Quantum Computing

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2 Linear Algebra

Lemma 2.1. Let A be a non-singular linear operator. If all the eigenvalues of A are ± 1 then $A^2 = I$

Proof. Since A is non-singular and normal with eigenvalues ± 1 then we can write $A = \sum \lambda_i |i\rangle\langle i|$ with $|i\rangle$ spanning the vector space. Then we have

$$\begin{aligned}
A^2 &= \left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_j \lambda_j |j\rangle\langle j|\right) \\
&= \sum_i \sum_j \lambda_i |i\rangle\langle i| \lambda_j |j\rangle\langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j |i\rangle\langle i|j\rangle\langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j |i\rangle \delta_{i,j} \langle j| \\
&= \sum_i \sum_j \lambda_i \lambda_j \delta_{i,j} |i\rangle\langle j| \\
&= \sum_i \lambda_i^2 |i\rangle\langle i| \\
&= \sum_i |i\rangle\langle i| \\
&= I
\end{aligned}$$

□

Lemma 2.2. Let $A = \sum_x |x\rangle\langle x| - \sum_y |y\rangle\langle y|$, where $\{|x\rangle, |y\rangle\}_{x,y}$ form an orthonormal basis. Then

$$f(\theta A) = \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} A.$$

Proof. We can write $A = \sum_x |x\rangle\langle x| - \sum_y |y\rangle\langle y|$. Let $X = \sum_x |x\rangle\langle x|$ and $Y = \sum_y |y\rangle\langle y|$. That means that $I = X + Y$ and $A = X - Y$. Then we have

$$\begin{aligned}
f(\theta)A &= \sum_x f(\theta)|x\rangle\langle x| + \sum_y f(-\theta)|y\rangle\langle y| \\
&= f(\theta)\sum_x |x\rangle\langle x| + f(-\theta)\sum_y |y\rangle\langle y| \\
&= f(\theta)X + f(-\theta)Y \\
&= \frac{f(\theta)}{2}X + \frac{f(\theta)}{2}Y + \frac{f(\theta)}{2}X - \frac{f(\theta)}{2}Y + \frac{f(-\theta)}{2}Y + \frac{f(-\theta)}{2}Y + \frac{f(-\theta)}{2}X - \frac{f(-\theta)}{2}X \\
&= \frac{f(\theta)}{2}(X+Y) + \frac{f(-\theta)}{2}(X+Y) + \frac{f(\theta)}{2}(X-Y) - \frac{f(-\theta)}{2}(X-Y) \\
&= \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}A
\end{aligned}$$

□

Exercise 2.44. Suppose that A is invertible and that $\{A, B\} = [A, B] = 0$. Show that B is 0.

Proof. $[A, B] = 0$ tells us that $AB = BA$ and $\{A, B\} = 0$ tells us that $AB = -BA$. Therefore we know that $BA = -BA$. Now multiplying on the right side by A^{-1}, we get $B = -B$. Thus $-2B = 0$ which implies that $B = 0$. □

Exercise 2.45. Show that $[A, B]^\dagger = [B^\dagger, A^\dagger]$.

Proof.

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= [B^\dagger, A^\dagger].
\end{aligned}$$

□

Exercise 2.46. Show that $[A, B] = -[B, A]$.

Proof.

$$\begin{aligned}
[A, B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B, A]
\end{aligned}$$

□

Exercise 2.47. Suppose that A and B are Hermitian. Show that $i[A, B]$ is Hermitian.

Proof. Suppose that $A = A^\dagger$ and $B = B^\dagger$. Then we have

$$\begin{aligned}(i[A,B])^\dagger &= -i[B^\dagger, A^\dagger] \\ &= -i[B,A] = i[A,B].\end{aligned}$$

Therefore $i[A,B]$ is Hermitian. □

Lemma 2.3. Let A be a diagonalizable matrix. Write $A = \sum_i \lambda_i |i\rangle$. Then $\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i|$.

