

Rephrasing the MacArthur's model

We analyze an example of ecological feedback, which turns out to be beneficial to describe resource-competition mechanisms.

The MacArthur's model was first analyzed in a low-dimensional setting, namely with few interacting species and a limited number of resources. It is equivalent to define a set of coupled differential equations and minimize a quadratic form. Interestingly, new critical features emerge when one tries to generalize this framework to the high-dimensional interaction space. In the following, we shall focus on a static formalism by means of disordered system techniques.

According to Tikhonov and Monasson's model [2], we introduce a **resource surplus**: $\Delta_\mu = \sum_{i=1}^N \sigma_{\mu i} h_i - \chi_\mu \dots \rightarrow$ **species' requirement** $i \in \{1, \dots, N\}$ **# resources**
metabolic strategies $\mu \in \{1, \dots, S\}$ **# species**

The resource availabilities depend both on the resource supply and the total demand: $h_i = \frac{R_i}{\sum_\mu n_\mu \sigma_{\mu i}}$

species abundances

The dynamics is defined by the following differential equation:

$$\frac{dn_\mu}{dt} \propto n_\mu \Delta_\mu \begin{cases} n_\mu > 0 & \& \Delta_\mu = 0 \\ n_\mu = 0 & \& \Delta_\mu < 0 \end{cases} \quad \text{At equilibrium two different scenarios are possible}$$

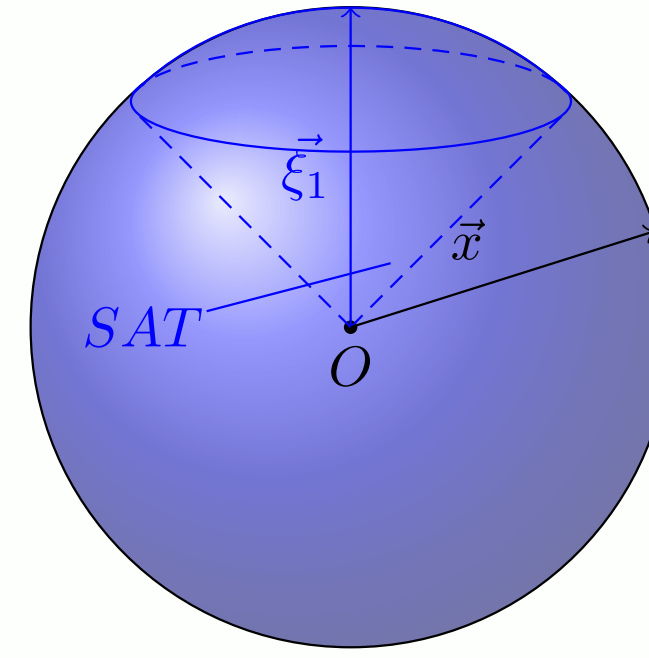
This model - even in its purely convex version - can be regarded as an example of constraint satisfaction problem: a positive resource surplus allows species to survive and multiply. In the opposite case, the solution defines an *unsustainable region*.

Similarities with the (convex) perceptron model

Interestingly, the equilibrium resource surplus distribution shows the same kind of behavior as the distribution of random gaps in the perceptron model (close to the SAT/UNSAT transition):

$$p(\Delta) = \frac{1}{\sqrt{2\pi\psi^2}} e^{-\frac{(\Delta+\lambda\psi)^2}{2\psi^2}} \theta(-\Delta) + E(\lambda)\delta(\Delta)$$

resource surplus distribution



$$g(h) = \alpha(1 - H(\sigma))\delta(h) + \frac{\alpha}{\sqrt{2\pi}} e^{-(h+\sigma)^2/2} \theta(h) \quad h_\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i^\mu x_i - \sigma$$

gap distribution in the RS phase

where $H(\sigma) = \int_{-\infty}^{\sigma} \mathcal{D}y$ is the complementary error function. As shown for the perceptron model, if σ is negative the above expression for the gap distribution is no longer valid. It provides a well-defined framework only in the convex (replica-symmetric) regime.

