Inverse Scattering
and
Chaotic Scattering

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Abstract

The topics of this work are how an expression for the potential can be obtained from information about the cross section and a qualitative discussion about scattering from two or three targets.

Contents

1 The Inverse Scattering Problem

When a scattering experiment is performed you obtain the scattering cross section or some information about it. Sometimes one wants to know how the force or the potential looks like and though it can be obtained by information about the cross section in the central force problem. If the cross section is known from measurements, $\theta(s)$ can be obtained by following formula

$$\sigma(\theta) = \frac{s(\theta)}{\sin\theta}\left|\frac{dS}{d\theta}\right| \tag{1}$$

θ is the scattering angle and s is the impact parameter. Then the following formula can be used to get an expression for $V(r)$

$$\theta = \pi - 2s\int_{r_0}^{\infty}\frac{dr}{r\sqrt{r^2\left(1-\frac{V(r)}{E}\right)-s^2}} \tag{2}$$

Start to define a new function

$$y(r) = r\sqrt{1-\frac{V(r)}{E}} \tag{3}$$

and let the integral part in (2) be called I. I can be written in terms of this new function as

$$I = 2s\int_{r_0}^{\infty}\frac{dr}{r\sqrt{y^2-s^2}} \tag{4}$$

where r_0 is the shortest radial distance. Assume that $y(r)$ is invertible so that $r(y)$ exists and change the variable of integration from r to y. Then

$$I = 2s\int_{s}^{\infty}\frac{r'(y)dy}{r(y)\sqrt{y^2-s^2}} = 2s\int_{s}^{\infty}\frac{dy}{\sqrt{y^2-s^2}}\frac{d}{dy}\ln r(y) \tag{5}$$

The lower limit in the y integral is s because the lower limit in the r integral is the root r_0 of the radical in the denominator, which is given by $y(r_0) = s$. According to (2), $\theta(s) = \pi - I$, but π can be written as

$$\pi = 2s\int_{s}^{\infty}\frac{dy}{y\sqrt{y^2-s^2}} = 2s\int_{s}^{\infty}\frac{dy}{\sqrt{y^2-s^2}}\frac{d}{dy}\ln y \tag{6}$$

so that

$$\theta(s) = \pi - I = 2s\int_{s}^{\infty}\frac{dy}{\sqrt{y^2-s^2}}\frac{d}{dy}\ln\frac{y}{r(y)} \tag{7}$$

Instead of calculating this integral directly define a new function from which $y(r)$ will be found

$$T(y) = \frac{1}{\pi}\int_{y}^{\infty}\frac{ds}{\sqrt{s^2-y^2}}\theta(s) \tag{8}$$

Eliminate $\theta(s)$ from this by inserting (7). That gives

$$T(y) = \frac{1}{\pi}\int_y^\infty \frac{2s}{\sqrt{s^2-y^2}}\left[\int_s^\infty \frac{du}{\sqrt{u^2-s^2}}\frac{d}{du}\ln\frac{u}{r(u)}\right]ds \qquad (9)$$

If the order of integration is changed, this double integral can be performed, but it requires care with the limits of integration. When the

