

Calculate Slope

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0\% \text{ or undefined}$$

Introduction to limits:

Limits: A limit describes the ~~false~~ value ~~of~~ function ~~as~~ approaches when the ~~is~~ input variable to a function approaches a specific value. In our case, the input variable is  $x_2$  and our function is (see above).

Calc for machine learning / Understanding Limits

→ Mission 07 Undefined Limit to defined limit

$$f(x) = -x^2 + 3x - 1$$

$$\lim_{x_2 \rightarrow 3} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_1) = 1 \\ x_1 = +3$$

expand:  
expand:

$$\lim_{x_2 \rightarrow 3} \frac{-(x_2)^2 + 3x_2 - 1 + 1}{x_2 - 3} \Rightarrow \frac{-x_2^2 + 3x_2}{x_2 - 3} \Rightarrow \frac{x_2(-x_2 + 3)}{x_2 - 3}$$

$$\Rightarrow \frac{-x_2(x_2 - 3)}{x_2 - 3} \Rightarrow -x_2$$

$$\lim_{x_2 \rightarrow 3} -x_2 = -3$$

Subchapter: Finding Extreme Points (maxima)

## 03 Differentiation

## 02 Introduction to Derivatives

**Confusion:** Derivative is defined as a function that can determine the slope of a tangent line for any  $x$  value along the function.

The widget is confusing as the green line is only tangent when  $h=0$ . I guess my confusion is because the axes are not labeled.

## 03 Differentiation

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad f(x) = -(x)^2 + 3x - 1$$

$$\lim_{h \rightarrow 0} \frac{(-(x+h)^2 + 3(x+h) - 1) - (-x^2 + 3x - 1)}{h}$$

$$\rightarrow -(x^2 + 2xh + h^2) = -x^2 - 2xh - h^2$$

$$\lim_{h \rightarrow 0} \frac{-x^2 - 2xh - h^2 + 3x + 3h + \cancel{x^2} - \cancel{3x} + \cancel{x}}{h}$$

$$\lim_{h \rightarrow 0} \frac{-2xh - h^2 + 3h}{h} = -2x - h + 3$$

to simplify this using direct substitution do the following see green line above

$$f(x) = -2x - 0 + 3 = -2x + 3$$

or

$y = -2x + 3$  this is the derivative which is expressed as  $y$  prime or...

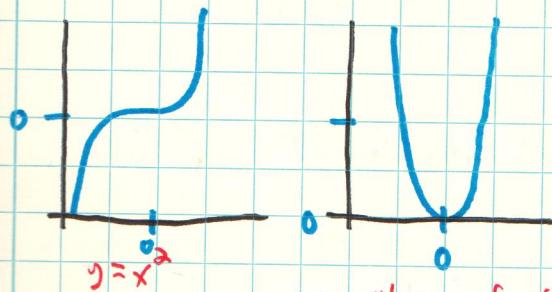
$$y' = -2x + 3 \quad \text{or} \quad f'(x) = -2x + 3$$

The last common notation is the following which can be read as "the derivative w/ respect to x is"

$$\frac{d}{dx} = [-x^2 + 3x - 1] = -2x + 3$$

04 Critical Points: a point on a curve where the gradient = 0

function:  ~~$y = x^2$~~   $y = x^2$



slope of the tangent @ different x values

-Critical points are interesting in data science when they represent extreme values, which can be split into two categories: maximum & minimum.

- 1) Maximum  $\rightarrow$  When slope transitions from positive to negative.
- 2) Minimum  $\rightarrow$  When slope transitions from negative to positive
- 3) When slope doesn't transition ~~when signs flip~~ than it is not an extreme. See the graphs above.

## 06 Power Rule

$$f'(x) = rx^{r-1}$$

So...