Proof. The proof is pretty straight forward. We start with

$$\begin{aligned}A^\dagger A &= \left(\sum_i \lambda_i^* |i\rangle \langle i|\right)\left(\sum_i \lambda_i |i\rangle \langle i|\right) \\ &= \sum_i |\lambda_i|^2 |i\rangle \langle i|.\end{aligned}$$

Therefore $\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i|$. □

Exercise 2.48. What is the polar decomposition of a positive matrix, P? Of a unitary matrix, U? or a Hermitian matrix, H?

Positive matrix. By the above lemma, for a positive matrix, P, $\sqrt{P^\dagger P} = P$. Therefore we have IP or PI as the polar decompositions. □

Unitary matrix. Suppose V is unitary. By the above lemma $\sqrt{V^\dagger V} = I$. Therefore $V = VI = IV$ is the polar decomposition. □

Hermitian matrix. Suppose that H is Hermitian. Write $H = \sum_i \lambda_i |i\rangle \langle i|$. Then $J = \sum_i |\lambda_i| |i\rangle \langle i|$. Let a_i be defined as $\lambda_i/|\lambda_i|$ when $\lambda_i \neq 0$ and 1 when $\lambda_i = 0$. Then $U = \sum_i v_i |i\rangle \langle i|$. □

Exercise 2.49. Express the polar decomposition of a normal matrix in the outer product representation.

Proof. The solution is similar to what we see in the Hermitian version of the above exercise. J has eigenvalues that are the absolute value of the eigenvalues of H. U would have eigenvalues that are the unit vectors of the eigenvalues of H (or 1 when eigenvalues are 0). □

Exercise 2.50. Find the left and right polar decompositions of the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Proof. It is easy to see that the eigenvalues of P is 1 repeated. Therefore P is positive. Therefore $U = I$ and $J = K = P$. □

Definition 2.4. We define H to be the Hadamard matrix.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 2.51. Verify that th Hadamard gate H is unitary.

Proof.

$$H^2 = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = I.$$

It is clear that $H^\dagger = H$. Thus $H \dagger H = I$ and H is unitary. □

Exercise 2.52. Verify that $H^2 = I$.

Proof. This was shown in the previous exercise. □

Exercise 2.53. What are the eigenvalues and eigenvectors of H?

Proof. To find the eigenvalues of H we solve the polynomial equation

$$\begin{aligned} 0 &= (1-\lambda)(-1-\lambda) - 1 \\ &= (\lambda-1)(\lambda+1) - 1 \\ &= \lambda^2 - 2. \end{aligned}$$

Thus our eigenvalues are $\lambda = \pm\sqrt{2}$. To find the eigenvectors we solve the system:

$$\begin{aligned} x + y &= \frac{\lambda}{\sqrt{2}}x \\ x - y &= \frac{\lambda}{\sqrt{2}}y \end{aligned}$$

Since $(0,1)$ is clearly not an eigenvector we can assume that $x = 1$. For $\lambda = 1$ we have $y = \frac{1-\sqrt{2}}{\sqrt{2}}$. For $\lambda = -1$ we get $y = \frac{-1+\sqrt{2}}{\sqrt{2}}$. Therefore we have the eigenvectors $(\sqrt{2}, 1-\sqrt{2})$ and $(\sqrt{2}, \sqrt{2}-1)$ with eigenvalues 1 and -1 respectively. □

Exercise 2.54. Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A+B)$.

Proof. We can write $A = \sum_i a_i |i\rangle\langle i|$ and $B = \sum_i b_i |i\rangle\langle i|$ for some orthonormal basis $|i\rangle$. Then we have

$$exp(A) = \sum_i e^{a_i} keti \langle i|,$$

$$exp(B) = \sum_i e^{b_i} keti \langle i|,$$

and so

$$\begin{aligned} exp(A)exp(B) &= \sum_i e^{a_i} e^{b_i} keti \langle i| \\ &= \sum_i e^{a_i+b_i} keti \langle i| \\ &= exp(A+B). \end{aligned}$$

□

Definition 2.5.

$$U(t_1, t_2) = exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right].$$

Exercise 2.55. Prove that $U(t_1, t_2)$ is unitary.

