Revisiting the Challenges of MaxClique

Raffaele Marino[1] and Scott Kirkpatrick[1]

[1]School of Computer Science and Engineering,
The Hebrew University of Jerusalem,
Edmond Safra Campus , Givat Ram,
Jerusalem, 91904, Israel

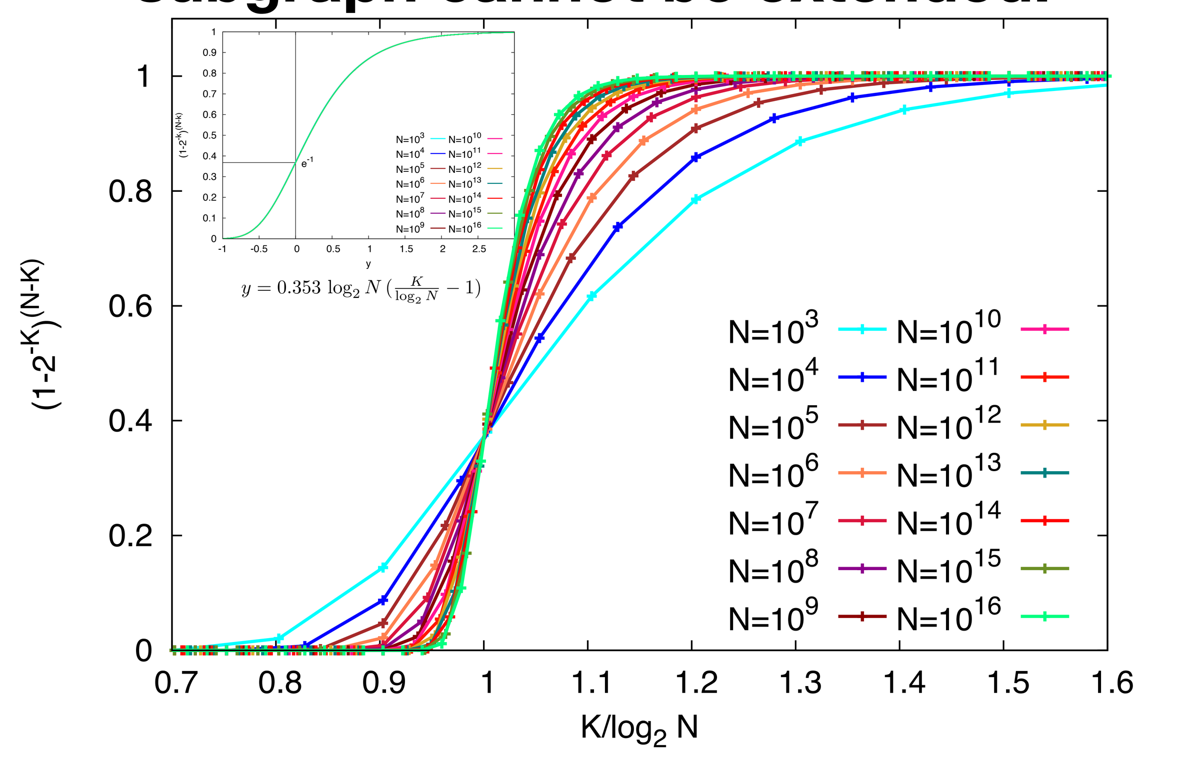
marino.raffaele@mail.huji.ac.il
kirk@cs.huji.ac.il

