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David W. K. Yeung
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Cooperative Stochastic Differential Games

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To Stella and Nina

Preface

The subject of this book is the theory and applications of cooperative stochastic differential games. Game theory is the study of decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other disciplines. The theory's foundations were laid some sixty years ago by von Neumann and Morgenstern (1944). One particularly complex and fruitful branch of game theory is dynamic or differential games (originated from Isaacs (1965)), which investigates interactive decision making over time. Since human beings live in time and decisions generally lead to effects over time, it is only a slight exaggeration to maintain that "life is a dynamic game". Applications of differential game theory in various fields have been accumulating at an increasing rate over the past fifty years.

Cooperative games hold out the promise of socially optimal and group efficient solutions to problems involving strategic actions. Formulation of optimal player behavior is a fundamental element in this theory. In dynamic cooperative games, a particularly stringent condition on the cooperative agreement is required: In the solution, the optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is known as *dynamic stability* or *time consistency.*

Since interactions between strategic behavior, dynamic evolution, and stochastic elements are considered simultaneously, cooperative stochastic differential games represent one of the most complex forms of decision making. In the field of cooperative stochastic differential games, little research has been published to date owing to difficulties in deriving tractable solutions. In the presence of stochastic elements, a very stringent condition – *subgame consistency* – is required for a credible cooperative solution. A cooperative solution is subgame consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. Conditions ensuring time consistency of cooperative solutions are generally analytically intractable.

This book expounds in greater detail the authors' recent contributions in research journals. In particular, Yeung and Petrosyan (2004) offered a generalized theorem for the derivation of the analytically tractable "payoff distribution procedure" leading to the realization of subgame consistent solutions. This work is not only theoretically interesting in itself, but would enable the hitherto intractable problems in cooperative stochastic differential games to be fruitfully explored.

In the case when payoffs are nontransferable in cooperative games, the solution mechanism becomes extremely complicated and intractable. A subgame consistent solution is developed for a class of cooperative stochastic differential games with nontransferable payoffs in Yeung and Petrosyan (2005). The previously intractable problem of obtaining subgame consistent cooperative solution has now been rendered tractable for the first time.

This book draws on the results of this research and expands on the analysis of cooperative schemes to dynamic interactive systems with uncertainty. It is the first volume devoted to cooperative stochastic differential games and complements two standard texts on cooperative differential games, one by Leitmann (1974) and the other by Petrosyan and Danilov (1979). We provide readers a rigorous and practically effective tool to study cooperative arrangements of conflict situations over time and under uncertainty. Cooperative game theory has been successfully applied in operations research, management, economics, politics and other areas. The extension of these applications to a dynamic environment with stochastic elements is likely to prove even more fruitful, so that the book will be of interest to game theorists, mathematicians, economists, policy-makers, corporate planners and graduate students.

We are grateful to our friends and colleagues who provided us with support and stimulus, in particular Georges Zaccour, Alain Haurie, Nikolay Zenkevich and John Nash. We are especially indebted to George Leitmann and Vladimir Mazalov for their encouragement and generous advance praise for the book (which we are extremely honored to receive).

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St Petersburg, *D.W.K. Yeung*
May 2005 *L.A. Petrosyan*

Contents

1

Introduction

The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, a game obtains when an individual pursues an objective(s) in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives: The problem is then to determine each individual's optimal decision, how these decisions interact to produce equilibria, and the properties of such outcomes. The foundations of game theory were laid some sixty years ago by von Neumann and Morgenstern (1944).

Theoretical research and applications in games are proceeding apace, in areas ranging from aircraft and missile control to market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment, and inventory management. In all these areas, game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision making under real world conditions. One particularly complex and fruitful branch of game theory is dynamic or differential games, which investigates interactive decision making over time. Since human beings live in time and decisions generally lead to effects over time, it is only a slight exaggeration to claim that "life is a dynamic game".

Advances in technology, communications, industrial organization, regulation methodology, international trade, economic integration and political reform have created rapidly expanding social and economic networks incorporating cross-personal and cross-country activities and interactions. From a decision- and policy-maker's perspective, it has become increasing important to recognize and accommodate the interdependencies and interactions of human decisions under such circumstances. The strategic aspects of decision making are often crucial in areas as diverse as trade negotiation, foreign and domestic investment, multinational pollution planning, market development and integration, technological R&D, resource extraction, competitive marketing, regional cooperation, military policies, and arms control.

Game theory has greatly enhanced our understanding of decision making. As socioeconomic and political problems increase in complexity, further advances in the theory's analytical content, methodology, techniques and applications as well as case studies and empirical investigations are urgently required. In the social sciences, economics and finance are the fields which most vividly display the characteristics of games. Not only would research be directed towards more realistic and relevant analysis of economic and social decision-making, but the game-theoretic approach is likely to reveal new and interesting questions and problems, especially in management science.

The origin of differential games traces back to the late 1940s. Rufus Isaacs modeled missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation variables (state and control), and formulated a fundamental principle called the tenet of transition. For various reasons, Isaacs's work did not appear in print until 1965. In the meantime, control theory reached its maturity in the *Optimal Control Theory* of Pontryagin et al. (1962) and Bellman's *Dynamic Programming* (1957). Research in differential games focused in the first place on extending control theory to incorporate strategic behavior. In particular, applications of dynamic programming improved Isaacs' results. Berkovitz (1964) developed a variational approach to differential games, and Leitmann and Mon (1967) investigated the geometry of differential games. Pontryagin (1966) solved differential games in open-loop solution in terms of the maximum principle. (Other developments in the Soviet literature up to the early 1970s are reviewed by Zaubermann (1975).)

Research in differential game theory continues to appear over a large number of fields and areas. Applications in economics and management science are surveyed in Dockner et al. (2000). In the general literature, derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in Berkovitz (1964) and Ho et. al. (1965); Case (1967, 1969) and Starr and Ho (1969a, 1969b) were the first to study open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games. While open-loop solutions are relatively tractable and easy-to-apply, feedback solutions avoid time inconsistency at the expense of reduced intractability. In following research, differential games solved in feedback Nash format were presented by Clemhout and Wan (1974), Fershtman (1987), Jørgensen (1985), Jørgensen and Sorger (1990), Leitmann and Schmitendorf (1978), Lukes (1971a, 1971b), Sorger (1989), and Yeung (1987, 1989, 1992, 1994).

Results obtained by Fleming (1969) in stochastic control made it possible to analyze differential game situations with uncertain stock dynamics. Explicit results for stochastic quadratic differential games were obtained by Basar (1977a, 1977b, 1980). Other stochastic games were solved by Clemhout and Wan (1985), Kaitala (1993), Jørgensen and Yeung (1996), and Yeung (1998, 1999). Stochastic differential games study decision-making in one of its most complex forms. In particular, interactions between strategic behavior, dynamic evolution and stochastic elements are considered simultaneously. This complexity generally leads to great difficulties in the derivation of solutions: As a result, such games have not been studied in any satisfactory extent. As noted above, Basar (*op. cit.*) was first to derive explicit solutions for stochastic quadratic differential games, and a small number of solvable stochastic differential games were published by Clemhout and Wan (*op. cit.*), Kaitala (*op. cit.*), Jørgensen and Yeung (*op. cit.*), and Yeung (*op. cit.*). Recently, Yeung (2001) introduced the paradigm of randomly-furcating stochastic differential games. As a result, it is now possible to study stochastic elements via branching payoffs under the format of differential games.

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. Formulation of optimal behavior for players is a fundamental element in this theory. In dynamic cooperative games, a stringent condition on cooperation and agreement is required: In the solution, the optimality principle must remain optimal throughout the game, at any instant of time along the optimal state trajectory determined at the outset. This condition is known as *dynamic stability* or *time consistency*. In other words, dynamic stability of solutions to any cooperative differential game involved the property that, as the game proceeds along an optimal trajectory, players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior throughout the game.

The question of dynamic stability in differential games has been rigorously explored in the past three decades. Haurie (1976) raised the problem of instability when the Nash bargaining solution is extended to differential games. Petrosyan (1977) formalized the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1982) introduced the notion of "imputation distribution procedure" for cooperative solution. Tolwinski et al. (1986) investigated cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosyan (1993) and Petrosyan and Zenkevich (1996) presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was introduced to construct time consistent solutions. Yeung and Petrosyan (2001) designed time consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosyan (2003) employed the regularization method to construct time consistent bargaining procedures. Petrosyan and Zaccour (2003) presented time consistent Shapley value allocation in a differential game of pollution cost reduction.

In the field of cooperative stochastic differential games, little research has been published to date, mainly because of difficulties in deriving tractable subgame-consistent solutions. Haurie et al. (1994) derived cooperative equilibria in a stochastic differential game of fishery with the use of monitoring and memory strategies. In the presence of stochastic elements, a more stringent condition – that of *subgame consistency* – is required for a credible cooperative solution. In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behavior would remain optimal.

As pointed out by Jørgensen and Zaccour (2002), conditions ensuring time consistency of cooperative solutions are generally stringent and intractable. A significant breakthrough in the study of cooperative stochastic differential games can be found in the recent work of Yeung and Petrosyan (2004). In particular, these authors developed a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" which would lead to subgame-consistent solutions. In offering analytical tractable solutions, Yeung and Petrossian's work is not only theoretically interesting in itself, but would enable hitherto insurmountable problems in cooperative stochastic differential games to be fruitfully explored.

When payoffs are nontransferable in cooperative games, the solution mechanism becomes extremely complicated and intractable. Recently, a subgame-consistent solution was constructed by Yeung and Petrosyan (2005) for a class of cooperative stochastic differential games with nontransferable payoffs. The problem of obtaining subgame-consistent cooperative solutions has been rendered tractable for the first time.

Stochastic dynamic cooperation represents perhaps decision-making in its most complex form. Interactions between strategic behavior, dynamic evolution and stochastic elements have to be considered simultaneously in the process, thereby leading to enormous difficulties in the way of satisfactory analysis. Despite urgent calls for cooperation in the politics, environmental control, the global economy and arms control, the absence of formal solutions has precluded rigorous analysis of this problem. One of the main objectives of this book is introduce readers to a rigorous and practical paradigm for studying cooperation in a dynamic stochastic environment.

The text is organized as follows. Chapter 2 introduces the problems and general methodology of differential game theory. The exposition begins with basic (and classic) dynamic optimization techniques involving dynamic programming, optimal control and stochastic control. General differential games and their solution concepts – open-loop Nash equilibria, closed-loop Nash equilibria and feedback Nash equilibria – are then introduced, followed by a general discussion of stochastic differential games, infinite horizon differential games and their solution methods. Most of the basic theorems and proofs are adopted from the classic work of Isaacs (1965), Bellman (1957), Pontryagin et al. (1962), Fleming (1969), Fleming and Rishel (1975) and Basar and Olsder (1995). Illustrative applications in duopolistic competition, competitive advertising and resource extraction end the chapter.

Chapter 3 examines cooperative differential games in characteristic function form. We begin by explaining imputation in a dynamic context, followed by the principle of dynamic stability and dynamically stable solutions. The payoff distribution procedure formulated by Petrosyan (1997) is discussed in detail and illustrated by a model of pollution control using the Shapley value as imputation.

Chapter 4 presents two-person cooperative differential games with discounting. We begin with the concepts of group rationality, optimal trajectory and individual rationality. Cooperative arrangements and dynamically stable cooperation are then introduced, and the notion of time consistency is formally defined. Exact formulae for equilibrating transitory compensation are obtained for specific optimality principles, together with an economic exegesis of transitory compensation. Infinite horizon cooperative differential games and applications in resource extraction follow, with particular emphasis on the problems of games with nontransferable payoffs and Pareto optimal trajectories under cooperation. In order to allow for the verification of individual rationality over the entire period of cooperation, individual payoffs under cooperation are obtained using Yeung's (2004) methodology. In particular, time consistent solutions for cooperative differential games with nontransferable payoffs are derived for the first time.

Chapter 5 covers two-person cooperative stochastic differential games. The challenging issues of cooperative arrangements under uncertainty are presented and defined, leading to an analysis of group rationality, optimal trajectory and individual rationality in a stochastic environment. We proceed to dynamically stable cooperation and the notion of subgame consistency introduced by Yeung and Petrosyan (2004). Transitory compensation and payoff distribution procedures under uncertainty are derived, together with cases involving Nash bargaining/Shapley value solution and proportional distributions. Explicit illustration of this technique is supplied in an application in cooperative resource extraction under uncertainty. An exegesis of transitory compensation under uncertainty follows, and subgame-consistent solutions are obtained for infinite horizon cooperative stochastic differential games.

Chapter 6 presents a class of multiperson games in cooperative technology development. In Chapter 3, we considered multiplayer cooperative differential games in characteristic function form. However, in many game situations it is unlikely that a subset of players would form a coalition to maximize joint payoffs while the remaining players would form an anti-coalition to harm this effort. As a result, the introduction of characteristic functions may not be realistic at all times. In this chapter we discuss a class of multiplayer cooperative stochastic differential which does not require the use of characteristic functions. The theoretical basis of a cooperative solution is first demonstrated under deterministic assumptions. Payoffs from different coalitions are computed with the Shapley value as solution imputation mechanism. The transitory compensation leading to the realization of the solution imputation is then derived. Extending the analysis to a stochastic environment, we obtain the stochastic dynamic Shapley value and corresponding transitory compensation. An application in cooperative R&D under uncertainty is presented, after which the chapter ends with an analysis of an infinite horizon version of this class of games.

Chapter 7 takes up the great analytical challenge of cooperative stochastic differential games with nontransferable payoffs, especially with regard to Pareto optimal trajectories, individual rationality and cooperative arrangement under uncertainty and nontransferable payoffs. We solve the crucial problem of deriving individual player's expected payoffs under cooperation using Yeung's (2004) methodology, and the notion and definition of subgame-consistency are analyzed along the lines presented in Yeung and Petrosyan (2005). Configurations of the solution set are mapped out in specific cases, and subgame-consistent solutions are presented along with numerical illustrations. Finally, applications and infinite horizon problems are discussed.

Problems are provided at the end of each chapter.

2

Deterministic and Stochastic Differential Games

In game theory, strategic behavior and decision making are modeled in terms of the characteristics of players, the objective or payoff function of each individual, the actions open to each player throughout the game, the order of such actions, and the information available at each stage of play. Optimal decisions are then determined under different assumptions regarding the availability and transmission of information, and the opportunities and possibilities for individuals to communicate, negotiate, collude, make threats, offer inducements, and enter into agreements which are binding or enforceable to varying degrees and at varying costs. Significant contributions to general game theory include von Neumann and Morgenstern (1944), Nash (1950, 1953), Vorob'ev (1972), Shapley (1953) and Shubik (1959a, 1959b). Dynamic optimization techniques are essential in the derivation of solutions to differential games.

2.1 Dynamic Optimization Techniques

Consider the dynamic optimization problem in which the single decision-maker:

$$\max_u \left\{ \int_{t_0}^{T} g\left[s, x\left(s\right), u\left(s\right)\right] ds + q\left(x\left(T\right)\right) \right\}, \tag{2.1}$$

subject to the vector-valued differential equation:

$$\dot{x}\left(s\right) = f\left[s, x\left(s\right), u\left(s\right)\right] ds, \quad x\left(t_0\right) = x_0, \tag{2.2}$$

where $x\left(s\right) \in X \subset R^m$ denotes the state variables of game, and $u \in U$ is the control.

The functions $f\left[s, x, u\right]$, $g\left[s, x, u\right]$ and $q\left(x\right)$ are differentiable functions.

Dynamic programming and optimal control are used to identify optimal solutions for the problem (2.1)–(2.2).

2.1.1 Dynamic Programming

A frequently adopted approach to dynamic optimization problems is the technique of dynamic programming. The technique was developed by Bellman (1957). The technique is given in Theorem 2.1.1 below.

Theorem 2.1.1. *(Bellman's Dynamic Programming) A set of controls* $u^*(t) = \phi^*(t,x)$ *constitutes an optimal solution to the control problem (2.1)–(2.2) if there exist continuously differentiable functions* $V(t,x)$ *defined on* $[t_0,T] \times R^m \to R$ *and satisfying the following Bellman equation:*

$$\begin{aligned}
-V_t(t,x) &= \max_u \{g[t,x,u] + V_x(t,x) f[t,x,u]\} \\
&= \{g[t,x,\phi^*(t,x)] + V_x(t,x) f[t,x,\phi^*(t,x)]\}, \\
V(T,x) &= q(x).
\end{aligned}$$

Proof. Define the maximized payoff at time t *with current state* x *as a* value function *in the form:*

$$\begin{aligned}
V(t,x) &= \max_u \left[\int_t^T g(s,x(s),u(s))\,ds + q(x(T)) \right] \\
&= \int_t^T g[s,x^*(s),\phi^*(s,x^*(s))]\,ds + q(x^*(T))
\end{aligned}$$

satisfying the boundary condition

$$V(T,x^*(T)) = q(x^*(T)),$$

and

$$\dot{x}^*(s) = f[s,x^*(s),\phi^*(s,x^*(s))], \quad x^*(t_0) = x_0.$$

If in addition to $u^*(s) \equiv \phi^*(s,x)$*, we are given another set of strategies,* $u(s) \in U$*, with the corresponding terminating trajectory* $x(s)$*, then Theorem 2.1.1 implies*

$$\begin{aligned}
&g(t,x,u) + V_x(t,x) f(t,x,u) + V_t(t,x) \le 0, \ \text{and} \\
&g(t,x^*,u^*) + V_{x^*}(t,x^*) f(t,x^*,u^*) + V_t(t,x^*) = 0.
\end{aligned}$$

Integrating the above expressions from t_0 *to* T*, we obtain*

$$\begin{aligned}
&\int_{t_0}^T g(s,x(s),u(s))\,ds + V(T,x(T)) - V(t_0,x_0) \le 0, \ \text{and} \\
&\int_{t_0}^T g(s,x^*(s),u^*(s))\,ds + V(T,x^*(T)) - V(t_0,x_0) = 0.
\end{aligned}$$

Elimination of $V(t_0,x_0)$ *yields*

$$\int_{t_0}^{T} g\left(s, x\left(s\right), u\left(s\right)\right) ds + q\left(x\left(T\right)\right) \leq \int_{t_0}^{T} g\left(s, x^*\left(s\right), u^*\left(s\right)\right) ds + q\left(x^*\left(T\right)\right),$$

from which it readily follows that u^* is the optimal strategy.

Upon substituting the optimal strategy $\phi^*\left(t, x\right)$ into (2.2) yields the dynamics of optimal state trajectory as:

$$\dot{x}\left(s\right) = f\left[s, x\left(s\right), \phi^*\left(s, x\left(s\right)\right)\right] ds, \qquad x\left(t_0\right) = x_0. \tag{2.3}$$

Let $x^*\left(t\right)$ denote the solution to (2.3). The optimal trajectory $\left\{x^*\left(t\right)\right\}_{t=t_0}^{T}$ can be expressed as:

$$x^*\left(t\right) = x_0 + \int_{t_0}^{t} f\left[s, x^*\left(s\right), \psi^*\left(s, x^*\left(s\right)\right)\right] ds. \tag{2.4}$$

For notational convenience, we use the terms $x^*\left(t\right)$ and x_t^* interchangeably. The value function $V\left(t, x\right)$ where $x = x_t^*$ can be expressed as

$$V\left(t, x_t^*\right) = \int_{t}^{T} g\left[s, x^*\left(s\right), \phi^*\left(s\right)\right] ds + q\left(x^*\left(T\right)\right).$$

Example 2.1.1. Consider the dynamic optimization problem:

$$\max_u \left\{ \int_0^T \exp\left[-rs\right] \left[-x\left(s\right) - cu\left(s\right)^2\right] ds + \exp\left[-rT\right] qx\left(T\right) \right\} \tag{2.5}$$

subject to

$$\dot{x}\left(s\right) = a - u\left(s\right)\left(x\left(s\right)\right)^{1/2}, \qquad x\left(0\right) = x_0, \qquad u\left(s\right) \geq 0, \tag{2.6}$$

where a, c, x_0 are positive parameters.

Invoking Theorem 2.1.1 we have

$$\begin{aligned} &-V_t\left(t, x\right) = \max_u \left\{ \left[-x - cu^2\right] \exp\left[-rt\right] + V_x\left(t, x\right) \left[a - ux^{1/2}\right] \right\}, \\ &\text{and} \\ &V\left(T, x\right) = \exp\left[-rT\right] qx. \end{aligned} \tag{2.7}$$

Performing the indicated maximization in (2.7) yields:

$$\phi\left(t, x\right) = \frac{-V_x\left(t, x\right) x^{1/2}}{2c} \exp\left[rt\right].$$

Substituting $\phi\left(t, x\right)$ into (2.7) and upon solving (2.7), one obtains:

$$V\left(t, x\right) = \exp\left[-rt\right] \left[A\left(t\right) x + B\left(t\right)\right],$$

where $A(t)$ and $B(t)$ satisfy:

$$\dot{A}(t) = rA(t) - \frac{A(t)^2}{4c} + 1,$$
$$\dot{B}(t) = rB(t) - aA(t),$$
$$A(T) = q \text{ and } B(T) = 0.$$

The optimal control can be solved explicitly as

$$\phi(t,x) = \frac{-A(t)\,x^{1/2}}{2c} \exp[rt].$$

Now, consider the infinite-horizon dynamic optimization problem with a constant discount rate:

$$\max_u \left\{ \int_{t_0}^{\infty} g[x(s), u(s)] \exp[-r(s-t_0)]\, ds \right\}, \tag{2.8}$$

subject to the vector-valued differential equation:

$$\dot{x}(s) = f[x(s), u(s)]\, ds, \qquad x(t_0) = x_0. \tag{2.9}$$

Since s does not appear in $g[x(s), u(s)]$ and the state dynamics explicitly, the problem (2.8)–(2.9) is an autonomous problem.

Consider the alternative problem:

$$\max_u \int_t^{\infty} g[x(s), u(s)] \exp[-r(s-t)]\, ds, \tag{2.10}$$

subject to

$$\dot{x}(s) = f[x(s), u(s)], \qquad x(t) = x. \tag{2.11}$$

The infinite-horizon autonomous problem (2.10)–(2.11) is independent of the choice of t and dependent only upon the state at the starting time, that is x.

Define the value function to the problem (2.8)–(2.9) by

$$V(t,x) = \max_u \left\{ \int_t^{\infty} g[x(s), u(s)] \exp[-r(s-t_0)]\, ds \,\middle|\, x(t) = x = x_t^* \right\},$$

where x_t^* is the state at time t along the optimal trajectory. Moreover, we can write

$$V(t,x) =$$
$$\exp[-r(t-t_0)] \max_u \left\{ \int_t^{\infty} g[x(s), u(s)] \exp[-r(s-t)]\, ds \,\middle|\, x(t) = x = x_t^* \right\}.$$

Since the problem

$$\max_u \left\{ \int_t^\infty g[x(s), u(s)] \exp[-r(s-t)]\, ds \middle| x(t) = x = x_t^* \right\}$$

depends on the current state x only, we can write:

$$W(x) = \max_u \left\{ \int_t^\infty g[x(s), u(s)] \exp[-r(s-t)]\, ds \middle| x(t) = x = x_t^* \right\}.$$

It follows that:

$$\begin{aligned} &V(t,x) = \exp[-r(t-t_0)]\, W(x), \\ &V_t(t,x) = -r\exp[-r(t-t_0)]\, W(x), \text{ and} \\ &V_x(t,x) = -r\exp[-r(t-t_0)]\, W_x(x). \end{aligned} \tag{2.12}$$

Substituting the results from (2.12) into Theorem 2.1.1 yields

$$rW(x) = \max_u \{ g[x,u] + W_x(x) f[x,u] \}. \tag{2.13}$$

Since time is not explicitly involved (2.13), the derived control u will be a function of x only. Hence one can obtain:

Theorem 2.1.2. *A set of controls $u = \phi^*(x)$ constitutes an optimal solution to the infinite-horizon control problem (2.10)–(2.11) if there exists continuously differentiable function $W(x)$ defined on $R^m \to R$ which satisfies the following equation:*

$$\begin{aligned} rW(x) &= \max_u \{ g[x,u] + W_x(x) f[x,u] \} \\ &= \{ g[x, \phi^*(x)] + W_x(x) f[x, \phi^*(x)] \}. \end{aligned}$$

Substituting the optimal control in Theorem 2.1.2 into (2.9) yields the dynamics of the optimal state path as:

$$\dot{x}(s) = f[x(s), \phi^*(x(s))]\, ds, \qquad x(t_0) = x_0.$$

Solving the above dynamics yields the optimal state trajectory $\{x^*(t)\}_{t \ge t_o}$ as

$$x^*(t) = x_0 + \int_{t_0}^t f[x^*(s), \psi^*(x^*(s))]\, ds, \quad \text{for } t \ge t_0.$$

We denote term $x^*(t)$ by x_t^*. The optimal control to the infinite-horizon problem (2.8)–(2.9) can be expressed as $\psi^*(x_t^*)$ in the time interval $[t_0, \infty)$.

Example 2.1.2. Consider the infinite-horizon dynamic optimization problem:

$$\max_u \int_0^\infty \exp[-rs] \left[-x(s) - cu(s)^2 \right] ds \tag{2.14}$$

subject to dynamics (2.6).

Invoking Theorem 2.1.2 we have

$$rW(x) = \max_u \left\{ \left[-x - cu^2\right] + W_x(x) \left[a - ux^{1/2}\right] \right\}. \tag{2.15}$$

Performing the indicated maximization in (2.15) yields:

$$\phi^*(x) = \frac{-V_x(x)\, x^{1/2}}{2c}.$$

Substituting $\phi(x)$ into (2.15) and upon solving (2.15), one obtains:

$$V(t,x) = \exp[-rt]\,[Ax + B],$$

where A and B satisfy:

$$0 = rA - \frac{A^2}{4c} + 1 \ \text{ and } \ B = \frac{-a}{r} A.$$

Solving A to be $2c\left[r \pm \left(r^2 + c^{-1}\right)^{1/2}\right]$. For a maximum, the negative root of A holds. The optimal control can be obtained as

$$\phi^*(x) = \frac{-Ax^{1/2}}{2c}.$$

Substituting $\phi^*(x) = -Ax^{1/2}/(2c)$ into (2.6) yields the dynamics of the optimal trajectory as:

$$\dot{x}(s) = a + \frac{A}{2c}(x(s)), \qquad x(0) = x_0.$$

Upon the above dynamical equation yields the optimal trajectory $\{x^*(t)\}_{t \ge t_0}$ as

$$x^*(t) = \left[x_0 + \frac{2ac}{A}\right] \exp\left(\frac{A}{2c} t\right) - \frac{2ac}{A} = x_t^*, \qquad \text{for } t \ge t_0.$$

The optimal control of problem (2.14)–(2.15) is then

$$\phi^*(x_t^*) = \frac{-A\left(x_t^*\right)^{1/2}}{2c}.$$

2.1.2 Optimal Control

The maximum principle of optimal control was developed by Pontryagin (details in Pontryagin et al (1962)). Consider again the dynamic optimization problem (2.1)–(2.2).

Theorem 2.1.3. *(Pontryagin's Maximum Principle) A set of controls* $u^*(s) = \zeta^*(s, x_0)$ *provides an optimal solution to control problem (2.1)–(2.2), and* $\{x^*(s), t_0 \le s \le T\}$ *is the corresponding state trajectory, if there exist costate functions* $\Lambda(s) : [t_0, T] \to R^m$ *such that the following relations are satisfied:*

$$\begin{aligned}
&\zeta^*(s, x_0) \equiv u^*(s) = \arg\max_u \{g[s, x^*(s), u(s)] + \Lambda(s) f[s, x^*(s), u(s)]\}, \\
&\dot{x}^*(s) = f[s, x^*(s), u^*(s)], \quad x^*(t_0) = x_0, \\
&\dot{\Lambda}(s) = -\frac{\partial}{\partial x}\{g[s, x^*(s), u^*(s)] + \Lambda(s) f[s, x^*(s), u^*(s)]\}, \\
&\Lambda(T) = \frac{\partial}{\partial x^*} q(x^*(T)).
\end{aligned}$$

Proof. First define the function (Hamiltonian)

$$H(t, x, u) = g(t, x, u) + V_x(t, x) f(t, x, u).$$

From Theorem 2.1.1, we obtain

$$-V_t(t, x) = \max_u H(t, x, u).$$

This yields the first condition of Theorem 2.1.1. Using u^* *to denote the payoff maximizing control, we obtain*

$$H(t, x, u^*) + V_t(t, x) \equiv 0,$$

which is an identity in x*. Differentiating this identity partially with respect to* x *yields*

$$\begin{aligned}
&V_{tx}(t, x) + g_x(t, x, u^*) + V_x(t, x) f_x(t, x, u^*) + V_{xx}(t, x) f(t, x, u^*) \\
&\quad + [g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] \frac{\partial u^*}{\partial x} = 0.
\end{aligned}$$

If u^* *is an interior point, then* $[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] = 0$ *according to the condition* $-V_t(t, x) = \max_u H(t, x, u)$*. If* u^* *is not an interior point, then it can be shown that*

$$[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] \frac{\partial u^*}{\partial x} = 0$$

(because of optimality, $[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)]$ *and* $\partial u^*/\partial x$ *are orthogonal; and for specific problems we may have* $\partial u^*/\partial x = 0$*). Moreover, the expression* $V_{tx}(t, x) + V_{xx}(t, x) f(t, x, u^*) \equiv V_{tx}(t, x) + V_{xx}(t, x)\dot{x}$ *can be written as* $[dV_x(t, x)](dt)^{-1}$*. Hence, we obtain:*

$$\frac{dV_x(t, x)}{dt} + g_x(t, x, u^*) + V_x(t, x) f_x(t, x, u^*) = 0.$$

By introducing the costate vector, $\Lambda(t) = V_{x^*}(t, x^*)$*, where* x^* *denotes the state trajectory corresponding to* u^**, we arrive at*

$$\frac{dV_x(t,x^*)}{dt} = \dot{\Lambda}(s) = -\frac{\partial}{\partial x}\left\{g\left[s,x^*(s),u^*(s)\right] + \Lambda(s) f\left[s,x^*(s),u^*(s)\right]\right\}.$$

Finally, the boundary condition for $\Lambda(t)$ is determined from the terminal condition of optimal control in Theorem 2.1.1 as

$$\Lambda(T) = \frac{\partial V(T,x^*)}{\partial x} = \frac{\partial q(x^*)}{\partial x}.$$

Hence Theorem 2.1.3 follows.

Example 2.1.3. Consider the problem in Example 2.1.1. Invoking Theorem 2.1.3, we first solve the control $u(s)$ that satisfies

$$\arg\max_u \left\{\left[-x^*(s) - cu(s)^2\right]\exp[-rs] + \Lambda(s)\left[a - u(s)\,x^*(s)^{1/2}\right]\right\}.$$

Performing the indicated maximization:

$$u^*(s) = \frac{-\Lambda(s)\,x^*(s)^{1/2}}{2c}\exp[rs]. \tag{2.16}$$

We also obtain

$$\dot{\Lambda}(s) = \exp[-rs] + \frac{1}{2}\Lambda(s)\,u^*(s)\,x^*(s)^{-1/2}. \tag{2.17}$$

Substituting $u^*(s)$ from (2.16) into (2.6) and (2.17) yields a pair of differential equations:

$$\begin{aligned} \dot{x}^*(s) &= a + \frac{1}{2c}\Lambda(s)\left(x^*(s)\right)\exp[rs], \\ \dot{\Lambda}(s) &= \exp[-rs] + \frac{1}{4c}\Lambda(s)^2\exp[rs], \end{aligned} \tag{2.18}$$

with boundary conditions:

$$x^*(0) = x_0 \text{ and } \Lambda(T) = \exp[-rT]\,q.$$

Solving (2.18) yields

$$\begin{aligned} \Lambda(s) = 2c&\left(\theta_1 - \theta_2\frac{q-2c\theta_1}{q-2c\theta_2}\exp\left[\frac{\theta_1-\theta_2}{2}(T-s)\right]\right)\exp(-rs) \\ &\div\left(1 - \frac{q-2c\theta_1}{q-2c\theta_2}\exp\left[\frac{\theta_1-\theta_2}{2}(T-s)\right]\right), \text{ and} \\ x^*(s) = \varpi(0,s)&\left[x_0 + \int_0^s \varpi^{-1}(0,t)\,a\;dt\right], \qquad \text{for } s\in[0,T], \end{aligned}$$

where

$$\theta_1 = r - \sqrt{r^2 + \frac{1}{c}} \text{ and } \theta_2 = r + \sqrt{r^2 + \frac{1}{c}};$$

$$\varpi(0,s) = \exp\left[\int_0^s H(\tau)\, d\tau\right], \text{ and}$$

$$H(\tau) = \left(\theta_1 - \theta_2 \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T - \tau)\right]\right)$$
$$\div \left(1 - \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T - \tau)\right]\right).$$

Upon substituting $\Lambda(s)$ and $x^*(s)$ into (2.16) yields $u^*(s) = \zeta^*(s, x_0)$ which is a function of s and x_0.

Now, consider the infinite-horizon dynamic optimization problem (2.8)–(2.9). The Hamiltonian function can be expressed as

$$H(t,x,u) = g(x,u)\exp\left[-r(t - t_0)\right] + \Lambda(t) f(x,u).$$

Define $\lambda(t) = \Lambda(t)\exp\left[r(t - t_0)\right]$ and the current value Hamiltonian

$$\begin{aligned}\hat{H}(t,x,u) &= H(t,x,u)\exp\left[r(t - t_0)\right] \\ &= g(x,u) + \lambda(t) f(x,u).\end{aligned} \tag{2.19}$$

Substituting (2.19) into Theorem 2.1.3 yields the maximum principle for the game (2.10)–(2.11).

Theorem 2.1.4. *A set of controls $u^*(s) = \zeta^*(s, x_t)$ provides an optimal solution to the infinite-horizon control problem (2.10)–(2.11), and $\{x^*(s),\ s \geq t\}$ is the corresponding state trajectory, if there exist costate functions $\lambda(s) : [t, \infty) \to R^m$ such that the following relations are satisfied:*

$$\zeta^*(s, x_t) \equiv u^*(s) = \arg\max_u \left\{g\left[x^*(s), u(s)\right] + \lambda(s) f\left[x^*(s), u(s)\right]\right\},$$
$$\dot{x}^*(s) = f\left[x^*(s), u^*(s)\right], \qquad x^*(t) = x_t,$$
$$\dot{\lambda}(s) = r\lambda(s) - \frac{\partial}{\partial x}\left\{g\left[x^*(s), u^*(s)\right] + \lambda(s) f\left[x^*(s), u^*(s)\right]\right\}.$$

Example 2.1.4. Consider the infinite-horizon problem in Example 2.1.2.

Invoking Theorem 2.1.4 we have

$$\zeta^*(s, x_t) \equiv u^*(s) =$$
$$\arg\max_u \left\{\left[-x^*(s) - cu(s)^2\right] + \lambda(s)\left[a - u(s) x^*(s)^{1/2}\right]\right\},$$
$$\dot{x}^*(s) = a - u^*(s)\left(x^*(s)\right)^{1/2}, \qquad x^*(t) = x_t,$$
$$\dot{\lambda}(s) = r\lambda(s) + \left[1 + \frac{1}{2}\lambda(s) u^*(s) x^*(s)^{-1/2}\right]. \tag{2.20}$$

Performing the indicated maximization yields

$$u^*(s) = \frac{-\lambda(s)\, x^*(s)^{1/2}}{2c}.$$

Substituting $u^*(s)$ into (2.20), one obtains

$$\begin{aligned}
\dot{x}^*(s) &= a + \frac{\lambda(s)}{2c} u^*(s)\, x^*(s), \quad x^*(t) = x_t, \\
\dot{\lambda}(s) &= r\lambda(s) + \left[1 - \frac{1}{4c}\lambda(s)^2\right]. \qquad (2.21)
\end{aligned}$$

Solving (2.21) in a manner similar to that in Example 2.1.3 yields the solutions of $x^*(s)$ and $\lambda(s)$. Upon substituting them into $u^*(s)$ gives the optimal control of the problem.

2.1.3 Stochastic Control

Consider the dynamic optimization problem in which the single decision-maker:

$$\max_u E_{t_0} \left\{ \int_{t_0}^{T} g^i[s, x(s), u(s)]\, ds + q(x(T)) \right\}, \qquad (2.22)$$

subject to the vector-valued stochastic differential equation:

$$dx(s) = f[s, x(s), u(s)]\, ds + \sigma[s, x(s)]\, dz(s), \quad x(t_0) = x_0, \qquad (2.23)$$

where E_{t_0} denotes the expectation operator performed at time t_0, and $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\, \sigma[s, x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$.

The technique of stochastic control developed by Fleming (1969) can be applied to solve the problem.

