



Algorithms: COMP3121/3821/9101/9801

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TOPIC 3: THE FAST FOURIER TRANSFORM

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n + 1$ distinct input values x_0, x_1, \dots, x_n :

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- It can be shown that if x_i are all distinct, then this matrix is invertible.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are distinct, given any values $P_A(x_0), P_A(x_1), \dots, P_A(x_n)$ the coefficients A_0, A_1, \dots, A_n are uniquely determined:

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 - 2 a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \dots, (x_n, P_A(x_n))\}$$

Our strategy to multiply polynomials fast:

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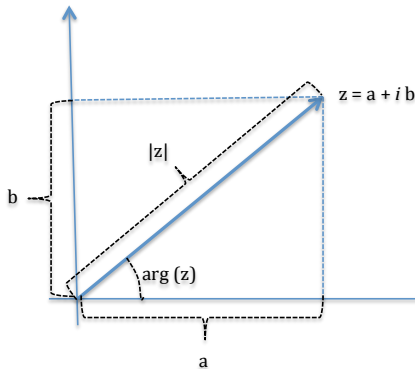
- Key Question:** What values should we take for x_0, \dots, x_{2n} to avoid “explosion” of size when we evaluate x_i^n while computing $P_A(x_i) = A_n x_i^n + \dots + A_0$?

Complex numbers revisited

Complex numbers $z = a + ib$ can be represented using their *modulus* $|z| = \sqrt{a^2 + b^2}$ and their *argument*, $\arg(z)$, which is an angle taking values in $(-\pi, \pi]$ and satisfying:

$$z = |z|e^{i \arg(z)} = |z|(\cos \arg(z) + i \sin \arg(z)),$$

see figure below.

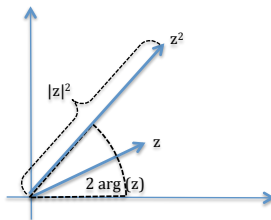


Complex numbers revisited

Recall that

$$z^n = \left(|z|e^{i \arg(z)}\right)^n = |z|^n e^{i n \arg(z)} = |z|^n (\cos(n \arg(z)) + i \sin(n \arg(z))),$$

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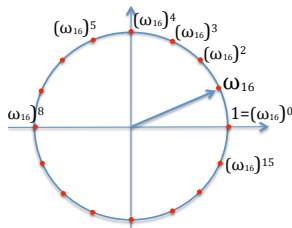
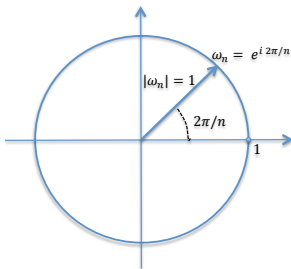
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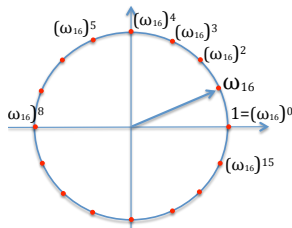
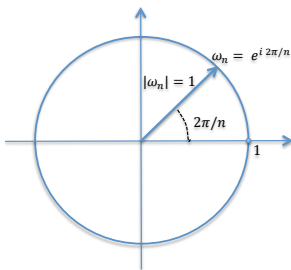
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Roots of unity of order 16

- A root of unity ω of order n is *primitive* if all other roots of unity of the same order can be obtained as its powers ω^k .

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- For a product of any two roots of unity ω_n^k and ω_n^m of the same order we have $\omega_n^k \omega_n^m = \omega_n^{k+m} = \omega_n^l$ where $0 \leq l < n$ and $l = (k+m) \bmod n$.

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- So in the set of all roots of unity of order n , i.e., $\{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\}$ we can multiply any two elements or raise an element to any power without going out of this set.

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- Note that this is not true for addition, i.e., the sum of two roots of unity is NOT another root of unity!

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- Thus, in particular, $(\omega_{2n}^k)^2 = (\omega_{2n}^2)^k = \omega_{2n}^{2k} = \omega_n^k$;
- So squares of the roots of unity of order $2n$ are just the roots of unity of order n .

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- We evaluate it at all complex roots of unity of order n , i.e., we compute $P_A(\omega_n^k)$ for all $0 \leq k \leq n-1$.
- The sequence of values $\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$, is called **the Discrete Fourier Transform (DFT)** of the sequence $A = \langle A_0, A_1, \dots, A_{n-1} \rangle$.

