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Gravitational Waves in Modified Gravity

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Abstract Modified gravity theories generally aim to solve part of the *cosmological constant problem* by providing self-accelerating cosmological solutions without a cosmological constant. Such modifications of general relativity also affect the evolution of gravitational waves in the proposed theory. Instead of focussing on a specific model, I introduce parametric modifications to the evolution equation of gravitational waves in both unimetric and bimetric settings and investigate their effect on the evolution of tensor perturbation modes. In particular, I argue that any modified gravity theory that exhibits *growing tensor modes* in cosmological evolution can be in tension with experiments. Therefore, parametric constraints for the physical viability of a general modified gravity theory can be found such that tensor modes remain within limits set by observations.

Zusammenfassung

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1. Introduction

For a long time, physicists believed space and time were both flat and fundamentally different concepts. In classical *Newtonian spacetime*, it is therefore straight forward to define notions such as the *length of a curve* and *simultaneity* [Tolish, 2010].

With Maxwell's theory of electromagnetism and experimental observations regarding the speed of light, however, many of these concepts had to be abandoned. Physicists realized the necessity to reconsider fundamental assumptions about space and time. Albert Einstein addressed many of these issues with his theory of special relativity in 1905 [Einstein, 1905]. But, particularly, one striking observation remained unexplained:

Newtonian mechanics postulates that particles accelerate under the influence of any force \mathbf{F} proportional to their *inertial mass* m_i with

$$m_i \cdot \mathbf{a} = \mathbf{F} \quad \text{where } \mathbf{a} \equiv \frac{d^2\mathbf{x}}{dt^2} \text{ denotes acceleration.} \quad (1.1)$$

Simultaneously, the gravitational force

$$|\mathbf{F}_G| = G_N \frac{m_{g,1}m_{g,2}}{|\mathbf{x}_1 - \mathbf{x}_2|^2} \quad \text{with the } \textit{gravitational constant} G_N \quad (1.2)$$

two particles \mathbf{x}_1 and \mathbf{x}_2 induce on each other is proportional to their *gravitational mass* m_g that is completely unrelated to the inertial mass in this theory. Strikingly, however, experiments find inertial and gravitational mass indistinguishable from another such that physicists identify both as *mass* $m = m_i = m_g$ instead. This *weak equivalence principle* has the well-known implication that any two objects will fall with the same velocity in vacuum, regardless their weight, and has been precisely tested to orders of 10^{-13} [Adelberger, 2001, Wagner et al., 2012].

The weak equivalence principle constitutes the basis of Einstein's 1915 theory of *general relativity* where gravity is not regarded as an external force but rather a

phenomenon of the geometry of spacetime itself that is curved by its matter content [Einstein, 1915a]. Therefore, the trajectories of freely moving particles appear affected by a gravitational force although they actually follow straight lines in curved spacetime, known as *geodesics*. In other words, gravity is only an inertial force observed in a cartesian reference frame that is induced by the curvature of spacetime in presence of mass.

General relativity not only succeeded in explaining phenomena at astronomical scales that stood in conflict with Newtonian gravity, such as the perihelion shift of Mercury and gravitational lensing [Einstein, 1915b], but also enabled the development of many high-precision technologies. Among them, navigational systems such as GPS are arguably the most prominent. Most remarkably, however, it allowed physicists to scientifically approach questions that concern the evolution of the entire universe for the first time in the field of *cosmology*. Assuming only that, at sufficiently large scales, our universe looks the same everywhere and in all directions, notions such as the *beginning* and *age of the universe* became computable in cosmological models that could be compared to experimental data. For instance, distance measurements of galaxies and stars provided evidence of a recent accelerated expansion of our universe [Perlmutter, 2003]. One of the most important cosmological observations remains the high precision measurements of the *cosmic microwave background* [Penzias and Wilson, 1965, Planck Collaboration et al., 2015a] that is thermal radiation originating in the early universe. These discoveries, together with measurements of the matter composition in our universe and many more, lead to the Λ CDM (Lambda Cold Dark Matter) standard model of cosmology. In this model, our universe went through a phase of exponential expansion, known as *cosmic inflation*, at very early times, followed by several cosmological eras where each is dominated by one of the different matter species in the universe. These include radiation, ordinary matter and the elusive *dark matter* and *dark energy*.

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Gravitational waves are an intrinsic part of general relativity and the standard model of cosmology that occur when the geometry of spacetime is disturbed in a tensorial manner. For instance, binary systems of stellar objects such as stars or black holes of different mass orbiting each other generate such perturbations that will propagate through spacetime. Gravitational waves that originate in the early universe and could be measured in the cosmic microwave background are known as *primordial gravitational waves*. This thesis begins with a detailed discussion of gravitational waves in general relativity in section 1.1. Since attempts to measure

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gravitational waves, either directly by dedicated ground-based experiments or indirectly by precise measurements of the cosmic microwave background, have not been successful yet, their amplitude is constrained to be extremely small by the data at hand [Abbott et al., 2009, Planck Collaboration et al., 2015a].

Although the Λ CDM standard model of cosmology agrees with observational data remarkably well, it does not offer an entirely conclusive model of our universe. One of the shortcoming this model exhibits is related to its dark energy component that provides the accelerated expansion of our universe at recent times. It is given by a free parameter Λ known as the *cosmological constant* in this theory. In fact, the *cosmological constant problem* connects unsolved phenomena of both quantum field theory and general relativity and remains one of the greatest mysteries in theoretical physics of our time. It is formulated in section 1.2 and also introduces the concept of modified theories of gravity as an approach for solving part of it. Such theories generally propose alterations to general relativity that have self-accelerating late-time cosmological solutions without relying on a cosmological constant.

Any theory of gravity that goes beyond general relativity must also predict gravitational waves that agree with the experimental constraints mentioned above. In particular, modified gravity theories where gravitational waves grow in amplitude in cosmological evolution can easily be in tension with experiments. The main idea of this thesis is to find properties of modified gravity theories such that growing gravitational waves, that could render the theory physically unviable, are avoided. To achieve this, I introduce appropriate parameters in the evolution equation of gravitational waves that can result from general modifications of gravity. I will explore the effect these parametric modifications have on the evolution of gravitational waves in modified gravity theories in chapter 2. In addition to numerical solutions, I will solve the parametrized evolution equation analytically for appropriate assumptions to find constraints for the parameters such that the theory does not exhibit gravitational waves that grow in amplitude. Subsequently in chapter 3, I will focus this parametric approach on the extended class of *bimetric* models where a second spacetime metric in our universe interacts with the gravity-inducing physical metric and both metrics can exhibit gravitational waves. Finally, chapter 4 summarizes the results of this thesis.

Throughout the thesis, I choose natural units with the speed of light

$$c \equiv 1 \tag{1.3}$$

and a mostly-positive metric signature $(-1, 1, 1, 1)$.

1.1. Gravitational Waves

In this section, I will first give a brief introduction to general relativity and standard FLRW cosmology. The evolution equation for gravitational waves, that is the main object of consideration for the remainder of this thesis, then follows from metric tensor perturbations in such a universe.

1.1.1. The Spacetime Metric in General Relativity

In general relativity, the *Einstein equations*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \quad (1.4)$$

$$\text{with the Einstein Tensor} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}, \quad (1.5)$$

$$\text{the Ricci Scalar} \quad \mathcal{R} = g^{\mu\nu}R_{\mu\nu}, \quad (1.6)$$

$$\text{the Ricci Tensor} \quad R_{\mu\nu} = R^\alpha_{\alpha\mu\nu}, \quad (1.7)$$

$$\text{the Riemann Tensor} \quad R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu} \quad (1.8)$$

$$\text{and the Christoffel Symbols} \quad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \quad (1.9)$$

relate the geometry of spacetime, encoded in the metric tensor $g_{\mu\nu}$, to the energy content of the universe $T_{\mu\nu}$. The *cosmological constant* Λ is a free parameter in this theory and is discussed in detail in [subsection 1.2.1](#).

We can obtain the Einstein equations through variation of the *Einstein-Hilbert action*

$$S[g] = \int d^4x \sqrt{-|g|} (\mathcal{R} - 2\Lambda) + \int d^4x \sqrt{-|g|} \mathcal{L}_{\text{matter}}(g, \Phi) \quad \text{with} \quad |g| = \det g \quad (1.10)$$

with respect to $g_{\mu\nu}$. A detailed derivation is done in [Appendix A.1](#). The matter Lagrangian denoted by $\mathcal{L}_{\text{matter}}(g, \Phi)$, where Φ represents all matter fields, is only *minimally coupled* to gravity through the measure $\sqrt{-|g|} d^4x$ that arises from the spacetime metric.

Given a stress-energy tensor $T_{\mu\nu}$, the Einstein equations constitute ten highly non-linear partial differential equations for the spacetime metric g since it is a symmetric, four-dimensional tensor. The metric, in turn, restores the notion of the length of a curve in spacetime and thus allows us to formulate postulates for the dynamics of matter in the universe. In particular, as a generalization of Newton's law, particles without any external forces acted upon will move on *geodesics* in spacetime where their curve γ is stationary with respect to the length functional $L[\gamma]$ such that

$$\delta L[\gamma] = 0. \quad (1.11)$$

Many solutions to the Einstein equations have been found for particular choices of matter distributions given by $T_{\mu\nu}$. To make an exact solution possible, they often make strong symmetry assumptions. The metric tensor is usually given in some choice of coordinates x^μ in its matrix notation or as line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1.12)$$

For instance, a trivial vacuum solution to the Einstein equations is the flat *Minkowskian* spacetime metric of special relativity

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{1}_3 \end{bmatrix}_{\mu\nu} \quad \text{or} \quad ds^2 = -dt^2 + \delta_{ij} dx^i dx^j \quad (1.13)$$

in cartesian coordinates.

