Modified gravity theories generally aim to solve part of the *cosmological constant problem* by providing self-accelerating cosmological solutions without a cosmological constant. Such modifications of general relativity also affect the evolution of gravitational waves in the proposed theory. Instead of focussing on a specific model, I introduce parametric modifications to the evolution equation of gravitational waves in both unimetric and bimetric settings and investigate their effect on the evolution of tensor perturbation modes. In particular, I argue that any modified gravity theory that exhibits *growing tensor modes* in cosmological evolution can be in tension with experiments. Therefore, parametric constraints for the physical viability of a general modified gravity theory can be found such that tensor modes remain within limits set by observations.

Contents

1. Introduction

Gravitational waves are an intrinsic part of general relativity and the standard model of cosmology. Since attempts to measure gravitational waves, either directly by dedicated ground-based experiments or indirectly by precise measurements of the cosmic microwave background, have not been successful yet, their amplitude is constrained to be extremely small by the data at hand. Any theory of gravity that goes beyond general relativity must also predict gravitational waves that agree with these experimental constraints. In particular, modified gravity theories where gravitational waves grow in amplitude in cosmological evolution can easily be in tension with experiments.

In fact, many modified gravity theories have been proposed to address the *cosmological constant problem* that remains one of the greatest mysteries in theoretical physics of our time. They generally aim to explain the accelerated expansion of our universe today without relying on a cosmological constant. The main idea of this thesis is to find properties of modified gravity theories such that growing gravitational waves are avoided that could render the theory physically unviable. To achieve this, I introduce appropriate parameters in the evolution equation of gravitational waves that can result from general modifications of gravity. Furthermore, I will focus this parametric approach on the extended class of *bimetric* models where a second spacetime metric in our universe interacts with the gravity-inducing physical metric and both metrics can exhibit gravitational waves.

This chapter first gives a brief introduction to gravitational waves in general relativity in section 1.1. Then, section 1.2 formulates the remarkably complex cosmological constant problem and introduces the concept of modified gravity theories as an approach for solving part of it. The main part of the thesis in chapter 2 and chapter 3 explores various parametric modifications to the evolution equation of gravitational waves in unimetric and bimetric modified gravity, respectively. Finally, chapter 4 summarizes the results of this thesis.

Throughout the thesis, I choose natural units with the speed of light

$$c \equiv 1 \tag{1.1}$$

and a mostly-positive metric signature $(-1, 1, 1, 1)$.

1.1. Gravitational Waves

I will first give a brief introduction to standard FRW cosmology in general relativity and derive the evolution equation for gravitational waves by considering perturbations to such a universe.

1.1.1. Gravity and Spacetime in General Relativity

For a long time, physicists believed space and time were both flat and fundamentally different concepts. In *Newtonian spacetime*, it is therefore straight forward to define notions such as the *length of a curve* and *simultaneity*.

cite [Tolish]

With Maxwell's theory of electromagnetism and experimental observations regarding the speed of light, however, many of these concepts had to be abandoned. Physicists realized the necessity to reconsider fundamental assumptions about space and time. Albert Einstein addressed many of these issues with his theory of special relativity in 1905, but, in particular, one striking observation remained unexplained:

summarize SR briefly, mention GR->SR for minkowskia

Newtonian mechanics postulates that particles accelerate under the influence of **any** force proportional to their *inertial mass* m_i. At the same time, the gravitational force a particle induces on another is proportional to its *gravitational mass* m_g that is completely unrelated to the former in this theory. Strikingly, however, experiments find inertial and gravitational mass indistinguishable from another — to this end, any two objects will fall with the same velocity, no matter their weight — and physicists identify both as *mass* $m = m_i = m_g$ instead. This *equivalence principle* constitutes the basis of Einstein's 1915 theory of *general relativity* where gravity is **not regarded as an external force** acting on particles in spacetime but rather a phenomenon of the geometry of spacetime itself, with **particles moving freely in curved spacetime**.

order of magnitude?

In general relativity, the *Einstein equations*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \mathrm{G_N} T_{\mu\nu} \tag{1.2}$$

with the *Einstein Tensor* $$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \mathcal{R}, \tag{1.3}$$

the *Ricci Scalar* $$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}, \tag{1.4}$$

the *Ricci Tensor* $$R_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu}, \tag{1.5}$$

the *Riemann Tensor* $$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu} + \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu} \tag{1.6}$$

and the *Christoffel Symbols* $$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} \right) \tag{1.7}$$

relate the geometry of spacetime, encoded in the metric tensor $g_{\mu\nu}$, to the energy content of the universe $T_{\mu\nu}$. The *cosmological constant* Λ is a free parameter in this theory and is discussed in detail in subsection 1.2.1.

We can obtain the Einstein equations through variation of the *Einstein-Hilbert action*

$$S[g] = \int \mathrm{d}^4x \sqrt{-|g|}\, (\mathcal{R} - 2\Lambda) + \int \mathrm{d}^4x \sqrt{-|g|}\, \mathcal{L}_{\text{matter}} \quad \text{with} \quad |g| = \det g \tag{1.8}$$

with respect to $g_{\mu\nu}$. A detailed derivation is done in Appendix A.1. The matter Lagrangian denoted by $\mathcal{L}_{\text{matter}}$ is only *minimally coupled* to gravity through the measure $\sqrt{-|g|}\,\mathrm{d}^4x$ that arises from the spacetime metric in general relativity.

Given a stress-energy tensor $T_{\mu\nu}$, the Einstein equations constitute ten highly non-linear partial differential equations for the spacetime metric g. The metric, in turn, restores the notion of the length of a curve in spacetime and thus allows us to formulate postulates for the dynamics of matter in the universe.

why ten? > symmetric

In particular, as a generalization of Newton's law, particles without any external forces acted upon will move on *geodesics* in spacetime, i.e. their curve is stationary with respect to the length functional.

explain better

A simple solution to the Einstein equations is the flat *Minkowskian* spacetime metric

in vacuum?

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{1}_3 \end{bmatrix}_{\mu\nu} \tag{1.9}$$

of special relativity.

include this here, or elaborate?

1.1.2. FRW Cosmology

In cosmology, we strive to understand **how the entire universe evolves**.

elaborate

As of the *cosmological principle*, it is reasonable to assume the universe is *spatially homogeneous and isotropic* at large scales. This gives rise to six spatial symmetries and leads to the *Friedmann-Robertson-Walker (FRW)* metric

include derivation with Killing vector fields?

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \quad \text{where} \quad \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{1.10}$$

$$\text{and} \quad \gamma_{ij}(r,\theta,\phi) = \begin{bmatrix} \frac{1}{1-\kappa r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2(\theta) \end{bmatrix}_{ij} \tag{1.11}$$

that is a particular solution to the Einstein equations describing a smooth, expanding universe. The *scale factor* $a(t)$ is the only freedom left after considering the symmetries and scales the time-independent metric γ_{ij} of a spatial subspace with constant curvature κ. See Appendix A.2 for a detailed derivation.

can the sub-space(?) be called spatial, or rather three-dim.?