The partition function eventually reads:

$$Z = \text{const.} \times \int d\vec{h} e^{-\beta \tilde{F}(\vec{h})} \prod_\mu \frac{1/\beta}{\chi_\mu - \vec{h} \cdot \vec{\sigma}_\mu}$$

from which it is immediate to figure out the leading behaviour near the boundary.

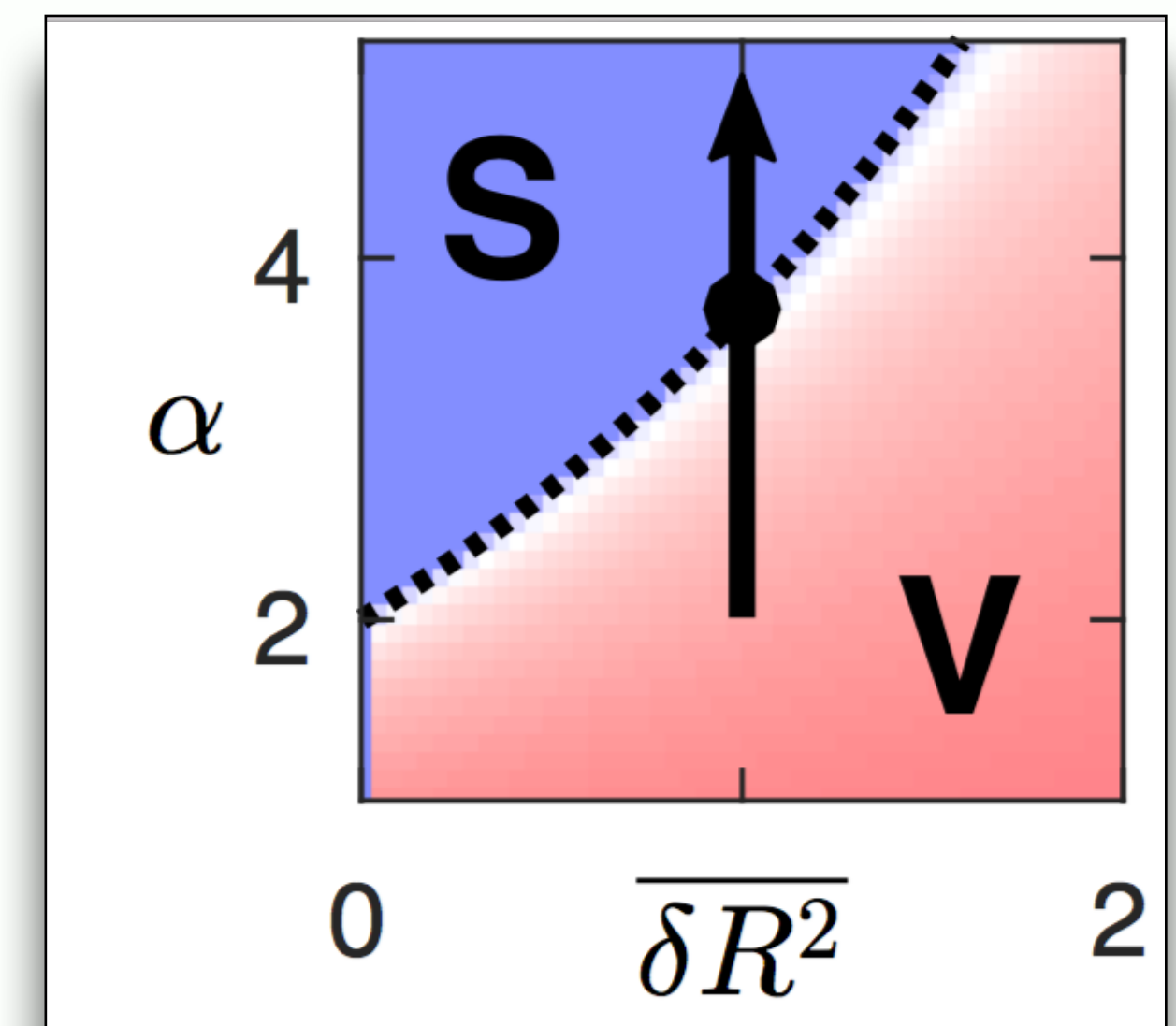
The different behavior depends on the finite (divergent) nature of the term $1/\Delta_\mu$. In particular, in the finite-temperature regime, all species have a finite abundance, while in the zero-temperature limit only a few group does not vanish, namely the species with Δ_μ exactly zero.