1.  $f(x) = x^2$ ,  $r$  would be 2 & its derivative would be:  $f'(x) = 2x$

$$2. f(x) = -(x)^2 + 3x - 1$$

$$f'(x) = -2x + 3 \text{ wow so much easier}$$

-----

The actual exercise, calculate the derivative

$$1. f(x) = x^5 \quad f'(x) = 5x^4$$

$$2. f(x) = x^9 \quad f'(x) = 9x^8$$

Next, look up the slope at  $x=2$  for (1)  
&  $x=0$  for (2)

$$1. f'(2) = 5(2)^4 = 80$$

$$2. f'(0) = 9(0)^8 = 0$$

## 07 Linearity of Differentiation

Consists of 2 rules:

i) The Sum Rule

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$$

ii) The Constant Factor rule

$$\frac{d}{dx} [c f(x)] = c \frac{d}{dx} [f(x)]$$

example:

$$\frac{d}{dx} [-x^3 + x^2] = \frac{d}{dx} [-x^3] + \frac{d}{dx} [x^2] = -3x^2 + 2x$$

The actual exercise...

calculate the derivative of

$$f(x) = x^5 - x \text{ , determine the slope when } x=1$$

$$\frac{d}{dx} = 5x^4 - 1 \quad f'(1) = 5(1)^4 - 1 = 4$$

$$f(x) = x^3 - x^2 \text{ , slope when } x=2$$

$$\frac{d}{dx} = 3x^2 - 2x \quad f'(2) = 3(2)^2 - 2(2) = 8$$

oops, I used the power rule & not the linearity of differentiation rule.

## 08 Practicing Finding Extreme Values

- Find the critical points for  $x^3 - x^2$

To do this I'll need to determine the derivative,  
set it equal to 0 & solve for x.

$$f(x) = x^3 - x^2 \quad \frac{d}{dx} = 3x^2 - 2x$$

this is the easy part  
I'm not sure how to  
solve for x, when there  
are more than 2 solutions

$$0 = 3x^2 - 2x$$

ok, I'm just going to try some stuff here.

$$3x^2 = 2x \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$$

wholy shit,  
that's one  
solution...

divide both  
sides by x

the other solution is probably  
the easy one, 0

$$\text{critical points} = \frac{2}{3} \text{ & } 0$$

- Determine if a critical point is a relative maximum or minimum

Recall if slope goes from (+) to (-) its a max  
& vice versa if its a min

## 01 Overview of Linear Algebra

## • Optimal Salary Problem

2. 1.  $y = 1000 + 30x$  base + rate

2.  $y = 100 + 50x$

base per week	$\$/\text{hour}$
\$ 1000/wk	\$ 30/hr
\$ 100/wk	\$ 50/hr

$y$  = the total earned each week

If we know how much we'd like to earn each week.

## 02 Solving Linear Systems By Elimination

The point where both functions intersect is called the solution to the system.

Using equations 1 & 2 above we can solve for  $x$ .

$$1000 + 30x = 100 + 50x \Rightarrow 1000 - 100 = 50x - 30x$$

$$900 = 20x$$

$$x = 45$$

The same can be done to solve for  $y$ .

$$y = 30(45) + 1000 = 2,350$$

$$y = 50(45) + 100 = 2,350$$

This illustrates that after 45 hours we will have earned the same amount of \$12,350.

03 Representing Functions In General Form &

04 Representing An Augmented Array Matrix In NumPy

In linear algebra we usually represent linear functions in the general form.

$$Ax + By = C$$

In the general form, variables & their coefficients are on the left side & the constant term is on the right. We can switch from point-slope form to the general form by rearranging the terms:

$$y = mx + b \Rightarrow mx - y = -b$$

function 1 in general form

$$30x - y = -1000$$

function 2 in general form

$$50x - y = -100$$

To represent both linear functions in a system we use an augmented matrix:

$$\left[ \begin{array}{cc|c} 30 & -1 & -1000 \\ 50 & -1 & -100 \end{array} \right]$$

## 06 (Matrix) Row Operations

1) Any two rows can be swapped

(continue from previous matrix)