New way for fast multiplication of polynomials

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- This produces the DFT of the (0 padded) sequence of their coefficients $(A_0, A_1, \dots, A_n, \underbrace{0, \dots, 0}_n)$.

New way for fast multiplication of polynomials

- To multiply two polynomials $P_A(x)$ and $P_B(x)$ of degree (at most) n we will evaluate them at the roots of unity of order $2n+1$ (instead of at $-n, \dots, -1, 0, 1, \dots, n$ as in Karatsuba's method!)
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- This produces the DFT of the (0 padded) sequence of their coefficients $(A_0, A_1, \dots, A_n, \underbrace{0, \dots, 0}_n)$.
- We will then multiply the corresponding values $P_A(\omega_{2n+1}^k)$ and $P_B(\omega_{2n+1}^k)$.
- We then use the inverse transformation for DFT, called IDFT, to recover the coefficients of the product polynomial from its values at these roots of unity.

New way for fast multiplication of polynomials

$$P_A(x) = A_0 + \dots + A_n x^n + 0 \cdot x^{n+1} + \dots + 0 \cdot x^{2n}; P_B(x) = B_0 + \dots + B_n x^n + 0 \cdot x^{n+1} + \dots + 0 \cdot x^{2n}$$

↓ DFT

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$$\{P_A(1), P_A(\omega_{2n+1}), P_A(\omega_{2n+1}^2), \dots, P_A(\omega_{2n+1}^{2n})\}; \quad \{P_B(1), P_B(\omega_{2n+1}), P_B(\omega_{2n+1}^2), \dots, P_B(\omega_{2n+1}^{2n})\}$$

↓ multiplication

$$\{P_A(1)P_B(1), P_A(\omega_{2n+1})P_B(\omega_{2n+1}), \dots, P_A(\omega_{2n+1}^{2n})P_B(\omega_{2n+1}^{2n})\}$$

↓ IDFT

$$P_C(x) = \sum_{j=0}^{2n} \underbrace{\left(\sum_{i=0}^j A_i B_{j-i} \right)}_{C_j} x^j = \sum_{j=0}^{2n} C_j x^j = P_A(x) \cdot P_B(x)$$

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$$P_A(\omega_{2n+1}^k) = A_0 + A_1\omega_{2n+1}^k + A_2\omega_{2n+1}^{2k} + \dots + A_n\omega_{2n+1}^{nk}$$

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- Thus, since m ranges from 0 to n , we would have to do $O(n^2)$ multiplications.
- Can we do it faster??
- This is precisely what the **Fast Fourier Transform (FFT)** does; it computes the values $P_A(\omega_n^k)$ for all k such that $0 \leq k < n$ in $O(n \log n)$ time.

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 - Exercise: Show that for every n which is not a power of two the smallest power of 2 larger or equal to n is smaller than $2n$.
 - *Hint*: consider n in binary. How many bits does the nearest power of two have?

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- Note that the degree of the polynomials $A^0(y)$ and $A^1(y)$ is $n/2 - 1$, while the degree of the polynomial $P_A(x)$ is $n - 1$. Thus, the degree of the polynomials $A^0(y)$ and $A^1(y)$ is only about one half of the degree of the polynomial $P_A(x)$.

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- We reduced evaluation of our polynomial $P_A(x)$ of degree $n - 1$ at inputs $x = \omega_n^0, x = \omega_n^1, x = \omega_n^2, \dots, x = \omega_n^{n-1}$ to evaluation of two polynomials $A^0(y)$ and $A^1(y)$ of degree $n/2 - 1$, at points $y = x^2$ for the same values of inputs x .

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- However, as x ranges through values $\{\omega_n^0, \omega_n^1, \omega_n^2, \dots, \omega_n^{n-1}\}$, the value of $y = x^2$ ranges through $\{\omega_{\frac{n}{2}}^0, \omega_{\frac{n}{2}}^1, \omega_{\frac{n}{2}}^2, \dots, \omega_{\frac{n}{2}}^{n-1}\}$, and there are only $n/2$ distinct such values.

