Algorithms: COMP3121/3821/9101/9801

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

TOPIC 1: RECURRENCES

Asymptotic notation

- **“Big Oh” notation:** $f(n) = O(g(n))$ is an abbreviation for:

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- **"Theta" notation:** $f(n) = \Theta(g(n))$ iff and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$; thus, $f(n)$ and $g(n)$ have the same asymptotic growth rate.

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- Since Merge(A, p, q, r) runs in linear time, the runtime $T(n)$ of Merge-Sort(A, p, r) satisfies

$$T(n) = 2T\left(\frac{n}{2}\right) + c\,n$$

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but it can be shown that assuming that n is a power of b is OK, and that the estimate produced is still valid for all n.

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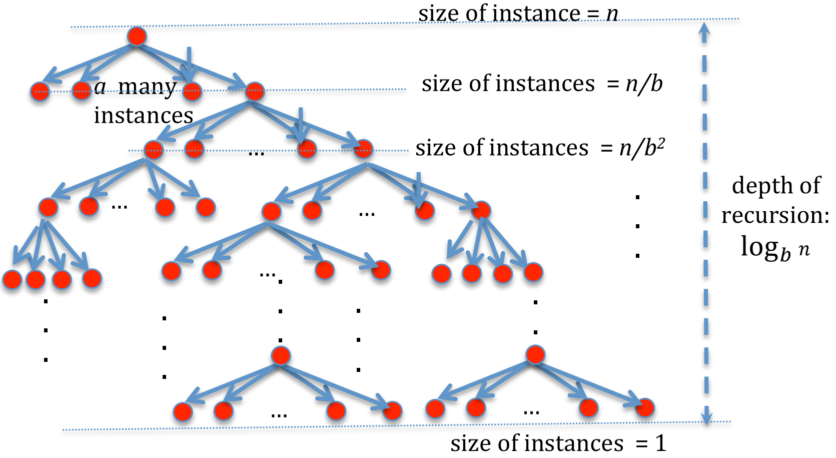

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- This is what the **Master Theorem** provides (when it is applicable).

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4. If none of these conditions hold, the Master Theorem is NOT applicable (in the form presented).

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- So whenever we have $f = \Theta(g(n) \log n)$ we do not have to specify what base the log is - all bases produce equivalent asymptotic estimates.

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  Thus, condition of case 2 is satisfied; and so,

$$T(n) = \Theta(n^{\log_2 2} \log n) = \Theta(n \log n).$$

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  - Thus, Case 3 applies, and $T(n) = \Theta(f(n)) = \Theta(n)$.
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  - Thus, in this case the Master Theorem does **not** apply!

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Note that so far we did not use any assumptions on $f(n)$...

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$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i O\left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon}$$

$$= O\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon}\right) = O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a - \varepsilon}}\right)\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a - \varepsilon}}\right)^i\right) = O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a} b^{-\varepsilon}}\right)^i\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a\, b^{\varepsilon}}{a}\right)^i\right) = O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} (b^{\varepsilon})^i\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{(b^{\varepsilon})^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right); \qquad \text{we are using } \sum_{i=0}^{m} q^m = \frac{q^{m+1} - 1}{q - 1}$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^{\varepsilon})^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

Master Theorem Proof:

Case 1 - continued:

$$\begin{aligned}\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) &= O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right) \\ &= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right) \\ &= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right)\end{aligned}$$

Master Theorem Proof:

Case 1 - continued:

$$
\begin{aligned}
\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) &= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\varepsilon}\right)^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right) \\
&= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^{\varepsilon} - 1}\right)
\end{aligned}
$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$
$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right)$$
$$= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right)$$
$$= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^\varepsilon - 1}\right)$$
$$= O\left(n^{\log_b a}\right)$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Since we had: $T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$ we get:

Master Theorem Proof:

Case 1 - continued:

$$\begin{aligned}
\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) &= O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right) \\
&= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a}\right)
\end{aligned}$$

