Smoothed analysis of the low-rank approach for smooth semidefinite programs

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Overview

Summary

Problem: Class of SDPs with equality constraints. Factorizing the optimization variable leads to a non-convex problem.

Assumptions:

– The optimization variable $X = YY^*$ with Y of size $n \times k$.

– The constraints regularly define a smooth manifold.

– $k \sim \sqrt{m}$ ($m = \#$ constraints).

Main Result: With high probability, approximate second-order stationary points (ASOSPs) for a randomly perturbed objective function are approximate global optima.

Approach: Smoothed analysis.

Motivating applications: Phase retrieval. Angular Synchronization.

Related work

– Burer and Monteiro [3] showed that if Y is a rank-deficient local optimum, then $X = YY^*$ is a global optimum.

– Under the same setting as us, Boumal et al. [2] showed that for almost all cost matrices, all second-order stationary points (SOSPs) are optimal.

– Bhojanapalli et al. [1] considered a quadratically penalized version of the SDP and showed that SOSPs for a random randomly perturbed objective function are global optima.

Contributions

– Approximate optimality conditions for ASOSPs of smooth SDPs.

Setting

Burer-Monteiro factorization

• We consider SDPs of the form,

$$\min_{X \in \mathbb{S}^{n\times n}} \langle C, X\rangle \text{ s.t. } \mathcal{A}(X) = b,\ X \succeq 0. \qquad \text{(SDP)}$$

– Solvable in poly time but enforcing PSD constraint can be expensive.

– Burer-Monteiro factorization: Set $X = YY^*$ and (SDP) becomes

$$\min_{Y \in \mathbb{K}^{n\times k}} \langle C, YY^*\rangle \text{ s.t. } \mathcal{A}(YY^*) = b. \qquad \text{(P)}$$

• **Advantages**:

– PSD constraint naturally enforced.

– Moreover, if SDP is compact, it always has a solution of rank r with $\dim \mathbb{S}^{r\times r} \le m$ (# constraints): can reduce dimension to $k \sim \sqrt{2m}$.

• **Issues**:

– The problem becomes non-convex.

– Besides, algorithms can only guarantee approximate second-order optimality conditions in a finite number of iterations: they return ASOSPs.

Smoothness assumption

– Search space of (P): $\mathcal{M}_k = \{Y \in \mathbb{K}^{n\times k} : \mathcal{A}(YY^*) = b\}$.

– For all values of k up to n s.t. it is non-empty, $\mathcal{M}_k$ defines a smooth manifold.

Benign non-convexity in Burer-Monteiro factorization

Assumptions

– The search space of X is compact.

– The search space of Y is a manifold.

Main theorem

– Randomly perturb the cost matrix C, w.h.p.:

– If $Y \in \mathbb{K}^{n\times k}$ with $k = \tilde{\Omega}(\sqrt{m})$ is an ASOSP for (P),

– Then, $X = YY^*$ is an approximate global optimum.

Proof sketch

Probabilistic argument

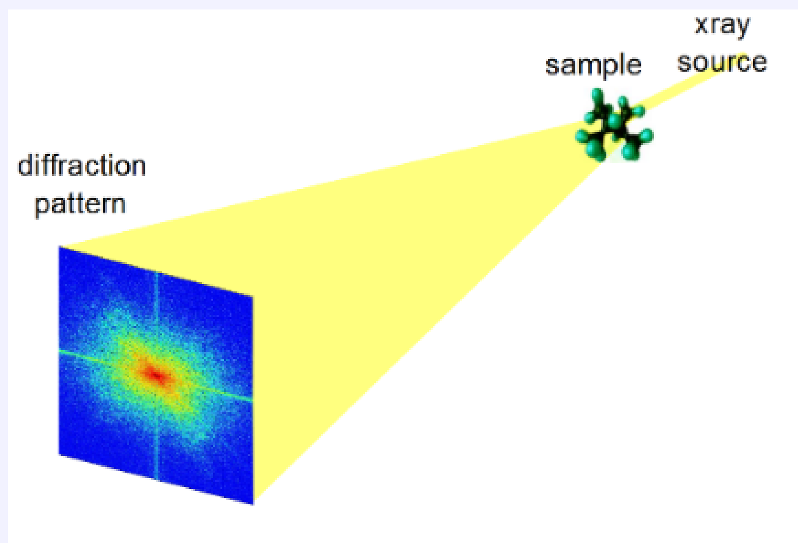
– Perturbing the cost matrix in (P) with a Gaussian Wigner matrix ⇒ any approximate first-order stationary point Y of the perturbed (P) is almost column-rank deficient w.h.p.