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1.1.2. Standard FLRW Cosmology

When we strive to understand how the entire universe evolves, we can impose a number of symmetry conditions that are based on the *cosmological principle*. The postulate states that the universe is *spatially homogeneous and isotropic* at the largest scales since there is no apparent reason to believe otherwise¹. This gives rise to six spatial symmetries and leads to the *Friedmann-Lemaître-Robertson-Walker (FLRW)*

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¹Of course, the cosmological principle can be questioned, and has been. See for example [Schwarz et al. \[2015\]](#) and references therein.

metric

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j \quad (1.14a)$$

$$\text{where } \gamma_{ij}(r, \theta, \phi) = \begin{bmatrix} \frac{1}{1-\kappa r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}_{ij} \quad (1.14b)$$

that is a particular solution to the Einstein equations describing a smooth, expanding universe. The *scale factor* $a(t)$ is the only freedom left after considering the symmetries and scales the time-independent metric γ_{ij} of a spatial, three-dimensional subspace with constant curvature κ . See [Appendix A.2](#) for a detailed derivation.

It is important to note that distances described by the FLRW coordinates are merely *coordinate distances* or *comoving distances* that remain constant even in an expanding universe. Comoving distances are scaled by the time-dependent scale factor $a(t)$ to obtain the physical distances. Similarly, one can define the *comoving* or *conformal time*

$$\eta = \int_0^t \frac{dt'}{a(t')} \quad (1.15)$$

as the comoving distance light (with $ds^2 = 0$) could have traveled since $t = 0$. The conformal time thus defines the causal structure in comoving coordinates and is also called the *comoving horizon*. It is often convenient to parametrize the evolution of the universe in conformal time η instead of cosmic time t . Another parametrization we will use are *e-foldings* $\log a$. For the remainder of this thesis, derivatives by conformal time and e-foldings will be denoted by dots and primes, respectively, as in

$$\frac{d}{d\eta} \equiv \cdot \quad \text{and} \quad \frac{d}{d \log a} \equiv '. \quad (1.16)$$

Given the particular form of the spacetime metric, the [Einstein tensor](#) (1.5) can be explicitly computed. This is done in [Appendix ??](#). When we also assume the universe is filled with matter that, at large scales, resembles a perfect fluid with stress-energy tensor

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \quad \text{in time direction} \quad u^\mu = (1, 0, 0, 0)^T \quad (1.17)$$

with *matter density* ρ , *pressure* p and *linear equation of state*

$$p = w\rho \quad \text{with equation of state parameter } w, \quad (1.18)$$

the **Einstein equations** (1.4) reduce to two ordinary, coupled differential equations

$$\frac{d^2 a}{dt^2} = -\frac{4\pi}{3} G_N (\rho + 3p) a + \frac{\Lambda}{3} \quad (1.19a) \quad \text{acceleration equation}$$

$$H^2 = \frac{8\pi}{3} G_N \rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \quad (1.19b) \quad \text{Friedmann equation}$$

with the *Hubble function*

$$H \equiv \frac{da}{dt} \frac{1}{a} \quad \text{or} \quad \mathcal{H} \equiv \frac{\dot{a}}{a} = aH \quad \text{in conformal time } \eta. \quad (1.20)$$

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The equation of state parameter w for various matter types is summarized in **Table 1.1**. For a universe filled with any such matter, we can solve the **Friedmann equations** (1.19) to find both the time evolution of the matter density

$$\rho \propto a^{-n(w)} \quad (1.21)$$

and of the scale factor

$$a(t) \propto \begin{cases} t^{\frac{2}{n(w)}} & \text{for } w \neq -1 \\ e^{Ht} & \text{for } w = -1 \end{cases} \quad \text{with } n(w) := 3(1+w). \quad (1.22)$$

This result is derived in **Appendix A.3** and already exhibits a wealth of cosmological implications. In a universe filled with matter of $w \neq -1$ we find $H \propto \frac{1}{t}$, for example, and can therefore identify it with a notion of the *age of the universe*. Furthermore, the relation $\rho \propto t^{-2}$ that follows from (1.21) in such a universe implies a singularity at $t = 0$ where the density becomes infinite. This is called the *big bang*.

	i	w	$n(w)$	$\tilde{n}(w)$
radiation	γ	$\frac{1}{3}$	4	1
dust	d	0	3	2
cosmological constant	Λ	-1	0	-1
spatial curvature	κ	$-\frac{1}{3}$	-2	∞

Table 1.1.: **Overview of cosmological properties for a universe filled with various matter types**

Multiple matter types in the universe combine to the total matter density

$$\rho(t) = \sum_i \rho_i(t) \quad \text{with equation of state} \quad p_i = w_i \rho_i \quad \text{each,} \quad (1.23)$$

where we can include the effect of a spatial curvature κ and a cosmological constant Λ through the definition of additional pseudo-densities

$$\rho_\kappa(t) = -\frac{3}{8\pi G_N} \frac{\kappa}{a^2} \quad \text{and} \quad \rho_\Lambda(t) = \frac{\Lambda}{8\pi G_N}. \quad (1.24)$$

When we then define the relative matter densities

explain

$$\Omega_i(t) := \frac{\rho_i(t)}{\rho_{\text{crit}}} \quad \text{with} \quad \rho_{\text{crit}} := \frac{3H_0^2}{8\pi G_N} \quad (1.25)$$

where H_0 denotes the value of the Hubble function today at cosmic time $t = t_0$, the **Friedmann equation** (1.19b) becomes

$$\frac{H^2}{H_0^2} = \sum_i \Omega_i(t) \quad (1.26)$$

and we find from (1.21) the relation

$$\Omega_\Lambda \propto a^2 \Omega_\kappa \propto a^3 \Omega_d \propto a^4 \Omega_\gamma. \quad (1.27)$$

Because the universe expands monotonically with time for any of these matter types, as given by (1.22), this result allows us to consider successive *cosmological regimes* or *eras* in an FLRW universe with a dominant matter type each. Radiation dominates in the early universe and is followed by a regime of dust domination (also called *matter domination*). The spatial curvature κ is measured to be zero very precisely today and is therefore assumed to vanish in the following. At recent times, the universe is in a regime dominated by a cosmological constant known as *de Sitter space* that is discussed in more detail in **subsection 1.2.1**.

1.1.3. Tensor Perturbations in an FLRW Universe

At smaller scales, the universe is not homogeneous and isotropic at all, of course. Galaxies, stars and planets, as well as radiation or, in fact, any energy content of the universe disturb the spacetime metric locally. It is therefore reasonable to consider perturbations δg around the smooth FLRW metric and their evolution.

When we assume an exact solution g to the unperturbed Einstein equations and consider a sufficiently small perturbation to the stress-energy tensor $\delta T_{\mu\nu}$, then the metric perturbation δg that solves

$$G_{\mu\nu}[g + \delta g] = T_{\mu\nu}[g] + \delta T_{\mu\nu}[g] \quad \text{for} \quad 8\pi G_N = 1 \quad (1.28)$$

will also be small and we obtain [Schulz, 2015]

$$\delta G_{\mu\nu}[g, \delta g] = \delta T_{\mu\nu}[g] \quad (1.29)$$

with $\delta G_{\mu\nu}[g, \delta g]$ linear in δg in linear perturbation theory.

Because of its symmetry condition, δg has 10 degrees of freedom that we can parametrize as [Schulz, 2015]

$$\delta g = -2A \, dx^0 \otimes dx^0 + B_i \left(dx^0 \otimes dx^i + dx^i \otimes dx^0 \right) + (2C\gamma_{ij} + 2E_{ij}) \, dx^i \otimes dx^j \quad (1.30)$$

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for small spatial scalar fields $A = A(x^0)$ and $C = C(x^0)$, a vector field $B_i = B_i(x^0)$ and a symmetric, trace-free tensor field $E_{ij} = E_{ij}(x^0)$.

notation

As of the *Helmholtz theorem*, the parameters uniquely decompose further into scalar, vector and tensor components as

$$\delta g = \delta g^{\text{scalar}} + \delta g^{\text{vector}} + \delta g^{\text{tensor}}. \quad (1.31)$$

This is shown in detail in [Appendix A.4](#) and allows us to study scalar, vector and tensor perturbations separately in linear perturbation theory.

It is shown in [Appendix A.4](#) that

$$\delta g_{ij}^{\text{tensor}} = h_{ij} \quad (1.32)$$

is a *symmetric, traceless, divergence-free tensor field* and can therefore be expressed in terms of two functions h_{\times} and h_{+} as

$$h_{ij} = \begin{bmatrix} h_{+} & h_{\times} & 0 \\ h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} \quad (1.33)$$

with an implicit choice of the z -axis in direction of the *wave vector* \mathbf{k} [Dodelson, 2003].