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- Once we get these $n/2$ values of $A^0(x^2)$ and $A^1(x^2)$ we need n additional multiplications with numbers ω_n^k to obtain the values of

$$\begin{aligned} P_A(\omega_n^k) &= A^0((\omega_n^k)^2) + \omega_n^k \cdot A^1((\omega_n^k)^2) \\ &= A^0(\omega_{n/2}^k) + \omega_n^k \cdot A^1(\omega_{n/2}^k). \end{aligned}$$

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- Thus, we have reduced a problem of size n to two such problems of size $n/2$, plus a linear overhead.

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- Note that by the Cancellation Lemma $\omega_n^{\frac{n}{2}} = \omega_{2^{\frac{n}{2}}}^{\frac{n}{2}} = \omega_2 = -1$;

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$$\begin{aligned} P_A(\omega_n^{\frac{n}{2}+m}) &= A^0((\omega_n^{\frac{n}{2}+m})^2) + \omega_n^{\frac{n}{2}+m} A^1((\omega_n^{\frac{n}{2}+m})^2) \\ &= A^0(\omega_n^{n+2m}) + \omega_n^{\frac{n}{2}} \omega_n^m A^1(\omega_n^{n+2m}) \\ &= A^0(\omega_n^n \omega_n^{2m}) + \omega_{2\frac{n}{2}} \omega_n^m A^1(\omega_n^n \omega_n^{2m}) \\ &= A^0(\omega_n^{2m}) + \omega_2 \omega_n^m A^1(\omega_n^{2m}) \\ &= A^0((\omega_n^m)^2) - \omega_n^m A^1((\omega_n^m)^2) \end{aligned}$$

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- Compare this with $P_A(\omega_n^m) = A^0((\omega_n^m)^2) + \omega_n^m A^1((\omega_n^m)^2)$

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- We can now write a pseudo-code for our FFT algorithm:

FFT algorithm

```
1: function FFT( $A$ )
2:    $n \leftarrow \text{length}[A]$ 
3:   if  $n = 1$  then return  $A$ 
4:   else
5:      $A^{[0]} \leftarrow (A_0, A_2, \dots, A_{n-2})$ ;
6:      $A^{[1]} \leftarrow (A_1, A_3, \dots, A_{n-1})$ ;
7:      $y^{[0]} \leftarrow \text{FFT}(A^{[0]})$ ;
8:      $y^{[1]} \leftarrow \text{FFT}(A^{[1]})$ ;
9:      $\omega_n \leftarrow e^{i\frac{2\pi}{n}}$ ;
10:     $\omega \leftarrow 1$ ;
11:    for  $k = 0$  to  $k = \frac{n}{2} - 1$  do;
12:       $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$ ;
13:       $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$ 
14:       $\omega \leftarrow \omega \cdot \omega_n$ ;
15:    end for
16:    return  $y$ 
17:  end if
18: end function
```

How fast is the Fast Fourier Transform?

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- The Master Theorem gives $T(n) = \Theta(n \log n)$.

Matrix representation of polynomial evaluation

- Evaluation of a polynomial $P_A(x) = A_0 + A_1x + \dots + A_{n-1}x^{n-1}$ at roots of unity ω_n^k of order n can be represented in the matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2 \cdot 2} & \dots & \omega_n^{2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} \quad (3)$$

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- From $P_A(1) = P_A(\omega_n^0)$, $P_A(\omega_n)$, $P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1})$, we get the coefficients from

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2 \cdot 2} & \dots & \omega_n^{2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} \quad (4)$$

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To see this, note that if we compute the product

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So,

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(5)

So we get:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2 \cdot 2} & \dots & \omega_n^{2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2 \cdot 2} & \dots & \omega_n^{-2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \\
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i.e.,

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 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2 \cdot 2} & \dots & \omega_n^{2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} = \\
 &= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2 \cdot 2} & \dots & \omega_n^{-2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}
 \end{aligned}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$$

back to the coefficient form

$$P_A(x) = A_0 + A_1x + A_2x^2 + A_{n-1}x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:

- 1 the root of unity ω_n is replaced by $\overline{\omega_n} = \omega_n^{-1} = e^{-i \frac{2\pi}{n}}$,

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we can use **the same** FFT algorithm with the only change that:

- 1 the root of unity ω_n is replaced by $\overline{\omega_n} = \omega_n^{-1} = e^{-i\frac{2\pi}{n}}$,
- 2 the resulting output values are divided by n .