$\xrightarrow{\text{Swap}}$   
R1 & R2

$$\begin{array}{c|cc|c} 30 & 1 & -1 & -100 \\ \hline 50 & 3 & -1 & -100 \end{array} \xrightarrow{\text{Swap R1 and R2}} \begin{array}{c|cc|c} 50 & 3 & -1 & -100 \\ \hline 30 & 1 & -1 & -100 \end{array}$$

2) Any row can be multiplied by a nonzero constant.

$$\begin{array}{c|cc|c} 30 & -1 & -100 \\ \hline 50 & -1 & -100 \end{array} \xrightarrow{2 \cdot R1} \begin{array}{c|cc|c} 60 & -2 & -200 \\ \hline 150 & -3 & -300 \end{array}$$

3) Any row can be added to another row

$$\begin{array}{c|cc|c} 30 & -1 & -100 \\ \hline 50 & -1 & -100 \end{array} \xrightarrow{R2 = R2 + R1} \begin{array}{c|cc|c} 30 & -1 & -100 \\ \hline 80 & -2 & -1100 \end{array}$$

## 07 Simplifying Matrix To Echelon Form

Two Steps to find the solution of a matrix:

- 1) Rearrange the matrix into echelon form. In this form the values on the diagonal locations are all equal to (1) and the values below the diagonal locations are all equal to (0).

$$\left[ \begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right]$$

First, divide R1 by 30 so that the diagonal value in the 1<sup>st</sup> row is 1:

$$\left[ \begin{array}{cc|c} 30 & -1 & -1000 \\ 50 & -1 & -100 \end{array} \right] \rightarrow R1 = R1 / 30 \rightarrow \left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 50 & -1 & -100 \end{array} \right]$$

Then, subtract 50 times the 1<sup>st</sup> row from the second:

$$\left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 50 & -1 & -100 \end{array} \right] \rightarrow R2 = R2 - (R1 \times 50)$$

$\left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 0 & \frac{20}{30} & \frac{47,000}{30} \end{array} \right]$

Next, multiply R2 by 1.5 to get the diagonal value in the second row to be (1).

$$\left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 0 & \frac{20}{30} & \frac{47,000}{30} \end{array} \right] \rightarrow R2 = R2 \cdot \frac{3}{2} \rightarrow \left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 0 & 1 & \frac{141,000}{60} \end{array} \right]$$

To complete the transformation we'll have to zero out the second value in the first row. To do this let's subtract ~~R2 x~~ add  $R2 \times \frac{1}{30}$  to R1.

$$\left[ \begin{array}{cc|c} 1 & -\frac{1}{30} & -\frac{1000}{30} \\ 0 & 1 & \frac{141,000}{60} \end{array} \right] \rightarrow R1 = R1 + (R2 \times \frac{1}{30}) \quad \downarrow 0 + \frac{1}{30} \mid \frac{141,000}{1800}$$

$$\left[ \begin{array}{cc|c} 1 & 0 & -\frac{2250}{30} \\ 0 & 1 & \frac{141,000}{60} \end{array} \right] \quad \text{not sure my math is right here, let me run it throughs python}$$

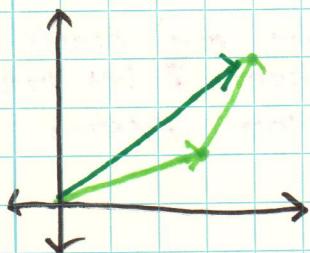
$$\Rightarrow \left[ \begin{array}{cc|c} 1 & 0 & -\frac{2250}{30} \\ 0 & 1 & \frac{2350}{30} \end{array} \right]$$

## 03 Vector Operations

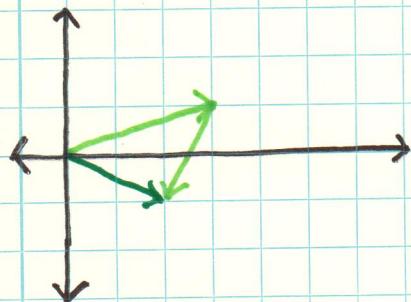
Vector Summation

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Stop putting the horizontal line in there!

