Algorithms: COMP3121/3821/9101/9801

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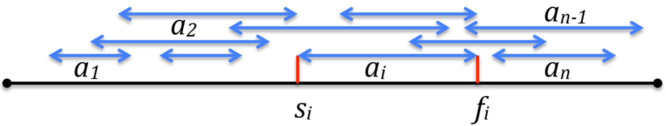
TOPIC 4: THE GREEDY METHOD

Activity selection problem

- **Instance:** A list of activities a_i, $(1 \leq i \leq n)$ with starting times s_i and finishing times f_i. No two activities can take place simultaneously.

- **Task:** Find a *maximum size* subset of compatible activities.



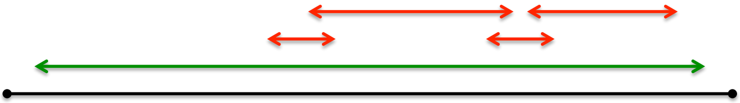
Attempt 1: always choose the shortest activity which does not conflict with the previously chosen activities, remove the conflicting activities and repeat?

- The above figure shows this does not work...
(chosen activities in green, conflicting in red)

Activity selection problem

- **Instance:** A list of activities a_i, $(1 \leq i \leq n)$ with starting times s_i and finishing times f_i. No two activities can take place simultaneously.
- **Task:** Find a *maximum size* subset of compatible activities.

- **Attempt 2:** Maybe among the activities which do not conflict with the previously chosen activities we should always choose the one with the earliest possible start?

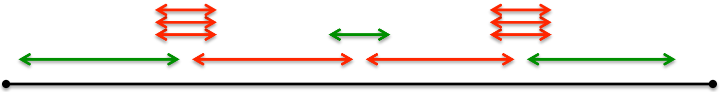


- The above figure shows this does not work either...

Activity selection problem.

- **Instance:** A list of activities a_i, $(1 \leq i \leq n)$ with starting times s_i and finishing times f_i. No two activities can take place simultaneously.
- **Task:** Find a *maximum size* subset of compatible activities.

- **Attempt 3:** Maybe we should always choose an activity which conflicts with the fewest possible number of the remaining activities? It may appear that in this way we minimally restrict our next choice....

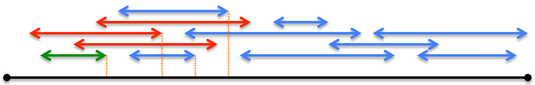


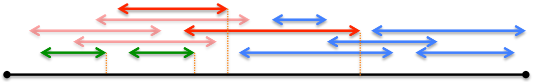
- As appealing the idea is, the above figure shows this again does not work ...

Activity selection problem

- **Instance:** A list of activities a_i, $(1 \leq i \leq n)$ with starting times s_i and finishing times f_i. No two activities can take place simultaneously.
- **Task:** Find a *maximum size* subset of compatible activities.

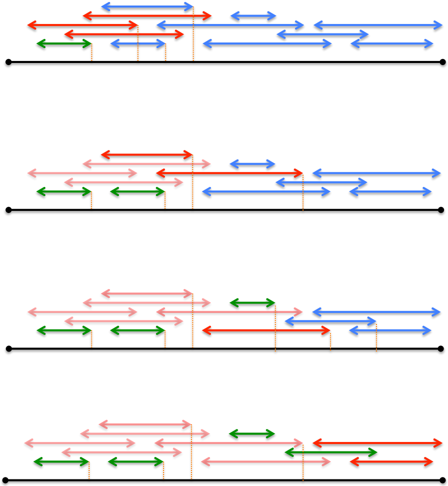
- **The correct solution:** Among the activities which do not conflict with the previously chosen activities always chose the one with the earliest end time.