It is important to note that distances described by the FRW coordinates are merely *coordinate distances* or *comoving distances* that remain constant even in an expanding universe. Comoving distances are scaled by the time-dependent scale factor $a(t)$ to obtain the physical distances. Similarly, one can define the *comoving* or *conformal time*

$$\eta = \int_0^t \frac{\mathrm{d}t'}{a(t')} \tag{1.12}$$

as the comoving distance light (with $\mathrm{d}s^2 = 0$) could have traveled since $t = 0$. The conformal time thus defines the causal structure in comoving coordinates and is also called the *comoving horizon*. It is often convenient to parametrize the evolution of the universe in conformal time η instead of cosmic time t. Another parametrization we will use are *e-foldings* $\log a$. For the remainder of this thesis, derivatives by conformal time and e-foldings will be denoted by dots and primes, respectively, as in

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \equiv^{\cdot} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\log a} \equiv'. \tag{1.13}$$

Given the particular form of the spacetime metric, the Einstein tensor (1.3) can be explicitly computed. This is done in Appendix **??**. When we also assume the universe is filled with matter that, at large scales, resembles a perfect fluid with

add reference

stress-energy tensor

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu} \quad \text{in time direction} \quad u^\mu = (1,0,0,0)^T \tag{1.14}$$

with *matter density* ρ, *pressure* p and linear *equation of state*

$$p = w\rho, \tag{1.15}$$

the Einstein equations (1.2) reduce to two ordinary, coupled differential equations

add reference

check lambda dependence

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} = -\frac{4\pi}{3}\mathrm{G_N}\,(\rho + 3p)\, a + \frac{\Lambda}{3} \tag{1.16a}$$

acceleration equation

$$\mathrm{H}^2 = \frac{8\pi}{3}\mathrm{G_N}\rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \tag{1.16b}$$

Friedmann equation

with the *Hubble function*

$$\mathrm{H} \equiv \frac{\mathrm{d}a}{\mathrm{d}t}\frac{1}{a} \quad \text{or} \quad \mathcal{H} \equiv \frac{\dot{a}}{a} = a\mathrm{H} \quad \text{in conformal time } \eta. \tag{1.17}$$

The *equation of state parameter* w for various matter types is summarized in Table 1.1. For a universe filled with any such matter, we can solve the Friedmann equations (1.16) to find both the time evolution of the matter density

$$\rho \propto a^{-n(w)} \tag{1.18}$$

and of the scale factor

$$a(t) \propto \begin{cases} t^{\frac{2}{n(w)}} & \text{for } w \neq -1 \\ \mathrm{e}^{\mathrm{H}t} & \text{for } w = -1 \end{cases} \quad \text{with} \quad n(w) \coloneqq 3\,(1+w)\,. \tag{1.19}$$

This result is derived in Appendix A.3 and already exhibits a wealth of cosmological implications. In a universe filled with matter of $w \neq -1$ we find $\mathrm{H} \propto \frac{1}{t}$, for example, and can therefore identify it with a notion of the *age of the universe*. Furthermore, the relation $\rho \propto t^{-2}$ that follows from (1.18) in such a universe implies a singularity at $t = 0$ where the density becomes infinite. This is called the *big bang*.

Multiple matter types in the universe combine to the total matter density

$$\rho(t) = \sum_i \rho_i(t) \quad \text{with equation of state} \quad p_i = w_i \rho_i \quad \text{each,} \tag{1.20}$$

	i	w	$n(w)$
radiation	γ	$\frac{1}{3}$	4
dust	d	0	3
cosmological constant	Λ	-1	0
spatial curvature	κ	$-\frac{1}{3}$	-2

Table 1.1.: Overview of cosmological properties for a universe filled with various matter types

where we can include the effect of a spatial curvature κ and a cosmological constant Λ through the definition of additional pseudo-densities

$$\rho_\kappa(t) = -\frac{3}{8\pi \mathrm{G_N}} \frac{\kappa}{a^2} \quad \text{and} \quad \rho_\Lambda(t) = \frac{\Lambda}{8\pi \mathrm{G_N}}. \tag{1.21}$$

When we then define the relative matter densities explain

$$\Omega_i(t) := \frac{\rho_i(t)}{\rho_{\text{crit}}} \quad \text{with} \quad \rho_{\text{crit}} := \frac{3\mathrm{H}_0^2}{8\pi \mathrm{G_N}} \tag{1.22}$$

where H_0 denotes the value of the Hubble function today at cosmic time $t = t_0$, the Friedmann equation (1.16b) becomes

$$\frac{\mathrm{H}^2}{\mathrm{H}_0^2} = \sum_i \Omega_i(t) \tag{1.23}$$

and we find from (1.18) the relation

$$\Omega_\Lambda \propto a^2 \Omega_\kappa \propto a^3 \Omega_{\mathrm{d}} \propto a^4 \Omega_\gamma. \tag{1.24}$$

Because the universe expands monotonically with time for any of these matter types, as given by (1.19), this result allows us to consider successive *cosmological regimes* in an FRW universe with a dominant matter type each. Radiation dominates in the early universe and is followed by a regime of dust domination (also called *matter domination*). The spatial curvature κ is measured to be zero very precisely today, thus eliminating the corresponding regime. Today, the universe is in a regime dominated by a cosmological constant (also called *de Sitter space*) that is discussed in more detail in subsection 1.2.1.

1.1.3. Tensor Perturbations in an FRW Universe

At smaller scales, the universe is not homogeneous and isotropic at all, of course. Galaxies, stars and planets, as well as radiation or, in fact, any energy content of the universe disturb the spacetime metric locally. It is therefore reasonable to consider perturbations δg around the smooth FRW metric and their evolution.

also primordial / early universe / inflation and such

When we assume an exact solution g to the unperturbed Einstein equations and consider a sufficiently small perturbation to the stress-energy tensor $\delta T_{\mu\nu}$, then the metric perturbation δg that solves

$$G_{\mu\nu}[g + \delta g] = T_{\mu\nu}[g] + \delta T_{\mu\nu}[g] \quad \text{for} \quad 8\pi \mathrm{G_N} = 1 \tag{1.25}$$

will also be small and we obtain

$$\delta G_{\mu\nu}[g, \delta g] = \delta T_{\mu\nu}[g] \tag{1.26}$$

..and lead to gravitational interaction

with $\delta G_{\mu\nu}[g, \delta g]$ linear in δg in linear perturbation theory.

Because of its symmetry condition, δg has 10 degrees of freedom that we can parametrize as [Schulz]

motivate better > bacci-galupi p. 22

$$\delta g = -2A\,\mathrm{d}x^0 \otimes \mathrm{d}x^0 + B_i\left(\mathrm{d}x^0 \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}x^0\right) + \left(2C\gamma_{ij} + 2E_{ij}\right)\mathrm{d}x^i \otimes \mathrm{d}x^j \tag{1.27}$$

for small spatial *scalar fields* $A = A(x^0)$ and $C = C(x^0)$, a *vector field* $B_i = B_i(x^0)$ and a symmetric, trace-free *tensor field* $E_{ij} = E_{ij}(x^0)$.

really?

As of the *Helmholtz theorem*, the parameters uniquely decompose further into scalar, vector and tensor components as

$$\delta g = \delta g^{\text{scalar}} + \delta g^{\text{vector}} + \delta g^{\text{tensor}}. \tag{1.28}$$

This is shown in detail in Appendix A.4 and allows us to study scalar, vector and tensor perturbations separately.

add reference or derive

notation

Gravitational waves now arise when we only consider **unsourced tensor perturbations** of the metric. This is analogous to the propagation of electromagnetic waves in vacuum, for example. In fact, scalar and vector perturbations can only arise from stress-energy perturbation. Therefore, only tensor perturbations remain for $\delta T_{\mu\nu} = 0$.

add info about scalar, vector, tensor > grav. waves

It is shown in Appendix A.4 that

$$\delta g_{ij}^{\text{tensor}} = h_{ij} \tag{1.29}$$

elaborate

add

is a *symmetric, traceless, divergence-free tensor field* and can therefore be expressed in terms of two functions $h_\times$ and h_+ as

$$h_{ij} = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} \tag{1.30}$$

with an implicit choice of the z-axis in direction of the *wave vector* $\boldsymbol{k}$ [Dodelson].

elaborate choice of axis, show div/trace-free with wave-vector k, "transverse-traceless TT gauge"

The perturbed line element becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2 \left(\gamma_{ij} + h_{ij}\right) \mathrm{d}x^i \,\mathrm{d}x^j \tag{1.31}$$

and with this explicit form of the metric we can compute the Einstein tensor perturbation in (1.26). This is done in Appendix ?? and we obtain

$$\delta G_{ij} = \delta R_{ij} = \frac{3a^2}{2}\mathrm{H}\frac{\mathrm{d}h_{ij}}{\mathrm{d}t} + \frac{a^2}{2}\frac{\mathrm{d}^2 h_{ij}}{\mathrm{d}t^2} + \frac{k^2}{2}h_{ij}. \tag{1.32}$$

The perturbed Einstein equations (1.26) that govern the evolution of the metric perturbations then become a wave equation

damped oscillator, not wave!?