Vector Subtraction

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



## 04 Scaling Vectors

$$3 \times \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

## 05 Vectors in NumPy

see code...

## 06. Dot Product

$$\vec{a} * \vec{b} = \sum_{i=1}^n a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

What this looks like visually

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = (1 \times 3) + (2 \times 0) + (1 \times 1) = 4$$

Unlike the other vector operations, the result of the ~~scalar~~ dot product is a scalar value not a vector.

To compute the dot product between 2 vectors, we need to use the `numpy.dot()` function. This function accepts `NumPy ndarray` objects as the required parameters. The main quirk on is that one of the two vectors need to be represented as a row vector while the other a column vector.

Linear

## 07 Linear Combination

Using vectors we can determine if a certain vector can be obtained by combining other vectors.

For example we may want to know if we can combine the vectors

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ & } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Here's what it looks like mathematically:

$$c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Being able to scale vectors using scalar multiplication then adding or subtracting these scaled vectors is known as linear combination. This concept is ~~very~~ crucial to being able to bring algebra to solve useful problems.

if  
 $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

## 08 Linear Combination & Vectors

Circle back to the salary problem represented as a matrix.

$$\left[ \begin{array}{cc|c} 30 & -1 & -1000 \\ 50 & -1 & -100 \end{array} \right]$$

We can now link this augmented matrix to the linear combination of vectors idea we just discussed.

We want to know if  $\begin{bmatrix} -1000 \\ -100 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 30 \\ 50 \end{bmatrix}$  &  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ . To find the solution to this system

We need to find the constants  $x$  &  $y$  where the following equation is true.

$$x \begin{bmatrix} 30 \\ 50 \end{bmatrix} + y \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1000 \\ -100 \end{bmatrix}$$

In the last mission, we solved the augmented matrix by using row operations to obtain the following form.

$$\left[ \begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \end{array} \right]$$

## 09 The Matrix Equation

The matrix equation is the representation of a linear system using only matrices & vectors. Here's the augmented matrix we started out with:

$$\left[ \begin{array}{cc|c} 30 & -1 & -1000 \\ 50 & -1 & -100 \end{array} \right]$$

This ~~the~~ augmented matrix is the ~~short hand~~ notation representation for the matrix equation:

$$\left[ \begin{array}{cc} 30 & -1 \\ 50 & -1 \end{array} \right] \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} -1000 \\ -100 \end{array} \right]$$

On the left side we're multiplying a matrix containing the coefficients w/ the vector containing the variables. The right side ~~also~~ contains the constant values. This separation of coefficients & variables from the constants should be familiar. This is exactly ~~the same~~ what we did in the general form as well.

It's common practice to use  $x_1, x_2, \dots, x_n$  instead of  $x$  &  $y$  to represent the individual values in the solution vector.

$$\left[ \begin{array}{cc} 30 & -1 \\ 50 & -1 \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[ \begin{array}{c} -1000 \\ -100 \end{array} \right]$$

This allows us to work with vectors w/ any number of elements (instead of just 26 for the number of letters in the English dictionary). We can now introduce the arithmetic representation of the matrix equation:

$$\vec{A}\vec{x} = \vec{b}$$

Where  $A$  represents the coefficient matrix,  $\vec{x}$  represents the solution vector, and  $\vec{b}$  represents the constants. Note that  $\vec{b}$  can't be a vector containing all zero's also known as zero vector & represented using  $\vec{0}$ .

Before we can work with this form of the system we need to learn about the following topics in the following mission.

1. The rules that describe how matrices can be combined
2. How to multiply a matrix w/ a vector
3. How to calculate the solution vector  $X$  w/o using Gaussian elimination.