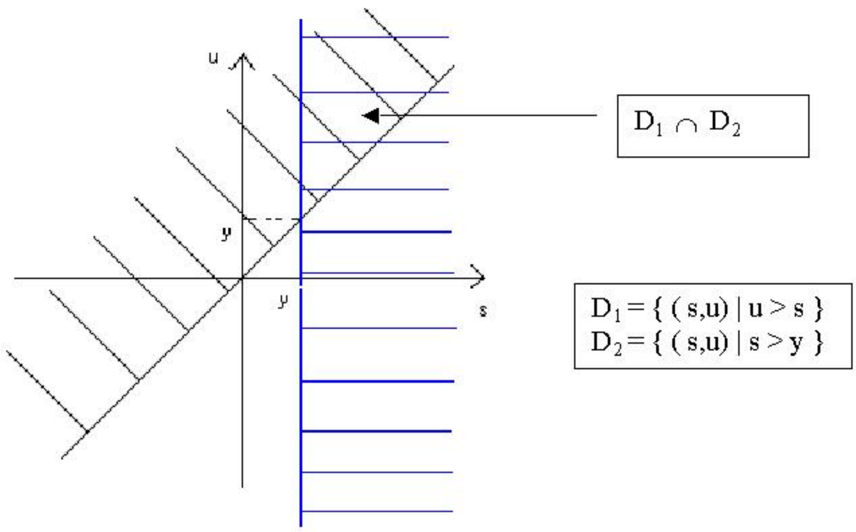

Figure 1: Illustration of $D_1 \cap D_2$

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order is changed the double integral becomes

$$T(y) = \frac{1}{\pi}\int_y^\infty \left[\frac{d}{du}\ln\frac{u}{r(u)}\right]du\int_y^u \frac{2sds}{\sqrt{(s^2-y^2)(u^2-s^2)}} \qquad (10)$$

The integral over s is quite tricky but turns out to equal π, so the result is an expression for $T(y)$ in terms of r and y

$$T(y) = \ln\left[\frac{r(y)}{y}\right] \qquad (11)$$

which can be inverted and yield

$$r(y) = y\exp\left[T(y)\right] \qquad (12)$$

Equation (3) can be rewritten so the potential is a function in terms of r and y

$$V = E\frac{r^2-y^2}{r^2} \qquad (13)$$

$T(y)$ is assumed to be the known function obtained from $\theta(s)$ so when you solve (12) for y in terms of r and the solution is inserted in (13) an expression for the potential, $V(r)$, is obtained.

2 Scattering by Two or More Targets

In the inverse scattering problem a process called one-on-one scattering is assumed. If this assumption is dropped it will become scattering by two or more stationary targets. The scattering is two-dimensional with a hard-core elastic interaction. The incoming beam is composed of noninteracting point particles (projectiles of radius zero), and the target is composed of infinite-mass hard disks of radius a. The projectile can now bounce back and forth between the disks and it leads to scattering angles that can be very irregular functions of initial data and even to orbits that are trapped and never leave the scattering region. So, the collisions are elastic and the angle of incidence equals the angle of reflection, so neither the energy nor the disk radius play any role in the scattering from each individual disk therefore put $a = 1$. The configuration manifold Q of the system is a plane with circular holes at the disk positions, regions that the projectile cannot enter. The four-dimensional TQ is obtained by attaching the two-dimensional tangent plane of the velocities above each point of Q. In general there would be four initial conditions, but as the energy plays no role and all that matters is the incident line of the projectile, the needed part of TQ is just a two-dimensional sub manifold. Different numbers of target particles lead to different geometries for TQ and the needed sub manifold.

2.1 Two Disk Scattering

Assume that the centers of the two disks are separated by a distance $D > 2$ (D is the ratio of the separation to the disk radius). A projectile

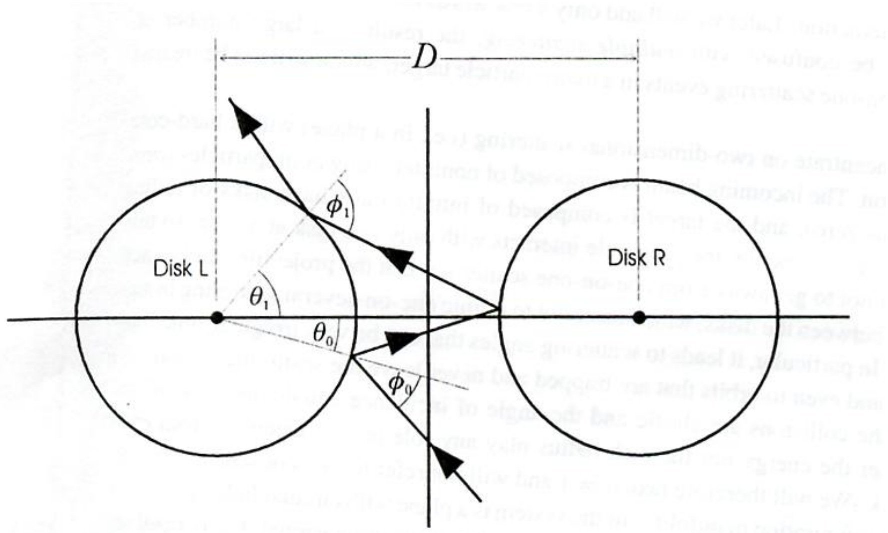

Figure 2: Two disk scattering, (Figure 4.5 from referens [1]).

