1 Simpson's Rule

Simpson's rule differs from the other numerical integration routines in that it requires that we partition the interval $[a, b]$ into an *even* number of subintervals.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

However, the width of each subinterval remains the same, calculated with

$$h = \frac{b-a}{n}.$$

Consider the case, shown in **Figure 1**, where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$.

The function f (in green) is evaluated at x_0, x_1, and x_2, producing the corresponding y-values, y_0, y_1, and y_2.

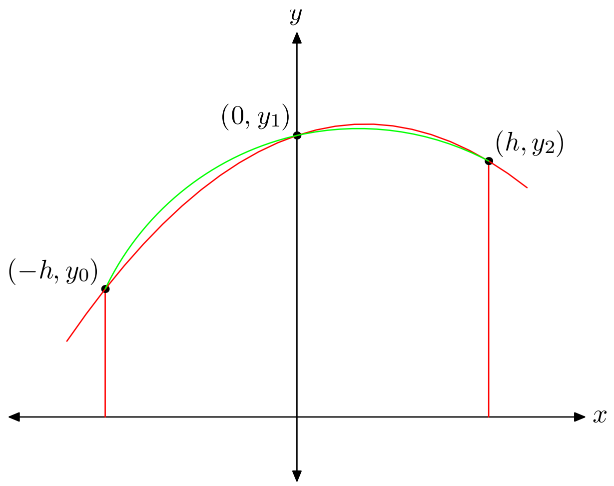

Figure 1 The parabola $y = Ax^2 + bx + c$ passes through the point $(-h, y_0)$, $(0, y_1)$, and (h, y_2).

Next the function f is approximated on the interval $[x_0, x_2]$ with a parabola, $y = Ax^2 + Bx + C$, draw in red in **Figure 1**. The area under the function on the interval $[x_0, x_2]$ is approximated by finding the area under the curve $y = Ax^2 + Bx + C$ on $[x_0, x_2]$.

However, the area under the parabola $y = Ax^2 + Bx + C$ on the interval $[-h, h]$ is given by the integral

$$\begin{aligned} A &= \int_{-h}^{h} (Ax^2 + Bx + C)\, dx, \\ &= \int_{-h}^{h} (Ax^2 + C)\, dx + \int_{-h}^{h} Bx\, dx. \end{aligned}$$

The first integrand is an even function, so

$$\begin{aligned}\int_{-h}^{h} (Ax^2 + C)\,dx &= 2\int_{0}^{h} (Ax^2 + C)\,dx,\\ &= 2\left(\frac{1}{3}Ax^3 + Cx\right)\Bigg|_0^h,\\ &= \frac{2}{3}Ah^3 + 2Ch,\\ &= \frac{h}{3}(2Ah^2 + 6C).\end{aligned}$$

The second integrand is an odd function, so

$$\int_{-h}^{h} Bx\,dx = 0.$$

Thus, the area under the parabola $y = Ax^2 + Bx + C$ on $[-h, h]$ is

$$A = \frac{h}{3}(2Ah^2 + 6C). \tag{1}$$

However, each of the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) lie on the parabola, so each must satisfy the equation of the parabola. Consequently,

$$\begin{aligned} y_0 &= Ah^2 - Bh + C,\\ y_1 &= C,\\ y_2 &= Ah^2 + Bh + C.\end{aligned}$$

Thus,

$$\begin{aligned} y_0 + 4y_1 + y_0 &= (Ah^2 - Bh + C) + 4C + (Ah^2 + Bh + C),\\ &= 2Ah^2 + 6C.\end{aligned}$$

Placing this result in Equation (1), the area under the parabola $y = Ax^2 + Bx + C$ on $[-h, h]$ is given by

$$A = \frac{h}{3}(y_0 + 4y_1 + y_0).$$

Note that this area result remains unchanged if we shift the region right or left. In **Figure 2**, the area under the curve is partitioned into six subintervals.

Parabolic arches are drawn through P_0, P_1, and P_2, then P_2, P_3, and P_4, and finally, through P_4, P_5, and P_6. If y_k is in the y-value of point P_k, $k = 0, 1, \ldots, 6$, then the total area under the parabolic arches is

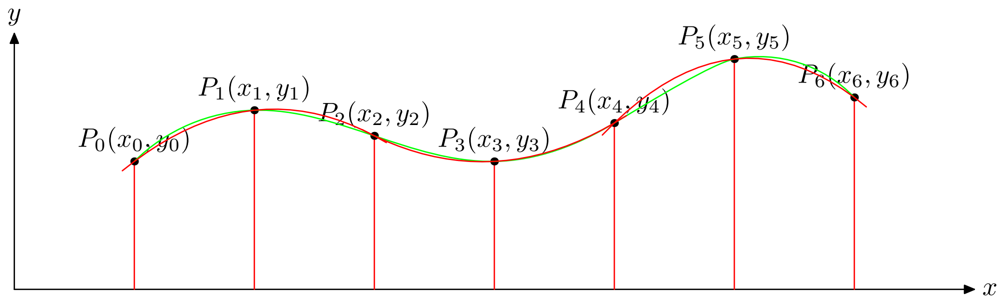

Figure 2 Simpson's rule requires an even number of subintervals.

$$S = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \frac{h}{3}(y_4 + 4y_5 + y_6).$$

Obviously, if we increase the number of subintervals to n, where n is even, the total area is now

$$S = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

If we factor $h/3$ from each term, this will greatly simplify our programming instructions.

$$S = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right] \qquad (2)$$

We are now ready to write a program that will execute Simpson's rule.

