FDP: NETWORK SCIENCE

Graph Models II

6 December 2018

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Outline

1. **Introduction**
2. **Watts-Strogatz Model**
   Properties of WS Graphs
3. **Barabasi-Albert Model**
   Properties of BA Graphs

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	Cellular Network	**WWW**
Nodes	proteins or metabolites	documents
Links	chemical reactions	hyperlinks
Purpose	production of chemicals cells needed to survive	information access and delivery
History	4 billion years of evolution	few decades old

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  - Random graph model fails to exhibit a high clustering coefficient, but it is small-world
  - WS model tries to explicitly model high local clustering
- Barabási-Albert (BA) Model [BA99]
  - BA model tries to capture the scale-free degree distributions of real-world graphs via a generative process

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Watts-Strogatz Small-world Graph Model [WS98]

- Model starts with a *regular graph* of degree $2k$ with n nodes arranged in a circular layout
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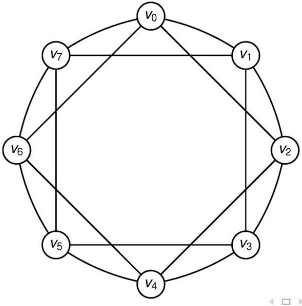
Watts–Strogatz Regular Graph: $n = 8$, $k = 2$



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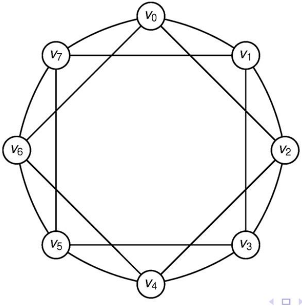

- Regular Graph has high clustering coefficient
- Surprisingly, adding a small amount of randomness leads to emergence of small-world phenomenon

Image courtesy of [ZJ14]

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- Node r_i has edges to $k - i$ of its immediate neighbours and to $k - 1$ of its left neighbours
- Due to the symmetry about v, a node l_i that is at a distance of i backbone hops from v to the left has the same number of edges
- Degree of any node that is i backbone hops away from v is given as

$$d_i = (k - i) + (k - l) = 2k - i - l \tag{2}$$

WS Regular Graph: Clustering Coefficient

- Each edge contributes to the degree of its two incident nodes, summing the degrees of all neighbours of v

$$2m_v = 2\left(\sum_{i=1}^{k} 2k - i - 1\right)$$

$$m_v = 2k^2 - \frac{k(k+1)}{2} - k$$

$$m_v = \frac{3}{2}k(k-1) \quad (3)$$

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$$\Rightarrow C_v = \frac{m_v}{M_v} = \frac{3k-3}{4k-2} \quad (5)$$

WS Regular Graph: Clustering Coefficient

- Clustering coefficient of node v given by

$$C(v) = \frac{m_v}{M_v} = \frac{3k-3}{4k-2} \tag{6}$$

- As k increases, clustering coefficient approaches $\frac{3}{4}$ because $C(G) = C(v) \to \frac{3}{4}$ as $k \to \infty$
- WS graph has high clustering coefficient

WS Regular Graph: Diameter

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- Since each node is connected to k neighbours on either side, furthest node reachable in at most $\frac{n/2}{k}$ hops

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- WS Regular Graph is not small-world

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- **Edge Rewiring:**
  - For each edge (u, v) , with probability r, replace v with another randomly chosen node avoiding loops and duplicate edges
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- **Edge Shortcuts:**
  - Add a *few* edges between random pairs of nodes, with probability r, per edge, of adding a successful edge
  - Total number of random shortcut edges added to the network are $mr = knr$
  - Total number of edges in the graph is $m + mr = (1 + r)m = (1 + r)kn$

Watts-Strogatz Graph: Shortcut Edges

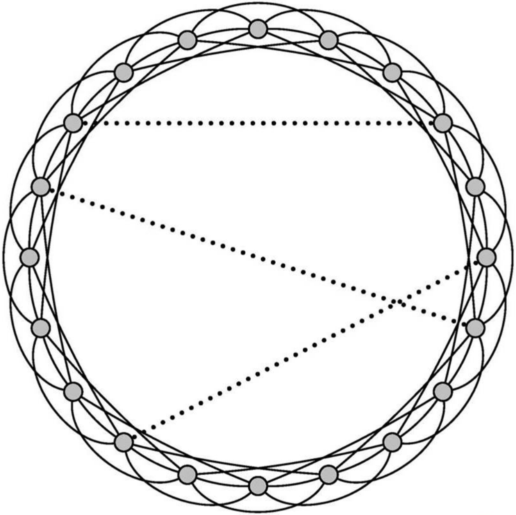


Image courtesy of [ZJ14]

n = 20, k = 3

Watts Strogatz Graphs: Degree Distribution

- Consider the shortcut approach, each vertex has degree at least $2k$
- Additionaly, the shortcut edge follow a Binomial distribution
- Each node can have $n' = n - 2k - 1$ additional shortcut edges, so we take n' as the number of independent trials to add edges
- Since a node has degree $2k$, with shortcut edge probability of r, we expect roughly $2kr$ shortcuts from that node, but the node can connect to at most $n - 2k - 1$ other nodes
- $\Rightarrow$ probability of success is

$$p = \frac{2kr}{n - 2k - 1} = \frac{2kr}{n'} \tag{7}$$

Watts Strogatz Graphs: Degree Distribution

- Let X denote random variable denoting number of shortcuts for each node
- Probability of a node with j shortcut edges is given as

$$f(j) = P(X = j) = \binom{n'}{j} p^j (1-p)^{n^j - j} \quad (8)$$

  - with $E[X] = n'p = 2kr$ and $p = \frac{2kr}{n-2k-1} = \frac{2kr}{n'}$
- Therefore, expected degree of each node in the network is

$$2k + E[X] = 2k + 2kr = 2k(1+r) \quad (9)$$

- Clear that the degree distribution of the WS graph does not adhere to a powerlaw
- WS networks are not scale-free

