

Pursuit Games

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Pursuit Games

An Introduction to the
Theory and Applications of
Differential Games
of Pursuit and Evasion

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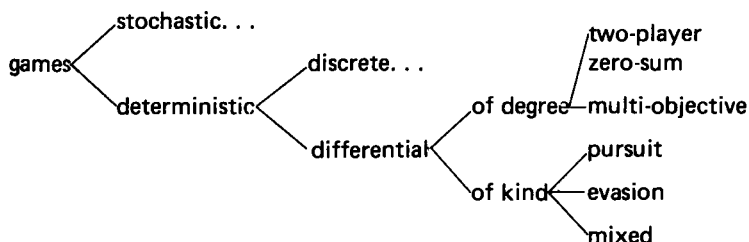
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Introduction

The purpose of this book is to present systematic methods for winning in differential games of pursuit and evasion, and to illustrate the scope and application of the developed procedures. These games appear in the abbreviated schema



Many interesting games involve the attainment of some primary objective (capture), followed by a minimization of cost (zero-sum); pursuit is one stage of a structured real game.

In treating a pursuit game one may address a series of problems of increasing complexity: decide whether there are any winning positions at all (absence of complete evasion); decide whether all initial points are in position to win (complete capture); find all winning positions (capture region); for each winning position construct a winning strategy ("solution"); find all winning strategies, or proceed to an optimization problem. (The terms used here will subsequently acquire technical meanings.) Analogously for evasion games, where the last two questions are often more difficult.

Since we desire to obtain general procedures, applicable to quite extensive classes of games (rather than to devise ingenious solutions to individual games), the Law of Conservation of Triviality will exact its price: only the first question is answered to any degree of completeness; while, for linear games, some progress is made on the fourth.

A test that might be applied to any theory aspiring to respectability is whether, in addition to its professed purpose, it also furnishes unforeseen solutions to problems not obviously within its scope—the degree of unexpectedness could be the measure; or whether (like many an established bureaucracy) it is entirely self-sufficient, even solving only the problems it itself creates.

It may be too early to require this of pursuit game theory. Nonetheless, one natural application is to control systems with persistent but unpredictable disturbances (e.g., in the Navigation Problem of 1.1, the pilot steers an aircraft between two points, against the effect of a variable wind current): the disturbance, error, or perturbing term may be viewed as the control action of a second player. A less obvious application is the unorthodox linearization of 3.2; there the nonlinearity, although perfectly deterministic, is usefully interpreted as the action of a completely fictitious opponent.

For these and similar applications it is quite essential that a solution to the games be provided (whereas other interesting questions, such as whether the game has Value, or what happens if both players use inconsistent strategies, are irrelevant). To say that we wish to solve games rather than to study them would, of course, be simplistic; on the other hand, it seems both necessary and healthy to emphasize the "practical" aspect of pursuit games.

Chapter I collects some basic examples. Linear games are treated in Chapters III–VI, nonlinear ones in Chapter VII; Chapters II and VIII are common. For linear games, in Chapters III–V progressively weaker assumptions are made on the winning strategies; Chapter VI contains the elements of an algebraic structure theory. The last four sections of Chapter III are appendices, introducing some of the needed concepts that do not properly belong to game theory.

The form is that of a graduate level text for students interested in applied mathematics and systems engineering. This may explain several features of the style. There are rather extensive exercise sections; these (together with examples in the text proper) serve to concretize and verify; they also contain minor or simple extensions of the results. The references are rudimentary, and obviously not in monograph style. There is considerable overlap, in that special results are obtained and their applications illustrated first, and more general ones later. Thus, the First Reciprocity Theorem is a special case of the Second (see 3.1 and 3.3); similarly for the results of 3.4 and 4.2, 5.1 and 7.3, 4.2 and 7.4. This even applies to concepts; e.g., the notion of player's control order appears naturally before it is formally introduced. It would have been far more elegant to proceed deductively, beginning with Chapter II, followed by Chapters VIII–III in reverse order. Finally, the proofs are rather detailed, and it was attempted to keep them at one level of difficulty; similarly for the worked examples, and the quite detailed hints to the exercises. The facile "Exercises: provide proofs for Theorems 1–8" has largely been avoided.

Cognoscenti may remark on the absence, in our treatment, of Value, Hamiltonians, Isaacs' equation, etc. Some of this is present implicitly (the time derivative $\delta\varphi$ of 7.3 is a Hamiltonian, the minimal time function of 2.3 may, with some distortion, be viewed as the Value), but there is good reason for the omission.

Pursuit and evasion games do not involve, either explicitly or implicitly, any cost (penalty function, payoff): the desired outcome is, simply, capture or its avoidance. One may, of course, inquire into questions of minimal capture time; but one is not forced into this (and, indeed, much of our development concerns winning strategies that are not time optimal). Similarly, in the Examples 1.1 and

3.2 mentioned earlier, and also in collision avoidance problems, it is rather unnatural to speak of antagonistic players ("zero-sum" games).

The discomforts of a Procrustean bedding of pursuit game theory may be illustrated on the case of Issacs' equation. This should probably be confined to cost-optimal games of fixed duration. Its indiscriminate use elsewhere is open to serious objections: these are easiest stated in the simplified case of a time-optimal linear control problem in n -space,

$$\dot{x} = Ax - u; \quad u(t) \in U; \quad \text{end: } x = 0.$$

The assertion is that the minimal time function $x \mapsto T(x)$ satisfies

$$\min_{u \in U} DT(x) \cdot (Ax - u) = -1$$

at points of differentiability of T ; and, if T satisfies a local Lipschitz condition, then indeed it is differentiable almost everywhere (Rademacher's theorem). One then obtains T by solving the partial differential equation.

Unfortunately, usually this is science fiction. For proper systems and compact U containing O , T is never differentiable at the origin. T is not locally Lipschitzian unless U is a neighborhood of O (whereupon U must have dimension n , precluding most of the interesting applications). The "almost everywhere" refers to n -dimensional Lebesgue measure; if U is a parallelepiped, then the set where T is not differentiable is an $(n - 1)$ -manifold S , and, far worse, *every* optimal trajectory meets S and remains there for an entire time interval.

Are there any general results which, for pursuit games, would be as incisive as is the Principle of Maximality for control problems? The "principle" of suboptimality in 2.3 is probably too shallow: however, there is a reasonable candidate in the principle of constant bearing navigation in 7.3.

In these days of mounting demands on one's time, some reviewers find it difficult to read much further than the introductions. For their convenience, all of Chapter I and Section 3.2 are devoted to examples and applications; numerous further examples, worked in the text, and some 200 exercises (with detailed hints where appropriate) serve to illustrate the material presented in the main text.

The present text was developed from a set of notes (Lectures on Linear Pursuit Games, Case Western Reserve University, 1973) for a one-semester seminar course. I am indebted to the participants for their patience and comments.

The study of differential games is in a period of explosive growth; thus is it difficult to gauge the moment at which the development achieved merits an account in book form without being quickly outdated by new results. I am most grateful to the publishers for having decided on this venture. In addition, their professional competence and assistance must be experienced to be really appreciated.

It is a pleasure to acknowledge support from Case Western Reserve University during a sabbatical leave in which work on the manuscript was completed. The Department of Mathematics and Statistics furnished the assistance of its capable

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staff. Miss Shirline Williams prepared the preliminary and final versions of the typescript with her usual, and obvious, competence.

O. Hájek
March, 1975

CONVENTIONS

R^+ is the positive real axis $[0, +\infty)$; R is Euclidean n -space, its elements are, consistently, column vectors; occasionally, S^{n-1} denotes the $(n-1)$ -dimensional unit sphere in R^n . Matrices (and vectors, matrix-valued functions, etc.) have real entries unless otherwise noted. Primes denote transposition; e.g., $x'y$ is the standard inner product of column vectors x, y . Dots, one or two, above a letter symbol denote differentiation with respect to an implied time variable.

For unions, intersections, etc., of sets the standard symbols are used, with an inverted slash for set-theoretic difference,

$$A \setminus B = \{x: x \in A, x \notin B\}$$

If the sets are in a fixed Euclidean space, then standard 'operator' notation is used, as in

$$A + B = \{a+b: a \in A, b \in B\}; \quad a + B = \{a+b: b \in B\},$$

$$A - B = \{a-b: a \in A, b \in B\}, \quad R^1 A = \{ta: t \in R^1, a \in A\},$$

etc.

The 'little and big O' symbols occasionally used have the usual meaning: $\sigma(\varphi(x))$ denotes a function ψ such that $\psi/\varphi \rightarrow 0$, and $\mathcal{O}(\varphi(x))$ one with ψ/φ bounded. E.g., $e^x = 1 + x + \sigma(x)$ and $e^x = 1 + x + \mathcal{O}(x^2)$ are both true, the second providing more information.

Integrals of set-valued mappings are defined as follows. If $t \mapsto F_t$ is a mapping from R^1 to the collection of all subsets of R^n , then $\int_a^b F_t dt$ denotes the set of all points of the form $\int_a^b f(t)dt$, where $f: [a, b] \rightarrow R^n$ is Lebesgue integrable and $f(t) \in F_t$ for almost all $t \in [a, b]$.

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It will be necessary to distinguish between a mapping, say f , and its value $f(x)$ at a point x ; however, we shall also use harmless abbreviations, such as $t \mapsto u(t)$, for the mapping u ; or $T(\cdot, \sigma)$ for the mapping whose value at any x is $T(x, \sigma)$.

Rarely, 'if and only if' is abbreviated to 'iff'. In definitions we conform to the curious custom of using an 'if' clause when 'if and only if' is meant, as in: a matrix A is said to be symmetric if $A = A'$. The meaning of QED is extended to denote the end of a proof. In corollaries to an assertion, the assumptions of the latter are to be taken over.

Internal references have the style 'Lemma 2', referring to the current section; 'Lemma 2 in 3.4' refers to the third chapter and fourth section.

CHAPTER I

EXAMPLES

This chapter consists of several examples, worked to various degree (e.g., not at all); it contains only one mathematical result, the Lemma in 1.4, and this does not even mention games.

In spite of this, the reader is strongly urged to tackle the chapter. Game theory cannot claim to any redeeming inner elegance or beauty: the applications are its only raison d'être. They also serve to interpret the meaning, and check on the utility, of the general results to be obtained. The Navigation Problem is the prototype of games treated in Chapters III and IV; and one-dimensional Rocket Chase of those in Chapters V and VII.

Generally applicable theorems tend to make unnecessary the considerable ingenuity needed to reduce, e.g., the Homicidal Chauffeur and Two Car games, to workable low dimension, where insight and a reasonable amount of analysis make a solution possible. Nevertheless, however much success one has with general theory, the examples retain their usefulness and freshness.

1.1 Navigation Problem

In 1931 E. Zermelo suggested and solved the following problem (in loose translation):

"In an unbounded plane let the wind be given as a vector-valued function of position and time; and consider an airplane whose speed is constant relative to the moving air mass. How

should the airplane be steered so as to arrive, in shortest time, at a given target from a given starting point?"

The equation of motion is readily seen to be

$$(1) \quad \dot{z} = w + e^{i\varphi}$$

in complex notation. Here $z(t)$ is the position of the airplane at time t , $w(z,t) = u + iv$ the wind velocity vector at point $z = x + iy$ and time t , and $\varphi(t)$ the pilots' steering action, interpreted as the angle to an appropriate semi-axis in R^2 . (The aircraft speed has been normalised to 1.) The pilot is to choose $t \mapsto \varphi(t)$ so as to arrive at point $b = z(T)$ from point $a = z(0)$ while minimising $T \geq 0$.

Zermelo uses the calculus of variations to arrive at his solution: optimal (and even extremal) $\varphi(\cdot)$ satisfies

$$(2) \quad \frac{d\varphi}{dt} = - \frac{\partial u}{\partial y} \cos^2 \varphi + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \cos \varphi \sin \varphi + \frac{\partial v}{\partial x} \sin^2 \varphi ;$$

thus $\frac{d\varphi}{dt} = - \frac{\partial u}{\partial y}$ if the reference axis is so chosen that $\varphi = 0$ at time t : pilot steers to the left ($\varphi > 0$) if the wind component in the direction of motion is smaller there.

This also indicates the mild implicit assumptions, and illustrates the reasonable 'information pattern' employed: at time t and position z , the pilot bases his decision on the knowledge of the wind velocity vector at the same time t (and in positions close to z).

We will change the problem somewhat. First, the term $e^{i\varphi}$ in (1) will be replaced by p subject to $|p| \leq 1$. Since it is intuitively obvious that pilot should use maximal available control to minimise arrival time, this modification is insignificant, while the advantages of having a convex set as control constraint set (the unit disc in place of the unit circle) are immense.

A major change will be the replacement of $w(z,t)$ by $w(t)$, independent of the state variable ("gleichformigen Winde-

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field"). Condition (2) then states that optimal steering maintains a constant direction; and this is a complete mystery: a wind gust unrestricted in magnitude can, of course, blow the airplane off any straight path. For appropriately bounded wind velocities, the conclusion does make sense (Exercise 1 in 3.1).

Another paradox may not be apparent at first glance. Optimal steering $\varphi(t)$ at time t is determined on the basis of $w(z,t)$; if the pilot could predict the future wind behaviour, why can no advantage be gained from such knowledge? An answer is provided in 5.4: this is indeed the case for a somewhat special class of games, to which the navigation problem belongs.

For Zermelo's problem and solution see

E. Zermelo: "Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung, Zeitschr. f. angew. Math. u. Mech. 11 (1931) 114-124.

The case of constant wind is an illustrative example in Section 1.17 of

G. Leitmann, An Introduction to Optimal Control, McGraw-Hill, New York, etc., 1966.

1.2 Simple Pursuit in the Plane

Two players move in the Euclidean plane R^2 with simple motion: each has a bound on his speed, but there are no further restrictions (e.g., abrupt directional changes are allowed). One, the pursuer, wishes to capture the other, quarry, i.e., attain perfect coincidence of their terminal positions.

The answer is obvious: if $\alpha > \beta$ holds for the pursuer's speed bound α and the quarry's β , then termination is assured in finite time, whatever the initial positions and action of quarry; on the other hand, in the case $\alpha \leq \beta$

quarry can avoid capture forever from any positions not in contact initially.

Let us discuss briefly some aspects of simple motion, first for a single player (for the fun of it, to make possible the analysis below, and to prepare an analogy for subsequent sections). If the player's position at time $t \in \mathbb{R}^1$ is denoted by $x(t) \in \mathbb{R}^2$, then the velocity vector is $\dot{x}(t)$, and the speed $|\dot{x}(t)|$. Thus the dynamical constraint is $|\dot{x}| \leq \alpha$; actually, another formulation is preferable, viz

$$(1) \quad \dot{x} = u; \text{ measurable } u: \mathbb{R}^1 \rightarrow \mathbb{R}^2, |u(t)| \leq \alpha.$$

What is really meant here is that $x(t) = x(0) + \int_0^t u(s) ds$ for some function $u(\cdot)$ as indicated. Just to simplify matters place the origin at $x(0)$, so that $x(0) = 0$.

Where can the player get to at time t ? The constraint on $u(\cdot)$ yields

$$|x(t)| \leq \int_0^t |u(s)| ds \leq \alpha t.$$

Conversely, any point y with $|y| \leq \alpha t$ can be 'attained' by control u , e.g.,

$$u(s) = \begin{cases} \alpha \frac{y}{|y|} & \text{for } 0 \leq s \leq |y| \\ 0 & \text{for } |y| < s \leq t. \end{cases}$$

Thus the attainability set $A_0(t) = \{y: |y| \leq \alpha t\}$ (unfortunately, current terminology reserves 'reachable set' for a different use). See fig. 1.

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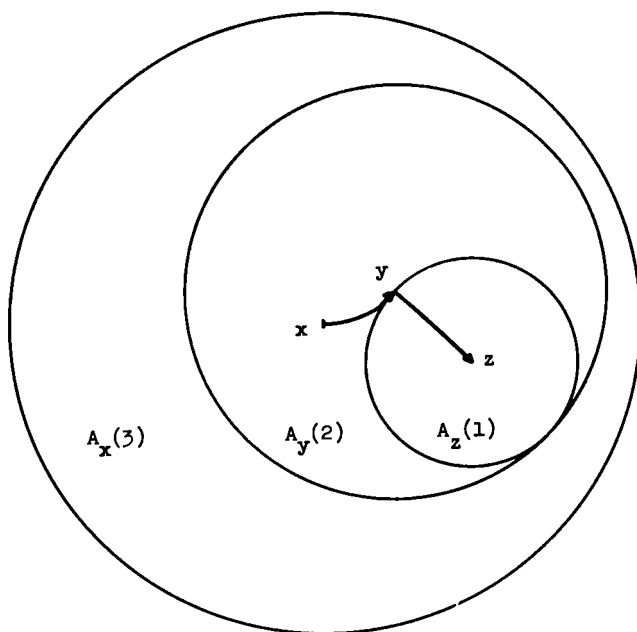


Fig. 1 Simple motion in the plane. Point x moves anywhere within $A_x(3)$ at time $t = 3$; if its position y at $t = 1$ or z at $t = 2$ is known, the possibilities reduce to $A_y(2)$ or $A_z(1)$.

Let us return to the game, in the case $\alpha > \beta$. Probably the first capture strategy that comes to mind is for the pursuer to aim, constantly, at the quarry, thereby following an appropriate curve of pursuit (Huygens' tractrix, if quarry's motion is uniform). It would be a formidable and boring task to obtain this curve explicitly; fortunately this may be avoided. If the pursuer's motion is $x: R^1 \rightarrow R^2$ and the quarry's $y: R^1 \rightarrow R^2$, the equations of motion are

$$(2) \quad \dot{x} = u, \quad \dot{y} = v$$

for suitable measurable controls $u, v: R^1 \rightarrow R^2$ with all

values $|u(t)| \leq \alpha$, $|v(t)| \leq \beta$. At any time denote the players' distance by $r = |x - y|$. Then

$$(3) \quad r\dot{r} = \frac{d}{dt} \frac{1}{2} |x - y|^2 = (x-y)'(\dot{x}-\dot{y}) = (x-y)'u - (x-y)'v.$$

The procedure suggested for the pursuer is to take

$$u = \dot{x} = \alpha \frac{y-x}{|y-x|} = \frac{\alpha}{r} (y-x).$$

Then, in (3),

$$r\dot{r} = \alpha \frac{r^2}{r} - (x-y)'v \leq -\alpha r + r\beta = -(\alpha-\beta)r$$

by Cauchy's inequality. Therefore, as long as $r > 0$, we have $\dot{r} \leq -(\alpha-\beta)$,

$$r(t) \leq r(0) - (\alpha-\beta)t = |x_0 - y_0| - (\alpha-\beta)t.$$

This shows that capture ($r = 0$) must occur, at some time θ with

$$(4) \quad \theta \leq \frac{|x_0 - y_0|}{\alpha - \beta}.$$

Exercises

1. For a mapping $x: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ the two conditions $\dot{x} = u$, $|u(t)| \leq \alpha$ almost everywhere; $x(t) = x(0) + \int_0^t u(s)ds$, $|u(t)| \leq \alpha$ a.e. are often treated as if they were synonymous. Prove carefully that this is so if 'x(.) absolutely continuous' is added to the first, but mere continuity is insufficient. (Hint: this involves some simple but rather delicate real analysis.)

Exercises 2 to 7 concern the case $\alpha > \beta$ of simple pursuit in the plane.

2. (Cutting Corners) The pursuer, starting at x_0 , first steers to quarry's initial position y_0 ; on reaching it, he (she?) repeats the process. Set up an accounting of times t_k and positions x_k, y_k , with the requirement $x_{k+1} = y_k$ determining t_{k+1} and an estimate of $|y_{k+1} - y_k|$.

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Prove that (4) holds.

3. (Follow the Leader) The pursuer first steers to quarry's initial position, and then follows his path, using excess speed to catch up. Show that, again, (4) holds.

4. (Neutralisation) Referring to (1), the pursuer first neutralises quarry's action, and then uses left-over capability to force termination: $u = v + w$ with w in the direction $y - x$ and magnitude $|\alpha - \beta|$. Prove that termination always occurs at time $|x_0 - y_0|/(\alpha - \beta)$ precisely (even if quarry cooperates).

5. (A Moveable Feast) Take a moving coordinate system, with origin at the quarry's position $y(t)$: new coordinate $z = y(t)$ satisfies $\dot{z} = u - v(t)$, with $z = 0$ as termination condition. The new control variable is $w = u - v(t)$, with constraint

$$w \in \{u - v(t) : |u| \leq \alpha\},$$

or, if constant bounds are desired, $|w| \leq \alpha - \beta$. Show that either choice is equivalent to one of the capture procedures already outlined. (Hint: in the second case let $u = v + w$ where w has magnitude $\alpha - \beta$ and direction $y_0 - x_0$.)

6. (Capture with Anticipation) The quarry announces his future policy $v: R \rightarrow R^2$ to the pursuer; based on this information, the pursuer proceeds with capture. Prove that (4) holds.

7. (Approximate Capture under Time Delays) At any time $t \geq 0$ the pursuer is allowed information on quarry's control $v(s)$ or position $y(s)$ only for $s \leq t - \delta$ ($\delta > 0$ is a given constant). Prove that approach to within a distance $\beta\delta$ can be guaranteed, and estimate the needed time.

8. Show that the case $\alpha \leq \beta$ is unfavourable to the pursuer: there is an extremely simple evasion control for the quarry.

9. Treat simple pursuit on a straight line; in particular, show that again capture always occurs if $\alpha > \beta$, and never otherwise. Prove that, by a 'neutralisation technique,' capture time can be prescribed, but not capture position.

10. Consider pursuit on a straight line, by two pursuers, of one quarry, all moving with simple motion. Prove that, from some initial positions, capture occurs even if the pursuers are both slower than the quarry, but the capture time cannot be prescribed; and sketch the corresponding set in R^3 for which 'bracketing' occurs.

Remarks

The reader is encouraged to experiment with other reasonable pursuit strategies. E.g., the method of cutting corners (Exercise 2) might be modified, with the pursuer only reaching half way at each step, bringing the method closer to the curve-of-pursuit strategy from the text (which then suggests further questions). Or, the pursuer might overshoot, e.g. by a factor 1.1, in an attempt to predict quarry's motion. Another analogue of this involves 'programmed' time instants $t_0 = 0 < t_1 < t_2 < \dots$, given in advance rather than evolved during play as in Exercise 2 (e.g., $t_k = k\delta$, or $t_k = \frac{1}{2^{k+1}} \frac{|x_0 - y_0|}{\alpha - \beta}$).

It is paradoxical that all the winning procedures proposed (text, and Exercises 2 to 6) have the same capture time bound. Of course, some of these can make use of quarry's mistakes, to a greater or lesser extent, and one is tempted to say that the curve-of-pursuit method is best in this respect, and neutralisation (Exercise 4) worst. It is highly plausible that there exist winning strategies which are incomparable, and possibly even irreducible. (The notions can

EXAMPLES

formalised in terms of the three types of termination time $T(\cdot)$ defined in 2.3.)

1.3 One-Dimensional Rocket Chase

Two players move on a straight line, the pursuer having a bound on his acceleration, the quarry a bound on his speed. The game ends when the pursuer attains a previously given distance from the quarry.

There is an obvious solution: the pursuer uses all his capabilities to move toward the quarry, who is then captured within a bounded time interval. (The precise time bound will depend on the parameters of the game, and on the initial positions.) Notwithstanding its almost rudimentary character, this game exhibits most of the phenomena present in far more general games, and will be repeatedly referred to in the subsequent text.

If $x: R^1 \rightarrow R^1$ describes the pursuer's motion, and $y: R^1 \rightarrow R^1$ that of the quarry, the equations of motion are

$$\ddot{x} = u, \dot{y} = v$$

for measurable $u, v: R^1 \rightarrow [-1, 1]$. (We are taking 1 as bound for both controls, and ϵ as the capture distance, $0 \leq \epsilon < +\infty$.) Thus the quarry moves on R^1 with simple motion, in the sense of 1.2. The pursuer's motion is described by

$$\begin{aligned} x(t) &= x(0) + \dot{x}(0)t + \int_0^t \int_0^s u(r) dr ds \\ &= x(0) + \dot{x}(0)t + \int_0^t (t-s)u(s) ds \end{aligned}$$

and suggested by some of the attainability sets in Fig. 1.

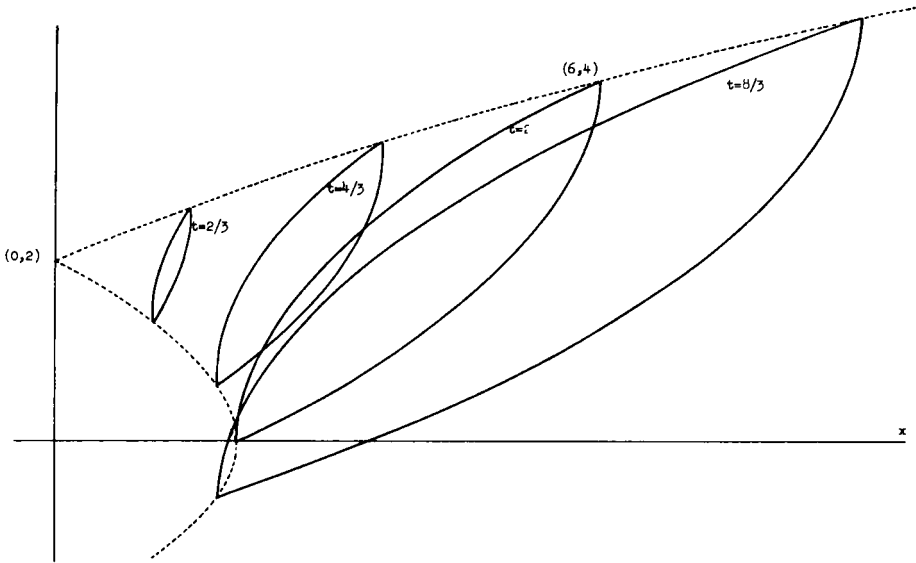


Fig. 1 Phase portrait of motion in $\ddot{x} = u$; attainability sets at $t = 2/3, 4/3, 2, 8/3$ for initial values $x = 0, \dot{x} = 2$. The vertex loci are parabolas $y^2 = 2(+x + 2)$; the lower describes motion in a doubling back manoeuvre.

The first-order version of the motion equation is the dynamical equation for the two-player system:

$$\dot{x}_1 = x_2, \dot{x}_2 = u, \dot{x}_3 = v.$$

Subsequently the matrix form of this will be treated,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

the termination condition $|x-y| \leq \epsilon$ translating to

$$(1, 0, -1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in [-\epsilon, \epsilon].$$

EXAMPLES

Thus the natural (immediate, 'realistic') phase space is R^3 . This is readily reduced to R^2 by introducing new variables $x = x_1 - x_3$, $y = x_2$: the resulting equations and termination condition are

$$(1) \quad \dot{x} = y - v, \dot{y} = u; \text{ end: } |x| \leq \epsilon.$$

For a preliminary orientation, assume that both players' controls are constant on some time interval. The differential equation for the trajectories, with $dy/dx = \dot{y}/\dot{x}$, is

$$(2) \quad \begin{aligned} (y-v) \frac{dy}{dx} &= u, \\ (y-v)^2 &= 2ux + \text{const.}; \end{aligned}$$

the point then moves along the parabola, upward for $u > 0$ and down if $u < 0$ (see fig. 2, with $u = v = 1$ in the left half plane, and $u = v = -1$ in the right).

Suppose now that, at some point to the left of the target, the quarry chooses a control other than $v = 1$, and pursuer sticks with $u = 1$. The motion then proceeds along another parabola (with axis $y = v$, see fig. 3 for $v = -1$). This suggests, quite correctly, that capture, even with $\epsilon = 0$, can be ensured from all initial positions, e.g. by taking $u = 1$ quite indifferent to quarry's action.

PURSUIT GAMES

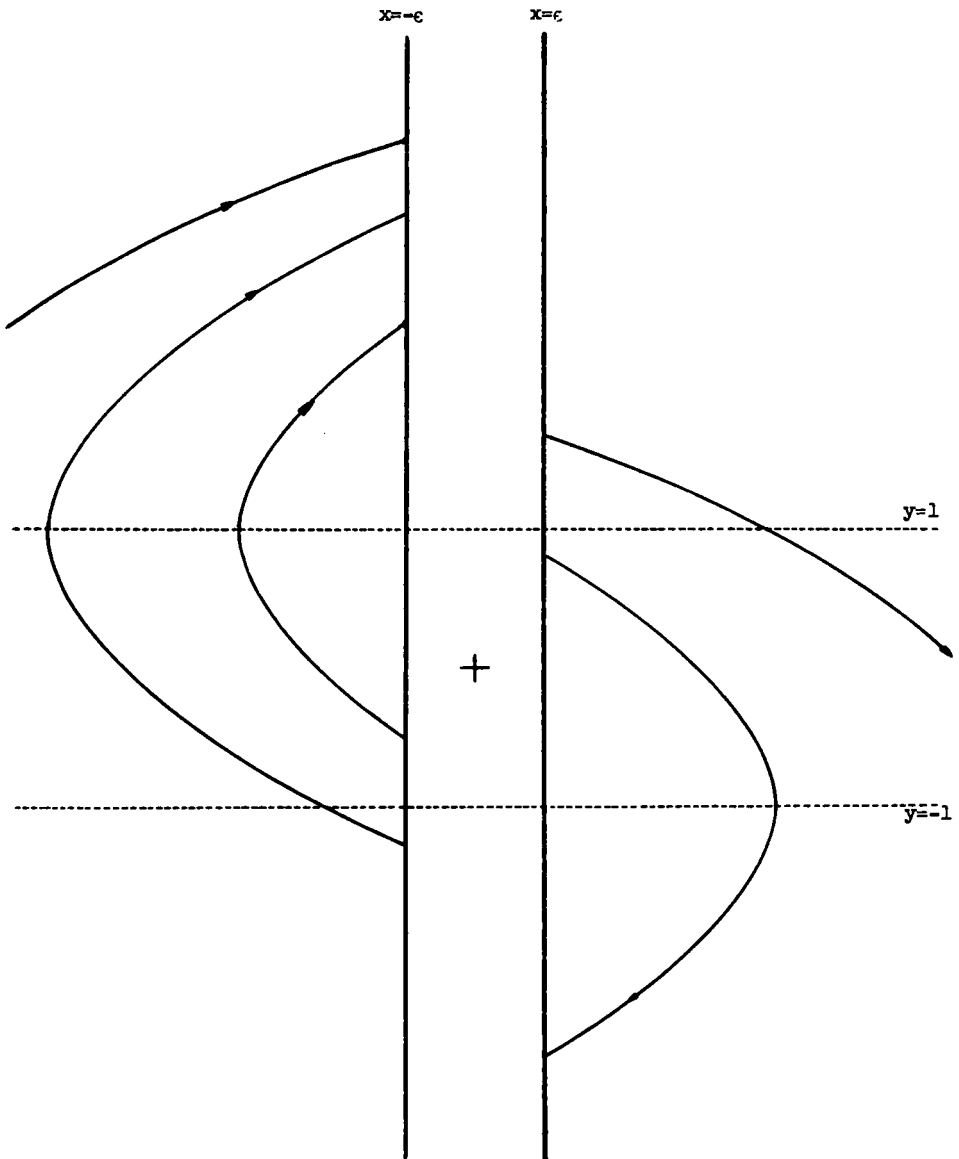


Fig. 2 Trajectories of $\dot{x} = y - v$, $\dot{y} = u$ with $u = v = \pm 1$ outside target $|x| \leq \epsilon$.

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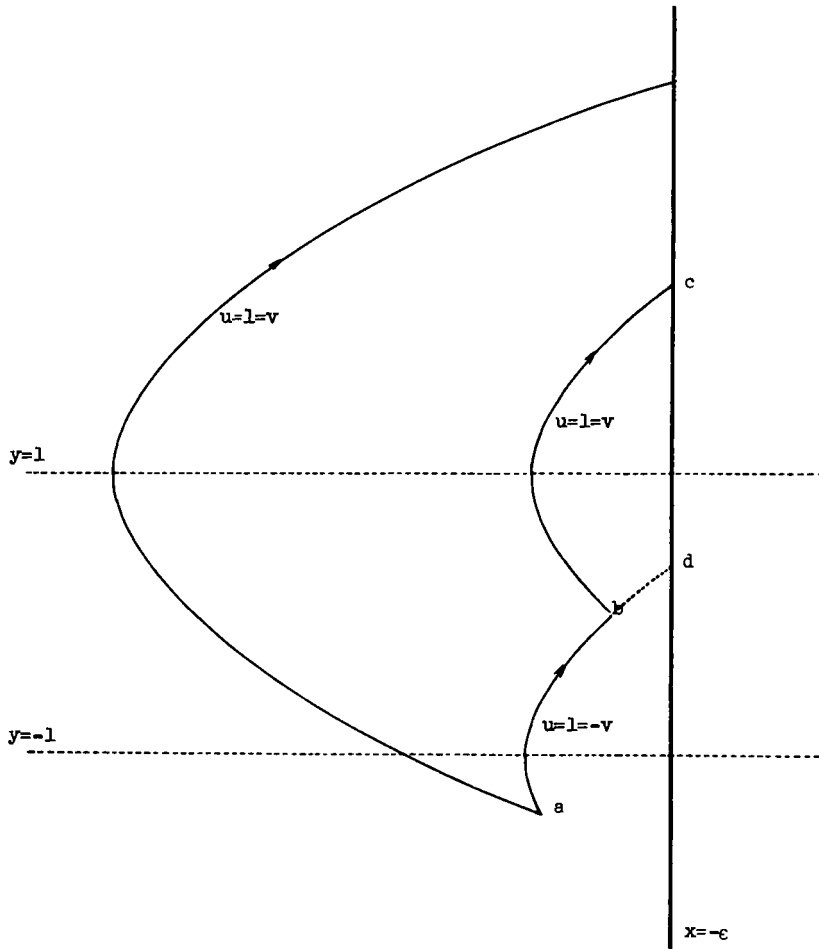


Fig. 3 Trajectories in $\dot{x} = y = v$, $y = u$. From point a , quarry mistakenly chooses $v = -1$, but reverses his choice at b ; capture occurs at c (later than it would have at d).

Exercises

1. Prove that, in (1),

$$x(t) = x_0 + y_0 t - \int_0^t v(s) ds + \int_0^t s u(t-s) ds.$$

(Hint: solve for y , then for x .)

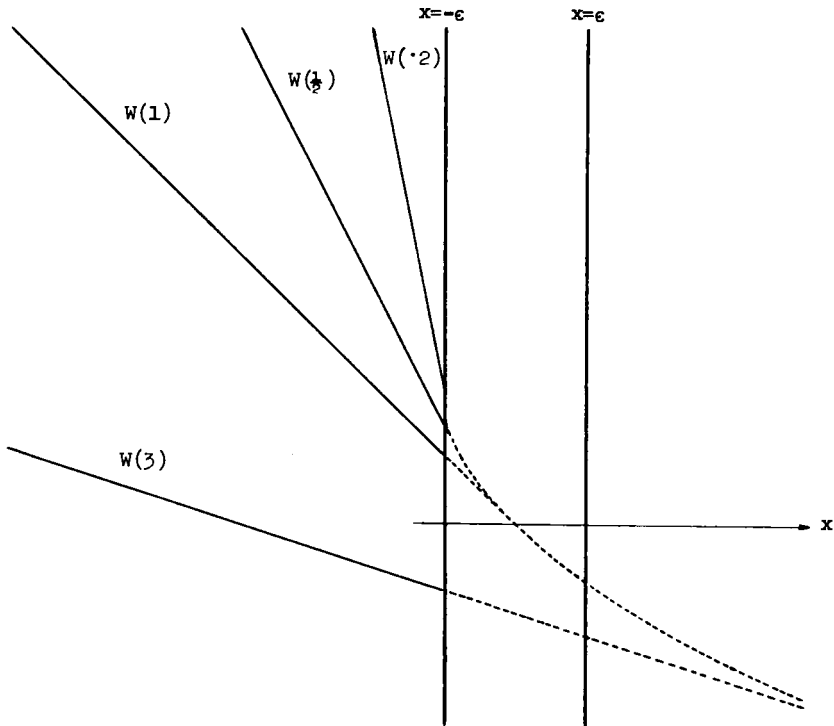


Fig. 4 Sets $W(t)$ of points in position to win at time $t > 0$, in $x \leq -\epsilon$. As $t \rightarrow 0$, the boundaries (having a parabolic envelope in $x \geq -\epsilon$) approach $x = -\epsilon$, $y \geq 1$.

2. On taking $u \equiv 1$, show that

$$x(t) \geq x_0 + (y_0 - 1)t + \frac{t^2}{2}$$

for any quarry control $v(\cdot)$. Conclude that, if $x_0 < 0$, then there exists $\theta > 0$ such that $x(t) = 0$ occurs for some $t \in [0, \theta]$; and actually obtain the least value of θ (in dependence on x_0, y_0). (Answer: $\sqrt{(y_0 - 1)^2 - 2x_0} - (y_0 - 1)$.)

3. The above exercise exhibits a capture time bound θ , obtained on taking a particular pursuer control. Show that

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it is actually independent of this: there exists a quarry control $v(\cdot)$, such that $x(t) < 0$ for $t \in [0, \theta)$ and all pursuer controls $u(\cdot)$ (here $x_0 < 0$ and y_0 are fixed).

4. Prove that, if a point in the left half plane is forced to the target, and ends up at the point $(-e, y_\theta)$, then in any case $y_\theta \geq -1$, and quarry can always attain $y_\theta \geq 1$.

5. Experiment with graphical solution for other shapes of target (e.g., a unit circle at the origin); granting that pursuer's responses are piecewise constant $u = \pm 1$, motion will be along the parabolas (2) with vertex ordinates $-1 \leq v \leq 1$, and $v = \pm 1$ as the interesting cases. Obtain, heuristically, the boundary of the set of winning positions.

6. Consider the two-dimensional version of the game treated, and, in particular, the motion $x: \mathbb{R}^1 \rightarrow \mathbb{R}^2$ of a player with $\ddot{x} = u$, $|u(t)| \leq 1$. If the origin is placed at $x(0)$, with the real axis in the direction of $\dot{x}(0)$, show that the set of possible positions at time $t \geq 0$ is a disc with radius $t^2/2$ and centre $|\dot{x}(0)|t$; that there is a doubling-back effect starting at time $\theta = |\dot{x}(0)|$, with initial position regained at doubled time 2θ . (Warning: this is not the control-theoretic concept of attainable set - in our case the latter is in \mathbb{R}^4 , not \mathbb{R}^2 - and actually the phenomenon cannot occur for these.)

7. For the game with n -dimensional equations of motion

$$\ddot{x} + A\dot{x} + Bx = u, \quad y = C\dot{y} + v$$

set up the appropriate first-order equation, in \mathbb{R}^{3n} . In the case $B = 0 = C$, with $x = y$ at termination, show that there is a reduction, with \mathbb{R}^{2n} as state space, and equations analogous to (1) (the special case $n = 3$, scalar matrix $A = \alpha I$, $B = 0$ is the so-called Isotropic Rocket Game).
(Answer:

$$\dot{x} = y - v, \quad \dot{y} = -Ay + u.)$$

on changing notation.)

Remarks

The game treated is the one-dimensional version (frictionless case) of the isotropic rocket game, and also of the homicidal chauffeur game; see

R. Isaacs: Differential Games, Wiley, New York, etc., 1967.

For general dimensions, this is called the Boy and Crocodile game in

J. F. Miščenko: Pursuit and evasion problems in differential games, Izvestia Akad. Nauk SSSR 5 (1971) 3-9.

1.4 Kelley's Game: Pursuit on a Sphere

Two players move on the two-sphere S^2 in R^3 , each with a fixed bound on his speed; the game ends at coincidence of positions.

The motivation is that "in a dogfight, the planes tend to move in a circular fashion" (in a written text one cannot adequately convey the appropriate hand motions). The simplification does away with one significant aspect of actual combat: that the roles of pursuer and quarry are not fixed, but may well switch back and forth.

The outcome is not too surprising (denote the pursuer's speed bound by α , the quarry's by β): if $\alpha > \beta$, the pursuer can force termination from any initial position, within a bounded time interval; in the case $\alpha < \beta$ the quarry can avoid capture at all times $t > 0$ (and the stand-off situation $\alpha = \beta$ is rather too sensitive to details in the specification of the players' strategies).

This is readily seen as follows. In the case $\alpha > \beta$ first assume that the players are not at diametrically

opposite points initially. Then there is a unique shorter arc γ of a great circle joining their positions. By a parallel shift along γ , move a neighbourhood of quarry's position to the pursuer's (this 'action at a distance' serves to identify the quarry control). The pursuer then uses the control $u = v + w$; the first component neutralises quarry's action, the second, w with magnitude $|w| = \alpha - \beta$ in the direction of γ , serves to decrease the players' distance (at a rate $\alpha - \beta$, see 1.3) until capture occurs. If their initial positions are opposite, then any constant control u with $|u| = \alpha > \beta$ applied over a short interval will achieve non-opposing positions. By a like reasoning, in the case $\alpha < \beta$ the quarry can maintain forever an initial distance from the pursuer.

The idea is probably clear enough, and will apply equally well to simple pursuit on an n -dimensional Riemannian manifold (thus, the 'diametrically opposite points' would be replaced by conjugate points). However, an ad hoc treatment of our special case will be useful later. Consider the motion of a single player over the unit sphere S^{n-1} of R^n . His motion is described by $x: R^1 \rightarrow R^n$; if $x(0) \in S^{n-1}$, then $x(t)$ will remain on S^{n-1} iff $|x(t)|^2$ is constant, i.e., x is perpendicular to $\dot{x} = dx/dt$. Further, motion will be 'simple' if the only further dynamical restriction is a magnitude bound on \dot{x} . We wish to express this as a relation between \dot{x} and suitable 'arbitrary' controls u .

LEMMA For any points $a \neq b$ on S^{n-1} there is a mapping $x, y \mapsto E(x, y)$, defined for x, y near a, b and with $(n, n-1)$ matrices as values, analytic in the coordinates of x, y , and such that

$$(1) \quad x'E(x, y) = 0, \quad E'(x, y) E(x, y) = I_{n-1},$$

$$(2) \quad y'E(x, y) = |\sin \varphi| (1, 0, \dots, 0),$$

where φ is the angle between x and y .

COROLLARY On the neighbourhood of any point on S^{n-1} there is an analytic mapping $x \mapsto E(x)$, whose values are $(n, n-1)$ matrices, and

$$(3) \quad x'E(x) = 0, \quad E'(x) E(x) = I_{n-1}, \quad E(tx) = E(x) \text{ for } t > 0.$$

(Proof) The corollary will follow on taking $y = b \neq \pm a$; positive homogeneity is ensured by extending $E(\cdot)$ in the obvious manner.

By assumption, the vectors a, b are independent, so that there is a basis for R^n of the form a, b, c_3, \dots, c_n . Then

$$(4) \quad x, y, c_3, \dots, c_n$$

remain independent if x, y are close enough to a, b . Apply the Gram-Schmidt orthogonalization process to the sequence (4), obtaining orthonormal vectors e_1, e_2, \dots, e_n . Since $|x| = 1$, we have $e_1 = x$, and, in the second step,

$$(5) \quad e_2 = \frac{y - (x'y)x}{|\sin \varphi|}$$

since

$$|y - (x'y)x|^2 = 1 - (x'y)^2 = 1 - \cos^2 \varphi.$$

Collect the column vectors e_2, \dots, e_n into the $(n, n-1)$ matrix $E(x, y)$; (1) holds since $(x, E(x, y))$ is orthonormal. The first coordinate of $y'E(x, y)$ is $y'e_2 = |\sin \varphi|$ from (5); the remaining coordinates are 0, since e_k is perpendicular to both $e_1 = x$ and e_2 , and hence to y also. QED

Returning to Kelley's game (actually, for dimensions $n \geq 2$), we may choose the state space description

$$\dot{x} = E(x, y)u, \quad \dot{y} = E(y, x)v.$$

The control values in R^{n-1} are constrained by $|u(t)| \leq \alpha$. $|v(t)| \leq \beta$; the initial positions are on the unit sphere. If φ is the angle between x, y , and $r = |\sin \varphi|$, then

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$$(6) \quad \begin{aligned} \dot{r}r &= \frac{1}{2} \frac{d}{dt} \sin^2 \varphi = \frac{1}{2} \frac{d}{dt} (1 - (x'y)^2) \\ &= -(x'\dot{y} + y'\dot{x}) = -x'E(y,x)v - y'E(x,y)u. \end{aligned}$$

Write $u = -v + w$ with $w \in R^{n-1}$ to be chosen subsequently, subject to $|w| \leq \alpha - \beta$. Then, using (2),

$$\begin{aligned} \dot{r}r &= (-x'E(y,x) + y'E(x,y))v - y'E(x,y)w \\ &= 0 - |\sin \varphi|(1, 0, \dots, 0)w \leq -r(\alpha - \beta) \end{aligned}$$

on taking $w' = (\alpha - \beta)(1, 0, \dots, 0)$. Thus $\dot{r} \leq -(\alpha - \beta) < 0$, and $\sin \varphi = 0$ can be attained in finite time (for obtuse angles φ an obvious modification must be employed).

Exercises

1. In the corollary show that the neighbourhood of the point $a \in S^{n-1}$ on which $E(\cdot)$ is defined may be taken as the entire hemisphere with apex at a . (Hint: choose b, c_3, \dots, c_n suitably.)

2. The analysis remains valid in the trivial case of $n = 2$ (pursuit on the circle). Note that here $E(x) = ix$ in complex notation, so that $x \mapsto E(x)$ is a well-defined mapping $S^1 \rightarrow R^2$. Furthermore, in (6),

$$\dot{r}r = -\text{Im } \overline{xy}(u - v)$$

3. Assume that a continuous $x \mapsto E(x)$ satisfies (3). Letting e_1 be the first basic unit vector in R^{n-1} , verify that $x \mapsto E(x)e_1$ defines a continuous (tangent) vector field. Conclude that $x \mapsto E(x)$ cannot be defined over all of S^2 in R^3 (you may use Poincaré's theorem on continuous vector fields, the Hedgehog Theorem.)

4. Extend the negative result of Exercise 3 to all dimensions $n \geq 3$. (Hint: intersect S^{n-1} with an appropriate linear space, thereby reducing to S^2 .)

5. For $n = 3$ use standard polar coordinates θ, φ of

a point $x \in \mathbb{R}^3$ with $|x| = 1$, and show that

$$E(x) = \begin{pmatrix} \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\cos\varphi & -\cos\theta \\ -\sin\varphi & 0 \end{pmatrix}$$

This satisfies conditions (3) for all θ, φ ; how does this match up with Exercise 3?

6. Verify that, in the preceding situation, the equation $\dot{x} = E(x)u$ reduces to

$$\dot{\varphi} = u_1, -\sin\varphi \cdot \dot{\theta} = u_2.$$

7. Carry out the preceding two exercises for $n = 4$.
(Answer: for polar coordinates $\theta_1, \theta_2, \theta_3$ on S^3 , with $c_k = \cos \theta_k$ and $s_k = \sin \theta_k$,

$$E(x) = \begin{pmatrix} c_1 s_2 c_3 & c_1 c_2 & s_1 \\ s_1 s_2 c_3 & s_1 c_2 & -c_1 \\ c_2 c_3 & -s_2 & 0 \\ -s_3 & 0 & 0 \end{pmatrix},$$

$$-s_2 s_3 \dot{\theta}_1 = u_3, s_3 \dot{\theta}_2 = u_2, \dot{\theta}_3 = u_1.)$$

8. The solution of Kelley's game proposed in the text would steer x to the point diametrically opposite y if, at the initial time, the angle φ is obtuse. Obtain a solution for this case.

1.5 Homicidal Chauffeur

A constituent part of several games is the motion $x: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ of a point in n -space subject to two dynamical constraints: the speed is constant, and the curvature is bounded a priori. Recall that the curvature κ of a parametrised curve $t \mapsto x(t)$ is the magnitude of

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$$\frac{dT}{ds} = \frac{dt}{ds} \cdot \frac{d}{dt} \frac{\dot{x}}{|\dot{x}|} = \frac{|\dot{x}|^2 \ddot{x} - (\dot{x}' \ddot{x}) \dot{x}}{|\dot{x}|^4};$$

a direct computation of κ^2 then yields

$$\kappa = \frac{|\ddot{x}|}{|\dot{x}|^2} |\sin \omega| \quad (\omega = \text{angle } \dot{x}, \ddot{x}).$$

Now, $|\dot{x}|$ is constant iff x remains perpendicular to \dot{x} , $\omega = \pm \pi/2$; then the constraint $\kappa \leq 1/\rho$ reduces to $|\ddot{x}| \leq s^2/\rho$ for $s = |\dot{x}|$. The perpendicularity requirement can be expressed in terms of the mapping $E(\cdot)$ (see Corollary in 1.4); we then obtain the control-theoretic formulation

$$(1) \quad \ddot{x} = E(\dot{x})u, \\ \text{measurable } u: R^1 \rightarrow R^{n-1}, |u(t)| \leq \frac{s^2}{\rho},$$

initial conditions including $|x(0)| = s$.

In the homicidal chauffeur game the pursuer has such a motion (parameters: s_1, ρ_1), the quarry has simple motion, with speed bound s_2 (as in 1.3); the game ends if $|x-y| \leq \epsilon$ for a given capture radius $\epsilon \geq 0$. The primary equations of motion are

$$(2) \quad \ddot{x} = E(\dot{x})u, \dot{y} = v,$$

with first-order version

$$\dot{x}_1 = x_2, \dot{x}_2 = E(x_2)u, \dot{x}_3 = v; \text{ end: } |x_1 - x_3| \leq \epsilon;$$

thus the natural phase space is $R^n \times S^{n-1} \times R^n$ with dimension $3n - 1$. An obvious reduction is obtained in terms of new coordinates $x = x_1 - x_3$, $y = x_2$ (not those of (2)):

$$(3) \quad \dot{x} = y - v, \dot{y} = E(y)u; \text{ end: } |x| \leq \epsilon,$$

control constraints

$$u: R^1 \rightarrow R^{n-1}, |u| \leq \frac{s_1^2}{\rho_1}; v: R^1 \rightarrow R^n, |v| \leq s_2,$$

and a requirement on initial values, $|y(0)| = s_1$. The reduced phase space is $R^n \times S^{n-1}$, with dimension $2n - 1$.

Exercises

1. (Two-Dimensional Homicidal Chauffeur) In 1.4 the case $n = 2$ was trivial: here it is not. Use a new variable θ to replace $y = s_1 e^{i\theta}$ in (3):

$$\dot{x}_1 = s_1 \cos \theta - v_1$$

$$\dot{x}_2 = s_1 \sin \theta - v_2$$

$$\dot{\theta} = u,$$

constraints: $|u| \leq s_1/\rho_1$, $v_1^2 + v_2^2 \leq s_2^2$, end: $x_1^2 + x_2^2 \leq \epsilon^2$.
(Hint: Exercise 2 in 1.5.)

2. The dimension $2n - 1 = 3$ above is still unnecessarily large. Use a new complex coordinate

$$(4) \quad z = e^{-i\theta}(x_1 + ix_2)$$

to obtain

$$(5) \quad \dot{z} = -izu + s_1 + v; \text{ end: } |z| \leq \epsilon$$

with real-valued u , $|u| \leq s_1/\rho_1$, and complex v , $|v| \leq s_2$.

3. Obtain the dimension-less version of (5),

$$\dot{z} = -izu + 1 + \frac{s_2}{s_1} v; \text{ end: } |z| \leq \frac{\epsilon}{\rho_1}$$

with unity control constraints.

4. (Three-dimensional Homicidal Chauffeur). In the version of (3) for $n = 3$ use polar coordinates θ, φ for y ; eliminate θ as in (4) to obtain the following dynamical equations for the state variables z, x_3, φ :

$$\dot{z} = -iz \frac{u_2}{\sin \varphi} + s_1 \sin \varphi + (v_1 + iv_2)$$

$$\dot{x}_3 = s_1 \cos \varphi - v_3$$

$$\dot{\varphi} = u_1.$$

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Here the control components satisfy

$$u_1^2 + u_2^2 \leq s_1^2/\rho_1, \quad v_1^2 + v_2^2 + v_3^2 \leq s_2^2.$$

and the termination condition is $|z|^2 + x_5^2 \leq \epsilon^2$.

Remarks

We have not even attempted a solution to the homicidal chauffeur game. For a considerable analysis (and a sanguinary interpretation) see

R. Issacs: Differential Games, Wiley, New York, etc., 1967,

in particular, Sections 2.2 (for our (5)), 9.1, 10.2, 10.3. It is shown there that, if $s_2 < s_1$ and θ is defined by $\sin\theta = s_2/s_1$, $0 < \theta < \pi/2$, then

$$1 + \frac{\epsilon}{\rho_1} \geq \theta + \cos\theta$$

is necessary (and the sharp inequality, sufficient) for capture to occur from all initial positions.

For dimension $n = 2$, the set of points $x(t)$ in the control system (1) (i.e., the projection of the 4-dimensional attainable set $A(t)$) has been studied in

E. J. Cockayne, G. W. C. Hall: Plane motion of a particle subject to curvature constraints, SIAM J. Control 13 (1975) 197-220.

1.6 Two Cars

In this game both players have the type of motion treated in (1) of 1.5: the equations are

$$\ddot{x} = E(\dot{x})u, \quad \ddot{y} = E(\dot{y})v,$$

with controls $u, v: \mathbb{R}^1 \rightarrow \mathbb{R}^{n-1}$ constrained by $|u| \leq s_1^2/\rho_1$, $|v| \leq s_2^2/\rho_2$, and with termination condition involving only x, y .

The corresponding first-order dynamical equation is

easily obtained, using $y_2 = \dot{y}_1 = \dot{x}$, $y_4 = \dot{y}_3 = \dot{y}$; the state space has dimension $4n - 2$, with the dynamical equation of formal dimension $4n$. In terms of new state variables

$$x_1 = y_1 - y_3, \quad x_2 = y_2, \quad x_3 = y_4,$$

this reduces to

$$(1) \quad \dot{x}_1 = x_2 - x_3, \quad \dot{x}_2 = E(x_2)u, \quad \dot{x}_3 = E(x_3)v,$$

with state space dimension $3n-2$. To apply this we must assume that the original termination condition depends on $x - y$ only (rather than on x, y separately).

Let us consider, specifically, the case of dimension $n = 2$, and proceed to lower the state space dimension (from 4 to 3) in the manner suggested by Exercises 1 and 2 in 1.5; for the moment this formal reason shall be our sole justification. The notation is simplified by using new (real valued) control variables p, q : $u = s_1 p$, $v = s_2 q$, with constraints

$$|p(t)| \leq \frac{s_1}{\rho_1}, \quad |q(t)| \leq \frac{s_2}{\rho_2}.$$

The one-dimensional 'circular' variables will be written as

$$(2) \quad x_2 = s_1 e^{i\varphi}, \quad x_3 = s_2 e^{i\psi};$$

then, from the latter two relations in (1),

$$(3) \quad \dot{\varphi} = p, \quad \dot{\psi} = q.$$

Finally, new state variables are introduced: complex $z = e^{-i\varphi} x_1$ and real $\theta = \varphi - \psi$. Then, from (1) to (3),

$$(4) \quad \dot{z} = -izp - s_1 + s_2 e^{-i\theta}, \quad \dot{\theta} = p - q.$$

Alternately, $w = e^{-i\psi} x_1$ and θ satisfy

$$(5) \quad \dot{w} = -iwq + s_1 e^{i\theta} - s_2, \quad \dot{\theta} = p - q.$$

In either set, the three real equations are easily obtained; (4) corresponds, modulo notational changes, to the 'kinematic equation', p. 238 of

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R. Issacs: Differential Games, Wiley, New York, etc., 1967.

This reveals another interpretation (readily verified by re-tracing the successive substitutions): (4) depicts the game on the pursuer's radar screen, i.e., in a movable coordinate system which has its origin at the pursuer's current position, and with real axis in the direction of his motion; similarly, (5) shows the game on the quarry's radar screen.

There is a further use for (4), (5). Suppose the target is not circular, but rather ship-shaped, with a chosen point as the quarry position. The bow will then constantly point in the quarry's direction of motion (say, side thrusters are not allowed for); the termination condition will then involve not only x, y but also \dot{y} . Analogously, if the pursuer is an airplane with fixed guns, the target set is a fan-shaped region with axis in the pursuer's direction of motion. In these cases use of (5) or (4) will profoundly simplify the termination condition.

Remarks

The game is treated in Section 9.2 of Issacs' book, for circular targets. More detail, together with extensive (and intriguing) plots of the results, appear in

T. L. Vincent, E. M. Cliff, W. J. Grantham, W. Y. Peng:
A problem of collision avoidance, EES Series
Report 39, Engineering Experiment Station,
The University of Arizona, 1972.

The results are summarised in

T. L. Vincent, E. M. Cliff, W. J. Grantham, W. Y. Peng:
Some aspects of collision avoidance, AIAA
Journal 12 (1974) 3-4,

T. L. Vincent: Avoidance of guided projectiles,
p. 267-279 in The Theory and Application of
Differential Games (ed. J. D. Grote), Reidel,
Dordrecht and Boston, 1975.

The case of a ship-shaped (or rather, lens shaped) target is treated in the last reference. A preliminary treatment, with the target replaced by a disc displaced in the direction opposite to the velocity vector, appears in

T. L. Vincent, W. Y. Peng: Ship collision avoidance,
Navy Workshop in Differential Games, Annapolis,
1973.

The case of perfect capture is treated in

E. Cockayne: Plane [sic] pursuit with curvature constraints, SIAM J. Appl. Math. 15 (1967)
1511-1516.

The assertion there is that pursuer can force capture from any initial position if, and only if,

$$(6) \quad s^2/\rho_1 \geq s_2^2/\rho_2.$$

In

G. T. Rublein: On pursuit with curvature constraints,
SIAM J. Control 10 (1972) 37-39

the method is extended to the case of dimension $n = 3$; sharp inequalities in (6) ensure perfect capture from all initial positions. See also the remarks following 7.3.

1.7 Unpleasant Examples

The reason for the section heading is that the examples described here, simple instances of far more general situations, cannot be treated by the methods to be developed later in the book (surely there is an obligation that the reader be made aware of this circumstance).

