Two-Sided Matching Theory

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Introduction to the Theory of Two-Sided Matching

To see which results are robust, we will look at some increasingly general models. Even before we look at complex design problems, we can get a head start at figuring out which are our most applicable results by doing this sort of theoretical sensitivity analysis.

Discrete models

1. One to one matching: the "marriage" model
2. many to one matching (with simple preferences) : the "college admissions" model
3. many to one matching with money and complex (gross substitutes) preferences

These lectures on theory follow Roth and Sotomayor (1990), and theorems are numbered as in the book.

One-to-One Matching: The Marriage Model

Gale and Shapley, AMM (1962)

A **marriage market** is a triple $(M, W, \succsim)$ with

- M is a finite set of **men**
- W is a finite set of **women**
- $\succsim = (\succsim_k)_{k \in M \cup W}$ is a **preference profile** where
  $\succsim_m$ is a preference relation over $W \cup \{m\}$
  $\succsim_w$ is a preference relation over $M \cup \{w\}$
  Let $\succ_k$ be the strict preference relation derived from $\succsim_k$
  $i \succ_k j$ means $i \succsim_k j$ and not $j \succsim_k i$.
- If agent $k \in M \cup W$ prefers to remain single rather than be matched to agent j, i.e. if $k \succ_k j$, then j is unacceptable to k. (If an agent is not indifferent between any two acceptable mates, or between being matched and unmatched, we'll say he/she has strict preferences. Some theorems below are valid only for strict preferences. If otherwise mentioned we assume that preferences are **strict**.)

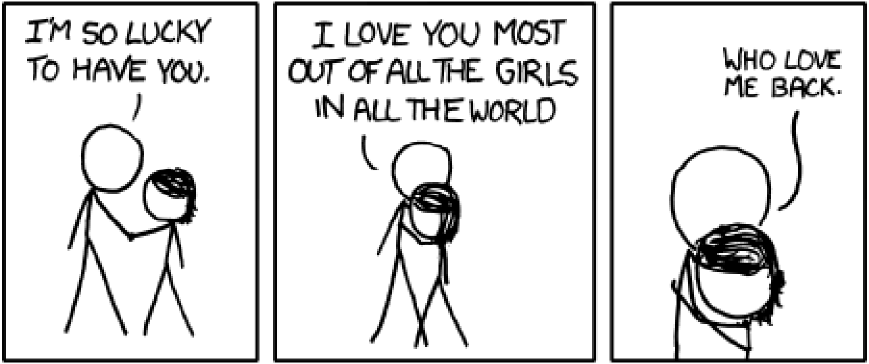
One-to-one Matching: The Marriage Model

- An outcome of the marriage market is a **matching** $\mu : M \cup W \to M \cup W$ such that $\mu(m) = w$ iff $\mu(w) = m$ and for all m and either $\mu(w) \in M \cup \{w\}$ and $\mu(m) \in W \cup \{m\}$.
- A matching μ is **blocked by an individual** $k \in M \cup W$ if k prefers being single to being matched with $\mu(k)$, i.e.$k \succ_k \mu(k)$.
- A matching μ is **blocked by a pair** $(m, w) \in M \times W$ if they each prefer each other to their partners under μ, i.e.

$$w \succ_m \mu(m) \text{ and } m \succ_w \mu(w).$$

- A matching is **stable** if it isn't blocked by any individual or pair of agents.
- A stable matching is Pareto-efficient.

Stability of Marriage:



(xkcd.com)

(Strong) Core, Weak Core, The Set of Stable Matchings

The **weak core** is the set of matchings μ such that there exists no matching ν and coalition $T \subseteq M \cup W$ such that for all $k \in T$, $\nu(k) \succ_k \mu(k)$ and $\nu(k) \in T$.

The **(strong) core** is the set of matchings μ such that there exists no matching ν and coalition $T \subseteq M \cup W$ such that for all $k \in T$, $\nu(k) \succsim_k \mu(k)$, for some $i \in T$, $\nu(i) \succ_i \mu(i)$, and $\nu(k) \in T$.

$\mathcal{S}$: set of stable matchings
$\mathcal{C}^W$: weak core
$\mathcal{C}^S$: (strong) core

Lemma

In the marriage model, when preferences are general, $\mathcal{C}^S \subseteq \mathcal{C}^W = \mathcal{S}$; when preferences are strict, $\mathcal{C}^S = \mathcal{C}^W = \mathcal{S}$.