Inverse Fast Fourier Transform (IFFT):

```
1: function IFFT*(A)
2:    $n \leftarrow \text{length}(A)$ 
3:   if  $n = 1$  then return  $A$ 
4:   else
5:      $A^{[0]} \leftarrow (A_0, A_2, \dots, A_{n-2})$ ;
6:      $A^{[1]} \leftarrow (A_1, A_3, \dots, A_{n-1})$ ;
7:      $y^{[0]} \leftarrow \text{IFFT}^*(A^{[0]})$ ;
8:      $y^{[1]} \leftarrow \text{IFFT}^*(A^{[1]})$ ;
9:      $\omega_n \leftarrow e^{-i\frac{2\pi}{n}}$ ; ⇐ different from FFT
10:     $\omega \leftarrow 1$ ;
11:    for  $k = 0$  to  $k = \frac{n}{2} - 1$  do;
12:       $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$ ;
13:       $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$ 
14:       $\omega \leftarrow \omega \cdot \omega_n$ ;
15:    end for
16:    return  $y$ ;
17:  end if
18: end function
```

```
1: function IFFT(A) ⇐ different from FFT
2:   return  $\text{IFFT}^*(A)/\text{length}(A)$ 
3: end function
```

More on DFT

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- However, what CLRS calls DFT, namely, the sequence

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- Taking this as the “forward operation” has an important conceptual advantage and is used more often than the textbook’s choice, especially in electrical engineering literature.

More on DFT

- Another “tweak” of DFT: note that

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2 \cdot 2} & \dots & \omega_n^{2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2 \cdot 2} & \dots & \omega_n^{-2 \cdot (n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \\ = n \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

implies:

More on DFT

$$\begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \vdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{\omega_n}{\sqrt{n}} & \frac{\omega_n^2}{\sqrt{n}} & \vdots & \frac{\omega_n^{n-1}}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{\omega_n^2}{\sqrt{n}} & \frac{\omega_n^{2 \cdot 2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{2 \cdot (n-1)}}{\sqrt{n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{\omega_n^{n-1}}{\sqrt{n}} & \frac{\omega_n^{2(n-1)}}{\sqrt{n}} & \vdots & \frac{\omega_n^{(n-1)(n-1)}}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \vdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{\omega_n^{-1}}{\sqrt{n}} & \frac{\omega_n^{-2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-(n-1)}}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{\omega_n^{-2}}{\sqrt{n}} & \frac{\omega_n^{-2 \cdot 2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-2 \cdot (n-1)}}{\sqrt{n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{\omega_n^{-(n-1)}}{\sqrt{n}} & \frac{\omega_n^{-2(n-1)}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-(n-1)(n-1)}}{\sqrt{n}} \end{pmatrix} \\
 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Thus, these two matrices are inverses of each other.

More on DFT

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- Given a sequence of numbers $\vec{A} = (A_0, A_1, \dots, A_{n-1})$ the Discrete Fourier Transform of this sequence is the sequence of the values of the polynomial

$$P_A^*(x) = \frac{1}{\sqrt{n}} P_A(x) = \frac{1}{\sqrt{n}} (A_0 + A_1x + \dots + A_{n-1}x^{n-1})$$

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for $x = \omega_n^{-k}$ for $k = 0, \dots, n-1$; i.e., the sequence of values $P_A^*(\omega_n^{-k})$:

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- Given a sequence of numbers $(A_0, A_1, \dots, A_{n-1})$ the **Inverse Discrete Fourier Transform** of this sequence is the sequence of the values of the same polynomial

$$P_A^*(x) = \frac{1}{\sqrt{n}} (A_0 + A_1 x + \dots + A_{n-1} x^{n-1})$$

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but for $x = \omega_n^k$ for $k = 0, \dots, n-1$; i.e., the sequence of values $P_A^*(\omega_n^k)$

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More on DFT

```
1: function FFT*(A)
2:    $n \leftarrow \text{length}[A]$ 
3:   if  $n = 1$  then return A
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5:      $A^{[0]} \leftarrow (A_0, A_2, \dots, A_{n-2})$ ;
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```

