RIGOROUS QUANTUM FIELD THEORY FUNCTIONAL INTEGRALS OVER THE p-ADICS I: ANOMALOUS DIMENSIONS

ABDELMALEK ABDESSELAM, AJAY CHANDRA, AND GIANLUCA GUADAGNI

ABSTRACT. In this article we provide the complete proof of the result announced in [4] about the construction of scale invariant non-Gaussian generalized stochastic processes over three dimensional p-adic space. The construction includes that of the associated squared field and our result shows this squared field has a dynamically generated anomalous dimension which rigorously confirms a prediction made more than forty years ago, in an essentially identical situation, by K. G. Wilson. We also prove a mild form of universality for the model under consideration. Our main innovation is that our rigourous renormalization group formalism allows for space dependent couplings. We derive the relationship between mixed correlations and the dynamical systems features of our extended renormalization group transformation at a nontrivial fixed point. The key to our control of the composite field is a partial linearization theorem which is an infinitedimensional version of the Kœnigs Theorem in holomorphic dynamics. This is akin to a nonperturbative construction of a nonlinear scaling field in the sense of F. J. Wegner infinitesimally near the critical surface. Our presentation is essentially self-contained and geared towards a wider audience. While primarily concerning the areas of probability and mathematical physics we believe this article will be of interest to researchers in dynamical systems theory, harmonic analysis and number theory. It can also be profitably read by graduate students in theoretical physics with a craving for mathematical precision while struggling to learn the renormalization group.

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1. Introduction

Although the constructive approach to quantum field theory (QFT) and the rigorous methods based on the renormalization group (RG) already have a long history, the work from that area concerning the construction of composite fields and the operator product expansion (OPE) is rather scarce. All we managed to find after a review of the literature are: the work of Feldman and Raczka [30] followed by Constantinescu [22] (see also [28]) on composite fields (up to ϕ^3) in the three-dimensional massive ϕ^4 model, and that of Iagolnitzer and Magnen [47, 48] on the OPE for the two-dimensional massive Gross-Neveu model. Note that we do not count $\mathcal{P}(\phi)_2$ theories since the renormalization of composite fields in that case requires nothing beyond what is already needed for the free field, namely, Wick ordering [38]. Neither are we concerned here with perturbative results on composite field renormalization and the OPE. For the latter the reader is referred to the excellent presentations in [79, 51, 52, 45] and references therein. Yet, for ϕ_3^4 and GN_2 which respectively are superrenormalizable and asymptotically free in the ultraviolet, the short distance behaviour is governed by a Gaussian RG fixed point. Thus the concerned composite fields do not exhibit anomalous scaling dimensions. In this article we construct a composite field with dynamically generated anomalous dimension governed by a nontrivial RG fixed point. Regarding similar anomalous dimensions for elementary rather than composite fields we should mention the previous works [29] and [9]. The first concerns a model believed to have anomalous scaling (see, e.g., the review [63, §2.7]). However, after a more than heroic effort, the author stopped at the construction of correlation functions. To get a hold on the anomalous dimension would have required the extra work consisting of more precise estimates on the quantity $\delta^*(N)$ in [29, p. 189] together with the short distance asymptotics of some correlation function. The second goes the full distance and proves the existence of anomalous dimension. However, the latter is governed by a line of fixed points and depends on the coupling, unlike the situation for the paradigmatic Wilson-Fisher fixed point [76]. We should from the onset warn the reader that our main theorem, given in §3, seems in contradiction with a statement made in [33, p. 277]. Whether this contradiction is real or only apparent remains to be seen. By apparent we mean a contradiction which could be explained away, e.g., by the difference between the models or the objects to which the conflicting statements apply as often happens when one takes limits in different orders.

The particular model studied in this article is what we call the p-adic BMS model in honor of Brydges, Mitter and Scoppola who initiated the rigorous study of its real counterpart [14]. This followed the earlier study of a similar model by Brydges, Dimock and Hurd [13]. The real BMS model formally is a Radon-Nikodym perturbation of a massless Gaussian measure $d\mu_{C_{-\infty}}(\phi)$ for a random scalar field ϕ on \mathbb{R}^3 with covariance

$$C_{-\infty} = (-\Delta)^{-\left(\frac{3+\epsilon}{4}\right)} \ .$$

Here $\epsilon > 0$ is a small bifurcation parameter. The Radon-Nikodym weight is heuristically given by a constant times

(1)
$$\exp\left(-\int_{\mathbb{R}^3} \left\{g\phi(x)^4 + \mu\phi(x)^2\right\} d^3x\right) .$$

Compared with the model studied in [13], the BMS model has several advantages. The first is that it is technically simpler since it does not require wave function renormalization when $\epsilon < 1$. It is also more physical. Indeed, the BMS family of models indexed by ϵ includes the massless ϕ_3^4 model at $\epsilon = 1$. Even for $\epsilon < 1$ the BMS model is also, in all likelihood, Osterwalder-Schrader positive and should therefore lead to the construction of a unitary QFT in Minkowski space. Note that this kind of models with fractional powers of the Laplacian represent the best one can presently tackle as far as the rigorous study of the phenomena associated to the Wilson-Fisher fixed point. Indeed, the so-called ϵ -expansion in the physics literature is usually based on dimensional regularization which, as far as we know, has not been defined rigorously and nonperturbatively. See however [43] for an intriguing conjecture which may lead to progress on this issue, although it needs some amendments as shown in [65].

In [14] the authors considered an ad hoc RG transformation for this model in the formal infinite volume limit and they proved the existence of a nontrivial fixed point. They also constructed its local stable manifold. This was followed by [2] where connecting orbits joining the trivial fixed point to the nontrivial one were constructed. In this article we study the natural p-adic analogue of the real BMS model considered in [14, 2]. Here the random field ϕ lives on \mathbb{Q}_p^3 instead of \mathbb{R}^3 but it still is a real-valued field. There already is a rather large body of literature on p-adic QFT models (see [59] and references therein). However, we have not seen in this literature an explicit rigorous nonperturbative construction of the self-similar scalar field studied in this article (or the very similar version on \mathbb{Q}_p instead of \mathbb{Q}_p^3), although the technology for doing that has been available for a long time, i.e., since the ground-breaking work of Bleher and Sinai [11, 12]. Indeed, the RG transformation naturally associated to a random field over the p-adics falls under the umbrella of hierarchical RGs. This connection was briefly pointed out in [10]. The only p-adic QFT model for which we have seen an explicit construction is that in [53]. Yet, the propagator in that model has a very mild singularity in the ultraviolet so that Wick ordering and hypercontractivity techniques are enough to do the construction. Furthermore, the issue of infinite volume limit [37, 40] is not settled.

In the next few paragraphs we will try to give a rough idea of our result and methods. The precise statement of our theorem which was already announced in [4] will be recalled in §3 after the necessary definitions, in particular regarding p-adic analysis, are presented in §2. For the sake of pedagogy and efficiency we will pretend to be working over \mathbb{R}^3 in what follows. This will delay having to come to terms with some oddities of the p-adic world such as: the size of L is L^{-1} , the lattice is $\mathbb{Q}_p^3/\mathbb{Z}_p^3$ while \mathbb{Z}_p^3 is a lattice cell, and so on. The reader should bear in mind that it is the p-adic analogues of the following statements which are addressed in this article. First introduce a number L > 1 which serves as a yardstick for measuring changes of scale. This is the analogue of L = 2 used in the dyadic decomposition methods in harmonic analysis. We then define a cut-off covariance C_r , for $r \in \mathbb{Z}$, by suppressing Fourier modes or momenta k with size greater than L^{-r} starting from the non-cut-off covariance

$$\hat{C}_{-\infty}(k) = \frac{1}{|k|^{\left(\frac{3+\epsilon}{2}\right)}} = \frac{1}{|k|^{3-2[\phi]}}$$

where the symbol $[\phi]$ stands for the quantity $\frac{3-\epsilon}{4}$, i.e., the scaling dimension of the free massless Gaussian. We also change the integration set in (1) from \mathbb{R}^3 to a finite volume Λ_s of linear size L^s , $s \in \mathbb{Z}$. As renormalization theory tells us to, we also replace the couplings g, μ in (1) by r-dependent quantities \tilde{g}_r and $\tilde{\mu}_r$. This dependence is also called a bare ansatz. Finally, we also replace the monomials $\phi(x)^4$, $\phi(x)^2$ by their Wick ordered analogues : $\phi^4 :_{C_r}(x)$, : $\phi^2 :_{C_r}(x)$. This corresponds to a triangular change of coordinates, namely, switching from the monomial basis to that of Hermite polynomials which is more convenient. These modifications result in a well defined probability measure $\mathrm{d}\nu_{r,s}(\phi)$ with moments or correlators

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{r,s}$$
.

We construct the wanted measure $d\nu$ as the limit of the $d\nu_{r,s}$ when the ultraviolet cut-off r is taken to $-\infty$ and the infrared cut-off s is taken to ∞ . The particular bare ansatz we use is of the form $\tilde{g}_r = L^{-(3-4[\phi])r}g$, $\tilde{\mu}_r = L^{-(3-2[\phi])r}\mu$ where g, μ are fixed quantitities. In fact the mass μ has to be fine-tuned to its so-called

critical value $\mu_c(g)$. The correlators $\langle \phi(x_1) \cdots \phi(x_n) \rangle_{r,s}$ are obtained as derivatives of a moment generating function $\mathcal{S}_{r,s}(\tilde{f})$ in terms of a test function \tilde{f} . This generating function is of the form

$$S_{r,s}(\tilde{f}) = \frac{\int d\mu_{C_r}(\phi) \ e^{-V_{r,s}(\phi) + \phi(\tilde{f})}}{\int d\mu_{C_r}(\phi) \ e^{-V_{r,s}(\phi)}}$$

where $\phi(\tilde{f})$ denotes the distributional pairing.

The first step of the analysis involves a 'rescaling to unit lattice' which is a change of variables in the field $\phi(x)$. Indeed, the latter sampled according to the Gaussian measure with covariance C_r has the same law as the field $L^{-r[\phi]}\phi(L^{-r}x)$ where the new ϕ is sampled according to the Gaussian measure with covariance C_0 . One then repeatedly applies an RG transformation to both the numerator and denominator of the resulting expression for $S_{r,s}(\tilde{f})$. In its naive form this transformation is based on the identity

(2)
$$\int d\mu_{C_0}(\phi) \ e^{-V(\phi)} = \int d\mu_{C_0}(\phi) \ e^{-V'(\phi)}$$

with a new potential V' given by

$$V'(\phi) = -\log\left(\int d\mu_{\Gamma}(\zeta) \ e^{-V(\zeta + L^{-[\phi]}\phi(L^{-1}\bullet))}\right)$$

where $\Gamma = C_0 - C_1$ is the fluctuation covariance involving Fourier modes in the shell $L^{-1} < |k| \le 1$. In fact we need to extract a field independent quantity δb in (2) which becomes

$$\int d\mu_{C_0}(\phi) \ e^{-V(\phi)} = e^{\delta b(V)} \int d\mu_{C_0}(\phi) \ e^{-V'(\phi)}$$

while the new potential rather is

$$V'(\phi) = \delta b(V) - \log \left(\int d\mu_{\Gamma}(\zeta) e^{-V(\zeta + L^{-[\phi]}\phi(L^{-1}\bullet))} \right) .$$

While the above transformation $V \to V'$ is well defined (in finite volume), it is difficult to exploit it. Indeed, this transformation is most useful where it is most singular, i.e., in infinite volume. In their seminal article [15], Brydges and Yau, based on earlier work in constructive QFT, found a solution to this difficulty which is to introduce a suitable lift:

$$\begin{array}{ccc} \vec{V} & \longrightarrow & \vec{V}' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

for the naive RG transformation $V \to V'$. Such a Brydges-Yau lift is highly nonunique and quite complicated but it has the advantage of providing a dynamical system which can be analyzed in the infinite volume limit using rigorous estimates. We define the Brydges-Yau lift relevant to the present model in §4. After the initial rescaling to unit lattice, the numerator of the ratio expressing $S_{r,s}(\tilde{f})$ gives rise to an initial vector $\vec{V}^{(r,r)}(\tilde{f})$ while the denominator produces a similar one $\vec{V}^{(r,r)}(0)$. The moment generating function is obtained from the log-moment generating function $S^{T}(\tilde{f})$ which itself is the limit of quantities roughly given by

$$\mathcal{S}_{r,s}^{\mathrm{T}}(\tilde{f}) = \sum_{r \leq q \leq s} \left\{ \delta b[\vec{V}^{(r,q)}(\tilde{f})] - \delta b[\vec{V}^{(r,q)}(0)] \right\}$$

where $\vec{V}^{(r,r)} \to \vec{V}^{(r,r+1)} \to \vec{V}^{(r,r+2)} \to \cdots$ denote the RG iterates of the initial vectors.

We remove the cut-offs by showing the convergence of the series over (logarithmic) scales q

(3)
$$\mathcal{S}^{\mathrm{T}}(\tilde{f}) = \sum_{q \in \mathbb{Z}} \left\{ \delta b[\vec{V}^{(-\infty,q)}(\tilde{f})] - \delta b[\vec{V}^{(-\infty,q)}(0)] \right\} .$$

This hinges on controlling the deviations from the bulk $\vec{V}^{(r,q)}(\tilde{f}) - \vec{V}^{(r,q)}(0)$ corresponding to the effect of a local perturbation due to the test function.

Most of the previous work in rigorous RG theory relies on a translation trick (completing the square) which reduces the RG for the numerator to that of the denominator (see, e.g., [8, Ch. 4]). However this creates technical difficulties under RG iteration since the effect of this trick is to add an \tilde{f} dependent term to the so-called background field (the ϕ on the right-hand side of (2)) which becomes more and more singular.

Since we would like to treat similar perturbations by higher smeared powers of the field, with a view towards composite field renormalization, a more systematic approach would be to define a more general RG map which handles the data $\vec{V}^{(r,q)}(\vec{f})$ produced by the numerator. This is one of the main technical innovations in this article: we define in §4 what we call the extended RG transformation which handles potentials with couplings g, μ which are allowed to vary with the position x of the $\phi(x)^m$ monomials. To first order of approximation this extended RG amounts to taking local averages of the previous couplings and multiplying them by the appropriate power-counting factor. This is somewhat dual to the usual block-spinning approach which acts on the field rather than the couplings. The convergence of (3) is proved by exhibiting suitable decay in both infinite directions. Very roughly one can introduce L^{q_+} which corresponds to the size of the support of the test function \tilde{f} in direct space, while L^{-q} corresponds to size of the support of the Fourier transform of \tilde{f} . Over the p-adics this is strictly true while only approximately so over the reals. The uncertainty principle can be seen as the relation $q_- \leq q_+$. Most of the contribution to the series (3) comes from the range of scales where the test function lives, i.e., for q between q_- and q_+ . For the ϕ perturbations, because of our specific choice of cut-offs, the contribution of scales $q < q_{-}$ is identically zero. However, when considering ϕ^2 perturbations smeared by another test function \tilde{j} this is no longer the case. In fact, a large part of the analysis is devoted to proving the needed decay in q after subtraction of a suitable linear term in \ddot{j} . Two sectors of the extended RG space are important for the analysis. The bulk RG corresponds to spatially uniform couplings (as for the denominator of $S_{r,s}(f)$) and is what is needed for the control of the ultraviolet region. Indeed, after the initial rescaling, the test functions become diluted over a very large volume and thus appear spatially constant, i.e., bulk-like. On the other hand, in the infrared region the deviations from the bulk appear point-like. There is no more dangerous L^3 factor responsible for the expanding or relevant RG directions and this fact is key to the needed decay in q for $q > q_{+}$.

In this article we construct the mixed correlations

$$\langle \phi(x_1) \cdots \phi(x_n) \ \phi^2(y_1) \cdots \phi^2(y_m) \rangle$$

as distributions even at coinciding points [28]. This is done by controlling a more complicated generating function which very roughly is given by

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) = \frac{\int \mathrm{d}\mu_{C_r}(\phi) \ e^{-V_{r,s}(\phi) + \phi(\tilde{f}) + \phi^2(\tilde{j})}}{\int \mathrm{d}\mu_{C_r}(\phi) \ e^{-V_{r,s}(\phi)}}$$

instead of $S_{r,s}(\tilde{f})$. In the initial rescaling or dilution, the test function \tilde{f} is weakened by a factor $L^{(3-[\phi])r}$ while the test function \tilde{j} should normally be weakened by a factor $L^{(3-2[\phi])r}$. We say normally because this would be the case in the absence of anomalous dimension. It turns out, however, the correct weakening factor one needs to use is $\alpha_{\rm u}^r$ where $\alpha_{\rm u}$ is the eigenvalue in the expanding direction at the nontrivial RG fixed point. The anomalous dimension is due to the fact α_u is strictly less than the analogous eigenvalue $L^{3-2[\bar{\phi}]} = L^{\frac{3+\epsilon}{2}}$ at the Gaussian fixed point. This strict inequality was known for a long time. Indeed, for a very similar hierarchical model, Bleher and Sinai obtained small ϵ asymptotic expansions for the eigenvalues at the fixed point [12, Thm. 3.1]. The previous fact follows from such results. The new and nontrivial part of our proof concerns the translation of this fact about the eigenvalue α_n into information about the ϕ^2 correlation functions. This we accomplish thanks to a, possibly new, infinite-dimensional version of the Theorem of Kænigs in holomorphic dynamics.

In [54, $\S 8$] Keenigs proves the following result. Let F(z) be an analytic function defined near zero such that F(0) = 0 and $F'(0) = \alpha$ with $0 < |\alpha| < 1$. Then the limit

$$\Psi(z) = \lim_{n \to \infty} \alpha^{-n} F^n(z)$$

exists and is analytic near zero. It also satisfies $\Psi'(0) = 1$ and therefore provides a conjugation of F to its linearization at zero. Indeed, one has the intertwining relation

$$\alpha \Psi = \Psi \circ F$$
.

A two-line proof (which however was a great source of inspiration to us) was given in [1, p. 6]. It amounts to showing the uniform absolute convergence of the telescopic sum with general term

$$\alpha^{-(n+1)}F^{n+1}(z) - \alpha^{-n}F^{n}(z).$$

In our situation we have an expanding eignenvalue and we do not know if our map, i.e., the RG is invertible (common wisdom says not since the RG is an irreversible process of erasing degrees of freedom). So we are in a situation where we rather have to construct the limit

$$\Psi(z) = \lim_{n \to \infty} F^n(\alpha^{-n}z)$$

instead. In fact, the result we establish is more general. It is the existence and analyticity of

$$\lim_{n\to\infty} RG^n(v + \alpha_{\mathbf{u}}^{-n}w)$$

for a transformation RG in an infinite-dimensional Banach space. This map has a fixed point v_* with a dimension one unstable manifold $W^{\rm u}$ with eigenvalue $\alpha_{\rm u}$. The previous statement applies to points v on the codimension one stable manifold W^s , while w can be any vector which is not too large. This realizes a partial linearization of the RG map and is tantamount to constructing a nonlinear scaling field in the sense of Wegner [72] infinitesimally close to the critical surface $W^{\rm s}$. Our result holds regardless of nonresonance conditions because the unstable manifold in the mass direction is one-dimensional. One should also note the similarity of the above construction of a conjugating function to that of the classical scattering analogue of Møller wave operators [46, 61, 66]. Indeed, Nelson in particular in his book popularized the scattering idea as a way to obtain such conjugations. He also attempted to derive Sternberg's Linearization Theorem in this way. Linearization results in infinitely many dimensions are rather rare. See however [42] which contains an interesting discussion of the relation between linearization and classical scattering. In essence, this is also related to the age old method of variation of constants [18]. Other authors attempted to obtain a partial linearization result (in the C^{∞} category) analogous to ours [20]. However, their proof is incorrect as they later acknowledged in [21, §12]. Our partial linearization theorem derived in §9 is in the analytic category. This is essential for the construction of our generalized stochastic processes. Indeed, this analyticity is inherited by the series (3), in the presence of the test function \tilde{j} . As a result we get n! bounds on the moments of both the elementary field ϕ and composite field ϕ^2 . These are needed for reconstructing the measures from the moments.

This allows us to produce generalized stochastic process which are translation, rotation and scale invariant by the subgroup $L^{\mathbb{Z}}$ of the full group of scale transformations. In a companion paper [5] we will show that our model has full scale invariance. This would open the door to the investigation of conformal invariance along the lines of [57, 56]. If one also makes progress on the OPE one could even contemplate the possibility of an exact solution (see [62, p. 70] and [44] for related work). Our article can be seen as taking some steps towards the *systematic* elaboration of a very general theory of 'local fields on local fields', to borrow a pun attributed to S. Evans. A brief sketch of such a theory was given in [3]. It is a natural continuation of a line of thought pursued by the Soviet School of probability theory and mathematical physics, see in particular [26, 27, 70]. Note that the presence of anomalous dimension (for ϕ^2) should make it clear that our field ϕ is not subordinated to a Gaussian field in the sense of [26, 58]. However proving this would require the beginning of the OPE (namely generating the ϕ^2 field by collapsing two ϕ 's) which we do not address in this article and leave for a future publication. We believe our article opens the way to a rigorous investigation of the OPE. The latter is of fundamental importance. Indeed, vertex operator algebras (see, e.g., [32]), chiral algebras [7] and factorization algebras [23] can be seen as mathematical constructions which try to capture the OPE structure in QFT.

Our primary motivation for considering the p-adic version of the BMS model is that it is a good toy model for the original version over the reals. In fact the model over \mathbb{R} (for ϵ small) is expected to exhibit similar features: absence of anomalous dimension ($\eta_{\phi} = 0$) for the elementary field ϕ and presence of anomalous dimension ($\eta_{\phi^2} > 0$) of order ϵ for the composite field ϕ^2 . Regarding this real model, P. K. Mitter [60] proved nonperturbatively but in finite volume that the elementary field has no anomalous scaling. He also did a formal perturbative calculation for the anomalous dimension of ϕ^2 (see also [31]). Regarding the p-adic model, one can argue that the prediction of the properties $\eta_{\phi} = 0$ and $\eta_{\phi^2} > 0$ was made forty years ago by Wilson himself in [75]. Indeed, the discussion in that article was in the framework of Wilson's approximate RG recursion which is what we now call the hierarchical RG. In other words, the situation considered in [75] is essentially identical to that of the p-adic BMS model. Using hierarchical models such as the p-adic model in order to shed light on Euclidean ones such as the real BMS model has been and will continue to be a fruitful approach. Indeed, some of Wilson's ideas on the RG were already present in the article [73]. Nevertheless, the

first systematic exposition of what is now known as the Wilson RG philosophy most likely is the article [74] which is about the approximate recursion. Only later came the adaptation of his methods to the model over \mathbb{R} which found its definitive presentation in the famous lectures [77]. Our methodology is to try to follow a similar path. In doing so we took great care in choosing, for our treatment of the p-adic case, methods which are known to work over the reals. We simply transposed such methods, in as natural a way as possible, to the p-adic setting. Indeed, the definition of our RG transformation in §4, the corresponding estimates in §6.3 as well as some of the dynamical systems techniques in §8 were directly adapted from [14, 2]. We believe the main nontrivial task which remains in order to extend the results of the present paper to the real case is to devise a proper analogue of the extended RG given in §4. Such a transformation should essentially reduce, in the special case of spatially uniform potentials, to the RG transformation in [14, 2]. This problem is a matter of harmonic analysis and is thematically similar to extending a result about Walsh series to Fourier series (see, e.g., [24]).

Our secondary motivation is to help facilitate the investigation of the connection between QFT and number theory. This is perhaps still at the speculative stage. Nevertheless, see [16, 17, 34, 35, 36, 55] for interesting work in this direction. We hope our article will help number theorists unterstand how the *rigorous* RG works when used for the construction and study of QFT functional integrals.

We will end this introduction by commenting on the length of this article. There are two reasons for this: the choice of methods and the high level of detail. As we said earlier we did not try to prove our result about the p-adic BMS model in the quickest and most direct manner. The potential for adaptability to the real situation was the overarching principle that guided our investigation. Also note that some of our results are stated in more generality than needed for the sole purpose of proving our main result which is Theorem 3. An example is Theorem 4. The benefit reaped from this methodological choice is that we will be able to reuse Theorem 4, exactly as stated, and in combination with the techniques from [2], in order to construct the generalized processes or QFTs corresponding to the RG orbits connecting the Gaussian and infrared fixed points. As for the amount of detail, we note the following. Interest in QFT by mathematicians is high while the community of people with a working knowledge of constructive QFT and rigorous RG theory is very small. This gives us a strong incentive to write this article so it is understandable to a wider audience. Our presentation is essentially self-contained and only uses very modest prerequisites to be mentioned in the next section. We believe our article can also be read profitably by students in theoretical physics who would like to see, on a simple example, what is the precise relation between the RG dynamical system near a fixed point and the behaviour of the correlation functions. Such a reader may skip the sections containing estimates such as §6.3. Last but not least, our article concerns matters on which much was published that contained errors. We needed the high amount of detail to be absolutely sure that our proof is correct.

2. Preliminaries

2.1. Generalities about p-adics. Here we breifly review the basic notions about p-adics which are needed in this article. More details can be found in [71, 6, 41]. See also [78] for a quick introduction to the p-adics which includes very helpful pictures. Let p be a prime number and consider the p-adic absolute value $|\cdot|_p$ on $\mathbb Q$ defined by $|x|_p = 0$ if x = 0 and $|x|_p = p^{-k}$ if $x = \frac{a}{b} \times p^k$ where $a, k \in \mathbb Z$ and b, a positive integer, are such that a, b are coprime and neither are divisible by p. The field $\mathbb Q_p$ of p-adic numbers is the completion of $\mathbb Q$ with respect to this absolute value. Every x in $\mathbb Q_p$ has a unique convergent series representation

$$x = \sum_{n \in \mathbb{Z}} a_n p^n$$

where the digits a_n belong to $\{0, 1, \dots, p-1\}$ and at most finitely many of them are nonzero for negative n. The absolute value of $x \neq 0$ can be recovered from this representation as $|x|_p = p^{-v_p(x)}$ where

$$v_p(x) = \min\{n \in \mathbb{Z} \mid a_n \neq 0\} .$$

Using the same representation one can define the fractional (or polar) part of x which is $\{x\}_p = \sum_{n<0} a_n p^n$. The closed unit ball $\mathbb{Z}_p = \{x \in \mathbb{Q}_p | |x| \le 1\}$ is a compact subring of \mathbb{Q}_p . From now on we will drop the p subscript from the absolute value. The additive Haar measure on \mathbb{Q}_p normalized so that \mathbb{Z}_p has measure one will simply be denoted by dx. In d dimensions, the p-adic norm of a vector $x = (x_1, \ldots, x_d) \in \mathbb{Q}_p^d$ is defined as $|x| = \max\{|x_1|, \ldots, |x_d|\}$. The product measure d^dx obtained from the previous one-dimensional

measure is invariant by the subgroup $GL_d(\mathbb{Z}_p)$ of $GL_d(\mathbb{Q}_p)$. The subgroup $GL_d(\mathbb{Z}_p)$ is defined as the set of $d \times d$ matrices which together with their inverses have entries in \mathbb{Z}_p . This subgroup is the maximal compact subgroup of $GL_d(\mathbb{Q}_p)$ (unique up to conjugacy) and is the natural analogue of the orthogonal group O(d) acting on \mathbb{R}^d . The use of the maximum in the definition of the norm is motivated by the resulting invariance with respect to $GL_d(\mathbb{Z}_p)$.

The space of real (resp. complex) test functions $S(\mathbb{Q}_p^d, \mathbb{R})$ (resp. $S(\mathbb{Q}_p^d, \mathbb{C})$) is the Schwartz-Bruhat space of compactly supported locally constant real-valued (resp. complex-valued) functions on \mathbb{Q}_p^d . If we do not specify the target, then we mean \mathbb{R} . Recall that a seminorm on $S(\mathbb{Q}_p^d)$ is a function $\mathcal{N}: S(\mathbb{Q}_p^d) \to [0, \infty)$ which satisfies the usual norm axioms except the requirement that $\mathcal{N}(f) = 0$ implies f = 0. The coarsest topology on $S(\mathbb{Q}_p^d)$ which makes all possible seminorms continuous is called the finest locally convex topology and it is the one we use. The space of distributions $S'(\mathbb{Q}_p^d)$ simply is the topological dual of $S(\mathbb{Q}_p^d)$ which turns out to be the algebraic dual. Note that $S(\mathbb{Q}_p^d)$ is a nonmetrizable topological vector space. Therefore the theory of denumerably Hilbert nuclear spaces does not apply to it.

The Fourier transform of a complex valued test function f is defined by

$$\widehat{f}(k) = \int_{\mathbb{Q}_n^d} f(x) \exp(-2i\pi \{k \cdot x\}_p) \, d^d x$$

where $k \cdot x = k_1 x_1 + \cdots + k_d x_d$ and the rational $\{k \cdot x\}_p$ is seen as a real number. One has that the characteristic function of \mathbb{Z}_p^d is fixed by the Fourier transform, that is $\widehat{\mathbb{1}}_{\mathbb{Z}_p^d} = \mathbb{1}_{\mathbb{Z}_p^d}$. From this it easily follows that the space $S(\mathbb{Q}_p^d, \mathbb{C})$ is stable by Fourier transform. One can also define the Fourier transform of distributions by duality.

One has an analogue of the nuclear theorem in this setting which allows one to identify an n-linear form $W: S(\mathbb{Q}_p^d) \times \cdots \times S(\mathbb{Q}_p^d) \to \mathbb{R}$ with a distribution in $S'(\mathbb{Q}_p^{nd})$. We believe the most expedient way of proving such results as well as the ones in §2.2 is by following B. Simon's philosophy of exploiting a topological vector space isomorphism with a very concrete space of sequences (see [67] and [69, §1.2]). Indeed, it is easy to see that $S(\mathbb{Q}_p^d)$ can be written as a countable union of an increasing sequence of finite-dimensional vector spaces $V_1 \subset V_2 \subset \cdots$. One can construct a basis $(f_n)_{n \in \mathbb{N}}$ by taking a basis of V_1 , then appending vectors needed to complete it into a basis of V_2 , etc. If one takes $(e_n)_{n \in \mathbb{N}}$ to be the Gram-Schmidt orthonormalization for the L^2 inner product $\langle \bullet, \bullet \rangle$ then the map

$$f \longmapsto (\langle e_n, f \rangle)_{n \in \mathbb{N}}$$

realizes such an isomorphism from $S(\mathbb{Q}_p^d)$ to $s = \bigoplus_{n \in \mathbb{N}} \mathbb{R}$. Namely, s is the space of almost finite sequences $(x_n)_{n \geq 0}$ of real numbers, also equipped with the finest locally convex topology. The dual is the space $s' = \mathbb{R}^{\mathbb{N}}$ of all sequences $(y_n)_{n \geq 0}$ with duality pairing given by $\sum_{n \geq 0} x_n y_n$. In this new setting it is very easy to prove statements such as: a bilinear form on s is automatically continuous. This follows from the, somewhat counterintuitive, remark that if $(a_{i,j})_{(i,j) \in \mathbb{N}^2}$ is an array of nonnegative numbers, there exists a sequence $b_n \geq 0$ such that $a_{(i,j)} \leq b_i b_j$ for all i and j. The same property also holds for multilinear maps.

Next we need to define some transformations which will allow us to give a precise formulation for the notions of translation, rotation and scale invariance. If one views a point x in \mathbb{Q}_p^d as a column vector then one has a left-action of $GL_d(\mathbb{Z}_p)$ on points simply by matrix multiplication. It results in left-actions on test functions f, distributions ϕ and more generally n-linear forms W on $S(\mathbb{Q}_p^d)$, using

$$(M \cdot f)(x) = f(M^{-1}x) ,$$

$$(M \cdot \phi)(f) = \phi(M^{-1} \cdot f) ,$$

$$(M \cdot W)(f_1, \dots, f_n) = W(M^{-1} \cdot f_1, \dots, M^{-1} \cdot f_n) .$$

Such objects are called rotation invariant if they are preserved by all $M \in GL_d(\mathbb{Z}_p)$. If one formally thinks of a distribution ϕ as a 'function' via the L^2 pairing

$$\phi(f) = \int_{\mathbb{Q}_p^d} \phi(x) f(x) \, \mathrm{d}^d x$$

then the choice of definition means " $(M \cdot \phi)(x) = \phi(M^{-1}x)$ ". Thus, a distribution ϕ is rotation invariant if " $\phi(M^{-1}x) = \phi(x)$ " for all M and x.

Likewise regarding translations, one can define for $y \in \mathbb{Q}_p^d$ the transformations

$$\tau_y(x) = x + y ,$$

$$\tau_y(f)(x) = f(x - y) ,$$

$$\tau_y(\phi)(f) = \phi(\tau_{-y}(f)) ,$$

$$\tau_y(W)(f_1, \dots, f_n) = W(\tau_{-y}(f_1), \dots, \tau_{-y}(f_n)) .$$

One then defines the notion of invariance by translation for such objects in the same way as before.

We now consider scaling transformations. Given $\lambda \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$, we write:

$$(\lambda \cdot f)(x) = f(\lambda^{-1}x) ,$$

$$(\lambda \cdot \phi)(f) = |\lambda|^d \phi(\lambda^{-1} \cdot f) .$$

This corresponds to the formal equation " $(\lambda \cdot \phi)(x) = \phi(\lambda^{-1}x)$ ". A distribution ϕ is called partially scale invariant with homogeneity $\alpha \in \mathbb{R}$ with respect to a subgroup H of the full scaling group $p^{\mathbb{Z}} \subset \mathbb{Q}_p^*$ if $\lambda \cdot \phi = |\lambda|^{-\alpha} \phi$ for all $\lambda \in H$. This formally means " $|\lambda|^{\alpha} \phi(\lambda^{-1} x) = \phi(x)$ ".

2.2. Probability measures on the space of distributions. The generalized stochastic processes we will be interested in are probability measures on the space of distributions $S'(\mathbb{Q}_p^d)$. The σ -algebra is the cylindrical one \mathcal{C} which is the smallest that makes the maps $\phi \to \phi(f)$ measurable, for all test functions f. Such a probability measure ν is called rotation invariant if for any $M \in GL_d(\mathbb{Z}_p)$ the push-forward (or direct image) of ν by the map $\phi \mapsto M \cdot \phi$ is ν itself. Invariance by translation is defined in the same way. A probability measure on $S'(\mathbb{Q}_p^d)$ is called partially scale invariant with homogeneity α with respect to the subgroup H if the push-forward of ν by the map $\phi \mapsto |\lambda|^{\alpha} (\lambda \cdot \phi)$ is ν itself, for all $\lambda \in H$.

We now need two theorems which allow us to construct and identify probability measures from their characteristic functions or their moments. The first one is the analogue of the Bochner-Minlos Theorem (see [69, $\S1.2$]) in the p-adic setting. The second is a reconstructions theorem with n! bounds for the Hamburger Moment Problem on $S'(\mathbb{Q}_p^d)$.

Theorem 1. Let Φ be a function $S(\mathbb{Q}_p^d) \to \mathbb{C}$ which satisfies

- (1) $\Phi(0) = 1$,
- (2) Φ is continuous,
- (3) For all $n \geq 1$, all test functions f_1, \ldots, f_n in $S(\mathbb{Q}_p^d)$ and all complex numbers z_1, \ldots, z_n ,

$$\sum_{a,b=1}^{n} \bar{z}_a z_b \ \Phi(f_b - f_a) \in [0,\infty) ;$$

then there exists a unique probability measure ν on the measurable space $(S'(\mathbb{Q}_p^d),\mathcal{C})$ such that for all $f \in$ $S(\mathbb{Q}_n^d)$ we have

$$\Phi(f) = \int_{S'(\mathbb{Q}_p^d)} d\nu(\phi) \ e^{i\phi(f)} \ .$$

Theorem 2. Let $(S_n)_{n\geq 0}$ be a sequence of distributions with $S_n\in S'(\mathbb{Q}_p^{nd})$ which satisfies

- (1) $S_0 = 1$,
- (2) for any n, S_n is invariant by the permutation group \mathfrak{S}_n ,
- (3) for all almost finite sequence of test functions $(h_n)_{n\geq 0}$ with $h_n\in S(\mathbb{Q}_p^{nd},\mathbb{C})$ one has

$$\sum_{n,m\geq 0} S_{n+m}(\overline{h_n}\otimes h_m) \in [0,\infty) ,$$

(4) For all finite dimensional complex subspace V of $S(\mathbb{Q}_p^d,\mathbb{C})$ there exists a semi-norm \mathcal{N}_V on $S(\mathbb{Q}_p^d,\mathbb{C})$ such that for all $n \geq 0$ and all f_1, \ldots, f_n in V one has

$$|S_n(f_1 \otimes \cdots \otimes f_n)| \leq n! \times \mathcal{N}_V(f_1) \times \cdots \times \mathcal{N}_V(f_n) ;$$

then there exists a unique probability measure with finite moments ν on the measurable space $(S'(\mathbb{Q}_p^d), \mathcal{C})$ such that for all $f_1, \ldots, f_n \in S(\mathbb{Q}_p^d, \mathbb{C})$ we have

$$S_n(f_1 \otimes \cdots \otimes f_n) = \int_{S'(\mathbb{Q}_p^d)} d\nu(\phi) \ \phi(f_1) \cdots \phi(f_n) \ .$$

Note that we used in the statement of the second theorem some obvious functorial properties of the nuclear theorem with respect to complexification. We do not give the proofs of these two theorems. We simply note that, via the isomorphism with s and s' and Kolmogorov's Extension Theorem, they reduce to their finite dimensional versions which are classical results in analysis. Finally, we will use some basic formulas for Gaussian integration and manipulation of Wick monomials. These are left as easy exercises for the reader. This material is also covered in [39, Ch. 9], [68, §I.1], [64, §2.2].

2.3. Banach space analyticity. A tool which we use a lot in this article is the theory of analytic maps in the complex Banach space context. See for instance [19] for an introduction or reminder. We also need to consider the analyticity of maps on $S(\mathbb{Q}_p^d,\mathbb{C})$. However when we do so we always restrict to a finite-dimensional subspace so we only need the usual notion of analyticity on \mathbb{C}^n . This allows us to avoid using the more involved theory of analyticity in locally convex spaces [25]. We only state here a lemma that is used many times in this article and which allows us to get Lipshitz estimates in an effortless manner. We use the notation $B(x_0, r)$ for the open ball of radius r centered at x_0 . We likewise use $\bar{B}(x_0, r)$ to denote the corresponding closed ball.

Lemma 1. Let X and Y be two complex Banach spaces. Suppose $r_1 > 0$ and $r_2 \ge 0$. Let $x_0 \in X$ and $y_0 \in Y$, and let f be an analytic map

$$f: B(x_0, r_1) \longrightarrow \bar{B}(y_0, r_2)$$
.

Let $\nu \in (0, \frac{1}{2})$, then for any $x_1, x_2 \in \bar{B}(x_0, \nu r_1)$

$$||f(x_1) - f(x_2)|| \le \frac{r_2(1-\nu)}{r_1(1-2\nu)} ||x_1 - x_2||.$$

Proof: Suppose $x_1 \neq x_2$ satisfy the hypothesis of the proposition. For $z \in \mathbb{C}$ define

$$g(z) = f\left(\frac{x_1 + x_2}{2} + z\frac{x_1 - x_2}{2}\right) - y_0.$$

We first find a bound on |z| which garantees that the argument of f is in the ball $B(x_0, r_1)$. Since $\nu < \frac{1}{2}$, we have

$$2r_1(1-\nu) > 2\nu r_1 \ge ||x_1 - x_0|| + ||x_2 - x_0|| \ge ||x_1 - x_2||.$$

Therefore

$$R_{\text{max}} = \frac{2r_1(1-\nu)}{||x_1 - x_2||} > 1 \ .$$

Now the open interval $(1, R_{\text{max}})$ is nonempty, and for any R in this interval as well as for any z with $|z| \leq R$ we have

$$\left\| \frac{x_1 + x_2}{2} + z \frac{x_1 - x_2}{2} - x_0 \right\| \le \nu r_1 + \frac{R}{2} \|x_1 - x_2\| < r_1.$$

Let γ be the circle of radius R around the origin in the complex plane. For such an $R \in (1, R_{\text{max}})$ we have by Cauchy's Theorem

$$f(x_1) - f(x_2) = g(1) - g(-1) = \frac{1}{\pi i} \oint_{\alpha} \frac{g(z)}{z^2 - 1} dz$$
.

Hence

$$||f(x_1) - f(x_2)|| \le \frac{1}{\pi} \times 2\pi Rr_2 \times \max_{|z|=R} \frac{1}{|z^2 - 1|} = \frac{2Rr_2}{R^2 - 1}.$$

We now minimize this bound with respect to $R \in (1, R_{\text{max}})$. Since $R \mapsto \frac{2R}{R^2 - 1}$ is decreasing on $(1, \infty)$,

$$\inf_{R \in (1,R_{\text{max}})} \; \frac{2R}{R^2 - 1} = \frac{2R_{\text{max}}}{R_{\text{max}}^2 - 1} \; .$$

Inserting the formula for R_{max} in the upper bound for $||f(x_1)-f(x_2)||$ and simplifying the resulting expression gives the desired Lipschitz estimate.

3. Formal statement of the results

Now let us pick d=3 and for $0 < \epsilon < 1$ let us denote the quantity $\frac{3-\epsilon}{4}$ by the symbol $[\phi]$. Let $L=p^l$ for some integer $l \geq 1$. For $r \in \mathbb{Z}$ (typically negative), we consider the bilinear form on $S(\mathbb{Q}_p^3)$ given by

$$C_r(f,g) = \int_{\mathbb{Q}_n^3} \frac{\widehat{f}(-k)\widehat{g}(k)1\{|k| \le L^{-r}\}}{|k|^{3-2[\phi]}} d^3k$$

where we use $\mathbb{1}\{\cdots\}$ for the characteristic function of the condition between braces. By Theorem 1, there is a unique probability measure μ_{C_r} on $S'(\mathbb{Q}_p^3)$ such that for any $f \in S(\mathbb{Q}_p^3)$

$$\left\langle e^{i\phi(f)}\right\rangle_{\mu_{C_r}} = \exp\left(-\frac{1}{2}C_r(f,f)\right)$$

where we used the statistical mechanics notation for the expectation with respect to ϕ sampled according to the measure μ_{C_r} . Note that one can write, with a slight abuse of notation

$$C_r(f,g) = \int_{\mathbb{Q}_p^{3\times 2}} C_r(x-y)f(x)g(y) d^3x d^3y$$

where the function C_r is explicitly given by

$$C_r(x) = \sum_{n=l_r}^{\infty} p^{-2n[\phi]} \left[\mathbb{1}_{\mathbb{Z}_p^3}(p^n x) - p^{-3} \mathbb{1}_{\mathbb{Z}_p^3}(p^{n+1} x) \right] .$$

The measure μ_{C_r} is supported on distributions given by bonafide functions which are locally constant at scale L^r , namely, constant on each coset in $\mathbb{Q}_p^3/(L^{-r}\mathbb{Z}_p)^3$, the latter quotient playing the role of the lattice of mesh L^r . Note that since $|p|=p^{-1}$ where the p on the left wears its p-adic hat while the one on the right is viewed as a real number, the volume of $(L^{-r}\mathbb{Z}_p)^3$ is L^{3r} in accordance with the intuitive image of a three-dimensional box with linear dimension L^r . For $s \in \mathbb{Z}$ (typically positive), we use the notation $\Lambda_s = \{x \in \mathbb{Q}_p^3 | |x| \le L^s\}$ and we also define the Wick powers

$$: \phi^2 :_{C_r} (x) = \phi(x)^2 - C_r(0) ,$$

:
$$\phi^4$$
:_{C_r} $(x) = \phi(x)^4 - 6 C_r(0) \phi(x)^2 + 3 C_r(0)^2$

and, given g > 0 as well as $\mu \in \mathbb{R}$, the potential

$$\tilde{V}_{r,s}(\phi) = \int_{\Lambda_s} \left\{ L^{-(3-4[\phi])r} \ g : \phi^4 :_{C_r} (x) + L^{-(3-2[\phi])r} \ \mu : \phi^2 :_{C_r} (x) \right\} \ \mathrm{d}^3 x \ .$$

By the previous remarks, the measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-\tilde{V}_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

is a well defined probability measure on $S'(\mathbb{Q}_p^3)$ with finite moments. The normalization factor $\mathcal{Z}_{r,s}$ is at least equal to one as can be seen from Jensen's inequality. We will denote expectations with respect to $\nu_{r,s}$ by $\langle \cdots \rangle_{r,s}$. Finally, given a locally constant ϕ at scale L^r , we define an element $N_r[\phi^2]$ of $S'(\mathbb{Q}_p^3)$ by letting it act on $j \in S(\mathbb{Q}_p^3)$ via

$$N_r[\phi^2](j) = Z_2^r \int_{\mathbb{O}^3} \left(Y_2 : \phi^2 :_{C_r} (x) - Y_0 L^{-2r[\phi]} \right) j(x) d^3x$$

where Z_2, Y_0, Y_2 are parameters used in the construction.

We will also use the notation

$$\bar{g}_* = \frac{(p^{\epsilon} - 1)}{36 L^{\epsilon} (1 - p^{-3})} .$$

The main result of this article is the following theorem.

Theorem 3.

 $\exists \rho > 0, \ \exists L_0, \ \forall L \geq L_0, \ \exists \epsilon_0 > 0, \ \forall \epsilon \in (0, \epsilon_0], \ one \ can \ find \ \eta_{\phi^2} > 0 \ and \ functions \ \mu(g), Y_0(g), Y_2(g) \ of \ g \ in the interval (\bar{g}_* - \rho \epsilon^{\frac{3}{2}}, \bar{g}_* + \rho \epsilon^{\frac{3}{2}}), \ such that if one sets \ \mu = \mu(g), \ Z_2 = L^{-\frac{1}{2}\eta_{\phi^2}}, \ Y_0 = Y_0(g) \ and \ Y_2 = Y_2(g)$ in the previous definitions, then for all collections of test functions $f_1, \ldots, f_n, j_1, \ldots, j_m$, the limits

$$\lim_{\substack{r \to -\infty \\ s \to \infty}} \langle \phi(f_1) \cdots \phi(f_n) N_r[\phi^2](j_1) \cdots N_r[\phi^2](j_m) \rangle_{r,s}$$

exist and do not depend on the order in which the $r \to -\infty$ and $s \to \infty$ limits are taken. Moreover, the resulting quantities or correlators henceforth similarly and formally denoted by dropping the r and s subscripts (and using squares, 4-th powers, etc., for repeats) satisfy the following properties:

- 1) They are invariant by translation and rotation.
- 2) They satisfy the partial scale invariance property

$$\langle \phi(\lambda \cdot f_1) \cdots \phi(\lambda \cdot f_n) \ N[\phi^2](\lambda \cdot j_1) \cdots N[\phi^2](\lambda \cdot j_m) \rangle =$$

$$|\lambda|^{(3-[\phi])n+(3-2[\phi]-\frac{1}{2}\eta_{\phi^2})m} \langle \phi(f_1) \cdots \phi(f_n) \ N[\phi^2](j_1) \cdots N[\phi^2](j_m) \rangle$$

for all $\lambda \in L^{\mathbb{Z}}$.

3) They satisfy the nontriviality conditions

$$\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^4 \rangle - 3\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle < 0 ,$$
$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_2^3})^2 \rangle = 1 .$$

- 4) The pure ϕ correlators are the moments of a unique probability measure ν_{ϕ} on $S'(\mathbb{Q}_p^2)$ with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with homogeneity $-[\phi]$ with respect to the scaling subgroup $L^{\mathbb{Z}}$.
- 5) The pure $N[\phi^2]$ correlators are the moments of a unique probability measure ν_{ϕ^2} on $S'(\mathbb{Q}_p^2)$ with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with homogeneity $-2[\phi] \frac{1}{2}\eta_{\phi^2}$ with respect to the scaling subgroup $L^{\mathbb{Z}}$.
- 6) The measures ν_{ϕ} and ν_{ϕ^2} satisfy a mild form of universality: they do not depend on g in the above-mentioned interval.

4. Definition of the extended RG

4.1. **Functional spaces.** In this section we will introduce the different spaces on which the RG transformation will act. The basic space we will use is $C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$, namely, the space of nine times continuously differentiable functions from \mathbb{R} to \mathbb{C} which, together with their derivatives up to order nine, are bounded. On this space we will use seminorms $||\cdot||_{\partial\phi,\psi,\theta}$ defined for $K \in C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ by

$$||K(\phi)||_{\partial\phi,\psi,\theta} = \sum_{j=0}^{9} \frac{\theta^{j}}{j!} \left| \frac{\mathrm{d}^{j}K}{\mathrm{d}\phi^{j}}(\psi) \right| .$$

Here $\partial \phi$ is merely a symbol which indicates the variable with respect to which the derivatives are taken. This will be especially useful when the function may depend on several such variables. By contrast, ψ is an argument of the seminorm. The derivatives are evaluated at $\phi = \psi$ and therefore the result depends on ψ . Finally $\theta \in [0, \infty)$ is a parameter used to properly calibrate this seminorm. We will mainly use two values for this parameter denoted by h and h_* to be specified later. As an example of use of the previous notation, we have $||\phi^2||_{\partial \phi, \psi, \theta} = |\psi|^2 + 2\theta|\psi| + \theta^2$. In the important special case where $\psi = 0$, we will abbreviate the notation into

$$|K(\phi)|_{\partial\phi,\theta} = |K(\phi)|_{\partial\phi,0,\theta}$$
.

A nice property of these seminorms is multiplicativity. Indeed for any two functions K_1 , K_2 in $C^9_{\rm bd}(\mathbb{R},\mathbb{C})$ we have

$$||K_1(\phi)K_2(\phi)||_{\partial\phi,\psi,\theta} \le ||K_1(\phi)||_{\partial\phi,\psi,\theta} \times ||K_2(\phi)||_{\partial\phi,\psi,\theta}$$

which is an easy consequence of the Leibniz rule and the choice of $\frac{1}{i!}$ weights.

To a parameter $\bar{g} > 0$ called a calibrator we associate a norm $||| \cdot |||_{\bar{g}}$ on the complex Banach space $C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ defined by

$$|||K|||_{\bar{g}} = \max \left\{ |K(\phi)|_{\partial \phi, h_*}, \bar{g}^2 \sup_{\phi \in \mathbb{R}} ||K(\phi)||_{\partial \phi, \phi, h} \right\} .$$

The parameter h > 0 will be a function of \bar{g} while $h_* > 0$ only depends on L and ϵ . We also introduce the notation $C^9_{\mathrm{bd,ev}}(\mathbb{R},\mathbb{C})$ for the closed subspace of $C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ made of functions K with the even symmetry $K(-\phi) = K(\phi)$.

We use the notation $\mathbb{L}_q = \mathbb{Q}_p^3/(L^{-q}\mathbb{Z}_p)^3$ for the lattice with mesh L^q . The unit lattice \mathbb{L}_0 will simply be denoted by \mathbb{L} . We will typically denote an element of the latter by Δ . We will call such an element a unit cube, a unit block or simply a box. Elements of \mathbb{L}_1 will be called L-blocks. These are all of the form $L^{-1}\Delta$ for some $\Delta \in \mathbb{L}$. We will denote by $[L^{-1}\Delta]$ the set of unit blocks contained in the L-block $L^{-1}\Delta$. For $x \in \mathbb{Q}_p^3$ we denote by $\Delta(x)$ the unique box in \mathbb{L} which contains x.

We will need to work with complex valued test functions on \mathbb{Q}_p^3 belonging to suitable finite-dimensional subspaces of $S(\mathbb{Q}_p^3,\mathbb{C})$. For any $q_-,q_+\in\mathbb{Z}$ such that $q_-\leq q_+$ we define the space $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$ of test functions with support in Λ_{q_+} and which are locally constant at scale L^{q_-} . Namely, a complex-valued test function f belongs to $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$ if and only if

$$\forall x \in \mathbb{Q}_p^3, \ f(x) \neq 0 \implies |x| \le L^{q_+}$$

and

$$\forall x, y \in \mathbb{Q}_p^3, |x - y| \le L^{q_-} \implies f(x) = f(y)$$
.

By a trivial compacity argument, it is clear that $S(\mathbb{Q}_p^3,\mathbb{C})$ is the union of all the subspaces $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$. The latter have complex dimension $L^{3(q_+-q_-)}$.

The most general RG transformation considered in this article is denoted by $RG_{\rm ex}$ and is called the extended RG map. It will be defined as a transformation $\vec{V} \mapsto \vec{V}'$ on the Banach space $\mathcal{E}_{\rm ex}$ defined as follows. An element of that space is an indexed family

$$\vec{V} = (V_{\Delta})_{\Delta \in \mathbb{L}}$$

where

$$V_{\Delta} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta}) \in \mathbb{C}^7 \times C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C}) \ .$$

Given suitable positive exponents $e_1, e_2, e_3, e_4, e_W, e_R$ we define the norm

$$||V_{\Delta}|| = \max \left\{ |\beta_{4,\Delta}|\bar{g}^{-e_4}, |\beta_{3,\Delta}|\bar{g}^{-e_3}, |\beta_{2,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_{1,\Delta}|\bar{g}^{-e_2}, |\beta_{1,\Delta}|\bar{g}^{-e_1}, |\beta_$$

$$|W_{5,\Delta}|\bar{g}^{-e_W}, |W_{6,\Delta}|\bar{g}^{-e_W}, |f_{\Delta}|L^{(3-[\phi])}, |||R_{\Delta}|||_{\bar{g}} \bar{g}^{-e_R}$$

for the components living in a box Δ , and

$$||\vec{V}|| = \sup_{\Delta \in \mathbb{L}} ||V_{\Delta}||$$

for the whole infinite vector. Now $\mathcal{E}_{\mathrm{ex}}$ is by definition the Banach space

$$\mathcal{E}_{\mathrm{ex}} = \left\{ \vec{V} \in \prod_{\Delta \in \mathbb{L}} \left(\mathbb{C}^7 \times C_{\mathrm{bd}}^9(\mathbb{R}, \mathbb{C}) \right) \mid ||\vec{V}|| < \infty \right\} .$$

We let \mathcal{E}_{bk} denote the closed subspace of \mathcal{E}_{ex} made of vectors $\vec{V} = (V_{\Delta})_{\Delta \in \mathbb{L}}$ such that V_{Δ} is constant with respect to Δ . This corresponds to uniform potentials which are relevent for the bulk RG evolution. We have a canonical isometric identification between \mathcal{E}_{bk} and the one-box space $\mathcal{E}_{1B} = \mathbb{C}^7 \times C_{bd}^9(\mathbb{R}, \mathbb{C})$. By definition $\mathcal{E} = \mathbb{C}^2 \times C_{bd,ev}^9(\mathbb{R}, \mathbb{C})$ and consists of elements of the form (g, μ, R) , with R even, to which one canonically associates

$$V = (g, 0, \mu, 0, 0, 0, 0, R)$$

in \mathcal{E}_{1B} . Finally we define \mathcal{E}_{pt} to be the closed subspace of \mathcal{E}_{ex} given by vectors $(V_{\Delta})_{\Delta \in \mathbb{L}}$ such that $V_{\Delta} = 0$ if $\Delta \neq \Delta(0)$. This corresponds to point-like perturbations living at the origin. It is easy to see that \mathcal{E}_{bk} and \mathcal{E}_{pt} are in direct sum inside \mathcal{E}_{ex} . We will later see the that $\mathcal{E}_{bk} \oplus \mathcal{E}_{pt}$ is stable by the extended RG.

4.2. Algebraic definition of RG_{ex} . We now proceed with the presentation of the formulas which express the map $\vec{V} \mapsto \vec{V}' = RG_{\text{ex}}[\vec{V}]$. We will also define a collection $\delta b[\vec{V}] = (\delta b_{\Delta}[\vec{V}])_{\Delta \in \mathbb{L}} \in \mathbb{C}^{\mathbb{L}}$ of field independent quantitites also called vacuum contributions.

We will need the following notations for scaling shifts. For a field ϕ living on \mathbb{Q}_p^3 and for any $q \in \mathbb{Z}$ we let

$$\phi_{\leadsto q}(x) = L^{-[\phi]q}\phi(L^q x) .$$

For a test function f which is typically paired with a ϕ field we write

$$f_{\to q}(x) = L^{-(3-[\phi])q} f(L^q x)$$
.

Finally for a test function j which is typically paired with a ϕ^2 field we write

$$j_{\Rightarrow q}(x) = L^{-(3-2[\phi])q} j(L^q x)$$
.

At the beginning of the RG analysis one needs to do a rescaling which transforms the ultraviolet scale L^r into $L^0=1$, i.e., unit scale. Quantities resulting from this initial rescaling will typically be denoted without tildes whereas the original or native quantities will usually be denoted by tildes. Given g and μ as in §3 we use the notation

$$\tilde{g}_r = L^{-(3-4[\phi])r} g$$
 and $\tilde{\mu}_r = L^{-(3-2[\phi])r} \mu$.

Then for a field $\tilde{\phi}$ which is locally constant at scale L^r one has

$$\tilde{V}_{r,s}(\tilde{\phi}) = \int_{\Lambda_s} d^3x \left[\tilde{g}_r : \tilde{\phi}^4(x) :_{C_r} + \tilde{\mu}_r : \tilde{\phi}^2(x) :_{C_r} \right]$$

according to the definition in §3.

For test functions \tilde{f} and \tilde{j} in $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$, our main quantity of interest will be

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \int_{S'(\mathbb{Q}_p^3)} d\mu_{C_r}(\tilde{\phi}) \exp\left(-\tilde{V}_{r,s}(\tilde{\phi}) + \tilde{\phi}(\tilde{f}) + Y_2 Z_2^r : \tilde{\phi}^2 :_{C_r} (\tilde{j}) - Y_0 Z_0^r \int_{\mathbb{Q}_p^3} \tilde{j}(x) d^3x\right)$$

where we used the pairing

$$: \tilde{\phi}^2 :_{C_r} (\tilde{j}) = \int_{\mathbb{Q}_2^3} : \tilde{\phi}^2 :_{C_r} (x) \tilde{j}(x) d^3x$$

and where $Z_2, Z_0 > 0$ and $Y_2, Y_0 \in \mathbb{R}$ are yet to be defined. Indeed, one can obtain the correlators

$$\left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) N_r [\tilde{\phi}^2](\tilde{j}_1) \cdots N_r [\tilde{\phi}^2](\tilde{j}_m) \right\rangle_{r,s} = \int_{S'(\mathbb{Q}_p^3)} d\nu_{r,s}(\tilde{\phi}) \ \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) N_r [\tilde{\phi}^2](\tilde{j}_1) \cdots N_r [\tilde{\phi}^2](\tilde{j}_m)$$

as multiple derivatives at $\tilde{f} = \tilde{j} = 0$ of the moment generating function

$$S_{r,s}(\tilde{f},\tilde{j}) = \frac{Z_{r,s}(\tilde{f},\tilde{j})}{Z_{r,s}(0,0)}$$
.

Note that if $\tilde{\phi}$ is distributed according to $d\mu_{C_r}$ then if we define ϕ so that $\tilde{\phi} = \phi_{\leadsto r}$ then we have that ϕ is distributed according to $d\mu_{C_0}$. We also have

$$\tilde{V}_{r,s}(\phi_{\leadsto r}) = \int_{\Lambda_{s,n}} d^3x \left[L^{(3-4[\phi])r} \tilde{g}_r : \phi^4(x) :_{C_0} + L^{(3-2[\phi])r} \tilde{\mu}_r : \phi^2(x) :_{C_0} \right] .$$

Therefore by a simple change of variables from $\tilde{\phi}$ to ϕ we have

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r} \int_{\mathbb{Q}_p^3} j(x) \, d^3x\right)$$

$$\times \int_{S'(\mathbb{Q}_{2}^{3})} d\mu_{C_{0}}(\phi) \exp\left(-V_{r,s}(\phi) + \phi(f) + Y_{2}Z_{2}^{r} : \phi^{2} :_{C_{0}}(j)\right)$$

where $f = \tilde{f}_{\rightarrow -r}$ and $j = \tilde{j}_{\Rightarrow -r}$ and

$$V_{r,s}(\phi) = \int_{\Lambda_{s-r}} d^3x \left[g : \phi^4(x) :_{C_0} + \mu : \phi^2(x) :_{C_0} \right].$$

When $r \leq q_{-} \leq q_{+} \leq s$, the latter functional integral can be written in the form

$$\int_{S'(\mathbb{Q}_p^3)} d\mu_{C_0}(\phi) \, \mathcal{I}_{s-r}[\vec{V}](\phi)$$

for a suitable vector \vec{V} in \mathcal{E}_{ex} . Here the integrand associated to such a vector is defined as follows. Note that such a vector can be written in a compact way as

$$\vec{V} = (\beta_4, ..., \beta_1, W_5, W_6, f, R)$$

where each entry in \vec{V} is an indexed collection of unit box dependent quantities, for example:

$$\beta_4 = (\beta_{4,\Delta})_{\Delta \in \mathbb{L}}$$
.

Note that we will also make the natural identification between functions on \mathbb{Q}_p^3 which are constant over unit blocks and \mathbb{L} -indexed vectors. For example if $f \in S(\mathbb{Q}_p^3, \mathbb{C})$ is constant over unit blocks then we can just as well think of f as the collection $(f_{\Delta})_{\Delta \in \mathbb{L}}$ where $f_{\Delta(x)} = f(x)$ for all $x \in \mathbb{Q}_p^3$. We will also use this device for the β 's and W's.

The correspondence between \vec{V} 's and integrands is given for any integer $t \geq 0$ as follows:

$$\mathcal{I}_t[\vec{V}](\phi) = \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_t}} \mathcal{I}_{\Delta}[\vec{V}](\phi)$$

with

$$\mathcal{I}_{\Delta}[\vec{V}](\phi) = e^{f_{\Delta}\phi_{\Delta}} \times \left\{ \exp\left[-\beta_{4,\Delta} : \phi_{\Delta}^4 :_{C_0} - \beta_{3,\Delta} : \phi_{\Delta}^3 :_{C_0} - \beta_{2,\Delta} : \phi_{\Delta}^2 :_{C_0} - \beta_{1,\Delta} : \phi_{\Delta} :_{C_0} \right] \times \left(1 + W_{5,\Delta} : \phi_{\Delta}^5 :_{C_0} + W_{6,\Delta} : \phi_{\Delta}^6 :_{C_0} \right) + R_{\Delta}(\phi_{\Delta}) \right\}.$$

The RG evolution of \vec{V} and the definition of the field independent quantities δb will gives us, for $t \geq 1$, the following identity:

$$\int_{S'(\mathbb{Q}_p^3)} d\mu_{C_0}(\phi) \,\, \mathcal{I}_t[\vec{V}](\phi) = \exp \left[\frac{1}{2} (f, \Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{t-1}}} \delta b_{\Delta}[\vec{V}] \right] \times \int_{S'(\mathbb{Q}_p^3)} d\mu_{C_0}(\phi) \,\, \mathcal{I}_{t-1} \left[RG_{\mathrm{ex}}[\vec{V}] \right] (\phi)$$

where we used the notation

$$(f, \Gamma f)_X = \int_{X^2} d^3x d^3y f(x) \Gamma(x-y) f(y)$$

for any measurable subset X of \mathbb{Q}_n^3 .

Each choice of r, \tilde{f}, \tilde{j} gives us a sequence of vectors $(\vec{V}^{(r,q)}(\tilde{f}, \tilde{j}))_{q \in \mathbb{Z}, r \leq q \leq s}$. The first such \vec{V} in the sequence is given by:

$$\vec{V}^{(r,r)}(\tilde{f},\tilde{j}) = (\beta_4, \beta_3, \beta_2, \beta_1, W_5, W_6, f, R)$$

where

$$\begin{array}{rcl} \beta_3 & = & 0 \\ \beta_1 & = & 0 \\ W_5 & = & 0 \\ W_6 & = & 0 \\ R & = & 0 \\ f & = & \tilde{f}_{\rightarrow (-r)} \\ \beta_4(x) & = & g \text{ for all } x \\ \beta_2(x) & = & \mu - Y_2 Z_2^r \ L^{(3-2[\phi])^r} \ \tilde{j}(L^{-r}x) \text{ for all } x \ . \end{array}$$

With these definitions we have that:

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r} \int_{\mathbb{Q}_p^3} j(x) \, \mathrm{d}^3 x\right) \times \int \mathrm{d}\mu_{C_0}(\phi) \, \mathcal{I}_{s-r}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})](\phi)$$

where from now on we will drop the domain $S'(\mathbb{Q}_p^3)$ of the functional integrals. We define $\vec{V}^{(r,q)}(\tilde{f},\tilde{j})$ for $r < q \le s$ by iterating the extended RG transformation q - r times, namely:

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = RG_{\text{ex}}^{q-r}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})].$$

Iterating the RG transformation gives us the following equation for $\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})$:

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r} \int_{\mathbb{Q}_p^3} j(x) \, d^3x\right)$$

$$\times \exp\left[\sum_{r \leq q \leq t} \left(\frac{1}{2} (f^{(r,q)}, \Gamma f^{(r,q)})_{\Lambda_{s-q}} + \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \delta b_{\Delta} [\vec{V}^{r,q}(\tilde{f},\tilde{j})]\right)\right] \times \int d\mu_{C_0}(\phi) \, \mathcal{I}_{s-t-1}[\vec{V}^{(r,t+1)}(\tilde{f},\tilde{j})](\phi)$$

which holds for every scale t with $r \leq t < s$. The notation $f^{(r,q)}$ stands for the f component of $\vec{V}^{(r,q)}(\tilde{f},\tilde{j})$. We stop the RG iterations at t = s - 1, i.e., when the finite volume Λ becomes a unit box. This gives

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r} \int_{\mathbb{Q}_p^3} j(x) \, d^3x\right) \times$$

$$\exp\left[\sum_{r \le q \le s-1} \left(\frac{1}{2} (f^{(r,q)}, \Gamma f^{(r,q)})_{\Lambda_{s-q}} + \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \delta b_{\Delta}[\vec{V}^{r,q}(\tilde{f})]\right)\right] \times \partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j})$$

where the boundary factor is:

$$\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \int d\mu_{C_0}(\phi) \, \mathcal{I}_0[\vec{V}^{(r,s)}(\tilde{f},\tilde{j})](\phi)$$

which reduces to an integral over a single real variable.

The full RG transformation $\vec{V} \rightarrow \vec{V}' = RG_{\rm ex}[\vec{V}]$ will be defined by specifying

$$\vec{V}' = (\beta_4', ..., \beta_1', W_5', W_6', f', R')$$

starting from the analogous unprimed quantities. We will also define the corresponding $\delta b = \delta b[\vec{V}]$ at the same time.

The easiest terms are given by:

(4)
$$f'_{\Delta'} = L^{3-[\phi]} \underset{\Delta \in [L^{-1}\Delta']}{\text{avg}} f_{\Delta}$$

where "avg" means the average.

We also have

$$W'_{6,\Delta'} = L^{3-6[\phi]} \underset{\Delta \in [L^{-1}\Delta']}{\text{avg}} W_{6,\Delta} + 8L^{-6[\phi]} \underset{\beta_4}{\bullet \bullet \bullet} \beta_4$$

and

$$W_{5,\Delta'}' = L^{3-5[\phi]} \underset{\Delta \in [L^{-1}\Delta']}{\operatorname{avg}} W_{5,\Delta} + 6L^{-5[\phi]} \bigvee_{W_6}^{f} + 12L^{-5[\phi]} \underset{\beta_4}{\underbrace{\hspace{1cm}}} + 48L^{-5[\phi]} \underset{\beta_4}{\underbrace{\hspace{1cm}}} f$$

The Feynman diagrams are given explicitly by

$$\oint_{\beta_4} = \oint_{(L^{-1}\Delta')^2} d^3x \ d^3y \ \beta_4(x) \ \Gamma(x-y) \ \beta_4(y)$$

$$\oint_{W_6} f = \oint_{(L^{-1}\Delta')^2} d^3x \ d^3y \ W_6(x) \ \Gamma(x-y) \ f(y)$$

and

$$\int_{\beta_4}^{f} \int_{\beta_4}^{f} = \int_{(L^{-1}\Delta')^3} d^3x d^3y d^3z \beta_4(x) \Gamma(x-y) \beta_4(y) \Gamma(y-z) f(z)$$

where we again used the correspondence between vectors indexed by unit cubes and functions on \mathbb{Q}_p^3 , i.e., we used $\beta_4(x) = \beta_{4,\Delta(x)}$, etc.

We will need the following intermediate quantities

$$\hat{\beta}_{k,\Delta'} = L^{3-k[\phi]} \underset{\Delta \in [L^{-1}\Delta']}{\text{avg}} \beta_{k,\Delta}$$

for $1 \le k \le 4$.

We will also write

$$V_{\Delta}(\phi) = \sum_{k=1}^{4} \beta_{k,\Delta} : \phi^{k} :_{C_{0}}$$
$$Q_{\Delta}(\phi) = W_{5,\Delta} : \phi^{5} :_{C_{0}} + W_{6,\Delta} : \phi^{6} :_{C_{0}}$$

and

$$K_{\Delta}(\phi) = Q_{\Delta}(\phi)e^{-V_{\Delta}(\phi)} + R_{\Delta}(\phi)$$
.

These are functions of a single variable $\phi = \phi_{\Delta}$. To lighten the notations we drop the reference to Δ for the field ϕ , when this causes no ambiguity. We have, using the decomposition of Gaussian measures

$$\int d\mu_{C_0}(\phi) \, \mathcal{I}_t[\vec{V}](\phi) = \int d\mu_{C_0}(\phi) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_t}} \left\{ e^{f_{\Delta}\phi_{\Delta}} \times \left[e^{-V_{\Delta}(\phi_{\Delta})} + K_{\Delta}(\phi_{\Delta}) \right] \right\}
= \int d\mu_{C_0}(\phi) \int d\mu_{\Gamma}(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_t}} \left\{ e^{f_{\Delta}\phi_{1,\Delta} + f_{\Delta}\zeta_{\Delta}} \times \left[e^{-V_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta})} + K_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta}) \right] \right\}$$

where $\phi_1 = \phi_{\leadsto 1}$.

We then organize the product according to the L-blocks containing Δ and use the independence of the ζ random variables living in different L-blocks to obtain

$$\int d\mu_{C_0}(\phi) \, \mathcal{I}_t[\vec{V}](\phi) = \int d\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{t-1}}} \left(\int d\mu_{\Gamma}(\zeta) \right)
= \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \in [L^{-1}\Delta']}} \left\{ e^{f_{\Delta}\phi_{1,\Delta} + f_{\Delta}\zeta_{\Delta}} \times \left[e^{-V_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta})} + K_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta}) \right] \right\} \right)
= \int d\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{t-1}}} \left(e^{f'_{\Delta'}\phi_{\Delta'}} \times \mathcal{B}_{\Delta'} \right)$$

where

$$\mathcal{B}_{\Delta'} = \int \mathrm{d}\mu_{\Gamma}(\zeta) \prod_{\Delta \in [L^{-1}\Delta']} \left\{ e^{f_{\Delta}\zeta_{\Delta}} \times \left[e^{-V_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta})} + K_{\Delta}(\phi_{1,\Delta} + \zeta_{\Delta}) \right] \right\} .$$

With a slight abuse of notation we define

$$\tilde{V}_{\Delta}(\phi_1) = \sum_{k=1}^{4} \beta_{k,\Delta} : \phi_1^k :_{C_1}$$

We also let

$$\hat{V}_{\Delta'}(\phi) = \sum_{k=1}^{4} \hat{\beta}_{k,\Delta'} : \phi^k :_{C_0}$$
.

Note that $\sum_{\Delta \in [L^{-1}\Delta']} \tilde{V}_{\Delta}(\phi_1) = \hat{V}_{\Delta'}(\phi)$ where ϕ is in fact the component $\phi_{\Delta'}$ of the field but we suppressed this from the notation. Now define

$$p_{\Delta} = p_{\Delta}(\phi_1, \zeta) = V_{\Delta}(\phi_1 + \zeta) - \tilde{V}_{\Delta}(\phi_1)$$

namely

$$p_{\Delta} = \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} a+b \leq 4 \\ a \geq 0 , b \geq 1 \end{array} \right\} \frac{(a+b)!}{a! \ b!} \ \beta_{a+b,\Delta} : \phi_1^a :_{C_1} \times : \zeta^b :_{\Gamma} .$$

Now let

$$P_{\Delta}(\phi_1,\zeta) = e^{-V_{\Delta}(\phi_1+\zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} .$$

We expand $\mathcal{B}_{\Delta'}$ by writing the factors as

$$e^{-V_{\Delta}(\phi_{1,\Delta}+\zeta_{\Delta})} + K_{\Delta}(\phi_{1,\Delta}+\zeta_{\Delta}) = e^{-\tilde{V}_{\Delta}(\phi_{1})} + P_{\Delta}(\phi_{1},\zeta) + K_{\Delta}(\phi_{1,\Delta}+\zeta_{\Delta}) .$$

This results in

$$\mathcal{B}_{\Delta'} = e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} - \hat{V}_{\Delta'}(\phi)} + \hat{K}_{\Delta'}(\phi)$$

where

$$\hat{K}_{\Delta'}(\phi) = \sum_{Y_P, Y_K} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f \zeta} \times \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \times \prod_{\Delta \in Y_P} \left[P_{\Delta}(\phi_1, \zeta) \right] \times \prod_{\Delta \in Y_K} \left[K_{\Delta}(\phi_1 + \zeta) \right]$$

where the sum is over pairs of disjoint subsets Y_P , Y_K of $[L^{-1}\Delta']$ such that at least one of them is nonempty. We now assume that we are given collections of numbers $\delta\beta_{k,\Delta'}$ for $0 \le k \le 4$ and $\Delta' \in \mathbb{L}$. We will write $\delta b_{\Delta'} = \delta\beta_{0,\Delta'}$ We therefore have

$$\int d\mu_{C_0}(\phi) \, \mathcal{I}_t[\vec{V}](\phi) = \exp\left(\frac{1}{2}(f, \Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{t-1}}} \delta b_{\Delta'}\right) \times
\int d\mu_{C_0}(\phi) \prod_{\Delta' \in \mathbb{L}} \left\{ e^{f'_{\Delta'}\phi_{\Delta'}} \times \left[e^{-\hat{V}_{\Delta'}(\phi_{\Delta'}) - \delta b_{\Delta'}} + \hat{K}_{\Delta'}(\phi_{\Delta'}) e^{-\delta b_{\Delta'} - \frac{1}{2}(f, \Gamma f)_{L^{-1}\Delta'}} \right] \right\}$$

Define

$$\delta V_{\Delta'}(\phi) = \sum_{k=0}^{4} \delta \beta_{k,\Delta'} : \phi^k :_{C_0}$$

and

$$V'_{\Delta'}(\phi) = \sum_{k=1}^{4} (\hat{\beta}_{k,\Delta'} - \delta \beta_{k,\Delta'}) : \phi^k :_{C_0}$$

so that

$$V_{\Delta'}'(\phi) = \hat{V}_{\Delta'}(\phi) - \delta V_{\Delta'}(\phi) + \delta b_{\Delta'} \ .$$

One can check that

$$\int d\mu_{C_0}(\phi) \, \mathcal{I}_t[\vec{V}](\phi) = \exp\left(\frac{1}{2}(f, \Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{t-1}}} \delta b_{\Delta'}\right) \times \int d\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{t-1}}} \left\{ e^{f'_{\Delta'}\phi_{\Delta'}} \times \left[e^{-V'_{\Delta'}(\phi_{\Delta'})} + K'_{\Delta'}(\phi_{\Delta'}) \right] \right\}$$

where

$$K'_{\Delta'}(\phi) = e^{-\delta b_{\Delta'} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left\{ \hat{K}_{\Delta'}(\phi) - e^{-\hat{V}_{\Delta'}(\phi) + \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \left(e^{\delta V_{\Delta'}(\phi)} - 1 \right) \right\} \ .$$

We now do the λ expansion which introduces a new complex parameter λ . Define

$$r_{1,\Delta} = r_{1,\Delta}(\phi_1, \zeta) = e^{-\tilde{V}_{\Delta}(\phi_1)} \left[e^{-p_{\Delta}} - 1 + p_{\Delta} - \frac{1}{2} p_{\Delta}^2 \right]$$

and let

$$P_{\Delta}(\lambda, \phi_1, \zeta) = e^{-\tilde{V}_{\Delta}(\phi_1)} \left[-\lambda p_{\Delta} + \frac{\lambda^2}{2} p_{\Delta}^2 \right] + \lambda^3 r_{1,\Delta}(\phi_1, \zeta)$$

so that

$$P_{\Delta}(\lambda, \phi_1, \zeta)|_{\lambda=1} = P_{\Delta}(\phi_1, \zeta)$$
.

We also define

$$K_{\Delta}(\lambda, \phi_1, \zeta) = \lambda^2 Q_{\Delta}(\phi_1 + \zeta) e^{-\tilde{V}_{\Delta}(\phi_1)} + \lambda^3 \left[Q_{\Delta}(\phi_1 + \zeta) \left(e^{-p_{\Delta}} - 1 \right) e^{-\tilde{V}_{\Delta}(\phi_1)} + R_{\Delta}(\phi_1 + \zeta) \right]$$

so that

$$K_{\Delta}(\lambda, \phi_1, \zeta)|_{\lambda=1} = K_{\Delta}(\phi_1 + \zeta)$$

We use the same expansion formula as before in order to define the λ -deformation

$$\hat{K}_{\Delta'}(\lambda,\phi) = \sum_{Y_P,Y_K} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \times \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \times \prod_{\Delta \in Y_P} \left[P_{\Delta}(\lambda,\phi_1,\zeta) \right] \times \prod_{\Delta \in Y_K} \left[K_{\Delta}(\lambda,\phi_1,\zeta) \right] \ .$$

This is a polynomial expression in λ with no constant term. We can write it as

$$\hat{K}_{\Delta'}(\lambda,\phi) = \mathsf{A}\lambda + \mathsf{B}\lambda^2 + \mathsf{C}\lambda^3 + \hat{K}_{\Delta'}^{\geq 4}(\lambda,\phi)$$

where $\hat{K}_{\Lambda'}^{\geq 4}(\lambda, \phi)$ contains the terms of order 4 or more.

We now assume that we are given collections of numbers $\delta \beta_{k,j,\Delta'}$ for $0 \le k \le 4, \ 1 \le j \le 3$ and $\Delta' \in \mathbb{L}$ such that

$$\delta \beta_{k,\Delta'} = \delta \beta_{k,1,\Delta'} + \delta \beta_{k,2,\Delta'} + \delta \beta_{k,3,\Delta'}$$
.

Define

$$\delta \beta_{k,\Delta'}(\lambda) = \lambda \ \delta \beta_{k,1,\Delta'} + \lambda^2 \delta \beta_{k,2,\Delta'} + \lambda^3 \delta \beta_{k,3,\Delta'}$$
.

In particular this defines $\delta b_{\Delta'}(\lambda) = \delta \beta_{0,\Delta'}(\lambda)$. We also let

$$\delta V_{\Delta'}(\lambda, \phi) = \sum_{k=0}^{4} \delta \beta_{k, \Delta'}(\lambda) : \phi^k :_{C_0} .$$

Using the same formula as before for $K'_{\Delta'}$, we define the corresponding λ -deformation:

$$K'_{\Delta'}(\lambda,\phi) = e^{-\delta b_{\Delta'}(\lambda) - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left\{ \hat{K}_{\Delta'}(\lambda,\phi) - e^{-\hat{V}_{\Delta'}(\phi) + \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \left(e^{\delta V_{\Delta'}(\lambda,\phi)} - 1 \right) \right\} \ .$$

We again expand this in λ up to order 3:

$$K'_{\Delta'}(\lambda,\phi) = \mathsf{A}'\lambda + \mathsf{B}'\lambda^2 + \mathsf{C}'\lambda^3 + O(\lambda^4) \ .$$

We now choose the order 1 counterterms $\delta \beta_{k,1,\Delta'}$ so that $\mathsf{A}'=0$. Namely, for any $k, 0 \leq k \leq 4$, we let

(5)
$$\delta \beta_{k,1,\Delta'} = -\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \int_{\beta_{k+b}}^{b} f(x) dx$$

where

$$f \sim \int_{\beta_{k+b}}^{b} f = \int_{(L^{-1}\Delta')^{b+1}} d^3x \ d^3y_1 \cdots d^3y_b \ \beta_{k+b}(x) \times \prod_{i=1}^{b} \left[\Gamma(x - y_i) \ f(y_i) \right] .$$

We define the order 2 counterterms $\delta \beta_{k,2,\Delta'}$ so that

$$\mathsf{B}' = e^{-\hat{V}_{\Delta'}(\phi)} Q'_{\Delta'}(\phi)$$

where

$$Q'_{\Delta'}(\phi) = W'_{5,\Delta'} : \phi^5_{\Delta'} :_{C_0} + W'_{6,\Delta'} : \phi^6_{\Delta'} :_{C_0}$$

and $W'_{5,\Delta'}$, $W'_{6,\Delta'}$ are new coefficients. The later will turn out to be the output quantities defined previously. This hinges on imposing the choice:

$$\delta\beta_{k,2,\Delta'} = \sum_{a_1,a_2,b_1,b_2,m} \mathbbm{1} \left\{ \begin{array}{c} a_i + b_i \leq 4 \\ a_i \geq 0 \ , \ b_i \geq 1 \\ 1 \leq m \leq \min(b_1,b_2) \end{array} \right\} \frac{(a_1 + b_1)! \ (a_2 + b_2)!}{a_1! \ a_2! \ m! \ (b_1 - m)! \ (b_2 - m)!}$$

$$\times \frac{1}{2}C(a_1, a_2|k) \times L^{-(a_1+a_2)[\phi]} \times C_0(0)^{\frac{a_1+a_2-k}{2}} \times b_{1-m} \cdot b_{1-m} \cdot b_{2-m}$$

(6)
$$+ \sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b=5 \text{ or } 6 \\ b \ge 0 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \quad f \qquad b \qquad f$$

where

$$\beta_{a_1+b_1}(x_1) \ \beta_{a_2+b_2}(x_2) \ \Gamma(x_1-x_2)^m \times \prod_{i=1}^{b_1-m} \left[\Gamma(x_1-y_i) \ f(y_i) \right] \times \prod_{i=1}^{b_2-m} \left[\Gamma(x_2-z_i) \ f(z_i) \right]$$

$$f \underbrace{\sum_{W_{k+b}}^{b}}_{f} = \int_{(L^{-1}\Delta')^{b+1}} d^3x \ d^3y_1 \cdots d^3y_b \ W_{k+b}(x) \times \prod_{i=1}^{b} [\Gamma(x-y_i) \ f(y_i)]$$

and where $C(a_1, a_2|k)$ are connection coefficients for Hermite polynomials. More precisely

$$C(a_1, a_2|k) = 1 \left\{ \begin{array}{c} |a_1 - a_2| \le k \le a_1 + a_2 \\ a_1 + a_2 + k \in 2\mathbb{Z} \end{array} \right\} \times \frac{a_1! \ a_2!}{\left(\frac{a_1 + a_2 - k}{2}\right)! \ \left(\frac{a_1 + k - a_2}{2}\right)! \ \left(\frac{a_2 + k - a_1}{2}\right)!} \ .$$

These satisfy the property

$$: \phi_{\Delta'}^{a_1} :_{C_0} \times : \phi_{\Delta'}^{a_2} :_{C_0} = \sum_{k} C(a_1, a_2 | k) \ C_0(0)^{\frac{a_1 + a_2 - k}{2}} : \phi_{\Delta'}^{k} :_{C_0} .$$

The order 3 counterterms $\delta\beta_{k,3,\Delta'}$ will be defined as $(\vec{\beta}, f)$ -dependent linear functions of R. This is a bit lengthy so we need a few preparatory steps before we can give the explicit formulas for these counterterms.

First notice that the quantity C splits as $C = C_0 + C_1$ where

$$C_{0} = -\frac{1}{6} \sum_{\substack{\Delta_{1}, \Delta_{2}, \Delta_{3} \in [L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)} \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ p_{\Delta_{1}} \ p_{\Delta_{2}} \ p_{\Delta_{3}}$$

$$-\frac{1}{2} \sum_{\substack{\Delta_{1}, \Delta_{2} \in [L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)} \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ p_{\Delta_{1}} \ p_{\Delta_{2}}^{2}$$

$$-\sum_{\substack{\Delta_{1}, \Delta_{2} \in [L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)} \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ p_{\Delta_{1}} \ Q_{\Delta_{2}}(\phi_{1} + \zeta)$$

$$+\sum_{\substack{\Delta_{1} \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_{1}}} e^{-\hat{V}_{\Delta'}(\phi)} \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ Q_{\Delta_{1}}(\phi_{1} + \zeta) \left(e^{-p\Delta_{1}} - 1\right)$$

$$+\sum_{\substack{\Delta_{1} \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_{1}}} \left(\prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_{1}}} e^{-\tilde{V}_{\Delta}(\phi_{1})}\right) \times \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ r_{1,\Delta_{1}}$$

and

$$\mathsf{C}_{1} = \sum_{\substack{\Delta_{1} \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_{1}}} \left(\prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_{1}}} e^{-\tilde{V}_{\Delta}(\phi_{1})} \right) \times \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ R_{\Delta_{1}}(\phi_{1} + \zeta) \ .$$

Note that we will not need the detailed evaluation of C_0 , but we simply need the remark that it is R-independent.

Define

$$\delta V_{j,\Delta'}(\phi) = \sum_{k=0}^{4} \delta \beta_{k,j,\Delta'} : \phi^k :_{C_0}$$

for $1 \leq j \leq 3$. Then the λ^3 coefficient of $K'_{\Delta'}(\lambda,\phi)$ is given by $\mathsf{C}' = \mathsf{C}'_0 + \mathsf{C}'_1$ where

$$\mathsf{C}_0' = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \; \mathsf{C}_0 - e^{-\hat{V}_{\Delta'}(\phi)} \left(\frac{1}{6} \delta V_{1,\Delta'}(\phi)^3 + \delta V_{1,\Delta'}(\phi) \; \delta V_{2,\Delta'}(\phi) \right) - e^{-\hat{V}_{\Delta'}(\phi)} \; Q_{\Delta'}'(\phi) \; \delta \beta_{0,1,\Delta'}(\phi)^3 + \delta V_{1,\Delta'}(\phi) \; \delta V_{2,\Delta'}(\phi) + e^{-\hat{V}_{\Delta'}(\phi)} \; Q_{\Delta'}'(\phi) \; \delta \beta_{0,1,\Delta'}(\phi)^3 + \delta V_{1,\Delta'}(\phi)^3 + \delta V_{1,\Delta$$

and

$$C'_1 = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} C_1 - e^{-\hat{V}_{\Delta'}(\phi)} \delta V_{3,\Delta'}(\phi)$$
.

We now suppose that we are given collections of numbers $\delta \beta_{k,3,\Delta',\Delta_1}$ for $0 \le k \le 4$, $\Delta' \in \mathbb{L}$ and $\Delta_1 \in [L^{-1}\Delta']$ such that

$$\delta \beta_{k,3,\Delta'} = \sum_{\Delta_1 \in [L^{-1}\Delta']} \delta \beta_{k,3,\Delta',\Delta_1}$$
.