Deterministic argument

– If Y ASOSP for (P) and almost column-rank deficient ⇒ $X = YY^*$ is an approximate global optimum for (SDP).

Example: Phase retrieval

Problem setting



Molecular imaging [Candes et al.,11]

Goal: Retrieve a signal $z \in \mathbb{C}^n$ from $b = |Az| \in \mathbb{R}^m_+$.

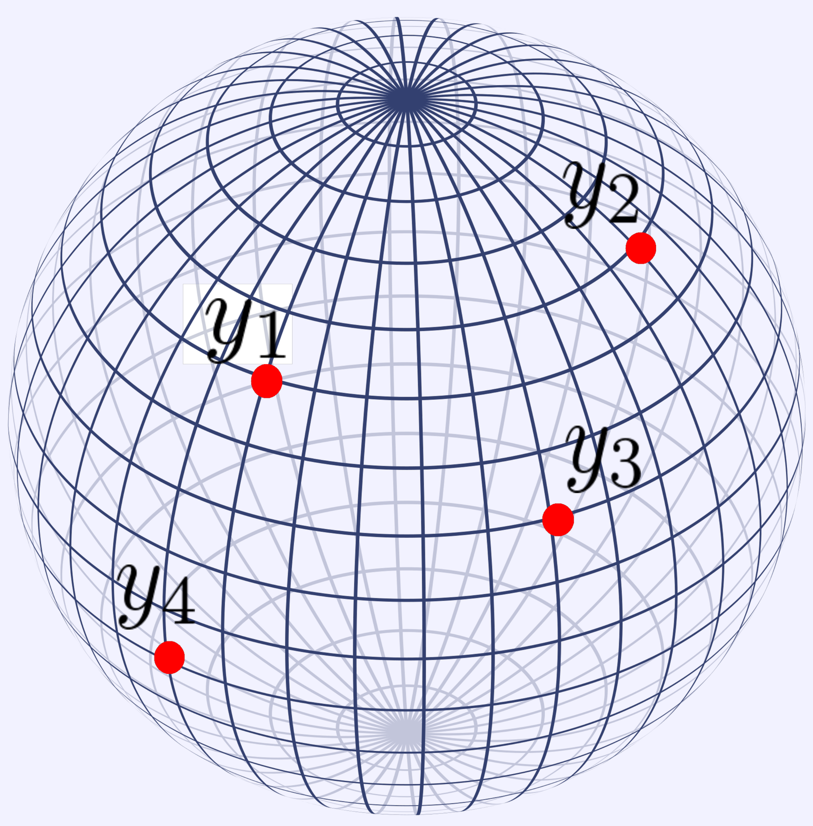
$$\min_{u \in \mathbb{C}^m} u^* C u \text{ s.t. } |u_i| = 1, \qquad \text{(PR)}$$

with $C = \mathrm{diag}(b)(I - AA^\dagger)\mathrm{diag}(b)$.