Theorem 2.1.5. *A set of controls $u^*(t) = \phi^*(t, x)$ constitutes an optimal solution to the problem (2.22)–(2.23), if there exist continuously differentiable functions $V(t, x) : [t_0, T] \times R^m \to R$, satisfying the following partial differential equation:*

$$\begin{aligned}
&-V_t(t, x) - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x)\, V_{x^h x^\zeta}(t, x) = \\
&\qquad\qquad \max_u \left\{ g^i[t, x, u] + V_x(t, x)\, f[t, x, u] \right\}, \text{ and} \\
&V(T, x) = q(x).
\end{aligned}$$

Proof. *Substitute the optimal control $\phi^*(t, x)$ into the (2.23) to obtain the optimal state dynamics as*

$$dx(s) = f[s, x(s), \phi^*(s, x(s))] ds + \sigma[s, x(s)] dz(s),$$
$$x(t_0) = x_0. \tag{2.24}$$

The solution to (2.24), denoted by $x^*(t)$*, can be expressed as*

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s, x^*(s), \psi_1^{(t_0)*}(s, x^*(s)), \psi_2^{(t_0)*}(s, x^*(s))\right] ds$$
$$+ \int_{t_0}^{t} \sigma[s, x^*(s)] dz(s). \tag{2.25}$$

We use X_t^* *to denote the set of realizable values of* $x^*(t)$ *at time* t *generated by (2.25). The term* x_t^* *is used to denote an element in the set* X_t^**.*

Define the maximized payoff at time t *with current state* x_t^* *as a* value function *in the form*

$$V(t, x_t^*) = \max_u E_{t_0}\left\{ \int_t^T g^i[s, x(s), u(s)] ds + q(x(T)) \middle| x(t) = x_t^* \right\}$$
$$= E_{t_0}\left\{ \int_t^T g[s, x^*(s), \phi^*(s, x^*(s))] ds + q(x^*(T)) \right\}$$

satisfying the boundary condition

$$V(T, x^*(T)) = q(x^*(T)).$$

One can express $V(t, x_t^*)$ *as*

$$V(t, x_t^*)$$
$$= \max_u E_{t_0}\left\{ \int_t^T g^i[s, x(s), u(s)] ds + q(x(T)) \middle| x(t) = x_t^* \right\}$$
$$= \max_u E_{t_0}\left\{ \int_t^{t+\Delta t} g^i[s, x(s), u(s)] ds + V(t + \Delta t, x_t^* + \Delta x_t^*) \middle| x(t) = x_t^* \right\}. \tag{2.26}$$

where

$$\Delta x_t^* = f[t, x_t^*, \phi^*(t, x_t^*)] \Delta t + \sigma[t, x_t^*] \Delta z_t + o(\Delta t),$$
$$\Delta z_t = z(t + \Delta t) - z(t), \text{ and } E_t[o(\Delta t)]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

With $\Delta t \to 0$*, applying Ito's lemma equation (2.26) can be expressed as:*

$$V(t, x_t^*) = \max_u E_{t_0}\left\{ g^i[t, x_t^*, u] \Delta t + V(t, x_t^*) + V_t(t, x_t^*) \Delta t \right.$$

$$+ V_{x_t}(t, x_t^*) f[t, x_t^*, \phi^*(t, x_t^*)] \Delta t + V_{x_t}(t, x_t^*) \sigma[t, x_t^*] \Delta z_t$$
$$\left. + \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x) V_{x^h x^\zeta}(t, x) \Delta t + o(\Delta t) \right\} . \tag{2.27}$$

Dividing (2.27) throughout by Δt, with $\Delta t \to 0$, and taking expectation yields

$$-V_t(t, x_t^*) - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x) V_{x^h x^\zeta}(t, x) =$$
$$\max_u \left\{ g^i[t, x_t^*, u] + V_{x_t}(t, x_t^*) f[t, x_t^*, \phi^*(t, x_t^*)] \right\}, \tag{2.28}$$

with boundary condition

$$V(T, x^*(T)) = q(x^*(T)) .$$

Hence Theorem 2.1.5.

Example 2.1.5. Consider the stochastic control problem

$$E_{t_0} \left\{ \int_{t_0}^{T} \left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}} u(s) \right] \exp[-r(s - t_0)] ds \right.$$
$$\left. + \exp[-r(T - t_0)] q x(T)^{1/2} \right\}, \tag{2.29}$$

subject to

$$dx(s) = \left[a x(s)^{1/2} - b x(s) - u(s) \right] ds + \sigma x(s) dz(s),$$
$$x(t_0) = x_0 \in X, \tag{2.30}$$

where c, a, b and σ are positive parameters.

Invoking Theorem 2.1.5 we have

$$-V_t(t, x) - \frac{1}{2} \sigma^2 x^2 V_{xx}(t, x) =$$
$$\max_u \left\{ \left[u^{1/2} - \frac{c}{x^{1/2}} u \right] \exp[-r(t - t_0)] + V_x(t, x) \left[a x^{1/2} - b x - u \right] \right\}, \text{ and}$$
$$V(T, x) \exp[-r(T - t_0)] q x^{1/2}. \tag{2.31}$$

Performing the indicated maximization in (2.31) yields

$$\phi^*(t, x) = \frac{x}{4 \left[c + V_x \exp[r(t - t_0)] x^{1/2} \right]^2} . \tag{2.32}$$

Substituting $\phi^*(t, x)$ from (2.32) into (2.31) and upon solving (2.31) yields the value function:

$$V(t,x)\exp\left[-r(t-t_0)\right]\left[A(t)\,x^{1/2}+B(t)\right],$$

where $A(t)$ and $B(t)$ satisfy:

$$\begin{aligned}\dot{A}(t) &= \left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]A(t)-\frac{1}{2\left[c+A(t)/2\right]}\\ &\qquad +\frac{c}{4\left[c+A(t)/2\right]^2}+\frac{A(t)}{8\left[c+A(t)/2\right]^2},\\ \dot{B}(t) &= rB(t)-\frac{a}{2}A(t),\\ A(T) &= q, \quad \text{and} \quad B(T)=0.\end{aligned}$$

The optimal control for the problem (2.29)–(2.30) can be obtained as

$$\phi^*(t,x)=\frac{x}{4\left[c+\frac{A(t)}{2}\right]^2}.$$

Now, consider the infinite-horizon stochastic control problem with a constant discount rate:

$$\max_u E_{t_0}\left\{\int_{t_0}^{\infty} g^i\left[x(s),u(s)\right]\exp\left[-r(s-t_0)\right]ds\right\}, \tag{2.33}$$

subject to the vector-valued stochastic differential equation:

$$dx(s)=f\left[x(s),u(s)\right]ds+\sigma\left[x(s)\right]dz(s), \qquad x(t_0)=x_0, \tag{2.34}$$

Since s does not appear in $g\left[x(s),u(s)\right]$ and the state dynamics explicitly, the problem (2.33)–(2.34) is an autonomous problem.

Consider the alternative problem:

$$\max_u E_t\left\{\int_t^{\infty} g^i\left[x(s),u(s)\right]\exp\left[-r(s-t)\right]ds\right\}, \tag{2.35}$$

subject to the vector-valued stochastic differential equation:

$$dx(s)=f\left[x(s),u(s)\right]ds+\sigma\left[x(s)\right]dz(s), \qquad x(t)=x_t. \tag{2.36}$$

The infinite-horizon autonomous problem (2.35)–(2.36) is independent of the choice of t and dependent only upon the state at the starting time, that is x_t.

Define the value function to the problem (2.35)–(2.36) by

$$V(t,x_t^*)=\max_u E_{t_0}\left\{\int_t^{\infty} g\left[x(s),u(s)\right]\exp\left[-r(s-t_0)\right]ds \,\middle|\, x(t)=x_t^*\right\},$$

where x_t^* is an element belonging to the set of feasible values along the optimal state trajectory at time t. Moreover, we can write

$$V\left(t, x_t^*\right) =$$
$$\exp\left[-r\left(t-t_0\right)\right] \max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}.$$

Since the problem

$$\max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}$$

depends on the current state x_t^* only, we can write

$$W\left(x_t^*\right) = \max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}.$$

It follows that:

$$\begin{aligned}
&V\left(t, x_t^*\right) = \exp\left[-r\left(t-t_0\right)\right] W\left(x_t^*\right), \\
&V_t\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W\left(x_t^*\right), \\
&V_{x_t}\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W_{x_t}\left(x_t^*\right), \text{ and} \\
&V_{x_t x_t}\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W_{x_t x_t}\left(x_t^*\right).
\end{aligned} \tag{2.37}$$

Substituting the results from (2.37) into Theorem 2.1.5 yields

$$rW\left(x\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}\left(t, x\right) W_{x^h x^\zeta}\left(t, x\right) = \max_u \left\{g\left[x, u\right] + W_x\left(x\right) f\left[x, u\right]\right\}. \tag{2.38}$$

Since time is not explicitly involved (2.38), the derived control u will be a function of x only. Hence one can obtain:

Theorem 2.1.6. *A set of controls $u = \phi^*\left(x\right)$ constitutes an optimal solution to the infinite-horizon stochastic control problem (2.33)–(2.34) if there exists continuously differentiable function $W\left(x\right)$ defined on $R^m \to R$ which satisfies the following equation:*

$$\begin{aligned}
rW\left(x\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}\left(t, x\right) W_{x^h x^\zeta}\left(t, x\right) &= \max_u \left\{g\left[x, u\right] + W_x\left(x\right) f\left[x, u\right]\right\} \\
&= \left\{g\left[x, \phi^*\left(x\right)\right] + W_x\left(x\right) f\left[x, \phi^*\left(x\right)\right]\right\}.
\end{aligned}$$

Substituting the optimal control in Theorem 2.1.6 into (2.34) yields the dynamics of the optimal state path as

$$dx\left(s\right) = f\left[x\left(s\right), \phi^*\left(x\left(s\right)\right)\right] ds + \sigma\left[x\left(s\right)\right] dz\left(s\right), \qquad x\left(t_0\right) = x_0.$$

Solving the above vector-valued stochastic differential equations yields the optimal state trajectory $\left\{x^*\left(t\right)\right\}_{t \geq t_0}$ as

$$x^*(t) = x_0 + \int_{t_0}^{t} f[x^*(s), \psi^*(x^*(s))]\, ds + \int_{t_0}^{t} \sigma[x^*(s)]\, dz(s). \qquad (2.39)$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (2.39). The term x_t^* is used to denote an element in the set X_t^*.

Given that x_t^* is realized at time t, the optimal control to the infinite-horizon problem (2.33)–(2.34) can be expressed as $\psi^*(x_t^*)$.

Example 2.1.6. Consider the infinite-horizon problem

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp[-r(s-t_0)]\, ds\right\}, \qquad (2.40)$$

subject to

$$\begin{aligned} dx(s) &= \left[ax(s)^{1/2} - bx(s) - u(s)\right] ds + \sigma x(s)\, dz(s), \\ x(t_0) &= x_0 \in X, \end{aligned} \qquad (2.41)$$

where c, a, b and σ are positive parameters.

Invoking Theorem 2.1.6 we have

$$\begin{aligned} &rW(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}(x) = \\ &\qquad \max_u \left\{\left[u^{1/2} - \frac{c}{x^{1/2}}u\right] + W_x(x)\left[ax^{1/2} - bx - u\right]\right\}. \end{aligned} \qquad (2.42)$$

Performing the indicated maximization in (2.42) yields the control:

$$\phi^*(x) = \frac{x}{4\left[c + W_x(x)\, x^{1/2}\right]^2}.$$

Substituting $\phi^*(t,x)$ into (2.42) above and upon solving (2.42) yields the value function

$$W(x) = \left[Ax^{1/2} + B\right],$$

where A and B satisfy:

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]A - \frac{1}{2[c + A/2]} + \frac{c}{4[c + A/2]^2} + \frac{A}{8[c + A/2]^2},$$

$$B = \frac{a}{2r}A.$$

The optimal control can then be expressed as

$$\phi^*(x) = \frac{x}{4[c + A/2]^2}.$$

Substituting $\phi^*(x) = x / \left\{4\left[c + A/2\right]^2\right\}$ into (2.41) yields the dynamics of the optimal trajectory as

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \frac{x(s)}{4\left[c + A/2\right]^2}\right] ds + \sigma x(s)\, dz(s),$$
$$x(t_0) = x_0 \in X.$$

Upon the above dynamical equation yields the optimal trajectory $\{x^*(t)\}_{t \geq t_0}$ as

$$x^*(t) = \varpi(t_0, t)^2 \left[x_0^{1/2} + \int_{t_0}^{t} \varpi^{-1}(t_0, s)\, H_1\, ds\right]^2,$$
$$\text{for } t \geq t_0, \tag{2.43}$$

where

$$\varpi(t_0, t) = \exp\left[\int_{t_0}^{t}\left[H_2 - \frac{\sigma^2}{8}\right] dv + \int_{t_0}^{t} \frac{\sigma}{2} dz(v)\right],$$
$$H_1 = \frac{1}{2}a, \text{ and } H_2 = -\left[\frac{1}{2}b + \frac{1}{4\left[c + A/2\right]^2} + \frac{\sigma^2}{8}\right].$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (2.43). The term x_t^* is used to denote an element in the set X_t^*. Given that x_t^* is realized at time t, the optimal control to the infinite-horizon problem (2.40)–(2.41) can be expressed as $\psi^*(x_t^*)$.

2.2 Differential Games and their Solution Concepts

One particularly complex – but fruitful – branch of game theory is dynamic or differential games, which investigates interactive decision making over time under different assumptions regarding pre-commitment (of actions), information, and uncertainty. The origin of differential games traces back to the late 1940s. Rufus Isaacs (whose work was published in 1965) formulated missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation variables (state and control), and established a fundamental principle: the *tenet of transition.* The seminal contributions of Isaacs together with the classic research of Bellman on dynamic programming and Pontryagin et al. on optimal control laid the foundations of deterministic differential games. Early research in differential games centers on the extension of control theory problems. Berkovitz (1964) developed a variational approach to differential games, Leitmann and Mon (1967) studies the geometric aspects of differential games, Pontryagin (1966) solved differential games solution with his maximum principle, while Zaubermann (1975) accounted for various developments in Soviet literature prior to the early 1970s.

Contributions to differential games continue to appear in many fields and disciplines. In particular, applications in economics and management sciences have been growing rapidly, with reference to which a detailed account of these applications can be found in Dockner et al. (2000).

Differential games or continuous-time infinite dynamic games study a class of decision problems, under which the evolution of the state is described by a differential equation and the players act throughout a time interval.

In particular, in the general n-person differential game, Player i seeks to:

$$\max_{u_i} \int_{t_0}^{T} g^i \left[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] ds + q^i\left(x(T)\right), \quad (2.44)$$
$$\text{for } \; i \in N = \{1, 2, \ldots, n\},$$

subject to the deterministic dynamics

$$\dot{x}(s) = f\left[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)\right], \qquad x(t_0) = x_0, \qquad (2.45)$$

where $x(s) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of Player i, for $i \in N$.

The functions $f[s, x, u_1, u_2, \ldots, u_n]$, $g^i[s, \cdot, u_1, u_2, \ldots, u_n]$ and $q^i(\cdot)$, for $i \in N$, and $s \in [t_0, T]$ are differentiable functions.

A set-valued function $\eta^i(\cdot)$ defined for each $i \in N$ as

$$\eta^i(s) = \left\{x(t), \quad t_0 \le t \le \epsilon_s^i\right\}, \qquad t_0 \le \epsilon_s^i \le s,$$

where ϵ_s^i is nondecreasing in s, and $\eta^i(s)$ determines the state information gained and recalled by Player i at time $s \in [t_0, T]$. Specification of $\eta^i(\cdot)$(in fact, ϵ_s^i in this formulation) characterizes the *information structure* of Player i and the collection (over $i \in N$) of these information structures is the *information structure* of the game.

A sigma-field N_s^i in S_0 generated for each $i \in N$ by the cylinder sets $\{x \in S_0, \, x(t) \in B\}$ where B is a Borel set in S^0 and $0 \le t \le \epsilon_s$. N_s^i, $s \ge t_0$, is called the *information field* of Player i.

A pre-specified class Γ^i of mappings $\upsilon_i : [t_0, T] \times S_0 \to S^i$, with the property that $u_i(s) = \upsilon_i(s, x)$ is n_s^i-measurable (i.e. it is adapted to the information field N_s^i). U^i is the strategy space of Player i and each of its elements υ_i is a permissible strategy for Player i.

Definition 2.2.1. *A set of strategies* $\{\upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\}$ *is said to constitute a* non-cooperative Nash equilibrium solution *for the n-person differential game (2.44)–(2.45), if the following inequalities are satisfied for all* $\upsilon_i(s) \in U^i$, $i \in N$*:*

$$\int_{t_0}^{T} g^1 \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^1\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^1 \left[s, x^{[1]}(s), \upsilon_1(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^1\left(x^{[1]}(T)\right),$$
$$\int_{t_0}^{T} g^2 \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^2\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^2 \left[s, x^{[2]}(s), \upsilon_1^*(s), \upsilon_2(s), \upsilon_3^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^2\left(x^{[2]}(T)\right),$$
$$\vdots \qquad \vdots$$
$$\vdots \qquad \vdots$$
$$\int_{t_0}^{T} g^n \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^n\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^n \left[s, x^{[n]}(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_{n-1}^*(s), \upsilon_n(s)\right] ds + q^n\left(x^{[n]}(T)\right);$$

where on the time interval $[t_0, T]$*:*

$$\dot{x}^*(s) = f\left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^*(t_0) = x_0,$$
$$\dot{x}^{[1]}(s) = f\left[s, x^{[1]}(s), \upsilon_1(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^{[1]}(t_0) = x_0,$$
$$\dot{x}^{[2]}(s) = f\left[s, x^{[2]}(s), \upsilon_1^*(s), \upsilon_2(s), \upsilon_3^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^{[2]}(t_0) = x_0,$$
$$\vdots \qquad \vdots$$
$$\vdots \qquad \vdots$$
$$\dot{x}^{[n]}(s) = f\left[s, x^{[n]}(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_{n-1}^*(s), \upsilon_n(s)\right], \quad x^{[n]}(t_0) = x_0.$$

The set of strategies $\{\upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\}$ *is known as a* Nash equilibrium *of the game.*

2.2.1 Open-loop Nash Equilibria

If the players choose to commit their strategies from the outset, the players' information structure can be seen as an *open-loop* pattern in which $\eta^i(s) = \{x_0\}$, $s \in [t_0, T]$. Their strategies become functions of the initial state x_0 and time s, and can be expressed as $\{u_i(s) = \vartheta_i(s, x_0), \text{ for } i \in N\}$. In particular, an open-loop Nash equilibria for the game (2.44) and (2.45) is characterized as:

Theorem 2.2.1. *A set of strategies* $\{u_i^*(s) = \zeta_i^*(s, x_0), \text{ for } i \in N\}$ *provides an open-loop Nash equilibrium solution to the game (2.44)–(2.45), and*

$\{x^*(s), t_0 \le s \le T\}$ is the corresponding state trajectory, if there exist m costate functions $\Lambda^i(s) : [t_0, T] \to R^m$, for $i \in N$, such that the following relations are satisfied:

$$\zeta_i^*(s, x_0) \equiv u_i^*(s) =$$
$$\arg\max_{u_i \in U^i} \left\{ g^i \left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s) \right] \right.$$
$$\left. + \Lambda^i(s) f \left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s) \right] \right\},$$
$$\dot{x}^*(s) = f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right], \quad x^*(t_0) = x_0,$$
$$\dot{\Lambda}^i(s) = -\frac{\partial}{\partial x^*} \left\{ g^i\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right] \right.$$
$$\left. + \Lambda^i(s) f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right] \right\},$$
$$\Lambda^i(T) = \frac{\partial}{\partial x^*} q^i(x^*(T)); \qquad \text{for } i \in N.$$

Proof. *Consider the i^{th} equality in Theorem 2.1.1, which states that $\upsilon_i^*(s) = u_i^*(s) = \zeta_i^*(s, x_0)$ maximizes*

$$\int_{t_0}^{T} g^i \left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s) \right] ds$$
$$+ q^i(x(T)),$$

over the choice of $\upsilon_i(s) \in U^i$ subject to the state dynamics:

$$\dot{x}(s) = f \left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s) \right],$$
$$x(t_0) = x_0, \qquad \text{for } i \in N.$$

This is standard optimal control problem for Player i, since $u_j^*(s)$, for $j \in N$ and $j \neq i$, are open-loop controls and hence do not depend on $u_i^*(s)$. These results then follow directly from the maximum principle of Pontryagin as stated in Theorem 2.1.3.

Derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in Berkovitz (1964) and Ho et al. (1965), with open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games being presented in Case (1967, 1969) and Starr and Ho (1969a and 1969b). A detailed account of applications of open-loop equilibria in economic and management science can be found in Dockner et al. (2000).

2.2.2 Closed-loop Nash Equilibria

Under the memoryless perfect state information, the players' information structures follow the pattern $\eta^i(s) = \{x_0, x(s)\}$, $s \in [t_0, T]$. The players' strategies become functions of the initial state x_0, current state $x(s)$ and current time s, and can be expressed as $\{u_i(s) = \vartheta_i(s, x, x_0), \text{ for } i \in N\}$. The following theorem provides a set of necessary conditions for any closed-loop no-memory Nash equilibrium solution to satisfy.

Theorem 2.2.2. *A set of strategies* $\{u_i(s) = \vartheta_i(s, x, x_0), \textit{ for } i \in N\}$ *provides a closed-loop no memory Nash equilibrium solution to the game (2.44)–(2.45), and* $\{x^*(s), t_0 \le s \le T\}$ *is the corresponding state trajectory, if there exist* N *costate functions* $\Lambda^i(s) : [t_0, T] \to R^m$, *for* $i \in N$, *such that the following relations are satisfied:*

$$\vartheta_i^*(s, x^*, x_0) \equiv u_i^*(s) =$$
$$\arg\max_{u_i \in U^i} \left\{ g^i\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right.$$
$$\left. +\Lambda^i(s) f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right\},$$
$$\dot{x}^*(s) = f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right], \quad x^*(t_0) = x_0,$$
$$\begin{aligned}\dot{\Lambda}^i(s) = -\frac{\partial}{\partial x^*} \{ & g^i\left[s, x^*(s), \vartheta_1^*(s, x^*, x_0), \vartheta_2^*(s, x^*, x_0), \ldots \right. \\ & \left. \ldots, \vartheta_{i-1}^*(s, x^*, x_0), u_i^*(s), \vartheta_{i+1}^*(s, x^*, x_0), \ldots, \vartheta_n^*(s, x^*, x_0)\right] \\ & +\Lambda^i(s) f\left[s, x^*(s), \vartheta_1^*(s, x^*, x_0), \vartheta_2^*(s, x^*, x_0), \ldots \right. \\ & \left. \ldots, \vartheta_{i-1}^*(s, x^*, x_0), u_i^*(s), \vartheta_{i+1}^*(s, x^*, x_0), \ldots, \vartheta_n^*(s, x^*, x_0)\right],\end{aligned}$$
$$\Lambda^i(T) = \frac{\partial}{\partial x^*} q^i(x^*(T)); \qquad \textit{for } i \in N.$$

Proof. Consider the i^{th} *equality in Theorem 2.2.2, which fixed all players' strategies (except those of the* i^{th} *player) at* $u_j^*(s) = \vartheta_j^*(s, x^*, x_0)$, *for* $j \ne i$ *and* $j \in N$, *and constitutes an optimal control problem for Player* i. *Therefore, the above conditions follow from the maximum principle of Pontryagin, and Player* i *maximizes*

$$\int_{t_0}^{T} g^i\left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] ds$$
$$+q^i(x(T)),$$

over the choice of $\upsilon_i(s) \in U^i$ *subject to the state dynamics:*

$$\dot{x}(s) = f\left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right],$$
$$x(t_0) = x_0, \qquad \textit{for } i \in N.$$

Note that the partial derivative with respect to x in the costate equations of Theorem 2.2.2 receives contributions from dependence of the other $n-1$ players' strategies on the current value of x. This is a feature absent from the costate equations of Theorem 2.2.1. The set of equations in Theorem 2.2.2 in general admits of an uncountable number of solutions, which correspond to "*informationally nonunique*" Nash equilibrium solutions of differential games under memoryless perfect state information pattern. Derivation of nonunique closed-loop Nash equilibria can be found in Basar (1977c) and Mehlmann and Willing (1984).

2.2.3 Feedback Nash Equilibria

To eliminate information nonuniqueness in the derivation of Nash equilibria, one can constrain the Nash solution further by requiring it to satisfy the feedback Nash equilibrium property. In particular, the players' information structures follow either a *closed-loop perfect state* (CLPS) pattern in which $\eta^i(s) = \{x(s), t_0 \le t \le s\}$ or a *memoryless perfect state* (MPS) pattern in which $\eta^i(s) = \{x_0, x(s)\}$. Moreover, we require the following feedback Nash equilibrium condition to be satisfied.

Definition 2.2.2. *For the n-person differential game (2.1)–(2.2) with MPS or CLPS information, an n-tuple of strategies $\{u_i^*(s) = \phi_i^*(s,x) \in U^i$, for $i \in N\}$ constitutes a* feedback Nash equilibrium solution *if there exist functionals $V^i(t,x)$ defined on $[t_0,T] \times R^m$ and satisfying the following relations for each $i \in N$:*

$$
\begin{aligned}
&V^i(T,x) = q^i(x),\\
&V^i(t,x) =\\
&\qquad \int_t^T g^i\left[s, x^*(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\right] ds + q^i\left(x^*(T)\right) \ge\\
&\int_t^T g^i\Big[s, x^{[i]}(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots\\
&\qquad \ldots, \phi_{i-1}^*(s,\eta_s), \phi_i(s,\eta_s), \phi_{i+1}^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\Big] ds + q^i\Big(x^{[i]}(T)\Big),\\
&\forall \phi_i(\cdot,\cdot) \in \Gamma^i, x \in R^n
\end{aligned}
$$

where on the interval $[t_0,T]$,

$$
\begin{aligned}
&\dot{x}^{[i]}(s) = f\Big[s, x^{[i]}(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots\\
&\qquad \ldots, \phi_{i-1}^*(s,\eta_s), \phi_i(s,\eta_s), \phi_{i+1}^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\Big], \quad x^{[1]}(t) = x;\\
&\dot{x}^*(s) = f\left[s, x^*(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\right], \quad x(s) = x;
\end{aligned}
$$

and η_s stands for either the data set $\{x(s), x_0\}$ or $\{x(\tau), \tau \le s\}$, depending on whether the information pattern is MPS or CLPS.

One salient feature of the concept introduced above is that if an n-tuple $\{\phi_i^*; i \in N\}$ provides a feedback Nash equilibrium solution (FNES) to an N-person differential game with duration $[t_0, T]$, its restriction to the time interval $[t, T]$ provides an FNES to the same differential game defined on the shorter time interval $[t, T]$, with the initial state taken as $x(t)$, and this being so for all $t_0 \leq t \leq T$. An immediate consequence of this observation is that, under either MPS or CLPS information pattern, feedback Nash equilibrium strategies will depend only on the time variable and the current value of the state, but not on memory (including the initial state x_0). Therefore the players' strategies can be expressed as $\{u_i(s) = \phi_i(s, x), \text{ for } i \in N\}$. The following theorem provides a set of necessary conditions characterizing a feedback Nash equilibrium solution for the game (2.44) and (2.45) is characterized as follows:

Theorem 2.2.3. *An n-tuple of strategies $\left\{u_i^*(s) = \phi_i^*(t, x) \in U^i, \text{ for } i \in N\right\}$ provides a feedback Nash equilibrium solution to the game (2.44)–(2.45) if there exist continuously differentiable functions $V^i(t, x) : [t_0, T] \times R^m \to R,\ i \in N$, satisfying the following set of partial differential equations:*

$$
\begin{aligned}
-V_t^i(t, x) &= \max_{u_i} \Big\{ g^i\big[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots \\
&\qquad \ldots, \phi_{i-1}^*(t, x), u_i(t, x), \phi_{i+1}^*(t, x), \ldots, \phi_n^*(t, x)\big] \\
&\quad + V_x^i(t, x) f\big[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots \\
&\qquad \ldots, \phi_{i-1}^*(t, x), u_i(t, x), \phi_{i+1}^*(t, x), \ldots, \phi_n^*(t, x)\big] \Big\} \\
&= \Big\{ g^i\left[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots, \phi_n^*(t, x)\right] \\
&\quad + V_x^i(t, x) f\left[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots, \phi_n^*(t, x)\right] \Big\}, \\
V^i(T, x) &= q^i(x), \quad i \in N.
\end{aligned}
$$

Proof. By Theorem 2.1.1, $V^i(t, x)$ is the value function associated with the optimal control problem of Player i, $i \in N$. Together with the i^{th} expression in Definition 2.2.1, the conditions in Theorem 2.2.3 imply a Nash equilibrium.

Consider the two-person zero-sum version of the game (2.44)–(2.45) in which the payoff of Player 1 is the negative of that of Player 2. Under either MPS or CLPS information pattern, a feedback saddle-point is characterized as follows.

Theorem 2.2.4. *A pair of strategies $\{\phi_i^*(t, x); i = 1, 2\}$ provides a feedback saddle-point solution to the zero-sum version of the game (2.44)–(2.45) if there exists a function $V : [t_0, T] \times R^m \to R$ satisfying the partial differential equation:*

$$-V_t(t, x) = \min_{u_1 \in S^1} \max_{u_2 \in S^2} \left\{ g\left[t, x, u_1(t), u_2(t)\right] + V_x f\left[t, x, u_1(t), u_2(t)\right] \right\}$$

$$= \max_{u_2 \in S^2} \min_{u_1 \in S^1} \{g[t,x,u_1(t),u_2(t)] + V_x f[t,x,u_1(t),u_2(t)]\}$$
$$= \{g[t,x,\phi_1^*(t,x),\phi_2^*(t,x)] + V_x f[t,x,\phi_1^*(t,x),\phi_2^*(t,x)]\},$$
$$V(T,x) = q(x).$$

Proof. This result follows as a special case of Theorem 2.2.3 by taking $n = 2$, $g^1(\cdot) = -g^2(\cdot) \equiv g(\cdot)$, *and* $q^1(\cdot) = -q^2(\cdot) \equiv q(\cdot)$, *in which case* $V^1 = -V^2 \equiv V$ *and existence of a saddle point is equivalent to interchangeability of the min max operations.*

The partial differential equation in Theorem 2.2.4 was first obtained by Isaacs (see, Isaacs, 1965), and is therefore called the Isaacs equation.

While open-loop solutions are relatively tractable and more widely applied, feedback solutions avoid the time-inconsistency problem at the expense of intractability. Examples of differential games solved in feedback Nash solution include Clemhout and Wan (1974), Fershtman (1987), Fershtman and Kamien (1987), Jørgensen (1985), Jørgensen and Sorger (1990), Leitmann and Schmitendorf (1978), Lukes (1971a and 1971b), Sorger (1989), and Yeung (1987, 1989, 1992 and 1994).

In general, feedback Nash equilibrium solution and open-loop Nash equilibrium solution do not coincide. A feedback Nash equilibrium solution of a differential game is a degenerate feedback Nash equilibrium solution if it coincides with the game's open-loop Nash equilibrium solution. In particular, the degenerate Nash equilibrium strategies depend on time only and therefore $\phi_i^*(t,x) = \varphi_i^*(t) = \vartheta_i^*(t,x_0)$, for $i \in N$.

Clemhout and Wan (1974), Leitmann and Schmitendorf (1978), Reinganum (1982a and 1982b), Dockner, Feichtinger and Jørgensen (1985), Jørgensen (1985) and Plourde and Yeung (1989), Yeung (1987 and 1992) identified and examined several classes of games in which the feedback Nash equilibrium and open-loop Nash equilibrium coincide. Fershtman (1987) proposed a technique to identify classes of games for which the open-loop Nash equilibrium is a degenerate feedback Nash equilibrium, while Yeung (1994) established a lemma demonstrating that for every differential game which yields a degenerate/non-degenerate FNE, there exists a class of non-degenerate/degenerate correspondences.

2.3 Application of Differential Games in Economics

In this section we consider application of differential games in competitive advertising.

2.3.1 Open-loop Solution in Competitive Advertising

Consider the competitive dynamic advertising game in Sorger (1989). There are two firms in a market and the profit of firm 1 and that of 2 are respectively:

$$\int_0^T \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2\right] \exp(-rs)\, ds + \exp(-rT) S_1 x(T)$$

and

$$\int_0^T \left[q_2 (1 - x(s)) - \frac{c_2}{2} u_2(s)^2\right] \exp(-rs)\, ds + \exp(-rT) S_2 [1 - x(T)], \tag{2.46}$$

where r, q_i, c_i, S_i, for $i \in \{1,2\}$, are positive constants,
$x(s)$ is the market share of firm 1 at time s, $[1 - x(s)]$ is that of firm 2's,
$u_i(s)$ is advertising rate for firm $i \in \{1,2\}$.

It is assumed that market potential is constant over time. The only marketing instrument used by the firms is advertising. Advertising has diminishing returns, since there are increasing marginal costs of advertising as reflected through the quadratic cost function. The dynamics of firm 1's market share is governed by

$$\dot{x}(s) = u_1(s) [1 - x(s)]^{1/2} - u_2(s) x(s)^{1/2}, \qquad x(0) = x_0, \tag{2.47}$$

There are saturation effects, since u_i operates only on the buyer market of the competing firm j.

Consider that the firms would like to seek an open-loop solution. Using open-loop strategies requires the firms to determine their action paths at the outset. This is realistic only if there are restrictive commitments concerning advertising. Invoking Theorem 2.2.1, an open-loop solution to the game (2.46)–(2.47) has to satisfy the following conditions:

$$\begin{aligned}
u_1^*(s) = \arg\max_{u_1} \Bigg\{ & \left[q_1 x^*(s) - \frac{c_1}{2} u_1(s)^2\right] \exp(-rs) \\
& + \Lambda^1(s) \left(u_1(s) [1 - x^*(s)]^{1/2} - u_2(s) x^*(s)^{1/2} \right) \Bigg\},
\end{aligned}$$

$$\begin{aligned}
u_2^*(s) = \arg\max_{u_2} \Bigg\{ & \left[q_2 (1 - x^*(s)) - \frac{c_2}{2} u_2(s)^2\right] \exp(-rs) \\
& + \Lambda^2(s) \left(u_1(s) [1 - x^*(s)]^{1/2} - u_2(s) x^*(s)^{1/2} \right) \Bigg\},
\end{aligned}$$

$$\dot{x}^*(s) = u_1^*(s) [1 - x^*(s)]^{1/2} - u_2^*(s) x^*(s)^{1/2}, \qquad x^*(0) = x_0,$$

$$\dot{\Lambda}^1(s) =$$

$$\left\{ -q_1 \exp(-rs) + \Lambda^1(s) \left(\frac{1}{2} u_1^*(s) [1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s) x^*(s)^{-1/2} \right) \right\},$$

$$\dot{\Lambda}^2(s) =$$

$$\left\{ q_2 \exp(-rs) + \Lambda^2(s) \left(\frac{1}{2} u_1^*(s) [1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s) x^*(s)^{-1/2} \right) \right\},$$

$$
\begin{aligned}
\Lambda^1(T) &= \exp(-rT) S_1, \\
\Lambda^2(T) &= -\exp(-rT) S_2.
\end{aligned}
\tag{2.48}
$$

Using (2.48), we obtain

$$
\begin{aligned}
u_1^*(s) &= \frac{\Lambda^1(s)}{c_1}\left[1 - x^*(s)\right]^{1/2} \exp(rs), \quad \text{and} \\
u_2^*(s) &= \frac{\Lambda^2(s)}{c_2}\left[x^*(s)\right]^{1/2} \exp(rs).
\end{aligned}
$$

Upon substituting $u_1^*(s)$ and $u_2^*(s)$ into (2.48) yields:

$$
\begin{aligned}
\dot{\Lambda}^1(s) &= \left\{-q_1 \exp(-rs) + \left(\frac{\left[\Lambda^1(s)\right]^2}{2c_1} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_2}\right)\right\}, \\
\dot{\Lambda}^2(s) &= \left\{q_2 \exp(-rs) + \left(\frac{\left[\Lambda^2(s)\right]^2}{2c_2} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_1}\right)\right\},
\end{aligned}
\tag{2.49}
$$

with boundary conditions

$$
\Lambda^1(T) = \exp(-rT) S_1 \quad \text{and} \quad \Lambda^2(T) = -\exp(-rT) S_2.
$$

The game equilibrium state dynamics becomes:

$$
\begin{aligned}
\dot{x}^*(s) &= \frac{\Lambda^1(s)\exp(rs)}{c_1}\left[1 - x^*(s)\right] - \frac{\Lambda^2(s)\exp(rs)}{c_2} x^*(s), \\
x^*(0) &= x_0.
\end{aligned}
\tag{2.50}
$$

Solving the block recursive system of differential equations (2.49)–(2.50) gives the solutions to $x^*(s)$, $\Lambda^1(s)$ and $\Lambda^2(s)$. Upon substituting them into $u_1^*(s)$ and $u_2^*(s)$ yields the open-loop game equilibrium strategies.

2.3.2 Feedback Solution in Competitive Advertising

A feedback solution which allows the firm to choose their advertising rates contingent upon the state of the game is a realistic approach to the problem (2.46)–(2.47). Invoking Theorem 2.2.3, a feedback Nash equilibrium solution to the game (2.46)–(2.47) has to satisfy the following conditions:

$$
\begin{aligned}
-V_t^1(t,x) &= \max_{u_1}\left\{\left[q_1 x - \frac{c_1}{2}u_1^2\right]\exp(-rt)\right. \\
&\qquad \left. + V_x^1(t,x)\left(u_1\left[1-x\right]^{1/2} - \phi_2^*(t,x)\, x^{1/2}\right)\right\}, \\
-V_t^2(t,x) &= \max_{u_2}\left\{\left[q_2(1-x) - \frac{c_2}{2}u_2^2\right]\exp(-rt)\right. \\
&\qquad \left. + V_x^2(t,x)\left(\phi_1^*(t,x)\left[1-x\right]^{1/2} - u_2 x^{1/2}\right)\right\}, \\
V^1(T,x) &= \exp(-rT) S_1 x, \\
V^2(T,x) &= \exp(-rT) S_2 (1-x).
\end{aligned}
\tag{2.51}
$$

Performing the indicated maximization in (2.51) yields:

$$\phi_1^*(t,x) = \frac{V_x^1(t,x)}{c_1}[1-x]^{1/2}\exp(rt) \quad \text{and}$$
$$\phi_2^*(t,x) = \frac{V_x^2(t,x)}{c_2}[x]^{1/2}\exp(rt).$$

Upon substituting $\phi_1^*(t,x)$ and $\phi_2^*(t,x)$ into (2.51) and solving (2.51) we obtain the value functions:

$$\begin{aligned} V^1(t,x) &= \exp[-r(t)][A_1(t)x + B_1(t)] \quad \text{and} \\ V^2(t,x) &= \exp[-r(t)][A_2(t)(1-x) + B_2(t)] \end{aligned} \tag{2.52}$$

where $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ satisfy:

$$\dot{A}_1(t) = rA_1(t) - q_1 + \frac{A_1(t)^2}{2c_1} + \frac{A_1(t)A_2(t)}{2c_2},$$
$$\dot{A}_2(t) = rA_2(t) - q_2 + \frac{A_2(t)^2}{2c_2} + \frac{A_1(t)A_2(t)}{2c_1},$$
$$A_1(T) = S_1,\ B_1(T) = 0,\ A_2(T) = S_2 \ \text{ and } \ B_2(T) = 0.$$

Upon substituting the relevant partial derivatives of $V^1(t,x)$ and $V^2(t,x)$ from (2.52) into (2.51) yields the feedback Nash equilibrium strategies

$$\phi_1^*(t,x) = \frac{A_1(t)}{c_1}[1-x]^{1/2} \quad \text{and} \quad \phi_2^*(t,x) = \frac{A_2(t)}{c_2}[x]^{1/2}. \tag{2.53}$$

2.4 Infinite-Horizon Differential Games

Consider the infinite-horizon autonomous game problem with constant discounting, in which T approaches infinity and where the objective functions and state dynamics are both autonomous. In particular, the game becomes:

$$\max_{u_i} \int_{t_0}^{\infty} g^i[x(s), u_1(s), u_2(s), \ldots, u_n(s)]\exp[-r(s-t_0)]\,ds,$$
$$\text{for } i \in N, \tag{2.54}$$

subject to the deterministic dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \ldots, u_n(s)], \quad x(t_0) = x_0, \tag{2.55}$$

where r is a constant discount rate.