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + 3\mathrm{H}\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{k^2}{a^2}h = 0 \quad \text{in cosmic time } t \tag{1.33a}$$

$$\text{or} \quad \ddot{h} + 2\mathcal{H}\dot{h} + k^2 h = 0 \quad \text{in conformal time } \eta \tag{1.33b}$$

for $h \in \{h_\times, h_+\}$. Its solutions are called *gravitational waves* and occur in two independent *polarizations* $h_\times$ and h_+. Neglecting the friction term, harmonic oscillations

$$h \propto \mathrm{e}^{\pm k\eta} \quad \text{with the } \textit{wavelength mode} \quad k \equiv |\boldsymbol{k}| \tag{1.34}$$

solve (1.33b).

Gravitational waves are damped by the expansion of the universe as exhibited by (1.33) where the friction term is proportional to the Hubble function. In an expanding universe, the Hubble function (1.17) is positive, thus damping the amplitude of gravitational waves. However, the tensor perturbations will remain constant at early times where the conformal time is still smaller than the wavelength scal $\frac{1}{k}$ of the gravitational wave. It begins to oscillate at its *horizon entry* $\eta \simeq \frac{1}{k}$ where cosmological scales on the order of the gravitational wave's wavelength move into causal contact. The horizon entry for larger-scale modes with smaller

k thus occurs later. This behaviour is presented in Figure 1.1 where the absolute value $|h(a)|$ of several tensor modes in a universe dominated by the cosmological regimes discussed in subsection 1.1.2 is plotted in double-logarithmic scale. Detailed information about the assumptions, initial conditions and parameters that were chosen to obtain the plot in Figure 1.1 are given in the context of a parametric modification to the evolution equation of gravitational waves (1.33) in section 2.1.

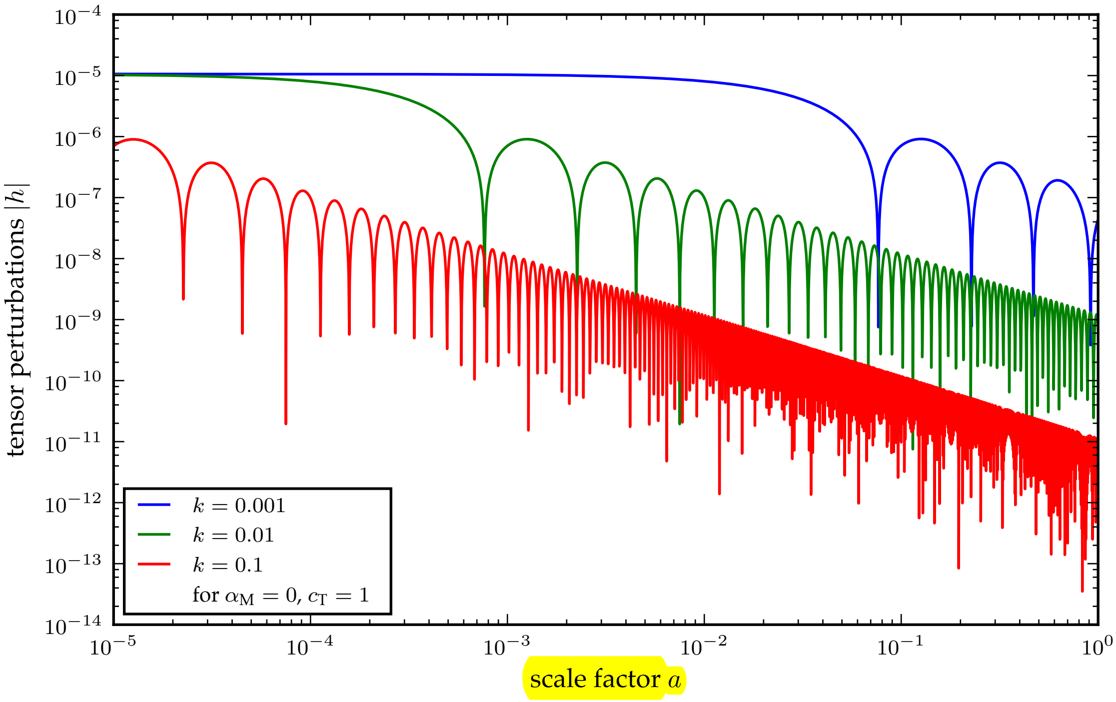

Figure 1.1.: **Tensor perturbations $|h(a)|$ for different wavelength modes k** In an expanding universe, tensor perturbations remain constant at early times until cosmological scales on the order of the gravitational wave's wavelength scale $\frac{1}{k}$ move into causal contact. This horizon entry occurs later for large-scale gravitational waves with smaller modes k.

observations limits

We will further discuss gravitational waves in the context of *modified gravity* in chapter 2 and the remainder of the thesis. However, to understand the reason why a modification of general relativity may be necessary, section 1.2 first formulates the *cosmological constant problem* and an approach for solving it.

1.2. The Cosmological Constant Problem

Both quantum field theory and general relativity are extremely well-tested theories and constitute the basis of modern physics, both in their respective fields. Whereas quantum field theory succeeds remarkably well in predicting particle physics phenomena, general relativity celebrates an equal success in large-scale cosmological observations. Both theories and particularly their interplay are not without mysteries, however.

better: not fully under- stood?

This section strives to formulate the *cosmological constant problem* and particularly emphasizes its problem of *radiative instability*. The quest for solutions will lead us to consider modified theories of gravity that will be the focus for the remainder of this thesis.

1.2.1. The Cosmological Constant

Lovelock's theorem states that general relativity as it emerges from the Einstein-Hilbert action 1.8 is, in fact, the **unique** metric theory in four spacetime dimensions that

add refer- ence

- gives rise to second-order equations of motion
- for only one symmetric rank-2 tensor
- that is local and lorentz-invariant.

Bianchi-identities?

This includes a free parameter Λ of the theory that is called the *cosmological constant*. Its value does not follow from the theory and thus it can only be constrained experimentally.

It appears in the Einstein equations 1.2 as a contribution

$$T_{\Lambda,\mu\nu} = -\Lambda g_{\mu\nu} \tag{1.35}$$

to the stress-energy tensor that corresponds to a **homogeneous energy density** penetrating the entire spacetime. In standard FRW cosmology, such a contribution results in an accelerating expansion of our universe as derived in detail in Appendix **??**.

In fact, cosmological observations clearly suggest such an accelerated expansion taking place at recent cosmic times. The ΛCDM (Λ *cold dark matter*) standard model of cosmology therefore includes a cosmological constant on the order of

add refer- ences

10^{-122} in Planck units and agrees with all cosmological observations remarkably well. In this theory, the physical origin of the homogeneous energy contribution that the cosmological constant represents remains a mystery, however, and is given the elusive name *dark energy*.

really?

good word?

add reference

The fact that the cosmological constant must be fixed by observations is not remarkable on its own, of course, since also the gravitational constant G_N must be determined experimentally. Theories such as the standard model of particle physics include a number of such free parameters. This poses an entirely different, somewhat philosophical problem that is often naively answered using anthropic arguments. This is discussed in detail in Appendix B.

1.2.2. Contributions to the Cosmological Constant

The cosmological constant problem arises when we consider both classical and quantum phenomena that, to our knowledge, should contribute to the cosmological constant.