A stable matching is also referred to as a core matching.

Men-Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step 0 If some preferences are not strict, arbitrarily break ties (e.g. if some m is indifferent between w and w', order them consecutively in an arbitrary manner).

Step 1 Each man m proposes to his 1st choice (if he has any acceptable choices).
Each woman rejects any unacceptable proposals and, if more than one acceptable proposal is received, "holds" the most preferred.
If no proposals are rejected, and match each woman to the man (if any) whose proposal she is "holding" and terminate the procedure.

⋮

Men-Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step k Any man who was rejected at step $k-1$ makes a new proposal to his most preferred acceptable mate who hasn't yet rejected him. (If no acceptable choices remain, he makes no proposal.)
Each woman "holds" her most preferred acceptable offer to date, and rejects the rest.
If no proposals are rejected, and match each woman to the man (if any) whose proposal she is "holding" and terminate the procedure.

Existence of Stable Matchings

Theorem (Gale and Shapley, AMM, 1962)

A stable matching exists for every marriage market.

Proof: Observation: Women get (weakly) better off as the process goes on, and men get (weakly) worse off as the process goes on.
· the deferred acceptance algorithm always stops.
· the matching it produces, μ, is always stable with respect to the strict preferences (i.e. after any arbitrary tie-breaking), why?
The matching μ cannot be blocked by any individual, since men do not make any offers to unacceptable women, and women immediately reject unacceptable offers.
Suppose there is a pair (m, w) blocking the matching. Then m made an offer to w in deferred acceptance algorithm and since they are not matched with each other w rejected m in favor of someone better. A contradiction that (m, w) blocks μ.
· μ is stable with respect to the original preferences.

Side-Optimal Stable Matchings

Theorem (Gale and Shapley, AMM, 1962)

When all men and women have strict preferences, there always exists an men-optimal stable matching (that every man likes at least as well as any other stable matching), and a women-optimal stable matching.

Furthermore, the matching μ_M produced by the deferred acceptance algorithm with men-proposing is the men-optimal stable matching. The women-optimal stable matching is the matching μ_W produced by the algorithm when the women propose.

Side-Optimal Stable Matchings

Proof:

- **Terminology:** w is **achievable** for m if there is some stable matching μ such that $\mu(m) = w$.
- **Proof by induction:**
  Inductive step: suppose that up to step k of the algorithm, no man has been rejected by an achievable partner, and that at step k woman w rejects man m (who is acceptable to w) and (therefore) holds on to some m'.
  Then w is not achievable for m. Why? Suppose μ is stable such that $\mu(m) = w$. Then $\mu(m')$ is achievable for m'. Then μ can't be stable: by the inductive step, (m', w) would be a blocking pair.
  Therefore every man in matched with the best acheivable partner under the outcome of the men-proposing deferred acceptance algorithm, μ_M meaning that μ_M is men-optimal stable matching.

Lattice Properties of Set of Stable Matchings

- Let $\mu \succ_M \mu'$ denote that all men like μ at least as well as μ', with at least one man having strict preference.
- Then $\succ_M$ is a partial order on the set of matchings, representing the common preferences of the men. Similarly, define $\succ_W$ as the common preference of the women.

Theorem (Knuth)

When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings: if μ and μ' are stable matchings, then all men like μ at least as well as μ' if and only if all women like μ' at least as well as μ. That is, $\mu \succ_M \mu'$ if and only if $\mu' \succ_W \mu$.