With this form of the metric we can compute the Einstein tensor perturbation in (1.29). This is done in Appendix ?? and we obtain

$$\delta G_{ij} = \delta R_{ij} = \frac{3a^2}{2} H \frac{dh_{ij}}{dt} + \frac{a^2}{2} \frac{d^2 h_{ij}}{dt^2} + \frac{k^2}{2} h_{ij}. \quad (1.34)$$

The **perturbed Einstein equations** (1.29) that govern the evolution of the metric perturbations then become a wave equation

$$\frac{d^2 h}{dt^2} + 3H \frac{dh}{dt} + \frac{k^2}{a^2} h = 0 \quad \text{in cosmic time } t \quad (1.35a)$$

$$\text{or } \ddot{h} + 2\mathcal{H}\dot{h} + k^2 h = 0 \quad \text{in conformal time } \eta \quad (1.35b)$$

for $h \in \{h_{\times}, h_{+}\}$. Its solutions are called *gravitational waves* and occur in two independent *polarizations* h_{\times} and h_{+} . They deform spacetime in a tensorial manner that is visualized in ??. Neglecting the friction term, harmonic oscillations

$$h \propto e^{\pm k\eta} \quad \text{with the wavelength mode } k \equiv |\mathbf{k}| \quad (1.36)$$

solve (1.35b).

Gravitational waves are damped by the expansion of the universe as exhibited by (1.35) where the friction term is proportional to the Hubble function. In an expanding universe, the **Hubble function** (1.20) is positive, thus damping the amplitude of gravitational waves. However, the tensor perturbations will remain constant at early times where the conformal time is still smaller than the wavelength scale $\frac{1}{k}$ of the gravitational wave. It begins to oscillate at its *horizon entry* $\eta \simeq \frac{1}{k}$ where cosmological scales on the order of the gravitational wave's wavelength move into causal contact. The horizon entry for larger-scale modes with smaller k thus occurs later. This behaviour is presented in **Figure 1.1** where the absolute value $|h(a)|$ of several tensor modes in a universe dominated by the cosmological regimes discussed in **subsection 1.1.2** is plotted in double-logarithmic scale. Detailed information about the assumptions, initial conditions and parameters that were chosen to obtain the plot in **Figure 1.1** are given in the context of a parametric modification to the **evolution equation of gravitational waves** (1.35) in **section 2.1**.

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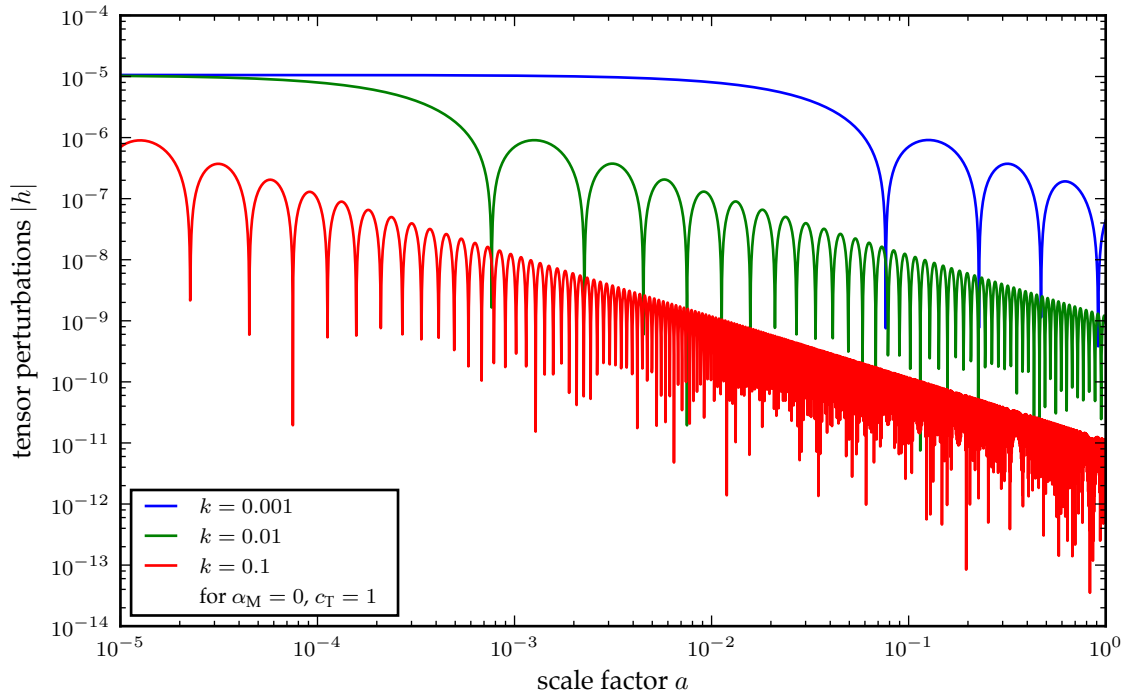


Figure 1.1.: **Tensor perturbations $|h(a)|$ for different wavelength modes k** In an expanding universe, tensor perturbations remain constant at early times until cosmological scales on the order of the gravitational wave's wavelength scale $\frac{1}{k}$ move into causal contact. This horizon entry occurs later for large-scale gravitational waves with smaller modes k .

1.1.4. Detection of Gravitational Waves

Since quadrupole motions, as depicted in ??, generate gravitational waves, they distort spacetime in the same tensorial manner when passing through. A gravitational wave therefore alternately stretches and compresses orthogonal directions in the plane transverse to its propagation direction [?]. The change in distance ΔL that two masses with an equilibrium distance L experience when the wave passes through can be measured to obtain its amplitude

$$h \propto \frac{\Delta L}{L}. \quad (1.37)$$

Since the expected amplitudes for typical astronomical sources are at most on the order of $h \sim 10^{-16}$ [Ju et al., 2000, ?], sophisticated technology is necessary to measure these tiny changes in distance.

For this reason, the detection of gravitational waves was far out of reach when Einstein predicted their existence in 1918 [Einstein, 1918]. The experimental search

for gravitational did not begin until several decades later with the pioneering work of Joseph Weber on *resonant bar detectors*² [Weber, 1960]. In 1984, the expected loss of energy through gravitational radiation was measured³ for the first time and provided a precision test of general relativity and the existence of gravitational waves [Weisberg and Taylor, 1984].

More recent resonant-bar detectors such as *ALLEGRO* and *NAUTILUS* [Astone et al., 1997, Pizzella, 1999] reached sensitivities on the order of 10^{-18} in their respective bandwidth [Ju et al., 2000]. Many modern gravitational wave detectors are based on laser interferometry, where the change in length of the interferometer's arms is detected as a change in interference intensity [Ju et al., 2000]. Among detectors of this type such as *VIRGO*, *GEO 600* and *TAMA 300*, the interferometer LIGO has reported a sensitivity on the order of 10^{-22} at a 100 Hz band without detecting any gravitational waves and is currently in the process of being upgraded to increase its sensitivity by a factor of order 10^3 [Abbott et al., 2009, Aasi et al., 2015].

Through increasingly sophisticated technology, detectors are just about to reach the sensitivity necessary for a direct detection of gravitational waves at a number of experiments around the world and in space. Since both the sensitivity and maximum wavelength scale of interferometers increase with the length of their arms, that in turn is limited by the curvature of earth, several space-based gravitational wave detectors have been proposed. The *LISA* project is planned to feature an interferometer that consists of a triangular constellation of satellites on a solar orbit trailing the earth and is scheduled to launch its proof-of-concept *Pathfinder* later this year [Vitale, 2014]. Since massive astronomical sources in our galaxy and beyond are expected to also emit gravitational radiation of wavelengths larger than any experimentally accessible length scale in our solar system, the *pulsar timing* method [Hobbs et al., 2009, and references therein] promises to provide sensitivity for such low-frequency gravitational waves.

In addition to astronomical sources of gravitational waves and the quest to detect them, our understanding of the early universe predicts the existence of *primordial*

²Any two masses connected by a spring can form a *resonant* gravitational wave detector that operates at the resonance frequency of the spring, since a passing gravitational waves that oscillates at the same frequency will induce vibrations between the masses. The original *Weber bar* detectors [Weber, 1960] consisted of large, cylindrical bars of aluminum with piezoelectric sensors to detect changes in length.

³The binary pulsar PSR 1913+16 that was observed to obtain these measurements had been discovered nearly two decades earlier by Hulse and Taylor [Hulse and Taylor, 1975], who received the 2003 Nobel Price in Physics for this discovery.

gravitational waves imprinted in the polarization of the cosmic microwave background (CMB). First discovered accidentally in 1964 [Penzias and Wilson, 1965] and measured to extreme precision ever since [Planck Collaboration et al., 2015a], the CMB is a thermal radiation of temperature 2.7260(13) K [Fixsen, 2009] originating at the cosmological epoch of *recombination*. At this point in cosmological evolution, the expanding universe became cold enough for neutral atoms to form from the hot hydrogen plasma the universe was filled with. This event is also known as *decoupling* or *last scattering*, since photons that previously continuously scattered through interaction with the free protons and electrons began to propagate freely through the now transparent universe.