```
1: function FFT(A)
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Back to fast multiplication of polynomials (convolution)

$$P_A(x) = A_0 + A_1x + \dots + A_nx^n$$

$$P_B(x) = B_0 + B_1x + \dots + B_nx^n$$

↓ DFT $O(n \log n)$

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$$\{P_A(1), P_A(\omega_{2n+1}), P_A(\omega_{2n+1}^2), \dots, P_A(\omega_{2n+1}^{2n})\}; \quad \{P_B(1), P_B(\omega_{2n+1}), P_B(\omega_{2n+1}^2), \dots, P_B(\omega_{2n+1}^{2n})\}$$

↓ multiplication $O(n)$

$$\{P_A(1)P_B(1), P_A(\omega_{2n+1})P_B(\omega_{2n+1}), \dots, P_A(\omega_{2n+1}^{2n})P_B(\omega_{2n+1}^{2n})\}$$

↓ IDFT $O(n \log n)$

$$P_C(x) = \sum_{j=0}^{2n} \underbrace{\left(\sum_{i=0}^j A_i B_{j-i} \right)}_{C_j} x^j = \sum_{j=0}^{2n} C_j x^j = P_A(x) \cdot P_B(x)$$

Thus, convolution can be computed in time $O(n \log n)$.

Interpretation of DFT

- The **scalar product** (also called *dot product*) of two vectors with real coordinates, $\vec{x} = (x_0, x_1, \dots, x_{n-1})$ and $\vec{y} = (y_0, y_1, \dots, y_{n-1})$, denoted by $\langle \vec{x}, \vec{y} \rangle$ is defined as

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$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=0}^{n-1} x_i \overline{x_i}} = \sqrt{\sum_{i=0}^{n-1} |x_i|^2}$$

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Interpretation of DFT

- Note that

$$\begin{aligned}\overline{\omega_n^k} &= \overline{e^{i\frac{2\pi k}{n}}} = \overline{\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}} = \cos \frac{2\pi k}{n} - i \sin \frac{2\pi k}{n} \\ &= \cos \frac{-2\pi k}{n} + i \sin \frac{-2\pi k}{n} = e^{-i\frac{2\pi k}{n}} = \omega_n^{-k}\end{aligned}$$

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- Thus, what we had before,

$$\begin{pmatrix} 1 & \omega_n^k & \omega_n^{2 \cdot k} & \dots & \omega_n^{k \cdot (n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-m} \\ \omega_n^{-2m} \\ \vdots \\ \omega_n^{-(n-1)m} \end{pmatrix} = \sum_{j=0}^{n-1} \omega_n^{(k-m)j} = \begin{cases} n & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases} \quad (6)$$

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simply means that for $k \neq m$ vectors $(1, \omega_n^k, \omega_n^{2 \cdot k}, \dots, \omega_n^{k \cdot (n-1)})$ and $(1, \omega_n^m, \omega_n^{2 \cdot m}, \dots, \omega_n^{m \cdot (n-1)})$ are mutually orthogonal and that their norm is equal to \sqrt{n} .

Interpretation of DFT

- If we define

$$\vec{e}_k = \frac{1}{\sqrt{n}} \left(1, \omega_n^{k \cdot 1}, \omega_n^{k \cdot 2}, \dots, \omega_n^{k \cdot (n-1)} \right)$$

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$$\begin{aligned} \|\vec{e}_k\| &= \sqrt{\langle \vec{e}_k, \vec{e}_k \rangle} = 1 \\ \langle \vec{e}_k, \vec{e}_m \rangle &= 0 \quad \text{for } k \neq m \end{aligned}$$

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and the DFT of this sequence we have

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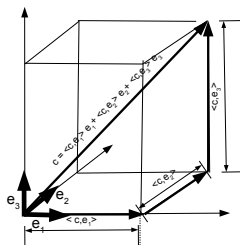
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- Thus, the DFT of a vector \vec{A} is simply the sequence of projections of \vec{A} onto the basis vectors \vec{e}_k , ($k = 0, \dots, n-1$).