2.4.1 Game Equilibrium Solutions

Now consider the alternative game to (2.54)–(2.55)

$$\max_{u_i} \int_t^{\infty} g^i \left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] \exp\left[-r(s-t)\right] ds,$$
$$\text{for } i \in N, \tag{2.56}$$

subject to the deterministic dynamics

$$\dot{x}(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right], \quad x(t) = x. \tag{2.57}$$

The infinite-horizon autonomous game (2.56)–(2.57) is independent of the choice of t and dependent only upon the state at the starting time, that is x.

In the infinite-horizon optimization problem in Section 2.1.1, the control is shown to be a function of the state variable x only. With the validity of the game equilibrium $\left\{u_i^*(s) = \phi_i^*(x) \in U^i, \text{ for } i \in N\right\}$ to be verified later, we define

Definition 2.4.1. *For the n-person differential game (2.54)–(2.55) with MPS or CLPS information, an n-tuple of strategies*

$$\left\{u_i^*(s) = \phi_i^*(x) \in U^i, \textit{ for } i \in N\right\}$$

constitutes a feedback Nash equilibrium solution *if there exist functionals $V^i(t,x)$ defined on $[t_0, \infty) \times R^m$ and satisfying the following relations for each $i \in N$:*

$$V^i(t,x) = \int_t^{\infty} g^i \left[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right] \exp\left[-r(s-t_0)\right] ds \geq$$
$$\int_t^{\infty} g^i \left[x^{[i]}(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_{i-1}^*(\eta_s) \phi_i(\eta_s) \phi_{i+1}^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right]$$
$$\times \exp\left[-r(s-t_0)\right] ds$$

$$\forall \phi_i(\cdot,\cdot) \in \Gamma^i, \quad x \in R^n,$$

where on the interval $[t_0, \infty)$,

$$\dot{x}^{[i]}(s) =$$
$$f\left[s^{[i]}(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_{i-1}^*(\eta_s) \phi_i(\eta_s) \phi_{i+1}^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right],$$
$$x^{[1]}(t) = x;$$
$$\dot{x}^*(s) = f\left[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right], \qquad x^*(s) = x;$$

and η_s stands for either the data set $\{x(s), x_0\}$ or $\{x(\tau), \tau \leq s\}$, depending on whether the information pattern in MPS or CLPS.

We can write

$$V^i(t,x) = \exp[-r(t-t_0)] \int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)]$$
$$\times \exp[-r(s-t)]\, ds,$$
$$\textit{for } x(t) = x = x_t^* = x^*(t).$$

Since

$$\int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)] \exp[-r(s-t)]\, ds$$

depends on the current state x only, we can write:

$$W^i(x) = \int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)] \exp[-r(s-t)]\, ds.$$

It follows that:

$$\begin{aligned} V^i(t,x) &= \exp[-r(t-t_0)]\, W^i(x), \\ V_t^i(t,x) &= -r\exp[-r(t-t_0)]\, W^i(x), \ \textit{and} \\ V_x^i(t,x) &= \exp[-r(t-t_0)]\, W_x^i(x), \ \textit{for } i \notin N. \end{aligned} \tag{2.58}$$

A feedback Nash equilibrium solution for the infinite-horizon autonomous game (2.56) and (2.57) can be characterized as follows:

Theorem 2.4.1. *An n-tuple of strategies $\left\{u_i^*(s) = \phi_i^*(\cdot) \in U^i;\ \textit{for } i \in N\right\}$ provides a feedback Nash equilibrium solution to the infinite-horizon game (2.56) and (2.57) if there exist continuously differentiable functions $W^i(x) : R^m \to R,\ i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned} &rW^i(x) = \\ &\max_{u_i} \left\{ g^i\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right] \right. \\ &\qquad \left. + W_x^i(x) f\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right] \right\} \\ &= \left\{ g^i[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)] \right. \\ &\qquad \left. + W_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)] \right\}, \qquad \textit{for } i \in N. \end{aligned}$$

Proof. By Theorem 2.1.2, $W^i(x)$ is the value function associated with the optimal control problem of Player i, $i \in N$. Together with the i^{th} expression in Definition 2.4.1, the conditions in Theorem 2.4.1 imply a Nash equilibrium.

Since time s is not explicitly involved the partial differential equations in Theorem 2.4.1, the validity of the feedback Nash equilibrium $\{u_i^* = \phi_i^*(x),$ for $i \in N\}$ are functions independent of time is obtained.

Substituting the game equilibrium strategies in Theorem 2.4.1 into (2.55) yields the game equilibrium dynamics of the state path as:

$$\dot{x}(s) = f\left[x(s), \phi_1^*(x(s)), \phi_2^*(x(s)), \ldots, \phi_n^*(x(s))\right], \qquad x(t_0) = x_0.$$

Solving the above dynamics yields the optimal state trajectory $\{x^*(t)\}_{t \geq t_0}$ as

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \ldots, \phi_n^*(x^*(s))\right] ds,$$
$$\text{for } \; t \geq t_0.$$

We denote term $x^*(t)$ by x_t^*. The feedback Nash equilibrium strategies for the infinite-horizon game (2.54)–(2.55) can be obtained as

$$\left[\phi_1^*(x_t^*), \phi_2^*(x_t^*), \ldots, \phi_n^*(x_t^*)\right], \qquad \text{for } \; t \geq t_0.$$

Following the above analysis and using Theorems 2.1.4 and 2.2.1, we can characterize an open loop equilibrium solution to the infinite-horizon game (2.56) and (2.57) as:

Theorem 2.4.2. *A set of strategies $\{u_i^*(s) = \zeta_i^*(s, x_t),$ for $i \in N\}$ provides an open-loop Nash equilibrium solution to the infinite-horizon game (2.56) and (2.57), and $\{x^*(s), \; t \leq s \leq T\}$ is the corresponding state trajectory, if there exist m costate functions $\Lambda^i(s) : [t, T] \to R^m$, for $i \in N$, such that the following relations are satisfied:*

$$\zeta_i^*(s, x) \equiv u_i^*(s) =$$
$$\arg\max_{u_i \in U^i} \left\{ g^i \left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right.$$
$$\left. + \lambda^i(s) f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right\},$$
$$\dot{x}^*(s) = f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right], \qquad x^*(t) = x_t,$$
$$\dot{\lambda}^i(s) = r\lambda(s) - \frac{\partial}{\partial x^*} \left\{ g^i \left[x^*(s), u_1^*(s), u_2^*(s), \ldots \right.\right.$$
$$\left.\left. \ldots, u_n^*(s)\right] + \lambda^i(s) f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right] \right\},$$
$$\textit{for } \; i \in N.$$

Proof. Consider the i^{th} equality in Theorem 2.4.2, which states that $\upsilon_i^*(s) = u_i^*(s) = \zeta_i^*(s, x_t)$ maximizes

$$\int_{t_0}^{\infty} g^i \left[x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] ds,$$

over the choice of $\upsilon_i(s) \in U^i$ subject to the state dynamics:

$$\dot{x}(s) = f\left[x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right],$$
$$x(t) = x_t, \qquad \textit{for } \ i \in N.$$

This is the infinite-horizon optimal control problem for Player i, since $u_j^*(s)$, for $j \in N$ and $j \neq i$, are open-loop controls and hence do not depend on $u_i^*(s)$. These results are stated in Theorem 2.1.4.

2.4.2 Infinite-Horizon Duopolistic Competition

Consider a dynamic duopoly in which there are two publicly listed firms selling a homogeneous good. Since the value of a publicly listed firm is the present value of its discounted expected future earnings, the terminal time of the game, T, may be very far in the future and nobody knows when the firms will be out of business. Therefore setting $T = \infty$ may very well be the best approximation for the true game horizon. Even if the firm's management restricts itself to considering profit maximization over the next year, it should value its asset positions at the end of the year by the earning potential of these assets in the years to come. There is a lag in price adjustment so the evolution of market price over time is assumed to be a function of the current market price and the price specified by the current demand condition. In particular, we follow Tsutsui and Mino (1990) and assume that

$$\dot{P}(s) = k\left[a - u_1(s) - u_2(s) - P(s)\right], \quad P(t_0) = P_0, \tag{2.59}$$

where $P(s)$ is the market price at time s, $u_i(s)$ is output supplied firm $i \in \{1, 2\}$, current demand condition is specified by the instantaneous inverse demand function $P(s) = [a - u_1(s) - u_2(s)]$, and $k > 0$ represents the price adjustment velocity.

The payoff of firm i is given as the present value of the stream of discounted profits:

$$\int_{t_0}^{\infty} \left\{P(s)\, u_i(s) - cu_i(s) - (1/2)\left[u_i(s)\right]^2\right\} \exp\left[-r(s - t_0)\right] ds,$$
$$\text{for } \ i \in \{1, 2\}, \tag{2.60}$$

where $cu_i(s) + (1/2)\left[u_i(s)\right]^2$ is the cost of producing output $u_i(s)$ and r is the interest rate.

Once again, we consider the alternative game

$$\max_{u_i} \int_{t_0}^{\infty} \left\{P(s)\, u_i(s) - cu_i(s) - (1/2)\left[u_i(s)\right]^2\right\} \exp\left[-r(s - t)\right] ds,$$
$$\text{for } \ i \in \{1, 2\}, \tag{2.61}$$

subject to

$$\dot{P}(s) = k\left[a - u_1(s) - u_2(s) - P(s)\right], \quad P(t) = P. \tag{2.62}$$

The infinite-horizon game (2.61)–(2.62) has autonomous structures and a constant rate. Therefore, we can apply Theorem 2.4.1 to characterize a feedback Nash equilibrium solution as:

$$\begin{aligned} rW^i(P) = \max_{u_i} \Big\{ \Big[Pu_i - cu_i - (1/2)(u_i)^2 \Big] \\ + W_P^i \left[k \left(a - u_i - \phi_j^*(P) - P \right) \right] \Big\}, \quad \text{for } i \in \{1,2\}. \end{aligned} \tag{2.63}$$

Performing the indicated maximization in (2.63), we obtain:

$$\phi_i^*(P) = P - c - kW_P^i(P), \quad \text{for } i \in \{1,2\}. \tag{2.64}$$

Substituting the results from (2.64) into (2.63), and upon solving (2.63) yields:

$$W^i(P) = \frac{1}{2}AP^2 - BP + C, \tag{2.65}$$

where

$$\begin{aligned} A &= \frac{r + 6k - \sqrt{(r+6k)^2 - 12k^2}}{6k^2}, \\ B &= \frac{-akA + c - 2kcA}{r - 3k^2A + 3k}, \quad \text{and} \\ C &= \frac{c^2 + 3k^2B^2 - 2kB(2c+a)}{2r}. \end{aligned}$$

Again, one can readily verify that $W^i(P)$ in (2.65) indeed solves (2.63) by substituting $W^i(P)$ and its derivative into (2.63) and (2.64).

The game equilibrium strategy can then be expressed as:

$$\phi_i^*(P) = P - c - k(AP - B), \quad \text{for } i \in \{1,2\}.$$

Substituting the game equilibrium strategies above into (2.59) yields the game equilibrium state dynamics of the game (2.59)–(2.60) as:

$$\dot{P}(s) = k\left[a - 2(c + kB) - (3 - kA)P(s)\right], \quad P(t_0) = P_0.$$

Solving the above dynamics yields the optimal state trajectory as

$$P^*(t) = \left[P_0 - \frac{k\left[a - 2(c+kB)\right]}{k(3-kA)} \right] \exp\left[-k(3-kA)t\right] + \frac{k\left[a - 2(c+kB)\right]}{k(3-kA)}.$$

We denote term $P^*(t)$ by P_t^*. The feedback Nash equilibrium strategies for the infinite-horizon game (2.59)–(2.60) can be obtained as

$$\phi_i^*(P_t^*) = P_t^* - c - k(AP_t^* - B), \quad \text{for } \{1,2\}.$$

An open loop solution for the game is left for the readers to do in Problem 2.4 in Section 2.8.

2.5 Stochastic Differential Games and their Solutions

One way to incorporate stochastic elements in differential games is to introduce stochastic dynamics. A stochastic formulation for quantitative differential games of prescribed duration involves a vector-valued stochastic differential equation

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]\,ds + \sigma[s, x(s)]\,dz(s),$$
$$x(t_0) = x_0. \tag{2.66}$$

which describes the evolution of the state and N objective functionals

$$E_{t_0}\left\{\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]\,ds + q^i(x(T))\right\},$$
$$\text{for } i \in N, \tag{2.67}$$

with $E_{t_0}\{\cdot\}$ denoting the expectation operation taken at time t_0, $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\,\sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_\varpi] = 0$ and $E[dz_\varpi dt] = 0$ and $E\left[(dz_\varpi)^2\right] = dt$, for $\varpi \in [1, 2, \ldots, \Theta]$; and $E[dz_\varpi dz_\omega] = 0$, for $\varpi \in [1, 2, \ldots, \Theta]$, $\omega \in [1, 2, \ldots, \Theta]$ and $\varpi \neq \omega$. Given the stochastic nature, the information structures must follow the MPS pattern or CLPS pattern or the *feedback perfect state* (FB) pattern in which $\eta^i(s) = \{x(s)\},\ s \in [t_0, T]$.

A Nash equilibrium of the stochastic game (2.66)–(2.67) can be characterized as:

Theorem 2.5.1. *An n-tuple of feedback strategies* $\left\{\phi_i^*(t,x) \in U^i;\ i \in N\right\}$ *provides a Nash equilibrium solution to the game (2.66)–(2.67) if there exist suitably smooth functions* $V^i : [t_0, T] \times R^m \to R$, $i \in N$, *satisfying the semilinear parabolic partial differential equations*

$$\begin{aligned}
-V_t^i - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(t,x)\,V_{x_h x_\zeta}^i &= \\
&\max_{u_i}\Big\{g^i\big[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots \\
&\qquad \ldots, \phi_{i-1}^*(t,x), u_i(t), \phi_{i+1}^*(t,x), \ldots, \phi_n^*(t,x)\big] \\
&+V_x^i(t,x)\,f\big[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots \\
&\qquad \ldots, \phi_{i-1}^*(t,x), u_i(t), \phi_{i+1}^*(t,x), \ldots, \phi_n^*(t,x)\big]\Big\} \\
&= \Big\{g^i[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_n^*(t,x)] \\
&\qquad +V_x^i(t,x)\,f[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_n^*(t,x)]\Big\}, \\
V^i(T,x) = q^i(x), \quad i \in N.
\end{aligned}$$

Proof. This result follows readily from the definition of Nash equilibrium and from Theorem 2.1.5, since by fixing all players' strategies, except the i^{th} one's, at their equilibrium choices (which are known to be feedback by hypothesis), we arrive at a stochastic optimal control problem of the type covered by Theorem 2.1.5 and whose optimal solution (if it exists) is a feedback strategy.

Consider the two-person zero-sum version of the game (2.66)–(2.67) in which the payoff of Player 1 is the negative of that of Player 2. Under either MPS or CLPS information pattern, a Nash equilibrium solution can be characterized as follows.

Theorem 2.5.2. *A pair of strategies $\left\{\phi_i^*(t,x) \in U^i;\ i=1,2\right\}$ provides a feedback saddle-point solution to the two-person zero-sum version of the game (2.66)–(2.67) if there exists a function $V : [t_0,T] \times R^m \to R$ satisfying the partial differential equation:*

$$\begin{aligned}
-V_t - \frac{1}{2}\sum_{h,\zeta=1}^{n} &\Omega^{h\zeta}(t,x)\,V_{x_h x_\zeta} \\
&= \min_{u_1\in S^1}\max_{u_2\in S^2}\left\{g\left[t,x,u_1,u_2\right] + V_x f\left[t,x,u_1,u_2\right]\right\} \\
&= \max_{u_2\in S^2}\min_{u_1\in S^1}\left\{g\left[t,x,u_1,u_2\right] + V_x f\left[t,x,u_1,u_2\right]\right\} \\
&= \left\{g\left[t,x,\phi_1^*(t,x),\phi_2^*(t,x)\right] + V_x f\left[t,x(t)\,\phi_1^*(t,x),\phi_2^*(t,x)\right]\right\}, \\
V(T,x) = q(x)&.
\end{aligned}$$

Proof. This result follows as a special case of Theorem 2.5.1 by taking $n=2$, $g^1(\cdot) = -g^2(\cdot) \equiv g(\cdot)$, and $q^1(\cdot) = -q^2(\cdot) \equiv q(\cdot)$, in which case $V^1 = -V^2 \equiv V$ and existence of a saddle point is equivalent to interchangeability of the min max operations.

Basar (1977a, 1977c and 1980) was first to derive explicit results for stochastic quadratic differential games. Subsequent examples of solvable stochastic differential games include Clemhout and Wan (1985), Kaitala (1993), Jørgensen and Yeung (1996 and 1999), and Yeung (1998, 1999 and 2001).

2.6 An Application of Stochastic Differential Games in Resource Extraction

Consider an economy endowed with a renewable resource and with $n \geq 2$ resource extractors (firms). The lease for resource extraction begins at time t_0 and ends at time T. Let $u_i(s)$ denote the rate of resource extraction of firm i at time s, $i \in N = \{1,2,\ldots,n\}$, where each extractor controls its rate of extraction. Let U^i be the set of admissible extraction rates, and $x(s)$ the size of the resource stock at time s. In particular, we have $U^i \in R^+$ for $x > 0$, and $U^i = \{0\}$ for $x = 0$. The extraction cost for firm $i \in N$ depends on the quantity of resource extracted $u_i(s)$, the resource stock size $x(s)$, and a parameter c.

In particular, extraction cost can be specified as $C^i = cu_i(s)/x(s)^{1/2}$. The market price of the resource depends on the total amount extracted and supplied to the market. The price-output relationship at time s is given by the following downward sloping inverse demand curve $P(s) = Q(s)^{-1/2}$, where $Q(s) = \sum_{i \in N} u_i(s)$ is the total amount of resource extracted and marketed at time s. A terminal bonus $wx(T)$ is offered to each extractor and r is a discount rate which is common to all extractors. Extractor i seeks to maximize the expected payoff:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[\left(\sum_{j=1}^{n} u_j(s)\right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s)\right] \exp\left[-r(t-t_0)\right] ds \right.$$
$$\left. + \exp\left[-r(T-t_0)\right] wx(T)^{1/2}\right\}, \qquad \text{for } i \in N, \tag{2.68}$$

subject to the resource dynamics:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \sum_{j=1}^{n} u_j(s)\right] ds + \sigma x(s)\, dz(s),$$
$$x(t_0) = x_0 \in X. \tag{2.69}$$

Invoking Theorem 2.5.1, a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t,x);\ i \in N\}$ constitutes a Nash equilibrium solution for the game (2.68)–(2.69), if there exist functionals $V^i(t,x) : [t_0, T] \times R \to R$, for $i \in N$, which satisfy the following set of partial differential equations:

$$-V_t^i(t,x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^i(t,x)$$
$$= \max_{u_i \in U^i}\left\{\left[u_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \phi_j^*(t,x) + u_i\right)^{-1/2} - \frac{c}{x^{1/2}} u_i(t)\right] \exp\left[-r(t-t_0)\right]\right.$$
$$\left. + V_x^i\left[ax^{1/2} - bx - \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_j^*(t,x) - u_i\right]\right\}, \text{ and}$$
$$V^i(T,x) = \exp\left[-r(T-t_0)\right] wx^{1/2}. \tag{2.70}$$

Applying the maximization operator on the right-hand-side of the first equation in (2.70) for Player i, yields the condition for a maximum as:

$$\left[\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \phi_j^* (t,x) + \frac{1}{2}\phi_i^* (t,x) \right) \left(\sum_{j=1}^{n} \phi_j^* (t,x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right]$$

$$\times \exp\left[-r\left(t-t_0\right)\right] - V_x^i = 0, \qquad \text{for } \ i \in N. \tag{2.71}$$

Summing over $i = 1, 2, \ldots, n$ in (2.71) yields:

$$\left(\sum_{j=1}^{n} \phi_j^* (t,x) \right)^{1/2} = \left(n - \frac{1}{2} \right) \left(\sum_{j=1}^{n} \left[\frac{c}{x^{1/2}} + \exp\left[r\left(t-t_0\right)\right] V_x^j \right] \right)^{-1}. \tag{2.72}$$

Substituting (2.72) into (2.71) produces

$$\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \phi_j^* (t,x) + \frac{1}{2}\phi_i^* (t,x) \right) \left(n - \frac{1}{2} \right)^{-3} \left(\sum_{j=1}^{n} \left[\frac{c}{x^{1/2}} + \exp\left[r\left(t-t_0\right)\right] V_x^j \right] \right)^{3}$$

$$- \frac{c}{x^{1/2}} - \exp\left[r\left(t-t_0\right)\right] V_x^i = 0, \quad \text{for } \ i \in N. \tag{2.73}$$

Re-arranging terms in (2.73) yields:

$$\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \phi_j^* (t,x) + \frac{1}{2}\phi_i^* (t,x) \right) =$$

$$\left(n - \frac{1}{2} \right)^{3} \frac{\left[c + \exp\left[r\left(t-t_0\right)\right] V_x^i x^{1/2} \right] x}{\left(\sum_{j=1}^{n} \left[c + \exp\left[r\left(t-t_0\right)\right] V_x^j x^{1/2} \right] \right)^{3}},$$

$$\text{for } \ i \in N. \tag{2.74}$$

Condition (2.74) represents a system of equations which is linear in $\{\phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_n^*(t,x)\}$. Solving (2.74) yields:

$$\phi_i^* (t,x) = \frac{x\left(2n-1\right)^2}{2\left[\sum_{j=1}^{n} \left[c + \exp\left[r\left(t-t_0\right)\right] V_x^j x^{1/2} \right] \right]^3}$$

$$\times \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[c + \frac{V_x^j x^{1/2}}{\exp\left[-r\left(t-t_0\right)\right]} \right] - \left(n - \frac{3}{2} \right) \left[c + \frac{V_x^i x^{1/2}}{\exp\left[-r\left(t-t_0\right)\right]} \right] \right\},$$

$$\text{for } \ i \in N. \tag{2.75}$$

Substituting $\phi_i^*(t,x)$ in (2.75) into (2.70) and upon solving yields:

Corollary 2.6.1. *The system (2.70) admits a solution*

$$V^i(t,x) = \exp[-r(t-t_0)]\left[A(t)\,x^{1/2} + B(t)\right], \tag{2.76}$$

where $A(t)$ and $B(t)$ satisfy:

$$\begin{aligned}
\dot{A}(t) &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]A(t) - \frac{(2n-1)}{2n^2}\left(c + \frac{A(t)}{2}\right)^{-1} \\
&\qquad + \frac{c(2n-1)^2}{4n^3}\left(c + \frac{A(t)}{2}\right)^{-2} + \frac{(2n-1)^2 A(t)}{8n^2\left(c + \frac{A(t)}{2}\right)^2}, \\
\dot{B}(t) &= rB(t) - \frac{a}{2}A(t), \\
A(T) &= w, \text{ and} \\
B(T) &= 0.
\end{aligned} \tag{2.77}$$

The first equation in (2.77) can be further reduced to:

$$\begin{aligned}
\dot{A}(t) = &\left\{\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)\frac{[A(t)]^3}{4} + \left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c\,[A(t)]^2\right. \\
&\left. + \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c^2 + \frac{(4n^2 - 8n + 3)}{8n^2}\right]A(t) - \frac{(2n-1)\,c}{4n^3}\right\} \\
&\div \left(c + \frac{A(t)}{2}\right)^2.
\end{aligned} \tag{2.78}$$

The denominator of the right-hand-side of (2.78) is always positive. Denote the numerator of the right-hand-side of (2.78) by:

$$F[A(t)] - \frac{(2n-1)\,c}{4n^3}. \tag{2.79}$$

In particular, $F[A(t)]$ is a polynomial function in $A(t)$ of degree 3. Moreover, $F[A(t)] = 0$ for $A(t) = 0$, and for any $A(t) \in (0,\infty)$,

$$\begin{aligned}
\frac{dF[A(t)]}{dA(t)} = &\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)\frac{3[A(t)]^2}{4} + 2\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c\,[A(t)] \\
&+ \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c^2 + \frac{(4n^2 - 8n + 3)}{8n^2}\right] > 0.
\end{aligned} \tag{2.80}$$

Therefore, there exists a unique level of $A(t)$, denoted by A^* at which

$$F\left[A^*\right] - \frac{(2n-1)\,c}{4n^3} = 0. \tag{2.81}$$

If $A(t)$ equals A^*, $\dot{A}(t) = 0$. For values of $A(t)$ less than A^*, $\dot{A}(t)$ is negative. For values of $A(t)$ greater than A^*, $\dot{A}(t)$ is positive. A phase diagram depicting the relationship between $\dot{A}(t)$ and $A(t)$ is provided in Figure 2.1a, while the time paths of $A(t)$ in relation to A^* are illustrated in Figure 2.1b.

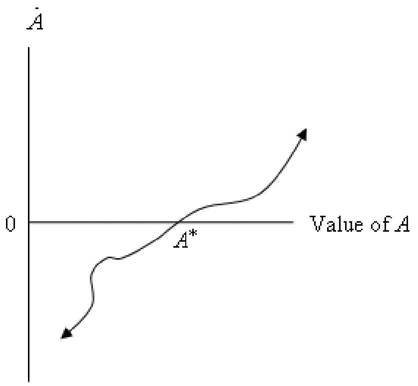

Fig. 2.1a. Phase diagram depicting the relationship between $\dot{A}$ and A.

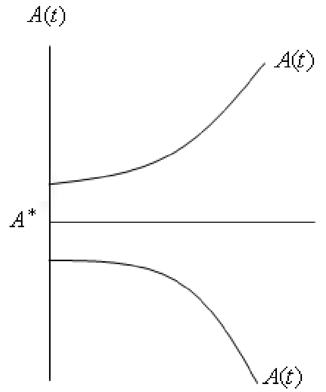

Fig. 2.1b. The time paths of $A(t)$ in relation to A^*.

For a given value of w which is less than A^*, the time path $\{A(t)\}_{t=t_0}^{T}$ will start at a value $A(t_0)$, which is greater than w and less than A^*. The value of $A(t)$ will decrease over time and reach w at time T. On the other hand, for a given value of w which is greater than A^*, the time path $\{A(t)\}_{t=t_0}^{T}$ will start at a value $A(t_0)$, which is less than w and greater than A^*. The value of $A(t)$ will increase over time and reach w at time T. Therefore $A(t)$ is a monotonic function and $A(t) > 0$ for $t \in [t_\tau, T]$.

Using $A(t)$, the solution to $B(t)$ can be readily obtained as:

$$B(t) = \exp(rt)\left(K - \int_{t_0}^{t} \frac{a}{2} A(s) \exp(-rs)\, ds\right), \tag{2.82}$$

where

$$K = \int_{t_0}^{T} \frac{a}{2} A(s) \exp(-rs)\, ds.$$

2.7 Infinite-Horizon Stochastic Differential Games

Consider the infinite-horizon autonomous game problem with constant discounting, in which T approaches infinity and where the objective functions and state dynamics are both autonomous. In particular, the game can be formulated as:

$$\max_{u_i} E_{t_0} \left\{ \int_{t_0}^{\infty} g^i \left[x(s), u_1(s), u_2(s), \ldots, u_n(s) \right] \exp\left[-r(s-t_0) \right] ds \right\},$$
$$\text{for } \; i \in N, \tag{2.83}$$

subject to the stochastic dynamics

$$dx(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s) \right] ds + \sigma \left[x(s) \right] dz(s),$$
$$x(t_0) = x_0. \tag{2.84}$$

Consider the alternative game

$$\max_{u_i} E_t \left\{ \int_t^{\infty} g^i \left[x(s), u_1(s), u_2(s), \ldots, u_n(s) \right] \right.$$
$$\left. \times \exp\left[-r(s-t) \right] ds \middle| x(t) = x_t \right\},$$
$$\text{for } \; i \in N, \tag{2.85}$$

subject to the stochastic dynamics

$$dx(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s) \right] ds + \sigma \left[x(s) \right] dz(s),$$
$$x(t) = x_t. \tag{2.86}$$

The infinite-horizon autonomous game (2.85)–(2.86) is independent of the choice of t and dependent only upon the state at the starting time, that is, x.

Using Theorem 2.1.6 and following the analysis leading to Theorem 2.4.1, a Nash equilibrium solution for the infinite-horizon stochastic differential game (2.85)–(2.86) can be characterized as:

Theorem 2.7.1. *An n-tuple of strategies $\left\{ u_i^*(s) = \phi_i^*(\cdot) \in U^i, \; \text{for } \; i \in N \right\}$ provides a Nash equilibrium solution to the game (2.85)–(2.86) if there exist continuously twice differentiable functions $W^i(x) : R^m \to R$, $i \in N$, satisfying the following set of partial differential equations:*

$$rW^i(x) - \frac{1}{2} \sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(x) W^i_{x_h x_\zeta}(x) =$$
$$\max_{u_i} \left\{ g^i \left[x, \phi_1^*(x), \phi_2^*(x), \ldots \right. \right.$$
$$\left. \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x) \right]$$
$$+ W_x^i(x) f \left[x, \phi_1^*(x), \phi_2^*(x), \ldots \right.$$
$$\left. \left. \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x) \right] \right\}$$
$$= \left\{ g^i \left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x) \right] \right.$$
$$\left. + W_x^i(x) f \left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x) \right] \right\}, \qquad \textit{for } \; i \in N.$$

Proof. This result follows readily from the definition of Nash equilibrium and from Theorem 2.1.6, since by fixing all players' strategies, except the i^{th} one's, at their equilibrium choices (which are known to be feedback by hypothesis), we arrive at a stochastic optimal control problem of the type covered by Theorem 2.1.6 and whose optimal solution (if it exists) is a feedback strategy.

Example 2.7.1. Consider the infinite-horizon game in which extractor i seeks to maximize the expected payoff:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[\left(\sum_{j=1}^{n}u_j(s)\right)^{-1/2}u_i(s)-\frac{c}{x(s)^{1/2}}u_i(s)\right]\exp\left[-r(s-t_0)\right]ds\right\},$$
$$\text{for } i\in N, \tag{2.87}$$

subject to the resource dynamics:

$$dx(s)=\left[ax(s)^{1/2}-bx(s)-\sum_{j=1}^{n}u_j(s)\right]ds+\sigma x(s)\,dz(s),$$
$$x(t_0)=x_0\in X. \tag{2.88}$$

Consider the alternative problem

$$E_{t}\left\{\int_{t}^{T}\left[\left(\sum_{j=1}^{n}u_j(s)\right)^{-1/2}u_i(s)-\frac{c}{x(s)^{1/2}}u_i(s)\right]\exp\left[-r(s-t)\right]ds\right\},$$
$$\text{for } i\in N, \tag{2.89}$$

subject to the resource dynamics:

$$dx(s)=\left[ax(s)^{1/2}-bx(s)-\sum_{j=1}^{n}u_j(s)\right]ds+\sigma x(s)\,dz(s),$$
$$x(t)=x\in X. \tag{2.90}$$

Invoking Theorem 2.7.1 we obtain, a set of feedback strategies $\{\phi_i^*(x),\ i\in N\}$ constitutes a Nash equilibrium solution for the game (2.89)–(2.90), if there exist functionals $V^i(x):R\rightarrow R$, for $i\in N$, which satisfy the following set of partial differential equations:

$$rW^i(x) - \frac{1}{2}\sigma^2 x^2 W^i_{xx}(x) = \max_{u_i \in U^i} \left\{ \left[u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j^*(x) + u_i \right)^{-1/2} - \frac{c}{x^{1/2}} u_i \right] \right.$$
$$\left. + W^i_x \left[a x^{1/2} - bx - \sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j^*(x) - u_i \right] \right\},$$
$$\text{for } \; i \in N, \tag{2.91}$$

Applying the maximization operator in (2.91) for Player i, yields the condition for a maximum as:

$$\left[\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(\sum_{j=1}^{n} \phi_j^*(x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right] - W^i_x = 0,$$
$$\text{for } \; i \in N. \tag{2.92}$$

Summing over $i = 1, 2, \ldots, n$ in (2.92) yields:

$$\left(\sum_{j=1}^{n} \phi_j^*(x) \right)^{1/2} = \left(n - \frac{1}{2} \right) \left(\sum_{j=1}^{n} \left[\frac{c}{x^{1/2}} + W^j_x \right] \right)^{-1}. \tag{2.93}$$

Substituting (2.93) into (2.92) produces

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(n - \frac{1}{2} \right)^{-3} \left(\sum_{j=1}^{n} \left[\frac{c}{x^{1/2}} + W^j_x \right] \right)^{3}$$
$$- \frac{c}{x^{1/2}} - W^i_x = 0, \qquad \text{for } \; i \in N. \tag{2.94}$$

Re-arranging terms in (2.94) yields:

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) = \left(n - \frac{1}{2} \right)^{3} \frac{\left[c + W^i_x x^{1/2} \right] x}{\left(\sum_{j=1}^{n} \left[c + W^j_x x^{1/2} \right] \right)^{3}},$$
$$\text{for } \; i \in N. \tag{2.95}$$

Condition (2.95) represents a system of equations which is linear in $\{\phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)\}$. Solving (2.95) yields:

$$\phi_i^*(x) = \frac{x(2n-1)^2}{2\left[\sum_{j=1}^{n}\left[c+W_x^j x^{1/2}\right]\right]^3}\left\{\sum_{\substack{j=1\\j\neq i}}^{n}\left[c+W_x^j x^{1/2}\right] - \left(n-\frac{3}{2}\right)\left[c+W_x^i x^{1/2}\right]\right\},$$

$$\text{for } i \in N. \tag{2.96}$$

Substituting $\phi_i^*(t,x)$ in (2.96) into (2.91) and upon solving yields:

Corollary 2.7.1. *The system (2.91) admits a solution*

$$W^i(x) - \left[Ax^{1/2} + B\right], \tag{2.97}$$

where A and B satisfy:

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] - \frac{(2n-1)}{2n^2}\left(c+\frac{A}{2}\right)^{-1} + \frac{c(2n-1)^2}{4n^3}\left(c+\frac{A}{2}\right)^{-2} + \frac{(2n-1)^2 A}{8n^2\left(c+\frac{A}{2}\right)^2},$$

$$B = \frac{a}{2r}A. \tag{2.98}$$

2.8 Problems

Problem 2.1. Consider the dynamic optimization problem

$$\int_{t_0}^{T}\left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp\left[-r(s-t_0)\right]ds + \exp\left[-r(T-t_0)\right]qx(T)^{1/2},$$

subject to

$$\dot{x}(s) = \left[ax(s)^{1/2} - bx(s) - u(s)\right], \qquad x(t_0) = x_0 \in X.$$

Use Bellman's techniques of dynamic programming to solve the problem.

Problem 2.2. Consider again the dynamic optimization problem

$$\int_{t_0}^{T}\left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp\left[-r(s-t_0)\right]ds + \exp\left[-r(T-t_0)\right]qx(T)^{1/2},$$

subject to

$$\dot{x}(s) = \left[ax(s)^{1/2} - bx(s) - u(s)\right], \qquad x(t_0) = x_0 \in X.$$

(a) If $c = 1$, $q = 2$, $r = 0.01$, $t_0 = 0$, $T = 5$, $a = 0.5$, $b = 1$ and $x_0 = 20$, use optimal control theory to solve the optimal controls, the optimal state trajectory and the costate trajectory $\{\Lambda(s)\}_{t=t_0}^{T}$.
(b) If $c = 1$, $q = 2$, $r = 0.01$, $t_0 = 0$, $T = 5$, $a = 0.5$, $b = 1$ and $x_0 = 30$, use optimal control theory to solve the optimal controls, the optimal state trajectory and the costate trajectory $\{\Lambda(s)\}_{t=t_0}^{T}$.

Problem 2.3. Consider the infinite-horizon problem

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u(s)^{1/2} - \frac{0.5}{x(s)^{1/2}}u(s)\right]\exp[-0.02(s-t_0)]\,ds\right\},$$

subject to

$$dx(s) = \left[x(s)^{1/2} - 1.5x(s) - u(s)\right]ds + 0.05x(s)\,dz(s), \quad x(t_0) = x_0 = 50.$$

Use the technique of stochastic to solve the problem.

Problem 2.4. Consider the game in Example 2.7.1 in which Player i maximizes

$$\int_{t_0}^{\infty}\left\{P(s)u_i(s) - cu_i(s) - (1/2)[u_i(s)]^2\right\}\exp[-r(s-t_0)]\,ds,$$
$$\text{for } i \in \{1.2\},$$

subject to

$$\dot{P}(s) = k[a - u_1(s) - u_2(s) - P(s)], \qquad P(t_0) = P_0.$$

Derive an open-loop solution to the game.

Problem 2.5. Consider the game

$$\max_{u_i}\left\{\int_0^{10}\left[10u_i(s) - \frac{u_i(s)^2}{x(s)}\right]\exp[-0.05s]\,ds + \exp(-0.5)\,2x(T)\right\},$$
$$\text{for } i \in \{1, 2, \ldots, 6\}$$

subject to

$$\dot{x}(s) = 15 - \frac{1}{2}x(s) - \sum_{j=1}^{6}u_j(s), \qquad x(0) = 25.$$

(a) Obtain an open-loop solution for the game.
(b) Obtain a feedback Nash equilibrium for the game.