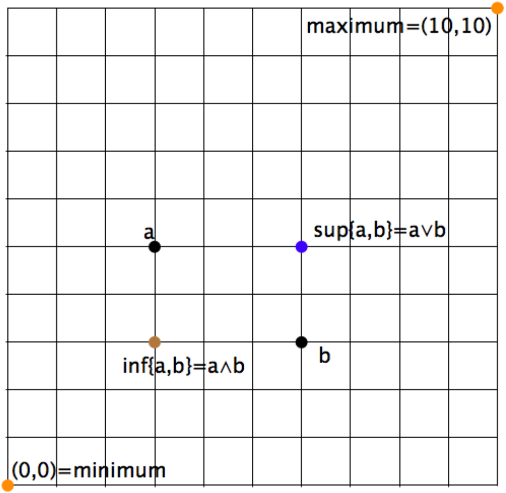
Proof: immediate from definition of stability.
So the best outcome for one side of the market is the worst for the other.
For any two matchings μ and μ', and for all men and women, define $\mu \vee_M \mu'$ (**join**) as the function that assigns each man his more preferred of the two matches, and each woman her less preferred:
$\cdot\ \mu \vee_M \mu'(m) = \mu(m)$ if $\mu(m) \succ_m \mu'(m)$ and
$\cdot \mu \vee_M \mu'(m) = \mu'(m)$ otherwise
$\cdot \mu \vee_M \mu'(w) = \mu(w)$ if $\mu'(w) \succ_w \mu(w)$ and
$\cdot \mu \vee_M \mu'(w) = \mu'(w)$ otherwise
Define $\mu \wedge_M \mu'$ (**meet**) analogously, by reversing the preferences.

Lattice:

A **lattice** is a partially ordered set with a unique maximum and unique minimum.
The **infimum** of two elements is defined by $\wedge$ (or meet) and the **supremum** of two elements is defined by $\vee$ (or join).
Observe that infimum and supremum of any two elements are well defined for a lattice.

Lattice:

Example: $\{0, 1, ..., 10\}^2$ and usual vector $\geq$ partial order



Lattice Theorem

Theorem (Lattice Theorem (Conway))

When all preferences are strict, if μ and μ' are stable matchings, then the functions $\mu \vee_M \mu'$ and $\mu \wedge_M \mu'$ are both matchings and they are both stable.

Lattice Theorem

So if we think of λ as asking men to point to their preferred mate from two stable matchings, and asking women to point to their **less** preferred mate, the theorem says that

- No two men point to the same woman
  - (this follows from the stability of μ and μ')
- Every woman points back at the man pointing to her;
  - (the direction [$\lambda(\text{m}) = \text{w}$ implies $\lambda(\text{w}) = \text{m}$] follows easily from stability, but the direction [$\lambda(\text{w}) = \text{m}$ implies $\lambda(\text{m}) = \text{w}$] takes a bit more work (See the Decomposition Lemma below).
- And the resulting matching is stable.
  - (again, immediately from the stability of μ and μ')

Theorem

In a market $(M, W, \succ)$ with strict preferences, the set of people who are single is the same for all stable matchings.

One strategy of proof: What can we say about the **number and identity** of men and women matched (and hence the number and identity unmatched) at μ_M and at μ_W?

$|\mu_M(W)| = |\mu_M(M)| \geq$

(by men optimality)

$|\mu_W(M)| = |\mu_W(W)| \geq$

(by women optimality)

$|\mu_M(W)|$

Theorem (Weak Pareto efficientity for Men)

There is no individually rational matching μ (stable or not) such that $\mu(m) \succ_m \mu_M(m)$ for all $m \in M$.

Proof: (using the deferred acceptance algorithm)

- If μ were such a matching it would match every man m to some woman w who had rejected him in the algorithm in favor of some other man m' (i.e. even though m was acceptable to w).
- Hence all of these women, $\mu(M)$, would have been matched under μ_M. That is, $\mu_M(\mu(M)) = M$.
- Hence all of M would have been matched under μ_M and $\mu_M(M) = \mu(M)$.
- But since all of M are matched under μ_M any woman who gets a proposal in the last step of the algorithm at which proposals were issued has not rejected any acceptable man, i.e. the algorithm stops as soon as every woman in $\mu_M(M)$ has an acceptable proposal.
- So such a woman must be single at μ (since every man prefers

Example

μ_M is not in general (strongly) Pareto efficient for men:

$M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$m_1 : w_2 \succ w_1 \succ w_3$

$m_2 : w_1 \succ w_2 \succ w_3$

$m_3 : w_1 \succ w_2 \succ w_3$

$w_1 : m_1 \succ m_2 \succ m_3$

$w_2 : m_3 \succ m_1 \succ m_2$

$w_3 : m_1 \succ m_2 \succ m_3$

$$\mu_M = ([m_1, w_1], [m_2, w_3], [m_3, w_2]) = \mu_W$$

But note that $\mu \succ_M \mu_M$ for

$$\mu = ([m_1, w_2], [m_2, w_3], [m_3, w_1])$$

The Preference Revelation Game

Let's consider strategic behavior in centralized matching mechanisms, in which participants submit a list of stated preferences. By the **revelation principle**, some of the results will apply to decentralized markets also, in which agents have different sets of strategies.