Proof. This is simple as $exp(x)$ is non-zero, so U is unitary. $\square$

Exercise 2.56. Use spectral decomposition to show that $K = -i\log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Proof. Let U be unitary. Then we can write $U = \sum \alpha_\phi |\phi\rangle \langle\phi|$ for some orthonormal basis $\langle\phi|$ and $\alpha_\phi \neq 0$. That means that $K = \sum -i\log(\alpha_\phi) |\phi\rangle \langle\phi|$. Since U is unitary then $\alpha_\phi = e^{i\theta_\phi}$ for some real θ_ϕ. Therefore we can simplify K to

$$K = \sum -i(i\theta_p hi) |\phi\rangle \langle\phi| = \sum \theta_\phi |\phi\rangle \langle\phi| .$$

Now sime θ_ϕ is real then K is Hermitian. Therefore any unitary operator is $\exp(iK)$ for some Hermitian K. $\square$

Exercise 2.58. Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

Proof. The average observed value of M is given by $E(M) = \langle\psi| M |\psi\rangle$. Then we have

$$\begin{aligned} E(M) &= \langle\psi| M |\psi\rangle \\ &= m \langle\psi|\rangle \psi \\ &= m. \end{aligned}$$

The standard deviation squared is given by

$$[\Delta(M)]^2 = \langle M^2\rangle - \langle M\langle^2.$$

We already saw that $\langle M\rangle = m$. And so we have

$$\begin{aligned} \langle M^2\rangle &= \langle\psi| M^2 |\psi\rangle \\ &= \langle\psi| mM |\psi\rangle \\ &= m^2 \langle\psi|\psi\rangle \\ &= m^2. \end{aligned}$$

Therefore the standard deviation is 0. $\square$

Exercise 2.59. Suppose we have a qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation of X?

Proof. The average value is given by $\langle 0| X |0\rangle = \langle 0|1\rangle = 0$.

The standard deviation squared is

$$\langle 0| X^2 |0\rangle = \langle 0|0\rangle = 1.$$

$\square$

Exercise 2.60. Show that $v \cdot \sigma$ has eigenvalues ± 1 and that the projectors onto the correspnding eigenspaces are given by $P_\pm = (I \pm v \cdot \sigma)/2$.

Proof. We have already seen that $v \cdot \sigma$ has eigenvalues ± 1. It is easy to verify that the eigenvectors of $v \cdot \sigma$ are

$$e_\pm = \begin{bmatrix} 1 \pm c \\ a \pm bi \end{bmatrix}.$$

Furthermore $P \pm e_\pm = e_\pm$. Therefore $P_\pm$ are the projectors. $\square$

Exercise 2.61. Calculate the probability of obtaining $+1$ for a measurement of $v \cdot \sigma$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement?

Proof. The value $p(+1)$ is given by $\langle 0| P_+ |0\rangle$. This gives $\frac{1}{2} \left\langle 0 \middle| \begin{bmatrix} 1 + c \\ a + bi \end{bmatrix} \right\rangle = \frac{1+c}{2}$.

After measurement the state of the system is

$$\frac{P_+ |0\rangle}{\sqrt{p(+1)}} = \begin{bmatrix} 1 + c \\ a + bi \end{bmatrix} \sqrt{\frac{2}{1 + c}}$$

$\square$

Exercise 2.62. Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Proof. Let M_m be the collection of measurement operators. We will show that $M_m = M_m^\dagger M_m$. Since $M_m = E_n$ for some n, then $M_m^\dagger M_m = E_n^\dagger E_n = E_n = M_m$. This means that M_m is positive. In that case $M_m^\dagger = M_m$ and we have M_m is projective as $M_m^2 = M_m$. $\square$

Exercise 2.63. Suppose a measurement is described by measurement operators, M_m. Show that there exist unitary operators U_m such that $M_m = U_m \sqrt{E_m}$, where E_m is the POVM associated to the measurement.

Proof. Write $M_m = \sum \lambda_i |i\rangle \langle i|$. Then $E_m = M_m^\dagger M_m = \sum |\lambda_i|^2 |i\rangle \langle i|$. Therefore $U_m = \sum_i \frac{\lambda_i}{|\lambda_i|} |i\rangle \langle i|$. $\square$