Although the CMB radiation is extremely homogeneous and isotropic, it exhibits both temperature fluctuations on the order of 10^{-5} as well as polarization anisotropies that are of particular interest when gravitational waves are concerned [Hu, 2008, Challinor, 2012]. Thompson scattering processes in regions with quadrupolar temperature anisotropies in the hot plasma are the primary source for the polarization of the CMB [Hu, 2008]. Since this polarization component originates from density perturbations, it can be described by a scalar field with vanishing curl, known as *E-mode*, that complements another *B-mode* scalar field with vanishing divergence. Whereas B-mode polarizations are not generated by scattering processes, both E- and B-modes are produced by gravitational waves that pass through the plasma at decoupling. Measurements of the CMB polarization can therefore provide evidence of primordial gravitational waves that are predicted by models of *inflation*⁴ [Starobinsky, 1979, Abbott and Wise, 1984]. However, secondary gravitational lensing effects can generate B-modes from primary E-mode polarizations, posing an additional challenge in the search for gravitational waves in the CMB B-mode polarization data [Challinor, 2012]. The abundance of tensor perturbations in the CMB is conventionally quantified by its *tensor-to-scalar ratio*

$$r_{0.05} \equiv \frac{A_T}{A_S} < 0.12 \quad \text{at 95 \% confidence} \quad (1.38)$$

where A_T and A_S denote the primordial tensor and scalar perturbation amplitude, respectively, the subscript in $r_{0.05}$ corresponds to the *pivot scale* $k_0 = 0.05 \text{ Mpc}^{-1}$ where the values are taken and the experimental bounds are obtained from Ade et al. [2015]. With this choice of parameters, the physical amplitude of primordial

⁴A detailed discussion of cosmological inflation is out of scope for this thesis but can be found in Baumann [2009], for instance.

tensor perturbations becomes

$$h_{\text{phys}} \propto \sqrt{A_{\text{T}} \left(\frac{k}{k_0} \right)^{n_{\text{T}}}} \quad (1.39)$$

where the amplitude is assumed to be weakly sensitive to the *tensor spectral index* n_{T} that obeys the theoretically motivated single-field inflation consistency relation [Planck Collaboration et al., 2015b]

$$n_{\text{T}} = -\frac{r}{8}. \quad (1.40)$$

We will further discuss gravitational waves in the context of *modified gravity* in chapter 2 and the remainder of the thesis. However, to understand the reason why a modification of general relativity may be necessary, section 1.2 first formulates the *cosmological constant problem* and an approach for solving it.

1.2. The Cosmological Constant Problem

Both quantum field theory and general relativity are extremely well-tested theories and constitute the basis of modern physics in their respective fields. Whereas quantum field theory succeeds remarkably well in predicting particle physics phenomena, general relativity celebrates an equal success in large-scale astronomical and cosmological observations. Both theories and particularly their interplay are not without mysteries, however.

This section strives to formulate the *cosmological constant problem* and particularly emphasizes its problem of *radiative instability*. The quest for solutions will lead us to consider modified theories of gravity that will be the focus for the remainder of this thesis.

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1.2.1. The Cosmological Constant

Lovelock's theorem [Lovelock, 1971, Navarro and Navarro, 2011, Clifton et al., 2012] states that general relativity as it emerges from the Einstein-Hilbert action 1.10 is, in fact, the unique metric theory in four spacetime dimensions that

- gives rise to second-order equations of motion

- for only one symmetric rank-2 tensor
- that is local and lorentz-invariant.

Bianchi-
identities?

This includes a free parameter Λ of the theory that is called the *cosmological constant*. Its value does not follow from the theory and thus it can only be constrained experimentally.

It appears in the Einstein equations 1.4 as a contribution

$$T_{\Lambda,\mu\nu} = -\Lambda g_{\mu\nu} \quad (1.41)$$

to the stress-energy tensor that corresponds to a homogeneous energy density penetrating the entire spacetime. In standard FLRW cosmology, such a contribution results in an accelerating expansion of our universe as derived in detail in Appendix ??.

In fact, cosmological observations clearly suggest such an accelerated expansion taking place at recent cosmic times [Perlmutter, 2003]. The Λ CDM (Λ cold dark matter) standard model of cosmology therefore includes a cosmological constant on the order of 10^{-122} in Planck units and agrees with all cosmological observations remarkably well. In this theory, the physical origin of the homogeneous energy contribution that the cosmological constant represents remains a mystery, however, and is given the elusive name *dark energy*.

really?

The fact that the cosmological constant must be fixed by observations is not remarkable on its own, of course, since also the gravitational constant G_N must be determined experimentally. Theories such as the standard model of particle physics include a number of such free parameters. This poses an entirely different, somewhat philosophical problem that is often naively answered using anthropic arguments. This is discussed in detail in Appendix B.

1.2.2. Contributions to the Cosmological Constant

The cosmological constant problem arises when we consider both classical and quantum phenomena that, to our knowledge, should contribute to the cosmological constant.

Already in basic quantum mechanics, the uncertainty principle requires every physical system to have a zero-point energy. This immediately carries over to

ref.

quantum field theory, where a (free) field is an infinite collection of coupled quantum mechanical harmonic oscillators. With a zero-point energy each, they combine to an infinite *vacuum energy* ρ_{vac} .

In quantum field theory, the vacuum energy is largely ignored as only differences in energy determine the dynamics of the system. In general relativity, however, all energy content gravitates, including the vacuum energy [Martin, 2012, Burgess, 2013, Padilla, 2015].

Remarkably, assigning an importance to not only energy differences, but also absolute energy values gives rise to another, entirely classical contribution to the stress-energy tensor, namely the *zero-point potential* V_0 . This is the energy

$$T_{\mu\nu} = -V_0 g_{\mu\nu} \quad (1.42)$$

where the kinetic energy vanishes and the potential assumes its minimum value V_0 . This zero-point potential is usually chosen arbitrarily when only energy differences are considered but must be taken into account in general relativity. Particularly in presence of phase transitions, one can generally not choose the zero-point potential such that it always vanishes [Martin, 2012].

The free parameter Λ in the Einstein equations therefore combines with both the quantum vacuum energy ρ_{vac} and the classical zero-point potential V_0 of every quantum field in the universe to an *effective* cosmological constant

$$\Lambda_{\text{eff}} = \Lambda + \rho_{\text{vac}} + V_0 \quad (1.43)$$

that we measure as dark energy. In comparison to the small value for Λ_{eff} we observe today, however, both contributions are extremely large [Martin, 2012]. This already suggests a severe *fine-tuning* problem where the value of the original cosmological constant Λ must precisely cancel the other contributions up to the small value we measure today. This is not the entire cosmological constant problem yet, though.

1.2.3. Radiative Instability

The full scope of the problem arises when we consider in more detail the vacuum energy that we found to be infinite before. The mechanism to make sense of divergencies like this in quantum field theory is the framework of *renormalization*.

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In the process to find a finite, *renormalized* value for ρ_{vac} one generally adds counterterms for every order in perturbation theory that each depend on an *arbitrary subtraction scale*.

Generally, successive orders are not significantly suppressed by a sufficiently small perturbation parameter Λ , however. In fact, for the standard model Higgs field the self-coupling parameter of perturbation Λ is of the order 10^{-1} and therefore every order in perturbation theory must be renormalized independently [Padilla, 2015]. This *radiative instability* requires us to fine-tune the cosmological constant **repeatedly** for every order in perturbation theory and thus makes it sensitive even to small-scale physics where we assume our theory to break down [Padilla, 2015].

This also prohibits us from finding an *effective theory* where the full structure of perturbation theory is encoded in one finite, renormalized value by means of a *Wilson effective action*. Again, we find the renormalized vacuum energy is unstable against changes in the unknown UV-regime of the theory [Padilla, 2015].

ref.

1.2.4. The New Cosmological Constant Problem and Modified Gravity

The cosmological constant problem is deeply rooted in our inability to find a renormalized vacuum energy that is stable against changes in its effective description. Therefore, an approach to the problem is to assume that some mechanism makes the vacuum energy vanish altogether instead and then find another theory that explains the non-zero cosmological constant we observe today.

In fact, unbroken *supersymmetry* would accomplish just that. In supersymmetry, bosons and fermions are related by a symmetry and share the same mass. Supersymmetric partners contribute to the vacuum energy with opposite signs, thus precisely canceling each other.

With the cosmological constant set to zero, the cosmological observation of accelerated expansion remains to be explained by a different mechanism. Dark energy models such as the *quintessence* theory postulate further contributions to the stress-energy tensor that have the same accelerating effect as a cosmological constant. A different approach is to modify the theory of gravity instead.

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Every attempt towards a modified theory of gravity has to take Lovelock's theorem into consideration and break at least one of its assumptions that make general relativity unique. The concept of the entire class of $f(\mathcal{R})$ theories, for example, is to replace the Ricci scalar \mathcal{R} in the **Einstein-Hilbert action** (1.10) by a function $f(\mathcal{R})$. $f(\mathcal{R})$ theories break Lovelock's assumption of only second-order equations of motion and are in many cases equivalent to an additional scalar field contribution to the matter Lagrangian that is non-minimally coupled to gravity [Clifton et al., 2012].

Alternatively, the hypothesis of a *massive graviton* requires a second, arbitrary *reference metric* f in addition to the physical metric g [de Rham, 2014]. The additional metric is necessary to construct a mass term because a single metric allows only trivial self-interaction terms $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu$ and $g^{\mu\nu}g_{\mu\nu} = 4$. By postulating a symmetric action where the reference metric f behaves dynamically just like g , we arrive at the theory of *massive bigravity*. The cosmology in this theory allows solutions with late-time acceleration without a cosmological constant [Könnig et al., 2014].

connect
to next
chapter

2. Parametrized Evolution of Gravitational Waves in Modified Gravity

Another review is the one we did for Euclid:
<http://arxiv.org/abs/1206.1225>

Many theories of modified gravity are discussed in the literature [Clifton et al., 2012, de Rham, 2014] and this thesis will not focus on one specific model. Instead, the remainder of the thesis will explore various parametric modifications to the evolution equation of gravitational waves (1.35) that can result from a modified gravity theory. For such a modified gravity theory to be physically viable, the metric tensor perturbations must remain within constraints set by the observations discussed in subsection 1.1.4. In particular, any theory that exhibits growing tensor modes in cosmological evolution should be regarded with serious doubt, as their amplitude would likely be large enough today to be excluded by experiments.