Strategic Behavior

Consider a marriage market $(M, W, \succsim)$ whose matching outcome will be determined by a centralized clearinghouse, based on a list of preferences that players will state ("reveal"). If the stated preference profile is $\succsim'$, the algorithm employed by the clearinghouse produces a matching $\phi[\succsim']$ assuming M and W are fixed. In general, the **matching mechanism** ϕ is defined for all $(M, W, \succsim')$.

If the matching produced is **always** a stable matching with respect to $\succsim'$, we'll say that ϕ is a stable matching mechanism.

No Incentive Compatible Mechanism

Theorem (Impossibility Theorem, Roth)

No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every agent.

Remark on proof: for an impossibility theorem, one example for which **no** stable matching mechanism induces a dominant strategy is sufficient.

Consider an example with 2 agents on each side, with preferences as follows:

$$m_1 : w_1 \succ w_2 \qquad\qquad w_1 : m_2 \succ m_1$$
$$m_2 : w_2 \succ w_1 \qquad\qquad w_2 : m_1 \succ m_2$$

In this example, what must an arbitrary stable mechanism do? I.e. what is the range of $\phi[\succsim]$ if ϕ is a stable mechanism?

No Incentive Compatible Mechanism

Given $\phi[\succsim]$, and the restriction that ϕ is a stable mechanism, can one of the players k engage in a profitable manipulation by stating some $\succsim'_k \neq \succsim_k$ such that k prefers $\phi[\succsim']$ to $\phi[\succsim]$?

Of course, this kind of proof of the impossibility theorem leaves open the possibility that situations in which some participant can profitably manipulate his preferences are rare. The following result suggests otherwise.

Theorem

When any stable mechanism is applied to a marriage market in which preferences are strict and there is more than one stable matching, then at least one agent can profitably misrepresent his or her preferences, assuming the others tell the truth. (This agent can misrepresent in such a way as to be matched to his or her most preferred achievable mate under the true preferences at every stable matching under the mis-stated preferences.)

Incentives facing the men when the men-optimal stable mechanism is used

Theorem (Dubins and Freedman 1980, Roth 1982)

The men-optimal stable mechanism makes it a dominant strategy for each man to state his true preferences.

Theorem (Dubins and Freedman 1980)

Let $\succ$ be the true preferences of the agents, and let $\succ'$ differ from $\succ$ in that some coalition M' of the men mis-state their preferences. Then there is no matching μ, stable for $\succ'$, which is preferred to μ_M by all members of M'.

Decomposition Lemma

One more observation about the marriage model that is used in the Lattice property's proof and later in the many-to-one matching model.

Suppose μ and μ' are stable matchings, and for some m, $w = \mu(m) \succ_m \mu'(m) = w'$. Then the stability of μ' immediately implies that $\mu'(w) \succ_w \mu(w) = m$.

But how about w'?

Lemma (Decomposition Lemma - Corollary 2.21, Knuth)

Let μ and μ' be stable matchings in $(M, W, \succ)$, with all preferences strict. Let M^{μ} (W^{μ}) be the set of men (women) who prefer μ to μ', and let $M^{\mu'}$ ($W^{\mu'}$) be those who prefer μ'. Then μ and μ' both map $M^{\mu'}$ onto W^{μ} and M^{μ} onto $W^{\mu'}$.

Decomposition Lemma

Proof: Suppose for some m, $w = \mu(m) \succ_m \mu'(m) = w'$. Then the stability of μ' immediately implies that $\mu'(w) \succ_w \mu(w) = m$. We've just observed above that $\mu(M^\mu)$ is contained in $W^{\mu'}$.
So $|M^\mu| \leq |W^{\mu'}|$.

Symmetrically, $\mu'(W^{\mu'})$ is contained in M^μ, so $|M^\mu| \geq |W^{\mu'}|$.

Since μ and μ' are one-to-one (and since M^μ and $W^{\mu'}$ are finite), both μ and μ' are onto. So, to answer the question posed at the top of the previous slide, a man or woman who prefers one stable matching to another is matched at **both** of them to a mate with the reverse preferences.

