Algorithms: COMP3121/3821/9101/9801

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TOPIC 2: FAST LARGE INTEGER MULTIPLICATION

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

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            X X X X   <- first input integer
          * X X X X   <- second input integer
            -------
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          X X X X      \ O(n^2) intermediate operations:
        X X X X        / O(n^2) elementary  multiplications
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- **Can we do it faster than in n^2 many steps??**

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$$\begin{aligned} AB &= A_1B_1 2^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \end{aligned}$$

function $\textsc{Mult}(A, B)$

  if $|A| = |B| = 1$ **then return** AB

  else

    $A_1 \leftarrow \text{MoreSignificantPart}(A)$;

    $A_0 \leftarrow \text{LessSignificantPart}(A)$;

    $B_1 \leftarrow \text{MoreSignificantPart}(B)$;

    $B_0 \leftarrow \text{LessSignificantPart}(B)$;

    $U \leftarrow A_0 + A_1$;

    $V \leftarrow B_0 + B_1$;

    $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0)$;

    $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1)$;

    $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V})$;

    return $W\,2^n + (Y - X - W)\,2^{n/2} + X$

  end if

end function

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- since $\quad 1.5 < \log_2 3 < 1.6$ we have

$$f(n) = c\,n = O(n^{\log_2 3 - \varepsilon}) \quad \text{for any} \quad 0 < \varepsilon < \log_2 3 - 1$$

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- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$$

without going through the messy calculations!

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- So,

$$AB = A_2B_2\,2^{4k} + (A_2B_1 + A_1B_2)2^{3k} + (A_2B_0 + A_1B_1 + A_0B_2)2^{2k} + \\ + (A_1B_0 + A_0B_1)2^k + A_0B_0$$

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$$\begin{aligned}
C_4 &= A_2B_2 \\
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- Can we get these with 5 multiplications only?
- Should we perhaps look at

$$(A_2 + A_1 + A_0)(B_2 + B_1 + B_0) = \\ A_0B_0 + A_1B_0 + A_2B_0 + A_0B_1 + A_1B_1 + A_2B_1 + \\ + A_0B_2 + A_1B_2 + A_2B_2 \quad ???$$

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- Not clear at all how to get $C_0 - C_4$ with 5 multiplications only ...

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- We form naturally corresponding polynomials:

$$P_A(x) = A_2\, x^2 + A_1\, x + A_0;$$
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- Note that

$$A = A_2\, (2^k)^2 + A_1\, 2^k + A_0 = P_A(2^k);$$
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- Thus, we compute

$$P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$$
$$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$

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- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$\begin{aligned}
P_A(-2) &= A_2(-2)^2 + A_1(-2) + A_0 = 4A_2 - 2A_1 + A_0 \\
P_A(-1) &= A_2(-1)^2 + A_1(-1) + A_0 = A_2 - A_1 + A_0 \\
P_A(0) &= A_2 0^2 + A_1 0 + A_0 = A_0 \\
P_A(1) &= A_2 1^2 + A_1 1 + A_0 = A_2 + A_1 + A_0 \\
P_A(2) &= A_2 2^2 + A_1 2 + A_0 = 4A_2 + 2A_1 + A_0.
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- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

$$\begin{aligned} P_B(-2) &= B_2(-2)^2 + B_1(-2) + B_0 = 4B_2 - 2B_1 + B_0 \\ P_B(-1) &= B_2(-1)^2 + B_1(-1) + B_0 = B_2 - B_1 + B_0 \\ P_B(0) &= B_2 0^2 + B_1 0 + B_0 = B_0 \\ P_B(1) &= B_2 1^2 + B_1 1 + B_0 = B_2 + B_1 + B_0 \\ P_B(2) &= B_2 2^2 + B_1 2 + B_0 = 4B_2 + 2B_1 + B_0. \end{aligned}$$

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\end{aligned}$$

- These evaluations involve only additions because $2A = A + A$; $4A = 2A + 2A$.