The first is the celebrated Lion and Man game; it is probably best to reproduce, with the kind permission of the publisher, two pages from

J. E. Littlewood: A Mathematician's Miscellany,
Methuen, London, 19

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'Lion and Man'

'A lion and a man in a closed arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?',¹

It was said that the 'weighing-pennies' problem wasted 10,000 scientist-hours of war-work, and that there was a proposal to drop it over Germany. This one,² though 25 years old, has recently swept the country; but most of us were teased no more than enough to appreciate a happy idea before arriving at the answer, 'L keeps on the radius OM'.

If L is off OM the asymmetry helps M. So L keeps on OM, M acts to conform, and irregularity on his part helps L. Let us then simplify to make M run in a circle C of radius a with angular velocity ω . Then L (keeping to the radius) runs in a circle touching C,³ at P, say, and M is caught in time less than π/ω . This follows easily from the equations of motion of L, namely $\dot{\theta} = \omega$, $\dot{r}^2 + r^2\omega^2 = a^2\omega^2$. It is, however, instructive to analyse the motion near P. For this

$$\dot{r} \geq (a-r)^{\frac{1}{2}}/K, \text{ and } t < \text{const.} + K \int (a-r)^{-\frac{1}{2}} dr.$$

The integral converges (as $r \rightarrow a$) with plenty to spare — plenty, one would guess, to cover the use of the simplifying hypothesis. ★ The professional will easily verify that when

M spirals outwards to a circle, and, with obvious notation (ω varying), we write $x = r_M - r_L$, we have

$-\dot{x} > (\omega^2 r_L)(x/\dot{r}_L) = X$, where $X/x \rightarrow \infty$. Then $t < \text{const.} + \int X^{-1} dx$, and in this the integral increases more slowly than $\int_x^x x^{-1} dx$: it is a generally safe guess in such a case that the integral converges. ★

1 'The curve of pursuit' (L running always straight at M) takes an infinite time, so the wording has its point.

2 Invented by R. Rado (unpublished).

3 The case when L starts at O is particularly obvious, on geometrical grounds.

All this notwithstanding, the 'answer' is wrong, and M can escape capture, (no matter what L does).⁴ This has just been discovered by Professor A. S. Besicovitch; here is the first (and only) version in print.

I begin with the case in which L does keep on OM; very easy to follow, this has all the essentials in it (and anyhow shows that the 'answer' is wrong). Starting from M's position M_0 at $t = 0$ there is a polygonal path $M_0M_1M_2\dots$ with the properties: (i) M_nM_{n+1} is perpendicular to OM_n , (ii) the total length is infinite, (iii) the path stays inside a circle round O inside the arena. In fact, if $\ell_n = M_{n-1}M_n$ we have $OM_n^2 = OM_0^2 + \sum \ell_m^2$, and all is secured if we take $\ell_n = cn^{-3/4}$, with a suitable c. Let M run along this path (L keeping, as agreed, on OM). Since M_0M_1 is perpendicular to L_0M_0 , L does not catch M while M is on M_0M_1 . Since L_1 is on OM_1 , M_1M_2 is perpendicular to L_1M_1 , and L does not catch M while M is on M_1M_2 . This continues for each successive M_nM_{n+1} , and for an infinite time since the total length is infinite.

★ I add a sketch, which the professional can easily complete, of the astonishingly concise proof for the quite general case. Given M_0 and L_0 , M selects a suitable O (to secure that the boundary does not interfere with what follows), and 'constructs' the polygon $M_0M_1M_2\dots$ described above, but runs along another one, $M_0M'_1M'_2\dots$, associated with it, but depending on what L does. $M_0M'_1$ is drawn perpendicular to L_0M_0 , N_0 is the foot of the perpendicular from O on $M_0M'_1$, and M'_1 is taken beyond N_0 from M_0 so that $N_0M'_1 = \ell_1 (= M_0M_1)$. If L is at L_1 when M is at M'_1 , $M'_1M'_2$ is drawn perpendicular to $L_1M'_1$, and M'_2 taken on it so that $N_1M'_2 = \ell_2$; and so on. Clearly $OM_n'^2 - OM_{n-1}'^2 \leq \ell_n^2$, $OM_n'^2 \leq OM_n^2$, and the new polygon is inside the same circle

⁴ I used the comma in 1. 9, [previous page] to mislead; it does not actually cheat.

as the old one. Since $M'_{n-1}M'_n \geq \ell_n$ the new polygon has again infinite length. And as before L fails to catch M .★

Little can be added - the exposition is eminently clear and concise (even though the discussion of convergence of $\int (a-r)^{\frac{1}{2}} dr = -2(a-r)^{\frac{1}{2}}$, and the last footnote, may leave the reader slightly uncomfortable).

For an account of subsequent developments, and further references, see

J. Flynn: Pursuit in the circle: lion versus man, pp. 99-124 in Differential Games and Control Theory (ed. E. O. Roxin, P. T. Liu, R. L. Sternberg), Dekker, New York, 1974.

Games similar to this will not be treated in this book. It is singular in that the players' capabilities are perfectly balanced; the confinement of play to the disc induces constraints on the state variable; and it is precisely the interplay of these two effects that makes the game interesting.

The next game, Obstacle Tag, is due to Isaacs. The two players move, with simple motion, in Euclidean n -space, neither being allowed to enter the 'obstacle', an open subset G of R^n ; equivalently, the state variables of both players are constrained to lie in the closed set $R^n \setminus G$.

This generalises several of the games already mentioned: simple pursuit, $G = \emptyset$; Kelley's game, $G = R^3 \setminus S^2$; and Lion and Man, $G = \{x \in R^2: |x| > 1\}$.

We shall only treat a very special case: $n = 2$, G is the interior of the unit disc in R^2 , and $\alpha > \beta$ holds for the pursuer's and quarry's speed bounds. The state constraint for, e.g., the pursuer can be treated by a somewhat clumsy equation of motion:

$$\dot{x} = u, \begin{cases} |u| \leq \alpha & \text{if } |x| > 1 \\ |u| \leq \alpha \text{ and } x'u \geq 0 & \text{if } |x| = 1. \end{cases}$$

A three stage procedure for capture (in our case $\alpha > \beta$) is easily described. The pursuer first carries out simple pursuit, until the first time that he meets the obstacle; then he moves as in simple pursuit on the circle (see Exercise 2 in 1.4) with $y/|y|$ as notional opponent; finally, he proceeds as in simple pursuit again.

Any one of these stages may be trivial, of time duration 0. It is readily verified that, from any initial positions, each pursuit stage terminates at a finite time, whatever the action of the quarry. Rather obviously the strategy is not time optimal, the first two stages being maintained far too long.

A paradoxial game, in R^1 , has dynamical equation

$$\dot{x} = (p-q)^2 - \frac{1}{8},$$

with players' controls constrained by $0 \leq p, q \leq 1$; the pursuer wishes to attain (and the quarry, to avoid) $x \leq 0$ from an initial position $x_0 > 0$, e.g., $x_0 = 1$. If pursuer chooses $p = q$, then

$$\dot{x} = -\frac{1}{8}, \quad x(t) = 1 - \frac{t}{8},$$

the state variable moves towards the origin, with the game ending at time $t = 8$. Apparently this is the best choice for the pursuer, in terms of minimising the termination time.

Second, suppose that the quarry chooses his control q thus:

$$q = 1 \text{ whenever } p \leq \frac{1}{2}, q = 0 \text{ if } p > \frac{1}{2}$$

Then $(p-q)^2 \geq 1/4$, best quarry's estimate, so that $\dot{x} \geq \frac{1}{8}$,

EXAMPLES

$x(t) \geq 1 + \frac{t}{8}$: the state variable constantly moves away from the origin.

What happens if both players simultaneously make the indicated choices? The paradox is not in this, since such incompatible strategies can be set up in almost all games (quarry counters pursuer's choice $p = q$ by $q = -p$), but in that the strategies are optimal.

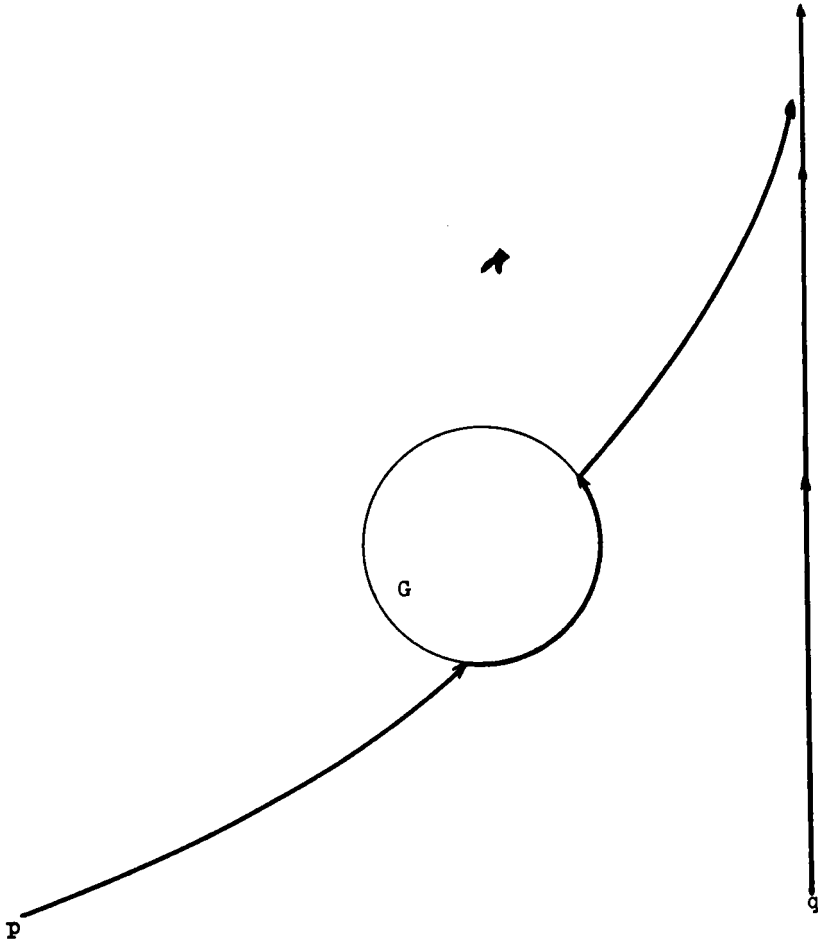


Fig. 1 Obstacle Tag. Three stages in the capture of quarry by pursuer avoiding a circular obstacle.

Exercise

Prove that the obvious analogue of Exercise 3 in 1.2 provides a solution to obstacle tag with minimal time to capture.

Remarks

The present author has the impression that the paradoxical game is due to L. D. Berkovitz, but has been unable to trace the reference. The obstacle tag game appears on p. 152 of

R. Issacs: Differential Games, Wiley, New York, etc., 1967.

PROBLEM Generalise 1.2, 1.4 and the Exercise above, and solve the simple pursuit problem on n -dimensional Riemannian manifolds with boundary. (The appropriate form of the equation of motion is $\dot{x} = u$, $u'G(x)u \leq \alpha^2$, where $G(\cdot)$ is the matrix defining the distance.)

CHAPTER II

BASIC CONCEPTS

This is a very dry chapter: it introduces, and comments upon, the concepts subsequently used: player controls, strategies, and winning positions. Other than this, there is some minor pedagogical content to a number of the exercises; and mathematical, in the two propositions from 2.3 -- these are needed to start off Chapters IV, V, and VII. The standard assumptions introduced in 2.1 are not really used until non-linear games are treated in Chapter VII. Thus a cursory first reading might be confined to the main text of 2.2, returning when and as need is felt.

2.1 Differential Equations: Miscellany

With minor exceptions, we shall be treating a game whose state space is Euclidean, say R^n for some n ; with dynamical equation of the form

$$(1) \quad \dot{x} = f(x, p, q);$$

players' control constraints

$$p(t) \in P, q(t) \in Q \quad (\text{for all } t),$$

the data being pursuer's constraint set P , and quarry's constraint set Q ; and a target set Ω (or termination condition $x \in \Omega$).

Usually the above will be abbreviated, as in

$$(2) \quad \begin{aligned} \dot{x} &= f(x, p, q); \quad p(t) \in P, q(t) \in Q; \\ \text{end: } x &\in \Omega; \text{ state space: } R^n. \end{aligned}$$

In (2) the data are f, P, Q, Ω , and n ; the symbols x, p, t, q are dummy variables (i.e., bound variables, just as the s in $\int_0^1 e^s ds$); and \dot{x} is a convention. The data constitute a partial description of the game; its specification will be completed implicitly, by studying only winning strategies, etc. The initial point is not part of the data: the same game is played from various initial positions.

Until the interplay between the two players' controls p, q becomes essential, one could well treat the pair as a single entity $u = (p, q)$, a control u with values in a given subset of some Euclidean space.

The following will be referred to as standard assumptions:

ASSUMPTIONS The players' constraint sets, P and Q , are nonvoid and compact subsets of R^l ; the target set Ω is a closed subset of R^n . The mapping $f: R^{n+2l} \rightarrow R^n$ is continuous, and satisfies two further requirements. First, a local Lipschitz condition, uniformly in the control variables: every point $x_0 \in R^n$ has a neighbourhood $U = \{x: |x - x_0| \leq \epsilon\}$, $\epsilon > 0$, such that

$$|f(x, p, q) - f(y, p, q)| \leq \lambda |x - y|$$

for some $\lambda \in R^1$ and all $x \in U, y \in U, p \in P$, and $q \in Q$. Second, the growth condition

$$\limsup_{|x| \rightarrow \infty} \sup_{p, q} \frac{x' f(x, p, q)}{|x|^2} < +\infty,$$

i.e.,

$$(3) \quad x' f(x, p, q) \leq \mu (1 + |x|^2)$$

for some $\mu \in R^1$ and all $x \in R^n, p \in P$, and $q \in Q$.

The reason for these requirements on f is the customary one: they ensure, in turn, local existence of solutions,

uniqueness (for the initial value problem), and global existence of solutions into positive time. Explicitly, for any initial position $x_0 \in R^n$, and any measurable $p: R^1 \rightarrow P$ and $q: R^1 \rightarrow Q$, there is a (generalised) solution $x(\cdot)$ of (1), defined on the entire half-axis $R^+ = [0, +\infty)$, and completely determined by the initial condition $x(0) = x_0$:

$$x(t) = x_0 + \int_0^t f(x(s), p(s), q(s)) ds \quad \text{for all } t \geq 0.$$

Equivalently, $x: R^+ \rightarrow R^n$ is absolutely continuous on each compact subinterval $[0, \theta]$, satisfies the differential equation almost everywhere,

$$\frac{dx(t)}{dt} = f(x(t), p(t), q(t)) \quad \text{for almost all } t \geq 0,$$

and has $x(0) = x_0$. This is admittedly lengthy, and (1) will be used as a convenient abbreviation; in the rare case that both $p(\cdot)$, $q(\cdot)$ are continuous, equation (1) actually holds, and one may substitute into it.

One important consequence of compactness of the constraint sets and of the growth condition is that solutions are acervate (grow in compact clusters):

LEMMA For any compact set $S \subset R^n$ and compact interval $[0, \theta] \subset R^1$, the set of all values $x(t)$ for $0 \leq t \leq \theta$ of solutions x of (1) with initial data in S and measurable $p: R^1 \rightarrow P$, $q: R^1 \rightarrow Q$, is bounded.

Occasionally it will be useful to treat other versions of (1). In

$$\dot{x} = f(x, p) - g(x, q)$$

the players' dynamical effects separate. Their controls appear linearly in

$$\dot{x} = a(x) + B(x)p + C(x)q$$

(where, e.g., each $B(x)$ is an (n, l) matrix). The allono-

mous version of (1) is

$$(4) \quad \dot{x} = f(x, t, p, q);$$

it is customary to interpret this via

$$\dot{x} = f(x, \xi, p, q), \quad \dot{\xi} = 1$$

i.e., an $(n+1)$ -dimensional instance of (1) (thus, once linearity is abandoned, there is little formal difference between autonomous and allonomous systems). Second-order games have

$$(5) \quad \ddot{x} = f(x, \dot{x}, p, q)$$

as n -dimensional equation of motion; as usual, this is interpreted as the $2n$ -dimensional case of (1),

$$\dot{x} = y, \quad \dot{y} = f(x, y, p, q).$$

Naturally, mixed types are easily concocted.

Sometimes the standard assumptions are unnecessarily confining. Thus, for autonomous linear equations

$$\dot{x} = Ax - p + q,$$

the solution is provided by the variation-of constants formula

$$(6) \quad x(t) = e^{At} \left(x - \int_0^t e^{-As} (p(s) - q(s)) ds \right)$$

whenever $p(\cdot)$ and $q(\cdot)$ are locally integrable, and there is no need to assume that their values are constrained to compact sets.

For an ordinary differential equation $\dot{x} = f(x)$ in R^n , the concept of associated vector field is quite familiar (formally, this is just the mapping $x \mapsto f(x)$). Corresponding to (2), the pursuer's vectogram at a point (x, q) is the set

$$x + f(x, P, q) = \{x + f(x, p, q) : p \in P\};$$

analogously, quarry's vectogram at (x, p) is $x + f(x, p, Q)$,

and the full vectogram at x is

$$x + f(x, P, Q) = \{x + f(x, p, q) : p \in P, q \in Q\}.$$

Exercises

1. Show that the growth condition appropriate to (5) is

$$\limsup_{|x|+|y| \rightarrow \infty} \sup_{p, q} \frac{y'(x+f(x, y, p, q))}{|x|^2 + |y|^2} < +\infty$$

2. Find, and then simplify, the version of the growth condition for (4).

3. Prove that, if f admits a linear estimate in x ,

$$(7) \quad |f(x, p, q)| \leq \mu_1 + \mu_2 |x|,$$

then (3) holds.

4. Show that (7), and hence also (3), holds if f satisfies a global Lipschitz condition,

$$|f(x, p, q) - f(y, p, q)| \leq \lambda |x - y|$$

for some $\lambda \in \mathbb{R}^1$ and all $x, y \in \mathbb{R}^n$, $p \in P$, $q \in Q$.

5. There is a good reason to prefer (3) over the simpler (7). Verify that van der Pol's equation,

$$\ddot{x} + 2\epsilon(x^2 - 1)x + x = 0 \quad (\text{const. } \epsilon > 0),$$

or rather its first-order equivalent in \mathbb{R}^2 , satisfies (3) but not (7). Moreover, as concerns development into negative time, obtained formally on replacing ϵ by $-\epsilon$, neither of the conditions is satisfied (in point of fact, points outside the unique limit cycle do have finite negative escape times).

Remarks

For the material on differential equations see, e.g.,

G. Sansone, R. Conti: *Equazioni differenziali non lineari*, Edizioni Cremonese, Roma 1956;
and revised translation, *Non-Linear Differential Equations*, Macmillan, New York, 1964.

According to this source, the growth condition (3) for global existence is due to L. Amerio.

Vectograms, in the sense of $f(x, P, q)$ instead of our
 $x + f(x, P, q)$

were introduced in Chapter 2 of

R. Issacs: Differential Games, Wiley, New York, etc.,
 1967.

A glance at the diagrams there shows that it is the present definition that was intended.

2.2 Controls, strategies, winning positions

The game will be indicated in the abbreviated form introduced in 2.1:

$$(1) \quad \begin{aligned} \dot{x} &= f(x, p, q): p(t) \in P, q(t) \in Q; \\ \text{end: } x &\in \Omega, \text{ state space: } R^n. \end{aligned}$$

This given, a pursuer control is any measurable mapping $p: R^1 \rightarrow P$, a quarry control is any measurable mapping $q: R^1 \rightarrow Q$; if P, Q are not presupposed compact, we also require that p, q be locally integrable (i.e., Lebesgue integrable on every bounded interval in R^1).

From the standard assumptions in 2.1 it follows that a solution $x: R^+ \rightarrow R^n$ is well and uniquely defined by specifying an initial value $x_0 \in R^n$ and player controls $p(\cdot)$, $q(\cdot)$: $x(\cdot)$ satisfies

$$(2) \quad x(t) = x_0 + \int_0^t f(x(s), p(s), q(s)) ds \text{ for } t \geq 0.$$

We will usually call $t \mapsto x(t)$ the state response, with initial position x_0 . From (2) and uniqueness it follows that $x(\cdot)$ is unchanged if the values of the player controls are altered on a set of measure zero.

Next, consider a mapping σ , from the collection of all quarry controls $R^1 \rightarrow Q$ to that of the pursuer controls. Its value at any $q: R^1 \rightarrow Q$ in its domain will be denoted by $\sigma[q]$; this is then a function $R^1 \rightarrow P$, and thus has values, denoted by $\sigma[q](t)$ in the usual manner. For example,

$$(3) \quad \sigma[q](t) = \int_0^t (t-s)q(s)ds.$$

The square-bracket notation is only a mnemotechnical device; it is, of course, inconsistent, and one ought to write $\sigma(q)(s)$; but in this book it is quite harmless.

The mapping σ will be called a pursuer strategy if it is non-anticipatory in the following sense: whenever two quarry controls q_k have $q_1 = q_2$ a.e. on some $(-\infty, \theta)$, then also $\sigma[q_1] = \sigma[q_2]$ a.e. on $(-\infty, \theta)$. (Other terms used synonymously: causal, hereditary, deterministic, satisfying the information constraint.) Thus, the mapping σ defined by (3) is non-anticipatory, while that determined by

$$\sigma[q](t) = \int_0^1 (t-s)q(s)ds$$

is not: its value at time t depends on the behaviour of q over $[0,1]$ entire, not only over $[0,t]$. The definition implicitly requires that σ preserve measurability: if q is measurable, then so is $\sigma[q]$. Sometimes $\sigma[q]$ will be called the pursuer response, to quarry control q .

A special type of strategy will be studied in Chapter III. Consider a mapping $\tau: Q \times R^1 \rightarrow P$ (thus, $\tau(q,t)$ is a point in P whenever q is a point in Q and $t \in R^1$). Now define an induced mapping σ , on the collection of quarry controls $q(\cdot)$, by setting

$$(4) \quad \sigma[q](t) = \tau(q(t), t).$$

If this σ preserves measurability, it is automatically non-anticipatory, and will then be called a stroboscopic strategy

(snap-decision, or memory-less); the term may then also be applied to τ . For example, if $t \mapsto u(t)$ is a mapping given in advance, then σ ,

$$\sigma[q](t) = q(t) + u(t),$$

is a stroboscopic strategy (if u is measurable and $Q + u(t) \subset P$ for all t); however, the strategy of (3) is not stroboscopic, since its value at time t strongly depends on the behaviour of $q(\cdot)$ at times s previous to t .

One further class of strategies may be mentioned here (for others, see the exercises): σ is indifferent if $\sigma[q]$ is independent of q .

We will always have to distinguish carefully between, say quarry controls q and quarry control values $q(t)$. Thus, in (1), $p(t) \in P$ is an abbreviation for $p(t) \in P$ for all $t \in \mathbb{R}_+^1$, and cannot be replaced by the careless $p(\cdot) \in P$ nor (horrible to even mention) $p(t) \subset P$. The reader not used to this will, obviously, have quite unnecessary difficulties with, e.g., (4).

Analogous definitions apply to quarry strategies, taking pursuer controls to quarry controls (always so named; but the qualifier in 'pursuer strategy' will often be dropped). On occasion we will speak of capture with (allowed) anticipation: this refers to mappings between the player controls which is not necessarily non-anticipatory.

Referring to the game (1), a point $x_0 \in \mathbb{R}^n$ is in position to win within time $\theta \in \mathbb{R}^+$ if there is a pursuer strategy σ which forces x_0 to the target within the time interval $[0, \theta]$ against all action of quarry. At length, and less picturesquely put, the condition is as follows. For any quarry control $q(\cdot)$, the state response $x(\cdot)$, from initial position x_0 , to the controls $p = \sigma[q]$ and q , meets the target, i.e.,

$$(5) \quad x(t) \in \Omega$$

for some $t \in [0, \theta]$; the termination times t may well vary with $q(\cdot)$ (but the definition requires that they all belong to the same time-interval $[0, \theta]$). It may be noticed that this excludes, purposely, the case that for each quarry control $q(\cdot)$ there is a finite time t at which (5) holds, but that $\sup t = +\infty$. It is useful not to be pinned down to one turn of phrase; we may speak of winning positions (within some unspecified time interval), of winning strategies forcing termination or capture within $[0, \theta]$, etc.

The set of all such winning positions x_0 is the winning set $W(0, \theta)$. We have

$$(6) \quad \Omega = W(0, 0) \subset W(0, s) \subset W(0, t) \subset \bigcup_{\theta \geq 0} W(0, \theta)$$

for $0 \leq s \leq t$; the largest term is the set of all winning positions.

If the termination times t in (5) can be taken equal, we speak of isochronous capture and strategy. Explicitly, the winning set $W(\theta)$ consists of all points x_0 which can be forced to the target at (rather than within) time $\theta \geq 0$: there is a strategy σ such that the corresponding state response endpoint $x(\theta) \in \Omega$. (Other terms used: games of fixed duration or horizon.) Analogously, if the strategy can be taken stroboscopic, we speak of stroboscopic capture. Obviously $W(0) = \Omega$, and

$$(7) \quad W(0, \theta) \supset \bigcup \{W(t) : 0 < t \leq \theta\}.$$

However, inequality can occur in (7) even in the extreme sense that one set is void while the other is not: i.e., nontrivial capture is possible, but not isochronous.

Exercises

These serve to introduce several further classes of

strategies. The condition that a mapping σ , taking quarry controls to pursuer controls, be non-anticipatory is that

$$(8) \quad q_1 = q_2 \text{ a.e. } S \text{ implies } \sigma[q_1] = \sigma[q_2] \text{ a.e. } S$$

hold for all sets S of the form $(-\infty, \theta)$.

Call σ (completely) deterministic if (8) holds for all unbounded intervals S , and almost stroboscopic if (8) holds for all measurable sets S .

1. For a given strategy σ , consider the collection \mathcal{g} of all subsets S of \mathbb{R}^1 for which (8) holds. Prove that if S_1, S_2, \dots belong to \mathcal{g} , then so do $\bigcup_{k=1}^{\infty} S_k$, $S_1 \cap S_2$ and also $S_1 \setminus N$, $S_1 \cup N$ for any set N of measure zero. (Hint: for the assertion on intersections, if $q_1 = q_2$ a.e. $S_1 \cap S_2$, let q be q_1 on S_1 and q_2 outside S_1 , and consider $\sigma[q]$.)

2. Prove that the following are equivalent: σ is deterministic; every open set S satisfies (8); there exists $\epsilon > 0$ such that (8) holds for all intervals of length ϵ .

3. Verify that, if (8) holds for all closed sets S , then σ is almost stroboscopic. (Hint: approximate a measurable set from below by F_σ sets, and use Exercise 1.)

Exercises 2 and 3 provide a hint for showing that deterministic strategies need not be almost stroboscopic. Define a mapping σ as follows. Its arguments are functions $q: \mathbb{R}^1 \rightarrow \mathbb{R}^1$, its values are functions $\sigma[q]: \mathbb{R}^1 \rightarrow \{0, 1\}$; and $\sigma[q](t) = 1$ if $\{s: q(s) \neq 0\} \cup N$ is a neighbourhood of t for some set N with measure zero, and 0 otherwise.

4. Prove that the mapping σ just described preserves measurability. (Hint: setting $S = \{s: q(s) \neq 0\}$, show that $\{t: \sigma[q](t) = 1\}$ is the countable union of sets where $\text{meas}(S \cap (t-k^{-1}, t+k^{-1})) = 2k^{-1}$.)

5. Show that σ is deterministic (e.g. using Exercise 2).

6. Choose any set $F \subset \mathbb{R}^1$ which is closed, nowhere dense, and has strictly positive measure; and let q be the characteristic function of F . Verify that the strategy σ defined above has $\sigma[q] \equiv 0$. This shows that σ is not almost stroboscopic: $q = 1$ on F but $\sigma[1] \neq \sigma[q]$ everywhere.

A strategy σ is said to have time lag ϵ at t if $q_1 = q_2$ a.e. $(-\infty, t-\epsilon)$ implies $\sigma[q_1] = \sigma[q_2]$ a.e. $(-\infty, t)$.

7. Check that σ is an indifferent strategy iff it has time lag $+\infty$ at all times t .

8. For a strategy σ and $t \in \mathbb{R}^1$ let $\epsilon(t)$ be the supremum of the time lags at t . Prove that $\epsilon(t) \geq 0$, and that $t \mapsto \epsilon(t)$ is measurable. (Hint: if $\epsilon(t) < \alpha$, then also $\epsilon(s) < \alpha$ for $s < t$ close to t ; thus $\{t: \epsilon(t) < \alpha\}$ is a union of non-degenerate intervals, hence a countable union of intervals, and therefore (open) \cup (countable).)

9. Show that the solution of simple pursuit described in Exercise 3 of 1.2 is effected by a strategy with time lag steadily converging to 0.

10. Prove carefully that equality holds in (7) if, and only if, capture is isochronous (in an appropriate sense).

11. Consider a non-anticipatory strategy σ , and two fixed player controls $p_0(\cdot)$, $q_0(\cdot)$. Define a second strategy, σ_0 , thus: for any quarry control $q(\cdot)$ take the largest α such that $q = q_0$ a.e. $(-\infty, \alpha)$, and then set

$$\sigma_0[q](s) = \begin{cases} p_0(s) & \text{for } s \geq \alpha \\ \sigma[q](s) & \text{for } s < \alpha \end{cases}$$

Prove that σ_0 is non-anticipatory. (Hint: first verify

the following: if, for three functions q_i , α_{ij} is the largest α with $q_i = q_j$ a.e. $(-\infty, \alpha)$, then two among α_{12} , α_{23} , α_{31} are equal and the third is not smaller.)

12. In a linear game $\dot{x} = Ax - p + q$, suppose that a pursuer 'feedback strategy' $F: R^n \rightarrow P$ is well-behaved, i.e., playable against all quarry controls, in the sense that

$$\dot{x} = Ax - F(x) + q, \quad x(t_0) = x_0$$

has a unique solution $x(\cdot)$ on $[t_0, +\infty)$ for any $x_0, t_0, q(\cdot)$. Show that there is a non-anticipatory strategy σ with the same effect: in detail, if the solution is written $x(t, q)$ (with x_0, t_0 fixed), then $F(x(t, q)) = \sigma[q](t)$.

Remarks

In the literature the term 'strategy' has been used (overused, perhaps) with quite disparate meanings. One respectable usage is that employed in this book, of a non-anticipative mapping from the set of quarry controls to that of the pursuer's; this is due to

P. P. Varaiya: The existence of solutions to a differential game, SIAM J. Control 5 (1967) 153-162.

A related but different usage appears in

R. Issacs: Differential Games, Wiley, New York, etc., 1967,

but is far older than the date suggests. Also see

A. Blaquière, F. Gérard, G. Leitmann: Quantitative and Qualitative Games, Academic Press, New York and London, 1969.

There the term refers to what might be described as the 'feedback strategies'; e.g. for the pursuer, a mapping $x \mapsto p$ from the state positions to pursuer's constraint set, with state responses $t \mapsto x(t)$ satisfying

$$(9) \quad \dot{x}(t) = f(x(t), p(x(t)), q(x(t))).$$

Among the advantages is that this latter notion seems particularly natural and appropriate (see, e.g., 1.2, 1.4, 1.7). Unfortunately, at the present stage of development of differential equation theory, the concept is almost completely intractable: in interesting cases $p(\cdot)$ is not continuous, so that the right-hand side in (9) depends discontinuously on the state variable, and thus we do not even have existence results (even if the equation is linear, and quarry absent). Uniqueness is in even worse shape: e.g., in R^1 , $\dot{x} = p(x) - q(x)$ with $q(x) = 0$ and $p(x) = \min(1, 3|x|^{2/3})$.

The thrust of our approach may be illustrated on the 'decoupling' of the analogous problem in linear control theory. For the control system $\dot{x} = Ax - u$ one first finds an optimal control $t \mapsto u(t)$, and only then, as a separate question, does one attempt to synthesise this by a feedback device $F: x \mapsto u$ such that optimal trajectories satisfy $\dot{x} = Ax - F(x)$ (i.e., $F(x(t)) = u(t)$ optimal). The object studied is then not (9) but

$$\dot{x}(t) = f(x(t), p(t), q(t));$$

and discontinuities in $t \mapsto p(t)$ no longer pose any conceptual problem.

The over-all effect is that the question of feedback strategies is avoided. If the game is such that one of our reciprocity theorems applies (see 3.1, 3.3, 3.7, 5.4) then the feedback problem is shifted into control theory, where, probably, it properly belongs.

Another question which we shall avoid is that suggested by the paradox in 1.7: the effect of both players using strategies simultaneously. Quite reasonable resolutions of this have been proposed (e.g., artificial time lags δ are

introduced at appropriate places in the information flow diagram, and subsequently one takes $\delta \rightarrow 0+$). The attitude adopted in this book is that, however this is resolved, the quarry chooses a control $t \mapsto q(t)$, which the pursuer counters, systematically, via a strategy. That the so posed problem is interesting is shown by the application to control under unpredictable perturbations, and the 'unorthodox linearisation' of 3.2; there the second player is purely notional - nature in the first case, and stupidity in the second.

This definitely prefers the pursuer to the quarry. Of course, the quarry's winning positions are precisely those which are not winning for the pursuer, so that a determination of the latter fixes the former. However, the specification of a successful quarry strategy involves a construction for all times $t \geq 0$: often far more difficult than that of a pursuer strategy, over a bounded time interval. Entering the plea of greater simplicity, we shall treat, almost exclusively, the pursuer's problem; our position is weakened by legitimate instances of interesting evader's problems, as in collision avoidance between aircraft treated in

T. L. Vincent, E. M. Cliff, W. J. Grantham, W. Y. Peng:
A problem of collision avoidance, Engineering
Experiment Station Report No. 39, The University of Arizona, 1972.

A situation in which the roles of the two players are symmetric might be indicated by the following. In addition to the dynamical equation and control constraint sets, let there be given three sets in state space: pursuer's target Ω , quarry's 'life-line' Σ , and state constraint Γ . The pursuer desires to reach Ω at finite time while remaining in $\Gamma \setminus \Sigma$ at all previous times; the quarry wishes to reach Σ , remaining in $\Gamma \setminus \Omega$ in the meanwhile. At present this

problem seems too difficult. One complication is that, in interesting cases, all of Ω , Σ , Γ are closed, whereupon $\Gamma \setminus \Sigma$, $\Gamma \setminus \Omega$ are not (trivialities excepted); and, barring some incisive new approach, even the simplified problem of a game - or even control system - with (pointwise) state constraints Δ is unrewarding unless Δ is, at least, closed.

2.3 Principle of Suboptimality

In the last section the basic terminology and notation was established. It is with reluctance that we introduce further notation here, namely

$$\pi(t, x, p, q), T(x, p, q), T(x, \sigma), T(x).$$

As usual, one must strike some balance between cumbersome (and, often, inaccurate) descriptive phrases, and a rising tide of symbols or abbreviations that the reader deciphers with increasing impatience.

The game is indicated by

$$(1) \quad \begin{aligned} \dot{x} &= f(x, p, q); p(t) \in P, q(t) \in Q; \\ \text{end: } x &\in \Omega; \text{ state space: } R^n. \end{aligned}$$

Under the standard assumptions, an initial position $x \in R^n$ and player controls $p(\cdot)$, $q(\cdot)$ determine uniquely a (generalised) solution to the differential equation on R^+ ; let us denote its value at $t \geq 0$ by $\pi(t, x, p, q)$. E.g., always $\pi(0, x, p, q) = x$; and, if the dynamical equation is linear, the variation-of-constants formula provides the explicit expression (6) of 2.1 for $\pi(\cdot)$.

Given (1), a position $x \in R^n$, and player controls $p(\cdot)$, $q(\cdot)$, define the first termination time

$$T(x, p, q) = \inf\{t \geq 0: \pi(t, x, p, q) \in \Omega\}.$$

The set involved may be empty, and then, of course, its infimum is $+\infty$. Since Ω is closed and the solution

$t \mapsto \pi(t, x, p, q)$ continuous, the infimum (if finite) is actually attained: thus

$$\begin{aligned}\pi(\theta, x, p, q) &\in \Omega \text{ for } \theta = T(x, p, q), \\ \pi(t, x, p, q) &\notin \Omega \text{ for } 0 \leq t \leq \theta.\end{aligned}$$

Given (1), a position $x \in R^n$ and (pursuer) strategy σ , one has the termination times $T(x, \sigma[q], q)$ for various quarry controls $q(\cdot)$; the 'worst' of these is

$$T(x, \sigma) = \sup_{q(\cdot)} T(x, \sigma[q], q).$$

Finally, we define the minimum time

$$T(x) = \inf_{\sigma} T(x, \sigma),$$

infimum taken over all strategies σ . Thus, if the intermediate concepts are eliminated,

$$T(x) = \inf_{\sigma} \sup_{q(\cdot)} \inf\{t: \pi(t, x, \sigma[q], q) \in \Omega\}.$$

As mentioned above, the innermost infimum is a minimum; so is the outermost, under reasonable conditions (see 8.1); no comment on the middle supremum.

It is almost immediate that $T(x)$ is finite iff x is a winning position (see Exercise 1); thus $x \mapsto T(x)$ is a real-valued function on the set of all winning positions. In analogy with control theory, $T(\cdot)$ will be called the minimal time function; it makes a convenient reference concept, but seldom is it available computationally.

PROPOSITION 1 Let x be a position, σ a strategy, and q a quarry control. For any finite time t with $0 \leq t \leq T(x, \sigma[q], q)$, the state response endpoint $y = \pi(t, x, \sigma[q], q)$ belongs to $W(0, \theta - t)$ for $\theta = T(x, \sigma)$, so that

$$(2) \quad T(y) \leq T(x, \sigma) - t.$$

PROPOSITION 2 Let $x \in W(\theta)$, with σ an isochronous strategy forcing x to termination at θ , and q a quarry control. For any time $t \in [0, \theta]$, the state response endpoint $y = \pi(t, x, \sigma[q], q)$ belongs to $W(\theta - t)$.

The assertion culminating with (2) will be called the principle of suboptimality (even though there is no reference to optimal strategies or controls). It should be noted that the entries y and t depend on the quarry control q . A proof for the first result only will be presented; it is readily modified to apply to the second. The underlying idea is extremely simple: formally, one decomposes integrals,

$$(3) \quad x + \int_0^\theta f = \left(x + \int_0^t f\right) + \int_t^\theta f ;$$

conceptually, a winning strategy σ_0 for y is obtained from the portion of σ over $[t, +\infty)$ not used (since σ is non-anticipatory) in reaching y from x - see fig. 1.

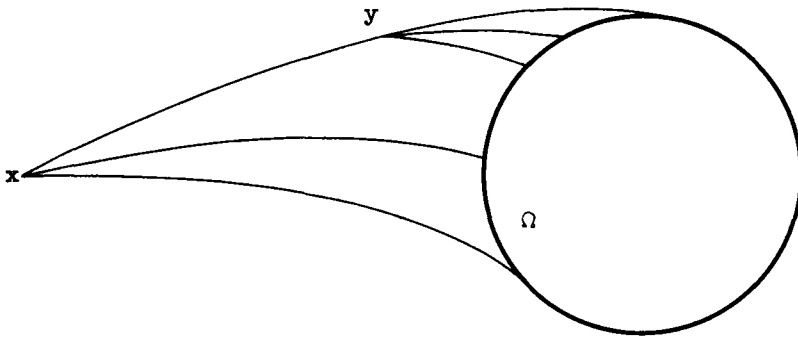


Fig. 1 Principle of Suboptimality (schematic). After reaching part way, y , to target Ω from x , the unused portion of the original strategy may be utilised to complete the motion; other options may also be available.

(Proof) Obviously (2) follows from $y \in W(0, \theta - t)$, and the strategy σ_0 needed to establish this is constructed as

follows. For any quarry control $q_0(\cdot)$ set

$$q_1(s) = \begin{cases} q(s) & \text{for } s \in [0, t) \\ q_0(s-t) & \text{for } s \geq t; \end{cases}$$

then let

$$(4) \quad \sigma_0[q_0](s) = \sigma[q_1](s+t) \text{ for } s \leq 0.$$

From (4), σ_0 is a pursuer strategy; we wish to verify that it forces y to termination within $\theta-t$.

In the nontrivial case $T(x, \sigma) < +\infty$, to the admissible quarry control q_1 as above (with q_0 arbitrary but fixed) there corresponds a finite termination τ ,

$$\tau = T(x, \sigma[q_1]q_1) \leq T(x, \sigma) = \theta,$$

at which the state response endpoint meets the target:

$$(5) \quad \Omega \ni \pi(\tau, x, \sigma[q_1], q_1) = x + \int_0^\tau f = (x + \int_0^t f) + \int_t^\tau f.$$

Since $q_1 = q$ on $[0, t)$, we have $\sigma[q_1] = \sigma[q]$ a.e. $[0, t)$, and therefore, in (2), the term $x + \int_0^\tau f = y$ satisfies

$$\begin{aligned} \Omega \ni y + \int_t^\tau f(x(s), \sigma[q_1](s), q_1(s)) ds \\ = y + \int_0^{\tau-t} f(x(s+t), \sigma[q_1](s+t), q_1(s+t)) ds \\ = y + \int_0^{\tau-t} f(y(s), \sigma_0[q_0](s), q_0(s)) ds, \end{aligned}$$

with $y(s) = x(s+t)$ an appropriate solution with initial point

$$y(0) = x(0+t) = y.$$

Since q_0 was arbitrary, we have that σ_0 forces to Ω , with termination time $\tau - t \leq \theta - t$. QED

COROLLARY 1 If σ is a winning strategy for a point

$x \notin \Omega$, then some point in the closure of the set of state response values

$$(6) \{ \pi(t, x, \sigma[q], q) : q(\cdot) \text{ quarry control, } 0 \leq t \leq T(x, \sigma[q], q) \}$$

belongs to $\lim_{t \rightarrow 0+} (W(0, t) \setminus \Omega)$.

(Proof) From monotonicity in (6) of 2.2, the sets $W(0, t) \setminus \Omega$ have coinciding limsup and liminf. Since σ is a winning strategy and $x \notin \Omega$,

$$(7) \quad 0 < T(x, \sigma) < +\infty.$$

Now choose quarry controls $q_k(\cdot)$ so that the corresponding termination times $\theta_k = T(x, \sigma[q_k], q_k)$ satisfy

$$\theta_k \leq T(x, \sigma), \quad \theta_k \rightarrow T(x, \sigma);$$

also choose times $t_k \geq 0$ (cf. (7)) with

$$(8) \quad t_k < \theta_k, \quad \theta_k - t_k \rightarrow 0;$$

and denote the state response endpoints

$$x_k = \pi(t_k, x, \sigma[q_k], q_k).$$

These x_k are in the set (6). From Proposition 1 and $t_k < \theta_k$ we have

$$x_k \in W(0, \theta_k - t_k) \setminus \Omega.$$

Hence and from (8), if a subsequence of the x_k converges, the limit will belong to $\lim_{t \rightarrow 0+} (W(0, t) \setminus \Omega)$. That the set (6) has compact closure, i.e., is bounded, follows from the lemma in 2.1 with, e.g., $\theta = T(x, \sigma)$. QED

COROLLARY 2 If $x \in W(\theta)$ with $\theta > 0$, and σ is an isochronous strategy forcing x to termination at time θ , then some point in the closure of the set (6) belongs to $\limsup_{t \rightarrow 0+} W(t)$.

(Proof: analogous to the preceding, using the second proposition.)

Exercises

1. Prove that

$$\{x: T(x) < \theta\} \subset W(0, \theta) \subset \{x: T(x) \leq \theta\}.$$

(For a more precise result see the proposition in 8.1).

2. Under the standard assumptions show that to each point $x \notin \Omega$ there corresponds $\epsilon > 0$ so that all $T(x, p, q) \geq \epsilon$; in particular, conclude that $T > 0$ outside Ω . (Hint: the continuous function $x(\cdot)$ is locally bounded; one can then obtain uniform bounds on $x(t) - x = \pi(t, x, p, q) - x$ for small $t \geq 0$.)

3. With the standard assumptions retained, prove that

$$\lim_{t \rightarrow 0^+} W(0, t) = \Omega, \quad \liminf_{y \rightarrow x} T(y) > 0 \text{ for } x \notin \Omega.$$

4. Check that, if compactness of the constraint sets is omitted, then $T(x, \sigma) = 0$ is possible for points $x \notin \Omega$. (Hint: simplify as far as possible; e.g., $P = \mathbb{R}^n$, $Q = 0$, linear dynamics, etc.)

Remarks

The two propositions are obviously related to Bellman's Principle of Optimality, suitably extended to differential games. For this, priority is claimed in

R. Isaacs: The past and some bits of the future,
pp. 1-11 in The Theory and Application
of Differential Games (ed. J. D. Grote),
Reidel, Dordrecht and Boston, 1975,

referring to a 'Tenet of Transition' in a 1951 Rand report entitled Games of Pursuit; also see p. 67 of the second printing of

BASIC CONCEPTS

R. Isaacs: Differential Games, Wiley, New York, etc.,
1967.

Another instance appears on p. 128 of

A. Friedman: Differential Games, Wiley-Interscience,
New York, 1971.

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CHAPTER III

STROBOSCOPIC AND ISOCHRONOUS CAPTURE

The main results of this chapter are in Sections 3.3 and 3.4. An instructive special case is treated first, in 3.1, and applied in 3.2 (another appears in 5.4). Sections 3.6 and 3.7 concern modifications of the basic problem, 3.5 exhibits some limitations of the method. Supporting material is presented in Sections 3.8 to 3.11, to be consulted as needed.

The thrust is in the Reciprocity Theorems of 3.1, 3.3 (and 3.7): a pursuit problem is reduced to one in control theory. The latter is usually simpler, because control theory has been developed much farther; a consequence is that evolution of practical control-theoretic methods will have an immediate impact within pursuit games.

The class of games to which the Reciprocity Theorems apply, and the type of answer they provide, may or may not be acceptable in practice. Thus, in most games one rarely steers to the origin of state space as in 3.1; and subsequent sections treat the more general problem and the ensuing complications. Or the Necessary Conditions (3.1, 3.4, 4.2) may not be satisfied: the conclusion is that winning strategies cannot then be isochronous. Finally, it may not be practicable to observe, and react to, the second player's control changes instantaneously; in 3.6 an attempt is made to treat the case of delays in the information flow.

We will address here one objection, sometimes brought up in connection with the otherwise valid distinction between state and control variables. The solution of the games treated

in 3.1 is as follows. From knowledge of the initial position alone, one constructs a vector-valued function $t \mapsto u(t)$; then, if the opponent chooses any control function $q(\cdot)$, the pursuer responds by taking the control $p(t) = u(t) + q(t)$, and thereby wins the game. The objection is to the instantaneous, stroboscopic, nature of the pursuer response; one might accept dependence on past values $q(s)$ with $s < t$, but the snap decision is too much.

Suppose that the pursuer is allowed to observe and retain the past history of his own control p , and of the state variable x governed by the dynamical equation $\dot{x} = Ax - p + q$ (warning: this is the point of the matter). Then, at each time t , he has available

$$\dot{x}(s) = \lim_{h \rightarrow 0+} \frac{x(s-h) - x(s)}{-h}$$

for all $s < t$, and thus also $q(s)$ from the dynamical equation, $q = \dot{x} - Ax + p$. Now, bounded measurable functions are locally integrable, and thus they are deterministic in the sense that past behaviour completely determines the present:

$$q(t) = \lim_{h \rightarrow 0+} \frac{1}{h} \int_{t-h}^t q(s) ds$$

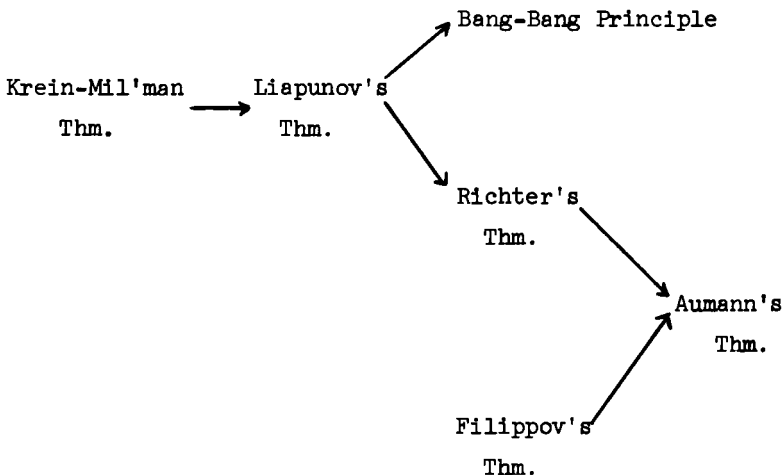
almost everywhere, by Lebesgue's Theorem. Therefore, at almost all times t the pursuer has available the current value $q(t)$ of his opponent's control choice for setting up his own policy. If this conclusion is found unacceptable, then either there must be realistic time delays involved, or the pursuer's information is incomplete, or the quarry has succeeded in constructing a non-measurable function.

Section 3.1 concerns games with forcing to the origin by stroboscopic and isochronous strategies. Although it is superseded by 3.3, (the First Reciprocity Theorem being a special case of the Second), some readers may find this manner of

exposition less painful. Section 3.2 presents one application, and, by implication, suggests others: that one might make up for lack of precision or over-simplification (e.g., linearisation) by treating their perturbation effects as the control of a notional opponent. Of course the attitude is plausible: the point is whether it is ever useful.

The material in the appendices 3.8 to 3.11 falls into two categories: superficial and less so. In 3.8 we emphasise one concept in convex set theory, the exterior normals, and then treat the relatively recent concept of Pontrjagin difference in some detail. The theorems of Liapunov, Richter and Aumann are presented in a rather weak form in 3.10, the reachable sets of control systems are treated cursorily in 3.11; however, Filippov's Lemma is painstakingly analysed in 3.9. The excuse is reference to the present purposes and requirements.

A part of the mathematical background of linear control theory may be summarised thus:



and all of these are of fundamental importance. For the more restricted purposes of this book, a modification of Filippov's

Lemma is used several times for the construction of strategies, and a weak version of Richter's Theorem is needed for Proposition 1 in §.5. It seemed appropriate to present as little of the above graph as possible.

3.1 Forcing to Origin

Consider the problem of forcing to the origin, in the game described by

$$\dot{x} = Ax - p + q, \quad p(t) \in P, \quad q(t) \in Q; \quad \text{state space: } R^n.$$

Specifically, we propose to study isochronous forcing to 0 by stroboscopic strategies (the assertion being proved will be formulated subsequently, assumptions including $P = \overline{P}$).

Assume that one has a stroboscopic strategy σ which forces an initial position x to 0 at time $\theta \geq 0$ against all actions of quarry. Reducing the definitions, we have that, for each integrable $q: [0, \theta] \rightarrow Q$, the mapping $t \mapsto p(t) = \sigma(q(t), t)$ is integrable $[0, \theta] \rightarrow P$, and the corresponding state response satisfies $x(\theta) = 0$. From the variation-of-constants formula, the last condition is

$$0 = e^{A\theta} \left(x - \int_0^\theta e^{-At} (\sigma(q(t), t) - q(t)) dt \right).$$

Since $e^{A\theta}$ is nonsingular,

$$(1) \quad x = \int_0^\theta e^{-At} (\sigma(q(t), t) - q(t)) dt.$$

Here there is considerable freedom in choosing $q(\cdot)$. Begin by fixing a reference point $q_0 \in Q$ (if, e.g., Q contains 0, it is tempting to take $q_0 = 0$). There is a corresponding constant quarry control; define

$$u(t) = \sigma(q_0, t) - q_0,$$

an integrable mapping $[0, \theta] \rightarrow R^n$. Then (1) yields

$$(2) \quad x = \int_0^{\theta} e^{-At} u(t) dt.$$

Second, take any point $q \in Q$, choose a time $s \in [0, \theta]$, and consider the admissible piecewise constant quarry control with value q in $[0, s]$ and value q_0 in $(s, \theta]$. Since σ is stroboscopic (and this is the crucial point of the reasoning), the pursuer response has $\sigma - q_0 = u$ in $(s, \theta]$, so that, in (1),

$$\begin{aligned} x &= \int_0^s e^{-At} (\sigma(q, t) - q) dt + \int_s^{\theta} e^{-At} (\sigma(q_0, t) - q_0) dt \\ &= \int_0^{\theta} e^{-At} u(t) dt - \int_0^s e^{-At} (u(t) + q - \sigma(q, t)) dt. \end{aligned}$$

We subtract from (2) to obtain

$$0 = \int_0^s e^{-At} (u(t) + q - \sigma(q, t)) dt.$$

Here the integrand is independent of s (as σ is non-anticipatory), and s is free to vary over $[0, \theta]$. Hence the integrand vanishes almost everywhere in $[0, \theta]$, and (since e^{-At} is nonsingular and σ has values in P)

$$(3) \quad \begin{aligned} \sigma(q, t) &= u(t) + q, \\ u(t) + q &\in P \end{aligned}$$

(almost all t).

Now, q is still free to vary over Q ; but we cannot directly conclude that $u(t) + Q \subset P$, since the set of exceptional times t may well depend on the point q chosen. Actually, the difficulty can be overcome; since the argument will be used again, we excerpt it as follows.

LEMMA If X and $Y_t = \bar{Y}_t$ are subsets of R^n , and $f: R^1 \rightarrow R^n$ has the property that, for each $x \in X$, $f(t) + x \in Y_t$ for almost all t , when $f(t) + X \subset Y_t$ a.e.

(Proof) X has a countable dense subset, say

$\{x_k: k = 1, 2, \dots\}$. For each k there is a set E_k of

measure zero such that $f(t) + x_k \in Y_t$ for all $t \notin E_k$. Then $E = \bigcup E_k$ still has measure zero, and $f(t) + x_k \in Y_t$ for $t \notin E$ and all k . To prove $f(t) + x \in Y_t$ for $t \notin E$ and all $x \in X$, merely take limits over an appropriate subsequence of the x_k converging to x . QED

Resuming the narrative, we now have $u(t) + Q \subset P$ a.e.; or, in terms of the Pontrjagin difference (see 3.8), $u(t) \in P \dot{-} Q$ a.e., so that u is an integrable mapping $[0, \theta] \rightarrow P \dot{-} Q$. Finally, (2) may be interpreted as stating that x belongs to the reachable set (cf. 3.11) at time θ of a linear control system, namely

$$\dot{y} = Ay - u; u(t) \in P \dot{-} Q.$$

It turns out that our steps can be re-traced, so that the property just established is a complete characterisation:

FIRST RECIPROCITY THEOREM With the game

$$(4) \quad \dot{x} = Ax - p + q; p(t) \in P, q(t) \in Q$$

in R^n associate the control system

$$\dot{y} = Ay - u; u(t) \in P \dot{-} Q;$$

and assume that P is closed. Then an initial position $x \in R^n$ can be forced to 0 at time $\theta \geq 0$ by a stroboscopic strategy σ within the game if, and only if, x can be steered to 0 at θ by an admissible control u within the control system.

Furthermore, either of σ and u can be determined from the other by the relation

$$(5) \quad \sigma(q, t) = u(t) + q.$$

(Proof) In one direction this has already been established (e.g., (3) is (5) a.e.). In the other, assume that an integrable control u steers x to 0 at θ within the control system, so that (2) holds. Define a stroboscopic

strategy σ by (5); for any integrable $q: [0, \theta] \rightarrow Q$,

$$p(t) = \sigma(q(t), t) = u(t) + q(t)$$

defines an integrable mapping p , with values in $(P \# Q) + Q \subset P$. To verify that σ forces as asserted, note that $p - q = u$, so that indeed

$$e^{A\theta} \left(x - \int_0^\theta e^{-At} (p(t) - q(t)) dt \right) = e^{A\theta} \left(x - \int_0^\theta e^{-At} u(t) dt \right) = 0$$

by (2). QED

There are many consequences of the reciprocity just established. The assertion is that the set of points which can be forced to 0 stroboscopically at time θ is precisely the reachable set $R(\theta)$ of the control system. Thus the set is convex (see Richter's Theorem); if $Q \subset P$, then $0 \in P \# Q$, whereupon the reachable sets $R(\theta)$ increase with θ . Most important, the minimal time problems coincide.

NECESSARY CONDITION In the game (4) let P be closed. If there are any points which can be forced to the origin stroboscopically at strictly positive time, then

$$Q - Q \subset P - P.$$

(Proof) If such points exist, then, at least, the constraint set of the control system must be nonvoid: $u \in P \# Q$ for some u . Take any points q_1, q_2 in Q ; then $u + q_k = p_k \in P$, and hence $q_1 - q_2 = p_1 - p_2 \in P - P$. QED

COROLLARY 1 In (4) assume that P, Q are nonvoid, convex, and symmetric. Then $Q \subset P$ is a necessary and sufficient condition for presence of positions which can be forced to 0 stroboscopically at strictly positive time.

(Proof) $P \# Q$ is convex and symmetric; thus it is nonvoid iff it contains 0, i.e., $Q \subset P$. Conversely, if $Q \subset P$, then $0 \in P \# Q$ is an admissible control, steering 0 to 0. QED

COROLLARY 2 If $Q \subset \text{Int } P$, then for every $t > 0$, the

set of positions which can be forced to 0 stroboscopically at time t is a neighbourhood of 0.

(Proof) $P \overset{*}{=} Q$ contains a neighbourhood of the origin; thus the control system is controllable, its reachable sets $R(t)$ are neighbourhoods of 0 and increase with $t > 0$. QED

Example A service trolley moves on a straight track; the operator actuates an accelerating/braking mechanism to bring the trolley to rest at the point of demand; the mechanism functions somewhat imperfectly, with a small and slowly varying error. It is desired to instruct the operator (or design a device to carry out automatically) to function with minimal time of travel if provided with the appropriate information.

Let $x(t)$ denote the trolley's position at time t , measured from the demand point. Its state is then completely determined by $x(t)$ and $\dot{x}(t)$; its equation of motion is $\ddot{x} = a - \epsilon$, with a the operator's action, and ϵ the error term; the constraints are

$$|a(t)| \leq \alpha, \quad |\epsilon(t)| \leq \epsilon_0,$$

and $\alpha - \epsilon_0 = \delta > 0$. The desired final state of the trolley is that of rest at the demand point, $x = 0 = \dot{x}$.