Many-to-one Matching: The College Admissions Model

A **college admissions market** $(W, F, q, \succ)$ consists of

- A finite set of **workers** W
- A finite set of **firms** F
- A finite non-negative **quota** q_f for each $f \in F$
- **Preference relation** of each $f \in F$ over subsets of W and f:$\succ_f$ (assume strict preferences)
- **Preference relation** of each $w \in W$ over $F \cup \{w\}$: $\succ_w$ (assume strict preferences)

Many-to-one Matching

We need to specify how firms' preferences over matchings, are related to their preferences over individual workers, since they hire groups of workers. The simplest model is:

- **Responsive preferences**: for any set of workers $S \subset W$ with $|S| < q_f$, and any workers w and $w' \in W \setminus S$, $S \cup \{w\} \succ_f S \cup \{w'\}$ if and only if $w \succ_f w'$, and $S \cup \{w\} \succ_f S$ if and only if w is acceptable to f.
- A matching μ is **individually irrational** if $\mu(w) = f$ for some worker w and firm f such that either the worker is unacceptable to the firm or the firm is unacceptable to the worker. An individually irrational matching is said to be blocked by the relevant individual.

Many-to-one Matching

- A matching μ is **block by a firm-worker pair** (f, w) if they each prefer each other to μ:
  - $w \succ_f w'$ for some $w' \in \mu(f)$ or $w \succ_f f$ if $|\mu(f)| < q_f$, and
  - $f \succ_w \mu(w)$
- As in the marriage model, a matching is **pairwise stable** if it isn't blocked by any individual or pair of agents.

Many-to-one Matching

- But now that firms employ multiple workers, it might not be enough to concentrate only on pairwise stability. The assumption of responsive preferences allows us to do this, however.
- A matching μ is **blocked by a group** A of firms and workers if there exists another matching μ' such that for all workers $w \in A$, and all firms $f \in A$
  1. $\mu'(w) \in A$
  2. $\mu'(w) \succ_w \mu(w)$
  3. $w' \in \mu'(f)$ implies $w' \in A \cup \mu(f)$, i.e. every firm in A is matched at μ' to new workers only from A, although it may continue to be matched with some of its "old" workers from μ. (This differs from the definition of core...)
  4. $\mu'(f) \succ_f \mu(f)$
- A matching is **(group) stable** if it is not blocked by a group of any size.

Lemma

When firm preferences are responsive, a matching is stable if and only if it is pairwise stable.

Proof: pairwise instability clearly implies instability.
Now suppose μ is blocked via group A and outcome μ'. Then there must be a worker w and a firm f such that w is in $\mu'(f)$ but not in $\mu(f)$ such that w and f block μ. (Otherwise it couldn't be that $\mu'(f) \succ_f \mu(f)$, since f has responsive preferences.) ■

(Strong) Core, Weak Core, and Set of Stable Matchings

Recall that
$\mathcal{S}$: Set of (group) stable matchings
$\mathcal{C}^W$: Weak core
$\mathcal{C}^S$: (Strong) core

Lemma

In the many-to-one matching model with strict preferences,
$\mathcal{S} = \mathcal{C}^S \subseteq \mathcal{C}^W$.

Firm Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step 1 Each firm f proposes to its q_f best choice workers (if there are less acceptable choices then all of them).

Each worker rejects any unacceptable proposals and, if more than one acceptable proposal is received, "holds" the most preferred.

If no proposals are rejected, and match each worker to the firm (if any) whose proposal she is "holding" and terminate the procedure.

$\vdots$

Firm Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step k Any firm who was rejected by ℓ workers at step $k-1$ makes $q_f - \ell$ new proposals to its most preferred acceptable workers who hasn't yet rejected it. (If less acceptable choices remain, it proposes to all of its remaining acceptable workers)
Each worker "holds" her most preferred acceptable offer to date, and rejects the rest.
If no proposals are rejected, match each worker to the firm (if any) whose proposal she is "holding" and terminate the procedure.