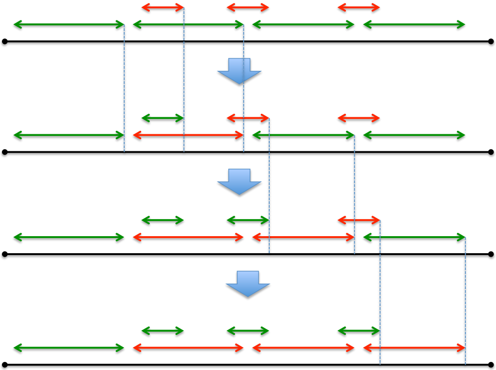
The Greedy Method

Activity selection problem



The Greedy Method

- Transforming any optimal solution to the greedy solution with equal number of activities: find the first place where the chosen activity violates the greedy choice and show that replacing that activity with the greedy choice produces a non conflicting selection with the same number of activities. Continue in this manner till you “morph” your optimal solution into the greedy solution, thus proving the greedy solution is also optimal.



The Greedy Method

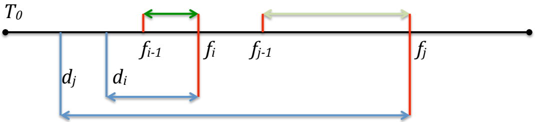
- What is the time complexity of the algorithm?
- We keep the list of activities and make two extra copies.
- We sort one of the two copies according to the finishing times, the other according to the starting time.
- This takes $O(n \log n)$ many steps.
- Every activity is handled only once, so the rest of the algorithm takes $O(n)$ time.
- Thus, the algorithm runs in time $O(n \log n)$.
- Think about the details how you would implement this algorithm efficiently.

Activity selection problem II

- **Instance:** A list of activities a_i, $(1 \leq i \leq n)$ with starting times s_i and finishing times $f_i = s_i + d$; thus, all activities are of the same duration. No two activities can take place simultaneously.
- **Task:** Find a subset of compatible activities of *maximal total duration*.
- **Solution:** Since all activities are of the same duration, this is equivalent to finding a selection with a largest number of non conflicting activities, i.e., the previous problem.
- **Question:** What happens if the activities are not all of the same duration?
- Greedy strategy no longer works - we will need a more sophisticated technique.

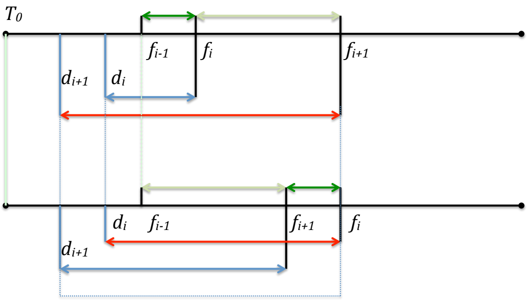
Minimising job lateness

- **Instance:** A start time T_0 and a list of jobs a_i, $(1 \leq i \leq n)$, with duration times t_i and deadlines d_i. Only one job can be performed at any time; all jobs have to be completed. If a job a_i is completed at a finishing time $f_i > d_i$ then we say that it has incurred lateness $l_i = f_i - d_i$.
- **Task:** Schedule all the jobs so that the lateness of the job with the largest lateness is minimised.
- **Solution:** Ignore job durations and schedule jobs in the increasing order of deadlines.
- **Optimality:** Consider any optimal solution. We say that jobs a_i and jobs a_j form an inversion if job a_i is scheduled before job a_j but $d_j < d_i$.



Minimising job lateness

- We will show that there exists a scheduling without inversions which is also optimal.
- Recall the Bubble Sort: if we manage to eliminate all inversions between adjacent jobs, eventually all the inversions will be eliminated.



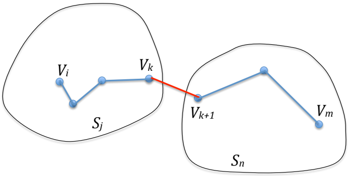
- Note that swapping adjacent inverted jobs reduces the larger lateness!

k-clustering of maximum spacing

- **Instance:** A complete graph G with weighted edges representing distances between the two vertices.
- **Task:** Partition the vertices of G into k disjoint subsets so that the minimal distance between two points belonging to different sets of the partition is as large as possible. Thus, we want a partition into k disjoint sets which are as far apart as possible.
- **Solution:** Sort the edges in an increasing order and start performing the usual Kruskal's algorithm for building a minimal spanning tree, but stop when you obtain k trees, rather than a single spanning tree.
- **Proof of optimality:** Let d be the distance associated with the first edge of the minimal spanning tree which was not added to our k trees; it is clearly the minimal distance between two vertices belonging to two of our k trees. Clearly, all the edges included in k many trees produced by our algorithm are of length smaller or equal to d.

k-clustering of maximum spacing

- Consider any partition $\mathcal{S}$ into k subsets different from the one produced by our algorithm. This means that there is a tree produced by our algorithm which contains vertices V_i and V_m from two different subsets from $\mathcal{S}$, say S_j and S_n.



