Quantum Computing

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4.2 Single qubit operators

Exercise 4.1. In Exercise 2.11, you computed the eigenvectors of the Pauli matrices. Find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices.

Proof. The eigenvectors of X are $\frac{1}{\sqrt{2}}[1,1]$ and $\frac{1}{\sqrt{2}}[1,-1]$. These correspond to $[1,0,0]$ and $[-1,0,0]$, respectively. The eigenvectors of Y are $\frac{1}{\sqrt{2}}[1,i]$ and $\frac{1}{\sqrt{2}}[1,-i]$. These correspond to $[0,1,0]$ and $[0,-1,0]$, respectively. Finally the eigenvectors, $[1,0]$ and $[0,1]$ of Z correspond to $[0,0,1]$ and $[0,0,-1]$ respectively. That is, the eigenvectors of X, Y, and Z, unsurprisingly, correspond to the antipodal points along the x, y, and z-axes respectively. □

Lemma 4.1. Let $|x\rangle$ and $|y\rangle$ be orthogonal then the Bloch representations are antipodal.

Proof. □

Exercise 4.2. Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(Aix) = \cos(x)I + i\sin(x)A.$$

User this result to verify Equations (4.4) through (4.6).

Proof. If $A^2 = I$ then $A = \lambda_x |x\rangle\langle x| + \lambda_y |y\rangle\langle x|$ where $\lambda_x, \lambda_y = \pm 1$. Therefore we have

$$\begin{aligned}
\exp(iAx) &= e^{ix\lambda_x}|x\rangle\langle x| + e^{ix\lambda_y}|y\rangle\langle y| \\
&= (\cos(x\lambda_x) + i\sin(x\lambda_x))|x\rangle\langle x| + (\cos(x\lambda_y) + i\sin(x\lambda_y))|y\rangle\langle y| \\
&= (\cos(x) + i\sin(x\lambda_x))|x\rangle\langle x| + (\cos(x) + i\sin(x\lambda_y))|y\rangle\langle y| \\
&= (\cos(x) + \lambda_x i\sin(x))|x\rangle\langle x| + (\cos(x) + \lambda_y i\sin(x))|y\rangle\langle y| \\
&= \cos(x)(|x\rangle\langle x| + |y\rangle\langle y|) + \lambda_x i\sin(x)|x\rangle\langle x| + \lambda_y i\sin(x)|y\rangle\langle y| \\
&= \cos(x)I + i\sin(x)A.
\end{aligned}$$

Where the λ_x and λ_y can be pulled out of the sin because they are ± 1. □

Exercise 4.3. Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_x(\pi/4)$.

Proof. □

Exercise 4.4. Express the Hadamard gate, H, as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ.

Solution. First observe that $H|0\rangle = |+\rangle$ and $H|+\rangle = |0\rangle$. From exercise **??** we know that $|0\rangle$ corresponds to the point $[0,0,1]$ and $|+\rangle$ corresponds to the point $[1,0,0]$. Now if we rotate $[0,0,1]$ along the x-axis by $\pi/2$ then we get the point $[0,1,0]$. Next if we rotate along the z-axis we get the point $[1,0,0]$. That is, $R_z(\pi/2)R_x(\pi/2)|0\rangle \simeq |+\rangle$. This gets us part of the way there.

On the other hand, if we rotate the point $[1,0,0]$ ($|+\rangle$) along the x-axis by $\pi/2$ we get the point $[1,0,0]$ back. If we follow that by a rotation along the z-axis then we get the point $[0,-1,0]$. To get $[0,0,1]$ from this piont we need to follow this up with a rotation along the x-axis by $\pi/2$. That is, $R_x(\pi/2)R_z(\pi/2)R_x(\pi/2)|+\rangle \simeq |0\rangle$.

Luckily it is also the case that $R_x(\pi/2)R_z(\pi/2)R_x(\pi/2)|0\rangle \simeq |+\rangle$. This means that

$$R_x(\pi/2)R_z(\pi/2)R_x(\pi/2) = e^{i\alpha}H$$

for some $\alpha \in \mathbb{R}$. Multiplying out $R_x(\pi/2)R_z(\pi/2)R_x(\pi/2)$ we have

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -i & -i \\ -i & i \end{bmatrix}.$$

Therefore we just need to multiply by $e^{i\pi/2}$ and hence

$$H = R_x(\pi/2)R_z(\pi/2)R_x(\pi/2)e^{i\pi/2}.$$

□

If $n = (n_x, n_y, n_z) \in \mathbb{R}^3$ is a real unit vector in three dimensions then we genearlize the previous definitions by defining a rotation by θ about the n axis by the equation

$$R_n(\theta) = \exp(i\theta n \cdot \sigma/2) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)(n_xX + n_yY + n_zZ), \tag{1}$$

where σ denotes the three component vector (X, Y, Z) of Pauli matrices.