The first-order version of the equation is

$$\dot{x} = y, \quad \dot{y} = a - \epsilon,$$

with the control constraints as above, and the origin as target. In applying the reciprocity theorem, the associated control system is

$$(6) \quad \dot{x} = y, \quad \dot{y} = u; \quad |u(t)| \leq \delta$$

(Indeed, $|u + \epsilon| \leq \alpha$ for ϵ with $|\epsilon| \leq \epsilon_0$ iff $|u| \leq \alpha - \epsilon_0 = \delta$.)

For the control system the time-optimal controls are well-known: they are piecewise constant, with values $\pm\delta$ only,

and at most one discontinuity. In point of fact there is a simple feed-back control: one solves the dynamical equations (6) for constant $u = \pm \delta$:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{u}{y}, \quad y^2 = 2u(x + \text{const.});$$

appropriate arcs of these parabolas which pass through the origin (viz, $y^2 = \pm 2\delta x$) then separate the (x, \dot{x}) plane into two domains, in which the feed-back control is constant, $f(x, \dot{x}) = \pm \delta$.

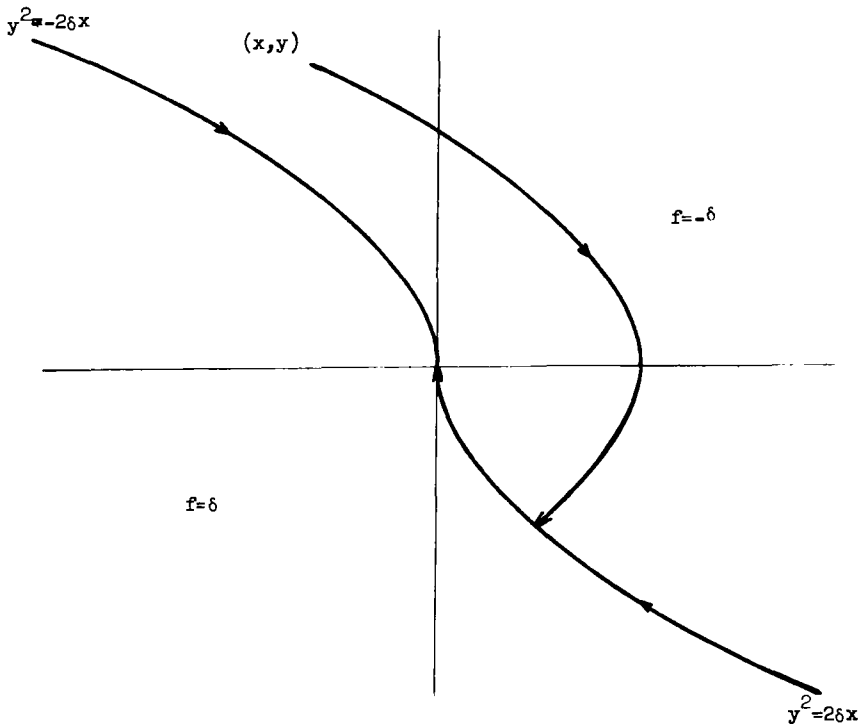


Fig. 1 Optimal trajectories and feed-back in control system 'reciprocal' to game.

Having this, we conclude, from the theorem, that the operator should use (or, a mechanism produce) the control

$$\dot{a} = \dot{\sigma} = \dot{u} + \dot{\epsilon} = f(x, \dot{x}) + \dot{\epsilon}(t)$$

with f as described. Apparently this will require observation, at each time instant, of the current values of the position and velocity of the trolley and of the error term.

Exercises

1. An idealised aircraft is to fly between points a, b (over a flat earth, at constant altitude); it can regulate its velocity arbitrarily, subject to a speed limit α ; the wind velocity vector has a constant component w and also a component unpredictable in direction but with a magnitude bound β . What arrival time can honestly be posted on the airline announcement board? (Partial answer: trivialities expected, arrival time can be guaranteed iff $\alpha - \beta \geq |w| \cdot |\sin \varphi|$, where φ is the angle between $a - b$ and w .)

2. Modify the following game so as to apply the reciprocity theorem. The two players in n -space have coinciding dynamics,

$$\dot{x} = Ax - p, \quad \dot{y} = Ay - q;$$

the termination condition is coincidence of positions, $x = y$.

3. Show that the strategies appearing in the reciprocity theorem satisfy

$$(7) \quad \sigma(\alpha_1 q_1 + \alpha_2 q_2, t) = \alpha_1 \sigma(q_1, t) + \alpha_2 \sigma(q_2, t)$$

whenever $q_k \in Q$, $\alpha_k \in \mathbb{R}^1$ and $\alpha_1 q_1 + \alpha_2 q_2 \in Q$. Conclude that $\sigma(\cdot, t)$ be extended over $\text{span } Q$ as an affine mapping.

4. Prove directly that, if a winning strategy satisfies (7), then σ has the form $\sigma(q, t) = u(t) + V(t)q$; that $V(t)q = q$ for all q and almost all t ; and that then u is as described in the theorem.

5. Formulate and prove an analogue of the reciprocity theorem for the case that the target is a given point b ,

possibly not the origin.

6. Obtain a version of the theorem if state constraints are present, e.g. in the form $x(t) \in C$ for all t , with given set $C \subset \mathbb{R}^n$, or in integral form.

7. Generalise the reciprocity theorem to the case of dynamical equation

$$\dot{x} = A(t)x + B(t)u + C(t)v + w(t),$$

control constraints $u(t) \in U$, $v(t) \in V$, and termination condition $x(\theta) = a(\theta)$. (You may wish to subsume the term $w(\cdot)$ into one of the players' controls.)

Remarks

Several authors have obtained the latter part of the proof of the theorem, even in the more general case of linear subspaces as targets. The first of a series was probably

L. S. Pontrjagin: On linear differential games 2,
Doklady Akad. Nauk SSSR 175 (1967)

(the method involving 'infinitesimal pursuer prediction'). A related result, actually involving nonlinear games, was announced in

E. O. Roxin: Some global problems in differential games, pp. 103-116 in Global Differentiable Dynamics (ed. O. Hájek, A. J. Lohwater, R. C. McCann) Lecture Notes in Mathematics 235, Springer, Berlin, etc., 1970.

In this formulation, the conclusion is a sufficient condition for general capture, so that stroboscopic and isochronous strategies need not be mentioned. The advantage of our presentation is that it specifies the sense in which the method is necessary, and second, that it isolates the associated control system. It was announced in

- O. Hájek: A relation between pursuit games and time-optimal control, NSF Regional Conference on Control Theory, Univ. of Maryland, Baltimore County, 1971

(but was not accepted for the conference proceedings issue of the SIAM J. Control).

In studying uniqueness of time-optimal strategies, the problem of whether the standard normality conditions for polytopes P, Q (and coefficient matrix A) imply normality for $P \neq Q$ was treated in

- E. Rechtschaffen: Unique winning policies for linear differential games (thesis), Case Western Reserve University, 1973

(yes for state space dimension $n \leq 2$, otherwise no in general). Results related to the allonomous case of Exercise 7 appeared in

- O. Hájek: Duality for differential games and optimal control, Math. Systems Theory 8 (1974) 1-7.

The example in the text is Example 2 in Chapter 1 of

- E. B. Lee, L. Markus: Foundations of Optimal Control Theory, Wiley, New York, etc., 1967.

According to the theorem, the principle of solution is the method of neutralisation: the pursuer neutralises quarry's action completely (in a suitable sense), and then uses left-over potential to steer to target.

This may seem trite; however, we shall see in 3.3 and 3.7 that this is exactly what happens for some classes of strategies, and in 5.4 that sometimes neutralisation cannot be avoided.

3.2 Unorthodox Linearisation

Suppose that a mathematical pendulum (not a linear oscillator) in a constant gravitational field can be controlled by a bounded external torque; and that it is desired to bring the bob to a stable equilibrium in shortest time.

Assume, for simplicity, that the motion is frictionless and planar; that the equation of motion and constraint are normalised,

$$(1) \quad \ddot{\theta} + \sin \theta = \tau, \quad -1 \leq \tau \leq 1;$$

and that the termination condition is $\theta = 0 = \dot{\theta}$ (disregarding the values $\theta = \pm 2\pi, \pm 4\pi, \dots$ appropriate to rapid oscillations). The problem belongs to nonlinear time-optimal control theory, and is readily treated by known methods - see, e.g., Example 2 in Chapter 7 of

E. B. Lee, L. Markus: Foundations of Optimal Control Theory, Wiley, New York, etc., 1967.

We propose to apply, somewhat surprisingly, linear game theory. The method, unorthodox linearisation, is easily described. First isolate the linear part of (1) as in

$$\ddot{\theta} + \alpha\theta = \tau - (\sin \theta - \alpha\theta);$$

here the reader may wish to think of $\alpha = 1$, but (as will be seen later) this can be improved upon. Then interpret the bracketed term, $\sin \theta - \alpha\theta = \rho$, as the action of a notional opponent,

$$\ddot{\theta} + \alpha\theta = \tau - \rho$$

$$(2) \quad |\tau(t)| \leq 1, \quad |\rho(t)| \leq \rho_0; \text{ end: } \theta = 0 = \dot{\theta}.$$

Here, of course, the quarry control bound ρ_0 is still to be determined; it will depend on α , but also on the initial position (obviously there is no uniform bound on ρ unless

$\alpha = 0$). Finally, solve the resulting linear game.

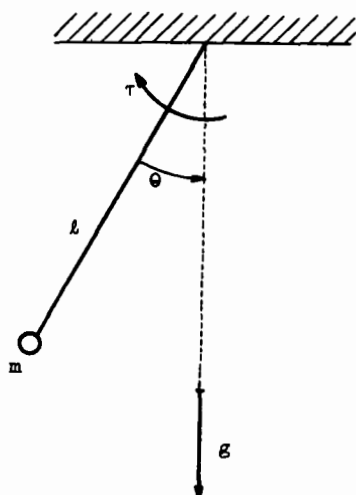


Fig. 1 Mathematical pendulum with torque control.

In our case the First Reciprocity Theorem can be applied, reducing the game problem to one in control theory again. The solution is as follows. Assume $\rho_0 < 1$; then there is a winning strategy $\tau = \rho + u$, where u is a control which steers to the origin within the control system

$$(3) \quad \ddot{\theta} + \alpha\theta = u; \quad |u(t)| \leq 1 - \rho_0.$$

Indeed, in the first-order version of (2) in R^2 , the players' constraint sets are

$$P = \left\{ \begin{pmatrix} 0 \\ \tau \end{pmatrix} : |\tau| \leq 1 \right\}, \quad Q = \left\{ \begin{pmatrix} 0 \\ \rho \end{pmatrix} : |\rho| \leq \rho_0 \right\},$$

so that

$$U = P \# Q = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} : |u| \leq 1 - \rho_0 \right\}.$$

The resulting control torque will steer to the origin (precisely, not just approximately); probably not optimally, but

perhaps suboptimally. One formulation is that one chooses to lose information by treating the 'deterministic' function $\sin\theta(t) - \alpha\theta(t)$ as the unpredictable action $\rho(t)$ of a fictional opponent while gaining the advantage of linearity. Another is that, while the optimal controls for (1) are quite rigid, the suboptimal ones are not, and one can adjust them so as to have the phase variable track a time-optimal trajectory of an approximate linear control system.

We shall compare the actual time-optimal solutions of (1) with the results of unorthodox linearisation (and, in Exercise 3, with 'ordinary' linearisation).

For the first, see the reference above. The state space description begins with $x = \theta$, $y = \dot{\theta}$:

$$\dot{x} = y, \dot{y} = -\sin x + \tau$$

For τ constant on some time interval one concludes that

$$\begin{aligned} \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{-\sin x + \tau}{y}, \\ (4) \quad y^2 &= 2(\tau x + \cos x + c) \end{aligned}$$

for constant c . The optimal controls τ always have extreme values, 1 below and -1 above the switching locus. This last consists of the extreme trajectories (4) ($\tau = \pm 1$) which enter the origin:

$$(5) \quad y = -\operatorname{sgn} x \sqrt{2(|x| + \cos x - 1)},$$

see fig. 2. On this locus, the optimal trajectories are described by (4), with orientation determined e.g. by $dx/dt = y$. Graphically, this is simplified if one more adjustment is made, replacing y by $\eta = y^2$, since then the various curves (4) are obtained by simple translation along the x -axis (fig. 3).

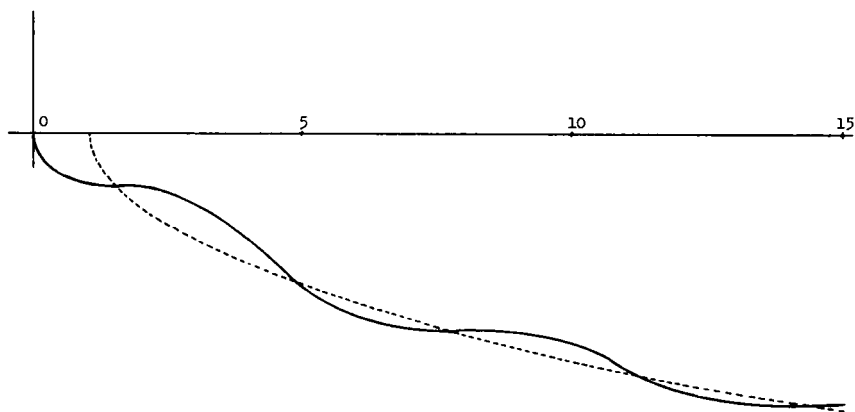


Fig. 2 Time-optimal switching curve
(lower part) for torque-controlled pendulum.

Dotted curve is $y^2 = 2(x-1)$.

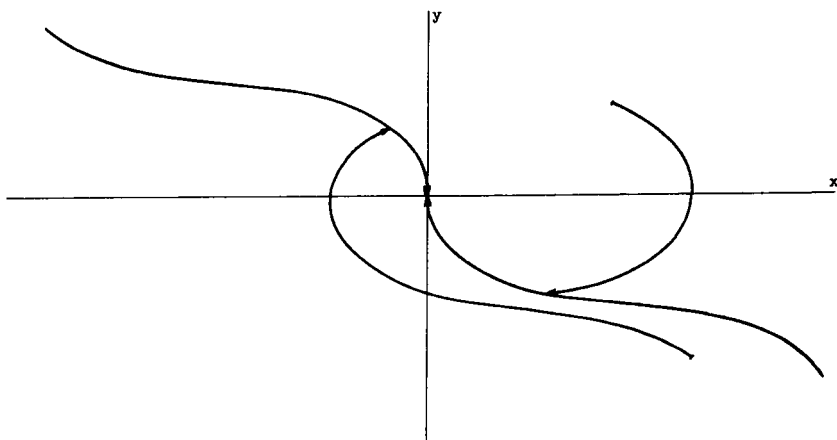


Fig. 3. Schematic time-optimal trajectories and switching curve for torque-controlled pendulum.

Let us carry this out in some detail for two initial positions in phase plane, $(\frac{1}{2}, 0)$ and $(1, 0)$. The coordinates of some points on the optimal trajectories are in tables 1, 2

below, with fig. 4 as summary.

Table 1

x	-y	t_-	t_+
.5	.0	.20049	.20415
.47	.29661	.05755	.05860
.45	.38176	.10727	.11078
.4	.53569	.08287	.08574
.35	.65083	.05560	.05721
.31121	.72549	.08472	.09175
.25	.66173	.07689	.08187
.2	.60012	.08617	.09147
.15	.52682	.10101	.10691
.1	.43589	.10314	.13635
.05	.31225	.11556	.11921
.02	.19900	.19900	.20306
0	0		

Points on optimal trajectory of system (1) through $(\frac{1}{2}, 0)$. Upper portion is on $y^2 = 2(-x + \cos x + c)$ with $c = -0.37758$; lower is on the switch curve.

Table 2

x	-y
1	0
.98	.27100
.97	.33166
.95	.42750
.9	.60218
.8	.84429
.72985	.97460
.6	.92232
.5	.86900
.4	.80132
.31121	.72549

Points on optimal trajectory of system (1) through $(1, 0)$. Upper portion is on $y^2 = 2(-x + \cos x + c)$ with $c = .45970$; lower is on the switch curve.

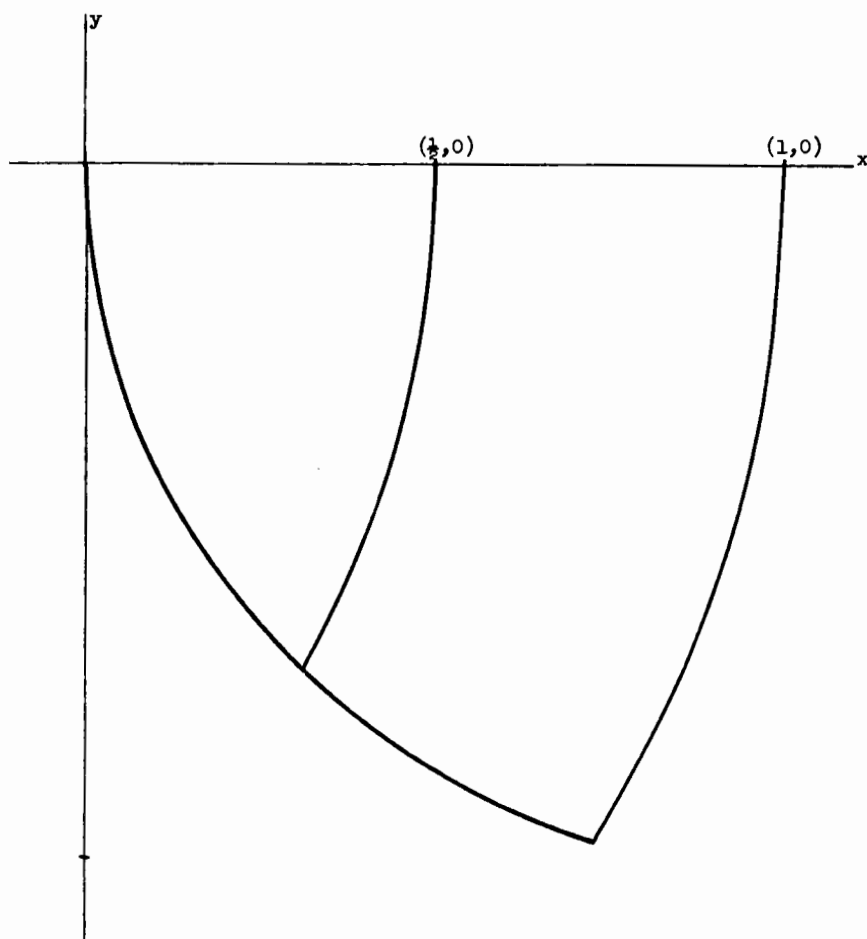


Fig. 4 Torque-controlled pendulum: time-optimal trajectories for points $(1/2, 0)$ and $(1, 0)$.

Next, for the initial point $(\frac{1}{2}, 0)$, the minimum time to reach the origin was computed. Referring to the state equation $\dot{y} = -\sin x - 1$ valid above the switch curve, the time t taken in moving from (x_1, y_1) to (x_2, y_2) satisfies

$$\frac{y_1 - y_2}{1 + \sin x_1} \leq t \leq \frac{y_1 - y_2}{1 + \sin x_2}$$

The extreme values appear in Table 1, with

$$\frac{y_2 - y_1}{1 - \sin x_2} \leq t \leq \frac{y_2 - y_1}{1 - \sin x_1}$$

for the lower part. The total time is then

$$(6) \quad T = 1.32270 \pm 0.02541.$$

Now let us examine the unorthodox linearisation (2), and the ensuing linear control system (3). The first task is to specify α . There is an obvious guess $\alpha = 1$, a bad one $\alpha = 0$; one naturally seeks to make $|\sin \theta - \alpha \theta|$ small. For the initial point $(\frac{1}{2}, 0)$, 'least squares' provides $\alpha = 0.97536$; the best uniform value is $\alpha = 0.96911$. Having this, one finds

$$\rho_0 = \max_{0 \leq \theta \leq \frac{1}{2}} |\sin \theta - \alpha \theta|,$$

determining the control system (3) completely. Actually, only the least squares value is put forward seriously as an improvement: the best uniform makes for a complicated computation, and is only included here for reference.

The optimal trajectories are, piecewise, elliptic arcs

$$(7) \quad \alpha x^2 + 2wx + y^2 = \text{const.}$$

with $w = 1 - \rho_0$; the switching curve is made up of translates of these (for $\alpha = 0$ this consists of two parabolic arcs). In the case of a single switch one readily determines the constant in (6) for an initial point $z_0 \in R^2$; then the intersection $z_1 \in R^2$ with the switch curve; and, finally, the times t_1, t_2 needed to reach, from

$$(8) \quad \begin{aligned} z_1 &= e^{At_1} \left(z_0 - \int_0^{t_1} e^{-As} ds \, bw(-1) \right), \\ 0 &= e^{At_2} \left(z_1 - \int_0^{t_2} e^{-As} ds \, bw \right). \end{aligned}$$

PURSUIT GAMES

For initial position $(\frac{1}{2}, 0)$ the results appear in Table 3, the second coordinate of (7) being used for both t_1 and t_2 .

Table 3 Minimal time T for initial point $(\frac{1}{2}, 0)$ via unorthodox linearisation.

α method	α	ρ_0	x_1	$-y_1$	$\sqrt{\alpha} t_1$	$\sqrt{\alpha} t_2$	T
bad guess	0	.47943					1.96009
first guess	1	.02057	.31381	.71850	.50710	.82357	1.33067
least squares	.97536	.00825	.31147	.72331	.50389	.80419	1.32450
best uniform	.96911	.00512	.31088	.72451	.50304	.79935	1.32298

In Table 4 there appear the corresponding results for several initial positions $(x_0, 0)$. Part of the computation was abandoned, perhaps precipitately, when it became clear from the values of (x_p, y_p) that the linearisation entailed two switches, i.e., an extra swing around the origin.

Table 4

x_0	α	ρ_0	x_p	y_p	x_1	$-y_1$	T
.5	.97536	.00825	1.01680	1.00420	.31147	.72331	1.32450
1	.90351	.06204	1.03813	.98677	.74082	.94544	2.08674
1.5	.79234	.19102	1.02100	.90883	1.30093	.87401	2.24097
2	.65310	.39690	.92344	.74628			
2.5	.51596	.69144	.59803	.43957			
3	.34568	.89592	.30110	.17703			

For initial positions $(x_0, 0)$, least squares method to obtain α . (x_p, y_p) are coordinates of the north pole of the 'first' switching ellipse; (x_1, y_1) are the coordinates of the intersection with the switch curve with the optimal trajectory; T is the termination time.

The conclusions are as follows. The upper estimate $T < 1.3245$ from Table 3 is well towards the centre of the confidence interval in (6). It was obtained, via (8), far easier than the 12-step integration procedure from Table 1: even a

4-step integration readily dissuades one from using this method. As concerns the trajectories, they are indistinguishable in the scale of Fig. 4 for $(\frac{1}{2}, 0)$, and barely distinguishable for $(1, 0)$.

A more general case may be indicated as follows. Assume given a control system in R^n ,

$$(10) \quad \dot{x} = f(x) - p; \quad p(t) \in P, \text{ end: } x = 0.$$

Rewrite

$$\dot{x} = Ax - p + (f(x) - Ax),$$

and consider the linear game this suggests,

$$(11) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \text{ end: } x = 0.$$

(There will be assumptions on the new data A, Q .)

PROPOSITION In (10) let $f: R^n \rightarrow R^n$ be continuous, and P closed nonvoid. Assume that a subset $\Theta \subset R^n$ satisfies two conditions: the estimate

$$(12) \quad f(x) - Ax \in Q \text{ whenever } x \in \Theta,$$

and invariance: there is a winning stroboscopic and isochronous strategy σ for (11) and an initial position $x_0 \in \Theta$ such that the corresponding state response $x(t)$ remains in Θ up to the termination time θ .

Then there exists a control $p(\cdot)$ admissible in (10) which steers x_0 to 0 at time θ .

Furthermore, the corresponding state response to $p(\cdot)$ in (11) is $x(\cdot)$ again; finally,

$$(13) \quad p(t) = \sigma(q(t), t) = u(t) + q(t)$$

for $q(t) = f(x(t)) - Ax(t)$, where $u(\cdot)$ is as described in the First Reciprocity Theorem.

(Proof) From (13), σ neutralises, so that $x(\cdot)$ is independent of quarry controls. Choose a constant in Q to obtain $x(\cdot)$; then take the admissible quarry control q ,

$q(t) = f(x(t)) - Ax(t)$, to conclude

$$\begin{aligned}\dot{x} &= Ax - \sigma(\cdot) + q = Ax - p + (f(x) - Ax) \\ &= f(x) - p\end{aligned}$$

almost everywhere. Thus indeed p steers the trajectory from $x(0) = x_0$ to $x(\theta) = 0$. QED

The conditions on Θ (and Q) can be eliminated entirely by the following formulation. Retaining the state response $x(\cdot)$ to σ in (11), let Θ be its trajectory arc, and V the corresponding image:

$$\Theta = \{x(t) : 0 \leq t \leq \theta\}, V = \{f(x) - Ax : x \in \Theta\}.$$

However, the original version is preferable. Thus, for the controlled pendulum and initial point $(\frac{1}{2}, 0)$ it is obvious that $\theta(t)$ decreases from $\frac{1}{2}$ to 0 monotonously (invariance), whereupon

$$|\sin \theta - \alpha \theta| \leq |\sin \frac{1}{2} - \alpha \frac{1}{2}|$$

(the estimate) for the values of α used there.

From the First Reciprocity Theorem, Q ought to be rather small to make $P \# Q$ large; hence, in (1), $f(x) - Ax$ should be small: linearisation.

Exercises

In the first three of these we examine the consequences of an 'ordinary' linearisation of (1). Assume that one installs a device which implements the feedback control appropriate to

$$(14) \quad \ddot{\theta} + \theta = \tau; \quad -1 \leq \tau \leq 1$$

1. Obtain the switching locus for (14). (Partial answer: semicircles with radius 1.)

2. For the initial points $(\frac{1}{2}, 0)$ and $(1, 0)$ find the switch on the locus from Exercise 1. (Hint: For orientation,

the lowest corners in fig. 4 have distance, from the centre (1,0), approximately 1.00039 and 1.01135.)

3. Prove that each trajectory near the origin (other than on the curve (4)) has an end-point, on the locus from Exercise 1, beyond which it cannot be continued.

Exercises 4 to 7 treat the controlled van der Pol equation

$$(15) \quad \ddot{x} + 2\epsilon(x^2 - 1)\dot{x} + x = u, \quad -1 \leq u(t) \leq 1.$$

Note that here the uncontrolled motion, corresponding to $u \equiv 0$, has an unstable focus at the origin.

4. There are two standard procedures for bringing (15) to a first-order system: the phase plane method, beginning with $\dot{x} = y$, and that in the Liénard plane, using the energy integral, $x + 2\epsilon\left(\frac{x^3}{3} - x\right) = y$. Comment on the advantages of these in respect to unorthodox linearisation.

5. The switching locus of (15) is not obtained by elementary integration. Obtain qualitative information about its disposition near the origin. (Answer: it enters the origin from the second and fourth quadrants, ultimately tangent to the y-axis.)

6. Apply unorthodox linearisation to (15) and show that an entire neighbourhood of the origin can be steered to the origin. (Hint: the associated linear control system is

$$\ddot{x} - 2\epsilon\dot{x} + x = w, \quad |w(t)| \leq 1 - \delta;$$

for small $t > 0$, its reachable sets are entirely within $x^2|y| \leq \delta$.)

7. Obtain specific estimates for Exercise 4. (Hints:

$$\int e^{-At} b dt = e^{-\epsilon s} \begin{pmatrix} \epsilon \sin \eta t + \eta \cos \eta t \\ \sin \eta t \end{pmatrix}$$

with $\eta = \sqrt{1-\epsilon^2}$. Estimate $R(t)$ and also the least distance from the origin of $x^2 y = \pm \delta$. Find an upper bound on t in

terms of ϵ, δ , and then vary δ over $(0,1)$.)

Remarks

Unorthodox linearisation was described in

O. Hájek: Pursuit games: a survey, pp. 281-291 in
The Theory and Application of Differential
Games (ed., J. D. Grote), Reidel, Dordrecht
and Boston, 1975.

The and its proof strongly depend on the target being the origin (so that σ completely neutralises q). Nevertheless it would be highly desirable to extend its scope to termination conditions of the form $Mx = 0$, as treated by the Second Reciprocity Theorem; and possibly, to general capture. For feedback controls there is no problem: in $\dot{x} = Ax - \sigma + q$ one simply takes $q = f(x) - Ax$.

3.3 Affine Targets

This section treats the game

(1) $\dot{x} = Ax - p + q$; $p(t) \in P$, $q(t) \in Q$; end: $x \in \Omega$
in R^n , with the target set an affine manifold

(2) $\Omega = \{x: Mx = Mb\}$

described by an (m,n) matrix M and a point $b \in R^m$.

There will appear an obvious analogy with the Reciprocity Theorem of 3.1 (which is, in fact, the special case $M = I$, $b = 0$). To unburden the formulation of the result below, the needed apparatus is introduced first. Let $N = \{x: Mx = 0\}$ be the null space of M , so that $\Omega = b + N$. Define a control system associated with (1),

(3) $\dot{y} = -Ay - v$; $v(t) \in V_t$

with constraint sets

$$V_t = (P + e^{-At}N) \pm Q.$$

SECOND RECIPROCITY THEOREM Assume that the constraint set P is compact. Then an initial position $x \in R^n$ can be forced to the affine manifold Ω stroboscopically and isochronously at time θ within the game (1) if, and only if,

$$(4) \quad x \in e^{-A\theta}(R(\theta) + \Omega),$$

where $R(\theta)$ is the reachable set at θ of the associated control system (3).

Furthermore,

$$(5) \quad \sigma(q, t) \equiv v(\theta - t) + q \text{ modulo } e^{-A(\theta-t)}N$$

(for all $q \in Q$, $t \in [0, \theta]$) can be used to determine a winning strategy σ from an admissible control for (3) (see (4)) and vice versa.

(Proof) This is an expansion of that of the First Reciprocity Theorem. First, assume that a stroboscopic strategy σ forces termination at time θ ,

$$Me^{A\theta}(x - \int_0^\theta e^{-At}(\sigma(q(t), t) - q(t))dt) = Mb.$$

Choose a fixed reference point $q_0 \in Q$, and define u by $u(t) = \sigma(q_0, t) - q_0$. Take arbitrarily a point $q \in Q$ and time $s \in [0, \theta]$; consider the admissible quarry control u with value q in $[0, s]$ and value q_0 in $(s, \theta]$: for $s = 0$ we obtain

$$(6) \quad Me^{A\theta}(x - \int_0^\theta e^{-At}u(t)dt) = Mb.$$

and for general s , after rearranging,

$$Me^{A\theta}(x - \int_0^\theta e^{-At}u(t)dt + \int_0^s e^{-At}(u(t) + q - \sigma(q, t))dt) = Mb.$$

Then

$$Me^{A\theta} \int_0^s e^{-At}(u(t) + q - \sigma(q, t))dt = 0$$

by subtraction; and, as s is free to vary over $[0, \theta]$,

$$Me^{A(\theta-t)}(u(t) + q - \sigma(q, t)) = 0 \text{ (almost all } t \text{)}.$$

Use the null space to reinterpret this as

$$(7) \quad u(t) + q \in \sigma(q, t) + e^{-A(\theta-t)}_N.$$

Now, σ has values in P ; using the Lemma from 3.1 and the Pontrjagin difference, we find that $v(t) = u(\theta-t)$ satisfies

$$v(t) \in (P + e^{-At}_N) * Q = V_t \text{ a.e.}$$

and is integrable together with u . Thus v is an admissible control for (3), and (6) yields

$$Me^{A\theta}x - M \int_0^\theta e^{As} v(s) ds = Mb,$$

so that (4) holds; finally, (7) provides (5).

Conversely, assume that (4) holds, and v is a control appropriate to the point in $R(\theta)$. Setting $u(t) = v(\theta-t)$ one obtains (6); and $v(t) \in V_t$ yields

$$u(t) + q \in P + e^{-A(\theta-t)}_N.$$

To construct the strategy σ apply Filippov's Lemma (3.9); there result measurability preserving mappings

$$\sigma: Q \times R^1 \rightarrow P, \quad v: Q \times R^1 \rightarrow N$$

such that

$$u(t) + q = \sigma(q, t) + e^{-A(\theta-t)}_v(q, t).$$

It now only remains to verify that σ is a winning strategy. For any quarry control $q(\cdot)$ we have $\sigma - q = e^{-A(\theta-t)}_v$, so that

$$\begin{aligned} & Me^{A\theta}(x - \int_0^\theta e^{-At}(\sigma(q(t), t) - q(t))dt) \\ &= Me^{A\theta}(x - \int_0^\theta e^{-At}u(t)dt) - Me^{A\theta}e^{-A\theta} \int_0^\theta v(q(t), t)dt \\ &= Mb + 0. \end{aligned} \quad \text{QED}$$

The theorem, though simple enough in principle, has quite complicated details. It seems appropriate to illustrate its workings on an elementary example.

Example We treat the game

$$\dot{x} = y - p, \dot{y} = q; \text{ end: } x = y$$

in R^2 , with both control values in $[-1,1]$. This resembles the reduced description of the one-dimensional rocket chase in 1.3 except for the target set.

A sketch of the players' vectograms for several points in R^2 suggests that points above the target $x = y$ can be forced upward: possibly away from the target initially, but ultimately toward it; and symmetrically below the target. Again this is reminiscent of rocket chase; but the resemblance ends there, see Exercise 3.

We have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M = (1, -1), b = 0; e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The constraint set of the associated control system is determined via Exercise 1:

$$(8) \quad Me^{At} \begin{pmatrix} x \\ y \end{pmatrix} = x + (t-1)y,$$

so that V_t consists of all points whose coordinates v_k satisfy

$$v_1 + (t-1)v_2 + (t-1)q = p \in [-1,1]$$

for all q , $|q| \leq 1$, i.e.,

$$|v_1 + (t-1)v_2| \leq 1 - |t-1|.$$

E.g.,

$$v_1 = v_2 \text{ for } t = 0, |v_1 - v_2/2| \leq 1/2 \text{ for } t = 1/2,$$

$$|v_1| \leq 1 \text{ for } t = 1, |v_1 + v_2/2| \leq 1/2 \text{ for } t = 3/2,$$

$$v_1 = -v_2 \text{ for } t = 2,$$

and $V_t = \emptyset$ for $t > 2$. Subsequent computations are simplified if we apply Exercise 2 (with $q_0 = 0$), and use only

$P \cap V_t$. These constraint sets are segments on the x-axis, symmetric about the origin; denoting the right endpoints by $\beta(t)$, we have

$$\beta(t) = t \text{ for } 0 \leq t \leq 1, \beta(t) = 2-t \text{ for } 1 \leq t \leq 2.$$

The associated control system may then be written as

$$\dot{x} = -y - \beta(t)w, \dot{y} = 0; |w(t)| \leq 1.$$

The reachable set $R(\theta)$ consists of all points

$$\int_0^\theta e^{At} \beta(t) \begin{pmatrix} w(t) \\ 0 \end{pmatrix} dt = \begin{pmatrix} \int_0^\theta \beta(t) w(t) dt \\ 0 \end{pmatrix}.$$

Thus $R(\theta)$ is a symmetric segment on the y-axis, with right endpoint $\gamma(t)$,

$$\gamma(t) = \begin{cases} \theta^2/2 & \text{for } 0 \leq \theta \leq 1 \\ 1-(\theta-2)^2/2 & \text{for } 1 \leq \theta \leq 2, \end{cases}$$

(obviously the extremal controls w are constants ± 1). The sets of winning positions from (4) are, using (8) and Exercise 1, determined by

$$(9) \quad |x + (\theta-1)y| \leq \gamma(\theta).$$

These are parallel strips, based on the segments $R(\theta)$:

$$\begin{aligned} x = y \text{ for } \theta = 0, \quad |x - y/2| &\leq \frac{1}{2} \text{ for } \theta = 1/2 \\ |x| &\leq \frac{1}{2} \text{ for } \theta = 1, \quad |x + y/2| \leq 7/8 \text{ for } \theta = 3/2 \\ |x + y| &\leq 1 \text{ for } \theta = 2. \end{aligned}$$

Exercises

1. Show that another description of the constraint set V_t in the theorem is that $v \in V_t$ iff

$$Me^{At} v \in (Me^{At} P) \# (Me^{At} Q);$$

and that (4) is equivalent to

$$Me^{A\theta} x \in MR(\theta) + Mb.$$

2. The constraint sets V_t need not be compact even if P is such. Verify that both parts of the proof carry through if V_t is replaced by the compact set

$$(P - q_0) \cap ((P + e^{-At}N) \# Q);$$

here q_0 is any fixed point in Q . (Hint: around (7), consider the set of values of u .)

3. In the example determine the set of all winning points, and decide whether it is open or closed. (Hint: find the enveloping curve to the parallel strips in (9). Partial answer: this comprises four arcs: the boundary of the set of winning positions consists of two curves, each being two half rays connected by a parabolic arc.)

4. Assume that the sets P, Q in (1) are nonvoid, compact, convex, and symmetric. Prove that the set of positions which can be forced to Ω stroboscopically at time θ is closed, convex, and symmetric about $e^{-A\theta}b$. (Hint: you will need Exercise 2.)

5. Obtain a version of the reciprocity theorem more closely related to that of 3.1 by treating the control system

$$\begin{aligned}\dot{x} &= Ax - u; u(t) \in U_\theta(t), \\ U_\theta(t) &= (P + e^{-A(\theta-t)}N) \# Q;\end{aligned}$$

note that this depends on the termination time θ as a further parameter. (Hint: in the proof of the theorem, use u rather than v . Partial answer: the control-theoretic condition on a winning position is $x \in R_\theta(t) + e^{-A\theta}\Omega$.)

6. Make appropriate simplifications in the preceding for the case that the null space N is invariant under A .

7. In the game (1) assume that the pursuer is allowed to choose a linear feedback component in his controls, replacing p by $Fx + u$ with, however, both values $Fx \in \text{span } P$ and $u(t) \in U \subset \text{span } P$. Show that the matrix F can be

chosen so as to make $N = \{x: Mx = 0\}$ invariant, under the new coefficient matrix $A - F$, iff $AN \subset N + \text{span } P$.

8. In 3.2 we treated a game induced by the one-dimensional equation $\ddot{x} + x = u - v$. Show that neither $x = 0$ nor $\dot{x} = 0$ yields an invariant target; apply the condition from the preceding exercise; if possible, design an appropriate feedback strategy component. (Answer: $F = \begin{pmatrix} 0 & 0 \\ -1 & \alpha \end{pmatrix}$.)

9. Generalise the reciprocity theorem to allonomous games, in the form treated in 3.1, Exercise 8, and with termination condition $Mx = Mb$. (Partial answer: in the approach of Exercise 6, the constraint set for the associated control system is

$$(B(t)U + X(t, \theta)N) \neq (C(t)V),$$

where $X(\cdot)$ is the fundamental matrix solution.)

Remarks

That the present formulation of the Second Reciprocity Theorem is preferable to the more involved one of Exercise 5, is due to E. Rechtschaffen (personal communication).

There are several dissimilarities with the First Reciprocity Theorem. First, the associated control system is not autonomous, since the constraint sets V_t may depend on time. It will be autonomous if A is a scalar matrix (e.g., $A = 0$, simple motion); or, more generally, if the null space N is invariant under A . Indeed, then $AN \subset N$, so that $e^{-At}N = N$ for all t , and

$$V_t = V = (P + N) \neq Q$$

(see Exercises 6-8, and also the exercise in 7.3 for further developments).

Second, it is possible to have V_t nonvoid for some t and empty for others, as actually happened in the example.

This will be examined in the next section. A less spectacular phenomenon is that even the dimension of the span of V_t may change discontinuously: in the illustration, the dimensions are, in turn, 0, 1, 0, -1.

Finally, the associated control system will steer one point to another, but, notwithstanding the reciprocity, a winning strategy does not force the initial position to a terminal point: it does force to the target set, but not to a single point in the target. This may be traced to the presence of the term w in (5), which may vary with the quarry control. More forcibly, however, the Necessary Condition from 3.1 for forcing to a singleton will usually fail in the games treated.

3.4 Necessary Conditions

We refer again to the game

$$(1) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q$$

with state space R^n , and target set Ω an affine manifold $\Omega = \{x: Mx = Mb\}$.

PROPOSITION In (1) let P be compact. A necessary and sufficient condition for presence of initial positions which can be forced to the target stroboscopically at time θ is that

$$(2) \quad (Me^{At}P) \neq (Me^{At}Q) \neq \emptyset \text{ whenever } 0 \leq t \leq \theta;$$

and a necessary condition is that

$$(3) \quad Me^{At}(Q-Q) \subset Me^{At}(P-P) \text{ whenever } 0 \leq t \leq \theta.$$

In case P is compact and convex, and both P, Q are symmetric, (2) may be replaced by

$$(4) \quad Me^{At}Q \subset Me^{At}P \text{ whenever } 0 \leq t \leq \theta.$$

(Proof) According to the Second Reciprocity Theorem of 3.3,

such positions exist iff $R(\theta) \neq \emptyset$, i.e.,

$$R(\theta) = \int_0^\theta e^{-At} V_t dt \neq \emptyset.$$

By Filippov's Lemma, this is equivalent to $V_t \neq \emptyset$ for almost all $t \in [0, \theta]$, and thence, for all $t \in [0, \theta]$.

(2) is now a reformulation of this (see Exercise 1 in 3.3), and the remaining assertions are then easily obtained (see 3.1, Necessary Condition and Corollary 1). QED

COROLLARY 1 In (1), if there exist positions which can be forced to target stroboscopically at one time θ , then there also exist such positions at all times $t \in [0, \theta]$.

The proposition involves the fundamental matrix solution e^{At} , so that the conditions are usually not directly verifiable; however, they are a source of working criteria.

COROLLARY 2 $M(Q-Q) \subset M(P-P)$ is necessary, and (if Q is compact) $MQ \subset \text{Int } MP$ sufficient, for presence of positions which can be forced to the target stroboscopically at strictly positive times.

(Proof) The first inclusion is the case $t = 0$ of (3). If the second holds, then, from compactness of MQ , there is a ball B about the origin such that $B + MQ \subset MP$. Since $t \mapsto e^{At}$ is continuous with value I at 0 , we will have $B_1 + Me^{At}Q \subset Me^{At}P$ for small $t > 0$ and a possibly smaller ball B_1 . Then $(Me^{At}P) \not\subset (Me^{At}Q) \supset B_1 \neq \emptyset$ for small $t > 0$, so that (2) holds. QED

A more precise result is presented below. The player control orders are defined in 6.1; under our assumptions, pursuer has control order k if $MA^j(P-P) = 0$ (the set consisting of the zero vector, not the empty set) holds for $j = 0, \dots, k-2$ but not for $j = k-1$.

NECESSARY CONDITION In (1) assume that P is compact. A necessary condition for presence of points which can be forced to termination stroboscopically at a strictly positive

time is that

(5) quarry control order \geq pursuer control order,

and, if k denotes the latter, that

$$(6) \quad MA^{k-1}(Q-Q) \subset MA^{k-1}(P-P).$$

(Proof) In (3) expand e^{At} into a power series. If k denotes the pursuer control order, then, for t in some interval $(0, \theta)$ with $\theta > 0$.

$$Me^{At}(Q-Q) \subset Me^{At}(P-P) = O(t^{k-1});$$

hence, by induction on $j = 0, \dots, k-2$, $MA^j(Q-Q) = 0$. This yields (5), and also (6) for $j = k-1$. QED

COROLLARY 3 Let P, Q be compact, convex, symmetric. If (5) holds and (6) is replaced by

$$(7) \quad MA^{k-1}Q \subset \text{Int } MA^{k-1}P,$$

then, for sufficiently small $\theta > 0$, the set of positions which can be forced to termination, stroboscopically at time θ , has nonvoid interior.

(Proof) Now we have

$$MA^jP = 0 = MA^jQ \quad \text{for } j = 0, \dots, k-2$$

by (5) and the assumptions on P, Q ; with (7), this ensures that $Me^{At}Q \subset \text{Int}(Me^{At}P)$ for small $t > 0$, again by power series expansion. It follows that the sets V_t have nonvoid interior (compactness used here). Hence so do the reachable sets $R(\theta)$, and, by the reciprocity theorem, the sets of winning positions also. QED

Example Consider the game with n -dimensional equations

$$\ddot{x} + A_1 \dot{x} + A_2 x = u, \quad \ddot{y} + B_1 \dot{y} + B_2 y = v;$$

$$u(t) \in U, \quad v(t) \in V; \quad \text{end: } x = y.$$

Both the pursuer's constraint set U and the quarry's V are

non-void, compact, convex, and symmetric.

The case of $A_2 = 0 = B_2$ is the linear analogue of the game of two cars (just as rocket chase is the linear version of the homicidal chauffeur game); also see 8.3. The formulation in terms of a first-order system, in R^{4n} , is simple: the result has

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ -A_2 & -A_1 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -B_2 & -B_1 \end{pmatrix} \quad P = \begin{pmatrix} 0 \\ U \\ 0 \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ V \end{pmatrix},$$

$$M = (I \ 0 \ -I \ 0).$$

The necessary condition of Corollary 2 is always satisfied, but no choice of the data will yield the sufficient condition (since both control orders are 2: thus, (5) holds). The containment relation (4) is

$$(8) \quad (I, 0) \exp \begin{pmatrix} 0 & I \\ -B_2 & -B_1 \end{pmatrix} t \begin{pmatrix} 0 \\ I \end{pmatrix} V \subset (I, 0) \exp \begin{pmatrix} 0 & I \\ -A_2 & -A_1 \end{pmatrix} t \begin{pmatrix} 0 \\ I \end{pmatrix} U.$$

Conditions (6) and (7) are $V \subset U$ and $V \subset \text{Int } U$ respectively (interior in R^n), independent of the coefficient matrices. Further developments appear in the exercises.

Exercises

1. For the n-dimensional rocket chase, verify that one of the necessary conditions fails.
2. In the example, write

$$C(t) = (I, 0) \exp \begin{pmatrix} 0 & I \\ -A_2 & -A_1 \end{pmatrix} t \begin{pmatrix} 0 \\ I \end{pmatrix},$$

for the matrix appearing in (12), and analogously for $D(t)$, with the matrices B_k . Re-interpret (8) as

$$V \subset D^{-1}(t)C(t)U$$

and obtain the first three terms of $D^{-1}C$. (Answer:

$$D^{-1}(t)C(t) = I + \frac{1}{2} (B_1 - A_1)t + \\ + \frac{1}{12} (B_1^2 + 2A_1^2 - 3A_1B_1 + 2(B_2 - A_2))t^2 + \dots)$$

3. In the preceding situation, assume that $V = \{x: x'Hx \leq \alpha\}$. Prove that (8) holds for small $t > 0$ (i.e., the condition from the lemma is satisfied) if $V \subset U$ and either

$$(A_1 - B_1)'H + H(A_1 - B_1)$$

is negative definite, or $A_1 = B_1$ and

$$(A_2 - B_2)'H + H(A_2 - B_2)$$

is negative definite.

4. Specify the preceding conditions in the case of the example, with both U, V balls about the origin.

5. Generalise the example to the following situation. The game is induced by differential equations

$$x^{(n)} + \sum_0^{n-1} A_k x^{(k)} = u, \quad y^{(m)} + \sum_0^{m-1} B_k y^{(k)} = v;$$

$$u(t) \in U, \quad v(t) \in V; \quad \text{end: } x = y$$

(all involving r -dimensional vectors, etc.). Apply Corollary 3, treating separately the cases $n > m$, $n = m$, $n < m$, and making suitable assumptions where appropriate.

3.5 General Targets

In this section we examine further consequences of adopting stroboscopic and isochronous capture. One is that the used portion of the target set Ω must be flat (Proposition 1); the implication is that, if the surface of Ω is well rounded, a successful strategy must force the state response well into Ω .

For a game of simple capture with several pursuers, the

target set obviously decomposes into linear subspaces. More generally, one might consider games whose target set is a countable union of simpler sets. A second result is that, in the context of stroboscopic and isochronous capture, this type of problem is irrelevant.

The game is

$$(1) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q$$

with R^n as state space.

PROPOSITION 1 If the player constraint sets are compact, then, for any initial position and stroboscopic strategy, the set of all state responses at each fixed end-time (to the various quarry choices) has convex closure.

(Proof) The set in question consists of all points

$$e^{At} \left(x - \int_0^t e^{-As} (\sigma(q(s), s) - q(s)) ds \right)$$

with $q(\cdot)$ varying and all other entries fixed. Apparently it is the image, under a parallel shift and nonsingular linear mapping (the effect of x and e^{At}), of the set of all points

$$\int_0^t e^{-As} (\sigma(q(s), s) - q(s)) ds. \quad \text{Each such point belongs to}$$

$$\int_0^t F_s, \quad \text{where}$$

$$F_s = \{ e^{-As} (\sigma(q, s) - q) : q \in Q \},$$

which has convex closure by Lemma 2 in §.10. Thus, to complete the proof, it is sufficient to verify the converse

inclusion. By definition, each point of $\int_0^t F_s$ is of the form $\int_0^t w(s) ds$ with measurable $s \mapsto w(s) \in F_s$; from Filippov's Lemma, there exists a measurable $s \mapsto q(s) \in Q$ such that

$$w(s) = e^{-As} (\sigma(q(s), s) - q(s))$$

(integrability follows from compactness of the constraint

sets). QED

COROLLARY In (1) assume that P, Q are compact, and that $Q - Q \not\subset P - P$. If the target set Ω contains no segments, then no position outside Ω can be forced to Ω stroboscopically and isochronously.

(Proof) By assumption, the only convex subsets of Ω are single points. From the Necessary Condition of 3.1, no initial position can be forced to a point target in positive game. QED

PROPOSITION 2 In game (1) assume that the constraint sets are compact, and that the closed target set $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ where each Ω_k is an intersection of sets Z which are analytic in the sense that

$$(2) \quad Z = \{x: \Phi(x) = 0\}, \quad \Phi: R^n \rightarrow R^1,$$

with $\Phi(x)$ analytic in the coordinates of $x \in R^n$. Then each initial position which is forced to Ω stroboscopically and isochronously is actually forced to an affine manifold entirely contained in one of the portions Ω_k of Ω .

(Proof) This will follow directly from Proposition 1 once we verify that, whenever a convex set C is contained in $\Omega = \bigcup_k \Omega_k$, the affine span of C is in some Ω_k .

Note that the affine span of $C \subset R^n$ is the affine span of at most $n+1$ points of C . We shall prove, by induction on m , that the affine span of any m points of C is entirely in some Ω_k . The assertion is true trivially for $m=0$ or 1 ; assume it holds for any m points, and consider $m+1$ points x_0, \dots, x_m in C . For a fixed $\alpha \in [0,1]$, the m points $\alpha x_j + (1-\alpha)x_0$ are in C ; by assumption, there exists an index k such that, for any t_1, \dots, t_m in R^1 with $\sum t_j = 1$, we have

$$(3) \quad \Omega_k \ni \sum t_j (\alpha x_j + (1-\alpha)x_0) = x_0 + \alpha \sum t_j (x_j - x_0).$$

Keeping the t_j variable, note that there are uncountably many values α but only countably many indices k . Thus (3) holds for some fixed k and infinitely many α 's (and, again, all $t_j \in R^1$ with $\sum t_j = 1$). Since the functions Φ which determine the sets Ω_k are analytic, (3) must hold for all $\alpha \in [0,1]$. Setting $\theta_0 = 1 - \alpha$, $\theta_j = \alpha t_j$ we find that, for the appropriate index k , $\sum_0^m \theta_j x_j \in \Omega_k$ for all θ_j with $\sum \theta_j = 1$ (the exceptional case $\theta_0 = 1$ is treated by an obvious limit argument). QED

COROLLARY In (1) assume that P, Q are compact and $Q - Q \not\subset P - P$. If the target set Ω is as in Proposition 2 but contains no straight lines (e.g., Ω is compact), then no point outside Ω can be forced to Ω stroboscopically and isochronously.

The remainder of this section is devoted to an elementary examination of affine subsets of analytic targets. The subject properly belongs to analytic geometry but, because of Proposition 2, it is of considerable interest for game theory. The very special case of quadratic hypersurfaces is treated in the exercises.

Consider a set $Z \subset R^n$ of the form (2), and any fixed point $a \in Z$. Expand Φ into a power series centered at a ; this converges absolutely, so one may group terms to obtain

$$\Phi(a + x) = \sum_{k=1}^{\infty} \Phi_k(x)$$

where $\Phi_k(x)$ is a k -ary form in the coordinates of $x \in R^n$ (we are suppressing notational dependence on a). Now, one may always write

$$(4) \quad \Phi_k(x) = \Psi_k(x, \dots, x)$$

where $\Psi_k(x_1, \dots, x_k)$ is a symmetric k -linear form in the k variables $x_i \in R^n$. In point of fact (omitting the indices k ; for details see the exercises),

$$(5) \quad \psi(x_1, \dots, x_k) = \frac{1}{k!} (\phi(x_1 + \dots + x_k) - \sum_i \phi(x_1 + \dots + \hat{x}_i + \dots + x_k) \\ + \sum_{i < j} \phi(x_1 + \dots + \hat{x}_i + \dots + \hat{x}_j + \dots + x_k) - \dots + (-1)^{k-1} \sum_l \phi(x_l))$$

(circumflexes indicating omitted terms).

LEMMA If L is a linear subspace of R^n , then the affine manifold $a + L$ is contained in Z iff

$$(6) \quad \psi_k(x_1, \dots, x_k) = 0 \text{ for all } k \text{ and } x_i \in L.$$

(Proof) First let $a + L \subset Z$; for any $x \in L$ and $t \in R^1$ we have $a + tx \in a + L \subset Z$, so that

$$0 = \phi(a + tx) = \sum_1^\infty t^k \phi_k(x).$$

But then all the coefficients $\phi_k(x) = 0$ whenever $x \in L$: thus (6) follows from (5).

Conversely, if (6) holds, then $\phi_k(x) = 0$ for all $x \in L$ from (4), so that $\phi(a + x) = 0$, i.e., $a + L \subset Z$. QED

As an illustration, let ϕ have degree 2, $\phi(x) = x'Hx + g'x + \phi_0$, and choose $a \in R^n$ with $\phi(a) = 0$; take H symmetric. Then

$$\phi(a + x) = (2a'H + g')x + x'Hx,$$

so that

$$\psi_1(x) = (2a'H + g')x, \quad \psi_2(x_1, x_2) = x_1'Hx_2.$$

Then a linear subspace L has $a + L \subset \phi^{-1}(0)$ iff

$$(2a'H + g')x = 0, \quad y'Hx = 0$$

for all $x, y \in L$. If, e.g., H is positive definite, this is possible only for $L = 0$.

We formalise part of this in the following definition: given a subset $\Omega \subset R^n$, an isotropic subspace of Ω is any affine manifold $a + L \subset \Omega$ which cannot be increased (i.e., such that the dimension of the linear subspace L is maximal).

In particular, every set Ω is the union of its isotropic subspaces; these need not be disjoint and may well be singletons.

Exercises

1. For n -dimensional rocket chase in reduced form,

$$\dot{x} = y - v, \dot{y} = u; \text{ end: } |x| = \epsilon$$

with quarry's constraint $v(t) \in V \neq 0$, and $\epsilon \geq 0$, prove that nontrivial isochronous and stroboscopic capture is impossible. (Hint: find the isotropic subspaces of the target set; apply, to each of these, the Necessary Condition from 3.4.)

2. Prove that the isotropic subspaces of $\bigcup_{k=1}^{\infty} \Omega_k$ are precisely the unions of isotropic subspaces of the Ω_k ; each isotropic subspace of $\Omega_1 \cap \Omega_2$ is the intersection of isotropic subspaces of Ω_k .

Exercises 3 to 6 establish formula (5), determining a k -linear ψ from its associated k -ary form ϕ , $\phi(x) = \psi(x, \dots, x)$. In any case, define ψ by (5).

3. Prove $\sum_{j=0}^k (-1)^j \binom{k}{j} (t-j)^k = k!$ for $k = 0, 1, \dots$ and complex t . (Hint: if $\Delta f(t) = f(t+1) - f(t)$ is the forward difference operator, and Δ is written as $E - I$, conclude a binomial formula for Δ^k , and then apply to $f(t) = t^k$, for which obviously $\Delta^k f = k!$)

4. Prove $\psi(x, \dots, x) = \phi(x)$. (Hint: previous exercise.)

5. Show that if $\phi(x)$ is a polynomial of degree $\leq k$ in the coordinates of $x \in \mathbb{R}^n$, and Δ_y is the forward difference operator with fixed step $y \in \mathbb{R}^n$, then $\Delta_y \phi(x)$ has degree $\leq k - 1$. Conclude that

$$\Delta_{x_1} \cdots \Delta_{x_{k-1}} \phi(x)$$

has degree ≤ 1 in the coordinates of x .

6. Complete the verification that $\psi(x_1, \dots, x_k)$ is a symmetric k -linear form.

7. Given the quadratic form $\Phi = \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 - \xi_5^2 + 0 \cdot \xi_6^2$ in R^6 , show that the rows of

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

span an isotropic subspace of $\Phi = 0$. (Do not omit verification of maximality.)

8. Extend to the form $\Phi = \sum_1^r \xi_j^2 - \sum_{r+1}^s \xi_k^2$ in R^n ($0 \leq r < s \leq n$). Conjecture a formula for the dimension of all isotropic subspaces, involving Φ only. (Partial answer: if Φ has full rank, the dimension is $(n - |\text{signature}|)/2$).

Remarks

The following converse to Exercise 8 can be proved for a nonsingular quadratic form Φ in n -space: any basis of an isotropic subspace of Φ can be augmented to a basis for R^n , relative to which Φ has the matrix

$$\begin{pmatrix} 0 & 0 & I \\ 0 & +I & 0 \\ I & 0 & 0 \end{pmatrix}$$

(if Φ is singular, there is an additional bordering by zero matrices).