We did introduce the friction term.
As I wrote you in previous notes,
you need to cite us.

introduce
better

In an $f(\mathcal{R})$ theory without anisotropic stress, for example, the evolution equation becomes [Xu, 2014, Hwang and Noh, 1996]

check
Xu ref.

$$\ddot{h} + \left(2 + \frac{d \log F}{d \log a}\right) \mathcal{H} \dot{h} + k^2 h = 0 \quad \text{with} \quad F := \frac{df(\mathcal{R})}{d\mathcal{R}}, \quad (2.1)$$

thus adding an additional friction term. Furthermore, the propagation speed of gravitational waves in a modified gravity theory can deviate from the speed of light c [Amendola et al., 2014, Raveri et al., 2014]. I investigate the effects such modifications have on the evolution of gravitational waves by introducing appropriate parameters in (1.35). A reasonable parametric modification motivated

by the above considerations is

Which considerations? Also here, if you use our parametrization, you need to say that it has been adopted/suggested in our paper and cite us

$$\frac{d^2 h}{dt^2} + (3 + \alpha_M) H \frac{dh}{dt} + \frac{c_T^2 k^2}{a^2} h = 0 \quad \text{in cosmic time } t, \quad (2.2a)$$

$$\ddot{h} + (2 + \alpha_M) \mathcal{H} \dot{h} + c_T^2 k^2 h = 0 \quad \text{in conformal time } \eta \quad (2.2b)$$

$$\text{or } h'' + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 2 + \alpha_M \right) h' + \frac{c_T^2 k^2}{\mathcal{H}^2} h = 0 \quad \text{in e-foldings } \log a \quad (2.2c)$$

where α_M denotes an additional friction term and c_T a deviation of the propagation speed from the speed of light. Both parameters may in general be time-dependent. The standard behaviour of general relativity discussed in subsection 1.1.3 is recovered for $\alpha_M = 0$ and $c_T = 1$. This chapter will explore the evolution of gravitational waves in a modified gravity theory that reduces to a modification of one or both of these parameters to find constraints such that growing tensor modes are avoided.

2.1. Constant additional friction

I will first consider a constant friction term α_M in the evolution equation (2.2).

Invert the text: first you explain what you are doing, which assumptions you are making, what you are solving and then you say that you plot the outcome. Then you need to explain each plot in detail within the text (axis, lines, what the plot tells us)

2.1.1. Numerical solution

Figure 2.1 shows the numerical solution of the parametrized evolution equation (2.2) for various values of α_M . The result was obtained with the Plot function in Mathematica where several assumptions were made.

the following

which corresponds to having the standard equation at early times (always explain in words your equations)

First, initial conditions were chosen such that

$$h(a \rightarrow 0) = 1 \quad \text{and} \quad h'(a \rightarrow 0) = 0. \quad (2.3)$$

Furthermore, a standard FLRW background was assumed where the Hubble function is given by the Friedmann equation (1.26)

$$H(t) = H_0 \sqrt{\sum_i \Omega_i(t)}. \quad (2.4)$$

write the limit like this?

Only matter and dark energy contributions Ω_d and Ω_Λ were considered here because they dominate in late-time cosmological regimes as discussed in [subsection 1.1.2](#). Values for H_0 and $\Omega_d(t_0)$ in

$$\Omega_d(t) = \Omega_d(t_0)a^{-3} \quad \text{and} \quad \Omega_\Lambda(t) = 1 - \Omega_d(t_0) \quad (2.5)$$

as well as for the [physical amplitude \(1.39\)](#)

$$h_{\text{phys}}(a) = \sqrt{A_T \left(\frac{k}{k_0} \right)^{n_T}} \cdot h(a) \quad (2.6)$$

of primordial gravitational waves were obtained from the [Planck Collaboration et al. \[2015b\]](#) results.

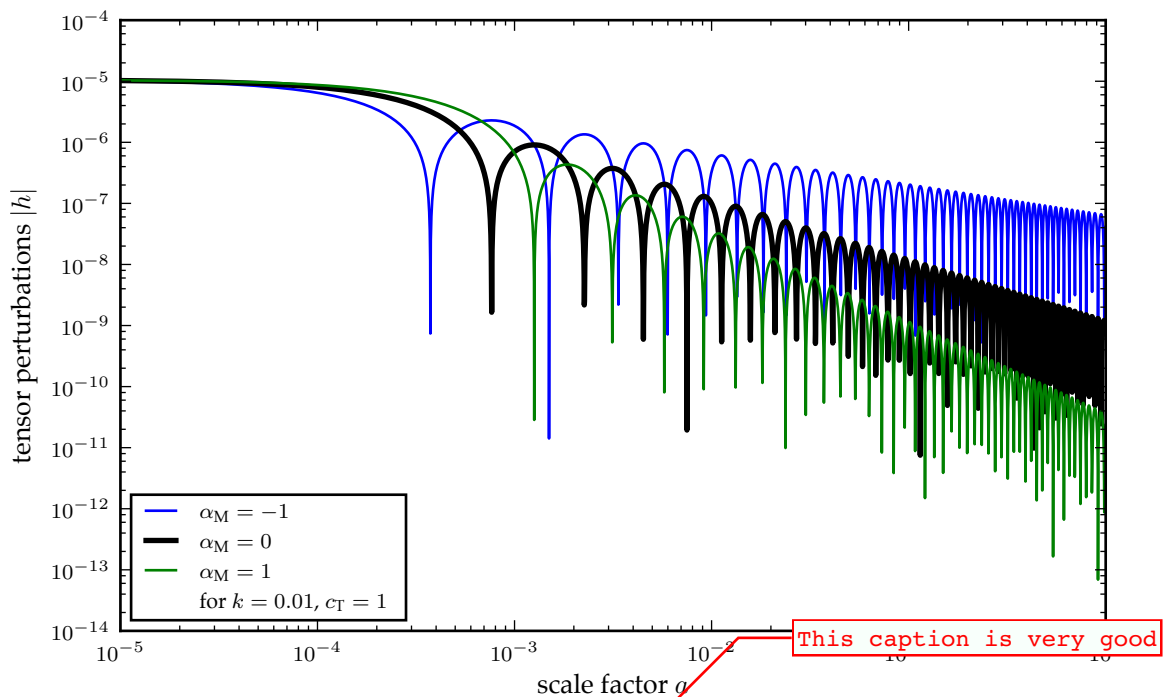


Figure 2.1.: **Tensor perturbations $|h(a)|$ with additional friction $\alpha_M = \text{const.}$** Increasing α_M introduces more friction, so that the amplitude of the gravitational wave decreases more rapidly, but also delays the horizon entry.

In [Figure 2.1](#), the absolute value $|h(a)|$ is plotted in ~~double~~-logarithmic scale for both positive and negative values of α_M . The amplitude of the gravitational wave clearly decreases more rapidly for larger α_M since additional friction is introduced. However, also the horizon entry is delayed for larger α_M thus introducing a competing effect. [This reproduces the behaviour found in Pettorino and Amendola](#)

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[2014].

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2.1.2. Analytic solution in cosmological regimes

The parametrized evolution equation (2.2) can ~~even~~ be solved analytically in cosmological regimes dominated by one of the matter types discussed in subsection 1.1.2. The expansion of an FLRW universe dominated by matter or radiation is given by (1.22) or

$$a(\eta) \propto \eta^{\tilde{n}(w)} \quad \text{with} \quad \tilde{n}(w) := \frac{2}{1+3w} \quad \text{in conformal time.} \quad (2.7)$$

This allows us to find an expression $\mathcal{H} = \frac{\dot{a}}{a} = \frac{\tilde{n}(w)}{\eta}$ for the Hubble function so that (2.2) becomes

$$\ddot{h} + (2 + \alpha_M) \frac{\tilde{n}(w)}{\eta} \dot{h} + c_T^2 k^2 h = 0. \quad (2.8)$$

This is a Bessel differential equation that has solutions in terms of Bessel functions as discussed in Appendix A.6. Thus, (2.8) is solved by

$$h(\eta) = \eta^{-p} [C_1 J_p(c_T k \eta) + C_2 Y_p(c_T k \eta)] \quad \text{with} \quad p = \tilde{n}(w) \left(1 + \frac{\alpha_M}{2}\right) - \frac{1}{2} \quad (2.9)$$

where $J_p(x)$ and $Y_p(x)$ denote the Bessel functions of first and second kind, respectively. Their asymptotic behaviour for large modes k or late times η is given in Appendix A.6 and both correspond to oscillations with an amplitude decreasing as $\eta^{-\frac{1}{2}}$. Therefore, the amplitude of gravitational waves with additional friction α_M behaves as

$$h(\eta) \propto \eta^{-p-\frac{1}{2}} = \eta^{-\tilde{n}(w)(1+\frac{\alpha_M}{2})} \quad (2.10a)$$

$$\text{or} \quad h(a) \propto a^{-\frac{1}{\tilde{n}(w)}(p-\frac{1}{2})} = a^{-(1+\frac{\alpha_M}{2})} \quad (2.10b)$$

times fast oscillation in this regime.