01 Basic Matrix Operations

$$A = \begin{bmatrix} 0.7 & 3 & 9 \\ 1.7 & 2 & 9 \\ 0.7 & 9 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 3 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 0.7 & 6 & 12 \\ 2.7 & 4 & 10 \\ 0.7 & 10 & 4 \end{bmatrix}$$

$$5A = \begin{bmatrix} 3.5 & 15 & 45 \\ 8.5 & 10 & 45 \\ 3.5 & 45 & 10 \end{bmatrix}$$

02 Matrix Vector Multiplication

$$A \cdot c = \begin{bmatrix} 0.7 & 3 & 9 \\ 1.7 & 2 & 9 \\ 0.7 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (0.7 \times 1) + (3 \times 2) + (9 \times 1) \\ (1.7 \times 1) + (2 \times 2) + (9 \times 1) \\ (0.7 \times 1) + (9 \times 2) + (2 \times 1) \end{bmatrix} = \begin{bmatrix} 15.7 \\ 14.7 \\ 20.7 \end{bmatrix}$$

- When we multiply a matrix by a vector we are essentially combining each row in the matrix w/ the column vector (pen dided)
- To multiply a matrix by a vector, the # of columns in the matrix needs to match the number of rows in the vector.
- To multiply a matrix with a vector in Numpy, we need to use the `numpy.dot()` function.
- The process is known as the dot product between each row in the matrix & the column vector.

03 Matrix Multiplication

$$A = \begin{bmatrix} 0.7 & 3 \\ 1.7 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} (0.7 \times 1 + 3 \times 2) & (0.7 \times 2 + 3 \times 3) \\ (1.7 \times 1 + 2 \times 2) & (1.7 \times 2 + 2 \times 3) \end{bmatrix} = \begin{bmatrix} 6.7 & 10.4 \\ 5.7 & 9.4 \end{bmatrix}$$

- In matrix multiplication we extend the dot product done in vector matrix multiplication, to perform a dot product between each row in the first matrix and each column in the second matrix.
- As with matrix vector multiplication the number of columns in the 1<sup>st</sup> matrix must match the number of rows in the second matrix.
- Order of operations matters.

$$A \times B \neq B \times A$$

## 04 Matrix Transpose

- The transpose of a matrix switches the rows and columns of a matrix. Think of this operation as a rotation.
- This is sometimes necessary because of the requirements ask for matrix multiplications.   
Transposing a matrix allows us to multiply matrices together ~~that~~ that, by default, don't overlap in # of rows & columns.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

### → Rules

1.) ~~The transpose~~ When taking the transpose of the sum of two matrices, we can distribute the transpose operation to each matrix.

$$\textcircled{a} \quad (A + B)^T = A^T + B^T$$

2) No description provided other than being a counter-intuitive.

$$(AB)^T = B^T A^T$$

Seems intuitive to me...?

3) Also:  $(A \times B)^T = A^T \times B^T$  I know this is a repeat of #1 but I proved it to myself & I get it now.

05 Identity Matrix

In the matrix equation that we discussed in the last mission, we're trying to solve for the vector  $\vec{x}$ .

$$A\vec{x} = \vec{b}$$

Right now, the matrix  $A$  multiplies the vector  $\vec{x}$  and we need a way to cancel  $A$ .

Let's look at the identity matrix, which we touched on briefly at the end of the first mission in this course. If you recall the identity matrix contains 1 along the diagonals and zero elsewhere. Here's what the  $2 \times 2$  identity matrix looks like, often represented as  $I_2$ :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When we multiply  $I_2$  w/ any vector containing 2 elements, the resulting vector matches the original vector exactly:

$$I_2 \vec{x} = \vec{x}$$

This is because each element in the vector is multiplied once by the diagonal 1 value in the identity matrix.

specific case

$$I_2 \vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot 2 \\ 0 \cdot 3 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

general case

$$I_2 \vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + 1 \cdot x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If we can transform matrix  $A$  & convert it to the identity matrix, then only the solution vector  $\vec{x}$  will remain.

Let's practice working w/ the identity matrix before exploring how to transform  $A$  into  $I$ .