Already in basic quantum mechanics, the uncertainty principle requires every physical system to have a zero-point energy. This immediately carries over to quantum field theory, where a (free) field is an infinite collection of coupled quantum mechanical harmonic oscillators. With a zero-point energy each, they combine to an infinite *vacuum energy* ρ_{vac}.

reference

add some derivation?

In quantum field theory, the vacuum energy is largely ignored as only differences in energy determine the dynamics of the system. In general relativity, however, all energy content gravitates, including the vacuum energy.

reference

how about Lamb shift, Casimir effect?

Remarkably, assigning an importance to not only energy differences, but also absolute energy values gives rise to another, entirely classical contribution to the stress-energy tensor, namely the *zero-point potential* V_0. This is the energy

$$T_{\mu\nu} = -V_0 g_{\mu\nu} \tag{1.36}$$

where the kinetic energy vanishes and the potential assumes its minimum value V_0. This zero-point potential is usually chosen arbitrarily when only energy differences are considered but must be taken into account in general relativity. Particularly in presence of phase transitions, one can generally not choose the zero-point potential such that it always vanishes.

The free parameter Λ in the Einstein equations therefore combines with both the quantum vacuum energy ρ_{vac} and the classical zero-point potential V_0 of every quantum field in the universe to an *effective* cosmological constant

$$\Lambda_{\text{eff}} = \Lambda + \rho_{\text{vac}} + V_0 \tag{1.37}$$

that we measure as dark energy. In comparison to the small value for Λ_{eff} we observe today, however, both contributions are extremely large [Martin, 2012]. This already suggests a severe *fine-tuning* problem where the value of the original cosmological constant Λ must precisely cancel the other contributions up to the small value we measure today. This is not the entire cosmological constant problem yet, though.

add reference

elaborate fine-tuning

1.2.3. Radiative Instability

The full scope of the problem arises when we consider in more detail the vacuum energy that we found to be infinite before. The mechanism to make sense of divergencies like this in quantum field theory is the framework of *renormalization*. In the process to find a find finite, *renormalized* value for ρ_{vac} one generally adds counterterms for every order in perturbation theory that each depend on an *arbitrary subtraction scale*.

Generally, successive orders are not significantly suppressed by a sufficiently small perturbation parameter Λ, however. In fact, for the standard model Higgs field the self-coupling parameter of perturbation Λ is of the order 10^{-1} and therefore every order in perturbation theory must be renormalized independently. This *radiative instability* requires us to fine-tune the cosmological constant **repeatedly** for every order in perturbation theory and thus makes it sensitive even to small-scale physics where we assume our theory to break down [Datta, 1996].

add reference

This also prohibits us from finding an *effective theory* where the full structure of perturbation theory is encoded in one finite, renormalized value by means of a *Wilson effective action*. Again, we find the renormalized vacuum energy is unstable against changes in the unknown UV-regime of the theory [Datta, 1996].

add reference

how about *why now?*

how about naturalness

1.2.4. The *New* Cosmological Constant Problem and Modified Gravity

The cosmological constant problem is deeply rooted in our inability to find a renormalized vacuum energy that is stable against changes in its effective description. Therefore, an approach to the problem is to assume that some mechanism makes the vacuum energy vanish altogether instead and then find another theory that explains the non-zero cosmological constant we observe today.

In fact, unbroken *supersymmetry* would accomplish just that. In supersymmetry, bosons and fermions are related by a symmetry and share the same mass. Supersymmetric partners contribute to the vacuum energy with opposite signs, however, thus precisely canceling each other.

With the cosmological constant set to zero, the cosmological observation of accelerated expansion remains to be explained by a different mechanism. Dark energy models such as the *quintessence* theory postulate further contributions to the stress-energy tensor that have the same accelerating effect as a cosmological constant. A different approach is to modify the theory of gravity instead.

other observables?

add reference

Every attempt towards a modified theory of gravity has to take Lovelock's theorem into consideration and break at least one of its assumptions that make general relativity unique. The concept of the entire class of $f(\mathcal{R})$ *theories*, for example, is to replace the Ricci scalar $\mathcal{R}$ in the Einstein-Hilbert action (1.8) by a function $f(\mathcal{R})$. $f(\mathcal{R})$ theories break Lovelock's assumption of only second-order equations of motion and are in many cases equivalent to an additional scalar field contribution to the matter Lagrangian that is non-minimally coupled to gravity.

add reference

Alternatively, the hypothesis of a *massive graviton* requires a second, arbitrary *reference metric* f in addition to the physical metric g. The additional metric is necessary to construct a mass term because a single metric allows only trivial self-interaction terms $g^{\mu\sigma}g_{\sigma\nu} = \delta^{\mu}_{\nu}$ and $g^{\mu\nu}g_{\mu\nu} = 4$. By postulating a symmetric action where the reference metric f behaves dynamically just like g, we arrive at the theory of *massive bigravity*. The cosmology in this theory allows solutions with late-time acceleration without a cosmological constant.

add reference

connect to next chapter

2. Parametrization of Modified Gravitational Wave Evolution

Many theories of modified gravity are discussed in the literature and this thesis will not focus on one specific model. Instead, the remainder of the thesis will explore various parametric modifications to the evolution equation of gravitational waves (1.33) that can result from a modified gravity theory. For such a modified gravity theory to be physically viable, the metric tensor perturbations must remain within constraints set by observations. In particular, any theory that exhibits **growing** tensor modes in cosmological evolution should be regarded with serious doubt, as their amplitude would likely be large enough today to be excluded by experiments.

add reference

mention observations

In an $f(\mathcal{R})$ theory without anisotropic stress, for example, the evolution equation becomes [Xu, Hwang and Noh, 1996]

add reference

$$\ddot{h} + \left(2 + \frac{\mathrm{d}\log F}{\mathrm{d}\log a}\right)\mathcal{H}\dot{h} + k^2 h = 0 \quad \text{with} \quad F := \frac{\mathrm{d}f(\mathcal{R})}{\mathrm{d}\mathcal{R}}, \tag{2.1}$$

thus adding an additional friction term. Furthermore, the propagation speed of gravitational waves in a modified gravity theory can deviate from the speed of light c [Amendola et al., 2014, Raveri et al., 2014]. I investigate the effects such modifications have on the evolution of gravitational waves by introducing appropriate parameters in (1.33). A reasonable parametric modification motivated by the above considerations is

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + (3 + \alpha_{\mathrm{M}})\,\mathrm{H}\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{{c_{\mathrm{T}}}^2 k^2}{a^2} h = 0 \quad \text{in cosmic time } t, \tag{2.2a}$$

$$\ddot{h} + (2 + \alpha_{\mathrm{M}})\,\mathcal{H}\dot{h} + {c_{\mathrm{T}}}^2 k^2 h = 0 \quad \text{in conformal time } \eta \tag{2.2b}$$

$$\text{or} \quad h'' + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 2 + \alpha_{\mathrm{M}}\right) h' + \frac{{c_{\mathrm{T}}}^2 k^2}{\mathcal{H}^2} h = 0 \quad \text{in e-foldings } \log a \tag{2.2c}$$

where α_{M} denotes an additional friction term and c_{T} a deviation of the propagation speed from the speed of light. Both parameters may in general be time-dependent.

The standard behaviour of general relativity discussed in subsection 1.1.3 is recovered for $\alpha_{\mathrm{M}} = 0$ and $c_{\mathrm{T}} = 1$.

2.1. Constant additional friction

I will first consider a constant friction term $\alpha_{\mathrm{M}} = const.$ in the parametrized evolution equation (2.2).

