Algorithms: COMP3121/3821/9101/9801

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

LECTURE 8: STRING MATCHING ALGORITHMS

String Matching algorithms

- Assume that you want to find out if a string $B = b_1 b_2 \dots b_m$ appears as a (contiguous) substring of a much longer string $A = a_1 a_2 \dots a_n$.

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- The "naive" string matching algorithm does not work well if B is much longer than what can fit in a single register; we need something cleverer.
- Assume that strings A and B are in an alphabet $\mathcal{A}$ with d many symbols in total.

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- We can identify each string with a sequence of integers by mapping each symbol s_i into a corresponding integer i:

$$\mathcal{A} = \{s_0, s_1, s_2, \dots, s_{d-1}\} \longrightarrow \{0, 1, 2, \dots, d-1\}$$

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- Now to every string $B = b_1 b_2 \ldots b_m$ we can associate an integer whose digits in base d are integers corresponding to each symbol in B:

$$h(B) = h(b_1 b_2 \ldots b_m) = d^{m-1} b_1 + d^{m-2} b_2 + \ldots + d \cdot b_{m-1} + b_m$$

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- This can be done efficiently using the Horner's rule:

$$h(B) = b_m + d(b_{m-1} + d(b_{m-2} + d(b_{m-3} + \ldots + d(b_2 + d \cdot b_1))) \ldots)$$

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- Next choose a large prime number p such that $(d+1)\,p$ fits in a single register and define the hash value of B as $H(B) = h(B) \bmod p$.

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- For each contiguous substring $A_s = a_s a_{s+1} a_{s+m-1}$ of string A we also compute its hash value as

$$H(A_s) = (d^{m-1}a_s + d^{m-2}a_{s+1} + \ldots + d^1 a_{s+m-2} + a_{s+m-1}) \bmod p$$

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- This is where recursion comes into play: we do not have compute the hash value $H(A_{s+1})$ of $A_{s+1} = a_{s+1} a_{s+2} \ldots a_{s+m}$ "from scratch", but we can compute it efficiently from the hash value $H(A_s)$ of $A_s = a_s a_{s+2} \ldots a_{s+m-1}$ as follows.

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Consequently, $H(A_{s+1}) = (d \cdot H(A_s) - (d^m \bmod p)a_s + a_{s+m}) \bmod p$.

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- Thus, since we chose p such that $(d+1)\,p$ fits in a register, all the values and the intermediate results for the above expression also fit in a single register.

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- Thus, since we chose p such that $(d + 1)\, p$ fits in a register, all the values and the intermediate results for the above expression also fit in a single register.

- The value of $H(A_s)$ can be computed in constant time independent of the length of the strings A and B.

String matching finite automata

- A string matching finite automaton for a string S with k symbols has $k+1$ many states $0, 1, \ldots k$ which correspond to the number of characters matched thus far and a transition function $\delta(s, c)$ where s is a state and c is a character, given by a table.

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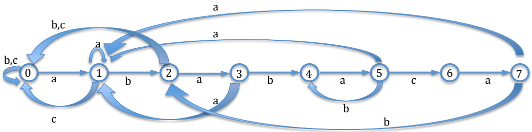
- To make things easier to describe, we consider the string $S = ababaca$. The table defining $\delta(s, c)$ would then be

	input			
state	a	b	c	
0	**1**	0	0	a
1	1	**2**	0	b
2	**3**	0	0	a
3	1	**4**	0	b
4	**5**	0	0	a
5	1	4	**6**	c
6	**7**	0	0	a
7	1	2	0	



state transition diagram for string *ababaca*

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- Thus, if a happens to be $P[k+1]$, then $m = k+1$ and so $\delta(k, a) = k+1$ and $P_k a = P_{k+1}$.

String matching with finite automata

- We can get by without precomputing $\delta(k, a)$ but instead compute it “on the fly”.
- We do that by matching the string against itself: we can recursively compute a function $\pi(k)$ which for each k returns the largest integer m such that the prefix P_m of P is a proper suffix of P_k.

The Knuth-Morris-Pratt algorithm

function $\text{Compute} - \text{Prefix} - \text{Function}(P)$

  $m \leftarrow \text{length}[P]$

  let $\pi[1..m]$ be a new array

  $\pi[1] = 0$

  $k = 0$

  for $q = 2$ to m **do**

    while $k > 0$ **and** $P[k+1] \neq P[q]$

    $k = \pi[k]$

    if $P[k+1] == P[q]$

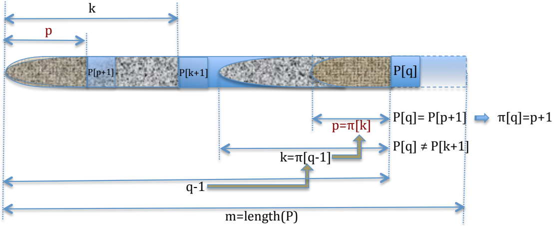
      $k = k + 1$

    $\pi[q] = k$

  end for

  return π

end function



Assume that length of P is m and that we have already found that $\pi[q-1] = k$; to compute $\pi[q]$ we check if $P[q] = P[k+1]$; if it is not; then $\pi[q] \neq k+1$ and we find $\pi[k] = p$; if now $P[q] = P[k+1]$ then $\pi[q] = p+1$.

The Knuth-Morris-Pratt algorithm

- We can now do our search for string P in a longer string T:

```
function KMP − Matcher(T, P)
    n ← length[T]
    m ← length[P]
    π = Compute − Prefix − Function(P)
    q = 0
    for i = 2 to n do
        while q > 0 and P[q + 1] ≠ T[i]
        q = π[q]
        if  P[q + 1] == T[i]
            q = q + 1
        if  q == m
        print pattern occurs with shift i − m
        q = π[q]
    end for
end function
```