The Karatsuba trick: slicing into 3 pieces

- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$ we can now obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$ with only 5 multiplications of large numbers:

$$\begin{aligned} P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2) \end{aligned}$$

$$\begin{aligned} P_C(-1) &= P_A(-1)P_B(-1) \\ &= (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(0) &= P_A(0)P_B(0) \\ &= A_0 B_0 \end{aligned}$$

$$\begin{aligned} P_C(1) &= P_A(1)P_B(1) \\ &= (A_0 + A_1 + A_2)(B_0 + B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(2) &= P_A(2)P_B(2) \\ &= (A_0 + 2A_1 + 4A_2)(B_0 + 2B_1 + 4B_2) \end{aligned}$$

The Karatsuba trick: slicing into 3 pieces

- Thus, if we represent the product $C(x) = P_A(x)P_B(x)$ in the coefficient form as $C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ we get

$$\begin{aligned}
C_4(-2)^4 + C_3(-2)^3 + C_2(-2)^2 + C_1(-2) + C_0 &= P_C(-2) = P_A(-2)P_B(-2)\\
C_4(-1)^4 + C_3(-1)^3 + C_2(-1)^2 + C_1(-1) + C_0 &= P_C(-1) = P_A(-1)P_B(-1)\\
C_4 0^4 + C_3 0^3 + C_2 0^2 + C_1 \cdot 0 + C_0 &= P_C(0) = P_A(0)P_B(0)\\
C_4 1^4 + C_3 1^3 + C_2 1^2 + C_1 \cdot 1 + C_0 &= P_C(1) = P_A(1)P_B(1)\\
C_4 2^4 + C_3 2^3 + C_2 2^2 + C_1 \cdot 2 + C_0 &= P_C(2) = P_A(2)P_B(2).
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C_40^4 + C_30^3 + C_20^2 + C_1 \cdot 0 + C_0 &= P_C(0) = P_A(0)P_B(0)\\
C_41^4 + C_31^3 + C_21^2 + C_1 \cdot 1 + C_0 &= P_C(1) = P_A(1)P_B(1)\\
C_42^4 + C_32^3 + C_22^2 + C_1 \cdot 2 + C_0 &= P_C(2) = P_A(2)P_B(2).
\end{aligned}$$

- Doing the simplifications we obtain

$$\begin{aligned}
16C_4 - 8C_3 + 4C_2 - 2C_1 + C_0 &= P_C(-2)\\
C_4 - C_3 + C_2 - C_1 + C_0 &= P_C(-1)\\
C_0 &= P_C(0)\\
C_4 + C_3 + C_2 + C_1 + C_0 &= P_C(1)\\
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- With the coefficients C_0, C_1, C_2, C_3, C_4 obtained, we can now form the polynomial $P_C(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4$.

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- We can now compute $P_C(2^k) = C_0 + C_1 2^k + C_2 2^{2k} + C_3 2^{3k} + C_4 2^{4k}$ in linear time, because computing $P_C(2^k)$ involves only binary shifts of the coefficients plus $O(k)$ additions.

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- Thus we have obtained $A \cdot B = P_A(2^k) P_B(2^k) = P_C(2^k)$ with only 5 multiplications!

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- Thus we have obtained $A \cdot B = P_A(2^k) P_B(2^k) = P_C(2^k)$ with only 5 multiplications!
- Here is the complete algorithm:

function $\textsc{Mult}(A, B)$

obtain A_0, A_1, A_2 and B_0, B_1, B_2 such that $A = A_2\, 2^{2\,k} + A_1\, 2^k + A_0$; $\quad B = B_2\, 2^{2\,k} + B_1\, 2^k + B_0$;

form polynomials $P_A(x) = A_2 x^2 + A_1 x + A_0$; $\quad P_B(x) = B_2 x^2 + B_1 x + B_0$;

$$
\begin{array}{ll}
P_A(-2) \leftarrow 4A_2 - 2A_1 + A_0 & P_B(-2) \leftarrow 4B_2 - 2B_1 + B_0 \\
P_A(-1) \leftarrow A_2 - A_1 + A_0 & P_B(-1) \leftarrow B_2 - B_1 + B_0 \\
P_A(0) \leftarrow A_0 & P_B(0) \leftarrow B_0 \\
P_A(1) \leftarrow A_2 + A_1 + A_0 & P_B(1) \leftarrow B_2 + B_1 + B_0 \\
P_A(2) \leftarrow 4A_2 + 2A_1 + A_0 & P_B(2) \leftarrow 4B_2 + 2B_1 + B_0
\end{array}
$$