The idea and details of Exercises 3 to 6 are due to Prof. L. Takács (private communication).

3.6 Time Delays

In a linear game consider the added complication that there are delays in the transmission of information: a time delay $\delta \geq 0$ in the pursuer's observation of quarry's action, and a time lag $\lambda \geq 0$ in the system reaction to pursuer's controls (it is convenient to treat these diverse effects

simultaneously). The first is adequately treated by the concept of the time-lag pursuer strategies, exercises in 2.2; for the stroboscopic strategies used here, this simplifies further, as suggested by $p(t) = \sigma(q(t - \delta), t)$. The second appears as a modification of the dynamical equation:

$$(1) \quad \dot{x} = Ax - p(t-\lambda) + q(t);$$

the remaining data are formally unchanged: players' control constraints $p(t) \in P$, $q(t) \in Q$, and an appropriate termination condition. Note that we are not allowing any time delays within the system itself (such as $\dot{x}(t) = Ax(t) + Bx(t-\theta) + \dots$); this would lead to problems of a quite different nature.

The formulation suggests that, if the game is played over an interval $[0, t]$, the pursuer is quite ineffective over the last interval $[t - \lambda, t]$, and cannot counter quarry's moves over a further interval of length δ . Referring to the variation-of-constants formula for the state response $x(t)$,

$$(2) \quad x(t) = e^{At} \left(x - \int_0^t e^{-As} (p(s-\lambda) - q(s)) ds \right),$$

the pursuer has no influence on the component

$$(3) \quad e^{At} \int_{t-\lambda-\delta}^t e^{-As} q(s) ds = \int_0^{\lambda+\delta} e^{As} q(t-s) ds.$$

As $q(\cdot)$ varies over the quarry controls, the points (3) make up the set

$$(4) \quad \Delta = \int_0^{\lambda+\delta} e^{As} Q ds,$$

independent of initial position and terminal time.

Now, if pursuer aims at a target set Ω , he can ensure at least the outcome $\Omega + \Delta$; or, if the outcome Ω is strictly required, pursuer may aim at the auxiliary target $\Omega \overset{*}{=} \Delta$, since then he can ensure the outcome

$$(\Omega \overset{*}{=} \Delta) + \Delta \subset \Omega.$$

As will be seen immediately, this quite plausible

reasoning does provide a moderately successful result. But it does not resolve the problem completely (the proposition provides a sufficient condition for winning the game, but not a necessary one); and obviously it is irrelevant if there is a constant winning pursuer strategy.

We refer to (1), data including $\lambda \geq 0$, $\delta \geq 0$, and (4).

PROPOSITION Assume that the player constraint sets are compact and contain 0. Then pursuer can force an initial position to target $\Omega + \Delta$ at time $t \geq 0$ if, within the control system

$$(5) \quad \dot{x} = Ax - u, \quad u(t) \in U = P \stackrel{*}{\subseteq} e^{A(\lambda+\delta)} Q,$$

x can be steered to Ω at t by an admissible control $u(\cdot)$. Furthermore, there is a stroboscopic winning strategy σ with delay δ ,

$$(6) \quad \sigma(q, s+\delta) = u(s+\lambda+\delta) + e^{A(\lambda+\delta)} q$$

for $q \in Q$, $0 \leq s + \lambda + \delta \leq t$.

(Proof) Since u steers x to Ω at t within the control system,

$$e^{At} \left(x - \int_0^t e^{-As} u(s) ds \right) \in \Omega.$$

Abbreviate $\lambda + \delta = \theta$; for any quarry control $q(\cdot)$ use (6) to define a pursuer response p ,

$$p(s) = \sigma(q(s - \delta), s) = u(s + \lambda) + e^{A\theta} q(s - \delta)$$

(interpreting $q(s) = 0$ for $s < 0$). Then the corresponding state response (2) has

$$\begin{aligned} x(t) &= e^{At} \left(x - \int_0^t e^{-As} p(s-\lambda) ds + \int_0^{t-\theta} e^{-As} q(s) ds + e^{At} \int_{t-\theta}^t e^{-As} q(s) ds \right) \\ &= e^{At} \left(x - \int_0^t e^{-As} u(s) ds \right) + \int_0^\theta e^{As} q(t-s) ds \in \Omega + \Delta. \quad \text{QED} \end{aligned}$$

If P and Q are convex and symmetric about 0, then a

necessary and sufficient condition for the constraint set U of the control system (5) to be non-void is that

$$(7) \quad e^{A(\lambda+\delta)} Q \subset P.$$

Neither of the time delays actually involves the state variable directly; this makes it possible to avoid the difficulties inherent in differential equations with time lag. Note, however, that a quarry control on $[0, t]$ determines the pursuer response on $[-\lambda, t - \lambda]$.

If quarry is ineffective, $Q = 0$, then $\Delta = 0$ also: paradoxically, a pursuer control forces an error-less outcome a time λ into the future.

Example Consider the game, in R^2 , with dynamical equation

$$\dot{x} = y + q, \quad \dot{y} = -x + p,$$

and player constraints $|p(t)| \leq 2$, $|q(t)| \leq 1$ (without the quarry constraint this would be related to $\ddot{x} + x = p + \dot{q}$). It follows from Corollary 1 in 3.1 that the origin of R^2 is not a possible target for stroboscopic and isochronous capture; it will turn out that non-stroboscopic isochronous capture, and even general capture, are also ruled out.

Assume that, nevertheless, it is at least desired to force the state variable close to the origin. Retracing our steps, the reason that stroboscopic and isochronous capture fails is that the necessary condition $Q \subset P$ is not satisfied: P is a segment on the x -axis, and Q one on the y -axis. Now, they could be aligned by a rotation by a right angle; and one naturally thinks of condition (7).

Introduce an artificial time lag δ in the quarry \rightarrow pursuer leg of the information pattern (keeping $\lambda = 0$) in such a way that $e^{A\delta} Q \subset P$. A simple computation yields $\delta = \pi/2$ (or, all odd multiples); and then

$$U = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} : |u| \leq 1 \right\},$$

with $\ddot{x} + x = u$ as the associated control system (5). It is easily established that all initial position can be steered to 0 within this control system. Thus, returning to our game, all positions can be forced to the set $0 + \Delta$ stroboscopically (with time lag $\pi/2$) and isochronously. Another simple computation describes the error term Δ , as the reachable set $R(\pi/2)$ of the associated control system: a lens-shaped set bounded by arcs of two circles, with radius 2 and centres at $\pm (1 + i)$.

1. (Stirred tank) A hopper feeds material to a long conveyor, which delivers the material into a large tank; solvent is continually added, and an equal volume of solution led off; the concentration is measured, as a guide for adjusting the hopper feed rate. Taking account of the time lag, and small system errors, obtain the dynamical equation of the system; use the variables suggested in the fig. 1. In terms of the chosen parameters, determine the error term. (Partial answer: $\dot{c} = kc + \frac{1}{V} a(t - t_0) + \epsilon(t)$.)

2. Consider external control u of a harmonic oscillator subject to small but unpredictable perturbations v , $\ddot{x} + x = u + v$, with constraints $|u(t)| \leq \alpha$, $|v(t)| \leq \beta$ and $\alpha > \beta \geq 0$. Given that the operator has time-delay $\delta \in (0, \pi]$ in reacting to the perturbations, how small can $x(t)$ be made? Assume $\delta \leq \pi$. (Hint: use the artificial time delay π ; answer: $|x(t)| \leq 2\beta$.)

3. Prove an analogue of the proposition, with a time-dependent constraint set in the control system: $U_t = P$ for $0 \leq t \leq \lambda + \delta$, $U_t = P \ast e^{A(\lambda+\delta)} Q$ for $t > \lambda + \delta$.

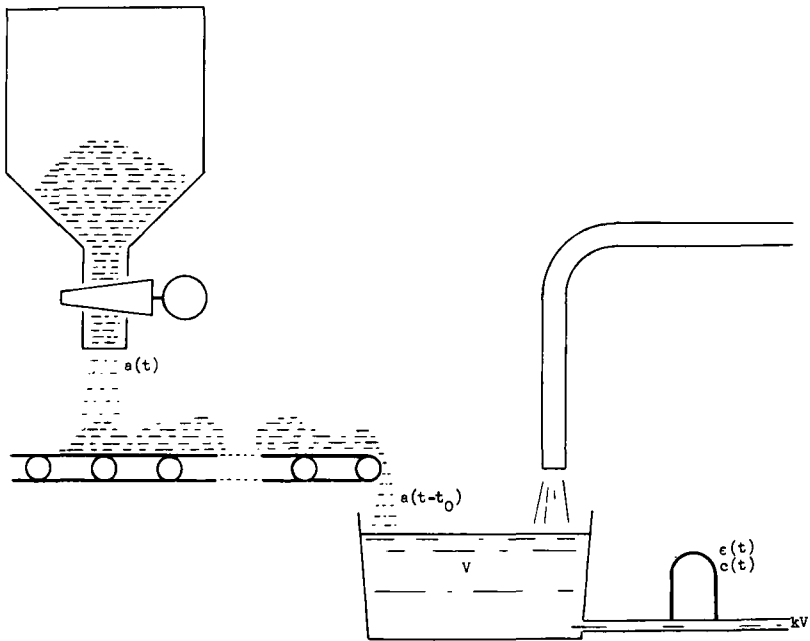


Fig. 1 Stirred Tank: control with time lag against system errors.

Remarks

The material is based on

O. Hájek: Pursuit games with delays, Funkcialaj Ekvacioj (to appear).

3.7 Holding

Suppose that one of the players in a game wishes to maintain some favourable configuration over an interval of time; pausing to catch one's breath is a pleasing but irrelevant interpretation. We modify slightly (but see Exercise 1), and replace 'maintaining state configuration' by the notion of holding within some (possibly auxiliary) target set.

Explicitly, a point $x \in \Omega$ is held within Ω by a strategy if the state response endpoints satisfy $x(t) \in \Omega$ for

all $t \geq 0$ and all quarry controls.

Capture with holding will consist, first, of forcing the state variable to the target within a bounded time interval $[0, \theta]$, and of subsequent holding within the target at all times after capture. Note that nothing changes if we require isochronous capture at time θ , and holding for all $t \geq \theta$. If stroboscopic strategies are to be used, then the results of 3.5 suggest that the affine manifolds are appropriate as targets.

Thus, consider the game in R^n ,

$$(1) \quad \dot{x} = Ax - p + q; p(t) \in P, q(t) \in Q; \text{ end: } Mx = Mb,$$

where M and b are given, and the target Ω is the indicated affine manifold. Anticipating the result below, we introduce the associated control system

$$(2) \quad \dot{y} = Ay - u; u(t) \in U; \text{ end: } Mx = Mb,$$

where A, M, b are as above, and

$$(3) \quad U = (P + L) \overset{*}{\cap} Q, L = \{x: MA^j x = 0 \text{ for } j = 0, 1, \dots\}.$$

Thus L is the largest A -invariant subspace of the null space $N = \{x: Mx = 0\}$ of M (cf. 6.3, Exercises 16 and 17); the obvious property $AL \subset L$ has the consequence that $e^{At}L = L$ for all $t \in R^1$.

THIRD RECIPROCITY THEOREM If pursuer's constraint set is compact, then the holding problems for the game (1) and the control system (2) are equivalent.

In detail, a position $x \in R^n$ can be forced to Ω within (1) at time $\theta \geq 0$ and held there subsequently, by a stroboscopic strategy σ , if, and only if, x can be steered to Ω within (2) at time θ and held there subsequently, by an admissible control u .

Moreover, the relation

$$(4) \quad \sigma(q, t) \equiv u(t) + q \text{ modulo } L$$

may be used to determine either of σ, u from the other.

(Proof) The techniques from the two reciprocity theorems will be put to a yet further use.

First, let σ be a successful strategy; select $q_0 \in Q$, and set $u(t) = \sigma(q_0, t) - q_0$; choose arbitrarily $\lambda \geq \theta$, $q \in Q$, $s \in [0, \lambda]$; and consider the piecewise constant quarry control with values q_0 in $[0, s]$, q in $(s, +\infty)$. The state response has $x(\lambda) \in \Omega$,

$$(5) \quad Me^{A\lambda} \left(x - \int_0^\lambda e^{-At} u(t) dt + \int_0^s e^{-At} (u(t) + q - \sigma(q, t)) dt \right) = Mb.$$

On subtracting the version of (5) with $s = 0$, we obtain

$$(6) \quad Me^{A\lambda} \int_0^s e^{-At} (u(t) + q - \sigma(q, t)) dt = 0.$$

Here one may keep s fixed while varying λ (the restrictions are $\lambda \geq \theta$, $\lambda \geq s$ only). By analyticity of $\lambda \mapsto e^{A\lambda}$, (6) holds for all $\lambda \in \mathbb{R}^1$ (and $s \geq 0$); therefore the integral term in (6) satisfies

$$\int_0^s \in \cap \{e^{-A\lambda} N : \lambda \in \mathbb{R}^1\} = L,$$

and thus the integrand

$$u(t) + q - \sigma(q, t) \in e^{At} L = L \text{ a.e.}$$

for all $q \in Q$. An application of the lemma from 3.1 yields (4); since q was arbitrary and σ has values in P ,

$$u(t) \in (P + L) \stackrel{*}{=} Q = U;$$

finally, (5) with $s = 0$ shows that $u(\cdot)$ steers x to the point $x(\lambda) \in \Omega$ for all $\lambda \geq \theta$.

Second, let u be an appropriate control within (2). Use (4) and Filippov's Lemma from 3.9 to define a stroboscopic strategy σ with values in P (by definition of U), and an auxiliary mapping w with values in L . Then the assumption

$$Me^{A\lambda}(x - \int_0^\lambda e^{-At}u(t)dt) = Mb \text{ for } \lambda \geq \theta$$

easily yields that σ is a successful strategy:

$$\begin{aligned} Me^{A\lambda}(x - \int_0^\lambda e^{-At}(\sigma(q(t), t) - q(t))dt) \\ = Mb + Me^{A\lambda} \int_0^\lambda e^{-At}w(\cdot)dt = Mb + 0. \quad \text{QED} \end{aligned}$$

LEMMA Assume that the data in (1) satisfy the following three requirements:

The target set $\Omega = \{x: Mx = 0\}$ with M of type (m, n) and rank m ;

P, Q are compact and both contained in an m -dimensional subspace $\{By: y \in R^m\}$ for an (n, m) matrix B ;

$\det M(\lambda I - A)^{-1}B$ does not vanish identically (e.g., $\det MB \neq 0$). In this situation, if a point $x \in \Omega$ can be held within Ω by a strategy σ , then σ is necessarily stroboscopic and of the form

$$(7) \quad \sigma[q](t) = q(t) + U(t)x \text{ a.e.}$$

with $U(\cdot)$ independent of x and q .

(Proof) Let σ be a strategy holding the point x_0 within Ω . The state responses $x(\cdot)$ will, therefore, have $Mx(t) = 0$ for all $t \geq 0$, so that their Laplace transforms $\hat{x}(s)$ satisfy

$$(8) \quad M\hat{x}(s) = 0.$$

From the differential equation,

$$(9) \quad s\hat{x}(s) - x_0 = A\hat{x}(s) - \hat{\sigma}[q](s) + \hat{q}(s)$$

(note that $q, \sigma[q]$ are measurable and bounded, so they do have Laplace transforms). We solve (9) for \hat{x} and replace in (8) (omitting most entries s):

$$\begin{aligned} M(sI - A)^{-1}(-\hat{\sigma} + \hat{q} + x_0) &= 0, \\ M(sI - A)^{-1}\hat{\sigma} &= M(sI - A)^{-1}\hat{q} + M(sI - A)^{-1}x_0. \end{aligned}$$

According to the second assumption one may write $\hat{\sigma} = B\hat{u}$,

$\hat{q} = B\hat{v}$, and obtain

$$(10) \quad M(sI - A)^{-1}B\hat{u} = M(sI - A)^{-1}B\hat{v} + M(sI - A)^{-1}x_0.$$

From the last assumption, the m -square matrix $M(sI - A)^{-1}B$ (entries are rational functions of s) has an inverse at least for large $|s|$, so that

$$\hat{u} = \hat{v} + (M(sI - A)^{-1}B)^{-1}M(sI - A)^{-1}x_0$$

which, pre-multiplied by B , yields

$$\hat{q} = \hat{q} + \hat{U}(s)x_0$$

with the incidental information

$$\hat{U}(s) = B(M(sI - A)^{-1}B)^{-1}(sI - A)^{-1}.$$

Inverse Laplace transforms now provide (7); $U(t)x$ is a function (rather than an operator) since both $\sigma[q], q$ are such.

QED

Exercises

1. In (1) show that holding a point $x \in \Omega$ within Ω over an interval $[0, \theta]$ with $\theta > 0$ is equivalent to holding over $[0, +\infty)$ entire. (Hint: the point is analyticity in the argument following (6).)

2. In the lemma, assume that P is one-dimensional, $P = \{bu: -1 \leq u \leq 1\}$, for a vector $b \in R^n$ such that $\text{rank}(A^{n-1}b, \dots, Ab, b) = n$. Prove that σ is, again, linear in x . (Hints: verify that $m'(sI - A)^{-1}b \neq 0$ for some row m' of $M \neq 0$, and solve (10) for that row.)

Remarks

In connection with the control-theoretic side, see references under 'core of target' in

E. B. Lee, L. Markus: Foundations of Optimal Control Theory, Wiley, New York, etc., 1967;

and also

O. Hájek: Cores of targets in linear control system,
Math. Systems Theory 8 (1974) 203-206.

It would be desirable to simplify the last assumption of the lemma, that $\det M(\lambda I - A)^{-1}B \neq 0$, at the very least for the observable case: an observed control system in R^n ,

$$\dot{x} = Ax - u, \quad \xi = Mx; \quad u(t) \in U,$$

is said to be observable if the partitioned (nm, n) matrix

$$\begin{pmatrix} M \\ MA \\ \dots \\ MA^{n-1} \end{pmatrix}$$

has maximal rank (i.e., n ; the condition is independent of U). From (3) it follows that observability is equivalent to $L = 0$. Compactness of the constraint sets may be omitted if, in the proof, one uses Mikusiński's operators.

A result analogous to the theorem holds for strategies which have the added property that $\sigma[q]$ is piecewise continuous whenever q is such

E. Rechtschaffen: Equivalences between differential games and optimal controls (preprint),
Pontificia Univ. Católica do Rio de Janeiro
(1975).

3.8 Convex Sets, Pontrjagin Difference

A set S in R^n is convex if it contains the entire segment between any two of its points,

$$\alpha x + (1-\alpha)y \in S \quad \text{whenever } x, y \in S, \quad 0 \leq \alpha \leq 1.$$

Given a triple S, a, c consisting of a set $S \subset R^n$, a point $a \in R^n$, and a vector $c \in R^n$, we shall say that c is an exterior normal to S at a if

$$(1) \quad c'x \leq c'a \text{ for all } x \in S.$$

The geometry of this is that the vector $x - a$ bears an obtuse angle to c (the cosine being nonpositive); and also that the 'supporting' hyperplane $\{x: c'x = c'a\}$ through a with normal c determines two half-spaces, of which $\{x: c'x \leq c'a\}$ contains the set S . In case S is a linear subspace, (1) implies that $c'tx \leq c'a$ for all $t \in \mathbb{R}^1$ and $x \in S$, so that the only possibility is $c'x = 0$: the concept then coincides with that of the orthogonal complement $S^\perp = \{y \in \mathbb{R}^n: y'x = 0 \text{ for all } x \in S\}$.

LEMMA 1 If a point b is not in a convex, closed, nonvoid set S , and $a \in S$ is the point closest to b , then the vector $c = b - a \neq 0$ is an exterior normal to S at a . Moreover, the corresponding supporting hyperplane separates S and b : in addition to (1), $c'a < c'b$.

(Proof) Convexity is not needed for the existence of a point $a \in S$ closest to b (but it does imply uniqueness). Take any $x \in S$; then, for $\alpha \in [0, 1]$, the point $\alpha x + (1-\alpha)a \in S$, so it is no closer to b than a :

$$|\alpha x + (1-\alpha)a - b| \geq |a - b|.$$

Rearrange and take squares,

$$2\alpha(a - b)'(x - a) + \alpha^2|x - a|^2 \geq 0;$$

divide by $2\alpha > 0$ and take $\alpha \rightarrow 0$,

$$(a - b)'(x - a) \geq 0.$$

Thus indeed $c = b - a$ satisfies (1). That $c \neq 0$ follows from $b \notin \bar{S} = S \ni a$, and then $|c|^2 > 0$ yields $c'b > c'a$.

QED

COROLLARY 1 Every closed, convex set in \mathbb{R}^n is the intersection of half-spaces $\{x: c'x \leq \alpha\}$.

(Proof) Excluding the trivial cases of \emptyset or \mathbb{R}^n , Lemma 1

shows that there exist half-spaces H containing S , and also that

$$S \subset \cap \{H: \text{half-space } H \supset S\}$$

is actually an equality. QED

By separability of R^n , one may always take the intersection countable. The polyhedra are the finite intersections of half-spaces.

COROLLARY 2 If a set-valued mapping $t \mapsto S_t$ has all values in a closed, convex set C , then

$$\frac{1}{t} \int_0^t S_\theta d\theta \subset C \quad \text{for } t > 0,$$

$$\frac{2}{t^2} \int_0^t \int_0^s S_\theta d\theta ds \subset C, \text{ etc.}$$

(Proof) By Corollary 1, it suffices to prove that the mean values are in the half-space $H = \{x: c'x \leq \alpha\}$ whenever $H \supset C$. This is an elementary exercise in integration: if $c'f(\theta) \leq \alpha$ a.e., then

$$c' \frac{1}{t} \int_0^t f(\theta) d\theta = \frac{1}{t} \int_0^t c'f(\theta) d\theta \leq \frac{1}{t} \int_0^t \alpha d\theta = \alpha;$$

analogously for the second moments $\frac{2}{t^2} \int_0^t \int_0^\delta S_\theta$. QED

LEMMA 2 Let there be given a nonvoid, convex set S . At every boundary point, S has a nontrivial exterior normal. If S is also compact, then every vector in R^n is an exterior normal to S , at some point of S .

(Proof) Consider any point $a \in \partial S$. Then there exists points $b_k \rightarrow a$ with $b_k \notin \bar{S}$; according to Lemma 1, the points $a_k \in S$ nearest to b_k have exterior normals $c_k = b_k - a_k \neq 0$. Now, also $a_k \rightarrow a$, since

$$|a_k - a| \leq |a_k - b_k| + |b_k - a| \leq |a - b_k| + |b_k - a| \rightarrow 0.$$

The unit vectors $d_k = c_k/|c_k|$ are also exterior normals to

S at a_k ,

$$(2) \quad d_k'x \leq d_k'a_k, \quad \text{for all } x \in S,$$

and, since they are on the compact unit sphere $\{x: |x| = 1\}$, they have a subsequence converging to some vector $d \neq 0$. On taking limits in (2) for fixed $x \in S$ we find that indeed

$$d'x \leq d'a \quad \text{for all } x \in S.$$

For the second assertion, note that the continuous real-valued function $x \mapsto c'x$, when restricted to the compact set S , attains its maximum at some point $a \in S$: thus (1) holds. QED

PROPOSITION 1 For subsets of R^n ,

$$A + X \supset B + X \quad \text{implies} \quad A \supset B$$

if A is closed and convex, and X compact, convex, nonvoid.

(Proof) Proceed by contradiction: the conclusion fails, and some point $b \in B$ is not in A . By Lemma 1, the point $a \in A$ closest to b yields $c = b - a$ as non-zero exterior normal to A at a . By Lemma 2, this c is an exterior normal to X at some point $x \in X$. Since $A + X \supset B + X$, we have $b + x = a_1 + x_1$ for some $a_1 \in A$ and $x_1 \in X$; thus

$$\begin{aligned} c'b &= c'a_1 + c'x_1 - c'x \leq c'a + 0, \\ 0 &\leq c'(a-b) = -|c|^2, \end{aligned}$$

contradicting $c \neq 0$. QED

For any set S in R^n there exists a least convex set containing S . It can be obtained either as the intersection of all convex sets C containing S , or as the set of all convex combinations of points in S ,

$$(3) \quad \left\{ \sum_{k=1}^r \alpha_k y_k : y_k \in S, \alpha_k \geq 0, \sum_{k=1}^r \alpha_k = 1, r = 1, 2, \dots \right\}.$$

The usual term is convex span (or hull) of S ; our notation will be $\text{cvx } S$.

If S is finite, $\text{conv } S$ is called a polytope; it can be shown that the polytopes are precisely the compact polyhedra. Obviously one cannot restrict $r \leq n$ in (3) (consider, e.g., the case $n = 2$). By Carathéodory's Theorem one may, however, always take $r = n + 1$; a consequence is that the convex span of a compact set is again compact (while, e.g., the convex span of the bell-shaped curve $y = 1/(1+x^2)$ in R^2 is not a closed set).

A point a of a convex set S is extreme if it is not an interior point of any true segment in S : $a = \alpha x + (1-\alpha)y$ with x, y in S and $0 < \alpha < 1$ implies that $x = a = y$. By the Kreĭn-Milman Theorem, every compact, convex set is the convex span of its extreme points.

If $\alpha_k \geq 0$ is not required in (3), the result is the affine manifold generated by S (and if $\sum \alpha_k = 1$ is also omitted, one obtains the linear span of, or linear space generated by, S). Each convex set has non-void interior within the generated affine manifold.

The reader is referred to many excellent books on convex sets, e.g.

H. G. Eggleston: Convexity, Cambridge University Press, New York, London, 1958;

B. Grunbaum: Convex Polytopes, Interscience, London, etc., 1967;

T. Bonnensen, W. Fenschel: Theorie der Konvexen Körper, Springer, Berlin, 1934.

The second topic of this section is the Pontrjagin difference operation. Consider the problem of cutting out a shaped profile by a milling machine with cutter head diameter 2δ ; it is desired to determine the positions of the cutter head centre, admitting that certain moderately sharp inner corners will not be shaped perfectly. If S is the set in R^2 to be milled off, and D denotes $\{d \in R^2: |d| \leq \delta\}$, then

the cutter head positions c must have $c + d \in S$ for all $d \in D$:

$$C = \{c \in R^2: c + D \subset S\}$$

A closely related problem is that of finding the curve at a distance δ from a given curve (above, the boundary of S).

Generalising only slightly, given two subsets P, Q in R^n , their Pontrjagin difference is

$$(4) \quad P \# Q = \{x: x + Q \subset P\},$$

the largest set X with $X + Q \subset P$. Note that here the containment may be proper, as in fig. 1. By inspection one obtains the alternative description

$$(5) \quad P \# Q = \bigcap_{q \in Q} (P - q).$$

Thus $P \# Q$ inherits various properties from P : if P is closed, or compact, convex, an affine manifold, then so is $P \# Q$. From (3), $P \# Q$ is symmetric if both P and Q are; thus we have

LEMMA 3 Assume that P, Q are convex, symmetric, and nonvoid; the $P \# Q$ is nonvoid if, and only if, $Q \subset P$.
(Proof: a convex symmetric set is nonvoid iff it contains 0.)

The following two results justify the difference notation.

PROPOSITION 2 (on Cancellation) For subsets of R^n we have

$$(6) \quad (P + X) \# (Q + X) = P \# Q$$

if P is closed, convex and X is compact, convex, and nonvoid.

(Proof) Referring to (4), equivalence is asserted for the two conditions

$$x + Q + X \subset P + X, \quad x + Q \subset P.$$

One implication is immediate; the other follows from Proposition 1. QED

COROLLARY 3 If U is closed, convex, and Q is compact, convex, and nonvoid, then

$$(7) \quad (U + Q) \overset{*}{=} Q = U.$$

Thus, if Q is a direct summand of P , in the sense that $P = U + Q$ for some closed, convex set U , then necessarily $U = P \overset{*}{=} Q$.

(Proof) (7) is a special case of (6), with O as one of the sets. The second assertion follows as $P \overset{*}{=} Q = (U + Q) \overset{*}{=} Q = U$. QED

A classical concept in convex set theory is that of asymptotic directions: a vector $c \in R^n$ is an asymptotic direction of a convex set S if the ray

$$x + R^+c = \{x + \theta c : \theta \geq 0\}$$

lies entirely in S . Note that necessarily $x \in S$; if S is closed (or open), then the condition is independent of the end point x , and S is compact iff O is its only asymptotic direction (or $S = \emptyset$).

PROPOSITION 3 If P is closed and convex, then $P \overset{*}{=} P$ is a closed wedge (W , with $W = \overline{W} = \text{cvx } W = R^+W$), and coincides with the set of asymptotic directions of P .

If P is closed, convex, and symmetric, then $P \overset{*}{=} P$ is the largest linear subspace L with $P + L \subset P$; moreover,

$$P = (P \overset{*}{=} P) + Q$$

with Q compact, convex, and symmetric.

(Proof) $P \overset{*}{=} P$ is closed and convex since P is such; if x, y are in $P \overset{*}{=} P$ then so is $x + y$:

$$(x + y) + P = x + (y + P) \subset x + P \subset P;$$

thus $P \overset{*}{=} P$ is a closed wedge. Each element of $P \overset{*}{=} P$ is an

asymptotic direction:

$$P + R^+(P \# P) = P + (P \# P) \subset P.$$

Conversely, if c is an asymptotic direction, then

$$P + R^+c \subset P, \text{ so that } c \in P \# P.$$

If P is, in addition, symmetric, then so is $P \# P$; thus $P \# P$ is a linear subspace, L , the largest with $P + L \subset P$. Set $Q = P \cap L^\perp$, so that Q is closed, convex, and symmetric; we propose to show that $P = L + Q$ and that Q is compact.

Obviously

$$L + Q \subset (P \# P) + P \subset P.$$

Conversely, every $x \in R^n$ can be written as $y + z$ with $y \in L$ and $z \in L^\perp$; here, if $x \in P$, then $z = y - x \in (P \# P) + P \subset P$, and so $z \in L^\perp \cap P = Q$. Finally, consider any asymptotic direction c of Q (we wish to show that $c = 0$). Then

$$0 + R^+c \subset Q \subset P,$$

c is an asymptotic direction of P also, $c \in P \# P = L$. Since simultaneously $c \in L^\perp$, this yields $c = 0$. QED

Exercises

1. The relation defined by the concept of exterior normal is a closed relation: show that, if $S \subset \limsup S_k$, $a_k \rightarrow a$, $c_k \rightarrow c$, and c_k is an exterior normal to S_k at a_k , then c is an exterior normal to S at a .

2. Prove that the exterior normals to S at a form a closed wedge.

3. Prove that the set of points of S at which c is an exterior normal forms a closed, convex set contained in ∂S and perpendicular to c . Then show that c is an exterior normal to S at all points of $c^\perp \cap \partial S$.

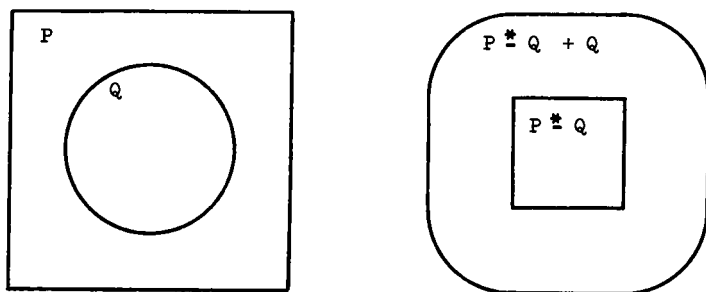


Fig. 1 Pontrjagin difference $P \# Q$ of sets P and Q ; the square with rounded corners is $(P \# Q) + Q$, properly contained in P .

4. Given a compact, convex set S , assume that either there is a unique exterior normal at each boundary point (S is smooth) or each supporting hyperplane intersects S at one point only (S is strictly convex). Prove that the correspondence between boundary points of S and unit exterior normals is continuous.

5. Show that asymptotic directions are independent of end-point: if S is closed and convex, $x + R^+c \subset S$ and $y \in S$, then also $y + R^+c \subset S$. (Hint: refer to fig. 2; obtain any $y + \theta c$ as the limit of convex combinations of points y and $x + tc$.)

6. Prove that a closed, convex, nonvoid set is compact iff 0 is its only asymptotic direction. (Hint: if one can find points $x_k \rightarrow \infty$ in the set, then any limit of their unit directions is an asymptotic direction.)

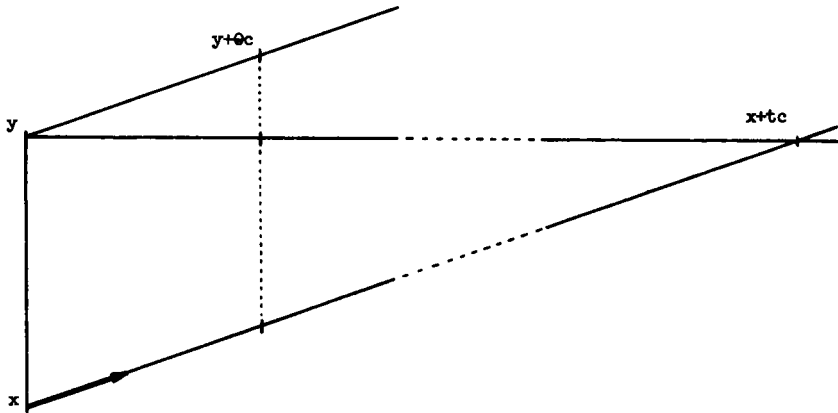


Fig. 2 Asymptotic directions are independent of end-point (Exercise 5).

7. Prove that

$$P \# (Q_1 + Q_2) = (P \# Q_1) \# Q_2.$$

8. Prove that

$$(P_1 + P_2) \# Q \supset P_1 + (P_2 \# Q),$$

and obtain simple counterexamples to equality.

9. Show that

$$(P + \alpha(P \# Q)) \# Q = (\alpha + 1)(P \# Q)$$

for compact, convex sets P, Q and $\alpha \geq 0$.

10. Show that the set $P \# Q$ is unchanged if P and Q are subjected to a translation $x \mapsto x + a$.

In exercises 11 and 12, the (m, n) matrix T is also interpreted as the linear mapping $x \mapsto Tx$.

11. For subsets P and $Q \neq \emptyset$ of R^n show that

$$(TP) \# (TQ) = T(P + \text{null } T) \# Q;$$

hence,

$$(TP) \dot{+} (TQ) = T(P \dot{+} Q)$$

if T is one-to-one, $\text{rank } T = n$. (Hint: if $x + TQ \subset TP$, then $P \not\vdash \emptyset$ and $x \in \text{range } T$.)

12. For subsets P, Q of R^m show that

$$T^{-1}(P \dot{+} Q) = T^{-1}(P) \dot{+} T^{-1}(Q \cap \text{range } T);$$

hence,

$$T^{-1}(P \dot{+} Q) = T^{-1}(P) \dot{+} T^{-1}(Q)$$

if T maps onto, $\text{rank } T = m$.

13. Assume that P is a linear subspace, and Q convex and symmetric. Prove that $P \dot{+} Q$ is P if $Q \subset P$, and \emptyset if not.

14. Show that, if P is a polyhedron or a polytope, then $P \dot{+} Q$ also has the corresponding property, without any assumption on Q .

15. This exercise consists of verifying the series of assertions below.

For the collection of compact, convex subsets of R^n , addition is a commutative and associative operation; thus one has an Abelian semigroup, with zero element O . The operation is also cancellative according to Proposition 2; thus there is a corresponding 'subtraction group'. The elements of this are pairs (P, Q) of compact, convex sets in R^n , and the pairs (P, Q) and $(P + U, Q + U)$ are equivalent; more generally

$$(8) \quad (P_1, Q_1) \sim (P_2, Q_2) \text{ iff } P_1 + Q_2 = P_2 + Q_1.$$

The group operation is as indicated in

$$(P_1, Q_1) + (P_2, Q_2) = (P_1 + P_2, Q_1 + Q_2),$$

independent of representation. The zero element is (O, O) , the element opposite to (P, Q) is (Q, P) ; the natural identification is $P \mapsto (P, O)$.

Multiplication by n -square matrices of sets extends to the pairs, defining a set of operators on the group: $A(P, Q) = (AP, AQ)$. Analogously, containment carries over, via an analogue of (8),

$$(P_1, Q_1) \geq (P_2, Q_2) \text{ iff } P_1 + Q_2 \supset P_2 + Q_1,$$

again independent of representation; and this behaves well under both the group addition and the set of operators. (An interesting project would be to treat the lattice-theoretic properties.)

There is a natural mapping $(P, Q) \mapsto P \overset{*}{\leftarrow} Q$ independent of representation, from the group to the semigroup (a left inverse to the embedding); it is isotone but probably not a semigroup homomorphism. There is a natural decomposition

$$(P, Q) = (P, 0) + (0, Q)$$

into 'real' and 'imaginary' parts; this defines homomorphisms from the group to the semigroup (which are not isotone).

For given n -square matrix A and real t , and for each pair (P, Q) , define a generalised reachable set

$$R_{P,Q}(t) = \left(\int_0^t e^{-As} P ds, \int_0^t e^{-As} Q ds \right);$$

this is independent of representation in the group, and the corresponding mapping is an isotone homomorphism (not invariant under the natural operators).

Remarks

The notion of the Pontrjagin difference is pre-historic: for intervals on the positive half-axis,

$$[0, \alpha] \overset{*}{\leftarrow} [0, \beta] = [0, \alpha - \beta];$$

and claims for priority may (and, probably, will) be made for analogues thereof. However, the concept (and notation) first appeared in

L. S. Pontrjagin: Linear differential games 2, Doklady Akad. Nauk SSSR 175 (1967) 721-723.

Several further properties of the operation were summarised in

E. Rechtschaffen, Unique winning policies for linear differential pursuit games (thesis), Case Western Reserve University, 1973.

One of these, our Exercise 11, has the consequence

$$\int_0^t (e^{-As}P) * (e^{-As}Q) = \int_0^t e^{-As}(P * Q).$$

This remained in the form on the left in Pontrjagin's paper; the second describes the essence of the First Reciprocity Theorem.

The idea of the 'subtraction group' in Exercise 15 is due to Mr. S. E. Clauss (personal communication)..

Experimentation with planar figures suggests that, if P is a polytope, then each face of $P * Q$ is parallel to one of P (even though not all faces of P need be utilised thus); e.g., fig. 1. The assertion is false already in R^3 (Rechtschaffen's thesis).

3.9 Measurable Selection

Consider the following problem. Given two sets P, Q in R^n , for every point $x \in P + Q$ there exists, by definition, a decomposition $x = y + z$ with $y \in P$ and $z \in Q$; this might be interpreted in terms of appropriate 'selection' mappings $P + Q \rightarrow P$, etc. The question is whether one can choose these mappings so as to preserve measurability: if $t \mapsto x$ is measurable, is $t \mapsto y$ also such? Apparently the selections will have to be made canonically, in some sense.

The definition of measurability of a set-valued mapping $t \mapsto F_t$, from R^1 to the collection of subsets of R^n , is that $\{t: F_t \cup C \neq \emptyset\}$ be Lebesgue measurable for every compact subset of C of R^n ; apparently this is a legitimate

generalisation of the classical concept.

In the result below it might seem natural to restrict oneself from the outset to nonvoid subsets of R^n ; curiously enough, this leads to irrelevant technical difficulties.

MEASURABLE SELECTION THEOREM There exists a mapping $S: \mathfrak{F} \rightarrow R^n$ on the collection \mathfrak{F} of all closed subsets of R^n , which is a selection,

$$(1) \quad S(F) \in F \text{ whenever } \emptyset \neq F = \overline{F} \subset R^n,$$

and preserves measurability: if $t \mapsto F_t$ is measurable, then so is $t \mapsto S(F_t)$.

(Proof) The underlying idea is extremely simple: one can choose a point in a given set by first selecting its first coordinate, then an appropriate second coordinate, etc. Second, one of the natural choices of a point within a compact subset of R^1 is the left-most one.

We exhibit S as the composition of mappings S_0, S_1, \dots, S_n . Here S_0 is the operation of taking points closest to origin,

$$S_0(F) = \{x \in F: |x| = \min_{y \in F} |y|\};$$

and, for $k \geq 1$, S_k is the operation of minimizing the k -th coordinate,

$$S_k(F) = \{x \in F: e_k'x = \min_{y \in F} e_k'y\}$$

(here e_k is the k -th unit basic vector; for $F = \emptyset$ take, e.g., $S_0(\emptyset) = 0$). Obviously $S_0(F)$ is a nonvoid, compact subset of F whenever $\emptyset \neq F = \overline{F} \subset R^n$, and similarly for $S_k(F)$ and compact $F \neq \emptyset$. Since S_k fixes the k -th coordinate, the composition $S = S_n \dots S_1 S_0$ is single-point-valued; thus indeed S is a selection in the sense of (1).

The proof is completed by verifying that each S_k preserves measurability; we shall carry this out for S_0 only.

Take any measurable $t \mapsto F_t$ and compact $C \subset \mathbb{R}^n$, and consider the set $\{t: S_0(F_t) \cap C \neq \emptyset\}$. The requirement unwinds thus: either $F_t = \emptyset$ and $0 \in C$, or C contains a point of F_t with minimal norm, i.e.,

$$\min_{F_t} |x| \leq \min_{F_t \cap C} |x|$$

Now, the second condition is 'measurable' (for the first, see Exercise 3), since the mappings appearing there are such.

E.g., for $t \mapsto \min_{F_t} |x|$ and real α , a time t satisfies

$\min_{F_t} |x| \leq \alpha$ iff F_t intersects the compact α -ball about the origin.

QED

FILIPPOV'S LEMMA Given continuous $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists a mapping σ with the following properties:

1. $\sigma(x, F) \in \mathbb{R}^n$ whenever $x \in \mathbb{R}^m$, $F = \overline{F} \subset \mathbb{R}^n$.
2. If $x \in f(F)$, then $f(\sigma(x, F)) = x$; thus $\sigma(\cdot, F): f(F) \rightarrow F$ is a right inverse to f .
3. If $t \mapsto x(t)$ and $t \mapsto F_t$ are measurable, and all $x(t) \in f(F_t)$, then $t \mapsto \sigma(x(t), F_t)$ is also measurable; thus there is a measurable solution $y(\cdot)$ to

$$(2) \quad x(t) = f(y(t)), \quad y(t) \in F_t.$$

(Proof) Set $\sigma(x, F) = S(f^{-1}(x) \cap F)$ where S is the measurable selection from the theorem. The first two assertions are verified easily. For the last one need only refer to the theorem again, once it is shown that $t \mapsto f^{-1}(x(t))$ is measurable. Indeed, $f^{-1}(x(t)) \cap C \neq \emptyset$ iff $x(t) \in f(C)$, with the sets $C, f(C)$ compact. QED

Example Let us return to the problem described in the first paragraph. We apply Filippov's Lemma, with addition as the continuous mapping $f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $F_t = P \times Q$. Then for any measurable $x: \mathbb{R}^1 \rightarrow P + Q$ we obtain measurable

$y: R^1 \rightarrow P$, $z: R^1 \rightarrow Q$ such that $x(t) = y(t) + z(t)$ (here y, z are the two coordinates of $\sigma(x, F) \in R^n \times R^n$).

Exercises

1. Prove $\int_0^1 (F_t + G_t) = \int_0^1 F_t + \int_0^1 G_t$ for integrals of set-valued mappings, assuming that the values F_t, G_t are closed, and, e.g., all F_t admit a common bound. (Hint: it is, of course, the 'difficult' inclusion that is the heart of the exercise. Recall also that the definition of $\int F_t$ involves integrable functions.)

2. Show that, if M is an (m, n) matrix, there is a measurable solution $x(\cdot)$ of $Mx(t) = a(t)$ for every measurable $a(\cdot)$ mapping into the row space of M . (Hint: you may wish to apply generalised inverses rather than Filippov's Lemma.)

3. Prove that, for measurable $t \mapsto F_t$, the set of times t with $F_t = \emptyset$ is measurable. (Hint: consider its complement, and use compact balls with radii $1, 2, \dots$)

4. If $t \mapsto F_t$ is measurable, then so is $t \mapsto F_t \cap C$ for fixed compact C (this is a detail needed in the proof of the Theorem; Exercise 8 generalises this). Show that if $t \mapsto F_t \cap C$ is measurable for each compact C , then so is $t \mapsto F_t$.

5. Show that $t \mapsto F_1(t) \cup F_2(t)$ is measurable if both $t \mapsto F_k(t)$ are such; and similarly for $t \mapsto f(F_t)$ with continuous $f: R^n \rightarrow R^n$.

6. In the definition of measurability, the test sets C may, equivalently, be replaced by closed sets. Prove the more useful result that they may be replaced by any basis of the open sets. (Hint: write an open set as the union of compact sets, and a compact set as the intersection of open, bounded sets.)

7. Show that if $F_k: \mathbb{R}^1 \rightarrow \mathcal{F}$ are measurable, then so are

$$t \mapsto \bigcup_{k=1}^{\infty} F_k(t), \quad t \mapsto F_1(t) \times F_2(t).$$

(Hint: Exercise 6.)

8. Prove that $t \mapsto F_1(t) \cap F_2(t)$ is measurable if both F_k are such. (Hint: $F_1 \cap (F_2 \cap C) \neq \emptyset$ iff $F_1 \times (F_2 \cap C)$ intersects the diagonal.)

9. If $x \mapsto f(x, t)$ is continuous and $t \mapsto f(x, t)$ measurable, then $t \mapsto f(C, t)$ is measurable for fixed compact C .

10. Obtain analogous concepts and results for the case of Borel-measurable set-valued mappings.

Remarks

Filippov's Lemma, essentially in the form (2) but for compact-valued maps, appeared in

A. F. Filippov, On certain questions in the theory of optimal control, SIAM J. Control 1 (1962) 76-84.

The extension to closed-valued mappings is easily effected by the preliminary operation S_0 in the proof of the theorem; it is needed to treat, e.g., the continuously moving linear subspaces in the Second Reciprocity Theorem. The point of isolating the Measurable Selection Theorem is that it shows, e.g. in (2), that $y(t)$ depends on $x(t)$ only, rather than on $x(\cdot)$. This is implicitly contained in Filippov's proof, and of fundamental importance for our purposes of constructing stroboscopic strategies. The modifications appeared in the proof of Theorem 2 in

O. Hájek, Duality for differential games and optimal control, Math. Systems Theory 8 (1974) 1-7.

3.10 Richter's Theorem

LEMMA 1 If $f: [0,1] \rightarrow \mathbb{R}^n$ is integrable, then the set

$$(1) \quad \left\{ \int_M f: \text{measurable } M \subset [0,1] \right\}$$

has compact and convex closure.

(Proof) The set is bounded, since each member has

$$\left| \int_M f \right| \leq \int_M |f| \leq \int_0^1 |f| < +\infty;$$

thus its closure is compact. Obviously - on choosing for u a characteristic function -

$$(2) \quad \left\{ \int_M f: \text{measurable } M \right\} \subset \left\{ \int_0^1 fu: \text{measurable } u: [0,1] \rightarrow [0,1] \right\};$$

equally obviously, the larger set is convex.

As the next step, we shall show that equality holds if f is a step function. Thus, let $f = \sum_{k=1}^{\infty} c_k v_k$, where the c_k are constants in \mathbb{R}^n , v_k is the characteristic function of the set

$$N_k = \{t: f(t) = c_k\},$$

and $[0,1] = \bigcup N_k$ is a disjoint measurable decomposition. Consider any member of the larger set in (2),

$$\int_0^1 fu = \sum c_k \int_{N_k} u,$$

and the k -th term there. There exists a measurable set

$M_k \subset N_k$ whose measure is $\int_{N_k} u$ precisely. Indeed, the real-

valued function $t \mapsto \text{meas}(N_k \cap (0,t))$ is continuous, with extreme values 0 and $\text{meas } N_k$:

$$0 \leq \int_{N_k} u \leq \int_{N_k} 1 = \text{meas } N_k;$$

thus, for some intermediate value $t \in [0,1]$, the function

value coincides with $\int u$. Then $M = \bigcup_{N_k} M_k$ is again a measurable disjoint decomposition, and

$$\int_0^1 fu = \sum_{N_k} c_k \int u = \sum_{M_k} c_k \int 1 = \int_M f.$$

this establishes the opposite containment in (2).

Returning to general mappings f , choose any $\epsilon > 0$ and point $\int_0^1 fu$ in the larger set. There exists a step function g with $|f - g| \leq \epsilon$. Then

$$\begin{aligned} \left| \int_0^1 fu - \int_0^1 gu \right| &\leq \int_0^1 |f-g|u \leq \epsilon \\ \int_0^1 gu &= \int_M g, \quad \left| \int_M g - \int_M f \right| \leq \int_M |g-f| \leq \epsilon \end{aligned}$$

for some measurable set $M \subset [0,1]$. Hence $\left| \int_0^1 fu - \int_M f \right| \leq 2\epsilon$;

as $\epsilon > 0$ was arbitrary, this shows that the smaller set in (1) is dense in the larger.

We have already shown that the smaller set has compact closure, and the larger is convex, and thus has convex closure. By density, the closures coincide. QED

Liapunov's Theorem is a stronger version of Lemma 1: in (1) the set itself is convex. Analogously, Richter's Theorem states that the set in (3) is convex; it is one of the rare results in which there are no assumptions, and a spectacular consequence appears.

LEMMA 2 For any mapping $t \mapsto S_t$ whose values are subsets of R^n , the set

$$(3) \quad \int_0^1 S_t = \left\{ \int_0^1 f(t)dt : f \text{ integrable, } f(t) \in S_t \text{ a.e.} \right\}$$

has convex closure.

(Proof) Take any two elements x, y in the closure of the

set in (3); we wish to show that $\alpha x + (1-\alpha)y$ is also in the closure for each $\alpha \in [0,1]$.

Choose an arbitrarily small $\epsilon > 0$, and find appropriate points $\int_0^1 f, \int_0^1 g$ ϵ -close to x, y respectively. Keeping these fixed, consider the set, in R^{2n} ,

$$\left\{ \int_M (f, g) = \left(\int_M f, \int_M g \right) : \text{measurable } M \subset [0,1] \right\}.$$

It contains both $(0,0)$ and $\int_0^1 (f, g)$, for the obvious choices of M . Thus, by Lemma 1, its closure must contain $\alpha \int_0^1 (f, g)$ for a measurable set M ,

$$\left| \int_M f - \alpha \int_0^1 f \right| \leq \epsilon, \quad \left| \int_M g - \alpha \int_0^1 g \right| \leq \epsilon.$$

Define $h: [0,1] \rightarrow R^n$ as f on M and g on $[0,1] \setminus M$. Then h is integrable, $h(t) \in S_t$ a.e., and

$$\int_0^1 h = \int_M f + \int_{[0,1] \setminus M} g = \int_M f + \int_0^1 g - \int_M g$$

is 2ϵ -close to

$$\alpha \int_0^1 f + \int_0^1 g - \alpha \int_0^1 g = \alpha \int_0^1 f + (1-\alpha) \int_0^1 g.$$

Hence, finally, $\int_0^1 h \in \int_0^1 S_t$ is 3ϵ -close to $\alpha x + (1-\alpha)y$. The assertion follows on taking ϵ to 0. QED

COROLLARY If S is compact, then the sets

$$\frac{1}{t} \int_0^t S ds, \quad \frac{2}{t^2} \int_0^t \int_0^s S dr ds$$

have the same closures for all $t > 0$, namely $\text{cvx } S$.

(Proof) First note that $\text{cvx } S$ is compact, by Carathéodory's

Theorem. Obviously $S \subset t^{-1} \int_0^t S ds$, and therefore

$$\text{cvx } S \subset \text{closure } \frac{1}{t} \int_0^t S ds$$

by Lemma 1.

In the opposite direction, $t^{-1} \int_0^t S ds \subset \text{cvx } S$ follows from Corollary 2 in 3.8. Similarly for the second moments; alternatively, one may use $\int_0^t \int_0^s S dr ds = \int_0^t (t-s) S ds$. QED

3.11 Reachable Sets

With a linear control system

$$(1) \quad \dot{x} = Ax - u; u(t) \in U$$

in R^n one may associate its reachable sets

$$\begin{aligned} R(t) &= \left\{ \int_0^t e^{-As} u(s) ds : \text{integrable } u: R^1 \rightarrow U \right\} \\ &= \int_0^t e^{-As} U ds. \end{aligned}$$

Properly speaking, one ought to refer to the mapping $t \mapsto R(t)$ (the 'performance' of (1)). These sets are important for two related reasons.

For any admissible control $u(\cdot)$, the solution to the equation in (1) with initial value x at $t = 0$ is

$$(2) \quad x(t) = e^{At} \left(x - \int_0^t e^{-As} u(s) ds \right)$$

by the variation-of-constants formula. Thus the set of all points to which x can be steered at time t within (1) (the set of attainability) is

$$A_x(t) = e^{At}(x - R(t));$$

the effect of linearity in (1) is that the second term is independent of x . Second, the initial position x can be steered to 0 at time t iff $x(t) = 0$ for some $u(\cdot)$ in

(2); since e^{At} is nonsingular, the condition reduces to $x \in R(t)$. More generally, a point x can be steered to a target $\Omega \subset R^n$ at time t iff

$$x \in R(t) + e^{-At}\Omega.$$

Obviously, if U in (1) is nonvoid, or convex, symmetric or bounded, then so is each $R(t)$; e.g., if α is a bound on U and $|A|$ a matrix norm for A , then each point x in $R(t)$ has

$$|x| \leq \int_0^t e^{|A|s} \alpha ds < +\infty.$$

If U is compact and convex, then, by an obvious weak compactness argument in the space of controls, each $R(t)$ is compact. In point of fact, convexity is not needed here (Richter's and Aumann's theorems).

In the next result, 'span' denotes the linear span (the set of all linear combinations of elements in the set, or equivalently, the intersection of all linear subspaces containing the given set).

PROPOSITION 1 In (1) assume that U is convex and symmetric. For strictly positive t the following sets coincide:

$$\text{span } R(t) = \text{span } \int_0^t e^{-As} U ds,$$

$$\text{span } \{e^{-As} u : s \in R^1, u \in U\},$$

$$\text{span } \{A^k u : k = 0, \dots, n-1, u \in U\},$$

$$\int_0^t e^{-As} \text{span } U ds.$$

Remarks The middle two terms do not depend on $t > 0$; the resulting concept is the controllability space of (1). The third term provides an alternate characterisation: the

smallest linear subspace L which is invariant under the coefficient matrix of (1) and contains U , $AL \subset L \supset U$. The last term justifies the appellation reachable set with 'unbounded controls'. If U is a parallelepiped,

$$U = \{Bv: v \text{ in unit cube of } R^m\},$$

then the columns of B span the set U ; the third term above shows that the controllability space is the column space of the (n, nm) matrix

$$(A^{n-1}B, A^{n-2}B, \dots, AB, B),$$

the 'controllability matrix' of (1).

(Proof) Denote the indicated sets by C_1, C_2, C_3, C_4 . The only obvious inclusions are

$$C_1 \subset C_4 \supset C_2.$$

The subsequent proof proceeds by the familiar technique of orthogonal complementation: if $X \subset Y$ in R^n , then $X^\perp \supset Y^\perp$, and $X^{\perp\perp} = \text{span } X$ (here $X^\perp = \{y \in R^n: y'x = 0 \text{ for all } x \in X\}$).

Consider any $c \in R(t)^\perp$,

$$(3) \quad c' \int_0^t e^{-As} u(s) ds = 0$$

for all admissible controls $u(\cdot)$. One possibility is a control with constant value $u \in U$ in $[0, \theta]$, and value 0 in $(\theta, t]$. Then from (3), $\int_0^\theta c' e^{-As} ds u = 0$, for all $u \in U$ and $\theta \in [0, t]$. By continuity, $c' e^{-A\theta} u = 0$; since the function of θ is analytic, we have $c' e^{-A\theta} u = 0$ for all $u \in U$ and all $\theta \in R^1$. This shows that $C_1^\perp \subset C_2^\perp$.

Repeated differentiation of $c' e^{-A\theta} u = 0$ yields $C_2^\perp \subset C_3^\perp$. If $c \in C_3^\perp$, then $c' A^k u = 0$ for $u \in U$ and all $k = 0, 1, \dots$, since, by the Cayley-Hamilton Theorem, A^n is a linear combination of I, A, \dots, A^{n-1} , and hence so are all A^k

for $k \geq n$. From this,

$$c'e^{-As}u = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} c'A^k u = 0$$

for $u \in U$ and $s \in \mathbb{R}^1$. This proves the second containment needed for $C_3^\perp = C_2^\perp$. Finally, $C_2^\perp \subset C_4^\perp$ is verified easily.

Thus all the C_i^\perp coincide; hence so do their orthogonal complements, the linear spaces C_i . QED

COROLLARY 1 The system (1) is said to be controllable if either of the following equivalent conditions obtains: some $R(t)$ has nonvoid interior; all $R(t)$ with $t > 0$ are neighbourhoods of 0; the controllability space coincides with \mathbb{R}^n .

(Proof) Every nonvoid, convex, symmetric set S is a neighbourhood of 0 within the linear space span S ; one merely applies this in the preceding result. QED

Symmetry is essential here: in the typical case of asymmetry, the linear span of $U\{R(t): t \geq 0\}$ is the entire space but 0 is on the boundary of $R(t)$ for small $t > 0$.

LEMMA 1 For $t, s \geq 0$ we have the addition formula

$$R(t + s) = R(t) + e^{-At}R(s).$$

(Proof) The two needed containment relations are obtained by reading

$$\int_0^{t+s} = \int_0^t + \int_t^{t+s} = \int_0^t e^{-Ar}u(r)dr + e^{-At} \int_0^s e^{-Ar}u(r+t)dr$$

in the two directions. QED

COROLLARY 2 If $0 \in U$, then $R(t) \subset R(s)$ for $0 \leq t \leq s$; if (1) is controllable, then even $R(t) \subset \text{Int } R(s)$ for $0 \leq t \leq s$.