We then have

$$\mathsf{C}_1' = \sum_{\substack{\Delta_1 \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} \left(\prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \times J_{\Delta',\Delta_1}(\phi)$$

where

$$J_{\Delta',\Delta_1}(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ R_{\Delta_1}(\phi_1 + \zeta)$$

(7)
$$-\left(\sum_{k=0}^{4} \delta \beta_{k,3,\Delta',\Delta_1} : \phi^k :_{C_0}\right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)} .$$

The quantities $\delta \beta_{k,3,\Delta',\Delta_1}$ are uniquely determined by imposing the following normalization conditions on the derivatives up to order 4:

$$J_{\Delta',\Delta_1}^{(\nu)}(0) = 0$$

for all $\Delta' \in \mathbb{L}$, $\Delta_1 \in [L^{-1}\Delta']$ and ν such that $0 \le \nu \le 4$. Write $J_{\Delta',\Delta_1}(\phi) = J_+(\phi) - J_-(\phi)$ where

$$J_{+}(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ R_{\Delta_{1}}(\phi_{1} + \zeta)$$

and

$$J_{-}(\phi) = \left(\sum_{k=0}^{4} \delta \beta_{k,3,\Delta',\Delta_{1}} : \phi^{k} :_{C_{0}}\right) \times e^{-\tilde{V}_{\Delta_{1}}(\phi_{1})}.$$

For any ν , $0 \le \nu \le 4$, we have

$$J_{+}^{(\nu)}(0) = L^{-\nu[\phi]} e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_{\Gamma}(\zeta) e^{\int_{L^{-1}\Delta'} f\zeta} R_{\Delta_{1}}^{(\nu)}(\zeta) .$$

Whereas

$$J_{-}(\phi) = u(\phi) e^{v(\phi)}$$

with

$$u(\phi) = u_4\phi^4 + u_3\phi^3 + u_2\phi^2 + u_1\phi + u_0$$

and

$$v(\phi) = v_4 \phi^4 + v_3 \phi^3 + v_2 \phi^2 + v_1 \phi + v_0$$

with coefficients explicitly given by

$$\begin{array}{rcl} u_4 & = & \delta\beta_4 \\ u_3 & = & \delta\beta_3 \\ u_2 & = & \delta\beta_2 - 6C\delta\beta_4 \\ u_1 & = & \delta\beta_1 - 3C\delta\beta_3 \\ u_0 & = & \delta\beta_0 - C\delta\beta_2 + 3C^2\delta\beta_4 \end{array}$$

and

$$\begin{array}{rcl} v_4 & = & -L^{-4[\phi]}\beta_4 \\ v_3 & = & -L^{-3[\phi]}\beta_3 \\ v_2 & = & -L^{-2[\phi]}\beta_2 + 6CL^{-4[\phi]}\beta_4 \\ v_1 & = & -L^{-[\phi]}\beta_1 + 3CL^{-3[\phi]}\beta_3 \\ v_0 & = & CL^{-2[\phi]}\beta_2 - 3C^2L^{-4[\phi]}\beta_4 \ . \end{array}$$

Note that we used the abbreviated notation $\delta\beta_k = \delta\beta_{k,3,\Delta',\Delta_1}$, $\beta_k = \beta_{k,\Delta_1}$ and $C = C_0(0)$. Using Maple we found for the Taylor expansion of $J_-(\phi)$ up to order 4:

$$J_{-}(\phi) = e^{v_0} \times \left\{ u_0 + (u_0 v_1 + u_1) \phi + \left(u_1 v_1 + u_0 v_2 + \frac{1}{2} u_0 v_1^2 + u_2 \right) \phi^2 \right.$$

$$\left. + \left(u_1 v_2 + \frac{1}{2} u_1 v_1^2 + u_2 v_1 + u_0 v_3 + u_0 v_1 v_2 + \frac{1}{6} u_0 v_1^3 + u_3 \right) \phi^3 \right.$$

$$\left. + \left(u_4 + u_1 v_3 + u_1 v_1 v_2 + \frac{1}{6} u_1 v_1^3 + u_0 v_4 + u_0 v_1 v_3 \right.$$

$$\left. + \frac{1}{2} u_0 v_2^2 + \frac{1}{2} u_0 v_2 v_1^2 + \frac{1}{24} u_0 v_1^4 + u_2 v_2 + \frac{1}{2} u_2 v_1^2 + u_3 v_1 \right) \phi^4 \right\} + O(\phi^5) .$$

Write $a_{\nu} = e^{-v_0} J_{+}^{(\nu)}(0)$. We therefore have to solve for u_0, \ldots, u_4 in the triangular polynomial system

$$a_{0} = u_{0}$$

$$a_{1} = u_{1} + u_{0} v_{1}$$

$$\frac{1}{2} a_{2} = u_{2} + u_{1} v_{1} + u_{0} v_{2} + \frac{1}{2} u_{0} v_{1}^{2}$$

$$\frac{1}{6} a_{3} = u_{3} + u_{1} v_{2} + \frac{1}{2} u_{1} v_{1}^{2} + u_{2} v_{1} + u_{0} v_{3} + u_{0} v_{1} v_{2} + \frac{1}{6} u_{0} v_{1}^{3}$$

$$\frac{1}{24} a_{4} = u_{4} + u_{1} v_{3} + u_{1} v_{1} v_{2} + \frac{1}{6} u_{1} v_{1}^{3} + u_{0} v_{4} + u_{0} v_{1} v_{3}$$

$$+ \frac{1}{2} u_{0} v_{2}^{2} + \frac{1}{2} u_{0} v_{2} v_{1}^{2} + \frac{1}{24} u_{0} v_{1}^{4} + u_{2} v_{2} + \frac{1}{2} u_{2} v_{1}^{2} + u_{3} v_{1}.$$

This is straightforward but leads to complicated intermediate formulas which we skip. We then replace the v's by their expressions in terms of the β 's. Finally we use the obtained formulas for the u's in order to get

$$\begin{array}{rcl}
\delta\beta_4 & = & u_4 \\
\delta\beta_3 & = & u_3 \\
\delta\beta_2 & = & u_2 + 6Cu_4 \\
\delta\beta_1 & = & u_1 + 3Cu_3 \\
\delta\beta_0 & = & u_0 + Cu_2 + 3C^2u_4 \ .
\end{array}$$

The final result, obtained with the help of Maple, and using the notation $d_k = L^{-k[\phi]}\beta_k$ is:

$$\begin{split} \delta\beta_4 &= \frac{1}{24} a_4 + \left(\frac{1}{6} d_1 - \frac{1}{2} C d_3\right) a_3 + \left(\frac{1}{4} d_1^2 - \frac{3}{2} C d_1 d_3 + \frac{9}{4} C^2 d_3^2 - 3 C d_4 + \frac{1}{2} d_2\right) a_2 \\ &+ \left(\frac{9}{2} C^2 d_1 d_3^2 - \frac{3}{2} C d_1^2 d_3 - 6 C d_1 d_4 - 3 C d_3 d_2 \right. \\ &+ 18 C^2 d_3 d_4 + \frac{1}{6} d_1^3 + d_1 d_2 - \frac{9}{2} C^3 d_3^3 + d_3\right) a_1 \\ &+ \left(d_4 - 6 C d_2 d_4 - \frac{1}{2} C d_1^3 d_3 + \frac{9}{4} C^2 d_1^2 d_3^2 - \frac{9}{2} C^3 d_1 d_3^3 - 3 C d_1^2 d_4 \right. \\ &+ \frac{9}{2} C^2 d_3^2 d_2 - 3 C d_3^2 + \frac{1}{2} d_1^2 d_2 + 18 C^2 d_1 d_3 d_4 - 3 C d_1 d_3 d_2 \\ &- 27 C^3 d_3^2 d_4 + 18 C^2 d_4^2 + d_3 d_1 + \frac{1}{24} d_1^4 + \frac{1}{2} d_2^2 + \frac{27}{8} C^4 d_3^4\right) a_0 \;, \end{split}$$

$$\delta\beta_3 = \frac{1}{6}a_3 + \left(\frac{1}{2}d_1 - \frac{3}{2}Cd_3\right)a_2 + \left(d_2 - 6Cd_4 + \frac{1}{2}d_1^2 - 3Cd_1d_3 + \frac{9}{2}C^2d_3^2\right)a_1 + \left(\frac{9}{2}C^2d_1d_3^2 - \frac{3}{2}Cd_1^2d_3 - 6Cd_1d_4 - 3Cd_3d_2 + 18C^2d_3d_4 + \frac{1}{6}d_1^3 + d_1d_2 - \frac{9}{2}C^3d_3^3 + d_3\right)a_0,$$

$$\begin{split} \delta\beta_2 &= \frac{1}{4}C\,a_4 + \left(-3\,C^2\,d_3 + C\,d_1\right)\,a_3 + \left(-9\,C^2\,d_1\,d_3 + 3\,C\,d_2 + \frac{3}{2}\,C\,d_1^{\,2} + \frac{1}{2} + \frac{27}{2}\,C^3\,d_3^{\,2} - 18\,C^2\,d_4\right)\,a_2 \\ &\quad + \left(108\,C^3\,d_3\,d_4 + C\,d_1^{\,3} - 36\,C^2\,d_1\,d_4 - 18\,C^2\,d_3\,d_2 + 6\,C\,d_1\,d_2 \right. \\ &\quad + d_1 + 3\,C\,d_3 - 9\,C^2\,d_1^{\,2}\,d_3 + 27\,C^3\,d_1\,d_3^{\,2} - 27\,C^4\,d_3^{\,3}\right)\,a_1 \\ &\quad + \left(\frac{27}{2}\,C^3\,d_1^{\,2}\,d_3^{\,2} - 27\,C^4\,d_1\,d_3^{\,3} - 36\,C^2\,d_2\,d_4 + \frac{1}{2}\,d_1^{\,2} + d_2 + \frac{1}{4}\,C\,d_1^{\,4} - 18\,C^2\,d_1^{\,2}\,d_4 + 27\,C^3\,d_3^{\,2}\,d_2 \right. \\ &\quad - 162\,C^4\,d_3^{\,2}\,d_4 + 108\,C^3\,d_1\,d_3\,d_4 + 3\,C\,d_1^{\,2}\,d_2 + 3\,C\,d_1\,d_3 - \frac{27}{2}\,C^2\,d_3^{\,2} + 3\,C\,d_2^{\,2} - 3\,C^2\,d_1^{\,3}\,d_3 \\ &\quad + 108\,C^3\,d_4^{\,2} + \frac{81}{4}\,C^5\,d_3^{\,4} - 18\,C^2\,d_1\,d_3\,d_2 \right)\,a_0 \;, \end{split}$$

$$\begin{split} \delta\beta_1 &= \frac{1}{2}\,C\,a_3 + \left(\frac{3}{2}\,C\,d_1 - \frac{9}{2}\,C^2\,d_3\right)\,a_2 + \left(3\,C\,d_2 + \frac{3}{2}\,C\,d_1^2 + \frac{27}{2}\,C^3\,d_3^2 - 18\,C^2\,d_4 + 1 - 9\,C^2\,d_1\,d_3\right)\,a_1 \\ &+ \left(-\frac{9}{2}\,C^2\,d_1^2\,d_3 + d_1 - \frac{27}{2}\,C^4\,d_3^3 + 54\,C^3\,d_3\,d_4 + 3\,C\,d_1\,d_2 \right. \\ &- 18\,C^2\,d_1\,d_4 - 9\,C^2\,d_3\,d_2 + \frac{1}{2}\,C\,d_1^3 + \frac{27}{2}\,C^3\,d_1\,d_3^2\right)a_0 \;, \end{split}$$

$$\begin{split} \delta\beta_0 &= \frac{1}{8}\,C^2\,a_4 + \left(-\frac{3}{2}\,C^3\,d_3 + \frac{1}{2}\,C^2\,d_1\right)\,a_3 \\ &+ \left(\frac{3}{4}\,C^2\,d_1^2 + \frac{27}{4}\,C^4\,d_3^2 + \frac{3}{2}\,C^2\,d_2 + \frac{1}{2}\,C - \frac{9}{2}\,C^3\,d_1\,d_3 - 9\,C^3\,d_4\right)\,a_2 \\ &+ \left(\frac{27}{2}\,C^4\,d_1\,d_3^2 - 9\,C^3\,d_3\,d_2 + 54\,C^4\,d_3\,d_4 + C\,d_1 - 18\,C^3\,d_1\,d_4 + \frac{1}{2}\,C^2\,d_1^3 \right. \\ &+ 3\,C^2\,d_1\,d_2 - \frac{9}{2}\,C^3\,d_1^2\,d_3 - \frac{27}{2}\,C^5\,d_3^3\right)\,a_1 \\ &+ \left(\frac{3}{2}\,C^2\,d_1^2\,d_2 - 9\,C^3\,d_1\,d_3\,d_2 - \frac{3}{2}\,C^3\,d_1^3\,d_3 + \frac{27}{4}\,C^4\,d_1^2\,d_3^2 - \frac{27}{2}\,C^5\,d_1\,d_3^3 - 18\,C^3\,d_2\,d_4 + \frac{1}{8}\,C^2\,d_1^4 \right. \\ &+ \frac{81}{8}\,C^6\,d_3^4 + C\,d_2 - 9\,C^3\,d_1^2\,d_4 + \frac{27}{2}\,C^4\,d_3^2\,d_2 - 81\,C^5\,d_3^2\,d_4 + 1 + 54\,C^4\,d_1\,d_3\,d_4 \\ &- \frac{9}{2}\,C^3\,d_3^2 + 54\,C^4\,d_4^2 - 3\,C^2\,d_4 + \frac{1}{2}\,C\,d_1^2 + \frac{3}{2}\,C^2\,d_2^2\right)a_0 \;. \end{split}$$

Also recall that

(8)
$$a_{i} = \exp \left[-CL^{-2[\phi]}\beta_{2} + 3C^{2}L^{-4[\phi]}\beta_{4} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} \right] \times L^{-i[\phi]} \times \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \ R_{\Delta_{1}}^{(i)}(\zeta) \ .$$

Note that we get formulas of the form

$$\delta \beta_k = \sum_{i=0}^4 M_{k,i} \ a_i$$

where the matrix elements $M_{k,i}$ are given by finite sums of the form

(9)
$$M_{k,i} = \sum \# C^{j} L^{-(l_1 + \dots + l_n)[\phi]} \beta_{l_1} \cdots \beta_{l_n}$$

with $j \ge 0$, $n \ge 0$, and $1 \le l_m \le 4$ for every m, $1 \le m \le n$. Here the symbol # stands for some purely numerical constants. Furthermore, the terms which appear satisfy the homogeneity constraint

$$(10) l_1 + \dots + l_n - 2j = k - i.$$

We also have a limitation on the range of allowed n's:

$$n \le (k-i) + 2 \left| \frac{4-k}{2} \right| .$$

This completes the definition of the $\delta \beta_{k,3,\Delta',\Delta_1}$ and therefore of the order 3 counterterms

$$\delta \beta_{k,3,\Delta'} = \sum_{\Delta_1 \in [L^{-1}\Delta']} \delta \beta_{k,3,\Delta',\Delta_1} .$$

We also have a complete definition of

$$K'_{\Delta'}(\lambda,\phi) = \lambda^2 e^{-\hat{V}_{\Delta'}(\phi)} Q'_{\Delta'}(\phi) + \lambda^3 \mathsf{C}'_0 + \lambda^3 \mathsf{C}'_1 + O(\lambda^4) \ .$$

We now define

$$\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R) = \mathsf{C}_1'$$

with the previous choices for the $\delta\beta_{k,3,\Delta',\Delta_1}$. This makes $\mathcal{L}^{(\vec{\beta},f)}$ a $(\vec{\beta},f)$ -dependent linear operator on the space where R lives.

Now the new couplings $\beta'_{k,\Delta'}$ as well as the quantities $\delta b_{\Delta'}$ are fully defined. We just need the new R. It is given by

$$R'_{\Delta'} = \mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R) + \xi_{R,\Delta'}(\vec{V})$$

where the formula for remainder term is

$$\begin{split} \xi_{R,\Delta'}(\vec{V})(\phi) &= \frac{1}{2\pi i} \oint_{\gamma_0} \frac{\mathrm{d}\lambda}{\lambda^4} \left. K'_{\Delta'}(\lambda,\phi) \right|_{R=0} \\ &+ \frac{1}{2\pi i} \oint_{\gamma_{01}} \frac{\mathrm{d}\lambda}{\lambda^4 (\lambda - 1)} K'_{\Delta'}(\lambda,\phi) \\ &+ \left(e^{-\hat{V}_{\Delta'}(\phi)} - e^{-V'_{\Delta'}(\phi)} \right) Q'_{\Delta'}(\phi) \end{split}$$

where γ_0 is any positively oriented contour around $\lambda = 0$, and γ_{01} is any positively oriented contour which encloses both $\lambda = 0$ and $\lambda = 1$. In [14] the three terms for the remainder are respectively denoted by R_{main} , R_3 , and R_4 . It is important to note that in the first term we set R = 0, which means that all $\delta \beta_{k,3,\Delta'}$ are set equal to zero. Also note that $\mathcal{L}^{(\vec{\beta},f)}(R)$ corresponds to the R_{linear} notation in [14].

To finish setting up the notation we write for $1 \le k \le 4$

$$\xi_{k,\Delta'}(\vec{V}) = -\delta\beta_{k,3,\Delta'}$$

whereas

$$\xi_{0,\Delta'}(\vec{V}) = \delta\beta_{0,3,\Delta'}$$

In this way the RG evolution for the couplings is

$$\beta'_{k,\Delta'} = \hat{\beta}_{k,\Delta'} - \delta \beta_{k,1,\Delta'} - \delta \beta_{k,2,\Delta'} + \xi_{k,\Delta'}(\vec{V})$$

for $1 \leq k \leq 4$. Likewise the collected terms which contribute to the progressive calculation of $\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})$ are

$$\delta b_{\Delta'}[\vec{V}] = \delta \beta_{0,1,\Delta'} + \delta \beta_{0,2,\Delta'} + \xi_{0,\Delta'}(\vec{V})$$

5. Preliminary estimates

5.1. Properties of covariances. In this section we collect some of the properties satisfied by the covariances and needed in the sequel. Recall that $L = p^l$ where l is an integer l > 0.

Lemma 2. The covariance Γ can be expressed pointwise as follows.

(1) If $|x| \le 1$ then

$$\Gamma(x) = \frac{1 - p^{-3}}{1 - p^{-2[\phi]}} (1 - L^{-2[\phi]}) .$$

(2) If $|x| = p^i$ with $1 \le i \le l$, then

$$\Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} + \frac{1-p^{-3+2[\phi]}}{1-p^{-2[\phi]}}(p^{-2i[\phi]} - p^{-2l[\phi]}) .$$

(3) If |x| > L then $\Gamma(x) = 0$.

Proof: Recall that

$$\Gamma(x) = \sum_{j=0}^{l-1} p^{-2j[\phi]} \left(\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) - p^{-3} \mathbb{1}_{\mathbb{Z}_p^3}(p^{j+1} x) \right) .$$

By Abel summation, or discrete integration by parts, this can be rewritten as

(11)
$$\Gamma(x) = \mathbb{1}_{\mathbb{Z}_p^3}(x) - p^{-3-2(l-1)[\phi]} \mathbb{1}_{\mathbb{Z}_p^3}(p^l x) + \sum_{j=1}^{l-1} p^{-2j[\phi]} (1 - p^{-3+2[\phi]}) \mathbb{1}_{\mathbb{Z}_p^3}(p^j x) .$$

Now we also have

$$1\!\!1_{\mathbb{Z}^3_p}(p^jx) = 1\!\!1\{|p^jx| \le 1\} = 1\!\!1\{|x| \le p^j\} = \sum_{i \le j} 1\!\!1\{|x| = p^i\} \ .$$

We insert the last expression into the sum in (11) and get after commuting the sums over i and j that

$$\Gamma(x) = \mathbb{1}_{\mathbb{Z}_p^3}(x) - p^{-3-2(l-1)[\phi]} \mathbb{1}_{\mathbb{Z}_p^3}(p^l x) + \sum_{i \in \mathbb{Z}} U_i \mathbb{1}\{|x| = p^i\}$$

where

$$U_i = \sum_{j \in \mathbb{Z}} \mathbb{1} \left\{ \begin{array}{c} 1 \leq j \leq l-1 \\ i \leq j \end{array} \right\} p^{-2j[\phi]} .$$

Now note that if $i \geq l$ then $U_i = 0$. Also, if $i \leq 0$ then

$$U_i = \frac{p^{-2[\phi]} - p^{-2l[\phi]}}{1 - p^{-2[\phi]}} .$$

Finally, if $1 \le i \le l-1$ then

$$U_i = \frac{p^{-2i[\phi]} - p^{-2l[\phi]}}{1 - p^{-2[\phi]}} \ .$$

As a result we have

$$\Gamma(x) = 1 \{ |x| \le 1 \} - p^{-3 + 2[\phi]} p^{-2l[\phi]} 1 \{ |x| \le p^l \} + \frac{1 - p^{-3 + 2[\phi]}}{1 - p^{-2[\phi]}} \sum_{i \le l - 1} 1 \{ |x| = p^i \} \left(p^{-2[\phi] \max(i, 1)} - p^{-2l[\phi]} \right)$$

from which the result follows by specialization to the different cases mentioned.

As a result of the previous lemma we have a precise control over the sign of the function Γ .

Lemma 3.

- (1) If $|x| < p^l \text{ then } \Gamma(x) > 0$.
- (2) If $|x| = p^l$ then $\Gamma(x) < 0$. (3) If $|x| > p^l$ then $\Gamma(x) = 0$.

Proof: Recall that $\epsilon \in (0,1]$ and therefore $[\phi] = \frac{3-\epsilon}{4} \in \left[\frac{1}{2},\frac{3}{4}\right)$. We also have $l \geq 1$ and of course the prime number p is at least 2. From Lemma 2 1), we then readily get that $\Gamma(x) > 0$ if $|x| \leq 1$. The case $|x| > p^l$ has already been considered. For $|x| = p^l$ the formula in Lemma 2 2) reduces to $\Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} < 0$. Finally when $|x| = p^i$, $2 \leq i \leq l-1$ then the formula in Lemma 2 2) shows that $\Gamma(x)$ decreases with i in that range. We only need look at the case i = l-1 where one has

$$\Gamma(x) = p^{-2(l-1)[\phi]} \left[1 - p^{-3} - p^{-3+2[\phi]} \right].$$

Simply using $p^{-3} \le \frac{1}{8}$ and $3 - 2[\phi] > \frac{3}{2}$, which implies $p^{-3+2[\phi]} < 2^{-\frac{3}{2}}$, we get $1 - p^{-3} - p^{-3+2[\phi]} > 0$ and thus $\Gamma(x) > 0$.

Corollary 1. The fluctuation covariance satisfies the L^1 bound

$$||\Gamma||_{L^1} < \frac{1}{\sqrt{2}} L^{3-2[\phi]}$$
.

Proof: Indeed, by $\Gamma = C_0 - C_1$ and the definitions of the C_r covariances in §3 we have that $\int_{\mathbb{Q}_p^3} d^3x \ \Gamma(x) = \widehat{\Gamma}(0) = 0$. In other words the positive part exactly cancels the negative part which is easy to compute since it only involves x's with $|x| = p^l$. Therefore

$$||\Gamma||_{L^{1}} = -2 \int_{\mathbb{Q}_{p}^{3}} d^{3}x \ \Gamma(x) \ \mathbb{1}\{|x| = p^{l}\}$$
$$= 2(1 - p^{-3})p^{-3 + 2[\phi]}L^{3 - 2[\phi]}.$$

We use $1 - p^{-3} < 1$ and again $p^{-3+2[\phi]} < 2^{-\frac{3}{2}}$ to conclude.

As for the unit cut-off covariance C_0 , the following easy property will useful in the sequel.

Lemma 4. When $\epsilon \in (0,1]$, we have $1 < C_0(0) < 2$.

Proof: Recall that

$$C_0(0) = \frac{1 - p^{-3}}{1 - p^{-2[\phi]}} = \frac{1 - p^{-3}}{1 - p^{-(\frac{3-\epsilon}{2})}}$$
.

Only using $p \geq 1$ and the given range for ϵ we get

$$p^{-\frac{3}{2}} \le p^{-\left(\frac{3-\epsilon}{2}\right)} \le p^{-1} \le \frac{1}{2}$$
.

Hence

$$1 < \frac{1 - p^{-3}}{1 - p^{-1}} \le C_0(0) \le \frac{1 - p^{-3}}{1 - p^{-\frac{3}{2}}} = 1 + p^{-\frac{3}{2}} < 2$$
.

We will also need some information on the L^{∞} and L^2 norms of Γ which are provided by the following two easy lemmas.

Lemma 5. We have the simple estimate

$$||\Gamma||_{L^{\infty}} \leq 2$$
.

Proof: If $|x| \le 1$, it follows from Lemmas 2 and 4 that $0 < \Gamma(x) < 2$. If |x| > L, then $\Gamma(x) = 0$. If |x| = L, then

$$|\Gamma(x)| = |-p^{-(3-2[\phi])}L^{-2[\phi]}| \le 1$$
.

Finally if $|x| = p^i$ with $1 \le i \le l - 1$, then by Lemma 3

$$|\Gamma(x)| = \Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} + \frac{1 - p^{-3+2[\phi]}}{1 - p^{-2l[\phi]}}(p^{-2i[\phi]} - p^{-2l[\phi]})$$

$$\leq \frac{1 - p^{-3 + 2[\phi]}}{1 - p^{-2[\phi]}} (p^{-2i[\phi]} - p^{-2l[\phi]}) \leq \frac{1 - p^{-3 + 2[\phi]}}{1 - p^{-2[\phi]}} \leq \frac{1 - p^{-3}}{1 - p^{-2[\phi]}} = C_0(0) < 2.$$

This shows $|\Gamma(x)| \leq 2$ in all cases.

Lemma 6. We have

$$\int_{\mathbb{Q}_2^3} |\Gamma(x)|^2 d^3x = \frac{(1 - p^{-3})(L^{\epsilon} - 1)}{p^{\epsilon} - 1} \longrightarrow (1 - p^{-3}) \times l$$

when $\epsilon \to 0$, with l defined by $L = p^l$ and the limit taken with L fixed.

Proof: By the Plancherel formula over the *p*-adics

$$\int_{\mathbb{Q}_p^3} |\Gamma(x)|^2 d^3x = \int_{\mathbb{Q}_p^3} |\widehat{\Gamma}(k)|^2 d^3k.$$

But

$$\widehat{\Gamma}(k) = \widehat{C}_0(k) - \widehat{C}_1(k) = \frac{\mathbb{1}\{L^{-1} < |k| \le 1\}}{|k|^{3-2[\phi]}}$$

and therefore

$$\int_{\mathbb{Q}_p^3} |\Gamma(x)|^2 d^3x = \int_{\mathbb{Q}_p^3} \frac{\mathbb{1}\{L^{-1} < |k| \le 1\}}{|k|^{6-4[\phi]}} d^3k$$

$$= \sum_{j=0}^{l-1} \int_{\mathbb{Q}_p^3} \frac{\mathbb{1}\{|k| = p^{-j}\}}{(p^{-j})^{6-4[\phi]}} d^3k$$

$$= \sum_{j=0}^{l-1} (1 - p^{-3}) p^{-3j} p^{j(6-4[\phi])}.$$

The result follows since $3-4[\phi]=\epsilon$ and of course the $\epsilon\to 0$ limit is trivial.

5.2. Gaussian integration bound.

Lemma 7. Let Δ' be a block in \mathbb{L} . Let the real parameter α satisfy $0 \leq \alpha \leq \frac{\sqrt{2}}{4}L^{-(3-2[\phi])}$. If f is a real-valued function on $L^{-1}\Delta'$ which is constant on unit cubes and such that $||f||_{L^{\infty}} \leq \frac{1}{2}L^{-\frac{1}{2}(3-2[\phi])}$, then for any finite set $X \subset [L^{-1}\Delta']$ we have the bound

$$\int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \prod_{\Delta \in X} e^{\alpha \zeta_{\Delta}^2} \le 2^{|X|} e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \ .$$

Proof:

First note that one can view the integral we would like to bound, I, as an expectation with respect to the centered Gaussian vector $(\zeta_{\Delta})_{\Delta \in [L^{-1}\Delta']}$ in \mathbb{R}^{L^3} with covariance $\mathbf{E}(\zeta_{\Delta_1}\zeta_{\Delta_2}) = \Gamma_{\Delta_1,\Delta_2} = \Gamma(x_1 - x_2)$ where x_1 is any point in Δ_1 and likewise for x_2 in Δ_2 . Let u_1,\ldots,u_{L^3} be an orthonormal basis which diagonalizes Γ (seen as an $L^3 \times L^3$ matrix). Let $\lambda_1,\ldots,\lambda_{L^3}$ be the corresponding eigenvalues and suppose we arranged the numbering so that $\lambda_1 \geq \lambda_2 \geq \cdots$. Note that the matrix Γ is singular and therefore only positive semi-definite, because of the property that $\int_{L^{-1}\Delta'} \zeta = 0$ almost surely. We therefore introduce $m = \max\{i | \lambda_i > 0\}$. We now have that ζ has the same law as $\sum_{i=1}^m a_i u_i$ where the a_i 's are independent centered Gaussian random variables with variance λ_i . Thus

$$I = \prod_{i=1}^{m} (2\pi\lambda_i)^{-\frac{1}{2}} \times \int_{\mathbb{R}^m} da_1 \dots da_m \exp \left[-\frac{1}{2} \sum_{i=1}^{m} \frac{a_i^2}{\lambda_i} + \sum_{\substack{\Delta \in [L^{-1}\Delta'] \\ 1 \le i \le m}} f_{\Delta} a_i u_{i,\Delta} + \alpha \sum_{\Delta \in X} \left(\sum_{i=1}^{m} a_i u_{i,\Delta} \right)^2 \right].$$

Since $X \subset [L^{-1}\Delta']$

$$\sum_{\Delta \in X} \left(\sum_{i=1}^m a_i u_{i,\Delta} \right)^2 \le \sum_{\Delta \in [L^{-1}\Delta']} \left(\sum_{i=1}^m a_i u_{i,\Delta} \right)^2 = \sum_{i=1}^m a_i^2$$

because of the orthonormality of the u's. Therefore a sufficient condition for the convergence of the integral is that $2\alpha\lambda_i < 1$ for all $i, 1 \le i \le m$. Granting this condition for now, we define $\tilde{f}_i = \sum_{\Delta \in [L^{-1}\Delta']} f_\Delta u_{i,\Delta}$

and use the standard 'completing the square' trick by writing

$$-\frac{1}{2}\sum_{i=1}^{m}\frac{a_i^2}{\lambda_i} + \sum_{i=1}^{m}a_i\tilde{f}_i = -\frac{1}{2}\sum_{i=1}^{m}\frac{1}{\lambda_i}(a_i - \lambda_i\tilde{f}_i)^2 + \frac{1}{2}\sum_{i=1}^{m}\lambda_i\tilde{f}_i^2$$

and changing variables to $a_i - \lambda_i \tilde{f}_i$. Hence

$$I = \prod_{i=1}^{m} (2\pi\lambda_i)^{-\frac{1}{2}} \times \int_{\mathbb{R}^m} da_1 \dots da_m \exp \left[-\frac{1}{2} \sum_{i=1}^{m} \frac{a_i^2}{\lambda_i} + \frac{1}{2} \sum_{i=1}^{m} \lambda_i \tilde{f}_i^2 + \alpha \sum_{\Delta \in X} \left(\sum_{i=1}^{m} (a_i + \lambda_i \tilde{f}_i) u_{i,\Delta} \right)^2 \right].$$

Note that

$$\sum_{i=1}^{m} \lambda_i \tilde{f}_i^2 = \sum_{i=1}^{m} \sum_{\Delta_1, \Delta_2 \in [L^{-1}\Delta']} \lambda_i f_{\Delta_1} f_{\Delta_2} u_{i, \Delta_1} u_{i, \Delta_2}$$

$$= \sum_{\Delta_1, \Delta_2 \in [L^{-1}\Delta']} f_{\Delta_1} f_{\Delta_2} \Gamma_{\Delta_1, \Delta_2}$$

$$= (f, \Gamma f)_{L^{-1}\Delta'}$$

by construction of the u's. We also have

$$\sum_{i=1}^{m} (a_i + \lambda_i \tilde{f}_i) u_{i,\Delta} = \zeta_{\Delta} + \sum_{i=1}^{m} \sum_{\Delta_1 \in [L^{-1} \Delta']} \lambda_i f_{\Delta_1} u_{i,\Delta_1} u_{i,\Delta}$$

$$= \zeta_{\Delta} + \sum_{\Delta_1 \in [L^{-1} \Delta']} \Gamma_{\Delta,\Delta_1} f_{\Delta_1}$$

$$= \zeta_{\Delta} + (\Gamma f)_{\Delta}$$

where we reverted to the use of the ζ_{Δ} variables of integration which have the same law as the quantities $\sum_{i=1}^{m} a_i u_{i\Delta}$, and where $(\Gamma f)(x)$ denotes $\int_{\mathbb{Q}_p^3} \mathrm{d}^3 y \ \Gamma(x-y) f(y)$. By the finite range property of Γ we have, for $x \in \Delta \in [L^{-1}\Delta']$, $(\Gamma f)(x) = (\Gamma f)_{\Delta} = \sum_{\Delta_1 \in [L^{-1}\Delta']} \Gamma_{\Delta,\Delta_1} f_{\Delta_1}$. As a result of the previous calculations

$$I = e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\alpha \sum_{\Delta \in X} ((\Gamma f)_{\Delta} + \zeta_{\Delta})^2} \ .$$

We now expand the square in the last exponential and we also introduce the covariance matrix Γ_X for the marginal random vector $\zeta|_X = (\zeta_\Delta)_{\Delta \in X}$ in order to write

$$I = e^{\frac{1}{2}(f,\Gamma f)_{L-1} \Delta' + \alpha(\Gamma f,\Gamma f)_X} \times \int d\mu_{\Gamma_X}(\zeta|_X) e^{\alpha(\zeta|_X,\zeta|_X) + 2\alpha(\Gamma f|_X,\zeta|_X)}$$

where the inner products are the ones of $l^2(X)$, namely $\langle w, w' \rangle = \sum_{\Delta \in X} w_\Delta w'_\Delta$ for vectors in $l^2(X)$ which are indexed by boxes in the finite set X.

Let $(v_i)_{1 \leq i \leq |X|}$ be an orthonormal basis diagonalizing the symmetric positive semi-definite matrix $\Gamma|_X$, with eigenvalues μ_i arranged so that $\mu_1 \geq \mu_2 \geq \cdots$ and let $n = \max\{i|\mu_i > 0\}$. As before, we have that the random vector $\zeta|_X$ has the same law as $\sum_{i=1}^n b_i v_i$ where the b_i are independent centered Gaussian random variables with variance μ_i . Following this change of variables of integration $\langle \zeta|_X, \zeta|_X \rangle$ becomes $\sum_{i=1}^n b_i^2$ whereas $\langle \Gamma f|_X, \zeta|_X \rangle$ becomes $\sum_{i=1}^n g_i b_i$ with $g_i = \sum_{\Delta \in X} (\Gamma f)_\Delta v_{i,\Delta}$. Hence

$$\int d\mu_{\Gamma_X}(\zeta|X) \ e^{\alpha(\zeta|X,\zeta|X) + 2\alpha(\Gamma f|X,\zeta|X)} = \prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi\mu_i}} \int_{\mathbb{R}} db_i \ e^{-\frac{b_i^2}{2\mu_i} + \alpha b_i^2 + 2\alpha g_i b_i} \right]
= \prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi\mu_i}} \times \sqrt{2\pi \left(\frac{1}{\mu_i} - 2\alpha\right)^{-1}} \times e^{\frac{1}{2} \left(\frac{1}{\mu_i} - 2\alpha\right)^{-1} (2\alpha g_i)^2} \right]
= \prod_{i=1}^n \left[\frac{1}{\sqrt{1 - 2\alpha\mu_i}} e^{2\alpha^2 \frac{\mu_i}{1 - 2\alpha\mu_i} g_i^2} \right]$$

provided $2\alpha\mu_i < 1$ for all $i, 1 \le i \le n$.

Now $\mu_i \leq ||\Gamma_X||$ where the latter quantity is the operator norm of Γ_X induced by the norm on $l^2(X)$ coming from the inner product $\langle \cdot, \cdot \rangle$. For v a real vector in $l^2(X)$, we have $||\Gamma_X v||^2 = \sum_{\Delta \in X} (\Gamma_X v)_{\Delta}^2 = \sum_{\Delta \in X} (\Gamma w)_{\Delta}^2$ where $w \in l^2([L^{-1}\Delta'])$ is the extension of v by zero outside X. Thus $||\Gamma_X v||^2 \leq \sum_{\Delta \in [L^{-1}\Delta']} (\Gamma_X w)_{\Delta}^2 = ||\Gamma w||^2 \leq ||\Gamma||^2 ||w||^2 = ||\Gamma||^2 ||v||^2$. As a result $||\Gamma_X|| \leq ||\Gamma||$ where the latter is the operator norm of the matrix Γ coming from the inner product norm of $l^2([L^{-1}\Delta'])$. However we have the bound $||\Gamma|| \leq ||\Gamma||_{L^1} = \int_{\mathbb{Q}_p^3} |\Gamma(x)| \mathrm{d}^3 x$. Indeed, given $w \in l^2([L^{-1}\Delta'])$ which we can identify with a function w(x) on \mathbb{Q}_p^3 with support in $L^{-1}\Delta'$ and which is constant on unit blocks, we have

$$\begin{split} ||\Gamma w||^2 &= \int_{\mathbb{Q}_p^3} [(\Gamma w)(x)]^2 \, \mathrm{d}^3 x \\ &= \int_{\mathbb{Q}_p^3 \times 3} \Gamma(x-y) \Gamma(x-z) w(y) w(z) \, \mathrm{d}^3 x \, \mathrm{d}^3 y \, \mathrm{d}^3 z \\ &\leq \int_{\mathbb{Q}_p^3 \times 3} |\Gamma(x-y)| \, |\Gamma(x-z)| \, |w(y)| \, |w(z)| \, \mathrm{d}^3 x \, \mathrm{d}^3 y \, \mathrm{d}^3 z \\ &\leq \int_{\mathbb{Q}_p^3 \times 3} |\Gamma(x-y)| \, |\Gamma(x-z)| \, \left(\frac{1}{2} |w(y)|^2 + \frac{1}{2} |w(z)|^2\right) \, \mathrm{d}^3 x \, \mathrm{d}^3 y \, \mathrm{d}^3 z \\ &= 2 \times \frac{1}{2} \times ||\Gamma||_{L^1}^2 ||w||_{L^2}^2 \, . \end{split}$$

Therefore from Corollary 1 we get $||\Gamma|| \le ||\Gamma||_{L^1} < \frac{1}{\sqrt{2}}L^{3-2[\phi]}$. Since the λ_i are bounded by $||\Gamma||$ (the case where $X = [L^{-1}\Delta']$), the hypothesis $\alpha \le \frac{\sqrt{2}}{4}L^{-3+2[\phi]}$ implies that the previous convergence requirement $2\alpha\lambda_i < 1$ is satisfied and also that not only $2\alpha\mu_i < 1$ holds but so does the stronger inequality $2\alpha\mu_i \le \frac{1}{2}$. From the latter we have $\frac{\mu_i}{1-2\alpha\mu_i} \le 2\mu_i$ and thus

$$\int d\mu_{\Gamma_X}(\zeta|_X) e^{\alpha(\zeta|_X,\zeta|_X) + 2\alpha\langle\Gamma f|_X,\zeta|_X\rangle} \leq \prod_{i=1}^n \left(\sqrt{2}e^{4\alpha^2\mu_i g_i^2}\right) \\
\leq 2^{\frac{|X|}{2}} \exp\left(\frac{\sqrt{2}}{4}L^{-(3-2[\phi])}\sum_{i=1}^n g_i^2\right)$$

where we used $n \leq |X|$, $\alpha \leq \frac{\sqrt{2}}{4}L^{-(3-2[\phi])}$ and $\mu_i < \frac{1}{\sqrt{2}}L^{3-2[\phi]}$. Besides, $g_i = \sum_{\Delta \in X} (\Gamma f)_{\Delta} v_{i,\Delta} = \langle v_i, (\Gamma f)|_X \rangle$ and therefore

$$\sum_{i=1}^n g_i^2 \le \sum_{i=1}^{|X|} \langle v_i, (\Gamma f)|_X \rangle^2 = \langle (\Gamma f)|_X, (\Gamma f)|_X \rangle = (\Gamma f, \Gamma f)_X.$$

But $(\Gamma f, \Gamma f)_X = \sum_{\Delta \in X} (\Gamma f)_{\Delta}^2$ and clearly $|(\Gamma f)_{\Delta}| \leq ||\Gamma||_{L^1}||f||_{L^{\infty}}$ so $(\Gamma f, \Gamma f)_X \leq |X| ||\Gamma||_{L^1}^2||f||_{L^{\infty}}^2$. Putting all the previous bounds together we see that the desired inequality holds provided

$$\exp\left[\frac{\sqrt{2}}{4}L^{3-2[\phi]}||f||_{L^{\infty}}^{2}\right] \le \sqrt{2}$$

which is true since, by hypothesis, $||f||_{L^{\infty}} \leq \frac{1}{2}L^{-\frac{1}{2}(3-2[\phi])}$ and $\frac{4}{\sqrt{2}} \times \frac{1}{2}\log 2 \simeq 0.980\ldots > \frac{1}{4}$.

5.3. Two easy lemmas. The following are simple bounds which will however be used many times in order to bound individual field factors using the exponential of a quartic or a quadratic expression. The quartic case will typically apply to background fields ϕ whereas quadratic bounds will typically apply to fluctuation fields ζ .

Note that the possibly complex ϕ^4 couplings β_4 will sit in an open ball of the form $|\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}$ with $\bar{g} > 0$. By elementary trigonometry it easily follows that $\frac{\Re \beta_4}{|\beta_4|} \ge \frac{\sqrt{3}}{2}$. We of course also have $\frac{1}{2} < \frac{\Re \beta_4}{\bar{g}} < \frac{3}{2}$.

Lemma 8. $\forall j \in \mathbb{N}, \ \forall \bar{g} > 0, \ \forall \gamma > 0, \ \forall \beta_4 \in \mathbb{C} \ such that \ |\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}, \ \forall \phi \in \mathbb{R} \ we have$

$$|\phi|^j \le \left(\frac{j}{4e}\right)^{\frac{j}{4}} (\gamma \Re \beta_4)^{-\frac{j}{4}} e^{\gamma (\Re \beta_4)\phi^4} \le \left(\frac{j}{2e}\right)^{\frac{j}{4}} (\gamma \bar{g})^{-\frac{j}{4}} e^{\gamma (\Re \beta_4)\phi^4}$$

with the convention $j^j = 1$ if j = 0.

Proof: The function $u^{\frac{j}{4}}e^{-u}$ for $u \geq 0$ is maximized when $u = \frac{j}{4}$. Simply apply this to $u = \gamma(\Re \beta_4)\phi^4$ and use $\frac{1}{2} < \frac{\Re \beta_4}{\bar{q}}$ for the second inequality.

Lemma 9. $\forall j \in \mathbb{N}, \ \forall \kappa > 0, \ \forall \zeta \in \mathbb{R} \ we \ have$

$$|\zeta|^j \le \left(\frac{j}{2e}\right)^{\frac{j}{2}} \kappa^{-\frac{j}{2}} e^{\kappa \zeta^2}$$

again with the convention $j^j = 1$ if j = 0.

The proof is similar.

5.4. **The key stability bound.** The following lemma is essential to our estimates since it provides bounds on the seminorms introduced in §4.1 for functions given by the exponential of a degree four polynomial in the real-valued field, with complex coefficients.

Lemma 10. Let $U(\phi) = a_4 \phi^4 + a_3 \phi^3 + a_2 \phi^2 + a_1 \phi + a_0$ where the possibly complex coefficients a_0, \ldots, a_4 satisfy $|a_4| > 0$, $\Re a_4 \ge \frac{\sqrt{3}}{2} |a_4|$, $|a_k| \le \frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2} \right) |a_k|^{\frac{k}{4}}$ for k = 1, 2, 3, and $|a_0| \le \log 2$. Then

(1) the condition

$$0 \le \theta \le \frac{\sqrt{2} - 1}{4} e^{-918785} \times |a_4|^{-\frac{1}{4}}$$

implies

$$||e^{-U(\phi)}||_{\partial \phi, \phi, \theta} < 2e^{-\frac{1}{2}(\Re a_4)\phi^4}$$

for all $\phi \in \mathbb{R}$;

(2) the condition

$$0 \le \theta \le \frac{(\sqrt{2} - 1)^2}{e} \times |a_4|^{-\frac{1}{4}}$$

implies

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \le 2$$
.

Proof: It follows from the definition of our seminorms that

$$||e^{-U(\phi)}||_{\partial \phi, \phi, \theta} = e^{-\Re U(\phi)} + \sum_{n=1}^{9} \frac{\theta^n}{n!} |D^n e^{-U(\phi)}|$$

where D denotes the differentiation operator $\frac{d}{d\phi}$. An easy induction provides the following explicit formula of Faa di Bruno type for the derivatives of functions of the form $e^{f(\phi)}$:

(12)
$$D^{n}e^{f(\phi)} = \sum_{k\geq 0} \frac{1}{k!} \sum_{\substack{m_{1},\dots,m_{k}\geq 1\\ \sum_{i=1}^{m}}} \frac{n!}{m_{1}!\cdots m_{k}!} \left(\prod_{i=1}^{k} D^{m_{i}}f(\phi)\right) e^{f(\phi)}.$$

This will be used in order to bound the quantities $|D^n e^{-U(\phi)}|$. First, let us introduce the notation $\alpha = \frac{\sqrt{3}}{2}$ and $r = \frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2} \right)$. We have

$$-\Re U(\phi) = -\sum_{k=0}^{4} (\Re a_k) \phi^k$$

$$\leq -\frac{1}{2} (\Re a_4) \phi^4 - \frac{\alpha}{2} |a_4| \phi^4 + \left(\sum_{k=1}^{3} |a_k| |\phi|^k\right) + |a_0|$$

from the hypothesis $\Re a_4 \ge \alpha |a_4|$. Using the assumption $|a_k| \le r|a_4|^{\frac{k}{4}}$ we then obtain

$$-\Re U(\phi) \le -\frac{1}{2}(\Re a_4)\phi^4 + \Omega_1(|a_4|^{\frac{1}{4}}|\phi|) + |a_0|$$

where $\Omega_1(x) = -\frac{\alpha}{2}x^4 + r(x^3 + x^2 + x)$. We first write a convenient upper bound on $\sup_{x \geq 0} \Omega_1(x)$. For $0 \leq x \leq 1$, we simply use $\Omega_1(x) \leq r(x^3 + x^2 + x) \leq 3r$. For $x \geq 1$, we write $\Omega_1(x) \leq -\frac{\alpha}{2}x^4 + 3rx^3$ and maximize the right-hand side over $[0, \infty)$. The maximum occurs at $x = \frac{9r}{2\alpha}$ and is equal to $\frac{3r}{4}\left(\frac{9r}{2\alpha}\right)^3 < 3r$. The last inequality used the fact $9r < 2\alpha$ which can be checked from the chosen numerical values of $r \simeq 0.0627\ldots$ and $\alpha \simeq 0.866\ldots$ As a result

$$e^{-\Re U(\phi)} < e^{-\frac{1}{2}(\Re a_4)\phi^4 + 3r + |a_0|}$$
.

We now use the formula (12) and write, for $1 \le n \le 9$,

$$D^{n}e^{-U(\phi)} = e^{-U(\phi)} \times \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{1 \le m_{1}, \dots, m_{k} \le 4 \\ \sum m_{i} = n}} \frac{n!}{m_{1}! \cdots m_{k}!} \times \prod_{i=1}^{k} (-D^{m_{i}}U(\phi)) .$$

Using the condition $\Sigma m_i = n$ for handling the θ exponents we get the bound

(13)
$$\frac{\theta^n}{n!} \left| D^n e^{-U(\phi)} \right| \le e^{-\Re U(\phi)} \times \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{1 \le m_1, \dots, m_k \le 4 \\ \sum m_i = n}} \prod_{i=1}^k \left[\frac{\theta^{m_i} \left| D^{m_i} U(\phi) \right|}{m_i!} \right] .$$

We now assume $\theta \leq \gamma_1 |a_4|^{-\frac{1}{4}}$ for some suitable $\gamma_1 \geq 0$ to be specified later. We insert this inequality in (13) and pull out ${\gamma_1}^{\sum m_i} = {\gamma_1}^n$ before throwing away the constraint $\sum m_i = n$ which results in

$$\frac{\theta^{n}}{n!} \left| D^{n} e^{-U(\phi)} \right| \leq e^{-\Re U(\phi)} \gamma_{1}^{n} \times \sum_{k=1}^{n} \frac{1}{k!} \left(\sum_{m=1}^{4} \frac{|a_{4}|^{-\frac{m}{4}}}{m!} |D^{m} U(\phi)| \right)^{k} \\
\leq \gamma_{1}^{n} \exp \left[-\Re U(\phi) + \sum_{m=1}^{4} \frac{|a_{4}|^{-\frac{m}{4}}}{m!} |D^{m} U(\phi)| \right].$$

The individual quantities in the last exponential are bounded in terms of $x = |a_4|^{\frac{1}{4}} |\phi|$ as follows:

$$|a_4|^{-\frac{1}{4}}|DU(\phi)| = |a_4|^{-\frac{1}{4}} \times |4a_4\phi^3 + 3a_3\phi^2 + 2a_2\phi + a_1|$$

$$\leq 4x^3 + 3rx^2 + 2rx + r,$$

$$\frac{|a_4|^{-\frac{2}{4}}}{2} |D^2 U(\phi)| = |a_4|^{-\frac{1}{2}} \times |6a_4\phi^2 + 3a_3\phi + a_2|$$

$$\leq 6x^2 + 3rx + r,$$

$$\frac{|a_4|^{-\frac{3}{4}}}{3!} |D^3 U(\phi)| = |a_4|^{-\frac{3}{4}} \times |4a_4 \phi + a_3|$$

$$\leq 4x + r,$$

whereas

$$\frac{|a_4|^{-\frac{4}{4}}}{4!} |D^4 U(\phi)| = 1.$$

Therefore

$$\sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^m U(\phi)| \le 4x^3 + (3r+6)x^2 + (5r+4)x + (3r+1)$$

and

$$-\Re U(\phi) + \sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^m U(\phi)| \le -\frac{1}{2} (\Re a_4) \phi^4 + \Omega_2(|a_4|^{\frac{1}{4}} |\phi|) + |a_0|$$

where

$$\Omega_2(x) = -\frac{\alpha}{2}x^4 + (r+4)x^3 + (4r+6)x^2 + (6r+4)x + (3r+1) \ .$$

We again find a convenient bound on $\sup_{x\geq 0}\Omega_2(x)$. Simply using r<1 and dropping the x^4 term we have, for $0\leq x\leq 1$, the bound $\Omega_2(x)\leq 5+10+10+4=29$. For $x\geq 1$, we have the crude bound $\Omega_2(x)\leq -\frac{\alpha}{2}x^4+29x^3$ and we proceed with maximizing the right-hand side over $x\in[0,\infty)$. When $\alpha=\frac{\sqrt{3}}{2}$

the maximum occurs at $x=\frac{87}{\sqrt{3}}$ and is equal to $\frac{29^4\times3\sqrt{3}}{4}\simeq 918784.97\ldots < 918785$. We denote the latter numerical constant by M. The previous considerations now give

$$||e^{-U(\phi)}||_{\partial \phi, \phi, \theta} \leq e^{-\frac{1}{2}(\Re a_4)\phi^4 + 3r + |a_0|}$$

$$+ \sum_{n=1}^{9} \gamma_1^n \exp\left[-\frac{1}{2}(\Re a_4)\phi^4 + M + |a_0|\right]$$

$$\leq e^{-\frac{1}{2}(\Re a_4)\phi^4} \times e^{|a_0|} \times \left[e^{3r} + e^M \times \frac{\gamma_1}{1 - \gamma_1}\right]$$

provided $\gamma_1 < 1$. If one requires the stronger condition $\gamma_1 \leq \frac{1}{2}$ then $e^{3r} + e^M \times \frac{\gamma_1}{1-\gamma_1} \leq e^{3r} + 2e^M \gamma_1$. From our choice for r we have $e^{3r} = \frac{1+\sqrt{2}}{2}$. If we now set $\gamma_1 = \frac{\sqrt{2}-1}{4}e^{-M}$ which clearly is less than $\frac{1}{2}$ then $e^{3r} + 2e^M \gamma_1 = \sqrt{2}$. On the other hand, by assumption on a_0 we have $e^{|a_0|} \leq \sqrt{2}$. The statement in 1) is therefore proved.

For the statement in 2) concerning the bound on $|e^{-U(\phi)}|_{\partial\phi,\theta} = ||e^{-U(\phi)}||_{\partial\phi,0,\theta}$, with derivatives taken at zero, we follow the same steps. However, the situation simplifies considerably. Indeed,

$$|e^{-U(\phi)}|_{\partial\phi,\theta} = e^{-\Re U(0)} + \sum_{n=1}^{9} \frac{\theta^n}{n!} \left| D^n e^{-U(\phi)} \right|_{\phi=0}$$

can be bounded as we did before, under the new hypothesis $\theta \leq \gamma_2 |a_4|^{-\frac{1}{4}}$ for suitable $\gamma_2 \in [0,1)$, by the estimate

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \le e^{-\Re a_0} + \frac{\gamma_2}{1-\gamma_2} \times \exp\left[-\Re U(0) + \sum_{m=1}^4 \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^m U(0)|\right].$$

Now

$$\sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^m U(0)| = \sum_{m=1}^{4} |a_4|^{-\frac{m}{4}} |a_m| \le 3r + 1.$$

If one imposes the condition $\gamma_2 \leq \frac{1}{2}$, then

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \le e^{|a_0|} \times [1 + 2\gamma_2 e^{3r+1}]$$
.

Because of the chosen value of r, one will have $1+2\gamma_2 e^{3r+1}=\sqrt{2}$ if one now sets $\gamma_2=\frac{(\sqrt{2}-1)^2}{e}\simeq 0.0631\ldots$ which is less than $\frac{1}{2}$. The statement in 2) then follows easily.

6. The main estimates on a single extended RG step

6.1. Statement of the theorem. Recall that $\epsilon \in (0,1]$, $L=p^l$ with $p \geq 2$ a prime number and where $l \geq 1$ is an integer. The symbol $[\phi]$ denotes the quantity $\frac{3-\epsilon}{4}$. We now introduce the numerical constants

$$c_1 = 2^{-\frac{9}{4}}(\sqrt{2} - 1)e^{-918785}$$
 and $c_2 = 2^{\frac{3}{4}}$

which are used to calibrate the parameters

$$h = c_1 \bar{g}^{-\frac{1}{4}}$$
 and $h_* = c_2 L^{\frac{3+\epsilon}{4}}$

for the seminorms we use. With these choices the norm

$$|||R|||_{\bar{g}} = \max \left\{ |R(\phi)|_{\partial \phi, h_*}, \bar{g}^2 \sup_{\phi \in \mathbb{R}} ||R(\phi)||_{\partial \phi, \phi, h} \right\}$$

is now unambiguously defined in terms of the calibrator \bar{g} . In the present article we will need to take \bar{g} of order ϵ , however the next theorem will be stated in greater generality as far as allowed values for this calibrator. Indeed, we intend to reuse the rather expensive theorem that follows for the construction of another quantum field theory on \mathbb{Q}_p^3 which corresponds to an RG trajectory joining the Gaussian and the infrared fixed points as in [2].

Theorem 4. $\exists B_{RL} \geq 0, \forall l \geq 1, \exists B_0, \dots, B_4, B_{R\xi} \geq 0,$

 $\forall \eta \in \left[0, \frac{1}{4}\right), \ \forall \eta_R \in \left[3\eta, \frac{3}{16} + \frac{9}{4}\eta\right], \\ \forall A_{\bar{g}} > 0, \ \exists \epsilon_0 > 0, \ \forall \epsilon \in (0, \epsilon_0], \ \forall \bar{g} \in (0, A_{\bar{g}}\epsilon] \ and \ \Delta' \in \mathbb{L}, \ then \ on \ the \ domain$

$$\forall \Delta \in [L^{-1}\Delta'], \begin{cases} |\beta_{4,\Delta} - \bar{g}| < \frac{1}{2}\bar{g} \\ |\beta_{k,\Delta}| < \bar{g}^{1-\eta} & \text{for } k = 1, 2, 3 \\ |W_{k,\Delta}| < \bar{g}^{2-2\eta} & \text{for } k = 5, 6 \\ |f_{\Delta}| < L^{-(3-[\phi])} \\ |||R_{\Delta}|||_{\bar{g}} < \bar{g}^{\frac{11}{4}-\eta_R} \end{cases}$$

the maps $\xi_{0,\Delta'},\ldots,\xi_{4,\Delta'},\,\mathcal{L}_{\Delta'}$ and $\xi_{R,\Delta'}$ are well-defined, analytic, send real data to real data and satisfy the bounds

$$|\xi_{k,\Delta'}(\vec{V})| \le B_k \max_{\Delta \in [L^{-1}\Delta']} |||R_{\Delta}|||_{\bar{g}} \quad \text{for } k = 0, \dots, 4,$$

 $|||\mathcal{L}_{\Delta'}^{\vec{\beta},f}(R)|||_{\bar{g}} \le B_{R\mathcal{L}} L^{3-5[\phi]} \max_{\Delta \in [L^{-1}\Delta']} |||R_{\Delta}|||_{\bar{g}},$

and

$$|||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}} \leq B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta}$$
.

Remark 1. In fact, except for the map $\xi_{R,\Delta'}$, the conclusions of the theorem are valid without restriction on the size of R. This is because $\xi_{0,\Delta'}, \ldots, \xi_{4,\Delta'}, \mathcal{L}_{\Delta'}$ are linear in R.

6.2. The standard hypotheses. In this section we collect a set of conditions which we call the standard hypotheses and which will be assumed throughout §6.3 and therefore will not be repeated in the local statements of the lemmas in that section. Finally in §6.4 we will show that it is possible to satisfy all these conditions and thus wrap up the proof of Theorem 4. The list of conditions repeats some statements made before and it is also redundant. We meant this list to collect in one place the various requirements for the validity of the lemmas in §6.3. We assume:

$$h = c_1 \bar{g}^{-\frac{1}{4}}$$
 with $c_1 = 2^{-\frac{9}{4}} (\sqrt{2} - 1) e^{-918785}$,

(14)
$$h_* = c_2 L^{\frac{3+\epsilon}{4}}$$
 with $c_2 = 2^{\frac{3}{4}}$,

(15)
$$0 < \bar{g} \le 1 , \ \eta \ge 0 , \ \eta_R \ge 0 , \ \eta < \frac{1}{4} ,$$

(16)
$$|\beta_{4,\Delta} - \bar{g}| < \frac{1}{2}\bar{g}, \ |\beta_{k,\Delta}| < \bar{g}^{1-\eta} \text{ for } k = 1, 2, 3,$$

(17)
$$|f_{\Delta}| < L^{-(3-[\phi])}, |W_{k,\Delta}| < \bar{g}^{2-2\eta} \text{ for } k = 5, 6,$$

(18)
$$|||R_{\Delta}|||_{\bar{g}} < \bar{g}^{\frac{11}{4} - \eta_R} \text{ where } |||R_{\Delta}|||_{\bar{g}} = \max \left\{ |R_{\Delta}(\phi)|_{\partial \phi, h_*}, \bar{g}^2 \sup_{\phi \in \mathbb{R}} ||R_{\Delta}(\phi)||_{\partial \phi, \phi, h} \right\} ,$$

(19)
$$2^{\frac{3}{4}} \times L^{\frac{3}{4}} \times \bar{g}^{\frac{1}{4} - \eta} \\ 19\sqrt{2} \times L^{\frac{3}{2}} \times \bar{g}^{\frac{1}{2} - \eta} \\ 2^{\frac{1}{4}} \times 7 \times L^{\frac{9}{4}} \times \bar{g}^{\frac{3}{4} - \eta} \right\} \leq \frac{1}{3} \log \left(\frac{1 + \sqrt{2}}{2} \right) ,$$

(20)
$$L^{\epsilon} \leq \frac{4}{3} , \ 20 \times L^{2} \times \bar{g}^{1-\eta} \leq \log 2 , \ \bar{g}^{\frac{1}{4}} \leq \frac{(\sqrt{2}-1)^{2}}{2e} \times L^{-1} ,$$

$$\bar{g} \le c_1^4 c_2^{-4} L^{-1} \ , \ \bar{g} \le c_1^{36} c_2^{-36} L^{-36} \ ,$$

(22)
$$\eta_R \ge 3\eta \; , \; \eta_R \le 1 + 3\eta \; ,$$

(23)
$$\exp\left(2\bar{g}^{\frac{1}{4}} + 18\bar{g}\right) \le 2 , \ 8L^4\bar{g}^{1-\eta} \le 1 ,$$

(24)
$$\eta_R \le \frac{3}{16} + \frac{9}{4}\eta \text{ and } \eta_R < \frac{3}{4}$$
.

We also assume the additional conditions

$$\begin{pmatrix}
2\mathcal{O}_{32} \times L^{15} \times \bar{g}^{\frac{1}{4} - \frac{\eta_R}{3}} \\
2\mathcal{O}_{35} \times L^{15} \times \bar{g}^{\frac{11}{12} - \frac{\eta_R}{3}} \\
3\mathcal{O}_{27} \times L^5 \times \bar{g}^{\frac{1}{4} - \eta} \\
\mathcal{O}_{28} \times L^9 \times \bar{g}^{\frac{11}{12} - \frac{\eta_R}{3}}
\end{pmatrix} \leq 1$$

which involve purely numerical constants \mathcal{O} to be specified in §6.3.

6.3. A long series of lemmas. Assuming the conditions stated in §6.2 we now embark on the following series of lemmas which will lead to the proof of Theorem 4. The estimates will involve a collection of numerical constants which are given explicitly and are numbered as $\mathcal{O}_1, \mathcal{O}_2$, etc. Since we are not aiming for optimal estimates, our motivation for keeping such constants explicit is to serve as an Ariadne thread for the reader in her/his journey through the following estimates which involve many interdependent parameters. The notations will continue those of §4.2.

Lemma 11. For all $t \in (0,1]$ and all unit cube Δ we have

$$\forall \phi \in \mathbb{R} , \quad ||e^{-tV_{\Delta}(\phi)}||_{\partial \phi, \phi, h} \leq 2^{-\frac{t}{2}(\Re \beta_{4,\Delta})\phi^4}$$

as well as

$$|e^{-tV_{\Delta}(\phi)}|_{\partial\phi,h_{\pi}} \leq 2$$
.

Proof: From the definition and by undoing the Wick ordering, we have

$$V_{\Delta}(\phi) = \sum_{k=1}^{4} \beta_{k,\Delta} : \phi^{k} :_{C_{0}}$$

$$= \beta_{4,\Delta} \left(\phi^{4} - 6C_{0}(0)\phi^{2} + 3C_{0}(0)^{2} \right)$$

$$+ \beta_{3,\Delta} \left(\phi^{3} - 3C_{0}(0)\phi \right)$$

$$+ \beta_{2,\Delta} \left(\phi^{2} - C_{0}(0) \right)$$

$$+ \beta_{1,\Delta} \phi$$

and therefore $tV_{\Delta}(\phi) = \sum_{k=0}^{4} a_k \phi^k$ with

$$\begin{array}{rcl} a_4 & = & t\beta_{4,\Delta} \ , \\ a_3 & = & t\beta_{3,\Delta} \ , \\ a_2 & = & t\left(\beta_{2,\Delta} - 6C_0(0)\beta_{4,\Delta}\right) \ , \\ a_1 & = & t\left(\beta_{1,\Delta} - 3C_0(0)\beta_{3,\Delta}\right) \ , \\ a_0 & = & t\left(-C_0(0)\beta_{2,\Delta} + 3C_0(0)^2\beta_{4,\Delta}\right) \ . \end{array}$$

We now simply check that the requirements in Lemma 10 are satisfied. From the standard hypothesis (16) we have $|a_4| = t|\beta_{4,\Delta}| > 0$ as well as $(\Re a_4) \times |a_4|^{-1} = (\Re \beta_{4,\Delta}) \times |\beta_{4,\Delta}| \ge \frac{\sqrt{3}}{2}$. We have, since $t \in (0,1]$,

$$|a_3| \times |a_4|^{-\frac{3}{4}} = t^{\frac{1}{4}} |\beta_{3,\Delta}| \times |\beta_{4,\Delta}|^{-\frac{3}{4}}$$

$$\leq \bar{g}^{1-\eta} \times \left(\frac{1}{2}\bar{g}\right)^{-\frac{3}{4}}$$

$$\leq \frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2}\right)$$

by (19). Likewise, using Lemma 4, we have

$$|a_{2}| \times |a_{4}|^{-\frac{2}{4}} \leq t^{\frac{1}{2}} [|\beta_{2,\Delta}| + 12|\beta_{4,\Delta}|] \times |\beta_{4,\Delta}|^{-\frac{1}{2}}$$

$$\leq \left(\bar{g}^{1-\eta} + 12 \times \frac{3}{2}\bar{g}\right) \times \left(\frac{1}{2}\bar{g}\right)^{-\frac{1}{2}}$$

$$\leq 19\sqrt{2} \times \bar{g}^{\frac{1}{2}-\eta}$$

since $\eta \geq 0$ and $\bar{g} \leq 1$. The standard hypothesis (19) then gives us the desired $\frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2} \right)$ upper bound. In the same way, we have

$$|a_1| \times |a_4|^{-\frac{1}{4}} \leq t^{\frac{3}{4}} [|\beta_{1,\Delta}| + 6|\beta_{3,\Delta}|] \times |\beta_{4,\Delta}|^{-\frac{1}{4}}$$

$$\leq 7\bar{g}^{1-\eta} \times \left(\frac{1}{2}\bar{g}\right)^{-\frac{1}{4}}$$

$$\leq \frac{1}{3}\log\left(\frac{1+\sqrt{2}}{2}\right)$$

by (19). Now

$$|a_0| \le 2|\beta_{2,\Delta}| + 12|\beta_{4,\Delta}| \le 2\bar{g}^{1-\eta} + 18\bar{g} \le \log 2$$

by (23). If one takes θ in Lemma 10 to be h, then the needed condition from part 1) of that lemma is equivalent to $|a_4| \leq 2\bar{g}$ which holds since $|a_4| = t|\beta_{4,\Delta}| \leq \frac{3}{2}\bar{g}$. Likewise if one takes $\theta = h_*$ and wants to use part 2) of the lemma then the required condition is $2^{\frac{3}{4}}L^{\left(\frac{3+\epsilon}{4}\right)} \leq \frac{(\sqrt{2}-1)^2}{e}|a_4|^{-\frac{1}{4}}$. Since $L \geq 2$ and $\epsilon \in (0,1]$, it is enough to have

$$|a_4|^{\frac{1}{4}} \le L^{-1}2^{-\frac{3}{4}} \times \frac{(\sqrt{2}-1)^2}{e}$$
.

Using $|a_4| \leq 2\bar{g}$, we see that the requirement follows from the standard hypothesis (20). The desired inequalities now follow from Lemma 10.

Lemma 12. For all $t \in (0,1]$ and all unit cube Δ we have

$$\forall \psi \in \mathbb{R} , \quad ||e^{-t\tilde{V}_{\Delta}(\psi)}||_{\partial \psi, \psi, h} \le 2^{-\frac{t}{2}(\Re \beta_{4,\Delta})\psi^4}$$

as well as

$$|e^{-t\tilde{V}_{\Delta}(\psi)}|_{\partial\psi,h_*} \leq 2$$
.

Proof: Recall that \tilde{V}_{Δ} is defined in the same way as V_{Δ} except that the Wick ordering is with respect to C_1 instead of C_0 . We again use Lemma 10 in order to prove the wanted result. The formulas for the a_k 's are exactly the same as in the previous lemma apart for changing $C_0(0)$ to $C_1(0) = L^{-2[\phi]}C_0(0)$ which is also bounded by 2. Since the latter property is the only thing we used about C_0 , the present lemma then follows in the same manner as the previous one.

Lemma 13. For all unit cube Δ' and for all subset $Y_0 \subset [L^{-1}\Delta']$ we have

$$\forall \phi \in \mathbb{R} , \left\| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial \phi, \phi, h} \le 2$$

as well as

$$\left| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right|_{\partial \phi, h_*} \le 2.$$

If $|Y_0| \ge \frac{L^3}{2}$ (which holds if $|Y_0| = L^3$ or $L^3 - 1$ because $L \ge 2$) then we have the improved bound

$$\forall \phi \in \mathbb{R} , \quad \left\| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial \phi, \phi, h} \le 2e^{-\frac{\tilde{g}}{16}\phi^4} .$$

Here ϕ_1 denotes the rescaled field $L^{-[\phi]}\phi$.

Proof: The argument is similar to the previous lemmas provided one keeps in mind that $\phi_1 = L^{-[\phi]}\phi$ but derivatives are with respect to ϕ . In the degenerate case where Y_0 is empty there is nothing to prove so we will assume that $|Y_0| > 0$. By definition

$$\prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} = e^{-U(\phi)}$$

where $U(\phi) = \sum_{k=0}^{4} a_k \phi^k$ with

$$a_{4} = \sum_{\Delta \in Y_{0}} L^{-4[\phi]} \beta_{4,\Delta}$$

$$a_{3} = \sum_{\Delta \in Y_{0}} L^{-3[\phi]} \beta_{3,\Delta}$$

$$a_{2} = \sum_{\Delta \in Y_{0}} \left(L^{-2[\phi]} \beta_{2,\Delta} - 6L^{-4[\phi]} C_{0}(0) \beta_{4,\Delta} \right)$$

$$a_{1} = \sum_{\Delta \in Y_{0}} \left(L^{-[\phi]} \beta_{1,\Delta} - 3L^{-3[\phi]} C_{0}(0) \beta_{3,\Delta} \right)$$

$$a_{0} = \sum_{\Delta \in Y_{0}} \left(-L^{-2[\phi]} C_{0}(0) \beta_{2,\Delta} + 3L^{-4[\phi]} C_{0}(0)^{2} \beta_{4,\Delta} \right) .$$

From

$$a_4 = |Y_0|L^{-4[\phi]}\bar{g} + \sum_{\Delta \in Y_0} L^{-4[\phi]}(\beta_{4,\Delta} - \bar{g})$$

we get

$$\left| a_4 - |Y_0| L^{-4[\phi]} \bar{g} \right| \le \frac{1}{2} \bar{g} |Y_0| L^{-4[\phi]}$$

and therefore

$$|a_4| \ge \frac{1}{2}\bar{g}|Y_0|L^{-4[\phi]} > 0$$
.

Since a_4 is $|Y_0|L^{-4[\phi]}$ times a barycenter of elements in the convex set $\{\beta \in \mathbb{C} | |\beta - \bar{g}| < \frac{1}{2}\bar{g}\}$, we easily see that $\Re a_4 \geq \frac{\sqrt{3}}{2}|a_4|$ holds. We also have

$$|a_3| \times |a_4|^{-\frac{3}{4}} \le |Y_0| L^{-3[\phi]} \bar{g}^{1-\eta \left(\frac{1}{2}|Y_0|\bar{g}L^{-4[\phi]}\right)^{-\frac{3}{4}}}$$

$$\le 2^{\frac{3}{4}} |Y_0|^{\frac{1}{4}} \bar{g}^{\frac{1}{4}-\eta} \le 2^{\frac{3}{4}} L^{\frac{3}{4}} \bar{g}^{\frac{1}{4}-\eta} \le \frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2}\right)$$

by (19). Likewise

$$|a_2| \times |a_4|^{-\frac{2}{4}} \le |Y_0| \left(L^{-2[\phi]} \bar{g}^{1-\eta} + L^{-4[\phi]} \times 12 \times \frac{3}{2} \bar{g} \right) \times \left(\frac{1}{2} |Y_0| \bar{g} L^{-4[\phi]} \right)^{-\frac{1}{2}}$$

$$\le 2^{\frac{1}{2}} |Y_0|^{\frac{1}{2}} \left(\bar{g}^{\frac{1}{2} - \eta} + 18L^{-2[\phi]} \bar{g}^{\frac{1}{2}} \right) \le 19\sqrt{2}L^{\frac{3}{2}} \bar{g}^{\frac{1}{2} - \eta}$$

which is bounded using (19). Also,

$$|a_{1}| \times |a_{4}|^{-\frac{1}{4}} \leq |Y_{0}| \left(L^{-[\phi]} \bar{g}^{1-\eta} + 6L^{-3[\phi]} \bar{g}^{1-\eta} \right) \times \left(\frac{1}{2} |Y_{0}| \bar{g}L^{-4[\phi]} \right)^{-\frac{1}{4}}$$

$$\leq 2^{\frac{1}{4}} \times 7 \times |Y_{0}|^{\frac{3}{4}} \bar{g}^{\frac{3}{4}-\eta} \leq 2^{\frac{1}{4}} \times 7 \times L^{\frac{9}{4}} \bar{g}^{\frac{3}{4}-\eta} \leq \frac{1}{3} \log \left(\frac{1+\sqrt{2}}{2} \right)$$

by (19). Finally,

$$|a_0| \le |Y_0| \left(2L^{-2[\phi]} \bar{g}^{1-\eta} + 12L^{-4[\phi]} \times \frac{1}{2} \bar{g} \right)$$

$$\le 20L^{3-2[\phi]} \bar{g}^{1-\eta} \le 20L^2 \bar{g}^{1-\eta} \le \log 2$$

by (20) and using $L^{\frac{3+\epsilon}{2}} \leq L^2$.

In order to apply Lemma 10 1) it is enough to have $|a_4| \leq 2\bar{g}$. Since

$$|a_4| \le L^{3-4[\phi]} \times \frac{3}{2}\bar{g} = \frac{3}{2}L^{\epsilon}\bar{g} ,$$

all we need is $L^{\epsilon} \leq \frac{4}{3}$ which is a condition in (20). As for the use of Lemma 10 2), a condition in (20), namely,

$$\bar{g}^{\frac{1}{4}} \le \frac{(\sqrt{2}-1)^2}{2e} L^{-1}$$

implies

$$|a_4|^{\frac{1}{4}} \le (2\bar{g})^{\frac{1}{4}} \le \frac{(\sqrt{2}-1)^2}{e} L^{-1} \times 2^{-\frac{3}{4}} \le \frac{(\sqrt{2}-1)^2}{e} L^{-\left(\frac{3+\epsilon}{4}\right)} \times 2^{-\frac{3}{4}}$$

which is the required relationship between $|a_4|$ and h_* .

As a result of the previous considerations and following the application of Lemma 10 we arrive at

(26)
$$\left\| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial \phi, \phi, h} \le 2e^{-\frac{1}{2}(\Re a_4)\phi^4} \le 2$$

since $\Re a_4 \ge \frac{\sqrt{3}}{2}|a_4| > 0$, as well as

$$\left| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right|_{\partial \phi_{\cdot}, h_{\star}} \leq 2.$$

Now we impose the stronger hypothesis $|Y_0| \ge \frac{L^3}{2}$. Then

$$\Re a_4 = \sum_{\Delta \in Y_0} L^{-4[\phi]} \Re \beta_{4,\Delta} \ge |Y_0| L^{-4[\phi]} \times \frac{1}{2} \bar{g}$$

$$\geq \frac{1}{4}L^{3-4[\phi]}\bar{g} \geq \frac{1}{4}\bar{g}$$

because $L^{\epsilon} > 1$. As a result, the crude bound by 2 in (26) can be amended to

$$\left\| \prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial \phi, \phi, h} \le 2e^{-\frac{1}{16}\bar{g}\phi^4}.$$

The special case $Y_0 = [L^{-1}\Delta']$ of the previous lemma, which is important for bounding $e^{-\hat{V}}$ expressions, will be stated as a separate lemma. It simply follows from the observation

$$\prod_{\Delta \in Y_0} e^{-\tilde{V}_{\Delta}(\phi_1)} = e^{-\hat{V}_{\Delta'}(\phi)}$$

when $Y_0 = [L^{-1}\Delta']$.

Lemma 14. For all unit boxes Δ' , we have

$$\forall \phi \in \mathbb{R}, \quad ||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial \phi, \phi, h} \le 2e^{-\frac{1}{16}\bar{g}\phi^4} \ .$$

and

$$|e^{-\hat{V}_{\Delta'}(\phi)}|_{\partial \phi, h_*} \leq 2$$
.

Lemma 15. For all unit cube Δ , for all $\kappa, \gamma \in (0,1]$ and for all $\phi, \zeta \in \mathbb{R}$ we have

$$||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \mathcal{O}_1 \kappa^{-2} \gamma^{-\frac{3}{4}} \bar{g}^{\frac{1}{4}-\eta} e^{\kappa \zeta_{\Delta}^2} e^{\gamma(\Re\beta_{4,\Delta})\phi_1^4}$$

where $O_1 = 54600$.

Proof: Recall that

$$p_{\Delta}(\phi_1, \zeta) = \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} a+b \leq 4 \\ a \geq 0 , b \geq 1 \end{array} \right\} \frac{(a+b)!}{a! \ b!} \ \beta_{a+b,\Delta} : \phi_1^a :_{C_1} \times : \zeta^b :_{\Gamma}$$

where in fact : $\phi_1^a:_{C_1}$ means : $\psi^a:_{C_1}|_{\psi=\phi_1}$ with $\phi_1=L^{-[\phi]}\phi$. We use

$$||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \le ||p_{\Delta}(\psi,\zeta)||_{\partial\psi,\phi_1,h}$$

because by the chain rule each derivative with respect to ϕ brings a factor $L^{-[\phi]} \leq 1$ times the corresponding derivative with respect to ψ . We therefore have

$$||p_{\Delta}(\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} a+b \leq 4 \\ a \geq 0 , b \geq 1 \end{array} \right\} \frac{(a+b)!}{a! \ b!} \ |\beta_{a+b,\Delta}| \times ||:\psi^{a}:_{C_{1}}||_{\partial\psi,\phi_{1},h} \times |:\zeta^{b}:_{\Gamma}|.$$

We undo the Wick ordering noting that $1 \le b \le 4$ and $0 \le a \le 3$. From

$$: \psi^{3} :_{C_{1}} = \psi^{3} - 3L^{-2[\phi]}C_{0}(0)\psi$$

$$: \psi^{2} :_{C_{1}} = \psi^{2} - L^{-2[\phi]}C_{0}(0)$$

$$: \psi^{1} :_{C_{1}} = \psi$$

$$: \psi^{0} :_{C_{1}} = 1$$

and the bound $C_0(0) < 2$ we get

$$||: \psi^a:_{C_1} ||_{\partial \psi, \phi_1, h} \le 7 \max_{0 \le k \le a} ||\psi^k||_{\partial \psi, \phi_1, h}$$

for $0 \le a \le 3$. Since, by Lemma 2, $\Gamma(0) = C_0(0)(1 - L^{-2[\phi]}) < C_0(0) < 2$, the explicit formulas

$$\begin{array}{rcl} :\zeta^4:_{\Gamma}&=&\zeta^4-6\Gamma(0)\zeta^2+3\Gamma(0)^2\\ :\zeta^3:_{\Gamma}&=&\zeta^3-3\Gamma(0)\zeta\\ :\zeta^2:_{\Gamma}&=&\zeta^2-\Gamma(0)\\ :\zeta^1:_{\Gamma}&=&\zeta\end{array}$$

similarly imply

$$|:\zeta^b:_{\Gamma}| \le 25 \max_{0 \le j \le b} |\zeta|^j$$

for $1 \le b \le 4$. The $|\beta|$'s are bounded by $\frac{3}{2}\bar{g}$ or $\bar{g}^{1-\eta}$. Since $\eta \ge 0$ and $\bar{g} \le 1$ we use the uniform bound by the worst case scenario $\frac{3}{2}\bar{g}^{1-\eta}$. Thus

$$||p_{\Delta}(\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq 7 \times 25 \times \sum_{a,b} \mathbb{1} \left\{ \begin{array}{l} a+b \leq 4 \\ a \geq 0 , b \geq 1 \end{array} \right\} \frac{(a+b)!}{a! \ b!} |\beta_{a+b,\Delta}|$$

$$\times \frac{3}{2} \bar{g}^{1-\eta} \times \max_{0 \leq k \leq a} ||\psi^{k}||_{\partial\psi,\phi_{1},h} \times \max_{0 \leq j \leq b} |\zeta|^{j} .$$

The binomial coefficients appearing in the sum add up to 26 and therefore using a maximum over a, b instead of a sum gives rise to the numerical coefficient $7 \times 25 \times \frac{3}{2} \times 26 = 6825$, namely,

$$||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \le 6825\bar{g}^{1-\eta} \times \max_{a,b} \left[\max_{0\le k \le a} ||\psi^k||_{\partial\psi,\phi_1,h} \times \max_{0\le j \le b} |\zeta|^j \right]$$

where the maximum is over pairs of integers which satisfy $a \ge 0$, $b \ge 1$ and $a + b \le 4$. By Lemma 9 and for $\kappa \in (0,1]$ and $b \le 4$ we have

$$\max_{0 \le j \le b} |\zeta|^j \le \kappa^{-2} e^{\kappa \zeta^2} \max_{0 \le j \le 4} \left(\frac{j}{2e}\right)^{\frac{j}{2}} \le \kappa^{-2} e^{\kappa \zeta^2}.$$

For $0 \le k \le 3 < 9$ we have

$$||\psi^k||_{\partial\psi,\psi,h} = (h+|\psi|)^k = \sum_{n=0}^k \binom{k}{n} (c_1\bar{g}^{-\frac{1}{4}})^{k-n}|\psi|^n.$$

We use Lemma 8 to write

$$|\psi|^n \le \left(\frac{n}{2e}\right)^{\frac{n}{4}} [\gamma \bar{g}]^{-\frac{n}{4}} e^{\gamma(\Re \beta_{4,\Delta})\psi^4}$$

We drop the numerical factor since $n \leq 3 < 2e$ and use $\gamma \in (0,1]$ to arrive at

$$||\psi^k||_{\partial\psi,\psi,h} \le \gamma^{-\frac{3}{4}} e^{\gamma(\Re\beta_{4,\Delta})\psi^4} \bar{g}^{-\frac{k}{4}} \times \sum_{n=0}^k \binom{k}{n} c_1^{k-n}.$$

The last sum reduces to $(1+c_1)^k < 2^k \le 8$. Therefore

$$\max_{0 \le k \le 3} ||\psi^k||_{\partial \psi, \phi_1, h} \le 8\gamma^{-\frac{3}{4}} \bar{g}^{-\frac{3}{4}} e^{\gamma(\Re \beta_{4, \Delta})\phi_1^4}$$

and the result follows.

Lemma 16. For all κ such that $0 < \kappa < 2^{-\frac{3}{2}}L^{-(3-2[\phi])}$ and for all $\zeta \in \mathbb{R}$ we have

$$|p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \leq \mathcal{O}_2\kappa^{-2}\bar{g}^{1-\eta}e^{\kappa\zeta_{\Delta}^2}$$

where $\mathcal{O}_2 = 6825$.

Proof: We proceed as in the previous lemma, thus arriving at the bound

$$|p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \leq |p_{\Delta}(\psi,\zeta)|_{\partial\psi,h_*}$$

$$\leq 6825\bar{g}^{1-\eta} \times \max_{a,b} \left[\max_{0 \leq k \leq a} |\psi^k|_{\partial \psi, h_*} \times \max_{0 \leq j \leq b} |\zeta|^j \right]$$

with the same conditions on a and b. However, now $|\psi^k|_{\partial\psi,h_*}=h_*^k$. By again using Lemma 9 we get

$$|p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \le 6825\bar{g}^{1-\eta} \times \max_{\substack{k,j\ge 0\\k+j\le 4}} \left[h_*^k \kappa^{-\frac{j}{2}} \left(\frac{j}{2e} \right)^{\frac{j}{2}} e^{\kappa\zeta^2} \right].$$

We drop the cumbersome factor $\left(\frac{j}{2e}\right)^{\frac{j}{2}} \leq 1$ and note that the hypothesis on κ ensures that $h_* \leq \kappa^{-\frac{1}{2}}$. This implies

$$h_{*}^{k}\kappa^{-\frac{j}{2}} < \kappa^{-\frac{k+j}{2}} < \kappa^{-2}$$

because of the condition $k+j \leq 4$ and $\kappa \leq 1$ which also follows from the hypothesis.

Lemma 17. For all $\kappa \in (0,3]$ we have

$$||r_{1,\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \le \mathcal{O}_3\kappa^{-6}e^{\kappa\zeta_\Delta^2}\bar{g}^{\frac{3}{4}-3\eta}$$

where $\mathcal{O}_3 = 2^{\frac{21}{4}} \times 3^{\frac{29}{4}} \times \mathcal{O}_1^3$.

Proof: By definition

$$r_{1,\Delta}(\phi_1,\zeta) = e^{-\tilde{V}_{\Delta}(\phi_1)} \left[e^{-p\Delta} - 1 + p_{\Delta} - \frac{1}{2} p_{\Delta}^2 \right]$$

where p_{Δ} is shorthand for $p_{\Delta}(\phi_1, \zeta)$. This is the third order Taylor remainder when expanding $e^{-\tilde{V}_{\Delta}(\phi_1)-sp_{\Delta}}$ at s=1 about 0. We can also write

$$r_{1,\Delta}(\phi_1,\zeta) = \int_0^1 dt \, \frac{(1-t)^2}{2} (-p_\Delta)^3 e^{-\tilde{V}_\Delta(\phi_1) - tp_\Delta} \, .$$

But

$$\tilde{V}_{\Delta}(\phi_1) + tp_{\Delta} = \tilde{V}_{\Delta}(\phi_1) + t \left(V_{\Delta}(\phi_1 + \zeta) - \tilde{V}_{\Delta}(\phi_1) \right)
= tV_{\Delta}(\phi_1 + \zeta) + (1 - t)\tilde{V}_{\Delta}(\phi_1) .$$

By the multiplicative property of the seminorms

$$||e^{-\tilde{V}_{\Delta}(\phi_{1})-tp_{\Delta}}||_{\partial\phi,\phi,h} = ||e^{tV_{\Delta}(\phi_{1}+\zeta)} \times e^{(1-t)\tilde{V}_{\Delta}(\phi_{1})}||_{\partial\phi,\phi,h}$$

$$\leq ||e^{tV_{\Delta}(\phi_{1}+\zeta)}||_{\partial\phi,\phi,h} \times ||e^{(1-t)\tilde{V}_{\Delta}(\phi_{1})}||_{\partial\phi,\phi,h}$$

and thus

$$||r_{1,\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \int_0^1 dt \, \frac{(1-t)^2}{2} ||(-p_\Delta)^3 e^{-\tilde{V}_\Delta(\phi_1) - tp_\Delta}||_{\partial\phi,\phi,h}$$
$$\leq \frac{1}{2} \int_0^1 dt \, \frac{(1-t)^2}{2} ||p_\Delta||_{\partial\phi,\phi,h}^3 \times ||e^{tV_\Delta(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \times ||e^{(1-t)\tilde{V}_\Delta(\phi_1)}||_{\partial\phi,\phi,h} .$$

Using the same inequality comparing derivatives with respect to ϕ versus $\psi = \phi_1$ as in (27) we obtain

$$||e^{tV_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \le ||e^{tV_{\Delta}(\psi+\zeta)}||_{\partial\psi,\phi_1,h} = ||e^{tV_{\Delta}(\psi)}||_{\partial\psi,\phi_1+\zeta,h}$$

(28)
$$\leq 2e^{-\frac{t}{2}(\Re\beta_{4,\Delta})(\phi_1+\zeta)^4} \leq 2$$

thanks to Lemma 11. Likewise

$$||e^{(1-t)\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \le ||e^{(1-t)\tilde{V}_{\Delta}(\psi)}||_{\partial\psi,\phi_1,h} \le 2e^{-\frac{(1-t)}{2}(\Re\beta_{4,\Delta})\phi_1^4}$$

by Lemma 12. Although $||p_{\Delta}||_{\partial \phi, \phi, h}$ does not depend on t, we bound it in a t dependent way using Lemma 15 with $\gamma = \frac{1-t}{6}$ and $\frac{\kappa}{3}$ instead of κ . Namely, we write

$$||p_{\Delta}||_{\partial \phi, \phi, h} \leq \mathcal{O}_{1} \times 9 \times \kappa^{-2} \times (1-t)^{-\frac{3}{4}} \times 6^{\frac{3}{4}} \times e^{\frac{\kappa}{3} \zeta_{\Delta}^{2}} e^{\frac{(1-t)}{6} (\Re \beta_{4,\Delta}) \phi_{1}^{4}} \times \bar{g}^{\frac{1}{4} - \eta}.$$

Altogether this results in the bound

$$||r_{1,\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \frac{1}{2}\mathcal{O}_1^3 \times 9^3 \times 6^{\frac{9}{4}}\kappa^{-6}e^{\kappa\zeta_{\Delta}^2} \times 4 \times \bar{g}^{\frac{3}{4}-3\eta} \times \int_0^1 (1-t)^{-\frac{1}{4}} dt$$

which is the desired result.

Lemma 18. For all κ such that $0 < \kappa \le 2^{-\frac{3}{2}} \times 3 \times L^{-(3-2[\phi])}$ and for all $\zeta \in \mathbb{R}$ we have

$$|r_{1,\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \leq \mathcal{O}_4\kappa^{-6}\bar{g}^{3-3\eta}e^{\kappa\zeta_\Delta^2}$$

where $\mathcal{O}_4 = 2 \times 3^5 \times \mathcal{O}_2^3$.

Proof: Proceeding as in the previous lemma we arrive at

$$|r_{1,\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \le \int_0^1 dt \, \frac{(1-t)^2}{2} |p_{\Delta}|_{\partial\phi,h_*}^3 \times |e^{tV_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} \times |e^{(1-t)\tilde{V}_{\Delta}(\phi_1)}|_{\partial\phi,h_*} \, .$$

We use

$$|e^{tV_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} = ||e^{tV_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,0,h_*} \le ||e^{tV_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,0,h}$$

since $h_* \leq h$. Indeed this follows from $c_2 L^{\frac{3+\epsilon}{4}} \leq c_2 L \leq c_1 \bar{g}^{-\frac{1}{4}}$, i.e., from (21) We can then reuse the bound (28) at $\phi = 0$, namely,

$$|e^{tV_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} \le 2$$
.

Now

$$|e^{(1-t)\tilde{V}_{\Delta}(\phi_1)}|_{\partial\phi,h_*} \le |e^{(1-t)\tilde{V}_{\Delta}(\psi)}|_{\partial\psi,h_*} \le 2$$

by Lemma 12. Finally, we use Lemma 16 with $\frac{\kappa}{3}$ instead of κ in order to get

$$|p_{\Delta}|_{\partial\phi,h_*} \leq \mathcal{O}_2 \times 9 \times \kappa^{-2} e^{\frac{\kappa}{3}\zeta_{\Delta}^2} \bar{g}^{1-\eta}$$
.