THE HEBREW UNIVERSITY OF JERUSALEM

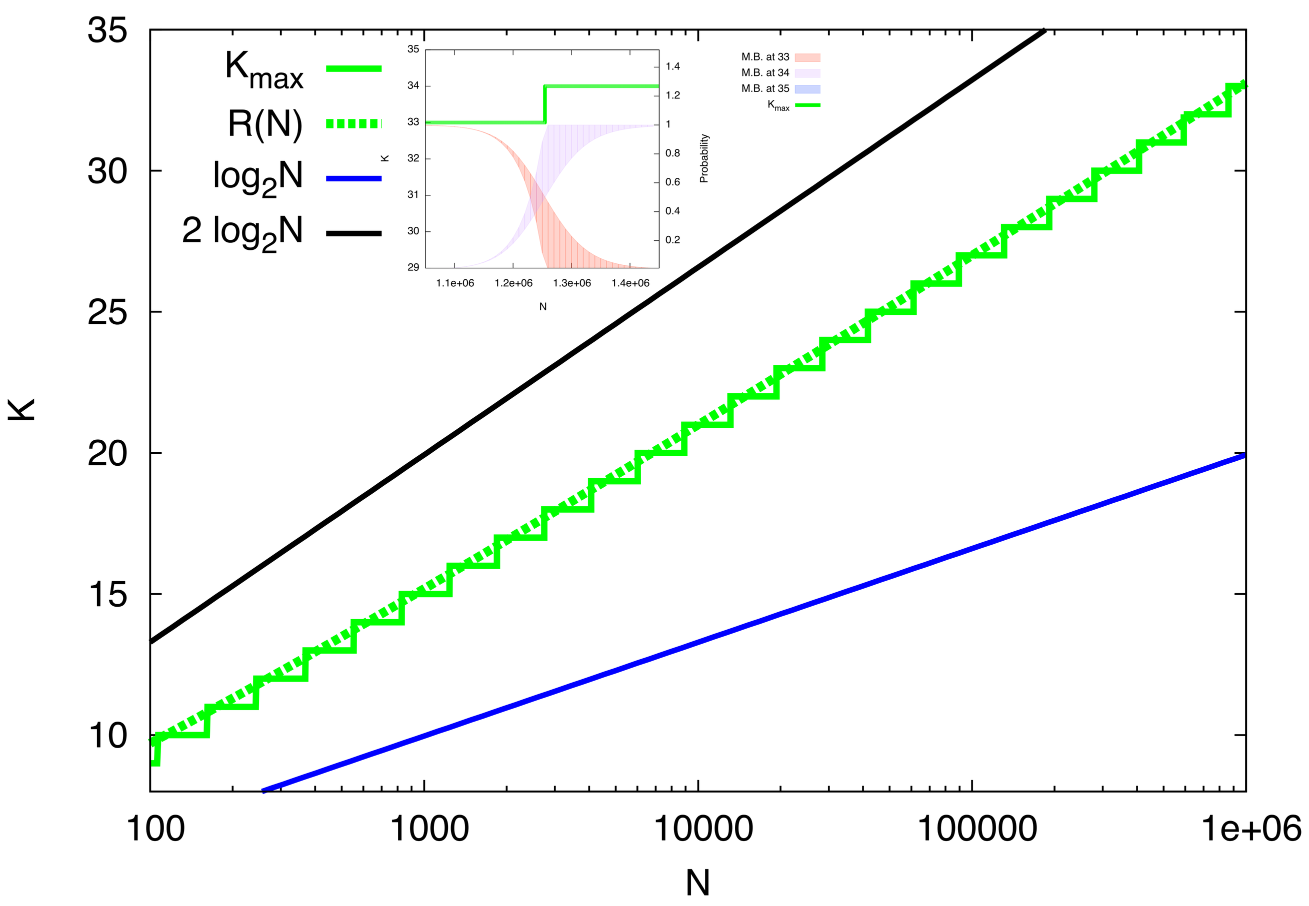
MaxClique Problem:

- E-R random graph G(N,p)
- Find the **maximum** complete subgraph. $\mathbb{E}[K] = \binom{N}{K} p^{\binom{K}{2}}$.

Probability that a complete subgraph cannot be extended:



Probability that a complete subgraph in G(N,0.5) cannot be extended as function of K/log_2N. The inset shows the universal form that all of these curves can be rescaled into.



Clique number as function of N. The green stair case has been obtained solving numerically $\mathbb{E}[K]=1$, while the green straight line ($R(N)$) using Stirling's approximation. $R(N)$:
$2\log_2 N - 2\log_2\log_2 N + 2\log_2 e/2 + 1$
The inset shows regions defined by upper and lower bounds, on the fractions of graphs of size N with maximum cliques of size K_{max} and $K_{max}+1$. Such a prediction was obtained from second moment arguments by Matula's [2]:

$$\left(\sum_{j=\max\{0,2K-N\}}^{K} \frac{\binom{N-K}{K-j}\binom{K}{j}}{\binom{N}{K}} p^{-\binom{j}{2}}\right)^{-1} \leq \mathrm{Prob}(K_{\max} \geq K) \leq \binom{N}{K} p^{\binom{K}{2}}$$