$$
\begin{array}{ll}
P_C(-2) \leftarrow \textsc{Mult}(P_A(-2), P_B(-2)); & P_C(-1) \leftarrow \textsc{Mult}(P_A(-1), P_B(-1)); \\
P_C(0) \leftarrow \textsc{Mult}(P_A(0), P_B(0)); & \\
P_C(1) \leftarrow \textsc{Mult}(P_A(1), P_B(1)); & P_C(2) \leftarrow \textsc{Mult}(P_A(2), P_B(2))
\end{array}
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\begin{array}{l}
C_0 \leftarrow P_C(0); \qquad C_1 \leftarrow \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12} \\
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form $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$; compute
$P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$

return $P_C(2^k) = A \cdot B$.

end function

The Karatsuba trick: slicing into 3 pieces

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- We have replaced a multiplication of two n bit numbers with 5 multiplications of $n/3$ bit numbers with an overhead of additions, shifts and the similar, all doable in linear time $c\,n$;

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  we have $a = 5,\ \ b = 3$, so we consider $n^{\log_b a} = n^{\log_3 5} \approx n^{1.465...}$

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  we have $a = 5,\ \ b = 3$, so we consider $n^{\log_b a} = n^{\log_3 5} \approx n^{1.465...}$
- Clearly, the first case of the MT applies and we get $T(n) = O(n^{\log_3 5}) < O(n^{1.47})$.

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- Recall that the original Karatsuba algorithm runs in time

$$n^{\log_2 3} \approx n^{1.58} > n^{1.47}.$$

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- Thus, we got a significantly faster algorithm.
- Then why not slice numbers A and B into even larger number of slices? Maybe we can get even faster algorithm?
- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $n+1$ many equal slices...

Generalizing Karatsuba's algorithm

The general case - slicing the input numbers A, B into $n + 1$ many slices

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- For simplicity, let A, B have $(n+1)k$ bits; (k can be arbitrarily large)
- Slice A, B into $n+1$ pieces each:

$$\begin{aligned} A &= A_n 2^{kn} + A_{n-1} 2^{k(n-1)} + \cdots + A_0 \\ B &= B_n 2^{kn} + B_{n-1} 2^{k(n-1)} + \cdots + B_0 \end{aligned}$$

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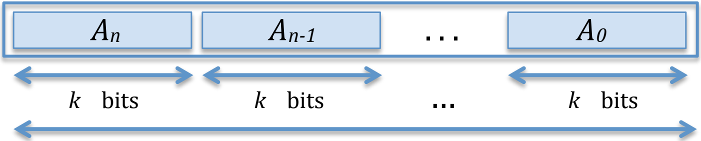

A divided into $n+1$ slices each slice k bits = $(n+1)\,k$ bits in total

- We form the naturally corresponding polynomials:

$$
\begin{aligned}
P_A(x) &= A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0 \\
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$$P_C(x) = \sum_{j=0}^{2n} C_j x^j$$

Generalizing Karatsuba's algorithm

- Example:

$$
\begin{aligned}
(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\
a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
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+(a_0b_1 + a_1b_0)x + a_0b_0
\end{aligned}
$$

- In general: for

$$
\begin{aligned}
P_A(x) &= A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0 \\
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- In general: for

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Generalizing Karatsuba's algorithm

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$$P_A(x) = A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0$$
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we have

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \left(\sum_{i+k=j} A_iB_k \right) x^j = \sum_{j=0}^{2n} C_j\, x^j$$

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- We need to find the coefficients $C_j = \sum_{i+k=j} A_iB_k$ without performing $(n+1)^2$ many multiplications necessary to get all products of the form A_iB_k.

A VERY IMPORTANT DIGRESSION:

If you have two sequences $\vec{A} = (A_0, A_1, \ldots, A_{n-1}, A_n)$ and $\vec{B} = (B_0, B_1, \ldots, B_{m-1}, B_m)$, and if you form the two corresponding polynomials

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then the sequence $\vec{C} = (C_0, C_1, \ldots, C_{n+m})$ of the coefficients of the product polynomial, with these coefficients given by

$$C_j = \sum_{i+k=j} A_i B_k, \quad \text{for} \quad 0 \leq j \leq n+m,$$

is **EXTREMELY IMPORTANT** and is called **the linear convolution** of sequences $\vec{A}$ and $\vec{B}$ and is denoted by $\vec{C} = \vec{A} \star \vec{B}$.