– Dropping the rank constraint:

$$\min_{X \in \mathbb{H}^{m\times m}} \langle C, X\rangle \text{ s.t. } \mathrm{diag}(X) = 1,\ X \succeq 0. \qquad \text{(PhaseCut)}$$

Burer-Monteiro factorization



Geometrically, each y_i is a point on the unit sphere in $\mathbb{C}^k$.

$$\min_{Y \in \mathbb{C}^{m\times k}} \langle CY, Y\rangle \text{ s.t. } y_i^* y_i = 1,\ \forall i. \qquad \text{(PhaseCut-BM)}$$

with $Y = [y_1^*, \ldots, y_m^*]$ & $y_i \in \mathbb{C}^k$.

• The main theorem applies to (PhaseCut-BM).

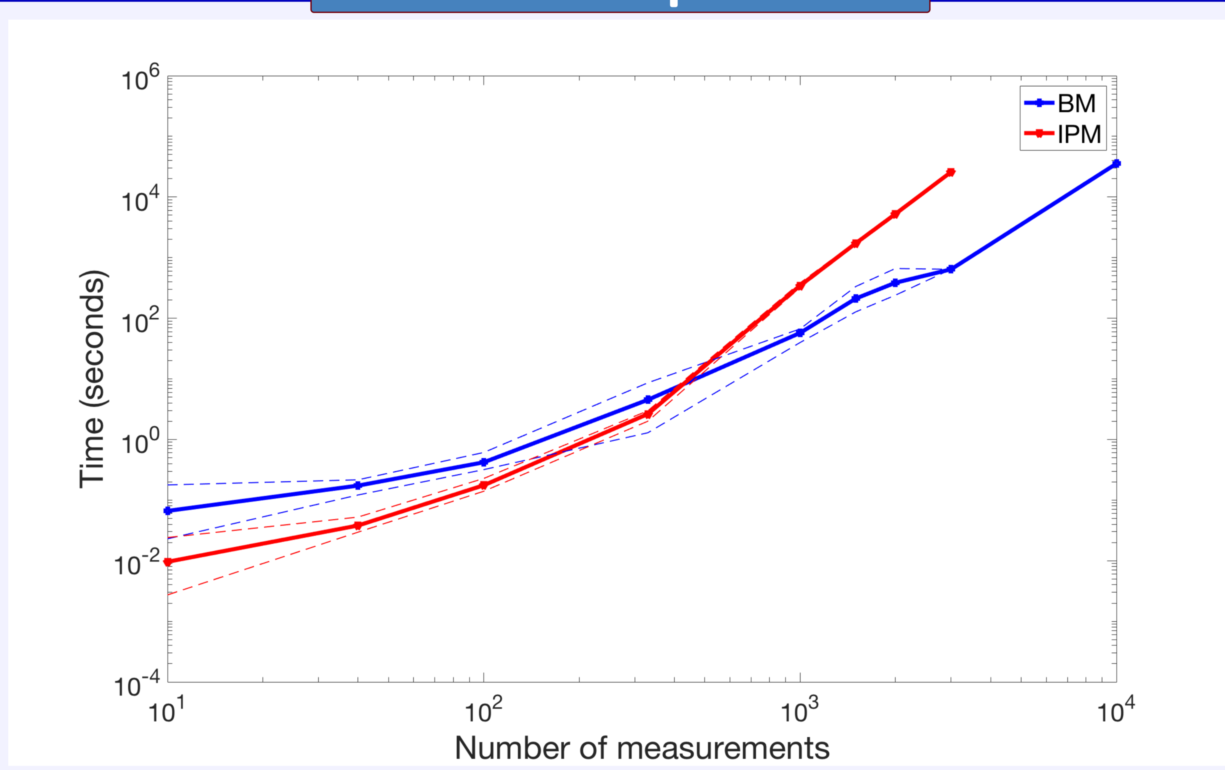
• Related work: Mei et al. [4].

– Holds for ASOSPs without perturbation.

– More general result since it holds for any k.

– When k large, non-informative on the optimality of ASOSPs.

Numerical Experiments



Computation time of the interior-point method (IPM) and of the Burer–Monteiro approach (BM) to solve (PhaseCut). As the number of measurements increases, BM outperforms IPM.

References

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