- Since V_i and V_m belong to the same tree, there is a path along the edges of that tree connecting V_i and V_m. Let V_k and V_{k+1} be two consecutive vertices on that path such that V_k belongs to S_j and V_{k+1} belongs to S_n. Note that $d(V_k, V_{k+1}) \leq d$ which implies that the minimal distance between clusters of $\mathcal{S}$ is smaller or equal to the minimal distance d between the k trees produced by our algorithm. Thus, such a partition cannot be a more optimal clustering than the one produced by our algorithm.

k-clustering of maximum spacing

- What is the time complexity of this algorithm?
- We have n^2 edges; thus sorting them by weight will take $O(n^2 \log n^2) = O(n^2 \log n)$.
- While running the (partial) Kruskal algorithm we use the UNIONFIND data structure and we make at most $2n^2$ calls of the FIND operation and at most n calls of the UNION operation.
- Each FIND operation takes $O(\log n)$ time and each UNION operation takes $O(1)$ steps.
- Thus, in total running the (partial) Kruskal algorithm also takes $O(n^2 \log n^2) = O(n^2 \log n)$.
- So the grand total for the whole algorithm is $O(n^2 \log n)$ many steps.

- Given two sequences of letters A and B, find if B is a subsequence of A in the sense that one can delete some letters from A and obtain the sequence B.
- There is a line of 111 stalls, some of which need to be covered with boards. You can use up to 11 boards, each of which may cover any number of consecutive stalls. Cover all the necessary stalls, while covering as few total stalls as possible.

Tape storrage

- **Instance:** A list of n files f_i of lengths l_i which have to be stored on a tape. Each file is equally likely to be needed. To retrieve a file, one must start from the beginning of the tape and scan it until the tape is found and read.

- **Task:** Order the files on the tape so that the average (expected) retrieval time is minimised.

- **Solution:** If the files are stored in order $l_1, l_2, \dots l_n$, then the expected time is proportional to

$$\begin{aligned} l_1 + (l_1 + l_2) + (l_1 + l_2 + l_3) + \dots + (l_1 + l_2 + l_3 + \dots + l_n) = \\ n l_1 + (n-1) l_2 + (n-2) l_3 + \dots + 2 l_{n-1} + l_n \end{aligned}$$

- This is minimised if $l_1 \leq l_2 \leq l_3 \leq \dots \leq l_n$.

Tape storrage II

- **Instance:** A list of n files f_i of lengths l_i and probabilities to be needed p_i, $\sum_{i=1}^{n} p_i = 1$, which have to be stored on a tape. To retrieve a file, one must start from the beginning of the tape and scan it until the tape is fund and read.
- **Task:** Order the files on the tape so that the expected retrieval time is minimised.
- **Solution:** If the files are stored in order $l_1, l_2, \ldots l_n$, then the expected time is proportional to

$$p_1 l_1 + (l_1 + l_2)p_2 + (l_1 + l_2 + l_3)p_3 + \ldots + (l_1 + l_2 + l_3 + \ldots + l_n)p_n$$

- We now show that this is minimised if the files are ordered in a decreasing order of values of the ratio p_i/l_i.

The Greedy Method

- Let us see what happens if we swap to adjacent files f_k and f_{k+1}.
- The expected time before the swap and after the swap are, respectively,

$$\begin{aligned} E = & p_1 l_1 + (l_1 + l_2) p_2 + (l_1 + l_2 + l_3) p_3 + (l_1 + l_2 + l_3 + \ldots + l_{k-1} + l_k) p_k + \ldots \\ & + (l_1 + l_2 + l_3 + \ldots + l_{k-1} + l_k + l_{k+1}) p_{k+1} + \ldots + (l_1 + l_2 + l_3 + \ldots + l_n) p_n \end{aligned}$$