This result immediately gives a constraint for the α_M parameter such that growing tensor modes are avoided. Since (2.10b) exhibits a growing amplitude of $h(a)$ for a positive exponent $-(1 + \frac{\alpha_M}{2}) > 0$ in cosmological expansion, any theory of gravity that modifies the evolution equation of gravitational waves (2.2) by an additional friction term $\alpha_M = \text{const.}$ should fulfill

$$\alpha_M \geq -2 \quad (2.11)$$

or else be regarded with serious doubt.

in order to avoid growing modes

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The numerical result obtained before reflects this behaviour as presented in Figure 2.2. The plot shows the evolution of tensor modes for both $\alpha_M = -2$ and $\alpha_M < -2$ using the same numerical solution as in Figure 2.1 as well as the slope of the amplitude obtained analytically in (2.10). As expected from the analytic considerations, the gravitational wave with $\alpha_M = -2$ remains stable whereas the gravitational wave with $\alpha_M < -2$ grows in amplitude.

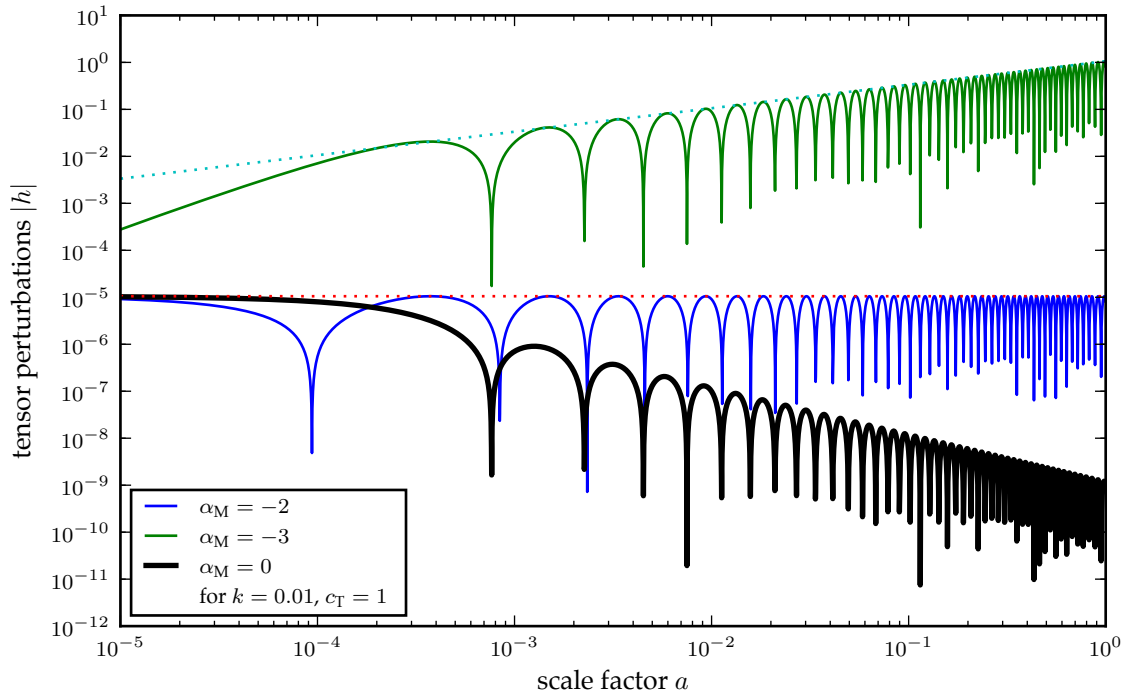


Figure 2.2.: **Tensor perturbations $|h(a)|$ with additional friction $\alpha_M \leq -2$** The numerical solution (solid lines) agrees with the slope of the amplitude obtained analytically (dotted lines). Gravitational waves with $\alpha_M = -2$ remain stable but grow in amplitude for $\alpha_M < -2$.

2.2. Deviating propagation speed

Again, you need to cite us and Raveri as people who did that

To explore the effect a modified propagating speed has on the evolution of gravitational waves in a modified gravity theory, I consider a parametric deviation c_T from the speed of light in the parametrized evolution equation (2.2). The deviation of the propagation speed as it appears in (2.2) changes the effective wavelength scale $\frac{1}{c_T k}$ such that the horizon entry of the gravitational wave occurs later

for smaller values of c_T . The horizon entry is discussed in detail in the context of general relativity in [subsection 1.1.2](#).

[Figure 2.3](#) shows the evolution of gravitational waves for several values of $c_T = \text{const.}$. The plots were obtained numerically with the `ParametricNDSolve` function in Mathematica where the same assumptions and initial conditions as in [section 2.1](#) were chosen.

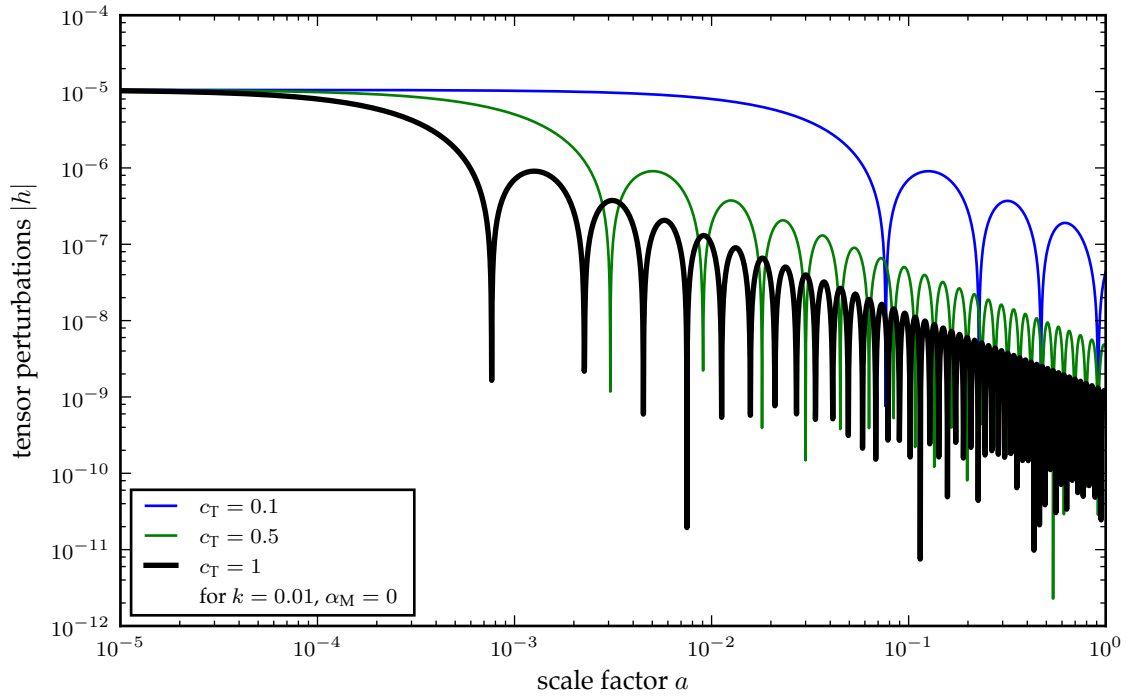


Figure 2.3.: **Tensor perturbations $|h(a)|$ with a modified propagation speed $c_T = \text{const.}$** Gravitational waves with lower propagation speed c_T exhibit a delayed horizon entry.

Because gravitational waves with lower propagation speed c_T exhibit a delayed horizon entry, their amplitude today is closer to their initial value than for gravitational waves that propagate faster. An additional friction contribution α_M as discussed in [section 2.1](#) therefore is less effective for gravitational waves with lower c_T that enter the horizon later and thus have less time until today to decrease or increase in amplitude. This suggests a degeneracy between a modified propagation speed and an additional friction contribution, as both affect the amplitude of tensor perturbations [measured by which](#). A positive α_M that results in a faster decrease in amplitude can be compensated by a lower propagation speed c_T that in turn delays the horizon entry and thus gives the tensor mode less time to decrease. The amount [growing tensor modes increase in amplitude is reduced](#)

by a lower propagation speed for the same reasons.

is $cT > 1$ possible?

2.3. Late-time additional friction

Many theories of modified gravity aim to solve the *new cosmological constant problem* presented in subsection 1.2.4. This requires such theories to model the accelerated expansion of our universe today as described in subsection 1.2.1. Proposed modifications to general relativity thus generally only affect late-time cosmological regimes where the Λ CDM standard model of cosmology relies on dark energy to accelerate the universe. Therefore, suitable parametrized modifications to the *evolution equation of gravitational waves* (2.2) are time-dependent and vanish in early-time cosmological regimes to reflect this behaviour of modified gravity theories.

add reference/example maybe specific $f(R)$ theory

In particular, I consider for the additional friction α_M the time-dependent parametrization

$$\alpha_M(t) = \alpha_{M0} \cdot a(t)^\beta \quad \text{with} \quad \alpha_{M0} = \text{const.} \quad \text{and} \quad \beta > 0. \quad (2.12)$$

In an expanding universe where $a(t) \in (0, 1]$ monotonically increases with time, the additional friction contribution α_M vanishes at early times to recover general relativity, but deviates from Λ CDM today.

Figure 2.4 shows numerical solutions to the *evolution equation* (2.2) with this particular parametrization for α_M for various values of both α_{M0} and β . It was obtained with the `ParametricNDSolve` function in Mathematica where the same initial conditions and assumption as in section 2.1 were chosen. The special case $\beta = 0$ corresponds to constant friction $\alpha_M(t) = \alpha_{M0}$ as investigated in section 2.1. Increasing β to positive values reduces the additional friction at early times where $a(t) < 1$. Larger values for β amplify this effect.

