Quantum Computing

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2 Linear Algebra

Lemma 2.1. Let A be a non-singular linear operator. If all the eigenvalues of A are ± 1 then $A^2 = I$

Proof. Since A is non-singular and normal with eigenvalues ± 1 then we can write $A = \sum \lambda_i |i\rangle \langle i|$ with $|i\rangle$ spanning the vector space. Then we have

$$\begin{aligned}
A^2 &= \left(\sum_i \lambda_i |i\rangle \langle i| \right) \left(\sum_j \lambda_j |j\rangle \langle j| \right) \\
&= \sum_i \sum_j \lambda_i |i\rangle \langle i| \lambda_j |j\rangle \langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j |i\rangle \delta_{i,j} \langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j \delta_{i,j} |i\rangle \langle j| \\
&= \sum_i \lambda_i^2 |i\rangle \langle i| \\
&= \sum_i |i\rangle \langle i| \\
&= I
\end{aligned}$$

□

Lemma 2.2. Let $A = \sum_x |x\rangle \langle x| - \sum_y |y\rangle \langle y|$, where $\{|x\rangle, |y\rangle\}_{x,y}$ form an orthonormal basis. Then

$$f(\theta A) = \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} A.$$

Proof. We can write $A = \sum_x |x\rangle\langle x| - \sum_y |y\rangle\langle y|$. Let $X = \sum_x |x\rangle\langle x|$ and $Y = \sum_y |y\rangle\langle y|$. That means that $I = X + Y$ and $A = X - Y$. Then we have

$$\begin{aligned}
f(\theta)A &= \sum_x f(\theta)|x\rangle\langle x| + \sum_y f(-\theta)|y\rangle\langle y| \\
&= f(\theta)\sum_x |x\rangle\langle x| + f(-\theta)\sum_y |y\rangle\langle y| \\
&= f(\theta)X + f(-\theta)Y \\
&= \frac{f(\theta)}{2}X + \frac{f(\theta)}{2}Y + \frac{f(\theta)}{2}X - \frac{f(\theta)}{2}Y + \frac{f(-\theta)}{2}Y + \frac{f(-\theta)}{2}Y + \frac{f(-\theta)}{2}X - \frac{f(-\theta)}{2}X \\
&= \frac{f(\theta)}{2}(X+Y) + \frac{f(-\theta)}{2}(X+Y) + \frac{f(\theta)}{2}(X-Y) - \frac{f(-\theta)}{2}(X-Y) \\
&= \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}A
\end{aligned}$$

□

Exercise 2.44. Suppose that A is invertible and that $\{A, B\} = [A, B] = 0$. Show that B is 0.

Proof. $[A, B] = 0$ tells us that $AB = BA$ and $\{A, B\} = 0$ tells us that $AB = -BA$. Therefore we know that $BA = -BA$. Now multiplying on the right side by A^{-1}, we get $B = -B$. Thus $-2B = 0$ which implies that $B = 0$. □

Exercise 2.45. Show that $[A, B]^\dagger = [B^\dagger, A^\dagger]$.

Proof.

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= [B^\dagger, A^\dagger].
\end{aligned}$$

□

Exercise 2.46. Show that $[A, B] = -[B, A]$.

Proof.

$$\begin{aligned}
[A, B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B, A]
\end{aligned}$$

□

Exercise 2.47. Suppose that A and B are Hermitian. Show that $i[A, B]$ is Hermitian.

Proof. Suppose that $A = A^\dagger$ and $B = B^\dagger$. Then we have

$$\begin{aligned}(i[A,B])^\dagger &= -i[B^\dagger, A^\dagger] \\ &= -i[B,A] = i[A,B].\end{aligned}$$

Therefore $i[A,B]$ is Hermitian. □

Lemma 2.3. Let A be a diagonalizable matrix. Write $A = \sum_i \lambda_i |i\rangle$. Then $\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i|$.