2.1.1. Numerical solution

Figure 2.1 shows the numerical solution of the parametrized evolution equation (2.2) for various values of α_{M}. The result was obtained with the `ParametricNDSolve` function in Mathematica where several assumptions were made:

First, initial conditions were chosen such that

$$h(a \to 0) = 1 \quad \text{and} \quad h'(a \to 0) = 0. \tag{2.3}$$

Furthermore, a standard FRW background was assumed where the Hubble function is given by the Friedmann equation (1.23)

write the limit like this?

$$\mathrm{H}(t) = \mathrm{H}_0 \sqrt{\sum_i \Omega_i(t)}. \tag{2.4}$$

Only matter and dark energy contributions Ω_{d} and Ω_{Λ} were considered here because they dominate in late-time cosmological regimes as discussed in subsection 1.1.2. Values for H_0 and $\Omega_{\mathrm{d}}(t_0)$ in

$$\Omega_{\mathrm{d}}(t) = \Omega_{\mathrm{d}}(t_0) a^{-3} \quad \text{and} \quad \Omega_{\Lambda}(t) = 1 - \Omega_{\mathrm{d}}(t_0) \tag{2.5}$$

were obtained from the **?** results given in **??**. Lastly, the physical amplitude for the gravitational waves

add reference to planck

$$h_{\text{phys}}(a) = \sqrt{A_{\mathrm{T}} \left(\frac{k}{k_0}\right)^{n_{\mathrm{T}}}} \cdot h(a) \tag{2.6}$$

was also computed from the **?** results given in **??**.

add reference

In Figure 2.1, the absolute value $|h(a)|$ is plotted in double-logarithmic scale for

explain phys. ampl.!

better

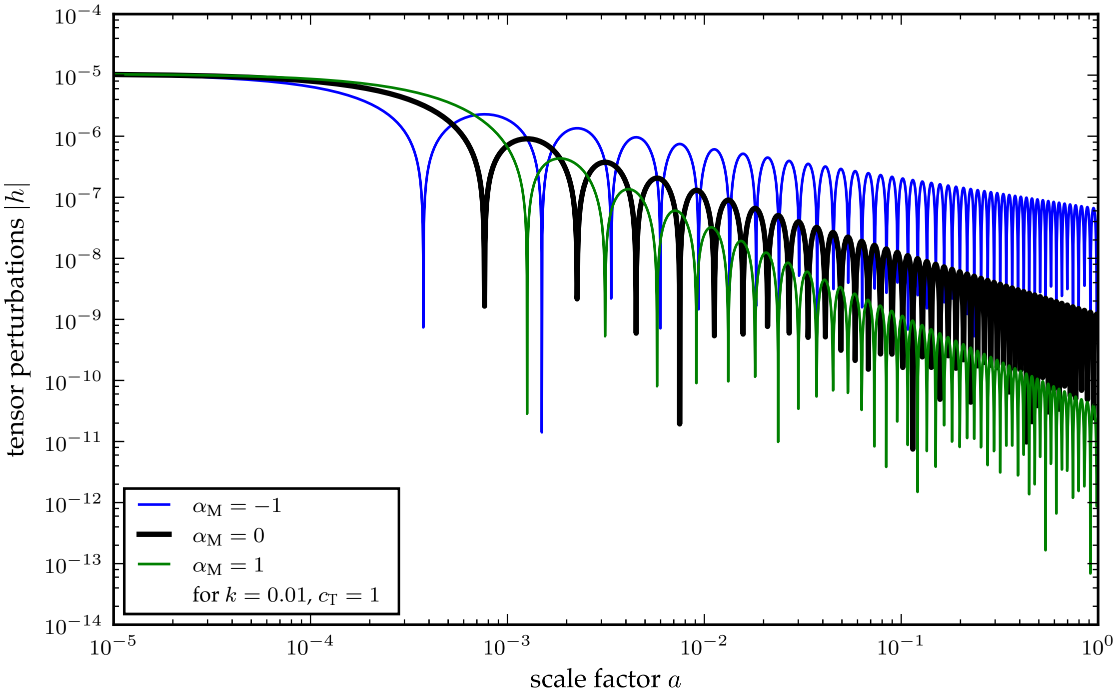

Figure 2.1.: **Tensor perturbations $|h(a)|$ with additional friction** $\alpha_{\mathrm{M}} = const.$ Increasing α_{M} introduces more friction, so that the amplitude of the gravitational wave decreases more rapidly, but also delays the horizon entry.

both positive and negative values of α_{M}. Clearly, the amplitude of the gravitational wave decreases more rapidly for larger α_{M} because additional friction is introduced. However, also the horizon entry is delayed for larger α_{M} thus introducing a competing effect. This reproduces the behaviour found in Pettorino and Amendola [2014].

write this here?

2.1.2. Analytic solution in cosmological regimes

The parametrized evolution equation (2.2) can even be solved analytically in cosmological regimes dominated by one of the matter types discussed in subsection 1.1.2. The expansion of an FRW universe dominated by matter or radiation is given by (1.19) or

$$a(\eta) \propto \eta^{\tilde{n}(w)} \quad \text{with} \quad \tilde{n}(w) := \frac{2}{1+3w} \quad \text{in conformal time.} \tag{2.7}$$

This allows us to find an expression $\mathcal{H} = \frac{\dot{a}}{a} = \frac{\tilde{n}(w)}{\eta}$ for the Hubble function so that (2.2) becomes

$$\ddot{h} + (2 + \alpha_{\mathrm{M}}) \frac{\tilde{n}(w)}{\eta} \dot{h} + {c_{\mathrm{T}}}^2 k^2 h = 0. \tag{2.8}$$

This is a Bessel differential equation that has solutions in terms of Bessel functions as discussed in Appendix A.6. Thus, (2.8) is solved by

$$h(\eta) = \eta^{-p}\left[C_1 J_p(c_{\mathrm{T}} k\eta) + C_2 Y_p(c_{\mathrm{T}} k\eta)\right] \quad \text{with} \quad p = \tilde{n}(w)\left(1 + \frac{\alpha_{\mathrm{M}}}{2}\right) - \frac{1}{2} \tag{2.9}$$

where $J_p(x)$ and $Y_p(x)$ denote the Bessel functions of first and second kind, respectively. Their asymptotic behaviour for large k or late times η is given in Appendix A.6 and both correspond to oscillations with an amplitude decreasing as $\eta^{-\frac{1}{2}}$. Therefore, the amplitude of gravitational waves with additional friction α_{M} behaves as

$$h(\eta) \propto \eta^{-p-\frac{1}{2}} = \eta^{-\tilde{n}(w)\left(1+\frac{\alpha_{\mathrm{M}}}{2}\right)} \tag{2.10a}$$

$$\text{or} \quad h(a) \propto a^{-\frac{1}{\tilde{n}(w)}\left(p-\frac{1}{2}\right)} = a^{-\left(1+\frac{\alpha_{\mathrm{M}}}{2}\right)} \tag{2.10b}$$

times fast oscillation in this regime.

This result immediately gives a constraint for the α_{M} parameter such that growing tensor modes are avoided. Since (2.10b) exhibits a growing amplitude of $h(a)$ for a positive exponent $-\left(1 + \frac{\alpha_{\mathrm{M}}}{2}\right) > 0$ in cosmological expansion, any theory of gravity that modifies the evolution equation of gravitational waves (2.2) by an additional friction term $\alpha_{\mathrm{M}} =$ *const.* should fulfill

$$\alpha_{\mathrm{M}} \geq -2 \tag{2.11}$$

or else be regarded with serious doubt.

”in tension with exp.”?

The numerical result obtained before reflects this behaviour as presented in Figure 2.2. The plot shows the evolution of tensor modes for both $\alpha_{\mathrm{M}} = -2$ and $\alpha_{\mathrm{M}} < -2$ using the same numerical solution as in Figure 2.1 as well as the slope of the amplitude obtained analytically in (2.10). As expected from the analytic considerations, the gravitational wave with $\alpha_{\mathrm{M}} = -2$ remains stable whereas the gravitational wave with $\alpha_{\mathrm{M}} < -2$ grows in amplitude.