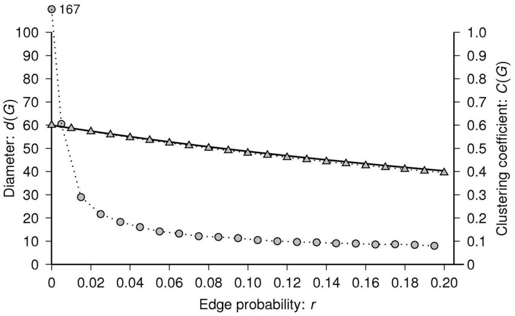
Watts Strogatz Graphs: Clustering Coefficient and Diameter

- Clustering Coefficient:

$$C(v) \approx \frac{3(k-1)}{(1+r)(4kr+2(2k-1))} = \frac{3k-3}{4k-2+2r(2kr+4k-1)} \quad (10)$$

- Thus for small values of r clustering coefficient remains high
- Diameter:
- Small values of shortcut edge probability r are enough to reduce the diameter from $O(n)$ to $O(logn)$

Watts-Strogatz Model: Diameter(circles) and Clustering Coefficient(triangles)



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- BA model yields a scale-free degree distribution based on *preferential attachment*

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- **Initialization:** BA model starts with G_0, with each node connected to its left and right neighbors in a circular layout. Thus $m_0 = n_0$
- **Growth and Preferential Attachment:** BA model derives a new graph G_{t+1} from G_t by adding exactly one new node u and adding $q \leq n_0$ new edges from u to q distinct nodes $v_i \in G_t$, where node v_j is chosen with probability $\pi_t(v_j)$ proportional to its degree in G_t, given as

$$\pi_i(v_t) = \frac{d_j}{\sum\limits_{v_j \in G_t} d_j} \qquad (11)$$

Example: BA Model
$n_0 = 3, q = 2, t = 12$

At $t = 0$, start with 3 vertices v_0, v_1, and v_2 fully connected (shown in gray).

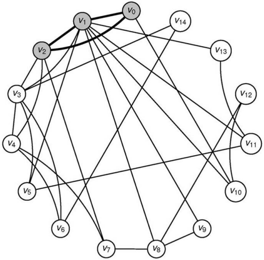
At $t = 1$, vertex v_3 is added, with edges to v_1 and v_2, chosen according to the distribution

$$\pi_0(v_i) = 1/3 \text{ for } i = 0, 1, 2$$

At $t = 2$, v_4 is added. Nodes v_2 and v_3 are preferentially chosen according to the probability distribution

$$\pi_1(v_0) = \pi_1(v_3) = \frac{2}{10} = 0.2$$

$$\pi_1(v_1) = \pi_1(v_2) = \frac{3}{10} = 0.3$$



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Barabasi-Albert Graphs: Degree Distribution

- Degree of BA Graphs is given by

$$f(k) = \frac{(q+2)(q+1)q}{(k+2)(k+1)k} \cdot \frac{2}{(q+2)} = \frac{2q(q+1)}{k(k+1)(k+2)} \quad (12)$$

  - For constant q and large k, degree distribution scales as $f(k) \approx k^{-3}$
  - BA model yields a power-law degree distribution wit $\gamma = 3$, especially for large degrees

Barabasi-Albert Graphs: Diameter

- **Diameter:** of BA graph scales as

$$d(G_t) = O\left(\frac{logn_t}{loglogn_t}\right) \tag{13}$$

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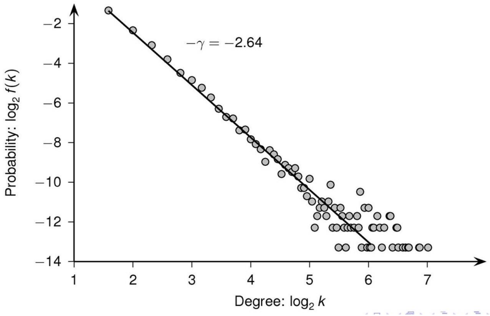
- **Clustering Coefficient:** Expected clustering coefficient scales as

$$E[C(G_t)] = O\left(\frac{(logn_t)^2}{n_t}\right) \tag{14}$$

  - which is only slightly better than for random graphs

Example: BA Model - Degree Distribution
$n_0 = 3, q = 2, t = 997$



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- Albert-Laszlo Barabasi and Reka Albert, *Emergence of Scaling in Random Networks*, Science **286** (1999), no. 5439, 509–512.
- Duncan J. Watts and Steven H. Strogatz, *Collective dynamics of 'small-world' networks*, Nature **393** (1998), 440–442.
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