collides with disk L at a point defined by the angle θ while moving at an angle ϕ with respect to the radial normal at that point. The projectile bounces off at the same angle ϕ on the other side of the normal. It hits disk R and then disk L again before leaving the target area. The time in the target area is called the dwell time. If it were possible for the projectile to travel exactly along the symmetry axis, it would be trapped forever and its dwell time would be infinite. But this exceptional periodic orbit can never occur because it requires that $(\theta,\ \phi) = (0,\ 0)$, and that is forbidden by time-reversal invariance, this means that given any possible orbit, another possible orbit can be constructed by replacing t by $-t$. The exceptional periodic orbit can be approached but never reached. On some orbits the $(\theta,\ \phi)$ point of successive hits comes close to $(0,\ 0)$ but the exceptional periodic orbit is a repeller of almost all orbits that approach it. $(\theta,\ \phi)$ eventually moves further and further from $(0,\ 0)$ and for this reason it is also called an unstable orbit. Suppose a projectile hits disk L at $\theta = -\pi/4$ with $\phi_1 =$

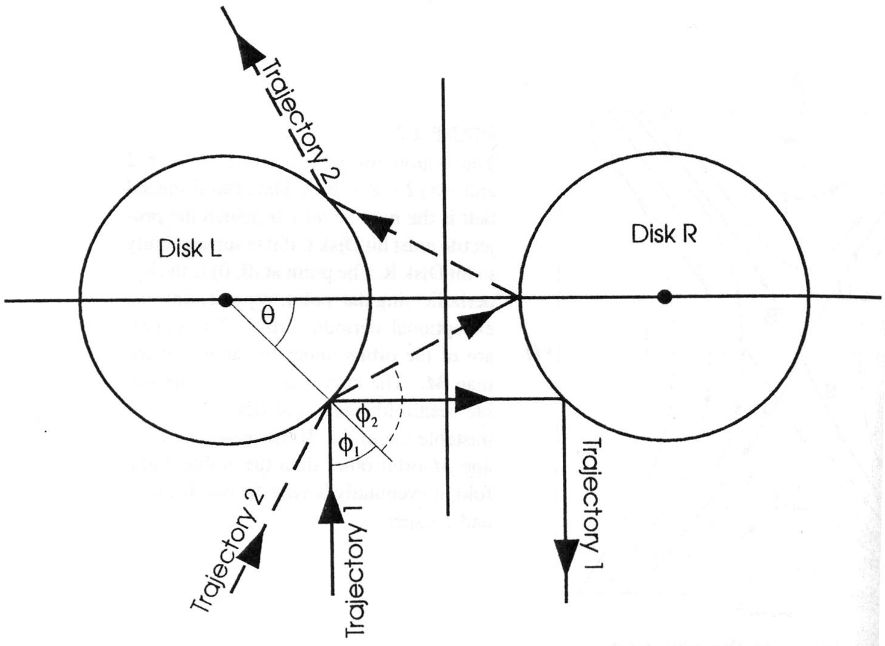

Figure 3: Two disk scattering, (Figure 4.6 from referens [1]).

$\pi/4$, trajectory 1 in *figure 3.* It will bounce off horizontally, hit disk R and leave the target area. Trajectory 2 hits disk L at the same θ but at ϕ_2 so after bouncing off disk L it hits disk R at the point of intersection with the line joining the centers of the disks. Symmetry then shows that the projectile will bounce off disk R and hit disk L directly above the first hit and leave in the direction given by the mirror image of ϕ_2. If the incident angle now vary between ϕ_1 and ϕ_2 there will be a range of ϕ values for which the projectile leave the region below the disks and a range of ϕ values for which it leave above. These are two open sets of trajectories and between them is an incident ϕ whose trajectory ends up neither above nor below. So the projectile is incident from outside the target area and although never quite reaching the periodic orbit it remains trapped and never leaves. That is a nonperiodic trapped orbit. The two-disk system can be solved exactly analytically.