Worker Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step 1 Each worker w proposes to her best acceptable firm.
Each firm f rejects any unacceptable proposals and, if more than q_f acceptable proposals are received, "holds" the most preferred q_f group of workers, if less than q_f acceptable proposals are received it temporarily holds all of them.
If no proposals are rejected, and match each firm to the worker (if any) whose proposal it is "holding" and terminate the procedure.

$\vdots$

Firm Proposing Deferred Acceptance Algorithm (the Gale-Shapley 1962 version)

Step k Any worker who was rejected at step $k-1$ makes a new proposal to her most preferred acceptable firm which hasn't yet rejected her.

Each firm "holds" its most preferred q_f acceptable offers from this step and the ones it is holding from the previous step, and rejects the rest, if there are more than q_f of them. Otherwise, it "holds" all of the acceptable offers from this step and the ones from the previous step. It rejects all unacceptable offers from this step.

If no proposals are rejected, match each worker to the firm (if any) whose proposal she is "holding" and terminate the procedure.

Both of these algorithms find stable matchings.

Worker proposing mechanism finds the **worker-optimal stable matching**.

Firm proposing mechanism finds the **firm-optimal stable matching**.

Lattice property of stable matchings still holds in a rather strong form.

Worker-optimal stable mechanism is still incentive compatible for workers in dominant strategies.

But why?

A related marriage market

Replace firm f by q_f positions of f denoted by $f^1, f^2, ..., f^{q_f}$. Each of these positions has f's preferences over individuals. Since each position f^i has a quota of 1, we do not need to consider preferences over groups of workers.

Each worker's preference list is modified by replacing f, wherever it appears on his list, by the string $f^1, f^2, ..., f^{q_f}$, in that order.

A matching μ of the many-to-one matching, corresponds to a matching μ' in the related marriage market in which the workers in $\mu(f)$ are matched, in the order which they occur in the preferences $\succ_f$, with the ordered positions of f that appear in the related marriage market. (If preferences are not strict, there will be more than one such matching.)

Lemma 5.6

Lemma

A matching of the many-to-one matching market is stable if and only if the corresponding matching of the related marriage market is stable.

(Note: some results from the marriage model will translate immediately)

Geographic Distribution

Theorem (Theorem 5.12)

When all preferences over individuals are strict, the set of workers employed and positions filled is the same at every stable matching.

The proof is immediate via the similar result for the marriage market and the construction of the corresponding marriage market (Lemma 5.6).

So any firm that fails to fill all of its positions at some stable matching will not be able to fill any more positions at any other stable matching. The next result shows that not only will such a firm fill the same number of positions, but it will fill them with exactly the same interns at any other stable matching.

Geographic Distribution

Theorem (Theorem 5.13: rural hospitals theorem - Roth 86)

*When preferences over individuals are strict, any firm that does not fill its quota at some stable matching **is assigned precisely the same set of workers** at every stable matching.*

Lemma 5.25 (Roth and Sotomayor)

Lemma

Suppose firms and workers have strict individual preferences, and let μ and μ' be stable matchings for $(W, F, q, \succ)$, such that $\mu(f) \neq \mu'(f)$ for some $f \in F$. Let $\hat{\mu}$ and $\hat{\mu}'$ be the stable matchings corresponding to μ and μ' in the related marriage market. If $\hat{\mu}(f^i) \succ_f \hat{\mu}'(f^i)$ for some position f^i of f then $\hat{\mu}(f^i) \succsim_f \hat{\mu}'(f^i)$ for all positions f^i of f.

Lemma 5.25 (Roth and Sotomayor)

Proof:
It is enough to show that $\hat{\mu}(f^j) \succ_f \hat{\mu}'(f^j)$ for all $j > i$. So suppose this is false. Then there exists an index j such that $\hat{\mu}(f^j) \succ_f \hat{\mu}'(f^j)$, but $\hat{\mu}(f^{j+1}) \precsim_f \hat{\mu}'(f^{j+1})$. Theorem 5.12 (constant employment) implies $\hat{\mu}'(f^j) \in W$. Let $w' \equiv \hat{\mu}'(f^j)$. By the **Decomposition Lemma** $f^j \equiv \hat{\mu}'(w') \succ_{w'} \hat{\mu}(w')$. Furthermore, $\hat{\mu}(w') \neq f^{j+1}$, since $w' \succ_f \hat{\mu}'(f^{j+1}) \succsim_f \hat{\mu}(f^{j+1})$ (where the first of these preferences follows from the fact that for any stable matching $\hat{\mu}'$ in the related marriage market, $\hat{\mu}(f^j) \succ_f \hat{\mu}'(f^{j+1})$ for all j). Therefore f^{j+1} comes right after f^j in the preferences of w' (or any w) in the related marriage market. So $\hat{\mu}$ is blocked via w' and f^{j+1}, contradicting (via Lemma 5.6) the stability of μ.
(This proof also establishes the rural hospitals theorem).