Problem 2.6. Consider the following stochastic dynamic advertising game. There are two firms in a market and the expected profit of firm 1 and that of 2 are respectively:

$$E_0 \left\{ \int_0^T \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs)\, ds + \exp(-rT) S_1 x(T) \right\}$$

and

$$E_0 \left\{ \int_0^T \left[q_2 (1 - x(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs)\, ds \right.$$
$$\left. + \exp(-rT) S_2 [1 - x(T)] \right\},$$

where r, q_i, c_i, S_i, for $i \in \{1, 2\}$, are positive constants,
$x(s)$ is the market share of firm 1 at time s, $[1 - x(s)]$ is that of firm 2's,
$u_i(s)$ is advertising rate for firm $i \in \{1, 2\}$.

The dynamics of firm 1's market share is governed by the stochastic differential equation:

$$dx(s) = \left\{ u_1(s) [1 - x(s)]^{1/2} - u_2(s) x(s)^{1/2} \right\} ds + \sigma x(s)\, dz(s),$$
$$x(0) = x_0,$$

where σ is a positive constant and $z(s)$ is a standard Wiener process.

Derive a Nash equilibrium for the above stochastic differential game.

Problem 2.7. Consider an infinite-horizon version of Problem 2.6 in which the expected profit of firm 1 and that of 2 are respectively:

$$E_0 \left\{ \int_0^\infty \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs)\, ds \right\}$$

and

$$E_0 \left\{ \int_0^\infty \left[q_2 (1 - x(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs)\, ds \right\},$$

where $x(s)$ is the market share of firm 1 at time s, and $u_i(s)$ is advertising rate for firm $i \in \{1, 2\}$.

The dynamics of firm 1's market share is governed by the stochastic differential equation:

$$dx(s) = \left\{ u_1(s) [1 - x(s)]^{1/2} - u_2(s) x(s)^{1/2} \right\} ds + \sigma x(s)\, dz(s),$$
$$x(0) = x_0,$$

where σ is a positive constant and $z(s)$ is a standard Wiener process.

Derive a Nash equilibrium for the above infinite-horizon stochastic differential game.

3

Cooperative Differential Games in Characteristic Function Form

The noncooperative games discussed in Chapter 2 fail to reflect all the facets of optimal behavior in an n-person game. In particular, equilibria in noncooperative games do not take into consideration Pareto efficiency or group optimality. In this chapter, we consider cooperative differential games in characteristic function form.

3.1 Cooperative Differential Games in Characteristic Function Form

We begin with the basic formulation of cooperative differential games in characteristic function form and the solution imputations.

3.1.1 Game Formulation

Consider a general N-person differential game in which the state dynamics has the form:

$$\dot{x}(s) = f\left[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)\right], \qquad x(t_0) = x_0, \tag{3.1}$$

The payoff of Player i is:

$$\int_{t_0}^{T} g^i\left[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] ds + q^i(x(T)), \tag{3.2}$$
$$\text{for } \; i \in N = \{1, 2, \ldots, n\},$$

where $x(s) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of Player i, for $i \in N$. In particular, the players' payoffs are transferable. Invoking Theorem 2.2.3 of Chapter 2, a feedback Nash equilibrium solution can be characterized if the players play noncooperatively.

Now consider the case when the players agree to cooperate. Let $\Gamma_c(x_0, T - t_0)$ denote a cooperative game with the game structure of $\Gamma(x_0, T - t_0)$ in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game $\Gamma_c(x_0, T - t_0)$ includes

(i) an agreement on a set of cooperative strategies/controls, and
(ii) a mechanism to distribute total payoff among players.

The solution optimality principle will remain in effect along the cooperative state trajectory path $\{x_s^*\}_{s=t_0}^T$. Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

To fulfill group rationality in the case of transferable payoffs, the players have to maximize the sum of their payoffs:

$$\sum_{j=1}^{N}\left\{\int_{t_0}^{T} g^j\left[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] ds + q^j\left(x(T)\right)\right\}, \tag{3.3}$$

subject to (3.1).

Invoking Pontryagin's Maximum Principle, a set of optimal controls $u^*(s) = \left[u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right]$ can be characterized as in Theorem 2.1.3 of Chapter 2. Substituting this set of optimal controls into (3.1) yields the optimal trajectory $\{x^*(t)\}_{t=t_0}^T$, where

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s, x^*(s) u^*(s)\right] ds, \qquad \text{for } t \in [t_0, T]. \tag{3.4}$$

For notational convenience in subsequent exposition, we use $x^*(t)$ and x_t^* interchangeably.

We denote

$$\sum_{j=1}^{n}\left\{\int_{t_0}^{T} g^j\left[s, x^*(s), u^*(s)\right] ds + q^j\left(x^*(T)\right)\right\}$$

by $v(N; x_0, T - t_0)$. Let $S \subseteq N$ and $v(S; x_0, T - t_0)$ stands for a characteristic function reflecting the payoff of coalition S. The quantity $v(S; x_0, T - t_0)$ yields the maximized payoff to coalition S as the rest of the players form a coalition $N \backslash S$ to play against S. Calling on the super-additivity property of characteristic functions, $v(S; x_0, T - t_0) \geq v(S'; x_0, T - t_0)$ for $S' \subset S \subseteq N$. Hence, it is advantageous for the players to form a maximal coalition N and obtain a maximal total payoff $v(N; x_0, T - t_0)$ that is possible in the game.

3.1.2 Solution Imputation

One of the integral parts of cooperative game is to explore the possibility of forming coalitions and offer an "agreeable" distribution of the total cooperative payoff among players. In fact, the characteristic function framework displays the possibilities of coalitions in an effective manner and establishes a basis for formulating distribution schemes of the total payoffs that are acceptable to participating players.

We can use $\Gamma_v(x_0, T-t_0)$ to denote a *cooperative differential game in characteristic function form.* The optimality principle for a cooperative game in characteristic function form includes

(i) an agreement on a set of cooperative strategies/controls

$$u^*(s) = [u_1^*(s), u_2^*(s), \ldots, u_n^*(s)], \quad \text{for } s \in [t_0, T], \text{ and}$$

(ii) a mechanism to distribute total payoff among players.

A set of payoff distributions satisfying the optimality principle is called a solution imputation to the cooperative game. We will now examine the solutions to $\Gamma_v(x_0, T-t_0)$. Denote by $\xi_i(x_0, T-t_0)$ the share of the player $i \in N$ from the total payoff $v(N; x_0, T-t_0)$.

Definition 3.1.1. *A vector*

$$\xi(x_0, T-t_0) = [\xi_1(x_0, T-t_0), \xi_2(x_0, T-t_0), \ldots, \xi_n(x_0, T-t_0)]$$

that satisfies the conditions:

(i)
$$\xi_i(x_0, T-t_0) \geq v(\{i\}; x_0, T-t_0), \quad \textit{for } i \in N, \textit{ and}$$

(ii)
$$\sum_{j \in N} \xi_j(x_0, T-t_0) = v(N; x_0, T-t_0)$$

is called an imputation of the game $\Gamma_v(x_0, T-t_0)$.

Part (i) of Definition 3.1.1 guarantees individual rationality in the sense that each player receives at least the payoff he or she will get if play against the rest of the players. Part (ii) ensures Pareto optimality and hence group rationality.

Theorem 3.1.1. *Suppose the function $w: 2^n \times R^m \times R^1 \to R^1$ is additive in $S \in 2^n$, that is for any S, $A \in 2^n$, $S \cap A = \emptyset$ we have $w(S \cup A;\ x_0, T-t_0) = w(S; x_0, T-t_0) + w(A; x_0, T-t_0)$. Then in the game $\Gamma_w(x_0, T-t_0)$ there is a unique imputation $\xi_i(x_0, T-t_0) = w(\{i\}; x_0, T-t_0)$, for $i \in N$.*

Proof. From the additivity of w we immediately obtain

$$\begin{aligned} &w(N; x_0, T-t_0) = \\ &\qquad w(\{1\}; x_0, T-t_0) + w(\{2\}; x_0, T-t_0) + \cdots + w(\{n\}; x_0, T-t_0), \end{aligned}$$

Hence Theorem 3.1.1 follows.

Games with additive characteristic functions are called inessential and games with superadditive characteristic functions are called essential. In an essential game $\Gamma_v(x_0, T - t_0)$ there is an infinite set of imputations. Indeed, any vector of the form

$$[v(\{1\}; x_0, T - t_0) + \alpha_1, v(\{2\}; x_0, T - t_0) + \alpha_2, \ldots$$
$$\ldots, v(\{n\}; x_0, T - t_0) + \alpha_n], \tag{3.5}$$

$$\text{for } \alpha_i \geq 0, \quad i \in N \text{ and}$$

$$\sum_{i \in N} \alpha_i = v(N; x_0, T - t_0) - \sum_{i \in N} v(\{i\}; x_0, T - t_0),$$

is an imputation of the game $\Gamma_v(x_0, T - t_0)$. We denote the imputation set of $\Gamma_v(x_0, T - t_0)$ by $E_v(x_0, T - t_0)$.

Definition 3.1.2. *The imputation $\xi(x_0, T - t_0)$ dominates the imputation $\eta(x_0, T - t_0)$ in the coalition S, or $\xi(x_0, T - t_0) \overset{S}{\succ} \eta(x_0, T - t_0)$, if*

(i)

$$\xi_i(x_0, T - t_0) \geq \eta_i(x_0, T - t_0), \quad i \in S; \text{ and}$$

(ii)

$$\sum_{i \in S} \xi_i(x_0, T - t_0) \leq v(S; x_0, T - t_0).$$

The imputation $\xi(x_0, T - t_0)$ is said to dominate the imputation $\eta(x_0, T - t_0)$, or $\xi(x_0, T - t_0) \succ \eta(x_0, T - t_0)$, if there does not exist any $S \subset N$ such that $\eta(x_0, T - t_0) \overset{S}{\succ} \xi(x_0, T - t_0)$ but there exists coalition $S \subset N$ such that $\xi(x_0, T - t_0) \overset{S}{\succ} \eta(x_0, T - t_0)$. It follows from the definition of imputation that domination in single-element coalition and coalition N, is not possible.

Definition 3.1.3. *The set of undominated imputations is called the core of the game $\Gamma_v(x_0, T - t_0)$ and is denoted by $C_v(x_0, T - t_0)$.*

Definition 3.1.4. *The set $L_v(x_0, T - t_0) \subset E_v(x_0, T - t_0)$ is called the Neumann-Morgenstern solution (the NM-solution) of the game $\Gamma_v(x_0, T - t_0)$ if*

(i)

$$\xi(x_0, T - t_0), \eta(x_0, T - t_0) \in L_v(x_0, T - t_0), \text{ implies}$$
$$\xi(x_0, T - t_0) \not\succ \eta(x_0, T - t_0),$$

(ii)

$$\text{for } \eta(x_0, T - t_0) \notin L_v(x_0, T - t_0) \text{ there exists}$$
$$\xi(x_0, T - t_0) \in L_v(x_0, T - t_0) \text{ such that}$$
$$\xi(x_0, T - t_0) \succ \eta(x_0, T - t_0).$$

Note that the NM-solution always contains the core.

Definition 3.1.5. *The vector*

$$\Phi^v(x_0, T - t_0) = \{\Phi_i^v(x_0, T - t_0),\ i = 1, \ldots, n\}$$

is called the Shapley value if it satisfies the following conditions:

$$\Phi_i^v(x_0, T - t_0) =$$
$$\sum_{S \subset N (S \ni i)} \frac{(n-s)!\,(s-1)!}{n!} \left[v(S; x_0, T - t_0) - v(S \backslash i; x_0, T - t_0) \right];$$
$$i = 1, \ldots, n.$$

The components of the Shapley value are the players' shares of the cooperative payoff. The Shapley value is unique and is an imputation (see Shapley (1953)). Unlike the core and NM-solution, the Shapley value represents an optimal distribution principle of the total gain $v(N; x_0, T - t_0)$ and is defined without using the concept of domination.

3.2 Imputation in a Dynamic Context

Section 3.1.2 characterizes the solution imputation at the outset of the game. In dynamic games, the solution imputation along the cooperative trajectory $\{x^*(t)\}_{t=t_0}^T$ would be of concern to the players. In this section, we focus our attention on the dynamic imputation brought about by the solution optimality principle.

Let an optimality principle be chosen in the game $\Gamma_v(x_0, T - t_0)$. The solution of this game constructed in the initial state $x(t_0) = x_0$ based on the chosen principle of optimality contains the solution imputation set $W_v(x_0, T - t_0) \subseteq E_v(x_0, T - t_0)$ and the conditionally optimal trajectory $\{x^*(t)\}_{t=t_0}^T$ which maximizes

$$\sum_{j=1}^n \left\{ \int_{t_0}^T g^j[s, x^*(s), u^*(s)]\, ds + q^j(x^*(T)) \right\}.$$

Assume that $W_v(x_0, T - t_0) \neq \emptyset$.

Definition 3.2.1. *Any trajectory* $\{x^*(t)\}_{t=t_0}^T$ *of the system (3.1) such that*

$$\sum_{j=1}^n \left\{ \int_{t_0}^T g^j[s, x^*(s), u^*(s)]\, ds + q^j(x^*(T)) \right\} = v(N; x_0, T - t_0)$$

is called a conditionally optimal trajectory in the game $\Gamma_v(x_0, T - t_0)$.

Definition 3.2.1 suggests that along the conditionally optimal trajectory the players obtain the largest total payoff. For exposition sake, we assume that such a trajectory exits. Now we consider the behavior of the set $W_v(x_0, T - t_0)$ along the conditionally optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$. Towards this end, in each current state $x^*(t) \equiv x_t^*$ the current subgame $\Gamma_v(x_t^*, T - t)$ is defined as follows. At time t with state $x^*(t)$, we define the characteristic function

$$v(S; x_t^*, T - t) = \begin{cases} 0, & S = \emptyset \\ Val\ \Gamma_S(x_t^*, T - t), & \text{if } S \subset N \\ K_N(x^*(t), u^*(\cdot), T - t) & S = N \end{cases} \tag{3.6}$$

where

$$K_N(x_t^*, u^*(\cdot), T - t) = \sum_{j=1}^{n} \left\{ \int_t^T g^j[s, x^*(s), u^*(s)]\, ds + q^j(x^*(T)) \right\}$$

is the total payoff of the players over the time interval $[t, T]$ along the conditionally optimal trajectory $\{x^*(s)\}_{s=t}^{T}$; and $Val\ \Gamma_S(x_t^*, T - t)$ is the value of the zero-sum differential game $\Gamma_S(x_t^*, T - t)$ between coalitions S and $N \backslash S$ with initial state $x^*(t) \equiv x_t^*$, duration $T - t$ and the S coalition being the maximizer.

The imputation set in the game $\Gamma_v(x_t^*, T - t)$ is of the form:

$$E_v(x_t^*, T - t) = \left\{ \xi \in R^n \,\middle|\, \xi_i \geq v(\{i\}; x_t^*, T - t),\ i = 1, 2, \ldots, n; \right.$$
$$\left. \sum_{i \in N} \xi_i = v(N; x_t^*, T - t) \right\}, \tag{3.7}$$

where

$$v(N; x_t^*, T - t) =$$
$$v(N; x_0, T - t_0) - \sum_{j=1}^{n} \left\{ \int_{t_0}^{t} g^j[s, x^*(s), u^*(s)]\, ds + q^j(x^*(T)) \right\}.$$

The quantity

$$\sum_{j=1}^{n} \left\{ \int_{t_0}^{t} g^j[s, x^*(s), u^*(s)]\, ds + q^j(x^*(T)) \right\}$$

denoted the cooperative payoff of the players over the time interval $[t_0, t]$ along the trajectory $\{x^*(s)\}_{s=t_0}^{T}$.

Consider the family of current games

$$\{\Gamma_v(x_t^*, T - t),\ t_0 \leq t \leq T\},$$

and their solutions $W_v(x_t^*, T - t) \subset E_v(x_t^*, T - t)$ generated by the same principle of optimality that yields the initially solution $W_v(x_0, T - t_0)$.

Lemma 3.2.1. *The set* $W_v(x_T^*, 0)$ *is a solution of the current game* $\Gamma_v(x_T^*, 0)$ *at time* T *and is composed of the only imputation*

$$q(x^*(T)) = \left\{q^1(x^*(T)), q^2(x^*(T)), \ldots, q^n(x^*(T))\right\}$$
$$= \left\{q^1(x_T^*), q^2(x_T^*), \ldots, q^n(x_T^*)\right\}.$$

Proof. Since the game $\Gamma_v(x_T^*, 0)$ *is of zero-duration, then for all* $i \in N$, $v(\{i\}; x_T^*, 0) = q^i(x_T^*)$. *Hence*

$$\sum_{i \in N} v(\{i\}; x_T^*, 0) = \sum_{i \in N} q^i(x_T^*) = v(N; x_T^*, 0),$$

and the characteristic function of the game $\Gamma_v(x_T^*, 0)$ *is additive for* S *and, by Theorem 3.1.1,*

$$E_v(x_T^*, 0) = q(x_T^*) = W_v(x_T^*, 0).$$

This completes the proof of Lemma 3.2.1.

3.3 Principle of Dynamic Stability

Formulation of optimal behaviors for players is a fundamental element in the theory of cooperative games. The players' behaviors satisfying some specific optimality principles constitute a solution of the game. In other words, the solution of a cooperative game is generated by a set of optimality principles (for instance, the Shapley value (1953), the von Neumann Morgenstern solution (1944) and the Nash bargaining solution (1953)). For dynamic games, an additional stringent condition on their solutions is required: the specific optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is known as *dynamic stability or time consistency*. Assume that at the start of the game the players adopt an optimality principle (which includes the consent to maximize the joint payoff and an agreed upon payoff distribution principle). When the game proceeds along the "optimal" trajectory, the state of the game changes and the optimality principle may not be feasible or remain optimal to all players. Then, some of the players will have an incentive to deviate from the initially chosen trajectory. If this happens, instability arises. In particular, the dynamic stability of a solution of a cooperative differential game is the property that, when the game proceeds along an "optimal" trajectory, at each instant of time the players are guided by the same optimality principles, and yet do not have any ground for deviation from the previously adopted "optimal" behavior throughout the game.

The question of dynamic stability in differential games has been explored rigorously in the past three decades. Haurie (1976) discussed the problem of instability in extending the Nash bargaining solution to differential games.

Petrosyan (1977) formalized mathematically the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1979 and 1982) introduced the notion of "imputation distribution procedure" for cooperative solution. Tolwinski et al. (1986) considered cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosyan and Zenkevich (1996) provided a detailed analysis of dynamic stability in cooperative differential games. In particular, the method of regularization was introduced to construct time consistent solutions. Yeung and Petrosyan (2001) designed a time consistent solution in differential games and characterized the conditions that the allocation distribution procedure must satisfy. Petrosyan (2003) used regularization method to construct time consistent bargaining procedures.

3.4 Dynamic Stable Solutions

Let there exist solutions $W_v(x_t^*, T-t) \neq \emptyset$, $t_0 \leq t \leq T$ along the conditionally optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$. If this condition is not satisfied, it is impossible for the players to adhere to the chosen principle of optimality, since at the very first instant t, when $W_v(x_t^*, T-t) \neq \emptyset$, the players have no possibility to follow this principle. Assume that at time t_0 when the initial state x_0 is the players agree on the imputation

$$\begin{aligned}&\xi(x_0, T-t_0) = \\ &\qquad [\xi_1(x_0, T-t_0), \xi_2(x_0, T-t_0), \ldots, \xi_n(x_0, T-t_0)] \in W_v(x_0, T-t_0)\,.\end{aligned}$$

This means that the players agree on an imputation of the gain in such a way that the share of the i^{th} player over the time interval $[t_0, T]$ is equal to $\xi_i(x_0, T-t_0)$. If according to $\xi(x_0, T-t_0)$ Player i is supposed to receive a payoff equaling $\varpi_i[\xi(x_0, T-t_0); x^*(\cdot), t-t_0]$ over the time interval $[t_0, t]$, then over the remaining time interval $[t, T]$ according to the $\xi(x_0, T-t_0)$ Player i is supposed to receive:

$$\begin{aligned}&\eta_i[\xi(x_0, T-t_0); x^*(t), T-t] = \\ &\qquad \xi_i(x_0, T-t_0) - \varpi_i[\xi(x_0, T-t_0); x^*(\cdot), t-t_0]\,.\end{aligned} \tag{3.8}$$

Theorem 3.4.1. *Let $\eta[\xi(x_0, T-t_0); x^*(t), T-t]$ be the vector containing*

$$\eta_i[\xi(x_0, T-t_0); x^*(t), T-t], \quad \textit{for } i \in \{1, 2, \ldots, n\}\,.$$

For the original imputation agreement (that is the imputation $\xi(x_0, T-t_0)$) to remain in force at the instant t, it is essential that the vector

$$\eta[\xi(x_0, T-t_0); x^*(t), T-t] \in W_v(x_t^*, T-t)\,, \tag{3.9}$$

and $\eta[\xi(x_0, T-t_0); x^*(t), T-t]$ is indeed a solution of the current game $\Gamma_v(x_t^*, T-t)$. If such a condition is satisfied at each instant of time $t \in$

$[t_0, T]$ along the trajectory $\{x^* (t)\}_{t=t_0}^{T}$, then the imputation $\xi (x_0, T - t_0)$ is dynamical stable.

Along the trajectory $x^* (t)$ over the time interval $[t, T]$, $t_0 \leq t \leq T$, the coalition N obtains the payoffs

$$v (N; x^* (t), T - t) = \sum_{j=1}^{n} \left\{ \int_t^T g^j [s, x^* (s), u^* (s)] \, ds + q^j (x^* (T)) \right\}. \quad (3.10)$$

Then the difference

$$v (N; x_0, T - t_0) - v (N; x^* (t), T - t) = \sum_{j=1}^{n} \left\{ \int_{t_0}^{t} g^j [s, x^* (s), u^* (s)] \, ds \right\}$$

is the payoff the coalition N obtains on the time interval $[t_0, t]$.

Dynamic stability or time consistency of the solution imputation $\xi (x_0, T - t_0)$ guarantees that the extension of the solution policy to a situation with a later starting time and along the optimal trajectory remains optimal. Moreover, group and individual rationalities are satisfied throughout the entire game interval. A payment mechanism leading to the realization of this imputation scheme must be formulated. This will be done in the next section.

3.5 Payoff Distribution Procedure

A payoff distribution procedure (PDP) proposed by Petrosyan (1997) will be formulated so that the agreed upon dynamically stable imputations can be realized. Let the payoff Player i receive over the time interval $[t_0, t]$ be expressed as:

$$\varpi_i [\xi (x_0 (\cdot), T - t_0); x^* (\cdot), t - t_0] = \int_{t_0}^{t} B_i (s) \, ds, \quad (3.11)$$

where

$$\sum_{j \in N} B_j (s) = \sum_{j \in N} g^j [s, x^* (s), u^* (s)], \quad \text{for } t_0 \leq s \leq t \leq T.$$

From (3.11) we get

$$\frac{d\varpi_i}{dt} = B_i (t). \quad (3.12)$$

This quantity may be interpreted as the instantaneous payoff of the Player i at the moment t. Hence it is clear the vector $B (t) = [B_1 (t), B_2 (t), \ldots, B_n (t)]$ prescribes distribution of the total gain among the members of the coalition N. By properly choosing $B (t)$, the players can ensure the desirable outcome that at each instant $t \in [t_0, T]$ there will be no objection against realization of the original agreement (the imputation $\xi (x_0, T - t_0)$).

Definition 3.5.1. *The imputation* $\xi(x_0, T - t_0) \in W_v(x_0, T - t_0)$ *is dynamically stable in the game* $\Gamma_v(x_0, T - t_0)$ *if the following conditions are satisfied:*

1. *there exists a conditionally optimal trajectory* $\{x^*(t)\}_{t=t_0}^{T}$ *along which* $W_v(x^*(t), T - t) \neq \emptyset,\ \ t_0 \leq t \leq T,$
2. *there exists function* $B(t) = [B_1(t), B_2(t), \ldots, B_n(t)]$ *integrable along* $[t_0, T]$ *such that*

$$\sum_{j \in N} B_j(t) = \sum_{j \in N} g^j[t, x^*(t), u^*(t)] \ \ \textit{for}\ \ t_0 \leq s \leq t \leq T, \ \ \textit{and}$$

$$\xi(x_0, T - t_0) \in \bigcap_{t_0 \leq t \leq T} \left(\varpi\left[\xi(x_0(\cdot), T - t_0); x^*(\cdot), t - t_0\right] \oplus W_v(x^*(t), T - t)\right)$$

where $\varpi[\xi(x_0(\cdot), T - t_0); x^*(\cdot), t - t_0]$ *is the vector of*

$$\varpi_i\left[\xi(x_0(\cdot), T - t_0); x^*(\cdot), t - t_0\right], \ \ \textit{for}\ \ i \in N;$$

and $W_v(x^*(t), T - t)$ *is a solution of the current game* $\Gamma_v(x^*(t), T - t)$*; and the operator* $\oplus$ *defines the operation: for* $\eta \in R^n$ *and* $A \subset R^n$*,* $\eta \oplus A = \{\pi + a |\, a \in A\}$ *.*

The cooperative differential game $\Gamma_v(x_0, T - t_0)$ has a dynamic stable solution $W_v(x_0, T - t_0)$ if all of the imputations $\xi(x_0, T - t_0) \in W_v(x_0, T - t_0)$ are dynamically stable. The conditionally optimal trajectory along which there exists a dynamically stable solution of the game $\Gamma_v(x_0, T - t_0)$ is called an *optimal trajectory.*

From Definition 3.5.1 we have

$$\xi(x_0, T - t_0) \in \left(\varpi\left[\xi(x_0(\cdot), T - t_0); x^*(\cdot), T - t_0\right] \oplus W_v(x^*(T), 0)\right)$$

where $W_v(x^*(T), 0) = q(x^*(T))$ is a solution of the game $\Gamma_v(x^*(T), 0)$. Therefore, we can write

$$\xi(x_0, T - t_0) = \int_{t_0}^{T} B(s)\, ds + q(x^*(T)).$$

The dynamically stable imputation $\xi(x_0, T - t_0) \in W_v(x_0, T - t_0)$ may be realized as follows. From Definition 3.5.1 at any instant $t_0 \leq t \leq T$ we have

$$\xi(x_0, T - t_0) \in \left(\varpi\left[\xi(x_0(\cdot), T - t_0); x^*(\cdot), t - t_0\right] \oplus W_v(x^*(t), T - t)\right), \tag{3.13}$$

where $\varpi[\xi(x_0(\cdot), T - t_0); x^*(\cdot), t - t_0] = \int_{t_0}^{t} B(s)\, ds$ is the payoff vector on the time interval $[t_0, t]$.

Player i's payoff over the same interval can be expressed as:

$$\varpi_i \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] = \int_{t_0}^{t} B\left(s\right) ds.$$

When the game proceeds along the optimal trajectory, over the time interval $[t_0, t]$ the players share the total gain $\int_{t_0}^{t} \sum_{j \in N} g^j \left[s, x^*\left(s\right), u^*\left(s\right)\right] ds$ so that the inclusion

$$\xi \left(x_0, T - t_0\right) - \varpi \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] \in W_v \left(x^* \left(t\right), T - t\right) \quad (3.14)$$

is satisfied. Condition (3.14) implies the existence of a vector $\xi \left(x_t^*, T - t\right) \in W_v \left(x^* \left(t\right), T - t\right)$ satisfying

$$\xi \left(x_0, T - t_0\right) = \varpi \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] + \xi \left(x_t^*, T - t\right).$$

Thus in the process of choosing $B\left(s\right)$, the vector of the gains to be obtained by the players at the remaining game interval $[t, T]$ has to satisfy:

$$\begin{aligned} \xi \left(x_t^*, T - t\right) &= \xi \left(x_0, T - t_0\right) - \varpi \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] \\ &= \int_{t}^{T} B\left(s\right) ds + q\left(x^*\left(T\right)\right) \end{aligned}$$

where

$$\sum_{j \in N} B_j\left(s\right) = \sum_{j \in N} g^j \left[s, x^*\left(s\right), u^*\left(s\right)\right] ds \;\text{ for }\; t \le s \le T, \;\text{ and}$$

$$\xi \left(x_t^*, T - t\right) \in W_v \left(x^* \left(t\right), T - t\right).$$

By varying the vector $\varpi \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right]$ restricted by the condition

$$\sum_{j \in N} \varpi_j \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] = \int_{t_0}^{t} \sum_{j \in N} g^j \left[s, x^*\left(s\right), u^*\left(s\right)\right] ds$$

the players ensure displacement of the set

$$\left(\varpi \left[\xi \left(x_0 \left(\cdot\right), T - t_0\right); x^* \left(\cdot\right), t - t_0\right] \oplus W_v \left(x^* \left(t\right), T - t\right)\right)$$

in such a way that (3.13) is satisfied.

For any vector $B\left(\tau\right)$ satisfying condition (3.13) and (3.14) at each time instant $t_0 \le t \le T$ the players are guided by the same optimality principle that leads to the imputation $\xi \left(x_t^*, T - t\right) \in W_v \left(x^* \left(t\right), T - t\right)$ throughout the game, and hence the players have no reason to dispute the previous agreement. In general, it is fairly easy to see that the vectors $B\left(\tau\right)$ satisfying conditions (3.13) and (3.14) may not be unique. Thus there exist multiple sharing methods satisfying the condition of dynamic stability.

Dynamic instability of the solution of the cooperative differential game leads to abandonment of the original optimality principle generating this solution, because none of the imputations from the solution set $W_v(x_0, T - t_0)$ remain optimal until the game terminates. Therefore, the set $W_v(x_0, T - t_0)$ may be called a solution to the game $\Gamma_v(x_0, T - t_0)$ only if it is dynamically stable. Otherwise the game $\Gamma_v(x_0, T - t_0)$ is assumed to have no solution.

3.6 An Analysis in Pollution Control

Consider the pollution model in Petrosyan and Zaccour (2003). Let N denote the set of countries involved in the game of emission reduction. Emission of player $i \in \{1, 2, \ldots, n\} = N$ at time $t\,(t \in [0, \infty))$ is denoted by $m_i(t)$. Let $x(t)$ denote the stock of accumulated pollution by time t. The evolution of this stock is governed by the following differential equation:

$$\frac{dx(t)}{dt} = \dot{x}(t) = \sum_{i \in I} m_i(t) - \delta x(t), \qquad \text{given } x(0) = x_0, \tag{3.15}$$

where δ denotes the natural rate of pollution absorption.

Each player seeks to minimize a stream of discounted sum of emission reduction cost and damage cost. The latter depends on the stock of pollution. We omit from now on the time argument when no ambiguity may arise, $C_i(m_i)$ denotes the emission reduction cost incurred by country i when limiting its emission to level m_i, and $D_i(x)$ its damage cost. We assume that both functions are continuously differentiable and convex, with $C'(m_i) < 0$ and $D'(x) > 0$. The optimization problem of country i is

$$\min_{m_i} J^i(m, x) = \int_0^\infty \exp(-rs)\{C_i(m_i(s)) + D_i(x(s))\}\, ds \tag{3.16}$$

subject to (3.15), where $m = (m_1, m_2, \ldots, m_n)$ and r is the common social discount rate.

This formulation is chosen due to the following motivations. First, a simple formulation of the ecological economics problem is chosen to put the emphasis on the cost sharing issue and the mechanism for allocating the total cost over time in a desirable manner. Second, obviously this model still captures one of the main ingredients of the problem, that is, each player's cost depends on total emissions and on inherited stock of pollution. Third, the convexity assumption and the sign assumption for the first derivatives of the two cost functions seem natural. Indeed, the convexity of $C_i(e_i)$ means that the marginal cost of reduction emissions is higher for lower levels of emissions (see Germain et al. (1998) for a full discussion). Fourth, for the sake of mathematical tractability it is assumed that all countries discount their costs at the same rate. Finally, again for tractability, it is worth noticing that this formulation implies that reduction of pollution can only be achieved by reducing emissions. For a case where pollution reduction can also be done through cleaning activities, one may refer to Krawczyk and Zaccour (1999).

3.6.1 Decomposition Over Time of the Shapley Value

A cooperative game methodology to deal with the problem of sharing the cost of emissions reduction is adopted. The steps are as follows:

(i) Computation of the characteristic function values of the cooperative game.
(ii) Allocation among the players of the total cooperative cost based on the Shapley value.
(iii) Allocation over time of each player's Shapley value having the property of being time-consistent.

The Shapley value is adopted as a solution concept for its fairness and uniqueness merits. The first two steps are classical and provide the individual total cost for each player as a lump sum. The third step aims to allocate over time this total cost in a time-consistent way. The definition and the computation of a time-consistent distribution scheme are dealt with below after introducing some notation.

Let the state of the game be defined by the pair (t, x) and denote by $\Gamma_v(x,t)$ the subgame starting at date t with stock of pollution x. Denote by $x^N(t)$ the trajectory of accumulated pollution under full cooperation (grand coalition N). In the sequel, we use $x^N(t)$ and x_t^N interchangeably. Let $\Gamma_v\left(x_t^N, t\right)$ denote a subgame that starts along the cooperative trajectory of the state. The characteristic function value for a coalition $K \subseteq N$ in subgame $\Gamma_v(x,t)$ is defined to be its minimal cost and is denoted $v(K; x, t)$. With this notation, the total cooperative cost to be allocated among the players is then $v(N; x, 0)$ which is the minimal cost for the grand coalition N and its characteristic function value in the game $\Gamma_v(x,0)$. Let $\Phi^v(x,t) = [\Phi_1^v(x,t), \Phi_2^v(x,t), \ldots, \Phi_n^v(x,t)]$ denote the Shapley value in subgame $\Gamma_v(x,t)$. Finally, denote by $\beta_i(t)$ the cost to be allocated to Player i at instant of time t and $\beta(t) = (\beta_1(t), \ldots, \beta_n(t))$.

Let the vector $B(t) = [B_1(t), B_2(t), \ldots, B_n(t)]$ denote an imputation distribution procedure (IDP) so that

$$\Phi_1^v(x,0) = \int_0^\infty \exp(-rt)\,\beta_i(t)\,dt, \qquad i = 1, \ldots, n. \tag{3.17}$$

The interpretation of this definition is obvious: a time function $B_i(t)$ qualifies as an IDP if it decomposes over time the total cost of Player i as given by the Shapley value component for the whole game $\Gamma_v(x,0)$, i.e. the sum of discounted instantaneous costs is equal to $\Phi_i^v(x,0)$.

The vector $B(t)$ is a time-consistent IDP if at $\left(x_t^N, t\right)$, $\forall t \in [0, \infty)$ the following condition holds:

$$\Phi_i^v(x_0, 0) = \int_0^t \exp(-r\tau)\,\beta_i(\tau)\,d\tau + \exp(-rt)\,\Phi_i^v\left(x_t^N, t\right). \tag{3.18}$$

To interpret condition (3.18), assume that the players wish to renegotiate the initial agreement reached in the game $\Gamma_v(x, 0)$ at (any) intermediate instant of time t. At this moment, the state of the system is $x^N(t)$, meaning that cooperation has prevailed from initial time until t, and that each Player i would have been allocated a sum of discounted stream of monetary amounts given by the first right-hand side term. Now, if the subgame $\Gamma_v\left(x_t^N, t\right)$, starting with initial condition $x(t) = x^N(t)$, is played cooperatively, then Player i will get his Shapley value component in this game given by the second right-hand side term of (3.18). If what he has been allocated until t and what he will be allocated from this date onward sum up to his cost in the original agreement, i.e. his Shapley value $\Phi_i^v(x_0, 0)$, then this renegotiation would leave the original agreement unaltered. If one can find an IDP $B(t) = [B_1(t), B_2(t), \ldots, B_n(t)]$ such that (3.18) holds true, then this IDP is time-consistent. An "algorithm" to build a time-consistent IDP in the case where the Shapley value is differentiable over time is suggested below. One can provide in such a case a simple expression for $B(t) = [B_1(t), B_2(t), \ldots, B_n(t)]$ having an economically appealing interpretation.

3.6.2 A Solution Algorithm

The first three steps compute the necessary elements to define the characteristic function given in the fourth step. In the next two, the Shapley value and the functions $B_i(t)$, $i = 1, 2, \ldots, n$, are computed.

Step 1: Minimize the total cost of the grand coalition.

The grand coalition solves a standard dynamic programming problem consisting of minimizing the sum of all players' costs subject to pollution accumulation dynamics, that is:

$$\min_{m_1, m_2, \ldots, m_n} \sum_{i \in N} \int_t^\infty \exp[-r(\tau - t)]\,\{C_i(m_i(\tau)) + D_i(x(\tau))\}\,d\tau$$

$$\text{s.t. } \dot{x}(s) = \sum_{i \in N} m_i(s) - \delta x(s), \qquad x(t) = x^N(t).$$

Denote by $W(N, x, t)$ the (Bellman) value function of this problem, where the first entry refers to the coalition for which the optimization has been performed, here the grand coalition N. The outcome of this optimization is a vector of emission strategy $m^N\left(x^N(\tau)\right) = \left[m_1^N\left(x^N(\tau)\right), \ldots, m_n^N\left(x^N(\tau)\right)\right]$, where $x^N(\tau)$ refers to the accumulated pollution under the scenario of full cooperation (grand coalition).

Step 2. Compute a feedback Nash equilibrium.

Since the game is played over an infinite time horizon, stationary strategies will be sought. To obtain a feedback Nash equilibrium, assuming differentiability of the value function, the following Isaacs-Bellman equations (in Theorem 2.4.1) must be satisfied

$$r\overline{V}^i(x) = \min_{m_i} \left\{ C_i(m_i) + D_i(x) + \overline{V}^i_x(x) \left[\sum_{i \in I} m_i - \delta x \right] \right\}, \qquad i \in N.$$

Denote by $m^*(x) = [m_1^*(x), m_2^*(x), \ldots, m_n^*(x)]$ any feedback Nash equilibrium of this noncooperative game. This vector can be interpreted as a business-as-usual emission scenario in the absence of an international agreement. It will be fully determined later on with special functional forms. For the moment, we need to stress that from this computation we can get the player's Nash outcome (cost-to-go) in game $\Gamma_v(x_0, 0)$, that we denote $V^i(0, x_0) = \overline{V}^i(x_0)$, and his outcome in subgame $\Gamma_v\left(x_t^N, t\right)$, that we denote $V^i\left(t, x_t^N\right) = \overline{V}^i\left(x_t^N\right)$.