Altogether this results in

$$|r_{1,\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \le \frac{1}{2} \left(\int_0^1 dt \, \frac{(1-t)^2}{2} \right) \times 4 \times \mathcal{O}_2^3 \times 3^6 \kappa^{-6} e^{\kappa \zeta_{\Delta}^2} \bar{g}^{3-3\eta}$$

which is the wanted bound.

Lemma 19. For all $\kappa \in (0,1]$, and all $\lambda \in \mathbb{C}$ which satisfies $|\lambda| \bar{g}^{\frac{1}{4}-\eta} \leq 1$, we have $\forall \phi, \zeta \in \mathbb{R}$

$$||P_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \le \mathcal{O}_5 \kappa^{-6} e^{\kappa \zeta_{\Delta}^2} \times |\lambda| \bar{g}^{\frac{1}{4} - \eta}$$

where $\mathcal{O}_5 = 2^{\frac{9}{2}}\mathcal{O}_1 + 2^7\mathcal{O}_1^2 + \mathcal{O}_3$.

Proof: By definition

$$P_{\Delta}(\lambda, \phi_1, \zeta) = e^{-\tilde{V}_{\Delta}(\phi_1)} \left[-\lambda p_{\Delta} + \frac{\lambda^2}{2} p_{\Delta}^2 \right] + \lambda^3 r_{1,\Delta}(\phi_1, \zeta)$$

and thus from the properties of the seminorm we get

$$||P_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq ||e^{-\tilde{V}_{\Delta}(\phi_{1})}||_{\partial\phi,\phi,h} \times \left[|\lambda| \ ||p_{\Delta}(\phi_{1},\zeta)||_{\partial\phi,\phi,h} + \frac{|\lambda|^{2}}{2} \ ||p_{\Delta}(\phi_{1},\zeta)||_{\partial\phi,\phi,h}\right] + |\lambda|^{3} ||r_{1,\Delta}(\phi_{1},\zeta)||_{\partial\phi,\phi,h}.$$

We bound $||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h}$ using Lemma 15 with $\gamma=\frac{1}{4}$ and with $\frac{\kappa}{2}$ instead of κ . We bound $||e^{-\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h}$ using Lemma 17. Finally, we bound $||r_{1,\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h}$ using Lemma 17. Put together, this results in

$$\begin{split} ||P_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} &\leq 2e^{-\frac{1}{2}(\Re\beta_{4,\Delta})\phi_{1}^{4}} \times \left[|\lambda|\mathcal{O}_{1}2^{2+\frac{3}{2}}\bar{g}^{\frac{1}{4}-\eta}\kappa^{-2}e^{\frac{\kappa}{2}\zeta_{\Delta}^{2}}e^{\frac{1}{4}(\Re\beta_{4,\Delta})\phi_{1}^{4}}\right] \\ & \frac{1}{2}|\lambda|^{2}\mathcal{O}_{1}^{2}2^{7}\bar{g}^{\frac{1}{2}-2\eta}\kappa^{-4}e^{\kappa\zeta_{\Delta}^{2}}e^{\frac{1}{2}(\Re\beta_{4,\Delta})\phi_{1}^{4}}\right] + |\lambda|^{3}\mathcal{O}_{3}\bar{g}^{\frac{3}{4}-3\eta}\kappa^{-6}e^{\kappa\zeta_{\Delta}^{2}}. \end{split}$$

Since $0 < \kappa \le 1$, κ^{-6} is the worst power of that kind. Since also $\Re \beta_{4,\Delta} > 0$, the worst exponential factor left is $e^{\kappa \zeta_{\Delta}^2}$. Hence

$$||P_{\Delta}(\lambda,\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \kappa^{-6} e^{\kappa\zeta_{\Delta}^2} \times \left[2^{\frac{9}{2}}\mathcal{O}_1|\lambda|\bar{g}^{\frac{1}{4}-\eta} + 2^7\mathcal{O}_1^2|\lambda|^2\bar{g}^{\frac{1}{2}-2\eta} + \mathcal{O}_3|\lambda|^3\bar{g}^{\frac{3}{4}-3\eta}\right] \ .$$

We then conclude using the hypothesis $|\lambda| \bar{g}^{\frac{1}{4}-\eta} \leq 1$.

Lemma 20. For all $\kappa \in (0, 2^{-\frac{1}{2}}L^{-(3-2[\phi])}]$, and all $\lambda \in \mathbb{C}$ which satisfies $|\lambda|\bar{g}^{1-\eta} \leq 1$, we have $\forall \zeta \in \mathbb{R}$

$$|P_{\Delta}(\lambda, \phi_1, \zeta)|_{\partial \phi, h_*} \le \mathcal{O}_6 \kappa^{-6} e^{\kappa \zeta_{\Delta}^2} \times |\lambda| \bar{g}^{1-\eta}$$

where $\mathcal{O}_6 = 8\mathcal{O}_2 + 16\mathcal{O}_2^2 + \mathcal{O}_4$.

Proof: Similarly to the proof of the previous lemma we have

$$|P_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \leq |e^{-\tilde{V}_{\Delta}(\phi_1)}|_{\partial\phi,h_*} \times \left[|\lambda| |p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} + \frac{|\lambda|^2}{2} |p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*}^2 \right]$$

$$+|\lambda|^3 |r_{1,\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*}$$
.

We bound $|p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*}$ by Lemma 16 with $\frac{\kappa}{2}$ instead of κ . We have

$$|e^{-\tilde{V}_{\Delta}(\phi_1)}|_{\partial \phi, h_*} \le |e^{-\tilde{V}_{\Delta}(\psi)}|_{\partial \psi, h_*} \le 2$$

thanks to Lemma 12. We also use Lemma 18 to bound $|r_{1,\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*}$. As a result we get

$$|P_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \leq 2 \left\lceil |\lambda|\mathcal{O}_2 \times 4 \times \bar{g}^{1-\eta}\kappa^{-2}e^{\frac{\kappa}{2}\zeta_{\Delta}^2} \right\rceil$$

$$\frac{1}{2}|\lambda|^2\mathcal{O}_2^2 2^4 \bar{g}^{2-2\eta}\kappa^{-4}e^{\kappa\zeta_\Delta^2}\bigg] + |\lambda|^3\mathcal{O}_4 \bar{g}^{3-3\eta}\kappa^{-6}e^{\kappa\zeta_\Delta^2}\ .$$

The hypothesis on κ implies that $\kappa \leq 1$ and therefore the worst negative power which appears is κ^{-6} . We also use the hypothesis $|\lambda|\bar{g}^{1-\eta} \leq 1$ to bound the square and cube of that quantity as in the previous lemma and the result follows.

Lemma 21. For all $\kappa \in (0,1]$, $\gamma \in (0,1]$, and $\phi, \zeta \in \mathbb{R}$ we have

$$||Q_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h}\leq \mathcal{O}_7\bar{g}^{\frac{1}{2}-2\eta}\gamma^{-\frac{3}{2}}e^{\gamma(\Re\beta_{4,\Delta})\phi_1^4}\kappa^{-3}e^{\kappa\zeta^2}$$

where

$$\mathcal{O}_7 = 331^2 \times 96 \times 2^6 \times \max_{0 \le j \le 6} \left(\frac{j}{2e}\right)^{\frac{j}{2}} \times \max_{0 \le n \le 6} \left(\frac{n}{2e}\right)^{\frac{n}{4}}.$$

Proof: We proceed as in the proof of Lemma 14. By definition and by the elementary properties of Wick monomials

$$Q_{\Delta}(\phi_1 + \zeta) = \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} 5 \le a + b \le 6 \\ a, b \ge 0 \end{array} \right\} \frac{(a+b)!}{a!b!} W_{a+b,\Delta} : \psi^a :_{C_1}|_{\psi = \phi_1} : \zeta^b :_{\Gamma} .$$

Therefore, again dominating ϕ derivatives by ψ derivatives and using (17), we get

$$||Q_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h} \leq \bar{g}^{2-2\eta} \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} 5 \leq a+b \leq 6 \\ a,b \geq 0 \end{array} \right\} \frac{(a+b)!}{a!b!} ||:\psi^a:_{C_1} ||_{\partial\psi,\phi_1,h} \times |:\zeta^b:_{\Gamma} |.$$

In addition to the Wick ordering formulas in the proof of Lemma 15 we have

$$\begin{array}{lll} :\psi^4:_{C_1}&=&\psi^4-6L^{-2[\phi]}C_0(0)\psi^2+3L^{-4[\phi]}C_0(0)^2\\ :\psi^5:_{C_1}&=&\psi^5-10L^{-2[\phi]}C_0(0)\psi^3+15L^{-4[\phi]}C_0(0)^2\psi\\ :\psi^6:_{C_1}&=&\psi^6-15L^{-2[\phi]}C_0(0)\psi^4+45L^{-4[\phi]}C_0(0)^2\psi^2-15L^{-6[\phi]}C_0(0)^3\\ \end{array}$$

as well as

$$\begin{array}{rcl} :\zeta^5:_{\Gamma} & = & \zeta^5 - 10\Gamma(0)\zeta^3 + 15\Gamma(0)^2\zeta \\ :\zeta^6:_{\Gamma} & = & \zeta^6 - 15\Gamma(0)\zeta^4 + 45\Gamma(0)^2\zeta^2 - 15\Gamma(0)^3 \ . \end{array}$$

Therefore when bounding these expressions using $\Gamma(0), C_0(0) \in [0, 2]$, the worst numerical factor coming from the sixth power case is $1 + 15 \times 2 + 45 \times 2^2 + 15 \times 2^3 = 331$. We therefore have

$$||: \psi^a :_{C_1} ||_{\partial \psi, \phi_1, h} \le 331 \max_{0 \le k \le a} ||\psi^k||_{\partial \psi, \phi_1, h}$$

 $|: \zeta^b :_{\Gamma} | \le 331 \max_{0 \le i \le b} |\zeta|^j$.

This result in the rather coarse bound

$$||Q_{\Delta}(\phi_{1}+\zeta)||_{\partial\phi,\phi,h} \leq 331^{2}\bar{g}^{2-2\eta} \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} 5 \leq a+b \leq 6 \\ a,b \geq 0 \end{array} \right\} \frac{(a+b)!}{a!b!} \left(\max_{0 \leq k \leq a} ||\psi^{k}||_{\partial\psi,\phi_{1},h} \right) \left(\max_{0 \leq j \leq b} |\zeta|^{j} \right)$$

$$\leq 331^{2} \times (2^{5}+2^{6})\bar{g}^{2-2\eta} \max_{a,b} \left[\left(\max_{0 \leq k \leq a} ||\psi^{k}||_{\partial\psi,\phi_{1},h} \right) \left(\max_{0 \leq j \leq b} |\zeta|^{j} \right) \right]$$

where the new numerical factor $2^5 + 2^6 = 96$ comes from the sum of binomial coefficients and the maximum is over pairs of nonnegative integers a, b such that a + b = 5 or 6. By Lemma 9, and given that $\kappa \in (0, 1]$, we have

$$\max_{0 \le j \le b} |\zeta|^j \le \max_{0 \le j \le 6} |\zeta|^j \le \kappa^{-3} e^{\kappa \zeta^2} \times \max_{0 \le j \le 6} \left(\frac{j}{2e}\right)^{\frac{j}{2}} \ .$$

For $0 \le k \le 6 < 9$ we still have

$$\begin{split} ||\psi^{k}||_{\partial\psi,\psi,h} &= (h + |\psi|)^{k} \\ &\leq \sum_{n=0}^{k} \binom{k}{n} (c_{1}\bar{g}^{-\frac{1}{4}})^{k-n} \left(\frac{n}{2e}\right)^{\frac{n}{4}} (\gamma\bar{g})^{-\frac{n}{4}} e^{\gamma(\Re\beta_{4,\Delta})\psi^{4}} \\ &\leq \left(\max_{0\leq n\leq 6} \left(\frac{n}{2e}\right)^{\frac{n}{4}}\right) \times (1+c_{1})^{k} \bar{g}^{-\frac{k}{4}} \gamma^{-\frac{k}{4}} e^{\gamma(\Re\beta_{4,\Delta})\psi^{4}} \end{split}$$

again by Lemma 8. We collect all these estimates and in the final result we bound $(1+c_1)^k$ by $2^k \le 2^6$ and the powers of \bar{g} , γ and κ by their worst case values, i.e., respectively $\bar{g}^{\frac{1}{2}-2\eta}$, $\gamma^{-\frac{3}{2}}$ and κ^{-3} .

Lemma 22. For all $\kappa \in (0, 2^{-\frac{3}{2}}L^{-(3-2[\phi])}]$, and $\zeta \in \mathbb{R}$ we have

$$|Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \le \mathcal{O}_8 \bar{g}^{2-2\eta} \kappa^{-3} e^{\kappa \zeta^2}$$

where

$$\mathcal{O}_8 = 331^2 \times 96 \times \max_{0 \le j \le 6} \left(\frac{j}{2e}\right)^{\frac{j}{2}} .$$

Proof: As in the previous lemma we have

$$|Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \leq \bar{g}^{2-2\eta} \sum_{a,b} \mathbb{1} \left\{ \begin{array}{c} 5 \leq a+b \leq 6 \\ a,b \geq 0 \end{array} \right\} \frac{(a+b)!}{a!b!} |: \psi^a :_{C_1} |_{\partial \psi, h_*} \times |: \zeta^b :_{\Gamma} |.$$

We again have $|: \zeta^b:_{\Gamma}| \leq 331 \max_{0 \leq j \leq b} |\zeta|^j$ as well as $|: \psi^a:_{C_1}|_{\partial \psi, h_*} \leq 331 \max_{0 \leq k \leq a} |\psi^k|_{\partial \psi, h_*}$, but now $|\psi^k|_{\partial \psi, h_*} = h_*^k$. As a result

$$|Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \le 331^2 \times 96 \times \bar{g}^{2-2\eta} \max_{a,b} \left[\left(\max_{0 \le k \le a} h_*^k \right) \times \left(\max_{0 \le j \le b} |\zeta|^j \right) \right]$$

with the maximum again over of nonnegative integers a, b such that a + b = 5 or 6. Therefore

$$|Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \le 331^2 \times 96 \times \bar{g}^{2-2\eta} \max_{\substack{j,k \ge 0\\j+k \le 6}} h_*^k |\zeta|^j$$

$$\leq 331^2 \times 96 \times \bar{g}^{2-2\eta} \times e^{\kappa \zeta^2} \max_{\substack{j,k \geq 0 \\ j+k \leq 6}} h_*^k \kappa^{-\frac{j}{2}} \left(\frac{j}{2e}\right)^{\frac{j}{2}}$$

after applying Lemma 9. From the hypothesis on κ and (14) we have that $h_* \leq \kappa^{-\frac{1}{2}}$ and therefore the quantities $h_*^k \kappa^{-\frac{j}{2}}$ are bounded by κ^{-3} and the result follows.

Lemma 23. For all $\phi, \zeta \in \mathbb{R}$ we have

$$||Q_{\Delta}(\phi_1+\zeta)e^{-V_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \leq \mathcal{O}_9\bar{g}^{\frac{1}{2}-2\eta}$$

where

$$\mathcal{O}_9 = 331 \times 2^{\frac{19}{2}} \times \max_{0 \le n \le 6} \left(\frac{n}{2e}\right)^{\frac{n}{4}}$$
.

Proof: By definition

$$Q_{\Delta}(\phi_1 + \zeta)e^{-V_{\Delta}(\phi_1 + \zeta)} = \sum_{k=5.6} W_{k,\Delta} \left(: \psi^k :_{C_0} e^{-V_{\Delta}(\psi)} \right) \Big|_{\psi = \phi_1 + \zeta}$$

and therefore

$$\begin{split} ||Q_{\Delta}(\phi_{1}+\zeta)e^{-V_{\Delta}(\phi_{1}+\zeta)}||_{\partial\phi,\phi,h} & \leq & \bar{g}^{2-2\eta}\sum_{k=5,6}||:\psi^{k}:_{C_{0}}e^{-V_{\Delta}(\psi)}||_{\partial\psi,\phi_{1}+\zeta,h} \\ & \leq & 2\times331\times\max_{0\leq k\leq6}||\psi^{k}||_{\partial\psi,\phi_{1}+\zeta,h}\;||e^{-V_{\Delta}(\psi)}||_{\partial\psi,\phi_{1}+\zeta,h} \end{split}$$

by undoing the Wick ordering as in the proof of Lemma 21. By proceeding as in the latter, but with the specific choice $\gamma = \frac{1}{2}$ in the step involving the application of Lemma 8, we get

$$||\psi^k||_{\partial\psi,\psi,h} \le \left(\max_{0 \le n \le 6} \left(\frac{n}{2e}\right)^{\frac{n}{4}}\right) \times \left(\frac{\bar{g}}{2}\right)^{-\frac{k}{4}} e^{\frac{1}{2}(\Re\beta_{4,\Delta})\psi^4} \times 2^6.$$

Hence, by Lemma 11 with t = 1, we get

$$||\psi^{k}||_{\partial\psi,\phi_{1}+\zeta,h}||e^{-V_{\Delta}(\psi)}||_{\partial\psi,\phi_{1}+\zeta,h} \leq \left(\max_{0\leq n\leq 6} \left(\frac{n}{2e}\right)^{\frac{n}{4}}\right) \times \bar{g}^{-\frac{k}{4}} \times 2^{\frac{k}{4}+7}.$$

Taking the worst case k = 6 gives the desired bound.

Lemma 24. If $0 < \kappa \le 2^{-\frac{3}{2}} L^{-(3-2[\phi])}$ then for all $\zeta \in \mathbb{R}$ we have

$$|Q_{\Delta}(\phi_1 + \zeta)e^{-V_{\Delta}(\phi_1 + \zeta)}|_{\partial \phi, h_*} \le \mathcal{O}_{10}\kappa^{-3}e^{\kappa\zeta^2}\bar{g}^{2-2\eta}$$

where $\mathcal{O}_{10} = 2 \times \mathcal{O}_9$.

Proof: We have

$$|Q_{\Delta}(\phi_{1}+\zeta)e^{-V_{\Delta}(\phi_{1}+\zeta)}|_{\partial\phi,h_{*}} \leq |Q_{\Delta}(\phi_{1}+\zeta)|_{\partial\phi,h_{*}} \times |e^{-V_{\Delta}(\phi_{1}+\zeta)}|_{\partial\phi,h_{*}}$$
$$\leq \mathcal{O}_{8}\kappa^{-3}e^{\kappa\zeta^{2}}\bar{g}^{2-2\eta} \times |e^{-V_{\Delta}(\phi_{1}+\zeta)}|_{\partial\phi,h_{*}}$$

by Lemma 22. The last factor is bounded in a coarse way using

$$|e^{-V_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} = ||e^{-V_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,0,h_*}$$

$$\leq ||e^{-V_{\Delta}(\psi)}||_{\partial\psi,\zeta,h_*} \leq ||e^{-V_{\Delta}(\psi)}||_{\partial\psi,\zeta,h}$$

since $h_* \leq h$ by (21). We finally use Lemma 11 with t=1 in order to write

$$|e^{-V_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} \le 2e^{-\frac{1}{2}(\Re\beta_{4,\Delta})\zeta^4} \le 2$$

and the result follows.

Lemma 25. For all $\kappa \in (0,1]$ and all $\phi, \zeta \in \mathbb{R}$ we have

$$\left\| \left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right| \right\|_{\partial \phi, \phi, h} \le \mathcal{O}_{11} \kappa^{-5} e^{\kappa \zeta^2} \bar{g}^{\frac{3}{4} - 3\eta}$$

with

$$\mathcal{O}_{11}=2^6\mathcal{O}_1\times\mathcal{O}_9+2^{\frac{43}{4}}\mathcal{O}_1\times\mathcal{O}_7\ .$$

Proof: Define

$$j(s) = e^{-(1-s)V_{\Delta}(\phi_1+\zeta) - s\tilde{V}_{\Delta}(\phi_1)} = e^{-V_{\Delta}(\phi_1+\zeta) + sp_{\Delta}}$$

for $s \in [0, 1]$. Thus

$$Q_{\Delta}(\phi_{1} + \zeta) \left(e^{-V_{\Delta}(\phi_{1} + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_{1})} \right) = Q_{\Delta}(\phi_{1} + \zeta)(j(0) - j(1))$$

$$= -Q_{\Delta}(\phi_{1} + \zeta) \int_{0}^{1} ds \ j'(s)$$

$$= -Q_{\Delta}(\phi_{1} + \zeta) \int_{0}^{1} ds \ p_{\Delta}(\phi_{1}, \zeta) e^{-(1-s)V_{\Delta}(\phi_{1} + \zeta) - s\tilde{V}_{\Delta}(\phi_{1})}$$

$$= A + B$$

with

$$A = -Q_{\Delta}(\phi_1 + \zeta) \int_0^{\frac{1}{2}} ds \ p_{\Delta}(\phi_1, \zeta) e^{-(1-s)V_{\Delta}(\phi_1 + \zeta) - s\tilde{V}_{\Delta}(\phi_1)}$$

and

$$B = -Q_{\Delta}(\phi_1 + \zeta) \int_{\frac{1}{2}}^{1} ds \ p_{\Delta}(\phi_1, \zeta) e^{-(1-s)V_{\Delta}(\phi_1 + \zeta) - s\tilde{V}_{\Delta}(\phi_1)} \ .$$

We bound these two pieces separately. Note that

$$A = -Q_{\Delta}(\phi_1 + \zeta)e^{-\frac{1}{2}V_{\Delta}(\phi_1 + \zeta)} \int_0^{\frac{1}{2}} ds \ p_{\Delta}(\phi_1, \zeta)e^{-(\frac{1}{2} - s)V_{\Delta}(\phi_1 + \zeta) - s\tilde{V}_{\Delta}(\phi_1)}$$

which implies the estimate

$$||A||_{\partial\phi,\phi,h} \leq ||Q_{\Delta}(\phi_1 + \zeta)e^{-\frac{1}{2}V_{\Delta}(\phi_1 + \zeta)}||_{\partial\phi,\phi,h} \times \int_0^{\frac{1}{2}} \mathrm{d}s \ ||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h}$$
$$\times ||e^{-(\frac{1}{2}-s)V_{\Delta}(\phi_1 + \zeta)}||_{\partial\phi,\phi,h} \times ||e^{s\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \ .$$

Repeating the proof of Lemma 23, but this time taking $\gamma = \frac{1}{4}$ instead of $\frac{1}{2}$ which results in an extra factor $\left(\frac{1}{2}\right)^{-\frac{6}{4}} = 2^{\frac{3}{2}}$, we obtain

$$||Q_{\Delta}(\phi_1+\zeta)e^{-\frac{1}{2}V_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \le \mathcal{O}_9 \times 2^{\frac{3}{2}} \times \bar{g}^{\frac{1}{2}-2\eta}$$
.

For $0 < s < \frac{1}{2}$, we get from Lemma 11 with $t = \frac{1}{2} - s$

$$||e^{-(\frac{1}{2}-s)V_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \leq ||e^{-(\frac{1}{2}-s)V_{\Delta}(\psi)}||_{\partial\psi,\phi_1+\zeta,h} \leq 2e^{-\frac{1}{2}(\frac{1}{2}-s)(\Re\beta_{4,\Delta})(\phi_1+\zeta)^4} \leq 2.$$

The above steps result in the bound

$$||A||_{\partial\phi,\phi,h} \leq 2^{\frac{5}{2}} \mathcal{O}_9 \bar{g}^{\frac{1}{2}-2\eta} \times \int_0^{\frac{1}{2}} \mathrm{d}s ||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} ||e^{s\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} .$$

By Lemma 12

$$||e^{s\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \leq ||e^{s\tilde{V}_{\Delta}(\psi)}||_{\partial\psi,\phi_1,h} \leq 2e^{-\frac{s}{2}(\Re\beta_{4,\Delta})\phi_1^4} \ .$$

Now we use Lemma 15 with $\gamma = \frac{s}{2}$ and with the present κ in order to derive

$$||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \mathcal{O}_1\kappa^{-2}\left(\frac{s}{2}\right)^{-\frac{3}{4}}\bar{g}^{\frac{1}{4}-\eta}e^{\kappa\zeta^2}e^{\frac{s}{2}(\Re\beta_{4,\Delta})\phi_1^4}.$$

This produces the bound

$$||A||_{\partial\phi,\phi,h} \le 2^{\frac{5}{2}}\mathcal{O}_9 \times 2 \times \mathcal{O}_1 \times 2^{\frac{3}{4}} \times \bar{g}^{\frac{3}{4}-3\eta} \times \kappa^{-2} e^{\kappa\zeta^2} \times \int_0^{\frac{1}{2}} \mathrm{d}s \ s^{-\frac{3}{4}} \ ,$$

namely,

$$||A||_{\partial\phi,\phi,h} \le 2^6 \times \mathcal{O}_1 \times \mathcal{O}_9 \times \bar{g}^{\frac{3}{4}-3\eta} \times \kappa^{-2} e^{\kappa\zeta^2}$$
.

We now take care of B. From the definition we readily obtain

$$||B||_{\partial \phi, \phi, h} \leq ||Q_{\Delta}(\phi_1 + \zeta)||_{\partial \phi, \phi, h} \int_{\frac{1}{2}}^{1} ds \ ||p_{\Delta}(\phi_1, \zeta)||_{\partial \phi, \phi, h} ||e^{-(1-s)V_{\Delta}(\phi_1 + \zeta)}||_{\partial \phi, \phi, h} ||e^{-s\tilde{V}_{\Delta}(\phi_1)}||_{\partial \phi, \phi, h} \ .$$

Since $\kappa \in (0,1]$, we have $\frac{\kappa}{2} \in (0,2]$ and therefore Lemma 21 with $\frac{\kappa}{2}$ instead of κ and with $\gamma = \frac{1}{8}$ gives us the estimate

$$||Q_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h} \leq \mathcal{O}_7 \bar{g}^{\frac{1}{2}-2\eta} \times 8^{\frac{3}{2}} e^{\frac{1}{8}(\Re\beta_{4,\Delta})\phi_1^4} \times 8\kappa^{-3} e^{\frac{\kappa}{2}\zeta^2} \ .$$

We also use Lemma 15 with $\frac{\kappa}{2}$ instead of κ and with $\gamma = \frac{1}{8}$ and get

$$||p_{\Delta}(\phi_1,\zeta)||_{\partial\phi,\phi,h} \leq \mathcal{O}_1 \times 4 \times \kappa^{-2} \times 8^{\frac{3}{4}} \times \overline{g}^{\frac{1}{4}-\eta} e^{\frac{\kappa}{2}\zeta^2} \times e^{\frac{1}{8}(\Re\beta_{4,\Delta})\phi_1^4} \ .$$

From Lemma 11 with t = 1 - s we obtain

$$||e^{-(1-s)V_{\Delta}(\phi_1+\zeta)}||_{\partial\phi,\phi,h} \leq ||e^{-(1-s)V_{\Delta}(\psi)}||_{\partial\psi,\phi_1+\zeta,h} \leq 2e^{-\frac{(1-s)}{2}(\Re\beta_{4,\Delta})(\phi_1+\zeta)^4} \leq 2.$$

Finally, the last ingredient is the use of Lemma 12 with t = s which results in

$$||e^{-s\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \leq ||e^{-s\tilde{V}_{\Delta}(\psi)}||_{\partial\psi,\phi_1,h} \leq e^{-\frac{s}{2}(\Re\beta_{4,\Delta})\phi_1^4} \leq 2e^{-\frac{1}{4}(\Re\beta_{4,\Delta})\phi_1^4}$$

since $s \geq \frac{1}{2}$. Altogether the previous bounds imply

$$||B||_{\partial\phi,\phi,h} \leq \int_{\frac{1}{2}}^{1} ds \, \mathcal{O}_{7} \times \bar{g}^{\frac{1}{2}-2\eta} \times 8^{\frac{3}{2}} \times 8 \times \kappa^{-3}$$
$$\times \mathcal{O}_{1} \times 4 \times \kappa^{-2} \times 8^{\frac{3}{4}} \times \bar{g}^{\frac{1}{4}-\eta} \times 2 \times e^{\kappa\zeta^{2}}$$
$$\leq 2^{\frac{43}{4}} \mathcal{O}_{1} \times \mathcal{O}_{7} \kappa^{-5} \bar{g}^{\frac{3}{4}-3\eta} e^{\kappa\zeta^{2}}.$$

Combining the bounds for A and B we obtain the desired estimate.

Lemma 26. If $0 < \kappa \le 2^{-\frac{3}{2}} L^{-(3-2[\phi])}$ then for all $\zeta \in \mathbb{R}$ we have

$$\left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right|_{\partial \phi, h_*} \le \mathcal{O}_{12} \kappa^{-5} e^{\kappa \zeta^2} \bar{g}^{3 - 3\eta}$$

where $\mathcal{O}_{12} = 2^7 \mathcal{O}_2 \times \mathcal{O}_8$.

Proof: The proof is simpler than that of the previous lemma because we do not need to split the quantity at hand. We directly bound the latter, namely,

$$Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) = -Q_{\Delta}(\phi_1 + \zeta) \int_0^1 \mathrm{d}s \ p_{\Delta}(\phi_1, \zeta) e^{-(1-s)V_{\Delta}(\phi_1 + \zeta) - s\tilde{V}_{\Delta}(\phi_1)}$$

by

$$\left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right|_{\partial \phi, h_*} \leq |Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \times \int_0^1 \mathrm{d}s \ |p_{\Delta}(\phi_1, \zeta)|_{\partial \phi, h_*}$$

$$\times |e^{-(1-s)V_{\Delta}(\phi_1+\zeta)}|_{\partial\phi,h_*} \times |e^{-s\tilde{V}_{\Delta}(\phi_1)}|_{\partial\phi,h_*}$$

From Lemma 22 with $\frac{\kappa}{2}$ instead of κ we have

$$|Q_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \le 8 \times \mathcal{O}_8 \kappa^{-3} e^{\frac{\kappa}{2}\zeta^2} \bar{g}^{2-2\eta} .$$

Likewise, from Lemma 16 with $\frac{\kappa}{2}$ instead of κ we get

$$|p_{\Delta}(\phi_1,\zeta)|_{\partial\phi,h_*} \leq 4 \times \mathcal{O}_2 \kappa^{-2} e^{\frac{\kappa}{2}\zeta^2} \bar{g}^{1-\eta}$$
.

We also have

$$|e^{-(1-s)V_{\Delta}(\phi_{1}+\zeta)}|_{\partial\phi,h_{*}} = ||e^{-(1-s)V_{\Delta}(\phi_{1}+\zeta)}||_{\partial\phi,0,h_{*}} \le ||e^{-(1-s)V_{\Delta}(\psi)}|_{\partial\psi,\zeta,h_{*}}$$

$$\le ||e^{-(1-s)V_{\Delta}(\psi)}|_{\partial\psi,\zeta,h} \le 2e^{-\frac{(1-s)}{2}(\Re\beta_{4,\Delta})\zeta^{4}} \le 2$$

by Lemma 11. Finally Lemma 12 provides the estimate

$$|e^{-s\tilde{V}_{\Delta}(\phi_1)}|_{\partial\phi,h_*} \le |e^{-s\tilde{V}_{\Delta}(\psi)}|_{\partial\psi,h_*} \le 2$$
.

Altogether the previous bounds imply

$$\left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right|_{\partial \phi, h_*} \le 8\mathcal{O}_8 \kappa^{-5} e^{\kappa \zeta^2} \bar{g}^{3 - 3\eta} \times 16 \times \mathcal{O}_2$$

from which the result follows.

Lemma 27. For all $K \in C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ and for all $\sigma \in \mathbb{R}$ we have

$$||K(\psi)||_{\partial\psi,\sigma,h_*} \le \mathcal{O}_{13}e^{h_*^{-2}\sigma^2} \times \left[|K(\psi)|_{\partial\psi,h_*} + h_*^9 h^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h_*}\right]$$

where

$$\mathcal{O}_{13} = 1 + 511 \times \max_{0 \le j \le 9} \left(\frac{j}{2e}\right)^{\frac{j}{2}}$$
.

Proof: Recall that by definition

$$||K(\psi)||_{\partial\psi,\sigma,h_*} = \sum_{n=0}^{9} \frac{h_*^n}{n!} |K^{(n)}(\sigma)|.$$

The term with n = 9 is bounded by writing

$$\frac{h_*^9}{9!}|K^{(9)}(\sigma)| = h_*^9 h^{-9} \times \frac{h^9}{9!}|K^{(9)}(\sigma)| \le h_*^9 h^{-9} \times \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, h_*} \ .$$

For terms with $0 \le n \le 8$ we use a Taylor expansion around zero of order 8-n so that the integral remainder involves (9-n)-th derivatives of $K^{(n)}$, i.e., 9-th derivatives of the original function K. Indeed, one can write

$$K^{(n)}(\sigma) = \sum_{m=0}^{8-n} \frac{\sigma^m}{m!} K^{(n+m)}(0) + \frac{1}{(8-n)!} \int_0^1 (1-s)^{8-n} \sigma^{9-n} K^{(9)}(s\sigma) \, ds$$

and therefore

$$|K^{(n)}(\sigma)| \le \sum_{m=0}^{8-n} \frac{|\sigma|^m}{m!} (n+m)! \ h_*^{-(n+m)} |K(\psi)|_{\partial \psi, h_*}$$

 $+ \frac{1}{(8-n)!} |\sigma|^{9-n} \times 9! \ h^{-9} \left(\sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, h} \right) \int_0^1 (1-s)^{8-n} \ \mathrm{d}s \ .$

We use Lemma 9 with $\kappa = h_*^{-2}$ in order to bound powers of $|\sigma|$ by

$$|\sigma|^m \leq \left(\frac{m}{2e}\right)^{\frac{m}{2}} \times h_*^m e^{h_*^{-2}\sigma^2}$$

which inserted in the previous inequality gives

$$\frac{h_*^n}{n!}|K^{(n)}(\sigma)| \le \left(\max_{0 \le j \le 9} \left(\frac{j}{2e}\right)^{\frac{j}{2}}\right) \times e^{h_*^{-2}\sigma^2}$$

$$\times \left[\sum_{m=0}^{8-n} \frac{(n+m)!}{n!m!} |K(\psi)|_{\partial \psi, h_*} + \frac{9! \ h_*^{9-n}}{n!(9-n)!} h_*^n h^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, h} \right] .$$

Putting together the bounds for the different values of n we obtain

$$||K(\psi)||_{\partial \psi, \sigma, h_*} \le h_*^9 h^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, h}$$

$$+e^{h_*^{-2}\sigma^2}\left(\max_{0\leq j\leq 9}\left(\frac{j}{2e}\right)^{\frac{j}{2}}\right)\sum_{n=0}^{8}\left[\left(\sum_{m=0}^{8-n}\frac{(n+m)!}{n!m!}\right)|K(\psi)|_{\partial\psi,h_*}+\frac{9!}{n!(9-n!)}h_*^9h^{-9}\sup_{\psi\in\mathbb{R}}||K(\psi)||_{\partial\psi,\psi,h}\right]\;.$$

The result as well as the given value of \mathcal{O}_{13} then follow since

$$\sum_{n=0}^{8} \sum_{m=0}^{8-n} \frac{(n+m)!}{n!m!} = \sum_{n=0}^{8} \frac{9!}{n!(9-n!)} = 2^9 - 1 = 511.$$

Lemma 28. For all $K \in C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C}), \ \beta_4 \in \mathbb{C} \ such \ that \ |\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}, \ \gamma \in (0,1] \ and \ \phi \in \mathbb{R} \ we \ have$

$$||K(\phi)||_{\partial \phi, \phi, h} \le \mathcal{O}_{14} \gamma^{-\frac{9}{4}} e^{\gamma(\Re \beta_4)\phi^4} \left[|K(\psi)|_{\partial \psi, h} + L^{-9[\phi]} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, L^{[\phi]} h} \right]$$

with

$$\mathcal{O}_{14} = 1 + ((1 + c_1^{-1})^9 - 1) \times \max_{0 \le j \le 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}}.$$

Proof: We proceed as in the proof of the previous lemma and write

$$\frac{h^9}{9!}|K^{(9)}(\phi)| = L^{-9[\phi]} \times \frac{(L^{[\phi]}h)^9}{9!}|K^{(9)}(\phi)| \le L^{-9[\phi]} \times \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, L^{[\phi]}h}$$

in order to handle the n=9 term in the sum defining $||K(\phi)||_{\partial\phi,\phi,h}$. For the other terms with $0 \le n \le 8$ one has, as before,

$$|K^{(n)}(\phi)| \leq \sum_{m=0}^{8-n} \frac{|\phi|^m}{m!} (n+m)! \ h^{-(n+m)} |K(\psi)|_{\partial \psi, h}$$

$$+ \frac{1}{(9-n)!} |\phi|^{9-n} \times 9! (L^{[\phi]}h)^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, L^{[\phi]}h} \ .$$

We this time use Lemma 8 in order to bound powers of $|\phi|$ by

$$|\phi|^m \le \left(\frac{m}{2e}\right)^{\frac{m}{4}} \gamma^{-\frac{m}{4}} \bar{g}^{-\frac{m}{4}} e^{\gamma(\Re\beta_4)\phi^4} .$$

Note that $\gamma^{-\frac{m}{4}} \leq \gamma^{-\frac{9}{4}}$ since $0 < \gamma \leq 1, 0 \leq n \leq 8$ and $0 \leq m \leq 9-n$. Besides $\bar{g}^{-\frac{m}{4}} = (c_1^{-1}h)^m$ and therefore

$$\frac{h^n}{n!}|K^{(n)}(\phi)| \le \left(\max_{0 \le j \le 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}}\right) \times \gamma^{-\frac{9}{4}} \times e^{\gamma(\Re\beta_4)\phi^4}$$

$$\times \left[\sum_{m=0}^{8-n} \frac{h^m c_1^{-m}}{m!} \frac{h^n}{n!} (n+m)! h^{-(n+m)} |K(\psi)|_{\partial \psi, h} + \frac{9!}{n!(9-n)!} h^n h^{9-n} c_1^{-(9-n)} (L^{[\phi]}h)^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial \psi, \psi, L^{[\phi]}h} \right].$$

Altogether this gives the estimate

$$||K(\phi)||_{\partial\phi,\phi,h} \leq L^{-9[\phi]} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h}$$

$$+ \left(\max_{0 \leq j \leq 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}}\right) \times \gamma^{-\frac{9}{4}} \times e^{\gamma(\Re\beta_4)\phi^4} \times \left\{ \left(\sum_{m=0}^{8-n} \binom{n+m}{m} c_1^{-m}\right) |K(\psi)|_{\partial\psi,h} + \binom{9}{n} c_1^{-(9-n)} L^{-9[\phi]} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h} \right\}.$$

The result with the given value for \mathcal{O}_{14} follows from this last inequality since

$$\sum_{n=0}^{8} \sum_{m=0}^{8-n} \binom{n+m}{m} c_1^{-m} = c_1 \left[(1+c_1^{-1})^9 - 1 \right] < (1+c_1^{-1})^9 - 1 = \sum_{n=0}^{8} \binom{9}{n} c_1^{-(9-n)}.$$

Lemma 29. *Let* $\kappa \in (0,1]$.

(1) If $|\lambda| \bar{q}^{\frac{1}{4} - \frac{1}{3}\eta_R} \le 1$ then $\forall \phi, \zeta \in \mathbb{R}$,

$$||K_{\Delta}(\lambda,\phi_1,\zeta)||_{\partial\phi,\phi,h} \le \mathcal{O}_{15}\kappa^{-5}e^{\kappa\zeta^2}\left(|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}\right)^2$$

where $\mathcal{O}_{15} = 2^{\frac{5}{2}}\mathcal{O}_7 + \mathcal{O}_{11} + 1$. (2) If R = 0 and $|\lambda|\bar{g}^{\frac{1}{4}-\eta} \leq 1$ then we have the improvement $\forall \phi, \zeta \in \mathbb{R}$,

$$||K_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \le \mathcal{O}_{16} \kappa^{-5} e^{\kappa \zeta^2} \left(|\lambda| \bar{g}^{\frac{1}{4} - \eta}\right)^2$$

where $\mathcal{O}_{16} = 2^{\frac{5}{2}}\mathcal{O}_7 + \mathcal{O}_{11}$.

Proof: By definition

$$K_{\Delta}(\lambda,\phi_1,\zeta) = \lambda^2 Q_{\Delta}(\phi_1+\zeta) e^{-\tilde{V}_{\Delta}(\phi_1)} + \lambda^3 Q_{\Delta}(\phi_1+\zeta) \left(e^{-V_{\Delta}(\phi_1+\zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) + \lambda^3 R_{\Delta}(\phi_1+\zeta) \ .$$

Thus

$$||K_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq |\lambda|^{2}||Q_{\Delta}(\phi_{1}+\zeta)||_{\partial\phi,\phi,h}||e^{-\tilde{V}_{\Delta}(\phi_{1})}||_{\partial\phi,\phi,h}$$
$$+|\lambda|^{3} \left| \left| Q_{\Delta}(\phi_{1}+\zeta) \left(e^{-V_{\Delta}(\phi_{1}+\zeta)} - e^{-\tilde{V}_{\Delta}(\phi_{1})} \right) \right| \right|_{\partial\phi,\phi,h} + |\lambda|^{3}||R_{\Delta}(\phi_{1}+\zeta)||_{\partial\phi,\phi,h} .$$

From Lemma 12 we have

$$||e^{-\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \le ||e^{-\tilde{V}_{\Delta}(\psi)}||_{\partial\psi,\phi_1,h} \le 2e^{-\frac{1}{2}(\Re\beta_{4,\Delta})\phi_1^4}$$
.

We use Lemma 21 with $\gamma = \frac{1}{2}$ and get

$$||Q_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h} \leq \mathcal{O}_7 \bar{g}^{\frac{1}{2}-2\eta} \times 2^{\frac{3}{2}} \times \kappa^{-3} e^{\kappa \zeta^2} e^{\frac{1}{2}(\Re\beta_{4,\Delta})\phi_1^4}.$$

As a result

$$|\lambda|^2||Q_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h}||e^{-\tilde{V}_{\Delta}(\phi_1)}||_{\partial\phi,\phi,h} \leq 2^{\frac{5}{2}}\mathcal{O}_7\bar{g}^{\frac{1}{2}-2\eta}\kappa^{-3}e^{\kappa\zeta^2}|\lambda|^2 \ .$$

From Lemma 25 we get

$$|\lambda|^3 \left| \left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right| \right|_{\partial \phi, \phi, h} \leq \mathcal{O}_{11} \bar{g}^{\frac{3}{4} - 3\eta} \kappa^{-5} e^{\kappa \zeta^2} |\lambda|^3.$$

Finally the last term is bounded using

$$||R_{\Delta}(\phi_1+\zeta)||_{\partial\phi,\phi,h} \le ||R_{\Delta}(\psi)||_{\partial\psi,\phi_1+\zeta,h} \le \sup_{\psi\in\mathbb{R}} ||R_{\Delta}(\psi)||_{\partial\psi,\psi,h}$$

$$\leq \bar{g}^{-2}|||R_{\Delta}|||_{\bar{g}} \leq \bar{g}^{-2} \times \bar{g}^{\frac{11}{4}-\eta_R} = \bar{g}^{\frac{3}{4}-\eta_R}$$

from (18). Collecting the previous estimates we arrive at

$$||K_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq \kappa^{-5}e^{\kappa\zeta^{2}}\left[2^{\frac{5}{2}}\mathcal{O}_{7}|\lambda|^{2}\bar{g}^{\frac{1}{2}-2\eta} + \mathcal{O}_{11}|\lambda|^{3}\bar{g}^{\frac{3}{4}-3\eta} + |\lambda|^{3}\bar{g}^{\frac{3}{4}-\eta_{R}}\right]$$

By the standard hypothesis (22), $\eta_R \geq 3\eta$ and since $0 < \bar{g} \leq 1$ we have $\bar{g}^{\frac{1}{4}-\eta} \leq \bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}$ and therefore

$$||K_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq \kappa^{-5}e^{\kappa\zeta^{2}}\left[2^{\frac{5}{2}}\mathcal{O}_{7}\left(|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_{R}}\right)^{2} + \mathcal{O}_{11}\left(|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_{R}}\right)^{3} + \left(|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_{R}}\right)^{3}\right].$$

from which part 1) follows. As for part 2), the R term being absent from the start, the bound on K reduces

$$||K_{\Delta}(\lambda,\phi_{1},\zeta)||_{\partial\phi,\phi,h} \leq \kappa^{-5}e^{\kappa\zeta^{2}}\left[2^{\frac{5}{2}}\mathcal{O}_{7}|\lambda|^{2}\bar{g}^{\frac{1}{2}-2\eta} + \mathcal{O}_{11}|\lambda|^{3}\bar{g}^{\frac{3}{4}-3\eta}\right].$$

which immediately yealds the desired result.

Lemma 30. (1) If $|\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R} \leq 1$ then $\forall \zeta \in \mathbb{R}$,

$$|K_{\Delta}(\lambda, \phi_1, \zeta)|_{\partial \phi, h_*} \le \mathcal{O}_{17} h_*^{10} e^{h_*^{-2} \zeta^2} \left(|\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3} \eta_R} \right)^2$$

where $\mathcal{O}_{17} = 2\mathcal{O}_8 + \mathcal{O}_{12} + 2\mathcal{O}_{13}$.

(2) If R = 0 and $|\lambda|\bar{g}^{1-\eta} \leq 1$ then we have the improvement $\forall \zeta \in \mathbb{R}$,

$$|K_{\Delta}(\lambda, \phi_1, \zeta)|_{\partial \phi, h_*} \le \mathcal{O}_{18} h_*^{10} e^{h_*^{-2} \zeta^2} (|\lambda| \bar{g}^{1-\eta})^2$$

where $O_{18} = 2O_8 + O_{12}$.

Proof: As before we start with

$$|K_{\Delta}(\lambda,\phi_{1},\zeta)|_{\partial\phi,h_{*}} \leq |\lambda|^{2}|Q_{\Delta}(\phi_{1}+\zeta)|_{\partial\phi,h_{*}}|e^{-\tilde{V}_{\Delta}(\phi_{1})}|_{\partial\phi,h_{*}}$$
$$+|\lambda|^{3}\left|Q_{\Delta}(\phi_{1}+\zeta)\left(e^{-V_{\Delta}(\phi_{1}+\zeta)}-e^{-\tilde{V}_{\Delta}(\phi_{1})}\right)\right|_{\partial\phi,h_{*}}+|\lambda|^{3}|R_{\Delta}(\phi_{1}+\zeta)|_{\partial\phi,h_{*}}.$$

Then by Lemma 22 with $\kappa=h_*^{-2}$

$$|Q_{\Delta}(\phi_1+\zeta)|_{\partial\phi,h_*} \leq \mathcal{O}_8 h_*^6 e^{h_*^{-2}\zeta^2} \bar{g}^{2-2\eta}$$
.

From Lemma 12 we have

$$|e^{-\tilde{V}_{\Delta}(\phi_1)}|_{\partial \phi, h_*} = ||e^{-\tilde{V}_{\Delta}(\phi_1)}||_{\partial \phi, 0, h_*} \le ||e^{-\tilde{V}_{\Delta}(\psi)}||_{\partial \psi, 0, h_*} = |e^{-\tilde{V}_{\Delta}(\psi)}|_{\partial \psi, h_*} \le 2.$$

By Lemma 26 with $\kappa = h_*^{-2}$ we have

$$\left| Q_{\Delta}(\phi_1 + \zeta) \left(e^{-V_{\Delta}(\phi_1 + \zeta)} - e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \right|_{\partial \phi, h_*} \le \mathcal{O}_{12} h_*^{10} e^{h_*^{-2} \zeta^2} \bar{g}^{3 - 3\eta}.$$

As a result of the estimates we have so far

$$|K_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \leq 2|\lambda|^2 \mathcal{O}_8 h_*^6 e^{h_*^{-2}\zeta^2} \bar{g}^{2-2\eta} + |\lambda|^3 \mathcal{O}_{12} h_*^{10} e^{h_*^{-2}\zeta^2} \bar{g}^{3-3\eta} + |\lambda|^3 |R_{\Delta}(\phi_1+\zeta)|_{\partial\phi,h_*}.$$

The last term will be estimated as follows. Note that

$$|R_{\Delta}(\phi_{1}+\zeta)|_{\partial\phi,h_{*}} = ||R_{\Delta}(\phi_{1}+\zeta)||_{\partial\phi,0,h_{*}} \leq ||R_{\Delta}(\psi+\zeta)|_{\partial\psi,0,h_{*}} = ||R_{\Delta}(\psi)||_{\partial\psi,\zeta,h_{*}}$$

$$\leq \mathcal{O}_{13} e^{h_*^{-2}\zeta^2} \left[|R_{\Delta}(\psi)|_{\partial \psi, h_*} + h_*^{-9} h^9 \sup_{\psi \in \mathbb{R}} ||R_{\Delta}(\psi)||_{\partial \psi, \psi, h} \right]$$

by Lemma 27. Hence

$$|R_{\Delta}(\phi_1 + \zeta)|_{\partial \phi, h_*} \le \mathcal{O}_{13} e^{h_*^{-2} \zeta^2} |||R_{\Delta}|||_{\bar{g}} \left(1 + \bar{g}^{-2} h_*^{-9} h^9\right)$$

Now

$$\bar{g}^{-2}h_*^{-9}h^9 = c_2^9c_1^{-9}L^{\frac{9}{4}(3+\epsilon)}\bar{g}^{\frac{1}{4}} \leq c_2^9c_1^{-9}L^9\bar{g}^{\frac{1}{4}} \leq 1$$

by the standard hypothesis (21). Also using (18) we now arrive at

$$|K_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \leq h_*^{10} e^{h_*^{-2}\zeta^2} \times \left[2\mathcal{O}_8 |\lambda|^2 \bar{g}^{2-2\eta} + \mathcal{O}_{12} |\lambda|^3 \bar{g}^{3-3\eta} + 2\mathcal{O}_{13} |\lambda|^3 \bar{g}^{\frac{11}{4}-\eta_R} \right] \ .$$

Since $\eta_R \geq 3\eta$ we have $\bar{g}^{1-\eta} \leq \bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R}$ and part 1) follows. When the R term is absent, the previous estimate on K reduces to

$$|K_{\Delta}(\lambda,\phi_{1},\zeta)|_{\partial\phi,h_{*}} \leq h_{*}^{10}e^{h_{*}^{-2}\zeta^{2}} \times \left[2\mathcal{O}_{8}|\lambda|^{2}\bar{g}^{2-2\eta} + \mathcal{O}_{12}|\lambda|^{3}\bar{g}^{3-3\eta}\right]$$

from which part 2) follows.

Lemma 31. If $|\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R} \leq 1$ then for all unit cube Δ' and $\phi \in \mathbb{R}$ we have

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}} \sum_{n=1}^{\infty} \left(\mathcal{O}_{19}L^{15}|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}\right)^n$$

where $\mathcal{O}_{19} = 2^{11} \max(\mathcal{O}_5, \mathcal{O}_{15})$.

Proof: Recall that by definition

$$\hat{K}_{\Delta'}(\lambda,\phi) = \sum_{Y_P,Y_K} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} \times \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \times \prod_{\Delta \in Y_P} \left[P_{\Delta}(\lambda,\phi_1,\zeta) \right] \times \prod_{\Delta \in Y_K} \left[K_{\Delta}(\lambda,\phi_1,\zeta) \right]$$

where (Y_P, Y_K) ranges over pairs of disjoint subsets of $[L^{-1}\Delta']$ such that not both are empty. It is easy to see that one therefore has the following bound on \hat{K} :

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq \sum_{Y_P,Y_K} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} \times \left| \left| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \right| \right|_{\partial\phi,\phi,h}$$

$$\times \prod_{\Delta \in Y_P} ||P_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \times \prod_{\Delta \in Y_K} ||K_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \ .$$

By Lemma13 we have

$$\left\| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \right\|_{\partial \phi, \phi, h} \le 2.$$

By Lemma 29 1) with $\kappa = h_*^{-2}$ we have

$$\prod_{\Delta \in Y_K} ||K_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \leq \prod_{\Delta \in Y_K} \left[\mathcal{O}_{15} h_*^{10} e^{h_*^{-2} \zeta^2} \left(|\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3} \eta_R} \right)^2 \right] \ .$$

Since $\eta_R \geq 3\eta$, it follows from the hypotheses that

$$|\lambda|\bar{g}^{\frac{1}{4}-\eta} \le |\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R} \le 1$$

and therefore Lemma 19 with $\kappa = h_*^{-2}$ implies

$$\prod_{\Delta \in Y_P} ||P_{\Delta}(\lambda, \phi_1, \zeta)||_{\partial \phi, \phi, h} \le \prod_{\Delta \in Y_P} \left[\mathcal{O}_5 h_*^{12} e^{h_*^{-2} \zeta^2} |\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3} \eta_R} \right] .$$

Thus

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq 2 \sum_{Y_P,Y_K} \prod_{\Delta \in Y_P} \left[\mathcal{O}_5 h_*^{12} |\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R} \right] \times \prod_{\Delta \in Y_K} \left[\mathcal{O}_{15} h_*^{10} \left(|\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R} \right)^2 \right] \times \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} (\Re f) \zeta} \prod_{\Delta \in Y_P \cup Y_K} e^{h_*^{-2} \zeta^2} \ .$$

We now use Lemma 7 with $\alpha = h_*^{-2} = \frac{\sqrt{2}}{4} L^{-(3-2[\phi])}$ in order to bound the last integral. Indeed, by the standard hypothesis (17)

$$||(\Re f)|_{L^{-1}\Delta'}||_{L^{\infty}} \le ||f|_{L^{-1}\Delta'}||_{L^{\infty}} < L^{-(3-[\phi])} < \frac{1}{2}L^{-\frac{1}{2}(3-2[\phi])}$$

since $L \geq 2$ implies $2L^{-\frac{3}{2}} \leq 2^{-\frac{1}{2}} < 1$. Hence

$$\int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} \prod_{\Delta \in Y_P \cup Y_K} e^{h_*^{-2}\zeta^2} \le 2^{|Y_P| + |Y_K|} e^{\frac{1}{2}(\Re f, \Gamma \Re f)_{L^{-1}\Delta'}}$$

and therefore

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}} \sum_{Y_P,Y_K} \prod_{\Delta\in Y_P} \left[2\mathcal{O}_5 h_*^{12} |\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R} \right] \times \prod_{\Delta\in Y_K} \left[2\mathcal{O}_{15} h_*^{10} \left(|\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R} \right)^2 \right] .$$

Using $h_* \leq 2^{\frac{3}{4}}L$ and dropping the square in the Y_K factors since $|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R} \leq 1$ we arrive at

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}} \sum_{Y_P,Y_K} \prod_{\Delta \in Y_P} \left[2^{10}\mathcal{O}_5 L^{12} |\lambda| \bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R} \right] \times \prod_{\Delta \in Y_K} \left[2^{\frac{17}{2}}\mathcal{O}_{15} L^{10} |\lambda| \bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R} \right]$$

$$\leq 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}}\sum_{Y_P,Y_{L'}}\rho^{|Y_P|+|Y_K|}$$

with

$$\rho = 2^{10} \times \max(\mathcal{O}_5, \mathcal{O}_{15}) \times L^{12} |\lambda| \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R}.$$

Now

$$\sum_{Y_P,Y_K} \rho^{|Y_P|+|Y_K|} = \sum_{n \geq 1} \rho^n \sum_{\substack{i,j \geq 0 \\ i+j=n}} \sum_{\substack{Y_P,Y_K \subset [L^{-1}\Delta'] \\ \text{disjoint}}} \mathbb{1}\{|Y_P| = i, |Y_K| = j\} \ .$$

Since the cardinality of $[L^{-1}\Delta']$ is L^3 , we have from elementary combinatorics

$$\sum_{Y_P, Y_K} \rho^{|Y_P| + |Y_K|} \leq \sum_{n=1}^{L^3} \rho^n \sum_{\substack{i,j \ge 0 \\ i+j=n}} \frac{(L^3)!}{i!j!(L^3 - n)!}$$

$$\leq \sum_{n=1}^{L^3} \rho^n \binom{L^3}{n} 2^n.$$

We use the very coarse bound

$$\begin{pmatrix} L^3 \\ n \end{pmatrix} = \frac{L^3(L^3 - 1) \cdots (L^3 - n + 1)}{n!} \le L^{3n}$$

which results in

$$\sum_{Y_P, Y_K} \rho^{|Y_P| + |Y_K|} \le \sum_{n=1}^{\infty} (2L^3 \rho)^n .$$

The latter inserted in the previous estimate for \hat{K} gives the desired inequality.

Lemma 32. If R = 0 and $|\lambda| \bar{g}^{\frac{1}{4} - \eta} \leq 1$ then for all unit cube Δ' and $\phi \in \mathbb{R}$ we have

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}}\sum_{n=1}^{\infty}\left(\mathcal{O}_{20}L^{15}|\lambda|\bar{g}^{\frac{1}{4}-\eta}\right)^{n}$$

where $\mathcal{O}_{20} = 2^{11} \max(\mathcal{O}_5, \mathcal{O}_{16})$.

Proof: One can repeat the last proof verbatim except that one must use part 2) of Lemma 29 instead of part 1). This accounts for \mathcal{O}_{16} featuring in the new constant instead of \mathcal{O}_{15} .

Lemma 33. If $|\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R} \leq 1$ then for all unit cube Δ' and $\phi \in \mathbb{R}$ we have

$$|\hat{K}_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}} \sum_{n=1}^{\infty} \left(\mathcal{O}_{21}L^{15}|\lambda|\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R} \right)^n$$

where $\mathcal{O}_{21} = 2^{11} \max(\mathcal{O}_6, \mathcal{O}_{17})$.

Proof: Again from the definition of \hat{K} one easily deduces the estimate

$$|\hat{K}_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \leq \sum_{Y_P,Y_K} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} \times \left| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \right|_{\partial\phi,h_*}$$

$$\times \prod_{\Delta \in Y_P} |P_{\Delta}(\lambda, \phi_1, \zeta)|_{\partial \phi, h_*} \times \prod_{\Delta \in Y_K} |K_{\Delta}(\lambda, \phi_1, \zeta)|_{\partial \phi, h_*} .$$

While

$$\left| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \notin Y_P \cup Y_K}} \left[e^{-\tilde{V}_{\Delta}(\phi_1)} \right] \right|_{\partial \phi, h_*} \le 2$$

by Lemma 13, we have

$$|P_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \leq \mathcal{O}_6 h_*^{12} e^{h_*^{-2}\zeta^2} |\lambda| \bar{g}^{1-\eta} \leq \mathcal{O}_6 h_*^{12} e^{h_*^{-2}\zeta^2} |\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$$

by Lemma 20. Indeed, $0 < \bar{g} \le 1$ and $\eta_R \ge 3\eta$ ensure that $|\lambda|\bar{g}^{1-\eta} \le |\lambda|\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R} \le 1$. Finally, Lemma 30 1) provides us with the last ingredient

$$|K_{\Delta}(\lambda,\phi_1,\zeta)|_{\partial\phi,h_*} \le \mathcal{O}_{17}h_*^{10}e^{h_*^{-2}\zeta^2}\left(|\lambda|\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R}\right)^2$$
.

The rest of the proof is exactly the same as that of Lemma 31.

Lemma 34. If R = 0 and $|\lambda|\bar{g}^{1-\eta} \leq 1$ then for all unit cube Δ' and $\phi \in \mathbb{R}$ we have

$$|\hat{K}_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le 2e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}} \sum_{n=1}^{\infty} (\mathcal{O}_{22}L^{15}|\lambda|\bar{g}^{1-\eta})^n$$

where $\mathcal{O}_{22} = 2^{11} \max(\mathcal{O}_6, \mathcal{O}_{18})$.

Proof: The argument is the same as in the last proof except for the use of Part 2) of Lemma 30 instead of Part 1). \Box

Lemma 35. For all $\Delta' \in \mathbb{L}$ and $\Delta_1 \in [L^{-1}\Delta']$, the quantity $J_+(\phi)$ defined in §4.2 satisfies the bound $|J_+(\phi)|_{\partial \phi, L^{[\phi]}h_*} \leq \mathcal{O}_{23}|||R_{\Delta_1}|||_{\bar{q}}$

where $\mathcal{O}_{23} = 4\mathcal{O}_{13} \times \exp\left(2^{-\frac{3}{2}}\right)$.

Proof: Recall that by definition

$$J_{+}(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} R_{\Delta_{1}}(\phi_{1} + \zeta)$$

and therefore one readily obtains

$$|J_{+}(\phi)|_{\partial\phi,L^{[\phi]}h_{*}} \leq e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} |R_{\Delta_{1}}(\phi_{1}+\zeta)|_{\partial\phi,L^{[\phi]}h_{*}} \ .$$

By the definitions of the seminorms and the chain rule one has

$$|R_{\Delta_1}(\phi_1+\zeta)|_{\partial\phi,L^{[\phi]}h_*} = ||R_{\Delta_1}(\phi_1+\zeta)||_{\partial\phi,0,L^{[\phi]}h_*} = ||R_{\Delta_1}(\psi+\zeta)||_{\partial\psi,0,h_*} = ||R_{\Delta_1}(\psi)||_{\partial\psi,\zeta,h_*}.$$

From Lemma 27 we then derive

$$|R_{\Delta_{1}}(\phi_{1}+\zeta)|_{\partial\phi,L^{[\phi]}h_{*}} \leq \mathcal{O}_{13}e^{h_{*}^{-2}\zeta_{\Delta_{1}}^{2}} \left[|R_{\Delta_{1}}(\psi)|_{\partial\psi,h_{*}} + h_{*}^{9}h^{-9} \sup_{\psi\in\mathbb{R}} ||R_{\Delta_{1}}(\psi)||_{\partial\psi,\psi,h} \right]$$

$$\leq \mathcal{O}_{13}e^{h_{*}^{-2}\zeta_{\Delta_{1}}^{2}} |||R_{\Delta_{1}}|||_{\bar{g}} \left(1 + h_{*}^{9}h^{-9}\bar{g}^{-2} \right)$$

$$\leq 2\mathcal{O}_{13}e^{h_{*}^{-2}\zeta_{\Delta_{1}}^{2}} |||R_{\Delta_{1}}|||_{\bar{g}}$$

by the standard hypothesis (21). As a result

$$|J_{+}(\phi)|_{\partial\phi,L^{[\phi]}h_{*}} \leq e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \times 2\mathcal{O}_{13} \times |||R_{\Delta_{1}}|||_{\bar{g}} \times \int d\mu_{\Gamma}(\zeta) e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} e^{h_{*}^{-2}\zeta_{\Delta_{1}}^{2}}.$$

The standard hypothesis (17) again allows to use Lemma 7 with $\alpha = h_*^{-2}$ to the effect that

$$|J_{+}(\phi)|_{\partial\phi,L^{[\phi]}h_{*}} \leq 4\mathcal{O}_{13} \times |||R_{\Delta_{1}}|||_{\bar{g}} \times \exp\left\{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'} + \frac{1}{2}(\Re f,\Gamma \Re f)_{L^{-1}\Delta'}\right\}$$

holds. Note that

$$\Re(f,\Gamma f)_{L^{-1}\Delta'} = (\Re f,\Gamma \Re f)_{L^{-1}\Delta'} - (\Im f,\Gamma \Im f)_{L^{-1}\Delta'}$$

and thus

$$|J_{+}(\phi)|_{\partial\phi,L^{[\phi]}h_{*}} \leq 4\mathcal{O}_{13} \times |||R_{\Delta_{1}}|||_{\bar{g}} \times \exp\left\{\frac{1}{2}(\Im f, \Gamma \Im f)_{L^{-1}\Delta'}\right\}.$$

But

$$|(\Im f, \Gamma \Im f)_{L^{-1}\Delta'}| \leq \int_{(L^{-1}\Delta')^2} d^3x d^3y |\Gamma x - y| |\Im f(x)| |\Im f(y)|$$

$$\leq ||f|_{L^{-1}\Delta'}||_{L^{\infty}}^2 \times L^3 \times ||\Gamma||_{L^1}$$

$$\leq L^{-2(3-[\phi])} \times L^3 \times \frac{L^{3-2[\phi]}}{\sqrt{2}}$$

$$\leq \frac{1}{\sqrt{2}}$$

because of the standard hypothesis (17), the finite range property of Γ and the bound in Corollary 1. Inserting this last inequality in the previous estimate for J_+ gives the wanted bound.

Lemma 36. For all $\Delta' \in \mathbb{L}$, $\Delta_1 \in [L^{-1}\Delta']$ and integer k such that $0 \le k \le 4$ the $\delta\beta$ quantities defined in §4.2 satisfy

$$|\delta \beta_{k,3,\Delta',\Delta_1}| \le \mathcal{O}_{24} \times \left(L^{[\phi]} h_*\right)^{-k} \times |||R_{\Delta_1}|||_{\bar{g}}$$

and

$$|\delta \beta_{k,3,\Delta'}| \leq \mathcal{O}_{24} \times L^{3-k[\phi]} \times \max_{\Delta_1 \in [L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}}$$

with

$$\mathcal{O}_{24} = 48 \times \mathcal{O}_{23} \times \sum_{i=0}^{4} \sum_{i,n,l} |\#_{k,i,j,n,l}| \, 2^{j} \left(\frac{3}{2}\right)^{n}$$

where $\#_{k,i,j,n,l}$ denote the numerical coefficients in the explicit formulas produced by Maple from §4.2.

Proof: Recall that

$$\delta \beta_{k,3,\Delta',\Delta_1} = \sum_{i=0}^4 M_{k,i} a_i$$

where

$$a_{i} = \exp \left[-C_{0}(0)L^{-2[\phi]}\beta_{2,\Delta_{1}} + 3C_{0}(0)^{2}L^{-4[\phi]}\beta_{4,\Delta_{1}} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} \right]$$

$$\times L^{-i[\phi]} \times \int d\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'} f\zeta} R_{\Delta_{1}}^{(i)}(\zeta)$$

$$= \exp \left[-C_{0}(0)L^{-2[\phi]}\beta_{2,\Delta_{1}} + 3C_{0}(0)^{2}L^{-4[\phi]}\beta_{4,\Delta_{1}} \right] \times J_{+}^{(i)}(0)$$

and

$$M_{k,i} = \sum_{j,n,l} \#_{k,i,j,n,l} C_0(0)^j L^{-(l_1 + \dots + l_n)[\phi]} \beta_{l_1,\Delta_1} \cdots \beta_{l_n,\Delta_1} .$$

From the standard hypotheses we have $|\beta_{2,\Delta_1}| < \bar{g}^{1-\eta} \le \bar{g}^{\frac{3}{4}}$ since $\eta < \frac{1}{4}$. We also have $|\beta_{4,\Delta_1}| < \frac{3}{2}\bar{g}$. Using $C_0(0) < 2$, $L^{-[\phi]} \le 1$ and the standard hypothesis (23) we then deduce the bounds

$$|a_i| \le |J_+^{(i)}(0)| \times \exp\left[2\bar{g}^{\frac{1}{4}} + 18\bar{g}\right] \le 2|J_+^{(i)}(0)|.$$

By definition of the seminorms

$$|J_{+}^{(i)}(0)| \le i! (L^{[\phi]}h_*)^{-i} |J_{+}(0)|_{\partial \phi, L^{[\phi]}h_*}$$
.

Since $i \leq 4$ we then get from the last inequality

$$|a_i| \le 48(L^{[\phi]}h_*)^{-i}|J_+(0)|_{\partial\phi,L^{[\phi]}h_*}$$
.

Now recall that the sum expressing the $M_{k,i}$ is quantified over $j \ge 0$, $n \ge 0$ and $l = (l_1, \ldots, l_n) \in \{1, \ldots, 4\}^n$. For the numerical coefficients $\#_{k,i,j,n,l}$ to be nonzero the constraint

$$l_1 + \dots + l_n - 2j = k - i$$

must be satisfied. The β_{l_{ν},Δ_1} are bounded by $\bar{g}^{1-\eta}$ or $\frac{3}{2}\bar{g}$ which can be replaced by a uniform worst case scenario bound of $\frac{3}{2}\bar{g}^{1-\eta}$. We can thus write

$$|M_{k,i}| \le \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j L^{-(l_1 + \dots + l_n)[\phi]} \times \left(\frac{3}{2}\bar{g}^{1-\eta}\right)^n.$$

We now consider two different cases in order to continue estimating the $|M_{k,i}|$.

1st case: Suppose $i \geq k$. Since the *l*'s are positive, we have $L^{-(l_1+\cdots+l_n)[\phi]} \leq 1$. We also use the coarse bound $\bar{g}^{1-\eta} \leq 1$ which results from the standard hypothesis $\eta < \frac{1}{4}$. We then simply write

$$|M_{k,i}| \le \sum_{i,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n$$
.

2nd case: Suppose i < k. Since $j \ge 0$, the previous constraint implies

$$l_1 + \cdots + l_n = 2j + k - i \ge k - i$$

and therefore $L^{-(l_1+\cdots+l_n)[\phi]} \leq L^{-(k-i)[\phi]}$. One can also infer that $n \geq 1$ since $l_1+\cdots+l_n \geq k-i > 0$ with the consequence that $(\bar{g}^{1-\eta})^n \leq \bar{g}^{1-\eta}$. The bound on $|M_{k,i}|$ which results from these remarks can reorganized as

$$|M_{k,i}| \le \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(h_* L^{[\phi]}\right)^{-(k-i)} \times \left(\frac{3}{2}\right)^n \bar{g}^{1-\eta} h_*^{k-i}.$$

Since $0 \le i < k \le 4$, $h_* \ge 1$ and $\epsilon \le 1$ we have

$$h_*^{k-i} \le h_*^4 = \left(2^{\frac{3}{4}} L^{\frac{3+\epsilon}{4}}\right)^4 \le 8L^4$$
.

The standard hypothesis (23) now allows us to write

$$|M_{k,i}| \le \left(h_* L^{[\phi]}\right)^{-(k-i)} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \left(\frac{3}{2}\right)^n$$

which is the wanted bound for $|M_{k,i}|$ in this second case.

We now combine the previous consideration and get

$$\begin{split} |\delta\beta_{k,3,\Delta',\Delta_1}| &\leq \sum_{i=k}^4 |M_{k,i}| \; |a_i| + \sum_{0 \leq i < k} |M_{k,i}| \; |a_i| \\ &\leq \sum_{i=k}^4 48 (L^{[\phi]}h_*)^{-i} |J_+(0)|_{\partial\phi,L^{[\phi]}h_*} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n \\ &+ \sum_{0 \leq i < k} 48 (L^{[\phi]}h_*)^{-k} |J_+(0)|_{\partial\phi,L^{[\phi]}h_*} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n \end{split}$$

Since $L^{[\phi]}$ and h_* are greater than 1 we have $(L^{[\phi]}h_*)^{-i} \leq (L^{[\phi]}h_*)^{-k}$ when $i \geq k$. We can then more conveniently write

$$|\delta\beta_{k,3,\Delta',\Delta_1}| \le 48(L^{[\phi]}h_*)^{-k}|J_+(0)|_{\partial\phi,L^{[\phi]}h_*} \times \sum_{i=0}^4 \sum_{i,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n$$

from which the desired follows thanks to Lemma 35. Finally the second bound on $|\delta\beta_{k,3,\Delta'}|$ follows simply by summing over $\Delta_1 \in [L^{-1}\Delta']$ and discarding the factors $h_*^{-k} \leq 1$.

Lemma 37. For all unit cube Δ' and all integer k such that $0 \le k \le 4$, we have

$$|\delta\beta_{k,1,\Delta'}| < \mathcal{O}_{25}\bar{q}^{1-\eta}L^{\frac{5}{2}}\mathbb{1}\{k < 3\}$$

where $\mathcal{O}_{25} = \frac{27}{2}$.

Proof: From the definition we get

$$|\delta \beta_{k,1,\Delta'}| \le \sum_{b} 1 \left\{ \begin{array}{c} k+b \le 4 \\ b \ge 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \ \left| \begin{array}{c} f \\ \vdots \\ \beta_{k+b} \end{array} \right.$$

where the Feynman diagram has been defined in §4.2. This already shows the vanishing when k=4. We now restrict to the case $k \leq 3$. We bound the f's by $||f|_{L^{-1}\Delta'}||_{L^{\infty}}$ and perform the integration over the corresponding points of evaluation in \mathbb{Q}_p^3 which give $||\Gamma||_{L^1}$ factors. We thus get the bound

$$\left| f \underbrace{\sum_{\beta_{k+b}}^{b} f} \right| \leq ||f|_{L^{-1}\Delta'}||_{L^{\infty}}^{b} \times ||\Gamma||_{L^{1}}^{b} \times L^{3} \times \max_{\Delta \in [L^{-1}\Delta']} |\beta_{k+b,\Delta}| .$$

$$\leq \frac{3}{2} \bar{g}^{1-\eta} L^{3} \times \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]} L^{-(3-[\phi])} \right)^{b} .$$

We discard the $\frac{1}{\sqrt{2}}$ factors and bound the remaining power of L, namely $L^{3-b[\phi]}$, by $L^{\frac{5}{2}}$ since $b \geq 1$ and $\epsilon \leq 1$. Hence

$$|\delta \beta_{k,1,\Delta'}| \le \sum_{b} 1 \left\{ \begin{array}{c} k+b \le 4 \\ b \ge 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \frac{3}{2} \bar{g}^{1-\eta} L^{\frac{5}{2}}$$

where we also discarded the factor $L^{-k[\phi]}$. Since

$$\max_{0 \le k \le 3} \sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \le 4 \\ b \ge 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} = 9$$

the lemma is proved.

Lemma 38. For all unit cube Δ' and all integer k such that $0 \le k \le 4$, we have

$$|\delta \beta_{k,2,\Delta'}| \le \mathcal{O}_{26} \bar{g}^{2-2\eta} L^5$$

with

$$\mathcal{O}_{26} = \frac{9}{2} \sum_{a_1, a_2, b_1, b_2, m} \mathbbm{1} \left\{ \begin{array}{l} a_i + b_i \leq 4 \\ a_i \geq 0 \; , \; b_i \geq 1 \\ 1 \leq m \leq \min(b_1, b_2) \end{array} \right\} \frac{(a_1 + b_1)! \; (a_2 + b_2)!}{a_1! \; a_2! \; m! \; (b_1 - m)! \; (b_2 - m)!} \\ \times C(a_1, a_2 | k) \times 2^{\frac{a_1 + a_2 - k}{2}} \\ + \sum_b \mathbbm{1} \left\{ \begin{array}{l} k + b = 5 \text{ or } 6 \\ b \geq 0 \end{array} \right\} \frac{(k + b)!}{k! \; b!}$$

where the $C(a_1, a_2|k)$ are the connection coefficients defined in S4.2.

Proof: From the definition we have

$$\begin{split} |\delta\beta_{k,2,\Delta'}| &\leq \sum_{a_1,a_2,b_1,b_2,m} \mathbbm{1} \left\{ \begin{array}{l} a_i + b_i \leq 4 \\ a_i \geq 0 \ , \ b_i \geq 1 \\ 1 \leq m \leq \min(b_1,b_2) \end{array} \right\} \frac{(a_1 + b_1)! \ (a_2 + b_2)!}{a_1! \ a_2! \ m! \ (b_1 - m)! \ (b_2 - m)!} \\ &\times \frac{1}{2} C(a_1,a_2|k) \times L^{-(a_1 + a_2)[\phi]} \times C_0(0)^{\frac{a_1 + a_2 - k}{2}} \times \left| \begin{array}{l} f \\ b_1 - m \\ f \end{array} \right] \frac{(a_1 + b_1)! \ (a_2 + b_2)!}{a_1! \ a_2! \ m! \ (b_1 - m)! \ (b_2 - m)!} \\ &+ \sum_b \mathbbm{1} \left\{ \begin{array}{l} k + b = 5 \text{ or } 6 \\ b \geq 0 \end{array} \right\} \frac{(k + b)!}{k! \ b!} L^{-k[\phi]} \left| \begin{array}{l} f \\ b \\ \vdots \\ W_{k + b} \end{array} \right. \end{split}$$

The W diagrams are bounded in the same way as in the previous lemma by

$$\left| \begin{array}{c} f \\ \vdots \\ W_{k+b} \end{array} \right| \leq \bar{g}^{2-2\eta} L^{3-b[\phi]} \leq \bar{g}^{2-2\eta} L^3 .$$

The diagrams with two internal β vertices are bounded using the same method which gives a factor of $||f|_{L^{-1}\Delta'}||_{L^{\infty}} \times ||\Gamma||_{L^{1}} \leq L^{-[\phi]}$ per f external vertex. The $|\beta|$'s are bounded in a uniform manner by $\frac{3}{2}\bar{g}^{1-\eta}$ and this results in the estimate

$$\left| \int_{b_1 - m} \int_{\beta_{a_1 + b_1}} f \int_{\beta_{a_2 + b_2}} f \right| \leq \frac{9}{4} \bar{g}^{2 - 2\eta} \times \left(L^{-[\phi]} \right)^{b_1 + b_2 - 2m} \times \int_{(L^{-1} \Delta')^2} d^3 x_1 d^3 x_2 |\Gamma(x_1 - x_2)|^m.$$

Since $b_1 + b_2 - 2m$ can take all integer values between 0 and 6 we simply discard the factor $(L^{-[\phi]})^{b_1+b_2-2m}$ in the bound. By the ultrametricity and the finite range property of Γ we have

$$\int_{(L^{-1}\Delta')^2} d^3x_1 d^3x_2 |\Gamma(x_1 - x_2)|^m = L^3 ||\Gamma||_{L^m}^m.$$

For the purposes of this lemma and for the relevant values of m, namely 1, 2 or 3, we use the blanket estimate

$$||\Gamma||_{L^m}^m \le ||\Gamma||_{L^\infty}^{m-1} \times ||\Gamma||_{L^1} \le 2^{m-1} \frac{1}{\sqrt{2}} L^{3-2[\phi]} \le 4L^2$$
.

We therefore have the estimates

 $\left| \begin{array}{c} f \\ \vdots \\ f \\ f \end{array} \right| \underbrace{\begin{array}{c} f \\ \vdots \\ \beta_{a_1+b_1} \\ \beta_{a_2+b_2} \\ f \\ \end{array}} \right| \leq 9\bar{g}^{2-2\eta} L^5$

and

$$\left|\begin{array}{cc} f & \ddots & f \\ \vdots & \ddots & f \\ W_{k+b} & \end{array}\right| \leq \bar{g}^{2-2\eta} L^3$$

which we insert in the previous bound on $|\delta\beta_{k,2,\Delta'}|$. We drop the $L^{-(a_1+a_2)[\phi]}$ and $L^{-k[\phi]}$ factors and use $C_0(0) < 2$ to arrive at the wanted result.

Lemma 39. Let $\mathcal{O}_{27} = 16 \times 25 \times [32\mathcal{O}_{25} + 40\mathcal{O}_{26} + 40\mathcal{O}_{24}]$. Provided λ satisfies $\mathcal{O}_{27}L^5|\lambda|\bar{g}^{\frac{1}{4}-\eta} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$ and $\phi \in \mathbb{R}$,

$$||e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} \le 3$$
.

Proof: By the multiplicative property of the seminorms and by Lemma 14 we have

$$||e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} \leq ||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial\phi,\phi,h} \times ||e^{\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h}$$

$$\leq 2e^{-\frac{\bar{g}}{16}\phi^4} \times \exp\left[||\delta V_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h}\right].$$

Now by definition

$$\delta V_{\Delta'}(\lambda, \phi) = \sum_{k=0}^{4} \delta \beta_{k,\Delta'}(\lambda) : \phi^{k} :_{C_{0}}$$

$$= \sum_{k=0}^{4} \left(\sum_{j=1}^{3} \lambda^{j} \delta \beta_{k,j,\Delta'} \right) : \phi^{k} :_{C_{0}}$$

and therefore

$$||\delta V_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq \sum_{\substack{0 \leq k \leq 4\\1 \leq j \leq 3}} |\lambda|^j |\delta\beta_{k,j,\Delta'}| ||: \phi^k:_{C_0} ||_{\partial\phi,\phi,h}.$$

As in the proof of Lemma 15 we have, for $0 \le k \le 4$,

$$||: \phi^{k}:_{C_{0}}||_{\partial \phi, \phi, h} \leq 25 \times \max_{0 \leq a \leq k} ||\phi^{a}||_{\partial \phi, \phi, h}$$
$$\leq 25 \times \max_{0 \leq a \leq k} (h + |\phi|)^{a}$$

For the definition of h one can write

$$|\bar{g}^{\frac{k}{4}}||:\phi^{k}:_{C_{0}}||_{\partial\phi,\phi,h} \le 25 \times \max_{0 \le a \le k} \bar{g}^{\frac{a}{4}}(h+|\phi|)^{a}$$

$$\le 25 \times \max_{0 \le a \le k} (c_{1}+\bar{g}^{\frac{1}{4}}|\phi|)^{a}$$

since $a \leq k$ and $\bar{g} \leq 1$. Since $c_1 < 1$ we has the more convenient bounds

$$|\bar{g}^{\frac{k}{4}}||:\phi^k:_{C_0}||_{\partial\phi,\phi,h} \le 25 \times \max_{0 \le a \le k} (1+\bar{g}^{\frac{1}{4}}|\phi|)^a$$

 $\le 25(1+\bar{g}^{\frac{1}{4}}|\phi|)^k$

which result in

$$||\delta V_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 25 \sum_{\substack{0 \le k \le 4\\1 \le j \le 3}} |\lambda|^j |\delta\beta_{k,j,\Delta'}| \times \bar{g}^{-\frac{k}{4}} (1+\bar{g}^{\frac{1}{4}}|\phi|)^k.$$

We now bound the contributions of each j separately. For j = 1, one has by applying Lemma 37

$$\sum_{0 \le k \le 4} |\delta \beta_{k,1,\Delta'}| \times \bar{g}^{-\frac{k}{4}} (1 + \bar{g}^{\frac{1}{4}} |\phi|)^k \le \mathcal{O}_{25} \bar{g}^{1-\eta} L^{\frac{5}{2}} \sum_{0 \le k \le 3} \bar{g}^{-\frac{k}{4}} (1 + \bar{g}^{\frac{1}{4}} |\phi|)^k \ .$$

For nonnegative numbers A and B one has the classic inequality

$$\frac{A+B}{2} \le \left(\frac{A^a + B^a}{2}\right)^{\frac{1}{a}}$$

for all $a \ge 1$ which can be conveniently rewritten as

$$(29) (A+B)^a \le 2^{a-1}(A^a+B^a) .$$

For $0 \le k \le 3$ we bound $\bar{g}^{-\frac{k}{4}}$ by $\bar{g}^{-\frac{3}{4}}$ and also write

$$(1+\bar{g}^{\frac{1}{4}}|\phi|)^k \le (1+\bar{g}^{\frac{1}{4}}|\phi|)^4 \le 8(1+\bar{g}\phi^4)$$

using (29) with a = 4. As a result we have

$$\sum_{0 \le k \le 4} |\delta \beta_{k,1,\Delta'}| \times \bar{g}^{-\frac{k}{4}} (1 + \bar{g}^{\frac{1}{4}} |\phi|)^k \le 4 \times 8 \times \mathcal{O}_{25} \bar{g}^{1-\eta} L^{\frac{5}{2}} \times (1 + \bar{g}\phi^4) .$$

We use a similar for the j=2 contribution where the sum over k goes from 0 to 4. Namely, bounding $\bar{g}^{-\frac{k}{4}}$ by \bar{g} , $(1+\bar{g}^{\frac{1}{4}}|\phi|)^4$ by $8(1+\bar{g}\phi^4)$ and using Lemma 38, we get

$$\sum_{0 \le k \le 4} |\delta \beta_{k,2,\Delta'}| \times \bar{g}^{-\frac{k}{4}} (1 + \bar{g}^{\frac{1}{4}} |\phi|)^k \le 5 \times 8 \times \mathcal{O}_{26} \bar{g}^{1-2\eta} L^5 \times (1 + \bar{g}\phi^4) .$$

The same procedure for the j = 3 contribution, this time using Lemma 36 and the standard hypothesis (18), gives

$$\sum_{0 \le k \le 4} |\delta \beta_{k,3,\Delta'}| \times \bar{g}^{-\frac{k}{4}} (1 + \bar{g}^{\frac{1}{4}} |\phi|)^k \le 5 \times 8 \times \mathcal{O}_{24} \bar{g}^{\frac{7}{4} - \eta_R} L^3 \times (1 + \bar{g}\phi^4) .$$

Hence one can collect the previous separate estimates into

$$||\delta V_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq 25(1+\bar{g}\phi^4) \left[32\mathcal{O}_{25}|\lambda|\bar{g}^{\frac{1}{4}-\eta}L^{\frac{5}{2}} + 40\mathcal{O}_{26}|\lambda|^2\bar{g}^{1-2\eta}L^5 + 40\mathcal{O}_{24}|\lambda|^3\bar{g}^{\frac{7}{4}-\eta_R}L^3 \right] \ .$$

Let $\rho = |\lambda| \bar{g}^{\frac{1}{4} - \eta}$ then clearly $|\lambda|^2 \bar{g}^{1-2\eta} = \rho^2 \bar{g}^{\frac{1}{2}} \le \rho^2$. Also because of the standard hypothesis $\eta_R \le 1 + 3\eta$ we have $|\lambda|^3 \bar{g}^{\frac{7}{4} - \eta_R} \le \rho^3$. Notice that since for instance $\mathcal{O}_{25} = \frac{27}{2}$, we clearly have $\mathcal{O}_{27} > 1$. Thus the hypothesis of the present lemma implies in particular that $\rho \le 1$. We therefore have the more convenient bound

$$||\delta V_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 25(1+\bar{g}\phi^4)L^5\rho \times [32\mathcal{O}_{25}+40\mathcal{O}_{26}+40\mathcal{O}_{24}]$$

and thus

$$||e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} \le 2e^{-\frac{\bar{g}}{16}\phi^4} \exp\left\{25L^5\rho\left[32\mathcal{O}_{25}+40\mathcal{O}_{26}+40\mathcal{O}_{24}\right](1+\bar{g}\phi^4)\right\} .$$

The hypothesis and the chosen definition of \mathcal{O}_{27} implies

$$\exp\left\{25L^{5}\rho\left[32\mathcal{O}_{25}+40\mathcal{O}_{26}+40\mathcal{O}_{24}\right](1+\bar{g}\phi^{4})\right\} \leq \exp\left\{\frac{1}{16}(1+\bar{g}\phi^{4})\right\}$$

which gives the desired bound

$$||e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} \le 2e^{\frac{1}{16}} < 3.$$

Lemma 40. Let

$$\mathcal{O}_{28} = 200 \times \left\{ \log \left(\frac{3}{2} \right) \right\}^{-1} \times [4\mathcal{O}_{25} + 5\mathcal{O}_{27} + 5\mathcal{O}_{24}].$$

Provided λ satisfies $\mathcal{O}_{28}L^9|\lambda|\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$,

$$|e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}|_{\partial\phi,h_*} \le 3$$
.

Proof: Again by the multiplicative property of the seminorms and by Lemma 14 we have

$$|e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}|_{\partial\phi,h_*} \leq 2\times \exp\left[|\delta V_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*}\right] \ .$$

We also have

$$|\delta V_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \leq \sum_{\substack{0 \leq k \leq 4\\1 \leq j \leq 3}} |\lambda|^j |\delta\beta_{k,j,\Delta'}| |: \phi^k :_{C_0} |_{\partial\phi,h_*}.$$

For $0 \le k \le 4$,

$$|:\phi^{k}:_{C_{0}}|_{\partial\phi,h_{*}} \leq 25 \times \max_{0 \leq a \leq k} |\phi^{a}|_{\partial\phi,h_{*}}$$

$$\leq 25 \times \max_{0 \leq a \leq k} h_{*}^{a}$$

$$\leq 25h_{*}^{k}$$

since $h_* \geq 1$. Using Lemmas 37, 38 and 36 we then immediately get

$$\begin{split} |\delta V_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} & \leq & 25 \left\{ |\lambda| \times \left(\sum_{k=0}^{3} \mathcal{O}_{25} \bar{g}^{1-\eta} L^{\frac{5}{2}} h_*^k \right) \right. \\ & + |\lambda|^2 \times \left(\sum_{k=0}^{4} \mathcal{O}_{26} \bar{g}^{2-2\eta} L^5 h_*^k \right) \\ & + |\lambda|^3 \times \left(\sum_{k=0}^{4} \mathcal{O}_{24} \bar{g}^{\frac{11}{4} - \eta_R} L^3 h_*^k \right) \right\} \; . \end{split}$$

We bound powers of h_* simply by $h_*^4 = 2^3 L^{\frac{3+\epsilon}{4}} \leq 8L^4$ and $L^{\frac{5}{2}}$ by L^3 . We thus easily get

$$|\delta V_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le 25 \times 8 \times L^9 \left\{ 4|\lambda|\mathcal{O}_{25}\bar{g}^{1-\eta} + 5|\lambda|^2 \mathcal{O}_{26}\bar{g}^{2-2\eta} + 5|\lambda|^2 3\mathcal{O}_{24}\bar{g}^{\frac{11}{4}-\eta_R} \right\}.$$

Let this time $\rho = |\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$. Since $\log\left(\frac{3}{2}\right) \simeq 0.405...$ and, e.g., $\mathcal{O}_{25} = \frac{27}{2}$, it is clear that $\mathcal{O}_{28} > 1$. Hence the hypothesis of the lemma implies $\rho \leq 1$. Besides, the standard hypothesis $\eta_R \geq 3\eta$ implies $\bar{g}^{1-\eta} \leq \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$. As a consequence we have

$$\delta V_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le 200 \times L^9 \rho \left\{ 4\mathcal{O}_{25} + 5\mathcal{O}_{26} + 5\mathcal{O}_{24} \right\}$$

from which the result follows easily.