(Proof) For $\epsilon = s - t$ we have $0 \in R(\epsilon)$, $0 \in e^{-At}R(\epsilon)$, and therefore

$$R(t) = R(t) + 0 \subset R(t) + e^{-At}R(\epsilon) = R(s).$$

Similarly in the second assertion. QED

According to Proposition 3 in 3.8, sets which are closed, convex, and symmetric may be written as the direct sum of a linear subspace and a compact set. Let us apply this decomposition to the control constraint set of (1):

PROPOSITION 2 If $U = V + L$ with V compact and L a linear subspace, then

$$R_U(t) = R_V(t) + L^*$$

where

$$R_U(t) = R(t) = \int_0^t e^{-As} U ds, \quad R_V(t) = \int_0^t e^{-As} V ds,$$

and L^* is the least A -invariant subspace containing L .

(Proof) From Exercise 1 in 3.9, $R_U(t) = R_V(t) + R_L(t)$. Since L is a linear subspace,

$$R_L(t) = \int_0^t e^{-As} L ds = \int_0^t e^{-As} \text{span } L ds;$$

thus, by Proposition 1 applied to L , $R_L(t) = L^*$. QED

LIMIT THEOREM (for reachable sets) If U is compact and convex, then

$$\lim_{t \rightarrow 0+} \frac{R(t)}{t} = U.$$

(Proof) This will follow from the estimate

$$(4) \quad \int_0^t e^{-As} ds U \subset R(t) \subset \int_0^t e^{-As} ds U + O(t^2)$$

on dividing by t and taking $t \rightarrow 0$: $t^{-1} \int_0^t e^{-As} \rightarrow e^0 = I$.

The first term in (4) describes points x steered to 0 by constant controls in U ; this establishes the first containment.

For the second, first write

$$\begin{aligned}\int_0^t e^{-As} u(s) ds &= \int_0^t e^{-As} ds v + \int_0^t e^{-As} (u(s) - v) ds \\ &= \int_0^t e^{-As} ds v + \int_0^t (u(s) - v) ds + \int_0^t (e^{-As} - I)(u(s) - v) ds.\end{aligned}$$

Given $u(\cdot)$ and $v \in U$, the first term belongs to $\int_0^t e^{-As} ds U$; the last is estimated by

$$\int_0^t (e^{|A|s} - 1) \delta ds = O(t^2)$$

with $\delta = \text{diam } U < +\infty$; the middle term vanishes if we take $v = t^{-1} \int_0^t u(s) ds$, and this $v \in U$ by Corollary 2 of 3.8. QED

COROLLARY 3 $\lim_{h \rightarrow 0+} \frac{1}{h} (R(t+h) * R(t)) = e^{-At} U$.

(Proof) From Lemma 1, and Corollary 3 in 3.8,

$$\frac{1}{h} (R(t+h) * R(t)) = e^{-At} \frac{R(h)}{h};$$

now merely apply Proposition 2. QED

In the Limit Theorem, boundedness of U is essential: if U contains a linear subspace L , then $R(t)$, and hence $R(t)/t$ also, contains the far larger space L^* as in Proposition 2. In Chapter 7 we will need a version applying to nonlinear equations; for these the concept of reachable set is not available.

Consider the control system in R^n

$$(5) \quad \dot{x} = f(x, u); u(t) \in U$$

Here $f: R^n \times R^m \rightarrow R^n$ is continuous, and U is a compact subset of R^m . For given initial position $x \in R^n$ and terminal time $\theta \geq 0$ define the attainable set $A_x(\theta)$, consisting of all state response endpoints $x(\theta)$, where $x(\cdot)$ is a (generalised) solution of (5) with initial value x ,

$$(6) \quad x(t) = x + \int_0^t f(x(s), u(s)) ds,$$

and $u: \mathbb{R}^1 \rightarrow U$ is measurable. (Thus the controls $u(\cdot)$ are varied; even for a fixed $u(\cdot)$ there may be several solutions $x(\cdot)$ to (6).)

LIMIT THEOREM (for attainable sets) With the above assumptions and notation, for each $x_0 \in \mathbb{R}^n$ we have

$$\lim_{\substack{x \rightarrow x_0 \\ t \rightarrow 0+}} \frac{1}{t} (A_x(t) - x) = \text{cvx } f(x_0, U).$$

(Proof) Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$(7) \quad f(x, U) \subset f(x_0, U) + B_\epsilon \text{ for } |x - x_0| \leq \delta,$$

where B_ϵ is the ϵ -ball about the origin in \mathbb{R}^m , since $x \mapsto f(x, u)$ is continuous at x_0 uniformly for $u \in U$. Again from continuity of f and compactness of U , there exists $\eta > 0$ so that the solutions $x(\cdot)$ have

$$|x(s) - x_0| \leq \delta \text{ for } 0 \leq s \leq \eta$$

if their initial values $x(0)$ are η -close to x_0 .

From (6), each point $y \in (A_x(t) - x)/t$ has

$$y = \frac{1}{t} \int_0^t f(x(s), u(s)) ds \in \frac{1}{t} \int_0^t f(x(s), U) ds$$

$$(8) \quad \subset \frac{1}{t} \int_0^t f(x_0, U) ds + \frac{1}{t} \int_0^t B_\epsilon ds$$

by (7) and Exercise 1 in 3.9, at least for small t and $|x - x_0|$. From the assumptions, $f(x_0, U)$ is compact, hence so is $\text{cvx } f(x_0, U)$ by Caratheodory's Theorem; thus Corollary 2 in 3.8, applied to the terms in (8) yields

$$y \in \text{cvx } f(x_0, U) + B_\epsilon.$$

Now, first take $x \rightarrow x_0$ and $t \rightarrow 0+$, and then let $\epsilon \rightarrow 0+$ to obtain

$$\limsup \frac{1}{t} (A(t, x) - x) \subset \text{cvx } f(x_0, U).$$

For the second containment use $x = x_0$ and the Corollary in 3.10. QED

COROLLARY 4 For each $\epsilon > 0$ there exists $\delta > 0$ such that, whenever

$$0 < t \leq \delta, \quad |x - x_0| \leq \delta,$$

then, for every $y \in A_x(t)$, there exists $z \in \text{cvx } f(x_0, U)$ with

$$(9) \quad \left| \frac{y-x}{t} - z \right| \leq \epsilon,$$

and conversely: for each such z there is a y with (9).

Exercises

The first three of these concern a rudimentary identification problem; given two linear control systems, (1) and also

$$(10) \quad \dot{x} = Bx - v; \quad v(t) \in V,$$

to determine under what circumstances they have coinciding reachable sets (i.e., the same performance). We will assume that both the constraint sets U, V are polytopes.

1. Prove that $U = V$, and, more generally, $e^{At}U = e^{Bt}U$.

2. Show that $e^{At}u = e^{Bt}u$ for each vertex u of U .

(Hint: $e^{-At}e^{Bt}$ depends continuously on t , and has value I at $t = 0$.)

3. Prove that $Ax = Bx$ for all x in the controllability space of (1). (Hint: In Exercise 2 differentiate repeatedly at 0; then apply Proposition 1.)

4. Deduce the Limit Theorem for reachable sets from that for the attainable sets.

5. In the notation preceding the last theorem, prove that the set of all points

$$\frac{2}{t^2} \int_0^t \int_0^s f(x(r), u(r)) dr ds$$

has \limsup in $\text{cvx } f(x_0, U)$ as $x \rightarrow x_0$ and $t \rightarrow 0+$.

6. In the Limit Theorems compactness is rather essential. Prove that, if $U = V + L$ with V compact and convex, and L a linear subspace, then

$$\lim_{t \rightarrow 0+} \frac{R(t)}{t} = V + L^*$$

where L^* is the least A -invariant subspace containing L .

Remarks

The addition formula for reachable sets belongs to control-theoretic folklore; e.g., one inclusion appeared within the proof of Corollary 15 of

O. Hájek, Geometric theory of time-optimal control,
SIAM J. Control 9 (1971) 339-350.

Our usual notation for control systems, $\dot{x} = Ax - u$, may seem stilted to readers used to $\dot{x} = Ax + Bu$. First, the unorthodox minus sign is the right one in cases where it really matters, if the constraint set is not symmetric; the formula for reachable sets is just the beginning of an unnecessary tracking of minus signs (similar remarks apply to game equations $\dot{x} = Ax - p + q$ or $\dot{x} = f(x, p) - g(x, q)$, etc). As concerns the control matrix B , there is no point in representing one general set as the linear image of another, unless there are further compelling reasons (and there are none in this book).

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CHAPTER IV

ISOCRONOUS CAPTURE

In this chapter we treat isochronous capture in linear games, without the additional restriction of Chapter III that the winning strategies be stroboscopic. It turns out that a large portion of the necessary conditions can be carried over to this more general case.

It is an open question whether omission of stroboscopia ever actually results in any real gain, in the sense of obtaining further winning positions or achieving faster capture (see the conjectures at the end of the chapter). Thus, a single theorem would render the entire chapter superfluous: in any case, the trend of the presented results is that, once one has decided on isochronous capture, little further harm is perpetrated by requiring that the strategies be stroboscopic.

The theorem in 4.1 is rather surprising: under moderately severe conditions on the shape and position of the players' constraint sets, but without any assumptions on the dynamics or on the target, all the finicky distinctions between strategies disappear, and actually nothing is gained even if the pursuer is granted complete information on the future behaviour of his opponent.

4.1 Winning Sets

In 2.2 the 'isochronous' winning sets $W(t)$ were introduced. For the case of a linear game

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } x &\in \Omega; \quad \text{state space: } R^n \end{aligned}$$

the definition may be paraphrased as follows. The set $W(t)$ consists of all positions $x \in R^n$ such that, for some strategy σ and all integrable $q: [0, t] \rightarrow Q$, we have

$$(2) \quad e^{At}(x - \int_0^t e^{-As}(\sigma[q](s) - q(s))ds) \in \Omega.$$

Next, pursuer's reachable set is

$$R_P(t) = \int_0^t e^{-As} P ds = \{ \int_0^t e^{-As} p(s) ds : \text{integrable } p: [0, t] \rightarrow P \};$$

and analogously for quarry's reachable set $R_Q(t) = \int_0^t e^{-As} Q ds$.

Somewhat more generally, for any subset $U \subset R^n$ let $R_U(t) = \int_0^t e^{-As} U ds$. (Observe that in this notation some of the dependence on the data is indicated, some suppressed.)

One important property, that $W(t)$ is closed if P is compact and convex, will be proved in 8.1.

LEMMA 1 If P, Q, Ω are convex and symmetric, then so is each $W(t)$; in particular, $0 \in W(t)$ whenever $W(t) \neq \emptyset$.

(Proof) This is quite straightforward. If positions x_1, x_2 in $W(t)$ are forced to Ω at t by strategies σ_1, σ_2 , and $0 \leq \alpha \leq 1$, then $\alpha\sigma_1 + (1-\alpha)\sigma_2$ forces $\alpha x_1 + (1-\alpha)x_2$ to Ω at t , and $\sigma[q] = -\sigma_1[-q]$ forces $-x_1$ to Ω at t . QED

LEMMA 2 For $0 \leq t \leq \theta$,

$$(3) \quad W(\theta) + R_Q(t) \subset R_P(t) + e^{-At}W(\theta-t).$$

(Proof) This reformulation of the principle of suboptimality is the counter-part to the addition formula for reachable sets.

Take any $x \in W(\theta)$, and an appropriate strategy σ . By Proposition 2 in 2.2, the point $y = \pi(t, x, \sigma[q], q) \in W(\theta-t)$. Here

$$y = e^{At}(x - \int_0^t e^{-As}(\sigma[q](s) - q(s))ds),$$

so that

$$x + \int_0^t e^{-As} q(s) ds \in \int_0^t e^{-As} \sigma[q](s) ds + e^{-At} W(\theta - t).$$

If $q(\cdot)$ is varied, we obtain (3). QED

COROLLARY 1 If some $W(\theta) \neq \emptyset$, then also $W(t) \neq \emptyset$ for all $t \in [0, \theta]$.

(Proof: in (3), the left side is nonvoid.)

COROLLARY 2 For all $t \geq 0$,

$$(4) \quad R_P \overset{*}{\subseteq} Q(t) + e^{-At} \Omega \subset W(t) \subset (R_P(t) + e^{-At} \Omega) \overset{*}{\subseteq} R_Q(t).$$

(Proof) The second inclusion is a re-interpretation of (3) with $\theta = t$. As for the first, if x is in the first member,

$$x = \int_0^t e^{-As} u(s) ds + e^{-At} y, u(s) \in P \overset{*}{\subseteq} Q, y \in \Omega,$$

then the stroboscopic strategy $\sigma = u + q$ satisfies (2),

$$e^{At}(x - \int_0^t e^{-As} (\sigma - q) ds) = e^{At}(0 + e^{-At} y) \in \Omega. \text{ QED}$$

In (4) the largest set has a pleasing geometric interpretation: see Exercise 2. The smallest set consists of points which can be forced to Ω at t by stroboscopic strategies which, furthermore, are required to have values in $P \overset{*}{\subseteq} Q$ as in the First Reciprocity Theorem (thus, by the Second Reciprocity Theorem, there will probably be further positions for stroboscopic isochronous capture). It is not implied that $P \overset{*}{\subseteq} Q \neq \emptyset$.

We will say that Q is a direct summand of P if $P = Q + U$ for some $U \subset R^n$. Here there is a unique choice, $U = P \overset{*}{\subseteq} Q$, if P, Q are as below (see Corollary 3 in 3.9).

THEOREM Assume that P, Q are compact and convex, and Ω is closed and convex. If Q is a direct summand of P , then

$$\begin{aligned} W(t) &= R_P \overset{*}{\subseteq} Q(t) + e^{-At} \Omega \\ &= (R_P(t) + e^{-At} \Omega) \overset{*}{\subseteq} R_Q(t). \end{aligned}$$

Thus capture with anticipation, capture, stroboscopic capture, and stroboscopic forcing to a single point of the target (all at given time t) are equivalent. Furthermore, for each $x \in W(t)$ there is a winning stroboscopic strategy as described in the First Reciprocity Theorem.

(Proof) The assertion is that the containment relations in (6) are equalities. From the assumption, $P = Q + \{P \# Q\}$, so that $R_P = R_Q + R_P \# Q$; then in (6)

$$\begin{aligned}(R_P(t) + e^{-At}\Omega) \# R_Q(t) &= (R_Q(t) + R_P \# Q(t) + e^{-At}\Omega) \# R_Q(t) \\ &= R_P \# Q(t) + e^{-At}\Omega\end{aligned}$$

by "cancellation". For the last assertion, see the remark following Corollary 2. QED

Exercises

1. Show that each point in the interior of Ω is in $W(t)$ for small $t > 0$.

2. Observe that the last member in (4) coincides with the set of positions which can be captured with anticipation at time t precisely.

3. Prove: $\limsup_{t \rightarrow 0+} \frac{W(t)}{t} \subset (P + \overline{R^+ \Omega}) \# Q$. (Hint: one possibility is to use the Limit Theorem for reachable sets.)

4. Assume P, Q, Ω are nonvoid, convex, and symmetric, and P is compact. Prove that, if $W(\theta) \neq \emptyset$ for some $\theta > 0$, then $0 \in \limsup W(t)/t$, and conclude $Q \subset P + \overline{R^+ \Omega}$ as a necessary condition. (Note: a sharper result appears in Proposition 1 of 4.2.)

5. Prove

$W(2t) \subset (R_P(t) + e^{-At}((R_P(t) + e^{-At}\Omega) \# R_Q(t))) \# R_Q(t)$,
and generate an analogous formula for $W(3t)$.

6. Setting $W^*(t) = e^{At}W(t)$, show

$$W^*(t+s) \# W^*(t) \subset e^{A(t+s)}(R_P(s) \# R_Q(s))$$

for $t, s \geq 0$; conclude that

$$\limsup_{s \rightarrow 0+} \frac{W^*(t+s) \# W^*(t)}{s} \subset e^{At}(P \# Q),$$

$$\limsup_{s \rightarrow 0+} \frac{W^*(s) \# \Omega}{s} \subset P \# Q.$$

7. Prove that, if $P \subset Q + V$ with V compact, then

$$W(t) \subset R_V(t) + e^{-At}\Omega$$

8. Suppose that P is the square $-1 \leq x, y \leq 1$ in R^2 , and Q a symmetric segment within P . Construct sets V such that $P \subset Q + V$. Observe that there are irreducible sets V , and exhibit at least two quite distinct ones.

9. Show that if $P \supset Q$ are both segments, then Q is a direct summand of P .

10. Let the target set Ω be the affine manifold $b + N$. Prove that

$$W(t) + e^{-At}N \subset W(t).$$

Carefully prove the following alternative: either all $W(t) = \emptyset$, or Ω is a single point, or all $W(t)$ are non-compact.

11. Assume that a winning set $W(t)$ contains an entire ray $x + R^+c$ ($x \in W(t)$, $c \in R^n$), $\Omega = b + N$ is an affine manifold, and P, Q are compact. Prove that necessarily $c \in e^{-At}N$. (Hint: in the relation $x + \theta c \in W(t)$ divide by θ and take $\theta \rightarrow +\infty$.)

Remarks

As mentioned in the proof of Lemma 2, the inclusion (3) is a set-theoretic version of the principle of suboptimality; and there is a corresponding statement involving optimal

strategies and minimal times. Exercise 5 extended to $W(kt)$ for general $k = 0, 1, \dots$ corresponds similarly to 'repeated min-max' constructions as in Section 1.3 of

A. Friedman: Differential Games, Wiley-Interscience, New York, etc., 1971.

Returning to Lemma 2, assume $\Omega = 0$ to simplify matters, and call (3) the hereditary property for the sets $W(\theta)$. According to (6), $W(t) \subset R_P^* R_P(t)$, but the latter sets do not have the hereditary property. Indeed, for the harmonic oscillator with unit segments for P and Q , (trivial cases excepted) reachable sets at time π coincide with the unit circle, so

$$R_P(t) \not\subset R_Q(\pi) = 0 \neq \emptyset,$$

but $R_P(t) \not\subset R_Q(t) = \emptyset$ for all $t \in (0, \pi)$.

CONJECTURE The isochronous winning sets for the game (1) are the largest sets with the hereditary property: if the set-valued mapping $t \mapsto S_t$ satisfies $S_0 = 0$ and

$$S_\theta + R_Q(t) \subset R_P(t) + e^{-At} S_{\theta-t} \text{ whenever } 0 \leq t \leq \theta,$$

then $S_t \subset W(t)$ for all $t \geq 0$. (Note that automatically $S_t \subset R_P(t) \not\subset R_Q(t)$.)

4.2 Necessary Conditions

Here we develop two variants of a necessary condition for isochronous capture. The first, Proposition 1, is obtained by simply omitting the qualifier 'stroboscopically' in the Necessary Condition from 3.4. The second, Propositions 2 and 3, is related to the results of 3.5 on unions of target sets.

The notation of 4.1, and the underlying game, is retained. It will be convenient to isolate a technical portion of the reasoning.

LEMMA Assume that a strategy σ and a terminal time

$\theta \geq 0$ have been fixed. For any two points q, q_0 in Q , and time $t \in [0, \theta]$, let q_t be the piecewise constant quarry control,

$$q_t = q_0 \text{ in } [0, \theta-t], \quad q_t = q \text{ in } (\theta-t, \theta];$$

$p_t = \sigma[q_t]$ the pursuer response; and, for an initial position x ,

$$(1) \quad y_t = e^{A\theta} \left(x - \int_0^\theta e^{-As} (p_t(s) - q_t(s)) ds \right)$$

the state response endpoint at θ . Then

$$(2) \quad y_t - y_0 = \int_0^t e^{As} ds (q - q_0) - \int_0^t e^{As} (p_t(t-s) - p_0(t-s)) ds \\ \in \int_0^t e^{As} ds (q - q_0) + (R_P^*(t) - R_P^*(t))$$

where $R_P^*(t) = e^{At} R_P(t)$ is the reachable set of the control system $\dot{x} = -Ax - p$, $p(t) \in P$.

(Proof) The essential point here is that σ is non-anticipatory, so that

$$p_t = \sigma[q_t] = \sigma[q_0] = p_0 \text{ a.e. } [0, \theta-t];$$

the verification is then straightforward, using

$$\int_0^{\theta-t} = \int_0^\theta - \int_{\theta-t}^\theta. \quad \text{QED}$$

An obvious but important observation is that the assertion is vacuous unless $\theta > 0$: if $\theta = 0$, the only possible $t = 0$, so that $y_t = y_0$ in (2).

NECESSARY CONDITION Assume that P is compact and convex, and the target set is contained in the affine manifold $\{x: Mx = Mb\}$. If nontrivial isochronous capture is possible, i.e., if $W(\theta) \neq \emptyset$ for some $\theta > 0$, then necessarily

$$M(Q - Q) \subset M(P - P).$$

Moreover,

(3) quarry control order \geq pursuer control order,

and, if k denotes the latter,

$$(4) \quad MA^{k-1}(Q - Q) \subset MA^{k-1}(P - P).$$

(Proof) By assumption, some strategy σ forces a point $x \in W(\theta)$ to Ω at time θ . Then, in (2), the endpoints y_t belong to Ω , so that $M(y_t - y_0) = 0$. Thus, if we premultiply by M and let the q 's range over Q ,

$$(5) \quad M \int_0^t e^{As} ds (Q - Q) \subset M(R_P^*(t) - R_P^*(t))$$

for all $t \in (0, \theta]$.

Here divide by t and take $t \rightarrow 0$. Using the Limit Theorem for the reachable sets R_P^* ,

$$(6) \quad M(Q - Q) \subset M(P - P).$$

In case the pursuer control order $k = 1$, this is the required result (4) precisely. For $k = 2$ we have

$$(7) \quad M(P - P) = 0,$$

hence $M(Q - Q) = 0$ from (5); in particular, (3) holds.

Returning to (5), points in the second member may be written

$$\begin{aligned} & M \int_0^t e^{As} (p_1(s) - p_2(s)) ds \\ &= M \int_0^t (p_1(s) - p_2(s)) ds + MA \int_0^1 s (p_1(s) - p_2(s)) ds + \dots \end{aligned}$$

with the first term vanishing according to (7). Hence

$$\frac{1}{t^2} M \int_0^t e^{As} (p_1(s) - p_2(s)) ds = MA \frac{1}{t^2} \int_0^1 s (p_1(s) - p_2(s)) ds + \dots,$$

with limit in $\frac{1}{2} MA(P - P)$. A simpler argument for the first

member of (5) then yields

$$\frac{1}{2} MA(Q - Q) \subset \frac{1}{2} MA(P - P),$$

i.e., (4). The case of general k is obtained by an obvious induction. QED

The last results treat target sets of the form

$$(8) \quad \Omega = \bigcup_{k=1}^r \Omega_k, \quad \Omega_k = \{x: F_k(x) = 0\}$$

where each $F_k: R^n \rightarrow R^m$ is of class C^1 in some neighbourhood of Ω_k . Then each F_k has a finite Taylor expansion near any $y \in \Omega_k$,

$$(9) \quad F_k(x) = 0 + DF_k(y)(x-y) + o(|x-y|) \text{ as } x \rightarrow y.$$

Here DF_k is the Jacobian matrix of F_k (of type (m_k, n) , entries are partial derivative of the coordinates of F_k). Let us also denote

$$N_k(y) = \{x: F_k(y)x = 0\},$$

the tangent space of $\{F_k = 0\}$ at y .

PROPOSITION 1 Assume that P is compact and convex, and Ω is as in (8) with $F_k \in C^1$ near Ω_k . If there exist winning positions $x \in W(\theta)$ with $\theta > 0$, and $y \in \Omega$ is a state response endpoint to a constant quarry control $q \in Q$, then

$$(10) \quad Q - q \subset P - P + \bigcup_{k=1}^r N_k(y).$$

(Proof) Referring to (1) and (2), let $q_0 = q$, $y_0 = y$. From (2), $y_t \rightarrow y$ as $t \rightarrow 0$, so that (9) will apply if $y \in \Omega_k$.

In (2), divide by $t > 0$. Obviously the term

$$\frac{1}{t} \int_0^t e^{As} ds (q_0 - q) \rightarrow q_0 - q.$$

As for the elements of the next term there, the Limit Theorem

applied to the reachable sets R_P^* yields a sequence $t = t_j \rightarrow 0+$ such that the entries converge to a point of $P - P$. But then $(y_t - y)/t$ will also have a finite limit, in $q_\theta - q + P - P$.

Finally, the points $y_t = y_{t_j}$ belong to $\Omega = \bigcup \Omega_k$. Then, for some fixed k and subsequence $t = t_i$, we have all $y_t \in \Omega_k$, whereupon $y = \lim y_t \in \Omega_k$ also. Then (9) holds for $x = y_t$, with $F_k(y_t) = 0$. Divide by t to obtain

$$0 = DF_k(y) \frac{y_t - y}{t} + \frac{|y_t - y|}{t} \sigma(1),$$

with the last term vanishing in the limit. Thus $\lim(y_t - y)/t \in N_k(y)$,

$$q_\theta - q \subset P - P + N_k(y) \subset P - P + \bigcup_{k=1}^r N_k(y).$$

Now merely let q_θ vary over Q . QED

Example (Capture with two pursuers) Consider a game with partitioned dynamical equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \\ -v \end{pmatrix},$$

termination condition $x_1 = y$ or $x_2 = y$, and player constraint sets U_1, U_2, V (nonvoid, compact, convex, symmetric). We need not assume that the dynamical equation is actually uncoupled (i.e., that A is appropriately block-diagonal).

The Necessary Condition is not directly applicable, since the target set $\Omega = L_1 \cup L_2$ is a union of linear subspaces, and hence is not convex (trivial cases expected). Enlarging Ω to its convex hull yields no information at all: $\text{cvx } \Omega$ is the entire state space.

In applying Proposition 1, let $y \in \Omega$ be a state response endpoint to any fixed constant $v \in V$. Then $N_k(y) = L_k$ is

independent of y , so that Proposition 1 yields

$$Q - Q \subset P - P + (L_1 \cup L_2)$$

(we have let q vary over Q also). By convexity and symmetry, then

$$Q \subset P + (L_1 \cup L_2) = (P + L_1) \cup (P + L_2).$$

Here

$$P = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} : u_k \in U_k \right\}, \quad Q = \left\{ \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} : v \in V \right\},$$

so that, in the notation of our game,

$$V \subset U_1 \cup U_2$$

is a necessary condition for isochronous capture. In particular, if $U_1 = U_2 = U$, then isochronous capture by the two pursuers is ruled out unless the necessary condition $V \subset U$ for a single pursuer (the other inactive) is satisfied.

Finally we treat one case where (10) provides no information because, in a sense, the players have control orders higher than 1.

PROPOSITION 2 Assume that P is a compact and convex, and $\Omega = \bigcup_{k=1}^r (b_k + L_k)$ a finite union of affine manifolds. If

$$(11) \quad Q - Q \subset \bigcap_{k=1}^r L_k \supset P - P,$$

then

$$(12) \quad A(Q - Q) \subset A(P - P) + \bigcup_{k=1}^r L_k$$

is a necessary condition for nontrivial isochronous termination.

(Proof) Choose matrices M_k so that $L_k = \{x: M_k x = 0\}$, whereupon (11) yields

$$M_k(Q - Q) = 0 = M_k(P - P)$$

for all k . Referring to (2), find a sequence $t = t_j \rightarrow 0$ and an index k such that all $y_t \in \Omega$ belong to $b_k + L_k$. Then so does $y = \lim y_t$, and $M_k(y_t - y) = 0$. In (2) pre-multiply by M_k , divide by t^2 and take $t = t_j \rightarrow 0$:

$$0 \in M_k \frac{A}{2} (q_\theta - q_0 + P - P)$$

(with k possibly depending on q_θ, q). In any case

$$A(q_\theta - q_0) \subset A(P - P) + \bigcup_{k=1}^r L_k$$

and one need only let q_θ, q_0 vary over Q to obtain (12).

QED

Exercises

1. In Proposition 1 omit the assumption on Ω , and conclude, in place of (10), that

$$Q - q \subset P - P + \Omega_y$$

where Ω_y , the tangent set of Ω at $y \in \Omega$, consisting of all finite limits of $\alpha_k(y_k - y)$ as $y_k \rightarrow y$, $y_k \in \Omega$, $\alpha_k \rightarrow +\infty$.

2. Prove that Ω_y is convex if Ω is such; show that, for general targets, the tangent set need not be convex.

3. Show that, in (10), one need only take unions over those indices k for which $y \in \Omega_k$.

4. Consider a game with two pursuers and one quarry, with n -dimensional equations of motion $\ddot{x}_k = u_k$, $\ddot{y} = v$, and termination condition $y = x_1$ or $y = x_2$. Show that $V \subset U_1 \cup U_2$ is a necessary condition on the constraint sets (compact, convex, symmetric) for nontrivial isochronous capture. (Intermediate answer: $M_1 A = (0 \ I \ 0 \ 0 \ 0 \ -I)$ in the notation from the proof of the lemma.)

5. Treat games analogous to the preceding, with one or two of the players having control order 1 rather than 2.

6. Consider yet another type of game with two pursuers and one quarry,

$$\dot{x} = Ax - p_1 - p_2 + q; p_k(t) \in P_k, q(t) \in Q,$$

with $x \in L_k$ as the k -th pursuer goal. Assume that L_k are linear subspaces, and P_k, Q nonvoid, compact, convex, symmetric. In terms of the first-order necessary conditions (i.e., (4) for $k = 1$) compare the following situations: the first pursuer decides to treat the other as added nuisance; both pursuers do this; the pursuers cooperate. (Warning: the last implies that both goals are reached simultaneously.)

Remarks

The exposition in 4.1 and 4.2 is based on two eponymous sections of

O. Hájek: Lectures on Linear Pursuit Games
(unpublished), Case Western Reserve University,
1973.

The reader of these lecture notes should be warned that a questionable assertion appears there: $Q \subset P + \overline{R^1 \Omega}$ is necessary for isochronous capture if P, Q are compact, convex and symmetric, but Ω is not necessarily convex. If true, this would provide a simpler apparatus for treating the example of several pursuers in 4.2; the proof is certainly false.

As for further developments, it would be desirable to treat the situation of (8), i.e., finite unions of C^1 targets, for higher player control orders than 1 and 2; even the case $r = 1$ is interesting. A more ambitious project is suggested by

CONJECTURE 1 Isochronous capture is stroboscopic: if a position can be forced to termination at time $t > 0$, then some stroboscopic strategy has the same effect.

CONJECTURE 2 For each non-anticipatory strategy, the

set of all state response endpoints, at given time and from given initial position, is convex (or at least has convex closure).

For the first, it might be useful to experiment with affine strategies,

$$\sigma[q](t) = u(t) + \int_0^t d_s W(t,s) \cdot q(s)$$

for given u , and matrix-valued W .

CHAPTER V

CAPTURE

In a game it may happen that some positions can be forced to the target, but not isochronously. A particularly simple case is the one-dimensional rocket chase of 1.3: every point can be forced to termination, but none isochronously (see Exercise 1 in 3.4 and the Necessary Condition of 4.2).

Thus, there is a need to study general capture. That this is not a simple extension of previous results is signalled by the fact that the sets of winning positions are not convex even when they naturally ought to be: compare Lemma 1 of 4.1 with Exercise 7 in 5.1. Our results here are, necessarily, much more modest.

Material analogous to that of Sections 4.2 and 3.4 is presented in 5.1 and 5.2. The remaining two sections concern special cases: target sets which are either very large, namely entire half-spaces, or special linear subspaces, with 0 as one possibility. In both situations it turns out that capture is necessarily isochronous,

$$W(0, \theta) = \bigcup_{0 \leq t \leq \theta} W(t)$$

in the notation of 2.2. The two cases are, in a sense, opposite extremes: in the first it sometimes happens that $W(0, \theta) = R^n$ for a finite time θ ; in the second, usually $W(0, t) = W(t)$.

5.1 Necessary Conditions

For the study of general capture, the appropriate concept of winning position, and winning set $W(0, \theta)$, is that

introduced in 2.2. For linear games

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } x &\in \Omega: \text{ state space: } R^n, \end{aligned}$$

the winning set $W(0, \theta)$ consists of all positions $x \in R^n$ to which there corresponds a non-anticipatory strategy σ such that

$$(2) \quad x(t) = e^{At} \left(x - \int_0^t e^{-As} (\sigma[q](s) - q(s)) ds \right) \in \Omega$$

for each quarry control $q(\cdot)$ and some termination time $t \in [0, \theta]$.

The attempt to obtain results parallel to Lemma 2 and Corollary 1 from 4.1 naturally leads to the concept of minimum time of winning, and to the principle of suboptimality in 2.3. The nontrivial winning positions x are those with $x \notin \Omega$ (usage differing from that of Chapter 4, where the term referred to $x \in W(\theta)$ with $\theta > 0$, but $x \in \Omega$ allowed).

LEMMA Assume that P, Q are compact, and Ω closed. If there exist nontrivial winning positions, then even

$$\lim_{t \rightarrow 0+} (W(0, t) \setminus \Omega) \neq \emptyset.$$

(Proof: direct consequence of Corollary 1 in 2.3.)

Consider again the one-dimensional homicidal chauffeur game; in particular, the disposition of its winning sets $W(0, t)$ for small times $t > 0$, fig. 4 in 1.3. We will show that, in an appropriate sense, this situation is quite general (see the Corollary, and Exercise 7). A necessary condition for nontrivial capture will be developed - obviously this must be weaker than the version in 4.2 applying to isochronous capture. Then, by modifying it slightly, a sufficient 'local' condition will be obtained in 5.2.

THEOREM In the game (1) assume that the players'

constraint sets are compact, convex, and symmetric, and that the target is contained within the affine manifold $\{x: Mx = Mb\}$. Then

$$(3) \quad MQ \subset MP + [-1,1] \text{ Max}$$

holds for every point $x \in \lim_{t \rightarrow 0+} (W(0,t) \setminus \Omega)$.

(Proof) First consider any point $x \in W(0,\theta) \setminus \Omega$, and quarry control value $q \in Q$. On choosing a winning strategy, for the constant quarry control q we have, from (2), that

$$Me^{At} \left(x + \int_0^t e^{-As} ds q \right) \in Mb + Me^{At} R_P(t)$$

with $t \in [0,\theta]$; furthermore, $t > 0$ since $x \notin \Omega$ (see Exercise 2 in 2.3). Analogously for the quarry control $-q$:

$$Me^{Ar} \left(x - \int_0^r e^{-As} ds q \right) \in Mb + Me^{Ar} R_P(r), \quad 0 \leq r \leq \theta.$$

Now subtract, divide by $t + r > 0$ and rearrange, using $\lambda = t/(t+r) \in [0,1]$:

$$(4) \quad \begin{aligned} & M \left(\lambda \frac{e^{At} - I}{t} - (1-\lambda) \frac{e^{Ar} - I}{r} \right) x + \\ & + M \left(e^{At} \lambda \frac{1}{t} \int_0^t e^{-As} ds q + e^{Ar} (1-\lambda) \frac{1}{r} \int_0^r e^{-As} ds q \right) \in \\ & M \left(e^{At} \lambda \frac{R_P(t)}{t} - e^{Ar} (1-\lambda) \frac{R_P(r)}{r} \right). \end{aligned}$$

Now return to the assertion; take a point $x \in \lim_{t \rightarrow 0+} (W(0,t) \setminus \Omega)$, find

$$x_k \in W(0, \theta_k) \setminus \Omega, \quad x_k \rightarrow x, \quad \theta_k \rightarrow 0+.$$

Apply (4) to the x_k ; the resulting capture times t_k, r_k converge to 0 since $\theta_k \rightarrow 0$. The weights $\lambda = \lambda_k \in [0,1]$ will have a convergent subsequence $\lambda_j \rightarrow \mu \in [0,1]$. The Limit Theorem for reachable sets then yields

$$M(\mu Ax - (1-\mu)Ax) + M(\mu q + (1-\mu)q) \in M(\mu P - (1-\mu)P),$$

$$(\mu-1)Mx + Mq \in MP$$

from the assumptions on P . Here x and q are independent, and $0 \leq \mu \leq 1$ yields constant bounds for the scalar factor, $-1 \leq \mu-1 \leq 1$. QED

NECESSARY CONDITIONS Under the assumptions of the theorem,

$$MQ \subset MP + [-1,1]c \text{ for some } c \in R^m, \\ \dim MQ \leq 1 + \dim MP$$

are necessary for nontrivial capture.

(Proof) If some $W(O, \theta) \not\subset \Omega$, then, by the lemma, there exist points $x \in \lim (W(O, \theta) \setminus \Omega$, and we may take $c = Mx$. QED

Example The n -dimensional equations of motion are

$$\ddot{x} + A\dot{x} + Bx = u, \dot{y} = Cy - v,$$

with pursuer's constraint set U and quarry's V . The termination condition is perfect capture, $x = y$.

The appropriate state space is R^{3n} (with coordinates x, \dot{x}, y); the matrix M is then $(I \ 0 \ -I)$; P consists of vectors with components $0, u, 0$, and Q of those with $0, 0, v$. Thus $MQ = V$, $MP = 0$, so that by the condition above, nontrivial capture cannot occur unless V is a segment. A very special case of this is the isotropic rocket game

$$\ddot{x} + \alpha x = u, \dot{y} = v$$

in R^2 , with the disc for quarry's constraint set: perfect capture is impossible.

COROLLARY If, in the theorem, the target set is the linear subspace $\{x: Mx = 0\}$, we have the following alternatives: either the necessary condition $MQ \subset MP$ for isochronous capture obtains, or, for small capture times, there are no nontrivial winning positions near the origin:

$$(5) \quad W(0, \epsilon) \cap G \subset \Omega$$

for some $\epsilon > 0$ and neighbourhood G of 0 .

(Proof) Indeed, either $x = 0$ is allowed in (3), or 0 does not belong to $\lim (W(0, t) \setminus \Omega)$, so that (5) holds. QED

Exercises

1. Show that, if P, Q, Ω are symmetric, then so are $W(0, t)$ and $W(0, t) \setminus \Omega$, and $T(x) = T(-x)$. If P, Q, Ω are compact, then each $W(0, t)$ is bounded, and an explicit bound is easily exhibited.

2. Assume that Ω is the affine manifold $\{x: Mx = Mb\}$, and set $N = \{x: Mx = 0\}$, $L = \{x: MA^j x = 0 \text{ for } j = 0, 1, \dots\}$. Prove that $W(0, t) + L \subset W(0, t)$; if $W(0, t)$ contains some $W(s) \neq \emptyset$, then even $W(0, t) + e^{-As}N \subset W(0, t)$.

3. Let Ω, N be as in Exercise 2, and P, Q compact. Show that, if $W(0, \theta)$ contains a ray $x + R^+c$ with $x \notin \Omega$, then $c \in e^{-At}N$ for some $t \in (0, \theta]$. (Hint: Exercise 11 in 4.1; be rather careful in checking that $t > 0$.)

4. Obtain a version of Lemma 2 from 4.1 applying to general capture: if P, Q are compact and Ω closed, then each point x in $W(0, \theta) \setminus \Omega$ satisfies

$$x + R_Q(t) \subset R_P(t) + e^{-At}W(0, \theta - t)$$

for small $t \geq 0$. (Hint: Exercise 2 in 2.3)

5. Discover unused assumptions in the theorem, and prove that, if Q contains a ray with direction d , then either $Md = 0$ or all $W(0, t) = \Omega$.

6. Sharpen the second alternative in the corollary to: there are no winning positions close to the set

$$\Omega \cap A^{-1}(\Omega) = \{x: Mx = 0 = MAX\}.$$

7. Prove that, if the first alternative is the corollary does not apply, then the winning sets $W(0, t)$ cannot be

convex for small $t > 0$ unless they are trivial. (Hint: Exercise 1.)

Remarks

The elements of the set $\lim_{t \rightarrow 0^+} (W(0,t) \setminus \Omega)$, which appeared naturally in the course of developing the Necessary Condition, will be studied in greater detail and generality in Chapter VII.

In the game with n -dimensional equations of motion

$$\ddot{x} = y - q, \dot{y} = p; \text{ end: } x = y$$

all positions are winning ones if $n = 1$ (the one-dimensional homicidal chauffeur), but, according to the example, for $n > 1$ there are no winning positions other than trivial ones. Though this might be blamed on the natural perversity of the number 1, probably a better reason is that the target set separates the state space if $n = 1$ but not otherwise; the case of separation will be treated in 5.3. (For a nonlinear situation see Example 1 in 7.3 and 7.2.)

The second alternative of the corollary requires that, in the terminal stage of successful capture, the state variable be relatively large; e.g., in one-dimensional rocket chase, the speed is at least 1 terminally. Actually, the phenomenon recurs in proposed winning strategies for specific games. See, e.g., references under 'swerve manoeuvre' in

R. Isaacs: *Differential Games*, Wiley, New York, etc., 1967.

CONJECTURE In the game (1) assume that P, Ω are compact, convex, and symmetric, and that Ω is a linear subspace. If a point x belongs to $\lim_{t \rightarrow 0^+} (W(0,t) \setminus \Omega)$, then so does the entire ray $[1, +\infty)x$.

A different type of necessary condition was presented in

L. S. Pontrjagin, E. F. Miščenko: The problem of escape of one controlled object from another, Doklady Akad. Nauk SSSR 189 (1969), 721-723;

in our notation it can be formulated as follows. In (1) let P, Q be compact and convex, and the target the linear subspace $\{x: Mx = 0\}$, with M of type (m, n) and $m \geq 2$. If

$$(6) \quad \begin{aligned} MA^j(Q - Q) = 0 = MA^j(P - P) \text{ for } 0 \leq j \leq k-2, \\ \text{Int } (MA^{k-1}Q \neq MA^{k-1}P) \neq \emptyset, \end{aligned}$$

then nontrivial capture is impossible (furthermore, to every position x with small Mx there corresponds an indifferent evasion strategy, and lower estimates of $|Mx(t)|$ are given). Note the unexpected order of Q and P in (6); and also the innocuous but significant assumption $m \geq 2$: in the case $m = 1$ nontrivial capture is usually possible according to 5.3.

5.2 Sufficient Conditions

In the preceding section a necessary condition for nontrivial capture was obtained, applying to the linear game

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q \\ \text{end: } &Mx = Mb; \text{ state space: } R^n \end{aligned}$$

(M is an (m, n) matrix). The condition may be rewritten as

$$MQ \subset M(P - Ax + [0, 2]Ax) \subset M(P - Ax - R^+d)$$

with $d = -Ax$; this is to hold for all points $x \in \lim(W(0, t) \setminus \Omega)$. The transition from this to a sufficient condition is modelled on the far simpler case of Corollary 2 in 3.4. The basic idea in the proof is 'constant bearing navigation,' properly formulated in Corollary 1.

THEOREM In the game (1) let the players' constraint sets be nonvoid and compact. If a point $x \in \Omega$ satisfies

$$(2) \quad MQ \subset \text{Int } M(P - Ax - R^+d)$$

(interior in R^m) for some vector $d \in R^n$, then there are non-trivial winning positions near x . More precisely, if also $Md \neq 0$, then

$$(3) \quad x + (0, \delta)d \subset \text{Int } (W(0, \epsilon) \setminus \Omega)$$

holds for small $\epsilon, \delta > 0$.

(Proof) Since MQ is compact, the term R^+d in (2) can be replaced by $[\alpha, \beta]d$ with $0 < \alpha < \beta < \infty$. For the same reason, the containment is preserved if the entries are varied slightly: x replaced by $y = x + \xi d$ (for $0 < \xi < \delta$ with δ small; note that then $y \notin \Omega$); and the term Q and $P - Ay$ perturbed:

$$Me^{A\theta}Q \subset Me^{A\theta}(P - Ay) - [\alpha, \beta]Md$$

for all $\theta \in [0, \eta]$ and $\eta > 0$ small enough. The point here is that $\theta \mapsto e^{A\theta}$ is continuous, with initial value I at $\theta = 0$.

Thus, for each $q \in Q$ and $\theta \in [0, \eta]$, there exists $p \in P$ and $\lambda \in [\alpha, \beta]$ such that

$$Me^{A\theta}q = Me^{A\theta}(p - Ay) - \lambda Md.$$

Now use Filippov's Lemma from 3.9 to describe these solutions p, λ as values of suitable mappings

$$\sigma: Q \times R^1 \rightarrow P, \varphi: Q \times R^1 \rightarrow [\alpha, \beta]$$

which preserve measurability. We propose to show that σ is a winning strategy (stroboscopic but not isochronous) for y as initial position.

If $0 \leq s \leq t \leq \eta$ and $q(\cdot)$ is any quarry control, then

$$Me^{A(t-s)}q(s) = Me^{A(t-s)}(\sigma(q(s), t-s) - Ay) - \varphi(q(s), t-s)Md.$$

Now integrate, $\int_0^t ds$, using $\int_0^t e^{A(t-s)}(-Ay)ds = e^{At}y - y$ and $y = x + \xi d$, $Mx = Mb$ to obtain

$$\begin{aligned} Me^{At}(y - \int_0^t e^{-As}(\sigma(q(s), t-s) - q(s))ds) &= My - \int_0^t \varphi(\cdot)ds \, Md \\ &= Mb + (\xi - \int_0^t \varphi(q(s), t-s)ds)Md. \end{aligned}$$

This has been assembled into the form

$$(4) \quad My(t) = Mb + \psi(t)Md,$$

where $y(\cdot)$ is the state response from initial position y , and ψ denotes the bracketed term. Obviously ψ is real-valued, continuous since $y(\cdot)$ is such and $Md \neq 0$. It remains to show that $\psi(t) = 0$ for small $t > 0$ (possibly depending on $q(\cdot)$). Since $\alpha \leq \varphi(\cdot) \leq \beta$, the function

$$\psi(t) = \xi - \int_0^t \varphi(\cdot) \text{ satisfies } \psi(0) = \xi > 0, \psi(t) \leq \xi - \alpha t.$$

Thus indeed $\psi(t) = 0$ for some $t \in (0, \epsilon]$, where $\epsilon = \min(\delta/\alpha, \eta)$.

This shows that the points $x + \xi d$ with $0 < \xi < \delta$ are in $W(0, \epsilon)$ and outside Ω . Letting d vary slightly so that (2) is preserved, we obtain an open set of winning positions, as asserted in (3). QED

An interesting effect appeared as (4) in the preceding proof: if $Mb = 0$ for simplicity, the observations $My(t)$ of the state responses maintain a constant direction (fig. 1).

COROLLARY 1 To the positions y on $x + (0, \delta)d$ there correspond winning strategies such that the state responses $y(\cdot)$ satisfy

$$My(t) \in Mb + R^+Md$$

for all times up to termination.

COROLLARY 2 If the target is a linear subspace $\{x: Mx = 0\}$ and a point x in it satisfies $MQ \subset \text{Int } M(P + R^1Ax)$, then all points θx with large $|\theta|$ belong to $\lim \text{Int}(W(0, t) \setminus \Omega)$.

$$\lim_{t \rightarrow 0^+} \text{Int}(W(0, t) \setminus \Omega)$$

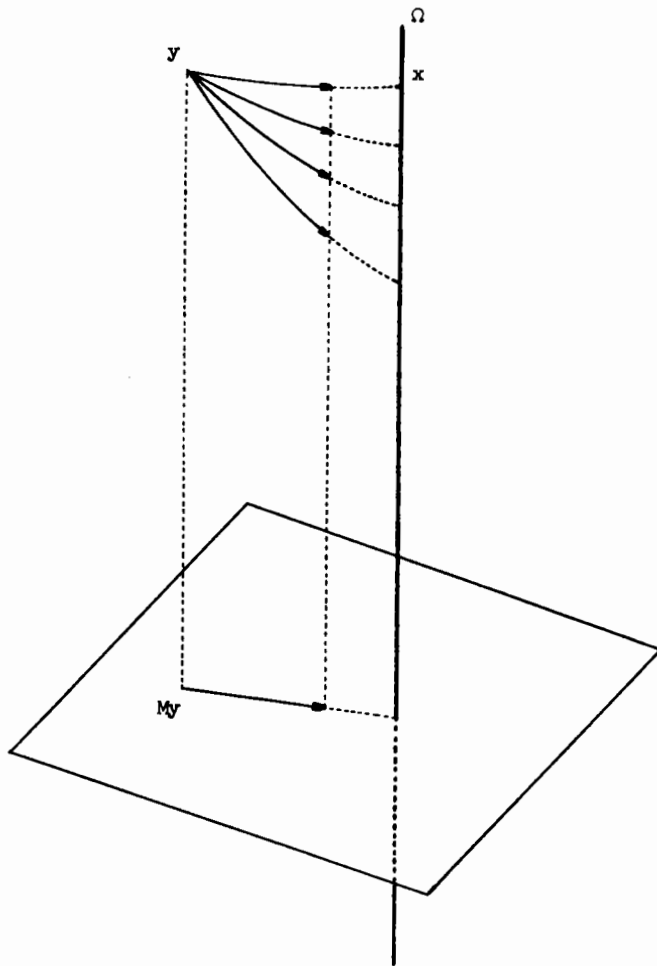


Fig. 1 (schematic) Constant Bearing Navigation: various state responses project onto a single direction.

(Proof) From compactness, $MQ \subset \text{Int } M(P + [\alpha, \beta]Ax)$ for finite $\alpha \leq \beta$; then

$$MQ \subset \text{Int } M(P - A\theta x - [\alpha - \theta, \beta - \theta]Ax)$$

yields (2) for the point θ , direction $d = Ax$, and all $\theta \leq \alpha$. Analogously for $-x$ in place of x . QED

Example Consider the two-dimensional game with highly mobile quarry,

$$\dot{x}_1 = x_2, \dot{x}_2 = u, \dot{y}_1 = v_1, \dot{y}_2 = v_2$$

and perfect capture $x_1 = y_1, x_2 = y_2$ as termination condition. The quarry's control constraint set is the unit disc, $v_1^2 + v_2^2 \leq 1$. The only advantage that the pursuer has is his slightly greater capability in one direction only:
 $|u(t)| \leq 1.1$.

The state space is R^4 , the observation matrix $M = (I, -I)$ has type $(2,4)$. With the notation from the theorem, the sets MP and MQ are as indicated in fig. 2; in particular, isochronous capture is impossible (see 4.2). Next,

$$MA = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

thus a point x in the target set, with coordinates x_1, x_2, x_3, x_4 has

$$MAx = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

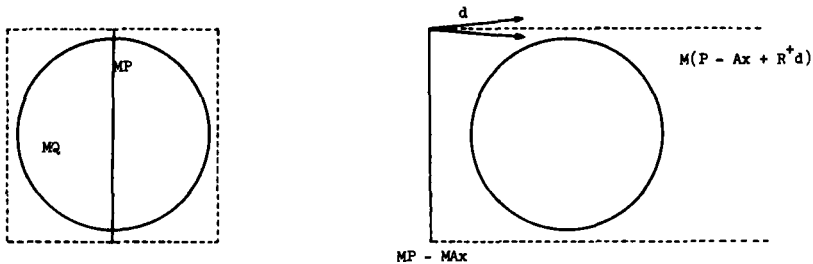


Fig. 2 Illustration of necessary, sufficient conditions for capture.

For the Necessary Condition of 5.1 we investigate all the shifts of MP by vectors between MAX and -MAX. Referring to fig. 2 again, these will cover MA if $|x_2| \geq 1$: the necessary condition for capture. The sufficient condition from the theorem is also illustrated there; $|x_2| > 1$ is needed, and the directions d are obvious. Mere presence of nontrivial winning positions is easiest obtained from Corollary 2.

Exercise

For the game with n-dimensional equations of motion

$$\dot{x} = Ax - p, \dot{y} = By - q; \text{ end: } x = y$$

find conditions for presence of nontrivial winning positions. (Remark: the case $A = B$ is treated separately as Exercise 2 of 3.1.)

Remarks

This section is based on

O. Hájek: The principle of constant observation
(submitted)

The form of the sufficient condition in the theorem suggests that the necessary condition from 5.1 is reasonably sharp, that there are no further independent necessary conditions (for presence of nontrivial winning positions; if one also demands complete capture, further requirements will appear).

The geometric interpretation (fig. 1) is that, in the terminal stage of capture, the pursuer adopts constant bearing navigation, of maintaining constant direction of the observed velocity vector $\dot{Mx}(t)$. E.g., (3) in 5.1 may be read as

$$\dot{Mx} = M(Ax - p + q) \in R^+Md$$

for $\dot{d} = -Ax$.

The principle extends to nonlinear games: see 7.3 and 7.2; the formulation applying to games without separation of the players is somewhat unexpected (Theorem 1 in 7.3).

Neither result applies well to second-order games, for instance to the example in 3.4. A necessary condition follows from more general considerations in 7.3: for the linear game

$$(5) \quad \ddot{x} + A\dot{x} + Bx = p - q; \text{ end: } Mx = Mb,$$

with assumptions on P and Q , this reads

$$MQ \subset MP + [-1, 1](A\dot{x} + Bx) + R^1 M\dot{x}$$

for points (x, \dot{x}) in $\lim(W(0, t) \setminus \Omega)$; in particular,

$$\dim MQ \leq 2 + \dim MP.$$

A sufficient condition for this game is attempted in the reference. It would be very desirable to obtain useful conditions for arbitrary player orders, as was carried out in the case of isochronous capture.

A second project concerns 'bracketing' of quarry by several weaker pursuers. Thus, let the n -dimensional ($n > 1$) equations of motion be

$$\dot{x}_k = u_k \quad (1 \leq k \leq r), \quad \dot{y} = v,$$

say with constraints $|u_k(t)| \leq 1$, $|v(t)| \leq 2$, and some $x_k = y$ at termination. Are there any nontrivial winning positions? Isochronous capture has been ruled out by the example in 4.2.

One conclusion from 5.1 and 5.2 is that many one-dimensional ('rectilinear', i.e., with rank $M = 1$) games are winnable, while most planar games are not unless they are of special form such as (5). This might be made more precise by defining an appropriate manifold, whose elements are games in fixed dimensions, and showing that the winnable games constitute a set which does, or does not, have nonvoid interior.

5.3 Large Targets

The games treated in this section,

$$(1) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q;$$

have rather large targets: entire half spaces of the state space.

Obviously there is a closely related situation where the target is the boundary hyperplane $\{x: c'x = \alpha\}$ (or the slab described by $|c'x| \leq \epsilon$, etc.) Indeed, every nontrivial position is forced to the hyperplane; conversely, forcing to the hyperplane is equivalent to forcing to one of the half-spaces

$$\{x: c'x \leq \alpha\}, \{x: c'x \geq \alpha\}$$

depending on the initial position. In the terminology of 6.1 the hyperplane targets correspond to game spaces of dimension 1; for these the Necessary Condition of 5.1 is no restriction (i.e., provides no information).

It will be convenient to introduce at the outset some common assumptions and notation.

ASSUMPTIONS In (1) the player constraint sets are non-void and compact; the normal vector $c \neq 0$.

Using Filippov's Lemma, choose player controls $\bar{p}(\cdot)$, $\bar{q}(\cdot)$ which maximise in the sense that, for all $t \in R^1$,

$$(2) \quad c'e^{At}\bar{p}(t) = \max_{p \in P} c'e^{At}p, \quad c'e^{At}\bar{q}(t) = \max_{q \in Q} c'e^{At}q.$$

Define $c_t \neq 0$ in R^n and $\alpha_t \in R^1$ by

$$(3) \quad c'_t = c'e^{At}, \quad \alpha_t = \alpha + \int_0^t c'_s(\bar{p}(s) - \bar{q}(s))ds.$$

Note that the integrand contains e^{As} , not our usual e^{-As} ; one reason for this is that the condition for state response endpoint $x(t) \in \Omega$ may usefully be rewritten as

$$(4) \quad \alpha \geq c'_t x(t) = c'_t x - \int_0^t c'_s p(t-s) ds + \int_0^t c'_s q(t-s) ds.$$

In this notation, (2) is

$$(5) \quad c'_t \bar{p}(t) = \max_{p \in P} c'_t p, \quad c'_t q(t) = \max_{q \in Q} c'_t q.$$

LEMMA 1 For each $t \geq 0$, the isochronous winning set

$$W(t) = \{x: c'_t x \leq \alpha_t\};$$

furthermore, $\sigma(s) = \bar{p}(t-s)$ defines an indifferent (hence, stroboscopic) isochronous winning strategy, independent of positions $x \in W(t)$.

(Proof) First take any $x \in R^n$ in the indicated half-space, and use the strategy σ . Then the state endpoint $x(t)$, responding to any quarry control $q(\cdot)$, has

$$\begin{aligned} c'_t x(t) &= c'_t x - \int_0^t c'_s \sigma(t-s) ds + \int_0^t c'_s q(t-s) ds \\ &\leq c'_t x - \int_0^t c'_s \bar{p}(s) ds + \int_0^t c'_s \bar{q}(s) ds \end{aligned}$$

(we have estimated the last term via (5)); continue, using (3), to obtain

$$= \alpha + (c'_t x - \alpha_t) \leq \alpha + 0.$$

Thus indeed σ forces x to Ω at t .

This establishes one inclusion in the assertion. For the other, take any $x \in W(t)$, and consider any winning isochronous strategy σ . Then $x(t) \in \Omega$, so that (4) holds with $p = \sigma[q]$ for each admissible quarry control. In particular, we may use $q(s) = \bar{q}(t-s)$, and estimate the second term via (5) to obtain

$$\alpha \geq c'_t x - \int_0^t c'_s \bar{p}(s) ds + \int_0^t c'_s \bar{q}(s) ds.$$

Thus indeed $\alpha_t \geq c'_t x$. QED

LEMMA 2 For each winning position x outside Ω , the minimum time $T(x)$ coincides with the first time $t \geq 0$ for which $c'_t x = \alpha_t$.

(Proof) We shall establish the two inequalities $T(x) \leq \min t \leq T(x)$. The first follows quickly from Lemma 1: if $c'_t x = \alpha_t$, then $x \in W(t)$, so that $T(x) \leq t$.