In this second case, in the low-temperature regime, the potential is dominated by a **logarithmic contribution**, which is reminiscent of the result we exactly derived for the perceptron model **close to the jamming line** [3]:

$$\log Z = \max \left[-\beta \tilde{F}(\vec{h}) - \sum_\mu \log |\Delta_\mu| \right]$$

$$G \simeq - \sum_\mu \theta(h_\mu) \log \left(\frac{h_\mu}{1-q} \right)$$

$$\alpha = \frac{S}{N}$$



Tikhonov and Monasson's phase diagram (PRL 118, 048103 (2017))

We recognize then two different phases:

- a **shielded phase S**, which corresponds to an *isostatic regime* in our *glassy phenomena jargon*;
- a **vulnerable phase V**, which should instead map into a *hypostatic regime*.

Spectrum of low-energy excitations

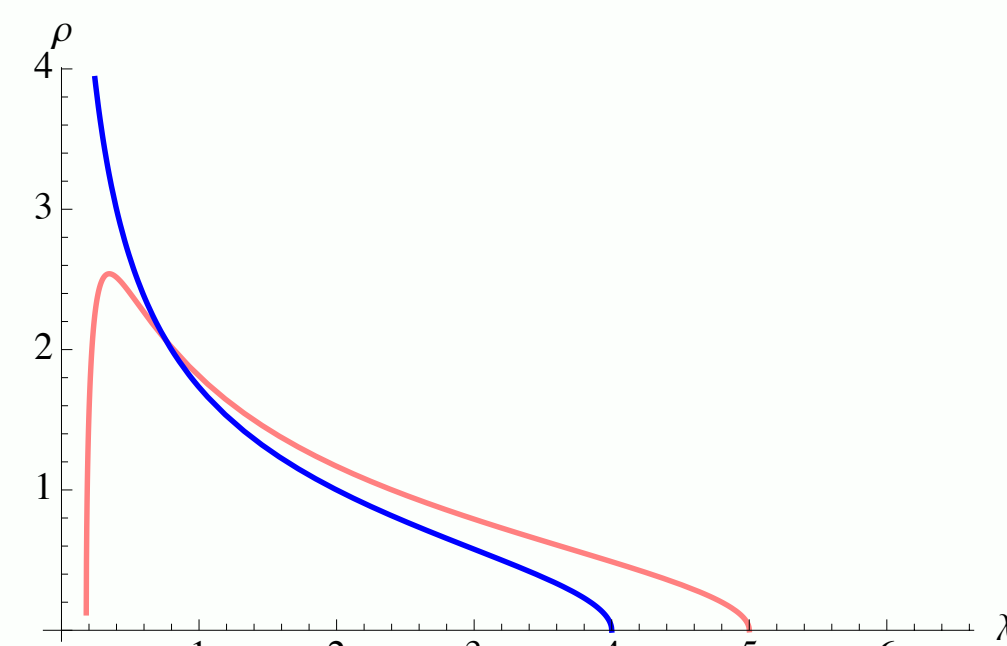
Once the Lyapunov function is defined, we can then study the density of fluctuations in both phases:

$$F(\{n_\mu\}) = \sum_i R_i \log \left(\sum_\mu n_\mu \sigma_{\mu i} \right) - \sum_\mu n_\mu \chi_\mu$$

The first derivatives with respect to the species abundances read:

$$\frac{\partial F}{\partial n_\mu} = \sum_i H(T_i) \frac{\partial T_i}{\partial n_\mu} - \chi_\mu = \Delta_\mu$$

$$\frac{\partial^2 F}{\partial n_\mu \partial n_\nu} = - \sum_i \sigma_{\mu i} \sigma_{\nu i} \frac{R_i}{(\sum_\rho n_\rho \sigma_{\rho i})^2} = - \sum_i \sigma_{\mu i} \sigma_{\nu i} \left(\frac{h_i^2}{R_i} \right)$$



