Quantum Computing

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1.2 Joint Measurements

The Pauli operators and how their measurements act on a set of qubits are well known. But most literature explains how to modify the qubits such that you can perform the Pauli Z measurement on just the first qubit to achieve the same result.

There is a table containing tensor products of Pauli matrices and operators that convert that operation to the simple Z measurement and operators that convert them back.

The only issue with this method is that it is harder to deal with the way that I've written my quantum simulator. This document will outline how my simulator performs joint measurements and prove that it achieves the same results. It is not complicated, but I felt the need to more thoroughly document it.

Fact 1.1. Pauli operators have eigenvalues ± 1 that split the quantum space into two equal subspaces.

Example 1.1. Let

$$|\psi\rangle = \frac{1}{2}|100\rangle + \frac{\sqrt{2}}{2}|011\rangle + \frac{1}{2}|101\rangle .$$

The Pauli Z gate has the following action:

$$\begin{aligned} |0\rangle &\rightarrow |0\rangle \\ |1\rangle &\rightarrow -|1\rangle \end{aligned}$$

That is $|0\rangle$ has 1 as its eigenvalue and $|1\rangle$ has -1 as its eigenvalue. Therefore if we want to measure ψ with respect to $Z \oplus I \oplus I$ it gets split as

$$\frac{1}{2}|100\rangle + \frac{1}{2}|101\rangle$$

and

$$\frac{\sqrt{2}}{2}|011\rangle .$$

Looking at the coefficients each has a probability of $1/2$. Of course after measurement we need to normalize (divide by the norm) the result in order to get a valid qubit array.

Similarly measuring with respect to $I \oplus I \oplus Z$ will split $|\psi\rangle$ as

$$\frac{1}{2}|100\rangle$$

and

$$\frac{\sqrt{2}}{2}|011\rangle + \frac{1}{2}|101\rangle,$$

The former with probability 1/4 and the latter with probability 3/4.

Lemma 1.2. Let $|\psi\rangle$ be a quantum state and φ a Pauli operator. Suppose $|\psi\rangle$ can be written $|\psi\rangle = |\alpha_+\rangle + |\alpha_-\rangle$, where α_+ has eigenvalue 1 and α_- has eigenvalue -1, then $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are unique.

Proof. First assume that $|\psi\rangle$ can be written two different ways, $|\psi\rangle = |\alpha_+\rangle + |\alpha_-\rangle$ and $|\psi\rangle = |\beta_+\rangle + |\beta_-\rangle$ where $|\alpha_+\rangle$ and $|\beta_+\rangle$ have eigenvalue 1 and $|\alpha_-\rangle$ and $|\beta_-\rangle$ have eigenvalue -1. I claim that $|\alpha_+\rangle = |\beta_+\rangle$ and $|\alpha_-\rangle = |\beta_-\rangle$. Since their sum is $|\psi\rangle$ then we necessarily have

$$\begin{aligned} |\alpha_+\rangle - |\beta_+\rangle &= |\beta_-\rangle - |\alpha_-\rangle \\ \varphi(|\alpha_+\rangle - |\beta_+\rangle) &= \varphi(|\beta_-\rangle - |\alpha_-\rangle) \\ |\alpha_+\rangle - |\beta_+\rangle &= |\alpha_-\rangle - |\beta_-\rangle. \end{aligned}$$

That is, $|\alpha_+\rangle - |\beta_+\rangle = -(|\alpha_+\rangle - |\beta_+\rangle)$ and so we must have $|\alpha_+\rangle - |\beta_+\rangle = 0$ and $|\alpha_+\rangle = |\beta_+\rangle$. Similarly we have $|\alpha_-\rangle = |\beta_-\rangle$. □

This means that if we find a way to write our quantum state as the sum of a vector with eigenvalue 1 and a vector with eigenvalue -1 then we are done.

Lemma 1.3. Let $|\psi\rangle$ be a state and φ any Pauli operator. Then $(|\psi\rangle + \varphi|\psi\rangle)/2$ has eigenvalue 1 or is 0. Similarly, $(|\psi\rangle - \varphi|\psi\rangle)/2$ has eigenvalue -1 or is 0.

Proof. The proof is pretty straightforward. Assume that $|\psi\rangle + \varphi|\psi\rangle \neq 0$. Then

$$\begin{aligned} \varphi(|\psi\rangle + \varphi|\psi\rangle) &= \varphi(|\psi\rangle) + \varphi(\varphi(|\psi\rangle)) \\ &= \varphi(|\psi\rangle) + |\psi\rangle \\ &= |\psi\rangle + \varphi|\psi\rangle. \end{aligned}$$

Hence it has eigenvalue 1.

Similarly if $|\psi\rangle - \varphi|\psi\rangle \neq 0$,

$$\begin{aligned} \varphi(|\psi\rangle - \varphi|\psi\rangle) &= \varphi(|\psi\rangle) - \varphi(\varphi(|\psi\rangle)) \\ &= \varphi(|\psi\rangle) - |\psi\rangle \\ &= -(|\psi\rangle - \varphi(|\psi\rangle)). \end{aligned}$$

Hence it has eigenvalue -1. □

This is how measurements are performed in my quantum emulator.