The proof of the second is considerably more complicated, since we cannot assume that the terminal times are independent of quarry's choices. Choose any point $x \notin \Omega$ in position to win, and then a strategy σ which forces x to Ω within a bounded time interval. Then (4) holds with $p = \sigma[q]$, and

$$t = T(x, p, q) \leq T(x, \sigma) < +\infty$$

for any quarry control $q(\cdot)$ (the T -notation was introduced in 2.3). Use (5) to estimate the first integral term; and, for any $\theta \in \mathbb{R}^1$, choose the quarry control q_θ , $q_\theta(s) = \bar{q}(\theta - s)$:

$$(6) \quad \alpha \geq c'_t x - \int_0^t c'_s \bar{p}(s) ds + \int_0^t c'_s \bar{q}(\theta - t + s) ds$$

for $t = T(x, \sigma[q_\theta], q_\theta)$. Thus the first time $t(\theta) \geq 0$ such that (6) holds satisfies

$$(7) \quad 0 \leq t(\theta) \leq T(x, \sigma[q_\theta], q_\theta) \leq T(x, \sigma).$$

(Observe that $t(\theta)$ depends on q_θ and hence on θ , but not on σ .)

For reasons possibly obvious from (6), we wish to find a fixed point $\theta = t(\theta)$ of $t(\cdot)$; first we show that the function $t(\cdot)$, bounded according to (7), is lower semi-continuous. For any $\theta_0 \in \mathbb{R}^1$ find a sequence $\theta_k \rightarrow \theta_0$ such that $t(\theta_k)$ converges to $\liminf_{\theta \rightarrow \theta_0} t(\theta) = t$. Taking limits in

(6) with $\theta = \theta_k$ and $t = t(\theta_k)$, we find that (6) also holds for $\theta = \theta_0$ and $t = t_0$, and hence the first time $t(\theta_0) \leq t_0$ as asserted (you may wish to rewrite the second integral term

as $\int_0^t c' e^{A(t-s)} \bar{q}(\theta-s) ds$ and note that $\bar{q}(\theta_k-s) \rightarrow \bar{q}(\theta-s)$

weakly: for any $\alpha \leq \beta$,

$$\begin{aligned} & \int_{\alpha}^{\beta} \bar{q}(\theta_k-s) ds - \int_{\alpha}^{\beta} \bar{q}(\theta-s) ds = \\ & = \left(\int_{\theta_k-\beta}^{\theta_k-\alpha} - \int_{\theta-\beta}^{\theta-\alpha} \right) \bar{q}(s) ds = \left(\int_{\theta-\alpha}^{\theta_k-\alpha} - \int_{\theta-\beta}^{\theta_k-\beta} \right) \bar{q}(s) ds \end{aligned}$$

has norm bounded by $2|\theta_k-\theta| \text{diam } Q \rightarrow 0$.

With this established, consider the set

$$\Theta = \{\theta \in \mathbb{R}^1: t(\theta) \leq \theta \leq T(x, \sigma)\}.$$

It is bounded, as all $t(\theta) \geq 0$; closed, since $t(\cdot)$ is lower semi-continuous; and nonvoid, since it contains at least the point $\theta = T(x, \sigma)$. It follows that $\theta_0 = \inf \Theta$ itself belongs to Θ , $t(\theta_0) \leq \theta_0$; and strict inequality is precluded by semi-continuity. We have shown that

$$t(\theta_0) = \theta_0 \text{ for some } \theta_0 \in [0, T(x, \sigma)].$$

Now, in (6) with $t = t(\theta)$, choose $\theta = \theta_0$; there results $\alpha_{\theta_0} \geq c'_{\theta_0} x$. Thus, in the original assertion, there exists a first t (note $\alpha < c'x$ as $x \notin \Omega$) and $t \leq T(x, \sigma)$. Finally, take infima over strategies σ to obtain $t \leq \inf T(x, \sigma) = T(x)$: the second inequality. QED

THEOREM In the game (1) assume that the players' constraint sets are nonvoid compacta, and the target set a non-trivial half-space. Then capture is isochronous and stroboscopic,

$$(8) \quad W(0, \theta) = \cup \{W(t): 0 \leq t \leq \theta\} = \{x: T(x) \leq \theta\}$$

and, in the previous notation,

$$(9) \quad W(t) = \{x: c'_t x \leq \alpha_t\}, \quad \partial W(t) = \{x: c'_t x = \alpha_t\}.$$

Furthermore, to each winning position x there corresponds an optimal pursuer strategy which is indifferent, isochronous, and independent of positions with the same minimum time:

$$\sigma(s) = \tilde{p}(T(x) - s).$$

(Proof) For (8), quite generally

$$\bigcup_{0 \leq t \leq \theta} W(t) \subset W(0, \theta) \subset \{x: T(x) \leq \theta\}$$

(see 5.1, (4) and Exercise 4). If $T(x) \leq \theta$, then either $x \in W(0) = \Omega$, or $x \notin \Omega$; in the second case $c'_t x = \alpha_t$ for $t = T(x)$ by Lemma 2, and then $x \in W(t)$ by Lemma 1, with $t = T(x) \leq \theta$.

The first relation in (9) is in Lemma 1, and implies the second (note $c \neq 0$ from the assumptions). The remaining assertions also follow from Lemmas 1 and 2. QED

One illustration is the treatment of the one-dimensional rocket chase in 1.3; our theorem now provides proper justification for the plausible arguments used there.

Example 1 In terms of coordinates in R^2 , the game has equations

$$\dot{x} = y + u + v, \quad \dot{y} = -x - u + v;$$

pursuer's control u and quarry's v are both constrained to the interval $[-1, 1]$; the termination condition is $x \leq 0$.

Explicitly, the data are as follows: $n = 2$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \alpha = 0$$

$$P = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} u : |u| \leq 1 \right\} \quad Q = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} v : |v| \leq 1 \right\}$$

Then

$$c'_t = c' e^{At} = (\cos t, \sin t),$$

$$c'_t \bar{p}(t) = \max_u c'_t \begin{pmatrix} -1 \\ 1 \end{pmatrix} u = |\sin t - \cos t|, c'_t \bar{q}(t) = |\sin t + \cos t|,$$

$$(10) \quad \alpha_t = \int_0^t (|\sin s - \cos s| - |\sin s + \cos s|) ds.$$

Let us first treat the case $t \leq \pi/4$, i.e., describe $W(0, \pi/4)$; then (10) reduces to

$$\alpha_t = \int_0^t -2 \sin s \, ds = -2(1 - \cos t),$$

so that $W(t)$ consists of all points (x, y) with

$$x \cos t + y \sin t \leq -2(1 - \cos t).$$

The boundary lines $\partial W(t)$ have an enveloping curve, an arc of the circle $(x-2)^2 + y^2 = 4$. It follows that the boundary of $\partial W(0, \pi/4)$ consists of two rays and a circular arc, see fig. 1.

For constant player controls u, v the trajectories satisfy $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$,

$$(x + u - v)dx + (y + u + v)dy = 0,$$

$$(x + u - v)^2 + (y + u + v)^2 = \text{const.}$$

Hence they are (arcs of) concentric circles; according to the dynamical equation, the motion is clockwise. In particular, within $W(0, \pi/4)$, for optimal $u = -1$ the centre is at $(2, 0)$ for the worst $v = 1$, and at $(0, 2)$ for cooperative $v = -1$.

What is qualitatively different from one-dimensional rocket chase is that the enveloping curve is outside Ω : $W(\pi/4)$ is properly contained in $W(0, \pi/4)$. One consequence is that the minimal time function is discontinuous in a spectacular manner. Consider a position $x \in \mathbb{R}^2$ outside $W(0, \pi/4)$ but close to the circular arc, and use controls $u = v = 1$. Along the resulting trajectory, $T(x(t)) > \pi/4$ up to the time t_0 when $x(t)$ meets the arc; then $T(x(t_0)) < \frac{\pi}{4}$, value

depending on which tangent, i.e., $\partial W(s)$, is reached.

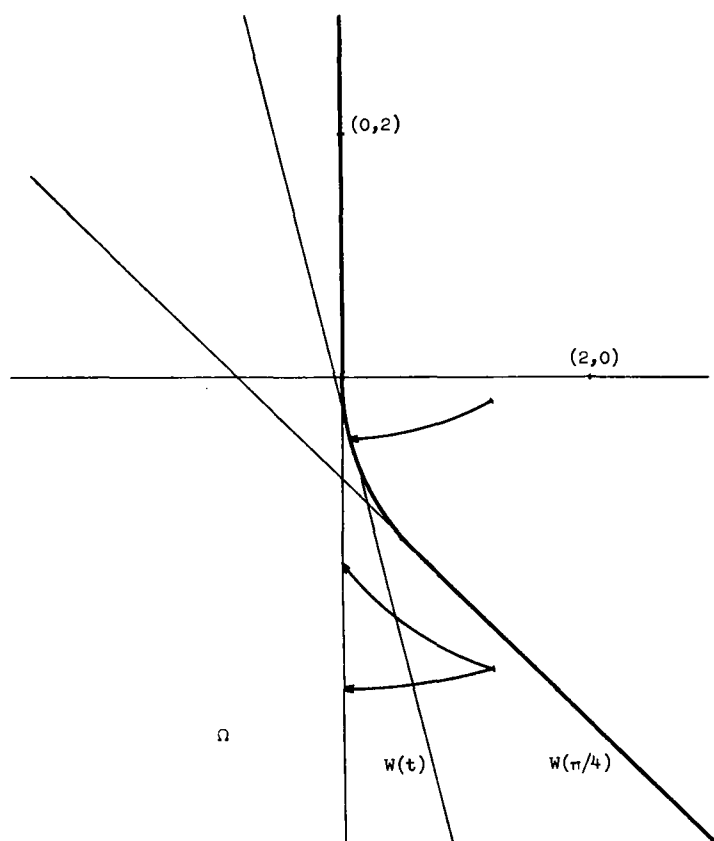


Fig. 1 Forcing to half-plane. The piecewise smooth curve is the boundary of $W(0, \pi/4)$; tangents to its circular arc are boundaries of isochronous $W(t)$ for $0 \leq t \leq \pi/4$.

Returning to (10), $W(t)$ is characterised by

$$(11) \quad x \cos t + y \sin t \leq \alpha_t.$$

At $t = \pi$ we obtain $x \geq 0$ (note that indeed the integrand in (10) has mean value 0 over $[0, \pi]$). It follows that every point in R^2 belongs to some $W(t)$ for $0 \leq t \leq \pi$: and the

conclusion $\sup T(x) = \pi$ is quite plausible geometrically.

The result for $W(0, \pi/4)$ suggest looking for circular enveloping curves. The condition that the boundary line of (11) be tangent to the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ is that r be its distance from the centre.

$$x_0 \cos t + y_0 \sin t - \alpha_t = \pm r$$

The circle will be common to the lines iff the expression on the left is constant; setting the derivative equal to zero,

$$(12) -x_0 \sin t + y_0 \cos t = \frac{d\alpha_t}{dt} = |\sin t - \cos t| - |\sin t + \cos t|.$$

Now, only $t \leq \pi$ need be treated, and $t \leq \pi/4$ already has; naturally we simplify by breaking up at $t = 3\pi/4$.

For $\pi/4 \leq t \leq 3\pi/4$ we have $d\alpha_t/dt = -2 \cos t$, (12) becomes $-x_0 \sin t + (y_0 + 2) \cos t = 0$, and the centre is $(x_0, y_0) = (0, -2)$; analogously, for $3\pi/4 \leq t \leq \pi$ the centre is $(-2, 0)$.

Example 2 The game is induced by the n -th order one-dimensional equation

$$(13) \quad x^{(n)} = \sum_{k=0}^{n-1} a_k x^{(k)} + u + v,$$

with control constraints $|u(t)| \leq \mu$, $|v(t)| \leq 1$, and termination condition $|x| \leq \epsilon$.

In the state space description, corresponding to the two conditions $x \leq \epsilon$ or

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & & 1 \\ a_0 & a_1 & a_2 & & a_{n-1} \end{pmatrix}, \quad P = \{bu: |u| \leq \mu\},$$

$$Q = \{bv: |v| \leq 1\},$$

$$\pm c' = (1, 0, \dots, 0), \quad b' = (0, \dots, 0, 1).$$

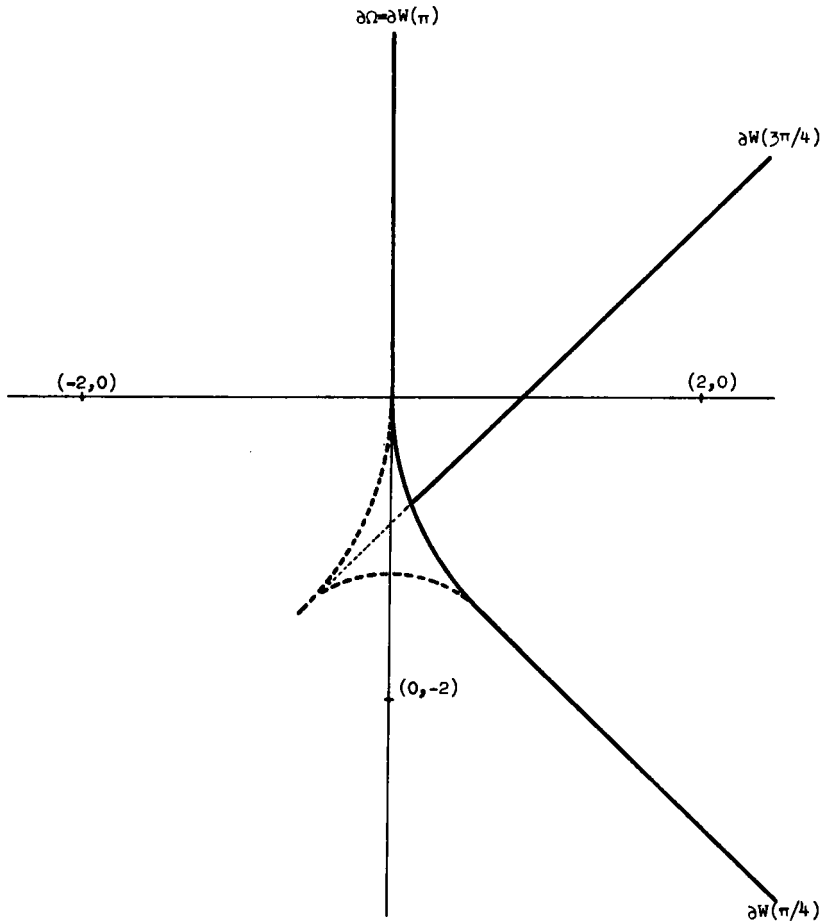


Fig. 2 Forcing to half-plane. The three circular arcs are (clockwise) envelopes of $\partial W(t)$ for t in $[0, \pi/4]$, $[\pi/4, 3\pi/4]$, $[3\pi/4, \pi]$.

The fundamental matrix solution $e^{At} = (\varphi_k^{(j-1)}(t))$, where $\varphi_1, \dots, \varphi_n$ form a fundamental system (of solutions of (13)) at 0, with $\varphi_k^{(j-1)}(0) = 0$ for $j \neq k$ and $\varphi_k^{(k-1)}(0) = 1$. Now,

CAPTURE

$$\begin{aligned}c'_t &= c'e^{At} = (\varphi_1(t), \dots, \varphi_n(t)), \\ \max_{p \in P} c'e^{At}p &= \max_{u \in U} c'e^{At}bu = \mu |\varphi_n(t)|, \\ \alpha_t &= \epsilon + (\mu-1) \int_0^t |\varphi_n(s)| ds.\end{aligned}$$

The equation characterising $\partial W(t)$, in terms of coordinates ξ_k of $x \in R^n$, is thus

$$(14) \quad \sum_{k=1}^n \xi_k \varphi_k(t) = \epsilon + (\mu-1) \int_0^t |\varphi_n(s)| ds.$$

From the initial values, $\varphi_n(s) > 0$ for small $s > 0$. Thus, for small $t > 0$, (14) is

$$(\xi_1 - \epsilon) + \sum_1^{n-1} \frac{\xi_{k+1}}{k!} t^k + \left(\frac{1}{n!} \sum_1^n \xi_k a_{k-1}^{\mu-1}\right) t^n + O(t^{n+1}) = 0,$$

where we have used

$$\varphi_k(t) = \frac{1}{(k-1)!} t^{k-1} + \frac{a_{k-1}}{n!} t^n + O(t^{n+1}).$$

The essential points, in the sense of Exercise 8 or 7.1, satisfy $\xi_1 = \epsilon, \xi_2 = 0$.

Exercises

1. In Example 1 of the main text, show that points (x, y) with $y \geq 0$ and small $x > 0$ have minimum times arbitrarily close to π . (Hint: check the positions of $\partial W(t)$ with $t \rightarrow \pi^-$.)

2. Again in Example 1, show that the two sets

$$\partial\{x: T(x) \leq \pi/4\}, \{x: T(x) = \pi/4\}$$

do not coincide.

3. Treat the game in R^2 with equations $\dot{x} = y + v$, $\dot{y} = -x - u$, constraint sets $[-1, 1]$, and $x \leq 0$ at termination. In particular, show that $\sup T(x) = \pi$; both \bar{p}, \bar{q} are constants (a.e.) in $[0, \pi]$; and the envelope of the bounding lines is entirely within Ω , $W(0, t) = W(t)$.

4. Treat analogously the game with one-dimensional equation $\ddot{x} + x = u + v$; $|u(t)| \leq \mu$, $|v(t)| \leq 1$, and $\dot{x} \leq 0$ at termination. (Partial answers: $\alpha_t = (\mu-1) \int_0^t |\cos s| ds$; $\sup T(x) = \pi$; the enveloping circle for $\partial W(t)$ with $0 \leq t \leq \pi/2$ is somewhat surprising. Hint: treat separately $\mu > 1$, $\mu = 1$, $0 \leq \mu < 1$.)

5. Show that, in (2), $t \mapsto c'e^{At} \bar{p}(t)$ is continuous, even though every maximizing \bar{p} may be discontinuous.

6. How do the principal assertions of this section change if the target is a hyperplane rather than a half-space? The interesting point is whether the strategies are isochronous, stroboscopic, indifferent.

7. If the hyperplanes $\partial W(t)$ with t near some t_0 have no envelope, they must be parallel but distinct. Show that necessarily c is an eigenvector of A' , corresponding to a real eigenvalue, and $c'(\bar{p} - \bar{q})$ is nonconstant near t_0 .

8. Prove that, if points x_t are in the intersection of $\partial \Omega$ and $\partial W(t)$, then every finite accumulations point $x_0 \in \partial \Omega$ (cf. essential points in 7.1) as $t \rightarrow 0$ satisfies

$$c'Ax_0 = \left. \frac{d\alpha_t}{dt} \right|_{t=0},$$

where the right hand side may equivalently be replaced by

$$[c'e^{At}(\bar{p}(t) - \bar{q}(t))]_{t=0} = \lim_{t \rightarrow 0+} c'(\bar{p}(t) - \bar{q}(t)).$$

Verify this for both examples and Exercises 2 and 3.

9. More generally, if x is in the intersection $\partial W(t)$ and an infinitesimally close $\partial W(t + dt)$, then

$$c'e^{At}(Ax - p(t) + q(t)) = 0.$$

(Hint: you may wish to use Exercise 5.)

10. Assume that Q is a polytope, and let $\Delta > 0$ be the discontinuity of $\bar{q}(\cdot)$: thus, for a constant $q \in Q$,

$c'e^{At} \bar{q}(t) = \max_{q \in Q} c'e^{At} q$ for $0 \leq t \leq \Delta$. Prove the following

(fourth) reciprocity theorem for (1): A point $x \in R^n$ can be forced to Ω within $[0, \Delta]$ iff x can be steered to Ω at a time in $[0, \Delta]$ within the control system

$$\dot{x} = Ax - u; u(t) \in P - \bar{q}.$$

11. Show that the mapping $t \mapsto \alpha_t$ defined in (3) is invariant under nonsingular linear transformations.

12. Assume that both P, Q are polytopes. Prove that all $W(0, t) = \Omega$ if, and only if, the normal c is a real eigenvector of A' and $c'(\bar{p}(0+) - \bar{q}(0+)) \leq 0$. (Hint: For small $t > 0$ necessarily $c'_t x \leq \alpha_t$ implies $c'x \leq \alpha$.)

13. Formulate and prove an assertion analogous to the preceding in the case that the target is a hyperplane.

14. For a game with dynamical equation $\dot{x} = f(x, p, q)$, the Isaacs' (main) equation is

$$(15) \quad \min_p \max_q D\phi(x) f(x, p, q) = 0$$

(here $D\phi = (\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n})$); its solutions $\phi(\cdot)$ define the

'semipermeable surfaces' $\phi(x) = \text{const}$. In the game from Example 1 find the semipermeable surfaces through the origin.

(Hint: from (15) at 0, one of $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ is 0, and the other is positive or negative; the four cases reduce to two.

Answer: quarter circles with centres at $(0, \pm 2)$.)

15. Proceeding as in Example 2, treat the game induced by two scalar equations

$$x^{(n)} + \sum_0^{n-1} a_k x^{(k)} = u, y^{(m)} + \sum_0^{m-1} b_k y^{(k)} = v,$$

with control constraints $|u(t)| \leq \mu$, $|v(t)| \leq \nu$, and termination condition $|x - y| \leq \epsilon$. In particular, describe the essential points.

16. Apply the Second Reciprocity Theorem to (1); in

particular show that the constraint set of the associated control system is

$$V_t = \{v \in \mathbb{R}^n: |c'_t v| \leq c'_t(\bar{p}(t) - \bar{q}(t))\}$$

if P, Q are compact, convex, and symmetric.

17. Show that the case that $W(t)$ increase monotonically is quite exceptional if $n \geq 3$. (Hint: compare fig. 1 with fig. 4 in 1.3; show that the intersections $W(t) \cap W(0)$ must be parallel, and then argue as in Exercise 7.)

Remarks

As suggested by the examples and exercises, for applying the method it is crucial that $\max_p c' e^{At} p$ be simple to compute. The geometric interpretation is that of a hyperplane with normal vector $e^{A't} c$ moving (as t varies) but remaining in contact with the set P : the points of contact maximize, and any measurable selection provides a maximizing mapping $\bar{p}(\cdot)$.

If P is a polytope, $p(\cdot)$ may be taken piecewise constant, with discontinuities having no finite accumulation point (whereupon $t \mapsto \alpha_t$ is C^1 -smooth and piecewise analytic). This assertion is an integral part of control theoretic lore; see, e.g.,

N. Levinson: Minimax, Liapunov and "bang-bang",
J. Diff. Equations 2 (1966) 218-241,

(where our present $c' e^{At}$ is even replaced by a piecewise analytic matrix). More information is available about the disposition of discontinuities of $\bar{p}(\cdot)$: under further assumptions there exists a constant $\epsilon > 0$ such that $\bar{p}(\cdot)$ has less than n discontinuities in each interval of length ϵ (counting multiplicities; there are no further restrictions). See

O. Hájek: Terminal manifolds and switching locus,
Math. Systems Theory 6 (1973) 289-301;

D. S. Yeung: Synthesis of time-optimal control
(thesis), Case Western Reserve University,
1974;

E. N. Chukwu and O. Hájek: Optimal control and
disconjugacy (submitted).

That the number involved is precisely $n - 1$ is of fundamental importance for construction of optimal feedback in control systems: it is the correct number of independent parameters for describing the piecewise smooth boundaries of reachable sets x in R^n .

It would be most welcome to have a simple method for obtaining enveloping hypersurfaces for the sets $\partial W(t)$: the classical one is rather unpleasant.

CONJECTURE For t in an interval where \bar{p} is constant, the hyperplanes $\partial W(t)$ have the envelope $F(x) = 0$, where F is of degree 2:

$$F(x) = x'Wx + w'x + \omega.$$

This is the obvious first guess from the examples treated; Exercise 9 might be useful; possibly $F(x) = (x - x_0)'A'A(x - x_0) - \rho$ for suitable $x_0 \in R^n$, $\rho \geq 0$:

PROBLEM Decide whether $f(x) = \bar{p}(T(x))$ is a feedback pursuer strategy, possibly even optimal.

In Chapter 8 of

R. Isaacs: Differential Games, Wiley, New York, etc.,
1967

there are proposed two procedures for finding 'game barriers' (these are put forward as working conjectures, with counter-examples, and not as universally valid results): that the barrier is the envelope of the sets $\{x: T(x) = \theta\}$ (the Envelope Principle p. 260); and that it may be obtained by

'passing a semipermeable surface' through the 'boundary of the useable portion' of the target set (p. 215). As consequence of the results in this section, neither prescription works for linear games with half-space targets: Example 1 (the envelope of the isochrones has a totally different function), and Exercise 14.

5.4 Invariant Targets

We shall now treat games

$$(1) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \quad \text{end: } x \in \Omega$$

which satisfy the following over-all

ASSUMPTIONS The player constraint sets P and Q are nonvoid, compact, convex, and symmetric; more important, the target set is a linear subspace invariant under the coefficient matrix A :

$$(2) \quad A\Omega \subset \Omega = \{x: Mx = 0\}.$$

Subspaces invariant under the coefficient matrix have already appeared, in 3.7; as mentioned there, there are obvious special cases: $\Omega = 0$ and all A , or $A = 0$ and all Ω . The condition $MQ \subset MP$ in the following theorem coincides with a necessary condition for isochronous capture (see 4.2).

THEOREM 1 Under the assumptions on (1), capture is isochronous. Furthermore, either $MQ \subset MP$, whereupon all $W(0,t) = W(t)$ and positions forced to Ω can be held there subsequently; or, in the trivial case $MQ \not\subset MP$, we have $W(t) = \emptyset$ for all $t > 0$, and $W(0,t) = W(0) = \Omega$.
(Proof) First assume that some $W(0,\theta) \not\subset \Omega$. By the Lemma and Exercise 6 in 5.1,

$$\emptyset \neq \lim_{t \rightarrow 0+} (W(0,t) \setminus \Omega) \subset \lim_{t \rightarrow 0+} W(0,t) = \Omega.$$

Thus there exists a point y which may be used in the

proposition of 5.1: $MQ \subset MP + [-1,1]MAy$; and, as $y \in \Omega$, the last term has $MAy \in MA\Omega \subset M\Omega = \emptyset$. Thus

$$(3) \quad MQ \subset MP, \quad Q \subset P + \Omega.$$

Now, assume that (3) holds. There exists a measurability-preserving mapping $\sigma: Q \rightarrow P$ such that $q \in \sigma(q) + \Omega$. Every point x in $W(0, \theta)$ is forced to Ω within $[0, \theta]$; once there, it can be held in Ω until time θ (and also subsequently) by σ used as stroboscopic strategy:

$$e^{At}(z - \int_0^t e^{-As}(\sigma(q(s)) - q(s))ds) \in e^{At}(z - \Omega) = \Omega$$

as soon as $z \in \Omega$. Thus $W(0, \theta) = W(\theta)$, and capture is isochronous. This shows that $W(0, \theta) = \Omega$ for all θ if (3) fails; and, if it does, then all $W(t) = \emptyset$ for $t > 0$ by the Necessary Condition from 4.2.

Second, assume that (3) does not hold. Then all $W(t) = \emptyset$ for $t > 0$ by the Necessary Condition from 4.2; and, as the counterpositive to the first argument, all $W(0, \theta) = \Omega$. Capture is again isochronous, but only trivially: only positions in Ω can be captured, at time 0. QED

LEMMA 1 For all $t \geq 0$ we have

$$(4) \quad R_V(t) \subset W(t) \subset (R_P(t) + \Omega) \# R_Q(t)$$

where $V = (P + \Omega) \# Q$. Referring to (1), $R_V(t)$ is the set of winning positions for stroboscopic capture at t ; $W(t) = W(0, t)$ if $MQ \subset MP$; and the last term in (4) is the set of winning positions, at t , for capture with anticipation.

(Proof) From the Assumptions, $e^{At}\Omega = \Omega$ for all t ; we will use this without further mention.

The second inclusion in (4), and the interpretation of the last member, is in Corollary 2 from 4.1. Naturally, $W(t)$ contains the points which can be forced to Ω stroboscopically at time t ; that this set is precisely $R_V(t)$ follows from

the Second Reciprocity Theorem in 3.3 (or, preferably, from Exercise 6 there). QED

COROLLARY 1 In the case $MQ \not\subset MP$ there is a constant evasion strategy for all positions outside the target (even if pursuer is allowed to anticipate).

(Proof) Using the Limit Theorem for reachable sets, it is easily shown that there exists a point $q_0 \in Q$ and $\epsilon > 0$ such that

$$M \int_0^t e^{-As} ds q_0 \not\subset M R_P(t) \text{ whenever } 0 < t \leq \epsilon;$$

in particular, the convex and symmetric set

$$(R_P(t) + \Omega) \stackrel{*}{\cap} R_Q(t) = \emptyset \text{ for } t \in [0, \epsilon].$$

Thus, if $x \notin \Omega$, then the state response endpoint, to quarry control q_0 and any pursuer control $p(\cdot)$, must have $x(t) \notin \Omega$ for $t \in (0, \epsilon]$, and also for $t = 0$. Now we need only apply the same argument to $x(\epsilon) \notin \Omega$ and continue over $[\epsilon, 2\epsilon]$, etc. QED

THEOREM 2 In the game (1), under the Assumptions, let MQ be a direct summand of MP : for some set S , $MP = MQ + S$. Then capture within $[0, t)$, stroboscopic and isochronous capture at t with subsequent holding, and capture with anticipation at t or within $[0, t]$, all coincide.

(Proof) First we reformulate the condition on the constraint sets. Necessarily

$$(4) \quad S = 0 + S \subset MQ + S = MP,$$

so that S is bounded; actually one may even take S compact, convex, and symmetric (replace S by \bar{S} , etc.). In particular, $0 \in S$, so that

$$MQ = MQ + 0 \subset MQ + S = MP$$

and

CAPTURE

$$(5) \quad W(t) = W(0, t)$$

by Lemma 1. From (4), $S \subset \text{range } M$, so that $S = MU$ with $U = M^{-1}S$ closed and convex. Then the condition yields $MP = M(Q + U)$, i.e.,

$$P + \Omega = Q + U + \Omega,$$

and, operating with $\# Q$,

$$U + \Omega = (Q + U + \Omega) \# Q = (P + \Omega) \# Q = V,$$

the set V from Lemma 1.

The main part of the proof consists in showing that the last member in (4) coincides with the first. Since Ω is a linear subspace and is A -invariant,

$$\begin{aligned} R_P + \Omega &= R_P + R_\Omega = R_{P+\Omega} = R_{Q+U+\Omega} \\ &= R_Q + R_V \end{aligned}$$

by the last two exhibited relations (we are omitting t in the notation for reachable sets). Thus the last member of (4)

$$(R_P + \Omega) \# R_Q = (R_Q + R_V) \# R_Q = R_V.$$

Finally, $V = (P + \Omega) \# Q$, so that the sets $(R_P(t) + \Omega) \# R_Q(t)$ increase with t : capture with anticipation within $[0, t]$ and at t coincide. The re-interpretations in Lemma 1 and (5) complete the proof. QED

COROLLARY 2 Assume that either the player constraint sets are one-dimensional, or the target is a hyperplane. Then either $MQ \subset MP$ and capture is isochronous and stroboscopic with subsequent holding; or $MQ \not\subset MP$ and there is a constant evasion strategy.

(Proof) In both cases the sets MQ, MP are segments; if $MQ \subset MP$, then MQ is a direct summand of MP . QED

There is one further case in which the first two sets in (4) (but not the third) coincide, so that capture is

stroboscopic and isochronous.

LEMMA 2 In (1) let P be a polytope and Q a segment in P , and $\Omega = 0$. If $P \not\equiv Q = 0$, then $W(t) = 0$ for all $t \geq 0$.

(Proof) First we show that the endpoint q of Q is a vertex of P . If not, then $q = (p_1 + p_2)/2$ with $p_1 \neq p_2$ in P ; also, $x = (p_1 - p_2)/2 \neq 0$ has $x \pm q \in P$, so that $x + Q \subset P$, although $x \neq 0$.

Each vertex of P , and q in particular, has the property that the set of exterior normals to P at q has nonvoid interior in R^n . Thus some $c \in R^n$ remains an exterior normal even if perturbed slightly. Therefore there exists $\epsilon > 0$ such that the vectors $e^{A's}c$ are exterior normals for $s \in [0, \epsilon]$:

$$c'e^{-As}p \leq c'e^{-As}q \text{ for } p \in P, s \in [0, \epsilon].$$

It follows that c is an exterior normal to $R_P(\epsilon)$ at the point $x = \int_0^\epsilon e^{-As} ds q \in R_Q(\epsilon)$.

Let us now show that $R_P(\epsilon) \not\equiv R_Q(\epsilon) = 0$. If $y = R_Q(\epsilon) \subset R_P(\epsilon)$, then in particular $y + x \in R_P(\epsilon)$. Thus

$$c'(y + x) \leq c'x, \quad c'y \leq 0.$$

By symmetry, also $c'(-y) \leq 0$, so that $c'y = 0$. If we now let c vary over the open set (the interior of the collection of exterior normals to P at q), it follows that $y = 0$.

From Lemma 1, $W(\epsilon) = 0$; using Lemma 2 in 4.1 (also see Exercise 5 there) one obtains $W(2\epsilon) = 0$, $W(3\epsilon) = 0$, etc. Finally, from Theorem 1 and monotonicity of $W(t) = W(0, t)$, all $W(t) = 0$. QED

Exercises

1. Verify that, if the coefficient matrix A is scalar,

$A = \alpha I$, then

$$R_P(t) = \frac{1 - e^{-\alpha t}}{\alpha} P$$

(with the obvious modification if $\alpha = 0$). Prove that capture is isochronous and stroboscopic for arbitrary $Q \subset P$.

2. Assume that A is simple with all eigenvalues real; and that P is a symmetric parallelepiped with edges parallel to eigenvectors a_k of A :

$$P = \{ \sum \alpha_k a_k : |\alpha_k| \leq \mu_k \}$$

for given $\mu_k \geq 0$. Prove that $R_P(t) = \int_0^t e^{-As} ds P$, and conclude that capture is isochronous and stroboscopic for arbitrary $Q \subset P$.

3. Consider the game with uncoupled players,

$$\dot{x} = Ax - p, \dot{y} = By - q; \text{ end: } x = y$$

Prove that the target is invariant (in the sense of this section) iff $A = B$.

CONJECTURE $W(t) = R_P \stackrel{*}{=} Q(t)$ if Q is a segment contained in the polytope P (without assuming $P \stackrel{*}{=} Q = 0$).

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CHAPTER VI

ALGEBRAIC THEORY

This chapter contains the elements of an algebraic theory of autonomous linear pursuit games - algebraic in the sense of depending on the shape of the constraint sets but not on their size.

The first section contains, primarily, definitions of the notions of game space, pursuer and quarry control orders, and uncoupled dynamics. In 6.2 we exhibit an alternate interpretation of one of the secondary concepts, min-max controllability; and establish two necessary algebraic conditions for winning. The last two sections treat games proper, as distinguished from their various state space descriptions. In particular, it is shown that the game space dimension, and the player control orders, are independent of particular state space representations. The concept of equivalence between game representations is rather rich: in contrast with algebraic isomorphism, it is highly sensitive to changes of the target set (the desired outcome of regulation).

The reader who is not interested in these topics may choose to omit all of this chapter, with one exception: the notion and formal concept of player control order is needed at several places in this book.

6.1 Game Space, Control Order

Consider the state description of a linear game,

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } x &\in \Omega; \text{ state space: } R^n \end{aligned}$$

(occasionally not all of the data n, A, P, Q, Ω need be specified). The example we shall refer to repeatedly is one-dimensional rocket chase (see 1.3), defined by one-dimensional equations $\ddot{x} = u, \dot{y} = v$. Here the state space is R^3 , the dynamical equation is

$$\dot{x}_1 = x_2, \dot{x}_2 = u, \dot{x}_3 = v;$$

the control values are in the interval $[-1, 1]$, and the termination condition is $x_1 = x_3$.

With the game (1) there is naturally associated the pursuer's control system

$$\dot{x} = Ax - p; p(t) \in P; \text{end: } x \in \Omega$$

(formally: replace Q in (1) by O), and analogously for quarry's control system. One can then bring control theory into the picture. In particular, there are available the concepts of pursuer's reachable sets

$$R_P(t) = \left\{ \int_0^t e^{-As} p(s) ds : \text{integrable } p: R^1 \rightarrow P \right\},$$

pursuer controllability space (see 3.11)

$$C_P = \text{span } R_P(t) \quad (t > 0),$$

etc. One interesting numerical characteristic may be introduced here: pursuer's controllability defect

$$\dim R^n - \dim C_P = n - \dim C_P.$$

For the example, in the transition to state space R^3 , one vital piece of information has been obscured: that, in some sense, the game is played (or, the players move) in a one-dimensional straight line. To recapture this notion we make the following definition: referring to (1), the game space is $(\Omega - \Omega)^\perp$, the orthogonal complement (hence a linear subspace of R^n) of the Pontrjagin difference of the target set with itself (cf. Proposition 3 in 3.8). Verification of

this on our example is a special case of the following result.

LEMMA 1 Assume that the nonvoid target set is of the form $\Omega = \{x: Tx \in \Sigma\}$, where Σ is a compact, convex set in R^m , and T is an (m,n) matrix; denote by $N = \{x: Tx = 0\}$ the null space of T . Then

$$(2) \quad \Omega \perp \Omega = N;$$

thus the game space is N^\perp , and is isomorphic to the range (i.e., column space) of T .

(Proof) One inclusion in (2) is simple: if $x \in N$, then $x + y \in \Omega$ for all $y \in \Omega$, since $T(x + y) = Ty \in \Sigma$; hence $x + \Omega \subset \Omega$, $x \in \Omega \perp \Omega$. For the other, assume $x \notin N$, $Tx \neq 0$; and aim to prove $x \notin \Omega \perp \Omega$, i.e., $x + y \notin \Omega$ for some $y \in \Omega$. Consider the compact, convex set $\Sigma \cap \text{range } T$; it is nonvoid, since it contains $T\Omega = \emptyset$. One can then find a point $Ty \in \Sigma$ (with $y \in \Omega$) at which $Tx \neq 0$ is an exterior normal. Then $Ty + Tx \notin \Sigma$, so that indeed $x + y \notin \Omega$.

Then (2) follows by taking orthogonal complements; and the last assertion by elementary linear space theory:

$$\dim(\text{row space } T) = \text{rank } T = n - \dim N = \dim N^\perp. \quad \text{QED}$$

In one-dimensional rocket chase, the primary equations of motion were $\ddot{x} = u$, $\dot{y} = v$; one is naturally led to say that the pursuer's order of control is 2, since he controls the second-order derivative of his state variable, while that of quarry is 1. This provides a rough measure of the participants' maneuverability: $y(\cdot)$ is continuous, while $x(\cdot)$ even has continuous first derivative, and so cannot negotiate sharp corners.

For the formalization, denote by $L = \text{span } \Omega \perp \Omega$ the orthogonal complement of the game space. Then the pursuers' control order is the largest integer $k \leq +\infty$ such that

$$(3) \quad A^j(P - P) \subset L \text{ for } 0 \leq j \leq k - 2.$$

Similarly for the quarry control order, with Q in place of P . The concept was designed so that Exercise 4 would hold. It is treated below and in the theorem of 7.4; and applied in Chapters III to VII.

LEMMA 2 If P contains O and Ω is as in Lemma 1, then (3) is equivalent to

$$TA^j P = 0 \quad \text{for } 0 \leq j \leq k - 2.$$

LEMMA 3 The pursuer control order $k \geq 1$; and, for games with state space R^n , either $k \leq n$ or $k = +\infty$.

(Proof) The condition is satisfied vacuously for $k \leq 1$. In the second assertion, if $A^j(P - P) \subset L$ holds for $j \leq n - 1$, then also for $j = n$ by the Cayley-Hamilton theorem, and thus also for all $j > n$. QED

In the rocket chase each player's control affects solely the appropriate component of the state variable; the only connection between the components is by means of the termination condition. In other games (e.g., 1.1) the players' controls are inextricably connected with the state variable. Some insight is provided by a Kálmán-type decomposition theorem:

DECOMPOSITION THEOREM Referring to the game (1), there is a nonsingular linear coordinate transformation and a

partitioning $x = \begin{pmatrix} x_1 \\ \dots \\ x_4 \end{pmatrix}$ of the new state variable x in which

the dynamical equation decomposes as follows:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 - B_{11}p + B_{12}q \\ \dot{x}_2 &= A_{22}x_2 + A_{24}x_4 - B_{21}p \\ \dot{x}_3 &= A_{33}x_3 + A_{34}x_4 + B_{32}q \\ \dot{x}_4 &= A_{44}x_4 \end{aligned}$$

for appropriate matrices A_{ij} , B_{ij} . Furthermore, $x_3 = 0 = x_4$ characterises the pursuer's controllability space, and

$x_2 = 0 = x_4$ that of the quarry.

(Proof) Begin with the players' controllability spaces, C_P and C_Q . By augmenting a basis in $C_P \cap C_Q$ one obtains bases for C_P and C_Q , and then for $C_P + C_Q$ and R^n . Relative to the latter, there is a partitioning such that C_P and C_Q are characterised as asserted; then $x_4 = 0$ describes $C_P + C_Q$, and $x_2 = x_3 = x_4 = 0$ characterises $C_P \cap C_Q$. (For this one must, of course admit the possibility of zero-dimensional vectors: e.g., $\dim x_1 = 0$ iff $C_P \cap C_Q = 0$.)

Now, observe that C_P is invariant under A , hence also under all e^{As} ; thus $e^{At}(x - \int_0^t e^{-As} p(s) ds) \in C_P$ whenever $x \in C_P$ and p maps into P ; and analogously for C_Q and $C_P + C_Q$. Consider any point x in $C_P + C_Q$ (thus, $x_4 = 0$), and constant controls $p \in P$, $q \in Q$. Then the state response, to initial position x and player controls p, q , will remain in $C_P + C_Q$ (hence, $x_4(\cdot) \equiv 0$); furthermore, it is differentiable and thus satisfies the differential equation everywhere (hence, $\dot{x}_4(\cdot) \equiv 0$):

$$\dot{x}_4 = \sum_{j=1}^4 A_{4j} x_j - B_{41} p + B_{42} q.$$

At $t = 0$ this yields that

$$0 = A_{41} x_1 + A_{42} x_2 + A_{43} x_3 + A_{44} 0 - B_{41} p + B_{42} q.$$

Here x_1, x_2, x_3 , and also p, q , are quite arbitrary. The choice $x = 0$ shows that

$$(4) \quad -B_{41} p + B_{42} q = 0 \quad (\text{all } p \in P, q \in Q),$$

so that $\sum_{j=1}^3 A_{4j} x_j = 0$ for all x_j , and hence $A_{41} = 0$,

$A_{42} = 0$, $A_{43} = 0$; finally, from (4), the game is not changed if we replace B_{41} and B_{42} by zero matrices. This yields

the desired form of equation for \dot{x}_4 ; a similar procedure simplifies those for \dot{x}_2, \dot{x}_3 . QED

In the situation of the preceding assertion, let n_k be the dimension of x_k for $k = 1, 2, 3, 4$; that these n_k are determined uniquely is shown in Exercise 7. For reasons that are probably obvious we will say that the game (or the players' dynamics) are uncoupled if $n_1 = 0$; in general

$$\dim R^n - \dim(C_P \cap C_Q) = n - n_1$$

is the degree of uncoupling. If $n_3 = 0$, i.e., $C_Q \subset C_P$, the game is said to be min-max controllable. As in control theory, we will say that the pursuer control system is controllable if $C_P = R^n$, i.e., $n_1 + n_2 = n$, or $n_3 = n_4 = 0$. The game is controllable if $n_4 = 0$, i.e., $n_1 + n_2 + n_3 = n$. Finally, the pursuer's controllability defect is

$$\dim R^n - \dim C_P = n - (n_1 + n_2),$$

and analogously for the quarry.

Exercises

1. For one-dimensional rocket chase, exhibit explicitly the state space description in R^3 (thus, find the $(3,3)$ matrix A , and the three subsets P, Q, Ω of R^3). Determine the game space and player control orders according to the definitions, and verify that they coincide with the intuitive notions. Find the players' controllability spaces, their intersection and sum; determine the degree of uncoupling and controllability defects. (Partial answer: the game is uncoupled.)

2. The same assignment for the reduced description

$$\dot{x} = y - v, \dot{y} = u; \text{ end: } x = 0$$

in R^2 . (Partial answer: the game is min-max controllable,

with 1 as degree of uncoupling.)

3. Consider an n -th order one-dimensional equation

$$(5) \quad x^{(n)} + \sum_0^{n-1} A_j x^{(j)} = u$$

with the forcing term u as control variable, and take $|x| \leq \epsilon$ as termination condition ($0 \leq \epsilon < +\infty$). Show that, in terms of the corresponding first-order equation in R^n , the control order is precisely n .

4. Extend the preceding result to the case of k -vectors x and k -square matrices A_j . Be specific about the control constraint set.

5. For a game involving k -vectors x and y , governed by (5) and, say,

$$y^{(m)} + \sum_0^{m-1} B_j y^{(j)} = v,$$

with termination condition $|x - y| \leq \epsilon$ ($0 \leq \epsilon < +\infty$), show that the game space is R^k .

6. In (1) let $\Omega = \{x: Tx \in \Sigma\}$ with Σ a convex (but not necessarily compact) set in R^m , and assume that $\sum_{j=0}^{k-2} TA^j(P - P) = 0$. Prove that the pursuer control order is k at least.

7. Show that the dimensions $n_k = \dim x_k$ in the Decomposition Theorem are independent of choice of basis, and of the partition procedure. (Hint: If $x \mapsto Tx$ with T n -square nonsingular, then $C_P \mapsto TC_P$, etc.)

8. Untangle the definitions and simplify the following assumption: the game (1) is both uncoupled and min-max controllable.

9. The game space is a subspace of the state space R^n . Under what circumstances does it coincide with R^n , or reduce to 0?

10. Show that, if the target set is hyperplane

$\{x \in R^n: c'x = \alpha\}$ for given $c \neq 0$ in R^n and $\alpha \in R^1$, then the game space is one-dimensional.

Remarks

The concept of game space is obviously an attempt to rescue a notion, inherent in direct descriptions of a game, from the complexities of the various state-space descriptions. The order of control appears to be a relatively new concept (curiously, since it is natural enough); it was presented in

O. Hájek: Strategy design in pursuit games, pp. 71-79, in Differential Games and Control Theory (ed. E. O. Roxin, P. T. Liu, R. L. Sternberg), Dekker, New York, 1974.

The Decomposition Theorem, in statement and proof, is an obvious analogue of well-known results in control theory. Here as there, it is a very pleasing result, but seems to have little factual (as opposed to conceptual) consequences. The term max-min controllability is due to Heymann, Stern, and Pachter (see the next section). As will be seen in 6.4, it is preferable to treat the controllability defect $n - \dim C_P$ rather than $\dim C_P$ directly; similarly for the degree of uncoupling.

PROBLEM In the control system, in R^n ,

$$\dot{x} = Ax - u; u(t) \in U; \text{ end: } Mx = 0$$

(with U a linear space) assume that the order of control is at least 2, i.e., $MU = 0$. In general the system need not have arisen from one such as $\ddot{y} + B\dot{y} + Cy = v$ (since then necessarily $2|n$). Obtain a complete analysis.

6.2 Min-Max Controllability

In this section the concept of min-max controllability is examined in greater detail. The setting is that of the game

$$(1) \quad \dot{x} = Ax - p + q; p(t) \in P, q(t) \in Q; \text{ end: } x = 0$$

Observe that the state response $x(t) = e^{At}(\dots)$, to player controls $p(\cdot)$ and $q(\cdot)$ and initial position x , meets the origin at time t iff

$$(2) \quad x + \int_0^t e^{-As} q(s) ds = \int_0^t e^{-As} p(s) ds$$

As in 2.2, let us say that an initial position x can be captured with anticipation if, for every quarry control $q(\cdot)$, there is a pursuer control $p(\cdot)$ such that (2) holds for some $t > 0$. This implicitly refers to a mapping taking quarry controls to pursuer controls; in contrast with the strategies (in the sense used in this book) it is not required to be non-anticipatory. If the termination time t can be taken independent of $q(\cdot)$, we will say that capture with anticipation is isochronous.

These concepts provide an alternate formulation of min-max controllability.

THEOREM In the game (1) assume that the players' constraint sets are linear subspaces. Then there exist points capturable with anticipation if, and only if, the game is min-max controllable. In the positive case these are precisely the points in pursuer's controllability space, and all of these are capturable isochronously at arbitrarily small times $t > 0$.

It is convenient to isolate certain parts of the proof.

LEMMA 1 In the game (1) assume $0 \in Q$. For every position $x \notin C_P$, 0 is an indifferent evasion strategy. (Proof: (2) with $q \equiv 0$ implies $x \in C_P$.)

LEMMA 2 In (1) assume that Q is closed, convex and symmetric. If the game is not min-max controllable, i.e. $C_Q \not\subset C_P$, then there is a quarry control which is an indifferent evasion strategy simultaneously for all initial

positions in C_P .

(Proof) Note first that $Q \neq \emptyset$ from the assumption $C_Q \not\subset C_P$. Second, necessarily $C_P^\perp \not\subset C_Q^\perp$ for the orthogonal complements of the subspaces, so that there exists a vector $c \in C_P^\perp$ with $c \notin C_Q^\perp$. Consider now the reduction of Q to the compact set

$$Q_0 = \{q \in Q: |q| \leq 1\},$$

which is again convex and symmetric. Obviously $C_Q = C_{Q_0}$, and of course $C_{Q_0} = \text{span } R_{Q_0}(t)$ for each $t > 0$; thus c is not perpendicular to the symmetric sets $R_{Q_0}(t)$,

$$0 < \max\{c'y: y \in R_{Q_0}(t)\} \leq \int_0^t \max_{q \in Q_0} c'e^{-As} q ds.$$

Now, one can find a maximizing measurable $q_0: R^1 \rightarrow Q_0 \subset Q$,

$$c'e^{-As} q_0(s) = \max_{q \in Q_0} c'e^{-As} q,$$

and then

$$\int_0^t c'e^{-As} q_0(s) ds > 0 \quad (\text{all } t > 0).$$

This is the desired quarry control. Indeed, if $x \in C_P$, then one cannot have (2) with $q = q_0$ for any $t > 0$, since c is perpendicular to both x and the right side, but not to the term involving the quarry. QED

COROLLARY 1 Under the assumptions of Lemma 2, if a point x is in position to win, then necessarily

$$(3) \quad C_Q \subset C_P \ni x.$$

COROLLARY 2 In the same situation, unless the pursuer control system is controllable, the set of winning points has empty interior.

(Proof of Theorem) Having the two lemmas, it only remains to show that, if (3) holds, then for arbitrarily small $t > 0$

and any quarry control $q(\cdot)$ there is a pursuer control $p(\cdot)$ such that (2) holds. This is immediate: since e.g. P is a linear subspace, $C_P = R_P(t)$ independently of $t > 0$. QED

Remark The preceding proof establishes the existence of an appropriate mapping $q(\cdot) \mapsto p(\cdot)$; here we present a constructive procedure. The assumptions are as in the theorem, and $C_Q \subset C_P \ni x$.

Without essential loss of generality, let the terminal time be 1. Choose bases, q_1, \dots, q_a in Q and c_1, \dots, c_b in C_P . Since $C_P = R_P(1)$, each $c_j = \int_0^1 e^{-As} p_j(s) ds$ for some integrable $p_j: [0,1] \rightarrow P$. Also, $e^{-At} q \in C_Q \subset C_P$, so there exist coefficients α such that

$$e^{-At} q_i = \sum_j \alpha_{ij}(t) c_j$$

(and actually the $\alpha_{ij}(\cdot)$ are continuous on $[0,1]$). Now consider any integrable $q: [0,1] \rightarrow Q$; then $q(t) = \sum \beta_i(t) q_i$ with integrable $\beta_i(\cdot)$. Finally,

$$\begin{aligned} \int_0^1 e^{-At} q(t) dt &= \sum_{ij} \int_0^1 \beta_i(t) \alpha_{ij}(t) dt c_j \\ &= \int_0^1 e^{-As} \left(\sum_{ij} \int_0^1 \beta_i(t) \alpha_{ij}(t) dt p_j(s) \right) ds, \end{aligned}$$

with the bracketed term being an integrable function, $\bar{p}(\cdot)$, having values in P .

All this was independent of x . Since $x \in C_P$, we have $x = \int_0^1 e^{-As} p_0(s) ds$ for suitable integrable $p_0(\cdot)$ (independent of $q(\cdot)$). Then (2) holds with $p = \bar{p} + p_0$ and $t = 1$.

Exercises

1. Carry out the construction described in the remark for the game

$$\dot{x} = y - v, \quad \dot{y} = u$$

in R^2 . (Partial answer: for the obvious bases, $\alpha_{11} \equiv 1$, $\alpha_{12} \equiv 0$; and one may choose, e.g., $p_1(\cdot)$ as 4 in $[0, 1/2]$, -4 in $[1/2, 1]$.)

2. Show that, if the bases are taken orthonormal, then

$$\bar{p}(s) = \int_0^1 q'(t) \sum_{ij} q_i q_j' e^{-A't} dt c_j p_j(s),$$

so that $q(\cdot) \mapsto \bar{p}(\cdot)$ is effected by a bounded linear operator.

3. Find an error in the following argument, purporting to show that, in the min-max controllable case, the strategy may be taken non-anticipatory. Again choose bases q_i, c_j , coefficients $\alpha_{ij}(\cdot)$ and $\beta_i(\cdot)$ as before; now, however, for each $t < 1$ express $c_j = \int_t^1 e^{-As} p_j(t, s) ds$. Then an interchange in the order of integration yields

$$\int_0^1 e^{-At} q(t) dt = \int_0^1 e^{-As} \left(\sum_{ij} \int_0^s \beta_i(t) \alpha_{ij}(t) p_j(t, s) dt \right) ds.$$

and the bracketed term depends only on values $q(t)$ with $t < s$.

4. In Lemma 1, C_p contains the linear span of the set of winning positions. Show that equality does not hold. (E.g., in R^1 , $\dot{x} = p + q$ with $|p(t)| \leq 1$, $|q(t)| \leq 2$.)

Remarks

The term min-max controllability was introduced in

M. Heymann, M. Pachter, R. J. Stern: Max-min control problems: a system theoretic approach
(to appear)

where a large part of the theorem was also presented; further results concern optimisation of $\int |p|^2 - |q|^2$ as cost.

6.3 Equivalent Descriptions

Let us return once again to one-dimensional rocket chase. Two state space descriptions of this have been given: one obtained by a natural procedure in R^3 , and then a reduced description in R^2 . The point to be made here is that a game may have several different but equally adequate descriptions, efficient or redundant to various degree: but the game itself should be independent of the descriptions.

This is resolved in the usual manner, by defining a game proper as an equivalence class of an appropriate equivalence relation. To begin we formalize the notion of 'reduced description'.

Given games described by

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } x &\in \Omega; \quad \text{state space: } R^n, \end{aligned}$$

$$(2) \quad \begin{aligned} \dot{y} &= By - v + w; \quad u(t) \in U, \quad v(t) \in V; \\ \text{end: } y &\in \Sigma; \quad \text{state space: } R^m, \end{aligned}$$

we will say that (2) is a reduction of (1) if there is a linear onto mapping

$$(3) \quad x \mapsto y = Tx: R^n \rightarrow R^m$$

of the state spaces, such that

$$(4) \quad TA = BT, \quad TP = U, \quad TQ = V, \quad \Omega = T^{-1}(\Sigma).$$

In (4) the first three conditions require that the state responses and controls of (1) map appropriately into those of (2); the last is the requirement that $x \in \Omega$ be equivalent to $y \in \Sigma$ whenever $y = Tx$. (Occasionally the mapping T will also be termed a reduction.)

With this settled it is simple to formalize other intuitive notions. Thus, (1) and (2) are linearly isomorphic if the onto mapping (3) can also be taken one-to-one. The

game (1) is a minimal representation if each of its reductions is linearly isomorphic to (1). Finally, (1) and (2) are equivalent representations (or descriptions) if there is a finite sequence of games, beginning with (1) and ending with (2), such that neighbouring terms are related by reduction; and a game proper is any equivalence class (modulo the relation of being equivalent representations).

All this is natural enough, and actually yields some results (e.g., existence of minimal representations). Nevertheless, without applicable results, it is merely a top-heavy pile of definitions; thus it is not at all clear whether the minimal representations of a game proper are all linearly isomorphic, nor whether minimal representations are controllable (the former is true, the latter false).

The key appears in focusing on one or the other of the games in the relation of reduction.

PROPOSITION 1 If T is a reduction of (1) to (2), then its null space $N = \{x: Tx = 0\}$ satisfies

$$(5) \quad AN \subset N, \quad \Omega + N \subset \Omega.$$

Conversely, if a linear subspace N of R^n has (5), relative to a game (1), then there exists a mapping T which reduces (1) to a game (2), and (2) is then determined uniquely up to linear isomorphism.

(Proof) First note that, if N is the null space of T , then

$$(6) \quad \Omega = T^{-1}(\Sigma) \text{ iff } T\Omega = \Sigma \text{ and } \Omega + N \subset \Omega.$$

Now, if T is a reduction, then (5) follows from the first and last conditions in (4).

Conversely, if (5) holds, then T is determined by N uniquely up to nonsingular left factors (since it is to be an onto mapping). Next, $Tx = 0$ implies $TAx = 0$ by (5), so that T is a right factor of TA , i.e., $TA = BT$ for suitable

B. The specification of (2) is then completed by setting

$$(7) \quad R^m = \text{range } T, \quad U = TP, \quad V = TQ, \quad \Sigma = T\Omega$$

(where we then use (6)). QED

COROLLARY 1 (1) is a minimal representation if, and only if, 0 is the only subspace N of R^n which satisfies (5); and also if, and only if, none of the minimal A -invariant subspaces N (thus, $\dim N$ is 1 or 2) satisfies $\Omega + N \subset \Omega$.

(Proof: the conditions are precisely that T be one-to-one.)

COROLLARY 2 If the target set Ω of (1) is compact, or, more generally, contains no straight lines, then (1) is minimal.