Interpretation of DFT

- In an n -dimensional vector space V with an orthonormal basis \mathbf{B} every vector \vec{A} can be represented as a linear combination of the basis vectors with coefficients equal to the projections of \vec{A} onto the basis vectors, i.e., the scalar product $\langle \vec{A}, \vec{e}_k \rangle$:

$$\vec{A} = \langle \vec{A}, \vec{e}_0 \rangle \vec{e}_0 + \langle \vec{A}, \vec{e}_1 \rangle \vec{e}_1 + \dots + \langle \vec{A}, \vec{e}_{n-1} \rangle \vec{e}_{n-1}$$



Representing vector c as a linear combination of the basis vectors e_1, e_2, e_3 with projections as coefficients

Interpretation of DFT

- Thus, in our case

$$\begin{aligned}\vec{A} &= \langle \vec{A}, \vec{e}_0 \rangle \vec{e}_0 + \langle \vec{A}, \vec{e}_1 \rangle \vec{e}_1 + \dots + \langle \vec{A}, \vec{e}_{n-1} \rangle \vec{e}_{n-1} \\ &= P_A^*(\omega_n^0) \vec{e}_0 + P_A^*(\omega_n^{-1}) \vec{e}_1 + \dots + P_A^*(\omega_n^{-(n-1)}) \vec{e}_{n-1}\end{aligned}$$

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- Looking at the k^{th} coordinate of both the left and the right side we get

$$\begin{aligned}A_k &= P_A^*(\omega_n^0) \frac{(\omega_n^0)^k}{\sqrt{n}} + P_A^*(\omega_n^{-1}) \frac{(\omega_n^{-1})^k}{\sqrt{n}} + \dots + P_A^*(\omega_n^{-(n-1)}) \frac{(\omega_n^{-(n-1)})^k}{\sqrt{n}} \\ &= \frac{1}{\sqrt{n}} \left(P_A^*(\omega_n^0) + P_A^*(\omega_n^{-1}) (\omega_n^k)^1 + \dots + P_A^*(\omega_n^{-(n-1)}) (\omega_n^k)^{n-1} \right)\end{aligned}$$

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- Thus, A_k is obtained by evaluating the polynomial

$$Q(x) = \frac{1}{\sqrt{n}} \left(P_A^*(1) + P_A^*(\omega_n^{-1})x + \dots + P_A^*(\omega_n^{-(n-1)})x^{n-1} \right)$$

at $x = \omega_n^k$.

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- But this is precisely the Inverse Discrete Fourier Transform of the sequence

$$\left(P_A^*(1), P_A^*(\omega_n^{-1}), \dots, P_A^*(\omega_n^{-(n-1)}) \right)$$

Interpretation of DFT

- Let us denote the usual orthonormal basis of \mathbb{C}^n by \mathcal{B} :

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and by \mathcal{F} the basis $\mathcal{F} = \{\vec{e}_0, \vec{e}_1, \dots, \vec{e}_{n-1}\}$ where $\vec{e}_k = \left(\frac{1}{\sqrt{n}}, \frac{\omega_n^{1 \cdot k}}{\sqrt{n}}, \frac{\omega_n^{2 \cdot k}}{\sqrt{n}}, \dots, \frac{\omega_n^{(n-1) \cdot k}}{\sqrt{n}} \right)$.

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- Thus, DFT is just a change of basis operation: it transforms the sequence of coordinates

$$(A_0, A_1, A_2, \dots, A_{n-1})_{\mathcal{B}}$$

of a vector \vec{A} in the basis \mathcal{B} into the sequence

$$(P_A^*(\omega_n^0), P_A(\omega_n^{-1}), \dots, P_A(\omega_n^{-(n-1)}))_{\mathcal{F}}$$

of the coordinates of the same vector in the basis \mathcal{F} .

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- This means that the samples $A(k)$ of the signal $A(t)$ at instants $t = 0, 1, \dots, n-1$ are represented as a linear combination of samples at these instants of sinusoids (also called pure harmonic oscillations) $\cos\left(\frac{2\pi m}{n}t\right)$ and $\sin\left(\frac{2\pi m}{n}t\right)$ of increasing frequencies $\frac{2\pi \cdot 1}{n}, \frac{2\pi \cdot 2}{n}, \dots, \frac{2\pi \cdot (n-1)}{n}$;

Interpretation of DFT

- Thus, in a sense the absolute value $|P_A^*(\omega_n^{-m})|$ of the m^{th} coordinate $P_A^*(\omega_n^{-m})$ in the basis \mathcal{F} tells us how much of the sinusoids $\cos(\frac{2\pi m}{n}t)$ and $\sin(\frac{2\pi m}{n}t)$ are present in the signal $A(t)$.

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