Lemma 41. Let

$$\mathcal{O}_{29} = 200 \times \left\{ \log \left(\frac{3}{2} \right) \right\}^{-1} \times [4\mathcal{O}_{25} + 5\mathcal{O}_{27}].$$

Under the extra assumption that R=0 and provided λ satisfies $\mathcal{O}_{29}L^9|\lambda|\bar{g}^{1-\eta}\leq 1$ we have, for all $\Delta'\in\mathbb{L}$,

$$|e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}|_{\partial\phi,h_*} \leq 3$$
.

Proof: The proof is the same as that of the previous lemma except for the absence of the $\beta_{k,3,\Delta'}$ terms. The only modification is to let $\rho = |\lambda| \bar{g}^{1-\eta}$ instead of $|\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$.

Lemma 42. Let $\mathcal{O}_{30} = \mathcal{O}_{25} + \mathcal{O}_{26} + \mathcal{O}_{24}$. Provided λ satisfies $\mathcal{O}_{30}L^5|\lambda|\bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$, $|\delta b_{\Delta'}(\lambda)| \leq 1$.

Proof: By definition

$$\delta b_{\Delta'}(\lambda) = \delta \beta_{0,\Delta'}(\lambda) = \lambda \delta \beta_{0,1,\Delta'} + \lambda^2 \delta \beta_{0,2,\Delta'} + \lambda^3 \delta \beta_{0,3,\Delta'}.$$

From Lemmas 37, 38 and 36 we get

$$\begin{array}{lcl} |\delta\beta_{0,1,\Delta'}| & \leq & \mathcal{O}_{25}L^{\frac{5}{2}}\bar{g}^{1-\eta} \\ |\delta\beta_{0,2,\Delta'}| & \leq & \mathcal{O}_{26}L^{5}\bar{g}^{2-2\eta} \\ |\delta\beta_{0,3,\Delta'}| & \leq & \mathcal{O}_{24}L^{3}\bar{g}^{\frac{11}{4}-\eta_R} \end{array}$$

which give

$$|\delta b_{\Delta'}(\lambda)| \le L^5 \left[\mathcal{O}_{25} |\lambda| \bar{g}^{1-\eta} + \mathcal{O}_{26} |\lambda|^2 \bar{g}^{2-2\eta} + \mathcal{O}_{24} |\lambda|^3 \bar{g}^{\frac{11}{4} - \eta_R} \right] .$$

Since clearly $\mathcal{O}_{30} > 1$, one can conclude as we did previously that

$$|\delta b_{\Delta'}(\lambda)| \le \mathcal{O}_{30} L^5 |\lambda| \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R} \le 1$$
.

Lemma 43. Let $\mathcal{O}_{31} = \mathcal{O}_{25} + \mathcal{O}_{26}$. Under the extra assumption that R = 0 and provided λ satisfies $\mathcal{O}_{31}L^5|\lambda|\bar{g}^{1-\eta} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$, $|\delta b_{\Delta'}(\lambda)| \leq 1$.

Proof: The proof is the same as that of the previous lemma, without the $\delta\beta_{0,3,\Delta'}$ term.

Lemma 44. Let

$$\mathcal{O}_{32} = \max(2\mathcal{O}_{19}, \mathcal{O}_{27}, \mathcal{O}_{30}) \text{ and } \mathcal{O}_{33} = 7 \times \exp\left(1 + \frac{\sqrt{2}}{2}\right)$$
.

Provided λ satisfies $\mathcal{O}_{32}L^{15}|\lambda|\bar{q}^{\frac{1}{4}-\frac{1}{3}\eta_R} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$ and $\phi \in \mathbb{R}$,

$$||K'_{\Lambda'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq \mathcal{O}_{33}$$
.

Proof: One can rewrite the definition of $K'_{\Lambda'}(\lambda,\phi)$ as

$$\begin{split} K'_{\Delta'}(\lambda,\phi) &= e^{-\delta b_{\Delta'}(\lambda)} e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \\ &\times \left\{ \hat{K}_{\Delta'}(\lambda,\phi) - e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \left(e^{-\hat{V}_{\Delta'}(\phi) + \delta V_{\Delta'}(\lambda,\phi)} - e^{-\hat{V}_{\Delta'}(\phi)} \right) \right\} \end{split}$$

from which one deduces

$$||K'_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le e^{|\delta b_{\Delta'}(\lambda)|} \times \exp\left[2^{-\frac{3}{2}}\right]$$

$$\times \left\{ ||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} + \exp\left[2^{-\frac{3}{2}}\right] \times \left(||e^{-\hat{V}_{\Delta'}(\phi) + \delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} + ||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial\phi,\phi,h} \right) \right\}.$$

Indeed, we previously showed $|(f, \Gamma f)_{L^{-1}\Delta'}| \leq \frac{1}{\sqrt{2}}$. We have $||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial \phi, \phi, h} \leq 2$ by Lemma 14. Clearly $\mathcal{O}_{32} \geq \mathcal{O}_{30} \geq \mathcal{O}_{25} > 1$ and therefore the hypothesis of the present lemma implies that of Lemma 31. The latter gives the bound

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 2 \exp\left[2^{-\frac{3}{2}}\right] \times \sum_{n=1}^{\infty} \left(\mathcal{O}_{19}L^{15}|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}\right)^n$$

where we used $|(\Re f, \Gamma \Re f)_{L^{-1}\Delta'}| \leq \frac{1}{\sqrt{2}}$. Since $0 \leq x \leq \frac{1}{2}$ implies $\sum_{n=1}^{\infty} x^n \leq 1$ and since the hypothesis implies $\mathcal{O}_{19}L^{15}|\lambda|\bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R} \leq \frac{1}{2}$, we have the simpler estimate

$$||\hat{K}_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 2\exp\left[2^{-\frac{3}{2}}\right].$$

From $\eta_R \geq 3\eta$ we get $\bar{g}^{\frac{1}{4}-\eta} \leq \bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}$. Since also $L^5 < L^{15}$, the hypothesis of the present lemma implies that of Lemma 39 which gives us $|\delta b_{\Delta'}(\lambda)| \leq 1$. Finally, since $\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R} \leq \bar{g}^{\frac{1}{4}-\frac{1}{3}\eta_R}$, the hypothesis of Lemma 42 is satisfied. This gives us the last needed ingredient

$$||e^{-\hat{V}_{\Delta'}(\phi)+\delta V_{\Delta'}(\lambda,\phi)}||_{\partial\phi,\phi,h} \leq 3.$$

Altogether we obtain

$$||K'_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq e \times \exp\left[2^{-\frac{3}{2}}\right] \times \left\{2\exp\left[2^{-\frac{3}{2}}\right] + \exp\left[2^{-\frac{3}{2}}\right](3+2)\right\} = \mathcal{O}_{33} \ .$$

Lemma 45. Let

$$\mathcal{O}_{34} = \max(2\mathcal{O}_{20}, \mathcal{O}_{27}, \mathcal{O}_{31})$$
.

Under the extra assumption that R=0 and provided λ satisfies $\mathcal{O}_{34}L^{15}|\lambda|\bar{g}^{\frac{1}{4}-\eta}\leq 1$ we have, for all $\Delta'\in\mathbb{L}$ and $\phi\in\mathbb{R}$,

$$||K'_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \leq \mathcal{O}_{33}$$
.

Proof: The proof is similar to that of the last lemma. The only modifications are as follows. We use Lemma 32 instead of Lemma 31, noting that $\mathcal{O}_{34} \geq \mathcal{O}_{31} \leq \mathcal{O}_{25} > 1$. We use Lemma 43 instead of 42.

Lemma 46. Let

$$\mathcal{O}_{35} = \max(2\mathcal{O}_{21}, \mathcal{O}_{28}, \mathcal{O}_{30})$$
.

Provided λ satisfies $\mathcal{O}_{35}L^{15}|\lambda|\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$ and $\phi \in \mathbb{R}$,

$$|K'_{\Lambda'}(\lambda,\phi)|_{\partial\phi,h_*} \leq \mathcal{O}_{33}$$
.

Proof: The proof is similar to that of Lemma 44. The only modifications are as follows. We use Lemma 33 instead of Lemma 31, noting that $\mathcal{O}_{35} \geq \mathcal{O}_{30} \leq \mathcal{O}_{25} > 1$. We use Lemma 40 instead of 39.

Lemma 47. Let

$$\mathcal{O}_{36} = \max(2\mathcal{O}_{22}, \mathcal{O}_{29}, \mathcal{O}_{31})$$
.

Under the extra assumption that R=0 and provided λ satisfies $\mathcal{O}_{36}L^{15}|\lambda|\bar{g}^{1-\eta} \leq 1$ we have, for all $\Delta' \in \mathbb{L}$ and $\phi \in \mathbb{R}$,

$$|K'_{\Lambda'}(\lambda,\phi)|_{\partial\phi,h_*} \leq \mathcal{O}_{33}$$
.

Proof: The proof is similar to that of Lemma 44. The only modifications are as follows. We use Lemma 34 instead of Lemma 31, noting that $\mathcal{O}_{36} \geq \mathcal{O}_{31} \leq \mathcal{O}_{25} > 1$. We use Lemma 41 instead of 39 and Lemma 43 instead of 42.

Recall from §4.2 that

(30)
$$\xi_{R,\Delta'}(\vec{V})(\phi) = \xi_{R,\Delta'}^{\text{main}}(\vec{V})(\phi) + \xi_{R,\Delta'}^{\text{higher}}(\vec{V})(\phi) + \xi_{R,\Delta'}^{\text{shift}}(\vec{V})(\phi)$$

where

$$\xi_{R,\Delta'}^{\rm main}(\vec{V})(\phi) = \frac{1}{2\pi i} \oint_{\gamma_0} \frac{{\rm d}\lambda}{\lambda^4} \left. K_{\Delta'}'(\lambda,\phi) \right|_{R=0} \ , \label{eq:energy_energy}$$

$$\xi_{R,\Delta'}^{\text{higher}}(\vec{V})(\phi) = \frac{1}{2\pi i} \oint_{\gamma_{01}} \frac{\mathrm{d}\lambda}{\lambda^4(\lambda-1)} K'_{\Delta'}(\lambda,\phi)$$

and

$$\xi_{R,\Delta'}^{\rm shift}(\vec{V})(\phi) = \left(e^{-\hat{V}_{\Delta'}(\phi)} - e^{-V'_{\Delta'}(\phi)}\right) Q'_{\Delta'}(\phi) \ .$$

The next few lemmas will provide bounds for each of these terms.

Lemma 48. For all unit cube Δ' we have that

$$|||\xi_{R,\Delta'}^{\text{main}}(\vec{V})|||_{\bar{g}} \le \mathcal{O}_{37} L^{45} \bar{g}^{\frac{11}{4} - 3\eta}$$

where $\mathcal{O}_{37} = \mathcal{O}_{33} \times \max[\mathcal{O}_{34}^3, \mathcal{O}_{36}^3]$.

Proof: We use the freedom to deform the contour of integration in order to pick for γ_0 the circle of radius ρ around the origin where

$$\rho = \left(\mathcal{O}_{34}L^{15}\bar{g}^{\frac{1}{4}-\eta}\right)^{-1} > 0 \ .$$

We then use

$$\left\| \frac{1}{2\pi i} \oint_{\gamma_0} \frac{\mathrm{d}\lambda}{\lambda^4} K'_{\Delta'}(\lambda, \phi) \big|_{R=0} \right\|_{\partial \phi, \phi, h} \le \rho^{-3} \sup_{\lambda \in \gamma_0} \left\| K'_{\Delta'}(\lambda, \phi) \big|_{R=0} \right\|_{\partial \phi, \phi, h}$$

$$< \mathcal{O}_{23} \rho^{-3}$$

by Lemma 45. Hence

$$||\xi_{R \ \Delta'}^{\text{main}}(\vec{V})(\phi)||_{\partial \phi, \phi, h} \leq \mathcal{O}_{33} \mathcal{O}_{34}^3 L^{45} \bar{g}^{\frac{3}{4} - 3\eta}.$$

The bound on $|\xi_{R,\Delta'}^{\mathrm{main}}(\vec{V})(\phi)||_{\partial\phi,h_*}$ is derived in the same manner using Lemma 47 and setting

$$\rho = \left(\mathcal{O}_{36}L^{15}\bar{g}^{1-\eta}\right)^{-1} > 0$$

for the contour radius. We get

$$|\xi_{R,\Delta'}^{\text{main}}(\vec{V})(\phi)||_{\partial\phi,h_*} \leq \mathcal{O}_{33}\mathcal{O}_{36}^3 L^{45} \bar{g}^{3-3\eta}$$

and therefore

$$|||\xi_{R,\Delta'}^{\mathrm{main}}(\vec{V})|||_{\bar{g}} \leq \mathcal{O}_{33} \times L^{45} \times \max \left[\mathcal{O}_{36}^3 \bar{g}^{3-3\eta}, \mathcal{O}_{36}^3 \bar{g}^{\frac{11}{4}-3\eta} \right]$$

by definition of the $|||\cdot|||_{\bar{g}}$ norm. Since $\bar{g} \leq 1$ the lemma follows.

Lemma 49. For all unit cube Δ' we have that

$$|||\xi_{R,\Lambda'}^{\text{higher}}(\vec{V})|||_{\bar{q}} \leq \mathcal{O}_{38}L^{60}\bar{g}^{\frac{11}{4}-3\eta}$$

where $\mathcal{O}_{38} = 2\mathcal{O}_{33} \times \max[\mathcal{O}_{32}^4, \mathcal{O}_{35}^4].$

Proof: We proceed as in the proof of the previous lemma. We take for the contour γ_{01} a circle of radius γ ge2 around the origin. This ensures that both 0 and 1 are enclosed by the contour and allows one to bound $\frac{1}{|\lambda-1|}$ by $\frac{2}{\rho}$. We first take

$$\rho = \left(\mathcal{O}_{32} L^{15} \bar{g}^{\frac{1}{4} - \frac{1}{3}\eta_R}\right)^{-1} \ge 2$$

because of standard hypothesis (25). Lemma 44 then results in the bound

$$||\xi_{R,\Delta'}^{\text{higher}}(\vec{V})(\phi)||_{\partial\phi,\phi,h} \le \mathcal{O}_{33} \times \frac{2}{\rho^4} = 2\mathcal{O}_{33} \times \mathcal{O}_{32}^4 L^{60} \bar{g}^{1-\frac{4}{3}\eta_R}$$
.

Likewise, if we pick

$$\rho = \left(\mathcal{O}_{35}L^{15}\bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}\right)^{-1}$$

then the latter is at least equal to 2 by the standard hypothesis (25) Therefore 46 results in the bound

$$|\xi_{R,\Delta'}^{\text{higher}}(\vec{V})(\phi)|_{\partial\phi,h_*} \leq 2\mathcal{O}_{33} \times \mathcal{O}_{35}^4 L^{60} \bar{g}^{\frac{11}{3} - \frac{4}{3}\eta_R}$$
.

Finally, we get

$$|||\xi_{R,\Delta'}^{\text{higher}}(\vec{V})|||_{\bar{g}} \leq 2\mathcal{O}_{33} \times L^{60} \times \max \left[\mathcal{O}_{35}^4 \bar{g}^{\frac{11}{3} - \frac{4}{3}\eta_R}, \mathcal{O}_{32}^4 \bar{g}^{3 - \frac{4}{3}\eta_R}\right] \ .$$

The conclusion of the proof is a matter of showing that the powers of \bar{g} involved are bounded by $\bar{g}^{\frac{11}{4}-3\eta}$. In other words, one needs to check the two inequalities

$$\frac{11}{3} - \frac{4}{3}\eta_R \ge \frac{11}{4} - 3\eta$$

and

$$3 - \frac{4}{3}\eta_R \ge \frac{11}{4} - 3\eta \ .$$

The first inequality follows from the second since $\frac{11}{3} > 3$. Finally, the second inequality is equivalent to the standard hypothesis (24) and therefore holds.

Lemma 50. For all unit cube Δ' and all $\phi \in \mathbb{R}$ we have that

$$||\xi_{R,\Delta'}^{\text{shift}}(\vec{V})(\phi)||_{\partial\phi,\phi,h} \leq \mathcal{O}_{39}L^{\frac{15}{2}}\bar{g}^{\frac{3}{4}-3\eta}$$

where

$$\mathcal{O}_{39} = 2^{15} \times 3^{\frac{5}{2}} \times 11 \times 103 \times (5 + \sqrt{2}) \times e^{\frac{1}{48}} \times \mathcal{O}_{27} \times \max_{0 \le n \le 6} \left(\frac{n}{4e}\right)^{\frac{n}{4}}.$$

Proof: Recall that

$$\hat{V}_{\Delta'}(\phi) = \sum_{k=1}^{4} \hat{\beta}_{k,\Delta'} : \phi^k :_{C_0}$$

while

$$V'_{\Delta'}(\phi) = \hat{V}_{\Delta'}(\phi) - \delta V_{\Delta'}(\phi) + \delta b_{\Delta'}$$

with

$$\delta V_{\Delta'}(\phi) = \delta V_{\Delta'}(\lambda, \phi)|_{\lambda=1}$$
 and $\delta b_{\Delta'} = \delta b_{\Delta'}(\lambda)|_{\lambda=1}$.

Let us introduce the notation

$$U_{\Delta'}(\phi) = \hat{V}_{\Delta'}(\phi) - V'_{\Delta'}(\phi) = \delta V_{\Delta'}(\phi) - \delta b_{\Delta'}(\phi)$$

so that

$$U_{\Delta'}(\phi) = \sum_{k=1}^{4} \delta \beta_{k,\Delta'} : \phi^k :_{C_0} = \sum_{k=1}^{4} \left(\sum_{j=1}^{3} \delta \beta_{k,j,\Delta'} \right) : \phi^k :_{C_0} .$$

The latter quantity is the same as $\delta V_{\Delta'}(\lambda, \phi)$ that was estimated in Lemma 39 except that the sum over k goes from 1 to 4 instead of from 0 to 4, and λ is now set equal to 1. By the same argument as in Lemma 39 we therefore get

$$||U_{\Delta'}(\lambda,\phi)||_{\partial\phi,\phi,h} \le 25(1+\bar{q}\phi^4)L^5\rho \times [32\mathcal{O}_{25}+40\mathcal{O}_{26}+40\mathcal{O}_{24}]$$

with $\rho = \bar{g}^{\frac{1}{4} - \eta} \le 1$ by the standard hypothesis (15). Note that for simplicity we did not take advantage of the smaller range of summation for k in order to improve the coefficients 32 and 40 in the last bound.

We now write

$$\begin{split} \xi_{R,\Delta'}^{\text{shift}}(\vec{V}) &= \left(e^{-\hat{V}_{\Delta'}(\phi)} - e^{-\hat{V}_{\Delta'}(\phi) + U_{\Delta'}(\phi)}\right) Q'_{\Delta'}(\phi) \\ &= -e^{-\hat{V}_{\Delta'}(\phi)} \left(e^{U_{\Delta'}(\phi) - 1}\right) Q'_{\Delta'}(\phi) \\ &= -e^{-\hat{V}_{\Delta'}(\phi)} Q'_{\Delta'}(\phi) \int_0^1 \mathrm{d}s \; U_{\Delta'}(\phi) \; e^{sU_{\Delta'}(\phi)} \end{split}$$

which implies the bound

$$||\xi_{R,\Delta'}^{\text{shift}}(\vec{V})||_{\partial\phi,\phi,h} \leq ||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial\phi,\phi,h}||Q'_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \times \int_0^1 \mathrm{d}s \; ||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \times \exp\left\{s||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h}\right\}$$

$$(31) \qquad \leq ||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial\phi,\phi,h} ||Q'_{\Delta'}(\phi)||_{\partial\phi,\phi,h} ||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h} e^{||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h}}.$$

Each of these four factors needs to be estimated separately. By Lemma 14 we have

(32)
$$||e^{-\hat{V}_{\Delta'}(\phi)}||_{\partial\phi,\phi,h} \le 2e^{-\frac{\bar{q}}{16}\phi^4}.$$

The last exponential expression will be needed in order to control each of the other three factors in (31). Indeed, we will show that the latter can be bounded using the exponential of $\frac{1}{3} \times \frac{\bar{g}}{16} \phi^4 = \frac{\bar{g}}{48} \phi^4$.

By the previous considerations,

(33)
$$||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \le 25(1+\bar{g}\phi^4)L^5\bar{g}^{\frac{1}{4}-\eta} \times [32\mathcal{O}_{25}+40\mathcal{O}_{26}+40\mathcal{O}_{24}] .$$

Since

$$48 \times 25 \times [32\mathcal{O}_{25} + 40\mathcal{O}_{26} + 40\mathcal{O}_{24}] = 3 \times \mathcal{O}_{27}$$

the standard hypothesis (25) ensures that

$$||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \le \frac{1}{48}(1+\bar{g}\phi^4)$$

and therefore

(34)
$$e^{||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h}} < e^{\frac{1}{48}} \times e^{\frac{\bar{g}}{48}\phi^4}$$

which is the first estimate of the kind we are seeking.

Next we note that (33) can be rewritten as

$$||U_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \leq \frac{1}{16}\mathcal{O}_{27}L^{5}\bar{g}^{\frac{1}{4}-\eta}(1+\bar{g}\phi^{4})$$

$$\leq \frac{1}{16}\mathcal{O}_{27}L^{5}\bar{g}^{\frac{1}{4}-\eta}(48+\bar{g}\phi^{4})$$

$$\leq 3\mathcal{O}_{27}L^{5}\bar{g}^{\frac{1}{4}-\eta}(1+\frac{\bar{g}}{48}\phi^{4})$$

$$\leq 3\mathcal{O}_{27}L^{5}\bar{g}^{\frac{1}{4}-\eta}e^{\frac{\bar{g}}{48}\phi^{4}}.$$

Finally we need a similar bound on the $Q'_{\Delta'}$ factor. Recall that

$$Q'_{\Delta'}(\phi) = W'_{5,\Delta'} : \phi_{\Delta'}^5 :_{C_0} + W'_{6,\Delta'} : \phi_{\Delta'}^6 :_{C_0}$$

Similarly to the proof of Lemma 21 by undoing the Wick ordering we have

$$||Q'_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \le |W'_{5,\Delta'}| \times 81 \times \left(\max_{0 \le a \le 5} ||\phi^a||_{\partial\phi,\phi,h}\right)$$
$$+|W'_{5,\Delta'}| \times 331 \times \left(\max_{0 \le a \le 6} ||\phi^a||_{\partial\phi,\phi,h}\right)$$

and thus

(35)

$$||Q'_{\Delta'}(\phi)||_{\partial\phi,\phi,h} \le 412 \times \max\left[|W'_{5,\Delta'}|,|W'_{6,\Delta'}|\right] \times \max_{0 \le a \le 6} ||\phi^a||_{\partial\phi,\phi,h}$$
.

Now for $0 \le k \le 6$ and for $\gamma > 0$ we have

$$||\phi^{k}||_{\partial\phi,\phi,h} = (h+|\phi|)^{k}$$

$$\leq \sum_{n=0}^{k} {k \choose n} \times \left(c_{1}\bar{g}^{-\frac{1}{4}}\right)^{k-n} \times \left(\frac{n}{4e}\right)^{\frac{n}{4}} \times (\gamma\bar{g})^{-\frac{n}{4}} e^{\gamma\bar{g}\phi^{4}}$$

by the first inequality in Lemma 8 with $\beta_4 = \bar{g}$. If also $\gamma \leq 1$, then we bound $\gamma^{-\frac{n}{4}}$ by $\gamma^{-\frac{k}{4}}$ and get

$$||\phi^k||_{\partial\phi,\phi,h} \le \left(\max_{0 \le n \le 6} \left(\frac{n}{4e}\right)^{\frac{n}{4}}\right) \times \bar{g}^{-\frac{k}{4}} \gamma^{-\frac{k}{4}} e^{\gamma \bar{g}\phi^4} \times (1+c_1)^k.$$

We now pick $\gamma = \frac{1}{48}$ and simply bound $(1+c_1)^k$ by $2^k \leq 2^6$. One then easily obtains

$$||Q_{\Delta'}'(\phi)||_{\partial \phi, \phi, h} \leq 412 \times \max\left[|W_{5, \Delta'}'|, |W_{6, \Delta'}'|\right] \times \left(\max_{0 \leq n \leq 6} \left(\frac{n}{4e}\right)^{\frac{n}{4}}\right) \times \bar{g}^{-\frac{3}{2}} 48^{\frac{3}{2}} \times 2^{6} \times e^{\frac{\bar{q}}{48}\phi^{4}} \ .$$

In order to continue we need to now bound the W' factors. From the definition in §4.2 we immediately get

$$|W'_{6,\Delta'}| \le L^{3-6[\phi]} \bar{g}^{2-2\eta} + 8L^{-6[\phi]} L^3 \left(\frac{3}{2}\bar{g}\right)^2 \times ||\Gamma||_{L^1}$$

Since $\eta \geq 0$, we get from Corollary 1

$$|W'_{6,\Delta'}| \le L^{3-6[\phi]} \bar{g}^{2-2\eta} \left(1 + 8\left(\frac{3}{2}\right)^2 \times \frac{1}{\sqrt{2}} L^{3-2[\phi]}\right) ,$$

namely,

$$|W_{6,\Delta'}'| \leq L^{3-6[\phi]} \bar{g}^{2-2\eta} (1 + 9\sqrt{2}L^{3-2[\phi]}) \leq L^{6-8[\phi]} \bar{g}^{2-2\eta} (1 + 9\sqrt{2}) \ .$$

Now from the assumption $\epsilon \leq 1$ we get $[\phi] = \frac{3-\epsilon}{4} \geq \frac{1}{2}$ and therefore $6-8[\phi] \leq 2$. As a result we simplify the last bound on W_6' into

$$|W'_{6,\Delta'}| \le L^2 \bar{g}^{2-2\eta} (1+9\sqrt{2}) .$$

Similarly, from the definition in §4.2, Corollary 1 and the standard hypotheses (16) and (17), we easily get

$$\begin{split} |W_{5,\Delta'}'| &\leq L^{3-5[\phi]} \bar{g}^{2-2\eta} + 6L^{-5[\phi]} \bar{g}^{2-2\eta} L^3 \frac{1}{\sqrt{2}} L^{3-2[\phi]} L^{-(3-[\phi])} \\ &+ 12L^{-5[\phi]} \frac{3}{2} \bar{g} L^3 \frac{1}{\sqrt{2}} L^{3-2[\phi]} \bar{g}^{1-\eta} + 48L^{-5[\phi]} \left(\frac{3}{2}\right)^2 \bar{g}^2 L^3 \frac{1}{2} L^{2(3-2[\phi])} L^{-(3-[\phi])} \\ &\leq L^{3-5[\phi]} \bar{g}^{2-2\eta} \left[1 + 3\sqrt{2}L^{-[\phi]} + 9\sqrt{2}L^{3-2[\phi]} + 54L^{3-3[\phi]}\right] \\ &\leq L^{6-7[\phi]} \bar{g}^{2-2\eta} \left[1 + 3\sqrt{2} + 9\sqrt{2} + 54\right] \end{split}$$

since $L^{-[\phi]}$ and $L^{3-3[\phi]}$ are bounded by $L^{3-2[\phi]}$. Since $\epsilon \leq 1$ implies $6-7[\phi] \leq \frac{5}{2}$ we then have

$$|W_{5,\Delta'}'| \leq L^{\frac{5}{2}} \bar{g}^{2-2\eta} \times 11(5+\sqrt{2})$$

which compared with the previous estimate on W_6' results in

$$\max \left[|W'_{5,\Delta'}|, |W'_{6,\Delta'}| \right] \le L^{\frac{5}{2}} \bar{g}^{2-2\eta} \times 11(5+\sqrt{2})$$
.

As a consequence we have

$$(36) \qquad ||Q'_{\Delta'}(\phi)||_{\partial \phi, \phi, h} \leq 412 \times L^{\frac{5}{2}} \bar{g}^{2-2\eta} \times 11(5+\sqrt{2}) \times \left(\max_{0 \leq n \leq 6} \left(\frac{n}{4e}\right)^{\frac{n}{4}}\right) \times \bar{g}^{-\frac{3}{2}} 48^{\frac{3}{2}} \times 2^{6} \times e^{\frac{\bar{g}}{48}\phi^{4}} \ .$$

Finally we use the bounds (31),(32),(34),(35) and (36) in order to derive the inequality

$$\begin{split} ||\xi_{R,\Delta'}^{\text{shift}}(\vec{V})||_{\partial\phi,\phi,h} & \leq & 2e^{-\frac{\bar{g}}{16}\phi^4} \\ & \times 412 \times L^{\frac{5}{2}} \bar{g}^{\frac{1}{2}-2\eta} \times 11(5+\sqrt{2}) \times \left(\max_{0 \leq n \leq 6} \left(\frac{n}{4e}\right)^{\frac{n}{4}}\right) \times 48^{\frac{3}{2}} \times 2^{6} \times e^{\frac{\bar{g}}{48}\phi^4} \\ & \times 3\mathcal{O}_{27} L^5 \bar{g}^{\frac{1}{4}-\eta} e^{\frac{\bar{g}}{48}\phi^4} \\ & \times e^{\frac{1}{48}} \times e^{\frac{\bar{g}}{48}\phi^4} \end{split}$$

which after cleaning up becomes the desired bound.

Lemma 51. For all unit cube Δ' we have that

$$|\xi_{R,\Lambda'}^{\text{shift}}(\vec{V})(\phi)|_{\partial\phi,h_*} \leq \mathcal{O}_{40}L^{\frac{35}{2}}\bar{g}^{\frac{35}{12}-2\eta-\frac{1}{3}\eta_R}$$

where

$$\mathcal{O}_{40} = 2^{\frac{13}{2}} \times 3 \times 11 \times 103 \times \log\left(\frac{3}{2}\right) \times (5 + \sqrt{2}) \times \mathcal{O}_{28}$$
.

Proof: As before we have

$$|\xi_{R,\Delta'}^{\rm shift}(\vec{V})(\phi)|_{\partial\phi,h_*} \leq |e^{-\hat{V}_{\Delta'}(\phi)}|_{\partial\phi,h_*}|Q'_{\Delta'}(\phi)|_{\partial\phi,h_*}|U_{\Delta'}(\phi)|_{\partial\phi,h_*}e^{|U_{\Delta'}(\phi)|_{\partial\phi,h_*}} \ .$$

Lemma 14 allows us to bound the first factor by

$$|e^{-\hat{V}_{\Delta'}(\phi)}|_{\partial\phi,h_*} \le 2$$
.

The quantity $U_{\Delta'}(\phi)$ is estimated in the same as $\delta V_{\Delta'}(\lambda, \phi)$ in Lemma 40 with $\lambda = 1$. except that the sum over k goes from 1 to 4 instead of from 0 to 4, and λ is now set equal to 1. Hence

$$|U_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le 400 \times L^9 \times [4\mathcal{O}_{25} + 5\mathcal{O}_{26} + 5\mathcal{O}_{24}] \times \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$$

or equivalently

$$|U_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le \log\left(\frac{3}{2}\right)\mathcal{O}_{28} \times L^9 \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R}$$
.

Now the standard hypothesis (25) implies

$$|U_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*} \le \log\left(\frac{3}{2}\right)$$

and thus

$$e^{|U_{\Delta'}(\lambda,\phi)|_{\partial\phi,h_*}} \le \frac{3}{2}$$
.

Finally we bound Q' as before by writing

$$|Q'_{\Delta'}(\phi)|_{\partial\phi,h_*} \leq 412 \times \max\left[|W'_{5,\Delta'}|,|W'_{6,\Delta'}|\right] \times \max_{0 \leq a \leq 6} |\phi^a|_{\partial\phi,h_*} ,$$

but for $0 \le a \le 6$,

$$|\phi^a|_{\partial\phi,h_*} = h_*^a \le h_*^6 = \left(2^{\frac{3}{4}}L^{\frac{3+\epsilon}{4}}\right)^6 \le 2^{\frac{9}{2}}L^6$$
.

This, together with the W' bounds from the proof of the previous lemma, provides us with the estimate

$$|Q'_{\Delta'}(\phi)|_{\partial\phi,h_*} \le 412 \times 2^{\frac{9}{2}} L^{\frac{17}{2}} \bar{g}^{2-2\eta} \times 11 \times (5+\sqrt{2})$$
.

Altogether we collect the bound

$$|\xi_{R,\Delta'}^{\text{shift}}(\vec{V})(\phi)|_{\partial\phi,h_*} \le 2 \times \frac{3}{2} \times \log\left(\frac{3}{2}\right) \mathcal{O}_{28} \times L^9 \bar{g}^{\frac{11}{12} - \frac{1}{3}\eta_R} \times 412 \times 2^{\frac{9}{2}} L^{\frac{17}{2}} \bar{g}^{2-2\eta} \times 11 \times (5 + \sqrt{2})$$

which after cleaning up becomes the desired bound.

We now combine the last two lemmas into a single more convenient result.

Lemma 52. For all unit cube Δ' we have

$$|||\xi_{R,\Delta'}^{\text{shift}}|||_{\bar{g}} \le \mathcal{O}_{41} L^{\frac{35}{2}} \bar{g}^{\frac{11}{4} - 3\eta}$$

where $\mathcal{O}_{41} = \max(\mathcal{O}_{39}, \mathcal{O}_{40})$.

Proof: From Lemmas 50 and 51 we immediately obtain

$$|||\xi_{R,\Delta'}^{\rm shift}|||_{\bar{g}} \leq \max \left[\mathcal{O}_{40} L^{\frac{35}{2}} \bar{g}^{\frac{35}{12} - 2\eta - \frac{1}{3}\eta_R} \right. , \,\, \mathcal{O}_{39} L^{\frac{15}{2}} \bar{g}^{\frac{11}{4} - 3\eta} \right] \,\, .$$

We have

$$\left(\frac{35}{12} - 2\eta - \frac{1}{3}\eta_R\right) - \left(\frac{11}{4} - 3\eta\right) = \frac{1}{6} + \eta - \frac{1}{3}\eta_R$$
$$\ge \frac{1}{6} + \eta - \frac{1}{3}\left(\frac{3}{16} + \frac{9}{4}\eta\right) = \frac{5}{48} + \frac{1}{4}\eta > 0$$

by the standard hypotheses (24) and (15). Therefore

$$\bar{q}^{\frac{35}{12} - 2\eta - \frac{1}{3}\eta_R} < \bar{q}^{\frac{11}{4} - 3\eta}$$

and the result follows.

Lemma 53. For all $\Delta' \in \mathbb{L}$ and $\Delta \in [L^{-1}\Delta']$ we have

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,L^{[\phi]}h_*} \le \mathcal{O}_{42}|||R_{\Delta_1}|||_{\bar{g}}$$

where $\mathcal{O}_{42} = \mathcal{O}_{23} + 250\mathcal{O}_{24}$.

Proof: By definition

$$J_{\Delta',\Delta_1}(\phi) = J_{+}(\phi) - J_{-}(\phi)$$

where

$$J_{+}(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int d\mu_{\Gamma}(\zeta) e^{\int_{L^{-1}\Delta'} f\zeta} R_{\Delta_{1}}(\phi_{1} + \zeta)$$

and

$$J_{-}(\phi) = \left(\sum_{k=0}^{4} \delta \beta_{k,3,\Delta',\Delta_{1}} : \phi^{k} :_{C_{0}}\right) \times e^{-\tilde{V}_{\Delta_{1}}(\phi_{1})} .$$

By Lemma 35 we have

$$|J_{+}(\phi)|_{\partial \phi, L^{[\phi]}h_{*}} \leq \mathcal{O}_{23}|||R_{\Delta_{1}}|||_{\bar{g}}$$
.

By Lemma 36 we also have

$$|\delta \beta_{k,3,\Delta',\Delta_1}| \leq \mathcal{O}_{24} L^{-k[\phi]} h_*^{-k} |||R_{\Delta_1}|||_{\bar{q}}.$$

We again use

$$|:\phi^k:_{C_0}|_{\partial\phi,L^{[\phi]}h_*} \le 25 \max_{\substack{0 \le a \le k \\ ac}} |\phi^a|_{\partial\phi,L^{[\phi]}h_*} \le 25 L^{k[\phi]} h_*^k$$

since $|\phi^a|_{\partial \phi, L^{[\phi]}h_*} = (L^{[\phi]}h_*)^a$ and $L^{[\phi]}h_* \geq 1$. Finally, by the chain rule

$$|e^{-\tilde{V}_{\Delta_1}(\phi_1)}|_{\partial \phi, L^{[\phi]}h_*} = |e^{-\tilde{V}_{\Delta_1}(\psi)}|_{\partial \psi, h_*} \le 2$$

by Lemma 12. As result we easily arrive at

$$|J_{-}(\phi)|_{\partial \phi} |_{L^{[\phi]}h_{\infty}} \leq 250\mathcal{O}_{24}|||R_{\Delta_{1}}|||_{\bar{q}}.$$

The latter as well as the previous inequality for J_{+} imply the desired estimate.

Lemma 54. For all $\Delta' \in \mathbb{L}$, $\Delta \in [L^{-1}\Delta']$ and $\phi \in \mathbb{R}$ we have

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \le \mathcal{O}_{43}\bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}$$

where

$$\mathcal{O}_{43} = \exp\left(\frac{\sqrt{2}}{2}\right) + 155\mathcal{O}_{24} \ .$$

Proof: Clearly, we have

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \le ||J_+(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} + ||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h}$$

and both terms will be bounded as follows. We first write

$$||J_{+}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leq e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_{\Gamma}(\zeta) \ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} ||R_{\Delta_{1}}(\phi_{1}+\zeta)||_{\partial\phi,\phi,L^{[\phi]}h}$$

and then use the chain rule as well as the definition of the $|||\cdot|||_{\bar{g}}$ norm in order to derive

$$||R_{\Delta_1}(\phi_1+\zeta)||_{\partial\phi,\phi,L^{[\phi]}h} = ||R_{\Delta_1}(\psi+\zeta)||_{\partial\psi,\phi_1,h} = ||R_{\Delta_1}(\psi)||_{\partial\psi,\phi_1+\zeta,h} \leq \bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}.$$

Besides, as shown before $|(f, \Gamma f)_{L^{-1}\Delta'}| \leq \frac{1}{\sqrt{2}}$. Hence

$$||J_{+}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \le \exp[2^{-\frac{3}{2}}]\bar{g}^{-2}|||R_{\Delta_{1}}|||_{\bar{g}} \int d\mu_{\Gamma}(\zeta) e^{\int_{L^{-1}\Delta'}(\Re f)\zeta}$$

$$\leq \exp[2^{-\frac{3}{2}}]\bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}e^{\frac{1}{2}(\Re f,\Gamma \Re f)_{L^{-1}\Delta'}}$$

by Lemma 7 with $X=\emptyset$ or simply exact computation. Again one easily gets that $|(\Re f, \Gamma \Re f)_{L^{-1}\Delta'}| \leq \frac{1}{\sqrt{2}}$ which results in

$$||J_{+}(\phi)||_{\partial \phi, \phi, L^{[\phi]}h} \leq \exp[2^{-\frac{1}{2}}]\bar{g}^{-2}|||R_{\Delta_{1}}|||_{\bar{g}} \ .$$

From the definition of $J_{-}(\phi)$ we immediately get

$$||J_{-}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leq \sum_{k=0}^{4} |\delta\beta_{k,3,\Delta',\Delta_{1}}| \times ||:\phi^{k}:_{C_{0}}||_{\partial\phi,\phi,L^{[\phi]}h}||e^{-\tilde{V}_{\Delta_{1}}(\phi_{1})}||_{\partial\phi,\phi,L^{[\phi]}h}.$$

By the chain rule and Lemma 12

$$||e^{-\tilde{V}_{\Delta_1}(\phi_1)}||_{\partial \phi, \phi, L^{[\phi]}h} = ||e^{-\tilde{V}_{\Delta_1}(\psi)}||_{\partial \psi, \phi_1, h} \le 2e^{-\frac{1}{2}(\Re\beta_{4,\Delta_1})\phi_1^4}.$$

Again by undoing the Wick ordering we have

$$||:\phi^k:_{C_0}||_{\partial\phi,\phi,L^{[\phi]}h} \le 25 \max_{0\le a\le k} ||\phi^a||_{\partial\phi,\phi,L^{[\phi]}h}.$$

But

$$||\phi^a||_{\partial \phi, \phi, L^{[\phi]}h} = (L^{[\phi]}h + |\phi|)^a \le (L^{[\phi]}h + |\phi|)^k$$

since $L^{[\phi]}h \geq 1$ as follows from $h \geq h_*$, i.e., from (21). Now

$$||: \phi^k:_{C_0} ||_{\partial \phi, \phi, L^{[\phi]}h} \le 25(L^{[\phi]}h + |\phi|)^k = 25\sum_{n=0}^k \binom{k}{n} \left(L^{[\phi]}c_1\bar{g}^{-\frac{1}{4}}\right)^{k-n} |\phi|^n$$

$$\leq 25 \sum_{n=0}^{k} \binom{k}{n} \left(L^{[\phi]} c_1 \bar{g}^{-\frac{1}{4}} \right)^{k-n} \left(\frac{n}{2e} \right)^{\frac{n}{4}} (\gamma \bar{g})^{-\frac{n}{4}} e^{\gamma (\Re \beta_4, \Delta_1) \phi^4}$$

by Lemma 8 and for any $\gamma > 0$. Here we choose $\gamma = \frac{1}{2}L^{-4[\phi]}$ which entails

$$||: \phi^{k}:_{C_{0}}||_{\partial \phi, \phi, L^{[\phi]}h} \leq 25 \times \left(\max_{0 \leq n \leq 4} \left(\frac{n}{2e}\right)^{\frac{n}{4}}\right) \times e^{\frac{1}{2}(\Re \beta_{4, \Delta_{1}})\phi_{1}^{4}} \times \sum_{n=0}^{k} {k \choose n} \left(L^{[\phi]}c_{1}\bar{g}^{-\frac{1}{4}}\right)^{k-n} \left(\frac{1}{2}L^{-4[\phi]}\bar{g}\right)^{-\frac{n}{4}}.$$

As a result of the previous considerations we arrive at

$$||J_{-}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leq 50 \times \left(\max_{0\leq n\leq 4} \left(\frac{n}{2e}\right)^{\frac{n}{4}}\right)$$
$$\times \sum_{k=0}^{4} |\delta\beta_{k,3,\Delta',\Delta_{1}}| \times L^{k[\phi]}\bar{g}^{-\frac{k}{4}} \left(\sum_{n=0}^{k} \binom{k}{n} c_{1}^{k-n} 2^{\frac{n}{4}}\right).$$

Since $n \leq 4$ we simply bound $\frac{n}{2e}$ by 1. We also use Lemma 36 in order to write

$$||J_{-}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leq 50 \times \mathcal{O}_{24}|||R_{\Delta_{1}}|||_{\bar{g}} \times \sum_{n=0}^{k} \bar{g}^{-\frac{k}{4}} h_{*}^{-\frac{k}{4}} (c_{1} + 2^{\frac{1}{4}})^{k}.$$

Now we bound h_*^{-1} by 1, $\bar{g}^{-\frac{k}{4}}$ by the worst case scenario $\bar{g}^{-1} \leq \bar{g}^{-2}$ and finally $c_1 + 2^{\frac{1}{4}}$ by 2. Since $1 + 2 + \cdots + 2^4 = 31$ we then obtain

$$||J_{-}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leq 50 \times 31 \times \mathcal{O}_{24}\bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}$$
.

The latter inequality, combined with the previous one for J_+ , gives us the desired result.

Lemma 55. For all unit cube $\Delta' \in \mathbb{L}$ we have

$$|||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)|||_{\bar{g}} \leq \mathcal{O}_{44} \times L^{3-5[\phi]} \times \max_{\Delta_1 \in [L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}}$$

where

$${\cal O}_{44} = 2^{10} \times {\cal O}_{14} \times ({\cal O}_{42} + {\cal O}_{43})$$
 .

Proof: Recall that

$$\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R) = \sum_{\substack{\Delta_1 \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} \left(\prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right) \times J_{\Delta',\Delta_1}(\phi) .$$

Hence

$$|\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)|_{\partial\phi,h_*} \leq \sum_{\substack{\Delta_1 \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} \left| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right|_{\partial\phi,h_*} \times |J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h_*} .$$

Now by Lemma 13 with $Y_0 = [L^{-1}\Delta'] \setminus \{\Delta_1\}$ we have

$$\left| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right|_{\partial \phi, h_{\tilde{\alpha}}} \leq 2.$$

By definition of the seminorms

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h_*} = \sum_{n=0}^9 \frac{h_*^n}{n!} |J_{\Delta',\Delta_1}^{(n)}(0)|.$$

However, by construction in §4.2, the derivatives $J_{\Delta',\Delta_1}^{(n)}(0)$ vanish when $0 \le n \le 4$. As a result

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h_*} = \sum_{n=5}^9 \frac{h_*^n}{n!} |J_{\Delta',\Delta_1}^{(n)}(0)| = \sum_{n=5}^9 L^{-n[\phi]} \frac{(h_*L^{[\phi]})^n}{n!} |J_{\Delta',\Delta_1}^{(n)}(0)|$$

$$\leq L^{-5[\phi]} \sum_{n=5}^9 \frac{(h_*L^{[\phi]})^n}{n!} |J^{(n)}_{\Delta',\Delta_1}(0)| = L^{-5[\phi]} |J_{\Delta',\Delta_1}(\phi)|_{\partial \phi,L^{[\phi]}h_*}$$

and thus by Lemma 53 we have

$$|\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)|_{\partial\phi,h_{*}} \leq 2L^{-5[\phi]} \sum_{\substack{\Delta_{1} \in [L^{-1}\Delta'] \\ \Delta_{1} \in [L^{-1}\Delta']}} \mathcal{O}_{42}|||R_{\Delta_{1}}|||_{\bar{g}}$$

$$\leq 2\mathcal{O}_{42}L^{3-5[\phi]} \max_{\substack{\Delta_{1} \in [L^{-1}\Delta'] \\ \Delta_{1} \in [L^{-1}\Delta']}} |||R_{\Delta_{1}}|||_{\bar{g}}$$

Likewise, we have

$$||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)||_{\partial\phi,\phi,h} \leq \sum_{\substack{\Delta_1 \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} \left\| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial\phi,\phi,h} \times ||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} .$$

If we let $Y_0 = [L^{-1}\Delta'] \setminus \{\Delta_1\}$, then $|Y_0| \ge \frac{L^3}{2}$ and by Lemma 13 we have

$$\left\| \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_{\Delta}(\phi_1)} \right\|_{\partial \phi, \phi, h} \leq 2e^{-\frac{\tilde{g}}{16}\phi^4}.$$

By Lemma 28 with $\beta_4 = \bar{g}$ and $\gamma = \frac{1}{16}$ one has

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} \leq \mathcal{O}_{14} 16^{\frac{9}{4}} e^{\frac{\bar{\theta}}{16}\phi^4} \left[|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h} + L^{-9[\phi]} \sup_{\psi \in \mathbb{R}} |J_{\Delta',\Delta_1}(\psi)|_{\partial\psi,\psi,L^{[\phi]}h} \right].$$

By the same argument utilizing the vanishing of the first few derivatives at the origin as before, with h instead of h_* , we get

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h} \leq L^{-5[\phi]}|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,L^{[\phi]}h}$$
.

Now by Lemma 53

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial \phi,L^{[\phi]}h} \leq \mathcal{O}_{42}|||R_{\Delta_1}|||_{\bar{q}}$$

whereas, by Lemma 54, one has

$$\sup_{\psi \in \mathbb{R}} |J_{\Delta',\Delta_1}(\psi)|_{\partial \psi,\psi,L^{[\phi]}h} \le \mathcal{O}_{43}\bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}.$$

We then arrive at the estimate

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} \leq \mathcal{O}_{14} \times 2^9 \times e^{\frac{\bar{g}}{16}\phi^4} |||R_{\Delta_1}|||_{\bar{g}} \times \left[L^{-5[\phi]}\mathcal{O}_{42} + L^{-9[\phi]}\bar{g}^{-2}\mathcal{O}_{42}\right] \ .$$

Using $L^{-5[\phi]}\bar{g}^{-2}$ as a common bound of $L^{-9[\phi]}\bar{g}^{-2}$ and $L^{-5[\phi]}$ we immediately get

$$||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)(\phi)||_{\partial\phi,\phi,h} \leq \sum_{\Delta_1 \in [L^{-1}\Delta']} 2^{10} \mathcal{O}_{14}(\mathcal{O}_{42} + \mathcal{O}_{43}) L^{-5[\phi]} \bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}}$$

and hence

$$\bar{g}^2 \times ||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)(\phi)||_{\partial \phi,\phi,h} \leq 2^{10} \mathcal{O}_{14}(\mathcal{O}_{42} + \mathcal{O}_{43}) L^{3-5[\phi]} \max_{\Delta_1 \in [L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}} \ .$$

The latter inequality, combined with the previous one for the $|\cdot|_{\partial \phi, h_*}$ seminorm, give

$$|||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)|||_{\bar{g}} \leq L^{3-5[\phi]} \left(\max_{\Delta_1 \in [L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}} \right) \times \max \left[2\mathcal{O}_{42}, 2^{10}\mathcal{O}_{14}(\mathcal{O}_{42} + \mathcal{O}_{43}) \right] .$$

Since clearly $\mathcal{O}_{14} > 1$, the last maximum reduces to the second term, i.e., the given value of \mathcal{O}_{44} .

Lemma 56. For all unit cube Δ' we have

$$|||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}} \leq \mathcal{O}_{45}L^{60}\bar{g}^{\frac{11}{4}-3\eta}$$

where $\mathcal{O}_{45} = \mathcal{O}_{37} + \mathcal{O}_{38} + \mathcal{O}_{41}$.

Proof: Since (30) implies

$$|||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}} \leq |||\xi_{R,\Delta'}^{\mathrm{main}}(\vec{V})|||_{\bar{g}} + |||\xi_{R,\Delta'}^{\mathrm{higher}}(\vec{V})|||_{\bar{g}} + |||\xi_{R,\Delta'}^{\mathrm{shift}}(\vec{V})|||_{\bar{g}} ,$$

Lemmas 48, 49 and 52 immediately imply the desired result.

6.4. Conclusion of the proof of the main estimates for $RG_{\rm ex}$. We choose $B_{R\mathcal{L}} = \mathcal{O}_{44}$ defined in Lemma 55. Then if one fixes $l \geq 1$ or equivalently L, we take $B_k = \mathcal{O}_{24}L^{3-k[\phi]}$, for $0 \leq k \leq 4$. We also set $B_{R\xi} = \mathcal{O}_{45}L^{60}$. The hypotheses on η and η_R in the statement of Theorem 4 easily imply the properties of η and η_R mentioned in the standard hypotheses (15), (22) and (24). Now once $A_{\bar{g}}$ has been chosen, the calibrator \bar{g} can be made as small as desired by taking ϵ small enough. It is a simple matter of going through the inequalities in §6.2 in order to check that all the standard hypotheses are satisfied for small ϵ and therefore all the lemma in the previous section hold. In particular, for $0 \leq k \leq 4$,

$$|\xi_{k,\Delta'}(\vec{V})| \le B_k \max_{\Delta \in [L^{-1}\Delta']} |||R_\Delta|||_{\bar{g}}$$

follows from the definition $\xi_{k,\Delta'}(\vec{V}) = -\delta \beta_{k,3,\Delta'}$ and Lemma 36. Likewise,

$$|||\mathcal{L}_{\Delta'}^{\vec{\beta},f}(R)|||_{\bar{g}} \leq B_{R\mathcal{L}}L^{3-5[\phi]} \max_{\Delta \in [L^{-1}\Delta']} |||R_{\Delta}|||_{\bar{g}}$$

is the result of Lemma 55. Finally,

$$|||\xi_{R,\Delta'}(\vec{V})|||_{\bar{q}} \le B_{R\varepsilon}\bar{q}^{\frac{11}{4}-3\eta}$$

has been established in Lemma 56.

The statement in Theorem 4 about sending real data to real data is obvious from the definition the RG map in §4.2. So is the one about analyticity now that the previous bounds on the outcome have been proved. The proof of Theorem 4 is now complete.

7. The bulk RG

In §4.1 we defined the complex Banach spaces \mathcal{E} , \mathcal{E}_{1B} , \mathcal{E}_{bk} and \mathcal{E}_{ex} . The transformation RG_{ex} defined in §4.2 is an analytic map from a domain in \mathcal{E}_{ex} (given in the hypotheses of Theorem 4) into \mathcal{E}_{ex} . In this section we will show that the subspace \mathcal{E}_{bk} is stable by this transformation and similarly for $\mathcal{E} = \mathbb{C}^2 \times C^9_{bd,ev}(\mathbb{R},\mathbb{C})$ seen as a subspace of \mathcal{E}_{bk} and therefore of \mathcal{E}_{ex} too. We will also derive simpler formulas for the transformation restricted to \mathcal{E} .

Proposition 1. The space \mathcal{E}_{bk} is invariant by RG_{ex} .

Proof: This is a trivial consequence of the translation covariance of the definition of $RG_{\rm ex}$ in §4.2. Let $(g, \mu, R) \in \mathcal{E}$. This corresponds to an element

$$\vec{V} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})_{\Delta \in \mathbb{L}}$$

in \mathcal{E}_{ex} via the specifications

$$\beta_{4,\Delta} = g$$

$$\beta_{3,\Delta} = 0$$

$$\beta_{2,\Delta} = \mu$$

$$\beta_{1,\Delta} = 0$$

$$W_{5,\Delta} = 0$$

$$W_{6,\Delta} = 0$$

$$f_{\Delta} = 0$$

$$R_{\Delta} = R$$

for all unit cubes Δ . We introduce the notations

$$\begin{array}{rcl} \xi_{4}(g,\mu,R) & = & \xi_{4,\Delta(0)}(\vec{V}) \\ \xi_{2}(g,\mu,R) & = & \xi_{2,\Delta(0)}(\vec{V}) \\ \xi_{0}(g,\mu,R) & = & \xi_{0,\Delta(0)}(\vec{V}) \\ \xi_{R}(g,\mu,R) & = & \xi_{R,\Delta(0)}(\vec{V}) \\ \mathcal{L}^{(g,\mu)}(R) & = & \mathcal{L}^{(\vec{\beta},f)}_{\Delta(0)}(R) \\ \delta b(g,\mu,R) & = & \delta b_{\Delta(0)}(\vec{V}) \end{array}$$

in terms of the previous vector \vec{V} . Note that we could have used any box Δ' instead of $\Delta(0)$, the one containing the origin.

Proposition 2. The space \mathcal{E} is invariant by the map RG_{ex} . The restricted transformation

$$RG: \quad \mathcal{E} \longrightarrow \quad \mathcal{E}$$

 $(g,\mu,R) \longmapsto (g',\mu',R')$

which we call the bulk RG is, more explicitly, given by

$$\begin{cases} g' = L^{\epsilon}g - A_1g^2 + \xi_4(g, \mu, R) \\ \mu' = L^{\frac{3+\epsilon}{2}}\mu - A_2g^2 - A_3g\mu + \xi_2(g, \mu, R) \\ R' = \mathcal{L}^{(g,\mu)}(R) + \xi_R(g, \mu, R) \end{cases}$$

where

$$\begin{array}{lcl} A_1 & = & 36L^{3-4[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ \mathrm{d}^3x \\ \\ A_2 & = & 48L^{3-2[\phi]} \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^3 \ \mathrm{d}^3x \right) + 144L^{3-4[\phi]} C_0(0) \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ \mathrm{d}^3x \right) \\ \\ A_3 & = & 12L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ \mathrm{d}^3x \ . \end{array}$$

In addition, the vacuum counter-term $\delta b = \delta b(g, \mu, R)$ is given by

$$\delta b = A_4 g^2 + A_5 \mu^2 + \xi_0(g, \mu, R)$$

where

$$A_4 = 12L^3 \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^4 d^3x \right) + 48L^{3-2[\phi]} C_0(0) \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^3 d^3x \right) + 72L^{3-4[\phi]} C_0(0)^2 \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^2 d^3x \right)$$

$$A_5 = L^3 \int_{\mathbb{Q}_p^3} \Gamma(x)^2 d^3x .$$

Proof: We compute the specialization of the map $\vec{V} \mapsto \vec{V}'$ defined in §4.2 to the present situation. Clearly since $f_{\Delta} = 0$, the new $f'_{\Delta'}$'s defined in (4) are identically zero. Likewise, since the W_6 are zero the equation for the new one reduces to

$$W'_{6,\Delta'} = 8L^{-6[\phi]} \int_{(L^{-1}\Delta')^2} d^3x \ d^3y \ \beta_4(x) \ \Gamma(x-y) \ \beta_4(y) = 8L^{-6[\phi]} g^2 \int_{(L^{-1}\Delta')^2} d^3x \ d^3y \ \Gamma(x-y) \ .$$

But for $x \in L^{-1}\Delta'$, by a simple change of variables z = x - y,

$$\int_{L^{-1}\Delta'} d^3 y \ \Gamma(x-y) = \int_{L^{-1}\Delta(0)} d^3 z \ \Gamma(z) = \int_{\mathbb{Q}_p^3} d^3 z \ \Gamma(z) = \widehat{\Gamma}(0) = 0$$

because of the finite range property and the vanishing property at zero momentum. Therefore $W'_{6,\Delta'}$ vanishes identically and so does $W'_{5,\Delta'}$ for similar reasons. Now one easily sees from the definition and specification

of the input \vec{V} that

$$\hat{\beta}_{4,\Delta'} = L^{\epsilon}g$$

$$\hat{\beta}_{3,\Delta'} = 0$$

$$\hat{\beta}_{2,\Delta'} = L^{\frac{3+\epsilon}{2}}\mu$$

$$\hat{\beta}_{1,\Delta'} = 0.$$

Then we consider the first corrections $\delta \beta_{k,1,\Delta'}$. These are all zero since in their defining equation (5) the constraint $b \geq 1$ implies that at least one f is present. However f = 0 identically and therefore the Feynman diagram in (5) vanishes. In sum, $\delta \beta_{k,1,\Delta'} = 0$ for all k such that $0 \leq k \leq 4$ and all unit cube Δ' .

We now move on to the computation of the second order corrections $\delta \beta_{k,2,\Delta'}$. Again since f and the W's are zero, the defining equation (6) for these quantities reduces to

$$\delta \beta_{k,2,\Delta'} = \sum_{a_1,a_2,m} \mathbb{1} \left\{ \begin{array}{c} a_i + m \le 4 \\ a_i \ge 0 , m \ge 1 \end{array} \right\} \frac{(a_1 + m)! (a_2 + m)!}{a_1! a_2! m!}$$

$$\times \frac{1}{2} C(a_1, a_2 | k) \times L^{-(a_1 + a_2)[\phi]} \times C_0(0)^{\frac{a_1 + a_2 - k}{2}} \times \beta_{a_1 + m} \beta_{a_2 + m} \times \int_{(L^{-1} \Delta')^2} d^3 x_1 d^3 x_2 \Gamma(x_1 - x_2)^m.$$

Indeed, nonvanishing imposes the absence of f external vertices and thus $b_1 = b_2 = m$. Note that since the $\beta_{\nu}(x)$ are constants with respect to the location x we pulled them out of the integral and suppressed the x dependence in the notation. Now one can rule out the value m = 1 which gives a vanishing contribution for the same reason as explained above when computing W'_6 . Another simplification is that $\delta\beta_{k,2,\Delta'}$ vanishes if k is odd. Indeed, if k is odd and if the connection coefficient $C(a_1, a_2|k)$ is nonzero, then $a_1 + a_2$ must be odd too and thus also $a_1 + m + a_2 + m$. Since this forces $a_i + m$ to be odd for one i = 1, 2 then the contribution in the above sum vanishes. This is because β_n is nonzero only for even values of n, namely 2 and 4. We now only have three cases to consider: k = 4, 2 and 0.

1st Case: Let k=4. Then the connection coefficients force $a_1+a_2 \geq 4$. Also $m \geq 2$ and $a_i+m \leq 4$ imply $0 \leq a_i \leq 2$ so the only possibility is $(a_1, a_2) = (2, 2)$ and m=2. We also have $\beta_{a_1+m} = \beta_{a_1+m} = \beta_4 = g$. It is easy to see that the formula reduces to $\delta\beta_{4,2,\Delta'} = A_1g^2$.

2nd Case: Let k = 2. Now the constraints $a_1 + a_2 \ge 2$, $m \ge 2$, $a_i + m \le 4$, $a_i \ge 0$ and $a_i + m \in \{2, 4\}$, without which the contribution would vanish, imply that the only possibilities for the triple (a_1, a_2, m) are (2, 2, 2), (2, 0, 2), (0, 2, 2) and (1, 1, 3). The second and third give the same contribution by symmetry. A quick computation shows that (2, 2, 2) contributes

$$144L^{3-4[\phi]}C_0(0)g^2\int\Gamma^2$$

where we used the shorthand

$$\int \Gamma^m = \int_{(L^{-1}\Delta')^2} d^3x_1 d^3x_2 \Gamma(x_1 - x_2)^m.$$

Likewise (2,0,2) and (0,2,2) together contribute

$$12L^{3-2[\phi]}g\mu\int\Gamma^2.$$

Finally, (1,1,3) contributes

$$48L^{3-2[\phi]}g^2\int\Gamma^3.$$

Hence

$$\delta\beta_{2,2,\Delta'} = A_2 g^2 + A_3 g\mu .$$

3rd Case: Let k = 0. Note that the connection coefficients also impose $0 = k \ge |a_1 - a_2|$ and thus the restriction $a_1 = a_2$. Considerations similar to those of the two previous cases show that the only possibilities for the triple (a_1, a_2, m) are (0, 0, 2), (0, 0, 4), (1, 1, 3) and (2, 2, 2). Again a quick computation shows that (0, 0, 2) contributes

$$L^3\mu^2 \int_{72} \Gamma^2 .$$

The triple (0,0,4) contributes

$$12L^3g^2\int\Gamma^4\ .$$

The triple (1,1,3) contributes

$$48L^{3-2[\phi]}C_0(0)g^2\int\Gamma^3$$
.

Finally, the triple (2, 2, 2) contributes

$$72L^{3-4[\phi]}C_0(0)^2g^2\int\Gamma^2$$
.

Hence,

$$\delta \beta_{0,2,\Delta'} = A_4 g^2 + A_5 \mu^2$$
.

We now show that $\delta\beta_{3,3,\Delta'}$ and $\delta\beta_{1,3,\Delta'}$ are zero because the function R is even. First note that the formula (8) for the a_i reduces to

$$a_i = \exp\left[-C_0(0)L^{-2[\phi]}\mu + 3C_0(0)^2L^{-4[\phi]}g\right] \times L^{-i[\phi]} \times \int d\mu_{\Gamma}(\zeta) R^{(i)}(\zeta) .$$

It is easy to see that $a_i = 0$ if i is odd. Indeed, the Gaussian measure $\mathrm{d}\mu_\Gamma$ is centered and therefore one can change ζ into $-\zeta$ without changing the integral. However, for odd i we have $R^{(i)}(-\zeta) = -R^{(i)}(\zeta)$ for the i-th derivative of the even function R. Thus $a_i = -a_i$ and the stated vanishing property holds. Now the $M_{k,i}$ in (9) are zero unless k and i have the same parity. Indeed, the β_{l_ν} 's are nonzero only if $l_\nu = 4$ or 2. This together with the constaint (10) imply the desired property. Therefore the $\delta\beta_{k,3,\Delta',\Delta_1}$ and thus also the $\beta'_{k,\Delta'}$ vanish for k=1 and 3.

Finally in order to complete the proof, all that is needed is to show that R' is an even function. First notice that Wick powers only involve lower or equal ordinary powers of the same parity. Since the inputs β_k are zero for k = 1 and 3, the $\tilde{V}_{\Delta}(\phi_1)$ are even functions of ϕ . The defining formula (7) for $J_{\Delta',\Delta_1}(\phi)$ reduces, in the present situation, to

$$J_{\Delta',\Delta_1}(\phi) = \left\{ \int \mathrm{d}\mu_{\Gamma}(\zeta) \ R(\phi_1 + \zeta) \right\} - \left(\delta \beta_{4,3,\Delta',\Delta_1} : \phi^4 :_{C_0} + \delta \beta_{2,3,\Delta',\Delta_1} : \phi^2 :_{C_0} + \delta \beta_{0,3,\Delta',\Delta_1} \right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)}$$

which is easily seen to be even thanks to the change of variable $\zeta \to -\zeta$ and the hypothesis that the input R is even. Therefore the quantity denoted by C'_1 in §4.2, namely, $\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)(\phi)$ is even. Clearly $V_{\Delta}(\phi)$ is even which results in the invariance of $p_{\Delta}(\phi_1,\zeta)$ with respect to changing the sign of both ϕ and ζ . Hence also $P_{\Delta}(\lambda,\phi_1,\zeta)$ has the same invariance property. Note that since the W's are zero, Q vanishes and thus $K_{\Delta}(\lambda,\phi_1,\zeta) = \lambda^3 R(\phi_1+\zeta)$ has that invariance too. It follows using the change of variable $\zeta \to -\zeta$ that $\hat{K}_{\Delta'}(\lambda,\phi)$ is even. Since $\delta V_{\Delta'}(\lambda,\phi)$ contains no : $\phi^3:_{C_0}$ nor : $\phi^1:_{C_0}$ it is even and as a result $K'_{\Delta'}(\lambda,\phi)$ is also even. Since the W' have been shown to vanish, the function $Q'_{\Delta'}$ also vanishes. As a consequence one can easily see that $\xi_{R,\Delta'}(\vec{V})(-\phi) = \xi_{R,\Delta'}(\vec{V})(\phi)$. Finally, $R'_{\Delta'} = R'_{\Delta(0)} = R'$ must be an even function of the field ϕ .

8. The infrared fixed point and local analysis of the bulk RG

8.1. **Preparation.** In this section we make some choices related to the particular application of Theorem 4 which will be needed in the sequel. Note that by Lemma 6, the quantity A_1 defined in Proposition 2 is given by

$$A_1 = 36L^{\epsilon} \times \frac{(1-p^{-3})(L^{\epsilon}-1)}{p^{\epsilon}-1} > 0$$
.

If one ignores the ξ_4 term in the bulk RG evolution equation for the ϕ^4 coupling g then the fixed point equation becomes $g = L^{\epsilon}g - A_1g^2$. In addition to the trivial solution g = 0, this equation has another solution $g = (L^{\epsilon} - 1)/A_1$. This is the approximate value of the g coordinate of the nontrivial infrared RG fixed point. We will in the remainder of this article choose the calibrator defining the norms to be this approximate fixed point value, namely, we set

$$\bar{g} = \frac{L^{\epsilon} - 1}{A_1} = \frac{p^{\epsilon} - 1}{36L^{\epsilon}(1 - p^{-3})}$$
.

In other words, $\bar{g} = \bar{g}_*$ where \bar{g}_* has been defined in §3. Clearly,

$$\frac{\bar{g}}{\epsilon} \longrightarrow \frac{\log p}{36(1 - p^{-3})}$$

when $\epsilon \to 0$ with L fixed. This motivates making the choice

$$A_{\bar{g}} = \frac{\log p}{36(1 - p^{-3})} + 1$$

when applying Theorem 4. This ensures that when ϵ is made small with L fixed, our choice of \bar{g} will satisfy the requirement $\bar{g} \in (0, A_{\bar{q}}]$ in Theorem 4. We now also choose L, once and for all, so that

(37)
$$B_{R\mathcal{L}}L^{3-5[\phi]} \le \frac{1}{2}$$

holds. Note that $3-5[\phi]=-\frac{3}{4}+\frac{5}{4}\epsilon$. If we add the harmless condition $\epsilon \leq \frac{1}{5}$ which we now assume, then $3-5[\phi] \leq -\frac{1}{2}$. Now we pick L large enough so that $B_{R\mathcal{L}}L^{-\frac{1}{2}} \leq \frac{1}{2}$ and therefore (37) holds. Note that, contrary to the other B quantities in Theorem 4, $B_{R\mathcal{L}}$ is independent of L and indeed is a purely numerical constant. This fact is of course crucial to the previous considerations. The choices for the parameters η , η_R as well as the exponents e in the definition of the Banach space norm of $\mathcal{E}_{\rm ex}$ in §4.1 will be specified later. Once these choice are made, the only free parameter in the problem is the bifurcation parameter ϵ . All the following results will be established in the regime when this ϵ is made sufficiently small.

We now apply Theorem 4 with the choices just mentioned and in concert with Proposition 2 to obtain that, provided ϵ is small enough, the bulk RG transformation is well-defined and analytic on the domain

$$|g - \bar{g}| < \frac{1}{2}, \ |\mu| < \bar{g}^{1-\eta}, \ |||R|||_{\bar{g}} < \bar{g}^{\frac{11}{4} - \eta_R}$$

and therein satisfies

$$\begin{aligned} |\xi_{4}(g,\mu,R)| &\leq & B_{4}|||R|||_{\bar{g}} \\ |\xi_{2}(g,\mu,R)| &\leq & B_{2}|||R|||_{\bar{g}} \\ |||\xi_{R}(g,\mu,R)|||_{\bar{g}} &\leq & B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta} \\ |||\mathcal{L}^{(g,\mu)}|||_{\bar{g}} &\leq & \frac{1}{2} \end{aligned}$$

where $|||\mathcal{L}^{(g,\mu)}|||_{\bar{g}}$ is the operator norm of the linear operator $\mathcal{L}^{(g,\mu)}$ (with respect to the R variable) corresponding to the norm $|||\cdot|||_{\bar{g}}$. Note that the statement on analyticity applies not only to the full map RG but also to the constituent pieces such as ξ_4 , ξ_2 , ξ_R and $\mathcal{L}^{(g,\mu)}(R)$.

In order to analyze the bulk RG transformation we slightly change our coordinate system from (g, μ, R) to $(\delta g, \mu, R)$ where $\delta g = g - \bar{g}$. In this new coordinate system, the bulk RG transformation, still denoted by RG for simplicity, becomes $(\delta g, \mu, R) \longmapsto RG(\delta g, \mu, R) = (\delta g', \mu', R')$ with

$$\begin{cases} \delta g &= (2 - L^{\epsilon}) \delta g + \tilde{\xi}_4(\delta g, \mu, R) \\ \mu' &= L^{\frac{3+\epsilon}{2}} \mu + \tilde{\xi}_2(\delta g, \mu, R) \\ R' &= \tilde{\mathcal{L}}^{(\delta g, \mu)(R)} + \tilde{\xi}_R(\delta g, \mu, R) \end{cases}$$

where

$$\begin{array}{lcl} \tilde{\xi}_{4}(\delta g,\mu,R) & = & -A_{1}\delta g^{2} + \xi_{4}(\bar{g} + \delta g,\mu,R) \\ \tilde{\xi}_{2}(\delta g,\mu,R) & = & -A_{2}(\bar{g} + \delta g) - A_{3}(\bar{g} + \delta g)\mu + \xi_{2}(\bar{g} + \delta g,\mu,R) \\ \tilde{\xi}_{R}(\delta g,\mu,R) & = & \xi_{R}(\bar{g} + \delta g,\mu,R) \\ \tilde{\mathcal{L}}^{(\delta g,\mu)}(R) & = & \mathcal{L}^{(\bar{g} + \delta g,\mu)(R)} \end{array}$$

as follows from an easy computation using the relation $A_1\bar{g} = L^{\epsilon} - 1$. We will commit a similar abuse of notation for the function δb . Namely, we will write $\delta b(\delta g, \mu, R)$ for what in fact is $\delta b(\bar{g} + \delta g, \mu, R)$. Note that the norm we will use on such elements $v = (\delta g, \mu, R) \in \mathcal{E}$ is the one induced by the norm of the larger space \mathcal{E}_{ex} defined in §4.1, namely,

$$||v|| = \max \left\{ |\delta g|\bar{g}^{-e_4}, |\mu|\bar{g}^{-e_2}, |||R|||_{\bar{g}}\bar{g}^{-e_R} \right\} \ .$$

We will assume the following constraints on the exponents defining the norms as well as the parameters η and η_R :

$$(38) e_4 \geq 1$$

$$(39) e_2 \geq 1 - \eta$$

$$(40) e_R \geq \frac{11}{4} - \eta_R$$

$$(41) e_R > e_4 + 1$$

$$\frac{11}{4} - 3\eta > e_R$$

$$(43)$$
 $e_2 < 2$.

The following lemma provides Lipschitz estimates that will be needed in the sequel.

Lemma 57. For ϵ small enough we have for all $v=(\delta g,\mu,R),\ v'=(\delta g',\mu',R')$ in \mathcal{E} such that ||v||, $||v'|| \leq \frac{1}{8}$,

$$|\xi_4(\bar{g} + \delta g, \mu, R) - \xi_4(\bar{g} + \delta g', \mu', R')| \le 2B_4\bar{g}^{e_R}||v - v'||$$

$$|\xi_2(\bar{g} + \delta g, \mu, R) - \xi_4(\bar{g} + \delta g', \mu', R')| \le 2B_2\bar{g}^{e_R}||v - v'||$$

$$|||\mathcal{L}^{(\bar{g}+\delta g,\mu)}(R) - \mathcal{L}^{(\bar{g}+\delta g',\mu')}(R')|||_{\bar{g}} \leq \frac{3}{4}\bar{g}^{e_R}||v - v'||$$

and

$$|||\xi_R(\bar{g}+\delta g,\mu,R)-\xi_R(\bar{g}+\delta g',\mu',R')|||_{\bar{g}} \leq 3B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta}||v-v'||$$
.

Proof: If $||v|| < \frac{1}{2}$, then since $\bar{g} \le 1$ for ϵ small and because of (38), (39) and (40), we have

Hence, by Theorem 4

$$|\xi_4(\bar{g}+\delta g,\mu,R)| \le B_4|||R|||_{\bar{g}} \le \frac{1}{2}B_4\bar{g}^{e_R}$$
.

Therefore the analytic map $v \mapsto \xi_4(\bar{g} + \delta g, \mu, R)$ satisfies the hypotheses of Lemma 1 with $r_1 = \frac{1}{2}$ and $r_2 = \frac{1}{2}B_4\bar{g}^{e_R}$. We pick $\nu = \frac{1}{4}$ which results in

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{3}{2}B_4\bar{g}^{e_R} \ .$$

With these choices, Lemma 1 implies the desired Lipschitz estimate where we replaced the numerical factor $\frac{3}{2}$ by 2 for a simpler looking formula. The proof of the Lipschitz estimate for ξ_2 is exactly the same apart from changing ξ_4 , B_4 to ξ_2 , B_2 respectively.

We now do the same for the analytic map $v \mapsto \mathcal{L}^{(\bar{g}+\delta g,\mu)}(R)$. For $||v|| < \frac{1}{2} = r_1$ we obtain, as before from Theorem 4 and from the choice we made when fixing L,

$$|||\mathcal{L}^{(\bar{g}+\delta g,\mu)}(R)|||_{\bar{g}} \le \frac{1}{2}|||R|||_{\bar{g}} \le \frac{1}{2}||v||_{\bar{g}^{e_R}} \le r_2$$

with $r_2 = \frac{1}{4}\bar{g}^{e_R}$. Lemma 1 with $\nu = \frac{1}{4}$ now immediately implies the wanted estimate. Remark that we do not bound the numerical factor $\frac{3}{4}$ by the nearest integer here since it is important that this factor be less than 1.

Finally, for ξ_R we again note that $||v|| < \frac{1}{2} = r_1$ implies

$$|||\xi_R(\bar{q} + \delta q, \mu, R)|||_{\bar{q}} \le r_2$$

with $r_2 = B_{R\xi}\bar{g}^{\frac{11}{4}-\eta}$. Again, Lemma 1 with $\nu = \frac{1}{4}$ does the rest.

8.2. The local stable manifold. In order to construct the nontrivial infrared fixed point we first construct its local stable manifold, then show that the RG transformation is contractive on it and finally obtain the fixed point using the Banach Fixed Point Theorem. We now proceed with the first step which is the construction of the stable manifold also using the Banach Fixed Point Theorem in a space of one-sided sequences, in the spirit of Irwin's method [50]. Let \mathcal{B}_+ be the Banach space of sequences

$$\vec{u} = (\mu_0, (\delta g_1, \mu_1, R_1), (\delta g_2, \mu_2, R_2), \ldots) \in \mathbb{C} \times \prod_{n \ge 1} \left[\mathbb{C}^2 \times C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C}) \right]$$

which have finite norm given by

$$||\vec{u}|| = \sup \{ |\delta g_j| \bar{g}^{-e_4} \text{ for } j \ge 1; |\mu_j| \bar{g}^{-e_2} \text{ for } j \ge 0; |||R_j|||_{\bar{g}} \bar{g}^{-e_R} \text{ for } j \ge 1 \}$$
.

We will define a map \mathfrak{m} on this space of sequences which depends on parameters δg_0 , R_0 serving as boundary conditions. Given δg_0 and R_0 , the image $\vec{u}' = \mathfrak{m}(u)$ is defined as follows. For $n \geq 1$, we let

$$\delta g_n' = (2 - L^{\epsilon})^n \delta g_0 + \sum_{j=0}^{n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j)$$

and

$$R'_{n} = \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{0}, \mu_{0})}(R_{0})$$

$$+ \sum_{j=0}^{n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1}, \mu_{j+1})} \left(\tilde{\xi}_{R}(\delta g_{j}, \mu_{j}, R_{j}) \right) .$$

For $n \geq 0$, we let

$$\mu'_n = -\sum_{j=n}^{\infty} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j) .$$

Given a sufficiently small $\rho > 0$ we now show that this map is well defined and analytic on the open ball $B(\vec{0}, \rho) \in \mathcal{B}_+$ in the regime of small ϵ (made small after fixing ρ).

Proposition 3. If $0 < \rho < \frac{1}{12}$, $|\delta g_0| < \frac{\rho}{12} \bar{g}^{e_4}$ and $|||R_0|||_{\bar{g}} < \frac{\rho}{8} \bar{g}^{e_R}$ then the map \mathfrak{m} is well defined, analytic on $B(\vec{0}, \rho)$ and takes its values in the closed ball $\bar{B}(\vec{0}, \frac{\rho}{4})$, provided ϵ is made sufficiently small after fixing ρ . Moreover, \mathfrak{m} is jointly analytic in \vec{u} and the implicit variables δg_0 and R_0 .

Proof: Recall the choice of constraints (38), (39), (40). Their purpose is to ensure that the hypothesis $||\vec{u}|| < \rho$ guarantees that all triples $(\delta g_j, \mu_j, R_j)$ featuring in the definition of $\mathfrak{m}(\vec{u})$ are in the domain of definition and analyticity of $\tilde{\mathcal{L}}$ and the $\tilde{\xi}$ specified in Theorem 4. Indeed, since ϵ which controls the size of \bar{g} will be made small, one may assume $\bar{g} \leq 1$ and thus $\bar{g}^{e_4} \leq \bar{g}$, $\bar{g}^{e_2} \leq \bar{g}^{1-\eta}$ and $\bar{g}^{e_R} \leq \bar{g}^{\frac{11}{4}-\eta_R}$. Note that for $\epsilon > 0$ small we have $0 < 2 - L^{\epsilon} < 1$. Hence for all $n \geq 1$,

$$|\delta g'| \le (2 - L^{\epsilon})^n |\delta g_0| + \sum_{j=0}^{n-1} (2 - L^{\epsilon})^{n-1-j} |\tilde{\xi}_4(\delta g_j, \mu_j, R_j)|$$
.

From the definition of $\tilde{\xi}_4$, the hypothesis and the bounds provided by Theorem 4 we have

$$|\tilde{\xi}_4(\delta g_j, \mu_j, R_j)| \le A_1 |\delta g_j|^2 + |\xi_4(\bar{g} + \delta g_j, \mu_j, R_j)| \le A_1 \rho^2 \bar{g}^{2e_4} + B_4 \rho \bar{g}^{e_R}$$

and consequently, using $(2-L^{\epsilon})^n \leq 1$ for the first term,

$$\bar{g}^{-e_4}|\delta g'| \leq (2 - L^{\epsilon})^n |\delta g_0| \bar{g}^{-e_4} + \sum_{j=0}^{n-1} (2 - L^{\epsilon})^{n-1-j} \left[A_1 \rho^2 \bar{g}^{e_4} + B_4 \rho \bar{g}^{e_R - e_4} \right] \\
\leq |\delta g_0| \bar{g}^{-e_4} + \left[A_1 \rho^2 \bar{g}^{e_4} + B_4 \rho \bar{g}^{e_R - e_4} \right] \times \frac{1 - (2 - L^{\epsilon})^n}{1 - (2 - L^{\epsilon})} \\
\leq |\delta g_0| \bar{g}^{-e_4} + \left[A_1 \rho^2 \bar{g}^{e_4} + B_4 \rho \bar{g}^{e_R - e_4} \right] \times \frac{1}{L^{\epsilon} - 1} \\
\leq |\delta g_0| \bar{g}^{-e_4} + \rho^2 \bar{g}^{e_4 - 1} + A_1^{-1} B_4 \rho \bar{g}^{e_R - e_4 - 1} \\
\leq |\delta g_0| \bar{g}^{-e_4} + \rho^2 \bar{g}^{e_4 - 1} + A_1^{-1} B_4 \rho \bar{g}^{e_R - e_4 - 1}$$

where in the last line we involued the relation $L^{\epsilon} - 1 = A_1 \bar{g}$. From the hypothesis on δg_0 we then have

$$\bar{g}^{-e_4}|\delta g'| \le \frac{\rho}{12} + \rho^2 \bar{g}^{e_4-1} + A_1^{-1} B_4 \rho \bar{g}^{e_R-e_4-1}$$
.

Using $\bar{g} \leq 1$, $e_4 \geq 1$ and $\rho < \frac{\rho}{12}$ we get

$$\rho^2 \bar{g}^{e_4 - 1} \le \frac{1}{12} \ .$$

We have $\lim_{\epsilon \to 0} A_1 = 36(1-p^{-3})l$, with $l \ge 1$. Since ϵ will be made as small as necessary we may assume, e.g., $A_{1,\min} \le A_1 \le A_{1,\max}$ where $A_{1,\min} = 35(1-p^{-3})l$ and $A_{1,\max} = 37(1-p^{-3})l$. Then

$$A_1^{-1}B_4\rho \bar{g}^{e_R-e_4-1} \le A_{1,\min}^{-1}B_4\rho \bar{g}^{e_R-e_4-1} < \frac{\rho}{12}$$

for ϵ or equivalently \bar{g} small enough because of the requirement (41). As a result $\bar{g}^{-e_4}|\delta g'| \leq \frac{\rho}{4}$.

We now bound R'_n using the property that the operator norms of the $\tilde{\mathcal{L}}$ is at most $\frac{1}{2}$. Indeed, from ξ_R bound provided by Theorem 4,

$$|||R'_n|||_{\bar{g}} \le 2^{-n}|||R_0|||_{\bar{g}} + \sum_{j=0}^{n-1} 2^{-(n-1-j)} B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta}$$

and therefore, bounding 2^{-n} simply by 1,

$$|\bar{g}^{-e_R}|||R'_n|||_{\bar{q}} \leq \bar{g}^{-e_R}|||R_0|||_{\bar{q}} + 2B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta-e_R}$$
.

Now by hypothesis $\bar{g}^{-e_R}|||R_0|||_{\bar{g}} < \frac{\rho}{8}$ and from (42) we see that

$$2B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta-e_R} \le \frac{\rho}{8}$$

when ϵ is small enough. Thus we also get $\bar{g}^{-e_R}|||R'_n|||_{\bar{g}} \leq \frac{\rho}{4}$.

Finally, we bound μ'_n noting that

$$|\mu'_n| \le \sum_{j=n}^{\infty} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} |\tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j)|.$$

But we have, using $|\delta g_j| < \frac{1}{2}\bar{g}$ and the ξ_{μ} bound from Theorem 4,

$$|\tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j)| \le |A_2| \frac{9}{4} \bar{g}^2 + + |A_3| \frac{3}{2} \bar{g} \times \rho \bar{g}^{e_2} + B_2 \rho \bar{g}^{e_R}$$
.

Bounding $L^{-\left(\frac{3+\epsilon}{2}\right)}$ by $L^{-\frac{3}{2}}$ we immediately obtain

$$|\bar{g}^{-e_2}|\mu'_n| \le \frac{L^{-\frac{3}{2}}}{1 - L^{-\frac{3}{2}}} \times \left[\frac{9}{4} |A_2| \bar{g}^{2-e_2} + + \frac{3}{2} |A_3| \rho \bar{g} + B_2 \rho \bar{g}^{e_R - e_2} \right].$$

However, from the definition of A_2 , we have

$$\begin{split} |A_2| & \leq 4L^{\frac{3+\epsilon}{2}} ||\Gamma||_{L^3} + 144L^{\epsilon} C_0(0) ||\Gamma||_{L^2} \\ & \leq \left[4L^2 ||\Gamma||_{L^{\infty}} + 144L^{\epsilon} \times 2 \right] \times ||\Gamma||_{L^2} \\ & \leq \left[4L^2 ||\Gamma||_{L^{\infty}} + 144L^{\epsilon} \times 2 \right] \times \frac{1}{36} L^{-\epsilon} A_1 \\ & \leq A_{2,\max} \end{split}$$

with

$$A_{2,\text{max}} = 2 \times [4 + 144] \times \frac{1}{36} \times A_{1,\text{max}}$$
.

Note that we used our previous bounds on $C_0(0)$ and $||\Gamma||_{L^{\infty}}$ by 2 as well as $0 < \epsilon \le 1$ in order to eliminate ϵ from the exponents of L. We also have from the definition of A_3 that

$$|A_3| \le 12L^2 ||\Gamma||_{L^2} \le A_{3,\text{max}}$$

with

$$A_{3,\text{max}} = 12L^2 \times \frac{1}{36} \times A_{1,\text{max}} .$$

We now get

$$|\bar{g}^{-e_2}|\mu'_n| \le \frac{L^{-\frac{3}{2}}}{1 - L^{-\frac{3}{2}}} \times \left[\frac{9}{4} A_{2,\max} \bar{g}^{2-e_2} + \frac{3}{2} A_{3,\max} \rho \bar{g} + B_2 \rho \bar{g}^{e_R - e_2} \right] \le \frac{\rho}{4}$$

for ϵ small because $e_2 < 2 < e_R$ as follows from (38), (41) and (43).

When showing the absolute convergence of the series for the μ'_n we proved that the map \mathfrak{m} is well defined. Analyticity follows easily from uniform absolute convergence. The previous estimates show that $||\vec{u}|| < \rho$ implies $||\mathfrak{m}(\vec{u})|| \leq \frac{\rho}{4}$.

Using Lemma 1 with $r_1 = \rho$, $r_2 = \frac{\rho}{4}$ and $\nu = \frac{1}{3}$ so that

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{1}{2}$$

we immediately see that, under the hypotheses of Proposition 3, the closed ball $\bar{B}\left(\vec{0}, \frac{\rho}{3}\right)$ is stable by \mathfrak{m} and is a contraction. More precisely, for any \vec{u}_1 and \vec{u}_2 in that ball, we have

$$||\mathfrak{m}(\vec{u}_1) - \mathfrak{m}(\vec{u}_2)|| \le \frac{1}{2} ||\vec{u}_1 - \vec{u}_2||$$
.

By the Banach Fixed Point Theorem we then have the existence of a unique fixed point denoted by \vec{u}_* for the map \mathfrak{m} in the ball $\bar{B}\left(\vec{0}, \frac{\rho}{3}\right)$. Using the representation of this fixed point as

$$\vec{u}_* = \sum_{n=0}^{\infty} \left[\mathfrak{m}^{n+1}(\vec{0}) - \mathfrak{m}^n(\vec{0}) \right]$$

and by uniform absolute convergence, it is easy to see that \vec{u}_* is analytic in the implicit data $(\delta g_0, R_0)$. In particular the μ_0 component of the sequence \vec{u}_* which we will denote by $\mu_s(\delta g_0, R_0)$ is analytic on the domain given by $|\delta g_0| < \frac{\rho}{12} \bar{g}^{e_4}$ and $|||R_0|||_{\bar{g}} < \frac{\rho}{8} \bar{g}^{e_R}$.