Since we had: $\quad T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \quad$ we get:

$$\begin{aligned}
T(n) &\approx n^{\log_b a} T(1) + O\left(n^{\log_b a}\right) \\
&= \Theta\left(n^{\log_b a}\right)
\end{aligned}$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

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$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$
\begin{aligned}
\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) &= \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a} \\
&= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right) \\
&= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right) \\
&= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right) \\
&= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1\right) \\
&= \Theta\left(n^{\log_b a} \lfloor \log_b n \rfloor\right)
\end{aligned}
$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

because $\log_b n = \log_2 n \cdot \log_b 2 = \Theta(\log_2 n)$. Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

because $\log_b n = \log_2 n \cdot \log_b 2 = \Theta(\log_2 n)$. Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

we get:

$$\begin{aligned} T(n) &\approx n^{\log_b a} T(1) + \Theta\left(n^{\log_b a} \log_2 n\right) \\ &= \Theta\left(n^{\log_b a} \log_2 n\right) \end{aligned}$$

Master Theorem Proof:

Case 3: $f(m) = \Omega(m^{\log_b a + \varepsilon})$ and $a\, f(n/b) \leq c\, f(n)$ for some $0 < c < 1$.

We get by substitution:

$$f(n/b) \leq \frac{c}{a}\, f(n)$$

$$f(n/b^2) \leq \frac{c}{a}\, f(n/b)$$

$$f(n/b^3) \leq \frac{c}{a}\, f(n/b^2)$$

$$\cdots$$

$$f(n/b^i) \leq \frac{c}{a}\, f(n/b^{i-1})$$

Master Theorem Proof:

Case 3: $f(m) = \Omega(m^{\log_b a+\varepsilon})$ and $a\, f(n/b) \leq c\, f(n)$ for some $0 < c < 1$.

We get by substitution:

$$f(n/b) \leq \frac{c}{a}\, f(n)$$
$$f(n/b^2) \leq \frac{c}{a}\, f(n/b)$$
$$f(n/b^3) \leq \frac{c}{a}\, f(n/b^2)$$
$$\dots$$
$$f(n/b^i) \leq \frac{c}{a}\, f(n/b^{i-1})$$

By chaining these inequalities we get

$$f(n/b^2) \leq \frac{c}{a}\, \underbrace{f(n/b)} \leq \frac{c}{a} \cdot \underbrace{\frac{c}{a}\, f(n)} = \frac{c^2}{a^2}\, f(n)$$

$$f(n/b^3) \leq \frac{c}{a}\, \underbrace{f(n/b^2)} \leq \frac{c}{a} \cdot \underbrace{\frac{c^2}{a^2}\, f(n)} = \frac{c^3}{a^3}\, f(n)$$

$$\dots$$

$$f(n/b^i) \leq \frac{c}{a}\, \underbrace{f(n/b^{i-1})} \leq \frac{c}{a} \cdot \underbrace{\frac{c^{i-1}}{a^{i-1}}\, f(n)} = \frac{c^i}{a^i}\, f(n)$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i}\, f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i}\, f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Case 3 (continued):

We got $$f(n/b^i) \le \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \le \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \le \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \le \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \le \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \le \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

Case 3 (continued):

We got
$$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

thus,

$$T(n) = \Theta(f(n))$$

Master Theorem Proof: Homework

Exercise 1: Show that condition

$$f(n) = \Omega(n^{\log_b a + \varepsilon})$$

follows from the condition

$$a\, f(n/b) \leq c\, f(n) \;\; \text{for some } 0 < c < 1.$$

Exercise 2: Estimate $T(n)$ for

$$T(n) = 2\, T(n/2) + n \log n$$

Note: we have seen that the Master Theorem does **NOT** apply, but the technique used in its proof still works! Just unwind the recurrence and sum up the logarithmic overheads.