2.2. Deviating propagation speed

To explore the effect a modified propagating speed has on the evolution of gravitational waves in a modified gravity theory, I consider a parametric deviation c_{T} from the speed of light in the parametrized evolution equation (2.2). The deviation of the propagation speed as it appears in (2.2) changes the effective wave-

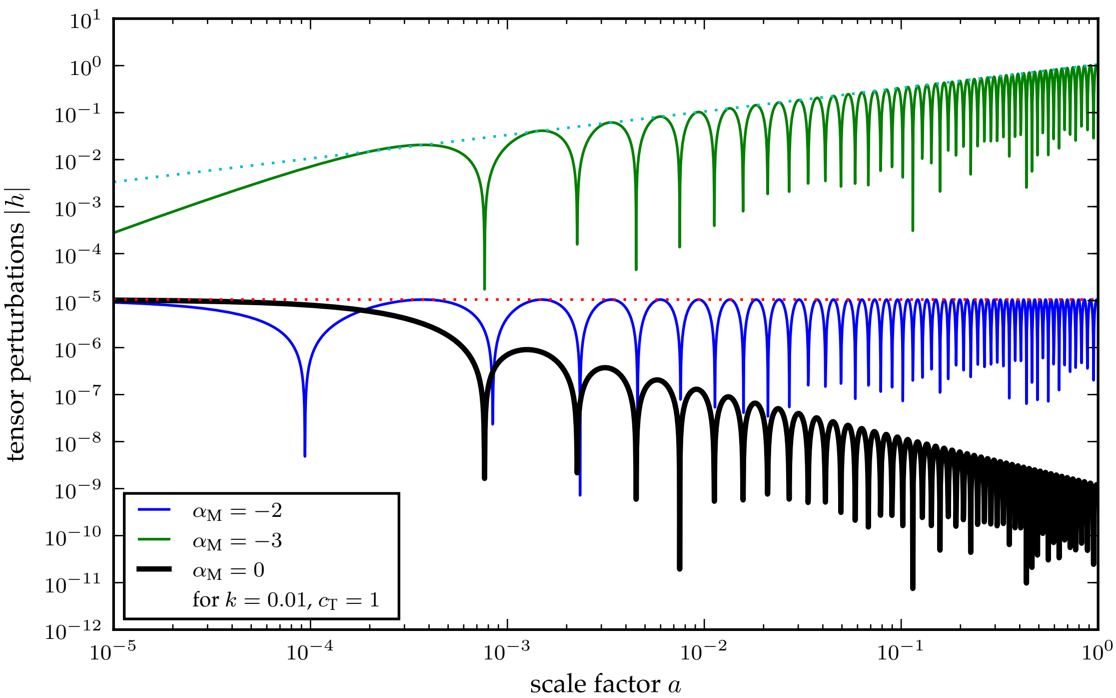

Figure 2.2.: **Tensor perturbations** $|h(a)|$ **with additional friction** $\alpha_M \leq -2$ The numerical solution (solid lines) agrees with the slope of the amplitude obtained analytically (dotted lines). Gravitational waves with $\alpha_M = -2$ remain stable but grow in amplitude for $\alpha_M < -2$.

length scale $\frac{1}{c_T k}$ such that the horizon entry of the gravitational wave occurs later for smaller values of c_T. The horizon entry is discussed in detail in the context of general relativity in subsection 1.1.2.

Figure 2.3 shows the evolution of gravitational waves for several values of $c_T =$ *const.*. The plots were obtained numerically with the `ParametricNDSolve` function in Mathematica and the same assumptions and initial conditions as chosen in section 2.1.

Because gravitational waves with lower propagation speed c_T exhibit a delayed horizon entry, their amplitude today is closer to their initial value than for gravitational waves that propagate faster. An additional friction contribution α_M as discussed in section 2.1 therefore is less effective for gravitational waves with lower c_T that enter the horizon later and thus have less time until today to decrease or increase in amplitude. This suggests a degeneracy between a modified propagation speed and an additional friction contribution, as both affect the amplitude of tensor perturbations measured today. A positive α_M that results in a faster decrease in amplitude can be compensated by a lower propagation speed c_T that in turn delays the horizon entry and thus gives the tensor mode less time to decrease. Also, the amount growing tensor modes increase in amplitude is reduced by a lower

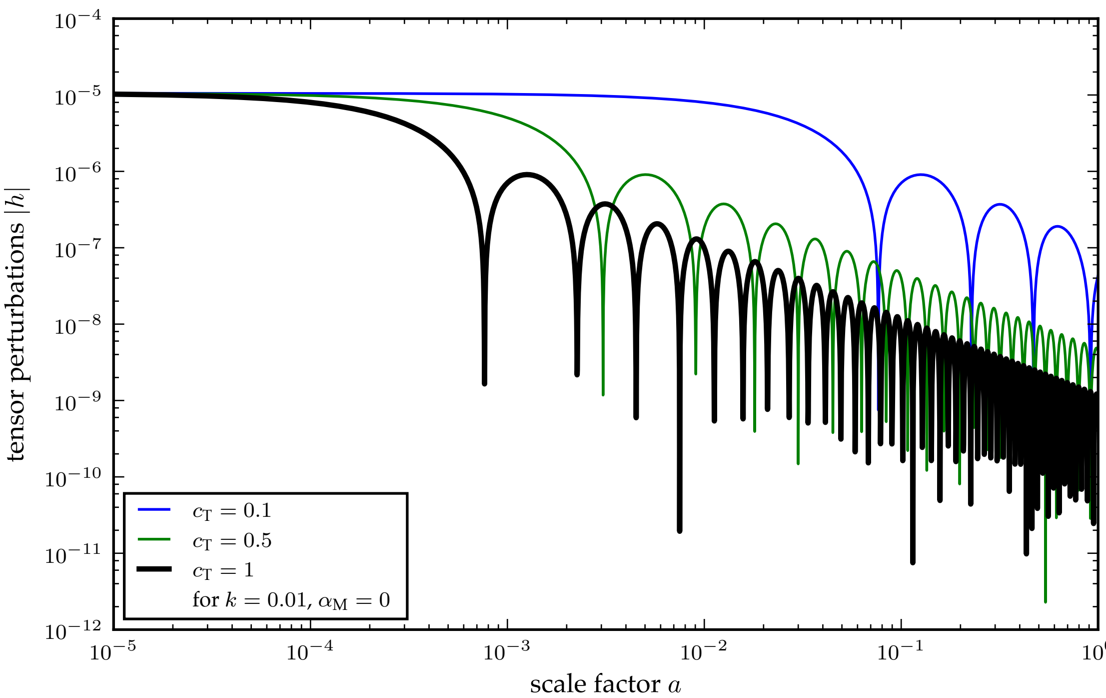

Figure 2.3.: **Tensor perturbations $|h(a)|$ with a modified propagation speed** $c_T =$ *const.* Gravitational waves with lower propagation speed c_T exhibit a delayed horizon entry.

propagation speed.

is cT > 1 possible?

2.3. Late-time additional friction

Many theories of modified gravity aim to solve the *new cosmological constant problem* presented in subsection 1.2.4. This requires such theories to model the accelerated expansion of our universe today as described in subsection 1.2.1. Proposed modifications to general relativity thus generally only affect late-time cosmological regimes where the ΛCDM standard model of cosmology relies on dark energy to accelerate the universe. Therefore, suitable parametrized modifications to the evolution equation of gravitational waves (2.2) are time-dependent and vanish in early-time cosmological regimes to reflect this behaviour of modified gravity theories.

add reference/example maybe specific f(R) theory

In particular, I consider for the additional friction parametrized by α_M the time-dependent parametrization

$$\alpha_M(t) = \alpha_{M0} \cdot a(t)^\beta \quad \text{with} \quad \alpha_{M0} = const. \quad \text{and} \quad \beta > 0. \tag{2.12}$$

In an expanding universe where $a(t) \in [0, 1]$ monotonically increases with time, the additional friction contribution α_{M} vanishes at early times to recover general relativity, but deviates from ΛCDM today.