Exercise 4.5. Prove that $(n \cdot \sigma)^2 = I$, and use this to verify equation **??**.

Proof. Write $n = (x, y, z)$. Then we have

$$\begin{aligned}
(n \cdot \sigma)^2 &= (xX + yY + zZ)^2 \\
&= x^2X^2 + xyXY + xzXZ + y^2Y^2 + xyYX + yzYZ + z^2Z^2 + xzZX + yzZY \\
&= x^2I + xyXY + xzXZ + y^2I + xyYX + yzYZ + z^2I + xzZX + yzZY \\
&= (x^2 + y^2 + z^2)I + xyXY + xzXZ + xyYX + yzYZ + xzZX + yzZY \\
&= (x^2 + y^2 + z^2)I + xyXY - xyXY + xzXZ - xzXZ + yzYZ - yzYZ \\
&= (x^2 + y^2 + z^2)I \\
&= I.
\end{aligned}$$

□

Exercise 4.6 (Boch sphere interpretation of rotations)**.** One reason why the $R_n(\theta)$ operators are reffered to as rotation operators in the following fact, which you are to prove. Suppose a single qubit has a state represented by the Bloch vector λ. Then the effect of the rotation $R_n(\theta)$ on the state is to rotate it by an angle θ about the n axis of the Bloch sphere. This fact explains the rather mysterious looking factor of two in the definition of the rotation matrices.

Proof. We will begin with a deeper dive into the Pauli rotations. For example $R_z(\theta)$ can be written as

$$\begin{aligned} R_z(2\theta) &= \cos(\theta)(|0\rangle\langle 0| + |1\rangle\langle 1|) + i\sin(\theta)(|0\rangle\langle 0| - |1\rangle\langle 1|) \\ &= (\cos(\theta) + i\sin(\theta))\,|0\rangle\langle 0| + (\cos(\theta) - i\sin(\theta))\,|1\rangle\langle 1| \\ &= e^{i\theta}\,|0\rangle\langle 0| + e^{-i\theta}\,|1\rangle\langle 1|\,. \end{aligned}$$

That is, $R_z(\theta)$ has an eigenvalue of $e^{i\theta/2}$ for $|0\rangle$ and an eigenvalue of $e^{-i\theta/2}$ for $|1\rangle$. In terms of the Bloch sphere this is what the eigenvalues are for the antipodal points $(0, 0, 1)$ and $(0, 0, -1)$ respectively – the points about which $R_z(\theta)$ rotates qubits. A similar calculation shows that the same holds true for each of $R_x(\theta)$ and $R_y(\theta)$.

Now we turn our attention to $n \cdot \sigma$. We already know that n is the Bloch representation of the eigenvector of $R_n(\theta)$. [Do we really?] By a similar calculation above we can see that $R_n(\theta)$ is a rotation about the n-axis. □

Exercise 4.7. Show that $XYX = -Y$ and use this to prove that $XR_y(\theta)X = R_y(-\theta)$.

Proof. A straight-forward calculation shows that $XYX = -Y$. We then have

$$\begin{aligned} XR_y(2\theta)X &= \cos(\theta)XIX + i\sin(\theta)XYX \\ &= \cos(\theta)I - i\sin(\theta)Y \\ &= \cos(-\theta)I + i\sin(-\theta)Y. \end{aligned}$$

Therefore $XR_y(\theta)X = R_y(-\theta)$. □

Exercise 4.8. An arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)R_n(\theta)$$

for some real number α and θ and a real three-dimensional unit vector n.

1. Prove this fact.
2. Find values for α, θ, and n giving the Hadamard gate H.
3. Find values for α, θ, and n giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

Proof of 1. Let $|x\rangle$ and $|y\rangle$ be orthonormal eigenvectors of U. Then we can write

$$U = \lambda_x |x\rangle \langle x| + \lambda_y |y\rangle \langle y|,$$

where $|\lambda_y| = 1$ and $|\lambda_x| = 1$. That means we can write $\lambda_x = e^{i\phi_x}$ and $\lambda_y = e^{i\phi_y}$ for some $\phi_x, \phi_y \in \mathbb{R}$. Let n be the Bloch representation of $|x\rangle$. By lemma **??** we know that $|y\rangle = -n$. This means that $|x\rangle$ and $|y\rangle$ are also eigenvectors of $R_n(\theta)$ for any θ. Let us write out and simplify $R_n(\theta)$.

$$\begin{aligned}
R_n(\theta) &= \cos\left(\frac{\theta}{2}\right) I + i\sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z) \\
&= \cos\left(\frac{\theta}{2}\right)(|x\rangle \langle x| + |y\rangle \langle y|) + i\sin\left(\frac{\theta}{2}\right)(|x\rangle \langle x| - |y\rangle \langle y|) \\
&= \left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right] |x\rangle \langle x| + \left[\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\right] |y\rangle \langle y| \\
&= e^{i\frac{\theta}{2}} |x\rangle \langle x| + e^{-i\frac{\theta}{2}} |y\rangle \langle y|.
\end{aligned}$$