Example (n-dimensional rocket chase) Consider the game with R^{3n} as state space, n-dimensional partitioned equation

$$(8) \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad \dot{x}_3 = v,$$

specified constraint sets in R^n , and termination condition

$|x_1 - x_3| \leq \epsilon$ (here $0 \leq \epsilon < +\infty$). The reduced description is

$$(9) \quad \dot{y}_1 = y_2 - v, \quad \dot{y}_2 = u$$

in R^{2n} , with the same constraint sets, and termination condition $|y_1| \leq \epsilon$. The reduction process $(x_1, x_2, x_3) \mapsto (x_1 - x_3, x_2)$ is effected by mapping via the partitioned matrix

$$(10) \quad T = \begin{pmatrix} I & 0 & -I \\ 0 & I & 0 \end{pmatrix}.$$

The null space of T ,

$$(11) \quad N = \left\{ \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} : a \in R^n \right\},$$

obviously satisfies $\Omega + N \subset \Omega$, and is indeed invariant under the coefficient matrix of the first state space description.

Now apply Corollary 1 to the reduced description. The coefficient matrix has all eigenvalues real (namely, 0). The

minimal invariant subspaces N are of the form $R \begin{pmatrix} 1 & a \\ 0 \end{pmatrix}$ with $a \neq 0$, and none of these satisfies $\Sigma + N \subset \Sigma$; this confirms the natural suspicion that the reduced description is minimal.

Now we return to the two games in the reduction relation, and focus on the 'smaller' one.

PROPOSITION 2 If T reduces (1) to (2), then, after a nonsingular coordinate transformation of the state space of (1),

$$(12) \quad T = (0, I), \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & B \end{pmatrix}, \quad \Omega = R^{n-m} \times \Sigma.$$

Conversely, if (12) holds and $TP = U$, $TQ = V$, then T is a reduction to (2) of a game (1) with data n, A, P, Q, Ω .

(Proof) If T, A, Ω, P, V are as described, then obviously T reduces (1) to (2). Conversely, let T reduce (1) to (2). If S is any n -square nonsingular matrix, then the isomorphism $x \mapsto S^{-1}x$ replaces T by TS (and changes the other data appropriately). Since T is of type (m, n) and maps onto R^m , we have $\text{rank } T = m$, and there exists an elementary n -square matrix S such that $TS = (0, I)$, where I is n -square and 0 is of type $(m, n-m)$. The remaining assertions in (12) follow from (4). QED

Note that, after the transformation, the differential equation of (1) partitions,

$$(13) \quad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}y - p_1 + q_1 \\ \dot{y} &= B y - p_2 + q_2 \end{aligned}$$

with the A_{11}, A_{12} essentially arbitrary; and that the game (2) is not subjected to an isomorphism.

COROLLARY 3 Given the game (2), for every $n = m, m+1, \dots$ there is a game with state space R^n which reduces to (2). For $n = m+1$ there are uncountably many non-isomor-

phic games with state space dimension n which reduce to (2). (Proof: Choose $\dot{\xi} = \alpha \xi$ as the first equation in (13); and observe that isomorphic games have their coefficient matrices similar.)

From Proposition 1 it follows readily that any two reductions of a single game have a common reduction (if N_1, N_2 are the null spaces, consider $N = N_1 + N_2$). Proposition 2 provides the means for proceeding in the opposite direction.

Consider two games,

$$(14_k) \quad \begin{aligned} \dot{x} &= A_k x - p + q; \quad p(t) \in P_k, \quad q(t) \in Q_k; \\ \text{end: } x &\in \Omega_k: \quad \text{state space } R^{n_k} \end{aligned}$$

(for $k = 1, 2$), and also a third game, with notation as in (2):

$$(15) \quad \begin{aligned} \dot{y} &= B y - u + v; \quad u(t) \in U, \quad v(t) \in V; \\ \text{end: } y &\in \Sigma; \quad \text{state space: } R^m. \end{aligned}$$

LEMMA If the two games (14_k) have (15) as a common reduction, then there is another game, in $R^{n_1+n_2+m}$, of which both (14_k) are reductions.

(Proof) Replace each of (14_k) by an isomorphic game as described in Proposition 2. Retaining some notation, for $k = 1, 2$ the dynamical equations of (14_k) are

$$\begin{aligned} \dot{x}_k &= A_{kk} x_k + A_{k3} y - p_k + q_k \\ \dot{y} &= B y - p_3 + q_3, \end{aligned}$$

the player constraints

$$\begin{pmatrix} p_k(t) \\ p_3(t) \end{pmatrix} \in P_k, \quad \begin{pmatrix} q_k(t) \\ q_3(t) \end{pmatrix} \in Q_k,$$

and the target sets are $R^{n_k-m} \times \Sigma$. Here the constraint sets satisfy

$$\{p_3 : \begin{pmatrix} p_k \\ p_3 \end{pmatrix} \in P_k\} = U,$$

and similarly for quarry.

Consider now the game in $R^{n_1+n_2 \rightarrow m}$ with partitioned dynamical equation

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{13}y - p_1 + q_1 \\ \dot{x}_2 &= A_{22}x_2 + A_{23}y - p_2 + q_2 \\ \dot{y} &= By - p_3 + q_3, \end{aligned}$$

where pursuer's constraint set

$$P = \left\{ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} : \begin{pmatrix} p_1 \\ p_3 \end{pmatrix} \in P_1, \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} \in P_2 \right\},$$

and similar for quarry; and with target set $R^{n_1-m} \times R^{n_2-m} \times \Sigma$. It is readily verified that

$$T_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad T_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

reduce this game to (1^4_k) for $k = 1, 2$. QED

COROLLARY 4 The minimal representations of a game proper are linearly isomorphic.

(Proof) Take any two minimal representations of a game proper; thus they are equivalent, and there is a chain of reduction-related games between them. Repeated application of the lemma provides a common game, of which both the minimal representations are reductions.

For $k = 1, 2$ let T_k be reductions to the minimal representations, and N_k their null spaces. Then $N_1 + N_2$ provides a yet further reduction (see Proposition 1); from minimality, $\dim N_k = \dim (N_1 + N_2)$, so that $N_1 = N_1 + N_2 = N_2$, and the minimal representations are linearly isomorphic

(Proposition 1 again). QED

Exercises

1. Prove that reduction is transitive. If this seems too easy, show that the non-isomorphic representations of a game proper form a modular lattice (in general non-distributive, of infinite length, with a least element, satisfying the descending chain condition, conditionally complete).

2. Verify that minimal representations need not be controllable. (E.g., in R^2 , $\dot{x} = u$, $\dot{y} = 0$, with the origin as target, and $[-1,1]$ and 0 as constraint sets.)

3. If two games are linearly isomorphic, then their coefficient matrices are similar. For which games is this necessary condition also sufficient?

4. Given a game in R^n as in (1), construct a new game in R^{2n} ,

$$\dot{x} = Ax - p, \dot{y} = Ay - q$$

with the same player constraints, and target set

$$\left\{ \begin{pmatrix} x+y \\ y \end{pmatrix} : x \in \Omega, y \in R^n \right\}$$

Show that the original game is a reduction, and the new game is uncoupled.

5. Referring to the game in the Second Reciprocity Theorem (see 3.3), show that, if the null space N of M is invariant under the coefficient matrix A , then there is an obvious reduction, to a game with singleton target. (This reduction applied to the situation of 5.3 would make possible far more tightly organised proofs there.)

6. Show directly that, if T reduces (1) to (2), then T maps eigenvectors of A either to 0 , or to those of B , whereupon the eigenvalues coincide. More precisely, if (2) is a reduction of (1), then the eigenvalues of B are among

those of A , counting multiplicities. (Hint: Proposition 2.)

7. In the context of Proposition 1, let N_k be two subspaces, both satisfying $AN_k \subset N_k$ and $\Omega + N_k \subset \Omega$. Show that, if $N_1 \subset N_2$, then one of the corresponding games is a reduction of the other.

8. In the preceding, the reducibility requirement $N_1 \subset N_2$ is not a necessary condition. (E.g., $n = 2$, $A = 0$; choose the N_k suitably.)

9. On the other hand, the reducibility condition of Exercise 7 is necessary sometimes. (Consider $n = 2$, A with distinct eigenvalues.)

10. Every game whose state space is R^1 is minimal; true or false? (Hint: It is first necessary to overcome one's aversion to calling R^0 a linear space. Then a leading question is: what is the necessary and sufficient condition on the target set for a game in R^1 to be minimal?)

11. Prove that all games with termination condition absent ($\Omega = R^n$ in (1)) are equivalent. (Hint: $\dot{x} = 0$.)

12. For the one-dimensional homicidal chauffeur game we already have a minimal representation. To check Corollary 4 on this example, find all reductions, using Proposition 1, and note which are minimal. (Determine all subspaces of the target, and, among these, all which are invariant under the coefficient matrix.)

The last group of exercises treats the construction of minimal representations (say, of the game (1)).

13. Verify that the following procedure will yield minimal representations: first, find the largest subspace N of R^n with $\Omega + N \subset \Omega$; second, determine the largest A -invariant subspace L of N (details in subsequent exercises); finally, reduce, using L and Proposition 1.

14. Prove: for every set $\Omega \subset R^n$ there is a unique

maximal subspace N of R^n with $\Omega + N \subset \Omega$. For every n -square matrix A and subspace N of R^n there is a unique largest subspace L with $AL \subset L \subset N$.

15. In the previous exercise, if Ω is nonvoid, convex, closed, and exhibited as the intersection of hyperplanes $\{x: c'_i x \leq \alpha_i\}$, then the subspace N is the solution set to the system $c'_i x = 0$. (Hint: for such sets Ω , $\Omega + N \subset \Omega$ is equivalent to $x + N \subset \Omega$ for some $x \in \Omega$.)

16. In Exercise 14, assume that A is simple; find all real eigenvalues α_i , and corresponding eigenspaces N_i ; then all non-real eigenvalues α_j , and the corresponding 2-dimensional minimal A -invariant subspaces N_j . Prove that the maximal space L is $\sum N_i + \sum N_j$, summation extended over subspaces of N .

17. More generally, L can be obtained from either of the formulas in

$$L = \bigcap_{j=0}^{\infty} A^{-j}(N) = \bigcap_{j=0}^{n-1} A^{-j}(N) = \bigcap_{t \in R^1} e^{At} N = \bigcap_{t \in [0,1]} e^{At} N.$$

Prove the equalities here, and show that they indeed provide the largest A -invariant subspace of N . (Hint: check the orthogonal complements.)

Remarks

If T reduces game (1) to (2), and σ is a winning pursuer (or evader) strategy for initial position x within (1), then there is an appropriate strategy for position Tx in game (2); this is obtained by lifting controls $v(\cdot)$, by a selection procedure, to controls q with $Tq = v$. Similarly in the converse direction, if one is provided with a winning strategy for (2). However, it might be thought desirable to avoid this, by requiring in addition that the reduction mapping T be one-to-one on the player constraint sets P

and Q (i.e., $N \cap (P + Q) = \emptyset$). The ensuing equivalence classes, of course, then change drastically; however only little modification is needed in the results of this section.

PROBLEM 1 Given two reductions of a game (1) with null spaces N_k (as in Exercise 7), obtain a useful criterion for deciding whether one is a reduction of the other, or at least is isomorphic.

PROBLEM 2 If possible, obtain a simple algorithm for finding the state space dimension of the minimal representation from the data of (1).

6.4 Invariants and Semi-Invariants

The results of the preceding section suggest that there are many equivalent representations of a single game proper. What is then common to all? The game proper itself, and its minimal representation; but that is not quite the intent of the question.

PROPOSITION 1 All representations of a game proper have the same player control orders, and also the same game space (up to a linear isomorphism).

(Proof) Of course one need only verify this for a single reduction. To fix notation, let the two games be as in

$$(1) \quad \begin{aligned} \dot{x} &= Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q, \\ \text{end: } x &\in \Omega; \quad \text{state space: } R^n, \end{aligned}$$

$$(2) \quad \begin{aligned} \dot{y} &= By - u + v; \quad u(t) \in U, \quad v(t) \in V; \\ \text{end: } y &\in \Sigma; \quad \text{state space: } R^m. \end{aligned}$$

Let an onto linear mapping $T: R^n \rightarrow R^m$ be a reduction of (1) to (2), so that

$$(3) \quad TA = BT, \quad TP = U, \quad TA = V, \quad T\Omega = \Sigma$$

where the null space N of T satisfies

$$(4) \quad TN \subset N, \quad \Omega + N \subset \Omega$$

(see Proposition 1 in 6.3).

For the pursuer control order (see 6.1) it is sufficient to show that the two inclusions

$$A^j(P - P) \subset \text{span}(\Omega^{\#} \Omega), \quad B^j(U - U) \subset \text{span}(\Sigma^{\#} \Sigma)$$

are equivalent for each integer $j \geq 0$. For this note, first, that $TA^jP = B^jTP = B^jU$ from (3). Second,

$$(5) \quad T(\Omega^{\#} \Omega) = \Sigma^{\#} \Sigma.$$

Indeed, $T(\Omega^{\#} \Omega) \subset T\Omega^{\#} T\Omega$ quite generally; and if $y = Tx$ has $y + \Sigma \subset \Sigma$, then $x + \Omega \subset \Omega + N \subset \Omega$ and so $x \in \Omega^{\#} \Omega$. Last, if $B^j(U - U) \subset \text{span} \Sigma^{\#} \Sigma$, then $TA^j(P - P) \subset T(\text{span} \Omega^{\#} \Omega)$, so that $A^j(P - P) \subset \text{span}(\Omega^{\#} \Omega) + N \subset \text{span}(\Omega^{\#} \Omega)$.

The assertion on the game spaces is merely that their dimensions coincide,

$$\dim(\Omega^{\#} \Omega)^{\perp} = \dim(\Sigma^{\#} \Sigma)^{\perp}.$$

Now,

$$(6) \quad \dim L = \dim TL + \dim(N \cap L)$$

for any subspace L (with N the null space of T). Thus

$$\begin{aligned} \dim(\Sigma^{\#} \Sigma)^{\perp} &= n - \dim(\Sigma^{\#} \Sigma) = m - \dim T(\Omega^{\#} \Omega) \\ &= (n - \dim(\Omega^{\#} \Omega)) - (\dim n - \dim(N \cap (\Omega^{\#} \Omega))) \\ &= \dim(\Omega^{\#} \Omega)^{\perp} - 0, \end{aligned}$$

since $N \subset N + (\Omega^{\#} \Omega) \subset \Omega^{\#} \Omega$. QED

PROPOSITION 2 The degree of uncoupling decreases (or remains unchanged) under reduction.

(Proof) We shall use repeatedly two results from elementary linear space theory: (6), and that, for linear subspaces L_1 and L_2 ,

$$(7) \quad \dim(L_1 \cap L_2) + \dim(L_1 + L_2) = \dim L_1 + \dim L_2.$$

(The first for obvious reasons; the second because sums map well while intersections do not.)

Retain the notation from the proof of the preceding result, including T for the reduction and N for its null-space; and let C_P, \dots, C_V be the appropriate controllability spaces.

Since $TA = BT$ and $TP = U$, we have $C_U = TC_P$ (note $TA^k = B^k T$ for $k = 0, 1, \dots$), and analogously $C_V = TC_Q$; hence $C_U + C_V = T(C_P + C_Q)$. Application of the dimension formulas then yields

$$\begin{aligned} \dim(C_U \cap C_V) &= \dim(C_P \cap C_Q) \\ &+ (\dim(N \cap (C_P + C_Q)) - \dim(N \cap C_P) - \dim(N \cap C_Q)) \end{aligned}$$

and also

$$m = \dim R^m = \dim TR^n = n - \dim N.$$

Thus the degrees of uncoupling satisfy

$$\begin{aligned} n - \dim(C_U \cap C_V) &= n - \dim(C_P \cap C_Q) - \delta, \\ \delta &= \dim N + \dim(N \cap (C_P + C_Q)) - \dim(N \cap C_P) - \dim(N \cap C_Q). \end{aligned}$$

Obvious inclusions yield, for the first two terms of δ ,

$$\begin{aligned} (8) \quad \dim N + \dim(N \cap (C_P + C_Q)) &\geq \dim(N \cap C_P \cap N \cap C_Q) \\ &+ \dim((N \cap C_P) + (N \cap C_Q)), \end{aligned}$$

so that $\delta \geq 0$ from (7). QED

Exercises

1. Prove that the degree of uncoupling is not changed iff $N \subset C_P \cap C_Q$ (notation as in the proof of Proposition 1). This is so, in particular, if both players' control systems are controllable.

2. Show that, for min-max controllable games, $\dim C_P - \dim C_Q$ decreases (or remains unchanged) under

reduction, and the reduced games are also min-max controllable. For which reductions does the characteristic remain constant?

3. Exercise 4 in 6.3, and also one-dimensional rocket chase with its reduction, suggest that the dimension $C_P \cap C_Q$ increases (or remains unchanged) under reduction. Provide a counterexample.

4. Prove that, at least, a game with uncoupled reduction is itself uncoupled.

5. Show that the players' controllability defects (e.g., $n - \dim C_P$) decrease, or are unchanged, by reduction. Conclude that the reductions of a controllable game are controllable.

6. Referring to (1), let L be the largest subspace of R^n with $\Omega + L \subset \Omega$. (see Exercise 1.3 in 6.3) Prove that $n - \dim L = \dim L^\perp$ is an invariant under reduction.

7. In (1) assume that the player constraint sets are convex and symmetric, and the target set Ω is a linear subspace. Prove that

$$\sup\{\theta: e^{At}Q \subset e^{At}P + \Omega \text{ for all } t \in [0, \theta]\}$$

is an invariant of all equivalent representations of (1). (Note: the characteristic is the largest $\theta \leq +\infty$ such that the condition from the proposition in 3.4 holds).

8. Let $L = \text{span } \Omega^\perp$ (the complement of the game space of (1)), and denote by r the first integer ≥ 1 such that $A^r L \subset \sum_{j=0}^{r-1} A^j L$ (the significance is that $\sum_{j=0}^{r-1} A^j L$ is the least A -invariant subspace containing L ; thus $r \leq n$ by the Cayley-Hamilton theorem). Prove that r is an invariant of the equivalent descriptions of (1).

9. Denote by m the (unknown) dimension of the minimal state space representation of (1). Prove that $k \leq m \leq n$, where k is the control order of either player. Actually,

k may be replaced by the largest integer such that

$$A^j(\text{span } P + \text{span } Q) \subset \text{span } \Omega \neq \Omega$$

for $j \leq k - 2$. Is it true that $r \leq m$, where r is as in the preceding exercise?

CHAPTER VII

NONLINEAR GAMES

This chapter treats games whose dynamics or termination conditions are not assumed to be linear. The project is, necessarily, less ambitious than the 'solution' of pursuit games, in the sense of providing general methods for the construction of successful strategies: we address the problem of detecting presence or absence of winning positions.

The procedure is somewhat indirect (thus, we do not exhibit explicit strategies). It is shown in 7.1 that non-trivial winning positions exists if, and only if, the target set boundary contains 'essential' points; and then computable conditions are presented for essential and inessential points (Sections 7.2 and 7.3; the idea was also applied in 5.1 and 5.2).

The concept of essential point may be defined in several approximately equivalent ways. One of these is adopted for the Definition in 7.1, and the others are then properties of the concept (Proposition 1 in 7.1, the Theorem and fig. 1 in 7.2). Analogous results for isochronous capture are presented in 7.4.

7.1 Essential Points

The game to be treated is indicated by

$$(1) \quad \begin{aligned} \dot{x} &= f(x, p, q); \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } x &\in \Omega; \text{ state space: } R^n, \end{aligned}$$

with the standard assumptions from 2.1.

DEFINITION Referring to game (1), the set $\text{ess } \Omega$ of essential points is

$$\text{ess } \Omega = \lim_{t \rightarrow 0+} (W(0,t) \setminus \Omega) = \bigcap_{t \geq 0} \overline{W(0,t) \setminus \Omega}$$

The limit exists (i.e., $\limsup = \liminf$) and the intersection formula holds because $W(0,t)$ increases with t : see (6) in 2.2.

To be specific, for the reduced version of one-dimensional rocket chase, the winning sets were determined in 1.3 (with a strict justification by the results of 5.3). The essential portion of the target set boundary consists of two symmetrically placed rays (see fig. 1)

One reason for the adjective 'essential' will appear in an alternate characterisation (Proposition 1) which we shall now develop. Recall the notation from 2.3: $t \mapsto \pi(t, x, p, q)$ for the solution initiating at the point x (state response to players' controls $p(\cdot), q(\cdot)$); and the corresponding first termination time $T(x, p, q)$.

Assume given an initial position x outside the target, and also a winning strategy σ forcing x to Ω within a compact time interval. Keeping these fixed, consider all points on the state responses to various quarry controls, but only up to the first intersection time. For a technical reason, the definition of the jet corresponding to x and σ is

$$(2) \quad J_{\sigma}(x) = \text{closure}\{\pi(t, x, \sigma[q], q) : \text{measurable } q: \mathbb{R}^1 \rightarrow Q, \\ 0 \leq t \leq T(x, \sigma[q], q)\}$$

To eliminate trivial jets, $x \notin \Omega$ is built into the definition.

The part of the target actually used by J is $J \cap \partial\Omega$, and one might then define the used part of the target as

$\cup(\overline{J \cap \partial \Omega})$, union over winning positions and jets. Thus, in our illustration, the used part of the target strip is somewhat larger than the essential portion, see. fig. 1.

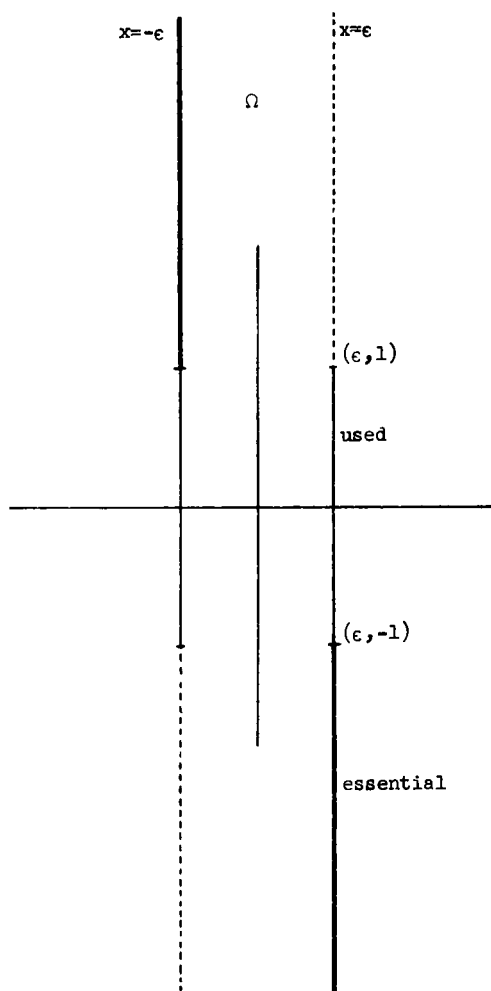


Fig. 1 Essential and used portions of target, for the game in 1.3.

Consider the closed sets S which intersect each jet; e.g., R^n , Ω , and the used portion of Ω are all such, but

one naturally inquires about sets S which are also as small as possible. Zorn's Lemma provides existence of irreducible sets S , since each jet is compact (see the lemma in 2.1; actually there is a theorem due to Cantor yielding a constructive proof, since R^n is separable). The surprising characterisation of essential points is that $\text{ess } \Omega$ is the only minimal set S :

PROPOSITION 1 Every jet intersects $\text{ess } \Omega$; every closed set which intersects all jets contains $\text{ess } \Omega$.

(Proof) The first assertion was obtained as Corollary 1 in 2.3.

Assume that the second fails: some point $x \in \text{ess } \Omega$ is not in a closed set S which otherwise intersects each jet. Choose a compact neighbourhood U of x disjoint with S . A standard argument from differential equation theory (continuous dependence on initial data) yields a second neighbourhood $V \subset U$ of x and a bound $\epsilon > 0$ such that

$$(3) \quad \pi(t, y, p, q) \in U \quad \text{for } y \in V, 0 \leq t \leq \epsilon$$

and any player controls p, q . Since $x \in \text{ess } \Omega$, there is a point $y \in V$ within $W(0, \epsilon) \setminus \Omega$, and therefore some strategy σ forces y to Ω within $[0, \epsilon]$. With this limit on the termination times, the entire jet $J_\sigma(y) \subset U$. By assumption, S intersects $J_\sigma(y)$ but is disjoint with U : a contradiction. QED

COROLLARY 1 $\text{ess } \Omega$ is a closed subset of $\partial \Omega$.

(Proof: $\partial \Omega$ is a closed set intersecting each jet.)

COROLLARY 2 Suppose that $\Omega = \bigcap_{k=1}^{\infty} \Omega_k$ with the Ω_k closed; then

$$\text{ess } \Omega \subset \liminf_{k \rightarrow \infty} \text{ess } \Omega_k.$$

(Proof: it is easily verified that the closed \liminf inter-

sects each jet.)

PROPOSITION 2 In the game (1), with standard assumptions, there are nontrivial winning positions if, and only if, there exist essential points.

(Proof) Of course, if all $W(0,t) = \Omega$, then $\text{ess } \Omega = \lim(W(0,t) \setminus \Omega) = \emptyset$. Conversely, if x is a nontrivial winning position, then $\text{ess } \Omega$ intersects its jet, and so $\text{ess } \Omega \neq \emptyset$. QED

Exercises

1. Prove that $\text{ess } \Omega_1 \subset \text{ess } \Omega_2$ if $\Omega_1 \subset \Omega_2$.
2. Show that $\text{ess}(\Omega_1 \cup \Omega_2) = \text{ess } \Omega_1 \cup \text{ess } \Omega_2$ if the Ω_k are closed and disjoint.

Remarks

The essential points have already appeared, rather naturally, in the Theorem of 5.1 and subsequent results (Proposition 1 in 2.3 is an ex post occurrence, as is the jet appearing in (6) there). Some historical remarks are delayed until the next section.

7.2 Essential Points on Large Targets

In this section we present and illustrate computable conditions for essentiality. The game

$$(1) \quad \begin{aligned} \dot{x} &= f(x,p,q); \quad p(t) \in P, \quad q(t) \in Q; \quad \text{end: } x \in \Omega; \\ \text{state space: } &R^n \end{aligned}$$

is retained, together with the standard assumptions from 2.1. However, our target sets are more special,

$$(2) \quad \Omega = \{x: \varphi(x) \leq 0\}$$

where $\varphi: R^n \rightarrow R^1$ is in C^1 (i.e., has continuous first partial derivatives). The gradient of φ is its Jacobian

row-vector, denoted as in $D\varphi = (\frac{\partial\varphi}{\partial\xi_1}, \dots, \frac{\partial\varphi}{\partial\xi_n})$, and $D\varphi(x)$ is its value at x . The target sets are large in the sense that the results become vacuous for points at which $D\varphi(x) = 0$.

We introduce one further useful notation: the formal time derivative, $\delta\varphi$, of φ along trajectories of the dynamical equation, is

$$(3) \quad \delta\varphi = D(x) \cdot f(x, p, q) = \sum_{k=1}^n \frac{\partial\varphi(x)}{\partial\xi_k} \cdot f_k(x, p, q).$$

THEOREM In game (1) let the standard assumptions obtain, and let the target set Ω be as described above. For a point $x \in \partial\Omega$ to be essential it is sufficient that

$$(4) \quad M_x = \min_{p \in P} \max_{q \in Q} \delta\varphi < 0$$

and necessary that

$$(5) \quad \mu_x = \max_{q \in Q} \min_{p \in P} \delta\varphi \leq 0.$$

(Proof) Change coordinates so that $x = 0$, and write $d' = D\varphi(0)$. Note that, since 0 is on the boundary of $\{\varphi \leq 0\}$, necessarily $\varphi(0) = 0$.

Consider first the case $M_x < 0$. Necessarily $d' \neq 0$, since otherwise $\delta\varphi = 0 = M_x$ according to (3) and (4). We shall prove that $t^2 d' \in W(0, t) \setminus \Omega$ for small $t > 0$, concluding that indeed

$$0 = \lim_{t \rightarrow 0} t^2 d' \in \limsup_{t \rightarrow 0} (W(0, t) \setminus \Omega) = \text{ess } \Omega.$$

By finite Taylor expansion,

$$\varphi(t^2 d') = 0 + d' t^2 d' + o(t^2),$$

so that $\varphi(t^2 d')/t^2 \rightarrow |d'|^2 > 0$; hence $\varphi(t^2 d') > 0$ and $t^2 d' \notin \Omega$ for small $t > 0$.

Next, find a point $\bar{p} \in P$ at which the outer minimum for M_x in (4) is attained:

$$(6) \quad d'f(0, \bar{p}, q) \leq M_x < 0 \quad \text{for } q \in Q,$$

We propose to interpret \bar{p} as a constant (hence, non-anticipatory) strategy. For any quarry control $q(\cdot)$ let $x(\cdot)$ be the corresponding state response with initial position $x(0) = t^2 d$. To show that $t^2 d \in W(0, t)$ it is sufficient to verify that $\varphi(x(t)) < 0$ for $0 < t < \epsilon$, with $\epsilon > 0$ independent of $q(\cdot)$. Again by Taylor expansion at 0,

$$(7) \quad \begin{aligned} \varphi(x(t)) &= 0 + d'x(t) + \sigma(|x(t)|), \\ \frac{\varphi(x(t))}{t} &= d' \frac{x(t)}{t} + \left| \frac{x(t)}{t} \right| \sigma(1) \end{aligned}$$

The point $x(t)$ is attained from its initial value $x(0) = t^2 d$ at time t ; from the Limit Theorem for attainable sets in 3.11, for small $t > 0$ the points

$$\frac{x(t)}{t} = \frac{x(t) - t^2 d}{t} + td$$

are close to the compact set $\text{cvx } f(0, \bar{p}, Q) + 0$. In particular, $x(t)/t$ is bounded, so the remainder term in (7) is

$O(1)\sigma(1) \rightarrow 0$ as $t \rightarrow 0$. Again from the Limit Theorem, the accumulation points $\varphi(x(t))/t$ in (7) belong to $d' \text{cvx } f(0, \bar{p}, Q)$, i.e., lie in $(-\infty, M_x] \subset (-\infty, 0)$. Since the estimates are uniform (cf. Corollary 4 in 3.11), we conclude that $\varphi(x(t))/t < 0$ indeed holds for small $t > 0$, estimate independent of $q(\cdot)$.

For the necessary condition, assume $0 \in \text{ess } \Omega$; thus there exist sequences of points $x = x_k$ and times $t = t_k$ with

$$0 \leftarrow x \in W(0, t) \setminus \Omega, \quad t \rightarrow 0+$$

(indices k will be suppressed where possible). We wish to show that (5) holds. Find a point $\bar{q} \in Q$ at which the outer maximum for μ_x is attained in (5). To each $x = x_k$ there corresponds a strategy σ_k which forces x to Ω within

$[0, t]$. Consider the pursuer controls $p = p_k = \sigma_k[\bar{q}]$, and let $x(\cdot) = x_k(\cdot)$ be the corresponding state responses. Thus

$$(8) \quad x = x(0) \notin \Omega \text{ and } \varphi(x) > 0, \quad x(s) \in \Omega \text{ and } \varphi(x(s)) \leq 0$$

for suitable $s = s_k \in (0, t_k]$. The Taylor expansions at x_k are

$$\varphi(x(s)) = \varphi(x) + d'_k(x(s) - x) + o(|x(s) - x|),$$

with $d'_k = D\varphi(x_k) \rightarrow D\varphi(0) = d'$; from (8), after dividing by $s > 0$,

$$(9) \quad 0 \geq d'_k \frac{x(s) - x}{s} + \left| \frac{x(s) - x}{s} \right| o(1).$$

From the Limit Theorem for attainable sets, the terms $(x(s) - x)/s$ are in a neighbourhood of $\text{cvx } f(0, p, \bar{q})$; thus they are bounded, the remainder term in (9) is $O(1)o(1) \rightarrow 0$, and the nonpositive accumulation points are in $d' \text{cvx } f(0, p, \bar{q})$. Hence indeed

$$0 \geq \min d' f(0, p, \bar{q}) = \mu_k. \quad \text{QED}$$

Condition (4) will be used to find essential points, and the negation $\mu_k > 0$ of (5) for the inessential points; they are easily visualised, as in fig. 1.

The test indices M_x, μ_x depend continuously on x . Always $\mu_k \leq M_x$, with equality if the players' control effects separate as in $f(x, p, q) = g(x, p) + h(x, q)$; e.g., in the linear case with $D\varphi(x) = d'$,

$$M_x = \mu_x = d'Ax - \max_{p \in P} d'p + \max_{q \in Q} d'q = d'Ax - d'(\bar{p} - \bar{q})$$

in the notation of 5.3. The assertion of the theorem is that

$$(10) \quad \{x \in \partial\Omega: Mx < 0\} \subset \text{ess } \Omega \subset \{x \in \Omega: \mu_k \leq 0\};$$

thus, the boundary of $\text{ess } \Omega$ relative to $\partial\Omega$ satisfies

$$\overset{*}{\partial} \text{ess } \Omega \subset \{x \in \partial\Omega: \mu_x \leq 0 \leq M_x\}.$$

The results of the theorem may be sharpened slightly:

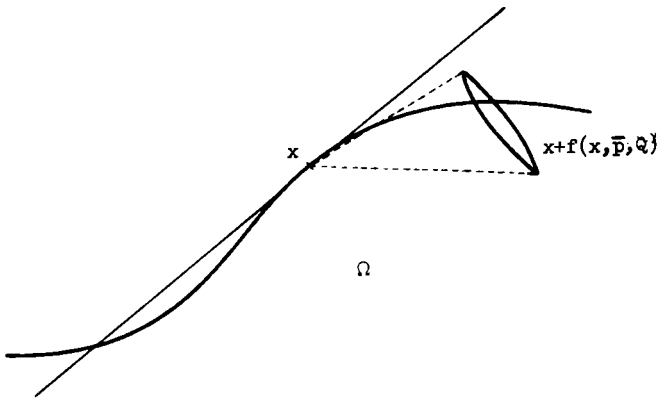
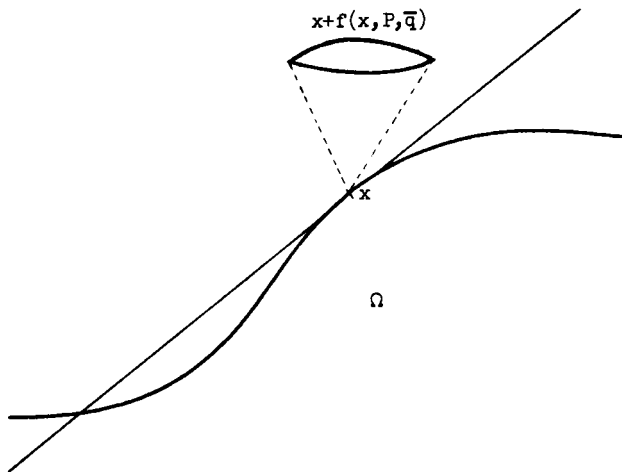


Fig. 1 Geometry of essential points. The essential point has some quarry vectogram in the target side of the tangent hyperplane (upper figure); the inessential point has a pursuer vectogram in the opposite side (lower figure).



COROLLARY If there is a sequence $y = y_k$ on $\partial\Omega$ with $M_y < 0$ and $y_k \rightarrow x$, then $x \in \text{ess } \Omega$. If $x \in \text{ess } \Omega$, then $\mu_y \leq 0$ for some sequence $y = y_k \rightarrow x$ outside Ω .

(Proof) The first assertion follows from the theorem, since $\text{ess } \Omega$ is closed. The second follows from Corollary 2 in 7.1 with $\Omega_k = \{x: \varphi(x) \leq k^{-1}\}$ and the theorem, since the additive constant does not effect $\delta\varphi$ nor μ_y . QED

Example 1 We will examine the essential points for the game indicated by uncoupled n -dimensional equations,

$$(8) \quad \ddot{x} = f(x, \dot{x}, u), \quad \dot{y} = g(y) - v;$$

pursuer's and quarry's constraints

$$u(t) \in U, \quad |v(t)| \leq 1;$$

and ϵ -capture as the termination condition,

$$|x-y| \leq \epsilon$$

(here $n \geq 1$, $U \neq \emptyset$, $\epsilon > 0$, and f, g satisfy appropriate versions of the standard assumptions from 2.1).

There is an interpretation in R^{3n} as state space, with coordinates partitioned into x, \dot{x}, y . An obvious choice for describing the termination condition by $\varphi \leq 0$ has

$$\varphi(x, \dot{x}, y) = \frac{1}{2} (|x - y|^2 - \epsilon^2).$$

Then

$$\begin{aligned} D\varphi &= (x' - y', 0, y' - x'), \quad \delta\varphi = (x - y)'(\dot{x} - g(y) + v), \\ M = \mu &= (x - y)'(\dot{x} - g(y)) + \epsilon \end{aligned}$$

The necessary condition $\mu \leq 0$ is also sufficient, and $\mu = 0$ describes the boundary of $\text{ess } \Omega$ in $\partial\Omega$. This follows from the Corollary: since $\epsilon > 0$ and thus $x - y \neq 0$, one can adjust \dot{x} to obtain $\mu < 0$ for points on $\partial\Omega$ close to any point satisfying $\mu \leq 0$.

To simplify further, for $z \in R^n$ with $|z| = 1$ set

$x = y - \epsilon z$; then z, \dot{x}, y (in $S^{n-1} \times R^n \times R^n$) are independent coordinates describing $\partial\Omega$, and the condition for essentiality is

$$(9) \quad z'(\dot{x} - g(y)) \geq 1.$$

Thus the essential point set will consist of entire half rays $z, \alpha \dot{x}, y$ with α large enough (positive if $z' \dot{x} > 0$, negative if $z' \dot{x} < 0$); and, in the remaining manifold $z' \dot{x} = 0$ of dimension $n^2 - n - 1$, the condition reduces to $z' g(y) \leq -1$.

One interesting point is that (9) is independent of $\epsilon > 0$; there is no interplay between ϵ and the other parameters in the game. It is not true that how close pursuer can approach quarry depends on the game data. The conclusion should not, however, be extrapolated via $\epsilon \rightarrow 0+$: we have already seen in 5.2 that for $\epsilon = 0$ and $n \geq 2$ there need not exist any winning positions.

A more forceful formulation is that, according to (9), the essential point problem for the original game is equivalent to that for

$$\ddot{x} = 0, \dot{y} = g(y) + v; u = 0; |v(t)| \leq 1; \text{ end: } |x-y| \leq \epsilon.$$

In particular, taking (8) is linear (e.g., the isotropic rocket game) provides no further simplification.

Example 2 (Homicidal Chauffeur Game) The reduced dynamical equation was obtained in Exercise 2 of 15; in real coordinates x, y of $z = x + iy$, this reads

$$\dot{x} = yu + s_1 + v_1, \dot{y} = -xu + v_2$$

with control constraints

$$|u| \leq \frac{s_1}{\rho_1}, v_1^2 + v_2^2 \leq s_2^2;$$

let $x^2 + y^2 \leq \epsilon^2$ be the termination condition. All the

parameters $s_1, s_2, \rho_1, \epsilon$ are to be strictly positive.

An obvious description of the target set is provided by

$\varphi(x, y) = \frac{1}{2} (x^2 + y^2 - \epsilon^2)$. Then, easily,

$$\begin{aligned}\delta\varphi &= s_1 x + (xv_1 + yv_2), \quad \mu = M = s_1 x + s_2 \epsilon, \\ \mu &= s_1 \epsilon \left(\cos \theta + \frac{s_2}{s_1} \right)\end{aligned}$$

with θ the polar angle ($x = \epsilon \cos \theta$).

The conclusions are as follows. The condition $s_1 \geq s_2$ is necessary for presence of nontrivial winning positions. If $s_1 > s_2$, then the essential points on the ϵ -circle are those with polar angle θ satisfying $\cos \theta \leq -s_2/s_1$, a left circular arc, quite independent of capture radius ϵ and pursuer's curvature bound $1/\rho_1$.

This agrees (modulo notation, and the endpoints of the essential arc) with the results on p. 279 of Isaacs' book. One interpretation is that the pursuer can overcome quarry's far greater capability of manoeuvring by even a slight excess of speed.

Exercises

1. In the first part of the proof of the theorem it was shown that, for small $t > 0$, the point $t^2 d \in W(0, t) \setminus \Omega$; expand slightly to prove that an entire neighbourhood of the point is in the set.

2. For allonomous games with dynamical equation $\dot{x} = f(t, x, p, q)$ and termination condition $\varphi(t, x) \leq 0$ the test expression analogous to M in (4) is

$$\frac{\partial \varphi(t, x)}{\partial t} + \min_p \max_q D_x \varphi(t, x) \cdot f(t, x, p, q),$$

Formulate the appropriate assumptions and prove this. (Hint: a short-cut is to reduce to an autonomous situation.)

3. In the reduced version of the one-dimensional

homicidal chauffeur game, find all pursuer and quarry vectorgrams at the point $(-\epsilon, 1.2)$ and also at $(-\epsilon, 0.8)$.

4. In the theorem omit the assumption on Ω , consider a point $x \in \partial\Omega$ and a ball B about x . Prove that if $\Omega \cap B$ contains a convex cone with vertex x and exterior normal d , and if $\min_p \max_q d'f(x, p, q) < 0$, then $x \in \text{ess } \Omega$. Prove that if $\Omega \cap B$ is convex, $x \in \text{ess } \Omega$ and d is an exterior normal, then $\max_q \min_p d'f(x, p, q) \leq 0$.

5. For the game in R^2

$$\dot{x} = y - q; \dot{y} = p; |p(t)| \leq 1 \geq |q(t)|; \text{ end: } |x| \leq 2 \geq |y|$$

use the geometric method suggested by Fig. 1 to find the essential and inessential portions of the target set boundary. (The answer is summarised in fig. 2.)

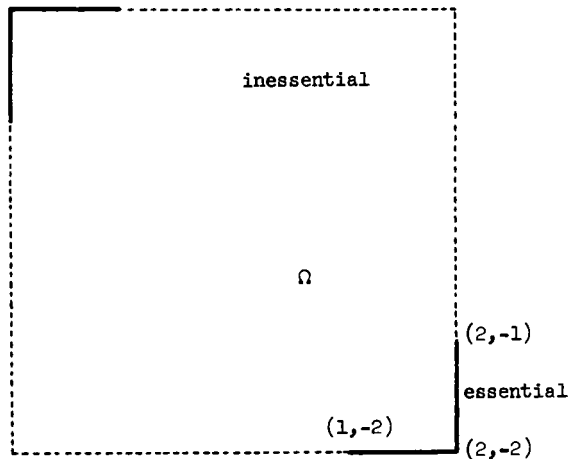


Fig. 2 The L-shaped sets form the essential portion of target in the game of Exercise 5.

6. Treat the two-dimensional game of two cars, with (4) of 1.6 for the dynamical equation. Show that, in any

configuration of the (strictly positive) parameters, there always exist nontrivial winning positions. (Partial answer: $\delta\varphi$ is independent of controls.)

7. Treat essential points in the case of an n -dimensional equation of order $k \geq 2$

$$x^{(k)} = f(x, \dots, x^{(k-1)}, p, q)$$

and $\varphi(x) \leq 0$ as termination condition; to simplify you may assume that $D\varphi(x) \neq 0$ on the boundary of the target set. (Check: as far as essentiality goes, $\dot{x} = 0$, $\varphi(x) \leq 0$ is an equivalent problem.)

8. In the special case $\varphi(x) = x'Wx - \epsilon$ of the preceding (W positive definite, $\epsilon > 0$) find the boundary of the essential point set on the surface of the target.

9. Consider the linear game with uncoupled players in n -space,

$$\dot{x} = Ax - p, \quad \dot{y} = By - q$$

player constraints $|p(t)| \leq \alpha$, $|q(t)| \leq \beta$, and $|x - y| \leq \epsilon$ at termination ($\alpha > 0$, $\beta > 0$, $\epsilon > 0$). Prove that nontrivial winning positions exist if $A \neq B$. (Hint: if there are no essential points, then $\mu > 0$ at all points of target boundary.)

10. For the paradoxial game of 1.7

$$\dot{x} = (p-q)^2 - \frac{1}{8}; \quad 0 \leq p(t), q(t) \leq 1; \quad \text{end: } x \leq 0$$

find μ_x and M_x ; prove directly that 0 is an essential point.

Remarks

The notion of essential point appears in

R. Isaacs: *Differential Games*, Wiley, New York, etc., 1967 (pp. 83, 215)

as point in the 'useable' part; in our notation, the definition adopted there is $M_x > 0$ (and $M_x < 0$ for the 'nonuseable' points). The disadvantage of this approach is that Proposition 2 from 7.1 does not hold. A formulation on p. 55 of

A. Blaquiére, F. Gérard, G. Leitman: Quantitative and Qualitative Games, Academic Press, New York and London, 1969

seems to suggest that the endpoints of time-optimal trajectories are 'useable'. A related concept, the usable points, appears on p. 88 of

A. Friedman: Differential Games, Wiley-Interscience, New York, etc., 1971.

The definition can probably be deciphered; a subsequent statement is that $\mu_x < 0$ is a sufficient condition in the case that the players' dynamics are separated.

This should not be confused with what we have called the actually used part of target (for which a sufficient condition would be $\min_p \min_q \delta\phi < 0$).

PROBLEM Is $\mu_x < 0$ sufficient for x to be an essential point?

7.3 Necessary Conditions for Small Targets

Our game in R^n is again

$$(1) \quad \dot{x} = f(x, p, q); \quad p(t) \in P, \quad q(t) \in Q; \quad \text{end: } x \in \Omega$$

with the standard assumptions from 2.1. The target sets will be described by

$$(2) \quad \Omega = \{x: F(x) = 0\}$$

where $F: R^n \rightarrow R^m$ is of class C^1 . The Jacobian matrix of F at x is denoted by

$$DF(x) = \left(\frac{\partial F_i(x)}{\partial x_j} \right); \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Every closed set Ω can be described in this manner, e.g. by taking as $F(x)$ the distance-square of x from Ω . The games to be treated have sparse targets: the results for points $x \in \partial\Omega$ become trivial if $DF(x)$ vanishes (e.g., $x \in \overline{\text{Int } \Omega}$). On the other hand, $m > 1$ is allowed.

THEOREM 1 In game (1) let the standard assumptions apply, with the target set as described above. For a point x to be essential it is necessary that there exist a ray $L = R^+_d$ in R^m such that

$$\text{conv } DF(x) \cdot f(x, P, q)$$

intersects L for every $q \in Q$.

Remark Neater versions of the condition appear for special cases, in the Necessary Conditions. It may not be obvious that this is an immediate generalisation of the necessary condition from the Theorem in 7.2. There $m = 1$, so there are only two rays, the semi-axes; the assertion is that either all $\delta F \leq 0$, or all $\delta F \geq 0$ (cf. Exercise 1).

(Proof) Consider first any winning position $x \in W(0, \theta) \setminus \Omega$, and let σ be a corresponding strategy; also consider any pair of quarry control values q_1, q_2 in Q . Then one has the pursuer responses $p_1 = \sigma[q_1]$, state responses $x(\cdot), y(\cdot)$ with initial value $x(0) = x = y(0)$, and termination times t, s in $(0, \theta]$. Finally, both $x = x(t)$ and $y = y(s)$ are in $\Omega = \{F = 0\}$, so that (Taylor's expansion, with integral remainder term)

$$\begin{aligned} (3) \quad 0 &= F(x) - F(y) = [F(y + \xi(x-y))]_{\xi=0}^{\xi=1} \\ &= \int_0^1 DF(y + \xi(x-y)) d\xi \cdot (x-y) \end{aligned}$$

We will subsequently divide by $t + s > 0$ and take limits appropriately. Note that if $x(t), y(s)$ have a common limit

x_0 , then the integral term in (3)

$$(4) \quad \int_0^1 DF(y(s) + \xi(x(t) - y(s))) d\xi \rightarrow \int_0^1 DF(x_0) d\xi = DF(x_0),$$

and the second factor

$$(5) \quad \frac{x(t) - y(s)}{t + s} = \lambda \frac{x(t) - x}{t} - (1 - \lambda) \frac{y(s) - x}{s}, \quad \lambda = \frac{t}{t + s} \in [0, 1].$$

With this prepared, assume $x_0 \in \text{ess } \Omega$. Then there is a sequence of points x_k and times θ_k such that

$$x_0 \leftarrow x_k \in W(0, \theta_k) \setminus \Omega, \quad 0 < \theta_k \rightarrow 0.$$

Therefore there exist strategies σ_k : for each point $q \in Q$, interpreted as a constant quarry control, there are corresponding pursuer responses $p_k = \sigma_k[q]$, state responses $x_k(\cdot)$ and termination times t_k . In detail,

$$\begin{aligned} x_k(t) &= x_k + \int_0^t f(x_k(s), p_k(s), q) ds, \\ x_k(0) &= x_k, \quad x_k(t) \in \Omega, \quad 0 < t_k \leq \theta_k \rightarrow 0. \end{aligned}$$

Since $x_k(t_k)$ is attained from x_k at t_k , the Limit Theorem for attainability sets (see 3.11) yields a subsequence x_j of the x_k such that one has convergence in

$$\frac{x_j(t_j) - x_j}{t_j} \rightarrow z \in \text{cvx } f(x_0, P, q).$$

Here, of course, the subsequence x_j will depend on the choice of the point $q \in Q$ above (this then determines $p_k(\cdot)$, $x_k(\cdot)$, t_k). The usual trick provides an almost universal subsequence. Take a countable dense set $\{q_i: i = 1, 2, \dots\}$ in Q , and apply the foregoing argument to q_1, q_2, \dots in turn. By taking subsequences of sequences (Cantor's 'second diagonal method'), one obtains a single subsequence x_j of x_k which works for all the q_i . In detail, for each i there exist

sequences of times t_{ij} , points $y_{ij} = x_{ij}(t_{ij})$, and also points z_i such that

$$(6) \quad 0 < t_{ij} \leq \theta_j \rightarrow 0, y_{ij} \in \Omega$$

$$\frac{y_{ij} - x_j}{t_{ij}} \rightarrow z_i \in \text{cvx } f(x_0, P, q_1) \text{ as } j \rightarrow \infty.$$

Since all $\text{cvx } f(x_0, P, q_1)$ are contained in the compact set $\text{cvx } f(x_0, P, Q)$, we also have from (6) that

$$y_{ij} = x_j + t_{ij} z_i \rightarrow x_0 + 0 \text{ as } j \rightarrow \infty, \text{ uniformly for } i.$$

Now apply the preparatory argument to $x_j \in W(0, \theta_j) \setminus \Omega$ and any pair q_i, q_ℓ in the dense subset. The $\lambda = \lambda_{ij}$ resulting from (5) have an accumulation point $\mu = \mu_{i\ell}$ in $[0, 1]$ as $j \rightarrow \infty$. Now take limits in (3), using (7) for (4), and (6) in (5):

$$(8) \quad 0 = DF(x_0)(\mu_{i\ell} z_i - (1 - \mu_{i\ell}) z_\ell)$$

holds for all i, ℓ and some $\mu_{i\ell} \in [0, 1]$.

The conclusion is now obvious geometrically. In detail, there are two alternatives. Either

$$DF(x_0)z_\ell = 0 \text{ for all } \ell = 1, 2, \dots;$$

or some $DF(x_0)z_\ell \neq 0$. Then $\mu_{i\ell} \neq 0$ from (8), and

$$DF(x_0)z_i = \frac{1 - \mu_{i\ell}}{\mu_{i\ell}} DF(x_0)z_\ell \text{ for all } i = 1, 2, \dots,$$

with the scalar factor $(1 - \mu)/\mu \geq 0$. Let L be the indicated ray, $L = R^+ 0$ or $L = F^+ DF(x_0)z_\ell$; the conclusion is that $DF(x_0)z_i$ belongs to both L and $DF(x_0) \cdot \text{cvx } f(x, P, q_1)$. This proves the assertion for all $q = q_1$ in a dense subset of Q ; the general case follows by continuity (L is closed, $\text{cvx } f(x, P, Q)$ compact). QED

A consequence is the following nonlinear generalisation of the conditions from 5.1:

NECESSARY CONDITIONS Appropriate versions of the standard assumptions are to apply: the target is $\Omega = \{x: F(x) = 0\}$ with a C^1 map $F: R^n \rightarrow R^m$; and let us write $D = DF(x)$.

If the players' dynamical effects separate,

$$\dot{x} = f(x, p) - g(x, q),$$

and $x \in \text{ess } \Omega$, then there is a ray $L = R^+ d$ from the origin in R^m such that

$$D \cdot f(x, Q) \subset \text{conv } D \cdot f(x, P) + L;$$

in particular,

$$\dim D \cdot g(x, Q) \leq 1 + \dim D \cdot f(x, D).$$

If the players' controls appear linearly,

$$\dot{x} = a(x) + B(x)p + C(x)q$$

and the players' constraint sets are compact, convex, and symmetric, then

$$D \cdot C(x)Q \subset D \cdot B(x)P + [-1, 1] D \cdot a(x)$$

is necessary for $x \in \text{ess } \Omega$.

(Proof) The assertion in the first case is a direct consequence of the theorem. The second case then appears as

$$DCQ \subset DA + DBP + R^+ d$$

for some $d \in R^m$, where x 's have been omitted. Take any $q \in Q$; then also $-q \in Q$, and thus

$$DCq = Da + DBp_1 + \alpha_1 d$$

$$-DCq = Da + DBp_2 + \alpha_2 d$$

for suitable $p_1 \in P$, $\alpha_1 \in R^+$. Multiply by μ and $(1 - \mu)$, and add:

$$DCq = (\alpha_1 - \alpha_2)Da + DB(\mu p_1 + (1 - \mu)p_2) + (\mu \alpha_1 - (1 - \mu)\alpha_2)d.$$

Now, it is always possible to choose μ in $[0, 1]$ so that the

last term vanishes. Then $\mu p_1 + (1-\mu)(-p_2) \in P$, and $-1 \leq 2\mu-1 \leq 1$:

$$DCq \in [-1,1]Da + DBP \quad (\text{all } q \in Q). \quad \text{QED}$$

Example 1 In Example 1 of 7.2, the termination condition was ϵ -approach with $\epsilon > 0$; here we shall treat the case $\epsilon = 0$ of perfect capture. The equations of motion are n -dimensional,

$$\ddot{x} = f(x, \dot{x}, u), \quad \dot{y} = g(y) - v,$$

compact players' control constraint sets U, V and $x = y$ as termination condition. (This is the nonlinear version of the example in 5.2.)

Use the notation established in 7.2. The theorem there, with real-valued $\varphi(x, \dot{x}, y) = \frac{1}{2} |x-y|^2$, provides no information, as the Jacobian vanishes on Ω . Theorem 1 with $F(x, \dot{x}, y) = x-y$ yields the following: if (x, \dot{x}, y) is essential, then $x = y$, and there is a ray R^+d in R^n such that

$$\dot{x} - g(y) + v \in R^+d.$$

Thus there are no winning positions unless V is a segment; and if V is a segment with endpoints $\pm v_0$, then each essential point $x, \dot{x}, y = x$ (if there are any at all) must have

$$\dot{x} = g(y) + \lambda v_0, \quad |\lambda| \geq 1.$$

Consider now a game whose n -dimensional equation is of order 2,

$$(9) \quad \ddot{x} = f(x, \dot{x}, p, q); \quad p(t) \in P, \quad q(t) \in Q; \quad \text{end: } F(x) = 0,$$

and, as indicated, the termination condition depends on the position only.

Of course, on passing to state space R^{2n} , this is a rather special case of a game of the form (1); one in which, however, Theorem 1 provides no information at all: the

necessary condition for (x, \dot{x}) to be an essential point is that $DF(x)\dot{x}$ belong to some ray. (The exceptional case of real-valued F is treated in Exercise 3.) Our aim is to obtain sharper results.

The second order version of the standard assumptions (see 2.1) will be used. The termination condition in (9) is $F(x) = 0$, independent of \dot{x} ; here $F: R^n \rightarrow R^m$ is to be of class C^1 , with Jacobian matrix $DF(x)$ of type (m, n) (i.e., not $m \times 2n$).

THEOREM 2 In the game (9) with the standard assumptions and target as above, a necessary condition for a position (x, \dot{x}) to be essential is that there exist a half-plane in R^m ,

$$L = R^+d + R^1DF(x)\dot{x},$$

with both 0 and $DF(x)\dot{x}$ on its boundary line, such that

$$cvx DF(x)f(x, \dot{x}, p, q)$$

intersects L for every $q \in Q$.

(Proof) The proof follows the outline of that of Theorem 1, and we shall mainly emphasise the needed modifications.

Consider any winning positions (x, \dot{x}) outside the target (i.e., $F(x) \neq 0$), forced to termination by a strategy σ within the time interval $[0, \theta]$. For any quarry control value $q_1 \in Q$ one has a corresponding pursuer response $p_1 = \sigma[q_1]$, and state response $(x(\cdot), \dot{x}(\cdot))$; if the initial position is the given one,

$$\begin{aligned} x(t) &= x + \int_0^t \dot{x}(t_1) dt_1 = x + \dot{x}t + \int_0^t \int_0^{t_1} \ddot{x}(t_2) dt_2 dt_1 \\ &= x + \dot{x}t + \int_0^t \int_0^{t_1} f(x(t_2), \dot{x}(t_2), p_1(t_2), q_1) dt_2 dt_1. \end{aligned}$$

For another quarry control value $q_2 \in Q$, pursuer response $p_2 = \sigma[q_2]$, state response $(y(\cdot), \dot{y}(\cdot))$, but the same initial position, we have

$$\begin{aligned}
y(s) &= x + \dot{x}s + \int_0^s \int_0^{s_1} \ddot{y}(s_2) ds_2 ds_1 \\
&= x + \dot{x}s + \int_0^s \int_0^{s_1} f(y(s_2), \dot{y}(s_2), p_2(s_2), q_2) ds_2 ds_1.
\end{aligned}$$

Now, (3) and (4) still hold. In place of (5),

$$\begin{aligned}
(10) \quad \frac{x(t) - y(s)}{t^2 + s^2} &= \frac{t-s}{t^2 + s^2} \dot{x} + \lambda \frac{1}{t^2} \int_0^t \int_0^{t_1} \ddot{x} - (1-\lambda) \frac{1}{s^2} \int_0^s \int_0^{s_1} \ddot{y}, \\
\lambda &= \frac{t^2}{t^2 + s^2} \in [0, 1].
\end{aligned}$$

According to Exercise 5 in 3.11 (a second-order version of the Limit Theorem for attainable sets) the term

$$\frac{1}{t^2} \int_0^t \int_0^{t_1} \ddot{x}(t_2) dt_2 dt_1 = \frac{1}{t^2} \int_0^t \int_0^{t_1} f(x(t_2), \dot{x}(t_2), p_1(t_2), q_1) dt_2 dt_1$$

has all its accumulation points (for $t \rightarrow 0+$) in $\frac{1}{2} \text{cvx } f(x, \dot{x}, p, q_1)$. In (10) the term $\frac{t-s}{t^2 + s^2} x \in R^1 x$, so that the finite accumulation points of

$$\int_0^1 DF(\dots) d\theta \frac{t-s}{t^2 + s^2} \dot{x} \text{ are in } R^1 DF(x) \dot{x}.$$

The process from the latter half of the proof of Theorem 1 is then applied: subsequences and limits in (3) divided by $t^2 + s^2$. The result, analogous to (8), is

$$(11) \quad DF(x)(\mu_{i_\ell} z_{i_\ell} - (1-\mu_{i_\ell}) z_\ell) \in R^1 DF(x) \dot{x}$$

with $\mu_{i_\ell} \in [0, 1]$ and $z_{i_\ell} \in \text{cvx } f(x, \dot{x}, p, q_1)$. Again the conclusion is plausible geometrically. A solid proof is by reduction to that from Theorem 1 on applying to (11) the orthogonal projection of R^m onto the hyperplane perpendicular to $DF(x) \dot{x}$: the right hand side projects to 0. QED

2nd ORDER NECESSARY CONDITIONS Assume appropriate versions of the assumptions above, and write $D = DF(x)$.