Figure 2.4 shows numerical solutions to the evolution equation (2.2) with this particular parametrization for α_{M} for various values of both α_{M0} and β. It was obtained with the `ParametricNDSolve` function in Mathematica where the same initial conditions and assumption as in section 2.1 were chosen. The special case $\beta = 0$ corresponds to constant friction $\alpha_{\mathrm{M}}(t) = \alpha_{\mathrm{M0}}$ as investigated in section 2.1. Increasing β to positive values reduces the additional friction at early times where $a(t) < 1$. Larger values for β amplify this effect.

include Figure 4.1 at all?

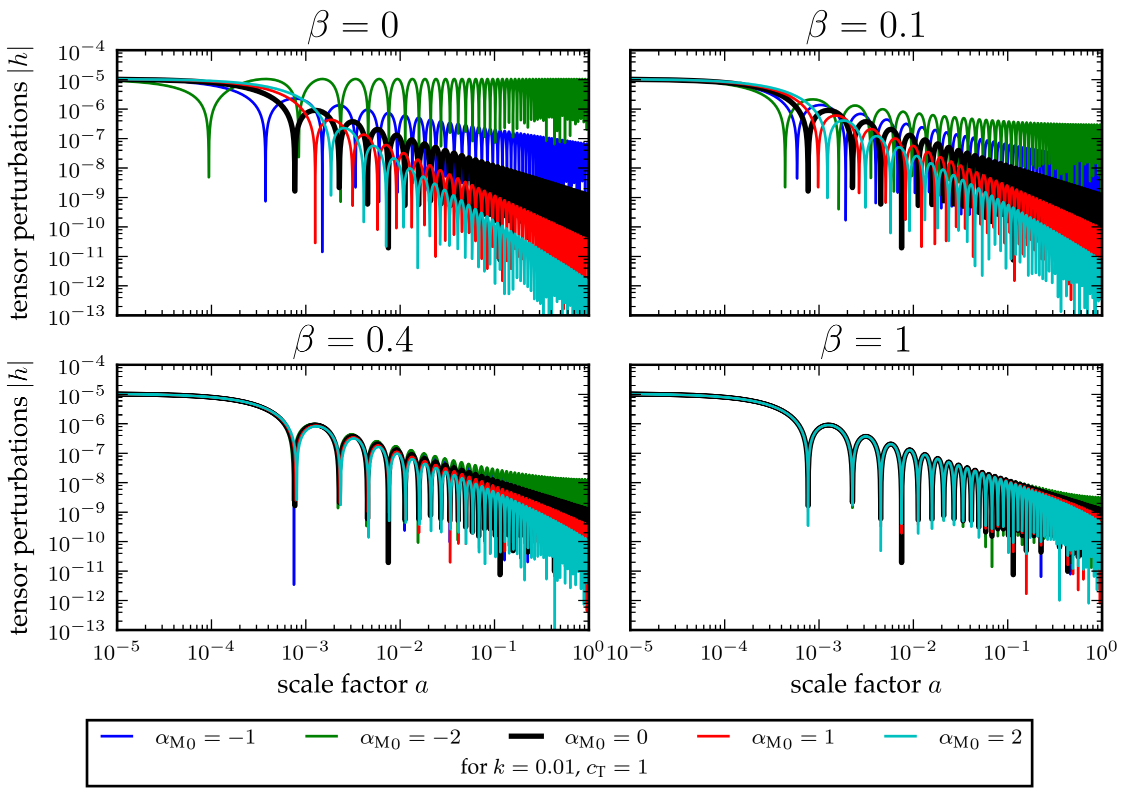

Figure 2.4.: **Tensor perturbations $|h(a)|$ with parametrized additional friction** $\alpha_{\mathrm{M}}(t) = \alpha_{\mathrm{M0}} \cdot a(t)^{\beta}$ For positive β, the evolution matches ΛCDM at early times and deviates at late times. Increasing $|\alpha_{\mathrm{M0}}|$ yields larger deviations from ΛCDM at late times.

At late times, tensor modes will increase or decrease in amplitude with a slope approximating the behaviour of constant additional friction α_{M0} discussed in section 2.1. However, tensor modes that grow in this regime according to (2.11) can be suppressed by a sufficiently large value of β remain physically viable.

3. Gravitational Waves in Parametrized Bigravity

Bigravity is a class of modified gravity theories introduced in subsection 1.2.4 where a second *reference metric* f in addition to the physical metric g is considered. Matter only couples to the physical metric, but both metrics are coupled such that the evolution equation of gravitational waves in this bimetric setting becomes [Amendola et al., 2015]

$$\ddot{h}_n + (2 + \alpha_{\mathrm{M}n})\,\mathcal{H}\dot{h}_n + \left(\mathcal{H}^2 m_n^2 + c_{\mathrm{T}n}^2 k^2\right) h_n = \mathcal{H}^2 q_n h_m \quad \text{in conformal time } \eta \tag{3.1a}$$

$$\text{or} \quad h_n'' + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 2 + \alpha_{\mathrm{M}n}\right) h_n' + \left(m_n^2 + \frac{c_{\mathrm{T}n}^2 k^2}{\mathcal{H}^2}\right) h_n = q_n h_m \quad \text{in e-foldings } \log a \tag{3.1b}$$

where the indices $n, m \in \{g, f\}$ with $n \neq m$ refer to the perturbations and parameters associated with the physical metric g and the reference metric f, respectively. The additional friction terms $\alpha_{\mathrm{M}n}$ and the deviations from the speed of light $c_{\mathrm{T}n}$ for each of the two metrics were discussed in chapter 2 in the context of unimetric modified gravity. Each metric also has an associated *mass parameter* m_n and a *coupling* q_n to the other metric. In the theory of bigravity, these parameters have specific, time-dependent forms that are given in Amendola et al. [2015].

3.1. Coupling to the reference metric

Since the amplitude of tensor perturbations of the physical metric g is severely constrained by cosmological observations, I neglect the coupling of the reference metric f that is sourced by h_g. Instead, I explore solutions of the parametrized bimetric evolution equation (3.1) with a non-zero coupling parameter $q_g = const.$. The non-standard behaviour of the reference metric that is governed by a choice of its associated parameters will therefore modify the evolution of the physical tensor perturbations h_g.

Figure 3.1 depicts several solutions of the parametrized bimetric evolution equation (3.1) that were obtained numerically with the `ParametricNDSolve` function in Mathematica. For both g and f, the same assumptions and initial conditions as in section 2.1 were chosen. In addition to the coupling $q_g = 1$, only the additional friction parameter of the f-metric was given a non-standard value such that its tensor perturbations grow in amplitude for $\alpha_{\mathrm{M}f} < -2$ as discussed in section 2.1.