2.2 Three Disk Scattering

With three disks the situation is more complicated. Here it is an infinite number of these unstable periodic orbits and what makes it even more complicated is the way in which the periodic orbits are distributed in the parameter space that is the equivalent of the $(\theta,\ \phi)$ region in two-disk scattering. They are distributed in what is called a Cantor set. The standard example of the middle third Cantor set is explained as follows. From the unit closed interval $I_0 = [0, 1]$ the open set of its center third

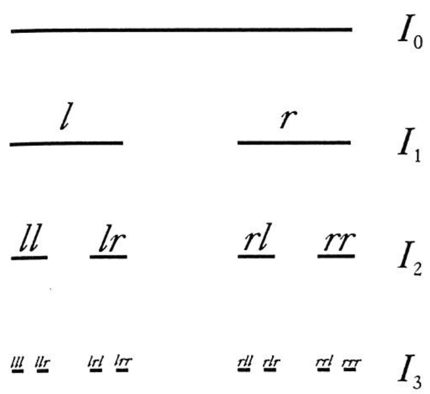

Figure 4: The first steps in the construction of the middle third Cantor set, (Figure 4.8 from referens [1]).

is removed, leaving the closed sets $l = [0,\ 1/3]$ and $r = [2/3,\ 1]$ (l=left and r=right). Define $I_1 = l \cup r$. Now remove the center thirds of l and r leaving the sets $ll = [0,\ 1/9]$, $lr = [2/9,\ 1/3]$ and so on. Define $I_2 = ll \cup lr \cup rl \cup rr$. Go on in this way always removing the center thirds of the remaining intervals.

The n:th step yields the set I_n a union of 2^n intervals designated by all possible permutations of n letters l and r, which is called n-strings. The middle third cantor set C is then defined as: What is left after an infinite number of steps. Since all that is removed in any step is an open set, some points remain and C is not empty. It can be shown that C has the power of the continuum, which means that there are as many points in C as there are points on the line, rational and irrational. The middle third Cantor set has another important property: it is self-similar, that is, after an infinite number of steps, each subinterval designated by a finite n-string looks like I_1 except that it is shorter by a factor of 3^n . Self-similarity is one of the features that are found in scattering from three disks arranged in an equilateral triangle. Each disk has radius 1

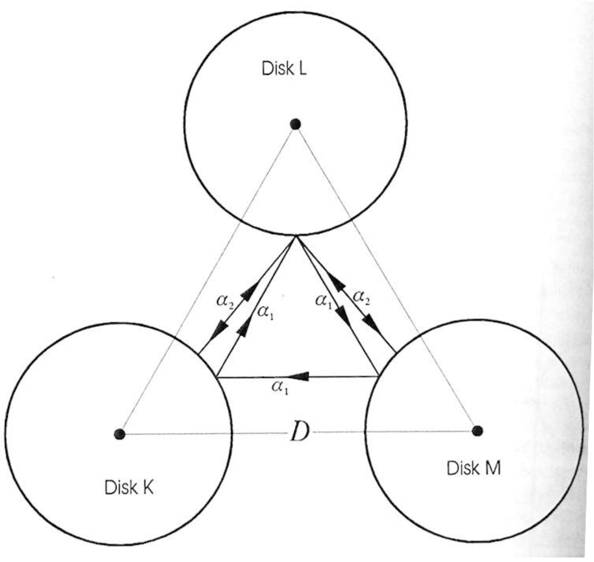

Figure 5: Hard-core scattering from three disks arranged equilaterally. Two periodic orbits are shown, (Figure 4.9 from referens [1]).

