Mat 354

Homework 12

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Due November 9, 2015

1. **Expected Probability**
   Consider a continuous random variable Y with (continuous) distribution function $F(y)$. In class it was shown that $E[F(Y)] = \frac{1}{2}$.

   a) Use similar tactics to derive an expression for $E[F^k(Y)]$ where $k > 1$.

   We note that a the product of a function with its derivative occurrs via the chain rule. Examining this situation, we see

$$\begin{aligned} E[F^k(Y)] &= \int_{ally} F^k(Y)F'(Y)dy \\ &= \left[\frac{1}{k+1}F^{k+1}(Y)\right]_{ally} \\ &= \frac{1}{k+1}(1) - \frac{1}{k+1}(0) \\ &= \frac{1}{k+1} \end{aligned}$$

   b) Determine $V[F(Y)]$

   By the above expression, we have

$$\begin{aligned} V[F(Y)] &= E(F^2(Y)) - E(F(Y))^2 \\ &= \frac{1}{3} - \frac{1}{2^2} \\ &= \frac{4}{12} - \frac{3}{12} \\ &= \frac{1}{12} \end{aligned}$$

c) Do you have a conjecture on the distribution of the random variable $W = F(Y)$?

I may be missing something, but the above appears to apply for any continuous random variable, so I don't see how I would know anything more about it.

d) Suppose instead Y is discrete with distribution function $F(y)$. Is $E[F(Y)] = 12$? Produce an argument for your assertion.

Nope. Roll a 6-sided die. The probability of any outcome is $\frac{1}{6}$ and the distribution function is

$$F(y) = \frac{1}{y}, y \in \{1, 2, 3, 4, 5, 6\}$$

And of course

$$E(F(Y)) = \sum_{y=1}^{6} F(y)p(y) = \frac{7}{12}$$

Of course, $\frac{7}{12} \neq \frac{1}{2}$.

2. **A-maze-ing**

In a lab, a strain of mice are selectively bred for intelligence. And...it turns out that the resulting mice are smarter than (other) mice. These mice are timed going through a maze to reach a reward of food. The time (in seconds) required for any mouse is a random variable Y with a density function given by

$$f(y) = \begin{cases} \frac{2b^2}{y^3} & b \leq y \\ 0 & b \nleq y \end{cases}$$

where b is the minimum time needed to traverse the maze.

a) Determine the expected time required.

$$\begin{aligned} E(y) &= \lim_{a \to \infty} \int_b^a y \frac{2b^2}{y^3} dy \\ &= \lim_{a \to \infty} \left[-\frac{2b^2}{y} \right]_b^a \\ &= \lim_{a \to \infty} \left(-\frac{2b^2}{a} \right) + 2b \\ &= 2b \end{aligned}$$

The expected time for a mouse to traverse the maze is twice the minimum time required.

b) Which mice have the larger expected time: the intelligent variety of this exercise, or the mice of Exercise 4.15? Explain.

First we calculate the expected time for the mice from Exercise 4.15:

$$\begin{aligned} E(y) &= \lim_{a\to\infty} \int_b^{\infty} y \frac{b}{y^2} dy \\ &= \lim_{a\to\infty} \left[b \ln(y) \right]_b^a \end{aligned}$$

This limit does not exist, which means $E(y)$ does not exist.

3. **(Finally) A Minimum**

A random variable Y (the time until the first occurrence of something) has exponential distribution with parameter 2.

$$F(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\frac{y}{2}} & y \geq 0 \end{cases} \qquad f(y) = \begin{cases} 0 & y < 0 \\ \frac{e^{-\frac{y}{2}}}{2} & y \geq 0 \end{cases} \qquad E(Y) = 2 \qquad V(Y) = 4$$

There are 5 independent runs of the experiment that result in the random variable Y. Let L denote the minimum result for the 5 runs. Determine the following.

a) The distribution function for $L : F_l(l) = P(L \leq l)$. (When handling the minimum, your first step is to attack the probability of the complementary event. Start with $P_l(L > l)$.)

It is apparent that $F_l(L) = 1 - P_l(L > l)$ and we see

$$\begin{aligned} 1 - P_l(L > l) &= 1 - P(Y > l)^5 \\ &= 1 - (1 - F(l))^5 \end{aligned}$$

Thus we have distribution function

$$F_l(l) = \begin{cases} 0 & l < 0 \\ 1 - e^{-5l/2} & 0 \leq l \end{cases}$$

b) The density for L.

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}l}(1-(1-F(l))^5) &= 5(1-F(l))^4F'(l)\\ &= 5(1-F(l))^4f(l)\end{aligned}$$

So the density function is given by

$$f_l(l) = \begin{cases} 0 & l < 0 \\ \frac{5}{2}e^{-5l/2} & 0 \leq l \end{cases}$$

c) The expected value and variance of L.

The following integrals were calculated using wolframalpha.com.

$$\begin{aligned}E(Y) &= \int_0^\infty l\left(\frac{5}{2}e^{-5l/2}\right)dl\\ &= \frac{2}{5} = 0.4\end{aligned}$$

$$\begin{aligned}E(Y^2) &= \int_0^\infty l^2\left(\frac{5}{2}e^{-5l/2}\right)dl\\ &= \frac{8}{25}\end{aligned}$$

Thus the variance is $V(Y) = \frac{4}{25} = 0.16$

d) Obtain values for the 0.005 and 0.995 quantiles of the distribution of the minimum. The probability is 0.99 that the minimum falls between these two values.

The expression for the p^{th} quantile is found by solving $F(l) = p$ for l:

$$\begin{aligned}1 - e^{-5l/2} &= p\\ \Longrightarrow l &= -\frac{2}{5}\ln(1-p)\end{aligned}$$

This implies

$$\phi_{0.005} = 0.002005 \qquad\qquad \phi_{0.995} = 2.119327$$