Theorem 5.27 (Roth and Sotomayor)

Theorem (Roth and Sotomayor)

Let preferences over individuals be strict, and let μ and μ' be stable matchings for $(W, F, q, \succsim)$. If $\mu(f) \succ_f \mu'(f)$ for some firm f, then $w \succ_f w'$ for all $w \in \mu(f)$ and $w' \in \mu'(f) \setminus \mu(f)$. That is, f prefers every worker in its assignment at μ to every worker who is in its assignment at μ' but not at μ.

Theorem 5.27 (Roth and Sotomayor)

Proof:

Consider the related marriage market and the stable matchings $\hat{\mu}$ and $\hat{\mu}'$ corresponding to μ and μ'. Let $q_f = k$, so that the positions of f are $f^1, \ldots, f^{q_f}$.

First observe that f fills its quota under μ and μ', since, if not, Theorem 5.13 (Rural hospitals) would imply that $\mu(f) = \mu'(f)$. So $\mu'(f) - \mu(f)$ is a nonempty subset of W, since $\mu(f) \neq \mu'(f)$. Let w' $= \hat{\mu}'(f^j)$ for some position f^j such that w' is not in $\mu(f)$. Then $\hat{\mu}(f^j) \neq \hat{\mu}'(f^j)$.

By Lemma 5.25 $\hat{\mu}(f^j) \succ_f \hat{\mu}'(f^j) = w'$.

The Decomposition Lemma implies $f^j \succ_{w'} \hat{\mu}(w')$.

So the construction of the related marriage market implies $f \succ_{w'} \mu(w')$, since $\mu(w') \neq f$.

Thus $w \succ_f w'$ for all $w \in \mu(f)$ by the stability of μ, which completes the proof.

Comments

- Consider a firm f with quota 2 and preferences over individuals $w_1 \succ_f w_2 \succ_f w_3 \succ_f w_4$. Suppose that at various matchings 1-4, f is matched to
  1. $\{w_1, w_4\}$,
  2. $\{w_2, w_3\}$,
  3. $\{w_1, w_3\}$, and
  4. $\{w_2, w_4\}$.
- Which matchings can be simultaneously stable for some responsive preferences over individuals?
- So long as all preferences over groups are responsive, matchings 1 and 2 cannot both be stable (Lemma 5.25), nor can matchings 3 and 4 (Theorem 5.27).

Strategic questions in the Many-to-one Matching Model:

Theorem (Roth)

A stable matching procedure which yields the worker-optimal stable matching makes it a dominant strategy for all workers to state their true preferences.

Proof: immediate from the related marriage market

Strategic questions in the Many-to-one Matching Model:

Theorem (Roth)

No stable matching mechanism exists that makes it a dominant strategy for all firms to state their true preferences.

Strategic questions in the Many-to-one Matching Model:

Proof:

Consider a market consisting of 3 firms and 4 workers. f_1 has a quota of 2, and both other firms have a quota of 1. The preferences are:

$w_1 : f_3 \succ f_1 \succ f_2$

$w_2 : f_2 \succ f_1 \succ f_3$

$w_3 : f_1 \succ f_3 \succ f_2$

$w_4 : f_1 \succ f_2 \succ f_3$

$f_1 : w_1 \succ w_2 \succ w_3 \succ w_4$

$f_2 : w_1 \succ w_2 \succ w_3 \succ w_4$

$f_3 : w_3 \succ w_1 \succ w_2 \succ w_4$

The unique stable matching is

$\{[f_1, w_3, w_4], [f_2, w_2], [f_3, w_1]\}$

But if f_1 instead submitted the preferences $w_1 \succ w_4$

the unique stable matching is

$\{[f_1, w_1, w_4], [f_2, w_2], [\mathrm{f}_3, \mathrm{w}_3]\}$.