.

Fig. source: [5]

Applications of STC

- To maintain connectivity in the network of power grids against physical attacks and natural disasters
- To have uninterrupted message flowing in a communication network
- To keep a social network integrated

Outline

Centrality Measures for Directed Networks

Direction of links matters while computing centrality

1. Citation network
   - Mentioning in a well cited article
2. Worldwide web
   - Stature of incoming link matters
3. Transportation routing network
   - Congestion control

Prestige

Also called eigenvector centrality for directed network

Used to measure the importance a node in directed network
Depends on the indegree, and recursively on the prestige of the nodes that point to it

Formal Definition for Prestige Score

If $p(v)$ is a positive real number, indicating the prestige score for node v, and A the adjacency matrix, then $p(v)$ is given by

$p(v) = \Sigma_u \mathcal{A}(u, v) * p(u)$
Equivalently... $p(v) = \Sigma_u \mathcal{A}^{\mathcal{T}}(v, u) * p(u)$

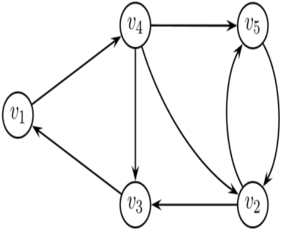
Prestige score at k^{th} iteration is $p_k = (\mathcal{A}^{\mathcal{T}})^k * p_0$, where p_0 is the initial prestige

- With increasing k, p_k converges to dominant eigenvector of $\mathcal{A}^{\mathcal{T}}$ as in EVC for undirected network
- Use Power iteration method used to compute p_k as done for EVC

Power iteration Method: Prestige score

1. $t \leftarrow 0$
2. $\mathcal{P}^0 \leftarrow 1$ //All nodes are of equal importance
3. Repeat
   1. $t \leftarrow t + 1$ //next iteration
   2. $\mathcal{P}^t \leftarrow \mathcal{A}^T \mathcal{P}^{t-1}$ //Eigenvector estimate
   3. $i \leftarrow \arg\max_j \{\mathcal{P}^t[j]\}$ // maximum value index
   4. $\lambda = \frac{\mathcal{P}^t[i]}{\mathcal{P}^{t-1}[i]}$ //Eigen Value estimate
   5. $\mathcal{P}^t = \frac{1}{\mathcal{P}^t[i]} \mathcal{P}^t$//scale Prestige vector
4. until $||\mathcal{P}^t - \mathcal{P}^{t-1} \leq \epsilon||$
5. $\mathcal{P} = \frac{1}{||\mathcal{P}^t||} \mathcal{P}^t$ // Normalize Prestige vector to get net score

Computations of Prestige Score ($\mathcal{P}$)



$$\mathcal{P}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{A}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathcal{P}^1 = A^T \mathcal{P}^0$$

$$\mathcal{P}^1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ 0.5 \\ 1 \end{bmatrix}$$

(scaling by $\lambda = 2$)

continuing...

$$\mathcal{P}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 1.5 \\ 0.5 \\ 1.5 \end{bmatrix} \rightarrow \begin{bmatrix} 0.67 \\ 1 \\ 1 \\ 0.33 \\ 1 \end{bmatrix}$$

(scaling by $\lambda = 1.5$)

$\vdots$

$$\mathcal{P}^7 = \begin{bmatrix} 1 \\ 1.46 \\ 1.46 \\ 0.69 \\ 1.46 \end{bmatrix} \rightarrow \begin{bmatrix} 0.68 \\ 1 \\ 1 \\ 0.47 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0.356 \\ 0.521 \\ 0.521 \\ 0.243 \\ 0.521 \end{bmatrix}$$

(scaling by $\lambda = 1.462$) at converging point

Normalizing Eigenvector by 1.9191 NET Prestige Score

Limitations of Prestige

- Prestige works well only if the directed network is strongly connected
- Nodes with no incoming edges are assigned null centrality score
- A node with high in-degree will have centrality zero if centrality for incoming nodes is zero

Solution is....

