Quantum Computing

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4.2 Single qubit operators

Exercise 4.1. In Exercise 2.11, you computed the eigenvectors of the Pauli matrices. Find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices.

Proof. □

Lemma 4.1. Let $|x\rangle$ and $|y\rangle$ be orthogonal then the Bloch representations are antipodal.

Proof. □

Exercise 4.2. Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(Ax) = \cos(x)I + i\sin(x)A.$$

User this result to verify Equations (4.4) through (4.6).

Proof. □

Exercise 4.3. Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_x(\pi/4)$.

Proof. □

Exercise 4.4. Express the Hadamard gate, H, as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ.

Proof. □

If $n = (n_x, n_y, n_z) \in \mathbb{R}^3$ is a real unit vector in three dimensions then we genearlize the previous definitions by defining a rotation by θ about the n axis by the equation

$$R_n(\theta) = \exp(i\theta n \cdot \sigma/2) = \cos\left(\frac{theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z), \tag{1}$$

where σ denotes the three component vector (X, Y, Z) of Pauli matrices.

Exercise 4.5. Prove that $(n \cdot \sigma)^2 = I$, and use this to verify equation 1.

Proof. □

Exercise 4.6 (Boch sphere interpretation of rotations)**.** One reason why the $R_n(\theta)$ operators are reffered to as rotation operators in the following fact, which you are to prove. Suppose a single qubit has a state represented by the Bloch vector λ. Then the effect of the rotation $R_n(\theta)$ on the state is to rotate it by an angle θ about the n axis of the Bloch sphere. This fact explains the rather mysterious looking factor of two in the definition of the rotation matrices.

Exercise 4.7. Show that $XYX = -Y$ and use this to prove that $XR_y(\theta)X = R_y(-\theta)$.

Proof. □

Exercise 4.8. An arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)R_n(\theta)$$

for some real number α and θ and a real three-dimensional unit vector n.

1. Prove this fact.
2. Find values for α, θ, and n giving the Hadamard gate H.
3. Find values for α, θ, and n giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

Proof of 1. Let $|x\rangle$ and $|y\rangle$ be orthonormal eigenvectors of U. Then we can write

$$U = \lambda_x |x\rangle \langle x| + \lambda_y |y\rangle \langle y|,$$

where $|\lambda_y| = 1$ and $|\lambda_x| = 1$. That means we can write $\lambda_x = e^{i\phi_x}$ and $\lambda_y = e^{i\phi_y}$ for some $\phi_x, \phi_y \in \mathbb{R}$. Let n be the Bloch representation of $|x\rangle$. By lemma 4.1 we know that $|y\rangle = -n$. This means that $|x\rangle$ and $|y\rangle$ are also eigenvectors of $R_n(\theta)$ for any θ. Let us write out and simplify $R_n(\theta)$.

$$\begin{aligned} R_n(\theta) &= \cos\left(\frac{\theta}{2}\right) I + i\sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z) \\ &= \cos\left(\frac{\theta}{2}\right)(|x\rangle\langle x| + |y\rangle\langle y|) + i\sin\left(\frac{\theta}{2}\right)(|x\rangle\langle x| - |y\rangle\langle y|) \\ &= \left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right]|x\rangle\langle x| + \left[\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\right]|y\rangle\langle y| \\ &= e^{i\frac{\theta}{2}}|x\rangle\langle x| + e^{-i\frac{\theta}{2}}|y\rangle\langle y|. \end{aligned}$$

Now choosing $\theta/2 = \phi_x - \phi_y$ and $\alpha = \phi_x + \phi_y$, we have

$$\begin{aligned} e^{i\alpha}R_n(\theta) &= e^{i\alpha}(e^{i\frac{\theta}{2}}|x\rangle\langle x| + e^{-i\frac{\theta}{2}}|y\rangle\langle y|) \\ &= e^{i\alpha}(e^{i(\phi_x-\phi_y)}|x\rangle\langle x| + e^{i(\phi_y-\phi_x)}|y\rangle\langle y|) \\ &= e^{i(\alpha+\phi_x-\phi_y)}|x\rangle\langle x| + e^{i(\alpha+\phi_y-\phi_x)}|y\rangle\langle y| \\ &= e^{i\phi_x}|x\rangle\langle x| + e^{i\phi_y}|y\rangle\langle y| \\ &= U. \end{aligned}$$

□

Theorem 4.2 ($Z - Y$ decomposition for a single qubit)**.**