We will now show that, for elements $v = (\delta g, \mu, R) \in \mathcal{E}$, the equation $\mu = \mu_s(\delta g, R)$ characterizes those on the stable manifold of the sought for fixed point. We now define a set $W^{s,\text{loc}}$ which will be our candidate for this local stable manifold. It will be defined in terms the radius ρ which is supposed to satisfy the hypothesis of Proposition 3. We let

$$W^{\rm s,loc} = \left\{ (\delta g, \mu, R) \in \mathcal{E} | \ |\delta g| \leq \frac{\rho}{13} \bar{g}^{e_4}, |||R|||_{\bar{g}} \leq \frac{\rho}{13} \bar{g}^{e_R}, \mu = \mu_{\rm s}(\delta g, R) \right\} \ .$$

We will also need the subset

$$W_{\rm int}^{\rm s,loc} = \left\{ (\delta g, \mu, R) \in \mathcal{E} |\ |\delta g| < \frac{\rho}{13} \bar{g}^{e_4}, |||R|||_{\bar{g}} < \frac{\rho}{13} \bar{g}^{e_R}, \mu = \mu_{\rm s}(\delta g, R) \right\} \ .$$

Proposition 4. For fixed $\rho \in (0, \frac{1}{12})$ and for ϵ small enough, an equivalent description of $W^{s,loc}$ is as the set of triples $(\delta g, \mu, R) \in \mathcal{E}$ that satisfy all of the following properties:

- $|\delta g| \leq \frac{\rho}{13} \bar{g}^{e_4}$,
- $|||R|||_{\bar{g}} \le \frac{\rho}{13} \bar{g}^{e_R}$,
- there exists a sequence $(\delta g_n, \mu_n, R_n)_{n\geq 0}$ in \mathcal{E} such that $\delta g_0 = \delta g$, $\mu_0 = \mu$, $R_0 = R$, $\forall n \geq 1$, $|\delta g_n| \leq \frac{\rho}{3} \bar{g}^{e_4}$ and $|||R_n|||_{\bar{g}} \leq \frac{\rho}{3} \bar{g}^{e_R}$, $\forall n \geq 0$, $|\mu_n| \leq \frac{\rho}{3} \bar{g}^{e_2}$, and $\forall n \geq 0$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$.

Proof: Suppose $(\delta g, \mu, R) \in W^{s,loc}$. We let $\delta g_0 = \delta g$ and $R_0 = R$ and consider the fixed point \vec{u}_* for the map \mathfrak{m} associated to the data $(\delta g_0, R_0)$ given by Proposition 3. This makes sense since $\frac{\rho}{13}$ is smaller than $\frac{\rho}{12}$ and $\frac{\rho}{8}$. Since

$$\vec{u}_* = (\mu_0, (\delta g_1, \mu_1, R_1), (\delta g_2, \mu_2, R_2), \ldots) \in \bar{B}\left(\vec{0}, \frac{\rho}{2}\right)$$

the $(\delta g_n, \mu_n, R_n)$, $n \ge 0$, are well-defined, belong to the domain of definition of the map RG and satisfy the wanted bounds. We just need to check that this sequence forms a trajectory for RG. From $\vec{u}_* = \mathfrak{m}(\vec{u}_*)$ we get, for all $n \ge 1$,

(44)
$$\delta g_n = (2 - L^{\epsilon})^n \delta g_0 + \sum_{j=0}^{n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) .$$

If n = 1, (44) reduces to the wanted equation, namely,

$$\delta g_1 = (2 - L^{\epsilon})\delta g_0 + \tilde{\xi}_4(\delta g_0, \mu_0, R_0) .$$

If $n \geq 2$, (44) can be rewritten as

$$\delta g_n = (2 - L^{\epsilon}) \left[(2 - L^{\epsilon})^{n-1} \delta g_0 + \sum_{j=0}^{n-2} (2 - L^{\epsilon})^{n-2-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) \right] + \tilde{\xi}_4(\delta g_{n-1}, \mu_{n-1}, R_{n-1})$$

$$= (2 - L^{\epsilon})\delta g_{n-1} + \tilde{\xi}_4(\delta g_{n-1}, \mu_{n-1}, R_{n-1})$$

by (44) for n-1 instead of n. Likewise, the R projections of the sequence fixed point equation $\vec{u}_* = \mathfrak{m}(\vec{u}_*)$ imply by similar manipulations that, for all $n \geq 1$,

$$R_n = \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})(R_{n-1})} + \tilde{\xi}_R(\delta g_{n-1}, \mu_{n-1}, R_{n-1})$$

Now for the μ 's we first write, for all $n \geq 0$,

(45)
$$\mu_n = -\sum_{j=n}^{\infty} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j)$$

as results from $\vec{u}_* = \mathfrak{m}(\vec{u}_*)$. Hence

$$\mu_n = -L^{-\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_{\mu}(\delta g_n, \mu_n, R_n) - L^{-\left(\frac{3+\epsilon}{2}\right)} \sum_{j=n+1}^{\infty} L^{-(j-(n+1)+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j)$$

$$=L^{-\left(\frac{3+\epsilon}{2}\right)}\tilde{\xi}_{\mu}(\delta g_n,\mu_n,R_n)-L^{-\left(\frac{3+\epsilon}{2}\right)}\mu_{n+1}$$

by (45) for n+1 instead of n. Thus

$$\mu_{n+1} = L^{\frac{3+\epsilon}{2}} \mu_n + \tilde{\xi}_{\mu}(\delta g_n, \mu_n, R_n) .$$

We therefore proved that for all $n \geq 0$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$ and consequently all the requirements in the statement of the proposition are satisfied.

We now prove the converse and assume that $(\delta q, \mu, R)$ satisfies the listed properties. We then define \vec{u} using the given RG trajectory $(\delta g_n, \mu_n, R_n)_{n>0}$, simply by setting

$$\vec{u} = (\mu_0, (\delta g_1, \mu_1, R_1), (\delta g_2, \mu_2, R_2), \ldots)$$
.

By hypothesis, one clearly has $\vec{u} \in \bar{B}\left(\vec{0}, \frac{\rho}{3}\right)$. For any $n \geq 1$, we have

$$\delta g_n = (2 - L^{\epsilon}) \delta g_{n-1} + \tilde{\xi}_4(\delta g_{n-1}, \mu_{n-1}, R_{n-1})$$
.

We apply this to n-1 instead of n and substitute in the first term of the previous equation only. We do the same for n-2 in the resuting equation and continue this backwards iteration. This immediately establishes (44). The same argument also shows the R parts of the sequence fixed point equation. As for the $\mu's$, we have for all $n \geq 0$

$$\mu_{n+1} = L^{\frac{3+\epsilon}{2}}\mu_n + \tilde{\xi}_{\mu}(\delta g_n, \mu_n, R_n)$$

which can be rewritten as

$$\mu_n = -L^{-\left(\frac{3+\epsilon}{2}\right)}\mu_{n+1} + L^{-\left(\frac{3+\epsilon}{2}\right)}\tilde{\xi}_2(\delta g_n, \mu_n, R_n) \ .$$

We apply this to n+1 instead of n and substitute in the first term of the previous equation. Iterating this procedure forward k times we get

$$\mu_n = -\sum_{j=n}^{n+k-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_2(\delta g_j, \mu_j, R_j) + L^{-k\left(\frac{3+\epsilon}{2}\right)} \mu_{n+k} .$$

Since by hypothesis the μ_i 's are bounded by $\frac{\rho}{3}\bar{g}^{e_2}$,

$$\lim_{k \to \infty} L^{-k\left(\frac{3+\epsilon}{2}\right)} \mu_{n+k} = 0$$

and the μ part of the sequence fixed point equation holds. We therefore proved $\vec{u} = \mathfrak{m}(\vec{u})$. By the uniqueness part of the Banach Fixed Point Theorem, \vec{u} and \vec{u}_* are equal and therefore so are their μ_0 components. Given the previous definition, this establishes $\mu = \mu_s(\delta g, R)$ and finally $(\delta g, \mu, R) \in W^{s, loc}$ as wanted.

Proposition 5. For fixed $\rho \in (0, \frac{1}{12})$ and for ϵ small enough, $W^{\mathrm{s,loc}}$ is stable by RG. In fact one has the stronger statement $RG\left(W^{\mathrm{s,loc}}\right) \subset W^{\mathrm{s,loc}}_{\mathrm{int}}$.

Proof: We use the characterization provided by Proposition 4 both ways. Let $(\delta g, \mu, R) \in W^{s,loc}$ and let $(\delta g_n, \mu_n, R_n)_{n\geq 0}$ be the trajectory provided by Proposition 4 such that $(\delta g_0, \mu_0, R_0) = (\delta g, \mu, R)$. We will apply the reverse direction of Proposition 4 to $(\delta g_1, \mu_1, R_1)$ together with $(\delta g_{n+1}, \mu_{n+1}, R_{n+1})_{n\geq 0}$, the shifted trajectory, in order to show $(\delta g_1, \mu_1, R_1) = RG(\delta g, \mu, R) \in W^{s,loc}$. All we need is to show the more restrictive inequalities $|\delta g_1| \leq \frac{\rho}{13} \bar{g}^{e_4}$ and $|||R_1|||_{\bar{g}} \leq \frac{\rho}{13} \bar{g}^{e_R}$. In fact we will prove the stronger estimates $|\delta g_1| < \frac{\rho}{13} \bar{g}^{e_4}$ and $|||R_1|||_{\bar{g}} \leq \frac{\rho}{13} \bar{g}^{e_R}$ which will show $(\delta g_1, \mu_1, R_1)$ belongs to $W^{s,loc}_{int}$ by definition of the latter.

From

$$R_1 = \tilde{\mathcal{L}}^{(\delta g_0, \mu_0)}(R_0) + \tilde{\xi}_R(\delta g_0, \mu_0, R_0)$$

we get

$$|||R_1|||_{\bar{g}} \le \frac{1}{2}|||R_0|||_{\bar{g}} + B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta}.$$

Hence

$$|\bar{g}^{-e_R}|||R_1|||_{\bar{g}} \le \frac{\rho}{26} + B_{R\xi}\bar{g}^{\frac{11}{4} - 3\eta - e_R} < \frac{\rho}{13}$$

when ϵ and therefore \bar{g} are small enough, because of the hypothesis (42).

From

$$\delta g_1 = (2 - L^{\epsilon}) \delta g_0 + \tilde{\xi}_4(\delta g_0, \mu_0, R_0) ,$$

i.e.,

$$\delta g_1 = (2 - L^{\epsilon})\delta g_0 - A_1 \delta g_0^2 + \xi_4(\bar{g} + \delta g_0, \mu_0, R_0) ,$$

we obtain, using $A_1 > 0$,

$$|\delta g_1| = (2 - L^{\epsilon})|\delta g_0| + A_1|\delta g_0|^2 + |\xi_4(\bar{g} + \delta g_0, \mu_0, R_0)|$$

$$\leq (2 - L^{\epsilon}) \left(\frac{\rho}{13} \bar{g}^{e_4} \right) + A_1 \left(\frac{\rho}{13} \bar{g}^{e_4} \right)^2 + B_4 \times \frac{\rho}{13} \bar{g}^{e_R} = \frac{\rho}{13} \bar{g}^{e_4} + \Omega$$

with

$$\Omega = -(L^{\epsilon} - 1)\frac{\rho}{13}\bar{g}^{e_4} + A_1\left(\frac{\rho}{13}\bar{g}^{e_4}\right)^2 + B_4 \times \frac{\rho}{13}\bar{g}^{e_R}.$$

Using the relation $A_1\bar{g}=L^{\epsilon}-1$, we have

$$\Omega = -A_1 \frac{\rho}{13} \bar{g}^{e_4+1} + A_1 \left(\frac{\rho}{13}\right)^2 \bar{g}^{2e_4} + B_4 \times \frac{\rho}{13} \bar{g}^{e_R} .$$

From (38) we have $\bar{g}^{2e_4} \leq \bar{g}^{e_4+1}$ and therefore

$$\Omega \le -A_1 \frac{\rho}{13} \bar{g}^{e_4+1} + A_1 \left(\frac{\rho}{13}\right)^2 \bar{g}^{e_4+1} + B_4 \frac{\rho}{13} \bar{g}^{e_R} = \frac{\rho}{13} \bar{g}^{e_4+1} \left[-A_1 \left(1 - \frac{\rho}{13}\right) + B_4 \bar{g}^{e_R - e_4 - 1} \right] .$$

Since $A_1 \geq A_{1,\text{min}}$ defined in the proof or Proposition 3 and since $\rho < \frac{1}{12}$ we have

$$A_1\left(1 - \frac{\rho}{13}\right) \ge \frac{155}{156}A_{1,\min} \ .$$

As a result

$$|\delta g_1| \le \frac{\rho}{13} \bar{g}^{e_4} + \frac{\rho}{13} \bar{g}^{e_4+1} \left[-\frac{155}{156} A_{1,\min} + B_4 \bar{g}^{e_R-e_4-1} \right].$$

Because of the assumption (38) we have

$$-\frac{155}{156}A_{1,\min} + B_4\bar{g}^{e_R - e_4 - 1} < 0$$

for ϵ small enough and thus $|\delta g_1| < \frac{\rho}{13} \bar{g}^{e_4}$ as desired.

8.3. A dichotomy lemma. We now prove an important lemma which gives quantitative growth or decay estimates which provide a separation between expanding and contracting directions. We first introduce some notation. Clearly $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ where

$$\mathcal{E}_1 = \{ (\delta g, 0, R) | \delta g \in \mathbb{C}, R \in C^9_{\text{bd,ev}}(\mathbb{R}, \mathbb{C}) \}$$

and

$$\mathcal{E}_2 = \{(0, \mu, 0) | \mu \in \mathbb{C}\} .$$

We denote by v_1 and v_2 the pieces of the unique decomposition $v = v_1 + v_2$ of an element $v \in \mathcal{E}$. Note that we will commit a slight abuse of notation by writing $v_1 = (\delta g, R)$ and $v_2 = \mu$ if $v = (\delta g, \mu, R)$ or, in other words, by making use of the identifications $\mathcal{E}_1 \simeq \mathbb{C} \times C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C})$ and $\mathcal{E}_2 \simeq \mathbb{C}$ as Banach spaces. In particular the norms we will be using all come from that of \mathcal{E} and thus ultimately from that of $\mathcal{E}_{\mathrm{ex}}$ in §4.1. For example, following up on the previous set-up we have

$$||v_1|| = \max \left[|\delta g|\bar{g}^{-e_4}, |||R|||_{\bar{g}}\bar{g}^{-e_R} \right] \text{ and } ||v_2|| = |\mu|\bar{g}^{-e_2}.$$

Finally if v is in the domain of definition for the map RG we write $RG_1(v) = [RG(v)]_1$ and $RG_2(v) = [RG(v)]_2$ for better readability. Our dichotomy lemma, in the spirit of [49, Lemma 2.2] is the following result.

Lemma 58. There exists $\epsilon_0 > 0$ and functions $c_1(\epsilon)$, $c_2(\epsilon)$, $c_3(\epsilon)$, $c_4(\epsilon)$, on $(0, \epsilon_0)$ which satisfy $0 < c_1(\epsilon) < 1$, $L^{\frac{3}{4}} \ge c_2(\epsilon) > 1$, $2L^{\frac{3}{2}} \ge c_3(\epsilon) \ge L^{\frac{3}{2}}$ and $0 < c_4(\epsilon) < 1$ (in fact $\lim_{\epsilon \to 0} c_4(\epsilon) = 0$) on that interval such that for all $v, v' \in \overline{B}\left(0, \frac{1}{8}\right) \subset \mathcal{E}$ the following statements hold:

- (1) unconditionally, $||RG_1(v) RG_1(v')|| \le c_1(\epsilon)||v v'||$;
- (2) if $L^{\frac{3}{4}}||v_2 v_2'|| \ge ||v_1 v_1'||$ then $||RG_2(v) RG_2(v')|| \ge c_2(\epsilon)||v v'||$;
- (3) unconditionally, $||RG_2(v) RG_2(v')|| \le c_3(\epsilon)||v v'||$;
- (4) unconditionally,

$$||RG_2(v) - RG_2(v') - L^{\frac{3+\epsilon}{2}}(v_2 - v_2')|| \le c_4(\epsilon)||v - v'||$$
.

More explicitly, the $c(\epsilon)$ functions are given by the formulas

$$c_{1}(\epsilon) = \max \left[1 - \frac{3}{4} (L^{\epsilon} - 1) + 2B_{4}\bar{g}^{e_{R} - e_{4}}, \frac{3}{4} + 3B_{R\xi}\bar{g}^{\frac{11}{4} - 3\eta - \eta_{R}} \right]$$

$$c_{2}(\epsilon) = L^{\frac{3}{4}} - \frac{9}{4} A_{2,\max} \bar{g}^{e_{4} - e_{2} + 1} - \frac{5}{4} A_{3,\max} \bar{g} - 2B_{2}\bar{g}^{e_{R} - e_{2}}$$

$$c_{3}(\epsilon) = L^{\frac{3+\epsilon}{2}} + \frac{9}{4} A_{2,\max} \bar{g}^{e_{4} - e_{2} + 1} + \frac{5}{4} A_{3,\max} \bar{g} + 2B_{2}\bar{g}^{e_{R} - e_{2}}$$

$$c_{4}(\epsilon) = \frac{9}{4} A_{2,\max} \bar{g}^{e_{4} - e_{2} + 1} + \frac{5}{4} A_{3,\max} \bar{g} + 2B_{2}\bar{g}^{e_{R} - e_{2}}.$$

Proof: Since $\frac{1}{8} < \frac{1}{2}$, $\bar{g} \le 1$ for ϵ small, and because of (38), (39) and (40), v and v' are in the domain of definition of RG as provided by Theorem 4. Let $v = (\delta g, \mu, R)$, $v' = (\delta g', \mu', R')$, $RG(v) = (\hat{\delta g}, \hat{\mu}, \hat{R})$ and $RG(v') = (\hat{\delta g}', \hat{\mu}', \hat{R}')$. From the formulas defining the bulk RG transformation we have

$$\widehat{\delta g} - \widehat{\delta g}' = (2 - L^{\epsilon})(\delta g - \delta g') - A_1(\delta g^2 - \delta g'^2) + \xi_4(\bar{g} + \delta g, \mu, R) - \xi_4(\bar{g} + \delta g', \mu', R')$$

and thus

$$|\widehat{\delta g} - \widehat{\delta g}'| \leq (2 - L^{\epsilon})|\delta g - \delta g'| + A_1|\delta g - \delta g'|(|\delta g| + |\delta g'|) + |\xi_4(\bar{g} + \delta g, \mu, R) - \xi_4(\bar{g} + \delta g', \mu', R')|.$$

Using Lemma 57 we therefore obtain

$$|\widehat{\delta g} - \widehat{\delta g}'| \le (2 - L^{\epsilon})|\delta g - \delta g'| + A_1|\delta g - \delta g'| \times 2 \times \frac{1}{8}\overline{g}^{e_4} + 2B_4\overline{g}^{e_R}||v - v'||.$$

By definition of the norm on \mathcal{E} we then get

$$\bar{g}^{-e_4} |\hat{\delta g} - \hat{\delta g}'| \le ||v - v'|| \times \left\{ 2 - L^{\epsilon} + \frac{1}{4} A_1 \bar{g}^{e_4} + 2B_4 \bar{g}^{e_R - e_4} \right\} \\
= ||v - v'|| \times \left\{ 1 - (L^{\epsilon} - 1) \left(1 - \frac{1}{4} \bar{g}^{e_4 - 1} \right) + 2B_4 \bar{g}^{e_R - e_4} \right\}$$

where we used $A_1\bar{g} = L^{\epsilon} - 1$. Also using (38) we get the simpler bound

$$|\bar{g}^{-e_4}|\hat{\delta g} - \hat{\delta g}'| \le ||v - v'|| \times \left\{1 - \frac{3}{4}(L^{\epsilon} - 1) + 2B_4\bar{g}^{e_R - e_4}\right\}.$$

We now turn to the R components and deduce from formulas for the bulk RG and Lemma 57 that

$$\begin{aligned} |||\widehat{R} - \widehat{R}'|||_{\bar{g}} &\leq |||\mathcal{L}^{(\bar{g} + \delta g, \mu)}(R) - \mathcal{L}^{(\bar{g} + \delta g', \mu')}(R')|||_{\bar{g}} + |||\xi_{R}(\bar{g} + \delta g, \mu, R) - \xi_{R}(\bar{g} + \delta g', \mu', R')|||_{\bar{g}} \\ &\leq \left(\frac{3}{4}\bar{g}^{e_{R}} + 3B_{R\xi}\bar{g}^{\frac{11}{4} - 3\eta}\right)||v - v'|| \ . \end{aligned}$$

This together with the previous estimate on the δg part and the definition of the $||\cdot||$ on \mathcal{E}_1 immediately implies Part 1) of the lemma with the given $c_1(\epsilon)$. What remains is to show that this quantity is in the interval (0,1) for ϵ sufficiently small. Note that \bar{g} is of the same order as ϵ in this regime. Therefore $L^{\epsilon} - 1 \sim \epsilon \log L$ as well as the assumptions (41) and (42) ensure that $c_1(\epsilon)$ has the wanted property.

We now tackle Part 2) and assume $L^{\frac{3}{4}}||v_2-v_2'|| \geq ||v_1-v_1'||$. For the formulas for RG we get

$$\widehat{\mu} - \widehat{\mu}' = L^{\frac{3+\epsilon}{2}}(\mu - \mu') - A_2 \left[(\bar{g} + \delta g)^2 - (\bar{g} + \delta g)^2 \right]$$

(46)
$$-A_3 [(\bar{g} + \delta g)\mu - (\bar{g} + \delta g')\mu'] + \xi_2(\bar{g} + \delta g, \mu, R) - \xi_2(\bar{g} + \delta g', \mu', R')$$

and therefore

$$|\widehat{\mu} - \widehat{\mu}'| \ge L^{\frac{3+\epsilon}{2}} |\mu - \mu'| - A_{2,\max} |\delta g - \delta g'| (2\bar{g} + |\delta g| + |\delta g'|) - A_{3,\max} |(\bar{g} + \delta g)\mu - (\bar{g} + \delta g')\mu'| - |\xi_2(\bar{g} + \delta g, \mu, R) - \xi_2(\bar{g} + \delta g', \mu', R')|$$

where $A_{2,\text{max}}$ and $A_{3,\text{max}}$ have been defined in the proof of Proposition 3. We use (38) and the hypothesis on v and v' in order to write

$$2\bar{g} + |\delta g| + |\delta g'| \le 2\bar{g} + \bar{g}^{e_4}||v|| + \bar{g}^{e_4}||v'|| \le \bar{g} \times \left[2 + \frac{1}{8} + \frac{1}{8}\right].$$

We also have, for the same reasons,

$$\begin{split} |(\bar{g} + \delta g)\mu - (\bar{g} + \delta g')\mu'| &= |\bar{g}(\mu - \mu') + \delta g(\mu - \mu') + (\delta g - \delta g')\mu'| \\ &\leq \bar{g} \times \bar{g}^{e_2}||v - v'|| + \bar{g}^{e_4}||v|| \times \bar{g}^{e_2}||v - v'|| + \bar{g}^{e_4}||v - v'|| \times \bar{g}^{e_2}||v|| \\ &\leq \bar{g}^{1+e_2}||v - v'|| \times \left[1 + \frac{1}{8} + \frac{1}{8}\right] \;. \end{split}$$

The last ingredients are the use of the ξ_2 Lispchitz estimate in Lemma 57 and the simplification $L^{\frac{3+\epsilon}{2}} \geq L^{\frac{3}{2}}$. Altogether, this gives

$$||RG_{2}(v) - RG_{2}(v')|| = \bar{g}^{-e_{2}}|\hat{\mu} - \hat{\mu}'| \ge L^{\frac{3}{2}}\bar{g} - e_{2}|\mu - \mu'| - \frac{9}{4}A_{2,\max}\bar{g}^{1+e_{4}-e_{2}}||v - v'||$$

$$- \frac{5}{4}A_{3,\max}\bar{g}||v - v'|| - 2B_{2}\bar{g}^{e_{R}-e_{2}}||v - v'||.$$

$$(47)$$

From the hypothesis $L^{\frac{3}{4}}||v_2 - v_2'|| \ge ||v_1 - v_1'||$ we get

$$||v - v'|| = \max[||v_1 - v_1'||, ||v_2 - v_2'||] \le L^{\frac{3}{4}} ||v_2 - v_2'|| = L^{\frac{3}{4}} \bar{g}^{-e_2} |\mu - \mu'|.$$

Therefore, losing a fraction $L^{\frac{3}{4}}$ of the initial $L^{\frac{3}{2}}$ factor, (47) implies Part 2) of the lemma with the given function $c_2(\epsilon)$. From (38) and (43) which also imply $1 + e_4 - e_2 > 0$ we see that $c_2(\epsilon) \to L^{\frac{3}{4}} > 1$ when $\epsilon \to 0$ as wanted.

For Part 3) we use (46)to write

$$|\widehat{\mu} - \widehat{\mu}'| \le L^{\frac{3+\epsilon}{2}} |\mu - \mu'| + A_{2,\max} |\delta g - \delta g'| (2\bar{g} + |\delta g| + |\delta g'|) + A_{3,\max} |(\bar{g} + \delta g)\mu - (\bar{g} + \delta g')\mu'| + |\xi_2(\bar{g} + \delta g, \mu, R) - \xi_2(\bar{g} + \delta g', \mu', R')|.$$

Then the same bounds as before on the last three terms give

$$||RG_2(v) - RG_2(v')|| = \bar{g}^{-e_2}|\widehat{\mu} - \widehat{\mu}'| \le L^{\frac{3+\epsilon}{2}}\bar{g}^{-e_2}|\mu - \mu'| + \frac{9}{4}A_{2,\max}\bar{g}^{1+e_4-e_2}||v - v'||$$

$$+ \frac{5}{4} A_{3,\max} \bar{g}||v - v'|| + 2B_2 \bar{g}^{e_R - e_2} ||v - v'|| \ .$$

Since $\bar{g}^{-e_2}|\mu - \mu'| \leq ||v - v'||$ holds by definition of the norm, the estimate in Part 3) follows. The bounds on $c_3(\epsilon)$ in the small ϵ regime are also immediate.

For Part 4), starting from (46) we transfer $L^{\frac{3+\epsilon}{2}}(\mu-\mu')$ to the left-hand side and use the same bounds to arrive at

$$\bar{g}^{-e_2}|\widehat{\mu} - \widehat{\mu}' - L^{\frac{3+\epsilon}{2}}(\mu - \mu')| \le \left[\frac{9}{4}A_{2,\max}\bar{g}^{1+e_4-e_2} + \frac{5}{4}A_{3,\max}\bar{g} + 2B_2\bar{g}^{e_R-e_2}\right] \times ||v - v'||$$

which is the desired result.

8.4. The infrared RG fixed point.

Lemma 59. If $v \neq v'$ belong to $W^{s,loc}$ then $||v_1 - v_1'|| > L^{\frac{3}{4}}||v_2 - v_2'||$.

Proof: Note that by the prevailing assumptions we have $\rho < \frac{1}{12} < \frac{1}{8}$ and thus Lemma 58 is applicable to all elements of $W^{\rm s,loc}$ and their RG iterates by stability of that set. We proceed by contradiction and suppose that $||v_1 - v_1'|| \le L^{\frac{3}{4}} ||v_2 - v_2'||$. Then by Lemma 58 Part 1) and 2)

$$||RG_1(v) - RG_1(v')|| \le c_1(\epsilon)||v - v'|| \le c_1(\epsilon)c_2(\epsilon)^{-1}||RG_2(v) - RG_2(v')||$$
.

From the bounds we have on $c_1(\epsilon)$ and $c_2(\epsilon)$ we trivially get $c_1(\epsilon)c_2(\epsilon)^{-1} < L^{\frac{3}{4}}$ and therefore

$$||RG_1(v) - RG_1(v')|| \le L^{\frac{3}{4}} ||RG_2(v) - RG_2(v')||,$$

i.e., the first iterates RG(v) and RG(v') satisfy the same hypothesis as v and v'. By an easy induction we then have

$$\forall n \geq 0, \quad ||RG_1^n(v) - RG_1^n(v')|| \leq L^{\frac{3}{4}} ||RG_2^n(v) - RG_2^n(v')||$$

for the higher iterates where $RG_1^n(\cdot)$ means $(RG^n(\cdot))_1$ and likewise for the second components. By Lemma 58 Part 2) we obtain, for all $n \ge 0$,

$$||RG_2^{n+1}(v) - RG_2^{n+1}(v')|| \ge c_2(\epsilon)||RG^n(v) - RG^n(v')|| \ge c_2(\epsilon)||RG_2^n(v) - RG_2^n(v')||.$$

Again by a trivial induction we get, for all $n \geq 0$,

$$||RG_2^n(v) - RG_2^n(v')|| \ge c_2(\epsilon)^n ||v_2 - v_2'||$$
.

But $c_2(\epsilon) > 1$, so if $||v_2 - v_2'|| > 0$ we have

$$\lim_{n \to \infty} ||RG_2^n(v) - RG_2^n(v')|| = \infty$$

which contradicts the stability and boundedness of the set $W^{s,loc}$. Therefore $||v_2 - v_2'|| = 0$ which also entails $||v_1 - v_1'|| = 0$ by the assumtion made at the beginning of this proof. This therefore leads to v = v' which is the desired contradition.

Lemma 60. For all $v, v' \in W^{s,loc}$ we have $||RG(v) - RG(v')|| \le c_1(\epsilon)||v - v'||$.

Proof: By the previous lemma and the stability of $W^{s,loc}$ we have

$$||RG_2(v) - RG_2(v')|| \le L^{-\frac{3}{4}} ||RG_1(v) - RG_1(v')|| \le ||RG_1(v) - RG_1(v')||$$

and therefore

$$||RG(v) - RG(v')|| = ||RG_1(v) - RG_1(v')||$$
.

As a result, the desired conclusion follows from Lemma 58 Part 1).

Proposition 6. The map RG is a contraction when restricted to $W^{s,loc}$ and thus has a unique fixed point $v_* = (\delta g_*, \mu_*, R_*)$ in that set. In fact v_* belongs to the smaller set $W^{s,loc}_{int}$.

Proof: Note that $W^{\mathrm{s,loc}}$ is a closed subset of the Banach space \mathcal{E} . Indeed, μ_{s} is analytic and thus continuous on an open domain containing that given by the condition $||(\delta g,R)|| \leq \frac{\rho}{13}$. Since $W^{\mathrm{s,loc}}$ is therefore a complete metric space for the distance coming from the $||\cdot||$ norm, and since RG restricted to this set is a contraction as follows form Lemma 60 and $c_1(\epsilon) < 1$, the Banach Fixed Point Theorem establishes the present lemma. The fixed point is in $W^{\mathrm{s,loc}}_{\mathrm{int}}$ since v_* is its own image by application of the stronger conclusion of Proposition 58.

8.5. The unstable manifold. We now construct the local unstable manifold following a procedure similar to that of §8.2. Let \mathcal{B}_{-} be the Banach space of sequences

$$\vec{u} = (\dots, (\delta g_{-2}, \mu_{-2}, R_{-2}), (\delta g_{-1}, \mu_{-1}, R_{-1}), \delta g_0, R_0) \in \prod_{n \le -1} \left[\mathbb{C}^2 \times C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C}) \right] \times \mathbb{C} \times C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C})$$

which have finite norm given by

$$||\vec{u}|| = \sup \{ |\delta g_j| \bar{g}^{-e_4} \text{ for } j \leq 0; |\mu_j| \bar{g}^{-e_2} \text{ for } j \leq -1; |||R_j||| \bar{g}^{-e_R} \text{ for } j \leq 0 \}$$
.

We will define a map \mathfrak{n} on this space of sequences which depends on the parameter μ_0 serving as boundary conditions. Given μ_0 , the image $\vec{u}' = \mathfrak{n}(u)$ is defined as follows. For $n \leq 0$, we let

$$\delta g_n' = \sum_{j \le n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j)$$

and

$$R'_n = \sum_{j \le n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1}, \mu_{j+1})} \left(\tilde{\xi}_R(\delta g_j, \mu_j, R_j) \right) .$$

For $n \leq -1$, we let

$$\mu'_n = L^{n\left(\frac{3+\epsilon}{2}\right)} \mu_0 - \sum_{j=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_{\mu}(\delta g_j, \mu_j, R_j) .$$

Given a sufficiently small $\rho' > 0$ we will show that this map is well defined and analytic on the open ball $B(\vec{0}, \rho') \in \mathcal{B}_{-}$ in the regime of small ϵ (made small after fixing ρ').

Proposition 7. If $0 < \rho' \le \frac{1}{8}$, $|\mu_0| < \frac{\rho'}{8} \bar{g}^{e_2}$ then the map \mathfrak{n} is well defined, analytic on $B(\vec{0}, \rho')$ and takes its values in the closed ball $\bar{B}(\vec{0}, \frac{\rho'}{4})$, provided ϵ is made sufficiently small after fixing ρ' . Moreover, \mathfrak{n} is jointly analytic in \vec{u} and the implicit variable μ_0 .

Proof: Again the choice of constraints (38), (39), (40) and the hypothesis $||\vec{u}|| < \rho' < \frac{1}{2}$ guarantees that all triples $(\delta g_j, \mu_j, R_j)$ featuring in the definition of $\mathfrak{n}(\vec{u})$ are in the domain of definition and analyticity of $\tilde{\mathcal{L}}$ and the $\tilde{\xi}$ coming from Theorem 4.

Hence for all $n \leq 0$,

$$|\delta g'| \le \sum_{j \le n-1}^{n-1} (2 - L^{\epsilon})^{n-1-j} |\tilde{\xi}_4(\delta g_j, \mu_j, R_j)|.$$

As in the proof of Proposition 3

$$|\tilde{\xi}_4(\delta g_j, \mu_j, R_j)| \le A_1 {\rho'}^2 \bar{g}^{2e_4} + B_4 {\rho'} \bar{g}^{e_R}$$

and consequently,

$$\bar{g}^{-e_4}|\delta g'| \leq \sum_{j\leq n-1} (2-L^{\epsilon})^{n-1-j} \left[A_1 {\rho'}^2 \bar{g}^{e_4} + B_4 {\rho'} \bar{g}^{e_R-e_4} \right]
\leq \left[A_1 {\rho'}^2 \bar{g}^{e_4} + B_4 {\rho'} \bar{g}^{e_R-e_4} \right] \times \frac{1}{L^{\epsilon}-1}
\leq \left[A_1 {\rho'}^2 \bar{g}^{e_4} + B_4 {\rho'} \bar{g}^{e_R-e_4} \right] \times \frac{1}{A_1 \bar{g}}
\leq {\rho'}^2 + A_{1 \min}^{-1} B_4 {\rho'} \bar{g}^{e_R-e_4-1}$$

where we used the relation $L^{\epsilon}-1=A_1\bar{g}$ and later the simple bound $\bar{g}^{e_4-1}\leq 1$ due to (38). The hypothesis on ρ' implies ${\rho'}^2\leq \frac{\rho'}{8}$ whereas (41) ensures that $A_{1,\min}^{-1}B_4\rho'\bar{g}^{e_R-e_4-1}\leq \frac{\rho'}{8}$ for ϵ small. Therefore, the previous estimates show that the series defining $\delta g'$ converges and that the latter satisfies the bound $\bar{g}^{-e_4}|\delta g'|\leq \frac{\rho'}{4}$ in the small ϵ regime.

We now bound R'_n using the property that the operator norms of the $\tilde{\mathcal{L}}$ is at most $\frac{1}{2}$. Indeed, similarly to the proof of Proposition 3,

$$|||R'_n|||_{\bar{g}} \le \sum_{j \le n-1} 2^{-(n-1-j)} B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta} = 2B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta} .$$

Hence

$$|\bar{g}^{-e_R}|||R'_n|||_{\bar{g}} \le 2B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta-e_R} \le \frac{\rho'}{4}$$

for ϵ small because of (42). We also showed by the same token the convergence in the Banach space \mathcal{E} of the series defining R'_n .

Finally, we bound μ'_n as in the proof of Proposition 3 by writing

$$|\mu'_n| \le L^{n\left(\frac{3+\epsilon}{2}\right)} |\mu_0| + \sum_{j=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} |\tilde{\xi}_2(\delta g_j, \mu_j, R_j)|$$

$$\le L^{n\left(\frac{3+\epsilon}{2}\right)} |\mu_0| + \sum_{j=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \times \left[\frac{9}{4} A_{2,\max} \bar{g}^2 + \frac{3}{2} A_{3,\max} \rho' \bar{g}^{e_2+1} + B_2 \rho' \bar{g}^{e_R} \right].$$

Using the simple bound $L^{n\left(\frac{3+\epsilon}{2}\right)} \leq 1$, since $n \leq -1$, as well as

$$\sum_{i=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \le \sum_{k=1}^{\infty} L^{-k\left(\frac{3+\epsilon}{2}\right)} \le \sum_{k=1}^{\infty} L^{-k\left(\frac{3}{2}\right)}$$

we obtain

$$\bar{g}^{-e_2}|\mu'_n| \leq \bar{g}^{-e_2}|\mu_0| + \frac{L^{-\frac{3}{2}}}{1 - L^{-\frac{3}{2}}} \times \left[\frac{9}{4} A_{2,\max} \bar{g}^{2-e_2} + \frac{3}{2} A_{3,\max} \rho' \bar{g} + B_2 \rho' \bar{g}^{e_R - e_2} \right].$$

The first term is bounded by $\frac{\rho'}{8}$ by hypothesis. Besides, the second also satisfies the same bound when ϵ is small enough because of $e_2 < 2 < e_R$ which follows from (38), (41) and (43). Hence $\bar{g}^{-e_2}|\mu'_n| \leq \frac{\rho'}{4}$.

Again, when showing the uniform absolute convergence of the series for the $\delta g'_n$ and R'_n we proved that the map \mathfrak{n} is well defined, analytic and satisfies the bound $||\mathfrak{n}(\vec{u})|| \leq \frac{\rho'}{4}$ when $||\vec{u}|| < \rho'$.

Again using Lemma 1 with $r_1 = \rho'$, $r_2 = \frac{\rho'}{4}$ and $\nu = \frac{1}{3}$ so that

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{1}{2}$$

we see that, under the hypotheses of Proposition 7, the closed ball $\bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$ is stable by \mathfrak{n} and is a contraction. More precisely, for any \vec{u}_1 and \vec{u}_2 in that ball, we have

$$||\mathfrak{n}(\vec{u}_1) - \mathfrak{n}(\vec{u}_2)|| \le \frac{1}{2} ||\vec{u}_1 - \vec{u}_2||.$$

By the Banach Fixed Point Theorem we have the existence of a unique fixed point which we again denote by \vec{u}_* for the map \mathfrak{n} in the ball $\bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$. Using the representation of this fixed point as

$$\vec{u}_* = \sum_{n=0}^{\infty} \left[\mathfrak{n}^{n+1}(\vec{0}) - \mathfrak{n}^n(\vec{0}) \right]$$

and by uniform absulote convergence, we see that \vec{u}_* is analytic in the implicit data μ_0 . In particular the δg_0 , R_0 components of the sequence \vec{u}_* which we will denote by $\delta g_{\rm u}(\mu_0)$, $R_{\rm u}(\mu_0)$ respectively are analytic on the domain given by $|\mu_0| < \frac{\rho'}{8} \bar{g}^{e_2}$.

As in section §8.2, the next step will be to show that, for elements $v=(\delta g,\mu,R)\in\mathcal{E}$, the equation $(\delta g,R)=(\delta g_{\mathrm{u}}(\mu),R_{\mathrm{u}}(\mu))$ characterizes those on the unstable manifold of the bulk RG fixed point v_* . We now define a set $W^{\mathrm{u,loc}}$ which will be our candidate for this local unstable manifold. It will be defined in terms the radius ρ' which is supposed to satisfy the hypothesis of Proposition 7. We let

$$W^{\mathrm{u,loc}} = \left\{ (\delta g, \mu, R) \in \mathcal{E} |\ |\mu| < \frac{\rho'}{8} \overline{g}^{e_2}, \delta g = \delta g_{\mathrm{u}}(\mu), R = R_{\mathrm{u}}(\mu) \right\} \ .$$

Proposition 8. For fixed $\rho' \in (0, \frac{1}{8}]$ and for ϵ small enough, an equivalent description of $W^{\mathrm{u,loc}}$ is as the set of triples $(\delta g, \mu, R) \in \mathcal{E}$ that satisfy all of the following properties:

$$\bullet \ |\mu| < \frac{\rho'}{8} \bar{g}^{e_2},$$

• there exists a sequence $(\delta g_n, \mu_n, R_n)_{n \leq 0}$ in \mathcal{E} such that $\delta g_0 = \delta g$, $\mu_0 = \mu$, $R_0 = R$, $\forall n \leq 0$, $|\delta g_n| \leq \frac{\rho'}{3} \bar{g}^{e_4}$ and $|||R_n|||_{\bar{g}} \leq \frac{\rho'}{3} \bar{g}^{e_R}$, $\forall n \leq -1$, $|\mu_n| \leq \frac{\rho'}{3} \bar{g}^{e_2}$, and $\forall n \leq -1$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$.

Proof: Suppose $(\delta g, \mu, R) \in W^{u,loc}$. We let $\mu_0 = \mu$ and consider the fixed point \vec{u}_* for the map \mathfrak{n} associated to the data μ_0 given by Proposition 7. We write

$$\vec{u}_* = (\dots, (\delta g_{-2}, \mu_{-2}, R_{-2}), (\delta g_{-1}, \mu_{-1}, R_{-1}), \delta g_0, R_0) \in \bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$$

and note that the $(\delta g_n, \mu_n, R_n)$, $n \leq -1$, are well-defined, belong to the domain of definition of the map RG and satisfy the wanted bounds. We need to check that this sequence, also including the n = 0 term, forms a trajectory for RG. Form $\vec{u}_* = \mathfrak{n}(\vec{u}_*)$ we get, for all $n \leq -1$,

$$L^{\left(\frac{3+\epsilon}{2}\right)}\mu_n + \tilde{\xi}_2(\delta g_n, \mu_n, R_n) = L^{\left(\frac{3+\epsilon}{2}\right)} \left[L^{n\left(\frac{3+\epsilon}{2}\right)}\mu_0 - \sum_{j=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_2(\delta g_j, \mu_j, R_j) \right] + \tilde{\xi}_2(\delta g_n, \mu_n, R_n)$$

$$= L^{(n+1)\left(\frac{3+\epsilon}{2}\right)}\mu_0 - \sum_{j=n+1}^{-1} L^{-(j-n)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_2(\delta g_j, \mu_j, R_j) .$$

The last quantity is equal to μ_{n+1} if $n \leq -2$ by the fixed point equation for the sequence \vec{u}_* . Otherwise if n = -1 the same quantity reduces to $\mu_0 = \mu_{n+1}$ because the sum is empty.

Likewise and still for $n \leq -1$ we have

$$(2 - L^{\epsilon})\delta g_n + \tilde{\xi}_4(\delta g_n, \mu_n, R_n) = (2 - L^{\epsilon}) \left[\sum_{j \le n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) \right] + \tilde{\xi}_4(\delta g_n, \mu_n, R_n)$$
$$= \sum_{j \le n} (2 - L^{\epsilon})^{n-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) = \delta g_{n+1} .$$

Similarly, the R projections of the sequence fixed point equation $\vec{u}_* = \mathfrak{n}(\vec{u}_*)$ imply by analogous manipulations that, for all $n \leq -1$,

$$\tilde{\mathcal{L}}^{(\delta g_n,\mu_n)(R_n)} + \tilde{\xi}_R(\delta g_n,\mu_n,R_n) = R_{n+1} .$$

We therefore proved that for all $n \leq -1$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$ and consequently all the requirements in the statement of the proposition are satisfied.

We now prove the converse and assume that $(\delta g, \mu, R)$ satisfies the listed properties. We then define \vec{u} using the given RG trajectory $(\delta g_n, \mu_n, R_n)_{n < 0}$, simply by setting

$$\vec{u} = (\dots, (\delta g_{-2}, \mu_{-2}, R_{-2}), (\delta g_{-1}, \mu_{-1}, R_{-1}), \delta g_0, R_0)$$
.

By hypothesis, we clearly have $\vec{u} \in \bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$. For any $n \leq 0$, we have

$$\delta g_n = (2 - L^{\epsilon}) \delta g_{n-1} + \tilde{\xi}_4(\delta g_{n-1}, \mu_{n-1}, R_{n-1})$$
.

We apply this to n-1 instead of n and substitute in the first term of the previous equation only. We do the same for n-2 in the resulting equation and continue this backwards iteration. We thus obtain for any $k \ge 1$,

$$\delta g_n = (2 - L^{\epsilon})^k \delta g_{n-k} + \sum_{j=n-k}^{n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) .$$

But $0 < 2 - L^{\epsilon} < 1$ and the sequence of δg 's is bounded and therefore $(2 - L^{\epsilon})^k \delta g_{n-k} \to 0$ when $k \to \infty$. Hence

$$\delta g_n = \sum_{j \le n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j) .$$

A similar argument for the R's gives for all $n \leq 0$ and all $k \geq 1$

$$R_{n} = \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{n-k}, \mu_{n-k})}(R_{n-k}) + \sum_{j=n-k}^{n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1}, \mu_{j+1})} \left(\tilde{\xi}_{R}(\delta g_{j}, \mu_{j}, R_{j}) \right) .$$

However,

$$|||\tilde{\mathcal{L}}^{(\delta g_{n-1},\mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{n-k},\mu_{n-k})}(R_{n-k})|||_{\bar{g}} \le \left(\frac{1}{2}\right)^k |||R_{n-k}|||_{\bar{g}} \le 2^{-k} \times \frac{\rho'}{3} \bar{g}^{e_R}$$

and thus this boundary term disappears when $k \to \infty$ and we then get

$$R_n = \sum_{j < n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1}, \mu_{j+1})} \left(\tilde{\xi}_R(\delta g_j, \mu_j, R_j) \right) .$$

As for the $\mu's$, we have for all $n \leq -1$

$$\mu_{n+1} = L^{\frac{3+\epsilon}{2}}\mu_n + \tilde{\xi}_{\mu}(\delta g_n, \mu_n, R_n)$$

or equivalently

$$\mu_n = -L^{-\left(\frac{3+\epsilon}{2}\right)}\mu_{n+1} + L^{-\left(\frac{3+\epsilon}{2}\right)}\tilde{\xi}_2(\delta g_n, \mu_n, R_n) .$$

Provided $n+1 \le -1$, we apply this to n+1 instead of n and substitute in the first of the previous equation. Iterating this procedure forward until one hits the boundary term μ_0 gives

$$\mu_n = L^{n(\frac{3+\epsilon}{2})} \mu_0 - \sum_{j=n}^{-1} L^{-(j-n+1)(\frac{3+\epsilon}{2})} \tilde{\xi}_2(\delta g_j, \mu_j, R_j) .$$

We therefore proved $\vec{u} = \mathfrak{n}(\vec{u})$. By the uniqueness part of the Banach Fixed Point Theorem, \vec{u} and \vec{u}_* are equal and therefore so are their δg_0 and R_0 components. This establishes $(\delta g, R) = (\delta g_{\rm u}(\mu), R_{\rm u}(\mu))$ and finally $(\delta g, \mu, R) \in W^{\text{u,loc}}$ as wanted.

Lemma 61. Provided ρ and ρ' are chosen so that $\rho < \frac{3}{8}\rho'$, we have $v_* \in W^{\mathrm{u,loc}}$ as well as the equations

$$\mu_* = \mu_s(\delta g_*, R_*) , \delta g_* = \delta g_u(\mu_*) , R_* = R_u(\mu_*) .$$

Proof: From $v_* \in W^{s,loc}$ we get

$$(48) |\delta g_*| \le \frac{\rho}{13} \bar{g}^{e_4} , |\mu_*| \le \frac{\rho}{3} \bar{g}^{e_2} , |||R_*|||_{\bar{g}} \le \frac{\rho}{13} \bar{g}^{e_R} .$$

We also know that $RG(v_*) = v_*$. Define $(\delta g_n, \mu_n, R_n) = v_*$ for all $n \leq 0$. Since this is an RG trajectory, all we need in order to prove $v_* = (\delta_0, \mu_0, R_0) \in W^{\mathrm{u,loc}}$ via Lemma 8 are the inequalities

$$|\delta g_*| \le \frac{\rho'}{3} \bar{g}^{e_4} \ , \ |\mu_*| < \frac{\rho'}{8} \bar{g}^{e_2} \ , \ |||R_*|||_{\bar{g}} \le \frac{\rho'}{3} \bar{g}^{e_R} \ .$$

The latter easily follow from (48) and the hypothesis $\rho < \frac{3}{8}\rho'$. Finally the three equations satisfied by v_* are tautological.

Lemma 62. If $v \neq v'$ belong to $W^{u,loc}$ then $||v_2 - v_2'|| > ||v_1 - v_1'||$.

Proof: By Proposition 8 there exists sequences $(w_n)_{n\leq 0}$ and $(w'_n)_{n\leq 0}$ in \mathcal{E} such that $w_0=v, w'_0=v',$ $w_{n+1} = RG(w_n), w'_{n+1} = RG(w'_n)$ for all $n \leq -1$ and such that the inequalities in Proposition 8 hold for both sequences. The latter imply that the corresponding points all are in the domain of application of Lemma 58 because $\frac{\rho'}{3} < \rho' \le \frac{1}{8}$ by assumption. For all $n \le 0$, $v \ne v'$ can be rewritten $RG^{(-n)}(w_n) \ne RG^{(-n)}(w_n')$ and thus $w_n \ne w_n'$. We proceed by contradiction and suppose $||v_1 - v_1'|| \ge ||v_2 - v_2'||$. This provides the n = 0instance of the property $\forall n \leq 0, \ ||w_{n,1} - w'_{n,1}|| \geq ||w_{n,2} - w'_{n,2}||$ which we prove by descending induction. Suppose the inequality is true for n. Then by Lemma 58 1)

$$||w_{n,1} - w'_{n,1}|| = ||RG_1(w_{n-1}) - RG_1(w'_{n-1})|| \le c_1(\epsilon)||w_{n-1} - w'_{n-1}||.$$

We now examine two possible cases.

1st Case: Suppose $L^{\frac{3}{4}}||w_{n-1,2}-w'_{n-1,2}|| \ge ||w_{n-1,1}-w'_{n-1,1}||$. By Part 2) of Lemma 58

(50)
$$||w_{n,2} - w'_{n,2}|| = ||RG_2(w_{n-1}) - RG_2(w'_{n-1})|| \ge c_2(\epsilon)||w_{n-1} - w'_{n-1}||.$$

Combining (49) and (50) we obtain

(51)
$$||w_{n,1} - w'_{n,1}|| \le c_1(\epsilon)c_2(\epsilon)^{-1}||w_{n,2} - w'_{n,2}||$$

However $c_1(\epsilon)c_2(\epsilon)^{-1} < 1$ makes (51) incompatible with the induction hypothesis unless $||w_{n,2} - w'_{n,2}|| = 0$. The latter implies, via (51), that $||w_{n,1} - w'_{n,1}|| = 0$ and therefore $w_n = w'_n$ which has been shown to be impossible. In fact, this 1st Case does not occur.

2nd Case: Suppose $L^{\frac{3}{4}}||w_{n-1,2}-w'_{n-1,2}|| < ||w_{n-1,1}-w'_{n-1,1}||$. Since $L^{\frac{3}{4}} > 1$, this immediately implies the induction hypothesis for n-1.

From the inequalities we just proved by induction and the definition of the norms we have, for all $n \leq 0$,

$$||w_n - w'_n|| = ||w_{n,1} - w'_{n,1}||$$
.

Thus (49) becomes

$$||w_{n-1,1} - w'_{n-1,1}|| \ge c_1(\epsilon)^{-1} ||w_{n,1} - w'_{n,1}||$$

for all $n \leq 0$. Trivial iteration gives

$$||w_{n,1} - w'_{n,1}|| \ge \left[c_1(\epsilon)^{-1}\right]^{-n} ||w_{0,1} - w'_{0,1}||$$

which contradicts the boundedness of the w and w' sequences in the $n \to -\infty$ limit because $c_1(\epsilon) < 1$, unless $||w_{0,1} - w'_{0,1}|| = 0$. Hence $||v_1 - v'_1|| = 0$ which also implies $||v_2 - v'_2|| = 0$ by the assumption made at the beginning. We then arrive at v = v' which is impossible.

Corollary 2. Under the hypotheses of Lemma 61 $W^{s,loc} \cap W^{u,loc} = \{v_*\}.$

Proof: We already know that the fixed point v_* belongs to the intersection. Suppose $v \neq v_*$ does too. Since v and v_* are distinct and belong to $W^{s,loc}$, then Lemma 59 and the fact $L^{\frac{3}{4}} \geq 1$ imply

$$||v_1 - v_{*,1}|| > ||v_2 - v_{*,2}|| \times L^{\frac{3}{4}} \ge ||v_2 - v_{*,2}||$$

However since v and v_* are distinct and belong to $W^{u,loc}$, Lemma 62 implies

$$||v_2 - v_{*,2}|| > ||v_1 - v_{*,1}||$$

which gives a contradiction.

We conclude this section by giving an analogue of Proposition 5 for the unstable manifold. Since this corresponds to an expanding direction for the bulk RG map, one cannot hope for the stability of $W^{\mathrm{u,loc}}$. However we will consider a smaller set $W^{\mathrm{u,loc}}_{\mathrm{tiny}}$ and show, under suitable additional hypotheses, that $RG(W^{\mathrm{u,loc}}_{\mathrm{tiny}}) \subset W^{\mathrm{u,loc}}$. For $\rho'' > 0$ to be suitably adjusted we first define

$$W_{\mathrm{small}}^{\mathrm{u,loc}} = \{ (\delta g_{\mathrm{u}}(\mu), \mu, R_{\mathrm{u}}(\mu)) | \ |\mu - \mu_*| < \rho'' \bar{g}^{e_2} \}$$

According to the prevailing hypotheses, as in the statement of Lemma 61, we have $\frac{\rho}{3} < \frac{\rho'}{8}$. This leaves the possibility of adding the new constraint $\rho'' < \frac{\rho'}{8} - \frac{\rho}{3}$ on the new parameter ρ'' . From the proof of Lemma 61 we get $|\mu_*| \leq \frac{\rho}{3}\bar{g}^{e_2}$ and therefore $|\mu - \mu_*| < \rho''\bar{g}^{e_2}$ implies

$$|\mu| \le |\mu - \mu_*| + |\mu_*| < \rho'' \bar{g}^{e_2} + \frac{\rho}{3} \bar{g}^{e_2} < \frac{\rho'}{8} \bar{g}^{e_2}$$
.

This garantees $W_{\rm small}^{\rm u,loc} \subset W^{\rm u,loc}$. Also note that $\rho' \leq \frac{1}{8}$ implies that $W^{\rm u,loc}$ and therefore $W_{\rm small}^{\rm u,loc}$ are contained in a domain where RG is well-defined and analytic. Therefore, the set RG(W) is also well-defined for any subset W of $W_{\rm small}^{\rm u,loc}$. We now define

$$W_{\rm tiny}^{\rm u,loc} = \{ (\delta g_{\rm u}(\mu), \mu, R_{\rm u}(\mu)) | |\mu - \mu_*| < \rho''' \bar{g}^{e_2} \}$$

where

$$\rho''' = \min \left\{ \rho'', \frac{3}{16} c_3(\epsilon)^{-1} \left(\frac{\rho'}{8} - \frac{\rho}{3} \right) \right\}$$

where $c_3(\epsilon)$ has been defined in Lemma 58. Our working assumptions are now that

$$0 < \rho < \frac{\rho'}{8}$$
 and $0 < \rho'' < \frac{1}{3} \left(\frac{\rho'}{8} - \rho \right)$

which are stronger than the previous ones and garantee $0 < \rho''' \le \rho''$ so $W_{\text{tiny}}^{\text{u,loc}}$ is indeed a subset of $W_{\text{small}}^{\text{u,loc}}$

Proposition 9. In the small ϵ regime, $W_{\rm tiny}^{\rm u,loc}$ satisfies

$$RG(W_{\mathrm{tiny}}^{\mathrm{u,loc}}) \subset W^{\mathrm{u,loc}}$$
.

Proof: Let $(\delta g, \mu, R) \in W_{\text{small}}^{\text{u,loc}}$ and consider the associated backwards trajectory $(\delta g_n, \mu_n, R_n)_{n \leq 0}$ produced by Proposition 8. Let $(\delta g_1, \mu_1, R_1) = RG(\delta g, \mu, R)$. We will show that the extended sequence $(\delta g_n, \mu_n, R_n)_{n \leq 1}$ satisfies the conditions stated in Proposition 8 (with suitable and obvious shift in indexation). For $n \leq -1$, the bounds we need are the ones we already have. For n = 0 the bounds we have are stronger than the ones we need. Indeed, $|\mu_0| < \frac{\rho'}{8} \bar{g}^{e_2}$ trivially implies $|\mu_0| \leq \frac{\rho'}{3} \bar{g}^{e_2}$. We now focus on the n = 1 case. By Lemma 58

$$||RG_1(\delta q_0, \mu_0, R_0) - RG_1(v_*)|| < c_1(\epsilon)||(\delta q_0, \mu_0, R_0) - v_*||$$

namely,

$$||(\delta g_1 - \delta g_*, R_1 - R_*)|| \le c_1(\epsilon) ||(\delta g_u(\mu) - \delta g_u(\mu_*), \mu - \mu_*, R_u(\mu) - R_u(\mu_*))||$$
.

Note that, by construction in Proposition 7, the analytic function $\delta g_{\rm u}$ satisfies the hypotheses of Lemma 1 with $r_1 = \frac{\rho'}{8} \bar{g}^{e_2}$ and $r_2 = \frac{\rho'}{3} \bar{g}^{e_4}$. If we choose $\nu = \frac{1}{3}$ then resulting Lipschitz estimate will give us

$$|\delta g_{\mathbf{u}}(\mu) - \delta g_{\mathbf{u}}(\mu_*)| \le \frac{16}{3} \bar{g}^{e_4 - e_2} |\mu - \mu_*|$$

provided both μ and μ_* are bounded by $\frac{\rho'}{24}\bar{g}^{e_2}$. However these two requirements are garanteed by the hypotheses $\rho < \frac{\rho'}{8}$ and $\rho''' \le \frac{1}{3}\left(\frac{\rho'}{8} - \rho\right)$ together with

$$|\mu_*| \le \frac{\rho}{3} \bar{g}^{e_2}$$
 and $|\mu| \le |\mu_*| + |\mu - \mu_*| < \frac{\rho}{3} \bar{g}^{e_2} + \rho''' \bar{g}^{e_2}$.

We therefore have

$$\bar{g}^{-e_4}|\delta g_{\mathrm{u}}(\mu) - \delta g_{\mathrm{u}}(\mu_*)| \leq \frac{16}{3}\bar{g}^{-e_2}|\mu - \mu_*|$$
.

By the same reasoning and use of Lemma 1 for the function $R_{\rm u}$ we also have

$$\bar{g}^{-e_R}|R_{\mathrm{u}}(\mu) - R_{\mathrm{u}}(\mu_*)| \le \frac{16}{3}\bar{g}^{-e_2}|\mu - \mu_*|$$
.

As a result

$$||(\delta g_{\mathbf{u}}(\mu) - \delta g_{\mathbf{u}}(\mu_*), \mu - \mu_*, R_{\mathbf{u}}(\mu) - R_{\mathbf{u}}(\mu_*))|| \le \frac{16}{3} \bar{g}^{-e_2} |\mu - \mu_*| < \frac{16}{3} \rho'''$$

and thus

$$||(\delta g_1 - \delta g_*, R_1 - R_*)|| < \frac{16}{3}c_1(\epsilon)\rho''' < \frac{16}{3}\rho'''$$
.

In view of $|\delta g_*| \leq \frac{\rho}{13}\bar{g}^{e_4}$ and $|||R_*|||_{\bar{g}} \leq \frac{\rho}{13}\bar{g}^{e_R}$ provided by the fact $v_* \in W^{s,loc}$ and by simple triangle inequalities we obtain

$$\begin{split} |\delta g_1| &< \left(\frac{16}{3}\rho''' + \frac{\rho}{13}\right)\bar{g}^{e_4} \;, \\ |||R_1|||_{\bar{g}} &< \left(\frac{16}{3}\rho''' + \frac{\rho}{13}\right)\bar{g}^{e_R} \;. \end{split}$$

Since $c_3(\epsilon) \geq L^{\frac{3}{4}} > 1$, the definition of ρ''' implies

$$\frac{16}{3}\rho''' + \frac{\rho}{13} < \frac{16}{3}c_3(\epsilon)\rho''' + \frac{\rho}{3} \le \frac{\rho'}{8} < \frac{\rho'}{3} .$$

Hence, $|\delta g_1| < \frac{\rho'}{3} \bar{g}^{e_4}$ and $|||R_1|||_{\bar{g}} < \frac{\rho'}{3} \bar{g}^{e_R}$.

By Lemma 58

$$||RG_2(\delta g_0, \mu_0, R_0) - RG_2(v_*)|| \le c_3(\epsilon)||(\delta g_0, \mu_0, R_0) - v_*||$$
.

By the same bound on the right-hand side as before we thus get

$$||RG_2(\delta g_0, \mu_0, R_0) - RG_2(v_*)|| \le c_3(\epsilon) \times \frac{16}{3} \bar{g}^{-e_2} |\mu - \mu_*| < \frac{16}{3} c_3(\epsilon) \rho'''.$$

Hence

$$|\mu_1 - \mu_*| < \frac{16}{3} c_3(\epsilon) \rho''' \bar{g}^{e_2}$$

which together with $|\mu_*| \leq \frac{\rho}{3} \bar{g}^{e_2}$ and the hypothesis on ρ''' implies $|\mu_1| < \frac{\rho'}{8}$.

We therefore proved the required shifted bounds on the sequence $(\delta g_n, \mu_n, R_n)_{n \leq 1}$ in order to conclude by reverse use of Proposition 8 that $(\delta g_1, \mu_1, R_1) \in W^{\mathrm{u,loc}}$.

8.6. Study of the differential of the RG map at the fixed point and quantitative transversality. We now study the differential $D_{v_*}RG$ of the map RG at the fixed point v_* in relation to the invariant linear subspaces \mathcal{E}^s and \mathcal{E}^u corresponding to the tangent spaces to the stable and unstable manifolds at the fixed point respectively. We first define \mathcal{E}^s as the kernel of the \mathbb{C} -linear form

$$(\delta g, \mu, R) \mapsto \mu - D_{v_{*,1}} \mu_{s}[\delta g, R]$$

where $D_{v_{*,1}}\mu_s$ is the differential of μ_s at $v_{*,1}=(\delta g_*,R_*)$. This linear form is clearly nonzero. It is also continuous by analyticity of μ_s . Therefore \mathcal{E}^s is a closed complex hyperplane in \mathcal{E} .

We likewise define \mathcal{E}^{u} as the kernel of the \mathbb{C} -linear map

$$\left\{ \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}_1 \\ (\delta g, \mu, R) & \longmapsto & (\delta g - D_{v_{*,2}} \delta g_{\mathbf{u}}[\mu], R - D_{v_{*,2}} R_{\mathbf{u}}[\mu]) \end{array} \right.$$

in terms of the differentials at $v_{*,2} = \mu_*$ of the analytic maps δg_u and R_u . Again, \mathcal{E}^u is a closed subspace of \mathcal{E} . In fact, it is easy to see that \mathcal{E}^u is equal to the complex line $\mathbb{C}e_u$ with

$$e_{\mathbf{u}} = (D_{v_{*,2}} \delta g_{\mathbf{u}}[1], 1, D_{v_{*,2}} R_{\mathbf{u}}[1])$$
.

Lemma 63. For all $v \in \mathcal{E}^u$ we have $||v_1|| \le ||v_2||$. For all $v \in \mathcal{E}^s$ we have $L^{\frac{3}{4}}||v_2|| \le ||v_1||$. As a consequence we have the direct sum decomposition $\mathcal{E} = \mathcal{E}^s \oplus \mathcal{E}^u$.

Proof: Define the complex curve parametrized by $\gamma(\mu) = (\delta g_{\mathbf{u}}(\mu_* + \mu), \mu_* + \mu, R_{\mathbf{u}}(\mu_* + \mu))$ for $\mu \in \mathbb{C}$ small. By Proposition 8, $\gamma(\mu) \in W^{\mathbf{u}, \text{loc}}$ for μ small. Since we also have $v_* \in W^{\mathbf{u}, \text{loc}}$, then Lemma 62 gives us

$$||\gamma(\mu)_1 - v_{*,1}|| \le ||\gamma(\mu)_2 - v_{*,2}||$$

for μ small. However by analyticity and therefore differentiability we have $\gamma(\mu) = v_* + \mu e_u + \mu \omega(\mu)$ where $\omega(\mu) \to 0$ when $\mu \to 0$. The previous inequality becomes

$$||\mu e_{u,1} + \mu \omega(\mu)_1|| \le ||\mu e_{u,2} + \mu \omega(\mu)_2||$$
.

For $\mu \neq 0$ we divide by $|\mu|$ and then let μ go to 0 which gives $||e_{\mathbf{u},1}|| \leq ||e_{\mathbf{u},2}||$ and therefore $||v_1|| \leq ||v_2||$ for all $v \in \mathcal{E}^{\mathbf{u}} = \mathbb{C}e_{\mathbf{u}}$.

Now for $v = (\delta q, \mu, R) \in \mathcal{E}^{s}$ we this time let, for $t \in \mathbb{C}$ small,

$$\gamma(t) = (\delta g_* + t \delta g, \mu_s(\delta g_* + t \delta g, R_* + t R), R_* + t R) .$$

Since $v_* \in W_{\mathrm{int}}^{\mathrm{s,loc}}$ and μ_s is analytic and therefore continuous we have $\gamma(t) \in W_{\mathrm{int}}^{\mathrm{s,loc}} \subset W^{\mathrm{s,loc}}$ for t small. Lemma 59 thus gives the inequality

$$L^{\frac{3}{4}}||\gamma(t)_2 - v_{*,2}|| \le ||\gamma(t)_1 - v_{*,1}||.$$

Again one can write

$$\gamma(t) = v_* + t(\delta g, D_{v_*, 1} \mu_s[v_1], R) + t\omega(t)$$

where the new function ω satisfies $\omega(t) \to 0$ when $t \to 0$. The previous inequality becomes

$$L^{\frac{3}{4}}||tD_{v_*,1}\mu_{\mathbf{s}}[v_1] + t\omega(t)_2|| \le ||tv_1 + t\omega(t)_1||.$$

Again dividing by |t| for $t \neq 0$ and then letting $t \rightarrow 0$ we get

$$L^{\frac{3}{4}}||D_{v_*,1}\mu_{\mathbf{s}}[v_1]|| \le ||v_1||,$$

i.e., $L^{\frac{3}{4}}||v_2|| \leq ||v_1||$ by the defining equation $v_2 = \mu = D_{v_*,1}\mu_s[v_1]$ of \mathcal{E}^s .

A vector v which satisfies both inequalities $L^{\frac{3}{4}}||v_2|| \leq ||v_1||$ and $||v_1|| \leq ||v_2||$ must clearly satisfy $||v_1|| = ||v_2|| = 0$ because $L^{\frac{3}{4}} > 1$. Namely, v must vanish. This shows $\mathcal{E}^s \cap \mathcal{E}^u = \{0\}$. Since $e_u \neq 0$ is in \mathcal{E}^u we get $e_u \notin \mathcal{E}^s$. This proves the direct sum property since we are considering a complex line spanned by e_u and a complex hyperplane.

Lemma 64. The subspace \mathcal{E}^{s} is invariant by $D_{v_*}RG$.

Proof: For $v = (\delta g, \mu, R) \in \mathcal{E}^{s}$ we again use the curve

$$\gamma(t) = (\delta g_* + t \delta g, \mu_s(\delta g_* + t \delta g, R_* + t R), R_* + t R) .$$

from the proof of the previous lemma and which satisfies $\gamma'(0) = v$. For t small $\gamma(t)$ is well-defined, takes values in $W_{\rm int}^{\rm s,loc} \subset W^{\rm s,loc}$, is analytic in t and belongs to an open set where RG is well-defined and analytic. Thus, $t \mapsto RG(\gamma(t))$ is analytic near t = 0. By Proposition 5, $RG(\gamma(t)) \in W_{\rm int}^{\rm s,loc}$ and therefore $RG(\gamma(t)) = \mu_{\rm s}(RG_1(\gamma(t)))$. We differentiate this at t = 0 using the chain rule and obtain

$$(D_{v_*}RG[\gamma'(0)])_2 = D_{v_*,1}\mu_s [(D_{v_*}RG[\gamma'(0)])_1],$$

i.e.,

$$(D_{v_*}RG[v])_2 = D_{v_*,1}\mu_s [(D_{v_*}RG[v])_1]$$
.

Hence $D_{v_*}RG[v] \in \mathcal{E}^{\mathrm{s}}$ by definition of \mathcal{E}^{s} .

Lemma 65. The subspace \mathcal{E}^{u} is invariant by $D_{v_*}RG$.

Proof: We reuse the curve

$$\gamma(\mu) = (\delta g_{u}(\mu_* + \mu), \mu_* + \mu, R_{u}(\mu_* + \mu))$$

from the proof of Lemma 63. Clearly, $\gamma(\mu)$ lies in $W_{\rm tiny}^{\rm u,loc}$ when μ is small. Therefore $RG(\gamma(\mu))$ is analytic and lies in $W^{\rm u,loc}$ when μ is small because of Proposition 9. By definition of $W^{\rm u,loc}$ we thus have

$$RG_1(\gamma(\mu)) = (\delta g_{\mathbf{u}}(RG_2(\gamma(\mu))), R_{\mathbf{u}}(RG_2(\gamma(\mu))))$$
.

We differentiate at $\mu = 0$ using $\gamma'(0) = e_{\rm u}$ and the chain rule. This gives

$$(D_{v_*}RG[e_{\mathbf{u}}])_1 = (D_{v_*,2}\delta g_{\mathbf{u}}[(D_{v_*}RG[e_{\mathbf{u}}])_2], D_{v_{*,2}}R_{\mathbf{u}}[(D_{v_*}RG[e_{\mathbf{u}}])_2]).$$

In other words $D_{v_*}RG[e_{\mathbf{u}}]$ satisfies the defining equation of $\mathcal{E}^{\mathbf{u}} = \mathbb{C}e_{\mathbf{u}}$ which therefore is invariant by the differential of RG at the fixed point v_* .

Lemma 66. The restriction $D_{v_*}RG|_{\mathcal{E}^u}$ is the multiplication by an eigenvalue α_u which is real and greater than 1. One also has the more precise estimate

$$|\alpha_{\mathrm{u}} - L^{\frac{3+\epsilon}{2}}| \le c_4(\epsilon)$$

where $c_4(\epsilon)$ has been defined in Lemma 58.

Proof: By Lemma 65 and the unidimensional property of \mathcal{E}^{u} we have $D_{v_*}RG[e_{\mathrm{u}}] = \alpha_{\mathrm{u}}e_{\mathrm{u}}$ for some possibly complex α_{u} . However, by Theorem 4, the map RG sends real data to real data. Thus, so does the map on sequences \mathfrak{m} used in Proposition 3. Therefore the corresponding fixed point \vec{u}_* in the space of sequences obtained by iteration starting from the null sequence $\vec{0}$ which is real is also real provided the implicit data $(\delta g_0, R_0)$ is too. As a result the map μ_{s} sends real data to real data. In other words, if $\delta g \in \mathbb{R}$ and if R is a real-valued even function then $\mu_{\mathrm{s}}(\delta g, R) \in \mathbb{R}$. Similar statements also hold for the functions δg_{u} and R_{u} used for the parametrization of the local unstable manifold $W^{\mathrm{u,loc}}$. It is also easy to see that the fixed point v_* is real. Finally, the eigenvalue α_{u} which coincides with the second or μ -component of $D_{v_*}RG[e_{\mathrm{u}}]$ is easily seen to be a real number.

Now again consider the curve $\gamma(\mu)$ as in the proof of Lemma 65. For μ small we have by Lemma 58

$$||RG_2(\gamma(\mu)) - RG_2(\gamma(0)) - L^{\frac{3+\epsilon}{2}}(\gamma(\mu) - \gamma(0))|| \le c_4(\epsilon)||\gamma(\mu) - \gamma(0)||$$
.

We divide by $|\mu| \neq 0$ and then take the limit when μ goes to 0. This results in

$$||(D_{v_*}RG[e_u])_2 - L^{\frac{3+\epsilon}{2}}e_u|| \le c_4(\epsilon)||e_u||,$$

i.e.,

$$|\alpha_{\mathbf{u}} - L^{\frac{3+\epsilon}{2}}| \times ||e_{\mathbf{u},2}|| \le c_4(\epsilon)||e_{\mathbf{u}}||$$
.

Since $e_{\rm u}$ belongs to $\mathcal{E}^{\rm u}$ we have by Lemma 63 the equality $||e_{\rm u},2|| = ||e_{\rm u}||$. Since $e_{\rm u}$ is nonzero we can simplify by $||e_{\rm u}||$ and we end up with the desired estimate. Finally, in the small ϵ regime, $c_4(\epsilon)$ goes to zero which readily implies $\alpha_{\rm u} > 1$.

Lemma 67. The restriction $D_{v_*}RG|_{\mathcal{E}^s}$ is a contraction on the subspace \mathcal{E}^s . More precisely, for every $v \in \mathcal{E}^s$, we have $D_{v_*}RG[v] \in \mathcal{E}^s$ and

$$||D_{v_*}RG[v]|| \le c_1(\epsilon)||v||$$

where $c_1(\epsilon) \in (0,1)$ has been defined in Lemma 58.

Proof: For $v = (\delta g, \mu, R) \in \mathcal{E}^s$ we again use the curve $\gamma(t)$ as in the proof of Lemma 64. For small t we have $\gamma(t) \in W^{s, loc}$. We can thus derive from Lemma 60 the inequality

$$||RG(\gamma(t)) - RG(\gamma(0))|| \le c_1(\epsilon)||\gamma(t) - \gamma(0)||.$$

We divide by $|t| \neq 0$ and take the $t \to 0$ limit in order to obtain

$$||D_{v_*}RG[\gamma'(0)]|| \le c_1(\epsilon)||\gamma'(0)||$$
,

i.e.,

$$||D_{v_*}RG[v]|| \le c_1(\epsilon)||v||$$

since, as one can easily see, $\gamma'(0) = v$. Stability has already been shown in Lemma 64.

8.7. Explicit equivalence of norms. For the needs of §9 we introduce another norm $||\cdot||_{\Diamond}$ on \mathcal{E} . Recall that the latter is the direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ and the original norm $||\cdot||$ behaves well with respect to this decomposition. Indeed, if $v = v_1 + v_2$ is the decomposition of a vector according to this direct sum, we have

$$||v|| = \max(||v_1||, ||v_2||)$$
.

The $||\cdot||_{\diamondsuit}$ is designed in order to satisfy a similar property with respect to the direct sum $\mathcal{E} = \mathcal{E}^{u} \oplus \mathcal{E}^{s}$. Using the notations P_{u} and P_{s} for the corresponding projections on the two subspaces \mathcal{E}^{u} and \mathcal{E}^{s} respectively, we let by definition

$$||v||_{\Diamond} = \max(||P_{\mathbf{u}}(v)||, ||P_{\mathbf{s}}(v)||)$$
.

Lemma 68. We have the explicit equivalence of norms

$$\frac{1}{2}||v|| \le ||v||_{\diamondsuit} \le 5||v||$$

for all $v \in \mathcal{E}$.

Proof: For such a v let us write for simplicity $v^{\rm u} = P_{\rm u}(v)$ and $v^{\rm s} = P_{\rm s}(v)$. We decompose all three vectors v, $v^{\rm u}$ and $v^{\rm s}$ according to the old direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ as

$$v = v_1 + v_2 v^{u} = v_1^{u} + v_2^{u} v^{s} = v_1^{s} + v_2^{s}$$

noting that we must then have the relations $v_1 = v_1^{\rm u} + v_1^{\rm s}$ and $v_2 = v_2^{\rm u} + v_2^{\rm s}$. Armed with this observation and the inequalities in Lemma 63 one easily checks that

$$\begin{split} ||v^{\mathbf{u}}|| &= & \max(||v^{\mathbf{u}}_1||, ||v^{\mathbf{u}}_2||) \\ &= & ||v^{\mathbf{u}}_2|| \\ &= & ||v_2 - v^{\mathbf{s}}_2|| \\ &\leq & ||v_2|| + ||v^{\mathbf{s}}_2|| \\ &\leq & ||v_2|| + L^{-\frac{3}{4}}||v^{\mathbf{s}}_1|| \\ &= & ||v_2|| + L^{-\frac{3}{4}}||v_1 - v^{\mathbf{u}}_1|| \\ &\leq & ||v_2|| + L^{-\frac{3}{4}}||v_1|| + L^{-\frac{3}{4}}||v^{\mathbf{u}}_1|| \\ &\leq & (1 + L^{-\frac{3}{4}})||v|| + L^{-\frac{3}{4}}||v^{\mathbf{u}}|| \end{split}$$

which results in

$$||v^{\mathbf{u}}|| \le \frac{1 + L^{-\frac{3}{4}}}{1 - L^{-\frac{3}{4}}} ||v||.$$

Similarly we have

$$\begin{split} ||v^{\mathrm{s}}|| &= & \max(||v^{\mathrm{s}}_1||, ||v^{\mathrm{s}}_2||) \\ &= & ||v^{\mathrm{s}}_1|| \\ &= & ||v_1 - v^{\mathrm{u}}_1|| \\ &\leq & ||v_1|| + ||v^{\mathrm{u}}_1|| \\ &\leq & ||v_1|| + ||v^{\mathrm{u}}_2|| \\ &= & ||v_1|| + ||v_2 - v^{\mathrm{s}}_2|| \\ &\leq & ||v_1|| + ||v_2|| + ||v^{\mathrm{s}}_2|| \\ &\leq & ||v_1|| + ||v_2|| + L^{-\frac{3}{4}}||v^{\mathrm{s}}_1|| \\ &\leq & 2||v|| + L^{-\frac{3}{4}}||v^{\mathrm{s}}|| \end{split}$$

which entails

$$||v^{\mathbf{s}}|| \le \frac{2}{1 - L^{-\frac{3}{4}}} ||v||$$
.

Since $L^{-\frac{3}{4}} < 1$ we get

$$||v||_{\diamondsuit} = \max(||v^{\mathrm{u}}||, ||v^{\mathrm{s}}||) \le \frac{2}{1 - L^{-\frac{3}{4}}} ||v|| \le \frac{2}{1 - 2^{-\frac{3}{4}}} ||v|| \le 5||v||$$

where we used the simplification $\frac{2}{1-2^{-\frac{3}{4}}} \simeq 4.933... < 5.$

The other inequality is much simpler. Indeed,

$$||v_1|| = ||v_1^{\mathbf{u}} + v_1^{\mathbf{s}}|| \le ||v_1^{\mathbf{u}}|| + ||v_1^{\mathbf{s}}|| \le ||v^{\mathbf{u}}|| + ||v^{\mathbf{s}}|| \le 2||v||_{\Diamond}$$

and

$$||v_2|| = ||v_2^{\mathbf{u}} + v_2^{\mathbf{s}}|| \le ||v_2^{\mathbf{u}}|| + ||v_2^{\mathbf{s}}|| \le ||v^{\mathbf{u}}|| + ||v^{\mathbf{s}}|| \le 2||v||_{\Diamond}$$
.

Hence

$$||v|| = \max(||v_1||, ||v_2||) \le 2||v||_{\triangle}$$

as desired.

9. Partial analytic linearization

The crucial ingredient for the proof of existence of anomalous dimension is an infinite-dimensional generalization of the Kænigs Linearization Theorem in one-dimensional holomorphic dynamics. This is the object of Theorem 5 below. As a preliminary step towards establishing this theorem, we prove some lemmas which give us some explicit control on the second differential of RG.

Lemma 69. In the small ϵ regime we have, for all v such that $||v|| < \frac{1}{4}$,

$$||D_v^2 RG|| \le 17.$$

Proof: In $(\delta g, \mu, R)$ coordinates we have:

$$RG[g, \mu, R] = RG^{\text{explicit}}(\delta g, \mu, R) + RG^{\text{implicit}}(\delta g, \mu, R)$$

where

$$RG^{\text{explicit}}(\delta g, \mu, R) = \begin{pmatrix} (2 - L^{\epsilon})\delta g - A_1 \delta g^2 \\ L^{\frac{3+\epsilon}{2}}\mu - A_2(\bar{g} + \delta g)^2 - A_3(\bar{g} + \delta g)\mu \\ 0 \end{pmatrix}^T$$

and

$$RG^{\text{implicit}}(\delta g, \mu, R) = \begin{pmatrix} \xi_4(g + \delta g, \mu, R) \\ \xi_2(\bar{g} + \delta g, \mu, R) \\ \mathcal{L}^{(\bar{g} + \delta g, \mu)}(R) + \xi_R(\bar{g} + \delta g, \mu, R) \end{pmatrix}^T$$

T meaning transpose.

An easy computation shows

$$D_v^2 R G^{\text{explicit}}[v', v''] = \begin{pmatrix} -2A_1 \delta g' \delta g'' \\ -2A_2 \delta g' \delta g'' - A_3 \delta g' \mu'' - A_3 \mu' \delta g'' \\ 0 \end{pmatrix}^T$$

where $v = (\delta g, \mu, R), v' = (\delta g', \mu', R'), v'' = (\delta g'', \mu'', R'').$

Here D_v^2 , the second differential at v, is seen as a bilinear map acting on pairs of vectors (v', v'').

It is immediate from the definition of the norm $||\cdot||$ that

$$||D_v^2 R G^{\text{explicit}}[v', v'']|| \le 2||v'|| \times ||v''|| \times \max \left[A_{1, \max} \bar{g}^{e_4}, A_{2, \max} \bar{g}^{2e_4 - e_2} + A_{3, \max} \bar{g}^{e_3} \right] .$$

On the other hand if $||v|| < \frac{1}{4}$ and v', v'' are nonzero, we can use Cauchy's formula to write

$$D_v^2 R G^{\text{implicit}}[v', v''] = \frac{1}{(2\pi i)^2} \oint \frac{d\lambda_1}{\lambda_1^2} \oint \frac{d\lambda_2}{\lambda_2^2} R G^{\text{implicit}}(v + \lambda_1 v' + \lambda_2 v'')$$

where the contours of integration are the positively oriented circles given by $|\lambda_1| = \frac{1}{8||v'||}$, $|\lambda_2| = \frac{1}{8||v''||}$

Since clearly $||v + \lambda_1 v' + \lambda_2 v''|| < \frac{1}{2}$ we are in the domain of analyticity specified by the specialization of Theorem 4 in §8.1. Thus we have

$$\begin{aligned} ||RG^{\text{implicit}}(v + \lambda_1 v' + \lambda_2 v'')|| &\leq \max \left[B_4 \bar{g}^{e_R - e_4} ||v + \lambda_1 v' + \lambda_2 v''||, B_2 \bar{g}^{e_R - e_2} ||v + \lambda_1 v' + \lambda_2 v''||, \\ & \frac{1}{2} ||v + \lambda_1 v' + \lambda_2 v''|| + B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta - e_R} \right] \\ &\leq \max \left[\frac{1}{2} B_4 \bar{g}^{e_R - e_4}, \frac{1}{2} B_2 \bar{g}^{e_R - e_2}, \frac{1}{4} + B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta - e_R} \right] \end{aligned}$$

and therefore

$$||D_v^2 R G^{\text{implicit}}[v',v'']|| \leq (8||v'||) \left(8||v''||\right) \times \max \left[\frac{1}{2} B_4 \bar{g}^{e_R-e_4}, \frac{1}{2} B_2 \bar{g}^{e_R-e_2}, \frac{1}{4} + B_{R\xi} \bar{g}^{\frac{11}{4}-3\eta-e_R}\right] .$$

In terms of the norm on bilinear forms induced by the vector space norm $||\cdot||$ we have:

$$\begin{split} ||D_v^2 R G|| \leq & 2 \max \left[A_{1,\max} \bar{g}^{e_4}, A_{2,\max} \bar{g}^{2e_4 - e_2} + A_{3,\max} \bar{g}^{e_4} \right] \\ & + 64 \max \left[\frac{1}{2} B_4 \bar{g}^{e_R - e_4}, \frac{1}{2} B_2 \bar{g}^{e_R - e_2}, \frac{1}{4} + B_{R\xi} \bar{g}^{\frac{11}{4} - 3\eta - e_R} \right] \leq 1 + \frac{64}{4} \leq 17 \; . \end{split}$$

In going to the last line we used the assumption on ϵ being sufficiently small and the inequalities for exponents indicated in §8.1.

For $v \in W^{s,loc}$ and n > 0 we define the continuous linear map

$$T_n(v) = \alpha_{\mathbf{u}}^{-n} D_v R G^n = \alpha_{\mathbf{u}}^{-n} D_{R G^{n-1}(v)} R G \circ \cdots \circ D_{R G(v)} R G \circ D_v R G.$$

It is well defined by the stability of $W^{s,loc}$ which lies in the domain of analyticity of RG.

On the same domain as RG (e.g., the domain $||v|| < \frac{1}{2}$) we define the map H by

$$H(v) = RG(v) - D_{v_*}RG[v]$$
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which is also analytic.

Note that $D_{v_*}RG = \alpha_{\mathbf{u}}P_{\mathbf{u}} + A_{\mathbf{s}}P_{\mathbf{s}}$ where $A_{\mathbf{s}} = D_{v_*}RG|_{\mathcal{E}_{\mathbf{s}}}$. Thus we have that $RG = \alpha_{\mathbf{u}}P_{\mathbf{u}} + A_{\mathbf{s}}P_{\mathbf{s}} + H$.

Lemma 70. In the small ϵ regime:

- (1) If $||v|| < \frac{1}{4}$ then $||D_v H|| \le 17||v v_*||$.
- (2) If v and w satisfy $||v|| < \frac{1}{4}$ and $||v+w|| < \frac{1}{4}$ then

$$||RG(v+w) - RG(v) - D_v RG[w]|| \le \frac{17}{2} ||w||^2$$
.

Proof: Note that by $||D_vH||$ we refer to the norm induced on linear operators on \mathcal{E} by the norm $||\cdot||$ on vectors in \mathcal{E} . Remark that $v_* \in W^{s, \text{loc}}$ implies $||v_*|| \leq \frac{\rho}{3} < \frac{1}{4}$ since $\rho < \frac{1}{12}$. Since the ball of vectors v with $||v|| < \frac{1}{4}$ is convex we can use the mean value theorem to deduce

$$||D_v H - D_{v_*} H|| \le ||v - v_*|| \times \sup_{0 \le t \le 1} ||D_{v_* + t(v - v_*)}^2 H||$$
.

However by construction $D_{v_*}H=0$ and $D^2H=D^2RG$ so Lemma 69 implies

$$||D_v H|| \le 17||v - v_*||.$$

By the mean value theorem, or Taylor's formula with integral remainder, we have

$$||RG(v+w) - RG(v) - D_v RG[w]|| \le \frac{1}{2} ||w||^2 \times \sup_{0 \le t \le 1} ||D_{v+tw}^2 RG||$$

and the desired inequality in Part 2) follows by Lemma 69.

We now give a lemma that shows boundedness for $||T_n(v)||_{\Diamond}$, namely, the operator norm induced by the norm $||\cdot||_{\Diamond}$ on vectors.

Lemma 71. For all $v \in W^{s,loc}$ and all $n \ge 0$ we have

$$||T_n(v)||_{\wedge} \leq \mathcal{C}_1(\epsilon),$$

where
$$C_1(\epsilon) = \exp\left[\frac{85}{\alpha_{\rm u}(1 - c_1(\epsilon))}\right]$$
.

Proof: From $T_n(v) = \alpha_u^{-n} D_v R G^n = \alpha_u^{-n} D_{R G^{n-1}(v)} R G \circ \cdots \circ D_{R G(v)} R G \circ D_v R G$ we immediately get

$$||T_n(v)||_{\diamondsuit} \le \prod_{k=0}^{n-1} ||\alpha_{\mathbf{u}}^{-1} D_{RG^k(v)} RG||_{\diamondsuit}.$$

But

$$D_{RG^k(v)}RG = \alpha_{\mathbf{u}}P_{\mathbf{u}} + A_{\mathbf{s}}P_{\mathbf{s}} + D_{RG^k(v)}H$$

results in

$$||\alpha_{\mathbf{u}}^{-1} D_{RG^{k}(v)} RG||_{\Diamond} \leq ||P_{\mathbf{u}} + \alpha_{\mathbf{u}}^{-1} A_{\mathbf{s}} P_{\mathbf{s}}||_{\Diamond} + ||\alpha_{\mathbf{u}}^{-1} D_{RG^{k}(v)} H||_{\Diamond}.$$

From the definition of $||\cdot||_{\Diamond}$, $P_{\rm u}$, $P_{\rm s}$, $\alpha_{\rm u} > 1$ and the contraction $A_{\rm s}$ we get $||P_{\rm u} + \alpha_{\rm u}^{-1} A_{\rm s} P_{\rm s}||_{\Diamond} \le 1$. Note that this would not work if we had used the $||\cdot||$ operator norm instead.