Step 3: Compute outcomes for all remaining possible coalitions.

The optimal total cost for any possible subset of players containing more than one player and excluding the grand coalition (there will be $2^n - n - 2$ subsets) is obtained in the following way. The objective function is the sum of objectives of players in the subset (coalition) considered. In the objective and in the constraints of this optimization problem, we insert for the left-out players the values of their decision variables (strategies) obtained at Step 2, that is their Nash values. Denote by $W(K, x, t)$ the value function for coalition K. This value is formally obtained as follows:

$$W(N, x, t) = \min_{m_i, i \in K} \sum_{i \in K} \left\{ \int_t^\infty \exp\left[-r(\tau - t)\right] \left\{ C_i(m_i(\tau)) + D_i(x(\tau)) \right\} d\tau \right\}$$

$$\text{s.t. } \dot{x}(s) = \sum_{i \in N} m_i(s) - \delta x(s), \qquad x(t) = x^N(t).$$

$$m_j = m_j^N \qquad \text{for } j \in I \backslash K.$$

Step 4: Define the characteristic function.

The characteristic function $v(K; x, t)$ of the cooperative game is defined as follows:

$$v(\{i\}; x, t) = V^i(x, t) = \overline{V}^i(x), \qquad i = 1, \ldots, n;$$
$$v(K; x, t) = W(K; x, t), \qquad K \subseteq I.$$

Step 5: Compute the Shapley value.

Denote by $\Phi^v(x, t) = [\Phi_1^v(x, t), \Phi_2^v(x, t), \ldots, \Phi_n^v(x, t)]$ the Shapley value in game $\Gamma_v(x, t)$. Component i is given by

$$\Phi_i^v(x,t) = \sum_{K \ni i} \frac{(n-k)!\,(k-1)!}{n!} \left[W(K;x,t) - W(K \backslash \{i\};x,t) \right],$$

where k denotes the number of players in coalition K. In particular, if cooperation is in force for the whole duration of the game, then the total cost of Player i would be given by his Shapley value in the game $\Gamma_v(x_0,0)$, that is

$$\Phi_i^v(x_0,0) = \sum_{K \ni i} \frac{(n-k)!\,(k-1)!}{n!} \left[W(K;x_0,0) - W(K \backslash \{i\};x_0,0) \right].$$

Justification for the use of this nonstandard definition of the characteristic function is provided in Section 3.6.3.

Step 6: Define a time consistent IDP.

Allocate to Player i, $i = 1,\dots,n$, at instant of time $t \in [0,\infty)$, the following amount:

$$B_i(t) = \Phi_i^v\left(x_t^N, t\right) - \frac{d}{dt}\Phi_i^v\left(x_t^N, t\right). \tag{3.19}$$

The formula (3.19) allocates at instant of time t to Player i a cost corresponding to the interest payment (interest rate times his cost-to-go under cooperation given by his Shapley value) minus the variation over time of this cost-to-go.

The following proposition shows that $\beta(t) = (\beta_1(t),\dots,\beta_n(t))$, as given by (3.19), is indeed a time-consistent IDP.

Proposition 3.6.1. *The vector* $\beta(t) = (\beta_1(t),\dots,\beta_n(t))$ *where* $\beta_i(t)$ *is given by (3.19) is a time-consistent IDP.*

Proof. We first show that it is an IDP, that is

$$\int_0^\infty \exp(-rt)\,\beta_i(t)\,dt = \Phi_i^v(x_0,0).$$

Multiply (3.19) by the discount factor $\exp(-rt)$ *and integrate*

$$\int_0^\infty \exp(-rt)\,\beta_i(t)\,dt = \int_0^\infty \exp(-rt)\left[r\Phi_i^v\left(x^N,t\right) - \frac{d}{dt}\Phi_i^v\left(x^N,t\right)\right] dt$$
$$= -exp(-rt)\,\Phi_i^v\left(x^N,t\right)\Big|_0^\infty = \Phi_i^v(x_0,0).$$

Repeating the above integration for $\Phi_i^v\left(x_t^N,t\right)$*, one can readily obtain*

$$\Phi_i^v(x_0,0) = \int_0^t \exp(-r\tau)\,\beta_i(\tau)\,d\tau + \exp(-rt)\,\Phi_i^v\left(x_t^N,t\right),$$

which satisfies the time-consistent property.

3.6.3 Rationale for the Algorithm and the Special Characteristic Function

This section discusses the rationale for the algorithm proposed in Section 3.6.2 and the nonstandard definition of the characteristic function adopted.

As Petorsyan and Zaccour pointed out while formulating the solution algorithm, a central element in formal negotiation theories is the status quo, which gives what a player would obtain if negotiation fails. It is a measure of the strategic force of a player when acting alone. The same idea can be extended to subsets of players. To measure the strategic force of a subset of players (coalition), one may call upon the concept of characteristic function, a mathematical tool that is precisely intended to provide such measure. All classical cooperative game solutions (core, the Shapley value, etc.) use the characteristic function to select a subset of imputations that satisfy the requirements embodied in the solution concept adopted. For instance, the core selects the imputations that cannot be blocked by any coalition whereas Shapley value selects one imputation satisfying some axioms, among them fairness. If the set of imputations is not a singleton, the players may negotiate to select one of them. In a dynamic game, the computed imputations usually correspond to the payoffs (here the sum of discounted costs) for the whole duration of the game. In this case, an interesting problem emerges which is how to allocate these amounts over time. One basic requirement is that the distribution over time is feasible, that is, the amounts allocated to each player sum up to his entitled total share (see the definition of an IDP). Obviously, one may construct an infinite number of intertemporal allocations that satisfy this requirement, but not all these streams are conceptually and intuitively appealing. The approach pursued here is to decompose the total individual cost over time so that if the players renegotiate the agreement at any intermediate instant of time along the cooperative state trajectory, then they would obtain the same outcomes.

It is also noted that the computation of the characteristic function values is not standard. The assumption that left-out players $(I \backslash K)$ stick to their feedback Nash strategies when the characteristic function value is computed for coalition K is made. Basically, there are few current options in game theory literature regarding this issue.

A first option is the one offered by Von Neumann and Morgenstern (1944) where they assumed that the left-out players maximize the cost of the considered coalition. This approach, which gives the minimum guaranteed cost, does not seem to be the best one in our context. Indeed, it is unlikely that if a subset of countries form a coalition to tackle an environmental problem, then the remaining countries would form an anti-coalition to harm their efforts. For instance, the Kyoto Protocol permits that countries fulfill jointly their abatement obligation (this is called "joint implementation" in the Kyoto Protocol). The question is then why, e.g., the European Union would wish to maximize abatement cost of say a collation formed of Canada and USA if they wish to take the joint implementation option? We believe that the Von Neumann-Morgenstern determination of the characteristic function has a great historic value and the advantage of being easy to compute but would not suit well the setting we are dealing with.

A second option is to assess the strengths of all nonempty coalitions using a variable or an index which is external to the game under investigation, as it is done in Filar and Gaertner (1997) who use trade flows to construct the characteristic function. This approach is valid to the extent that the main idea behind the characteristic function is to provide a measure of power of all possible coalitions and trade flows are obviously one way of doing it. We wish here to stick to the more traditional endogenous way (e.g., using elements of the game itself) of computing the characteristic function value.

A third option is to assume that the characteristic function value for a coalition is its Nash equilibrium total cost in the noncooperative game between this coalition and the other players acting individually or forming an anti-coalition (see e.g. Germain et al., 1998).

One "problem" with this approach is computation. Indeed, this approach requires solving $2^n - 2$ dynamic equilibrium problems (that is, as many as the number of nonvoid coalitions and excluding the grand one). Here we solve only one equilibrium problem, all others being standard dynamic optimization problems. Therefore, the computational burden is not at all of the same magnitude since solving a dynamic feedback equilibrium problem is much harder than dealing with a dynamic optimization one.

Now, assume that Nash equilirbia exist for all partitions of the set of players (clearly this is far from being automatically guaranteed). First, recall that we aim to compute the Shapley value for Player i. This latter involves his marginal contributions, which are differences between values of the characteristic function of the form $v(K, S, t) - v(K \setminus \{i\}, S, t)$. In the equilibrium approach, these values correspond to Nash outcomes of a game between players in coalition K and the remaining players in $I \setminus K$ (acting individually or collectively is not an issue at this level). If in any of the $2^n - 2$ equilibrium problems that must be solved the equilibrium is not unique, then we face an extremely hard selection problem. In our approach, the coalition computes its value with the assumption that left-out players will continue to adopt a noncooperative emission strategies. In the event that our Step 2 gives multiple equilibria, we could still compute the Shapley value for each of them without having to face a selection problem.

Finally global environmental problems involve by definition all countries around the globe. Although few of them are heavy weights in the sense that their environmental policies can make a difference on the level of pollution accumulation, many countries can be seen as nonatomistic players. It is intuitive to assume that probably these countries will follow their business-as-usual strategy, i.e., by sticking to their Nash emissions, even when some (possibly far away) countries are joining effort.

3.7 Illustration with Specific Functional Forms

Consider the following specification of (3.16) in which

$$C_i(m_i) = \frac{\gamma}{2}\left[m_i - \overline{m}_i\right]^2, \quad 0 \leq m_i \leq \overline{m}_i, \quad \gamma > 0 \text{ and } i \in \{1,2,3\};$$
$$D_i(x) = \pi x, \qquad \pi > 0.$$

Computation of optimal cost of grand coalition (Step 1)

The value function $W(N,x,t)$ must satisfy the Bellman equation:

$$rW(N,x,t) = \min_{m_1,m_2,m_3}\left\{\sum_{i=1}^{3}\left(\frac{\gamma}{2}\left[m_i - \overline{m}_i\right]^2 + \pi x\right) + W_x(N,x,t)\left[\sum_{i=1}^{3} m_i - \delta x\right]\right\}. \tag{3.20}$$

Performing the indicated minimization in (3.20) yields:

$$m_i^N = \overline{m}_i - \frac{1}{\gamma}W_x(N,x,t), \qquad \text{for } i \in \{1,2,3\}.$$

Substituting m_i^N in (3.20) and upon solving yields

$$W(N,x,t) = \overline{W}(N,x) = \frac{3\pi}{r(r+\delta)}\left\{\left[\sum_{i=1}^{3}\overline{m}_i - \frac{3^2\pi}{2\gamma(r+\delta)}\right] + rx\right\}, \text{ and} \tag{3.21}$$

$$m_i^N = \overline{m}_i - \frac{3\pi}{\gamma(r+\delta)}, \text{ for } i \in \{1,2,3\}. \tag{3.22}$$

The optimal trajectory of the stock of pollution can be obtained as:

$$x^N(t) = \exp(-\delta t)\,x(0) + \frac{1}{\delta}\left\{\left[\sum_{i=1}^{3} m_i^N\right]\left[1 - \exp(-\delta t)\right]\right\}. \tag{3.23}$$

Computation of feedback Nash equilibrium (Step 2)

To solve a feedback Nash equilibrium for the noncooperative game (3.15)–(3.16), we follow Theorem 2.4.1. and obtain the Bellman equation:

$$r\overline{V}^i(x) = \min_{m_i}\left\{\frac{\gamma}{2}\left[m_i - \overline{m}_i\right]^2 + \pi x + \overline{V}_x^i(x)\left[\sum_{\substack{j\in[1,2,3]\\ i\neq j}} m_j^* + m_i - \delta x\right]\right\},$$
$$\text{for } i \in \{1,2,3\}. \tag{3.24}$$

Performing the indicated minimization yields

$$m_i^* = \overline{m}_i - \frac{1}{\gamma}\overline{V}_x^i(x), \qquad \text{for } i \in \{1,2,3\}. \tag{3.25}$$

Substituting (3.25) into (3.24) and upon solving yield:

$$\overline{V}^i(x) = \frac{\pi}{r(r+\delta)}\left\{\frac{\pi}{2\gamma(r+\delta)} + \sum_{i=1}^{3}\overline{m}_i - \frac{3\pi}{\gamma(r+\delta)} + rx\right\},$$
$$\text{for } i \in \{1,2,3\}. \tag{3.26}$$

The Nash equilibrium feedback level of emission can then be obtained as:

$$m_i^* = \overline{m}_i^* - \frac{\pi}{\gamma(r+\delta)}, \qquad \text{for } i \in \{1,2,3\}. \tag{3.27}$$

The difference between Nash equilibrium emissions and those obtained for the grand coalition is that player takes into account the sum of marginal damage costs of all coalition members and not only his own one.

Computation of optimal cost for intermediate coalitions (Step 3)

The value function $W(K,x,t)$ for any coalition K of two players must satisfy the following Bellman equation:

$$rW(K,x,t) =$$
$$\min_{m_1, i\in K}\left\{\sum_{i\in K}\left(\frac{\gamma}{2}[m_i - \overline{m}_i]^2 + \pi x\right) + W_x(K,x,t)\left[\sum_{i\in K} m_i + m_j^* - \delta x\right]\right\}, \tag{3.28}$$

where $j \notin K$.

Following similar procedure adopted for solving for the grand coalition, one can obtain:

$$W(K,x,t) = \overline{W}(K,x) =$$
$$\frac{2\pi}{r(r+\delta)}\left\{\sum_{i\in K}\overline{m}_i - \frac{4\pi}{2\gamma(r+\delta)} - \frac{\pi}{\gamma(r+\delta)} + rx\right\}. \tag{3.29}$$

The corresponding emission of coalition K is:

$$m_i^K = \overline{m}_i - \frac{2\pi}{\gamma(r+\delta)}, \qquad i \in K. \tag{3.30}$$

Definition of the characteristic function (Step 4)

The characteristic function values are given by

$$v(\{i\};x,t) = V^i(x,t) = \overline{V}^i(x) =$$

$$\frac{\pi}{r(r+\delta)}\left\{\frac{\pi}{2\gamma(r+\delta)} + \sum_{i=1}^{3}\overline{m}_i - \frac{3\pi}{\gamma(r+\delta)} + rx\right\}, \quad i = 1,2,3;$$

$$v(K;x,t) = W(K,x,t) = \overline{W}(K,x) =$$

$$\frac{2\pi}{r(r+\delta)}\left\{\sum_{i\in K}\overline{m}_i - \frac{4\pi}{2\gamma(r+\delta)} - \frac{\pi}{\gamma(r+\delta)} + rx\right\},$$

$$K \subseteq \{1,2,3\}.$$

Computation of the Shapley value (Step 5)

Assuming symmetric $\overline{m}_i$, the Shapley value of the game can be expressed as:

$$\begin{aligned}\Phi_i^v(x,t) &= \sum_{K\ni i}\frac{(n-k)!\,(k-1)!}{n!}\left[v(K;x,t) - v(K\backslash\{i\};x,t)\right] \\ &= \frac{1}{2r(r+\delta)}\left\{2\pi\left(\sum_{i=1}^{3}\overline{m}_i + \rho S\right) - \frac{9\pi^2}{\gamma(r+\delta)}\right\}, \\ &\qquad i = 1,2,3. \end{aligned} \tag{3.31}$$

Computation of IDP functions (Step 6)

To provide a allocation that sustains the Shapley value $\Phi_i^v(x,t)$ over time along the optimal trajectory $x^N(t)$ in (3.23), we recall from (3.19) that the IDP functions are given by

$$B_i(t) = \Phi_i^v\left(x_t^N,t\right) - \frac{d}{dt}\Phi_i^v\left(x_t^N,t\right).$$

Straightforward calculations lead to

$$B_i(t) = \pi x^N(t) + \frac{9\pi^2}{2\gamma(r+\delta)^2}, \qquad i = 1,2,3. \tag{3.32}$$

To verify that $B_i(t)$ indeed brings about the Shapley value of Player i we note that

$$\begin{aligned}\Phi_i^v\left(x^N,0\right) &= \frac{1}{2r(r+\delta)}\left\{2\pi\left(\sum_{i=1}^{3}\overline{m}_i + rx(0)\right) - \frac{9\pi^2}{\gamma(r+\delta)}\right\}, \\ &\quad i = 1,2,3. \end{aligned} \tag{3.33}$$

Multiply both sides of (3.32) by the discount factor and integrate

$$\int_0^\infty \exp(-rt) B_i(t)\, dt = \int_0^\infty \exp(-rt) \left[\pi x^N(t) + \frac{9\pi^2}{2\gamma (r+\delta)^2} \right] dt,$$
$$i = 1, 2, 3, \tag{3.34}$$

where from (3.22)–(3.23)

$$x^N(t) = \exp(-\delta t)\, x(0) + \frac{1}{\delta} \left\{ \left[\sum_{j=1}^{3} \left(\overline{m}_j - \frac{3\pi}{2\gamma (r+\delta)} \right) \right] [1 - \exp(-\delta t)] \right\}.$$

Substituting $x^N(t)$ into (3.34) yields:

$$\int_0^\infty \exp(-rt)\, \beta_i(t)\, dt =$$
$$\int_0^\infty \exp[-(r+\delta)t]\, \pi x_0 dt + \int_0^\infty \exp(-rt) \frac{\pi}{\delta} \left(\sum_{i=1}^{3} \overline{m}_i - \frac{9\pi}{\gamma (r+\delta)} \right) dt$$
$$+ \int_0^\infty \exp[-(r+\delta)t] \frac{\pi}{\delta} \left(\sum_{i=1}^{3} \overline{m}_i - \frac{9\pi}{\gamma (r+\delta)} \right) dt$$
$$+ \int_0^\infty \exp(-rt) \frac{9\pi^2}{2\gamma (r+\delta)^2} dt.$$

Upon integrating

$$\int_0^\infty \exp(-rt)\, \beta_i(t)\, dt =$$
$$\frac{1}{2r(r+\delta)} \left\{ 2\pi \left(\sum_{i=1}^{3} \overline{m}_i + r x(0) \right) - \frac{9\pi^2}{\gamma (r+\delta)} \right\} = \Phi_i^v(x_0, 0),$$
$$\text{for } i = 1, 2, 3.$$

3.8 Problems

Problem 3.1. Consider a game with the following characteristic functions:

$$v(\{i\}) = 0, \quad \forall i \in \{1,2\};$$
$$v(\{3\}) = 50;$$
$$v(\{1,2\}) = 100; \; v(\{1,3\}) = 200; \; v(\{2,3\}) = 300;$$
$$v(\{1,2,3\}) = 500.$$

(a) Compute the Shapley value
(b) Compute the Shapley value if $v(\{2,3\}) = 200$ instead of 300

Problem 3.2. Consider a game with the following characteristic functions:

$$\begin{aligned}
&v(\{i\}) = 0, \quad \forall i \in \{1,2,3\}; \; v(\{4\}) = 0.15; \\
&v(\{i,j\}) = 0.1, \quad \forall i,j \in \{1,2,3\}, \quad i \neq j; \\
&v(\{i,4\}) = 0.2, \quad \forall i \in \{1,2,3\}; \\
&v(\{i,j,k\}) = 0.5, \quad \forall i,j,k \in \{1,2,3\}, \quad i \neq j \neq k; \\
&v(\{1,2,4\}) = 0.55; \; v(\{1,3,4\}) = 0.6; \; v(\{2,3,4\}) = 0.65; \\
&v(\{1,2,3,4\}) = 1.
\end{aligned}$$

(a) Compute the Shapley value.
(b) Compute the corresponding Shapley value if $v(\{4\}) = 0.2$ instead of 0.15.

Problem 3.3. Consider the case in which the payoffs to different coalition $S \in \{1,2,3,4\}$ are:

$$\begin{aligned}
&v(\{i\}) = w_i, \; i = 1,2,3,4; \\
&v(\{i,j\}) = w_{ij}, \quad i,j \in \{1,2,3,4\}, \quad i < j; \\
&v(\{i,j,k\}) = w_{ijk}, \quad i,j,k \in \{1,2,3,4\}, \quad i < j < k; \\
&v(\{1,2,3,4\}) = W;
\end{aligned}$$

such that

$$\begin{aligned}
&w_i + w_j \leq w_{ij}, \quad \forall i,j \in \{1,2,3,4\}, \quad i \neq j; \\
&w_{ij} + w_k \leq w_{ijk}, \quad \forall i,j,k \in \{1,2,3,4\}, \quad i \neq j \neq k; \\
&w_{ij} + w_{kl} \leq W, \quad \forall i,j,k,l \in \{1,2,3,4\}, \quad i \neq j \neq k \neq l; \\
&w_{ijk} + w_l \leq W, \quad \forall i,j,k,l \in \{1,2,3,4\}, \quad i \neq j \neq k \neq l; \\
&w_i \geq 0, \quad \forall i \in \{1,2,3,4\}.
\end{aligned}$$

Compute the Shapley value.

Problem 3.4. Consider a time-continuous version of the ecological economics model involving three countries. Emission of country $i \in \{1,2,3\}$ at time $t\,(t \in [0,\infty))$ is denoted by $m_i(t)$. Let $x(t)$ denote the stock of accumulated pollution by time t. The evolution of this stock is governed by the following differential equation:

$$\frac{dx(t)}{dt} = \dot{x}(t) = \sum_{i=1}^{3} m_i(t) - \delta x(t), \qquad \text{given } \; x(0) = 100,$$

where $\delta = 0.05$ denotes the natural rate of pollution absorption.

Each country seeks to minimize a stream of discounted sum of emission reduction cost and damage cost.

$$\min_{m_i} J^i(m,x) = \int_0^{\infty} \exp(-rs)\left\{C_i(m_i(s)) + D_i(x(s))\right\} ds,$$

where

$$C_i(m_i) = \frac{1}{2}[m_i - \overline{m}_i]^2, \qquad 0 \le m_i \le \overline{m}_i, \qquad \gamma > 0,$$
$$D_i(x) = x.$$

The countries agree to cooperate and share the cost of emissions reduction based on the Shapley value. Following Petrosyan and Zaccour's way of defining characteristic functions, derive a cooperative solution and its imputation distribution procedures (IDP).

4

Two-person Cooperative Differential Games with Discounting

In this chapter, we consider two-person cooperative differential games with discounting.

4.1 Game Formulation and Noncooperative Outcome

Consider the two-person nonzero-sum differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \in R^m$, with permissible state trajectories $\{x(s), t_0 \leq s \leq T\}$. The state dynamics of the game is characterized by the vector-valued differential equations:

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s)], \quad x(t_0) = x_0. \tag{4.1}$$

At time instant $s \in [t_0, T]$, the instantaneous payoff of Player i, for $i \in \{1, 2\}$, is denoted by $g^i[s, x(s), u_1(s), u_2(s)]$, and, when the game terminates at time T, Player i receives a terminal payment of $q^i(x(T))$. Payoffs are transferable across players and over time. Given a time-varying instantaneous discount rate $r(s)$, for $s \in [t_0, T]$, values received t time after t_0 have to be discounted by the factor $\exp\left[-\int_{t_0}^{t} r(y)\, dy\right]$. Hence at time t_0, the present value of the payoff function of Player i, for $i \in \{1, 2\}$, is given as:

$$\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s)] \exp\left[-\int_{t_0}^{s} r(y)\, dy\right] ds$$
$$+ \exp\left[-\int_{t_0}^{T} r(y)\, dy\right] q^i(x(T)). \tag{4.2}$$

Consider the case when the players act noncooperatively. We use $\Gamma(x_0, T - t_0)$ to denote the game (4.1)–(4.2). A feedback Nash equilibrium solution is sought so that time inconsistency problems are precluded. Invoking Theorem 2.2.3, we can characterize a feedback solution of $\Gamma(x_0, T - t_0)$ as follows.

Theorem 4.1.1. *A set of strategies* $\left\{u_1^{(t_0)*}(t) = \phi_1^{(t_0)*}(t,x),\ u_2^{(t_0)*}(t) = \phi_2^{(t_0)*}(t,x)\right\}$ *constitutes a feedback Nash equilibrium solution to the game* $\Gamma(x_0, T-t_0)$, *if there exist functionals* $V^{(t_0)1}(t,x) : [t_0,T] \times R^m \to R$ *and* $V^{(t_0)2}(t,x) : [t_0,T] \times R^m \to R$, *satisfying the following Isaacs-Bellman equations:*

$$-V_t^{(t_0)i}(t,x) = \max_{u_i}\left\{g^i\left[t,x,u_i,\phi_j^{(t_0)*}(t,x)\right]\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right.$$
$$\left.+V_x^{(t_0)i}(t,x)\,f\left[t,x,u_i,\phi_j^{(t_0)*}(t,x)\right]\right\},$$

and

$$V^{(t_0)i}(T,x) = \exp\left[-\int_{t_0}^{T} r(y)\,dy\right]q^i(x),$$
$$i \in \{1,2\} \quad \textit{and} \quad j \in \{1,2\} \quad \textit{and} \quad j \neq i.$$

The feedback strategies in a Nash equilibrium are *Markovian* in the sense that they are functions of current time t and current state x, and hence independent of other past values of the state (see Basar and Olsder (1995)).

Consider the alternative game $\Gamma(x_\tau, T-\tau)$ with payoff structure (4.1) and dynamics (4.2) starting at time $\tau \in [t_0,T]$ with initial state $x_\tau \in X$. Following Theorem 4.1.1, we denote a set of feedback strategies that constitutes a Nash equilibrium solution to the game $\Gamma(x_\tau, T-\tau)$ by $\left\{\phi_1^{(\tau)*}(t,x), \phi_2^{(\tau)*}(t,x)\right\}$, and the corresponding value function of player $i \in \{1,2\}$ by $V^{(\tau)i}(t,x_t) : [\tau,T] \times R^n \to R$. The functions $V^{(\tau)1}(t,x)$ and $V^{(\tau)2}(t,x)$ then satisfy the equations:

$$-V_t^{(\tau)i}(t,x) = \max_{u_i}\left\{g^i\left[t,x,u_i(t,x),\phi_j^{(\tau)*}(t,x)\right]\exp\left[-\int_{\tau}^{t} r(y)\,dy\right]\right.$$
$$\left.+V_x^{(\tau)i}(t,x)\,f\left[t,x,u_i(t,x),\phi_j^{(\tau)*}(t,x)\right]\right\},$$

and

$$V^{(\tau)i}(T,x) = \exp\left[-\int_{\tau}^{T} r(y)\,dy\right]q^i(x),$$
$$i \in \{1,2\} \quad \text{and} \quad j \in \{1,2\} \quad \text{and} \quad j \neq i. \tag{4.3}$$

Remark 4.1.1. Note that the equilibrium feedback strategies are Markovian in the sense that they depend on current time and current state. Comparing the Bellman-Isaacs equations in (4.3) for different values of $\tau \in [t_0,T]$, one can readily observe that:

$$\phi_i^{(\tau)*}(s,x(s)) = \phi_i^{(t_0)*}(s,x(s)),\ s \in [\tau,T],$$

$$V^{(\tau)i}(\tau, x_\tau) = \exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] V^{(t_0)i}(\tau, x_\tau), \text{ and}$$
$$V^{(t)i}(t, x_t) = \exp\left[\int_{\tau}^{t} r(y)\,dy\right] V^{(\tau)i}(t, x_t),$$
$$\text{for } t_0 \le \tau \le t \le T \text{ and } i \in \{1, 2\}.$$

In a Nash equilibrium of the game $\Gamma(x_\tau, T-\tau)$, the present value of Player i's payoff over the time interval $[t, T]$, for $x(t) = x_t$ and $t \in [\tau, T]$, can be expressed as:

$$V^{(\tau)i}(t, x_t) =$$
$$\int_t^T g^i\left[s, x(s), \phi_1^{(\tau)*}(s, x(s)), \phi_2^{(\tau)*}(s, x(s))\right] \exp\left[-\int_\tau^s r(y)\,dy\right] ds$$
$$+ \exp\left[-\int_\tau^T r(y)\,dy\right] q^i(x(T)) \Bigg| \; x(t) = x_t, \qquad i \in \{1, 2\}.$$

The game equilibrium dynamics can be obtained as:

$$\dot{x}(s) = f\left[s, x(s), \phi_1^{(\tau)*}(s, x(s)), \phi_2^{(\tau)*}(s, x(s))\right], \quad x(t) = x_\tau. \tag{4.4}$$

Example 4.1.1. Consider a resource extraction game, in which two extractors are awarded leases to extract a renewable resource over the time interval $[t_0, T]$. The resource stock $x(s) \in X \subset R$ follows the dynamics:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s), \quad x(t_0) = x_0 \in X, \tag{4.5}$$

where $u_1(s)$ is the harvest rate of extractor 1 and $u_2(s)$ is the harvest rate of extractor 2. The instantaneous payoff at time $s \in [t_0, T]$ for Player 1 and Player 2 are respectively:

$$\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s)\right] \text{ and } \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s)\right],$$

where c_1 and c_2 are constants and $c_1 \neq c_2$.

At time T, each extractor will receive a termination bonus:

$$qx(T)^{1/2},$$

which depends on the resource remaining at the terminal time.

Payoffs are transferable between Player 1 and Player 2 and over time. Given the discount rate r, values received t time after t_0 have to be discounted by the factor $\exp[-r(t-t_0)]$.

At time t_0, the payoff function of Player 1 and Player 2 are respectively:

$$\int_{t_0}^{T} \left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] \exp[-r(t-t_0)]\, ds$$
$$+ \exp[-r(T-t_0)]\, q x(T)^{1/2},$$

and

$$\int_{t_0}^{T} \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \exp[-r(t-t_0)]\, ds$$
$$+ \exp[-r(T-t_0)]\, q x(T)^{1/2}. \tag{4.6}$$

Let $\left[\phi_1^{(t_0)*}(t,x), \phi_2^{(t_0)*}(t,x)\right]$, for $t \in [t_0, T]$ denote a set of strategies that provides a feedback Nash equilibrium solution to the game $\Gamma(x_0, T-t_0)$, and $V^{(t_0)i}(t,x) : [t_0,T] \times R^n \to R$ denote the value function of player $i \in \{1,2\}$ that satisfies the Isaacs-Bellman equations (see Theorem 4.1.1)):

$$-V_t^{(t_0)i}(t,x) = \max_{u_i} \left\{ \left[u_i(t)^{1/2} - \frac{c_i}{x^{1/2}} u_i(t) \right] \exp[-r(t-t_0)] \right.$$
$$\left. + V_x^{(t_0)i}(t,x) \left[a x^{1/2} - bx - u_i(t) - \phi_j^{(t_0)*}(t,x) \right] \right\}, \text{ and}$$
$$V^{(t_0)i}(T,x) = \exp[-r(T-t_0)]\, q x(T)^{1/2},$$
$$\text{for } i \in \{1,2\} \text{ and } j \in \{1,2\} \text{ and } j \neq i. \tag{4.7}$$

Performing the indicated maximization in (4.7) yields:

$$\phi_i^{(t_0)*}(t,x) = \frac{x}{4\left[c_i + V_x^{(t_0)i} \exp[r(t-t_0)]\, x^{1/2}\right]^2},$$
$$\text{for } i \in \{1,2\}. \tag{4.8}$$

Proposition 4.1.1. *The value function of player $i \in \{1,2\}$ in the game $\Gamma(x_0, T-t_0)$ is:*

$$V^{(t_0)i}(t,x) = \exp[-r(t-t_0)] \left[A_i(t)\, x^{1/2} + B_i(t) \right],$$

where for $i, j \in \{1,2\}$ and $i \neq j$, $A_i(t)$, $B_i(t)$, $A_j(t)$ and $B_j(t)$ satisfy:

$$\dot{A}_i(t) = \left[r + \frac{b}{2} \right] A_i(t) - \frac{1}{2[c_i + A_i(t)/2]} + \frac{c_i}{4[c_i + A_i(t)/2]^2}$$
$$+ \frac{A_i(t)}{8[c_i + A_i(t)/2]^2} + \frac{A_i(t)}{8[c_j + A_j(t)/2]^2},$$

$$\dot{B}_i(t) = r B_i(t) - \frac{a}{2} A_i(t) \quad \text{and} \quad A_i(T) = q, \quad \text{and} \quad B_i(T) = 0.$$

Proof. Substituting $\phi_1^{(t_0)*}(t,x)$ and $\phi_2^{(t_0)*}(t,x)$ into (4.7) and upon solving (4.7) one obtains Proposition 4.1.1.

Using the results in Proposition 4.1.1, the game equilibrium strategies can be obtained as:

$$\phi_1^{(t_0)*}(t,x) = \frac{x}{4\left[c_1 + A_1(t)/2\right]^2}, \text{ and } \phi_2^{(t_0)*}(t,x) = \frac{x}{4\left[c_2 + A_2(t)/2\right]^2}. \tag{4.9}$$

Consider the alternative game $\Gamma(x_\tau, T-\tau)$ with payoff structure (4.5) and dynamics (4.6) starting at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$. Following the above analysis, the value function $V^{(\tau)i}(t,x) : [\tau, T] \times R \to R$, for $i \in \{1,2\}$ and $\tau \in [t_0, T]$, for the subgame $\Gamma(x_\tau, T-\tau)$ can be obtained as:

Proposition 4.1.2. *The value function of player $i \in \{1,2\}$ in the game $\Gamma(x_\tau, T-\tau)$ is:*

$$V^{(\tau)i}(t,x) = \exp\left[-r(t-\tau)\right]\left[A_i(t)\,x^{1/2} + B_i(t)\right],$$

where for $i,j \in \{1,2\}$ and $i \neq j$, $A_i(t)$, $B_i(t)$, $A_j(t)$ and $B_j(t)$ are the same as those in Proposition 4.1.1.

Proof. Follow the proof of Proposition 4.1.1.

The Nash equilibrium strategies of Player 1 and Player 2 in the subgame $\Gamma(x_\tau, T-\tau)$ are respectively:

$$\phi_1^{(\tau)*}(t,x) = \frac{x}{4\left[c_1 + A_1(t)/2\right]^2}, \text{ and } \phi_2^{(\tau)*}(t,x) = \frac{x}{4\left[c_2 + A_2(t)/2\right]^2}. \tag{4.10}$$

Note that the conditions in Remark 4.1.1. prevail.

4.2 Cooperative Arrangement

Now consider the case when the players agree to cooperate. Let $\Gamma_c(x_0, T-t_0)$ denote a cooperative game with the game structure of $\Gamma(x_0, T-t_0)$ in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game $\Gamma_c(x_0, T-t_0)$ includes

(i) an agreement on a set of cooperative strategies/controls, and
(ii) a mechanism to distribute total payoff among players.

The solution optimality principle will remain in effect along the cooperative state trajectory path $\{x_s^*\}_{s=t_0}^T$. Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

4.2.1 Group Rationality and Optimal Trajectory

Since payoffs are transferable, group rationality requires the players to maximize their joint payoff. Consider the cooperative game $\Gamma_c(x_0, T-t_0)$. To achieve group rationality, the players have to agree to act so that the sum of the payoffs is maximized. The players must then solve the following optimal control problem:

$$\max_{u_1,u_2}\left\{\int_{t_0}^{T}\sum_{j=1}^{2} g^j\left[s,x(s),u_1(s),u_2(s)\right]\exp\left[-\int_{t_0}^{s} r(y)\,dy\right]ds\right.$$
$$\left.+\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j\left(x(T)\right)\right\}, \tag{4.11}$$

subject to (4.1).

Denote the control problem (4.11) and (4.1) by $\Psi(x_0, T-t_0)$. Both optimal control and dynamic programming can be used to solve the problem. For the sake of comparison with other derived results and expositional convenience, dynamic programming technique is adopted. Using Theorem 2.1.1, we obtain:

Theorem 4.2.1. *A set of controls*

$$\left\{\left[\psi_1^{(t_0)*}(t,x),\psi_2^{(t_0)*}(t,x)\right],\ \textit{for}\ \ t\in[t_0,T]\right\}$$

provides an optimal solution to the control problem $\Psi(x_0, T-t_0)$ *if there exists continuously differentiable function* $W^{(t_0)}(t,x):[t_0,T]\times R^m\to R$ *satisfying the following Bellman equation:*

$$-W_t^{(t_0)}(t,x) =$$
$$\max_{u_1,u_2}\left\{\sum_{j=1}^{2} g^j\left[t,x,u_1,u_2\right]\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]+W_x^{(t_0)}f\left[t,x,u_1,u_2\right]\right\},$$
$$W^{(t_0)}(T,x)=\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j(x).$$

Hence the players will adopt the cooperative control $\left\{\left[\psi_1^{(t_0)*}(t,x),\right.\right.$ $\left.\left.\psi_2^{(t_0)*}(t,x)\right], \text{ for } t \in [t_0,T]\right\}$. In a cooperative framework, the issue of non-uniqueness of the optimal controls can be resolved by agreement between the players on a particular set of controls. Substituting this set of control into (4.1) yields the dynamics of the optimal (cooperative) trajectory as

$$\dot{x}(s) = f\left[s, x(s), \psi_1^{(t_0)*}(s, x(s)), \psi_2^{(t_0)*}(s, x(s))\right], \qquad x(t_0) = x_0. \tag{4.12}$$

Let $x^*(t)$ denote the solution to (4.12). The optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$ can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s, x^*(s), \psi_1^{(t_0)*}(s, x^*(s)), \psi_2^{(t_0)*}(s, x^*(s))\right] ds. \tag{4.13}$$

For notational convenience, we use the terms $x^*(t)$ and x_t^* interchangeably.

The cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the time interval $[t_0,T]$ can be expressed more precisely as:

$$\left\{\left[\psi_1^{(t_0)*}(t, x^*(t)), \psi_2^{(t_0)*}(t, x^*(t))\right], \text{ for } t \in [t_0,T]\right\}. \tag{4.14}$$

Note that for group optimality to be achievable, the cooperative controls (4.14) must be exercised throughout time interval $[t_0,T]$.

To verify whether the players would find it optimal to adopt the cooperative controls (4.14) throughout the cooperative duration, we consider a cooperative game $\Gamma_c(x_\tau^*, T-\tau)$ with dynamics (4.1) and payoffs (4.2) which begins at time $\tau \in [t_0,T]$ and initial state x_τ^*. At time τ, the optimality principle ensuring group rationality requires the players to solve the problem:

$$\max_{u_1,u_2}\left\{\int_\tau^T \sum_{j=1}^{2} g^j[s, x(s), u_1(s), u_2(s)] \exp\left[-\int_\tau^s r(y)\,dy\right] ds\right.$$
$$\left. + \exp\left[-\int_\tau^T r(y)\,dy\right] \sum_{j=1}^{2} q^j(x(T))\right\}, \tag{4.15}$$

subject to $\dot{x}(s) = f[s, x(s), u_1(s), u_2(s)]$, $x(\tau) = x_\tau^*$.