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- Convolutions are bread-and-butter of signal processing, and for that reason it is **extremely important** to find fast ways of multiplying two polynomials of possibly very large degrees.
- In signal processing these degrees can be greater than 1000.
- This is the main reason for us to study methods of fast computation of convolutions (aside of finding products of large integers, which is what we are doing at the moment).

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n+1$ distinct input values $x_0, x_1, \ldots, x_n$:

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- It can be shown that, if x_i are all distinct then this matrix is invertible.
- Such a matrix is called *the Vandermonde matrix*.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are all distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_n)$ the coefficients $A_0, A_1, \ldots, A_n$ of the polynomial $P_A(x)$ are uniquely determined:

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  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

Coefficient vs value representation of polynomials- ctd.

- If we fix the inputs $x_0, x_1, \ldots, x_n$ then commuting between a representation of a polynomial $P_A(x)$ via its coefficients and a representation via its values at these points is done via the following two matrix multiplications, with matrices made up from **constants**:

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- Thus, for fixed input values $x_0, \ldots, x_n$ this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

1. Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

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$$P_A(x)P_B(x) \leftrightarrow \{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\}$$

Our strategy to multiply polynomials fast:

1. Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

convert them into value representation at $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_{2n}, P_A(x_{2n}))\}$$

$$P_B(x) \leftrightarrow \{(x_0, P_B(x_0)), (x_1, P_B(x_1)), \ldots, (x_{2n}, P_B(x_{2n}))\}$$

- **Note:** since the product of the two polynomials will be of degree $2n$ we need the values of $P_A(x)$ and $P_B(x)$ at $2n+1$ points, rather than just $n+1$ points!

2. Multiply these two polynomials point-wise, using $2n+1$ multiplications only.

$$P_A(x)P_B(x) \leftrightarrow \{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\}$$

3. Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

Fast multiplication of polynomials - continued

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- Remember that $n+1$ is the number of slices we split the input numbers A, B.

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- Thus, all the values

$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \cdots + A_0 : \quad -n \leq m \leq n,$$

$$P_B(m) = B_n m^n + B_{n-1} m^{n-1} + \cdots + B_0 : \quad -n \leq m \leq n.$$

can be found in time linear in the number of bits of the input numbers!

Fast multiplication of polynomials - ctd.

- We now perform $2n + 1$ **multiplications of large numbers** to obtain

$$P_A(-n)P_B(-n), \ \dots\ , \ P_A(-1)P_B(-1), \quad P_A(0)P_B(0), \quad P_A(1)P_B(1), \ \dots\ , \ P_A(n)P_B(n)$$

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- For $P_C(x) = P_A(x)P_B(x)$ these products are $2n+1$ many values of $P_C(x)$:

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- We now have:

$$\begin{aligned}
C_{2n}(-n)^{2n} + C_{2n-1}(-n)^{2n-1} + \dots + C_0 &= P_C(-n)\\
C_{2n}(-(n-1))^{2n} + C_{2n-1}(-(n-1))^{2n-1} + \dots + C_0 &= P_C(-(n-1))\\
&\vdots\\
C_{2n}(n-1)^{2n} + C_{2n-1}(n-1)^{2n-1} + \dots + C_0 &= P_C(n-1)\\
C_{2n}n^{2n} + C_{2n-1}n^{2n-1} + \dots + C_0 &= P_C(n)
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- This is just a system of linear equations, that can be solved for $C_0, C_1, \ldots, C_{2n}$:

$$\begin{pmatrix} 1 & -n & (-n)^2 & \ldots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \ldots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \ldots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