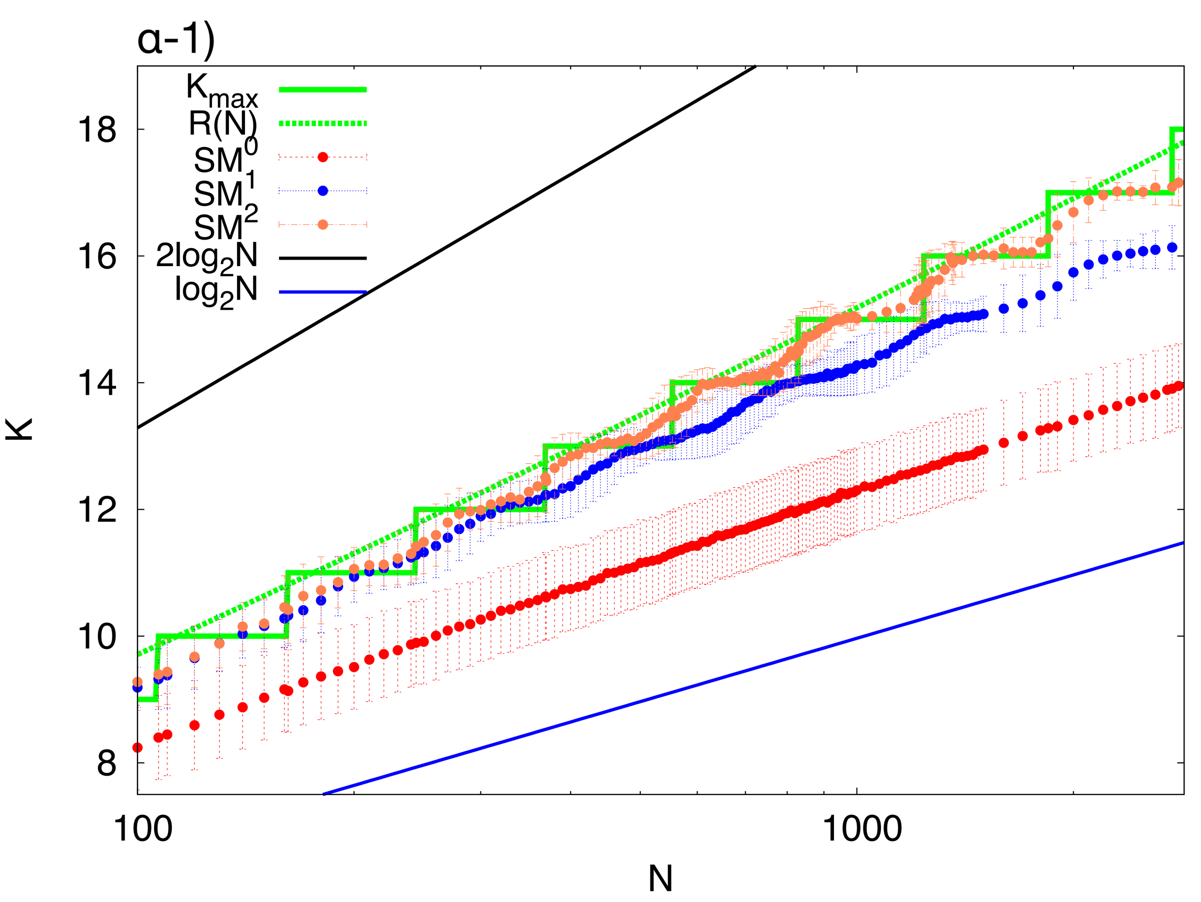
MaxClique challenges:

- In the asymptotic limit, no deterministic polynomial algorithm can find a clique of size *(1+ε)log$_2$N, ε > 0.*
- *For a Hidden Clique of size $\beta N^{0.5}$, how small should β be in such a way that we can find it?*
- Jerrum 1992 [1]: *"Given a graph G(N,0.5) selected randomly from the set of N vertex graphs that contain a clique of size at least $N^{0.49}$, find a clique of size at least 1.01 log$_2$N with probability 1/2, using a deterministic polynomial algorithm."*