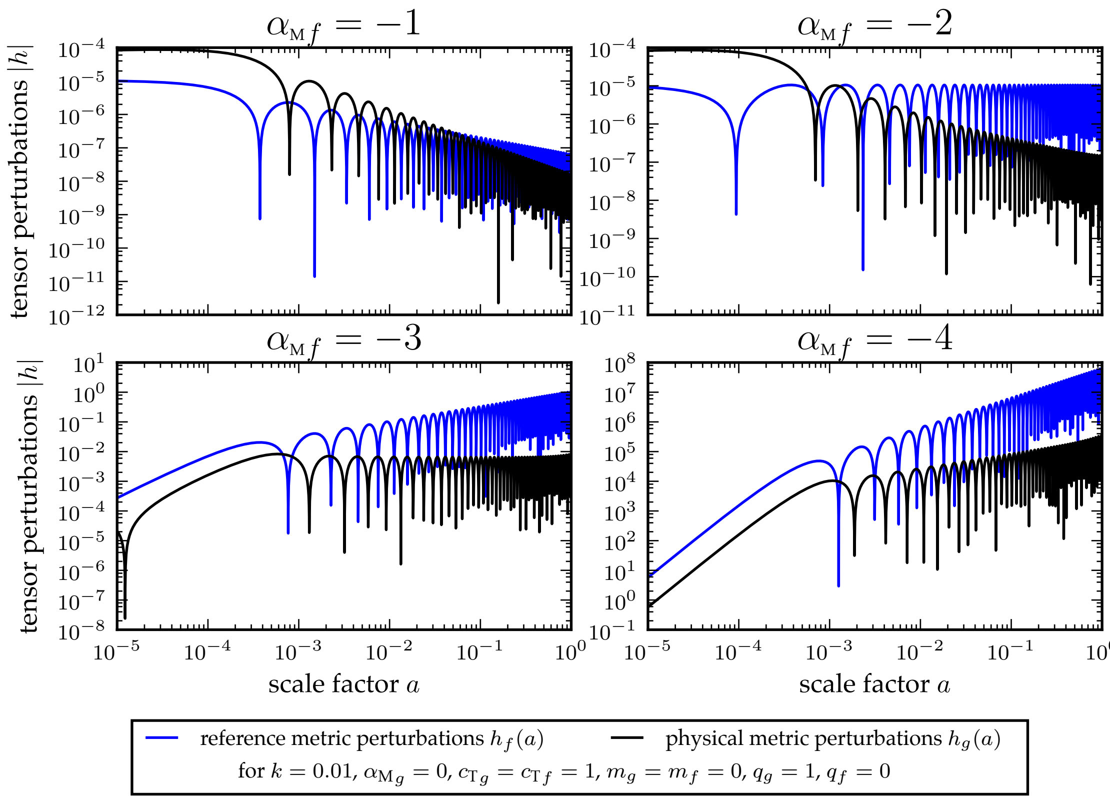

Figure 3.1.: **Physical and reference metric perturbations with parametrized coupling** $q_g = 1$ The amplitude of the physical metric perturbations follows the slope of the reference metric perturbations at late times. Note that the coupling q_g and the additional friction $\alpha_{\mathrm{M}f}$ of the f-metric are the only non-standard parameters.

Because matter only couples to the physical metric g whereas the reference metric f is not directly observable, growing tensor modes for the reference metric are not in tension with experiments. However, the coupling q_g also leads to non-standard behaviour of the physical metric, although its other associated parameters remain unmodified. Figure 3.1 shows that the physical metric perturbations

a coupling between the metrics must

follow the slope of the reference metric perturbations at late times and also exhibit a growing amplitude for sufficiently negative $\alpha_{\mathrm{M}f}$ when coupled to the reference metric. In the specific bigravity model investigated in Amendola et al. [2015], a similar behaviour is found for explicit, time-dependent expressions for the parameters that were considered constant here.

3.2. Non-zero mass parameter

The numerical solutions of the parametrized bimetric evolution equation (3.1) obtained in section 3.1 also allow for a brief discussion of the mass parameter m_n for both metrics $n \in \{g, f\}$. Figure 3.2 shows the evolution of the physical metric perturbations $|h_g(a)|$ for various values of its mass parameter m_g and all other parameters set to their standard value. Because the mass parameter appears as a positive contribution to the effective wavelength mode $c_{\mathrm{T}}k$ in (3.1), larger values of m_n will advance the horizon entry of the tensor mode. A degeneracy between the mass parameter and a deviating propagation speed c_{T} is to be expected.

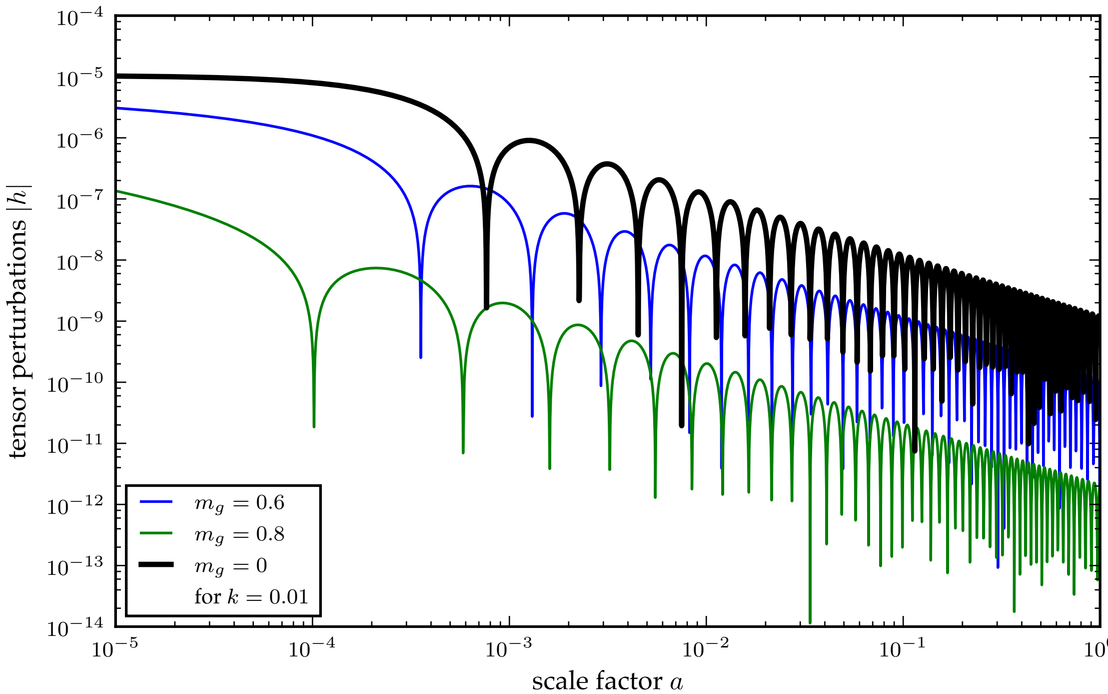

Figure 3.2.: **Tensor perturbations with non-zero mass parameter m_g**

3.3. Decoupling at early times

Bimetric modified gravity theories, like unimetric theories, usually deviate from ΛCDM only at late times to provide an accelerated expansion of the universe without a cosmological constant. This is discussed in more detail in section 2.3. For physically viable bimetric theories it is therefore reasonable to assume that the metrics decouple in early-time cosmological regimes where the physical metric g behaves unmodified but the reference metric can exhibit arbitrary dynamics.

add reference

A specific bimetric model is investigated in Amendola et al. [2015], for example, where the physical and reference metric decouple at early times. The authors found that for sub-horizon modes in radiation and matter dominated eras also the mass parameters become negligible in this model. Therefore, the evolution of tensor perturbations for both metrics reduce to the unimetric parametrized evolution equation (2.2) with parameters [Amendola et al., 2015]

$$\alpha_{\mathrm{M}g} = 0 \qquad c_{\mathrm{T}g} = 1 \tag{3.2a}$$

$$\alpha_{\mathrm{M}f} = -3(1+w) \qquad c_{\mathrm{T}f} = \frac{(3w+1)^2}{4} \tag{3.2b}$$

where the physical metric remains entirely standard such that the amplitude of tensor perturbations h_g falls like $\frac{1}{a}$ according to (2.10). However, the reference metric tensor perturbations h_f with $\alpha_{\mathrm{M}f} < -2$ in (3.2b) grow in amplitude like a^1 in radiation domination and $a^{\frac{1}{2}}$ in matter domination.

When the coupling q_g becomes relevant at appropriate late times, the behaviour explored in section 3.1 must be taken into consideration such that the physical metric perturbations do not exceed observational constraints by coupling to growing reference metric tensor modes. A detailed analysis of tensor perturbation can provide crucial insight about the physical viability of bimetric theories.

move to summary