We begin by providing the calculator with some initial data. We assume that the integrand of

$$\int_a^b f(x)dx$$

is currently loaded in the `Y=` Menu. For example if we wish to integrate

$$\int_1^2 \frac{1}{x} dx,$$

then we load $y = 1/x$ in `Y1`.

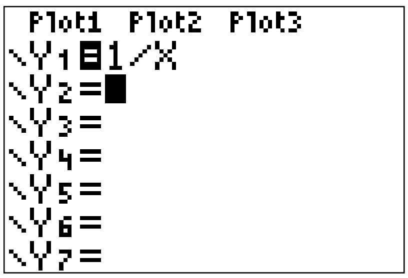

Figure 3 Entering the integrand in the `Y=` Menu.

The calculator will need to know the limits of integration and the number of subintervals in the partition. We also clear the home screen before prompting the user for this data.

```
Clr Home
Prompt A
Prompt B
Prompt N
```

The interval must be partitioned into n equal subintervals, so we must program the calculator to reject an odd value for n. We also should tell the user why the computation cannot be performed as entered.

```
If int(N/2)\ne N/2

Then
Disp "N MUST BE EVEN"
Stop
End
```

We immediately calculate the width Δx of each subinterval and store it in the variable `H`.

```
(B-A)/N->H
```

Because we begin with the endpoint of the first subinterval, we store the contents of the variable `A` in `X`.

```
A->X
```

We will store the sum in the variable `S`. It is important that we initialize this variable to zero before we begin.

```
0->S
```

We now add the function values in Equation (2). This is accomplished by proceeding one step at a time, moving from left to right. This calls for the use of a `For` loop.

```
For(K,1,N,1)
S+Y1(X)+4*Y1(X+H)+Y1(X+2*H)->S
X+2*H->X
End
```

This loop construct warrants some further explanation. The command `For(K,1,N,1)` initializes `K` to 1, then executes the loop. It increments the value of `K` by 1, then checks to see if `K` is less than or equal to `N`. If `K` is less than or equal to `N`, the loop is executed again. If `K` is greater than `N`, execution of the loop is halted and the program continues by executing the line following the `End` command, which terminates the loop. In the body of the loop, two things happen. First, the integrand, which was stored in `Y1`, is evaluated at `X`, `X+H`, and `X+2*H`, added to the current value of `S`, then `S` is updated to contain this current sum. This is accomplished with the command `S+Y1(X)+4*Y1(X+H)+Y1(X+2*H)->S`. Next, `X` is incremented by `2*H` by executing the command `X+2*H->X`. This is necessary because Simpson's rule uses 2 subintervals to approximate the area under the parabolic arch.

Once the loop terminates, `S` contains the sum $(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)$. By Equation (3), we must still multiply the function by $h/3$.

```
S*H/3->S
```

Finally, display the results.

```
Disp "METHOD: SIMPSON"
Disp "SUM:"
Disp S
```

A complete listing of the program follows in **Appendix 2**.

We need to test our program. Here is some data you can use to determine whether or not your program is running properly.

No. of Subintervals	Sum
10	0.6931502307
20	0.6931473747
30	0.6931472190
40	0.6931471927

Table 1 Integrating $1/x$ on the interval $[1, 2]$.

Enter the integrand of

$$\int_1^2 \frac{1}{x} dx$$

in the `Y=` Menu, as shown in **Figure 4**. Execute the program. Enter the upper and lower bounds of the integral when prompted. Enter $n = 5$ for the number of trapezoids or subintervals. This is shown in Figure ??.

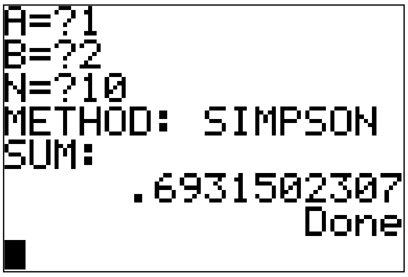

Figure 4 Entering the interval $[1, 2]$ and the number of subintervals ($n = 10$).

Note that this result agrees with the first row of **Table 1**. Use the program to verify the results in the next two rows of **Table 1**.

2 Appendix

Here is a complete program listing.

```
Clr Home
Prompt A
Prompt B
Prompt N
If int(N/2)\ne N/2

Then
Disp "N MUST BE EVEN"
Stop
End
(B-A)/N->H
A->X
0->S
For(K,1,N/2,1)
S+Y1(X)+4*Y1(X+H)+Y1(X+2*H)->S
X+2*H->X
End
S*H/3->S
Disp "METHOD: SIMPSON"
Disp "Sum:"
Disp S
```