At late times, tensor modes will increase or decrease in amplitude with a slope approximating the behaviour of constant additional friction α_{M0} discussed in section 2.1. However, tensor modes that grow in this regime according to (2.11) can be suppressed by a sufficiently large value of β to remain physically viable.

include Figure 4.1 with varying β for const α_M at all?

explain more

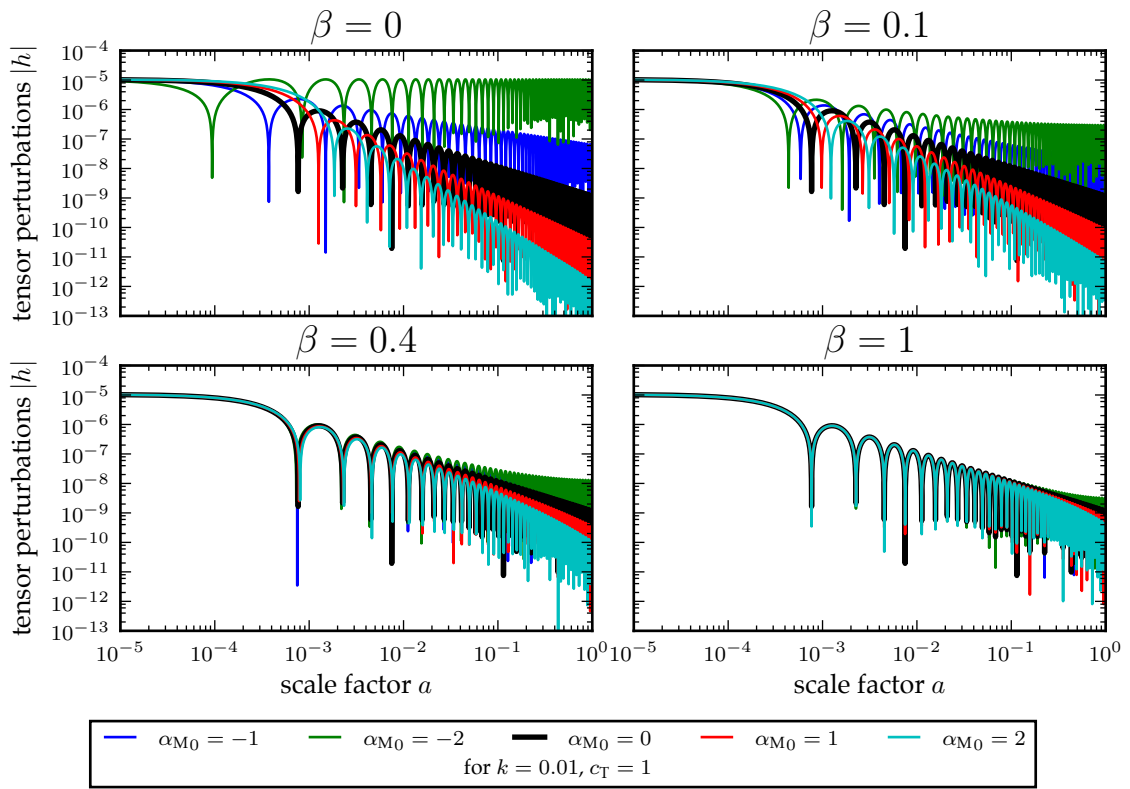


Figure 2.4.: **Tensor perturbations $|h(a)|$ with parametrized additional friction $\alpha_M(t) = \alpha_{M0} \cdot a(t)^\beta$** For positive β , the evolution matches Λ CDM at early times and deviates at late times. Increasing $|\alpha_{M0}|$ yields larger deviations from Λ CDM at late times.

This part is already quite good

3. Gravitational Waves in Parametrized Bigravity

Bigravity is a class of modified gravity theories introduced in [subsection 1.2.4](#) where a second *reference metric* f in addition to the physical metric g is considered with its own dynamics to obtain a theory of *massive gravity*. Only the physical g -metric induces physical gravity, but both metrics are coupled such that the evolution equation of gravitational waves in this bimetric setting becomes [[Amendola et al., 2015](#)]

$$\ddot{h}_n + (2 + \alpha_{Mn}) \mathcal{H} \dot{h}_n + (\mathcal{H}^2 m_n^2 + c_{Tn}^2 k^2) h_n = \mathcal{H}^2 q_n h_m \quad \text{in conformal time } \eta \quad (3.1a)$$

$$\text{or} \quad h_n'' + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 2 + \alpha_{Mn} \right) h_n' + \left(m_n^2 + \frac{c_{Tn}^2 k^2}{\mathcal{H}^2} \right) h_n = q_n h_m \quad \text{in e-foldings } \log a \quad (3.1b)$$

where the indices $n, m \in \{g, f\}$ with $n \neq m$ refer to the perturbations and parameters associated with the physical metric g and the reference metric f , respectively. The additional friction terms α_{Mn} and the deviations from the speed of light c_{Tn} for each of the two metrics were discussed in [chapter 2](#) in the context of unimetric modified gravity. Each metric also has an associated *mass parameter* m_n and a *coupling* q_n to the other metric. In the theory of bigravity, these parameters have specific, time-dependent forms that are given in [Amendola et al. \[2015\]](#). The effect the additional metric and mass parameters have on the evolution of gravitational waves will be discussed in this chapter.

3.1. Coupling to the reference metric

Since strict experimental bounds apply to the amplitude of physical tensor perturbations h_g , we can expect the source term $q_f h_g$ of the reference metric f to be negligible with respect to the other terms [Amendola et al., 2015]. Thus, I explore solutions of the parametrized bimetric evolution equation (3.1) with $q_f = 0$ but with a non-zero coupling parameter $q_g = \text{const.}$. The non-standard behaviour of the reference metric that is governed by a choice of its associated parameters will therefore modify the evolution of the physical tensor perturbations h_g .

Figure 3.1 depicts several solutions of the parametrized bimetric evolution equation (3.1) that were obtained numerically with the `ParametricNDSolve` function in Mathematica. For both g and f , the same assumptions and initial conditions as in section 2.1 were chosen. In addition to the coupling $q_g = 1$, only the additional friction parameter of the f -metric was given a non-standard value such that its tensor perturbations grow in amplitude for $\alpha_{Mf} < -2$ as discussed in section 2.1.

Because matter only couples to the physical metric g whereas the reference metric f is not directly observable, tensor modes for the reference metric are not constrained by experiments. However, the coupling q_g also leads to non-standard behaviour of the physical metric, although its other associated parameters remain unmodified. Figure 3.1 shows that the physical metric perturbations follow the slope of the reference metric perturbations at late times and also exhibit a growing amplitude for sufficiently negative α_{Mf} when coupled to the reference metric. In the specific bigravity model investigated in Amendola et al. [2015], a similar behaviour is found for explicit, time-dependent expressions for the parameters that were considered constant here.

elaborate?

3.2. Non-zero mass parameter

The numerical solutions of the parametrized bimetric evolution equation (3.1) obtained in section 3.1 also allow for a brief discussion of the mass parameter m_n for both metrics $n \in \{g, f\}$. Figure 3.2 shows the evolution of the physical metric perturbations $|h_g(a)|$ for various values of its mass parameter m_g and all other parameters set to their standard value. Because the mass parameter appears as an additive contribution to the effective wavelength mode $c_T k$ in (3.1), larger values

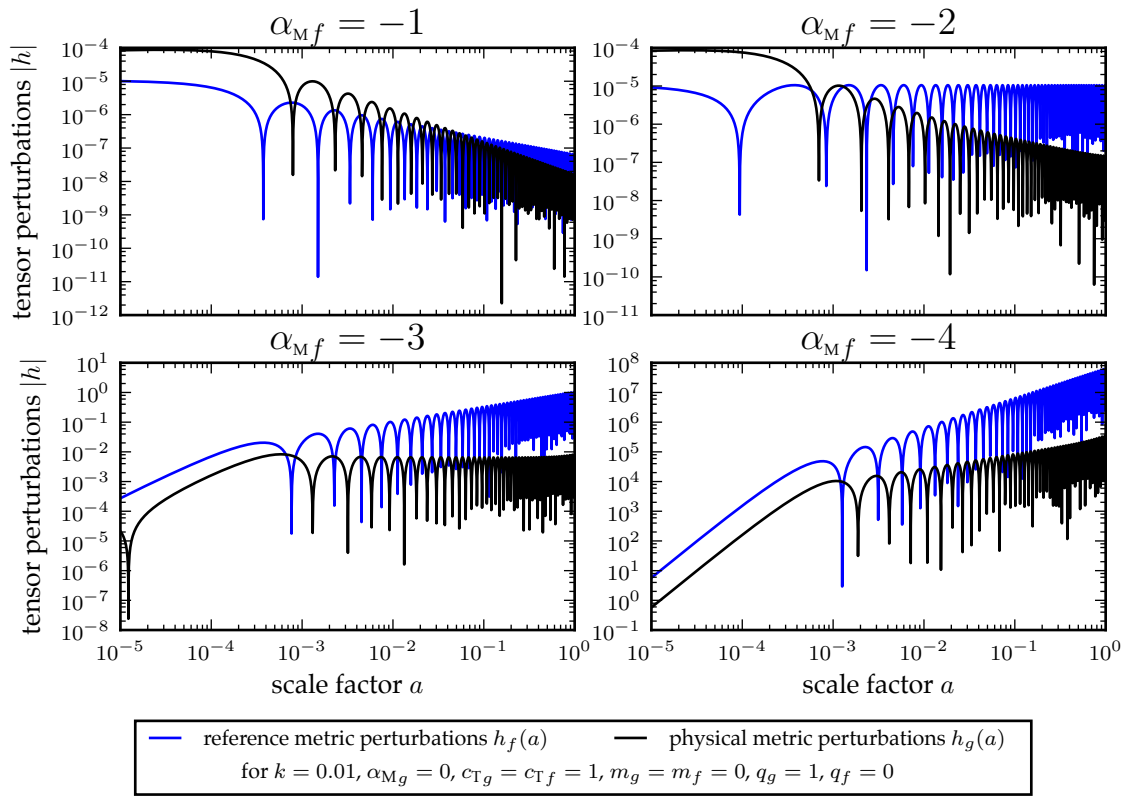


Figure 3.1.: **Physical and reference metric perturbations with parametrized coupling $q_g = 1$** The amplitude of the physical metric perturbations follows the slope of the reference metric perturbations at late times. Note that the coupling q_g and the additional friction α_{Mf} of the f -metric are the only non-standard parameters. The coupling is also the cause of the initial offset that appears in the plots in the depicted regime despite the same choice of initial conditions for both metrics.

of m_n will advance the horizon entry of the tensor mode. A degeneracy between the mass parameter and a deviating propagation speed c_T is to be expected. elaborate?