We can create any  $I_n$  identity matrix using the `numpy.identity()` function. This function only has 1 required parameter,  $n$ , which specifies the  $n \times n$  identity matrix we want.

## 06 Matrix Inverse

Now we can revisit canceling  $A$  by transforming it to the identity matrix. Multiplying the inverse of a matrix by a matrix does this task.

The matrix inverse is similar to the idea of the multiplicative inverse. For example, let's say we want to solve for  $x$  in the equation:  $5x = 10$ . To do so we need to multiply both sides by the multiplicative inverse of  $5$ , which is  $5^{-1}$  or  $(\frac{1}{5})$ .

To solve for the vector  $\vec{x}$  in the matrix equation, we need to multiply both sides by the inverse of  $A$ :

$$A^{-1} \times A\vec{x} = A^{-1} \times \vec{b}$$

This simplifies to  $I\vec{x} = A^{-1}\vec{b}$  and we're left w/ the formula for calculating the solution vector:

$$\vec{x} = A^{-1}\vec{b}$$

Matrix Calculation for a  $2 \times 2$  matrix:

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term  $ad-bc$  is known as the determinant & is often written as  $\det(A) = ad-bc$  or as  $|A| = ad-bc$ . Because we're dividing by the determinant when calculating the matrix inverse, a  $2 \times 2$  matrix is only invertible if the determinant is not = 0.

In this & the next step we'll focus on finding the matrix inverse when  $A$  is a  $2 \times 2$  matrix. We'll get to higher dimensional matrix later.

### Q7 Solving the matrix equation

Now that we know how to calculate the matrix inverse, we can solve our system using the matrix equation  $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 30 & -1 \\ 50 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1000 \\ -100 \end{bmatrix}$$

We start by left multiplying  $A^{-1}$  on both sides:

$$\begin{bmatrix} 30 & -1 \\ 50 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 30 & -1 \\ 50 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 & -1 \\ 50 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1000 \\ -100 \end{bmatrix}$$

This simplifies to:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 30 & -1 \\ 50 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1000 \\ -100 \end{bmatrix}$$

`numpy.linalg.inv()` can be used to calculate the inverse of an np array.

## 08 Determinant For Higher Dimensions

### More Terminology

**Square matrices:** Matrices that contain the same number of rows & columns.  
We can only compute the determinant & matrix inverse for square matrices.

**Minor matrices:** The determinant of a higher-dimensional system involves down the full matrix into minor matrices.

### General Example

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \times \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \times \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \times \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \times \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \times \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} + c \times \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

### specific case

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \times \det \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} + 3 \times \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

We can use `numpy.linalg.det()` to accomplish this.

## 01 Introduction

So far we've explored two different ways to find a solution to  $A\vec{x} = \vec{b}$  when  $\vec{b}$  isn't a vector containing all zeros.

- 1.) Gaussian Elimination, involving row operations to transform the augmented representation of a linear system to echelon form, and then finally to reduced row echelon form.
- 2.) Compute the matrix inverse of  $A$  & left multiplying both sides of the equation to find  $\vec{x}$ .

These techniques will not help us in the following situations:

- 1.) The solution set for a linear system does not exist.
- 2.) The solution set for a linear system isn't a single vector.
- 3.)  $\vec{b}$  is equal to  $\vec{0}$ .

This mission will explore these situations.

## 02 Inconsistent Systems

**Inconsistent Systems:** An inconsistent system will have two or more equations that don't overlap in values, making it impossible to find a solution.