We denote the control problem (4.15) by $\Psi(x_\tau^*, T-\tau)$. Using Theorem 4.2.1 we can obtain the following result. A set of controls $\left\{\left[\psi_1^{(\tau)*}(t,x), \psi_2^{(\tau)*}(t,x)\right],\right.$ $\left.\text{for } t \in [\tau,T]\right\}$ provides an optimal solution to the control problem $\Psi(x_\tau^*, T-\tau)$ if there exists continuously differentiable function $W^{(\tau)}(t,x)$: $[\tau,T] \times R^m \to R$ satisfying the following Bellman equation:

$$-W_t^{(\tau)}(t,x) =$$
$$\max_{u_1,u_2}\left\{\sum_{j=1}^{2} g^j[t,x,u_1,u_2]\exp\left[-\int_\tau^t r(y)\,dy\right] + W_x^{(\tau)} f[t,x,u_1,u_2]\right\},$$

with boundary condition

$$W^{(\tau)}(T,x) = \exp\left[-\int_\tau^T r(y)\,dy\right]\sum_{j=1}^{2} q^j(x).$$

Remark 4.2.1. Comparing the Bellman equations for $\Psi(x_\tau^*, T-\tau)$ and $\Psi(x_0, T-t_0)$, one can readily show that

$$\psi_i^{(\tau)*}(t,x_t^*),\psi_i^{(t_0)*}(t,x_t^*) \text{ at the point } (t,x_t^*), \text{ for } t_0 \le \tau \le t \le T.$$

Invoking Remark 4.2.1, one can show that the cooperative control for the game $\Gamma_c(x_\tau^*, T-\tau)$ over the time interval $[\tau, T]$ is identical to the cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the same period. Therefore, the optimal state trajectory of the game $\Gamma_c(x_\tau^*, T-\tau)$ is a continuation of the optimal state trajectory of the original game $\Gamma_c(x_0, T-t_0)$.

Moreover, along the optimal trajectory $\{x^*(s)\}_{s=t_0}^T$, we have the following result

Remark 4.2.2.

$$W^{(t_0)}(t,x_t^*) =$$
$$\left\{\int_t^T \sum_{j=1}^{2} g^j\left[s,x(s),\psi_1^{(t_0)*}(s,x^*(s)),\psi_2^{(t_0)*}(s,x^*(s))\right]\exp\left[-\int_{t_0}^s r(y)\,dy\right]ds\right.$$
$$\left.+\exp\left[-\int_{t_0}^T r(y)\,dy\right]\sum_{j=1}^{2} q^j(x^*(T))\right\}$$
$$= \exp\left[-\int_{t_0}^t r(y)\,dy\right]$$
$$\left\{\int_t^T \sum_{j=1}^{2} g^j\left[s,x(s),\psi_1^{(\tau)*}(s,x^*(s)),\psi_2^{(\tau)*}(s,x^*(s))\right]\exp\left[-\int_t^s r(y)\,dy\right]ds\right.$$
$$\left.+\exp\left[-\int_t^T r(y)\,dy\right]\sum_{j=1}^{2} q^j(x^*(T))\right\}$$
$$= \exp\left[-\int_{t_0}^t r(y)\,dy\right]W^{(\tau)}(t,x_t^*), \qquad \text{for } i\in\{1,2\}.$$

Example 4.2.1. Consider the optimal control problem $\Psi(x_0, T - t_0)$ which involves the maximization of the sum of the payoffs of Player 1 and Player 2 in Example 4.1.1:

$$\int_{t_0}^{T} \left(\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] \right.$$
$$\left. + \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \right) \exp[-r(t - t_0)] \, ds$$
$$+ 2 \exp[-r(T - t_0)] \, q x(T)^{1/2}, \tag{4.16}$$

subject to (4.5).

Let $\left[\psi_1^{(t_0)*}(t,x), \psi_2^{(t_0)*}(t,x)\right]$ denote a set of controls that provides a solution to the optimal control problem $\Psi(x_0, T - t_0)$, and $W^{(t_0)}(t,x) : [t_0, T] \times R^n \to R$ denote the value function that satisfies the equations (see Theorem 4.2.1):

$$-W_t^{(t_0)}(t,x)$$
$$= \max_{u_1,u_2} \left\{ \left(\left[u_1^{1/2} - \frac{c_1}{x^{1/2}} u_1 \right] + \left[u_2^{1/2} - \frac{c_2}{x^{1/2}} u_2 \right] \right) \exp[-r(t - t_0)] \right.$$
$$\left. + W_x^{(t_0)}(t,x) \left[a x^{1/2} - b x - u_1 - u_2 \right] \right\}, \text{ and}$$
$$W^{(t_0)}(T,x) = 2 \exp[-r(T - t_0)] \, q x^{1/2}. \tag{4.17}$$

Performing the indicated maximization we obtain:

$$\psi_1^{(t_0)*}(t,x) = \frac{x}{4\left[c_1 + W_x^{(t_0)} \exp[r(t - t_0)] \, x^{1/2}\right]^2}, \text{ and}$$
$$\psi_2^{(t_0)*}(t,x) = \frac{x}{4\left[c_2 + W_x^{(t_0)} \exp[r(t - t_0)] \, x^{1/2}\right]^2}.$$

Substituting $\psi_1^{(t_0)*}(t,x)$ and $\psi_2^{(t_0)*}(t,x)$ above into (4.17) yields the value function

$$W^{(t_0)}(t,x) = \exp[-r(t - t_0)] \left[\hat{A}(t) x^{1/2} + \hat{B}(t) \right],$$

where

$$\dot{\hat{A}}(t) = \left[r + \frac{b}{2} \right] \hat{A}(t) - \frac{1}{2\left[c_1 + \hat{A}(t)/2\right]} - \frac{1}{2\left[c_2 + \hat{A}(t)/2\right]}$$
$$+ \frac{c_1}{4\left[c_1 + \hat{A}(t)/2\right]^2} + \frac{c_2}{4\left[c_2 + \hat{A}(t)/2\right]^2}$$

$$+\frac{\hat{A}(t)}{8\left[c_1+\hat{A}(t)/2\right]^2}+\frac{\hat{A}(t)}{8\left[c_2+\hat{A}(t)/2\right]^2},$$

$$\dot{\hat{B}}(t)=r\hat{B}(t)-\frac{a}{2}\hat{A}(t),\ \hat{A}(T)=2q,\ \text{and}\ \hat{B}(T)=0.$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^{(t_0)*}(t,x)=\frac{x}{4\left[c_1+\hat{A}(t)/2\right]^2},\ \text{and}\ \psi_2^{(t_0)*}(t,x)=\frac{x}{4\left[c_2+\hat{A}(t)/2\right]^2}.$$

Substituting these control strategies into (4.5) yields the dynamics of the state trajectory under cooperation:

$$\dot{x}(s)=ax(s)^{1/2}-bx(s)-\frac{x(s)}{4\left[c_1+\hat{A}(s)/2\right]^2}-\frac{x(s)}{4\left[c_2+\hat{A}(s)/2\right]^2},\quad x(t_0)=x_0. \tag{4.18}$$

Solving (4.18) yields the optimal cooperative state trajectory for $\Gamma_c(x_0,T-t_0)$ as:

$$x^*(s)=\varpi(t_0,s)^2\left[x_0^{1/2}+\int_{t_0}^{s}\varpi^{-1}(t_0,t)H_1dt\right]^2,\quad \text{for}\ s\in[t_0,T], \tag{4.19}$$

where

$$\varpi(t_0,s)=\exp\left[\int_{t_0}^{s}H_2(\tau)d\tau\right],H_1=\frac{1}{2}a,\ \text{and}$$

$$H_2(s)=-\left[\frac{1}{2}b+\frac{1}{8\left[c_1+\hat{A}(s)/2\right]^2}+\frac{1}{8\left[c_2+\hat{A}(s)/2\right]^2}\right].$$

The cooperative control for the game $\Gamma_c(x_0,T-t_0)$ over the time interval $[t_0,T]$ along the optimal trajectory can be expressed more precisely as:

$$\psi_1^{(t_0)*}(t,x_t^*)=\frac{x_t^*}{4\left[c_1+\hat{A}(t)/2\right]^2},\ \text{and}\ \psi_2^{(t_0)*}(t,x_t^*)=\frac{x_t^*}{4\left[c_2+\hat{A}(t)/2\right]^2}. \tag{4.20}$$

Following the above analysis, the value function of the optimal control problem $\Psi(x_\tau,T-\tau)$ can be obtained as

$$W^{(\tau)}(t,x)=\exp[-r(t-\tau)]\left[\hat{A}(t)x^{1/2}+\hat{B}(t)\right],$$

and the corresponding optimal controls are

$$\psi_1^{(\tau)*}(t,x) = \frac{x}{4\left[c_1 + \hat{A}(t)/2\right]^2}, \text{ and } \psi_2^{(\tau)*}(t,x) = \frac{x}{4\left[c_2 + \hat{A}(t)/2\right]^2}.$$

Note that the conditions in Remark 4.2.1 prevail.

In the subgame $\Gamma(x_\tau^*, T-\tau)$ along the optimal state trajectory $\{x^*(s)\}_{s=t}^T$, the optimal controls are:

$$\psi_1^{(\tau)*}(t,x_t^*) = \frac{x_t^*}{4\left[c_1 + \hat{A}(t)/2\right]^2}, \text{ and } \psi_2^{(\tau)*}(t,x_t^*) = \frac{x_t^*}{4\left[c_2 + \hat{A}(t)/2\right]^2}.$$

The corresponding optimal trajectory becomes:

$$x^*(s) = \varpi(\tau,s)^2 \left[(x_\tau^*)^2 + \int_\tau^s \varpi^{-1}(\tau,t) H_1 dt\right]^2, \quad \text{for } s \in [\tau, T], \tag{4.21}$$

where

$$\varpi(\tau,s) = \exp\left[\int_\tau^s H_2(\varsigma)\, d\varsigma\right], \quad H_1 = \frac{1}{2}a, \text{ and}$$

$$H_2(s) = -\left[\frac{1}{2}b + \frac{1}{8\left[c_1 + \hat{A}(s)/2\right]^2} + \frac{1}{8\left[c_2 + \hat{A}(s)/2\right]^2}\right].$$

Condition (4.21) is identical to (4.19) in the interval $[\tau, T]$. In fact, the $x^*(s)$ in (4.21) is a subset of that in (4.19). Hence along the optimal path $\{x^*(s)\}_{s=t_0}^T$ group rationality is maintained at every instant $t \in [t_0, T]$.

4.2.2 Individual Rationality

Assume that at time t_0 when the initial state is x_0 the agreed upon optimality principle assigns an imputation vector $\xi(x_0, T-t_0) = \left[\xi^1(x_0, T-t_0),\ \xi^2(x_0, T-t_0)\right]$. This means that the players agree on an imputation of the gain in such a way that the share of the i^{th} player over the time interval $[t_0, T]$ is equal to $\xi^i(x_0, T-t_0)$.

Individual rationality requires that

$$\xi^i(x_0, T-t_0) \geq V^{(t_0)i}(t_0, x_0), \qquad \text{for } i \in \{1,2\}.$$

Using the same optimality principle, at time τ when the state is x_τ^* the same optimality principle assigns an imputation vector $\xi(x_\tau^*, T-\tau) =$

$\left[\xi^1\left(x_\tau^*, T-\tau\right), \xi^2\left(x_\tau^*, T-\tau\right)\right]$. This means that the players agree on an imputation of the gain in such a way that (viewed at time τ) the share of the ith player over the time interval $[\tau, T]$ is equal to $\xi^i\left(x_\tau^*, T-\tau\right)$. Individual rationality is satisfied if:

$$\xi^i\left(x_\tau^*, T-\tau\right) \geq V^{(\tau)i}\left(\tau, x_\tau^*\right), \qquad \text{for } i \in \{1, 2\}.$$

In a dynamic framework, individual rationality has to be maintained at every instant of time $\tau \in [t_0, T]$ along the optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$.

4.3 Dynamically Stable Cooperation and the Notion of Time Consistency

As mentioned in Section 3.3 in Chapter 3, a stringent requirement for solutions of cooperative differential games is time consistency or dynamic stability. In particular, the dynamic stability of a solution of a cooperative differential game is the property that, when the game proceeds along an optimal trajectory, at each instant of time the players are guided by the same optimality principles, and hence do not have any ground for deviation from the previously adopted "optimal" behavior throughout the game. In this section, the notion of time consistency in two-person cooperative differential games with transferable payoffs is presented.

As state before, the solution optimality principle for a cooperative game $\Gamma_c\left(x_0, T-t_0\right)$ includes

(i) an agreement on a set of cooperative strategies/controls and
(ii) a mechanism to distribute total payoff among players.

Consider the cooperative game $\Gamma_c\left(x_0, T-t_0\right)$ in which the players agree to act to maximize their joint payoff and adopt a certain mechanism governing the sharing of the players' payoffs. To achieve group rationality, the players adopt the cooperative controls $\left[\psi_1^{(t_0)*}(t, x), \psi_2^{(t_0)*}(t, x)\right]$ characterized by Theorem 4.2.1. The optimal cooperative state trajectory follows the path $\{x^*(s)\}_{s=t_0}^{T}$ in (4.13).

At time t_0, with the state being x_0, the term $\xi^{(t_0)i}\left(t_0, x_0\right)$ denotes the share/imputation of total cooperative payoff (received over the time interval $[t_0, T]$) to Player i guided by the agreed-upon optimality principle.

Now, consider the cooperative game $\Gamma_c\left(x_\tau^*, T-\tau\right)$ in which the game starts at time $\tau \in [t_0, T]$ with initial state x_τ^*, and the same agreed-upon optimality principle as above is adopted. Let $\xi^{(\tau)i}\left(\tau, x_\tau^*\right)$ denote the share/imputation of total cooperative payoff given to Player i over the time interval $[\tau, T]$.

The vectors $\xi^{(\tau)}\left(\tau, x_\tau^*\right) = \left[\xi^{(\tau)1}\left(\tau, x_\tau^*\right), \xi^{(\tau)2}\left(\tau, x_\tau^*\right)\right]$, for $\tau \in [t_0, T]$, are valid imputations if the following conditions are satisfied.

Definition 4.3.1. *The vector $\xi^{(\tau)}(\tau, x_\tau^*)$ is an imputation of the cooperative game $\Gamma_c(x_\tau^*, T-\tau)$, for $\tau \in [t_0, T]$, if it satisfies:*

(i) $\xi^{(\tau)}(\tau, x_\tau^*) = \left[\xi^{(\tau)1}(\tau, x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*)\right]$, *is a Pareto optimal imputation vector,*

(ii) $\xi^{(\tau)i}(\tau, x_\tau^*) \geq V^{(\tau)i}(\tau, x_\tau^*)$, *for* $i \in \{1, 2\}$.

In particular, part (i) of Definition 4.3.1 ensures Pareto optimality, while part (ii) guarantees individual rationality.

Following Petrosyan (1997) and Yeung and Petrosyan (2004), we formulate a payoff distribution over time so that the agreed imputations can be realized. Let the vectors $B^\tau(s) = [B_1^\tau(s), B_2^\tau(s)]$ denote the instantaneous payoff of the cooperative game at time $s \in [\tau, T]$ for the cooperative game $\Gamma_c(x_\tau^*, T-\tau)$. In other words, Player i, for $i \in \{1, 2\}$, obtains a payoff equaling $B_i^\tau(s)$ at time instant s. A terminal value of $q^i(x_T^*)$ is received by Player i at time T.

In particular, $B_i^\tau(s)$ and $q^i(x_T^*)$ constitute a payoff distribution for the game $\Gamma_c(x_\tau, T-\tau)$ in the sense that $\xi^{(\tau)i}(\tau, x_\tau^*)$ equals:

$$\begin{aligned}\Bigg\{\Bigg(\int_\tau^T B_i^\tau(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds \\ + q^i(x_T^*) \exp\left[-\int_\tau^T r(y)\,dy\right]\Bigg)\Bigg|\; x(\tau) = x_\tau^*\Bigg\}, \qquad (4.22)\\ \text{for } i \in \{1, 2\} \text{ and } \tau \in [t_0, T].\end{aligned}$$

Moreover, for $i \in \{1, 2\}$ and $t \in [\tau, T]$, we use the term $\xi^{(\tau)i}(t, x_t^*)$ which equals

$$\begin{aligned}\Bigg\{\Bigg(\int_t^T B_i^\tau(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds \\ + q^i(x_T^*) \exp\left[-\int_\tau^T r(y)\,dy\right]\Bigg)\Bigg|\; x(t) = x_t^*\Bigg\}, \qquad (4.23)\end{aligned}$$

to denote the present value of Player i' cooperative payoff over the time interval $[t, T]$, given that the state is x_t^* at time $t \in [\tau, T]$, for the game which starts at time τ with state x_τ^*.

Definition 4.3.2. *The vector* $\xi^{(\tau)}(\tau, x_\tau^*) = \left[\xi^{(\tau)1}(\tau, x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*)\right]$, *as defined by (4.22) and (4.23), is a time consistent imputation of* $\Gamma_c(x_\tau^*, T-\tau)$, *for* $\tau \in [t_0, T]$ *if it satisfies* $\xi^{(\tau)}(\tau, x_\tau^*)$ *is a Pareto Optimal imputation vector, for* $t \in [\tau, T]$ *and* $\xi^{(\tau)i}(t, x_t^*) \geq V^{(t)i}(t, x_t^*)$, *for* $i \in \{1, 2\}$, $t \in [\tau, T]$ *and the condition that*

$$\xi^{(\tau)i}(t, x_t^*) = \exp\left[-\int_\tau^t r(y)\,dy\right] \xi^{(t)i}(t, x_t^*),$$
$$\textit{for } \tau \leq t \leq T, \;\; i \in \{1, 2\}.$$

Time consistency as defined in Definition 4.3.2 guarantees that the solution imputations throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and along the optimal trajectory remains optimal. Moreover, group and individual rationality are satisfied throughout the entire game interval.

As pointed out by Jørgensen and Zaccour (2002), conditions ensuring dynamically stable or time consistency of cooperative solutions could be quite stringent and analytically intractable. Though a dynamic imputation principle $\xi^{(\tau)}(\tau, x_\tau^*)$ satisfying Definition 4.3.2 is dynamically stable or time consistent, a payment mechanism leading to the realization of this imputation scheme must be formualted. The recent work of Yeung and Petrosyan (2004) developed a generalized theorem for the derivation of analytically tractable "payoff distribution procedure" of time consistent solution. We shall adopt their work in a deterministic framework. This will be done in the next section.

4.4 Equilibrating Transitory Compensation

4.4.1 Time Consistent Payoff Distribution Procedures

A payoff distribution procedure (PDP) of the cooperative game as specified in (4.22) and (4.23) must be formulated so that the agreed imputations can be realized.

For Definition 4.3.2 to hold, it is required that $B_i^\tau(s) = B_i^t(s)$, for $i \in \{1,2\}$ and $\tau \in [t_0, T]$ and $t \in [t_0, T]$ and $\tau \neq t$. Adopting the notation $B_i^\tau(s) = B_i^t(s) = B_i(s)$ and applying Definition 4.3.2, the PDP of the time consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau^*)$ has to satisfy the following condition.

Corollary 4.4.1. *The PDP with $B(s)$ and $q(x^*(T))$ corresponding to the time consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau^*)$ must satisfy the following conditions:*

(i)
$$\sum_{j=1}^{2} B_i(s) = \sum_{j=1}^{2} g^j\left[s, x_s^*, \psi_1^{(\tau)*}(s, x_s^*), \psi_2^{(\tau)*}(s, x_s^*)\right],$$
$$\textit{for } s \in [t_0, T];$$

(ii)
$$\int_\tau^T B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds + q^i(x^*(T)) \exp\left[-\int_\tau^T r(y)\,dy\right]$$
$$\geq V^{(\tau)i}(\tau, x_\tau^*),$$
$$\textit{for } i \in \{1,2\} \textit{ and } \tau \in [t_0, T]; \textit{ and}$$

(iii)

$$\xi^{(\tau)i}(\tau, x_\tau^*) = \int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds$$
$$+ \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right] \xi^{(\tau+\Delta t)i}(\tau+\Delta t, x_\tau^* + \Delta x_\tau^*),$$
$$\text{for } \tau \in [t_0, T] \text{ and } i \in \{1,2\};$$

where

$$\Delta x_\tau^* = f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right] \Delta t + o(\Delta t),$$

and $o(\Delta t)/\Delta t \to 0$ *as* $\Delta t \to 0$.

Consider the following condition concerning $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$:

Condition 4.4.1. *For $i \in \{1,2\}$ and $t \geq \tau$ and $\tau \in [t_0, T]$, the terms $\xi^{(\tau)i}(t, x_t^*)$ are functions that are continuously twice differentiable in t and x_t^*.*

If the imputations $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$, satisfy Condition 4.4.1, one can obtain the following relationship:

$$\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds$$
$$= \xi^{(\tau)i}(\tau, x_\tau^*) - \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right] \xi^{(\tau+\Delta t)i}(\tau+\Delta t, x_\tau^* + \Delta x_\tau^*)$$
$$= \xi^{(\tau)i}(\tau, x_\tau^*) - \xi^{(\tau)i}(\tau+\Delta t, x_\tau^* + \Delta x_\tau^*),$$
$$\text{for all } \tau \in [t_0, T] \text{ and } i \in \{1,2\}. \tag{4.24}$$

With $\Delta t \to 0$, condition (4.24) can be expressed as

$$B_i(\tau)\Delta t = -\left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right]\Delta t$$
$$-\left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right]\Delta t$$
$$-o(\Delta t). \tag{4.25}$$

Dividing (4.25) throughout by Δt, with $\Delta t \to 0$, yields

$$B_i(\tau) =$$
$$-\left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] - \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right]. \tag{4.26}$$

Therefore, one can establish the following theorem.

Theorem 4.4.1. *If the solution imputations $\xi^{(\tau)i}(\tau, x_\tau^*)$, for $i \in \{1,2\}$ and $\tau \in [t_0, T]$, satisfy Definition 4.3.2 and Condition 4.4.1, a PDP with a terminal payment $q^i(x_T^*)$ at time T and an instantaneous payment at time $\tau \in [t_0, T]$:*

$$B_i(\tau) =$$
$$-\left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] - \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right],$$
$$\text{for } i \in \{1,2\},$$

yields a time consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

4.4.2 Time Consistent Solutions under Specific Optimality Principles

In this section, we consider time consistent solutions under specific optimality principles. Consider a cooperative game $\Gamma_c(x_0, T - t_0)$ in which the players agree to maximize the sum of their payoffs and divide the total cooperative payoff satisfying the Nash bargaining outcome – that is, they maximize the product of individual gains in excess of the noncooperative payoffs. This scheme also coincides with the Shapley value in two-player games. The imputation scheme has to satisfy:

Proposition 4.4.1. *In the game $\Gamma_c(x_0, T - t_0)$, an imputation*

$$\xi^{(t_0)i}(t_0, x_0) = V^{(t_0)i}(t_0, x_0) + \frac{1}{2}\left[W^{(t_0)}(t_0, x_0) - \sum_{j=1}^{2} V^{(t_0)j}(t_0, x_0)\right],$$

is assigned to Player i, for $i \in \{1,2\}$;
and in the subgame $\Gamma_c(x_\tau^*, T - \tau)$, for $\tau \in (t_0, T]$, an imputation

$$\xi^{(\tau)i}(\tau, x_\tau^*) = V^{(\tau)i}(\tau, x_\tau^*) + \frac{1}{2}\left[W^{(\tau)}(\tau, x_\tau^*) - \sum_{j=1}^{2} V^{(\tau)j}(\tau, x_\tau^*)\right],$$

is assigned to Player i, for $i \in \{1,2\}$.

Note that each player will receive a payoff equaling his noncooperative payoff plus half of the gains in excess of noncooperative payoffs over the period $[\tau, T]$, for $\tau \in [t_0, T]$.

One can readily verify that $\xi^{(\tau)i}(\tau, x_\tau^*)$ satisfies Definition 4.3.2. Moreover, employing Remarks 4.1.1 and 4.2.1, one has:

$$\xi^{(t)i}\left(t,x_t^*\right)=$$
$$\exp\left[\int_\tau^t r\left(y\right)dy\right]\left\{V^{(\tau)i}\left(t,x_t^*\right)+\frac{1}{2}\left[W^{(\tau)}\left(t,x_t^*\right)-\sum_{j=1}^{2}V^{(\tau)j}\left(t,x_t^*\right)\right]\right\}$$
$$=\exp\left[\int_\tau^t r\left(y\right)dy\right]\xi^{(\tau)i}\left(t,x_t^*\right),$$
$$\text{for } \ t_0\leq\tau\leq t. \tag{4.27}$$

Hence, $\xi^{(\tau)i}\left(\tau,x_\tau^*\right)$ as in Proposition 4.4.1 is a time consistent imputations for the cooperative game $\Gamma_c\left(x_0,T-t_0\right)$. Using Theorem 4.4.1 one obtains:

Corollary 4.4.2. *A PDP with a terminal payment $q^i\left(x\left(T\right)\right)$ at time T and an instantaneous imputation rate at time $\tau\in[t_0,T]$:*

$$\begin{aligned}B_i\left(\tau\right)=\frac{-1}{2}&\left[\left[V_t^{(\tau)i}\left(t,x_t\right)\Big|_{t=\tau}\right]\right.\\&\left.+\left[V_{x_t}^{(\tau)i}\left(t,x_t\right)\Big|_{t=\tau}\right]f\left[\tau,x_\tau,\psi_1^{(\tau)*}\left(\tau,x_\tau\right),\psi_2^{(\tau)*}\left(\tau,x_\tau\right)\right]\right]\\-\frac{1}{2}&\left[\left[W_t^{(\tau)}\left(t,x_t\right)\Big|_{t=\tau}\right]\right.\\&\left.+\left[W_{x_t}^{(\tau)}\left(t,x_t\right)\Big|_{t=\tau}\right]f\left[\tau,x_\tau,\psi_1^{(\tau)*}\left(\tau,x_\tau\right),\psi_2^{(\tau)*}\left(\tau,x_\tau\right)\right]\right]\\+\frac{1}{2}&\left[\left[V_t^{(\tau)j}\left(t,x_t\right)\Big|_{t=\tau}\right]\right.\\&\left.+\left[V_{x_t}^{(\tau)j}\left(t,x_t\right)\Big|_{t=\tau}\right]f\left[\tau,x_\tau,\psi_1^{(\tau)*}\left(\tau,x_\tau\right),\psi_2^{(\tau)*}\left(\tau,x_\tau\right)\right]\right],\end{aligned}$$
$$\textit{for } \ i,j\in\{1,2\} \ \ \textit{and} \ \ i\neq j, \tag{4.28}$$

yields a time consistent solution to the cooperative game $\Gamma_c\left(x_0,T-t_0\right)$, in which the players agree to divide their cooperative gains according to Proposition 4.4.1.

4.5 An Illustration in Cooperative Resource Extraction

In this section, we illustrate the derivation of PDP of time consistent solutions in which the players agree to divide their cooperative gains according to Proposition 4.4.1 in a resource extraction game. Consider the game in Example 4.1.1 in which the two extractors agree to maximize the sum of their payoffs and divide the total cooperative payoff according to Proposition 4.4.1.

Using the results in Example 4.1.1, Example 4.2.1 and Theorem 4.4.1 we obtain:

Corollary 4.5.1. *A PDP with a terminal payment $q^i\left(x\left(T\right)\right)$ at time T and an instantaneous imputation rate at time $\tau\in[t_0,T]$:*

$$B_i(\tau) = \frac{-1}{2}\Bigg\{ \left(\left[\dot{A}_i(\tau)(x_\tau^*)^{1/2} + \dot{B}_i(\tau) \right] + r\left[A_i(\tau)(x_\tau^*)^{1/2} + B_i(\tau) \right] \right)$$

$$+ \left[\frac{1}{2} A_i(\tau)(x_\tau^*)^{-1/2} \right]$$

$$\times \left[a(x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}(\tau)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}(\tau)/2\right]^2} \right] \Bigg\}$$

$$-\frac{1}{2}\Bigg\{ \left(\left[\dot{\hat{A}}(\tau)(x_\tau^*)^{1/2} + \dot{\hat{B}}(\tau) \right] + r\left[\hat{A}(\tau)(x_\tau^*)^{1/2} + \hat{B}(\tau) \right] \right)$$

$$+ \left[\frac{1}{2} \hat{A}(\tau)(x_\tau^*)^{-1/2} \right]$$

$$\times \left[a(x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}(\tau)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}(\tau)/2\right]^2} \right] \Bigg\}$$

$$+\frac{1}{2}\Bigg\{ \left(\left[\dot{A}_j(\tau)(x_\tau^*)^{1/2} + \dot{B}_j(\tau) \right] + r\left[A_j(\tau)(x_\tau^*)^{1/2} + B_j(\tau) \right] \right)$$

$$+ \left[\frac{1}{2} A_j(\tau)(x_\tau^*)^{-1/2} \right]$$

$$\times \left[a(x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}(\tau)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}(\tau)/2\right]^2} \right] \Bigg\},$$

$$\textit{for } i, j \in \{1, 2\} \quad \textit{and } i \neq j, \tag{4.29}$$

yields a time consistent solution to the cooperative game $\overline{\Gamma}_c(x_0, T - t_0)$ in which the players agree to divide their cooperative gains according to Proposition 4.4.1.

4.6 An Economic Exegesis of Transitory Compensations

In this section, we examine the economic explanation of equilibrating transitory compensation in Theorem 4.4.1. Consider a cooperative scheme $\Gamma_c(x_0, T - t_0)$ in which the players agree to maximize the sum of their payoffs and divide the total cooperative payoff according to a certain imputation mechanism. For instance, the imputation mechanism may be required to satisfy the Nash bargaining scheme as in Section 4.4.2.

In a more general setting, $\xi^{(\tau)i}(\tau, x_\tau^*)$ may be expressed as a function of the cooperative payoff and the individual noncoopeative payoffs. In particular

$$\xi^{(\tau)i}(\tau, x_\tau^*) = \omega^{(\tau)i}\left[W^{(\tau)}(\tau, x_\tau^*), V^{(\tau)i}(\tau, x_\tau^*), V^{(\tau)j}(\tau, x_\tau^*)\right], \text{ and}$$
$$\xi^{(\tau)i}(t, x_t^*) = \omega^{(\tau)i}\left[W^{(\tau)}(t, x_t^*), V^{(\tau)i}(t, x_t^*), V^{(\tau)j}(t, x_t^*)\right],$$
$$\text{for } i \in \{1, 2\}. \tag{4.30}$$

If $\omega^{(\tau)i}(t, x_t^*)$ is continuously differentiable in $W^{(\tau)}(t, x_t^*)$, $V^{(\tau)i}(t, x_t^*)$, and $V^{(\tau)j}(t, x_t^*)$, then Condition 4.4.1 is satisfied because the latter three expressions are continuously differentiable in t and x_t^*. Moreover, $\omega_W^{(\tau)i}(t, x_t^*) \geq 0$, $\omega_{V^i}^{(\tau)i}(t, x_t^*) \geq 0$ and $\omega_{V^j}^{(\tau)i}(t, x_t^*) \leq 0$.

Using Theorem 4.4.1 we obtain the equilibrating transition formula:

Formula 4.6.1.

$$\begin{aligned}
B_i(\tau) = {} & \omega_W^{(\tau)i}(\tau, x_\tau^*) \sum_{j=1}^{2} g^j\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \\
& + \omega_{V^i}^{(\tau)i}(\tau, x_\tau^*) \Big\{ g^i\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \\
& + \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau} \right] \\
& \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \Big\} \\
& + \omega_{V^j}^{(\tau)i}(\tau, x_\tau^*) \Big\{ g^j\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \\
& + \left[\xi_{x_t^*}^{(\tau)j}(t, x_t^*)\Big|_{t=\tau} \right] \\
& \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \Big\}.
\end{aligned} \tag{4.31}$$

Proof. See Appendix to Chapter 4.

Formula 4.6.1 provides the components of the equilibrating transitory compensation in economically interpretable terms. $\omega_W^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of total cooperative payoff that Player i is entitled to received according to agreed upon optimality principle. $\omega_{V^i}^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of his own payoff that Player i is entitled to received according to agreed upon optimality principle. $\omega_{V^j}^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of the other player's payoff that Player i is entitled to received according to agreed upon optimality principle.

The term $\sum_{j=1}^2 g^j[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)]$ is the instantaneous cooperative payoff and $g^i[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)]$ is the instantaneous noncooperative payoffs of Player i. The term

$$\left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right]$$
$$\times\left(f[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)] - f[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)]\right)$$

reflects the instantaneous effect on Player i's noncooperative payoff when the change in the state variable x_τ^* follows the cooperative trajectory governed by (4.13) instead of the noncooperative path (4.4).

Therefore the compensation $B_i(\tau)$ Player i receives at time τ is the sum of

(i) Player i's agreed upon marginal share of total cooperative profit,
(ii) Player i's agreed upon marginal share of his own nonccoperative profit plus the instantaneous effect on his noncooperative payoff when the change in the state variable x_τ^* follows the optimal trajectory instead of the noncooperative path, and
(iii) Player i's agreed upon marginal share of Player j's nonccoperative profit plus the instantaneous effect on Player j's noncooperative payoff when the change in the state variable x_τ^* follows the cooperative trajectory instead of the noncooperative path.

4.7 Infinite-Horizon Cooperative Differential Games

In many game situations, the terminal time of the game, T, is either very far in the future or unknown to the players. For example, the value of a publicly listed firm is the present value of its discounted future earnings. Nobody knows when the firm will be out of business. As argued by Dockner et al. (2000), in this case setting $T = \infty$ may very well be the best approximation for the true game horizon. Important examples of this kind of problems include renewable resources extraction, environmental management, and the pricing of corporate equities. In this section, games with infinite horizon are considered.

Consider the two-person nonzero-sum differential game with objective

$$\int_{t_0}^{\infty} g^i \left[x\left(s\right), u_1\left(s\right), u_2\left(s\right)\right] \exp\left[-r\left(s-t_0\right)\right] ds,$$
$$\text{for } \ i \in \{1,2\} \tag{4.32}$$

and state dynamics

$$\dot{x}\left(s\right) = f\left[x\left(s\right), u_1\left(s\right), u_2\left(s\right)\right], \qquad x\left(t_0\right) = x_0. \tag{4.33}$$

Since s does not appear in $g^i\left[x\left(s\right), u_1\left(s\right), u_2\left(s\right)\right]$ the objective function and the state dynamics, the game (4.32)–(4.33) is an autonomous problem. Consider the alternative game $\Gamma\left(x\right)$:

$$\max_{u_i} \int_{t}^{\infty} g^i \left[x\left(s\right), u_1\left(s\right), u_2\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds,$$
$$\text{for } \ i \in \{1,2\}$$

subject to

$$\dot{x}\left(s\right) = f\left[x\left(s\right), u_1\left(s\right), u_2\left(s\right)\right], \quad x\left(t\right) = x.$$

The infinite-horizon autonomous problem $\Gamma\left(x\right)$ is independent of the choice of t and dependent only upon the state at the starting time, that is x.

Invoking Theorem 2.4.1 a noncooperative feedback Nask equilibrium solution can be characterized as:

Theorem 4.7.1. *A set of strategies $\left\{\phi_1^*\left(x\right), \phi_2^*\left(x\right)\right\}$ constitutes a feedback Nash equilibrium solution to the game $\Gamma\left(x\right)$, if there exist functionals $V^1\left(x\right): R^m \to R$ and $V^2\left(x\right): R^m \to R$, satisfying the following set of partial differential equations:*

$$rV^i\left(x\right) = \max_{u_i} \left\{g^i\left[x, u_i, \phi_j^*\left(x\right)\right] + V_x^i\left(x\right) f\left[x, u_i, \phi_j^*\left(x\right)\right]\right\},$$
$$i \in \{1,2\} \quad \textit{and} \quad j \in \{1,2\} \quad \textit{and} \quad j \neq i.$$

In particular,

$$V^i\left(x\right) = \int_{t}^{\infty} g^i \left[x\left(s\right), \phi_1^*\left(s\right), \phi_2^*\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds$$

represents the current-value payoffs of Player i at current time $t \in [t_0, \infty]$, given that the state is x at t.

Now consider the case when the Players i agree to cooperatively. Let $\Gamma_c\left(x\right)$ denote a cooperative game with the game structure of $\Gamma\left(x\right)$ with the initial state being x. The players agree to act according to an agreed upon optimality principle.

To achieve group rationality the players agree to maximize the sum of their payoffs, that is

$$\max_{u_1,u_2}\left\{\int_t^T\sum_{j=1}^2 g^j\left[x(s),u_1(s),u_2(s)\right]\exp\left[-r(s-t)\right]ds\right\},\tag{4.34}$$

subject to $\dot{x}(s)=f\left[x(s),u_1(s),u_2(s)\right]$ and $x(t)=x$.

Following Theorem 2.1.2, we obtain:

Theorem 4.7.2. *A set of controls $\{[\psi_1^*(x),\psi_2^*(x)]\}$ provides a solution to the optimal control problem associated with $\Gamma_c(x)$ if there exists continuously differentiable function $W(x):R^m\to R$ satisfying the infinite-horizon Bellman equation:*

$$rW(x)=\max_{u_1,u_2}\left\{\sum_{j=1}^2 g^j\left[x,u_1,u_2\right]+W_x f\left[x,u_1,u_2\right]\right\}.$$

Hence the players will adopt the cooperative control $[\psi_1^*(x),\psi_2^*(x)]$ characterized in Theorem 4.7.2. Note that these controls are functions of the current state x only. Substituting this set of control into (4.33) yields the dynamics of the optimal (cooperative) trajectory as;

$$\dot{x}(s)=f\left[x(s),\psi_1^*(x(s)),\psi_2^*(x(s))\right],\qquad x(t)=x.$$

Consider the case at time t_0, where $x(t_0)=x_0$, we have

$$\dot{x}(s)=f\left[x(s),\psi_1^*(x(s)),\psi_2^*(x(s))\right],\qquad x(t_0)=x_0.\tag{4.35}$$

Let $x^*(t)$ denote the solution to (4.35). For notational convenience, we use the terms $x^*(t)$ and x_t^* interchangeably.