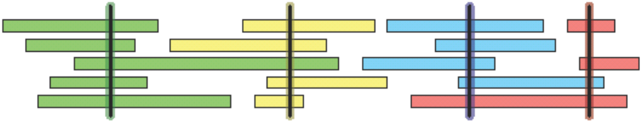
and

$$\begin{aligned} E' = & p_1 l_1 + (l_1 + l_2) p_2 + (l_1 + l_2 + l_3) p_3 + (l_1 + l_2 + l_3 + \ldots + l_{k-1} + l_{k+1}) p_{k+1} + \ldots \\ & + (l_1 + l_2 + l_3 + \ldots + l_{k-1} + l_{k+1} + l_k) p_k + \ldots + (l_1 + l_2 + l_3 + \ldots + l_n) p_n \end{aligned}$$

- Thus, $E - E' = l_k p_{k+1} - l_{k+1} p_k$, which is positive just in case $l_k p_{k+1} > l_{k+1} p_k$, i.e., if $p_k / l_k < p_{k+1} / l_{k+1}$.
- Consequently, $E > E'$ if and only if $p_k / l_k < p_{k+1} / l_{k+1}$, which means that the swap decreases the expected time just in case $p_k / l_k < p_{k+1} / l_{k+1}$, i.e., if there is an inversion: a file f_{i+1} with a larger ratio p_{k+1} / l_{k+1} has been put after a file f_i with a smaller ratio p_k / l_k.
- For as long as there are inversions there will be inversions of consecutive files and swapping will reduce the expected time. Consequently, the optimal solution is the one with no inversions.

The Greedy Method

- Let X be a set of n intervals on the real line. We say that a set P of points stabs X if every interval in X contains at least one point in P; see the figure below. Describe and analyse an efficient algorithm to compute the smallest set of points that stabs X. Assume that your input consists of two arrays $X_L[1..n]$ and $X_R[1..n]$, representing the left and right endpoints of the intervals in X.



A set of intervals stabbed by four points (shown here as vertical segments)

The Greedy Method

- Assume you are given n sorted arrays of different sizes. You are allowed to merge any two arrays into a single new sorted array and proceed in this manner until only one array is left. Design an algorithm that achieves this task and uses minimal total number of moves of elements of the arrays. Give an informal justification why your algorithm is optimal.

- Along the long, straight road from Loololong to Goolagong houses are scattered quite sparsely, sometimes with long gaps between two consecutive houses. Telstra must provide mobile phone service to people who live alongside the road, and the range of Telstras cell base station is 5km. Design an algorithm for placing the minimal number of base stations alongside the road, that is sufficient to cover all houses.

Using the Greedy Method to derive various properties of the object constructed

- You are given a connected graph with weighted edges. Find a spanning tree such that the largest weight of all of its edges is as small as possible.

- You are given a connected graph with weighted edges with all weights distinct. Prove that such a graph has a unique spanning tree.

- Assume that you are given a complete graph G with weighted edges such that all weights are distinct. We now obtain another complete weighted graph G' by replacing all weights $w(i,j)$ of edges $e(i,j)$ with new weights $w(i,j)^2$.
  1. Assume that p is a shortest path from a vertex u to a vertex v in G. Does p necessarily remain the shortest path from u to v in the new graph G'?
  2. Assume that T is the minimal spanning tree of G. Does T necessarily remain the minimal spanning tree for the new graph G'?

Using the Greedy Method to derive various properties of the object constructed

- Assume that you are given a complete weighted graph G with n vertices $v_1, \ldots, v_n$ and with the weights of all edges distinct and positive.
- Assume that you are also given the minimal spanning tree T for G.
- You are also given an additional new vertex v_{n+1} and weights $w(n+1, j)$ of all new edges $e(n+1, j)$ between the new vertex v_{n+1} and all old vertices $v_j \in G$, $1 \leq j \leq n$.
- Design an algorithm which produces a minimum spanning tre T' for the new graph containing the additional vertex v_{n+1} and which runs in time $O(n \log n)$.