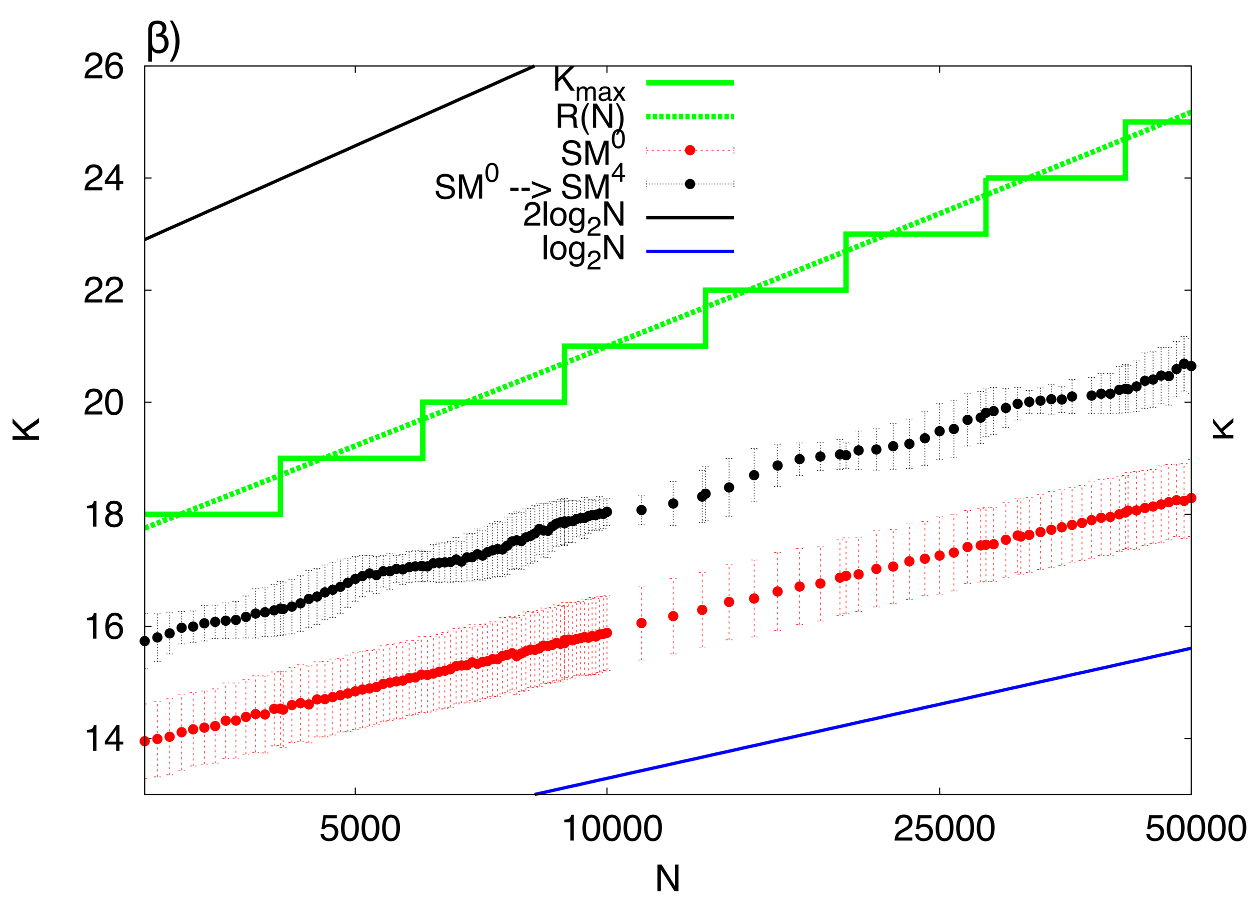
We use the probabilistic approach in a novel way to provide a more insightful test of constructive algorithms for MaxClique Problem. We show that improvements over existing methods are possible, meeting the *challenges* for practical problems with *N* as large as 10^{10} and perhaps longer.