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- Thus the coefficients C_i can be obtained in linear time.
- So here is the algorithm we have just described:

function $\textsc{Mult}(n, A, B)$

if $|A| = |B| = 1$ **then return** AB

else

obtain $n+1$ slices $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ such that

$$A = A_n 2^{n\,k} + A_{n-1} 2^{(n-1)\,k} + \ldots + A_0$$

$$B = B_n 2^{n\,k} + B_{n-1} 2^{(n-1)\,k} + \ldots + B_0$$

form polynomials

$$P_A(x) = A_n x^n + A_{n-1} x^{(n-1)} + \ldots + A_0$$

$$P_B(x) = B_n x^n + B_{n-1} x^{(n-1)} + \ldots + B_0$$

for $m = -n$ to $m = n$ **do**

compute $P_A(m)$ and $P_B(m)$;

$P_C(m) \leftarrow \textsc{Mult}(n, P_A(m) P_B(m))$

end for

compute $C_0,\ C_1, \ldots\ C_{2n}$ via

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -n & (-n)^2 & \ldots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \ldots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \ldots & n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix}.$$

form $P_C(x) = C_{2n} x^{2n} + \ldots + C_0$ and compute $P_C(2^k)$

return $P_C(2^k) = A \cdot B$

end if

end function

How fast is our algorithm?

- For each m such that $-n \leq m \leq n$, the value of $P_A(m)$ is a sum of $n+1$ values;
- Each value is a product of a k-bit number A_j with a constant m^j, $-n \leq m \leq n$:

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- The number of bits of each $P_A(m)$ for $-n \leq m \leq n$ is at most

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- But $s = \log_2(n+1) + n \log_2 n$ is constant; Thus,
- **Each $P_A(m)$ has $k+s$ bits, where s is a constant independent of k.**

Generalizing Karatsuba's algorithm

- Recall that

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \underbrace{\left(\sum_{i+m=j} A_i B_m \right)}_{C_j} x^j = \sum_{j=0}^{2n} C_j \, x^j, \quad \text{where} \quad C_j = \sum_{i+m=j} A_i B_m$$

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- Thus, C_j **has at most** $2k + \log_2(n+1)$ **many bits**

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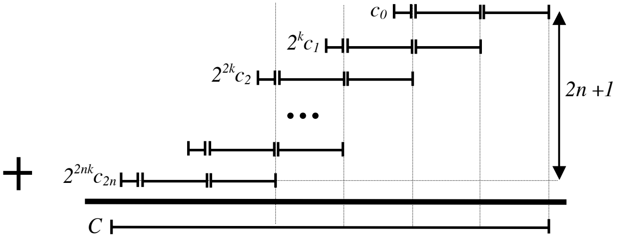

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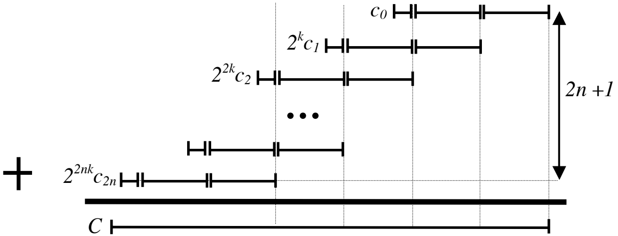

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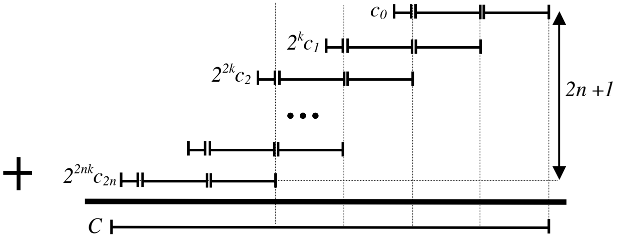

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- Thus, **evaluation of $P_C(2^k)$ takes $O(k)$ many steps.**

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- so we get:

$$T(N) = \Theta\left(N^{\log_b a}\right) = \Theta\left(N^{\log_{n+1}(2n+1)}\right)$$

- Note that

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- Thus, we would have to slice the input numbers into $2^{10} = 1024$ pieces!!

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- Consequently, slicing the input numbers in more than just a few slices results in a hopelessly slow algorithm, despite the fact that the asymptotic bounds improve as we increase the number of slices!
- The moral is: **In practice, asymptotic estimates are useless if the size of the constants hidden by the O-notation are not estimated and found to be reasonably small!!!**

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- After we study the FFT we will have a guest lecture by a Dolby engineer to demonstrate to you some cool applications of FFT.