Assume that at time $t\,(\geq t_0)$ when the initial state is x_t^* the agreed upon optimality principle assigns an imputation vector $\xi(x_t^*)=\left[\xi^1(x_t^*),\xi^2(x_t^*)\right]$. This means that the players agree on an imputation of the gains in such a way that the share of the ith player is equal to $\xi^i(x_t^*)$.

Individual rationality requires that

$$\xi^i(x_t^*)\geq V^i(x_t^*),\qquad \text{for } i\in\{1,2\}.$$

Following Petrosyan (1997) and Yeung and Petrosyan (2004), we use $B(s)=[B_1(s),B_2(s)]$ denote the instantaneous payoff of the cooperative game at time $s\in[t_0,\infty)$ for the cooperative game $\Gamma_c\left(x_{t_0}^*\right)$.

In particular, along the cooperative trajectory $\{x^*(t)\}_{t\geq t_0}$

$$\xi^i(x_\tau^*)=\int_\tau^\infty B_i(s)\exp\left[-r(s-\tau)\right]ds,\quad \text{for } i\in\{1,2\},\text{ and}$$

$$\xi^i(x_t^*)=\int_t^\infty B_i(s)\exp\left[-r(s-t)\right]ds,\quad \text{for } i\in\{1,2\}\text{ and } t\geq\tau.$$

We further define

$$\gamma^i(\tau;\tau,x_\tau^*) = \int_\tau^\infty B_i(s)\exp[-r(s-\tau)]\,ds = \xi^i(x_\tau^*), \text{ and}$$
$$\gamma^i(\tau;t,x_t^*) = \int_t^\infty B_i(s)\exp[-r(s-\tau)]\,ds,$$
$$\text{for } i \in \{1,2\} \text{ and } \tau \in [t_0,\infty).$$

Note that

$$\begin{aligned}\gamma^i(\tau;t,x_t^*) &= \exp[-r(t-\tau)]\int_t^\infty B_i^\tau(s)\exp[-r(s-t)]\,ds,\\ &= \exp[-r(t-\tau)]\,\xi^i(x_t^*)\\ &= \exp[-r(t-\tau)]\,\gamma^i(t;t,x_t^*),\\ &\quad\text{for } i \in \{1,2\}.\end{aligned} \tag{4.36}$$

The condition in (4.36) guarantees time consistency of the solution imputations throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and along the optimal trajectory remains optimal. Moreover, group and individual rationality are also required to be satisfied throughout the entire game interval.

Following the analysis in Section 4.4.1, we have

$$\begin{aligned}\gamma^i(\tau;\tau,x_\tau^*) &= \int_\tau^{\tau+\Delta t} B_i(s)\exp[-r(s-\tau)]\,ds\\ &\quad + \exp[-r(\Delta t)]\,\gamma^i(\tau+\Delta t;\tau+\Delta t,x_\tau^*+\Delta x_\tau^*),\\ &\quad\text{for } \tau\in[t_0,T] \text{ and } i\in\{1,2\};\end{aligned} \tag{4.37}$$

where

$$\Delta x_\tau^* = f[x_\tau^*,\psi_1^*(x_\tau^*),\psi_2^*(x_\tau^*)]\,\Delta t + o(\Delta t), \text{ and } o(\Delta t)/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

Since $x_\tau^* + \Delta x_\tau^* = x_{\tau+\Delta t}^*$, from (4.37), we have

$$\begin{aligned}\gamma^i\left(\tau;\tau+\Delta t,x_{\tau+\Delta t}^*\right) &= \exp[-r(\Delta t)]\,\xi^i\left(x_{\tau+\Delta t}^*\right)\\ &= \exp[-r(\Delta t)]\,\gamma^i\left(\tau+\Delta t;\tau+\Delta t,x_{\tau+\Delta t}^*\right)\\ &= \exp[-r(\Delta t)]\,\gamma^i(\tau+\Delta t;\tau+\Delta t,x_\tau^*+\Delta x_\tau^*).\end{aligned} \tag{4.38}$$

Therefore (4.37) becomes

$$\begin{aligned}\gamma^i(\tau;\tau,x_\tau^*) &= \int_\tau^{\tau+\Delta t} B_i(s)\exp[-r(s-\tau)]\,ds + \gamma^i\left(\tau;\tau+\Delta t,x_{\tau+\Delta t}^*\right),\\ &\quad\text{for } \tau\in[t_0,T] \text{ and } i\in\{1,2\}.\end{aligned} \tag{4.39}$$

One can obtain the following relationship:

$$\int_{\tau}^{\tau+\Delta t} B_i(s) \exp[-r(s-\tau)]\, ds = \gamma^i(\tau;\tau,x_\tau^*) - \gamma^i\left(\tau;\tau+\Delta t, x_{\tau+\Delta t}^*\right),$$
$$\text{for all } \tau \in [t_0,T] \text{ and } i \in \{1,2\}. \tag{4.40}$$

With $\Delta t \to 0$, condition (4.40) can be expressed as:

$$\begin{aligned} B_i(\tau)\,\Delta t = &-\left[\gamma_t^i(\tau;t,x_t^*)\big|_{t=\tau}\right]\Delta t \\ &-\left[\gamma_{x_t^*}^i(\tau;t,x_t^*)\Big|_{t=\tau}\right] f\left[x_\tau^*,\psi_1^*(x_\tau^*),\psi_2^*(x_\tau^*)\right]\Delta t - o(\Delta t). \end{aligned} \tag{4.41}$$

Dividing (4.41) throughout by Δt, with $\Delta t \to 0$, yield

$$B_i(\tau) = -\left[\gamma_t^i(\tau;t,x_t^*)\big|_{t=\tau}\right] - \left[\gamma_{x_t^*}^i(\tau;t,x_t^*)\Big|_{t=\tau}\right] f\left[x_\tau^*,\psi_1^*(x_\tau^*),\psi_2^*(x_\tau^*)\right]. \tag{4.42}$$

Using (4.36) we have $\gamma^i(\tau;t,x_t^*) = \exp[-r(t-\tau)]\,\xi^i(x_t^*)$ and $\gamma^i(\tau;\tau,x_\tau^*) = \xi^i(x_\tau^*)$. Then (4.42) can be used to obtain:

Theorem 4.7.3. *An instantaneous payment at time $\tau \in [t_0,T]$ equaling*

$$B_i(\tau) = r\xi^i(x_\tau^*) - \xi_{x_\tau^*}^i(x_\tau^*)\, f\left[x_\tau^*,\psi_1^*(x_\tau^*),\psi_2^*(x_\tau^*)\right],$$
$$\textit{for all } \tau \in [t_0,T] \textit{ and } i \in \{1,2\}, \tag{4.43}$$

yields a time consistent solution to the cooperative game $\Gamma_c(x_0)$.

$B_i(\tau)$ yields the transitory compensation that sustains a time consistent solution to the cooperative game $\Gamma_c(x_0)$. Since $B_i(\tau)$ is a function of the current state x_τ^* only, we can write $B_i(\tau)$ as $B_i(x_\tau^*)$.

Then, we consider time consistent solutions under specific optimality principles. Consider a cooperative game $\Gamma_c(x_0)$ in which the players agree to maximize the sum of their payoffs and divide the total cooperative payoff satisfying the Nash bargaining outcome. Hence the imputation scheme has to satisfy:

Proposition 4.7.1. *In the game $\Gamma_c(x_0)$, at time t_0 an imputation*

$$\xi^i(x_0) = V^i(x_0) + \frac{1}{2}\left[W(x_0) - \sum_{j=1}^{2} V^j(x_0)\right],$$

is assigned to Player i, for $i \in \{1,2\}$;
and at time $\tau \in (t_0,\infty)$, an imputation

$$\xi^i(x_\tau^*) = V^i(x_\tau^*) + \frac{1}{2}\left[W(x_\tau^*) - \sum_{j=1}^{2} V^j(x_\tau^*)\right],$$

is assigned to Player i, for $i \in \{1,2\}$.

Using Theorem 4.7.3, one can obtain a PDP with an instantaneous at time $\tau \in [t_0, \infty)$

$$B_i(\tau) = B_i(x_\tau^*) = \frac{1}{2}\left\{rV^i(x_\tau^*) - V^i_{x_\tau^*}(x_\tau^*) f[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)]\right\}$$
$$+\frac{1}{2}\{rW(x_\tau^*) - W_{x_\tau^*}(x_\tau^*) f[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)]\}$$
$$-\frac{1}{2}\left\{rV^j(x_\tau^*) - \xi^j_{x_\tau^*}(x_\tau^*) f[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)]\right\},$$
$$\text{for } i, j \in \{1,2\} \text{ and } i \neq j, \tag{4.44}$$

yields a time consistent solution to the cooperative game $\Gamma_c(x_0)$, in which the players agree to divide their cooperative gains according to Proposition 4.7.1.

Example 4.7.1. Consider the resource extraction game in Example 4.1.1 in which the game horizon is infinity. At time t_0, the payoff function of Player 1 and Player 2 are respectively:

$$\int_{t_0}^{\infty}\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}}u_1(s)\right]\exp[-r(t-t_0)]\,ds, \text{ and}$$
$$\int_{t_0}^{\infty}\left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}}u_2(s)\right]\exp[-r(t-t_0)]\,ds. \tag{4.45}$$

The resource stock $x(s) \in X \subset R$ follows the dynamics in (4.5).

A Nash equilibrium solution of the game (4.5) and (4.45), can then be characterized as follows.

Invoking Theorem 4.7.1 a noncooperative feedback Nash equilibrium solution can be characterized by:

$$rV^i(x) = \max_{u_i}\left\{u_i^{1/2} - \frac{c_i}{x^{1/2}}u_i + V_x^i(x)\left[ax^{1/2} - bx - u_i - \phi_j^*(x)\right]\right\},$$
$$\text{for } i, j \in \{1,2\} \text{ and } i \neq j. \tag{4.46}$$

Performing the indicated maximization in (4.46) yields:

$$\phi_i^*(x) = \frac{x}{4\left[c_i + V_x^i(x)\,x^{1/2}\right]^2}, \text{ for } i \in \{1,2\}.$$

Substituting $\phi_1^*(x)$ and $\phi_2^*(x)$ above into (4.46) and upon solving (4.46) one obtains the value function of player $i \in \{1,2\}$ as:

$$V^i(t,x) = \left[A_i x^{1/2} + B_i\right],$$

where for $i, j \in \{1,2\}$ and $i \neq j$, A_i, B_i, A_j and B_j satisfy:

$$\left[r+\frac{b}{2}\right]A_i-\frac{1}{2\left[c_i+A_i/2\right]}+\frac{c_i}{4\left[c_i+A_i/2\right]^2}$$
$$+\frac{A_i}{8\left[c_i+A_i/2\right]^2}+\frac{A_i}{8\left[c_j+A_j/2\right]^2}=0,\text{ and}$$

$$B_i=\frac{a}{2}A_i.$$

The game equilibrium strategies can be obtained as:

$$\phi_1^*(x)=\frac{x}{4\left[c_1+A_1/2\right]^2},\quad\text{and}\quad\phi_2^*(x)=\frac{x}{4\left[c_2+A_2/2\right]^2}.$$

Consider the case when these two extractors agree to maximize the sum of their payoffs and divide the total cooperative payoff according to Proposition 4.7.1. The players have to solve the control problem of maximizing

$$\int_{t_0}^{\infty}\left(\left[u_1(s)^{1/2}-\frac{c_1}{x(s)^{1/2}}u_1(s)\right]\right.$$
$$\left.+\left[u_2(s)^{1/2}-\frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\exp\left[-r(t-t_0)\right]ds \qquad (4.47)$$

subject to (4.5).

Invoking Theorem 4.7.2 we obtain:

$$rW(x)=\max_{u_1,u_2}\left\{\left(\left[u_1^{1/2}-\frac{c_1}{x^{1/2}}u_1\right]+\left[u_2^{1/2}-\frac{c_2}{x^{1/2}}u_2\right]\right)\right.$$
$$\left.+W_x(x)\left[ax^{1/2}-bx-u_1-u_2\right]\right\}.$$

Following similar procedures, one can obtain:

$$W(x)=\left[\hat{A}x^{1/2}+\hat{B}\right],$$

where

$$\left[r+\frac{b}{2}\right]\hat{A}-\frac{1}{2\left[c_1+\hat{A}/2\right]}-\frac{1}{2\left[c_2+\hat{A}/2\right]}+\frac{c_1}{4\left[c_1+\hat{A}/2\right]^2}$$
$$+\frac{c_2}{4\left[c_2+\hat{A}/2\right]^2}+\frac{\hat{A}}{8\left[c_1+\hat{A}/2\right]^2}+\frac{\hat{A}}{8\left[c_2+\hat{A}/2\right]^2}=0,\text{ and}$$

$$\hat{B}=\frac{a}{2r}\hat{A}.$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^*(x) = \frac{x}{4\left[c_1 + \hat{A}/2\right]^2} \quad \text{and} \quad \psi_2^*(x) = \frac{x}{4\left[c_2 + \hat{A}/2\right]^2}. \tag{4.48}$$

Substituting these control strategies into (4.5) yields the dynamics of the state trajectory under cooperation:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \frac{x(s)}{4\left[c_1 + \hat{A}/2\right]^2} - \frac{x(s)}{4\left[c_2 + \hat{A}/2\right]^2}, \qquad x(t_0) = x_0. \tag{4.49}$$

Solving (4.49) yields the optimal cooperative state trajectory for $\Gamma_c(x_0)$ as:

$$x^*(s) = \left[\frac{a}{2H} + \left(x_0^{1/2} - \frac{a}{2H}\right)\exp\left[-H(s - t_0)\right]\right]^2,$$

where

$$H = -\left[\frac{b}{2} + \frac{1}{8\left[c_1 + \hat{A}/2\right]^2} + \frac{1}{8\left[c_2 + \hat{A}/2\right]^2}\right].$$

Using (4.44) we obtain:

$$B_i(\tau) = B_i(x_\tau^*) =$$

$$\frac{1}{2}\left\{r\left[A_i(x_\tau^*)^{1/2} + B_i\right] + r\left[\hat{A}(x_\tau^*)^{1/2} + \hat{B}\right] - r\left[A_j(x_\tau^*)^{1/2} + B_j\right]\right\}$$

$$-\frac{1}{4}\left\{A_i(x_\tau^*)^{-1/2} + \hat{A}(x_\tau^*)^{-1/2} - A_j(x_\tau^*)^{-1/2}\right\}$$

$$\times\left[a(x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_1 + \hat{A}/2\right]^2} - \frac{x_\tau^*}{4\left[c_2 + \hat{A}/2\right]^2}\right],$$

$$\text{for } i, j \in \{1, 2\} \text{ and } i \neq j. \tag{4.50}$$

4.8 Games with Nontransferable Payoffs

The payoffs following from a cooperative scheme are transferable if there exists an acceptable medium of exchange. In reality, there are many important problems in which payoffs are not transferable – consider situations which involve national security, political stability, religious authority and sovereignty. In the case where payoffs are nontransferable, the solution for dynamic cooperation becomes more complicated and even intractable. In this section we consider cooperative differential games with nontransferable payoffs.

Consider a two-person nonzero-sum stochastic differential game with state dynamics (4.1) and payoffs (4.2). However, the payoffs are not transferable across players. The noncooperative outcomes of the nontransferable payoffs case is identical to that of the nontransferable payoffs case stated in Section 4.1.

4.8.1 Pareto Optimal Trajectories under Cooperation

Under cooperation with nontransferable payoffs, the players negotiate to establish an agreement (optimality principle) on a set cooperative controls. The player's payoffs will be determined by this set of controls. A necessary condition is that this optimality principle must satisfy group rationality and individual rationality. To achieve group rationality, the Pareto optimality of outcomes must be validated.

Consider the cooperative game $\Gamma_c\left(x_0, T-t_0\right)$ in which the payoffs are nontransferable. Pareto optimal outcomes for $\Gamma_c\left(x_0, T-t_0\right)$ can be identified by choosing a weight $\alpha_1 \in (0,\infty)$ that solves the following control problem (see Yeung and Petrosyan (2005)):

$$\begin{aligned}
&\max_{u_1,u_2}\left\{J^1\left(t_0,x_0\right)+\alpha_1 J^2\left(t_0,x_0\right)\right\} \equiv \\
&\quad \max_{u_1,u_2}\left\{\int_{t_0}^{T}\left(g^1\left[s,x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]\right.\right. \\
&\qquad \left.+\alpha_1 g^2\left[s,x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]\right)\exp\left[-\int_{t_0}^{s} r\left(y\right)dy\right]ds \\
&\qquad \left.+\left[q^1\left(x\left(T\right)\right)+\alpha_1 q^2\left(x\left(T\right)\right)\right]\exp\left[-\int_{t_0}^{T} r\left(y\right)dy\right]\right| x\left(t_0\right)=x_0\Bigg\}, \quad (4.51)
\end{aligned}$$

subject to the dynamics (4.1). Note that the optimal control strategies for the problem $\max_{u_1,u_2}\left\{J^1\left(t_0,x_0\right)+\alpha_1 J^2\left(t_0,x_0\right)\right\}$ is identical to those for the problem $\max_{u_1,u_2}\left\{J^2\left(t_0,x_0\right)+\alpha_2 J^1\left(t_0,x_0\right)\right\}$ when $\alpha_1 = 1/\alpha_2$.

In $\Gamma_c\left(x_0, T-t_0\right)$, let α_1^0 be the selected weight according an agreed upon optimality principle. Using Theorem 2.1.1, we obtain:

Theorem 4.8.1. *A set of controls* $\left\{\left[\psi_1^{\alpha_1^0(t_0)}\left(t,x\right),\psi_2^{\alpha_1^0(t_0)}\left(t,x\right)\right],\ \text{for}\ t\in\left[t_0,T\right]\right\}$ *provides an optimal solution to the control problem* $\max_{u_1,u_2}\left\{J^1\left(t_0,x_0\right)+\alpha_1^0 J^2\left(t_0,x_0\right)\right\}$ *if there exists continuously differentiable function* $W^{\alpha_1^0(t_0)}\left(t,x\right):\left[t_0,T\right]\times R^m\rightarrow R$ *satisfying the following Bellman equation:*

$$-W_t^{\alpha_1^0(t_0)}(t,x) =$$
$$\max_{u_1,u_2}\left\{\left(g^1[t,x,u_1,u_2]+\alpha_1^0 g^2[t,x,u_1,u_2]\right)\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right.$$
$$\left.+W_x^{\alpha_1^0(t_0)} f[t,x,u_1,u_2]\right\},$$

$$W^{\alpha_1^0(t_0)}(T,x) =$$
$$\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j(x)\left[q^1(x)+\alpha_1 q^2(x)\right]\exp\left[-\int_{t_0}^{T} r(y)\,dy\right].$$

Substituting $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ into (4.1) yields the dynamics of the Pareto optimal trajectory associated with weight α_1^0:

$$\dot{x}(s) = f\left[s, x(s), \psi_1^{\alpha_1^0(t_0)}(s,x(s)), \psi_2^{\alpha_1^0(t_0)}(s,x(s))\right], \qquad x(t_0)=x_0. \tag{4.52}$$

The solution to (4.52), denoted by $x^{\alpha_1^0}(t)$, can be expressed as:

$$x^{\alpha_1^0}(t) = x_0 + \int_{t_0}^{t} f\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right] ds,$$
$$\text{for } t\in[t_0,T]. \tag{4.53}$$

The path $\left\{x^{\alpha_1^0}(t)\right\}_{t=t_0}^{T}$ yields the optimal trajectory of the problem

$$\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1^0 J^2(t_0,x_0)\right\}.$$

The terms $x^{\alpha_1^0}(t)$ and $x_t^{\alpha_1^0}$ will be used interchangeably.

For group optimality to be achievable, the cooperative controls

$$\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$$

must be adopted throughout time interval $[t_0,T]$.

Now, consider the cooperative game $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ for $\tau\in[t_0,T]$. Let α_1^τ be the weight selected according to the originally agreed upon optimality principle. We use

$$\left\{\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right],\ t\in[\tau,T]\right\}$$

to denote a set of optimal controls, and $W^{\alpha_1^\tau(\tau)}(t,x):[\tau,T]\times R^n\to R$ the corresponding value function in Theorem 4.8.1.

One can readily verify that

$$\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right] = \left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right] \text{ when } \alpha_1^\tau=\alpha_1^0.$$

Remark 4.8.1. Gorup optimality can only be maintained if $\alpha_1^\tau=\alpha_1^0$ is the chosen weight in $\Gamma_c(x_\tau,T-\tau)$ for all $\tau\in[t_0,T]$.

4.8.2 Individual Player's Payoffs under Cooperation

In order to verify individual rationality in a cooperative scheme, we have to derive individual player's payoff functions under cooperation along the optimal trajectory. To do this, we first substitute the optimal controls $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ into the objective functions (4.2) to derive the players' expected payoff in $\Gamma_c(x_0, T-t_0)$ with α_1^0 being chosen as the cooperative weight. We follow Yeung (2004) and define:

Definition 4.8.1. *We define Player i's cooperative payoff over the interval* $[t,T]$ *as:*

$$
\begin{aligned}
&\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = \\
&\int_t^T g^i\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right] \\
&\qquad \times \exp\left[-\int_{t_0}^s r(y)\,dy\right] ds \\
&\quad + \exp\left[-\int_{t_0}^T r(y)\,dy\right] q^i\left(x^{\alpha_1^0}(T)\right), \\
&\qquad \textit{for } \; i \in \{1,2\}
\end{aligned}
$$

where

$$
\begin{aligned}
&\dot{x}^{\alpha_1^0}(s) = f\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right], \\
&x^{\alpha_1^0}(t) = x_t^{\alpha_1^0}.
\end{aligned}
$$

Note that for $\Delta t \to 0$, we can express $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right)$ as:

$$
\begin{aligned}
&\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = \\
&\int_t^{t+\Delta t} g^i\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right] \\
&\qquad \times \exp\left[-\int_{t_0}^s r(y)\,dy\right] ds \\
&\quad + \hat{W}^{\alpha_1^0(t_0)i}\left(t+\Delta t, x^{\alpha_1^0} + \Delta x^{\alpha_1^0}\right),
\end{aligned}
\tag{4.54}
$$

where $x^{\alpha_1^0} = f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \Delta t.$

Applying Taylor's Theorem, we have

$$
\begin{aligned}
&\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = \\
&\quad g^i\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \exp\left[-\int_{t_0}^{t} r(y)\, dy\right] \Delta t \\
&\qquad + \hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) + \hat{W}_t^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) \Delta t \\
&\qquad + \hat{W}_x^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \Delta t \\
&\qquad + o(\Delta t), \qquad \text{for } i \in \{1, 2\}.
\end{aligned} \tag{4.55}
$$

Canceling terms, performing the expectation operator, dividing throughout by Δt and taking $\Delta t \to 0$, we obtain:

$$
\begin{aligned}
&-\hat{W}_t^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = \\
&\quad g^i\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \exp\left[-\int_{t_0}^{t} r(y)\, dy\right] \\
&\qquad + \hat{W}_x^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right], \\
&\qquad\quad \text{for } i \in \{1, 2\}.
\end{aligned} \tag{4.56}
$$

Boundary conditions require:

$$
\hat{W}^{\alpha_1^0(t_0)i}\left(T, x_T^{\alpha_1^0}\right) = \exp\left[-\int_{t_0}^{T} r(y)\, dy\right] q^i\left(x_T^{\alpha_1^0}\right), \qquad \text{for } i \in \{1, 2\}. \tag{4.57}
$$

Theorem 4.8.2. ***(Yeung's (2004) deterministic version)*** *If there exist continuously differentiable functions* $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) : [t_0, T] \times R^m \to R$, $i \in \{1, 2\}$, *satisfying*

$$
\begin{aligned}
&-\hat{W}_t^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = \\
&\quad g^i\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \exp\left[-\int_{t_0}^{t} r(y)\, dy\right] \\
&\qquad + \hat{W}_x^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right],
\end{aligned}
$$

and

$$
\hat{W}^{\alpha_1^0(t_0)i}\left(T, x_T^{\alpha_1^0}\right) = \exp\left[-\int_{t_0}^{T} r(y)\, dy\right] q^i\left(x_T^{\alpha_1^0}\right),
$$

$$
\text{for } i \in \{1, 2\},
$$

then $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right)$ *gives Player i's cooperative payoff over the interval* $[t, T]$ *with* α_1^0 *being the cooperative weight.*

For the sake of subsequent comparison, we repeat the above analysis for the cooperative game $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ which starts at time τ with initial state $x_\tau^{\alpha_1^0}$ and chosen cooperative weight $\alpha_1^\tau = \alpha_1^0$. One can readily verify that

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right)\exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1, 2\}. \tag{4.58}$$

Remark 4.8.2. To maintain individual rationality throughout the game, the chosen α_1^0 has to satisfy

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right)\exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1, 2\}.$$

4.8.3 Time Consistent Solutions

In Section 4.3, the notion of time consistency is introduced and applied to cooperative differential games with transferable payoffs. The principle of time consistency ensures the solution optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset. In addition, the two essential properties of cooperation – Pareto optimality and individual rationality – are required to be present. In the present framework, a time consistent solution to the nontransferable payoffs game $\Gamma_c(x_0, T-t_0)$ requires:

Condition 4.8.1.

(i) The imputation vector

$$\left[\hat{W}^{\tau(\alpha_1^\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\tau(\alpha_1^\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right], \text{ for } \tau \in [t_0, T],$$

be Pareto optimal;

(ii)

$$\hat{W}^{\alpha_1^\tau(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1, 2\} \text{ and } \tau \in [t_0, T]; \text{ and}$$

(iii)

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right)\exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^\tau(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1, 2\} \text{ and } \tau \in [t_0, T].$$

Part (i) of Condition 4.8.1 guarantees Pareto optimality. Part (ii) ensures that individual rationality is satisfied thorughout the game horizon along the originally chosen optimal trajectory. Part (iii) guarantees the time consistency of the solution imputations throughout the game interval along the originally chosen optimal trajectory.

Consider that at time t_0 the players reach an agreed-upon optimality principle for $\Gamma_c(x_0, T-t_0)$ governing

(i) the choice a weight α_1^0 leading to a set of cooperative controls

$$\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right], \text{ and}$$

(ii) an imputation

$$\left[\hat{W}^{\alpha_1^0(t_0)1}(t_0,x_0), \hat{W}^{\alpha_1^0(t_0)2}(t_0,x_0)\right]$$

will then follow.

At a subsequent time $\tau \in (t_0, T]$, using the same optimality principle for $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$, the players agree to adopt

(i) a weight α_1^τ leading to a set of cooperative controls

$$\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right], \text{ and}$$

(ii) an imputations

$$\left[\hat{W}^{\alpha_1^\tau(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\alpha_1^\tau(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right]$$

will then follow.

Theorem 4.8.3. *A solution optimality principle under which the players agree to choose the same weight α_1^0 in all the games $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ such that*

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \quad \textit{and} \quad \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)$$

yields a time consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$.

Proof. *Given that the same weight α_1^0 will be chosen for all the subgames $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$, the cooperative control $\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$ will be adopted throughout time interval $[t_0, T]$. Group optimality is assured and the imputation vector*

$$\xi^{(\tau)}(x_\tau, T-\tau) = \left[\hat{W}^{\tau(\alpha_1^0)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\tau(\alpha_1^0)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right], \ \textit{for}\ \tau \in [t_0, T],$$

is indeed Pareto optimal. Hence part (i) of Condition 4.8.1 is satisfied.

With

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } \tau \in [t_0, T],$$

individual rationality is fulfilled throughout the game horizon along the optimal trajectory $\left\{x_\tau^{\alpha_1^0}\right\}_{\tau=t_0}^{T}$. *Part (ii) of Condition 4.8.1 is satisfied.*

Moreover, from (4.58), we have

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1, 2\}.$$

Part (iii) of Condition 4.8.1 is satisfied.

4.8.4 An Illustration

Consider the two-person differential game in Example 4.1.1 in which the payoffs are not transferable. The players negotiate to establish an agreement (optimality principle) on how to play the cooperative game and hence how to distribute the resulting payoff. A necessary condition is that this optimality principle must satisfy group rationality and individual rationality.

Pareto Optimal Trajectories

Pareto optimal outcomes for $\Gamma_c(x_0, T - t_0)$ can be identified by choosing a weight $\alpha_1^0 \in (0, \infty)$ that solves the following control problem:

$$\max_{u_1, u_2}\left\{J^1(t_0, x_0) + \alpha_1^0 J^2(t_0, x_0)\right\} \equiv$$
$$\max_{u_1, u_2}\left\{\int_{t_0}^{T}\left(\left[[u_1(s)]^{1/2} - \frac{c_1}{x(s)^{1/2}}u_1(s)\right]\right.\right.$$
$$\left. +\alpha_1\left[[u_2(s)]^{1/2} - \frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\exp[-r(s - t_0)]\,ds$$
$$\left. + \exp[-r(T - t_0)]\left(q_1 x(T)^{1/2} + \alpha_1^0 q_2 x(T)^{1/2}\right)\right| x(t_0) = x_0\Bigg\}, \tag{4.59}$$

subject to dynamics (4.5).

Let $\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$, for $t \in [t_0, T]$ denote a set of controls that provides a solution to the optimal control problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\}$, and $W^{(t_0)}(t,x) : [t_0,T] \times R^n \to R$ denote the value function that satisfies the equations (see Theorem 4.8.1):

$$
\begin{aligned}
&-W_t^{\alpha_1^0(t_0)}(t,x) = \\
&\quad \max_{u_1,u_2}\left\{\left(\left[u_1^{1/2} - \frac{c_1}{x^{1/2}}u_1\right] + \alpha_1^0\left[u_2^{1/2} - \frac{c_2}{x^{1/2}}u_2\right]\right)\exp\left[-r(t-t_0)\right]\right. \\
&\qquad \left. + W_x^{\alpha_1^0(t_0)}(t,x)\left[ax^{1/2} - bx - u_1 - u_2\right]\right\}, \\
&W^{\alpha_1^0(t_0)}(T,x) = \exp\left[-r(T-t_0)\right]\left[q_1 x^{1/2} + \alpha_1^0 q_2 x^{1/2}\right].
\end{aligned}
\tag{4.60}
$$

Performing the indicated maximization yields:

$$
\begin{aligned}
\psi_1^{\alpha_1^0(t_0)}(t,x) &= \frac{x}{4\left[c_1 + x^{1/2}W_x^{\alpha_1^0(t_0)}(t,x)\exp\left[r(t-t_0)\right]\right]^2} \text{ and} \\
\psi_2^{\alpha_1^0(t_0)}(t,x) &= \frac{x}{4\left[c_2 + x^{1/2}W_x^{\alpha_1^0(t_0)}(t,x)\exp\left[r(t-t_0)\right]/\alpha_1\right]^2}, \\
&\text{for } \ t \in [t_0, T].
\end{aligned}
\tag{4.61}
$$

Substituting $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ from (4.61) into (4.60) yields the value function

$$
\begin{aligned}
W^{\alpha_1^0(t_0)}(t,x) &= \exp\left[-r(t-t_0)\right]\left[A^{\alpha_1^0}(t)\,x^{1/2} + B^{\alpha_1^0}(t)\right], \\
&\text{for } \ t \in [t_0, T],
\end{aligned}
\tag{4.62}
$$

where $A^{\alpha_1^0}(t)$ and $B^{\alpha_1^0}(t)$ satisfy:

$$
\begin{aligned}
&\dot{A}^{\alpha_1^0}(t) = \left[r + \frac{b}{2}\right]A^{\alpha_1^0}(t) - \frac{1}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]} - \frac{\alpha_1^0}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1\right]}, \\
&\dot{B}^{\alpha_1^0}(t) = rB^{\alpha_1^0}(t) - \frac{a}{2}A^{\alpha_1^0}(t), \\
&A^{\alpha_1^0}(T) = q_1 + \alpha_1^0 q_2, \text{ and } \ B^{\alpha_1^0}(T) = 0.
\end{aligned}
$$

Substituting the partial derivative $W_x^{\alpha_1^0(t_0)}(t,x)$ from (4.62) into $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ one obtains the optimal controls of the problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\}$. Substituting these controls into (4.5) yields the dynamics of the Pareto optimal trajectory associated with a weight α_1^0 as:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \frac{x(s)}{4\left[c_1 + A^{\alpha_1^0}(s)/2\right]^2} - \frac{x(s)}{4\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]^2},$$
$$x(t_0) = x_0. \tag{4.63}$$

Solving (4.63) yields the Pareto optimal trajectory associated with weight α_1^0 as:

$$x^{\alpha_1^0}(s) = \left\{\Phi(t_0, s)\left[x_0^{1/2} + \int_{t_0}^{s} \Phi^{-1}(t_0, t)\frac{a}{2}dt\right]\right\}^2,$$
$$\text{for } s \in [t_0, T], \tag{4.64}$$

where $\Phi(t_0, s) = \exp\left[\int_{t_0}^{s} H_2(\tau)\, d\tau\right]$, and

$$H_2(s) = -\left[\frac{b}{2} + \frac{1}{8\left[c_1 + A^{\alpha_1^0}(s)/2\right]^2} + \frac{1}{8\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]^2}\right].$$

The cooperative control associated with weight α_1^0 over the time interval $[t_0, T]$ can be expressed precisely as:

$$\psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right) = \frac{x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \text{ and}$$
$$\psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right) = \frac{x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2},$$
$$\text{for } t \in [t_0, T]. \tag{4.65}$$

Individual Player's Payoffs

Substitute $\psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)$ and $\psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)$ from (4.65) into the players' payoff functions over the interval $[t, T]$ as:

$$\hat{W}^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right) =$$
$$\int_t^T \left[\frac{x^{\alpha_1^0}(s)^{1/2}}{2\left[c_1 + A^{\alpha_1^0}(s)/2\right]} - \frac{c_1 x^{\alpha_1^0}(s)^{1/2}}{4\left[c_1 + A^{\alpha_1^0}(s)/2\right]^2}\right]\exp\left[-r(s - t_0)\right] ds$$
$$+ \exp\left[-r(T - \tau)\right] q_1 x^{\alpha_1^0}(T)^{1/2};$$

and

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) =$$

$$\int_t^T \left[\frac{x^{\alpha_1^0}(s)^{1/2}}{2\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]} - \frac{c_2 x^{\alpha_1^0}(s)^{1/2}}{4\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]^2} \right] \exp\left[-r(s-t_0)\right] ds$$

$$+\exp\left[-r(T-\tau)\right] q_2 \left(x_T^{\alpha_1^0}\right)^{1/2}.$$

Invoking Theorem 4.8.2, we know if there exist continuously differentiable functions $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) : [t_0, T] \times R^m \to R$, $i \in \{1,2\}$, satisfying

$$-\hat{W}_t^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right) =$$

$$\left[\frac{\left(x_t^{\alpha_1^0}\right)^{1/2}}{2\left[c_1 + A^{\alpha_1^0}(t)/2\right]} - \frac{c_1\left(x_t^{\alpha_1^0}\right)^{1/2}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \right] \exp\left[-r(t-t_0)\right]$$

$$+\hat{W}_x^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right) \left[a\left(x_t^{\alpha_1^0}\right)^{1/2} - bx_t^{\alpha_1^0} - \frac{x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \right.$$

$$\left. - \frac{x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1\right]^2} \right],$$

$$\hat{W}^{\alpha_1^0(t_0)1}\left(T, x_T^{\alpha_1^0}\right) \exp\left[-r(T-\tau)\right] q_1 x^{\alpha_1^0}(T)^{1/2}; \text{ and}$$

$$-\hat{W}_t^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) =$$

$$\left[\frac{\left(x_t^{\alpha_1^0}\right)^{1/2}}{2\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]} - \frac{c_2\left(x_t^{\alpha_1^0}\right)^{1/2}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2} \right] \exp\left[-r(t-t_0)\right]$$

$$+\hat{W}_x^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) \left[a\left(x_t^{\alpha_1^0}\right)^{1/2} - bx_t^{\alpha_1^0} - \frac{x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \right.$$

$$\left. - \frac{x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1\right]^2} \right],$$

$$\hat{W}^{\alpha_1^0(t_0)2}\left(T, x_T^{\alpha_1^0}\right) \exp\left[-r(T-\tau)\right] q_2 \left(x_T^{\alpha_1^0}\right)^{1/2}; \qquad (4.66)$$

then $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right)$ gives Player i's cooperative payoff over the interval $[t, T]$ with α_1^0 being the cooperative weight.

Proposition 4.8.1. *The function* $\hat{W}^{\alpha_1^0(t_0)1}(t,x):[\tau,T]\times R\to R$ *satisfying (4.66) can be solved as:*

$$\hat{W}^{\alpha_1^0(t_0)1}\left(t,x_t^{\alpha_1^0}\right)=\exp\left[-r\left(t-t_0\right)\right]\left[\hat{A}_1^{\alpha_1^0}(t)\,x^{1/2}+\hat{B}_1^{\alpha_1^0}(t)\right], \tag{4.67}$$

where

$$\begin{aligned}\dot{\hat{A}}_1^{\alpha_1^0}(t)&=\left[r+\frac{b}{2}\right]\hat{A}_1^{\alpha_1^0}(t)-\frac{1}{2\left[c_1+A^{\alpha_1^0}(t)/2\right]}+\frac{c_1}{4\left[c_1+A^{\alpha_1^0}(t)/2\right]^2}\\&\quad+\frac{\hat{A}_1^{\alpha_1^0}(t)}{8\left[c_1+A^{\alpha_1^0}(t)/2\right]^2}+\frac{\hat{A}_1^{\alpha_1^0}(t)}{8\left[c_2+A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2},\\\dot{\hat{B}}_1^{\alpha_1^0}(t)&=r\hat{B}_1^{\alpha_1^0}(t)-\frac{a}{2}\hat{A}_1^{\alpha_1^0}(t),\ \text{and}\ \hat{A}_1^{\alpha_1^0}(T)=q_1,\ \text{and}\ \hat{B}_1^{\alpha_1^0}(T)=0.\end{aligned}$$

Proof. *Upon calculating the derivatives*

$$\hat{W}_t^{\alpha_1^0(t_0)1}\left(t,x_t^{\alpha_1^0}\right)\quad\text{and}\quad\hat{W}_{x_t}^{\alpha_1^0(t_0)1}\left(t,x_t^{\alpha_1^0}\right)$$

from (4.67) and then substituting them into (4.66) yield Proposition 4.8.1.