Results

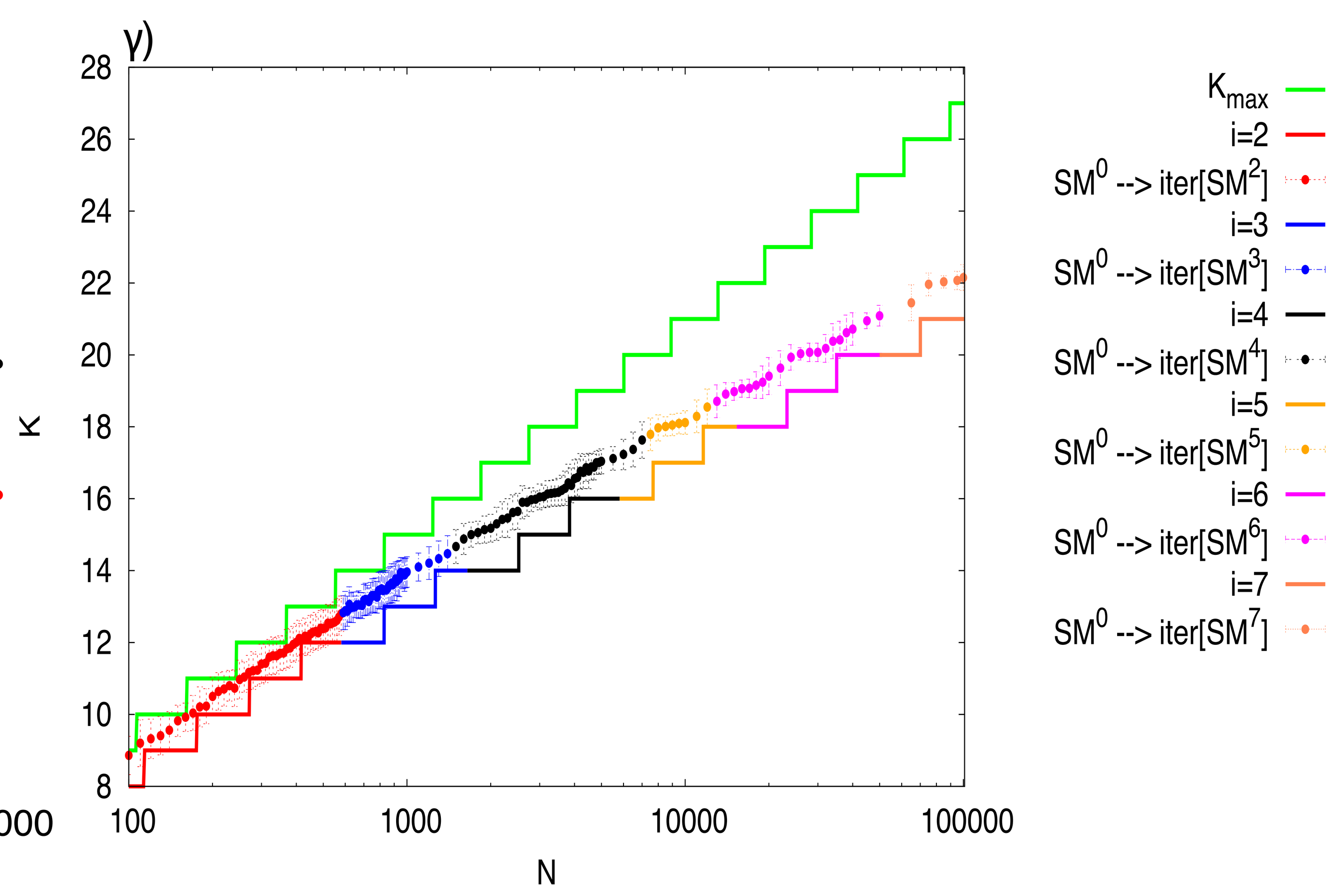
SM^0, SM^1, SM^2 :