Using the Greedy Method to derive various properties of the object constructed

- Once again, along the long, straight road from Loololong (on the West) to Goolagong (on the East) houses are scattered quite sparsely, sometimes with long gaps between two consecutive houses. Telstra must provide mobile phone service to people who live alongside the road, and the range of Telstras cell base station is 5km.
- One of Telstra's engineer started with the house closest to Loololong and put a tower 5km away to the East. He then found the westmost house not already in the range of the tower and placed another tower 5 km to the East of it and continued in this way till he reached Goolagong.
- His junior associate did exactly the same but starting from the East and moving westwards and claimed that his method required fewer towers.
- Is there a placement of houses for which the associate is right?

The Greedy Method

- There are n radio towers for broadcasting tsunami warnings. You are given the coordinates of each tower and its radius of range. When a tower is activated, all towers within the radius of range of the tower will also activate, and those can cause other towers to activate and so on.
- You need to equip some of these towers with seismic sensors so that when these sensors activate the towers where these sensors are located all towers will eventually get activated and send a tsunami warning.
- The goal is to design an algorithm which finds the fewest number of towers you must equip with seismic sensors.
- Someone has proposed the following two algorithms:
    - Algorithm 1: Find the unactivated tower with the largest radius (if multiple with the same radius, pick the any of them). Activate this tower. Find and remove all activated towers. Repeat.
    - Algorithm 2: Find the unactivated tower with the largest number of towers within its range. If there is none, activate the leftmost tower. Repeat.

1. Give examples which show that neither Algorithm 1 nor Algorithm 2 solve the problem correctly.
2. Design an algorithm which correctly solves the problem.

The Greedy Method

- Assume you have \$2, \$1, 50c, 20c, 10c and 5c coins to pay for your lunch. Design an algorithm that, given the amount that is a multiple of 5c, pays it with a minimal number of coins.

- Assume denominations of your $n + 1$ coins are $1, c, c^2, c^3, \ldots, c^n$ for some integer $c > 1$. Design a greedy algorithm which, given any amount, pays it with a minimal number of coins.

- Give an example of a set of denominations containing the single cent coin for which the greedy algorithm does not always produce an optimal solution.

The Greedy Method - Huffman code

- Assume you are given a set of symbols, for example the English alphabet plus punctuation marks and a blank space (to be used between words).
- You want to encode these symbols using binary strings, so that sequences of such symbols can be decoded in an unambiguous way.
- One way of doing so is to reserve bit strings of equal and sufficient length, given the number of distinct symbols to be encoded; say if you have 26 letters plus 6 punctuation symbols, you would need strings of length 5 ($2^5 = 32$).
- To decode a piece of text you would partition the bit stream into groups of 5 bits and use a lookup table to decode the text.
- However this is not an economical way: all the symbols have codes of equal length but the symbols are not equally frequent.
- One would prefer an encoding in which frequent symbols such as 'a', 'e', 'i' or 't' have short codes while infrequent ones, such as 'w','x' and 'y' can have longer codes.
- However, if the codes are of variable length, then how can we partition a bitstream UNIQUELY into segments each corresponding to a code?
- One way of insuring unique readability of codes from a single bitstream is to ensure that no code of a symbol is a prefix of a code for another symbol.
- Codes with such property are called *the prefix codes.*

The Greedy Method - Huffman code

- We can now formulate the problem:

  Given the frequencies (probabilities of occurrences) of each symbol, design an optimal prefix code, i.e., a prefix code such that the expected length of an encoded text is as small as possible.

- Note that this amounts to saying that the *average* number of bits per symbol in an "average" text is as small as possible.

- See the textbook for details; from now on, we will do things on the blackboard which is camera recorded and available on the YouTube.

0-1 knapsack problem

- **Instance:** A list of weights w_i and values v_i for **discrete items** a_i, $1 \leq i \leq n$, and a maximal weight limit W of your knapsack.
- **Task:** Find a subset S of all items available such that its weight does not exceed W and its value is maximal.
- Can we always choose the item with the highest value per unit weight?
- Assume there are just three items with weights and values: (10kg, \$60), (20kg, \$100), (30kg, \$120) and a knapsack of capacity W = 50kg.
- Greedy would choose (10kg, \$60) and (20kg, \$100), while the optimal solution is to take (20kg, \$100) and (30kg, \$120)!
- So when does the Greedy Strategy work??
- Unfortunately there is no easy rule...