If players' dynamic effects separate,

$$\ddot{x} = f(x, \dot{x}, p) - g(x, \dot{x}, q).$$

then a necessary condition for (x, \dot{x}) to be essential is that

$$D \cdot g(x, \dot{x}, Q) \subset \text{cvx } D \cdot f(x, \dot{x}, P) + R^1 D \cdot \dot{x} + R^+ d$$

for some $d \in R^m$; in particular,

$$(12) \quad \dim D \cdot g(x, \dot{x}, Q) \leq 2 + \dim D \cdot f(x, \dot{x}, P).$$

If the players' controls appear linearly,

$$\ddot{x} = a(x, \dot{x}) + B(x, \dot{x})p + C(x, \dot{x})q,$$

and the players' constraint sets are compact, convex, and symmetric, the necessary condition is (omitting variables x, \dot{x})

$$DCQ \subset DBP + [-1, 1]Da + R^1 D\dot{x}.$$

(Proof: again, the first assertion is a reformulation in our special case; the second is obtained hence as in the proof of the first-order Necessary Conditions.)

Example 2 (Two cars in n dimensions) The formulation appears in 1.6; let us now treat perfect capture. The equations of motion are

$$\ddot{x} = E(\dot{x})u, \quad \ddot{y} = E(\dot{y})v;$$

$$|u(t)| \leq \frac{s_1}{\rho_1}, \quad |v(t)| \leq \frac{s_2}{\rho_2}; \quad \text{end: } x = y.$$

Here the s_k, ρ_k are strictly positive parameters, x, y vary in R^n and u, v in R^{n-1} ; there are initial constraints $|\dot{x}| = s_1, |\dot{y}| = s_2$; the mapping $E(\cdot)$ was described in 1.4.

The reduction to the situation of Theorem 2 is rather obvious; e.g., the termination condition is

$$(I, -I) \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

The Necessary Condition (second order, second version) is

$$(13) \quad E(\dot{y})V \subset E(\dot{x})U + 0 + R^1(\dot{x} - \dot{y}).$$

To eliminate the last term, choose a unit vector c perpendicular to both \dot{x} and \dot{y} (we must assume $n \geq 3$), and let $\Phi: R^n \rightarrow R^n$ be the orthogonal projection onto the linear space c^\perp perpendicular to c . Then, by construction of $E(\cdot)$,

$$\Phi E(\dot{x}) = E(\dot{x}), \quad \Phi E(\dot{y}) = E(\dot{y}),$$

and $\Phi(\dot{x} - \dot{y}) = 0$; therefore, from (13),

$$E(\dot{y})V \subset E(\dot{x})U.$$

Now, both U, V are balls in R^n with centre at the origin; thus necessarily \dot{x} is parallel to \dot{y} , and the radii satisfy $s_1^2/\rho_1 \geq s_2^2/\rho_2$.

It follows that the game with the objective of forcing \dot{x} to be parallel to \dot{y} must also be winnable. Setting $\xi = \dot{x}/s_1$, $\eta = \dot{y}/s_2$, the equations of motion are $\dot{\xi} = E(\xi)p$, $\dot{\eta} = E(\eta)q$, with termination condition $\xi = \eta$, control constraints

$$|p| = \left| \frac{u}{s_1} \right| \leq \frac{s_1}{\rho_1}, \quad |q| \leq \frac{s_2}{\rho_2}$$

(and initial requirements $|\xi| = 1 = |\eta|$). Of course, this is Kelley's game, 1.4; and $s_1/\rho_1 \geq s_2/\rho_2$ is necessary for non-trivial capture.

Conclusions: if the dimension $n \geq 3$, then the inequalities

$$\frac{s_1}{s_2} \geq \frac{\rho_1}{\rho_2} \leq \left(\frac{s_1}{s_2} \right)^2$$

are necessary for presence of nontrivial winning positions (these are not independent: e.g., the second follows from the first if $s_1 \geq s_2$).

Exercises

The first three of these concern the case $m = 1$, i.e., that $\Omega = \{x: \varphi(x) = 0\}$ with $\varphi: R^n \rightarrow R^1$ of class C^1 ; the

version of the assumptions from 2.1 is to apply.

1. Prove that a sufficient condition for $x \in \partial\Omega$ to be essential is that

$$\min_p \max_q \delta\varphi < 0 \text{ or } \max_p \min_q \delta\varphi > 0, \quad []$$

and a necessary condition is

$$\max_q \min_p \delta\varphi \leq 0 \text{ or } \min_q \max_p \delta\varphi \geq 0.$$

(Hint: apply 7.2 to φ and $-\varphi$)

2. Treat essential points in the game

$$\dot{x} = f(x) - p + q; \quad |p(t)| \leq \alpha, \quad |q(t)| \leq \beta; \quad \text{end: } c'x = \gamma$$

($\alpha \geq 0, \beta \geq 0, |c| = 1$). Show that a target set point x is essential iff $|c'f(x)| \geq \beta - \alpha$, except possibly in the case that c is a left eigenvalue of $Df(x)$; in any case, all points of the target are essential if $\alpha > \beta$.

3. Treat essential points in the game induced by the second-order n -dimensional equation

$$\ddot{x} = f(x, \dot{x}, p, q); \quad \text{end: } \varphi(x) = 0$$

with nonvoid compact control constraint sets. Show that, for each x with $\varphi(x) = 0$ and $D\varphi(x) \neq 0$, the point (x, \dot{x}) in state space, with arbitrary $\dot{x} \in \mathbb{R}^n$, is essential. (Hint: $\text{ess } \Omega$ is closed.)

4. Obtain the version of the Necessary Conditions in the autonomous case (e.g., $\dot{x} = f(x, t, p) - g(x, t, q)$, end: $F(t, x) = 0$).

5. Using the notation from 1.4 or Example 2, apply the Necessary Conditions to Kelley's game.

6. Consider the game in the Example of 3.4. Prove that, if the constraint sets are balls about the origin in \mathbb{R}^n and $n \geq 3$, then the necessary condition $V \subset U$ for isochronous capture is also necessary for general capture. (Hint: In

$V \subset U + L$ with L linear and two-dimensional, consider vectors perpendicular to L .)

7. Treat the (two-dimensional) homicidal chauffeur game with perfect capture; e.g., in the formulation of Example 2 from 7.2, with termination condition $x = 0 = y$. (Answer: nontrivial capture is impossible.)

8. Treat the game of two cars with perfect capture, in the formulation of (4) of 1.6. (Answer: the Necessary Condition is inconclusive.)

Remarks

A fascinating sequence of necessary conditions has been obtained in the linear case (see 4.2, 5.1, (12)):

isochronous capture... $\dim MQ \leq \dim MP$

capture... $\dim MQ \leq 1 + \dim MP$

second-order games... $\dim MQ \leq 2 + \dim MP$

under appropriate assumptions. Here the first may be extraneous; nevertheless there remains a tempting conjecture to be investigated.

Computationally feasible sufficient conditions for essential points were obtained for large targets in 7.2, and for linear games (and 'small' targets) in 5.2. It would be highly desirable to extend the latter to non-linear games, possibly first for dynamical equations with the players' effects separating. One obstacle in the direct approach is that Filippov's Lemma applied to the nonlinear analogue of the proof of Theorem 2 in 5.2 only provides state-dependent strategies.

PROBLEM In Example 2 with $n = 3$, Cockayne asserts that $s_1 > s_2$, $s_1^2/\rho_1 > s_2^2/\rho_2$ is sufficient for capture from all initial positions (see reference in 1.6); but then our necessary condition $s_1/\rho_1 \geq s_2/\rho_2$ need not be satisfied,

e.g. with

$$s_1 = 3, s_2 = 1, p_1 = 4, p_2 = 1.$$

7.4 Isochronous Capture

Sections 7.2 and 7.3 may be interpreted as the extension, to the nonlinear situation, of the results established in 5.1 and 5.2. It is natural to ask whether there is an analogous generalisation of the treatment of isochronous capture in Chapter 4.

The game is indicated by

$$(1) \quad \begin{aligned} \dot{x} &= f(x, p, q); \quad p(t) \in P, \quad q(t) \in Q; \\ \text{end: } F(x) &= 0; \quad \text{state space: } R^n, \end{aligned}$$

with the standard assumptions. The termination condition involves $F: R^n \rightarrow R^m$ of class C^1 .

THEOREM Assume that, in the game just described, non-trivial isochronous capture is possible ($W(0) \neq \emptyset$ for some $\theta > 0$). Then

$$(2) \quad \limsup_{t \rightarrow 0+} W(t) \neq \emptyset$$

and, for every point x in the limsup, the sets

$$\text{conv } DF(x)f(x, P, q) \quad (q \in Q)$$

have a point in common.

(Proof) This is analogous to, but simpler than, that of Theorem 1 in 7.3. Corollary 2 in 2.3 states that (2) holds; this will be used in place of Proposition 2 from 7.1.

Take any point $x_0 \in \limsup_{t \rightarrow 0+} W(t)$:

$$(3) \quad x_k \rightarrow x_0, \quad x_k \in W(t_k), \quad 0 < t_k \rightarrow 0.$$

Let σ_k be an isochronous strategy, forcing x_k to termination at time t_k . For each element q_1 of a countable set $\{q_1, q_2, \dots\}$ dense in Q let $p_{1k} = \sigma_k[q_1]$ (thus $p_{1k}: R^1 \rightarrow P$,

while the q_i are constants). Also, let $x_{ik}(\cdot)$ be the state response, initiating at position x_k , to the controls p_{ik} , q_i : finally, set $y_{ik} = x_{ik}(t_k)$.

We have (omitting indices for the moment) that

$$\frac{y-x}{t} = \frac{x(t)-x(0)}{t} = \frac{1}{t} \int_0^t f(x(s), p(s), q) ds.$$

From the Limit Theorem for attainable sets (3.11),

$$(4) \quad \frac{y_{ij}-x_j}{t_j} \rightarrow z_i \in \text{cvx } f(x_0, P, q_i) \quad \text{as } j \rightarrow \infty$$

for each index i and suitable subsequence x_j of the x_k . By Cantor's diagonal method, there is a single subsequence x_j , common for all q_i .

Second, the y_{ij} are endpoints in the target set. Thus, for any indices i, ℓ, j

$$0 = \frac{1}{t_j} (F(y_{ij}) - F(y_{\ell j})) = \int_0^1 DF(y_{\ell j} + \xi(y_{ij} - y_{\ell j})) d\xi \frac{y_{ij} - y_{\ell j}}{t_j}.$$

Since the points y converge to x_0 (use (4) with (3)), the entire integral term converges to $DF(x_0)$. By adding and subtracting x_j in the second term, the limit is $z_i - z_\ell$ according to (4):

$$DF(x_0)(z_i - z_\ell) = 0 \quad (\text{all } i, \ell).$$

Take, say, $\ell = 1$: for all $i = 1, 2, \dots$,

$$(5) \quad DF(x_0)z_1 = DF(x_0)z_i \in \text{cvx } DF(x_0)f(x_0, P, q_i)$$

Since the q_i form a set dense in Q , we find that (5) holds for any $q \in Q$ in place of q_i . QED

COROLLARY If the players' dynamical effects separate, $\dot{x} = f(x, p) - g(x, q)$, then the necessary condition becomes

$$(6) \quad DF(x)(g(x, Q) - g(x, Q)) \subset DF(x)(\text{cvx } f(x, P) - \text{cvx } f(x, P)).$$

If their controls appear linearly, as in $\dot{x} = a(x) + B(x)p + C(x)q$, with compact, convex, and symmetric constraint sets, then the necessary condition is

$$(7) \quad DF(x)C(x)Q \subset DF(x)B(x)P.$$

Example Let us revisit the homicidal chauffeur game, with the notation of Example 2 in 7.2 (including capture radius $\epsilon > 0$). The form (7) is not applicable, but (6) is, with

$$\begin{aligned} DF(x)g(x, Q) &= \{\epsilon s_2 \sin(\theta + v) : |v| \leq \pi\} = [-\epsilon s_2, \epsilon s_2], \\ DF(x)f(x, P) &= s_1 \epsilon \sin \theta \end{aligned}$$

for the point $x \in \Omega$ with polar angle θ . Condition (6) is thus

$$(8) \quad [-2\epsilon s_2, 2\epsilon s_2] \subset \{0\} :$$

isochronous capture is impossible.

Exercises

1. In the homicidal chauffeur game, common sense tells us that once isochronous capture is impossible for capture radius $\epsilon > 0$, it must also be impossible for $\epsilon = 0$, perfect capture; on the other hand, (8) would be satisfied for $\epsilon = 0$. Settle the matter. (Partial answer: $g(x, Q)$ is a circle with radius s_2 , $f(x, P)$ is a single point.)

2. Treat the game of two cars for isochronous capture. (Answer: our conditions are inconclusive.)

3. Obtain a concise necessary condition for the linear game $\dot{x} = Ax - p + q$ with nonlinear termination condition $|Mx| \leq \epsilon$. (Hint: for $F(x) = \frac{1}{2} (|Mx|^2 - \epsilon^2)$, the Jacobian is $x'M'M$.)

4. The dolichobrachistochrone game has equations of motion

$$\dot{x} = \sqrt{y} \cos u + \frac{\omega}{2} (v-1), \quad \dot{y} = \sqrt{y} \sin u + \frac{\omega}{2} (v-1)$$

(constant $\omega > 0$); the pursuer's constraint is $|u| \leq \pi$, the quarry's $|v| \leq 1$; the termination condition is $x = 0$.

Ignoring the state constraint $y \geq 0$, decide on the possibility of general or isochronous capture. (Answers: $y \geq \omega^2/4$ is necessary for isochronous capture, and $y > 0$ sufficient for capture.)

Remarks

This section was motivated by the suggestion on p. 191 of

A. Friedman: *Differential Games*, Wiley-Interscience,
New York, etc., 1971

to study isochronous capture in the homicidal chauffeur game. Notwithstanding the negative aspect of our solution in the example, it would be useful to have companion sufficient conditions. An obvious candidate to parallel (7) is

$$DF(x)C(x)Q \subset \text{Int } DF(x)B(x)P .$$

CHAPTER VIII

STRATEGIES

The first two sections are concerned with optimality: in reasonable circumstances, there exist time-optimal strategies.

This is a non-constructive existence theorem pure and simple, and thus is a departure from our claim that the material is to contribute to the solution of games rather than to their study. The result is interesting, and generalises readily to a cost-optimal problem for a class of nonlinear games (see 8.2); however, this alone would be an irrelevant excuse. More important is that the proof yields results which are simple and important. Finally, it applies directly to 5.3 and 5.4: once there is assurance that some optimal strategy exists, the methods developed there provide relatively simple constructions of other optimal strategies.

The last section treats games for which the known procedures are either too complicated (e.g., the Second Reciprocity Theorem does cover Example 2) or inapplicable; and a solution may be highly desirable and not ruled out a priori by failure of the necessary conditions. If one is then willing to accept strategies which, although forcing capture, are not necessarily optimal, one has the situation treated there.

8.1 Compactness Lemma

In linear control theory, the shortest proof that reachable sets are closed uses a weak compactness argument on the control functions. Here we present an extension of this reasoning to game theory: the bare bones in Lemma 1, the

main interpretation as a compactness property of pursuer's strategies, and finally the development of the consequences.

LEMMA 1 To each sequence of mappings f_1, f_2, \dots from a set X to a compact metrisable space Y there corresponds a mapping $f: X \rightarrow Y$ which is their limit, in the following sense: for any countable subset $X_0 \subset X$ there is a subsequence f_{k_j} with

$$(2) \quad f_{k_j}(x) \rightarrow f(x) \text{ for every } x \in X_0.$$

(Proof) By Tychonov's Theorem, there is a subnet, say $i \mapsto f_{k(i)}$, for i in a directed set I of indices, which converges pointwise to a mapping $f: X \rightarrow Y$. (Unfortunately, no subnet of this needs be a sequence). Now take any countable set of points x_1, x_2, \dots in X . Then $\lim_i f_{k(i)}(x_r) = f(x_r)$ for each $r = 1, 2, \dots$. Interpreting this in the metrisable space $Y \times Y \times \dots = \prod_{r=1}^{\infty} Y$, we find that the net $i \mapsto y_i$ converges to y , where these points are in the product space and have the appropriate r -th coordinates:

$$y_i = \{f_{k(i)}(x_r): r = 1, 2, \dots\}, \quad y = \{f(x_r): r = 1, 2, \dots\}.$$

Now choose an index i_1 so that y_{i_1} has distance $\leq 1/4$ from y ; then take an index i_2 so that y_{i_2} has distance $\leq (1/4)^2$ from y and, in addition, that $k(i_1) < k(i_2)$; etc. The integer indices $k_j = k(i_j)$ have $k_1 < k_2 < \dots$; the points $y_{i_j} \rightarrow y$, so that their coordinates satisfy (2) for each $x = x_r$. QED

In order to apply this result to a sequence of strategies in a game, it is necessary to recognise or adjust the needed elements. If σ is a strategy, and quarry controls $q_1 = q_2$ almost everywhere, then also $\sigma[q_1] = \sigma[q_2]$ a.e.; thus the

strategies carry over to mappings between the appropriate function spaces, with the almost-everywhere identification. Next, bounded pursuer controls (or rather, their equivalence classes) belong to the Hilbert space H of measurable mappings $p: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ with

$$\int_{-\infty}^{\infty} e^{-|t|} |p(t)|^2 dt < +\infty.$$

If the constraint set P is compact and convex, the pursuer controls form a closed, bounded, convex subset Y of H . By a celebrated theorem, Y is compact in the weak topology (H is a Hilbert space, hence it is reflexive, and the weak-star topology coincides with the weak topology). Finally, as H is separable and Y bounded, the weak topology for Y is metrisable.

COMPACTNESS OF STRATEGIES If the pursuer's constraint set is compact and convex, then to each sequence of strategies $\sigma_1, \sigma_2, \dots$ there corresponds a strategy σ which is their limit in the following sense: for any countable set of quarry controls q_1, q_2, \dots there is a subsequence σ_{k_j} such that

$$\sigma_{k_j}[q_r] \rightarrow \sigma[q_r] \text{ weakly as } j \rightarrow \infty.$$

Thus, for any fixed $\theta \geq 0$, bounded measurable $F: \mathbb{R}^1 \rightarrow \mathbb{R}^{mn}$, and $r = 1, 2, \dots$, we have

$$\int_0^\theta F(t) \sigma_{k_j}[q_r](t) dt \rightarrow \int_0^\theta F(t) \sigma[q_r](t) dt$$

as $j \rightarrow \infty$.

(Proof) As indicated above, we reduce to the situation of Lemma 1 by taking for X the set of all quarry controls (without any structure), and for Y the subset, of H , consisting of the pursuer controls modulo equality a.e., equipped with the weak topology. The result is a mapping $f = \sigma$

such that $\sigma[q]: R^1 \rightarrow P$ is measurable for each quarry control q , and satisfying the limit relation as asserted. It only remains to show that σ is non-anticaptory (so that σ is indeed a strategy). Assume that $q_1 = q_2$ a.e. $(-\infty, \alpha)$ for quarry controls q_r . Then $\sigma_{k_j}[q_1] = \sigma_{k_j}[q_2]$ a.e. $(-\infty, \alpha)$; also, there is a subsequence such that, for both $r = 1, 2$, we have $\sigma_{k_j}[q_r] \rightarrow \sigma[q_r]$ weakly (in $(-\infty, \alpha)$ also). Hence their limits satisfy $\sigma[q_1] = \sigma[q_2]$ a.e. $(-\infty, \alpha)$. QED

Non-Example In the Compactness Lemma one cannot take a subsequence common to all quarry controls. Consider the sequence $\sigma_1, \sigma_2, \dots$,

$$\sigma_k[q](t) = \cos kq(t),$$

for measurable $q: R^1 \rightarrow [-1, 1]$ (however, we will only need constant q 's). Now, if constant functions converge weakly, then their limit is constant a.e., and is the ordinary limit of the constants. Thus, if a single subsequence $k = k_j$ leads to convergence for all $q \in [-1, 1]$, then $\cos k_j q \rightarrow \varphi(q)$; here φ is measurable, and, by the Bounded Convergence Theorem,

$$\int_{\alpha}^{\beta} \varphi(q) dq \leftarrow \int_{\alpha}^{\beta} \cos k_j q dq = \frac{1}{k_j} (\sin k_j \beta - \sin k_j \alpha) \rightarrow 0$$

for all real α, β . Thus $\varphi = 0$ a.e., or, $\cos k_j q \rightarrow 0$ for almost all q . For these q 's, say $q \in Q_0$, we also have

$$\cos k_j q/2 = 2\cos^2 k_j q - 1 \rightarrow -1.$$

Thus the set $\{q/2: q \in Q_0\}$ has measure 0, although Q_0 has full measure: a contradiction.

The consequences we wish to draw concern, first, a linear game

$$(3) \quad \dot{x} = Ax - p + q; \quad p(t) \in P, \quad q(t) \in Q; \quad \text{end: } x \in \Omega.$$

In 2.2 there was introduced the notation $W(0, \theta)$ and $W(\theta)$ for winning sets. Generalising slightly, for a subset

$\Theta \subset R^1$ let $W(\Theta)$ denote the set of positions which can be forced to the target with a termination time in the set $\Theta \subset R^1$.

LEMMA 2 In the game (3) assume that P is compact and convex, and Ω closed. For any sequence of sets Θ_k within a common compact interval we have

$$(4) \quad \limsup W(\Theta_k) \subset W(\limsup \Theta_k).$$

(Proof) Consider any point $x \in \limsup W(\Theta_k)$; thus, for a subsequence of the indices (to be denoted by k 's again),

$$x_k \in W(\Theta_k), \quad x_k \rightarrow x.$$

To each x_k there corresponds an appropriate strategy σ_k (forcing to Ω within Θ_k); by Compactness of Strategies there is a 'limit' strategy σ . It is plausible that σ is a winning strategy for x , and we proceed to verify that indeed σ forces x to Ω within the time set $\limsup \Theta_k$ (thus proving (4)).

Take any quarry control $q(\cdot)$; then $\sigma_{k_j}[q] \rightarrow \sigma[q]$ weakly for some sequence $k_1 < k_2 < \dots$ (which we will denote by k again). As each x_k wins within Θ_k ,

$$(5) \quad e^{At_k} \left(x_k - \int_0^{t_k} e^{-As} \sigma_k[q](s) ds + \int_0^{t_k} e^{-As} q(s) ds \right) \in \Omega$$

for some $t_k \in \Theta_k$. As all Θ_k are contained in a compact interval, by choosing a yet further subsequence (but retaining notation) we ensure that $t_k \rightarrow t \in \limsup \Theta_k$. Now take limits in (5) to obtain

$$e^{At} \left(x - \int_0^t e^{-As} \sigma[q](s) ds + \int_0^t e^{-As} q(s) ds \right) \in \Omega.$$

Indeed, t_k and x_k conerge appropriately; in the integral

term write $\int_0^{t_k} = \int_0^t + \int_t^{t_k}$, and use weak convergence for the first term, and compactness of P for the second; the last

integral term in (5) converges since q is locally integrable; finally, Ω is closed. QED

PROPOSITION In the game (3) let P be compact and convex, and Ω closed. Then the winning sets $W(0, \theta)$ and $W(\theta)$ are closed;

$$(6) \quad W(0, \theta) = \{x: T(x) \leq \theta\};$$

the minimum time function $T(\cdot)$ is lower semicontinuous,

$$T(x) \leq \liminf_{y \rightarrow x} T(y).$$

(Proof) To show, that, e.g., $W(0, \theta)$ is closed, apply Lemma 2 with $\Theta_k = [0, \theta]$:

$$\overline{W(0, \theta)} = \lim_k W(0, \theta) \subset W(0, \theta).$$

Exercise 1 of 2.3 states that

$$(7) \quad \{x: T(x) < \theta\} \subset W(0, \theta) \subset \{x: T(x) \leq \theta\}.$$

To complete (6) continue these inclusions

$$\begin{aligned} \{x: T(x) \leq \theta\} &\subset \bigcap_{k=1}^{\infty} \{x: T(x) \leq \theta + k^{-1}\} \\ &\subset \bigcap_{k=1}^{\infty} W(0, \theta + k^{-1}) = \lim \dots \subset W(0, \theta) \end{aligned}$$

by the first relation in (7) and Lemma 2.

Finally, T is lower semicontinuous since, by (6), its 'sublevel sets' $\{T \leq \theta\}$ are closed. QED

COROLLARY For every winning position x there is a time-optimal winning strategy, forcing x to Ω within the minimal interval $[0, T(x)]$.

(Proof) This is now quite straightforward. By assumption, $T(x) < +\infty$; thus x belongs to $\{y: T(y) \leq T(x)\} = W(0, T(x))$, and hence there is an appropriate strategy. QED

Exercises

Care must be taken in using the Compactness Lemma: it is highly sensitive to even minor changes, as in the non-example of the text, and the exercise below.

1. For locally integrable $q: R^1 \rightarrow R^1$ and each $h = 1, 1/2, 1/3, \dots$ define $\sigma_h[q]$ by

$$\sigma_h[q](t) = \frac{q(t-h) - q(t)}{-h}.$$

Obviously σ_h is a non-anticipatory strategy. Compactness of Strategies yields a strategy σ such that, for each $q(\cdot)$ as above, $\sigma_h[q] \rightarrow \sigma[q] = p$ weakly for a subsequence of the h 's. Hence, for any real $\alpha < \beta$,

$$\begin{aligned} \int_{\alpha}^{\beta} p &\leftarrow \int_{\alpha}^{\beta} \sigma_h[q] = \frac{1}{h} \int_{\alpha}^{\beta} (q(t-h) - q(t)) dt \\ &= \frac{1}{h} \left(- \int_{\alpha-h}^{\beta-h} + \int_{\alpha}^{\beta} \right) = \frac{1}{h} \int_{\beta-h}^{\beta} q - \frac{1}{h} \int_{\alpha-h}^{\alpha} q. \end{aligned}$$

Keeping q fixed, for almost all α, β the integrals have known limits as $h \rightarrow 0$:

$$\int_{\alpha}^{\beta} p(t) dt = q(\beta) - q(\alpha)$$

Thus every locally integrable q is absolutely continuous. The exercise is to find the error.

2. In the assertion on Compactness of Strategies, prove that if all σ_k are bilaterally deterministic, then so is σ ; and similarly for almost stroboscopic, isochronous, in-different strategies. What can be said about σ if all σ_k are stroboscopic, or if each has time-lag δ_k ?

3. (Principle of Suboptimality) For the linear game (3) show that, whenever $0 \leq t \leq T(x)$, there exists a point y , on some state response initiating at x , such that $T(y) \leq T(x) - t$.

Remarks

For a result analogous to Compactness of Strategies, see

E. Roxin: On Varaiya's definition of a differential game, Proc. U.S. - Japan Seminar on Diff. and Funct. Eqs. (ed. Harris, Sibuya), Benjamin, New York, 1968.

The proof suggested there contains errors which have not succumbed to repeated correction and adjustment of assertion. The result was intended to generalise a theorem due to Varaiya on compactness of state responses (rather than strategies). An exposition of the latter begins on p. 43 of

A. Friedman: Differential Games, Wiley-Interscience, New York, etc., 1971

It may be noted that this properly concerns one-person games, i.e., control systems.

It seems that the assertion in the text is the 'right' one. A rather clumsy proof was presented in

O. Hájek: Lectures on Linear Pursuit Games (unpublished), Case Western Reserve University, 1973.

The formulation of our compactness result is somewhat cumbersome because of the insistence on convergence of sequences rather than of general nets. There is a good reason for this: the strategies appear under integral signs; and integrals are continuous, at best, under convergence of sequences, but not for convergence of nets. E.g., if c_x is the mapping $R^1 \rightarrow R^1$ with value 1 at x and 0 elsewhere, then obviously $\int \sup_x c_x = 1 \neq 0 = \sup_x \int c_x$.

8.2 Optimal Strategies

In the preceding section, the existence of time-optimal strategies appeared as an easy corollary, in the course of

establishing some elementary properties of the minimum time function. Naturally, the existence proof itself may be dissected from this; an analysis reveals that it applies to a considerably more general situation.

Consider a non-linear game in R^n , with dynamical equation

$$(1) \quad \dot{x} = f(x, q) + G(x, q)p,$$

given target set $\Omega \subset R^n$, and cost functional $C(t, x, p, q)$ (where the arguments x, p, q are functions). A play of the game proceeds from an initial position, until the target set is first reached, possibly not time-optimally; the cost is then computed, with the pursuer attempting to minimize, and the quarry, to maximize. The data include the elements already mentioned, and also player constraint sets P, Q .

As is usual in these complex situations, an unwieldy list of assumptions and specifications obstructs the way to the main result. Most of the assumptions are in the category of book-keeping; the interesting ones are that pursuer controls appear linearly in the system dynamics (1); and that the cost $C(t, x, p, q)$ increase with t . The definitions only serve to pin down the obvious notion of cost-optimal pursuer strategy.

First, the conditions. In (1), f and G are mappings

$$f: R^n \times Q \rightarrow R^n, G: R^n \times Q \rightarrow R^{nm}$$

satisfying appropriate versions of the standard assumptions from 2.1. The players' constraint sets P, Q are compact subsets of R^m , with P convex. The target $\Omega \subset R^n$ is closed. The cost functional $C(t, x, p, q)$ is a well-defined real number whenever $t \geq 0$, $x: R^1 \rightarrow R^n$ is continuous, and $p: R^1 \rightarrow P$ and $q: R^1 \rightarrow Q$ are measurable. Furthermore, $C(t, x, p, q)$ is left-continuous in t , and continuous in x , uniformly with respect to the other variables. Finally, $C(t, x, p, q)$ is non-decreasing in t , and quasi-convex in p :

ignoring the other variables,

$$(3) \quad C(\alpha p_1 + (1-\alpha)p_2) \leq \max C(p_1), C(p_2) \text{ for } 0 \leq \alpha \leq 1.$$

Second, the specifications. The notation from 2.3 will be needed again. Given a winning strategy σ for an initial position x_0 , set $C(\sigma, q) = C(t, x, p, q)$ where

$$p = \sigma[q], \quad x(s) = \pi(s, x_0, \sigma[q], q), \quad t = T(\sigma, q),$$

and then

$$C(\sigma) = \sup_{q(\cdot)} C(\sigma, q), \quad \gamma = \inf_{\sigma} C(\sigma),$$

supremum taken over all quarry controls $q(\cdot)$, and infimum over winning pursuer strategies. A cost-optimal strategy σ is then one which has $C(\sigma) = \gamma$.

THEOREM Under the listed assumptions, to each position which can be forced to termination within a bounded time interval there corresponds a cost-optimal strategy.

(Proof) Assume that the position $x_0 \in R^n$ can be forced to Ω within the time interval $[0, \lambda]$, $\lambda < +\infty$. Choose strategies σ_k such that

$$(4) \quad T(\sigma_k, q) \leq \lambda \text{ (all } q), \quad C(\sigma_k) \rightarrow \gamma;$$

by Compactness of Strategies (8.1) there is a 'limit' strategy σ with the properties described there. It seems natural to expect that σ is cost-optimal: we wish to verify that σ forces x_0 to Ω and provides the infimal cost γ . It will be sufficient to prove

$$(5) \quad T(\sigma, q) \leq \lambda, \quad C(\sigma, q) \leq \limsup_k C(\sigma_k, q)$$

for all q , since $C(\sigma_k, q) \leq C(\sigma_k) \rightarrow \gamma \leq C(\sigma)$.

Keeping the quarry control $q(\cdot)$ fixed, set

$$p_k = \sigma_k[q], \quad p = \sigma[q];$$

let $x_k(\cdot)$ be the corresponding solutions of the dynamical equation (1), all with the same initial value x_0 ; finally, denote the first termination times

$$\theta_k = T(\sigma_k, q), \quad \theta = T(\sigma, q)$$

($\theta_k \leq \lambda$ by (4)), but, as yet, we do not even have $\theta < +\infty$. Now write

$$(6) \quad \begin{aligned} C(\sigma_k, q) - C(\sigma, q) &= (C(\theta_k, x_k, p_k, q) - C(\theta, x_k, p_k, q)) \\ &\quad + (C(\theta, x_k, p_k, q) - C(\theta, x, p_k, q)) \\ &\quad + (C(\theta, x, p_k, q) - C(\theta, x, p, q)). \end{aligned}$$

If necessary, take subsequences in such a fashion that each bracketed term here has a limit, that $p_k \rightarrow p$ weakly (cf. 8.1) and that the θ_k converge, $\theta_k \rightarrow \theta_0 \in [0, \lambda]$.

In the following lemmas (with present notation and assumptions retained) we will show that each of the brackets in (6) has a non-negative limit, and complete the proof of (5).

LEMMA 1 The solutions $x_k \rightarrow x$ uniformly in the interval $[0, \lambda]$. Furthermore,

$$(7) \quad \theta = T(\sigma, q) \leq \lambda,$$

$$(8) \quad \lim_k (C(\theta, x_k, p_k, q) - C(\theta, x, p_k, q)) = 0.$$

(Proof) The solutions satisfy the appropriate integral version of (1); we may then subtract and rearrange, omitting reference to $q(\cdot)$:

$$\begin{aligned} x_k(t) - x(t) &= \int_0^t (f(x_k(s)) - f(x(s)) + G(x_k(s)) - G(x(s))p_k(s)) ds \\ &\quad + \int_0^t G(x(s))(p_k(s) - p(s)) ds. \end{aligned}$$

Since $p_k \rightarrow p$ weakly, the last integral term converges to 0 as $k \rightarrow \infty$, uniformly in $[0, \lambda]$. The Lipschitz property (2) then yields the following estimate for $\xi_k = |x_k - x|$:

$$(9) \quad \xi_k(t) \leq \text{const.} \int_0^t \xi_k(s) ds + \sigma(1),$$

constant involving the Lipschitz constant μ and a bound on $P \ni p_k(s)$. It follows that $\xi_k \rightarrow 0$, and $x_k \rightarrow x$, e.g. by Gronwall's inequality ('Bellman's Lemma'). For a direct proof, rewrite (9) as $\xi_k \leq \alpha \int \xi_k + \epsilon_k$, multiply by the integrating factor $e^{-\alpha t}$ and integrate:

$$e^{-\alpha t} \xi_k \leq \frac{e^{-\alpha t} - 1}{-\alpha} \epsilon_k.$$

Then $x_k(\theta_k) \in \Omega$ yields $x(\theta_0) \in \Omega$ in the limit; thus the first intersection time θ for $x(\cdot)$ satisfies (7), the first relation in (5). Finally, (8) follows from $x_k \rightarrow x$ and a continuity assumption on the cost functional. QED

$$\text{LEMMA 2} \quad \lim_k (C(\theta_k, x_k, p_k, q) - C(\theta, x_k, p_k, q)) \geq 0.$$

(Proof) Omitting repeated variables, the assertion is that $\lim C(\theta_k) \geq C(\theta)$. Since $\theta_k \rightarrow \theta_0$, we have $\theta_k > \theta_0 - \epsilon$ for $\epsilon > 0$ and large k . Thus

$$\lim C(\theta_k) \geq \lim_{\epsilon \rightarrow 0+} C(\theta_0 - \epsilon) = C(\theta_0)$$

by monotonicity and left-continuity; the assertion now follows from (8). QED

$$\text{LEMMA 3} \quad \lim_k (C(\theta, x, p_k, q) - C(\theta, x, p, q)) \geq 0.$$

(Proof) Using abbreviated notation again, we wish to show that $C(p) \leq \lim C(p_k)$. Since $p_k \rightarrow p$ weakly, p is the 'strong' limit of finite convex combinations of the p_k . Also $p_{k+1} \rightarrow p$ weakly, so the convex combinations need only involve p_k with $k \geq 2$; etc. Thus

$$p = \lim_r \sum_{k=r}^{s_k} \alpha_{kr} p_r, \quad \alpha_{kr} \geq 0, \quad \sum_{r=k}^{s_k} \alpha_{kr} = 1$$

for suitable α_{kr} and $s_k \geq k$. Since C is continuous and quasi-convex in p ,

$$\begin{aligned} C(p) &= \lim_k C(\sum \alpha_{kr} p_r) \leq \limsup_k \max_{k \leq r \leq s_k} C(p_r) \\ &= \limsup_k C(p_k) = \lim C(p_k). \end{aligned} \quad \text{QED}$$

(Proof of Theorem, conclusion) The limit relations in the lemmas, applied to (6) in suitable order, yield $\lim (C(\sigma_k, q) - C(\sigma, q)) \geq 0$; this, together with (7), complete the verification of (5). QED

Exercises

1. Determine whether the following cost functionals satisfy the requirements of the theorem:

$$\begin{aligned} &\int_0^t |p(s)|^2 ds; \quad \varphi(x(t)) + \int_0^t (p'(s)W(s)p(s) + w'(s)p(s))ds, \\ &\int_0^t (|p(s)|^2 - |q(s)|^2)ds; \quad \int_0^t (|p(s)|^2 + \alpha^2 - |q(s)|^2)ds; \end{aligned}$$

least distance to target,

$$\inf_{t \geq 0} \text{dist}(x(t), \Omega);$$

least sustained distance from target

$$\inf_{t \geq 0} \sup_{s \geq t} \text{dist}(x(s), \Omega).$$

2. Show that Lemma 1 fails if the requirement that pursuer's control appear linearly is omitted. (Example: one-dimensional state space $\dot{x} = |p|$ with $-1 \leq p(t) \leq 1$; initial condition $x(0) = 0$. Construct a sequence $p_k(\cdot)$ which converges to 0 weakly but has $|p_k(t)| = 1$ a.e.)

Remarks

In the definition of cost-optimality it was not assumed that the optimal strategy σ forces to target within a compact time interval, but only that the termination times $T(\sigma, q)$

be finite for each quarry control. Nevertheless, (5) shows that the strategy studied in the proof does have this property.

The requirement that the cost be computed at the first termination time is treated, in the theorem, by requiring that $C(t, x, p, q)$ increase with t ; as seen in Exercise 1, this precludes several natural cases. The monotonicity condition could be dropped if the pursuer were allowed to choose, among the times at which the state response meets the target, that at which the cost is to be ascertained.

8.3 Design of Strategies

Given an intractable game with target Ω and state space, R^n , a plausible attitude for finding some solutions is to decompose, to select intermediate targets (gambits)

$$(1) \quad \Omega_0, \Omega_1, \dots, \Omega_k, \quad \Omega_k = \Omega, \Omega_0 = \text{state space},$$

in such a manner that first, some or all positions in each Ω_{k-1} can be forced to Ω_k , and second, that these intermediate end games be considerably simpler than the original game. Obviously this is open-ended: to obtain an actual procedure one must specify the classes of gambits and end games to be used.

As one facet of an example, suppose that in all the intermediate end games but the last we require that capture be isochronous. Refer to fig. 1; and recall that forcing to a target set is not the same as forcing to a point within the target (see the remarks in 3.3). Thus, depending on the action of the quarry, at time t_1 the position $x_1 \in \Omega_1$ will end at various points y_1, t_2, \dots within Ω_2 . Each of these can be forced to Ω_3 isochronously; e.g., y_1 at t_2 , y_2 at s_2 , etc. Since $t_2 \neq s_2$ in general, the durations $t_1 + t_2 \neq t_1 + s_2$ for the composite motion of x_1 to Ω_3 : forcing to Ω_3 is not isochronous. This negative result has positive

impact: hopefully some games with general capture can be assembled in this fashion from isochronous games.

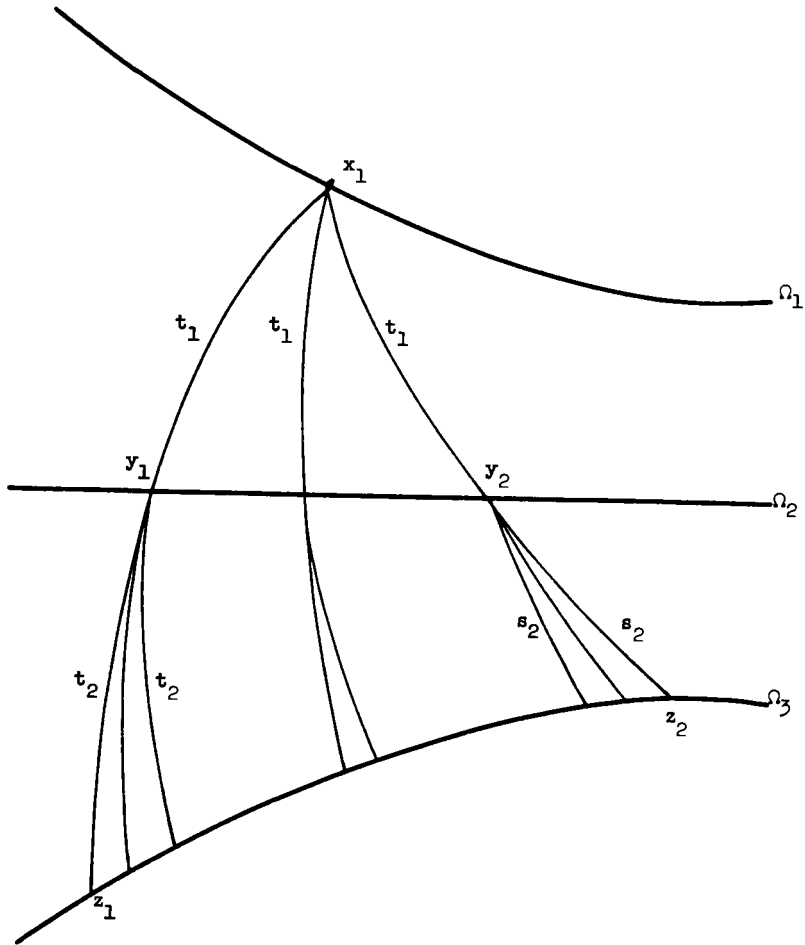


Fig. 1 (schematic) Gambits Ω_k . Composition of isochronous forcing to Ω_2 with that to Ω_3 results in non-isochronous capture.

Some remarks are in place here. One should allow for the possibility that the finite sequence of gambits in (1) is replaced by a sequence infinite in one or both directions;

thus, in Example 1, the appropriate situation is suggested by

$$\dots, \Omega_j, \dots, \Omega_1, \Omega_0 = 0$$

If, in (1), not all points in one gambit can be forced to the next, then one might at least retrace the path backward from $\Omega = \Omega_\ell$, thus obtaining winning positions in $\Omega_0 = \mathbb{R}^n$. The term end game may seem inappropriate for all but the last stage. However, at our level of vagueness, logically nothing prevents us from considering a game with $\Omega_{\ell-1}$ as last target, for which the forcing from $\Omega_{\ell-2}$ to $\Omega_{\ell-1}$ is the end game. Finally, the trivial case $\ell = 1$ of (1) amounts to the enquiry whether the initially given game is some elementary end game.

Obviously the idea is quite familiar. Closer to our interests, I am told it is considerably developed in puzzle-solving in the context of artificial intelligence, and in the study of large and heterogenous systems in systems engineering. Even in value-oriented treatment of game theory, some of the apparatus can be interpreted, in the present terms, as the use of end-games of fixed short duration, alternately indifferent and having anticipation allowed (and multitudinous ramifications of this).

Example 1 Consider again the games of 5.3, with half-spaces as targets; retain the notation developed there. To recapitulate the method, for given initial position x one first solves $c'_t x = \alpha_t$ for $t \geq 0$, and then uses the indifferent strategy σ , $\sigma(s) = \bar{p}(t-s)$, to force to Ω at time t against all action of quarry. Taking t minimal in the first step amounts to a pre-computation of the minimal time $T(x)$.

A simpler procedure may be applied if the players' constraint sets are polytopes. Then $\bar{u} = \bar{p} - \bar{q}$ needed for α_t can be taken piecewise constant, with discontinuities having no finite accumulation points:

$$0 = \theta_0 < \theta_1 < \dots$$

(see Remarks in 5.3). Let Ω_j be the hyperplane $W(\theta_j)$, readily described by its boundary $\{x: c'_t x = \alpha_t\}$ for $t = \theta_j$. From the theorem in 5.2 applied to Ω_{j-1} , each position x in Ω_j can be forced to Ω_{j-1} within a time $\delta_j = \theta_j - \theta_{j-1}$ by a constant strategy $\bar{p}(t)$, $t = (\theta_j + \theta_{j-1})/2$; there is no need to determine $T(x)$. Composing these end-games, we find that any winning position in one of the hyperplanes $W(\theta_j)$ can be forced to Ω , within a time $\delta_1 + \dots + \delta_j = \theta_j$ using j elementary end games. The computation of $T(x)$ is replaced by determination of j in $x \in W(\theta_j)$.

What loss does the simplification entail? In Example 1 of 5.3 (please refer to Fig. 2) the gambits are

$$W(\pi), W(3\pi/4), W(\pi/4), W(0).$$

In the method just described, some positions are forced to $W(\pi/4)$ (and subsequently to Ω), although obviously those in the curved portion of $W(0, \pi/4)$ could have been forced to Ω directly and faster; thus the method is not time-optimal.

Second, although it seems plausible that all winning positions can be treated thus, the equality needed for this,

$$\bigcup_{t \geq 0} W(0, t) = \bigcup_{j=0}^{\infty} W(\theta_j),$$

has not been established.

Section 5.4 suggests another candidate for gambits: the linear subspaces invariant under the coefficient matrix. Before presenting an illustration of this, let us dispose of one possible idea.

If the coefficient matrix A is simple with real eigenvalues, there are n independent real eigenvectors c_1, \dots, c_n of matrix A' (the 'left eigenvectors' of A ; we are taking R^n as state space). The orthogonal complement c_1^\perp is an invariant subspace of dimension $n - 1$; forcing to it is

isochronous and stroboscopic, and positions can be held in it thereafter (Corollary 2 in 5.3). Thus one may repeat the process: while holding in c_1^{\perp} one can force to $c_1^{\perp} \cap c_2^{\perp}$, etc., stepping down the dimensions. The sequence (1) is

$$R^n, c_1^{\perp}, c_1^{\perp} \cap c_2^{\perp}, \dots, c_1^{\perp} \cap \dots \cap c_n^{\perp} = 0;$$

and the suspicion that the idea is too beautiful to last under scrutiny is confirmed by the last target: only if $Q \subset P$ can one hope to force to origin. Speaking somewhat vaguely, each instance of holding uses up one degree of freedom in the pursuer's control capability; unless $\dim P = n$ at least one of the holding manoeuvres is impossible (see Exercises 2 and 3).

Example 2 (Linear analogue of two-car game) The game is induced by two n -dimensional second order equations,

$$(2) \quad \ddot{x} = A\dot{x} - u, \ddot{y} = B\dot{y} - v; \text{ end: } x = y$$

with the player constraint sets U, V non-void, compact, convex, and symmetric. Various assumptions will be placed on the data; the first is the sufficient condition for isochronous capture, $V \subset \text{Int } U$ (see 4.2, and the Example in 3.4). In case the constraint sets are balls and $n \geq 3$, $V \subset U$ is necessary for general capture (Exercise 6 in 7.3).

There is a standard description in state space R^{4n} , the coordinates being x, \dot{x}, y, \dot{y} (cf. 3.4; we will not need the obvious reduction to R^{3n} via $x - y, \dot{x}, \dot{y}$).

In attempting the method described earlier, let us first look for invariant subspaces. These are associated with the eigenvalues of the full coefficient matrix (namely, 0 and those of A, B); but Example 1 tends to discourage this approach. There are two obvious $3n$ -dimensional invariant subspaces, with equations

$$\dot{x} - Ax = 0, \dot{y} - By = 0.$$

Now, we are interested in the ultimate behaviour of the variable $x - y$; isolating this from the above, we are led to consider

$$(3) \quad x - y - A^{-1}x + B^{-1}y = 0$$

as the equation of a subspace, L ; it is readily verified that L is indeed invariant under the complete coefficient matrix. With this as the gambit, the sequence (1) is

$$R^{4n}, L, \Omega;$$

let us study the $\ell = 2$ end games involved.

The first consists in forcing to the invariant target L (cf. 5.4). It is formally simpler to use the reduction described in Exercise 5 of 6.3 (note that this is not a reduction of the original game: in connection with Proposition 1 in 6.3, here $\Omega + L \not\subset \Omega$). The new coordinate, according to (3), is $z = x - y - A^{-1}x + B^{-1}y$, and the reduced game is indicated by

$$(4) \quad \dot{z} = A^{-1}u - B^{-1}v; \text{ end: } z = 0; \text{ state space: } R^n.$$

The coefficient matrix is 0; thus capture is isochronous and stroboscopic (Exercise 1 in 5.4), and nontrivial iff $B^{-1}V \subset A^{-1}U$. The winning sets are

$$W(0, t) = W(t) = t(A^{-1}U \# B^{-1}V),$$

and

$$R^+(A^{-1}U \# B^{-1}V) = \text{span } (A^{-1}U \# B^{-1}V)$$

is the set of winning positions. In particular, complete capture occurs iff $B^{-1}V \subset \text{Int } A^{-1}U$, which we shall assume.

This translates easily into the original coordinates. A summary of assumptions to this point is that A, B are invertible and

$$(5) \quad V \subset \text{Int } U, B^{-1}V \subset \text{Int } A^{-1}U.$$

Next, consider the second end game, of forcing the outcome $x - y$ from an initial position in L . Directly from (2),

$$\dot{x}(t) = e^{At}(x_0 - \int_0^t e^{-As}u(s)ds).$$

and similarly for $y(t)$. By a second integration and rearrangement, we find

$$(6) \quad \begin{aligned} x(t) - y(t) = & (x_0 - y_0 - A^{-1}\dot{x}_0 + B^{-1}\dot{y}_0) \\ & + (e^{At}A^{-1}\dot{x}_0 - e^{Bt}B^{-1}\dot{y}_0) \\ & - \int_0^t ((\int_0^{t-s} e^{Ar}dr)u(s) - (\int_0^{t-s} e^{Br}dr)v(s))ds \end{aligned}$$

(e.g. by Laplace transform methods, the integral of a convolution is the convolution of one factor with the integral of the second). Since the initial values satisfy (3), the first bracket vanishes. (This is the purpose of the first step as shown in Exercise 5, it is not advisable to hold with- L .) The termination condition $x = y$ may now be rewritten as

$$(7) \quad \begin{aligned} \int_0^t (\int_0^{t-s} e^{Ar}dr)u(s)ds = & \int_0^t (\int_0^{t-s} e^{Br}dr)v(s)ds + \\ & + (e^{At}A^{-1}\dot{x}_0 - e^{Bt}B^{-1}\dot{y}_0). \end{aligned}$$

If A, B are stability matrices, the last bracket approaches 0 as $t \rightarrow +\infty$, for any \dot{x}_0, \dot{y}_0 . The solution procedure is then to split the pursuer's control into two components, appropriately compensating the right hand members in (7).

As our final assumption, let

$$(8) \quad A(I - e^{At})^{-1}(I - e^{Bt})B^{-1}V \subset \text{Int } U \quad \text{for all } t > 0$$

(note that the stability matrix A has no pure imaginary eigenvalues, so $I - e^{At}$ is nonsingular for $t > 0$).

To simplify notation, define

$$(9) \quad F(t) = A(I - e^{At})^{-1}(I - e^{Bt})B^{-1} = \left(\int_0^t e^{Ar} dr\right)^{-1} \left(\int_0^t e^{Br} dr\right)$$

for $t > 0$, and let

$$F(0) = \lim_{t \rightarrow 0} F(t) = I, \quad F(\infty) = \lim_{t \rightarrow \infty} F(t) = AB^{-1}$$

From (5) and (8) we have $F(t)V \subset \text{Int } U$ for $0 \leq t \leq \infty$, so that, by continuity and compactness, there is a ball G about the origin in R^n with

$$(10) \quad F(t)V + G \subset U \quad \text{for } t \geq 0$$

Consider (7) again. Choose $t > 0$ so that

$$e^{At}A^{-1}\dot{x}_0 - e^{Bt}B^{-1}\dot{y}_0 \in \int_0^t (e^{As} - I)A^{-1}G \, ds$$

(the left-hand member approaches 0, the right hand one is a neighbourhood of 0 increasing with $t > 0$). This determines $t > 0$ and also an integrable mapping $t \mapsto g(t) \in G$ with

$$(11) \quad e^{At}A^{-1}\dot{x}_0 - e^{Bt}B^{-1}\dot{y}_0 = \int_0^t (e^{As} - I)A^{-1}g(s)ds \\ = \int_0^t \left(\int_0^{t-s} e^{Ar} dr\right) g(t-s)ds.$$

Finally, define a stroboscopic strategy σ ,

$$\sigma(v, x) = F(t-s)v + g(t-s);$$

its values are in U according to (10). From (9) and (11), the pursuer control $s \mapsto u(s) = \sigma(v(s), s)$ will satisfy the termination condition (7).

Conclusions: If (8) holds, together with the two 'limit relations' (5), and A, B are stability matrices, then complete capture occurs for the game (2): any initial positions can be forced to coincidence (actually, by isochronous and stroboscopic strategies).

Example 3 (The two car game in three dimension) The

game is as described in 1.6; we shall merely present the highlights of Rublein's solution (cf. references there) of the case $n = 3$ and

$$(12) \quad \frac{s_1}{s_2} > 1, \left(\frac{s_1}{s_2} \right)^2 > \frac{\rho_1}{\rho_2} ;$$

as in 1.6, x denotes the position of the pursuer, and y that of the quarry.

The author shows that (merely from $s_1 > s_2$), every position can be forced to one with $|y - x|$ arbitrarily large, and then to one with $|y - x|$ large and \dot{x} parallel to $y - x$. Then, from (12) he shows that the following outcome can be forced:

$$(13) \quad (y - x) \cdot \dot{x} > 0; \dot{x}, \dot{y} \text{ and } y - x \text{ are coplanar;} \\ x, y \text{ have equal components in } (y - x)^\perp .$$

Another formulation of the latter two conditions in (13) is that the direction of $y - x$ can be held constant, so that the final stage of the game is, essentially, one-dimensional.

That this procedure is an instance of our gambits and end games is obvious (e.g., $\ell = 4$ in (1)); note that holding occurs, but not at all stages.

Exercises

The first three of these are related to Example 1.

1. Verify the following assertions for a game in R^2 ,

$$\dot{x} = x + y + u, \dot{y} = -y + v,$$

with $[-1, 1]$ as common constraint set. The eigenvalues are 1 and -1, with corresponding eigenvectors $a' = (2, 1)$, $b' = (0, 1)$. The fundamental matrix solution is

$$e^{At} = \begin{pmatrix} e^t & , \sinh t \\ 0 & , e^{-t} \end{pmatrix}$$

2. In the game above with target set a^\perp , show that the winning set $W(0,t) = W(t)$ is the strip described by

$$0 \leq 2x + y \leq 1 - e^{-t} < 1.$$

(Intermediate result: in the notation of 5.2, $c'e^{At}(\bar{p}-\bar{q}) = e^t$.)

If a position has been forced to a^\perp , it can be held there by taking $u = -v/2$, ultimately reaching the set

$$\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \theta : |\theta| \leq 1 + \epsilon \right\}$$

for any $\epsilon > 0$, but not for $\epsilon = 0$.

3. In the game of Exercise 1 with target b^\perp , show that the winning set $W(0,t) = W(t)$ is described by

$$y \geq 1 - e^t > -\infty.$$

However, no point in the target can be held there; exhibit some evasion strategy.

The last group of exercises concerns the game of Example 2.

4. Show that a point forced into L can be held there subsequently by the stroboscopic pursuer strategy $u = AB^{-1}v$.

5. Prove that, if pursuer does choose to hold within L , then

$$x(t) - y(t) = \sigma(1) + \int_0^t (e^{B(t-s)} - e^{A(t-s)})B^{-1}v(s)ds$$

from (6), and this can be made large by quarry ($A = B$ and other trivialities expected.)

6. Treat the case of scalar matrices $A = \alpha I$, $B = \beta I$ ($\alpha > 0 < \beta$). Prove that one of the limit relations (5) implies the other and also (8), in both cases $\alpha \gtrless \beta$.

7. Show that for complete capture in the second end game alone, the following weaker version of the requirements is sufficient: A, B are stability matrices, $U \subset \text{Int } V$, and, in place of (8),

$$A(I - e^{At})^{-1}(I - e^{Bt})B^{-1}V \subset U.$$

(Hint: use the 'interior containment' at $t = 0$ to neutralise the $\sigma(1)$ terms in (7).)

Remarks

The idea (without the examples) appeared in

O. Hájek: Strategy design in pursuit games, pp. 71-79
in Differential Games and Control Theory
(ed. E. O. Roxin, P. T. Liu, R. L. Sternberg),
Dekker, New York, 1974.

PROBLEM Extend the method of solving Example 2 to the
general case of the example in 3.4.

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