Hence

$$||T_n(v)||_{\diamondsuit} \le \prod_{k=0}^{n-1} (1 + \alpha_{\mathbf{u}}^{-1} ||D_{RG^k(v)}RG||_{\diamondsuit}).$$

Now from Lemmas 68,70 and 60 we derive

$$\begin{split} ||D_{RG^k(v)}H||_{\diamondsuit} &= \sup_{w \neq 0} \frac{||D_{RG^k(v)}H[w]||_{\diamondsuit}}{||w||_{\diamondsuit}} \leq \frac{5}{\frac{1}{2}} \sup_{w \neq 0} \frac{||D_{RG^k(v)}H[w]||}{||w||} \\ &\leq 170||RG^k(v) - v_*|| \\ &\leq 170c_1(\epsilon)^k ||v - v_*|| \; . \end{split}$$

Since $||v - v_*|| \le ||v|| + ||v_*|| \le \frac{1}{4} + \frac{1}{4} \le \frac{1}{2}$ we have:

$$||T_n(v)||_{\diamondsuit} \le \exp\left(\sum_{k=0}^{n-1} \alpha_{\mathbf{u}}^{-1} \times 170 \times \frac{1}{2} c_1(\epsilon)^k\right) \le C_1(\epsilon)$$

as wanted.

We now extract geometric decay in n from $||P_sT_n(v)||_{\diamondsuit}$.

Lemma 72. For all $v \in W^{s,loc}$ and n > 0

$$||P_{s}T_{n}(v)||_{\Diamond} \leq C_{2}(\epsilon)c_{1}(\epsilon)^{\frac{n}{2}}$$

where
$$C_2(\epsilon) = C_1(\epsilon) \left[1 + \frac{85C_1(\epsilon)}{c_1(\epsilon)(1 - c_1(\epsilon))} \right]$$

Proof: We write

$$P_{s}T_{n}(v) = P_{s}(M + N_{n-1}) \circ \cdots \circ (M + N_{0})$$
where $M = P_{u} + \alpha_{u}^{-1}A_{s}P_{s}$
and $N_{k} = \alpha_{u}^{-1}D_{RG^{k}(v)}H$.

Let m be such that $0 \le m \le n$, then

$$P_{s}T_{n}(v) = P_{s}(M + N_{n-1}) \cdots (M + N_{m})T_{m}(v)$$
.

Then by Lemma 71

$$||P_{s}T_{n}(v)||_{\Diamond} \leq C_{1}(\epsilon)||P_{s}(M+N_{n-1})\circ\cdots\circ(M+N_{m})||_{\Diamond}.$$

Now

$$||P_{s}(M+N_{n-1}) \circ \cdots \circ (M+N_{m})||_{\diamondsuit} \leq ||P_{s}(M+N_{n-1}) \circ \cdots \circ (M+N_{m}) - P_{s}M^{n-m}||_{\diamondsuit} + ||P_{s}M^{n-m}||_{\diamondsuit}$$

$$\leq ||(M+N_{n-1}) \circ \cdots \circ (M+N_{m}) - M^{n-m}||_{\diamondsuit} + ||P_{s}M^{n-m}||_{\diamondsuit}.$$

In going to the last line we used that $||P_s||_{\Diamond} \leq 1$ (in fact one has $||P_s||_{\Diamond} = 1$).

It is easy to see that

$$\begin{split} P_{\mathrm{s}}M^{n-m} = & P_{\mathrm{s}} \left(P_{\mathrm{u}} + \alpha_{\mathrm{u}}^{-(n-m)} A_{\mathrm{s}}^{n-m} P_{\mathrm{s}} \right) \\ = & \alpha_{\mathrm{u}}^{-(n-m)} A_{\mathrm{s}}^{n-m} P_{\mathrm{s}} \ . \end{split}$$

Note that $||w|| = ||w||_{\Diamond}$ if $w \in \mathcal{E}^{s}$. Then one easily gets from Lemma 67

$$||P_{\mathbf{s}}M^{n-m}||_{\diamondsuit} \le \alpha_{\mathbf{u}}^{-(n-m)}c_1(\epsilon)^{n-m}$$
.

On the other hand, we can write

$$(M + N_{n-1}) \circ \cdots \circ (M + N_m) - M^{n-m} = N_{n-1} \circ (M + N_{n-2}) \circ \cdots \circ (M + N_m) + M \circ N_{n-2} \circ (M + N_{n-3}) \circ \cdots \circ (M + N_m)$$

$$\vdots$$

$$+ M^{n-m-2} \circ N_{m+1} \circ (M + N_m) + M^{n-m-1} \circ N_m$$

$$= N_{n-1} \circ T_{n-m-1} (RG^m(v)) + M \circ N_{n-2} \circ T_{n-m-2} (RG^m(v))$$

$$\vdots$$

$$+ M^{n-m-2} \circ N_{m+1} \circ T_1 (RG^m(v)) + M^{n-m-1} \circ N_m \circ T_0 (RG^m(v)) .$$

In the first equality we are expanding the product and ordering terms with respect to the leftmost factor of N_{\bullet} that appears.

Remembering that $||M||_{\Diamond} \leq 1$ and using Lemma 71 one gets

$$||(M+N_{n-1})\circ\cdots\circ(M+N_m)-M^{n-m}||_{\diamondsuit}\leq \mathcal{C}_1(\epsilon)\left[||N_{n-1}||_{\diamondsuit}+\cdots+||N_m||_{\diamondsuit}\right].$$

Thus one has

$$||P_{s}T_{n}(v)||_{\Diamond} \leq \mathcal{C}_{1}(\epsilon)\alpha_{u}^{-(n-m)}c_{1}(\epsilon)^{n-m} + \mathcal{C}_{1}(\epsilon)^{2} \left[||N_{n-1}||_{\Diamond} + \cdots + ||N_{m}||_{\Diamond}\right].$$

We now note that the proof of Lemma 71 tells us that:

(52)
$$||N_k||_{\Diamond} \le 170c_1(\epsilon)^k ||v - v_*|| \le 85c_1(\epsilon)^k.$$

Using this in the previous inequality gives the bound

$$||P_{s}T_{n}(v)||_{\Diamond} \leq C_{1}(\epsilon)[\alpha_{u}^{-1}c_{1}(\epsilon)]^{n-m} + 85C_{1}(\epsilon)^{2}\frac{c_{1}(\epsilon)^{m}}{1 - c_{1}(\epsilon)}.$$

Now take $m = \left\lfloor \frac{n}{2} \right\rfloor$. Then $m > \frac{n}{2} - 1$ and one has

$$c_1(\epsilon)^m \le c_1(\epsilon)^{\frac{n}{2}-1}.$$

One also has $n - m \ge \frac{n}{2}$ so that

$$\left[\alpha_{\mathbf{u}}^{-1}c_{1}(\epsilon)\right]^{n-m} \leq \left[\alpha_{\mathbf{u}}^{-1}c_{1}(\epsilon)\right]^{\frac{n}{2}} \leq c_{1}(\epsilon)^{\frac{n}{2}}$$

Note that we used the fact that $\alpha_{\rm u} > 1$. Now inserting these two bounds into our last bound for $||P_{\rm s}T_n(v)||_{\diamondsuit}$ gives

$$||P_{s}T_{n}(v)||_{\diamondsuit} \leq c_{1}(\epsilon)^{\frac{n}{2}} \left[\mathcal{C}_{1}(\epsilon) + \frac{85\mathcal{C}_{1}(\epsilon)^{2}}{c_{1}(\epsilon)(1 - c_{1}(\epsilon))} \right].$$

Now we bound differences of the form $||T_{n+1}(v) - T_n(v)||_{\diamond}$.

Lemma 73. For $v \in W^{s, loc}$ and $n \ge 0$

$$||T_{n+1}(v) - T_n(v)||_{\diamondsuit} \leq \mathcal{C}_3(\epsilon)c_1(\epsilon)^{\frac{n}{2}}$$
where $\mathcal{C}_3(\epsilon) = 85\mathcal{C}_1(\epsilon) + (1 + \alpha_n^{-1}c_1(\epsilon))\mathcal{C}_2(\epsilon)$.

Proof: Using the same notation as earlier we have

$$T_{n+1}(v) - T_n(v) = (M + N_n - I) \circ T_n(v)$$

= $N_n \circ T_n(v) - (I - M) \circ T_n(v)$,

but $M = P_{\rm u} + \alpha_{\rm u}^{-1} A_{\rm s} P_{\rm s}$ and $I = P_{\rm u} + P_{\rm s}$ so we have

$$I - M = (I - \alpha_{\rm u}^{-1} A_{\rm s}) \circ P_{\rm s} .$$

Hence

$$||(I - M) \circ T_n(v)||_{\diamondsuit} = ||(I - \alpha_{\mathbf{u}}^{-1} A_{\mathbf{s}} P_{\mathbf{s}}) P_{\mathbf{s}} T_n(v)||_{\diamondsuit}$$

$$\leq ||I - \alpha_{\mathbf{u}}^{-1} A_{\mathbf{s}} P_{\mathbf{s}}||_{\diamondsuit} \times ||P_{\mathbf{s}} T_n(v)||_{\diamondsuit}$$

$$\leq (1 + \alpha_{\mathbf{u}}^{-1}) \times (\mathcal{C}_2(\epsilon) c_1(\epsilon)^{\frac{n}{2}}).$$

In going to last line we used Lemma 72 to bound $||P_sT_n(v)||_{\diamondsuit}$. Now by the estimate in (52) and Lemma 71 we have:

$$||N_n T_n(v)||_{\Diamond} \leq 85c_1(\epsilon)^n \times C_1(\epsilon)$$
.

Thus we have the bound

$$||T_{n+1}(v) - T_n(v)||_{\diamondsuit} \le (1 + \alpha_{\mathbf{u}}^{-1} c_1(\epsilon)) \mathcal{C}_2(\epsilon) c_1(\epsilon)^{\frac{n}{2}} + 85 \mathcal{C}_1(\epsilon) c_1(\epsilon)^n$$

and the lemma follows.

The last lemma implies that $T_{\infty}(v) = \lim_{n \to \infty} T_n(v)$ exists and is a continuous linear operator on \mathcal{E} . We also have the following as consequences of Lemma 73 and Lemma 71:

$$P_{s}T_{\infty}(v) = 0$$
 and $||T_{\infty}(v)||_{\Diamond} \leq C_{1}(\epsilon)$.

At the heart of the proof of Theorem 5 is a somewhat involved telescopic sum argument which will reappear many times in the remainder of this section in slightly different forms. It first features in the following lemma which merely ensures that quantities of interest are well defined.

Lemma 74. The following holds in the small ϵ regime. For all $v \in W^{s,loc}$ and $w \in \mathcal{E}$ such that

$$||w|| \le \frac{1}{240\mathcal{C}_1(\epsilon)}$$

we have that, for all integers $n, a, b \ge 0$ such that $a + b \le n$, the following expression is well defined:

$$RG^a \left(RG^b(v) + D_v RG^b[\alpha_{\mathbf{u}}^{-n} w] \right).$$

We also have the bound

$$||RG^a\left(RG^b(v) + D_vRG^b[\alpha_{\mathbf{u}}^{-n}w]\right)|| \le \frac{1}{8}.$$

Proof: Note that since $v \in W^{s,\text{loc}}$ and $W^{s,\text{loc}}$ is stable under RG we have that $RG^k(v)$ is well defined for all $k \geq 0$. We also have that $RG^k(v) \in W^{s,\text{loc}}$ so we by definition get the bound

$$||RG^k(v)|| \le \frac{\rho}{3} \le \frac{1}{36}$$

since $\rho < \frac{1}{12}$.

For $0 \le k \le n$ we have that $D_v RG^k[\alpha_u^{-n}w]$ is well defined and in fact equal to $\alpha_u^{-(n-k)}T_k(v)[w]$. Noting that $\alpha_u > 1$ gives the estimate

$$||D_v RG^k[\alpha_u^{-n}w]|| \le ||T_k(v)[w]|| \le 2||T_k(v)[w]||_{\Diamond}$$

 $\le 2||T_k(v)||_{\Diamond}||w||_{\Diamond}$
 $\le 2C_1(\epsilon) \times 5||w||$.

Thus we have that

$$||D_v R G^k[\alpha_{\mathbf{u}}^{-n} w]|| \le 10 C_1(\epsilon) ||w|| \le \frac{1}{24}.$$

We prove the assertion of the lemma by looking at various cases while applying induction on a + b.

For our first case assume a = 0. We are then looking at

$$RG^{a}(RG^{b}(v) + D_{v}RG^{b}[\alpha_{u}^{-n}w]) = RG^{b}(v) + D_{v}RG^{b}[\alpha_{u}^{-n}w].$$

The right hand side is well defined by the previous remarks on the two pieces composing it. We also have the bound

$$||RG^b(v) + D_v RG^b[\alpha_u^{-n}w]|| \le \frac{1}{24} + \frac{1}{24} = \frac{1}{12}.$$

This proves the claims of our lemma whenever a=0 and also covers the induction base case a+b=0.

For the second case assume a + b > 0. Note that if a = 0 then we are again under the previous case for which the assertions has been proved. Therefore we assume a > 0. By our induction hypothesis we have that

$$RG^{a-1}\left(RG^b(v) + D_vRG^b[\alpha_u^{-n}w]\right)$$
 is well defined

and we also have the bound

$$||RG^{a-1}\left(RG^b(v) + D_v RG^b[\alpha_{\mathbf{u}}^{-n}w]\right)|| \le \frac{1}{8}.$$

This places it within the domain of RG (which is defined on vectors of norm less than $\frac{1}{2}$).

Thus

$$RG^a\left(RG^b(v) + D_vRG^b[\alpha_{\mathbf{u}}^{-n}w]\right)$$
 is well defined.

By the same argument the following quantities are also well defined:

$$RG^{a-1} \left(RG^{b+1}(v) + D_v RG^{b+1} [\alpha_{\mathbf{u}}^{-n} w] \right),$$

$$RG^{a-2} \left(RG^{b+2}(v) + D_v RG^{b+2} [\alpha_{\mathbf{u}}^{-n} w] \right),$$

$$\vdots$$

$$RG \left(RG^{b+a-1}(v) + D_v RG^{b+a-1} [\alpha_{\mathbf{u}}^{-n} w] \right).$$

We write the telescopic sum

(53)
$$RG^{a}\left(RG^{b}(v) + D_{v}RG^{b}[\alpha_{\mathbf{u}}^{-1}w]\right) = RG^{a+b}(v) + D_{v}RG^{a+b}[\alpha_{\mathbf{u}}^{-n}w]$$

$$+ \sum_{j=0}^{a-1} \left\{ RG^{j+1}\left(RG^{b+a-j-1}(v) + D_{v}RG^{b+a-j-1}_{v}[\alpha_{\mathbf{u}}^{-n}w]\right) - RG^{j}\left(RG^{b+a-j}(v) + D_{v}RG^{b+a-j}[\alpha_{\mathbf{u}}^{-n}w]\right) \right\} .$$

Note that Part 1) and Part 3) of Lemma 58 can be combined into a single Lipschitz estimate

$$||RG(w') - RG(w'')|| \le c_3(\epsilon)||w' - w''||$$

for all w', w'' in \mathcal{E} such that $||w'||, ||w''|| \leq \frac{1}{8}$.

For $j \geq 1$ within the telescoping sum our induction hypothesis tells us that we can repeatedly use our Lipschitz estimate j times. At every step the arguments of the map RG will be within $\bar{B}\left(0,\frac{1}{8}\right) \subset \mathcal{E}$. Thus we have:

$$\begin{aligned} &||RG^{j+1}\left(RG^{b+a-j-1}(v)+D_{v}RG^{b+a-j}[\alpha_{\mathbf{u}}^{-n}w]\right)-RG^{j}\left(RG^{b+a-j-1}(v)+D_{v}RG^{b+a-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right)||\\ &\leq c_{3}(\epsilon)^{j}||RG\left(RG^{b+a-j}(v)+D_{v}RG^{b+a-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right)-\left(RG^{b+a-j}(v)+D_{v}RG^{b+a-j}[\alpha_{\mathbf{u}}^{-n}w]\right)|| \ .\end{aligned}$$

Note that the bound above holds for j = 0 as well so we can apply this estimate to all the terms of the telescoping sum:

$$\begin{split} \left\| \sum_{j=0}^{a-1} \left\{ RG^{j+1} \left(RG^{b+a-j-1}(v) + D_v RG_v^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right) - RG^j \left(RG^{b+a-j}(v) + D_v RG^{b+a-j} [\alpha_{\mathbf{u}}^{-n} w] \right) \right\} \right\| \\ \leq \sum_{j=0}^{a-1} c_3(\epsilon)^j \left\| \left| RG \left(RG^{b+a-j-1}(v) + D_v RG^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right) - RG \left(RG^{b+a-j-1}(v) \right) - D_{RG^{b+a-j-1}(v)} RG \left[D_v RG^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right] \right\|. \end{split}$$

Above we used the chain rule for Frechet differentials.

By the earlier remarks we know that $||RG^{b+a-j-1}(v)|| \le \frac{1}{24}$ and $||D_vRG_{b+a-j-1}[\alpha_{\mathbf{u}}^{-n}w]|| \le \frac{1}{24}$. Thus the quantities appearing in the sum above can be estimated using Lemma 70 which tells us that

$$\begin{aligned} & \left| \left| RG \left(RG^{b+a-j-1}(v) + D_v RG^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right) \right. \\ & \left. - RG \left(RG^{b+a-j-1}(v) \right) - D_{RG^{b+a-j-1}(v)} RG \left[D_v RG^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right] \right| \right| \\ & \leq \frac{17}{2} \left| \left| D_v RG^{b+a-j-1} [\alpha_{\mathbf{u}}^{-n} w] \right| \right|^2 \\ & = \frac{17}{2} \left[\alpha_{\mathbf{u}}^{-n+(b+a-j-1)} \left| \left| T_{b+a-j-1}(v) [w] \right| \right| \right]^2 \\ & \leq \frac{17}{2} \left[\alpha_{\mathbf{u}}^{-n+(b+a-j-1)} \times 10 \times \left| \left| T_{b+a-j-1}(v) \right| \left| \phi \right| \left| w \right| \right| \right]^2 \\ & \leq 50 \times 17 \alpha_{\mathbf{u}}^{-2(n-a-b+j+1)} \mathcal{C}_1(\epsilon)^2 ||w||^2 \, . \end{aligned}$$

Inserting all of our bounds into (53) yields the inequality

$$||RG^{a}\left(RG^{b}(v) + D_{v}RG^{b}\left[\alpha_{u}^{-n}w\right]\right)|| \leq \frac{1}{24} + \frac{1}{24} + \sum_{i=0}^{a-1} \left[c_{3}(\epsilon)^{j}850\alpha_{u}^{-2(j+1)}C_{1}(\epsilon)^{2}||w||^{2}\right].$$

Above we used that $\alpha_{\rm u} > 1$ and $n \ge a + b$ which means $\alpha_{\rm u}^{-2(n-a-b)} \le 1$.

We would like to sum the geometric series but for this we need to show that $\alpha_{\rm u}^{-2}c_3(\epsilon) < 1$ which we now do.

We have that $\alpha_{\rm u} \geq L^{\frac{3+\epsilon}{2}} - c_4(\epsilon) > 0$ for ϵ small by Lemma 66 and $c_3(\epsilon) = L^{\frac{3+\epsilon}{2}} + c_4(\epsilon)$ by the definitions given in Lemma 58. We also know that $\lim_{\epsilon \to 0} c_4(\epsilon) = 0$ so

$$\alpha_{\rm u}^2 - c_3(\epsilon) \ge (L^{\frac{3+\epsilon}{2}} - c_4(\epsilon))^2 - (L^{\frac{3+\epsilon}{2}} + c_4(\epsilon)) \to L^3 - L^{\frac{3}{2}}$$

when $\epsilon \to 0$.

We note that for $L \ge 2$ one has $L^3 - L^{\frac{3}{2}} > \frac{1}{2}L^3$. It then follows that for ϵ sufficiently small one has

(54)
$$\alpha_{\mathbf{u}}^2 - c_3(\epsilon) \ge \frac{1}{2}L^3.$$

We have shown $\alpha_{11}^{-2}c_3(\epsilon) < 1$. Thus

$$||RG^{a}\left(RG^{b}(v) + D_{v}RG^{b}\left[\alpha_{\mathbf{u}}^{-n}w\right]\right)|| \leq \frac{1}{24} + \frac{1}{24} + 850C_{1}(\epsilon)^{2}||w||^{2} \frac{\alpha_{\mathbf{u}}^{-2}}{1 - \alpha_{\mathbf{u}}^{-2}c_{3}(\epsilon)}$$
$$\leq \frac{1}{24} + \frac{1}{24} + 850 \times \left(\frac{1}{240}\right)^{2} \times \frac{2}{L^{3}} < \frac{1}{24} + \frac{1}{24} + \frac{1}{24} = \frac{1}{8}.$$

In going to the last line we have used the lemma's assumption that $||w|| \leq \frac{1}{240C_1(\epsilon)}$ along with the fact that $\frac{\alpha_{\rm u}^{-2}}{1-\alpha_{\rm u}^{-2}c_3(\epsilon)}=\frac{1}{\alpha_{\rm u}^2-c_3(\epsilon)}\leq \frac{2}{L^3}$. This proves the bound asserted by the lemma.

Note that the constant $C_1(\epsilon)$ featuring in the domain definition for w is a very bad one. Indeed it essentially blows up as $\exp(\epsilon^{-1})$ when $\epsilon \to 0$. This is because the previous lemma uniformly covers all starting points v in the local stable manifold $W^{s,loc}$ on which the convergence to the fixed point v_* is very slow. In the special case $v = v_*$ and $w \in \mathcal{E}^u$ one can obtain significantly better estimates which is what we do next.

Lemma 75. The following holds in the small ϵ regime. For all $w \in \mathcal{E}^{u}$ such that

$$||w|| \le \frac{1}{24}$$

we have that, for all integers $n, a, b \ge 0$ such that $a + b \le n$, the following expression is well defined:

$$RG^a\left(RG^b(v_*) + D_{v_*}RG^b[\alpha_{11}^{-n}w]\right)$$
.

We also have the bound

$$||RG^a(RG^b(v_*) + D_{v_*}RG^b[\alpha_{\mathbf{u}}^{-n}w])|| \le \frac{1}{8}.$$

Proof: The proof is exactly the same as that of the Lemma 74 except for the following modifications. When estimating $D_v RG^k[\alpha_{"}^{-n}w]$ we now have the tremendous simplification

$$D_{v_*}RG^k[\alpha_{\mathbf{u}}^{-n}w] = \alpha_{\mathbf{u}}^{-n}(D_{v_*}RG)^k[w] = \alpha_{\mathbf{u}}^{k-n}w$$

by Lemma 66 and the hypothesis $w \in \mathcal{E}^{\mathrm{u}}$. When we estimated the quantity $||T_{b+a-j-1}(v)[w]||$ we had to pay a factor of $10C_1(\epsilon)$. Now we simply note that $T_{b+a-j-1}(v)[w]=w$ and therefore we obtain the same bounds without this bad factor.

We now attack the main linearization theorem.

We proceed by showing that for $v \in W^{s,loc}$ and w sufficiently small the following sum converges:

$$\sum_{n=0}^{\infty} ||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w)||$$

as results for the next lemma

Lemma 76. In the small ϵ regime, for all $v \in W^{s,loc}$ and all w with $||w|| \leq \frac{1}{240C_1(\epsilon)}$

we have, for all $n \geq 0$,

$$||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w)|| \le C_{4}(\epsilon)c_{1}(\epsilon)^{\frac{n}{4}}$$

where
$$C_4(\epsilon) = \frac{\alpha_u^2}{12c_3(\epsilon)} + \frac{1}{2} + \frac{C_3(\epsilon)}{24C_1(\epsilon)}$$

Proof: By Lemma 74 with b=0 and a=n the quantities involved in the sum are all well defined if $||w|| \le \frac{1}{240C_1(\epsilon)}$.

We now proceed similarly to the proof of Lemma 74. For $n \ge 1$ and $0 \le k \le n$ we write the telescoping sum

$$\begin{split} RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w) - RG^{k}(RG^{n-k}(v) + D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w]) \\ &= \sum_{j=k}^{n-1} \left\{ RG^{j+1} \left(RG^{n-j-1}(v) + D_{v}RG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w] \right) - RG^{j} \left(RG^{n-j}(v) + D_{v}RG^{n-j}[\alpha_{\mathbf{u}}^{-n}w] \right) \right\} \; . \end{split}$$

Since $k \leq j \leq n-1$ the arguments of RG remain small enough to allow for repeated use of Lipschitz estimates giving the bound:

$$\begin{split} &||RG^{j+1}\left(RG^{n-j-1}(v) + D_vRG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right) - RG^{j}\left(RG^{n-j}(v) + D_vRG^{n-j}[\alpha_{\mathbf{u}}^{-n}w]\right)||\\ &\leq c_3(\epsilon)^{j}||RG\left(RG^{n-j-1}(v) + D_vRG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right) - RG^{n-j}(v) - D_vRG^{n-j}[\alpha_{\mathbf{u}}^{-n}w]||\\ &= c_3(\epsilon)^{j}||RG\left(RG^{n-j-1}(v) + D_vRG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right) - RG^{n-j}(v) - D_{RG^{n-j-1}(v)}RG\left[D_vRG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w]\right]||\\ &\leq c_3(\epsilon)^{j}\frac{17}{2}||D_vRG^{n-j-1}[\alpha_{\mathbf{u}}^{-n}w]||^2\\ &\leq c_3(\epsilon)^{j}\frac{17}{2}\left[\alpha_{\mathbf{u}}^{-(j+1)}||T_{n-j-1}(v)[w]||\right]^2\\ &\leq 850\mathcal{C}_1(\epsilon)^2||w||^2\times\alpha_{\mathbf{u}}^{-2}\times\left[c_3(\epsilon)\alpha_{\mathbf{u}}^{-2}\right]^{j}\;. \end{split}$$

Using this estimate in the earlier telescoping sum expression gives the bound

$$\begin{aligned} ||RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w) - RG^{k}(RG^{n-k}(v) + D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w])|| \\ &\leq 850C_{1}(\epsilon)^{2}||w||^{2}\alpha_{u}^{-2}\sum_{j=k}^{n-1}\left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{j} \\ &\leq 850C_{1}(\epsilon)^{2}||w||^{2}\left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{k}\frac{\alpha_{\mathbf{u}}^{-2}}{1 - \alpha_{\mathbf{u}}^{-2}c_{3}(\epsilon)} \leq \frac{1}{24}\left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{k} .\end{aligned}$$

For the last inequality we proceeded just as we did at the end of the proof of Lemma 74. Note that this bound holds for any $n, k \ge 0$ with $k \le n$ so we get a valid estimate if we replace n with n + 1:

$$||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{k}(RG^{n+1-k}(v) + D_{v}RG^{n+1-k}[\alpha_{\mathbf{u}}^{-(n+1)}w])|| \leq \frac{1}{24} \left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{k}.$$

As a result, the triangle inequality gives

$$\begin{split} ||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w)|| \\ &\leq \frac{1}{12} \left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2} \right)^{k} \\ &+ \left| \left| RG^{k} \left(RG^{n+1-k}(v) + D_{v}RG^{n+1-k}[\alpha_{\mathbf{u}}^{-(n+1)}w] \right) - RG^{k} \left(RG^{n-k}(v) + D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w] \right) \right| \right| \; . \end{split}$$

We again repeatedly used the Lipschitz estimate to bound the second term on the bottom line. Indeed, Lemma 74 guarantees that the arguments of the outermost RG's remain in the domain of validity of our Lipschitz estimate. This gives the bound

$$\left| \left| RG^{k} \left(RG^{n+1-k}(v) + D_{v}RG^{n+1-k}[\alpha_{\mathbf{u}}^{-(n+1)}w] \right) - RG^{k} \left(RG^{n-k}(v) + D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w] \right) \right| \right|$$

$$\leq c_{3}(\epsilon)^{k} ||RG^{n+1-k}(v) + D_{v}RG^{n+1-k}[\alpha_{\mathbf{u}}^{-(n+1)}w] - RG^{n-k}(v) - D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w]||$$

$$\leq c_{3}(\epsilon)^{k} \left[||RG^{n+1-k}(v) - RG^{n-k}(v)|| + ||D_{v}RG^{n+1-k}[\alpha_{\mathbf{u}}^{-(n+1)}w] - D_{v}RG^{n-k}[\alpha_{\mathbf{u}}^{-n}w]||\right]$$

$$= c_{3}(\epsilon)^{k} \left[||RG^{n-k}(RG(v)) - RG^{n-k}(v)|| + \alpha_{\mathbf{u}}^{-k}||T_{n-k+1}(v)[w] - T_{n-k}(v)[w]||\right]$$

$$\leq c_{3}(\epsilon)^{k} \left[c_{1}(\epsilon)^{n-k} ||RG(v) - v|| + \alpha_{\mathbf{u}}^{-k} \times 10 \times ||T_{n-k+1}(v) - T_{n-k}(v)||_{\diamondsuit}||w||\right]$$

$$\leq \frac{1}{2}c_{3}(\epsilon)^{k} c_{1}(\epsilon)^{n-k} + c_{3}(\epsilon)^{k} \alpha_{\mathbf{u}}^{-k} \times 10 \times C_{3}(\epsilon) c_{1}(\epsilon)^{\frac{n-k}{2}} \times \frac{1}{240C_{s}(\epsilon)}.$$

In going to the fifth line from the fourth line we used Lemma 60 since both v and RG(v) are in $W^{\rm s,loc}$. In going from the fifth line to the last line we used $||RG(v)-v|| \leq ||RG(v)|| + ||v|| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ when bounding the first term. For the second term we used Lemma 73 and our working assumption on the size of ||w||.

We now arrive at

$$||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w)|| \leq \frac{1}{2} \left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{k} + \frac{1}{2}c_{3}(\epsilon)^{k}c_{1}(\epsilon)^{n-k} + \frac{C_{3}(\epsilon)}{24C_{1}(\epsilon)}c_{3}(\epsilon)^{k}\alpha_{u}^{-k}c_{1}(\epsilon)^{\frac{n-k}{2}}\right)||C_{3}(\epsilon)^{k}\alpha_{\mathbf{u}}^{-k}c_{1}(\epsilon)^{n-k} + \frac{C_{3}(\epsilon)^{k}\alpha_{\mathbf{u}}^{-k}c_{1}(\epsilon)^{\frac{n-k}{2}}}{24C_{1}(\epsilon)^{n-k}} + \frac{C_{3}(\epsilon)^{n-k}\alpha_{\mathbf{u}}^{-k}c_{1}(\epsilon)^{\frac{n-k}{2}}}{24C_{1}(\epsilon)^{n-k}} + \frac{C_{3}(\epsilon)^{n-k}\alpha_{\mathbf{u}}^{-k}c_{1}(\epsilon)^{\frac{n-k}{2}}}{24C_{1}(\epsilon)^{n-k}} + \frac{C_{3}(\epsilon)^{n-k}\alpha_{\mathbf{u}}^{-k}c_{1}(\epsilon)^{n-k}}{24C_{1}(\epsilon)^{n-k}} + \frac{C_{3}(\epsilon)^{n-k}\alpha_{\mathbf{u}$$

We now choose k adequately as a function of n. We will set

$$k = \lfloor \sigma n \rfloor$$
 for suitable $\sigma \in [0, 1]$.

We then have that $0 \le k \le n$. From previous arguments we know that $0 < c_3(\epsilon)\alpha_{\rm u}^{-2} < 1$. Then since $k > \sigma n - 1$ we have

$$(c_3(\epsilon)\alpha_{\mathbf{u}}^{-2})^k \le (c_3(\epsilon)\alpha_{\mathbf{u}}^{-2})^{\sigma n-1}.$$

Since $c_3(\epsilon) \geq 1$, $c_1(\epsilon) < 1$, and $k \leq \sigma n$ we have

$$c_3(\epsilon)^k c_1(\epsilon)^{n-k} < c_3(\epsilon)^{\sigma n} c_1(\epsilon)^{n-\sigma n}$$
.

Since $c_3(\epsilon)\alpha_{\rm u}^{-1} \geq 1$ which is a consequence of $\alpha_{\rm u} \leq L^{\frac{3+\epsilon}{2}} + c_4(\epsilon) = c_3(\epsilon)$ and since $\sqrt{c_1(\epsilon)} \leq 1$, the inequality $k \leq \sigma n$ implies

$$c_3(\epsilon)^k \alpha_{ij}^{-k} c_1(\epsilon)^{\frac{n-k}{2}} \le c_3(\epsilon)^{\sigma n} \alpha_{ij}^{-\sigma n} c_1(\epsilon)^{\frac{n-\sigma n}{2}}.$$

Using these three statements in our previous bound we see that

$$||RG^{n+1}(v + \alpha_{\mathbf{u}}^{-(n+1)}w) - RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w)|| \leq \left[\frac{\alpha_{\mathbf{u}}^{2}}{12c_{3}(\epsilon)} + \frac{1}{2} + \frac{\mathcal{C}_{3}(\epsilon)}{24\mathcal{C}_{1}(\epsilon)}\right] \times \gamma^{n}$$
with $\gamma = \max\left[c_{3}(\epsilon)^{\sigma}\alpha_{\mathbf{u}}^{-2\sigma}, c_{3}(\epsilon)^{\sigma}c_{1}(\epsilon)^{1-\sigma}, c_{3}(\epsilon)^{\sigma}\alpha_{\mathbf{u}}^{-\sigma}c_{1}(\epsilon)^{\frac{1-\sigma}{2}}\right]$.

Recall that for ϵ small we have

$$\alpha_{\rm u}^2 > c_3(\epsilon) \ge \alpha_{\rm u} > 1 > c_1(\epsilon) > 0$$
.

Therefore when $\sigma \in [0, 1]$ one has

$$c_3(\epsilon)^{\sigma} \alpha_{\mathbf{u}}^{-\sigma} c_1(\epsilon)^{\frac{1-\sigma}{2}} \leq \left[c_3(\epsilon)^{\sigma} c_1(\epsilon)^{1-\sigma} \right]^{\frac{1}{2}}.$$

Hence

$$\gamma \leq \max \left[c_3(\epsilon)^{\sigma} \alpha_{\mathbf{u}}^{-2\sigma}, c_3(\epsilon)^{\sigma} c_1(\epsilon)^{1-\sigma}, \left(c_3(\epsilon)^{\sigma} c_1(\epsilon)^{1-\sigma} \right)^{\frac{1}{2}} \right] < 1.$$

The last inequality holds provided $c_3(\epsilon)^{\sigma}\alpha_{\mathrm{u}}^{-2\sigma} < 1$ and $c_3(\epsilon)^{\sigma}c_1(\epsilon)^{1-\sigma} < 1$ which can be guaranteed by choosing σ so that

$$0 < \sigma < \frac{-\log c_1(\epsilon)}{\log c_3(\epsilon) - \log c_1(\epsilon)}.$$

For simplicity, we pick

$$\sigma = \frac{1}{2} \times \frac{-\log c_1(\epsilon)}{\log c_3(\epsilon) - \log c_1(\epsilon)} \in \left(0, \frac{1}{2}\right) .$$

We then have

$$c_3(\epsilon)^{\sigma} c_1(\epsilon)^{1-\sigma} = c_1(\epsilon)^{\frac{1}{2}}.$$

So

$$\gamma \leq \max \left[c_3(\epsilon)^\sigma \alpha_{\mathrm{u}}^{-2\sigma}, c_1(\epsilon)^{\frac{1}{2}}, c_1(\epsilon)^{\frac{1}{4}} \right] = c_1(\epsilon)^{\frac{1}{4}} \text{ in the small } \epsilon \text{ regime.}$$

Indeed the strict inequality

$$c_3(\epsilon)^{\sigma}\alpha_{\mathrm{u}}^{-3\sigma} < c_1(\epsilon)^{\frac{1}{4}} = c_3(\epsilon)^{\frac{\sigma}{2}}c_1(\epsilon)^{\frac{1-\sigma}{2}}$$
 is successively equivalent to

$$\begin{split} \alpha_{\mathrm{u}}^{-2} &< c_3(\epsilon)^{-\frac{\sigma}{\frac{1}{2}-\sigma} \times \frac{1-\sigma}{2} - \frac{\sigma}{2}} ,\\ \alpha_{\mathrm{u}}^2 &> c_3(\epsilon)^{\frac{1-\sigma}{1-2\sigma} + \frac{1}{2}} ,\\ 2\log \ \alpha_{\mathrm{u}} &> \left(1 + \frac{1}{2(1-2\sigma)}\right) \log \ c_3(\epsilon) . \end{split}$$

However, Lemma 66 gives

$$2 \log \alpha_{\rm u} \to 3 \log L \text{ when } \epsilon \to 0$$

while

$$\left(1 + \frac{1}{2(1 - 2\sigma)}\log c_3(\epsilon)\right) = \frac{3}{2}\log c_3(\epsilon) - \frac{1}{2}\log c_1(\epsilon) \to \frac{9}{4}\log L < 3\log L \text{ when } \epsilon \to 0$$

as is readily checked from the definitions in Lemma 58. Therefore $c_3(\epsilon)\alpha_{\mathrm{u}}^{-2\sigma} < c_1(\epsilon)^{\frac{1}{4}}$ in the small ϵ regime and the result is proved.

The last lemma also has an improved version in the special case $v = v_*, w \in \mathcal{E}^{\mathrm{u}}$.

Lemma 77. In the small ϵ regime, for all $w \in \mathcal{E}^{\mathrm{u}}$ with $||w|| \leq \frac{1}{24}$

we have, for all $n \geq 0$,

$$||RG^{n+1}(v_* + \alpha_{u}^{-(n+1)}w) - RG^n(v_* + \alpha_{u}^{-n}w)|| \le C'_4(\epsilon)c_1(\epsilon)^{\frac{n}{4}}$$

where
$$C'_4(\epsilon) = \frac{\alpha_{\rm u}^2}{12c_3(\epsilon)} + \frac{1}{2}$$
.

Proof: The proof is the same as that of Lemma 76 except that we use Lemma 75 instead of Lemma 74. We bound $||T_{n-j-1}(v_*)[w]||$ simply by $||w|| \leq \frac{1}{24} < \frac{2}{17}$ and do not pay a bad factor $10C_1(\epsilon)$. Also $T_{n-k+1}(v_*)[w] - T_{n-k}(v_*)[w] = 0$ so the new constant $C_4'(\epsilon)$ does not have the third term of $C_4(\epsilon)$.

We now are in a position to state and prove our partial linearization theorem.

Theorem 5. For $v \in W^{s,loc}$ and $||w|| \leq \frac{1}{240C_1(\epsilon)}$ the quantity

$$\Psi(v,w) = \lim_{n \to \infty} RG^n(v + \alpha_{\mathbf{u}}^{-n}w)$$
 exists in \mathcal{E}

and defines a function of (v, w) with the following properties:

- (1) Ψ is continuous in the domain $v \in W^{s,loc}$ and $||w|| \leq \frac{1}{240C_1(\epsilon)}$. Over this set one has the uniform bound $||\Psi(v,w)|| \leq \frac{1}{8}$.
- (2) Ψ is jointly analytic in v_1 and w in the domain $||v_1|| < \frac{\rho}{13}$, $||w|| < \frac{1}{240C_1(\epsilon)}$ where we have implied the use of the parameterization

$$v_1 \mapsto v = (v_1, v_2) = (v_1, \mu_s(v_1)) \text{ of } W_{\text{int}}^{s, \text{loc}}.$$

(3) For all $v \in W^{s,loc}$, w such that $||w|| \leq \frac{1}{240C_1(\epsilon)\alpha_u}$ we have the intertwining relation

$$RG(\Psi(v, w)) = \Psi(v, \alpha_{\mathbf{u}}w).$$

(4) For all $v \in W^{s,loc}$, w such that $||w|| \leq \frac{1}{2400C_1(\epsilon)^2}$, and all integers $q \geq 0$, we have

$$\Psi(v, w) = \Psi(RG^q(v), T_q(v)[w]).$$

(5) For all $v \in W^{s,loc}$ and w such that $||w|| \leq \frac{1}{2400C_1(\epsilon)^2}$, we have

$$\Psi(v, w) = \Psi(v_*, T_{\infty}(v)[w]).$$

Proof: Parts 1) and 2) are immediate consequences of the $\frac{1}{8}$ bound in Lemma 74 and the uniform absolute convergence proved in Lemma 76.

For Part 3) note

$$\Psi(v, \alpha_{\mathbf{u}}w) = \lim_{n \to \infty} RG\left(RG^{n-1}(v + \alpha_{\mathbf{u}}^{-(n-1)}w)\right)$$

and the continuity of RG in the ball of radius $\frac{1}{8}$.

For Part 4) we use the $c_3(\epsilon)$ Lipschitz estimate and Lemma 70 which are justified by Lemma 74 in order to write for fixed q and $n \ge 0$:

$$\begin{split} &||RG^{n+q}(v+\alpha_{\mathbf{u}}^{-(n+q)}w)-RG^{n}(RG^{q}(v)+D_{v}RG^{q}[\alpha_{\mathbf{u}}^{-(n+q)}w])||\\ &\leq c_{3}(\epsilon)^{n}||RG^{q}(v+\alpha_{\mathbf{u}}^{-(n+q)}w)-RG^{q}(v)-D_{v}RG^{q}[\alpha_{\mathbf{u}}^{-(n+q)}w]||\\ &\leq c_{3}(\epsilon)^{n}\sum_{j=0}^{q-1}||RG^{j+1}\left(RG^{q-(j+1)}(v)+D_{v}RG^{q-(j+1)}[\alpha_{\mathbf{u}}^{-(n+q)}w]\right)-RG^{j}\left(RG^{q-j}(v)+D_{v}RG^{q-j}[\alpha_{\mathbf{u}}^{-(n+q)}w]\right)||\\ &\leq c_{3}(\epsilon)^{n}\sum_{j=0}^{q-1}c_{3}(\epsilon)^{j}||RG(RG^{q-(j+1)}(v)+D_{v}RG^{q-(j+1)}[\alpha_{\mathbf{u}}^{-(n+q)}w])-RG^{q-j}(v)-D_{v}RG^{q-j}[\alpha_{\mathbf{u}}^{-(n+q)}w]||\\ &=c_{3}(\epsilon)^{n}\sum_{j=0}^{q-1}c_{3}(\epsilon)^{j}||RG(RG^{q-(j+1)}(v)+D_{v}RG^{q-(j+1)}[\alpha_{\mathbf{u}}^{-(n+q)}w])-RG^{q-j}(v)\\ &-D_{RG^{q-(j+1)}(v)}RG\left[D_{v}RG^{q-(j+1)}[\alpha_{\mathbf{u}}^{-(n+q)}w]\right]||\\ &\leq c_{3}(\epsilon)^{n}\sum_{j=0}^{q-1}c_{3}(\epsilon)^{j}\times\frac{17}{2}\times||D_{v}RG^{q-(j+1)}[\alpha_{\mathbf{u}}^{-(n+q)}w]||^{2}\\ &\leq c_{3}(\epsilon)^{n}\sum_{j=0}^{q-1}c_{3}(\epsilon)^{j}\times\frac{17}{2}\times\left[\alpha_{\mathbf{u}}^{-(n+j+1)}\times10\times||T_{q-(j+1)}(v)||_{\Diamond}\times||w||\right]^{2}\;. \end{split}$$

We now note that we can extract a factor of $(c_3(\epsilon)\alpha_{\mathrm{u}}^{-2})^n$ which will drive the expression to 0 as $n \to \infty$.

$$\begin{split} \Psi(v,w) &= \lim_{n \to \infty} RG^{n+q}[v + \alpha_{\mathbf{u}}^{-(n+q)}w] \\ &= \lim_{n \to \infty} RG^{n}\left[RG^{q}(v) + \alpha_{\mathbf{u}}^{-n}\left(\alpha_{\mathbf{u}}^{-q}D_{v}RG^{q}[w]\right)\right] \\ &= &\Psi(RG^{q}(v), T_{q}(v)[w]) \end{split}$$

since $\alpha_{\mathbf{u}}^{-q} D_{v} R G^{q}[w] = T_{q}(v)[w]$ has norm bounded by

$$||T_q(v)|| \times ||w|| \le 10 ||T_q(v)||_{\diamondsuit} \times \frac{1}{2400C_1(\epsilon)^2} \le \frac{1}{240C_1(\epsilon)}$$

from Lemma 71.

Part 5) follows from Part (4), Lemmas 60 and 73 when taking the $q \to \infty$ limit.

At this point it could seem possible that we went through all this trouble in order to define a conjugation Ψ which in fact is identically zero. Our next theorem will rule this out thanks to the consideration of the special case $v = v_*$ and $w \in \mathcal{E}^u$.

Theorem 6. On the domain $||w|| < \frac{1}{24}$ of the one-dimensional space \mathcal{E}^{u} the limit

$$\lim_{n\to\infty} RG^n(v_* + \alpha_{\mathbf{u}}^{-n}w)$$

exists and defines an analytic function of w which will be denoted by $\Psi(v_*,w)$ since it coincides with the previous one on the common domain of definition. On the domain $B\left(0,\frac{1}{24}\right)\cap\mathcal{E}^u$, this function satisfies the bound

$$||\Psi(v_*, w)|| \le \frac{1}{8}$$

as well as

$$||\Psi(v_*, w) - v_* - w|| \le \frac{17}{8} ||w||^2$$
.

In particular, the differential with respect to w at w=0 is the identity on \mathcal{E}^{u} . On the domain $B\left(0,\frac{1}{24}\right)\cap\mathcal{E}^{\mathrm{u}}$ we also have the intertwining relation

$$RG(\Psi(v_*, \alpha_{\mathbf{u}}^{-1}w)) = \Psi(v_*, w) .$$

For w small enough in \mathcal{E}^u we have $\Psi(v_{*,w}) \in W_{\mathrm{tiny}}^{u,\mathrm{loc}}$

Proof: Lemma 75 garantees that the quantities $RG^n(v_* + \alpha_{\mathbf{u}}^{-n}w)$ are well defined and bounded in norm by $\frac{1}{8}$. Lemma 77 shows the limit exists and is analytic in w. Finally the same telescopic sum argument as in the proof of Lemma 76, with k = 0, gives the estimate

$$||RG^{n}(v_{*} + \alpha_{\mathbf{u}}^{-n}w) - (RG^{n}(v_{*}) + D_{v_{*}}RG[\alpha_{\mathbf{u}}^{-n}w])|| \leq \sum_{j=0}^{n-1} c_{3}(\epsilon)^{j} \times \frac{17}{2} \left[\alpha_{\mathbf{u}}^{-(j+1)}||T_{n-j-1}(v_{*})[w]||\right]^{2}$$

which in the present situation simply boils down to

$$||RG^{n}(v_{*} + \alpha_{\mathbf{u}}^{-n}w) - v_{*} - w|| \leq \sum_{j=0}^{n-1} c_{3}(\epsilon)^{j} \times \frac{17}{2} \left[\alpha_{\mathbf{u}}^{-(j+1)}||w||\right]^{2}$$

from which the wanted estimate follows easily. The intertwining relation follows as in Part 3) of Theorem 5. Using this intertwining relation to construct the backwards RG trajectory $\Psi(v_*, \alpha_{\rm u}^n w)$, for $n \leq 0$, and thanks to the criterion in Proposition 8 one easily see that $\Psi(v_*, w)$ is in the local unstable manifold $W^{\rm u,loc}$. By continuity of $\Psi(v_*, \bullet)$ at zero one also gets the stronger conclusion that $\Psi(v_*, w) \in W_{\rm tiny}^{\rm u,loc}$.

Before concluding this section we state a lemma which is on the same theme as Lemmas 74 and 76 and which will be needed in the sequel.

Lemma 78. In the small ϵ regime for all $v \in W^{s, loc}$ and all w with $||w|| \leq \frac{1}{240C_1(\epsilon)}$ we have

$$||RG^n(v + \alpha_{\mathbf{u}}^{-n}w) - RG^n(v)|| \le 11\mathcal{C}_1(\epsilon)||w||.$$

Proof: By the same telescopic sum argument as in the beginning of the proof of Lemma 76, with k = 0, we get

$$||RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w) - RG^{n}(v) - D_{v}RG^{n}[\alpha_{\mathbf{u}}^{-n}w]|| \leq \sum_{j=0}^{n-1} 850C_{1}(\epsilon)^{2}||w||^{2} \times \alpha_{\mathbf{u}}^{-2} \times (c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2})^{j}$$
$$\leq 850C_{1}(\epsilon)^{2}||w||^{2} \times \frac{1}{4}$$

where we used (54) and $L \geq 2$. As a result we have

$$||RG^{n}(v + \alpha_{\mathbf{u}}^{-n}w) - RG^{n}(v)|| \le ||T_{n}(v)[w]|| + \frac{425}{2}C_{1}(\epsilon)^{2}||w||^{2}$$

$$\le 10C_{1}(\epsilon)||w|| + \frac{425}{2}C_{1}(\epsilon)||w|| \times \frac{1}{240}$$

because of our hypothesis on ||w||. Since $2 \times 240 > 425$ the lemma follows.

10. Control of the deviation from the bulk

10.1. Algebraic considerations. We now pick up the thread from §4.2 where we consider for test functions $\tilde{f}, \tilde{j} \in S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$ the quantity

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) = \frac{\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\mathcal{Z}_{r,s}(0,0)}$$

which is the moment generating function with UV and IR cutoffs r and s respectively.

Introduce

$$\begin{split} \mathcal{S}_{r,s}^{\mathrm{T}}(\tilde{f},\tilde{j}) &= -Y_0 Z_0^r \int_{\mathbb{Q}_p^3} \tilde{j}(x) \ d^3x + \frac{1}{2} \sum_{r \leq q < s} \left(f^{(r,q)}, \Gamma f^{(r,q)} \right)_{\Lambda_{s-q}} \\ &+ \sum_{r \leq q < s} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_\Delta \left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) \right] - \delta b_\Delta \left[\vec{V}^{(r,q)}(0,0) \right] \right) \\ &+ \mathrm{Log} \left(\frac{\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial \mathcal{Z}_{r,s}(0,0)} \right) \end{split}$$

where Log is the principal logarithm with argument in $(-\pi, \pi]$.

We will show that it is indeed a well defined quantity which boils down to making sure all the RG iterates $\vec{V}^{(r,q)}$ are in the domain of definition and analyticity for $RG_{\rm ex}$ provided by Theorem 4. One also needs to check that $\frac{\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial \mathcal{Z}_{r,s}(0,0)}$ is well defined and nonzero.

Once this is verified then it immediately follows from the considerations in §4.2 that

$$S_{r,s}(\tilde{f}, \tilde{j}) = \exp\left(S_{r,s}^{\mathrm{T}}(\tilde{f}, \tilde{j})\right)$$
.

The brunt of the remaining work is controlling the $r \to -\infty$ and $s \to \infty$ limits of the log-moment generating function $\mathcal{S}_{r,s}^{\mathrm{T}}(\tilde{f},\tilde{j})$.

Recall that for the denominator, i.e. when $\tilde{f}, \tilde{j} = 0$, the initial condition for the $RG_{\rm ex}$ iterations is

$$\vec{V}^{(r,r)}(0,0) = (g,0,\mu_{\rm c}(g),0,0,0,0,0)$$

with $\mu_{\rm c}(g) = \mu_{\rm s}(g-\bar{g},0)$ by definition.

If ι is the affine isometric injection $\mathcal{E} \to \mathcal{E}_{\mathrm{ex}}$ which sends $(\delta g, \mu, R)$ to the vector

$$\vec{V} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})_{\Delta \in \mathbb{L}}$$

where for all $\Delta \in \mathbb{L}$

$$\beta_{4,\Delta} = \bar{g} + \delta g$$

$$\beta_{3,\Delta} = 0$$

$$\beta_{2,\Delta} = \mu$$

$$\beta_{1,\Delta} = 0$$

$$W_{5,\Delta} = 0$$

$$W_{6,\Delta} = 0$$

$$f_{\Delta} = 0$$

$$R_{\Delta} = R$$

then $\vec{V}^{r,r}(0,0) = \iota(v)$ with $v = (\delta g, \mu_s(\delta g, 0), 0)$ where $\delta g = g - \bar{g}$.

By construction $v \in W^{s,loc}$ and therefore all of its iterates are well defined and we have

$$\vec{V}^{(r,q)}(0,0) = \iota\left(RG^{q-r}(v)\right) \longrightarrow \iota(v_*)$$
 where $r \to -\infty$ with q fixed.

The purpose of this section is to derive estimates which control the deviations from this bulk trajectory due to the test functions \tilde{f} and \tilde{j} . We will break up the log-moment generating function into five pieces which will be analyzed separately.

Namely, we write

$$\begin{split} \mathcal{S}_{r,s}^{\mathrm{T}}(\tilde{f},\tilde{j}) = & \mathcal{S}_{r,s}^{\mathrm{T,FR}}(\tilde{f},\tilde{j}) + \mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) + \mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j}) \\ & + \mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) + \mathcal{S}_{r,s}^{\mathrm{T,BD}}(\tilde{f},\tilde{j}) \end{split}$$

where

$$\begin{split} \mathcal{S}_{r,s}^{\mathrm{T,FR}}(\tilde{f},\tilde{j}) &= \frac{1}{2} \sum_{r \leq q < s} \left(f^{(r,q)}, \Gamma f^{(r,q)} \right)_{\Lambda_{s-q}} \\ \mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) &= -Y_0 Z_0^r \int_{\mathbb{Q}_p^3} \tilde{j}(x) \ d^3x + \sum_{r \leq q < q_-} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_\Delta [\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta [\vec{V}^{(r,q)}(0,0)] \right) \\ \mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j}) &= \sum_{q_- \leq q < q_+} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_\Delta [\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta [\vec{V}^{(r,q)}(0,0)] \right) \\ \mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) &= \sum_{q_+ \leq q < s} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_\Delta [\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta [\vec{V}^{(r,q)}(0,0)] \right) \\ \text{and} \\ \mathcal{S}_{r,s}^{\mathrm{T,BD}}(\tilde{f},\tilde{j}) &= \mathrm{Log} \left(\frac{\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial \mathcal{Z}_{r,s}(0,0)} \right). \end{split}$$

The subscript "FR" stands for the free contribution. Indeed, an easy exercise shows that

$$\lim_{\substack{r \to -\infty \\ s \to \infty}} \mathcal{S}_{r,s}^{\mathrm{T,FR}}(\tilde{f}, \tilde{j}) = \frac{1}{2} \left(\tilde{f}, C_{-\infty} \tilde{f} \right)$$

which corresponds to the free massless measure without cut-offs, i.e., the Gaussian measure with covariance $C_{-\infty}$.

The quantity $\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j})$ collects the ultraviolet contributions while $\mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j})$ contains the infrared contributions. Most of the influence of the test functions is felt in the middle regime $q_{-} \leq q < q_{+}$, hence the abbreviation "MD". Finally $\mathcal{S}_{r,s}^{\mathrm{T,BD}}(\tilde{f},\tilde{j})$ corresponds to the a boundary term left after the RG iterations have shrunk the confining volume Λ down to a single unit cube.

The analysis will make use of the following observations with are of an algebraic or combinatorial nature. Since the RG runs from UV scales to IR scales we will first have a closer look at the terms featuring in $\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j})$.

From the definition of RG_{ex} in §4.2 one sees that this map is given by a collection of independent operations performed locally.

Indeed the output $(\beta'_{4,\Delta'}, \ldots, \beta'_{1,\Delta'}, W'_{5,\Delta'}, W'_{6,\Delta'}, f'_{\Delta'}, R_{\Delta'})$ as well as the output $\delta b_{\Delta'}$ produced for a cube Δ' only involves the data $(\beta_{4,\Delta}, \ldots, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})_{\Delta \in [L^{-1}\Delta']}$.

In other words, RG_{ex} is made up of independent copies of a map $(\mathcal{E}_{1B})^{\times L^3} \longrightarrow \mathcal{E}_{1B}$.

Let $\widetilde{\Delta} \in \mathbb{L}_{q_-}$ so that \widetilde{f} and \widetilde{j} are constant on $\widetilde{\Delta}$ taking the values $\widetilde{f}_{\widetilde{\Delta}}$ and $\widetilde{j}_{\widetilde{\Delta}}$ respectively. If $\widetilde{\Delta} \not\in \Lambda_{q_+}$ then $\widetilde{f}_{\widetilde{\Delta}} = \widetilde{j}_{\widetilde{\Delta}} = 0$.

First let us see what happens for the first iteration, i.e., q = r.

If a unit cube Δ is in $\Lambda_{s-r} \setminus \Lambda_{q_+-r}$ then the Δ component of $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ of $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ is exactly the same as that of the bulk $\vec{V}^{(r,r)}(0,0) = \iota(\delta g,\mu,0)$ with $\mu = \mu_s(\delta g,0)$.

If $\Delta \in \Lambda_{q_+-r}$ then there is a unique $\widetilde{\Delta} \in \mathbb{L}_{q_-}$, $\widetilde{\Delta} \subset \Lambda_{q_+}$ such that $\Delta \subset L^r \widetilde{\Delta}$ In this case:

$$\vec{V}^{(r,r)}(\tilde{f},\tilde{j}) = (g,0,\mu - Y_2 Z_2^r L^{(3-2[\phi])r} \tilde{j}_{\widetilde{\Lambda}}, 0, 0, 0, L^{(3-[\phi])r} \tilde{f}_{\widetilde{\Lambda}}, 0).$$

Now we choose Z_2 so that $Z_2 = \alpha_{\rm u} L^{-(3-2[\phi])}$ and thus

$$\vec{V}^{(r,r)}_{\Delta}(\tilde{f},\tilde{j}) = (g,0,\mu - Y_2\alpha_{\mathrm{u}}^r\tilde{j}_{\widetilde{\Delta}},0,0,0,L^{(3-[\phi])r}\tilde{f}_{\widetilde{\Delta}},0).$$

If $q = r < q_-$ then all immediate neighbors Δ carry the same data. Here by neighbors we mean the $L^3 - 1$ other unit cubes contained in the same L-block $L^{-1}\Delta'$ as Δ .

Therefore the computation producing $\delta b_{\Delta'}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})]$ as well as $\vec{V}_{\Delta'}^{(r,r+1)}(\tilde{f},\tilde{j})$ is the same as the RG acting on the space $\mathcal{E}_{\rm bk}$. In fact the computation reduces to the map RG on the even smaller subspace \mathcal{E} , except for the presence of the f-component $L^{(3-[\phi])r}\tilde{f}_{\tilde{\Delta}}$.

The key observation is that this component evolves by averaging without influencing or being influenced by the other variables.

This again results from the property that $\int_{L^{-1}\Delta'} \Gamma(x-y) d^3y = 0$ for all $x \in L^{-1}\Delta'$, as in the proof of Proposition 2.

Indeed for the explicit diagrams in the RG transformation the possible effect of f is through legs attached to f-vertices of valence 1 which precisely contribute a factor of the type $\int_{L^{-1}\Delta'} \Gamma(x-y) \mathrm{d}^3 y = 0$ because f is constant over the L-block $L^{-1}\Delta'$.

For the other \mathcal{L} or ξ terms, observe that one has $e^{\int_{L^{-1}\Delta'}f\zeta}=1$ because f is constant on $L^{-1}\Delta'$ and $\int_{L^{-1}\Delta'}\zeta=0$ almost surely by the property of the fluctuation covariance Γ .

As a result

$$\vec{V}_{\Delta'}^{(r,r+1)}(\tilde{f},\tilde{j}) = (g',0,\mu',0,0,0,L^{(3-[\phi])(r+1)}\tilde{f}_{\widetilde{\Delta}},R')$$

where

$$(g' - \bar{g}, \mu', R') = RG(g - \bar{g}, \mu - \alpha_{\mathfrak{U}}^{r} Y_{2} \tilde{j}_{\widetilde{\Lambda}}, 0)$$

and also

$$\delta b_{\Delta'}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})] = \delta b(g - \bar{g}, \mu - \alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}}, 0).$$

The same decoupling applies to subsequent iterates $\vec{V}^{(r,q+1)}(\tilde{f},\tilde{j}) = RG_{\rm ex}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})]$ as long as $q < q_-$, i.e. as long as $f^{(r,q)}$ is constant over each individual L-block.

Hence, in the quantity

$$\sum_{r \leq q < q_{-}} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_{\Delta} [\vec{V}^{(r,q)}(\tilde{f}, \tilde{j})] - \delta b_{\Delta} [\vec{V}^{(r,q)}(0,0)] \right)$$

appearing in $\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j})$, only boxes $\Delta \subset \Lambda_{q_+-r-1}$ will contribute and these can be organized according to $\widetilde{\Delta} \in \mathbb{L}_{q_-}$, $\widetilde{\Delta} \subset \Lambda_{q_+}$ such that $L^{q+1}\widetilde{\Delta}$ contains Δ . All $L^{3(q_--q-1)}$ boxes Δ which satisfy that condition for given $\widetilde{\Delta}$ produce the same contribution.

In other words, the previous expression can be rewritten as

$$\sum_{\widetilde{\Delta} \in \mathbb{L}_{q_{-}}} \sum_{r \leq q < q_{-}} L^{3(q_{-} - q - 1)} \left(\delta b \left[RG^{q - r} \left(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}} \right) \right] - \delta b \left[RG^{q - r} (v) \right] \right)$$

$$\widetilde{\Delta} \in \Lambda_{q_{-}}$$

where
$$v = (\delta g, \mu_s(\delta g, 0), 0)$$
 with $\delta g = g - \bar{g}$
and $e_{\phi^2} = (0, 1, 0) \in \mathcal{E}$.

Here e_{ϕ^2} gives the direction of pure : ϕ^2 : perturbations in the bulk.

We are thus reduced to $L^{3(q_+-q_-)}$ separate and independent bulk RG trajectories as considered in §7, one for each $\widetilde{\Delta}$. Also note that the effect of \widetilde{f} is completely absent form the UV regime contribution.

By also organizing the explicit extra linear term in \tilde{j} according to boxes $\tilde{\Delta}$ of size L^{q_-} we can write

$$\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) = \sum_{\substack{\widetilde{\Delta} \in \mathbb{L}_{q_{-}} \\ \widetilde{\Delta} \subset \Lambda_{q_{+}}}} \mathcal{K}_{\widetilde{\Delta}}$$

with

$$\mathcal{K}_{\widetilde{\Delta}} = -Y_0 Z_0^r L^{3q_-} \tilde{j}_{\widetilde{\Delta}} + \sum_{r \leq q < q_-} L^{3(q_- - q - 1)} \left(\delta b \left[R G^{q - r} \left(v - \alpha_{\mathbf{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right) \right] - \delta b \left[R G^{q - r} (v) \right] \right) .$$

We now look at the middle regime and note that

$$\mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j}) = \sum_{q_{-} \leq q < q_{+}} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_{+}-q-1}}} \left(\delta b_{\Delta}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta}[\vec{V}^{(r,q)}(0,0)] \right).$$

Here we replaced the s that appeared earlier with q_+ when describing the summation over boxes Δ . Indeed, if $\Delta \subset \Lambda_{s-q-1}$ is outside the rescaling Λ_{q_+-q-1} of the set Λ_{q_+} containing the supports of the \tilde{f} and \tilde{j} , then the effect of Δ is nil.

What we need here is a more precise description of the vector $\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})$ delivered by the RG evolution in the UV regime. This involves a fusion of $L^{3(q_+-q_-)}$ data living in \mathcal{E} into a single vector in $\mathcal{E}_{\rm ex}$.

For $m \ge 0$ we introduce the reinjection map

$$\mathcal{J}_m: S_{0,m}(\mathbb{Q}_p^3, \mathbb{C}) \times \left(\prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_m}} \mathcal{E}\right) \longrightarrow \mathcal{E}_{\mathrm{ex}}$$
$$\left(F, (\delta g_\Delta, \mu_\Delta, R_\Delta)_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_m}}, (\delta g, \mu, R)\right) \mapsto \vec{V}' = \left(\beta'_{4,\Delta}, \dots, \beta'_{1,\Delta}, W'_{5,\Delta}, W'_{6,\Delta}, f'_{\Delta'}, R'_{\Delta'}\right)_{\Delta \in \mathbb{L}}$$

defined as follows.

We let

$$\beta'_{4,\Delta} = \begin{cases} \bar{g} + \delta g_{\Delta} & \text{if } \Delta \subset \Lambda_m \\ \bar{g} + \delta g & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$\beta'_{2,\Delta} = \begin{cases} \mu_{\Delta} & \text{if } \Delta \subset \Lambda_m \\ \mu & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$R'_{\Delta} = \begin{cases} R_{\Delta} & \text{if } \Delta \subset \Lambda_m \\ R & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$\beta'_{3,\Delta} = \beta'_{1,\Delta} = W'_{5,\Delta} = W'_{6,\Delta} = 0$$

and finally f'_{Δ} is defined by

$$f'_{\Delta(x)} = F(x)$$
 for all $x \in \mathbb{Q}_p^3$.

Namely, via the correspondance between L-indexed vectors and functions that are constant on unit cubes, f' = F. Recall indeed that F is assumed constant on unit cubes and with support contained in Λ_m .

Now it is easy to see from the previous considerations that

$$\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_{+}-q_{-}}\left(\tilde{f}_{\rightarrow(-q_{-})},\left(RG^{q_{-}-r}\left(v - \alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{L^{-q_{-}}\Delta}e_{\phi^{2}}\right)\right)_{\Delta\subset\Lambda_{q_{+}-q_{-}}},RG^{q_{-}-r}(v)\right).$$

Note in particular that $f^{(r,q_-)} = \left(\tilde{f}_{\to(-r)}\right)_{\to(q_--r)} = \tilde{f}_{\to(-q_-)}.$

We also have the special case

$$\begin{split} \vec{V}^{(r,q_{-})}(0,0) = & \mathcal{J}_{q_{+}-q_{-}}\left(0, \left(RG^{q_{-}-r}(v)\right)_{\substack{\Delta \subset \Lambda_{q_{+}-q_{-}}}}, RG^{q_{-}-r}(v)\right) \\ = & \iota\left(RG^{q_{-}-r}(v)\right) \in \mathcal{E}_{\mathrm{bk}} \subset \mathcal{E}. \end{split}$$

Finally, in the infrared regime

$$\begin{split} \mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) &= \sum_{q_{+} \leq q < s} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \in \Lambda_{s-q-q}}} \left(\delta b_{\Delta} \left[RG^{q-q_{+}} \left(\vec{V}^{(r,q_{+})}(\tilde{f},\tilde{j}) \right) \right] - \delta b_{\Delta} \left[RG^{q-q_{+}} \left(\vec{V}^{(r,q_{+})}(0,0) \right) \right] \right) \\ & \text{where } \vec{V}^{(r,q_{+})}(\tilde{f},\tilde{j}) = RG^{q_{+}-q_{-}} \left(\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) \right) \; . \end{split}$$

Since $\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})$ agrees with $\vec{V}^{(r,q_-)}(0,0)$ on all unit cubes $\Delta \not\subset \Lambda_{q_+-q_-}$, it is easy to see that

$$RG^{q_+-q_-}\left(\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})\right)$$
 agrees with $RG^{q_+-q_-}\left(\vec{V}^{(r,q_-)}(0,0)\right)$ on all unit cubes $\Delta \not\subset \Lambda_0 = \Delta(0)$ the unit cube containing the origin.

Thus

$$\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_+)}(0,0) \in \mathcal{E}_{\mathrm{pt}}$$

or

$$\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) \in \iota(\mathcal{E}) \oplus \mathcal{E}_{\mathrm{pt}} \subset \mathcal{E}_{\mathrm{bk}} \oplus \mathcal{E}_{\mathrm{pt}}$$
.

This property remains true for the next iterates since the only difference with the bulk now only happens in $\Delta(0)$.

Therefore no summation over Δ is needed in the formula for $\mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j})$ which thus reduces to

$$\mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) = \sum_{q_{\perp} \leq q \leq s} \left(\delta b_{\Delta(0)} \left[RG^{q-q_{+}} \left(\vec{V}^{(r,q_{+})}(\tilde{f},\tilde{j}) \right) \right] - \delta b_{\Delta(0)} \left[RG^{q-q_{+}} \left(\vec{V}^{(r,q_{+})}(0,0) \right) \right] \right).$$

After these prepatory steps we can now address the estimates needed in order to take the $r \to -\infty$ and $s \to \infty$ limits.

10.2. The ultraviolet regime. We first need an analogue of Lemma 69 for the function δb .

Lemma 79. For ϵ small, for all v such that $||v|| < \frac{1}{4}$ we have

$$||D_v^2 \delta b|| = \sup_{v',v'' \neq 0} \frac{\left|D_v^2 \delta b[v',v'']\right|}{||v'|| \times ||v''||} \le 2.$$

Proof: Recall that

$$\delta b(\delta g, \mu, R) = \delta b^{\text{explicit}}(\delta g, \mu, R) + \delta b^{\text{implicit}}(\delta g, \mu, R)$$

where

$$\delta b^{\text{explicit}}(\delta g, \mu, R) = A_4(\bar{g} + \delta g)^2 + A_5 \mu^2$$

and

$$\delta b^{\rm implicit}(\delta g,\mu,R) = \xi_0(\bar{g}+\delta g,\mu,R) \ . \label{eq:deltable}$$

using the same notations as in Lemma 69 we have

$$D_v^2 \delta b^{\text{explicit}}[v', v''] = 2A_4 \delta g' \delta g'' + 2A_5 \mu' \mu''.$$

Now, using $C_0(0), ||\Gamma||_{L^{\infty}} \leq 2$, we have

$$\begin{aligned} |A_4| &\leq 12L^3 ||\Gamma||_{L^{\infty}}^2 ||\Gamma||_{L^2}^2 + 48L^{\frac{3+\epsilon}{2}} \times 2 \times ||\Gamma||_{L^{\infty}} ||\Gamma||_{L^2}^2 + 72L^{\epsilon} \times 4 \times ||\Gamma||_{L^2}^2 \\ &\leq \left[48L^3 + 192L^{\frac{3+\epsilon}{2}} + 288L^{\epsilon} \right] ||\Gamma||_{L^2}^2 \\ &= \left(48L^3 + 192L^{\frac{3+\epsilon}{2}} + 288L^{\epsilon} \right) \frac{1}{36}L^{-\epsilon}A_1 \leq A_{4,\max} \end{aligned}$$

with

$$A_{4,\text{max}} = \left(48L^3 + 192L^{\frac{3}{2}} + 288\right) \times \frac{1}{36}A_{1,\text{max}}.$$

Likewise $|A_5| \leq A_{5,\text{max}}$ with

$$A_{5,\text{max}} = L^3 \times \frac{1}{36} A_{1,\text{max}}.$$

Thus

$$\left| D_v^2 \delta b^{\text{explicit}}[v',v''] \right| \leq 2 ||v'|| \times ||v''|| \left[2 A_{4,\max} \bar{g}^{2e_4} + 2 A_{5,\max} \bar{g}^{2e_2} \right].$$

Now if $||v|| < \frac{1}{4}$ then we can use Cauchy's formula

$$D_v^2 \delta b^{\text{implicit}}[v', v''] = \frac{1}{(2i\pi)^2} \oint \frac{d\lambda_1}{\lambda_1^2} \oint \frac{d\lambda_2}{\lambda_2^2} \ \delta b^{\text{implicit}}[v + \lambda_1 v' + \lambda_2 v'']$$

where the contours are given by $|\lambda_1| = \frac{1}{8||v'||}$, $|\lambda_2| = \frac{1}{8||v''||}$. Then we get from Theorem 4

$$\begin{split} \left| D_v^2 \delta b^{\text{implicit}}[v', v''] \right| \leq & 8||v'|| \times 8||v''|| \times B_0 \bar{g}^{e_R} \times \sup_{\lambda_1, \lambda_2} ||v + \lambda_1 v' + \lambda_2 v''|| \\ \leq & 64||v'|| \times ||v''|| \times B_0 \bar{g}^{e_R} \times \frac{1}{2} \; . \end{split}$$

Combining both bounds we obtain

$$\left| D_v^2 \delta b[v',v''] \right| \le ||v'|| \times ||v''|| \left[4A_{4,\max} \bar{g}^{2e_4} + 4A_{5,\max} \bar{g}^{2e_2} + 32B_0 \bar{g}^{e_R} \right].$$

Since $e_4, e_2, e_R > 0$ the lemma follows by making \bar{g} , i.e., ϵ small enough.

Lemma 80. For ϵ small and for all v such that $||v|| < \frac{1}{4}$

$$||D_v \delta b|| = \sup_{v' \neq 0} \frac{|D_v \delta b[v']|}{||v'||} \le 1.$$

Proof: Now for $||v|| \le \frac{1}{2}$

$$D_v \delta b^{\text{explicit}}[v'] = 2A_4(\bar{q} + \delta q)\delta q' + 2A_5\mu\mu'$$

$$|D_v \delta b^{\text{explicit}}[v']| \le 2A_{4,\max} \times \frac{3}{2}\bar{g} \times \bar{g}^{e_4}||v'|| + 2A_{5,\max} \times \frac{1}{2}\bar{g}^{e_2} \times \bar{g}^{e_2}||v'||.$$

If furthermore $||v|| < \frac{1}{4}$ then we can write Cauchy's formula

$$D_v \delta b^{\text{implicit}}[v'] = \frac{1}{2i\pi} \oint \frac{d\lambda}{\lambda^2} \ \delta b^{\text{implicit}}(v + \lambda v')$$
 on the contour $|\lambda| = \frac{1}{4||v'||}$

and deduce

$$|D_v \delta b^{\text{implicit}}[v']| \le 4||v'|| \times B_0 \bar{g}^{e_R} \times \frac{1}{2}$$
.

Thus for $||v|| < \frac{1}{4}$,

$$||D_v \delta b|| \le 3A_{4,\max} \bar{g}^{e_4+1} + A_{5,\max} \bar{g}^{2e_2} + 2B_0 \bar{g}^{e_R}.$$

Again the last expression can be made as small as we want provided ϵ is small enough.

Using the mean value theorem or Taylor's formula with integral remainder we immediately obtain as before the following lemma.

Lemma 81. For ϵ small

(1) For all v with $||v|| < \frac{1}{4}$

$$||D_v \delta b - D_{v_*} \delta b|| \le 2||v - v_*||$$

(2) For all v, w such that $||v||, ||w|| < \frac{1}{4}$

$$||\delta b(v+w) - \delta b(v)|| \le ||w||$$

and

$$||\delta b(v+w) - \delta b(v) - D_v \delta b[w]|| \le ||w||^2.$$

We now resume the analysis of the expression for $\mathcal{S}_{r,s}^{T,UV}(\tilde{f},\tilde{j})$ derived in the last section. Adding and subtracting terms linear in $\tilde{j}_{\widetilde{\Delta}}$ we write

$$\mathcal{K}_{\widetilde{\Delta}} = \tilde{j}_{\widetilde{\Delta}} \left[-Y_0 Z_0^r L^{3q_-} + \sum_{r \leq q < q_-} L^{3(q_- - q - 1)} D_v \left(\delta b \circ R G^{q - r} \right) \left[-\alpha_{\mathbf{u}}^r Y_2 e_{\phi^2} \right] \right] + \sum_{r \leq q < q_-} L^{3(q_- - q - 1)} \mathcal{K}_{\widetilde{\Delta}, q}$$
where
$$\mathcal{K}_{\widetilde{\Delta}, q} = \delta b \left[R G^{q - r} \left(v - \alpha_{\mathbf{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right) \right] - \delta b \left[R G^{q - r} (v) \right] + D_v \left(\delta b \circ R G^{q - r} \right) \left[\alpha_{\mathbf{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right] .$$

Now $\mathcal{K}_{\widetilde{\Delta},q} = \mathcal{K}'_{\widetilde{\Delta},a} + \mathcal{K}''_{\widetilde{\Delta},a}$ where

$$\mathcal{K}_{\widetilde{\Delta},q}' = \delta b \left[RG^{q-r} \left(v - \alpha_{\mathrm{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}} \right) \right] - \delta b \left[RG^{q-r}(v) \right] - D_{RG^{q-r}(v)} \delta b \left[RG^{q-r} \left(v - \alpha_{\mathrm{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}} \right) - RG^{q-r}(v) \right]$$
 and

$$\mathcal{K}_{\widetilde{\Delta},q}'' = D_{RG^{q-r}(v)} \delta b \left[RG^{q-r} \left(v - \alpha_{\mathbf{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right) - RG^{q-r}(v) + D_v RG^{q-r} \left[\alpha_{\mathbf{u}}^r Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right] \right] \ .$$

We already know $||RG^{q-r}(v)|| < \frac{1}{4}$.

If the same is true for $RG^{q-r}\left(v-\alpha_{\rm u}^{r}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right)$ then Lemmas 81 and 80 imply

$$||\mathcal{K}_{\widetilde{\Delta},q}'|| \leq ||RG^{q-r}\left(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}}\right) - RG^{q-r}(v)||^{2}$$

and

$$||\mathcal{K}_{\widetilde{\Delta},q}''|| \leq ||RG^{q-r}\left(v - \alpha_{\mathbf{u}}^{r}\mathbf{Y}_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right) - RG^{q-r}(v) + D_{v}RG^{q-r}\left[\alpha_{\mathbf{u}}^{r}\mathbf{Y}_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right]||.$$

We assume $||\alpha_{\mathbf{u}}^{q_{-}-1}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}|| \leq \frac{1}{240C_{1}(\epsilon)}$ which implies $||\alpha_{\mathbf{u}}^{q}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}|| \leq \frac{1}{240C_{1}(\epsilon)}$ for all $q < q_{-}$.

Lemma 74 guarantees that

$$RG^{q-r}\left(v-\alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right)=RG^{q-r}\left(v+\alpha_{\mathbf{u}}^{-(q-r)}\left(-\alpha_{\mathbf{u}}^{q}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right)\right)$$

is well defined and has norm at most $\frac{1}{8}$.

The telescoping sum argument at the beginning of the proof of lemma 76 with n=q-1, k=0, and $w=-\alpha_{\rm u}^q Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2}$ gives

$$\begin{split} &||RG^{q-r}\left(v-\alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right)-RG^{q-r}(v)+D_{v}RG^{q-r}\left[\alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}\right]||\\ &=||RG^{q-r}\left(v+\alpha_{\mathbf{u}}^{-(q-r)}w\right)-RG^{q-r}(v)-D_{v}RG^{q-r}\left[\alpha_{\mathbf{u}}^{-(q-r)}w\right]||\\ &\leq\sum_{i=0}^{q-r}850\mathcal{C}_{1}(\epsilon)^{2}||w||^{2}\alpha_{\mathbf{u}}^{-2}\left(c_{3}(\epsilon)\alpha_{\mathbf{u}}^{-2}\right)^{i}\\ &\leq850\mathcal{C}_{1}(\epsilon)^{2}||w||^{2}\frac{1}{\alpha_{\mathbf{u}}^{2}-c_{3}(\epsilon)}\\ &\leq\frac{1700}{L^{3}}\mathcal{C}_{1}(\epsilon)^{2}||w||^{2}\\ &=1700L^{-3}\mathcal{C}_{1}(\epsilon)^{2}\alpha_{\mathbf{u}}^{-2(q-q-1)}||\alpha_{\mathbf{u}}^{q--1}Y_{2}\tilde{j}_{\widetilde{\Delta}}e_{\phi^{2}}||^{2}\\ &\leq\frac{1700}{240^{2}\times8}\alpha_{\mathbf{u}}^{-2(q-q-1)}<\alpha_{\mathbf{u}}^{-2(q-q-1)}\end{split}$$

where we used the bound in (54) to go from the fourth to the fifth line as well as $L \ge 2$ in the last line. On the other hand,

$$RG^{q-r}\left(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}}\right) - RG^{q-r}(v) = RG^{q-r}\left(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}}\right) - RG^{q-r}(v) + D_{v} RG^{q-r}\left[\alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}}\right] + T_{q-r}(v)[w] .$$

So by the previous bound, Lemma 71 and Lemma 68 we obtain

$$\begin{split} ||RG^{q-r}\left(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{\widetilde{\Delta}} e_{\phi^{2}}\right) - RG^{q-r}(v)|| &\leq \alpha_{\mathbf{u}}^{-2(q_{-}-q-1)} + 10\mathcal{C}_{1}(\epsilon)||w|| \\ &\leq \alpha_{\mathbf{u}}^{-2(q_{-}-q-1)} + 10\mathcal{C}_{1}(\epsilon) \times \alpha_{\mathbf{u}}^{-(q_{-}-q-1)} \frac{1}{240\mathcal{C}_{1}(\epsilon)} \\ &\leq \frac{25}{24} \alpha_{\mathbf{u}}^{-(q_{-}-q-1)} \; . \end{split}$$

Hence $||\mathcal{K}_{\widetilde{\Delta},q}''|| \le \alpha_{\mathbf{u}}^{-2(q_--q_-1)}$ and $||\mathcal{K}_{\widetilde{\Delta},q}'|| \le \left(\frac{25}{24}\right)^2 \alpha_{\mathbf{u}}^{-2(q_--q_-1)}$. With these two bounds in hand we can write the estimate $||\mathcal{K}_{\widetilde{\Delta},q}|| \le 3\alpha_{\mathbf{u}}^{-2(q_--q_-1)}$ for simplicity.

 Y_2 is a strictly positive quantity that will be fixed later and we have that $||e_{\phi^2}|| = ||(0,1,0)|| = \bar{g}^{-e_2}$. So the previous construction and bounds work if

(55)
$$||\tilde{j}||_{L^{\infty}} \leq \left\lceil 240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q-1}Y_2\bar{g}^{-e_2}\right\rceil^{-1} .$$

We will later also show $L^3\alpha_{\mathrm{u}}^{-2}<1$ which will imply that $\sum_{r\leq q< q_-}L^{3(q_--q_-1)}||\mathcal{K}_{\widetilde{\Delta},q}||$ is summable with uniform bounds with respect to the UV cut-off r.

We now analyze the quantity

$$\Omega_r = -Y_0 Z_0^r L^{3q_-} + \sum_{r < q < q_-} L^{3(q_- - q - 1)} D_v \left(\delta b \circ RG^{q - r} \right) \left[-\alpha_{\mathbf{u}}^r Y_2 e_{\phi^2} \right].$$

We change the summation index to n = q - r and rewrite the differential using the chain rule and get

$$\begin{split} &\Omega_{r} = L^{3q_{-}} \left(-Y_{0}Z_{0}^{r} - Y_{2} \sum_{n=0}^{q_{-}-r-1} L^{-3(n+r+1)} \alpha_{\mathbf{u}}^{r} D_{RG^{n}(v)} \delta b \left[D_{v} RG^{n}[e_{\phi^{2}}] \right] \right) \\ &= L^{3q_{-}} \left(-Y_{0}Z_{0}^{r} - Y_{2} \sum_{n=0}^{q_{-}-r-1} L^{-3(n+r+1)} \alpha_{\mathbf{u}}^{r+n} D_{RG^{n}(v)} \delta b \left[T_{n}(v)[e_{\phi^{2}}] \right] \right) \\ &= L^{3q_{-}} \left(-Y_{0}Z_{0}^{r} - Y_{2}L^{-3}(L^{-3}\alpha_{\mathbf{u}})^{r} \sum_{n=0}^{q_{-}-r-1} (L^{-3}\alpha_{\mathbf{u}})^{n} \Xi_{n} \right) \end{split}$$

with
$$\Xi_n = D_{RG^n(v)} \delta b \left[T_n(v) [e_{\phi^2}] \right]$$
.

Note that from Lemma 66 we have $L^{-3}\alpha_u < 1$. But from Lemmas 71, 68 and 80 we have

$$\begin{split} \left| D_{RG^n} \delta b \left[T_n(v) [e_{\phi^2}] \right] \right| \leq & ||D_{RG^n(v)} \delta b|| \times 10 \times ||T_n(v)||_{\diamondsuit} \times ||e_{\phi^2}|| \\ \leq & 10 \mathcal{C}_1(\epsilon) \bar{g}^{-e_2} \ . \end{split}$$

We then see that Ξ_n is bounded uniformly with respect to n. Hence

$$\Upsilon = \sum_{n=0}^{\infty} (L^{-3}\alpha_{\mathbf{u}})^n \; \Xi_n \text{ converges}$$

and we can write

$$\Omega_r = L^{3q_-} \left(-Y_0 Z_0^r - Y_2 L^{-3} (L^{-3} \alpha_{\mathbf{u}})^r \Upsilon + Y_2 L^{-3} (L^{-3} \alpha_{\mathbf{u}})^r \sum_{n=q_--r}^{\infty} (L^{-3} \alpha_{\mathbf{u}})^n \Xi_n \right) .$$

Since $L^{-3}\alpha_{\rm u} < 1$ and $r \to -\infty$ we choose Y_0, Y_2 , and Z_0 so that the dangerous first two terms cancel.

Namely, we set:

$$Z_0 = L^{-3}\alpha_{\rm u} ,$$

$$Y_0 = -L^{-3}Y_2\Upsilon .$$

Then

$$\Omega_r = L^{3q_-} Y_2 L^{-3} (L^{-3} \alpha_{\mathbf{u}})^r \sum_{n=q_--r}^{\infty} (L^{-3} \alpha_{\mathbf{u}})^n \Xi_n$$
$$= Y_2 L^{-3} \alpha_{\mathbf{u}}^{q_-} \sum_{k=0}^{\infty} (L^{-3} \alpha_{\mathbf{u}})^k \Xi_{k+q_--r}$$

after changing the summation index to $k = n - q_- + r$.

Provided one shows that $\lim_{n\to\infty} \Xi_n = \Xi_\infty$ exists, the discrete dominated convergence theorem will immediately imply

$$\lim_{r \to -\infty} \Omega_r = \frac{Y_2 L^{-3} \alpha_{\mathbf{u}}^{q} \Xi_{\infty}}{1 - L^{-3} \alpha_{\mathbf{u}}} .$$

Now

$$\begin{split} \left| \Xi_n - D_{v_*} \delta b \left[T_{\infty}(v) [e_{\phi^2}] \right] \right| &\leq \left| D_{RG^n(v)} \delta b \left[T_n(v) [e_{\phi^2}] \right] - D_{v_*} \delta b \left[T_n(v) [e_{\phi^2}] \right] \right| + \left| D_{v_*} \delta b \left[T_n(v) [e_{\phi^2}] - T_{\infty}(v) [e_{\phi^2}] \right] \\ &\leq 2 ||RG^n(v) - v_*|| \times 10 \mathcal{C}_1(\epsilon) ||e_{\phi^2}|| + ||T_n(v) - T_{\infty}(v)|| \times ||e_{\phi^2}|| . \end{split}$$

Above we used Lemmas 81, 80 and 71. Finally, Proposition 6 and Lemma 73 ensure that the limit of the Ξ_n exists and is given by Ξ_{∞} .

As a consequence of the previous considerations and Theorem 5 we see that

$$\lim_{\substack{r \to -\infty \\ s \to \infty}} \mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f}, \tilde{j}) = \mathcal{S}^{\mathrm{T,UV}}(\tilde{f}, \tilde{j}) \text{ with }$$

$$\begin{split} \mathcal{S}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) &= \sum_{\substack{\tilde{\Delta} \in \mathbb{L}_{q_{-}} \\ \tilde{\Delta} \subset \Lambda_{q_{+}}}} \left\{ \tilde{j}_{\tilde{\Delta}} \frac{Y_{2} \alpha_{\mathrm{u}}^{q_{-}}}{L^{3} - \alpha_{\mathrm{u}}} D_{v_{*}} \delta b \left[T_{\infty}(v) [e_{\phi^{2}}] \right] \right. \\ &\left. + \sum_{q < q_{-}} L^{3(q_{-} - q - 1)} \left(\delta b \left(\Psi(v, -\alpha_{\mathrm{u}}^{q} Y_{2} \tilde{j}_{\tilde{\Delta}} e_{\phi^{2}}) \right) - \delta b(v_{*}) + \alpha_{\mathrm{u}}^{q} Y_{2} \tilde{j}_{\tilde{\Delta}} D_{v_{*}} \delta b [T_{\infty}(v) (e_{\phi^{2}})] \right) \right\} \,. \end{split}$$

The latter is easily seen to be analytic in \tilde{j} in the domain $||\tilde{j}||_{L^{\infty}} < \left[240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_--1}Y_2\bar{g}^{-e_2}\right]^{-1}$ of $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$.

Note that there is no dependence on \tilde{f} for this piece. In fact the finite cut-off quantity $\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j})$ does not depend on \tilde{f} nor s.

10.3. **The middle regime.** From here onwards we make additional requirements on the exponents defining our norms:

$$(56) 1 - \eta \le e_1 \le e_2 \le e_3 \le e_4 < 2 - 2\eta$$

(57)
$$e_1, e_2 \le 1$$

$$(58) 2 - 2\eta < e_W < \min(e_3, 1) + e_4$$

We also introduce the notation \bar{V} for the approximate fixed point in \mathcal{E}_{bk} . Namely we set $\bar{V}_{\Delta} = (\bar{g}, 0, \dots, 0)$ for all $\Delta \in \mathbb{L}$. We note that RG_{ex} is well defined and analytic on $B(\bar{V}, \frac{1}{2})$.

We will next establish some very coarse bounds on the expansion of deviations which will be enough for the control of the middle regime. The next lemmas all assume that one is in the small ϵ regime. The first one is a refinement of Lemma 37.