Let's explore the following example:

$$\left[ \begin{array}{cc|c} 8 & 4 & 5 \\ 4 & 2 & 5 \end{array} \right]$$

Reduce by dividing both rows by 2

~~$$\left[ \begin{array}{cc|c} 4 & 2 & 5 \\ 4 & 2 & 5 \end{array} \right]$$~~

Subtract the top row from the bottom row ( $R_2 - R_1$ ) & divide the 1<sup>st</sup> row by 4 ( $R_1/4$ )

$$\left[ \begin{array}{cc|c} 4 & 2 & 5 \\ 4 & 2 & 5 \end{array} \right] \rightarrow R_2 = R_2 - R_1 \rightarrow \left[ \begin{array}{cc|c} 4 & 2 & 5 \\ 0 & 0 & 0 \end{array} \right] \rightarrow R_1 = \frac{R_1}{4} \rightarrow \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 1.25 \\ 0 & 0 & 0 \end{array} \right]$$

The augmented matrix ends up with zeros for the coefficients in  $R_2$ .

By plotting these two systems it will be clear why no solution exists.

02 Inconsistent Systems

Extract slope-intercept form from the following matrix:

$$\left[ \begin{array}{cc|c} 8 & 4 & 5 \\ 4 & 2 & 5 \end{array} \right]$$

1<sup>st</sup> attempt w/o looking anything up.

$$y = mx + b \Rightarrow x\cancel{m} - y = \cancel{-b}$$

$$8x - 4y = 5 \Rightarrow 4y = -8x - 5 \Rightarrow y = -2x - \frac{5}{4}$$

$$4x - 2y = 5 \Rightarrow 2y = -4x - 5 \Rightarrow y = -2x - \frac{5}{2}$$

I can already see why this won't have a solution, these functions will never intersect.

→ Close but no cigar, ignore the signs, and drop the coefficients and constants as is from the matrix...

Try again:

$$8x + 4y = 5 \Rightarrow 4y = -8x + 5 \Rightarrow y = -2x + \frac{5}{4}$$

$$4x + 2y = 5 \Rightarrow 2y = -4x + 5 \Rightarrow y = -2x + \frac{5}{2}$$

03 Singular Matrix

If we calculate the determinant from the coefficient matrix from the previous step we see that the result is 0. Recall that this means that the matrix has no inverse. When this is the case the matrix is **Singular**.

04 Possible Solutions For Nonhomogeneous Systems

**Nonhomogeneous Systems:** Systems where the coefficient matrix constants vector  $\vec{b}$  doesn't contain all zeros.

$$\begin{bmatrix} 8 & 4 & 1 & 5 \\ 4 & 2 & 1 & 5 \end{bmatrix}$$

**Homogeneous Systems:** Systems where the constants vector  $\vec{b}$  is equal to the zero vector.

$$\begin{bmatrix} 8 & 4 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{bmatrix}$$

The distinction is that ~~nonhomogeneous systems~~ always have a solution, the zero vector.

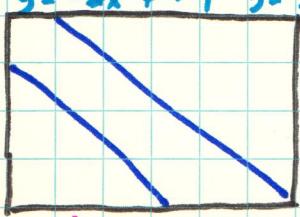
**Square, Nonhomogeneous Systems:** For square, nonhomogeneous systems, there are 3 possible solutions.

- 1.) no solution
- 2.) a single solution
- 3.) infinitely many solutions

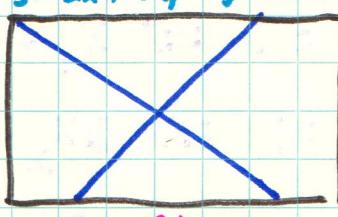
$$y = -2x + \frac{5}{4} \quad y = -2x - \frac{5}{2}$$

$$y = -2x + \frac{5}{4} \quad y = 2x + \frac{5}{2}$$

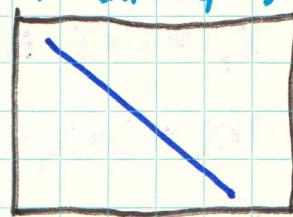
$$y = -2x + \frac{5}{4} \quad y = -2x + \frac{5}{4}$$



No Solution



One Solution



Infinite Solution

04 Possible Solutions For Nonhomogeneous Systems

Rectangular, Nonhomogeneous System

For a rectangular (nonsquare), ~~not~~ nonhomogeneous systems there are 2 possible solutions:

- 1.) no solution
- 2.) Infinitely many solutions

Example:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{array} \right]$$

Transform the matrix into echelon form.