In the simplest case (all availabilities fixed to the same value, say 1),

the stability matrix reduces to a simple Wishart matrix with a corresponding Marchenko-Pastur eigenvalue distribution: $\rho(\lambda) \sim \sqrt{\frac{4-\lambda}{\lambda}}$

In the most general case, for which the availabilities are not constant, we need to evaluate the full distribution and the associated moments.

We expect that the V-phase should have a gapped spectrum (as in analogous models that display a hypostatic regime).

Stability analysis: the replicon mode

To better investigate the behaviour of the spectral density, we proceed with the computation of the replicon mode in the S and the generic V phases.

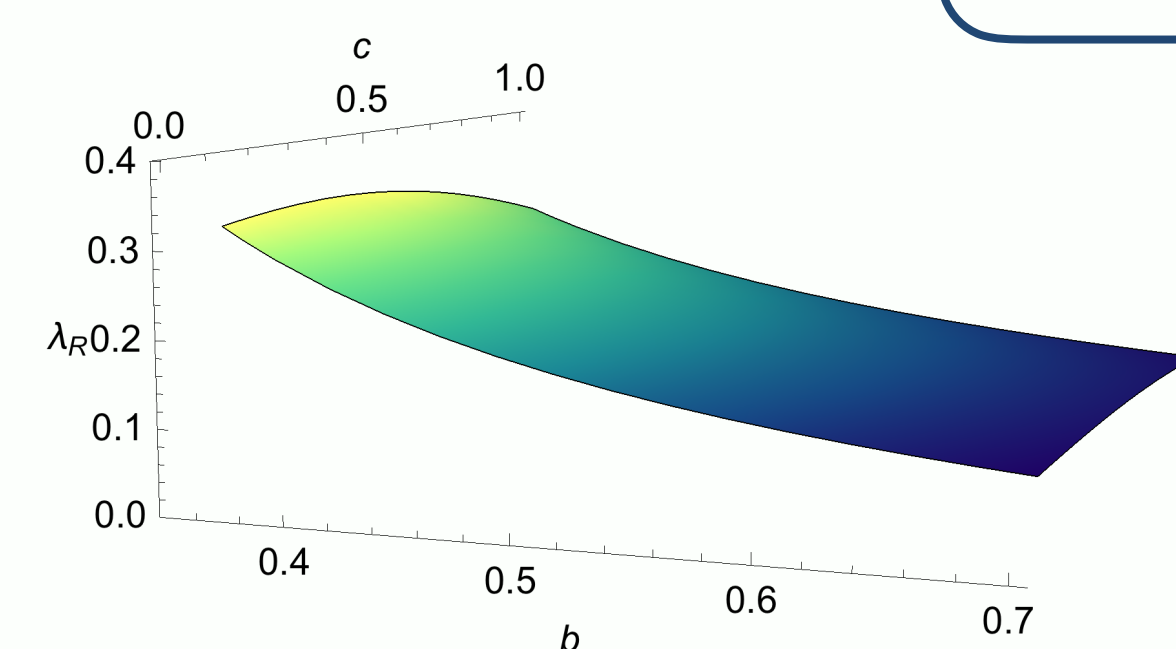
The diagonalization of the **mass matrix** implies to the leading order:

$$\frac{d^2 S}{dQ_{ab} dQ_{cd}} \equiv \mathcal{M}_{ab,cd} \longrightarrow \mathcal{M}_{ab,cd} \equiv \frac{\partial^2 S}{\partial Q_{ab} \partial Q_{cd}} = \frac{P}{4} p^2 (1-p)^2 \langle \hat{\Delta}_a \hat{\Delta}_b, \hat{\Delta}_c \hat{\Delta}_d \rangle_c$$

and, then, in the RS approximation: $\lambda_{\text{repl}} \equiv M_{ab,ab} - 2M_{ab,bc} + M_{ab,cd}$.