Katz Centrality

Assigns each node a small amount of centrality for free, regardless of its position in the network

Each node contributes to the importance of other nodes if referring to them

Makes a node important if it is linked from other important nodes or if it is highly linked

Formal definition of Katz Centrality

Katz centrality $\mathcal{K}_i$ of node v_i is

$$\mathcal{K}_i = \alpha \sum_j \mathcal{A}_{j,i} \mathcal{K}_j + \beta$$

where α and β are constants.

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In matrix form :

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where
α: assigns weight-age as per path length since endorsements devalue over long chains of links
β: a vector of elements having same positive values

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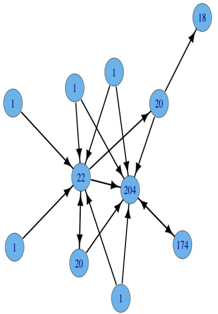
β: a vector of elements having same positive values

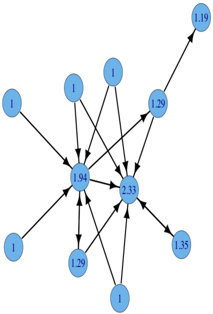
Centrality value is the sum of two components
- **An endogenous component ($\alpha x \mathcal{A}$) that considers network topology**
- **An exogenous component (β) that is independent of the network structure.**

Examples of Katz Centrality

$\alpha = 0.85$ and $\beta = 1$



$\alpha = 0.15$ and $\beta = 1$



(Source: https://www.sci.unich.it/ francesc/teaching/network/katz.html)

Applications of Katz Centrality

- **Each node contributes to the importance of others**
  –Each author is important in co-authorship network

- **Assigns a high score to a node with many neighbours regardless of their position in undirected network**
  –A hub node in communication network

PageRank

Used for measuring the importance of website pages

Used by the Google web search engine to rank websites

Named after Larry Page, one of the founders of Google

Assigns a numerical value to each node (web page) of a hyperlinked set of documents (www) that represents its relative importance within the set

Working of PageRank

Like prestige, the pagerank of a node v, recursively depends on the pagerank of other nodes that point to it

Based on normalized prestige combined with a random jump/walk assumption

Random walk on the web graph

1. Pick a page at random
2. With probability 1- α follow an outgoing link page
3. With probability α jump to a random page from current page i.e. random surfing

Fig. source: https://en.wikipedia.org/wiki/PageRank

About the PageRank Algorithm

- Let α be the probability that the web surfer jumps from the current node u to any other random node v

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•Let α be the probability that the web surfer jumps from the current node *u* to any other random node *v*

• *Then* $(1 - \alpha)$ *is the probability that the user follows an existing link from u*

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• If $od(u)$ is the number of out-links on a page, then $\frac{1}{od(u)}$ is the probability of following an out-link from that page *u*

Adjacency matrix must account for these probabilistic actions while surfing the web

Following the Outgoing Hyperlink

1. Start with an initial pagerank $p_0(u)$ for a node u, such that $\sum_u p_0(u) = 1$
   $p_0 = [\frac{1}{n} \cdots \frac{1}{n}]$

2. Compute initial pagerank of node v while taking into account probability of following a hyperlink from node u is
   $p_0(v) = \sum_u \frac{A(uv)}{od(u)} p_0(u)$
   $= \sum_u \frac{A^T(vu)}{od(u)} p_0(u) = \sum_u N^T(vu) p_0(u)$

   where N is the normalized adjacency matrix of the graph

3. Pagerank vector is computed for following outgoing link as $p = N^T * p$

Random Surfing Component

1. Account for random jumps by considering all are adjacent to each other.e. use $N \times N$ matrix of 1's

$$A_r = 1_{N\times N} = \begin{bmatrix} 1 & 1 & \dots & & 1 \\ 1 & & \dots & & 1 \\ & & & \dots & \\ & & \dots & & \\ 1 & 1 & & 1 & 1 \end{bmatrix}$$

2. As the probability of jumping to any of the N nodes is equal, then considering the out-degree of node u as $od(u) = n, \forall u$,

   Probability of jumping from u to any node v is $1/n$

Accounting for random jump, pagerank vector must use this normalized adjacency matrix; Thus

$p = N_r^T * p$, where $N_r = \frac{1}{n} A_r$

Net PageRank Score

- Random Surfing N_r with probability α
- Linked Surfing N with probability $(1-\alpha)$
- Final adjacency matrix for computing pagerank is probabilistic and is