and the triangle connecting their centers has sides D that is the ratio of the side of the triangle to the disk radius. There are many periodic orbits, for example between K and L and between L and M. Another are going round and round between the three disks (α_1 *in figure 5*) and there are many others. Here we consider projectiles incident from below and perpendicular to the base line connecting disks K and M. Unlike central-force scattering, this system is not rotationally invariant, so the incident angle is needed to describe the trajectory completely: the full sub manifold of initial conditions is two dimensional. But for simplicity the one-dimensional sub manifold which consists only of the perpendicular trajectories is considered. Let I_0 be the interval between disk K and M:s center. If now $D > 4$ it is gaps between the disks (*see figure 6*) where the projectile misses all three disks. Remove these gaps from I_0 and let I_1 be what is left. Within I_1 there are subintervals for which the projectile goes on to hit a second disk and other subintervals for which it leaves the target area after one collision. Remove the one-hit subintervals from I_1 and let I_2 be what is left. I_2 represents all the trajectories that makes at least two hits before leaving the target area. Go on in this way and what is left at the n:th stage is I_n, which represents all the trajectories that makes at least n hits before leaving the target area, with $I_0 \supset I_1 \supset \cdots \supset I_n$. Labelling the trajectories belonging to I_n gives more similarities with the Cantor set. Each trajectory starts with a hit on one of the disks and every subsequent hit can be labelled by l or r depending on whether it hits the one to its left or to its right. For ex. an I_4 - trajectory can be labelled by the 4-string $Llrr$ (the projectile first hit disk L then M then L and at last K) then

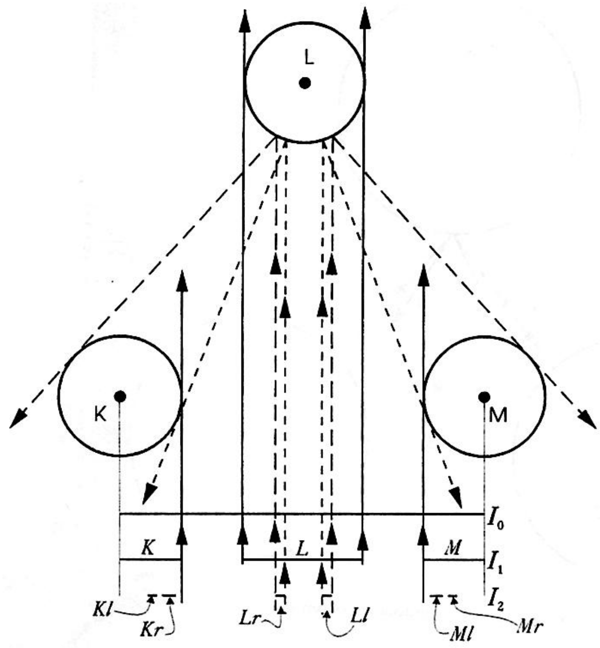

Figure 6: The first two steps in constructing the Cantor set for three disk scattering, (Figure 4.10 from referens [1]).

it have to leave the target area for it belongs to I_4. The I_∞ trajectories belong to strings that are infinitely long and by the same arguments as for the middle third Cantor set, the set of these ∞-strings can be shown not to be empty. This system of three disks with the hard-core interaction has been the subject of several computer and analytic investigations. What is often studied is the dependence of dwell time on initial data. Also the dependence of scattering angle on initial data has been studied and there it is found discontinuities in it. This means that the scattering angle cannot in general be predicted with complete accuracy, as extremely small variations in the initial data may cause wild changes in the scattering angle for orbits that lie close to points in I_∞. This is one reason among others why three-disk scattering is called chaotic or irregular.

In this work only a one-dimensional sub manifold of initial conditions has been considered. A full treatment would require dealing not only with variations in the impact parameter, but also in the incident angle. The full treatment shows that at all angles the scattering singularities lie in a Cantor set. A quantitative analysis of the three-disk scattering is not amenable to analytic methods but requires extensive use of computers.

References

[1] *Classical Dynamics*, J.V. José, E.J. Saletan, Cambridge University Press 1998, ISBN 0521636361.

[2] *www.wikipedia.org*