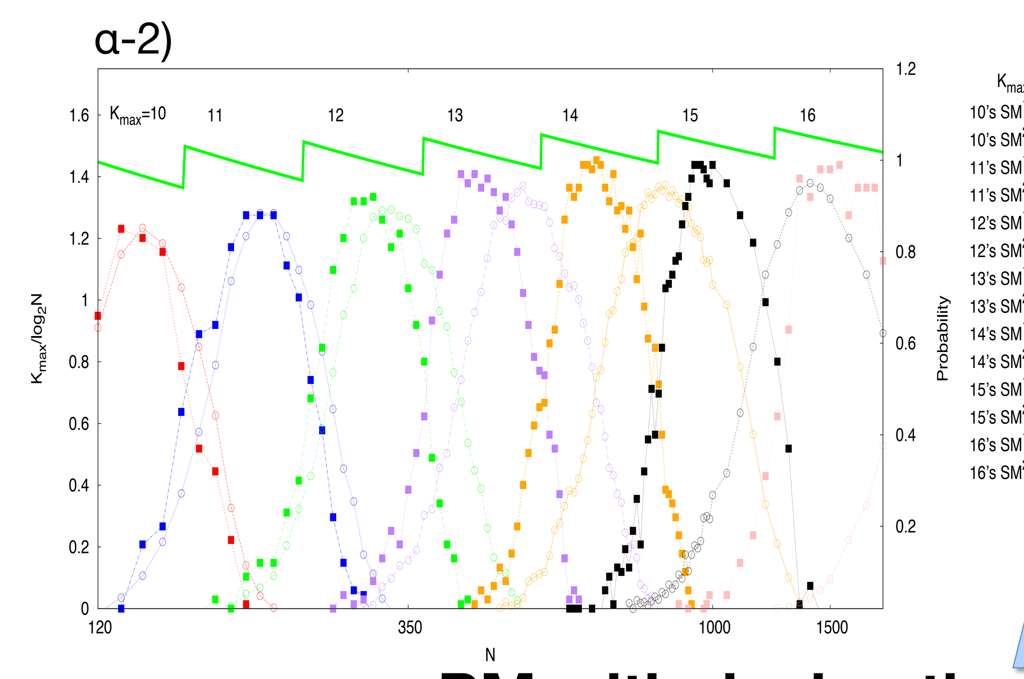


$SM^0 \rightarrow SM^4$:

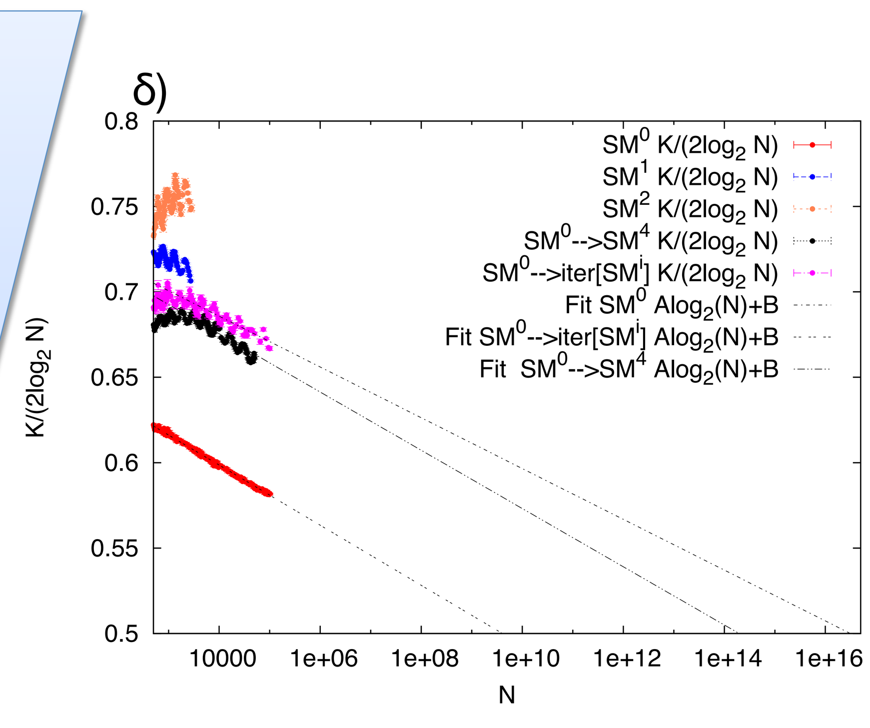


$SM^0 \rightarrow$ iter[SM^i]:

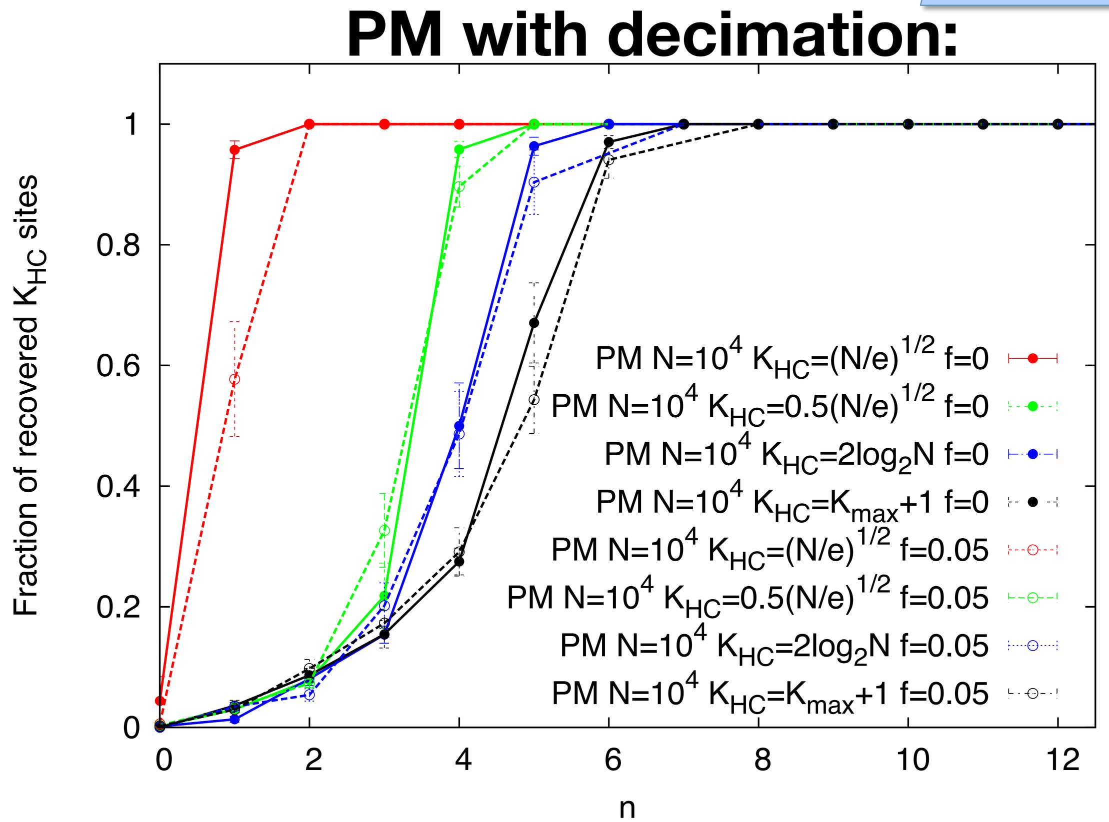




- α-1) The picture shows the sizes of the maximal clique from ***E-R random graphs G(N, 0.5)***, using the class of algorithms ***SM^i***, with i=*0*, *1*, *2*, (**red**, **blue** and **coral** points respectively). The inset compares the fraction of graphs predicted to have each value of K_{max}, with experimental probability obtained using ***SM^1*** and ***SM^2***. The computational complexity is ***$O(N^{i+2})$***.
- α-2) The picture shows the fraction of graphs found by ***SM^i***, with ***i=1, 2*** (*open-circles* and *solid-squares*, respectively). The fraction seen for each value of K_{max} receives a different color.
- β) The picture compares the size of the maximal cliques found on ***E-R graphs***, using ***SM^0*** (**red** points) and the algorithm ***SM^0 --> SM^4*** (**black** points). The computational complexity is bounded by ***$O(N^2 logN)$***.
- γ) Improvement of ***SM^0 --> SM^4*** using an iterative procedure, ***SM^0 --> iter[SM^i]***. The multi-colored staircase represents the expected maximum values of completed subgraphs conditioned by the fact that we are starting with arbitrary complete subgraphs of size *i*.
- δ) We extrapolate the meeting points with Jerrum's challenge at ***N*** as large as ***10^{10}*** or bigger.



PM with decimation:



Power Method with decimation (PM with decimation):

We have used a naïve power method for finding hidden cliques on ***E-R graph G(N,p)***. We have learned that using the hint to restrict attention to a smaller subgraph containing the hidden clique, power method with decimation is able to recover it. The picture shows results obtained by the algorithm for different size of the hidden clique, i.e. ***$(N/e)^{0.5}$***, ***$0.5(N/e)^{0.5}$***, ***2log$_2$N*** and ***K_{max} +1***, into graphs ***G(N=10^4,p=0.5)***. Surprisingly, power method with decimation needs only few sites to recover completely the hidden cliques.

References: [1] M. Jerrum, Random Structures and Algorithms, Vol. 3, n. 4 (1992) - [2] Matula D.W., Department of Computer Science, Southern Methodist University, (1976).