Lemma 82. Suppose that $\vec{V} \in B(\bar{V}, \frac{1}{2})$. Then for k = 1, 2, 3, 4 one has the following bound for all $\Delta' \in \mathbb{L}$

$$\left| \delta \beta_{k,1,\Delta'}[\vec{V}] \right| \bar{g}^{-e_k} \le 1 \{ 1 \le k < 4 \} \mathbf{O}_1 L^{\frac{5}{2}},$$

where $\mathbf{O}_1 = \frac{27}{4}$.

Proof: From the definition we get

$$\left|\delta\beta_{k,1,\Delta'}[\vec{V}]\right| \leq \sum_{b} \mathbb{1}\left\{\begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array}\right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \left[\begin{array}{c} f \\ \vdots \\ \beta_{k+b} \end{array}\right]$$

The fact that $\delta \beta_{k,1,\Delta'}[\vec{V}]$ vanishes for k=4 is immediate.

We bound the Feynman Diagrams appearing in the formula above:

In the last line we used $b \ge 1$ and $\epsilon \le 1$ so $3 - b[\phi] \le \frac{5}{2}$. We also dropped the factor of $\frac{1}{\sqrt{2}}$. We used (56) to bound $\max_{k+1 \le j \le 4} \left(\max_{\Delta \in [L^{-1}\Delta']} |\beta_{j,\Delta}| \right)$ by $\max\left(\frac{3}{2}\bar{g}, \frac{1}{2}\bar{g}^{e_{k+1}}\right)$.

We use this bound on Feynman diagrams to get the following bound valid for k=1 and 2:

$$\begin{split} \left| \delta \beta_{k,1,\Delta'}[\vec{V}] \right| \bar{g}^{-e_k} \leq & \bar{g}^{-e_k} \sum_b \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \ \left(\frac{3}{4} L^{\frac{5}{2}} \bar{g}^{\min(e_{k+1},1)} \right) \\ \leq & \frac{3}{4} L^{\frac{5}{2}} \sum_b \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \end{split}$$

In going to the second line we used that for k = 1, 2 one has $\min(e_{k+1}, 1) \ge e_k$, this is a consequence of (56) and (57). We also dropped the factors of $L^{-k[\phi]}$.

For k = 3 which forces b = 1 we only have one diagram to estimate:

$$\left| \begin{array}{c} \begin{array}{c} \begin{array}{c} \\ \\ \\ \\ \end{array} \right| \\ = \left| \begin{array}{c} \\ \\ \\ \end{array} \right| \\ \leq \left| \begin{array}{c} \\ \\ \\ \end{array} \right| \\ \leq \left| \left| f \right| \\ \\ \\ \\ \end{array} \right| \\ \leq \left| \left| f \right| \\ \\ \\ \\ \\ \end{array} \right| \\ \left| \left| L^{-1} \Delta' \right| \\ \\ \\ \\ \\ \\ \end{array} \right| \\ \left| \left| L^{\infty} \times ||\Gamma||_{L^{1}} \times L^{3} \times \max_{\Delta \in [L^{-1} \Delta']} |\beta_{4,\Delta} - \bar{g}| \\ \\ \\ \leq \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]} \times \frac{1}{2} L^{-(3-[\phi])} \right) L^{3} \times \frac{1}{2} \bar{g}^{e_{4}} \\ \\ \leq \frac{1}{4} L^{\frac{5}{2}} \bar{g}^{e_{4}} \end{aligned}$$

In going to the second line we used that

$$\int_{\bar{q}}^{f} = 0$$

since Γ integrates to 0.

Therefore we have:

$$\begin{split} |\delta\beta_{3,1,\Delta'}[\vec{V}]|\bar{g}^{-e_3} \leq & \bar{g}^{-e_3} \sum_b \mathbbm{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \ L^{\frac{5}{2}} \bar{g}^{e_4} \\ \leq & \frac{1}{4} L^{\frac{5}{2}} \sum_{l} \mathbbm{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \end{split}$$

In going to the second line we used that $e_4 \ge e_3$ which is a consequence of (56).

We now observe that:

$$\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \leq 9$$

This proves the lemma.

Lemma 83. Suppose that $\vec{V} \in B(\bar{V}, \frac{1}{2})$. Then one has the following bounds for the W_5' and W_6' components of $\vec{V}' = RG_{\rm ex}[\vec{V}]$.

For all $\Delta' \in \mathbb{L}$ and for k = 5 or 6

$$|W_{k,\Delta'}|\,\bar{g}^{-e_W} \le \mathbf{O}_2 L^{\frac{5}{2}},$$

where $\mathbf{O}_2 = 14$.

Proof: For k = 5 we have:

$$\left|W_{5,\Delta'}'\right| \leq L^{3-5[\phi]} \max_{\Delta \in [L^{-1}\Delta']} \left|W_{5,\Delta'}\right| + 6L^{-5[\phi]} \left| \begin{array}{c} f \\ W_6 \end{array} \right| + 12L^{-5[\phi]} \left| \begin{array}{c} \bullet \\ \beta_4 \end{array} \right| + 48L^{-5[\phi]} \left| \begin{array}{c} f \\ \beta_4 \end{array} \right| .$$

We bound each of the diagrams:

$$\begin{aligned} 6L^{-5[\phi]} \left| \begin{array}{c} { \swarrow f \\ W_6 } \end{array} \right| \leq & 6L^{-5[\phi]} ||f|_{L^{-1}\Delta'}||_{L^{\infty}} \times ||\Gamma||_{L^{1}} \times L^{3} \times \max_{\Delta \in [L^{-1}\Delta']} |W_{6,\Delta}| \\ \leq & 6L^{-5[\phi]} \times \frac{1}{2} L^{-(3-[\phi])} \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]} \right) L^{3} \times \frac{1}{2} \bar{g}^{e_{W}} \\ \leq & \frac{3}{2} L^{3-6[\phi]} \bar{g}^{e_{W}} \ . \end{aligned}$$

In going to the last line we dropped the factor of $\frac{1}{\sqrt{2}}$. We continue to bound the other two diagrams:

$$\begin{split} 12L^{-5[\phi]} \left| \begin{array}{c} \bullet & \bullet \\ \beta_4 & \beta_3 \end{array} \right| = & 12L^{-5[\phi]} \left| \begin{array}{c} \bullet & \bullet \\ (\beta_4 - \bar{g}) \, \beta_3 \end{array} + \begin{array}{c} \bullet & \bullet \\ \bar{g} & \beta_3 \end{array} \right| \\ = & 12L^{-5[\phi]} \left| \begin{array}{c} \bullet & \bullet \\ (\beta_4 - \bar{g}) \, \beta_3 \end{array} \right| \\ = & 12L^{-5[\phi]} \left| \int_{(L^{-1}\Delta')^2} \mathrm{d}^3x \, \mathrm{d}^3y \, \left(\beta_4(x) - \bar{g}\right) \Gamma(x - y) \beta_3(y) \right| \\ \leq & 12L^{-5[\phi]} \times \max_{\Delta \in [L^{-1}\Delta']} \left| \beta_{4,\Delta} - \bar{g} \right| \times ||\Gamma||_{L^1} \times \max_{\Delta \in [L^{-1}\Delta']} \left| \beta_{3,\Delta} \right| \times L^3 \\ \leq & 12L^{-5[\phi]} \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]} \right) L^3 \times \frac{1}{4} \bar{g}^{e_3 + e_4} \leq 3L^{6-7[\phi]} \bar{g}^{e_3 + e_4} \leq 3L^{\frac{5}{2}} \bar{g}^{e_3 + e_4} \, . \end{split}$$

In going to the second line we used the fact that Γ integrates to zero. In the last line we used $\epsilon \leq 1$ so that $6-7[\phi] \leq \frac{5}{2}$. We now move to the third diagram.

$$48L^{-5[\phi]} \left| \begin{array}{c} f \\ \beta_4 \end{array} \right| = 48L^{-5[\phi]} \left| \begin{array}{c} f \\ (\beta_4 - \bar{g}) \end{array} \right| \beta_4 + \frac{f}{\bar{g}} \beta_4$$

$$= 48L^{-5[\phi]} \left| \begin{array}{c} f \\ (\beta_4 - \bar{g}) \end{array} \right| \beta_4$$

$$\leq 48L^{-5[\phi]} \times ||f|_{L^{-1}\Delta'}||_{L^{\infty}} \times ||\Gamma||_{L^1}$$

$$\times \max_{\Delta \in [L^{-1}\Delta']} |\beta_{4,\Delta}| \times ||\Gamma||_{L^1} \times L^3 \times \max_{\Delta \in [L^{-1}\Delta']} |\beta_{4,\Delta} - \bar{g}|$$

$$\leq 48L^{-5[\phi]} \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]} \right)^2 \times \frac{1}{2} L^{-(3-[\phi])} \times L^3 \left(\frac{3}{4} \bar{g}^{1+e_4} \right) \leq 9L^{2\epsilon} \bar{g}^{1+e_4} .$$

Above we used that $6 - 8[\phi] = 2\epsilon$. Putting this together with our assumption on the size of the unprimed W_5 gives us:

$$|W_{5,\Delta'}| \le \frac{1}{2}\bar{g}^{e_W} + \frac{3}{2}\bar{g}^{e_W} + 12L^{\frac{5}{2}}\bar{g}^{\min(1,e_3)+e_4}$$

where we bounded $L^{3-5[\phi]}$ and $L^{3-6[\phi]}$ by 1 as well as $L^{2\epsilon}$ by $L^{\frac{5}{2}}$. We also recall that $\min(1, e_3) + e_4 \ge e_W$ (assumed in (58)) to end up with estimate:

$$|W_{5,\Delta'}| \bar{g}^{-e_W} \le 14L^{\frac{5}{2}}$$
.

This proves the lemma for the case k = 5. For k = 6 we have:

$$|W'_{6,\Delta'}| \le L^{-6[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} |W_{6,\Delta}| + 8L^{-6[\phi]} \Big| \xrightarrow{\beta_4} \beta_4 \Big|$$

We bound the diagram above:

$$8L^{-6[\phi]} \left| \begin{array}{c} \bullet \\ \beta_4 \end{array} \right| = 8L^{-6[\phi]} \left| \begin{array}{c} \bullet \\ \bar{g} \end{array} \right| + \left(\begin{array}{c} \bullet \\ \beta_4 - \bar{g} \right) \left(\beta_4 - \bar{g} \right) + 2 \end{array} \right| \\
= 8L^{-6[\phi]} \left| \begin{array}{c} \bullet \\ (\beta_4 - \bar{g}) \end{array} \right| \\
\leq 8L^{-6[\phi]} \times ||\Gamma||_{L^1} \times L^3 \times \left(\begin{array}{c} \max \\ \Delta \in [L^{-1}\Delta'] \end{array} \right) \beta_{4,\Delta} - \bar{g}| \right)^2 \\
\leq 2L^{2\epsilon} \bar{g}^{2e_4} .$$

We plug this back into our earlier estimate for $|W_{6,\Delta'}|$ to get:

$$|W_{6,\Delta'}| \le \frac{1}{2} L^{3-6[\phi]} \bar{g}^{e_W} + 2L^{2\epsilon} \bar{g}^{2e_4}$$
.

We again bound $L^{3-6[\phi]}$ by 1. We also bound $L^{2\epsilon}$ by 2 in the small ϵ regime. We also note that $2e_4 \ge e_W$ (this is a consequence of (58) and (56)). This leaves us with the bound:

$$|W_{6,\Delta'}|\bar{g}^{-e_W} \le \frac{9}{2}$$

This finishes the proof of the lemma.

Lemma 84. Suppose that \vec{V} in $B(\bar{V}, \frac{1}{2})$. Then one has the following bound for the R' component of $\vec{V}' = RG_{\text{ex}}(\vec{V})$: for all $\Delta' \in \mathbb{L}$

$$|||R'_{\Delta'}|||_{\bar{g}}\bar{g}^{-e_R} \le \frac{3}{8}.$$

Proof: We use estimates from (4).

$$|||R'_{\Delta}|||_{\bar{g}}\bar{g}^{-e_{R}} \leq \left[|||\mathcal{L}_{\Delta'}^{\vec{\beta},f}(R)|||_{\bar{g}} + |||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}}\right]\bar{g}^{-e_{R}}$$

$$\leq \left[\mathcal{B}_{R\mathcal{L}}L^{3-5[\phi]} \max_{\Delta \in [L^{-1}\Delta']} |||R_{\Delta}|||_{\bar{g}} + |||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}}\right]\bar{g}^{-e_{R}}$$

$$\leq \left[\frac{1}{4}\bar{g}^{e_{R}} + B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta}\right]\bar{g}^{-e_{R}}$$

$$\leq \frac{1}{4} + B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta-e_{R}} \leq \frac{3}{8}$$
(59)

In going to the third line we used that L has been fixed to guarantee $\mathcal{B}_{R\mathcal{L}}L^{3-5[\phi]} \leq \frac{1}{2}$. This was done in (37).

In the last line we used that $\frac{11}{4} - 3\eta - e_R > 0$ which was assumed in (42). Thus by requiring ϵ is sufficiently small we can guarantee $B_{R\xi}\bar{g}^{\frac{11}{4} - 3\eta - e_R} \leq \frac{1}{8}$.

Lemma 85. Suppose that \vec{V} in $B(\bar{V}, \frac{1}{2})$. Then one has the following bound for the β'_4 component of $\vec{V}' = RG_{\rm ex}[\vec{V}]$: For all $\Delta' \in \mathbb{L}$

$$\left|\beta_{4,\Delta'}' - \bar{g}\right| \bar{g}^{-e_4} \le \mathbf{O}_3$$

where $O_3 = 434 + O_{26}$ with O_{26} defined in the statement of Lemma 38.

Proof: Due to our assumption on \vec{V} for any $\Delta \in \mathbb{L}$ we can write $\beta_{4,\Delta} = \bar{g} + \delta g_{\Delta}$ where $|\delta g_{\Delta}| < \frac{1}{2}\bar{g}^{e_4}$. We substitute this into the flow equation to get the following:

$$\beta'_{4,\Delta'} = L^{3-4[\phi]} \bar{g} + L^{-4[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} \delta g_{\Delta} - \delta \beta_{4,2,\Delta'} [\vec{V}] + \xi_{4,\Delta'} [\vec{V}]$$

$$= L^{3-4[\phi]} \bar{g} - 36L^{-4[\phi]} \underbrace{\sum_{\bar{g} + \delta g = \bar{g} + \delta g} - \widetilde{\delta \beta}_{4,2,\Delta'} [\vec{V}] + \xi_{4,\Delta'} [\vec{V}] + L^{-4[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} \delta g_{\Delta} .$$

We have used the fact that $\delta\beta_{4,1,\Delta}[\vec{V}] = 0$. In the formula above $\widetilde{\delta\beta}_{4,2,\Delta'}[\vec{V}]$ is defined to be $\delta\beta_{4,2,\Delta'}[\vec{V}]$ with the graph that we have made explicit removed:

$$\begin{split} \widetilde{\delta\beta}_{4,2,\Delta'} \left[\vec{V} \right] &:= \sum_{a_1,a_2,b_1,b_2,m} \mathbbm{1} \left\{ \begin{array}{l} a_i + b_i \leq 4 \\ a_i \geq 0 \ , \ b_i \geq 1 \end{array} \right\} \frac{(a_1 + b_1)! \ (a_2 + b_2)!}{a_1! \ a_2! \ m! \ (b_1 - m)! \ (b_2 - m)!} \\ &\times \frac{1}{2} C(a_1,a_2|4) \times L^{-(a_1 + a_2)[\phi]} \times C_0(0)^{\frac{a_1 + a_2 - 4}{2}} \times \begin{array}{l} b \\ b_1 - m \end{array} \right. \\ &+ \sum_b \mathbbm{1} \left\{ \begin{array}{l} 4 + b = 5 \text{ or } 6 \\ b \geq 0 \end{array} \right\} \frac{(k + b)!}{k! \ b!} L^{-k[\phi]} \begin{array}{l} f \\ \vdots \\ W_{k+b} \end{array}$$

Indeed, first note that there is no graph with m=3. This is because this would imply $a_1, a_2 \leq 1$ which contradicts $a_1 + a_2 \geq 4$ imposed by the nonvanishing of the connection coefficient $C(a_1, a_2|4)$. Also the removed graph is the only one with m=2. This is because $b_1, b_2 \geq 2$ implies $a_1, a_2 \leq 4-2=2$, but the connection coefficient requires $a_1 + a_2 \geq 4$ so we are forced to have $a_1 = a_2 = 2$ which implies $b_1, b_2 \leq 2$ and therefore $b_1 = b_2 = 2$.

We note that we can decompose the graph above as follows:

$$\underbrace{\overline{g} + \delta g}_{\overline{g} + \delta g} = \underbrace{\overline{g}}_{\overline{g}} + 2 \underbrace{\overline{g}}_{\delta g} + \underbrace{\delta g}_{\delta g} + \underbrace{\delta g}_{\delta g}$$

We now use the fact that \bar{g} is an approximate fixed point:

$$\bar{g} = L^{\epsilon} \bar{g} - A_1 \bar{g}^2 = L^{3-4[\phi]} \bar{g} - 36L^{-4[\phi]}$$
 \bar{g}

Using this we can write:

$$\beta'_{4,\Delta'} = \overline{g} + L^{-4[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} \delta g_{\Delta}$$

$$-36L^{-4[\phi]} \left(2 \underbrace{\overline{g}}_{\delta g} + \underbrace{\delta g}_{\delta g} \right)$$

$$-\widetilde{\delta \beta}_{4,2,\Delta'}[\vec{V}] + \xi_{4,\Delta'}[\vec{V}] .$$

We now describe how to bound the second and third lines of (61). By the same arguments as used in Lemma 38 the contribution of the two graphs on the second line can each be bounded by $4L^5\bar{g}^{2-2\eta}$ as follows from the very coarse bounds $\bar{g} \leq \bar{g}^{1-\eta}$ and $|\delta g| \leq \bar{g}^{1-\eta}$. This gives us:

$$\left|36L^{-4[\phi]}\left(2 \left| \begin{array}{ccc} \\ \overline{g} \end{array} \right| \delta g + \left| \begin{array}{ccc} \\ \delta g \end{array} \right| \delta g \right| \right| \leq 36 \left[2 \left| \begin{array}{ccc} \\ \overline{g} \end{array} \right| \delta g \right| + \left| \begin{array}{ccc} \\ \delta g \end{array} \right| \delta g \right| \left| \begin{array}{cccc} \\ \end{array} \right|$$

$$<36 \times 3 \times 4L^5 \bar{q}^{2-2\eta} = 432L^5 \bar{q}^{2-2\eta}$$
.

Note that in the first line we dropped the factor of $L^{-4[\phi]}$. The quantity $\delta \beta_{4,2,\Delta'}[\vec{V}]$ on the third line of (61) can be bounded by $\mathcal{O}_{26}L^5\bar{g}^{2-2\eta}$ as in Lemma 38 (we are overestimating since we are summing over fewer graphs). We combine this with the estimate on $\xi_{4,\Delta'}[\vec{V}]$ from Theorem 4 to get

$$\left| \beta_{4,\Delta'}' - \bar{g} \right| \bar{g}^{-e_4} \le \frac{1}{2} L^{3-4[\phi]} + (432 + \mathcal{O}_{26}) L^5 \bar{g}^{2-2\eta-e_4} + B_4 \bar{g}^{e_R-e_4}$$

$$\le \frac{1}{2} L^{3-4[\phi]} + (432 + \mathcal{O}_{26}) + 1$$

$$\le 1 + (432 + \mathcal{O}_{26}) + 1$$

Note that in going to the second line that we used $2-2\eta-e_4>0$, this is a consequence of (56). Indeed this allows to have $L^5\bar{g}^{2-2\eta-e_4}\leq 1$ in the small ϵ regime. We also used that $e_4< e_R$ (a consequence of (41)), thus we can guarantee $B_4\bar{g}^{e_R-e_4}\leq 1$ for ϵ sufficiently small. In going to the third we used the bound $L^{3-4[\phi]}=L^{\epsilon}\leq 2$ for ϵ small.

Lemma 86. Suppose that \vec{V} in $B(\bar{V}, \frac{1}{2})$. Then one has the following bound for the β'_k components of $\vec{V}' = RG_{\text{ex}}[\vec{V}]$ when k = 1, 2, 3: for all $\Delta' \in \mathbb{L}$

$$|\beta'_{k,\Lambda'}|\bar{g}^{-e_k} \leq \mathbf{O}_4 L^{\frac{5}{2}}$$

where $O_4 = O_1 + 2$.

Proof: From the flow equations one has:

$$|\beta'_{k,\Delta'}| \le \left| L^{-k[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} \beta_{k,\Delta} \right| + \left| \delta \beta_{k,1,\Delta'}[\vec{V}] \right| + \left| \delta b_{k,2,\Delta'}[\vec{V}] \right| + \left| \xi_{k,\Delta'}[\vec{V}] \right|$$

$$\le L^{3-k[\phi]} \bar{g}^{e_k} + \mathbf{O}_1 L^{5/2} \bar{g}^{e_k} + \mathcal{O}_{26} L^5 \bar{g}^{2-2\eta} + \frac{1}{2} B_k \bar{g}^{e_R} .$$

The bound on the third term on the right hand side of the first line is from Lemma 38 and the bound on the last term of the first line is from Theorem (4). We used Lemma 82 to bound $\left|\delta\beta_{k,1,\Delta'}[\vec{V}]\right|$. Then for k=1,2,3 we have:

$$|\beta'_{k,\Delta'}|\bar{g}^{-e_k} \leq L^{3-k[\phi]} + \mathbf{O}_1 L^{5/2} \bar{g}^{e_{k+1}-e_k} + \mathcal{O}_{26} L^5 \bar{g}^{2-2\eta-e_k} + \frac{1}{2} B_k \bar{g}^{e_R-e_k}$$
$$\leq L^{\frac{5}{2}} + \mathbf{O}_1 L^{5/2} + 1$$

In going to the second line we used that $e_{k+1} \ge e_k$ which is a consequence of (56). We also used that $e_R > e_k$ and $2 - 2\eta > e_k$ which come from (56) and (41). Thus the sum of the last two terms on the first line can be made smaller than 1 by requiring that ϵ be sufficiently small. We also used that for $\epsilon \le 1$ and $k \ge 1$ one has $3 - k[\phi] \le \frac{5}{2}$.

Lemma 87. RG_{ex} is well defined and analytic on $B(\bar{V}, \frac{1}{2})$. Additionally one has the following uniform bound for $\vec{V} \in B(\bar{V}, \frac{1}{2})$:

(62)
$$||RG_{\text{ex}}[\vec{V}] - \bar{V}|| \le \mathbf{O}_5 L^{\frac{5}{2}}$$

where $\mathbf{O}_5 = \max(\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4)$.

Proof:

The fact that the map is well defined and is analytic comes from the Theorem 4 and inspection of the formulas for $\delta \beta_{k,j,\Delta}$ for j=1,2 and k=1,2,3,4. We now establish the uniform bound.

Let $\vec{V}' = RG_{\rm ex}[\vec{V}]$. We have the sufficient estimates on β_k' for k = 1, 2, 3, 4 from Lemmas 85 and 86. We have sufficient estimates on W_k' for k = 5 and 6 from Lemma 83. A sufficient estimate on R' comes from Lemma 84. All that is left is estimating f'.

Note that for any $\Delta' \in \mathbb{L}$ we have

$$\begin{split} |f'_{\Delta'}|\,L^{3-[\phi]} \leq & L^{3-[\phi]}L^{3-[\phi]} \max_{\Delta \in [L^{-1}\Delta']} |f_{\Delta}| \\ \leq & L^{3-[\phi]}L^{3-[\phi]} \left(\frac{1}{2}L^{-(3-[\phi])}\right) = \frac{1}{2}L^{3-[\phi]} \leq \frac{1}{2}L^{\frac{5}{2}} \;. \end{split}$$

On the last line we used the assumption $\epsilon \leq 1$ which implies $3-[\phi] \leq \frac{5}{2}$. Finally note that $\frac{3}{8} < \frac{1}{2} < 14 = \mathbf{O}_2$ to get the formula for the constant \mathbf{O}_5 .

Proposition 10. For any $\vec{V}^1, \vec{V}^2 \in \bar{B}(\bar{V}, \frac{1}{6})$ one has:

$$||RG_{\text{ex}}[\vec{V}^1] - RG_{\text{ex}}[\vec{V}^2]|| \le \mathbf{O}_6 L^{\frac{5}{2}} ||\vec{V}^1 - \vec{V}^2||,$$

where $\mathbf{O}_6 = 4\mathbf{O}_5$.

Proof:

By Lemma 87 we know that RG_{ex} is an analytic map taking $B(\bar{V}, \frac{1}{2})$ into $\bar{B}(\bar{V}, \mathbf{O}_5 L^{\frac{5}{2}})$. We get the desired inequality by applying Lemma 1 with the choice $\nu = \frac{1}{2}$.

After the previous estimates we now return to the analysis of the $r \to -\infty$ and $s \to \infty$ limits of $\mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j})$ which in fact does not depend on s such that $s \geq q_+$. Since the summation range $q_- \leq q < q_+$ is fixed and finite, all we need is to show that RG_{ex} remain in the domains of definition and analyticity, despite the temporary expansion with rate controlled by Lemma 87 and Proposition 10.

The quantity of interest, as delivered by $\S 10.1$, is

$$\mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j}) = \sum_{q_{-} \leq q < q_{+}} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_{+}-q-1}}} \left(\delta b_{\Delta}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta}[\vec{V}^{(r,q)}(0,0)] \right)$$

where

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_{-}} \left(\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) \right)$$

with

$$\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_{+}-q_{-}}\left(\tilde{f}_{\rightarrow(-q_{-})},\left(RG^{q_{-}-r}\left(v - \alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{L^{-q_{-}}\Delta}e_{\phi^{2}}\right)\right)_{\Delta \subset \Lambda_{q_{+}-q_{-}}},RG^{q_{-}-r}(v)\right).$$

It follows from our definitions for the norms and the reinjection map $\mathcal J$ that

$$||\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_{-})}(0,0)||$$

$$= \max \left\{ ||\tilde{f}_{\to(-q_{-})}||_{L^{\infty}}, \max_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_{+}-q_{-}}}} ||RG^{q_{-}-r}(v - \alpha_{\mathbf{u}}^{r}Y_{2}\tilde{j}_{L^{-q_{-}}\Delta}e_{\phi^{2}}) - RG^{q_{-}-r}(v)|| \right\}.$$

We also have

$$||\tilde{f}_{\to(-q_-)}||_{L^{\infty}} = L^{(3-[\phi])q_-}||\tilde{f}||_{L^{\infty}}.$$

We slightly strengthen the requirement in (55) by imposing

$$||\tilde{j}||_{L^{\infty}} \le [240C_1(\epsilon)\alpha_1^{q_-}Y_2\bar{g}^{-e_2}]^{-1}$$

which implies

$$||-\alpha_{\mathbf{u}}^{q_{-}}Y_{2}\tilde{j}_{L^{-q_{-}}\Delta}e_{\phi^{2}}|| \leq \frac{1}{240C_{1}(\epsilon)}$$

for all $\Delta \in \mathbb{L}$ such that $\Delta \subset \Lambda_{q_+-q_-}$. Thus by Lemma 78

$$||RG^{q_{-}-r}(v - \alpha_{\mathbf{u}}^{r} Y_{2} \tilde{j}_{L^{-q_{-}} \Delta} e_{\phi^{2}}) - RG^{q_{-}-r}(v)|| \leq 11 C_{1}(\epsilon) || - \alpha_{\mathbf{u}}^{q_{-}} Y_{2} \tilde{j}_{L^{-q_{-}} \Delta} e_{\phi^{2}}||$$

$$\leq 11 C_{1}(\epsilon) \alpha_{\mathbf{u}}^{q_{-}} Y_{2} \bar{q}^{-e_{2}} \times ||\tilde{j}||_{L^{\infty}}$$

and therefore

$$||\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_{-})}(0,0)|| \leq \max \left\{ L^{(3-[\phi])q_{-}}||\tilde{f}||_{L^{\infty}}, 11\mathcal{C}_{1}(\epsilon)\alpha_{\mathbf{u}}^{q_{-}}Y_{2}\bar{g}^{-e_{2}} \times ||\tilde{j}||_{L^{\infty}} \right\} .$$

On the other hand, minding the \bar{g} shift for β_4 components only, we easily see that

$$||\vec{V}^{(r,q_{-})}(0,0) - \bar{V}|| = ||\iota(RG^{q_{-}-r}(v)) - \bar{V}|| = ||RG^{q_{-}-r}(v)||$$

where the latter quantity can be computed as in section §8, i.e., via the norm inherited by \mathcal{E} from \mathcal{E}_{ex} and expressed in $(\delta g, \mu, R)$ coordinates.

By construction of $W^{s,loc}$, $||RG^{q-r}(v)|| \leq \frac{\rho}{3}$ with $\rho \in (0, \frac{1}{12})$ as yet unspecified. We thus have

$$||\vec{V}^{(r,q_-)}(0,0) - \bar{V}|| \le \frac{1}{12}$$
.

Provided we also have

$$\left(\mathbf{O}_{6}L^{\frac{5}{2}}\right)^{q_{+}-q_{-}}\times \max\left\{L^{(3-[\phi])q_{-}}||\tilde{f}||_{L^{\infty}},11\mathcal{C}_{1}(\epsilon)\alpha_{\mathrm{u}}^{q_{-}}Y_{2}\bar{g}^{-e_{2}}\times ||\tilde{j}||_{L^{\infty}}\right\}\leq \frac{1}{12}$$

then a trivial inductive application of Proposition 10 will garantee that for all $q, q_- \leq q \leq q_+$,

$$||\vec{V}^{(r,q_{-})}(\tilde{f},\tilde{j}) - \bar{V}|| \le \frac{1}{12}$$

so one remains, throughout the iterations, in the domain of definition and analyticity of $RG_{\rm ex}$ as well as the δb functions.

As a result of Theorem 5 we then immediately obtain, regardless of the order of limits,

$$\lim_{\substack{r \to -\infty \\ r,s}} \mathcal{S}_{r,s}^{T,MD}(\tilde{f},\tilde{j}) = \mathcal{S}^{T,MD}(\tilde{f},\tilde{j})$$

where

$$\mathcal{S}^{T,MD}(\tilde{f},\tilde{j}) = \sum_{\substack{q_{-} \leq q < q_{+} \\ \Delta \subset \Lambda_{q_{+}-q-1}}} \left(\delta b_{\Delta}[\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta}[\iota(v_{*})] \right)$$

with

$$\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q-} \left(\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j}) \right)$$

for

(63)
$$\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_+-q_-}\left(\tilde{f}_{\to(-q_-)}, (\Psi v, -\alpha_{\mathbf{u}}^{q_-} Y_2 \tilde{j}_{L^{-q_-} \Delta} e_{\phi^2})_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_+-q_-}}}, v_*\right).$$

Analyticity of $\mathcal{S}^{T,MD}(\tilde{f},\tilde{j})$ is also immediate.

For the purposes of the next section we also note that $\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})$ satisfies the bound (64)

$$||\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_+)}(0,0)|| \le \left(\mathbf{O}_6 L^{\frac{5}{2}}\right)^{q_+ - q_-} \times \max\left\{L^{(3-[\phi])q_-}||\tilde{f}||_{L^{\infty}}, 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-e_2} \times ||\tilde{j}||_{L^{\infty}}\right\}.$$

10.4. The infrared regime. In this section we are concerned with showing that essentially the differential of RG_{ex} at any suitable $\vec{V}_{\text{bk}} \in \mathcal{E}_{\text{bk}}$ in any direction $\dot{V} \in \mathcal{E}_{\text{pt}}$ is a contraction.

We will introduce new notation to facilitate the lemmas below. For $\vec{V}_{bk} \in \mathcal{E}_{bk}$ we write:

$$\vec{V}_{\rm bk} = \{V_{\rm bk}\}_{\Delta \in \mathbb{L}} = \{(\beta_{4,\rm bk}, \dots, \beta_{1,\rm bk}, W_{5,\rm bk}, W_{6,\rm bk}, f_{\rm bk}, R_{\rm bk})\}_{\Delta \in \mathbb{L}} .$$

Note that we do need to burden the notation with Δ subscripts since the quantities above are independent of the box Δ by definition of being in \mathcal{E}_{bk} .

Similarly for $\dot{V} \in \mathcal{E}_{pt}$ we write:

$$\dot{V} = \left\{ \dot{V}_{\Delta} \right\}_{\Delta \in \mathbb{L}} = \left\{ (\dot{\beta}_{4,\Delta}, \dots, \dot{\beta}_{1,\Delta}, \dot{W}_{5,\Delta}, \dot{W}_{6,\Delta}, \dot{f}_{\Delta}, \dot{R}_{\Delta}) \right\}_{\Delta \in \mathbb{L}}.$$

Note that $\dot{V}_{\Delta} = 0$ for $\Delta \neq \Delta(0)$. We also recall that $RG_{\rm ex}[\vec{V}_{\rm bk} + \dot{V}] - RG_{\rm ex}[\vec{V}_{\rm bk}] \in \mathcal{E}_{\rm pt}$ and so in our estimates we are only concerned with the $\Delta(0)$ component of $RG_{\rm ex}[\vec{V}_{\rm bk} + \dot{V}] - RG_{\rm ex}[\vec{V}_{\rm bk}] \in \mathcal{E}_{\rm pt}$.

Lemma 88. Let $\vec{V}_{bk} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{pt}$. Then one has the bound for k = 1, 2, 3, 4 and all $\Delta' \in \mathbb{L}$:

$$\left| \delta \beta_{k,1,\Delta(0)} \left[\vec{V}_{\rm bk} + \dot{V} \right] - \delta \beta_{k,1,\Delta(0)} \left[\vec{V}_{\rm bk} \right] \right| \bar{g}^{-e_k} \le 1 \!\! 1 \!\! \left\{ 1 \le k \le 3 \right\} \! \mathbf{O}_7 L^{-\frac{9}{4}} ||\dot{V}||^2,$$
 where $\mathbf{O}_7 = \left(6 + 21 \times 2^{\frac{3}{2}} \right)$.

Proof: We again note that the vanishing for k=4 follows by inspection of the definition of $\delta\beta_{k,1,\Delta(0)}$. We now observe that $\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\rm bk}]$ vanishes. Indeed, by definition we have

$$\delta \beta_{k,1,\Delta(0)}[\vec{V}_{\rm bk}] = -\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]}$$

$$\int_{\beta_{k+b} \ \rm bk}^{b} f_{\rm bk}$$

However one has that

$$f_{\rm bk}$$
 b $f_{\rm bk}$ $= 0$.

This is because we have at least one integration vertex of degree 1 which has been assigned a coupling $f_{\rm bk}$ which is constant over the integration region $L^{-1}\Delta(0)$. Using ultrametricity and the fact that Γ integrates to 0 allows one to show that after integrating any of the $f_{\rm bk}$ vertices the entire integral vanishes. So

$$\delta \beta_{k,1,\Delta(0)}[\vec{V}_{\rm bk}] = 0$$
.

We now turn to $\delta \beta_{k,1,\Delta(0)}[\vec{V}_{\rm bk}+\dot{V}]$. From the definition we have:

$$\delta\beta_{1,k,\Delta(0)}[\vec{V}_{\rm bk} + \dot{V}] = -\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]}$$

$$\beta_{k+b,\rm bk} + \dot{\beta}_{k+b}$$
the assumption that $h \geq 1$ we have:

Under the assumption that $b \ge 1$ we have:

$$f_{bk} + \dot{f} \qquad b \qquad f_{bk} + \dot{f}$$

$$\beta_{k+b,bk} + \dot{\beta}_{k+b} \qquad = \sum_{j=0}^{b} {b \choose j} \qquad f_{bk} \qquad f$$

$$f_{bk} + \dot{f} \qquad j \qquad b - j$$

$$f_{bk} + \dot{\beta}_{k+b,bk} \qquad f$$

In the sum above only the j=0 term can be non-vanishing, all other diagrams will have at least one integration vertex of degree 1 with a bulk variable assigned to it. We substitute this back into our formula for $\delta \beta_{k,1,\Delta(0)}$ and perform more manipulations:

(65)
$$\delta \beta_{k,1,\Delta(0)}[\vec{V}_{bk} + \dot{V}] = -\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 1 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]}$$

$$= -(k+1)L^{-k[\phi]} \int_{\beta_{k+1,bk} + \dot{\beta}_{k+1}} \dot{f}$$

$$-\sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 2 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]}$$

$$\dot{f} \int_{\beta_{k+1,bk} + \dot{\beta}_{k+1}} \dot{f} \int_{\beta_{k+1,bk}$$

where we have isolated the b=1 term. Note that for k=3 the sum on the last line is empty. We now bound the diagrams appearing above:

(67)
$$\begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1,\text{bk}} + \dot{\beta}_{k+1} \end{vmatrix} \leq \begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1} \end{vmatrix} + \begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1,\text{bk}} \end{vmatrix}$$

$$= \begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} \dot{f} \\ \dot{\beta}_{k+1} \end{vmatrix}$$

$$\leq 2 \left(L^{-(3-[\phi])} ||\dot{V}|| \right) \left(||\dot{V}|| \bar{g}^{e_{k+1}} \right)$$

$$\leq 2 L^{-\frac{9}{4}} ||\dot{V}||^2 \bar{g}^{e_{k+1}} .$$

In going to the third to last line we used local constancy at unit scale and the fact that all the couplings were supported at $\Delta(0)$ so we did not really do any integration. In going to the second to last line we used the bound $|\Gamma(0)| \leq 2$ which comes from Lemma 5. In going to the last line we used the bound $-(3-[\phi]) \leq -\frac{9}{4}$.

For k = 3 we immediately have the bound:

$$\left| \delta \beta_{3,1,\Delta(0)} \right| \bar{g}^{-e_3} \le 4L^{-3[\phi]} \times 2L^{-\frac{9}{4}} ||\dot{V}||^2 \bar{g}^{e_4} \bar{g}^{-e_3}$$

$$< 8L^{-\frac{9}{4}} ||\dot{V}||^2$$

Note that in going to the last line we dropped the factor of $L^{-3[\phi]}$ and used $e_4 \ge e_3$. This proves the lemma for the case k = 3. We now bound the remaining diagrams to prove the lemma for the cases k = 1 and k = 2. Before note that in these two cases k + b = 3 or 4 because we also assume $k \ge 2$.

If k + b = 4 then, because of the domain hypotheses for our lemma and noting the \bar{g} shift for the β_4 component of the bulk, we must have

$$|\beta_{k+b,\text{bk}}| + |\dot{\beta}_{b+k}| \le \bar{g} + \frac{1}{4}\bar{g}^{e_4} + \frac{1}{4}\bar{g}^{e_4} \le \frac{3}{2}\bar{g} \le \frac{3}{2}\bar{g}^{e_k}$$
.

This is because of our assumptions $e_1, e_2 \leq 1 \leq e_4$.

If k + b = 3 then

$$|\beta_{k+b,\text{bk}}| + |\dot{\beta}_{b+k}| \le \frac{1}{4}\bar{g}^{e_3} + \frac{1}{4}\bar{g}^{e_3} \le \frac{3}{2}\bar{g}^{e_k}$$

because of the assumption $e_1 \le e_2 \le e_3$. So in all relevent cases we can use $\frac{3}{2}\bar{g}^{e_k}$ as a bound, as we do next.

(68)
$$\begin{vmatrix}
\dot{f} & \dot{b} & \dot{f} \\
\dot{b} \geq 2
\end{vmatrix} \leq 1 \begin{cases}
k+b \leq 4 \\
b \geq 2
\end{cases} \times |\dot{f}(0)|^{b} \\
\times (|\beta_{b+k,bk}| + |\dot{\beta}_{b+k}|) \times \int_{\mathbb{Q}_{p}^{3}} d^{3}x |\Gamma(x)|^{b} \\
\leq (L^{-(3-[\phi])}||\dot{V}||)^{2} \times \frac{3}{2}\bar{g}^{e_{k}} \times 2^{5/2}L^{3-2[\phi]} \\
\leq 3 \times 2^{3/2}L^{-3}||\dot{V}||^{2}\bar{g}^{e_{k}} .$$

For the bound on the first line we used the fact that all the \dot{f} vertices are pinned to the origin and the only integration occurs at the $\beta_{b+k,bk} + \dot{\beta}_{b+k}$ vertex which has been left with b copies of the fluctuation covariance.

In going to the second to last line we used the bound $|\dot{f}(0)|^b \le |\dot{f}(0)|^2$ since $b \ge 2$ and $|\dot{f}(0)| \le 1$. For that same line also used the following bound which is valid for $2 \le b \le 4$:

$$\int_{\mathbb{Q}_p^3} d^3x |\Gamma(x)|^b \le ||\Gamma||_{L^1} ||\Gamma||_{L^\infty}^{b-1}$$

$$\le \left(\frac{1}{\sqrt{2}} L^{3-2[\phi]}\right) 2^{b-1}$$

$$\le 2^{5/2} L^{3-2[\phi]}.$$

Note that we have used fluctuation covariance bounds of Corollary 1 and Lemma 5. Thus we can use (67) to get the following bound for k = 1 and k = 2:

(69)
$$\left| \sum_{b} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 2 \end{array} \right\} \frac{(k+b)!}{k! \ b!} \ L^{-k[\phi]} \right| \stackrel{\dot{f}}{\underset{\beta_{k+b,\text{bk}}}{\longrightarrow}} \stackrel{\dot{b}}{\underset{\beta_{k+b}}{\longrightarrow}} \dot{f} \\ \times 3 \times 2^{3/2} L^{-3} ||\dot{V}||^2 \bar{g}^{e_k} \\ \leq 21 \times 2^{3/2} L^{-3} ||\dot{V}||^2 \bar{g}^{e_k} \ .$$

Note in going to the last line we dropped the factors of $L^{-k[\phi]}$ and used that

$$\max_{k=1,2} \sum_{l} \mathbb{1} \left\{ \begin{array}{c} k+b \leq 4 \\ b \geq 2 \end{array} \right\} \ \frac{(k+b)!}{k! \ b!} = 7 \ .$$

Finally by inserting the bound (67) and (69) into (65) we get the following bound for k = 1 and k = 2:

$$\left| \delta \beta_{k,1,\Delta(0)} [\vec{V}_{bk} + \dot{V}] \right| \bar{g}^{-e_k} \le (k+1) \times 2L^{-\frac{9}{4}} ||\dot{V}||^2 + 21 \times 2^{3/2} L^{-3} ||\dot{V}||^2$$

$$\le \left(6 + 21 \times 2^{\frac{3}{2}} \right) L^{-\frac{9}{4}} ||\dot{V}||^2 .$$

In going to the last line we simply bounded L^{-3} by $L^{-\frac{9}{4}}$. This proves the lemma for k=1 and k=2 which finishes the proof.

Lemma 89. For k=1,2,3,4 and for all $\Delta' \in \mathbb{L}$ one has that $\delta b_{k,2,\Delta'}[\bullet]$ and $\xi_{k,\Delta'}[\bullet]$ are analytic functions on $B(\bar{V},\frac{1}{2})$ taking values in \mathbb{C} . In particular one has the following bounds for any $\vec{V}^1,\vec{V}^2 \in \bar{B}(\bar{V},\frac{1}{6})$

(70)
$$\left| \delta \beta_{k,2,\Delta'} \left[\vec{V}^1 \right] - \delta \beta_{k,2,\Delta'} \left[\vec{V}^2 \right] \right| \bar{g}^{-e_k} \le \frac{1}{100} ||\vec{V}^1 - \vec{V}^2||,$$

(71)
$$\left| \xi_{k,\Delta'} \left[\vec{V}^1 \right] - \xi_{k,\Delta'} \left[\vec{V}^2 \right] \right| \bar{g}^{-e_k} \le \frac{1}{100} ||\vec{V}^1 - \vec{V}^2||.$$

Proof: The statement of analyticity for $\delta \beta_{k,2,\Delta'}$ immediate from the formulas that define $\delta \beta_{k,2,\Delta'}$. To establish the bound (70) we first use Lemma 38 which gives us the following uniform bound for all $\vec{V} \in B(\bar{V}, \frac{1}{2})$:

$$\left|\delta \beta_{k,2,\Delta'} \left[\vec{V} \right] \right| \leq \mathcal{O}_{26} L^5 \bar{g}^{2-2\eta} \ .$$

Thus by applying Lemma 1 with $\nu = \frac{1}{3}$ we have:

$$\left|\delta\beta_{k,2,\Delta'}\left[\vec{V}^1\right] - \delta\beta_{k,2,\Delta'}\left[\vec{V}^2\right]\right|\bar{g}^{-e_k} \leq 4\mathcal{O}_{26}L^5\bar{g}^{2-2\eta-e_k}||\vec{V}^1 - \vec{V}^2||\ .$$

Note that we have $2-2\eta-e_k>0$ as a consequence of (56), thus by making epsilon sufficiently small we can guarantee $2\mathcal{O}_{26}L^5\bar{g}^{2-2\eta-e_k}\leq \frac{1}{100}$ which proves (70).

For $\xi_{k,\Delta'}$ we have both analyticity and the following uniform bound for $\vec{V} \in B(\bar{V}, \frac{1}{2})$ as consequences of Theorem 4:

$$\left| \xi_{k,\Delta'}[\vec{V}] \right| \le B_k \times \frac{1}{2} \bar{g}^{e_R} .$$

We again use Lemma 1 with $\nu = \frac{1}{3}$ to get:

$$\left| \xi_{k,\Delta'}[\vec{V}^1] - \xi_{k,\Delta'}[\vec{V}^1] \right| \bar{g}^{-e_k} \le 2B_k \bar{g}^{e_R - e_k} ||\vec{V}^1 - \vec{V}^2|| .$$

Note that $e_R > e_k$ because of the assumptions (41) and (56), thus by requiring that ϵ be sufficiently small we can guarantee $2B_k \bar{g}^{e_R - e_k} \leq \frac{1}{100}$ which proves (71).

Given $\vec{V}_{bk} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{pt}$ we define:

$$RG_{\rm dv}[\vec{V}_{\rm bk}, \dot{V}] = RG_{\rm ex}[\vec{V}_{\rm bk} + \dot{V}] - RG_{\rm ex}[\vec{V}_{\rm bk}].$$

Note that, as a subspace of $\mathcal{E}_{\rm ex}$, the space $\mathcal{E}_{\rm bk} \oplus \mathcal{E}_{\rm pt}$ is invariant by $RG_{\rm ex}$. Since $\vec{V}_{\rm bk} + \dot{V} \in \mathcal{E}_{\rm bk} \oplus \mathcal{E}_{\rm pt}$ one has a unique decomposition $RG_{\rm ex}[\vec{V}_{\rm bk} + \dot{V}] = \vec{V}_{\rm bk}' + \dot{V}'$ with $\vec{V}_{\rm bk}' \in \mathcal{E}_{\rm bk}$ and $\dot{V}' \in \mathcal{E}_{\rm pt}$. Using the locality of $RG_{\rm ex}$ it is not hard to see that $\vec{V}_{\rm bk}' = RG_{\rm ex}[\vec{V}_{\rm bk}]$ and $\dot{V}' = RG_{\rm dv}[\vec{V}_{\rm bk}, \dot{V}]$. In particular $RG_{\rm ex}[\bullet, \bullet]$ takes values in $\mathcal{E}_{\rm pt}$.

Lemma 90. Suppose that $\vec{V}_{bk} \in B(\bar{V}, \frac{1}{12}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in B(0, \frac{1}{12}) \cap \mathcal{E}_{pt}$. Define $\dot{V}' = RG_{dv}[\vec{V}_{bk}, \dot{V}]$ and for k = 1, 2, 3, 4 let $\dot{\beta}'_k$ be the corresponding components of \dot{V}' .

We then have the following bound for k = 1, 2, 3, 4

$$\left|\dot{\beta}'_{k,\Delta(0)}\right|\bar{g}^{-e_k} \le \frac{4}{5}||\dot{V}|| + \mathbf{O}_7||\dot{V}||^2$$

where O_7 has been defined in Lemma 88.

Proof: By definition we have:

(72)
$$\dot{\beta}'_{k,\Delta(0)} = L^{-k[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{\beta}_{k,\Delta} \\
+ \left(\delta \beta_{k,1,\Delta(0)} \left[\vec{V}_{bk} \right] - \delta \beta_{k,1,\Delta(0)} \left[\vec{V}_{bk} + \dot{V} \right] \right) \\
+ \left(\delta \beta_{k,2,\Delta(0)} \left[\vec{V}_{bk} \right] - \delta \beta_{k,2,\Delta(0)} \left[\vec{V}_{bk} + \dot{V} \right] \right) \\
+ \left(\xi_{k,\Delta(0)} \left[\vec{V}_{bk} + \dot{V} \right] - \xi_{k,\Delta(0)} \left[\vec{V}_{bk} \right] \right) .$$

Since $\dot{\beta}_{k,\Delta}$ is supported on $\Delta = \Delta(0)$ the sum on the first line only has one non-zero term. We then have the bound:

$$\left| L^{-k[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{\beta}_{k,\Delta} \right| \le L^{-k[\phi]} ||\dot{V}|| \bar{g}^{e_k}$$

$$\le 2^{-\frac{1}{2}} ||\dot{V}|| \bar{g}^{e_k} .$$

In going to the last line we are assuming that $L \geq 2$ and $\epsilon \leq 1$ so that we have $L^{-k[\phi]} \leq L^{-[\phi]} \leq 2^{-\frac{1}{2}}$. We then combine the estimate above with the lemmas 88, 89, and 90 to get the bound:

$$\left| \dot{\beta}'_{k,\Delta(0)} \right| \bar{g}^{-e_k} \le 2^{-\frac{1}{2}} ||\dot{V}|| + \mathbf{O}_7 \mathbb{1} \{ 1 \le k \le 3 \} L^{-\frac{9}{4}} ||\dot{V}||^2 + \frac{1}{100} ||\dot{V}|| + \frac{1}{100} ||\dot{V}|| .$$

We get the bound of this lemma by dropping the factor of $L^{-\frac{9}{4}}$ and the indiciator function in the fourth term while also observing that $2^{-\frac{1}{2}} + \frac{1}{100} + \frac{1}{100} < \frac{4}{5}$.

Lemma 91. Suppose that $\vec{V}_{bk} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{pt}$. Let $\dot{V}' = RG_{dv}[\vec{V}_{bk}, \dot{V}]$ and for k = 5, 6 let \dot{W}'_k be the corresponding components of \dot{V}' .

We then have the following bound for k = 5, 6

$$\left|\dot{W}'_{k,\Delta(0)}\right| \le 2^{-\frac{5}{2}} ||\dot{V}|| + \mathbf{O}_8 ||\dot{V}||^2,$$

where $\mathbf{O}_8 = \left(18 + \frac{9}{\sqrt{2}}\right)$.

Proof: For k = 5 we have:

$$\dot{W}'_{5,\Delta(0)} = L^{-5[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{5,\Delta}
+ 48L^{-5[\phi]} \left(\int_{\beta_{4,bk} + \dot{\beta}_{4}}^{f_{bk} + \dot{\beta}_{4}} f_{bk} + \dot{\beta}_{4} - \int_{\beta_{4,bk}}^{f_{bk}} f_{bk} \right)
+ 6L^{-5[\phi]} \left(\int_{W_{6,bk} + \dot{W}_{6}}^{f_{bk} + \dot{f}} f_{bk} - \int_{W_{6,bk}}^{f_{bk}} f_{bk} \right)
+ 12L^{-5[\phi]} \left(\int_{\beta_{4,bk} + \dot{\beta}_{4}}^{\beta_{3,bk} + \dot{\beta}_{3}} f_{3,bk} - \int_{\beta_{4,bk}}^{\beta_{3,bk}} f_{3,bk} \right) .$$

As before using that $\dot{W}_{5,\Delta}$ is supported on $\Delta = \Delta(0)$ gives us the bound:

$$\left| L^{-5[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{5,\Delta} \right| \le L^{-5[\phi]} \bar{g}^{e_W} ||\dot{V}||.$$

We now bound the various graphs appearing in (73). We again note that when a graph has an integration vertex of degree one that has been assigned a bulk variable the graph will vanish. This tells us that:

We use this same observation to break up the non-vanishing graphs and show that their contribution is second order in $||\dot{V}||$. For example:

$$\beta_{4,bk} + \dot{\beta}_{4} \quad \beta_{4,bk} + \dot{\beta}_{4} = \dot{\beta}_{4} \quad \beta_{4,bk} + \dot{\beta}_{4} + \dot{\beta}_{4,bk} \quad \beta_{4,bk} + \dot{\beta}_{4} + \dot{\beta}_{4,bk} \quad \beta_{4,bk} + \dot{\beta}_{4}$$

$$+ \dot{\beta}_{4,bk} \quad \beta_{4,bk} + \dot{\beta}_{4} + \dot{\beta}_{4} \quad \beta_{4,bk} + \dot{\beta}_{4}$$

$$= \dot{\beta}_{4} \quad \beta_{4,bk} + \dot{\beta}_{4}$$

$$= \dot{\beta}_{4} \quad \beta_{4,bk} + \dot{\beta}_{4}$$

after expanding the two outer vertices of valence one.

We then have

(74)
$$\begin{vmatrix}
f_{bk} + |\dot{f}| \\
\beta_{4} + \dot{\beta}_{4} + \dot{\beta}_{4}
\end{vmatrix} = \begin{vmatrix}
\dot{f} \\
\dot{\beta}_{4} + \dot{\beta}_{4}
\end{vmatrix} \times \begin{vmatrix}
\dot{f} \\
\dot{\beta}_{4,bk} + \dot{\beta}_{4}
\end{vmatrix} \\
\leq |\dot{f}(0)| \times |\dot{\beta}_{4,bk} + \dot{\beta}_{4}| \\
\leq |\dot{f}(0)| \times |\dot{f}_{4,bk} + \dot{f}_{4}| \\
\leq |\dot{f}(0)| \times |\dot{f}_{4,bk} + \dot{f}_{4,bk} + \dot{f}_{4,bk} + \dot{f}_{4,bk} + \dot{f}_{4,bk} + \dot{f}_{4,bk} \\
\leq |\dot{f}(0)| \times |\dot{f}_{4,bk} + \dot{f}_{4,bk} + \dot{f}$$

Note that in going to the second to last line we again used the bound:

$$\int_{\mathbb{Q}_p^3} d^3 x \, |\Gamma(x)|^n \le 2^{n - \frac{3}{2}} L^{3 - 2[\phi]} .$$

Proceeding similarly for the other graphs we have:

(75)
$$\begin{vmatrix} f_{\text{bk}} + \dot{f} \\ W_{6,\text{bk}} + \dot{W}_{6} \end{vmatrix} = \begin{vmatrix} \dot{f}(0) \\ \dot{W}_{6} \end{vmatrix} = |\dot{f}(0)| \times |\dot{W}_{6,\Delta(0)}| \times |\Gamma(0)| \\ \leq L^{-(3-[\phi])} \bar{g}^{e_{W}} |\Gamma(0)| \times ||\dot{V}||^{2} \\ \leq 2\bar{g}^{e_{W}} ||\dot{V}||^{2}.$$

In going to the last line we used the bound $|\Gamma(0)| \leq 2$ which is a consequence of Corollary 1. We also dropped the factor of $L^{-(3-[\phi])} \leq L^{-\frac{9}{4}} \leq 1$. We continue to the last graph we need to bound for \dot{W}_5' :

(76)
$$\begin{vmatrix} \beta_{3,\text{bk}} + \dot{\beta}_{3} \\ \beta_{4,\text{bk}} + \dot{\beta}_{4} \end{vmatrix} \leq \begin{vmatrix} \dot{\beta}_{3} \\ \dot{\beta}_{4} \end{vmatrix} = \begin{vmatrix} \dot{\beta}_{4,\Delta(0)} \\ \times \begin{vmatrix} \dot{\beta}_{3,\Delta(0)} \\ & \leq 2 ||\dot{V}||^{2} \bar{g}^{e_{4} + e_{3}} \end{vmatrix}.$$

Using the bounds (74), (75), and (76) in (73) gives us the bound:

$$\begin{split} \left| \dot{W}_{5,\Delta(0)} \right| \bar{g}^{-e_W} \leq & L^{-5[\phi]} ||\dot{V}|| + L^{-5[\phi]} \left[48 \times 3 \times 2^{-\frac{1}{2}} \bar{g}^{1+e_4-e_W} + 6 \times 2 + 12 \times 2 \bar{g}^{e_4+e_3-e_W} \right] ||\dot{V}||^2 \\ \leq & 2^{-\frac{5}{2}} ||\dot{V}|| + 2^{-\frac{5}{2}} \left[48 \times 3 \times 2^{-\frac{1}{2}} + 6 \times 2 + 12 \times 2 \right] ||\dot{V}||^2 \\ = & 2^{-\frac{5}{2}} ||\dot{V}|| + \left(18 + \frac{9}{\sqrt{2}} \right) ||\dot{V}||^2 \; . \end{split}$$

In going to the second line we used the fact that $\epsilon \leq 1$ and $L \geq 2$ to bound $L^{-5[\phi]} \leq 2^{-\frac{5}{2}}$. We also used that $1 + e_4 - e_W \geq 0$ and $1 + e_4 - e_W \geq 0$, these are both consequences of (58). This proves the lemma for k = 5.

For k = 6 we have:

(77)
$$\dot{W}'_{6,\Delta(0)} = L^{-6[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{6,\Delta} + 8L^{-6[\phi]} \left(\begin{array}{c} \beta_{4,bk} + \dot{\beta}_{4} \\ \beta_{4,bk} + \dot{\beta}_{4} \end{array} \right) \begin{pmatrix} \beta_{4,bk} \\ \beta_{4,bk} \end{pmatrix} .$$

Proceeding as last time we see:

$$\left| L^{-6[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{6,\Delta} \right| \le L^{-6[\phi]} ||\dot{V}|| \bar{g}^{e_W}$$

and

$$\int_{\beta_{4,bk}}^{\beta_{4,bk}} = 0, \qquad \int_{\beta_{4,bk} + \dot{\beta}_{4}}^{\beta_{4,bk} + \dot{\beta}_{4}} = \dot{\beta}_{4}$$

which simplifies the right-hand side of (77). We now bound the contributing graph:

(78)
$$\begin{vmatrix} \dot{\beta}_4 \\ \dot{\beta}_4 \end{vmatrix} = |\dot{\beta}_{4,\Delta(0)}|^2 \times |\Gamma(0)| \\ \leq 2||\dot{V}||^2 \bar{g}^{2e_4} .$$

Inserting (78) along with the our earlier bound into (77) gives us:

$$\begin{split} \left| \dot{W}_{6,\Delta(0)}' \right| e^{-e_W} &\leq L^{-6[\phi]} ||\dot{V}|| + 8L^{-6[\phi]} \times 2||\dot{V}||^2 \bar{g}^{2e_4 - e_W} \\ &\leq 2^{-3} ||\dot{V}|| + 2||\dot{V}||^2 \; . \end{split}$$

In going to the last line we used our assumption that $\epsilon \leq 1$ and $L \geq 2$ to bound $L^{-6[\phi]} \leq 2^{-3}$. We also used that $2e_4 - e_W \geq 0$ which is a consequence of (58) and (56). This proves the bound of our lemma for the case k = 6 which finishes the proof.

Lemma 92. For any $\Delta' \in \mathbb{L}$ let $R'_{\Delta'}[\vec{V}]$ be the corresponding component of $RG_{\mathrm{ex}}[\vec{V}]$.

Let $\vec{V}^1, \vec{V}^2 \in \bar{B}(\bar{V}, \frac{1}{20})$. Then one has the following bound:

$$|||R'_{\Delta'}[\vec{V}^1] - R'_{\Delta}[\vec{V}^2]|||_{\bar{g}} \ \bar{g}^{-e_R} \leq \frac{27}{32}||\vec{V}^1 - \vec{V}^2|| \ .$$

Proof: By Theorem 4 and Lemma 84 we have that $R'_{\Delta'}[\bullet]$ is an analytic function from $B(\bar{V}, \frac{1}{2}) \subset \mathcal{E}_{\mathrm{ex}}$ into $\bar{B}(0, \frac{3}{8}\bar{g}^{e_R}) \subset C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C})$ where we are using the norm $|||\bullet|||_{\bar{g}}$ on $C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C})$. One can then use Lemma 1 with $\nu = \frac{1}{10}$ to get the bound:

$$\begin{split} |||R'_{\Delta'}[\vec{V}^1] - R'_{\Delta}[\vec{V}^2]|||_{\bar{g}} &\leq \frac{\frac{3}{8}\bar{g}^{e_R}\left(1 - \frac{1}{10}\right)}{\frac{1}{2}\left(1 - \frac{2}{10}\right)}||\vec{V}^1 - \vec{V}^2|| \\ &= \frac{27}{32}\bar{g}^{e_R}||\vec{V}^1 - \vec{V}^2|| \; . \end{split}$$

This proves the lemma.

Lemma 93. Suppose that $\vec{V}_{bk} \in \bar{B}(\bar{V}, \frac{1}{40}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in \bar{B}(0, \frac{1}{40}) \cap \mathcal{E}_{pt}$. Let $\dot{V}' = RG_{dv}[\vec{V}_{bk}, \dot{V}]$. Then one has the following bound:

$$||\dot{V}'|| \le \frac{27}{32} ||\dot{V}|| + \mathbf{O}_9 ||\dot{V}||^2$$

where $\mathbf{O}_9 = \max(\mathbf{O}_7, \mathbf{O}_8)$.

Proof: Note that $\dot{V}'_{\Delta'}$ is supported on $\Delta' = \Delta(0)$. The necessary estimates for the $\dot{\beta}'$ and \dot{W}' components of \dot{V}' come from Lemmas 90 and 91. For the R bound we note that by Lemma 92 one has:

$$\begin{aligned} |||\dot{R}'_{\Delta(0)}|||_{\bar{g}}\bar{g}^{-e_{R}} &= |||R'_{\Delta(0)}[\vec{V}_{\rm bk} + \dot{V}] - R'_{\Delta(0)}[\vec{V}_{\rm bk}]|||_{\bar{g}}\bar{g}^{-e_{R}} \\ &\leq \frac{27}{32}||\dot{V}|| \ . \end{aligned}$$

The last component we must estimate is \dot{f}' which can be done easily.

$$\begin{split} |\dot{f}'_{\Delta(0)}|L^{-(3-[\phi])} = & L^{-[\phi]} \left| \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{f}_{\Delta} \right| L^{-(3-[\phi])} \\ = & L^{-[\phi]} |\dot{f}_{\Delta(0)}| L^{-(3-[\phi])} \\ \leq & L^{-[\phi]} ||\dot{V}|| \\ \leq & \frac{27}{32} ||\dot{V}|| \; . \end{split}$$

In going to the second line we used the fact that \dot{f}_{Δ} is supported on $\Delta = \Delta(0)$. In going to the last line our assumptions that $\epsilon \leq 1$ and $L \geq 2$ give us that $L^{-[\phi]} \leq 2^{-\frac{1}{2}} < \frac{27}{32}$. The lemma is then proved.

Proposition 11. Suppose that $\vec{V}_{bk} \in \bar{B}(\bar{V}, \frac{1}{40}) \cap \mathcal{E}_{bk}$ and $\dot{V} \in \bar{B}(0, \mathbf{O}_{10}) \cap \mathcal{E}_{pt}$ where $\mathbf{O}_{10} = \min(\frac{1}{40}, \frac{3}{32}\mathbf{O}_9^{-1})$. Let $\dot{V}' = RG_{dv}[\vec{V}_{bk}, \dot{V}]$ Then one has the following bound:

$$||\dot{V}'|| \le \frac{15}{16} ||\dot{V}||$$
.

Proof: This proposition is a direct consequence of Lemma 93.

For the control of the infrared contributions to the log-moment generating function we will finally need a very coarse Lipschitz estimate on the δb functions.

Lemma 94. For all \vec{V}^1 , \vec{V}^2 in $\bar{B}\left(\bar{V}, \frac{1}{6}\right)$ we have

$$|\delta b_{\Delta(0)}[\vec{V}^1] - \delta b_{\Delta(0)}[\vec{V}^2]| \le 4||\vec{V}^1 - \vec{V}^2|| \ .$$

Proof: By our assumptions on exponents, $||\vec{V} - \bar{V}|| < \frac{1}{2}$ implies one is in the domain of applicability of Theorem 4 as well as all the lemmas that led to its proof. In particular Lemma 42 with $\lambda = 1$ gives us the bound $|\delta b_{\Delta(0)}[\vec{V}]| \leq 1$ provided $\mathcal{O}_{30}L^5\bar{g}^{\frac{11}{12}-\frac{1}{3}\eta_R} \leq 1$. However we can take the latter for granted since we are in the small ϵ regime and $\frac{11}{12} - \frac{1}{3}\eta_R > 0$. Now Lemma 1 with $\nu = \frac{1}{3}$ immediately produces the desired estimate.

Now recall from §10.1 that

$$\mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) = \sum_{q_{+} < q < s} \left(\delta b_{\Delta(0)} \left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) \right] - \delta b_{\Delta(0)} \left[\vec{V}^{(r,q)}(0,0) \right] \right)$$

where

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_+} \left(\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) \right) .$$

With a view to lighten the notation we write

$$\vec{V}^{(r,q)}(\tilde{f}, \tilde{j}) = \vec{V}_{hk}^{(r,q)} + \dot{V}^{(r,q)}$$

where

$$\vec{V}_{\rm bk}^{(r,q)} = \vec{V}^{(r,q)}(0,0) = \iota(RG^{q-r}(v)) \in \mathcal{E}_{\rm bk}$$

and

$$\dot{V}^{(r,q)} = \vec{V}^{(r,q)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q)}(0,0) \in \mathcal{E}_{\mathrm{pt}} \ .$$

We will control the latter via Proposition 11.

First note that

$$||\vec{V}_{\mathrm{bk}}^{((r,q))} - \bar{V}|| = ||RG^{q-r}(v)|| \le \frac{\rho}{3}.$$

To make this at most $\frac{1}{40}$ we add the new requirement on ρ :

$$\rho \le \frac{3}{40} \ .$$

If we can ensure that $||\dot{V}^{(r,q_+)}|| \leq \mathbf{O}_{10}$ then a trivial inductive use of Proposition 11 will imply that

$$||\dot{V}^{(r,q)}|| \leq \mathbf{O}_{10} \times \left(\frac{15}{16}\right)^{q-q_+}$$

for all q, such that $q_+ \leq q \leq s$. We again include the value s although it does not belong to what we called the infrared regime in order to pass the baton to the next section about controlling the boundary term. In view of (64), we now impose the new domain condition

$$\left(\mathbf{O}_{6}L^{\frac{5}{2}} \right)^{q_{+}-q_{-}} \times \max \left\{ L^{(3-[\phi])q_{-}} ||\tilde{f}||_{L^{\infty}}, 11\mathcal{C}_{1}(\epsilon)\alpha_{\mathbf{u}}^{q_{-}}Y_{2}\bar{g}^{-e_{2}} \times ||\tilde{j}||_{L^{\infty}} \right\} \leq \mathbf{O}_{10} .$$

Now Proposition 11 followed by Lemma 94 imply that for any q with $q_+ \leq q < s$ we have

$$\left|\delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})\right] - \delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(0,0)\right]\right| \leq 4\mathbf{O}_{10} \times \left(\frac{15}{16}\right)^{q-q_+}.$$

Hence we get the uniform absolute convergence of the sum over q needed to say

$$\lim_{\substack{r \to -\infty \\ s \to \infty}} \mathcal{S}_{r,s}^{T,IR}(\tilde{f},\tilde{j}) = \mathcal{S}^{T,IR}(\tilde{f},\tilde{j})$$

with

$$\mathcal{S}^{T,IR}(\tilde{f},\tilde{j}) = \sum_{q=q_{\perp}}^{\infty} \left(\delta b_{\Delta(0)} [\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta(0)} [\iota(v_*)] \right)$$

where

$$\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_{-}} \left(\vec{V}^{(-\infty,q_{-})}(\tilde{f},\tilde{j}) \right)$$

and $\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j})$ has been defined in (63). The limit $\mathcal{S}^{T,IR}(\tilde{f},\tilde{j})$ is analytic and the order of the $r\to -\infty$, $s\to \infty$ limits is immaterial.

10.5. The boundary term. Let $\vec{V} \in \mathcal{E}_{ex}$ and simply denote by

$$(\beta_4, \beta_3, \beta_2, \beta_1, W_5, W_6, f, R) \in \mathbb{C}^7 \times C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C})$$

its component at $\Delta = \Delta(0)$. We let

$$\partial \mathcal{Z}[\vec{V}] = \int d\mu_{C_0}(\phi) \ e^{f\phi} \times \left\{ \exp\left(-\beta_4 : \phi^4 :_{C_0} - \beta_3 : \phi^3 :_{C_0} - \beta_2 : \phi^2 :_{C_0} - \beta_1 : \phi :_{C_0}\right) \right.$$
$$\left. \times (1 + W_5 : \phi^5 :_{C_0} + W_6 : \phi^6 :_{C_0}) + R(\phi) \right\}$$

which reduces to an integral over a single real variable still denoted by ϕ . Let $\partial \mathcal{Z}_* = \partial \mathcal{Z}[\iota(v_*)]$ which is the value at the infrared fixed point. We have

$$\partial \mathcal{Z}_* = \int d\mu_{C_0}(\phi) \left\{ \exp\left(-g_* : \phi^4 :_{C_0} - \mu_* : \phi^2 :_{C_0}\right) + R_*(\phi) \right\}$$

with $g_* = \bar{g} + \delta g_*$. Recall that g_* , μ_* , R_* are real. Note that by Jensen's inequality and the basic properties of Wick ordering on has the lower bound

$$\int d\mu_{C_0}(\phi) \exp\left(-g_*: \phi^4:_{C_0} - \mu_*: \phi^2:_{C_0}\right) \ge \exp\left(-\int d\mu_{C_0}(\phi) \left(g_*: \phi^4:_{C_0} + \mu_*: \phi^2:_{C_0}\right)\right) = 1.$$

Besides

$$\left| \int d\mu_{C_0}(\phi) \ R_*(\phi) \right| \le \sup_{\phi \in \mathbb{R}} |R_*(\phi)| \le \sup_{\phi \in \mathbb{R}} ||R_*(\phi)||_{\partial \phi, \phi, h}$$

$$\le \bar{g}^{-2} |||R_*|||_{\bar{g}} \le \bar{g}^{e_R - 2} \frac{\rho}{13} .$$

Since $e_R > e_4 + 1 \ge 2$, $\bar{g} \le 1$ and $\rho < \frac{3}{40}$, we clearly have $\partial \mathcal{Z}_* \ge \frac{1}{2}$. Now if $||\vec{V} - \bar{V}|| < \frac{1}{2}$ it is easy to see that $|\partial \mathcal{Z}[\vec{V}]| \le C_5(\epsilon)$ with

$$C_5(\epsilon) = \int d\mu_{C_0}(\phi) \ e^{\frac{1}{2}L^{3-[\phi]}|\phi|} \times \\ \left\{ \exp\left[-\frac{1}{2}\bar{g}\phi^4 + \frac{3}{4}\bar{g}^{1-\eta} \left(|\phi|^3 + 13\phi^2 + 7|\phi| + 14 \right) \right] \right. \\ \left. \times \left(1 + \frac{1}{2}\bar{g}^{2-2\eta} \left(|\phi|^5 + 20|\phi|^3 + 60|\phi| \right) + \frac{1}{2}\bar{g}^{2-2\eta} \left(\phi^6 + 30\phi^4 + 180\phi^2 + 120 \right) \right) \\ \left. + \frac{1}{2}\bar{g}^{e_R - 2} \right\} \ .$$

Indeed, by undoing the Wick ordering

$$-\Re \left[\beta_4:\phi^4:_{C_0}+\beta_3:\phi^3:_{C_0}+\beta_2:\phi^2:_{C_0}+\beta_1:\phi:_{C_0}\right] = -\bar{g}\phi^4 - Y(\phi)$$

with

$$Y(\phi) = \Re(\beta_4 - \bar{g})\phi^4$$

$$+ (\Re\beta_3)\phi^3$$

$$+ (\Re\beta_2 - 6C_0(0)\Re\beta_4)\phi^2$$

$$+ (\Re\beta_1 - 3C_0(0)\Re\beta_3)\phi$$

$$+ (-C_0(0)\Re\beta_2 + 3C_0(0)^2\Re\beta_4) .$$

Using $|\Re(\beta_4 - \bar{g})| < \frac{1}{2}\bar{g}^{e_4} \le \frac{1}{2}\bar{g}$ for the fourth degree monomial and $|\Re\beta_k| \le \frac{3}{2}\bar{g}^{1-\eta}$ for k = 1, 2, 3, 4 when bounding the lower degree monomials, and finally using $C_0(0) \le 2$ we obtain

$$|Y(\phi)| \le \frac{1}{2}\bar{g}\phi^4 + \frac{3}{4}\bar{g}^{1-\eta}(|\phi|^3 + 13\phi^2 + 7|\phi| + 14)$$
.

The bounds on $W_k: \phi^k:_{C_0}$, for k=5,6 are similar and use the explicit Wick ordering formulas given in the proof of Lemma 21.

Since $\partial \mathcal{Z}[\vec{V}]$ is clearly analytic in the domain $||\vec{V} - \bar{V}|| < \frac{1}{2}$, Lemma 1 with $\nu = \frac{1}{3}$ tell us that for all \vec{V}^1 , \vec{V}^2 in $\vec{B}(\bar{V}, \frac{1}{6})$ one has the Lipschitz estimate

$$|\partial \mathcal{Z}[\vec{V}^1] - \partial \mathcal{Z}[\vec{V}^2]| \le 4C_5(\epsilon)||\vec{V}^1 - \vec{V}^2||$$
.

We now have, using the outcome of the discussion for the infrared regime

$$\begin{aligned} |\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) - \partial \mathcal{Z}_{*}| &= |\partial \mathcal{Z}[\vec{V}^{(r,s)}(\tilde{f},\tilde{j})] - \partial \mathcal{Z}[\iota(v_{*})]| \\ &\leq 4\mathcal{C}_{5}(\epsilon) \times \left[||\vec{V}^{(r,s)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,s)}(0,0)|| + ||\vec{V}^{(r,s)}(0,0) - \iota(v_{*})|| \right] \\ &\leq 4\mathcal{C}_{5}(\epsilon) \times \left[||\dot{V}^{(r,s)}|| + ||RG^{s-r}(v) - v_{*}|| \right] \\ &\leq 4\mathcal{C}_{5}(\epsilon) \times \left[\mathbf{O}_{10} \times \left(\frac{15}{16} \right)^{s-q_{+}} + c_{1}(\epsilon)^{s-r} ||v - v_{*}|| \right] . \end{aligned}$$

One of course has a similar and simpler estimate for the quantity $\partial \mathcal{Z}_{r,s}(0,0)$ appearing in the denominator of the boundary ratio. Namely, the \mathbf{O}_{10} term is absent. Bounding $c_1(\epsilon)^{s-r}$ by $c_1(\epsilon)^{s-q_+}$ and using the previous lower bound $\partial \mathcal{Z}_* \geq \frac{1}{2}$ we see that

$$\frac{\partial \mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial \mathcal{Z}_{r,s}(0,0)} \longrightarrow 1$$

when $s \to \infty$, uniformly in $r \le q_-$. Therefore the boundary term $\mathcal{S}^{T,BD}$ disappears when $r \to -\infty$, $s \to \infty$ regardless of the order of limits.

11. Construction of the limit measures and invariance properties

As a consequence of what we have shown in the previous section we see that

$$S_{r,s}(\tilde{f}, \tilde{j}) = \exp\left(S_{r,s}^{\mathrm{T}}(\tilde{f}, \tilde{j})\right)$$

converges uniformly to the analytic function

$$\mathcal{S}(\tilde{f},\tilde{j}) = \exp\left(\mathcal{S}^{\mathrm{T}}(\tilde{f},\tilde{j})\right)$$

in a suitable neighborhood of $\tilde{f} = \tilde{j} = 0$ in $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$, when $r \to -\infty$ and $s \to \infty$. Using the multivariate Cauchy formula it is immediate that the cut-off correlators

$$\left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) \ N_r[\tilde{\phi}^2](\tilde{j}_1) \cdots N_r[\tilde{\phi}^2](\tilde{j}_m) \right\rangle_{r,s} = \frac{1}{(2i\pi)^{n+m}} \oint \cdots \oint \prod_{j=1}^n \frac{\mathrm{d}z_j}{z_j^2} \prod_{k=1}^m \frac{\mathrm{d}u_k}{u_k^2} \ \mathcal{S}_{r,s}(z_1\tilde{f}_1 + \cdots + z_n\tilde{f}_n, u_1\tilde{j}_1 + \cdots + u_m\tilde{j}_m)$$

converge to the similar integrals with S instead of $S_{r,s}$. The contours of integration are governed by the domain condition (79). We define our mixed correlators by

$$\left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) N[\tilde{\phi}^2](\tilde{j}_1) \cdots N[\tilde{\phi}^2](\tilde{j}_m) \right\rangle = \frac{1}{(2i\pi)^{n+m}} \oint \cdots \oint \prod_{j=1}^n \frac{\mathrm{d}z_j}{z_j^2} \prod_{k=1}^m \frac{\mathrm{d}u_k}{u_k^2} \mathcal{S}(z_1 \tilde{f}_1 + \cdots + z_n \tilde{f}_n, u_1 \tilde{j}_1 + \cdots + u_m \tilde{j}_m)$$

which are multilinear in the \tilde{f} 's and \tilde{j} 's. Because of the uniform bounds on $\mathcal{S}_{r,s}^{\mathrm{T}}$, and therefore on \mathcal{S}^{T} , proved in the last section and thanks to Cauchy's formula, it is immediate that the pure $\tilde{\phi}$ or $N[\tilde{\phi}^2]$ correlators will satisfy Condition 4) in Theorem 2. The other conditions are satisfied by the cut-off correlators $\langle \cdots \rangle_{r,s}$ as joint moments of random variables obtained from the probability measures $\nu_{r,s}$. As these properties are preserved in the limit $r \to -\infty$ and $s \to \infty$ we can use Theorem 2 to affirm the existence and uniqueness of the measures ν_{ϕ} and ν_{ϕ^2} mentioned in Theorem 3. By the uniqueness part of Theorem 2, the invariance properties of the measures ν_{ϕ} and ν_{ϕ^2} follow from those of the moments. Hence it is enough to show Parts 1) and 2) of Theorem 3. These are easier to prove from the functional integral definitions of the cut-off correlators.

Indeed, one can trivially check that for $M \in GL_3(\mathbb{Z}_p)$ one has

$$\left\langle \tilde{\phi}(M \cdot \tilde{f}_1) \cdots \tilde{\phi}(M \cdot \tilde{f}_n) \ N_r[\tilde{\phi}^2](M \cdot \tilde{j}_1) \cdots N_r[\tilde{\phi}^2](M \cdot \tilde{j}_m) \right\rangle_{r,s} = \left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) \ N_r[\tilde{\phi}^2](\tilde{j}_1) \cdots N_r[\tilde{\phi}^2](\tilde{j}_m) \right\rangle_{r,s}$$

because $d\mu_{C_r}$ is invariant by rotation and $M \cdot \Lambda_s = \Lambda_s$.

Also if $y \in \mathbb{Q}_p^3$ with $|y| \leq L^s$ then

$$\left\langle \tilde{\phi}(\tau_y \tilde{f}_1) \cdots \tilde{\phi}(\tau_y \tilde{f}_n) \ N_r[\tilde{\phi}^2](\tau_y \tilde{j}_1) \cdots N_r[\tilde{\phi}^2](\tau_y \tilde{j}_m) \right\rangle_{r,s} = \left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) \ N_r[\tilde{\phi}^2](\tilde{j}_1) \cdots N_r[\tilde{\phi}^2](\tilde{j}_m) \right\rangle_{r,s}$$

because Λ_s is unchanged by this translation as results from ultrametricity.

Finally, by changing variables from $\tilde{\phi}$ to $\tilde{\phi}_{\sim 1}$, one has

$$\left\langle \tilde{\phi}(L \cdot \tilde{f}_1) \cdots \tilde{\phi}(L \cdot \tilde{f}_n) \ N_r[\tilde{\phi}^2](L \cdot \tilde{j}_1) \cdots N_r[\tilde{\phi}^2](L \cdot \tilde{j}_m) \right\rangle_{r,s} =$$

$$\left\langle \tilde{\phi}(\tilde{f}_1) \cdots \tilde{\phi}(\tilde{f}_n) \ N_r[\tilde{\phi}^2](\tilde{j}_1) \cdots N_r[\tilde{\phi}^2](\tilde{j}_m) \right\rangle_{r+1,s+1} \times \left[L^{-(3-[\phi])} \right]^n \times \left[L^{-(3-2[\phi])} Z_2^{-1} \right]^m .$$

Noting that $|L| = L^{-1}$ and $Z_2 = L^{-\frac{1}{2}\eta_{\phi^2}}$ by definition of η_{ϕ^2} , and from the existence of the $r \to -\infty$, $s \to \infty$ limits, we see that the property in Part 3) of Theorem 3 holds for $\lambda = L$. Thus it holds for the subgroup $L^{\mathbb{Z}}$ it generates.

A trivial consequence of these invariance properties is that

$$\langle N[\tilde{\phi}^2](\tilde{j})\rangle = 0$$

identically. Namely, there is no one-point function. Indeed, it is enough to show this for $\tilde{j} = \mathbb{1}_{\mathbb{Z}_p^3}$. In that case, by translation invariance followed by scale invariance

$$\begin{split} \langle N[\tilde{\phi}^2](\mathbbm{1}_{\mathbb{Z}_p^3})\rangle &= L^3 \langle N[\tilde{\phi}^2](\mathbbm{1}_{(L\mathbb{Z}_p)^3})\rangle \\ &= L^3 \times L^{-3+2[\phi]+\frac{1}{2}\eta_{\phi^2}} \times \langle N[\tilde{\phi}^2](\mathbbm{1}_{\mathbb{Z}_p^3})\rangle \\ &= L^3 \alpha_{_{\mathrm{II}}}^{-1} \times \langle N[\tilde{\phi}^2](\mathbbm{1}_{\mathbb{Z}^3})\rangle \;. \end{split}$$

By Lemma 66 it is clear that $L^3\alpha_{\rm u}^{-1}>1$ for ϵ small and the vanishing follows. We leave it as an exercise to show this same fact explicitly using the $\mathcal{S}^{\rm T,UV}+\mathcal{S}^{\rm T,IR}$ expression for $\langle N[\tilde{\phi}^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle$. This hinges on showing that the vector in $\mathcal{E}_{\rm pt}$ corresponding to an $e_{\rm u}$ perturbation in the box $\Delta(0)$ only is an eigenvector of $D_{\iota(v_*)}RG_{\rm ex}$ with eigenvalue $L^{-3}\alpha_{\rm u}$. One has a similar statement for the evaluation of $D_{\iota(v_*)}\delta b_{\Delta(0)}$ on that vector.

12. Nontriviality and proof of existence of anomalous dimension

12.1. The two-point and four-point functions of the elementary field. We have constructed the generalized random field $\tilde{\phi}$ via constructing and proving the analyticity of $\mathcal{S}^{\mathrm{T}}(\tilde{f},0)$, the cumulant generating function. We now show that the process $\tilde{\phi}$ is not Gaussian. In particular we show that in the small ϵ regime one has

$$\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) = \langle \tilde{\phi}(\mathbb{1}_{\mathbb{Z}_p^3})^4 \rangle - 3\langle \tilde{\phi}(\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle < 0.$$

We establish the inequality above by expanding $\mathcal{S}^{\mathrm{T}}(z1\!\!1_{\mathbb{Z}_p^3},0)$ and isolating a part that explicitly contains first order pertubation theory. We will calculate the derivative by hand for this explicit part and use Cauchy bounds to estimate the contribution of the remainder. From now on we will drop the tildes from the notation for the fields $\tilde{\phi}$ and $N[\tilde{\phi}^2]$ but we will still use tildes for test functions if needed.

Since $z\mathbb{1}_{\mathbb{Z}_p^3} \in S_{0,0}(\mathbb{Q}_p^3,\mathbb{C})$ we can set $q_- = q_+ = 0$. From section §10 and in particular the domain condition (79) we know that $S^T(z\mathbb{1}_{\mathbb{Z}_p^3},0)$ is an analytic function for z such that $|z| < \mathbf{O}_{10}$. This condition is assumed

throughout this section. We will repeatedly make use of the fact that for z in this domain $|z| \le 1$ which follows from $O_{10} \le \frac{1}{40}$. In particular for z in that domain we have

$$\mathcal{S}^{\mathrm{T}}(z1\!\!1_{\mathbb{Z}^3_p},0) = \!\!\mathcal{S}^{\mathrm{T},\mathrm{FR}}(z1\!\!1_{\mathbb{Z}^3_p},0) + \mathcal{S}^{\mathrm{T},\mathrm{UV}}(z1\!\!1_{\mathbb{Z}^3_p},0) + \mathcal{S}^{\mathrm{T},\mathrm{MD}}(z1\!\!1_{\mathbb{Z}^3_p},0) + \mathcal{S}^{\mathrm{T},\mathrm{IR}}(z1\!\!1_{\mathbb{Z}^3_p},0) \ .$$

For our choice of test function we have:

$$\begin{split} \mathcal{S}^{\mathrm{T,FR}}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) &= \frac{1}{2} z^2 \left(1\!\!1_{\mathbb{Z}_p^3}, C_{-\infty} 1\!\!1_{\mathbb{Z}_p^3} \right) \\ \mathcal{S}^{\mathrm{T,UV}}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) &= 0 \text{ since } \tilde{j} = 0 \\ \\ \mathcal{S}^{\mathrm{T,MD}}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) &= 0 \text{ since } q_- = q_+ = 0 \\ \\ \mathcal{S}^{\mathrm{T,IR}}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) &= \sum_{q=0}^{\infty} \left(\delta b_{\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) \right] - \delta b_{\Delta(0)} [\vec{V}_*] \right) \end{split}$$

where
$$\vec{V}_* = \iota(v_*) = \vec{V}^{(-\infty,q)}(0,0)$$
.

By previous considerations we know that up to scale $q_-=0$ the test function $\tilde{f}=z1\!\!1_{\mathbb{Z}_p}$ does not influence the evolution of the other parameters, thus for scales $q\leq q_-=0$ all components of $\vec{V}^{(-\infty,q)}(z1\!\!1_{\mathbb{Z}_p^3},0)$ other than the f component take their fixed point value. Additionally we know that for scales $q\geq q_+=0$ the vector $\vec{V}^{(-\infty,q)}$ deviates from \vec{V}_* only at $\Delta=\Delta(0)$.

We write

$$\vec{V}^{(-\infty,q)}(z1\!\!1_{\mathbb{Z}_p^3},0) = \left((\beta_{4,\Delta}^{(q)},\dots,\beta_{1,\Delta}^{(q)},W_{5,\Delta}^{(q)},W_{6,\Delta}^{(q)},f_{\Delta}^{(q)},R_{\Delta}^{(q)}) \right)_{\Delta \in \mathbb{L}}.$$

Keeping our previous observations in mind for k = 1, 2, 3, 4 we decompose $\beta_{k,\Delta}^{(q)}$ as follows:

$$\begin{split} \beta_{4,\Delta}^{(q)} &= \begin{cases} g_* + \beta_4^{(q, \exp)} + \beta_4^{(q, imp)} & \text{if } \Delta = \Delta(0) \\ g_* & \text{if } \Delta \neq \Delta(0) \end{cases} \\ \beta_{3,\Delta}^{(q)} &= \begin{cases} \beta_3^{(q, \exp)} + \beta_3^{(q, imp)} & \text{if } \Delta = \Delta(0) \\ 0 & \text{if } \Delta \neq \Delta(0) \end{cases} \\ \beta_{2,\Delta}^{(q)} &= \begin{cases} \mu_* + \beta_2^{(q, \exp)} + \beta_2^{(q, imp)} & \text{if } \Delta = \Delta(0) \\ \mu_* & \text{if } \Delta \neq \Delta(0) \end{cases} \\ \beta_{1,\Delta}^{(q)} &= \begin{cases} \beta_1^{(q, \exp)} + \beta_1^{(q, imp)} & \text{if } \Delta = \Delta(0) \\ 0 & \text{if } \Delta \neq \Delta(0) \end{cases} \end{split}$$

Here "exp" and "imp" are abbreviations for explicit and implicit. The quantities $\beta_k^{(q, \exp)}$ and $\beta_k^{(q, imp)}$ will be defined inductively starting from q = 0. We start with the following intital condition:

for
$$k = 1, 2, 3, 4$$
 we set $\beta_k^{(0, \exp)} = \beta_k^{(0, imp)} = 0$.

Now we prepare to give the inductive part of the definition. Recall that for k = 1, 2, 3, 4 the evolution of our couplings is given by

$$\beta_{k,\Delta(0)}^{(q+1)} = L^{-k[\phi]} \left(\sum_{\Delta \in [L^{-1}\Delta(0)]} \beta_{k,\Delta}^{(q)} \right) - \delta \beta_{k,1,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right]$$

$$- \delta \beta_{k,2,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] + \xi_{k,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] .$$

We introduce some more short hand. For k=1,2,3,4 we define β_k^* to be the corresponding component of $\vec{V}_* \in \mathcal{E}_{bk}$. In particular $\beta_4^* = g_*$, $\beta_3^* = 0$, $\beta_2^* = \mu_*$, and $\beta_1^* = 0$. These are also seen as constant vectors in $\mathbb{C}^{\mathbb{L}}$.