Start by subtracting  $R_2 - R_1$ ,

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{array} \right] \rightarrow R_2 = R_2 - R_1 \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & -2 & 1 \end{array} \right]$$

Next divide  $R_2/2$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & -2 & 1 \end{array} \right] \rightarrow R_2 = \frac{R_2}{2} \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0.5 \end{array} \right]$$

Convert back to equation form:

$$x_1 + 0x_2 + 3x_3 = 1 \Rightarrow x_1 + 3x_3 = 1$$

$$0x_1 + 1x_2 - 1x_3 = 1 \Rightarrow x_2 - x_3 = 1$$

If we solve for ~~each~~ each variable we get:

$$\begin{aligned} x_1 &= 1 - 3x_3 \\ x_2 &= 1 + 3x_3 \end{aligned}$$

## 04 Possible Solutions For Nonhomogeneous Systems.

$x_3$  is known as a free variable, because it is allowed to vary freely. You'll notice that both  $x_1$  &  $x_2$  are expressed in terms of  $x_3$ . This system has infinitely many solutions, because for any real number we plug in for  $x_3$ , we'll get different values for  $x_1$  &  $x_2$ .

Said another way, there are infinite solutions to this system because there's an infinite number of values that  $x_3$  could be that make the system true.

This set of infinite solutions is known as a **Solution Space**. Let's test this using some random examples.

## 05 Homogeneous Systems

**Trivial Solution:** We can solve any homogeneous system  $A\vec{x} = \vec{0}$ , by setting  $\vec{x}$  to the zero vector. This is known as the trivial solution. This isn't very useful were interested in determining if infinitely many solutions exist or not. Here's an example:

Reduce to echelon form

$$\begin{array}{l} \left[ \begin{array}{ccc|c} 6 & 10 & -8 & 0 \\ -6 & -4 & 8 & 0 \\ 3 & \frac{1}{2} & -4 & 0 \end{array} \right] \xrightarrow{R_2 = R_2 + R_1} \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 36 & 0 & 0 \\ 3 & \frac{1}{2} & -4 & 0 \end{array} \right] \xrightarrow{R_3 = \frac{R_3}{2}} \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 3 & \frac{1}{2} & -4 & 0 \end{array} \right] \\ \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad R_3 = R_3 - R_1 \end{array}$$

$$\begin{array}{l} \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xleftarrow{R_2 = R_2 - (\frac{R_1}{3})} \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad R_1 = R_1 - \frac{R_2}{3} \end{array}$$

$$\begin{array}{l} \left[ \begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{4}{3} & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 = \frac{R_2}{3}} \left[ \begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad R_1 = R_1 - \left( \frac{R_2}{3} \right) \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

05 Homogeneous Systems

If we convert this to equation form we get

~~$x_1 = \frac{4}{3}x_3$~~

$$x_1 - \frac{4}{3}x_3 = 0 \Rightarrow x_1 = \frac{4}{3}x_3$$

$$x_2 = 0$$

$$0 = 0$$

In this system  $x_3$  is a free variable,  $x_3$  can be any real number. Thus there is an infinite number of solutions to this ~~sys~~ system

**Parametric Vector Form:** When the solution is a solution space (and not just a unique set of values), it's common to rewrite it into parametric vector form

$$\vec{x} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Here's where the values in the parametric vector form come from:

1.) Since  $x_3$  is the free variable w/ respect to  $x_5$ .

2.) The solution to each variable ~~is the~~ makes up the vector:

$$\left\{ \begin{array}{l} x_1: \frac{4}{3} \\ x_2: 0 \\ x_3: \text{is the free variable, that's why we put } 1 \end{array} \right.$$

05 Homogeneous Systems

The Exercise...

- Create a function that uses the solution functions to calculate  $X_1$  &  $X_2$
- Plug these values into the original linear system and check if the values turn out to be true