Proof. The proof is pretty straight forward. We start with

$$\begin{aligned}A^\dagger A &= \left(\sum_i \lambda_i^* |i\rangle \langle i|\right)\left(\sum_i \lambda_i |i\rangle \langle i|\right) \\ &= \sum_i |\lambda_i|^2 |i\rangle \langle i|.\end{aligned}$$

Therefore $\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i|$. □

Exercise 2.48. What is the polar decomposition of a positive matrix, P? Of a unitary matrix, U? or a Hermitian matrix, H?

Positive matrix. By the above lemma, for a positive matrix, P, $\sqrt{P^\dagger P} = P$. Therefore we have IP or PI as the polar decompositions. □

Unitary matrix. Suppose V is unitary. By the above lemma $\sqrt{V^\dagger V} = I$. Therefore $V = VI = IV$ is the polar decomposition. □

Hermitian matrix. Suppose that H is Hermitian. Write $H = \sum_i \lambda_i |i\rangle \langle i|$. Then $J = \sum_i |\lambda_i| |i\rangle \langle i|$. Let a_i be defined as $\lambda_i / |\lambda_i|$ when $\lambda_i \neq 0$ and 1 when $\lambda_i = 0$. Then $U = \sum_i v_i |i\rangle \langle i|$. □

Exercise 2.49. Express the polar decomposition of a normal matrix in the outer product representation.

Proof. The solution is similar to what we see in the Hermitian version of the above exercise. J has eigenvalues that are the absolute value of the eigenvalues of H. U would have eigenvalues that are the unit vectors of the eigenvalues of H (or 1 when eigenvalues are 0). □

Exercise 2.50. Find the left and right polar decompositions of the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Proof. It is easy to see that the eigenvalues of P is 1 repeated. Therefore P is positive. Therefore $U = I$ and $J = K = P$. □

Definition 2.4. We define H to be the Hadamard matrix.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 2.51. Verify that th Hadamard gate H is unitary.

Proof.

$$H^2 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

It is clear that $H^\dagger = H$. Thus $H \dagger H = I$ and H is unitary. □

Exercise 2.52. Verify that $H^2 = I$.

Proof. This was shown in the previous exercise. □

Exercise 2.53. What are the eigenvalues and eigenvectors of H?

Proof. To find the eigenvalues of H we solve the polynomial equation

$$\begin{aligned} 0 &= (1-\lambda)(-1-\lambda) - 1 \\ &= (\lambda - 1)(\lambda + 1) - 1 \\ &= \lambda^2 - 2. \end{aligned}$$

Thus our eigenvalues are $\lambda = \pm\sqrt{2}$. To find the eigenvectors we solve the system:

$$\begin{aligned} x + y &= \frac{\lambda}{\sqrt{2}} x \\ x - y &= \frac{\lambda}{\sqrt{2}} y \end{aligned}$$

Since $(0, 1)$ is clearly not an eigenvector we can assume that $x = 1$. For $\lambda = 1$ we have $y = \frac{1-\sqrt{2}}{\sqrt{2}}$. For $\lambda = -1$ we get $y = \frac{-1+\sqrt{2}}{\sqrt{2}}$. Therefore we have the eigenvectors $(\sqrt{2}, 1-\sqrt{2})$ and $(\sqrt{2}, \sqrt{2}-1)$ with eigenvalues 1 and -1 respectively. □

Exercise 2.54. Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A + B)$.

Proof. We can write $A = \sum_i a_i |i\rangle\langle i|$ and $B = \sum_i b_i |i\rangle\langle i|$ for some orthonormal basis $|i\rangle$. Then we have

$$exp(A) = \sum_i e^{a_i} keti \langle i|,$$

$$exp(B) = \sum_i e^{b_i} keti \langle i|,$$

and so

$$\begin{aligned} exp(A)exp(B) &= \sum_i e^{a_i} e^{b_i} keti \langle i| \\ &= \sum_i e^{a_i + b_i} keti \langle i| \\ &= exp(A + B). \end{aligned}$$

□

Definition 2.5.

$$U(t_1, t_2) = exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right].$$

Exercise 2.55. Prove that $U(t_1, t_2)$ is unitary.

Proof. This is simple as $exp(x)$ is non-zero, so U is unitary. $\square$