Quantum Computing

Korben Rusek

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1 Linear Algebra

1.3 The Pauli matrices

$$\sigma_0 = I$$

$$\sigma_1 = \sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 = \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1.6 Adjoints and Hermitian operators

Definition 1.1 (Hermitian)**.** An operator A is Hermitian if $A = A^\dagger$.

Theorem 1.2. Two eigenvectors of a Hermitian operator with different eigenvalues are orthogonal.

Definition 1.3. A matrix is normal if $AA^\dagger = A^\dagger A$.

Theorem 1.4. A normal matrix is Hermitian iff it has real eigenvalues.

Definition 1.5 (Unitary)**.** A matrix U is unitary if $U^\dagger U = I$.

Definition 1.6 (Positive and Positive definite)**.** A positive operator A is defined to be an operator such that for any vector $|v\rangle$, $(|v\rangle, A|v\rangle)$ is a real, non-negative number. If $(|v\rangle, A|v\rangle) > 0$ then A is positive definite.

1.7 Tensor products

1.8 Operator functions

1.9 The commutator and anti-commutator

Definition 1.7 (Commutator)**.** The *commutator* between two operators A and B is defined to be

$$[A, B] = AB - BA.$$

If $[A, B] = 0$ then we say that A commutes with B.

Definition 1.8 (Anti-commutator)**.** The *anti-commutator* between to operators A and B is defined to be

$$\{A, B\} = AB + BA.$$

If $\{A, B\} = 0$ then we say that A anti-commutes with B.

Definition 1.9 (Simultaneously Diagonalizable)**.** Hermitian operators A and B are said to be *simultaneously diagonalizable* if there exist some orthonormal set of vectors $\langle i|$ such that $A = \sum a_i |i\rangle \langle i|$ and $B = \sum b_i |i\rangle \langle i|$.

Theorem 1.10 (Simultaneous diagonalization theorem)**.** Suppose A and B are Hermitian operators. Then $[A, B] = 0$ iff A and B are simultaneously diagonalizable.

Here are some facts

$$[X, Y] = 2iZ; [Y, Z] = 2iX; [Z, X] = 2iY$$
$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

Theorem 1.11. Suppose A and B are Hermitian. Then $i[A, B]$ is also Hermitian.

1.10 The polar and singular value decompositions

Theorem 1.12 (Polar decompositions)**.** Let A be a linear operator on a vector space V. Then there exists unitary U and positive operators J and K such that

$$A = UJ = KU,$$

where the unique positive operators J and K satisfying these equations are $J = \sqrt{A^\dagger A}$ and $K = \sqrt{AA^\dagger}$. Moreover, if A is invertible then U is unique.

Theorem 1.13 (Singular value decomposition)**.** Let A be a square matrix. Then there exist unitary matrices U and V, and a diagonal matrix D with non-negative entries such that

$$A = UDV.$$

The diagonal elements of D are known as the *singular values* of A.

2 The postulates of quantum mechanics

2.1 State Space

Postulate 2.1. Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

2.2 Evolution

Postulate 2.2. The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\phi\rangle$ of the system at time t_1 is related to the state $|\phi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$|\phi'\rangle = U |\phi\rangle .$$

The X Pauli matrix is often referred to as the not gate or the *bit flip* gate. It will send $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

The Z Pauli matrix is called the *phase flip* gate. It leaves $|0\rangle$ invariant but sends $|1\rangle$ to $-|1\rangle$.

Definition 2.1 (Hadamard Gate)**.** An interesting unitary operator is the *Hadamard gate*. This had the matrix representation

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Postulate 2.2 (Revised)**.** The time evolution of the state of a closed quantum system is described by the *Schrödinger equation*,

$$i\hbar \frac{d|\phi\rangle}{dt} = H |\phi\rangle .$$

In this equation, $\hbar$ is a physical constant known as *Planck's constant* whose value must be experimentally determined. The exact value is not important to us. In practice, it is common to absorb the factor $\hbar$ into H, effectively setting $\hbar = 1$. H is a fixed Hermitian operator known as the *Hamiltonian* of the closed system.

2.3 Quantum measurement

We introduce Postulate 3 in order to describe the effects that measurement have on a system.

Postulate 2.3. Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acing on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\phi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\phi| M_m^\dagger M_m |\phi\rangle ,$$

and the state of the system after the measurement is

$$\frac{M_m |\phi\rangle}{\sqrt{\langle\phi| M_m^\dagger M_m |\phi\rangle}}.$$

The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I.$$

The completeness equation expresses the fact that probabilities sum to one:

$$I = \sum p(m) = \sum \langle\phi| M_m^\dagger M_m |\phi\rangle .$$