We now use the fact that \vec{V}_* is a fixed point of RG_{ex} to arrive at the following formula:

(80)
$$\beta_{k,\Delta(0)}^{(q+1)} = \beta_k^* + L^{-k[\phi]} \left(\beta_k^{(q,\exp)} + \beta_k^{(q,\operatorname{imp})} \right) \\ - \delta \beta_{k,1,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] \\ + \left(\delta \beta_{k,2,\Delta(0)} \left[\vec{V}_* \right] - \delta \beta_{k,2,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] \right) \\ - \left(\xi_{k,\Delta(0)} \left[\vec{V}_* \right] - \xi_{k,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] \right).$$

Above we have used the fact that $\delta b_{k,1,\Delta}\left[\vec{V}_*\right]=0$. We now decompose $\delta \beta_{k,1,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right]$. For $0 \leq k < l \leq 4$ and $\beta, f \in \mathbb{C}^{\mathbb{L}}$ define

$$F_{k,l}[\beta, f] = \binom{l}{k} \int_{(L^{-1}\Delta(0))^{l-k}} d^3 a \ d^3 b_1 \cdots d^3 b_{l-k} \ \beta(a) \times \prod_{i=1}^{l-k} [\Gamma(a-b_i)f(b_i)].$$

With this notation we have:

$$\delta \beta_{k,1,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] = -\sum_{l=k+1}^4 L^{-k[\phi]} F_{k,l} \left[\beta_l^{(q)}, f^{(q)} \right].$$

With this notation we define the evolution for $\beta_k^{(q, \exp)}$ and $\beta_k^{(q, imp)}$ as follows:

(81)
$$\beta_k^{(q+1),\exp} = L^{-k[\phi]} \beta_k^{(q,\exp)} + \sum_{l=k+1}^4 L^{-k[\phi]} F_{k,l} \left[\beta_l^* + \beta_l^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right]$$

(82)

$$\begin{split} \beta_k^{(q+1),\text{imp}} &= L^{-k[\phi]} \beta_k^{(q,\text{imp})} + \sum_{l=k+1}^4 L^{-k[\phi]} F_{k,l} \left[\beta_l^{(q,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] \\ &+ \left(\delta \beta_{k,2,\Delta(0)} [\vec{V}_*] - \delta \beta_{k,2,\Delta(0)} [\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0)] \right) + \left(\xi_{k,\Delta(0)} [\vec{V}_*] - \xi_{k,\Delta(0)} [\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0)] \right) \; . \end{split}$$

Here we have designated $\mathbb{1}_{\Delta(0)} : \mathbb{L} \to \mathbb{C}$ as the indicator function of $\{\Delta(0)\}$.

We also impose a splitting of the difference of vacuum renormalizations at $\Delta(0)$. For $q \geq 0$ we have:

$$\delta b_{\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) \right] - \delta b_{\Delta(0)} [\vec{V}_*] = \delta b^{(q, \exp)} + \delta b^{(q, \mathrm{imp})} \ .$$

We define

(83)
$$\delta b^{(q,\exp)} = -\sum_{l=1}^{4} F_{0,l} \left[\beta_l^* + \beta_l^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] ,$$

$$\delta b^{(q,\text{imp})} = -\sum_{l=1}^{4} F_{0,l} \left[\beta_l^{(q,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(q)} \right]$$

$$+ \left(\delta \beta_{0,2,\Delta(0)} [\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0)] - \delta \beta_{0,2,\Delta(0)} [\vec{V}_*] \right)$$

$$+ \left(\xi_{0,\Delta(0)} [\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0)] - \xi_{0,\Delta(0)} [\vec{V}_*] \right) .$$
(84)

We now derive explicit formulas for $\beta_k^{(q, \exp)}$ and $\delta b^{(q, \exp)}$.

Lemma 95. Given the previous inductive definitions for $\beta_k^{(q, exp)}$ for $q \ge 0$ and k = 1, 2, 3, 4 we have the following explicit formulas:

$$\begin{split} \beta_4^{(q, \exp)} &= 0 \\ \beta_3^{(q, \exp)} &= 0 \\ \beta_2^{(q, \exp)} &= 6qL^{-2q[\phi]}z^2g_*||\Gamma||_{L^2}^2 \\ \beta_1^{(q, \exp)} &= z^3g_*L^{-q[\phi]} \left[4\frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}} \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ \Gamma(x)^3 \right) + 12 \left(\sum_{n=0}^{q-1} nL^{-2n[\phi]} \right) ||\Gamma||_{L^2}^2 \times \Gamma(0) \right] \ . \end{split}$$

For $q \geq 0$ we also have

$$\begin{split} \delta b^{(q, \exp)} &= -\,z^4 g_* \left[L^{-4q[\phi]} \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3 x \ \Gamma(x)^4 \right) + 6 L^{-4q[\phi]} q ||\Gamma||_{L^2}^2 \Gamma(0)^2 + 12 L^{-2q[\phi]} \left(\sum_{n=0}^{q-1} n L^{-2n[\phi]} \right) ||\Gamma||_{L^2}^2 \Gamma(0)^2 \right. \\ &+ \left. 4 L^{-2q[\phi]} \frac{1 - L^{-2q[\phi]}}{1 - L^{-2[\phi]}} \Gamma(0) \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3 x \ \Gamma(x)^3 \right) \right] - z^2 \mu_* L^{-2[\phi]q} ||\Gamma||_{L^2}^2 \ . \end{split}$$

Proof: We first note that below one often sees expressions of the form $\int_{L^{-1}\Delta(0)} \Gamma(x)^n$. In the statement of the theorem we extended the integration to all of \mathbb{Q}_p^3 , we can do this since Γ is supported on $L^{-1}\Delta(0)$.

For $\beta_4^{(q, \exp)}$ the result is immediate after recalling that $\beta_4^{(0, \exp)} = 0$ and noticing the evolution for this parameter reduces to multiplication by $L^{-4[\phi]}$.

For
$$\beta_3^{(q,\text{exp})}$$
 we have

$$\beta_3^{(q,\exp)} = \sum_{n=0}^{q-1} L^{-3[\phi](q-n)} F_{3,4} [\beta_4^* + \beta_4^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}]$$

$$= \sum_{n=0}^{q-1} L^{-3[\phi](q-n)} F_{3,4} [g_*, f^{(n)}]$$

$$= \sum_{n=0}^{q-1} 0.$$

The last line follows from ultrametricity and the fact that Γ integrates to 0. In particular $F_{j,j+1}\left[\beta_j^*, f^{(\cdot)}\right]$ will always vanish.

For $\beta_2^{(q,\exp)}$ we have

$$\begin{split} \beta_2^{(q, \exp)} &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} \left(F_{2,4} \left[\beta_4^* + \beta_4^{(n, \exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] + F_{2,3} \left[\beta_3^* + \beta_3^{(n, \exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right) \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} F_{2,4} \left[g_*, f^{(n)} \right] \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} 6 \left(\int_{(L^{-1}\Delta(0))^3} \mathrm{d}^3 a \, \, \mathrm{d}^3 b_1 \, \, \mathrm{d}^3 b_2 \, g_* \prod_{i=1,2} \left[\Gamma(a-b_i) L^{-n[\phi]} z \mathbb{1}_{\mathbb{Z}_p}(b_i) \right] \right) \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} 6 z^2 g_* L^{-2n[\phi]} \left(\int_{L^{-1}\Delta(0)} \mathrm{d}^3 a \, \Gamma(a)^2 \right) \\ &= \sum_{n=0}^{q-1} L^{-2q[\phi]} 6 z^2 g_* ||\Gamma||_{L^2}^2 \end{split}$$

from which the formula for $\beta_2^{(q, \exp)}$ follows. Note that above we used the fact that $f^{(n)} = L^{-n[\phi]} \mathbbm{1}_{\Delta(0)}$ as a vector in $\mathbb{C}^{\mathbb{L}}$ or $L^{-n[\phi]} \mathbbm{1}_{\mathbb{Z}_p^3}$ as function on \mathbb{Q}_p^3 .

For $\beta_1^{(q,\text{exp})}$ we have

$$\beta_{1}^{(q,\exp)} = \sum_{n=0}^{q-1} L^{-(q-n)[\phi]} \left(F_{1,4} \left[\beta_{4}^{*} + \beta_{4}^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] + F_{1,3} \left[\beta_{3}^{*} + \beta_{3}^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right)$$

$$+ F_{1,2} \left[\beta_{2}^{*} + \beta_{2}^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right)$$

$$= \sum_{n=0}^{q-1} L^{-(q-n)[\phi]} \left(F_{1,4} \left[g_{*}, f^{(n)} \right] + F_{1,2} \left[\mu_{*} + \beta_{2}^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right).$$

Looking at the terms involved one sees

$$F_{1,4}\left[g_*, f^{(n)}\right] = 4g_* z^3 L^{-3n[\phi]} \left(\int_{L^{-1}\Delta(0)} d^3x \ \Gamma(x)^3\right)$$

and

$$\begin{split} F_{1,2} \left[\mu_* + \beta_2^{(n, \exp)} \, \mathbbm{1}_{\Delta(0)}, f^{(n)} \right] = & F_{1,2} \left[\mu_*, f^{(n)} \right] + F_{1,2} \left[\beta_2^{(n, \exp)} \, \mathbbm{1}_{\Delta(0)}, f^{(n)} \right] \\ = & F_{1,2} \left[\beta_2^{(n, \exp)} \, \mathbbm{1}_{\Delta(0)}, f^{(n)} \right] \\ = & 2L^{-n[\phi]} z \Gamma(0) \times \left(6nL^{-2n[\phi]} z^2 g_* ||\Gamma||_{L^2}^2 \right). \end{split}$$

The formula for $\beta_1^{(q,\exp)}$ then follows.

We now move on to $\delta b^{(q,\exp)}$. To keep things lighter we have left out terms with a vanishing contribution:

$$\delta b^{(q,\exp)} = -F_{0,4} \left[g_*, f^{(q)} \right] - F_{0,2} \left[\beta_2^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] - F_{0,2} \left[\mu_*, f^{(q)} \right] - F_{0,1} \left[\beta_1^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] .$$

We calculate each of the terms appearing above:

$$F_{0,4} \left[g_*, f^{(q)} \right] = z^4 g_* L^{-4q[\phi]} \left(\int_{L^{-1}\Delta(0)} d^3 x \; \Gamma(x)^4 \right)$$

$$F_{0,2} \left[\beta_2^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] = z^2 L^{-2q[\phi]} \Gamma(0)^2 \times \left[6q L^{-2q[\phi]} z^2 g_* ||\Gamma||_{L^2}^2 \right]$$

$$F_{0,2} \left[\mu_*, f^{(q)} \right] = z^2 L^{-2q[\phi]} ||\Gamma||_{L^2}^2 \mu_*$$

$$F_{0,1} \left[\beta_1^{(q,\exp)} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] = z L^{-q[\phi]} \Gamma(0) \times \left\{ z^3 g_* L^{-q[\phi]} \left[4 \frac{1 - L^{-2q[\phi]}}{1 - L^{-2[\phi]}} \left(\int_{\mathbb{Q}_p^3} d^3 x \; \Gamma(x)^3 \right) + 12 \left(\sum_{n=0}^{q-1} n L^{-2[\phi]n} \right) ||\Gamma||_{L^2}^2 \times \Gamma(0) \right] \right\}.$$

This proves the formula for $\delta b^{(q,\exp)}$.

We now calculate running bounds for the $\beta_k^{(q,\text{imp})}$.

Lemma 96. In the small ϵ regime one has the following bounds for $q \geq 0$

$$\begin{aligned} |\beta_{4}^{(q,\text{imp})}| \leq & \mathbf{O}_{11} \times q \times L^{8} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^{q} \\ |\beta_{3}^{(q,\text{imp})}| \leq & 17 \times \mathbf{O}_{11} \times q \times L^{8} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^{q} \\ |\beta_{2}^{(q,\text{imp})}| \leq & 253 \times \mathbf{O}_{11} \times q \times L^{8} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^{q} \\ |\beta_{1}^{(q,\text{imp})}| \leq & 2497 \times \mathbf{O}_{11} \times q \times L^{8} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^{q} \\ |\delta b^{(q,\text{imp})}| \leq & \mathbf{O}_{12} \times L^{8} \times \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^{q} \end{aligned}$$

where $\mathbf{O}_{11} = (4\mathcal{O}_{26} + 1)$ and $\mathbf{O}_{12} = 319617 \times \mathbf{O}_{11}$.

Proof: We note that for all $q \ge 0$ one has $\vec{V}^{(-\infty,q)}(z1\!\!1_{\mathbb{Z}_p^3},0), \vec{V}_* \in \bar{B}(0,\frac{1}{6})$. Thus by the proof of Lemma 89 we have the following bounds for all $q \ge 0$ and for k = 0, 1, 2, 3, 4.

$$\begin{split} \left| \delta \beta_{k,2,\Delta(0)} \left[\vec{V}_* \right] - \delta \beta_{k,2,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] \right| \leq & 4 \mathcal{O}_{26} L^5 \bar{g}^{2-2\eta} || \vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_* || \\ \left| \xi_{k,\Delta(0)} \left[\vec{V}_* \right] - \xi_{k,\Delta(0)} \left[\vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \right] \right| \leq & 2 B_k \bar{g}^{e_R} || \vec{V}^{(-\infty,q)}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_* || . \end{split}$$

We also note that by applying the bound of Proposition 11 q-times one has:

$$\begin{split} ||\vec{V}^{(-\infty,q)}(z1\!\!1_{\mathbb{Z}_p^3},0) - \vec{V}_*|| = &||\vec{V}^{(-\infty,q)}(z1\!\!1_{\mathbb{Z}_p^3},0) - \vec{V}^{(-\infty,q)}(0,0)|| \\ \leq & \left(\frac{15}{16}\right)^q ||\vec{V}^{(-\infty,0)}(z1\!\!1_{\mathbb{Z}_p^3},0) - \vec{V}^{(-\infty,0)}(0,0)|| \\ \leq & \left(\frac{15}{16}\right)^q \; . \end{split}$$

Now we note that by (41) and (38) one has $e_R > e_4 + 1 \ge 2$. Thus $e_R > 2 - 2\eta$ so in the ϵ small regime one has:

$$\begin{split} \left| \delta \beta_{k,2,\Delta(0)} [\vec{V}_*] - \delta \beta_{k,2,\Delta(0)} [\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0)] \right| + \left| \xi_{k,\Delta(0)} [\vec{V}_*] - \xi_{k,\Delta(0)} [\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0)] \right| \\ & \leq (4\mathcal{O}_{26} + 1) \, L^5 \bar{g}^{2-2\eta} ||\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \\ & = \mathbf{O}_{11} L^5 \bar{g}^{2-2\eta} ||\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \\ & \leq \mathbf{O}_{11} L^5 \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^q \; . \end{split}$$

We start with estimating $\beta_4^{(q,\text{imp})}$:

$$\begin{split} |\beta_4^{(q,\text{imp})}| \leq & L^{4[\phi]} \sum_{n=0}^{q-1} L^{-4(q-n)[\phi]} \left(\left| \delta \beta_{4,2,\Delta(0)} [\vec{V}_*] - \delta \beta_{4,2,\Delta(0)} [\vec{V}^{(-\infty,n)}(z \mathbbm{1}_{\mathbb{Z}_p^3}, 0)] \right| \right) \\ & + \left| \xi_{4,\Delta(0)} [\vec{V}_*] - \xi_{4,\Delta(0)} [\vec{V}^{(-\infty,n)}(z \mathbbm{1}_{\mathbb{Z}_p^3}, 0)] \right| \right) \\ \leq & L^{4[\phi]} \mathbf{O}_{11} \times L^5 \times \bar{g}^{2-2\eta} \sum_{n=0}^{q-1} L^{-4(q-n)[\phi]} \left(\frac{15}{16} \right)^n \\ \leq & L^{4[\phi]} \mathbf{O}_{11} \times L^5 \bar{g}^{2-2\eta} \sum_{n=0}^{q-1} \left(\frac{15}{16} \right)^{(q-n)} \left(\frac{15}{16} \right)^n \\ \leq & \mathbf{O}_{11} \times q L^{5+4[\phi]} \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q . \end{split}$$

In going to the second to last line we used the fact that for $L \geq 2$ and $\epsilon \leq 1$ we have the following inequality: $L^{-4[\phi]} \leq L^{-[\phi]} \leq 2^{-\frac{1}{2}} < \left(\frac{15}{16}\right)$. Then by bounding $L^{5+4[\phi]} \leq L^8$ we get the desired bound for $\beta_4^{(q,\text{imp})}$.

For $\beta_3^{(q,\text{imp})}$ we have

$$\begin{split} |\beta_{3}^{(q,\text{imp})}| \leq & L^{3[\phi]} \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left(\left| \delta \beta_{3,2,\Delta(0)} [\vec{V}_{*}] - \delta \beta_{3,2,\Delta(0)} [\vec{V}^{(-\infty,n)}(z 1\!\!1_{\mathbb{Z}_{p}^{3}}, 0)] \right| \right. \\ & + \left| \xi_{3,\Delta(0)} [\vec{V}_{*}] - \xi_{3,\Delta(0)} [\vec{V}^{(-\infty,n)}(z 1\!\!1_{\mathbb{Z}_{p}^{3}}, 0)] \right| \right) \right] \\ & + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4} \left[\beta_{4}^{(n,\text{imp})} 1\!\!1_{\Delta(0)}, f^{(n)} \right] \right| \right] \\ \leq & L^{3[\phi]} \mathbf{O}_{11} L^{5} \bar{g}^{2-2\eta} \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left(\frac{15}{16} \right)^{n} \right] + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4} \left[\beta_{4}^{(n,\text{imp})} 1\!\!1_{\Delta(0)}, f^{(n)} \right] \right| \right] \\ \leq & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^{q} + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4} \left[\beta_{4}^{(n,\text{imp})} 1\!\!1_{\Delta(0)}, f^{(n)} \right] \right| \right] \, . \end{split}$$

In the above expressions the first term was bounded just as it was for $\beta_4^{(q,\text{imp})}$. We now try to estimate the summands appearing inside of the second term. We will use $|z| \leq 1$.

$$\begin{aligned}
\left| F_{3,4} \left[\beta_4^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| &\leq 4L^{-n[\phi]} |\beta_4^{(n,\text{imp})}| \times |\Gamma(0)| \\
&\leq 8L^{-n[\phi]} \mathbf{O}_{11} n L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n \\
&\leq 16 \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n .
\end{aligned}$$

In going to the second to last line we used the bound $|\Gamma(0)| \leq ||\Gamma||_{L^{\infty}} \leq 2$. In going to the last line note that for $\epsilon \leq 1$ and $L \geq 2$ one has $nL^{-n[\phi]} \leq n2^{-\frac{n}{2}} \leq \frac{2}{e \times \log(2)} \leq 2$. Inserting this into our previous inequality gives us

$$\begin{aligned} |\beta_3^{(q,\text{imp})}| \leq & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^q + 16 \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \sum_{n=0}^{q-1} \left[L^{-3(q-n)[\phi]} \left(\frac{15}{16}\right)^n \right] \\ \leq & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^q + 16 \mathbf{O}_{11} q L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^q \\ \leq & 17 \mathbf{O}_{11} q L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16}\right)^q . \end{aligned}$$

Not that in going to the second line we used the bound $L^{-3(q-n)[\phi]} \leq \left(\frac{15}{16}\right)^{(q-n)}$.

We start on $\beta_2^{(q,\text{imp})}$ by making the following estimates:

$$\begin{split} \left| F_{2,4} \left[\beta_4^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| &\leq 6 \times \left| \beta_4^{(n,\text{imp})} \right| \times \Gamma(0)^2 \times L^{-2n[\phi]} \\ &\leq 24 \times \mathbf{O}_{11} n L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n L^{-2n[\phi]} \\ &\leq 48 \times \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n \; . \end{split}$$

Similarly one gets the bound

$$\left| F_{2,3} \left[\beta_3^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \le 204 \times \mathbf{O}_{11} L^6 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n$$

The bound for $\beta_2^{(q,\text{imp})}$ then proceeds along familiar lines. One uses the same arguments to prove the estimate for $\beta_1^{(q,\text{imp})}$. In particular

$$\begin{aligned}
& \left| F_{1,4} \left[\beta_4^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \le 64 \times \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n \\
& \left| F_{1,3} \left[\beta_3^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \le 408 \times \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n \\
& \left| F_{1,2} \left[\beta_2^{(n,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \le 2024 \times \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^n .
\end{aligned}$$

To bound $\delta b^{(q,\text{imp})}$ we first make the following estimate. For k=1,2,3,4 one has:

$$\begin{aligned}
\left| F_{0,k} \left[\beta_k^{(q,\text{imp})} \mathbb{1}_{\Delta(0)}, f^{(q)} \right] \right| &\leq L^{-kq[\phi]} \times \Gamma(0)^k \times |\beta_k^{(q,\text{imp})}| \\
&\leq L^{-q[\phi]} \times 2^4 \times 2497 \times \mathbf{O}_{11} \times qL^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q \\
&\leq 79904 \times \mathbf{O}_{11} L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q .
\end{aligned}$$

We then have

$$\begin{split} |\delta b^{(q,\text{imp})}| &\leq \left| \delta \beta_{0,2,\Delta(0)}[V_*] - \delta \beta_{0,2,\Delta(0)}[\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0)] \right| \\ &+ \left| \xi_{0,\Delta(0)}[V_*] - \xi_{0,\Delta(0)}[\vec{V}^{(-\infty,q)}(z 1\!\!1_{\mathbb{Z}_p^3}, 0)] \right| \\ &+ \left[\sum_{k=1}^4 \left| F_{0,k} \left[\beta_k^{(q,\text{imp})} 1\!\!1_{\Delta(0)}, f^{(q)} \right] \right| \right] \\ &\leq \mathbf{O}_{11} \times L^5 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q + 4 \times 79904 \times \mathbf{O}_{11} \times L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q \\ &\leq 319617 \times \mathbf{O}_{11} \times L^8 \bar{g}^{2-2\eta} \left(\frac{15}{16} \right)^q \; . \end{split}$$

This gives the desired bound.

Lemma 97. In the ϵ small regime and on the domain $\{z \in \mathbb{C} \mid |z| < \mathbf{O}_{10}\}$ one has the decomposition $\mathcal{S}^{\mathrm{T}}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) = S^{\mathrm{T,exp}}(z) + S^{\mathrm{T,imp}}(z)$.

All three of the above functions are analytic on the above domain. Additionally, over this domain one has the following explicit formula

$$\begin{split} \mathcal{S}^{\mathrm{T,exp}}(z) &= -\sum_{q=0}^{\infty} \left\{ z^4 g_* \left[L^{-4q[\phi]} \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3 x \ \Gamma(x)^4 \right) + 6 L^{-4q[\phi]} q ||\Gamma||_{L^2}^2 \times \Gamma(0)^2 \right. \\ &+ 12 L^{-2q[\phi]} \left(\sum_{n=0}^{q-1} n L^{-2n[\phi]} \right) ||\Gamma||_{L^2}^2 \times \Gamma(0)^2 \\ &+ 4 L^{-2q[\phi]} \frac{1 - L^{-2q[\phi]}}{1 - L^{-2[\phi]}} \Gamma(0) \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3 x \ \Gamma(x)^3 \right) \right] + z^2 \mu_* L^{-2q[\phi]} ||\Gamma||_{L^2}^2 \right\} \\ &+ \frac{z^2}{2} \left(1\!\!1_{\mathbb{Z}_p^3}, C_{-\infty} 1\!\!1_{\mathbb{Z}_p^3} \right) \end{split}$$

and the following uniform bound

$$|\mathcal{S}^{\mathrm{T,imp}}(z)| \leq \mathbf{O}_{13} L^8 \bar{g}^{2-2\eta}.$$

where $O_{13} = 16 \times O_{12}$.

Proof: From earlier definitions we have that

$$S^{\mathrm{T}}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) = \frac{z^2}{2} \left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty} \mathbb{1}_{\mathbb{Z}_p^3} \right) + \sum_{q=0}^{\infty} \left(\delta b^{(q, \exp)} + \delta b^{(q, \mathrm{imp})} \right) .$$

We define

$$\begin{split} \mathcal{S}^{\mathrm{T,exp}}(z) &= \frac{z^2}{2} \left(1\!\!1_{\mathbb{Z}_p^3}, C_{-\infty} 1\!\!1_{\mathbb{Z}_p^3} \right) + \sum_{q=0}^{\infty} \delta b^{(q, \mathrm{exp})} \ . \\ \mathcal{S}^{\mathrm{T,imp}}(z) &= \sum_{q=0}^{\infty} \delta b^{(q, \mathrm{imp})} \ . \end{split}$$

The explicit formula given for $S^{T,exp}(z)$ comes from substitution of the explicit formula for the $\delta b^{(q,exp)}$ from Lemma 95. Since $[\phi] > 0$ for $\epsilon \in (0,1]$ it is not hard to see that infinite sum in the expression for $S^{T,exp}(z)$ is uniformly absolutely summable on our domain. Analyticity follows from the explicit formula.

On the other hand we have

$$\begin{aligned} |\mathcal{S}^{\mathrm{T,imp}}(z)| &\leq \sum_{q=0}^{\infty} |\delta b^{(q,\mathrm{imp})}| \\ &\leq \mathbf{O}_{12} \times L^6 \times \bar{g}^{2-2\eta} \sum_{q=0}^{\infty} \left(\frac{15}{16}\right)^q \\ &\leq 16 \times \mathbf{O}_{12} \times L^8 \times \bar{g}^{2-2\eta} \ . \end{aligned}$$

We have then proved the desired uniform bound and we have uniform absolute convergence yielding analyticity as well. \Box

Lemma 98. In the small ϵ regime one has

$$\left| \frac{d^2}{dz^2} \right|_{z=0} \mathcal{S}^{\mathrm{T}}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) - U_2 \right| \le \mathbf{O}_{14} L^8 \bar{g}^{2-2\eta}$$

where

$$U_2 = \left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right) - 2||\Gamma||_{L^2}^2 \times \frac{1}{1 - L^{-2[\phi]}} \times \mu_*$$

and

$$\left| \frac{d^4}{dz^4} \right|_{z=0} \mathcal{S}^{\mathrm{T}}(z 1\!\!1_{\mathbb{Z}_p^3}, 0) - U_4 \right| \le \mathbf{O}_{15} L^8 \bar{g}^{2-2\eta}$$

where

$$\begin{split} U_4 &= -24g_* \sum_{q=0}^{\infty} \left[L^{-4q[\phi]} \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ \Gamma(x)^4 \right) + 6L^{-4q[\phi]} q ||\Gamma||_{L^2}^2 \Gamma(0)^2 + 12L^{-2q[\phi]} \left(\sum_{n=0}^{q-1} nL^{-2n[\phi]} \right) ||\Gamma||_{L^2}^2 \Gamma(0)^2 \right. \\ &+ 4L^{-2q[\phi]} \frac{1 - L^{-2q[\phi]}}{1 - L^{-2[\phi]}} \Gamma(0) \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ \Gamma(x)^3 \right) \right]. \end{split}$$

Here we have used the following numerical constants: $\mathbf{O}_{14} = 8 \times \mathbf{O}_{10}^{-2} \mathbf{O}_{13}$ and $\mathbf{O}_{15} = 384 \times \mathbf{O}_{10}^{-4} \mathbf{O}_{13}$.

Proof: We note that for j = 2, 4 we have that $U_j = \frac{d^j}{dz^j}\Big|_{z=0} \mathcal{S}^{\mathrm{T,exp}}(z)$.

By the previous lemma the bounds above will follow if we have the necessary bounds on $\left|\frac{d^j}{dz^j}\right|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z)$. By Cauchy's formula we have

$$\frac{d^{j}}{dz^{j}}\Big|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z) = \frac{j!}{2i\pi} \oint \frac{d\lambda}{\lambda^{j+1}} \, \mathcal{S}^{\mathrm{T,imp}}(\lambda)$$

Here we are integrating around the contour $|\lambda| = \frac{1}{2}\mathbf{O}_{10}$. Utilizing the uniform bound on $\mathcal{S}^{T,\text{imp}}(z)$ from the previous lemma we get the estimate:

$$\left| \frac{d^j}{dz^j} \right|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z) \right| \leq j! \times 2^j \mathbf{O}_{10}^{-j} \times \mathbf{O}_{13} \times L^8 \times \bar{g}^{2-2\eta} .$$

This proves the lemma.

Proposition 12. In the small ϵ regime

$$\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z1_{\mathbb{Z}_p^3},0) \le -\frac{1}{4}\bar{g} < 0.$$

Proof: We observe that since $\hat{\Gamma}(k) \geq 0$ one has

$$\Gamma(0) = \int_{\mathbb{Q}_p^3} d^3k \ \hat{\Gamma}(k) \ge 0$$
$$\int_{\mathbb{Q}_p^3} d^3x \ \Gamma(x)^3 = \left(\hat{\Gamma} * \hat{\Gamma} * \hat{\Gamma}\right)(0) \ge 0 \ .$$

In the above expression * denotes convolution. It then follows by only keeping the first q=0 term that

$$U_4 \le -24g_* \int_{\mathbb{Q}_p^3} d^3x \ \Gamma(x)^4$$

$$\le -24g_* \int_{\mathbb{Z}_p^3} d^3x \ \Gamma(x)^4$$

$$= -24g_* \Gamma(0)^4$$

$$= -24g_* \times \left[\frac{1 - p^{-3}}{1 - p^{-2[\phi]}} \left(1 - L^{-2[\phi]} \right) \right]^4.$$

In going to the last line we used Lemma 2. Now we note that $p, L \ge 2$ and $\epsilon \le 1$ implies that $-2[\phi] \le -1$

$$\begin{split} U_4 & \leq -24 g_* \times \left[\frac{1 - \frac{1}{2^3}}{1} \times \left(1 - \frac{1}{2} \right) \right]^4 \\ & = -24 \left(\frac{7}{16} \right)^4 g_* \\ & \leq -12 \left(\frac{7}{16} \right)^4 \bar{g} \\ & \leq -\frac{1}{3} \bar{g} \ . \end{split}$$

Note that in going to the third line we used that $g_* > \frac{1}{2}\bar{g}$. Now using the previous lemma we have:

$$\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \le U_4 + \mathbf{O}_{15} L^8 \bar{g}^{2-2\eta}
\le -\frac{1}{3} \bar{g} + \mathbf{O}_{15} L^8 \bar{g}^{2-2\eta} .$$

Since $2-2\eta > e_4 \ge 1$ we can take ϵ sufficiently small to guarantee that $\mathbf{O}_{15}L^8\bar{g}^{2-2\eta} \le \frac{1}{12}\bar{g}$. This proves the proposition.

12.2. The two-point function for the composite field. We now study the ϕ^2 correlation when smeared with the characteristic function of \mathbb{Z}_p^3 , i.e., the quantity

$$\begin{split} \frac{d^2}{dz^2}\Big|_{z=0} \mathcal{S}^T(0, z \mathbbm{1}_{\mathbb{Z}_p^3}) &= \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle - \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3}) \rangle^2 \\ &= \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle \end{split}$$

since the one-point function is identically zero.

Here $q_{-}=q_{+}=0$ so there is no contribution from the middle regime. Thus

$$\begin{split} \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle &= \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{UV}} + \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{IR}} \\ \text{where } \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{UV}} &= \frac{d^2}{dz^2} \Big|_{z=0} \mathcal{S}^{\mathrm{T,UV}}(0,z\mathbbm{1}_{\mathbb{Z}_p^3}) \\ \text{and } \langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{IR}} &= \frac{d^2}{dz^2} \Big|_{z=0} \mathcal{S}^{\mathrm{T,IR}}(0,z\mathbbm{1}_{\mathbb{Z}_p^3}). \end{split}$$

Clearly, since we can derive term-by-term in the sum over q and since the constant and linear parts disappear

$$\begin{split} \langle N[\phi^2] (\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{UV}} = & \frac{d^2}{dz^2} \Big|_{z=0} \sum_{q<0} L^{-3(q+1)} \delta b \left[\Psi(v, -\alpha_{\mathrm{u}}^q Y_2 z e_{\phi^2}) \right] \\ = & Y_2^2 \times \left(\sum_{q<0} L^{-3(q+1)} \alpha_{\mathrm{u}}^{2q} \right) \times \frac{d^2}{dz^2} \Big|_{z=0} \delta b \left[\Psi(v, z e_{\phi^2}) \right] \end{split}$$

by the chain rule. This also uses $L^3\alpha_{\rm u}^{-2}<1$ which will be proved shortly.

We will use the more convenient notation $\Psi_v(w)$ instead of $\Psi(v,w)$.

Now for w small we have by Theorem 5

$$\Psi_v(w) = \Psi_{v_*} \left(T_{\infty}(v)[w] \right).$$

By the remark following Lemma 73

$$P_s T_{\infty}(v)[e_{\phi^2}] = 0$$

i.e. $T_{\infty}(v)[e_{\phi^2}]$ is in \mathcal{E}^{u} and therefore is proportional to e_{u} .

We define \varkappa_{ϕ^2} as the proportionality constant, i.e., by

$$T_{\infty}(v)[e_{\phi^2}] = \varkappa_{\phi^2} e_{\mathbf{u}}.$$

Hence

$$\Psi(v, ze_{\phi^2}) = \Psi_{v_*}(z\varkappa_{\phi^2}e_{\mathbf{u}})$$

and as a result

$$\langle N[\phi^2](1\!\!1_{\!\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}} = Y_2^2 \varkappa_{\phi^2}^2 \langle N[\phi^2](1\!\!1_{\!\mathbb{Z}_p^3})^2\rangle_{\mathrm{reduced}}^{\mathrm{UV}}$$

with

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle_{\text{reduced}}^{\text{UV}} = \frac{1}{\alpha_{\text{u}}^2 - L^3} \times D_0^2(\delta b \circ \Psi_{\nu_*})[e_{\text{u}}, e_{\text{u}}].$$

On the other hand we easily see that

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle^{\mathrm{IR}} = \sum_{q>0} \frac{d^2}{dz^2} \Big|_{z=0} \delta b_{\Delta(0)} \left[RG_{\mathrm{ex}}^q \left(\vec{V}^{(-\infty,0)}(0,z \mathbb{1}_{\mathbb{Z}_p^3}) \right) \right]$$

where

$$\vec{V}^{(-\infty,0)}(0,z1_{\mathbb{Z}_n^3}) = \mathcal{J}_0\left(0,(\Psi_v(-Y_2ze_{\phi^2})),v_*\right).$$

We define the affine isometric map $\varpi : \mathcal{E} \to \mathcal{E}_{pt}$ which sends $v = (\delta g, \mu, R)$ to $\vec{V} = (V_{\Delta})_{\Delta \in \mathbb{L}} = \varpi(v)$ such that

$$V_{\Delta} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})$$

is zero for $\Delta \neq \Delta(0)$ and equal to

$$(\delta g - \delta g_*, 0, \mu - \mu_*, 0, 0, 0, 0, R - R_*)$$

for $\Delta = \Delta(0)$.

It easily follows from the definitions that

$$\vec{V}^{(-\infty,0)}(0,z\mathbb{1}_{\mathbb{Z}_p^3}) = \iota(v_*) + \varpi \circ \Psi_v(-Y_2 z e_{\phi^2})$$
$$= \iota(v_*) + \varpi \circ \Psi_{v_*}(-Y_2 \varkappa_{\phi^2} z e_{\mathbf{u}})$$

for z small.

Hence by the chain rule

$$\langle N[\phi^2](1\!\!1_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{IR}} = Y_2^2 \varkappa_{\phi^2}^2 \langle N[\phi^2](1\!\!1_{\mathbb{Z}_p^3})^2\rangle_{\mathrm{reduced}}^{\mathrm{IR}}$$

where

$$\langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2\rangle_{\mathrm{reduced}}^{\mathrm{IR}} = \sum_{q\geq 0} \frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\Delta(0)} \left[\iota(v_*) + RG_{\mathrm{dv},\iota(v_*)}^q \circ \varpi \circ \Psi_{v_*}(ze_{\mathrm{u}})\right]$$

where we introduced the more convenient notation $RG_{\mathrm{dv},\vec{V}_{\mathrm{bk}}}[\dot{V}]$ for $RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}},\dot{V}]$ of section §10.4.

In what follows we will show that when $\epsilon \to 0$, $\langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle_{\text{reduced}}^{\text{IR}}$ remains bounded while $\langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2 \rangle^{\text{UV}}$ blows up.

This will need new constraints on the norm exponents which we list redundantly as:

$$\begin{aligned} e_4 &> 1 \\ e_4 &> e_2 \\ e_4 &> 2e_2 - 1 \\ e_R &> e_2 + 1 \\ e_R &> 2e_2 \ . \end{aligned}$$

We first introduce the subspace $\mathcal{E}_{ex,ev}$ of \mathcal{E}_{ex} .

It is the space of vectors

$$(\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})_{\Delta \in \mathbb{L}}$$

such that for all $\Delta \in \mathbb{L}$,

$$\beta_{3,\Delta} = \beta_{1,\Delta} = W_{5,\Delta} = W_{6,\Delta} = f_{\Delta} = 0$$

and $R_{\Delta} \in C^9_{\text{bd ev}}(\mathbb{R}, \mathbb{C}).$

Using the same line of reasoning as in the proof of Proposition 2 or in §10.1 it is easy to see that $\mathcal{E}_{\text{ex,ev}}$ is invariant by RG_{ex} .

Lemma 99. In the small ϵ regime and for $\vec{V} \in B(\bar{V}, \frac{1}{2}) \cap \mathcal{E}_{ex,dv}$ we have for all $\Delta' \in \mathbb{L}$

$$|\delta b_{\Delta'}[\vec{V}]| \le \mathbf{O}_{16} L^5 \bar{g}^{2e_2}$$

where

$$\mathbf{O}_{16} = 1 + 9 \sum_{a_1, a_2, m} \mathbb{1} \left\{ \begin{array}{l} a_i + m = 2 \text{ or } 4 \\ a_i \ge 0, \ m \ge 1 \end{array} \right\} \times C(a_1, a_2 | 0) \times 2^{\frac{a_1 + a_2}{2}}.$$

Proof: Recall that

$$\delta b_{\Delta'}[\vec{V}] = \delta \beta_{0,1,\Delta'} + \delta \beta_{0,2,\Delta'} + \xi_{0,\Delta'}(\vec{V}).$$

Since there are no f's we have $\beta_{0,1,\Delta'}=0$. Similarly the $\delta\beta_{0,2,\Delta'}$ contirbution reduces to

$$\delta\beta_{0,2,\Delta'} = \sum_{a_1,a_2,m} \mathbb{1} \left\{ \begin{array}{l} a_i + m = 2 \text{ or } 4 \\ a_i \ge 0, \ m \ge 1 \end{array} \right\} \frac{(a_1 + m)!(a_2 + m)!}{a_1!a_2!m!} \times \frac{1}{2} C(a_1, a_2|0)$$
$$\times L^{-(a_1 + a_2)[\phi]} C_0(0)^{\frac{a_1 + a_2}{2}} \times \int_{(L^{-1}\Delta')^2} d^3x_1 d^3x_2 \ \beta_{a_1 + m}(x_1) \beta_{a_2 + m}(x_2) \ \Gamma(x_1 - x_2)^m.$$

We use the bound

$$\left| \int_{(L^{-1}\Delta')^2} \mathrm{d}^3 x_1 \mathrm{d}^3 x_2 \, \beta_{a_1+m}(x_1) \beta_{a_2+m}(x_2) \, \Gamma(x_1 - x_2)^m \right| \leq L^3 ||\Gamma||_{L^{\infty}}^{m-1} \times ||\Gamma||_{L^1} \\ \times \sup_{x \in L^{-1}\Delta'} |\beta_{a_1+m}(x)| \times \sup_{x \in L^{-1}\Delta'} |\beta_{a_2+m}(x)|.$$

We bound the supremums by noting that β_{a_1+m} can only be β_2 or β_4 . Since \bar{V} has no β_2 component

$$|\beta_2(x)| \le ||\vec{V} - \bar{V}||\bar{g}^{e_2} \le \frac{1}{2}\bar{g}^{e_2}.$$

On the other hand

$$|\beta_4(x)| \leq \bar{g} + ||\vec{V} - \bar{V}||\bar{g}^{e_4} \leq \frac{3}{2}\bar{g} \leq \frac{3}{2}\bar{g}^{e_2}$$

since $e_4 > 1 \ge e_2$. As a result the previous integral is bounded by

$$L^{3}||\Gamma||_{L^{\infty}}^{m-1}\times||\Gamma||_{L^{1}}\times\frac{9}{4}\bar{g}^{2e_{2}}\leq L^{3}\times2^{m-1}\times\frac{1}{\sqrt{2}}L^{3-2[\phi]}\times\frac{9}{4}\bar{g}^{2e_{2}}\leq18L^{5}\bar{g}^{2e_{2}}$$

where we used $\epsilon \leq 1$ so $3-2[\phi] \leq 2$, and $m \leq 4$ while dropping $\sqrt{2}$. Finally $|\xi_{0,\Delta'}(\vec{V})| \leq \frac{1}{2}B_4\bar{g}^{e_R}$ by Theorem 4. Noting that $\frac{1}{2}B_4\bar{g}^{e_R-2e_2} \leq 1$ for ϵ small the lemma follows.

Lemma 100. For $\vec{V}^1, \vec{V}^2 \in \bar{B}(\bar{V}, \frac{1}{6}) \cap \mathcal{E}_{ex,dv}$, we have the Lipschitz estimate

$$\left| \delta b_{\Delta(0)} [\vec{V}^1] - \delta b_{\Delta(0)} [\vec{V}^2] \right| \leq 4 \mathbf{O}_{16} L^5 \bar{g}^{2e_2} ||\vec{V}^1 - \vec{V}^2|| \ .$$

Proof: This is an immediate consequence of the previous lemma and Lemma 1 with $\nu = \frac{1}{2}$.

Since we are computing second derivatives there is no harm in writing

$$\langle N[\phi^2](\mathbbm{1}_{\mathbb{Z}_p^3})^2\rangle_{\mathrm{reduced}}^{\mathrm{IR}} = \sum_{q>0} \frac{d^2}{dz^2}\Big|_{z=0} \left\{ \delta b_{\Delta(0)} \left[\iota(v_*) + RG_{\mathrm{dv},\iota(\mathbf{v}_*)}^q \circ \varpi \circ \Psi_{v_*}(ze_{\mathbf{u}}) \right] - \delta b_{\Delta(0)} \left[\iota(v_*) \right] \right\}.$$

If z is small enough so that

$$||\Psi(ze_{\mathbf{u}}) - v_*|| \le \mathbf{O}_{10}$$

which is the same as saying that $||\varpi \circ \Psi_{v_*}(ze_{\mathbf{u}})|| \leq \mathbf{O}_{10}$, then Proposition 11 along with the last lemma will imply

$$\left|\delta b_{\Delta(0)}\left[\iota(v_*) + RG^q_{\mathrm{dv},\iota(v_*)} \circ \varpi \circ \Psi_{v_*}(ze_{\mathrm{u}})\right] - \delta b_{\Delta(0)}\left[\iota(v_*)\right]\right| \leq 4\mathbf{O}_{16}L^5\bar{g}^{2e_2}\left(\frac{15}{16}\right)^q \times \mathbf{O}_{10}.$$

Let $z_{\text{max}} > 0$ be such that $|z| \le z_{\text{max}}$ implies $||\Psi_{v_*}(ze_{\text{u}}) - v_*|| \le \mathbf{O}_{10}$. Then by extracting the derivatives with Cauchy's formula we easily arrive at the bound

$$\left| \langle N[\phi^2] (\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{IR}} \right| \le 4 \mathbf{O}_{10} \mathbf{O}_{16} L^5 \bar{g}^{2e_2} \times \frac{1}{1 - \frac{15}{1c}} \times 2! \times z_{\mathrm{max}}^{-2}$$

Now from Theorem 6 $||ze_{\mathbf{u}}|| < \frac{1}{24}$ implies

$$||\Psi_{v_*}(ze_{\mathbf{u}}) - v_*|| \le ||ze_{\mathbf{u}}|| \left(1 + \frac{17}{18} \times \frac{1}{24}\right)$$

 $\le 2||ze_{\mathbf{u}}||$

for simplicity. So $z_{\text{max}} = \frac{1}{2}\mathbf{O}_{10}||e_{\text{u}}||^{-1}$ works because $\frac{1}{2}\mathbf{O}_{10} \leq \frac{1}{80} < \frac{1}{24}$. Also by lemma 63, $||e_{\text{u}}|| = \bar{g}^{-e_2}$. Hence in the small ϵ regime we have the bound

$$\left| \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{IR}} \right| \le 512 \mathbf{O}_{10}^{-1} \times \mathbf{O}_{16} \times L^5 \ .$$

Namely, the infrared contribution remains finite when $\epsilon \to 0$.

We now examine the ultraviolet contribution more closely. From Theorem 6 the small z expansion of $\Psi_{v_*}(ze_{\mathrm{u}})$ is of the form

(85)
$$\Psi_{v_*}(ze_{u}) = v_* + ze_{u} + z^2\Theta + O(z^3)$$

for some vector Θ to be determined shortly. Using the decomposition in Lemma 79

$$D_0^2(\delta b \circ \Psi_{v_*})[e_{\mathbf{u}}, e_{\mathbf{u}}] = \frac{d^2}{dz^2}\Big|_{z=0} \delta b^{\text{explicit}} \left(\Psi_{v_*}(ze_{\mathbf{u}})\right) + \frac{d^2}{dz^2}\Big|_{z=0} \delta b^{\text{implicit}} \left(\Psi_{v_*}(ze_{\mathbf{u}})\right).$$

If $|z| \leq \frac{1}{30}\bar{g}^{e_2}$ then as before we get

$$\begin{aligned} ||\Psi_{v_*}(ze_{\mathbf{u}})|| &\leq ||v_*|| + 2||ze_{\mathbf{u}}|| \\ &\leq \frac{1}{40} + \frac{1}{15} < \frac{1}{2}. \end{aligned}$$

So by Theorem 4

$$\left|\delta b^{\text{implicit}}\left(\Psi_{v_*}(ze_{\mathbf{u}})\right)\right| \leq \frac{1}{2}B_0\bar{g}^{e_R}.$$

Cauchy's formula then immediately implies

$$\left|\frac{d^2}{dz^2}\right|_{z=0} \delta b^{\mathrm{implicit}} \left(\Psi_{v_*}(ze_{\mathrm{u}})\right) \right| \leq 2! \left(\frac{1}{30} \bar{g}^{e_2}\right)^{-2} \times \frac{1}{2} B_0 \bar{g}^{e_R}.$$

Since $e_R > 2e_2$, we must have

$$\lim_{\epsilon \to 0} \frac{d^2}{dz^2} \bigg|_{z=0} \delta b^{\text{implicit}} \left(\Psi_{v_*}(ze_{\mathbf{u}}) \right) = 0.$$

Now recall that

$$\begin{split} \delta b^{\text{explicit}} &= A_4 \bar{g}^2 + \delta b_{\text{I}}^{\text{explicit}}(\delta g, \mu, R) + \delta b_{\text{II}}^{\text{explicit}}(\delta g, \mu, R) \\ &\text{where } \delta b_{\text{I}}^{\text{explicit}}(\delta g, \mu, R) = 2 A_4 \bar{g} \delta g + A_4 \delta g^2 \\ &\text{and } \delta b_{\text{II}}^{\text{explicit}}(\delta g, \mu, R) = A_5 \mu^2 \ . \end{split}$$

Note that the $A_4\bar{g}^2$ term disappears in the computation of derivatives while $\delta b_{\rm I}^{\rm explicit}$ can be treated as we treated $\delta b^{\rm implicit}$. Indeed by Cauchy's formula and Theorem 4

$$\left| \frac{d^2}{dz^2} \right|_{z=0} \delta b_{\rm I}^{\rm explicit} \left(\Psi_{v_*}(ze_{\rm u}) \right) \right| \leq 2! \left(\frac{1}{30} \bar{g}^{e_2} \right)^{-2} \times A_{4, \rm max} \left[2 \times \bar{g} \times \frac{1}{2} \bar{g}^{e_4} + \left(\frac{1}{2} \bar{g}^{e_4} \right)^2 \right].$$

Since $e_4 > 2e_2 - 1$ and $e_4 > e_2$ we have

$$\lim_{\epsilon \to 0} \frac{d^2}{dz^2} \Big|_{z=0} \delta b_{\rm I}^{\rm explicit} \left(\Psi_{v_*}(ze_{\rm u}) \right) = 0.$$

As a result of the formula $e_{\rm u}=(\delta g'_{\rm u}(\mu_*),1,R'_{\rm u}(\mu_*))$ and the expansion (85) we easily compute

$$\frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\rm II}^{\rm explicit} \left(\Psi_{v_*}(ze_{\rm u})\right) = 2A_5 \left(1 + 2\mu_* \Theta_{\mu}\right)$$

where Θ_{μ} is the μ component of $\Theta \in \mathcal{E}$.

We determine the latter using the intertwining relation in Theorem 6 for small z.

We have by an easy calculation using (85)

$$RG\left(\Psi_{v_*}(ze_{\mathbf{u}})\right) = v_* + D_{v_*}RG[e_{\mathbf{u}}] + z^2 \left(D_{v_*}RG[\Theta] + \frac{1}{2}D_{v_*}^2RG[e_{\mathbf{u}}, e_{\mathbf{u}}]\right) + O(z^3) .$$

But this is the same as

$$\Psi_{v_*}(\alpha_{\mathbf{u}}ze_{\mathbf{u}}) = v_* + z\alpha_{\mathbf{u}}e_{\mathbf{u}} + z^2\alpha_{\mathbf{u}}^2\Theta + O(z^3).$$

Thus

(86)
$$\alpha_{\mathbf{u}}^{2}\Theta = D_{v_{*}}RG[\Theta] + \frac{1}{2}D_{v_{*}}^{2}RG[e_{\mathbf{u}}, e_{\mathbf{u}}].$$

On the other hand $\Psi_{v_*} \in W^{\mathrm{u,loc}}$ for z small and therefore

$$[\Psi_{v_*}(ze_{\mathbf{u}})]_{\delta g} = \delta g_{\mathbf{u}} \left([\Psi_{v_*}(ze_{\mathbf{u}})]_{\mu} \right)$$

and

$$\left[\Psi_{v_*}(ze_{\mathbf{u}})\right]_R = R_{\mathbf{u}} \left(\left[\Psi_{v_*}(ze_{\mathbf{u}})\right]_{\mu} \right)$$

where $[\cdots]_{\delta g}$, $[\cdots]_{\mu}$, and $[\cdots]_R$ refer to the δg , μ , and R components respectively.

Expanding these relations up to second order imply

$$\Theta = (\Theta_{\delta g}, \Theta_{\mu}, \Theta_{R}) = \Theta_{\mu} e_{\mathbf{u}} + \frac{1}{2} c_{\mathbf{u}}$$
where $c_{\mathbf{u}} = (\delta q_{\mathbf{u}}''(\mu_{*}), 0, R_{\mathbf{u}}''(\mu_{*})).$

Taking the μ component of (86) we see that

$$\alpha_{\mathbf{u}}^2 \Theta_{\mu} = \Theta_{\mu} \alpha_{\mathbf{u}} + \frac{1}{2} \left[D_{v_*} RG[c_{\mathbf{u}}] \right]_{\mu} + \frac{1}{2} \left[D_{v_*}^2 RG[e_{\mathbf{u}}, e_{\mathbf{u}}] \right]_{\mu}$$

where we have used $[e_{\rm u}]_{\mu}=1$ and $D_{v_*}RG[e_{\rm u}]=\alpha_{\rm u}e_{\rm u}.$ Since $\alpha_{\rm u}$ we have

$$\Theta_{\mu} = \frac{1}{2\alpha_{\text{u}}(\alpha_{\text{u}} - 1)} \left\{ \left[D_{v_*} RG[c_{\text{u}}] \right]_{\mu} + \left[D_{v_*}^2 RG[e_{\text{u}}, e_{\text{u}}] \right]_{\mu} \right\} .$$

Now $|\mu - \mu_*| < \rho'' \bar{g}^{e_2}$ implies $|\delta g_u(\mu)| \le \frac{\rho'}{3} \bar{g}^{e_4}$ and $|||R_u(\mu)|||_{\bar{g}} \le \frac{\rho'}{3} \bar{g}^{e_R}$. Using $|\mu - \mu_*| = \frac{1}{2} \rho'' \bar{g}^{e_2}$ as a contour of integration, Cauchy's formula implies the following estimates:

$$\begin{split} |\delta g_{\mathrm{u}}'(\mu_*)| &\leq \frac{2\rho'}{3\rho''} \bar{g}^{e_4 - e_2} \\ |\delta g_{\mathrm{u}}''(\mu_*)| &\leq \frac{8\rho'}{3(\rho'')^2} \bar{g}^{e_4 - 2e_2} \\ |||R_{\mathrm{u}}'(\mu_*)|||_{\bar{g}} &\leq \frac{2\rho'}{3\rho''} \bar{g}^{e_R - e_2} \\ |||R_{\mathrm{u}}''(\mu_*)|||_{\bar{g}} &\leq \frac{8\rho'}{3(\rho'')^2} \bar{g}^{e_R - 2e_2}. \end{split}$$

As a result

$$||c_{\mathbf{u}}|| = \max\left\{|\delta g_{\mathbf{u}}''(\mu_*)|\bar{g}^{-e_4}, |||R_{\mathbf{u}}''(\mu_*)|||_{\bar{g}}\bar{g}^{-e_R}\right\} \le \frac{8\rho'}{3(\rho'')^2}\bar{g}^{-2e_2}$$

From the explicit formulas in the proof of Lemma 69 and following the same notation

$$[D_v RG[v']]_{\mu} = L^{\frac{3+\epsilon}{2}} \mu' - 2A_2(\bar{g} + \delta g)\delta g' - A_3(\bar{g} + \delta g)\mu' - A_3\mu \delta g' + \left[D_v RG^{\text{implicit}}[v']\right]_{\mu}.$$
For $v = v_*$ and $v' = c_u$ this gives

$$[D_{v_*}RG[v']]_{\mu} = -2A_2(\bar{g} + \delta g_*)\delta g_{\mathbf{u}}''(\mu_*) - A_3\mu_*\delta g_{\mathbf{u}}''(\mu_*) + [D_{v_*}RG^{\mathrm{implicit}}[c_{\mathbf{u}}]]_{\mu}$$

The infinitesimal version of the ξ_2 Lipschitz estimate in Lemma 57 immediately implies

$$||[D_{v_*}RG^{\text{implicit}}]_{\mu}|| \le 2B_2\bar{g}^{e_R}$$

for the operator norm induced on linear maps from \mathcal{E} to \mathbb{C} by the norm $||\cdot||$ on \mathcal{E} and the modulus on \mathbb{C} . As a result we have

$$\left| [D_{v_*} RG[c_{\mathbf{u}}]]_{\mu} \right| \leq \left(2A_{2,\max} \times \frac{3}{2} \bar{g} + A_{3,\max} \times \frac{1}{2} \bar{g}^{e_2} \right) \times \frac{8\rho'}{3(\rho'')^2} \bar{g}^{e_4 - 2e_2} + 2B_2 \bar{g}^{e_R} \times \frac{8\rho'}{3(\rho'')^2} \bar{g}^{-2e_2} \ .$$

Since $e_4 > e_2$, $e_4 > 2e_2 - 1$, and $e_R > 2e_2$ we have

$$\lim_{\epsilon \to 0} \left[D_{v_*} RG[c_{\mathbf{u}}] \right]_{\mu} = 0 .$$

Also from the formulas in Lemma 69

$$\left[D_{v_*}^2 RG[e_{\rm u},e_{\rm u}]\right]_{\mu} = -2A_2\delta g_{\rm u}'(\mu_*)^2 - 2A_3\delta g_{\rm u}'(\mu_*) + \left[D_{v_*}^2 RG^{\rm implicit}[e_{\rm u},e_{\rm u}]\right]_{\mu}$$

and therefore

$$\left|\left[D_{v_*}^2RG[e_{\mathbf{u}},e_{\mathbf{u}}]\right]_{\mu}\right| \leq 2A_{2,\max}\left(\frac{2\rho'}{3\rho''}\bar{g}^{e_4-e_2}\right)^2 + 2A_{3,\max}\left(\frac{2\rho'}{3\rho''}\bar{g}^{e_4-e_2}\right) + \left|\left[D_{v_*}^2RG^{\mathrm{implicit}}[e_{\mathbf{u}},e_{\mathbf{u}}]\right]_{\mu}\right|.$$

The argument of Lemma 69 only applied to the μ component of RG^{implicit} gives

$$||D_{v_{\pi}}^{2}RG_{\mu}^{\text{implicit}}|| \le 32B_{2}\bar{g}^{e_{R}-e_{2}}$$

for the norm of the second differential.

Since $||e_{\mathbf{u}}|| = \bar{g}^{-e_2}$ we obtain

$$\left| \left[D_{v_*}^2 R G^{\text{implicit}}[e_{\mathbf{u}}, e_{\mathbf{u}}] \right]_{\mu} \right| \le 32 B_2 \bar{g}^{e_R - 3e_2}.$$

Since $|\mu_*| \leq \frac{1}{2}\bar{g}^{e_2}$ and $e_R > 2e_2$ we have

$$\lim_{\epsilon \to 0} \mu_* \left[D_{v_*}^2 RG^{\text{implicit}}[e_{\mathbf{u}}, e_{\mathbf{u}}] \right]_{\mu} = 0.$$

Since also $\lim_{\epsilon \to 0} \mu_* = 0$ and $\lim_{\epsilon \to 0} \alpha_{\rm u} = L^{\frac{3}{2}} > 1$ we have enough to affirm

$$\lim_{\epsilon \to 0} \mu_* \Theta_\mu = 0.$$

Thus

$$\lim_{\epsilon \to 0} \frac{d^2}{dz^2} \Big|_{z=0} D_0^2 (\delta b \circ \Psi_{v_*})[e_{\mathbf{u}}, e_{\mathbf{u}}] = 2 \lim_{\epsilon \to 0} A_5 = 2L^3 (1 - p^{-3}) \times l > 0 \text{ by Lemma 6.}$$

We now study the $\epsilon \to 0$ asymptotics of α_u more closely. One way to get a precise hold on this eigenvalue is to note that

$$\alpha_{\mathbf{u}} = [D_{v_*} RG[e_{\mathbf{u}}]]_{\mu}.$$

Then by the formula in (87) we have

$$\alpha_{\rm u} = L^{\frac{3+\epsilon}{2}} - 2A_2(\bar{g} + \delta g_*)\delta g'_{\rm u}(\mu_*) - A_3(\bar{g} + \delta g_*) - A_3\mu_*\delta g'_{\rm u}(\mu_*) + \left[D_{v_*}RG^{\rm implicit}[e_{\rm u}]\right]_{\mu}$$
 since $e_{\rm u} = (\delta g'_{\rm u}(\mu_*), 1, R'_{\rm u}(\mu_*)).$

As before

$$\begin{split} \left| \left[D_{v_*} R G^{\text{implicit}}[e_{\mathbf{u}}] \right]_{\mu} \right| &\leq ||D_{v_*} R G^{\text{implicit}}|| \times ||e_{\mathbf{u}}|| \\ &\leq 2B_2 \bar{g}^{e_R} \times \bar{g}^{-e_2}. \end{split}$$

But $e_R > e_2 + 1$ and \bar{g} is of order ϵ so

$$\left[D_{v_*}RG^{\text{implicit}}[e_{\mathbf{u}}]\right]_{\mu} = o(\epsilon).$$

We have

$$|-2A_2(\bar{g}+\delta g_*)\delta g'_{\mathrm{u}}(\mu_*)| \le 2A_{2,\max} \times \frac{3}{2}\bar{g} \times \frac{2\rho'}{3\rho''}\bar{g}^{e_4-e_2}$$

so this is also an $o(\epsilon)$ term because $e_4 > e_2$.

Likewise

$$|-A_3\delta g_*| \le A_{3,\max} \times \frac{1}{2}\bar{g}^{e_4}$$

so this is $o(\epsilon)$ because $e_4 > 1$.

Finally,

$$|-A_3\mu_*\delta g'_{\mathrm{u}}(\mu_*)| \le A_{3,\max} \times \frac{1}{2}\bar{g}^{e_2} \times \frac{2\rho'}{3\rho''}\bar{g}^{e_4-e_2}$$

so this is an $o(\epsilon)$ term too.

As a result we have

$$\begin{split} \alpha_{\mathrm{u}} &= L^{\frac{3+\epsilon}{2}} - A_{3}\bar{g} + o(\epsilon) \\ &= L^{\frac{3+\epsilon}{2}} - 12 \times L^{\frac{3+\epsilon}{2}} \times \frac{A_{1}}{36L^{\epsilon}}\bar{g} + o(\epsilon) \\ &= L^{\frac{3+\epsilon}{2}} \left(1 - \frac{1}{3} \left(\frac{L^{\epsilon} - 1}{L^{\epsilon}}\right)\right) + o(\epsilon) \end{split}$$

from the relations between A_3 , A_1 , and \bar{g} . It is now a simple calculus exercise to derive

$$\eta_{\phi^2} = \frac{2}{3}\epsilon + o(\epsilon)$$

where η_{ϕ^2} is defined by

$$L^{\frac{1}{2}\eta_{\phi^2}} = Z_2^{-1} = L^{\frac{3+\epsilon}{2}}\alpha_n^{-1}.$$

We also easily get

$$L^{3}\alpha_{\mathrm{u}}^{-2} = 1 - \frac{\log(L)}{3}\epsilon + o(\epsilon)$$

which proves the earlier statement that

$$L^3 \alpha_{\rm u}^{-2} < 1$$

in the small ϵ regime which was crucial for the convergence and analyticity in the ultraviolet regime.

Another byproduct is

$$\frac{1}{\alpha_v^2 - L^3} \sim \frac{3L^{-3}}{\log(L)} \times \frac{1}{\epsilon}$$

and therefore

$$\langle\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle_{\mathrm{reduced}}^{\mathrm{UV}} \sim \frac{6(1-p^{-3})}{\log(p)} \times \frac{1}{\epsilon}$$

when $\epsilon \to 0$.

Since $\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\text{reduced}}^{\text{IR}}$ remains bounded, the quantity

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_n^3})^2 \rangle_{\text{reduced}} = \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_n^3})^2 \rangle_{\text{reduced}}^{\text{UV}} + \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_n^3})^2 \rangle_{\text{reduced}}^{\text{IR}}$$

is strictly positive for ϵ small enough.

Provided $\varkappa_{\phi^2} \neq 0$ we can then impose by definition

$$Y_2 = |\varkappa_{\phi^2}|^{-1} \times \left\{ \langle N[\phi^2] (\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\text{reduced}} \right\}^{-\frac{1}{2}}$$

and thus force the normalization

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_n^3})^2 \rangle = 1.$$

We now address the issue of showing $\varkappa_{\phi^2} \neq 0$. While most of the proof so far relied on quantitative estimates, here we had to use a more qualitative approach. This is because of the slow convergence to the fixed point on the stable manifold and the fact that we do not have much freedom of choice for our starting point v. The latter has to be on the R=0 bare surface and therefore we cannot choose it as close to v_* as we would like to.

Recall that $W_{\text{int}}^{\text{s,loc}}$ is parametrized as

$$v_1 \mapsto (v_1, \mu_{\mathbf{s}}(v_1))$$

for $||v_1|| < \frac{\rho}{13}$ in \mathcal{E}_1 . For $v \in W^{s, loc}_{int}$ we consider the tangent space $T_v W^s$ defined as the kernel of the linear form

$$(w_1, w_2) \mapsto w_2 - D_{v_1} \mu_s[w_1]$$

via the identification $\mathcal{E}_2 \simeq \mathbb{C}$.

This linear form is continuous and does not vanish identically, so T_vW^s is a closed complex hyperplane in \mathcal{E} . If $w \in \mathcal{E}$ satisfies $w \notin T_vW^s$ then we have a direct sum decomposition $\mathcal{E} = \mathbb{C} \oplus T_vW^s$.

We have the following infinitesimal version of Parts 1) and Parts 2) of Lemma 58 and Lemma 59.

Lemma 101. For all $v \in W_{\text{int}}^{s,\text{loc}}$ we have:

1) for all $w \in \mathcal{E}$,

$$||(D_v RG[w])_1|| \le c_1(\epsilon)||w||$$

2) for all $w \in \mathcal{E}$, such that $L^{\frac{3}{4}}||w_2|| \geq ||w_1||$,

$$||(D_v RG[w])_2|| \ge c_2(\epsilon)||w||$$

3) for all $w \in T_v W^s$,

$$||w_1|| \ge L^{\frac{3}{4}}||w_2||$$
.

Proof: Consider the complex curve $\gamma(t) = v + tw$ for t small which ensures that $\Gamma(t) \in \bar{B}\left(0, \frac{1}{8}\right)$. Lemma 58 Part 1) implies

$$||RG_1(\gamma(t)) - RG_1(\gamma(0))|| \le c_1(\epsilon)||tw||.$$

Dividing by |t| and taking $t \to 0$ we immediately get $||(D_v(RG[w])_1|| \le c_1(\epsilon)||w||$.

Now if $L^{\frac{3}{4}}||w_2|| \geq ||w_1||$ then we have

$$L^{\frac{3}{4}}||\gamma(t)_2 - \gamma(0)_2|| > ||\gamma(t)_1 - \gamma(0)_1||$$

and thus

$$||RG_2(\gamma(t)) - RG_2(\gamma(0))|| \ge c_2(\epsilon)||tw||$$

by Lemma 58 Part 2). Taking the $t \to 0$ limit as before we obtain

$$||(D_v RG[w])_2|| \ge c_2(\epsilon)||w||.$$

For the third part we use Lemma 59 to write

$$||(v_1 + tw_1) - v_1|| \ge L^{\frac{3}{4}} ||\mu_s(v_1 + tw_1) - \mu_s(v_1)||$$

for t small. Dividing by |t| and taking $t \to 0$ gives

$$||w_1|| \ge L^{\frac{3}{4}} ||D_{v_1} \mu_{\mathbf{s}}[w_1]|| = L^{\frac{3}{4}} ||w_2||$$

since $w \in T_v W^s$.

Lemma 102. For all $v \in W_{\text{int}}^{s,\text{loc}}$ and $w \in \mathcal{E}$ we have the implication

$$L^{\frac{3}{4}}||w_2|| > ||w_1|| \Rightarrow D_v RG[w] \notin T_{RG(v)}W^s$$
.

Proof: We proceed by contradiction. Suppose

$$L^{\frac{3}{4}}||w_2|| > ||w_1|| \text{ and } D_v RG[w] \in T_{RG(v)}W^{\mathrm{s}}.$$

Then by Lemma 101 Parts 1), 2), 3) we have

$$c_1(\epsilon)||w|| \ge ||(D_v RG[w])_1||,$$

$$||(D_v RG[w])_2|| \ge c_2(\epsilon)||w||$$

and

$$||(D_v RG[w])_1|| \ge L^{\frac{3}{4}} ||(D_v RG[w])_2||$$

respectively. As a result

$$c_1(\epsilon)||w|| \ge L^{\frac{3}{4}}c_2(\epsilon)||w||.$$

But $c_1(\epsilon) < 1 < L^{\frac{3}{4}}c_2(\epsilon)$ so ||w|| = 0 which contradicts the strict inequality $L^{\frac{3}{4}}||w_2|| > ||w_1||$.

Lemma 103. For all $v \in W_{\mathrm{int}}^{\mathrm{s,loc}}$ and $w \in T_v W^{\mathrm{s}}$

$$T_1(v)[w] \in T_{RG(v)}W^{\mathrm{s}}$$

and

$$T_{\infty}(v)[w] \in T_{v_*}W^{\mathrm{s}}$$
.

Proof: Consider the curve $t \mapsto (v_1 + tw_1, \mu_s(v_1 + tw_1))$ in $W_{\text{int}}^{s,\text{loc}}$ for t small. By Proposition 5 and the parametrization of $W_{\text{int}}^{s,\text{loc}}$ we have

$$RG_2(v_1 + tw_1, \mu_s(v_1 + tw_1)) = \mu_s \left(RG_1(v_1 + tw_2, \mu_s(v_1 + tw_1)) \right).$$

Differentiating this at t = 0 gives

$$\left(D_v RG[(w_1, D_{v_1} \mu_{\mathbf{s}}[w_1])]\right)_2 = D_{RG_1(v)} \mu_{\mathbf{s}} \left[\left(D_v RG[(w_1, D_{v_1} \mu_{\mathbf{s}}[w_1])]\right)_1 \right],$$

i.e.,

$$(D_v RG[w])_2 = D_{RG_1(v)} \mu_s [(D_v RG[w])_1].$$

Hence $D_v RG[w]$ belongs to $T_{RG(v)}W^s$ and so does $T_1(v)[w] = \alpha_u^{-1}D_v RG[w]$. By iteration this immediately implies

$$T_n(v)[w] \in T_{RG^n(v)}W^{\mathrm{s}}$$

for all integer $n \geq 0$.

Namely, we have

$$(T_n(v)[w])_2 = D_{(RG^n(v))_1} \mu_s [(T_n(v)[w])_1].$$

Using continuity, the remark following Lemma 73, and the fact that $RG^n(v) \to v_*$, we can take the $n \to \infty$ limit in the previous equality and obtain

$$(T_{\infty}(v)[w])_2 = D_{v_{*,1}}\mu_s [(T_{\infty}(v)[w])_1].$$

This proves $T_{\infty}(v)[w] \in T_{v_*}W^s = \mathcal{E}^s$ by definition of \mathcal{E}^s .

Lemma 104. For all $v \in W_{\text{int}}^{s,\text{loc}}$ and $w \in T_v W^s$

$$D_0\Psi_v[w] = 0,$$

where the differential is with respect to the w variable at w=0 for the function $\Psi_v(\bullet)=\Psi(v,\bullet)$.

Proof: By Theorem 5 Part 5)

$$\Psi_v = \Psi_{v_*} \circ T_{\infty}(v)$$

and thus by the chain rule

$$D_0 \Psi_v[w] = D_0 \Psi_{v_*} [T_{\infty}(v)[w]].$$

However by the previous lemma $T_{\infty}(v)[w] \in \mathcal{E}^{s}$ so $P_{u}T_{\infty}(v)[w] = 0$. But we also have $P_{s}T_{\infty}(v)[w] = 0$ as a follow up to Lemma 73.

As a result, $T_{\infty}(v)[w] = 0$ and consequently $D_0 \Psi_v[w] = 0$.

Lemma 105. For all $v \in W_{\text{int}}^{s,\text{loc}}$, if $D_0\Psi_v = 0$ then $D_0\Psi_{RG(v)} = 0$.

Proof: By Theorem 5 Part 4)

$$\Psi_v = \Psi_{RG(v)} \circ T_1(v)$$

near w = 0. Differentiating at zero gives

(88)
$$D_0 \Psi_v = D_0 \Psi_{RG(v)} \circ T_1(v).$$

Pick some vector $u \in \mathcal{E}$ satisfying the hypothesis of lemma 102. For instance e_{ϕ^2} works since $L^{\frac{3}{4}}||e_{\phi^2,2}|| = L^{\frac{3}{4}}\bar{g}^{-e_2} > ||e_{\phi^2,1}|| = 0$. By the same lemma $T_1(v)[u] \notin T_{RG(v)}W^s$ and therefore $\mathcal{E} = \mathbb{C}T_1(v)[u] \oplus T_{RG(v)}W^s$. Let $w \in \mathcal{E}$. We decompose it as $w = \lambda T_1(v)[u] + w'$ with $w' \in T_{RG(v)}W^s$. Then by (88):

$$D_0 \Psi_{RG(v)}[w] = \lambda D_0 \Psi_v[u] + D_0 \Psi_{RG(v)}[w']$$

by the hypothesis and the previous lemma for RG(v) instead of v. Hence the differential $D_0\Psi_{RG(v)}$ vanishes.

Iterating the last lemma we see that if $D_*\Psi_v = 0$ then $D_0\Psi_{RG^n(v)} = 0$ for all $n \geq 0$. By the joint analyticity in Theorem 5 we can take the $n \to \infty$ limit which gives $D_0\Psi_{v_*} = 0$ and therefore

$$\frac{d}{dz}\Big|_{z=0}\Psi_{v_*}(ze_{\mathbf{u}}) = 0$$

which contradicts (85) and $e_{\rm u} \neq 0$.

We have proved $D_0\Psi_v \neq 0$ for all $v \in W_{\mathrm{int}}^{\mathrm{s,loc}}$. Now since e_{ϕ^2} satisfies $L^{\frac{3}{4}}||e_{\phi^2,2}|| > ||e_{\phi^2,1}||$ we know that $e_{\phi^2} \notin T_v W^{\mathrm{s}}$ by Lemma 101. Thus $\mathcal{E} = \mathbb{C}e_{\phi^2} \oplus T_v W^{\mathrm{s}}$.

Recall that $D_0\Psi_v = D_0\Psi_{v_*} \circ T_\infty(v)$ so $D_0\Psi_v[e_{\phi^2}] = \varkappa_{\phi^2}D_0\Psi_{v_*}[e_{\mathrm{u}}]$ by definition of \varkappa_{ϕ^2} . If the latter vanishes then $D_0\Psi_v$ vanishes on $\mathbb{C}e_{\phi^2}$ and therefore on all of \mathcal{E} by Lemma 104. This contradicts $D_0\Psi_v \neq 0$. We have now finally proved $\varkappa_{\phi^2} \neq 0$.

The remaining items to be settled are the mini-universality result and the choice of parameters η , e, etc.

The mini-universality should be clear at this point: the generating function $\mathcal{S}^T(\tilde{f}, \tilde{j})$ does not depend on the starting point $v = (g - \bar{g}, \mu_c(g), 0) \in W_{\mathrm{int}}^{\mathrm{s,loc}}$ for the RG iterations. Indeed using $\Psi_v = \Psi_{v_*} \circ T_{\infty}(v)$ we see that the effect of v is entirely in the multiplying factor \varkappa_{ϕ^2} which however always comes in the combination $Y_2 \varkappa_{\phi^2}$. By our choice of normalization, $Y_2 \varkappa_{\phi^2}$ is defined in terms of the reduced $N[\phi^2]$ two-point function which only involves data at the fixed point v_* .

Finally, to complete our very long proof of Theorem 3 we have to pick a choice of parameters which satisfies all the required inequalities. We pick

$$\eta = 0
\eta_R = \frac{1}{8}
e_1 = e_2 = e_3 = 1
e_4 = \frac{3}{2}
e_W = 2
e_R = \frac{21}{8}
\rho' = \frac{1}{8}
\rho = \frac{1}{128}
\rho'' = \frac{1}{768} .$$

We leave it to the reader to check that these choices indeed satisfy all the previously stated inequalities. Note that ρ''' was already defined and was only needed in the local analysis at the fixed point. The proof of Theorem 3 is now complete.

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References

- [1] M. Abate, Discrete holomorphic local dynamical systems. In: *Holomorphic Dynamical Systems*, Ed.: G. Gentili, J. Guenot and G. Patrizio, pp. 1-55, Lecture Notes in Math., **1998**, Springer, Berlin, 2010.
- [2] A. Abdesselam, A complete renormalization group trajectory between two fixed points. Comm. Math. Phys. 276 (2007), no. 3, 727-772.

- [3] A. Abdesselam, A massless quantum field theory over the p-adics (joint work with Ajay Chandra, Gianluca Guadagni). Oberwolfach Rep. 8 (2011), no. 1, 785–788.
- [4] A. Abdesselam, A. Chandra and G. Guadagni, Rigorous quantum field theory functional integrals over the *p*-adics: research announcement. Preprint arXiv:1210.7717[math.PR], 2012.
- [5] A. Abdesselam, A. Chandra and G. Guadagni, Rigorous quantum field theory functional integrals over the p-adics II: full scale invariance. In preparation.
- [6] S. Albeverio, A. Yu. Khrennikov and V. M. Shelkovich, Theory of p-adic distributions: linear and nonlinear models. London Mathematical Society Lecture Note Series, 370. Cambridge University Press, Cambridge, 2010.
- [7] A. Beilinson and V. Drinfeld, Chiral Algebras. American Mathematical Society Colloquium Publications, 51. American Math. Soc., Providence, RI, 2004.
- [8] G. Benfatto and G. Gallavotti, Renormalization Group. Physics Notes, 1, Princeton University Press, Princeton, NJ, 1995.
- [9] G. Benfatto, G. Gallavotti, A. Procacci and B. Scoppola, Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface. Comm. Math. Phys. **160** (1994), no. 1, 93-171.
- [10] P. M. Bleher, Construction of non-Gaussian self-similar random fields with hierarchical structure. Comm. Math. Phys. 84 (1982), no. 4, 557-578.
- [11] P. M. Bleher and Ja. G. Sinai, Investigation of the critical point in models of the type of Dyson's hierarchical models. Comm. Math. Phys. 33 (1973), no. 1, 23-42.
- [12] P. M. Bleher and Ya. G. Sinai, Critical indices for Dyson's asymptotically-hierarchical models. Comm. Math. Phys. 45 (1975), no. 3, 247-278.
- [13] D. C. Brydges, J. Dimock and T. R. Hurd, A non-Gaussian fixed point for ϕ^4 in 4ϵ dimensions. Comm. Math. Phys. 198 (1998), 111–156.
- [14] D. C. Brydges, P. K. Mitter and B. Scoppola, Critical $(\Phi^4)_{3,\varepsilon}$. Comm. Math. Phys. **240** (2003), no. 1-2, 281-327.
- [15] D. C. Brydges and H. T. Yau, Grad φ perturbations of massless Gaussian fields. Comm. Math. Phys. **129** (1990), no. 2, 351–392.
- [16] J.-F. Burnol, The explicit formula and a propagator. Unpublished preprint arXiv:math/9809119[math.NT], 1998.
- [17] J.-F. Burnol, On Fourier and zeta(s). Forum Math. 16 (2004), no. 6, 789-840.
- [18] T. Carletti, A. Margheri and M. Villarini, Normalization of Poincaré singularities via variation of constants. Publ. Mat. 49 (2005), no. 1, 197212.
- [19] S. B. Chae, Holomorphy and calculus in normed spaces. With an appendix by Angus E. Taylor. Monographs and Textbooks in Pure and Applied Mathematics, 92, Marcel Dekker, Inc., New York, 1985.
- [20] P. Collet and J.-P. Eckmann, The ε -expansion for the hierarchical model. Comm. Math. Phys. 55 (1977), no. 1, 67-96.
- [21] P. Collet and J.-P. Eckmann, A Renormalization Group Analysis of The Hierarchical Model in Statistical Mechanics. Lecture Notes in Physics, 74, Springer, Berlin-New York, 1978.
- [22] F. Constantinescu, Nontriviality of the scattering matrix for weakly coupled Φ_3^4 models. Ann. Physics **108** (1977), no. 1, 37-48.
- [23] K. Costello and O. Gwilliam, Factorization Algebras in Quantum Field Theory, book in progress. Available at http://www.math.northwestern.edu/~costello/factorization.pdf
- [24] C. Demeter, M. Lacey, T. Tao and C. Thiele, The Walsh model for M_2^* Carleson. Rev. Mat. Iberoam. **24** (2008), no. 3, 721-744.
- [25] S. Dineen, Complex analysis in locally convex spaces. North-Holland Mathematics Studies, 57, Notas de Matemática, 83. North-Holland Publishing Co., Amsterdam-New York, 1981.
- [26] R. L. Dobrushin, Gaussian and their subordinated self-similar random generalized fields. Ann. Probab. 7 (1979), no. 1, 1-28.
- [27] R. L. Dobrushin, Automodel generalized random fields and their renorm group. In: Multicomponent Random Systems, Ed.: R. L. Dobrushin and Ya. G. Sinai, pp. 153-198, Adv. Probab. Related Topics 6, Marcel Dekker, New York, 1980.
- [28] J.-P. Eckmann and H. Epstein, Time-ordered products and Schwinger functions. Comm. Math. Phys. 64 (1978/79), no. 2, 95-130.
- [29] P. A. Faria da Veiga, Construction de Modèles Non Renormalisables en Théorie Quantique des Champs. Ph. D. Thesis, Université Paris-Sud, Orsay, 1991.
- [30] J. S. Feldman and R. Rączka, The relativistic field equation of the $\lambda \Phi_4^3$ quantum field theory. Ann. Physics **108** (1977), no. 1, 212-229.
- [31] M. E. Fisher, S.-K. Ma and B. G. Nickel, Critical exponents for long-range interactions. Phys. Rev. Lett. 29 (1972), no. 14, 917-920.
- [32] E. Frenkel and D. Ben-Zvi, Vertex Algebras and Algebraic Curves. Mathematical Surveys and Monographs, 88. American Math. Soc., Providence, RI, 2001.
- [33] K. Gawędzki and A. Kupiainen, Non-Gaussian scaling limits. Hierarchical model approximation. J. Statist. Phys. 35 (1984), no. 3-4, 267-284.
- [34] A. Gerasimov, D. Lebedev and S. Oblezin, Archimedean L-factors and topological field theories I. Commun. Number Theory Phys. 5 (2011), no. 1, 57-100.
- [35] A. Gerasimov, D. Lebedev and S. Oblezin, Archimedean *L*-factors and topological field theories II. Commun. Number Theory Phys. 5 (2011), no. 1, 101-133.

- [36] A. Gerasimov, D. Lebedev and S. Oblezin, Parabolic Whittaker functions and topological field theories I. Commun. Number Theory Phys. 5 (2011), no. 1, 135-201.
- [37] J. Glimm and A. Jaffe, The $\lambda \phi_2^4$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian. J. Mathematical Phys. 13 (1972), 1568-1584.
- [38] J. Glimm and A. Jaffe, Two and three body equations in quantum field models. Comm. Math. Phys. 44 (1975), no. 3, 293-320.
- [39] J. Glimm and A. Jaffe, Quantum Physics. A Functional Integral Point of View. Second edition, Springer, New York, 1987.
- [40] J. Glimm, A. Jaffe and T. Spencer, The Wightman axioms and particle structure in the $\mathcal{P}(\varphi)_2$ quantum field model. Ann. of Math. (2) **100** (1974), 585-632.
- [41] D. Goldfeld and J. Hundley, Automorphic representations and L-functions for the general linear group. Volume I. With exercises and a preface by Xander Faber. Cambridge Studies in Advanced Mathematics, 129. Cambridge University Press, Cambridge, 2011.
- [42] A. F. Grünbaum, Linearization for the Boltzmann equation. Trans. Amer. Math. Soc. 165 (1972), 425-449.
- [43] R. Gurau, J. Magnen and V. Rivasseau, Tree quantum field theory. Ann. Henri Poincaré 10 (2009), no. 5, 867-891.
- [44] D. Harlow, S. H. Shenker, D. Stanford and L. Susskind. Tree-like structure of eternal inflation: A solvable model. Phys. Rev. D 85 (2012), 063516.
- [45] S. Hollands and C. Kopper, The operator product expansion converges in perturbative field theory. Comm. Math. Phys. **313** (2012), no. 1, 257-290.
- [46] W. Hunziker, The S-matrix in classical mechanics. Comm. Math. Phys. 8 (1968), no. 4, 282-299.
- [47] D. Iagolnitzer and J. Magnen, Bethe-Salpeter kernel and short distance expansion in the massive Gross-Neveu model. Comm. Math. Phys. 119 (1988), no. 4, 567-584.
- [48] D. Iagolnitzer and J. Magnen, Large momentum properties and Wilson short distance expansion in nonperturbative field theory. Comm. Math. Phys. 119 (1988), no. 4, 609-626.
- [49] M. C. Irwin, A classification of elementary cycles. Topology 9 (1970), no. 1, 35-47.
- [50] M. C. Irwin, On the stable manifold theorem. Bull. London Math. Soc. 2 (1970), no. 2, 196-198.
- [51] G. Keller and C. Kopper, Perturbative renormalization of composite operators via flow equations. I. Comm. Math. Phys. 148 (1992), no. 3, 445-467.
- [52] G. Keller and C. Kopper, Perturbative renormalization of composite operators via flow equations. II. Short distance expansion. Comm. Math. Phys. 153 (1993), no. 2, 245-276.
- [53] A. N. Kochubei and M. R. Sait-Ametov, Interaction measures on the space of distributions over the field of p-adic numbers. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), no. 3, 389-411.
- [54] G. Kœnigs, Recherches sur les substitutions uniformes, Bull. Sci. Math. Astro. (2) 7 (1883), no. 1, 340–357.
- [55] E. Leichtnam, Scaling group flow and Lefschetz trace formula for laminated spaces with p-adic transversal. Bull. Sci. Math. 131 (2007), no. 7, 638-669.
- [56] È. Yu. Lerner, The hierarchical Dyson model and p-adic conformal invariance. Theoret. and Math. Phys. 97 (1993), no. 2, 1259-1266.
- [57] È. Yu. Lerner and M. D. Missarov, p-adic conformal invariance and the Bruhat-Tits tree. Lett. Math. Phys. 22 (1991), no. 2, 123-129.
- [58] P. Major, Multiple Wiener-Itô Integrals. With Applications to Limit Theorems. Lecture Notes in Mathematics, 849, Springer, Berlin, 1981.
- [59] M. D. Missarov, p-adic renormalization group solutions and the Euclidean renormalization group conjectures. p-Adic Numbers Ultrametric Anal. Appl. 4 (2012), no. 2, 109-114.
- [60] P. K. Mitter, personal communication, 2004.
- [61] E. Nelson, Topics in Dynamics. I: Flows. Mathematical Notes. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1969.
- [62] A. Pordt and C. Wieczerkowski, Nonassociative algebras and nonperturbative field theory for hierarchical models. Unpublished preprint arXiv:hep-lat/9406005, 1994.
- [63] B. Rosenstein, B. J. Warr and S. H. Park, Dynamical symmetry breaking in four-fermion interaction models. Phys. Rep. 205 (1991), no. 2, 59–108.
- [64] M. Salmhofer, Renormalization. An Introduction. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1999.
- [65] A. D. Scott and A. D. Sokal, Complete monotonicity for inverse powers of some combinatorially defined polynomials. Preprint arXiv:1301.2449[math.CO], 2013.
- [66] B. Simon, Wave operators for classical particle scattering, Comm. Math. Phys. 23 (1971), 37-48.
- [67] B. Simon, Distributions and their Hermite expansions. J. Mathematical Phys. 12 (1971), 140-148.
- [68] B. Simon, The P(φ)₂ Euclidean (Quantum) Field Theory. Princeton University Press, Princeton, NJ, 1974.
- [69] B. Simon, Functional integration and quantum physics. Pure and Applied Mathematics, 86, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), New York-London, 1979.
- [70] Ya. G. Sinai, Self-similar probability distributions. Theor. Probability Appl. 21 (1976), 64–80.
- [71] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, p-adic analysis and mathematical physics. Series on Soviet and East European Mathematics, 1. World Scientific Publishing Co., River Edge, NJ, 1994.
- [72] F. J. Wegner, Corrections to scaling laws. Phys. Rev. B 5 (1972), no. 11, 4529-4536.
- [73] K. G. Wilson, Model Hamiltonians for local quantum field theory. Phys. Rev. 140 (1965), no. 2B, B445-B457.

- [74] K. G. Wilson, Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior. Phys. Rev. B 4 (1971), no. 9, 3184-3205.
- [75] K. G. Wilson, Renormalization of a scalar field theory in strong coupling. Phys. Rev. D 6 (1972), no. 2, 419-426.
- [76] K. G. Wilson and M. E. Fisher, Critical Exponents in 3.99 Dimensions. Phys. Rev. Lett 28 (1972), no. 4, 240-243.
- [77] K. G. Wilson and J. Kogut, The renormalization group and the ϵ expansion. Phys. Rep. 12 (1974), no. 2, 75-199.
- [78] A. V. Zabrodin, Non-Archimedean strings and Bruhat-Tits trees. Comm. Math. Phys. 123 (1989), no. 3, 463-483.
- [79] W. Zimmermann, Local operator products and renormalization in quantum field theory. In: Lectures on Elementary Particles and Quantum Field Theory, Ed.: S. Deser, M. Grisaru and H. Pendleton, pp. 395–589, M.I.T. Press, Cambridge, MA and London, England, 1970.

Abdelmalek Abdesselam, Department of Mathematics, P. O. Box 400137, University of Virginia, Charlottesville, VA 22904-4137, USA

 $E\text{-}mail\ address{:}\ \mathtt{malek@virginia.edu}$

AJAY CHANDRA, DEPARTMENT OF MATHEMATICS, P. O. BOX 400137, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904-4137, USA

 $E ext{-}mail\ address: ac2yx@virginia.edu}$

Gianluca Guadagni, Department of Mathematics, P. O. Box 400137, University of Virginia, Charlottesville, VA 22904-4137, USA

 $E\text{-}mail\ address{:}\ \texttt{gg5d@virginia.edu}$