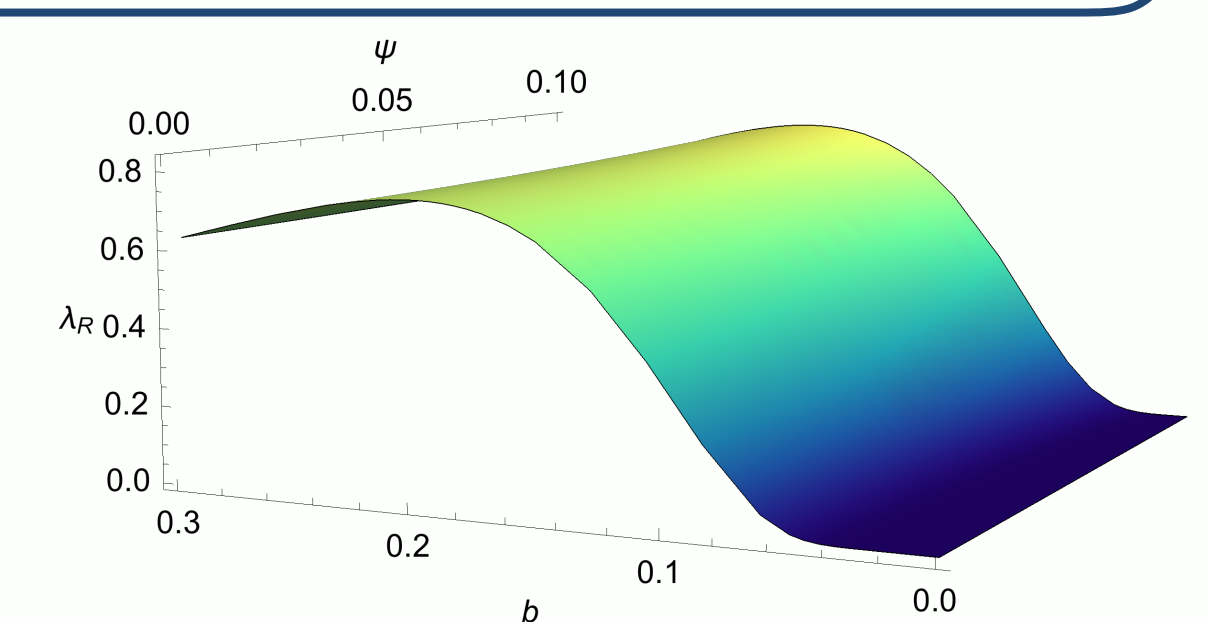
The resulting expression is :

$$\lambda_{\text{repl}} \propto \int \mathcal{D}z \frac{4e^{-\frac{2(c+\psi z)^2}{b}} \left[b + 2\sqrt{2\pi}b(c+\psi z)e^{\frac{(c+\psi z)^2}{2b}} + 2\pi(c+\psi z)^2 e^{\frac{(c+\psi z)^2}{b}} \right]}{b^3 \pi^2 \left[1 + \text{Erf} \left(\frac{c+\psi z}{\sqrt{2b}} \right) \right]^4}$$



Vulnerable (V) phase

It turns out to be finite and positive-definite in the entire V phase, while it vanishes upon approaching the V/S transition line. The S phase is clearly characterized by a marginal stability condition.



Shielded (S) phase

References

- [1] R. MacArthur, PNAS **64**, 4 (1969).
- [2] M. Tikhonov, R. Monasson, PRL **118**, 048103 (2017).
- [3] A. Altieri, S. Franz, G. Parisi, J. Stat. Mech. 093301 (2016).
- [4] A. Altieri, S. Franz, *arXiv pre-print*, arXiv:1805.06412 (2018).

- An immediate extension - of interest in different guises and especially for the jamming phenomenology - results in the definition of a slightly modified version allowing to recover a non-convex critical regime.
- It would imply a non-trivial behaviour also in terms of symmetry breaking. What are the consequences of a full-RSB phase for ecological / biological systems?

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