Now choosing $\theta/2 = \phi_x - \phi_y$ and $\alpha = \phi_x + \phi_y$, we have

$$\begin{aligned}
e^{i\alpha} R_n(\theta) &= e^{i\alpha}(e^{i\frac{\theta}{2}} |x\rangle \langle x| + e^{-i\frac{\theta}{2}} |y\rangle \langle y|) \\
&= e^{i\alpha}(e^{i(\phi_x - \phi_y)} |x\rangle \langle x| + e^{i(\phi_y - \phi_x)} |y\rangle \langle y|) \\
&= e^{i(\alpha + \phi_x - \phi_y)} |x\rangle \langle x| + e^{i(\alpha + \phi_y - \phi_x)} |y\rangle \langle y| \\
&= e^{i\phi_x} |x\rangle \langle x| + e^{i\phi_y} |y\rangle \langle y| \\
&= U.
\end{aligned}$$

□

Solution for 2. The eigenvector of the Hadamard gate is □

Theorem 4.2 ($Z - Y$ decomposition for a single qubit)**.** Suppose U is a unitary operation on a single qubit. Then there exist real numbers α, β, γ, and δ such that

$$U = e^{i\alpha} R_z(\beta) R_x(\gamma) R_z(\delta).$$

Exercise 4.9. Explain why any single qubit unitary operator may be written in the form

$$U = \begin{bmatrix} e^{i(\alpha - \beta/2 - \delta/2)} \cos\frac{\gamma}{2} & -e^{i(\alpha - \beta/2 + \delta/2)} \sin\frac{\gamma}{2} \\ e^{i(\alpha + \beta/2 - \delta/2)} \sin\frac{\gamma}{2} & e^{i(\alpha + \beta/2 + \delta/2)} \cos\frac{\gamma}{2} \end{bmatrix}.$$

Exercise 4.10 ($Z-Y$ decomposition for a single qubit)**.** Give a decomposition analogous to Theorem **??**.

Exercise 4.11. Suppose m an n are non-parallel real unit vectors in three dimensions. Use Theorem **??** to show that an arbitrary single qubit unitary U may be written

$$U = e^{i\alpha} R_n(\beta) R_m(\gamma) R_n(\delta),$$

for appropriate choices of α, β, γ, and δ.

Corollary 4.3. Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some overall phase factor.

Exercise 4.12. Give A, B, C, and α for the Hadamard gate.

Exercise 4.13 (Circuit Identities)**.** It is useful to be able to simplify circuits by inspection, using well-known identities. Prove the following identities:

$$HXH = Z; \quad HYH = -Y; \quad HZH = X.$$

Exercise 4.14. Use the previous exercise to show that $HTH = R_x(\pi/4)$, up to a global phase.

Exercise 4.15 (Composition of single qubit operations)**.** The Bloch representation gives a nice way to visualize the effect of composing two rotations.

1. Prove that if a rotation through an angle β_1 about the axis n_1 is followed by a rotation through an angle β_2 about the axis n_2, then the overall rotation is through an angle β_{12} about the axis n_{12}, given by

$$\begin{aligned} c_{12} &= c_1 c_2 - s_1 s_2 n_1 \cdot n_2 \\ s_{12} n_{12} &= s_1 c_2 n_1 + c_1 s_2 n_2 - s_1 s_2 n_2 \times n_1, \end{aligned}$$

   where $c_i = \cos(\beta_i/2)$, $s_i = \sin(\beta_i/2)$, $c_{12} = \cos(\beta_{12}/2)$, and $s_{12} = \sin(\beta_{12}/2)$.

2. Show that if $\beta_1 = \beta_2$ and $n_1 = z$ these equations simplify to

$$\begin{aligned} c_{12} &= c^2 - s^2 z \cdot n_2 \\ s_{12} n_{12} &= sc(z + n_2) - s^2 n^2 \times z, \end{aligned}$$

   where $c = c_1$ and $s = s_1$.

Exercise 4.25 (Fredkin gate construction)**.** Recall that the Fredkin (controlled-swap) gate performs the transform

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

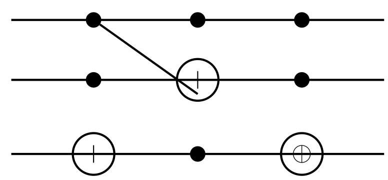
1. Give a quantum circuit which uses three Toffoli gates to construct the Fredkin gate (Hint: think of the swap gate construction – you can control each gate, one at a time).

2. Show that the first and last Toffoli gates can be replaced by CNOT gates.

3. Now replace the middle Toffoli gate with the circuit in Figure 4.8 to obtain a Fredkin gate construction using only six two-qubit gates.

4. Can you come up with an even simpler construction, with only five two-qubit gates?

Proof of 1.



□