Mat 354

Homework 15

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1. **EXC3** Y_1 and Y_2 are jointly distributed with density $f(y_1, y_2) = 4y_2^2$ $0 \le y_1 \le y_2 \le 1$

   i. Determine $P(\max\{Y_1, Y_2\} < 1/2)$

   This probability is found by integrating the density as y_1 goes from 0 to y_2 and y_2 goes from 0 to 1/2:

$$\begin{aligned} P(\max\{Y_1, Y_2\} < 1/2) &= \int_0^{1/2} \int_0^{y_2} 4y_2^2 dy_1 dy_2 \\ &= \int_0^{1/2} 4y_2^3 dy_2 \\ &= [y_2^4]_0^{1/2} \\ &= \frac{1}{16} \end{aligned}$$

   ii. Determine $P(Y_1 + Y_2 < 1/2)$

   This is really the probability that the max of the two is less than 1/4. If that is not obvious to you, think about it. The only way for their sum to be 1/2 is for one to be at least 1/4. Thus we find this probability in the exact same manner as the first problem, just with different bounds:

$$\begin{aligned} P(Y_1 + Y_2 < 1/2) &= \int_0^{1/4} \int_0^{y_2} 4y_2^2 dy_1 dy_2 \\ &= \int_0^{1/4} 4y_2^3 dy_2 \\ &= [y_2^4]_0^{1/4} \\ &= \frac{1}{256} \end{aligned}$$

iii. Determine $P(Y_1Y_2 < 1/2)$

This is the probability that that max of the two is, you guessed it, the square root of 1/2. Same problem. Different bounds. (Gee, I really hope I'm not wrong because I'm getting awfully cocky)

$$\begin{aligned}
P(Y_1Y_2 < 1/2) &= \int_0^{\sqrt{1/2}} \int_0^{y_2} 4y_2^2 dy_1 dy_2 \\
&= \int_0^{\sqrt{1/2}} 4y_2^3 dy_2 \\
&= [y_2^4]_0^{\sqrt{1/2}} \\
&= \frac{1}{1/4}
\end{aligned}$$

iv. Determine $P(Y_1/Y_2 < 1/2)$

Now we have something interesting! Y_1/Y_2 is less than 1/2 if $Y_1 < Y_2/2$. Of course, this will work for any values of Y_1 and Y_2 that satisfy this inequality, and so Y_2 can go anywhere from 0 to 1. Technically we should remove the case where $Y_2 = 0$, but the probability of that happening is basically 0 since we are dealing with continous variables, so we don't have to worry about it:

$$\begin{aligned}
P(Y_1/Y_2 < 1/2) &= \int_0^1 \int_0^{y_2/2} 4y_2^2 dy_1 dy_2 \\
&= \int_0^1 2y_2^3 dy_2 \\
&= [\frac{1}{2}y_2^4]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

v. Determine $P(Y_2 - Y_1 < 1/2)$

Just isolate Y_1 to get the upper bound:

$$\begin{aligned}
P(Y_1 - Y_2 < 1/2) &= \int_0^1 \int_0^{y_2 - 2} 4y_2^2 dy_1 dy_2 \\
&= \int_0^1 4y_2^3 - 2y_2^2 dy_2 \\
&= [y_2^4 - \frac{2}{3}y_2^3]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

I do wonder if this is actually right versus using 1/2 to 1 for the bounds of the outer integral. Y_2 cannot be less than 1/2 for this outcome, but perhaps the inner integral takes care of that? The probability then is 17/48 which is just a little bit higher than 1/3.

vi. Determine $P(\min\{Y_1, Y_2\} < 1/2)$

Since $Y_2 \geq Y_1$, this is really the $P(Y_1 < 1/2)$:

$$\begin{aligned}
P(\min\{Y_1, Y_2\} < 1/2) &= \int_0^1 \int_0^{1/2} 4y_2^2 dy_1 dy_2 \\
&= \int_0^1 2y_2^2 dy_2 \\
&= [\frac{2}{3} y_2^3]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

I apologize for the next three being wrong. If I had had more time maybe I could figure them out, but I did not have more time.

vii. Determine the marginal distribution for Y_1
Well, in this case Y_1 can be anywhere from 0 up to a given y_1, and Y_2 can be anywhere from y_1 to 1. So the marginal distribution must be:

$$\begin{aligned}
F_1(y_1) &= \int_{y_1}^1 \int_0^{y_1} 4y_2^2 dy_1 dy_2 \\
&= \int_{y_1}^1 4y_2^2 y_1 dy_2 \\
&= \left[\frac{4}{3} y_2^3 y_1\right]_{y_1}^1 \\
&= \frac{4}{3} y_1 - \frac{4}{3} y_1^4
\end{aligned}$$

viii. Determine $P(Y_1 < 1/2)$
According to my answer in the previous problem, this probability is:

$$\begin{aligned}
P(Y_1 < 1/2) &= F_1(1/2) \\
&= \frac{4}{3}\left[1/2 - 1/16\right] \\
&= \frac{7}{12}
\end{aligned}$$

ix. Determine the marginal distribution for Y_2 Here Y_2 ranges from 0 to y_2, and Y_1 ranges from 0 to Y_2:

$$
\begin{aligned}
F_2(y_2) &= \int_0^{y_2} \int_0^{y_2} 4y_2^2 dy_1 dy_2 \\
&= \int_0^{y_2} 4y_2^3 dy_2 \\
&= y_2^4
\end{aligned}
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