  $M = (\alpha N_r + (1-\alpha)N)$ where α is the small probability for random surfing

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Pagerank algorithm transforms deterministic graph to probabilistic graph (with adjacency matrix M) to simulate behaviour of web surfers

Net PageRank Score

Pagerank is eigenvector centrality of this probabilistic graph
Formally, $p = M^T p$
=$(\alpha N_r^T + (1-\alpha)N^T)p$
$= \alpha N_r^T * p + (1-\alpha)N^T * p$

The final pagerank vector depends upon normalized pagerank vector (using out-links) and random jump vector (random surfing).

- **If there is no outgoing edge from** u **then** $od(u) = 0$ **the only choice is to simply jump to another random node, i.e.** $\alpha = 1$
- **Row corresponding to that node is set in M to**

$[1/n \quad \ldots \quad\quad \ldots \quad 1/n]$

Compute the dominant eigenvector of M^T using power iteration method to obtain pagerank as done for PRESTIGE

Hyperlink Induced Topic Search (HITS) - algorithm to rank web pages

Pagerank of a node is a global value computed using pages traversed irrespective of query posed by the user.

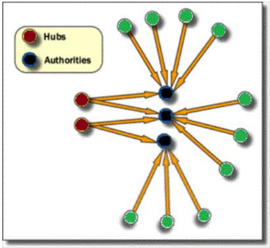
Need is to have query-specific pagerank or prestige of a page

Hyperlink Induced Topic Search - algorithm to rank web pages

- Based on concept of HUB and AUTHORITY
- A good *Hub* represents a page that points to many other pages, and a good *Authority* represents a page that is linked by many different hubs
- Hub score (h_u): indicates to how many pages of importance does the page point to
- Authority score (a_u): indicates how many good pages (high rank) point to it

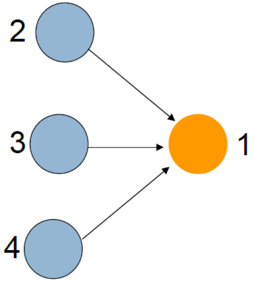
Page with high authority has many pages with high hub score pointing to it

Page with high hub score points to many pages that have high authority

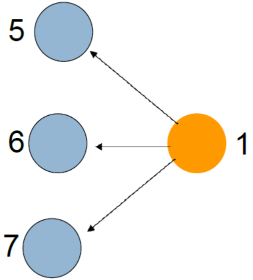


HUB and AUTHORITY

$$a(1) = h(2) + h(3) + h(4)$$



$$h(1) = a(5) + a(6) + a(7)$$



HITS Algorithm: Input - Query q, Output - Ranked pages for q

1. Set S = All pages relevant to q
2. S = S $\cup$ pages pointing to those in S
   incoming links to pages in S
3. S = S $\cup$ pages pointed by those in S *outgoing links from pages in S*
4. Eliminate pages originating from the same host
5. Compute Hub and Authority scores

Computations for Hub and Authority score

- Hub scores are influenced by authority scores of *heads* of outgoing edges, i.e. $h(v) = \sum_u A(v, u)a(u)$
- Authority scores are influenced by hub scores of *tails* of incoming edges, i.e. $a(v) = \sum_u A^T(v, u)h(u)$
- In matrix notation $a' = A^T h$ and $h' = Aa$
- Writing recursively
  $a_k = (A^T)h_{k-1} = A^T(Aa_{(k-1)}) = (A^T A)a_{(k-1)}$ and
  $h_k = Aa_{k-1} = A(A^T h_{k-1}) = (AA^T)h_{k-1}$
- As $k \to \infty$ Authority score converges to the dominant eigenvector of $A^T A$, whereas the hub score converges to the dominant eigenvector of AA^T

Algorithm for HITS

1. Starting with an initial authority vector a (all ones), compute vector $h = Aa$
2. Compute $a = A^T h$ to complete one iteration
3. Iterate until both a and h converge

SUMMARY

1. **Selection of a centrality measure is context specific**
2. **Select centrality measure suitable for undirected/directed network**
3. **Choose local measure in case neighborhood topology is to be considered else those measures which consider entire network structure**

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THANKS!!