3.3. Decoupling at early times

Bimetric modified gravity theories, like unimetric theories, usually deviate from Λ CDM only at late times to provide an accelerated expansion of the universe without a cosmological constant. This is discussed in more detail in [section 2.3](#). For physically viable bimetric theories it is therefore reasonable to assume that the metrics decouple in early-time cosmological regimes. This allows the physical

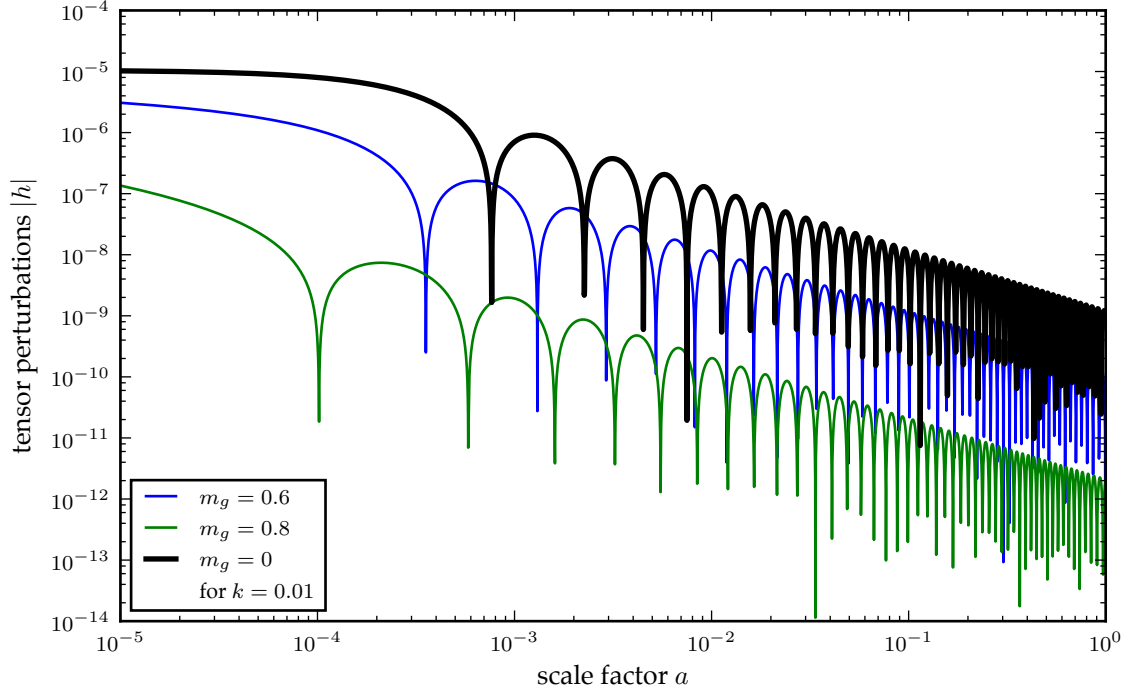


Figure 3.2.: **Tensor perturbations with non-zero mass parameter m_g** Due to the additive contribution to the effective wavelength mode $c_T k$, increasing the mass parameter m_g will advance the horizon crossing of the gravitational wave.

metric g to behave unmodified whereas the reference metric can exhibit arbitrary dynamics.

A specific bimetric model is investigated in Amendola et al. [2015], for example, where the physical and reference metric decouple at early times. The authors found that for sub-horizon modes in radiation and matter dominated eras also the mass parameters become negligible in this model. Therefore, the evolution of tensor perturbations for both metrics reduce to the **unimetric parametrized evolution equation (2.2)** with parameters [Amendola et al., 2015]

$$\alpha_{Mg} = 0 \qquad c_{Tg} = 1 \qquad (3.2a)$$

$$\alpha_{Mf} = -3(1 + w) \qquad c_{Tf} = \frac{(3w + 1)^2}{4} \qquad (3.2b)$$

where the physical metric remains entirely standard such that the amplitude of tensor perturbations h_g falls like $\frac{1}{a}$ according to (2.10). However, the reference metric tensor perturbations h_f with $\alpha_{Mf} < -2$ in (3.2b) grow in amplitude like a^1 in radiation domination and $a^{\frac{1}{2}}$ in matter domination.

4. Summary

Many theories of modified gravity were proposed to explain the late-time accelerated expansion of our universe without a cosmological constant. The latter is a free parameter in general relativity that resembles a homogeneous energy density penetrating the entire spacetime and is known as *dark energy* in the Λ CDM standard model of cosmology. To our knowledge, other energy components such as the classical zero-point potential and the vacuum energy of all quantum fields should also contribute to the effective cosmological constant. Since a radiatively stable vacuum energy can not be found, a severe fine-tuning problem exists that is known as the *cosmological constant problem*. An approach for solving this problem is to assume a mechanism that sets the cosmological constant to zero and to find a modified theory of gravity that allows self-accelerating cosmological solutions.

Any modified gravity theory must respect the experimental bounds for gravitational waves in order to be considered physically viable. Theories that predict tensor perturbations with growing amplitudes in cosmological expansion can therefore easily be in tension with observations. Common modifications to the evolution of gravitational waves include additional friction contributions and a deviating propagation speed that I investigated in [chapter 2](#) by introducing appropriate parameters α_M and c_T . Numerical solutions showed a more rapid decay of tensor perturbation amplitudes and the competing effect of a delayed horizon entry for larger additional friction contributions $\alpha_M = \text{const.}$. An analytic solution could also be found in a universe dominated by matter or radiation. The slope of the tensor perturbation amplitude in this regime is given by [\(2.10\)](#)

$$h(a) \propto a^{-(1+\frac{\alpha_M}{2})} \quad (4.1)$$

such that gravitational waves remain stable for [\(2.11\)](#)

$$\alpha_M \geq -2. \quad (4.2)$$

However, the effect an additional friction contribution has on the amplitude of gravitational waves today is reduced by a decrease in propagation speed c_T as shown in [section 2.2](#). Since the horizon entry of tensor modes is delayed for smaller c_T , they have less time until today to decrease or increase in amplitude.

Generally, time-dependent additional friction and propagation speed parameters such as the parametrization for α_M explored in [section 2.3](#) can suppress the modifications at early times so that the model only recently deviates from Λ CDM. A thorough analysis of the evolution of tensor modes in a specific modified gravity theory is necessary to determine whether their amplitudes remain within experimental bounds. Particularly, theories that predict late-time additional friction in conflict with the [stability condition \(2.11\)](#) can easily be in tension with observational data.

To formulate theories of massive gravity, a second, arbitrary *reference metric* f must be coupled to the physical metric g . This additional metric is given dynamics similar to g in the theory of *bigravity* so that both can exhibit gravitational waves. Since the metrics are coupled, the parametrized evolution equation of tensor perturbations is extended by general mass and coupling parameters m_n and q_n for both metrics $n \in \{g, f\}$ in [chapter 3](#). A non-zero mass parameter will advance the horizon entry of the tensor mode and therefore lead to an expected degeneracy with a deviating propagation speed c_T . The parameter q_n denotes the strength of the coupling to the other metric's tensor perturbations h_m .

Without any coupling, both metrics evolve independently. Since only the g -metric induces physical gravity, the f -metric perturbations are unobservable. Therefore, they are not subject to experimental constraints and can exhibit arbitrary behaviour including growing tensor modes. However, through the coupling parameters q_n , the unobservable reference metric can interact with the physical metric to modify its evolution. In particular, the amplitude of the physical metric perturbations will follow the slope of the reference metric perturbations at late times for sufficiently large negative friction α_{Mf} as shown in [section 3.1](#).

A coupling between the metrics can lead to any modifications to observable gravity. The coupling q_g becomes relevant at appropriate late times, the behaviour explored in [section 3.1](#) must be taken into consideration such that the physical metric perturbations do not exceed observational

Avoid saying vague statements in the conclusions. What does it mean appropriate in this context? Give limits/numbers you found. How big can the coupling be before spoiling tensor modes?

constraints by coupling to growing reference metric tensor modes. Therefore, a detailed analysis of tensor perturbation modes can provide crucial insight about the physical viability of bimetric theories.