Proposition 4.8.2. *The function* $\hat{W}^{\alpha_1^0(t_0)2}(t,x):[\tau,T]\times R\to R$ *satisfying (4.66) can be solved as:*

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t,x_t^{\alpha_1^0}\right)=\exp\left[-r\left(t-t_0\right)\right]\left[\hat{A}_2^{\alpha_1^0}(t)\,x^{1/2}+\hat{B}_2^{\alpha_1^0}(t)\right], \tag{4.68}$$

where

$$\begin{aligned}\dot{\hat{A}}_2^{\alpha_1^0}(t)&=\left[r+\frac{b}{2}\right]\hat{A}_2^{\alpha_1}(t)-\frac{1}{2\left[c_2+A^{\alpha_1}(t)/2\alpha_1^0\right]}+\frac{c_2}{4\left[c_2+A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}\\&\quad+\frac{\hat{A}_2^{\alpha_1^0}(t)}{8\left[c_1+A^{\alpha_1^0}(t)/2\right]^2}+\frac{\hat{A}_2^{\alpha_1^0}(t)}{8\left[c_2+A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2},\\\dot{\hat{B}}_2^{\alpha_1^0}(t)&=r\hat{B}_2^{\alpha_1^0}(t)-\frac{a}{2}\hat{A}_2^{\alpha_1^0}(t),\ \ \hat{A}_2^{\alpha_1^0}(T)=q_2,\ \text{and}\ \hat{B}_2^{\alpha_1^0}(T)=0.\end{aligned}$$

Proof. *Upon calculating the derivatives*

$$\hat{W}_t^{\alpha_1^0(t_0)2}\left(t,x_t^{\alpha_1^0}\right)\quad\text{and}\quad\hat{W}_{x_t}^{\alpha_1^0(t_0)2}\left(t,x_t^{\alpha_1^0}\right)$$

from (4.68) and then substituting them into (4.66) yield Proposition 4.8.2.

Using (4.58) or alternatively repeating the above, one can obtain:

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) \exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)1}\left(t, x_t^{\alpha_1^0}\right)$$
$$= \exp\left[-r\left(t-\tau\right)\right]\left[\hat{A}_1^{\alpha_1^0}\left(t\right)\left(x_t^{\alpha_1^0}\right)^{1/2} + \hat{B}_1^{\alpha_1^0}\left(t\right)\right]$$

and

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) \exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)2}\left(t, x_t^{\alpha_1^0}\right)$$
$$= \exp\left[-r\left(t-\tau\right)\right]\left[\hat{A}_2^{\alpha_1^0}\left(t\right)\left(x_t^{\alpha_1^0}\right)^{1/2} + \hat{B}_2^{\alpha_1^0}\left(t\right)\right]. \tag{4.69}$$

In order to fulfill individual rationality, the choice of α_1^0 must satisfy the condition

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right).$$

4.8.5 A Proposed Solution

In this section, we propose a time consistent solution to the cooperative game $\Gamma_c\left(x_0, T - t_0\right)$. Invoking Theorem 4.8.3, a solution optimality principle under which the players agree to choose the same weight α_1^0 in all the games $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ and

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)$$

would yields a time consistent solution to the cooperative game $\Gamma_c\left(x_0, T - t_0\right)$.

The condition

$$\hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right), \text{ for } i \in \{1,2\}$$

implies the restriction on the choice of α_1^0 such that:

$$\left[\hat{A}_i^{\alpha_1^0}\left(\tau\right)\left(x_\tau^{\alpha_1^0}\right)^{1/2} + \hat{B}_i^{\alpha_1^0}\left(\tau\right)\right] \geq \left[A_i\left(\tau\right)\left(x_\tau^{\alpha_1^0}\right)^{1/2} + B_i\left(\tau\right)\right], \tag{4.70}$$
$$\text{for } i \in \{1,2\} \text{ and } \tau \in \left[t_0, T\right].$$

Note that from Propositions 4.8.1 and 4.8.2 one can obtain

$$\hat{B}_i^{\alpha_1^0}\left(\tau\right) = \left(a/2r\right)\hat{A}_i^{\alpha_1^0}\left(\tau\right),$$

and from Proposition 4.1.1 one can obtain $B_i\left(\tau\right) = \left(a/2r\right)A_i\left(\tau\right)$. Therefore, if $\hat{A}_i^{\alpha_1^0}\left(\tau\right) \geq A_i\left(\tau\right)$, then

$$\hat{B}_i^{\alpha_1^0}\left(\tau\right) \geq B_i\left(\tau\right), \quad \text{for } i \in \{1,2\} \text{ and } \tau \in \left[t_0, T\right].$$

For (4.70) to hold it is necessary that

$$\hat{A}_i^{\alpha_1^0}(\tau) \geq A_i(\tau), \quad \text{for } i \in \{1,2\} \quad \text{and} \quad \tau \in [t_0, T]. \tag{4.71}$$

We define:

Definition 4.8.2. *We denote the set of α_1^t that satisfies*

$$\hat{A}_i^{\alpha_1^t}(t) \geq A_i(t), \quad \textit{for } i \in \{1,2\}$$

at time $t \in [t_0, T)$ by S_t. We use $\underline{\alpha}_1^t$ to denote the lowest value of α_1 in S_t, and $\overline{\alpha}_1^t$ the highest value. In the case when t tends to T, we use $\overline{\alpha}_1^{T-}$ to stand for $\lim_{t\to T^-}\overline{\alpha}_1^t$, and $\overline{\alpha}_1^{T-}$ for $\lim_{t\to T^-}\underline{\alpha}_1^t$.

Definition 4.8.3. *We define the set $S_\tau^T = \bigcap_{\tau\leq t<T} S_t$, for $\tau \in [t_0, T)$.*

S_t is a set of α_1 that satisfies individual rationality at time $t \in [t_0, T)$ and S_τ^T is a set of α_1 that satisfies individual rationality throughout the interval $[\tau, T)$. In general $S_\tau^T \neq S_t^T$ for $\tau, t \in [t_0, T)$ where $\tau \neq t$.

Typical Configurations of S_t

To find out typical configurations of the set S_t for $t \in [t_0, T)$ of the game $\Gamma_c(x_0, T-t_0)$, we perform extensive numerical simulations with a wide range of parameter specifications for a, b, σ, c_1, c_2, q_1, q_2, T, r, x_0. We calculate the time paths of $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ for $t \in [t_0, T]$ from the Example 4.1.1. Then we select weights α_1^t and calculate the time paths of $\hat{A}_1^{\alpha_1^t}(t)$, $\hat{A}_2^{\alpha_1^t}(t)$, $\hat{B}_1^{\alpha_1^t}(t)$ and $\hat{B}_2^{\alpha_1^t}(t)$ in Propositions 4.8.1 and 4.8.2, for $t \in [t_0, T]$. At each time instant $t \in [t_0, T]$, we derive the set of α_1^t that yields $\hat{A}_i^{\alpha_1^t}(t) \geq A_i(t)$, for $i \in \{1,2\}$, to construct the set S_t, for $t \in [t_0, T)$.

We denote the locus of the values of $\underline{\alpha}_1^t$ along $t \in [t_0, T)$ as curve $\underline{\alpha}_1$ and the locus of the values of $\overline{\alpha}_1^t$ as curve $\overline{\alpha}_1$. In particular, three typical characteristics prevail:

(i) The curves $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are continuous and always move in the same direction over the entire game duration: either both increase monotonically or both decrease monotonically.

(ii) The set $S_t = \left[\underline{\alpha}_1^t, \overline{\alpha}_1^t\right]$, and

$$\hat{W}^{t(\overline{\alpha}_1^t)1}(t,x) = V^{(t)1}(t,x) \quad \text{and} \quad \hat{W}^{t(\underline{\alpha}_1^t)2}(t,x) = V^{(t)2}(t,x),$$
$$\text{for } t \in [t_0, T).$$

(iii) The set $S_{t_0}^T$ can be nonempty or empty.

These typical configurations are plotted in Figures 4.1–4.3.

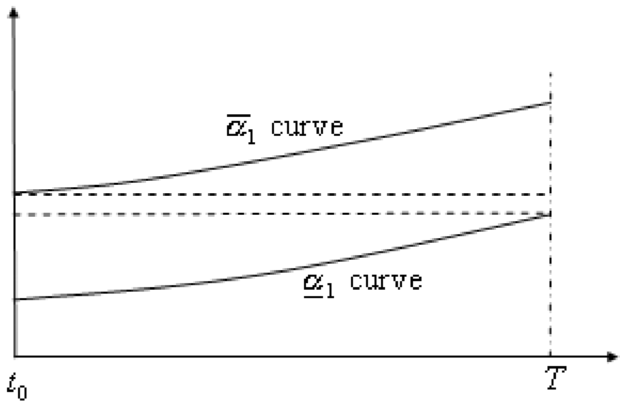

Fig. 4.1. The set $S_t = \left[\underline{\alpha}_1^t, \overline{\alpha}_1^t\right]$, with $S_{t_0}^T \neq \emptyset$.

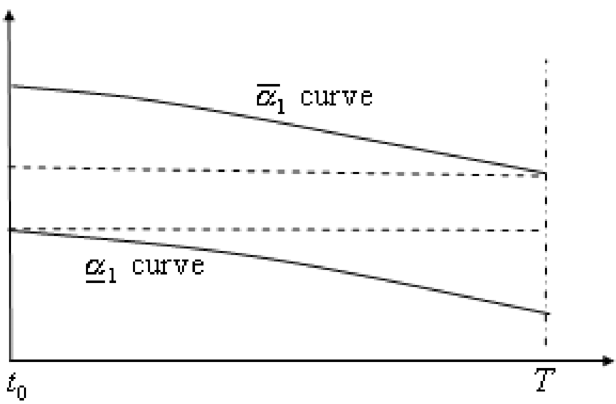

Fig. 4.2. The set $S_t = \left[\underline{\alpha}_1^t, \overline{\alpha}_1^t\right]$, with $S_{t_0}^T \neq \emptyset$.

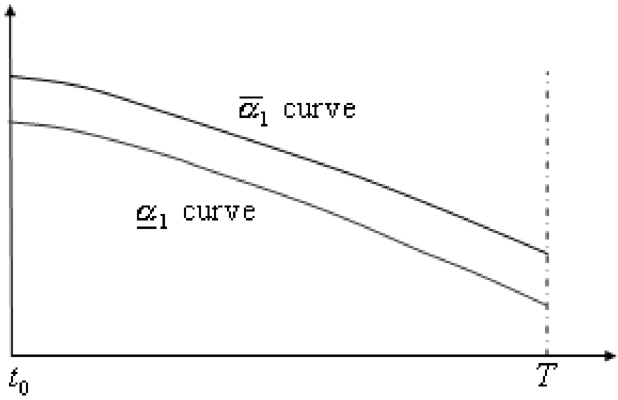

Fig. 4.3. The set $S_t = \left[\underline{\alpha}_1^t, \overline{\alpha}_1^t\right]$, with $S_{t_0}^T \neq \emptyset$.

Remark 4.8.3. Consider the case when $S_{t_0}^T \neq \emptyset$. If both $\underline{\alpha}_1$ and $\overline{\alpha}_1$ decrease monotonically, the condition $\underline{\alpha}_1^{T-} \notin S_\tau^T$ and $\overline{\alpha}_1^{T-} \in S_\tau^T$ for $\tau \in [t_0, T)$ prevails. If both $\underline{\alpha}_1$ and $\overline{\alpha}_1$ increase monotonically, the condition $\underline{\alpha}_1^{T-} \in S_\tau^T$ and $\overline{\alpha}_1^{T-} \notin S_\tau^T$ for $\tau \in [t_0, T)$ prevails.

A proposed time consistent solution to $\Gamma_c(x_0, T - t_0)$ is presented below.

Theorem 4.8.4. *If $S_{t_0}^T \neq \emptyset$, an optimality principle under which the players agree to choose the weight*

$$\alpha_1^* = \begin{cases} \underline{\alpha}_1^{T-}, & \text{if } \underline{\alpha}_1^{T-} \in S_\tau^T \text{ and } \overline{\alpha}_1^{T-} \notin S_\tau^T, \text{ for } \tau \in [t_0, T], \\ \overline{\alpha}_1^{T-}, & \text{if } \overline{\alpha}_1^{T-} \in S_\tau^T \text{ and } \underline{\alpha}_1^{T-} \notin S_\tau^T, \text{ for } \tau \in [t_0, T], \end{cases} \tag{4.72}$$

yields a time consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

Proof. If any one of the two mutually exclusive conditions governing the choice of α_1^* in (4.72) prevails, according to the optimality principle in Theorem 4.8.4, a unique α_1^* will be chosen for all the subgames $\Gamma_c\left(x_\tau^{\alpha_1^*}, T - \tau\right)$, for $t_0 \leq \tau \leq t \leq T$. The vector

$$\xi^{(\tau)}\left(x_\tau^{\alpha_1^*}, T - \tau\right) = \left[\hat{W}^{\tau(\alpha_1^*)1}(\tau, x_\tau), \hat{W}^{\tau(\alpha_1^*)2}(\tau, x_\tau)\right],$$
$$\text{for } \tau \in [t_0, T],$$

yields a Pareto optimal pair of imputations. Hence part (i) of Condition 4.8.1 is satisfied.

Since

$$\alpha_1^* \in S_{t_0}^T, \left[\hat{A}_i^{\alpha_1^*}(t)\left(x_t^{\alpha_1^*}\right)^{1/2} + \hat{B}_i^{\alpha_1^*}(t)\right] \geq \left[A_i(t)\left(x_t^{\alpha_1^*}\right)^{1/2} + B_i(t)\right],$$
$$\text{for } i \in \{1, 2\}.$$

Hence part (ii) of Condition 4.8.1 is satisfied.

Finally from (4.69),

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)\exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)1}\left(t, x_t^{\alpha_1^0}\right),$$
$$\textit{for } i \in \{1, 2\}, \quad \tau \in [t_0, T].$$

Part (iii) of Condition 4.8.1 is satisfied.

Hence Theorem 4.8.4 follows.

4.9 Appendix to Chapter 4

Proof of Formula 4.6.1

Using Theorem 4.4.1 and (4.30), we obtain:

$$\begin{aligned} B_i(\tau) = & -\omega_W^{(\tau)i}(\tau, x_\tau^*)\left\{\left[W_t^{(\tau)}(t, x_t^*)\Big|_{t=\tau}\right]\right. \\ & \left. + \left[W_{x_t^*}^{(\tau)}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right]\right\} \\ & -\omega_{V^i}^{(\tau)i}(\tau, x_\tau^*)\left\{\left[V_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right]\right. \\ & \left. + \left[V_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right]\right\} \\ & -\omega_{V^j}^{(\tau)i}(\tau, x_\tau^*)\left\{\left[V_t^{(\tau)j}(t, x_t^*)\Big|_{t=\tau}\right]\right. \\ & \left. + \left[V_{x_t^*}^{(\tau)j}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right]\right\}. \end{aligned} \tag{4.73}$$

Invoking Bellman's equation, we obtain:

$$\begin{aligned} -\left[W_t^{(\tau)}(t, x_t^*)\Big|_{t=\tau}\right] = & \sum_{j=1}^{2} g^j\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \\ & + \left[W_{x_t^*}^{(\tau)}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right]. \end{aligned} \tag{4.74}$$

Invoking the Isaacs-Bellman equation we obtain:

$$\begin{aligned} -\left[V_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] = & \, g^i\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \\ & + \left[V_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right], \\ & \text{for } i \in \{1, 2\}. \end{aligned} \tag{4.75}$$

Substituting (4.74) and (4.75) into (4.73), we obtain:

$$
\begin{aligned}
B_i(\tau) = {} & \omega_W^{(\tau)i}(\tau, x_\tau^*) \sum_{j=1}^{2} g^j \left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \\
& + \omega_{V^i}^{(\tau)i}(\tau, x_\tau^*) \Big\{ g^i \left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \\
& + \left[V_{x_t^*}^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] \\
& \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \Big\} \\
& + \omega_{V^j}^{(\tau)i}(\tau, x_\tau^*) \Big\{ g^j \left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \\
& + \left[V_{x_t^*}^{(\tau)j}(t, x_t^*) \Big|_{t=\tau} \right] \\
& \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \Big\}.
\end{aligned}
$$

Hence Formula 4.6.1 follows. Q.E.D.

4.10 Problems

Problem 4.1. Consider a resource extraction game in which two extractors are awarded leases to extract a renewable resource over the time interval $[t_0, T]$. The resource stock $x(s) \in X \subset R$ follows the dynamics:

$$
\dot{x}(s) = a - bx(s) - u_1(s) - u_2(s), \qquad x(t_0) = x_0 \in X,
$$

where a and b are constants, $u_1(s)$ is the harvest rate of extractor 1, and $u_2(s)$ is the harvest rate of extractor 2.

At time t_0, the payoff function of extractor 1 and extractor 2 are respectively:

$$
\int_{t_0}^{T} \left[u_1(s) - \frac{c_1}{x(s)} u_1^2(s) \right] \exp\left[-r(t - t_0)\right] ds + \exp\left[-r(T - t_0)\right] q x(T),
$$

and

$$
\int_{t_0}^{T} \left[u_2(s) - \frac{c_2}{x(s)} u_2^2(s) \right] \exp\left[-r(t - t_0)\right] ds + \exp\left[-r(T - t_0)\right] q x(T).
$$

where q, c_1 and c_2 are constants and $c_1 \neq c_2$. Payoffs are transferable and r is the interest rate.

(a) Derive a feedback Nash equilibrium for the game.
(b) If the extractors agree to cooperative, derive the optimal extraction strategies.

(c) Derive the optimal state trajectory under cooperation.

Problem 4.2. Consider the game in Problem 4.1, if the extractors agree to share the cooperative profit satisfying the Nash bargaining outcome – that is, they maximize the product of individual gains in excess of the noncooperative payoffs. The imputation scheme has to satisfy the condition that (see Proposition 4.4.1)

$$\xi^{(\tau)i}\left(\tau, x_\tau^*\right) = V^{(\tau)i}\left(\tau, x_\tau^*\right) + \frac{1}{2}\left[W^{(\tau)}\left(\tau, x_\tau^*\right) - \sum_{j=1}^{2} V^{(\tau)j}\left(\tau, x_\tau^*\right)\right],$$

is assigned to extractor i, for $i \in \{1, 2\}$, and in the subgame $\Gamma_c\left(x_\tau^*, T - \tau\right)$, for $\tau \in [t_0, T]$.

(a) Formulate a payoff distribution procedure (PDP) of the cooperative game so that the agreed imputations can be realized.
(b) Show that Pareto optimality and individual rationality are satisfied.

Problem 4.3. Using the PDP obtained in part (a) of Problem 4.2, express the compensation $B_i\left(\tau\right)$ extractor i receives at time τ in terms of

(i) extractor i's agreed upon marginal share of total cooperative profit,
(ii) extractor i's agreed upon marginal share of his own nonccoperative profit plus the instantaneous effect on his noncooperative payoff when the change in the state variable x_τ^* follows the optimal trajectory instead of the noncooperative path, and
(iii) extractor i's agreed upon marginal share of extractor j's nonccoperative profit plus the instantaneous effect on extractor j's noncooperative payoff when the change in the state variable x_τ^* follows the cooperative trajectory instead of the noncooperative path.

Problem 4.4. Consider the infinite-horizon game in which Player 1 and Player 2 maximizes respectively:

$$\int_{t_0}^{\infty}\left[u_1\left(s\right) - \frac{c_1}{x\left(s\right)}u_1^2\left(s\right)\right]\exp\left[-r\left(t - t_0\right)\right]ds,$$

and

$$\int_{t_0}^{\infty}\left[u_2\left(s\right) - \frac{c_2}{x\left(s\right)}u_2^2\left(s\right)\right]\exp\left[-r\left(t - t_0\right)\right]ds.$$

The players' payoffs are transferable and the resource stock $x\left(s\right) \in X \subset R$ follows the dynamics:

$$\dot{x}\left(s\right) = a - bx\left(s\right) - u_1\left(s\right) - u_2\left(s\right), \qquad x\left(t_0\right) = x_0 \in X.$$

(a) Derive a feedback Nash equilibrium for the game.
(b) If the players agree to cooperative, derive the optimal extraction strategies.

(c) Derive the optimal state trajectory under cooperation.
(d) If the players agree to share the cooperative profit satisfying the Nash bargaining outcome – that is, they maximize the product of individual gains in excess of the noncooperative payoffs, formulate a payoff distribution procedure (PDP) of the cooperative game so that the agreed imputations can be realized.

Problem 4.5. Consider the game in Problem 4.4. However, the players' payoffs are not transferable.

(a) If the players agree to cooperate, describe the Pareto optimal outcomes and the corresponding optimal trajectories.
(b) Derive individual player's payoff functions under cooperation along the optimal trajectory.
(c) Present a time consistent solution which satisfies the axiom of symmetry.

5

Two-person Cooperative Stochastic Differential Games

An essential characteristic of time – and hence decision making over time – is that though the individual may, through the expenditure of resources, gather past and present information, the future is inherently unknown and therefore (in the mathematical sense) uncertain. There is no escape from this fact, regardless of what resources the individual should choose to devote to obtaining data, information, and forecasting. An empirically meaningful theory of games must therefore incorporate time-uncertainty in an appropriate manner. This development establishes a framework or paradigm for modeling game-theoretic situations with stochastic dynamics and uncertain environments over time. This chapter consider two-person cooperative stochastic differential games.

5.1 Game Formulation and Noncooperative Outcome

Consider the two-person nonzero-sum stochastic differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \in R^m$, with permissible state trajectories $\{x(s), t_0 \leq s \leq T\}$. The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$dx(s) = f[s, x(s), u_1(s), u_2(s)]\,ds + \sigma[s, x(s)]\,dz(s), \quad x(t_0) = x_0, \quad (5.1)$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\,\sigma[s, x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. $u_i \in U_i \subset compR^l$ is the control vector of Player i, for $i \in \{1, 2\}$.

At time instant $s \in [t_0, T]$, the instantaneous payoff of Player i, for $i \in \{1, 2\}$, is denoted by $g^i[s, x(s), u_1(s), u_2(s)]$, and, when the game terminates at time T, Player i receives a terminal payment of $q^i(x(T))$. Payoffs are transferable across players and over time. Given a time-varying instantaneous discount rate $r(s)$, for $s \in [t_0, T]$, values received t time after t_0 have to be discounted by the factor $\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]$. Hence at time t_0, the payoff function of Player i can be expressed as

$$E_{t_0}\left\{\int_{t_0}^{T} g^i\left[s, x(s), u_1(s), u_2(s)\right]\exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds\right.$$
$$\left.+\exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i\left(x(T)\right)\right\}, \tag{5.2}$$
$$\text{for } \; i \in \{1,2\},$$

where E_{t_0} denotes the expectation operator performed at time t_0.

We use $\Gamma(x_0, T-t_0)$ to denote the stochastic differential game (5.1)–(5.2). Invoking Theorem 2.5.1 a noncooperative Nash equilibrium solution of the game $\Gamma(x_0, T-t_0)$ can be characterized as follows.

Theorem 5.1.1. *A set of feedback strategies* $\left[\phi_1^{(t_0)*}(t,x), \phi_2^{(t_0)*}(t,x)\right]$ *provides a Nash equilibrium solution to the game* $\Gamma(x_0, T-t_0)$*, if there exist continuously differentiable functions* $V^{(t_0)i}(t,x) : [t_0,T]\times R^n \to R$*,* $i \in \{1,2\}$*, satisfying the following (Fleming-Bellman-Isaacs) partial differential equations:*

$$-V_t^{(t_0)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, V_{x^h x^\zeta}^{(t_0)i}(t,x) =$$
$$\max_{u_i}\left\{g^i\left[t, x, u_i, \phi_j^{(t_0)*}(t,x)\right]\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right.$$
$$\left.+V_x^{(t_0)i}(t,x)\, f\left[t, x, u_i, \phi_j^{(t_0)*}(t,x)\right]\right\},$$

and

$$V^{(t_0)i}(T,x) = \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i(x),$$
$$i \in \{1,2\} \;\; \textit{and} \;\; j \in \{1,2\} \;\; \textit{and} \;\; j \neq i.$$

The feedback strategies in Theorem 5.1.1 are *Markovian* in the sense that they are functions of current time t and current state x, and hence independent of past values of state (see Basar and Olsder (1995)).

This implies the property: At time τ with the state being $x_\tau \in X$, the set of Nash equilibrium strategies can be obtained as

$$\left\{u_i^{(t_0)*}(\tau) = \phi_i^{(t_0)*}(\tau, x_\tau), \text{ for } \; i \in \{1,2\}\right\}.$$

Consider the alternative game $\Gamma(x_\tau, T-\tau)$ with payoff structure (5.1) and dynamics (5.2) starting at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$.

Let $\left\{u_i^{(\tau)*}(t) = \phi_i^{(\tau)*}(t,x) \in \Phi^i, \text{ for } i \in \{1,2\} \text{ and } t \in [\tau,T]\right\}$ denote a set of feedback strategies that constitutes a Nash equilibrium solution to the game $\Gamma(x_\tau, T-\tau)$, and $V^{(\tau)i}(t,x_t) : [\tau,T] \times R^n \to R$ denote the value function of player $i \in \{1,2\}$ that satisfies the corresponding Bellman-Isaacs-Fleming equations

$$
\begin{aligned}
&-V_t^{(\tau)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, V_{x^h x^\zeta}^{(\tau)i}(t,x) = \\
&\qquad \max_{u_i}\left\{ g^i\left[t,x,u_i,\phi_j^{(\tau)*}(t,x)\right] \exp\left[-\int_\tau^t r(y)\,dy\right] \right. \\
&\qquad\qquad \left. + V_x^{(\tau)i}(t,x)\, f\left[t,x,u_i,\phi_j^{(\tau)*}(t,x)\right] \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&V^{(\tau)i}(T,x) = \exp\left[-\int_\tau^T r(y)\,dy\right] q^i(x), \\
&\qquad i \in \{1,2\} \text{ and } j \in \{1,2\} \text{ and } j \neq i.
\end{aligned}
\tag{5.3}
$$

Remark 5.1.1. Note that the equilibrium feedback strategies are Markovian in the sense that they depend on current time and current state. Comparing the Bellman-Isaacs-Fleming equations in (5.3) for different values of $\tau \in [t_0,T]$, one can readily observe that:

$$
\begin{aligned}
&\phi_i^{(\tau)*}(s,x(s)) = \phi_i^{(t_0)*}(s,x(s)),\ s \in [\tau,T], \\
&V^{(\tau)i}(\tau,x_\tau) = \exp\left[\int_{t_0}^\tau r(y)\,dy\right] V^{(t_0)i}(\tau,x_\tau), \text{ and} \\
&V^{(t)i}(t,x_t) = \exp\left[\int_\tau^t r(y)\,dy\right] V^{(\tau)i}(t,x_t), \\
&\qquad \text{for } t_0 \le \tau \le t \le T \text{ and } i \in \{1,2\}.
\end{aligned}
$$

Definition 5.1.1. *In a Nash equilibrium of the game $\Gamma(x_\tau, T-\tau)$, the value function $V^{(\tau)i}(t,x_t)$, which measures the expected present value of Player i's payoff in the time interval $[t,T]$ when $x(t) = x_t$, can be expressed as:*

$$
\begin{aligned}
&V^{(\tau)i}(t,x_t) = \\
&E_\tau\left\{\int_t^T g^i\left[s,x(s),\phi_1^{(\tau)*}(s,x(s)),\phi_2^{(\tau)*}(s,x(s))\right] \exp\left[-\int_\tau^s r(y)\,dy\right] ds \right. \\
&\qquad \left. + \exp\left[-\int_\tau^T r(y)\,dy\right] q^i(x(T)) \,\middle|\, x(t) = x_t \right\} \\
&= E_\tau\left\{\left(\int_t^T g^i\left[s,x(s),\phi_1^{(\tau)*}(s,x(s)),\phi_2^{(\tau)*}(s,x(s))\right] \exp\left[-\int_t^s r(y)\,dy\right] ds \right.\right.
\end{aligned}
$$

$$+\exp\left[-\int_t^T r(y)\,dy\right] q^i(x(T))\Bigg)\exp\left[-\int_\tau^t r(y)\,dy\right] ds \Bigg|\, x(t)=x_t\Bigg\}$$

$$= \exp\left[-\int_\tau^t r(y)\,dy\right] ds\, V^{(t)i}(t, x_t),$$

$$\textit{for } \tau \in [t_0, T],\ t \in [\tau, T] \ \textit{ and } i \in \{1, 2\}.$$

The game equilibrium dynamics of $\Gamma(x_\tau, T-\tau)$ can be obtained as:

$$dx(s) = f\left[s, x(s), \phi_1^{(\tau)*}(s, x(s)), \phi_2^{(\tau)*}(s, x(s))\right] ds + \sigma[s, x(s)]\,dz(s),$$
$$x(\tau) = x_\tau. \tag{5.4}$$

Example 5.1.1. Consider a stochastic version of the game in Example 4.1.1 in which two extractors are awarded leases to extract a renewable resource over the time interval $[t_0, T]$. The resource stock $x(s) \in X \subset R$ follows the dynamics:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s)\right] ds + \sigma x(s)\,dz(s),$$
$$x(t_0) = x_0 \in X, \tag{5.5}$$

where $u_i(s)$ is the harvest rate of extractor $i \in \{1, 2\}$. The instantaneous payoffs at time $s \in [t_0, T]$ for Player 1 and Player 2 are, respectively,

$$\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s)\right] \text{ and } \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s)\right],$$

where c_1 and c_2 are constants and $c_1 \neq c_2$. At time T, each extractor will receive a termination bonus $qx(T)^{1/2}$. Payoffs are transferable between players and over time. Given the constant discount rate r, values received at time t are discounted by the factor $\exp[-r(t-t_0)]$.

At time t_0, the expected payoff of Player i is:

$$E_{t_0}\Bigg\{\int_{t_0}^T \left[u_i(s)^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s)\right] \exp[-r(t-t_0)]\,ds$$
$$+\exp[-r(T-t_0)]\,qx(T)^{1/2}\Bigg\}, \tag{5.6}$$

$$\text{for } i \in \{1, 2\}.$$

Let $\left[\phi_1^{(t_0)*}(t, x), \phi_2^{(t_0)*}(t, x)\right]$ for $t \in [t_0, T]$ denote a set of strategies that provides a feedback Nash equilibrium solution to the game $\Gamma(x_0, T-t_0)$, and $V^{(t_0)i}(t, x) : [t_0, T] \times R^n \to R$ denote the value function of player $i \in \{1, 2\}$ that satisfies the Isaacs-Bellman-Fleming equations:

$$
\begin{aligned}
&-V_t^{(t_0)i}(t,x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^{(t_0)i}(t,x) = \\
&\qquad \max_{u_i}\left\{\left[u_i^{1/2} - \frac{c_i}{x^{1/2}}u_i\right]\exp\left[-r(t-t_0)\right]\right. \\
&\qquad\qquad \left. + V_x^{(t_0)i}(t,x)\left[ax^{1/2} - bx - u_i - \phi_j^{(t_0)*}(t,x)\right]\right\}, \text{ and} \\
&V^{(t_0)i}(T,x) = \exp\left[-r(T-t_0)\right] q x(T)^{1/2}, \\
&\qquad\qquad \text{for } i \in \{1,2\} \text{ and } j \in \{1,2\} \text{ and } j \neq i.
\end{aligned} \tag{5.7}
$$

Performing the indicated maximization in (5.7) yields:

$$
\phi_i^{(t_0)*}(t,x) = \frac{x}{4\left[c_i + V_x^{(t_0)i}\exp\left[r(t-t_0)\right]x^{1/2}\right]^2}, \text{ for } i \in \{1,2\}. \tag{5.8}
$$

Proposition 5.1.1. *The value functions of player $i \in \{1,2\}$ in the game $\Gamma(x_0, T-t_0)$ is:*

$$
V^{(t_0)i}(t,x) = \exp\left[-r(t-t_0)\right]\left[A_i(t)x^{1/2} + B_i(t)\right],
$$

where for $i,j \in \{1,2\}$ and $i \neq j$, $A_i(t)$, $B_i(t)$, $A_j(t)$ and $B_j(t)$ satisfy:

$$
\begin{aligned}
\dot{A}_i(t) &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]A_i(t) - \frac{1}{2\left[c_i + A_i(t)/2\right]} + \frac{c_i}{4\left[c_i + A_i(t)/2\right]^2} \\
&\qquad + \frac{A_i(t)}{8\left[c_i + A_i(t)/2\right]^2} + \frac{A_i(t)}{8\left[c_j + A_j(t)/2\right]^2}, \\
\dot{B}_i(t) &= rB_i(t) - \frac{a}{2}A_i(t), \\
A_i(T) &= q, \text{ and } B_i(T) = 0.
\end{aligned}
$$

Proof. *Substituting $\phi_1^{(t_0)*}(t,x)$ and $\phi_2^{(t_0)*}(t,x)$ from (5.8) into (5.7), and upon solving (5.7) one obtains Proposition 5.1.1.*

Using the results in Proposition 5.1.1, the game equilibrium strategies can be obtained as:

$$
\phi_1^{(t_0)*}(t,x) = \frac{x}{4\left[c_1 + A_1(t)/2\right]^2}, \text{ and } \phi_2^{(t_0)*}(t,x) = \frac{x}{4\left[c_2 + A_2(t)/2\right]^2}. \tag{5.9}
$$

Consider the alternative game $\Gamma(x_\tau, T-\tau)$ with payoff structure (5.5) and dynamics (5.6) starting at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$. Following the above analysis, the value function $V^{(\tau)i}(t,x) : [\tau, T] \times R \rightarrow R$, for $i \in \{1,2\}$ and $\tau \in [t_0, T]$, for the subgame $\Gamma(x_\tau, T-\tau)$ can be obtained as:

Proposition 5.1.2. *The value function of player $i \in \{1,2\}$ in the game $\Gamma(x_\tau, T-\tau)$ is:*

$$V^{(\tau)i}(t,x) = \exp[-r(t-\tau)]\left[A_i(t)\, x^{1/2} + B_i(t)\right],$$

where for $i,j \in \{1,2\}$ and $i \neq j$, $A_i(t)$, $B_i(t)$, $A_j(t)$ and $B_j(t)$ are the same as those in Proposition 5.1.1.

Proof. *Follow the proof of Proposition 5.1.1.*

The Nash equilibrium strategies of Player 1 and Player 2 in the subgame $\Gamma(x_\tau, T-\tau)$ are respectively:

$$\phi_1^{(\tau)*}(t,x) = \frac{x}{4\left[c_1 + A_1(t)/2\right]^2}, \text{ and } \phi_2^{(\tau)*}(t,x) = \frac{x}{4\left[c_2 + A_2(t)/2\right]^2}. \tag{5.10}$$

Note that the conditions in Remark 5.1.1. prevail.

5.2 Cooperative Arrangement under Uncertainty

Now consider the case when the players agree to cooperate. Let $\Gamma_c(x_0, T-t_0)$ denote a cooperative stochastic differential game with the game structure of $\Gamma(x_0, T-t_0)$ in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game $\Gamma_c(x_0, T-t_0)$ includes

(i) an agreement on a set of cooperative strategies/controls, and
(ii) a mechanism to distribute total payoff among players.

The solution optimality principle will remain in effect along the cooperative state trajectory path $\{x_s^*\}_{s=t_0}^{T}$. Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

5.2.1 Group Rationality and Optimal Trajectory

Since payoffs are transferable, group rationality requires the players to maximize their joint payoff. Consider the cooperative game $\Gamma_c(x_0, T-t_0)$. To achieve group rationality, the players have to agree to act so that the sum of the expected payoffs is maximized. The players must then solve the following stochastic control problem:

$$\max_{u_1,u_2} E_{t_0}\left\{\int_{t_0}^{T}\sum_{j=1}^{2} g^j\left[s,x(s),u_1(s),u_2(s)\right]\exp\left[-\int_{t_0}^{s} r(y)\,dy\right]ds\right.$$
$$\left.+\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j\left(x(T)\right)\right\}, \tag{5.11}$$

subject to (5.1).

Denote the control problem (5.11) and (5.1) by $\Psi(x_0,T-t_0)$. Using Theorem 2.1.5, we obtain:

Theorem 5.2.1. *A set of controls* $\left[\psi_1^{(t_0)*}(t,x),\psi_2^{(t_0)*}(t,x)\right]$ *provides an optimal solution to the stochastic control problem* $\Psi(x_0,T-t_0)$*, if there exists continuously differentiable function* $W^{(t_0)}(t,x):[t_0,T]\times R^n\to R$ *satisfying the following partial differential equation:*

$$-W_t^{(t_0)}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,W_{x^h x^\zeta}^{(t_0)}(t,x)=$$
$$\max_{u_1,u_2}\left\{\sum_{j=1}^{2} g^j\left[t,x,u_1,u_2\right]\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]+W_x^{(t_0)}f\left[s,x,u_1,u_2\right]\right\},$$
$$W^{(t_0)}(T,x)=\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j(x).$$

Hence the players will adopt the cooperative control $\left\{\left[\psi_1^{(t_0)*}(t,x),\psi_2^{(t_0)*}(t,x)\right],\text{ for } t\in[t_0,T]\right\}$. One again, in a cooperative framework, the issue of non-uniqueness of the optimal controls can be resolved by agreement between the players on a particular set of controls. Substituting this set of control into (5.1) yields the dynamics of the optimal (cooperative) trajectory as:

$$dx(s)=f\left[s,x(s),\psi_1^{(t_0)*}(s,x(s)),\psi_2^{(t_0)*}(s,x(s))\right]ds+\sigma\left[s,x(s)\right]dz(s),$$
$$x(t_0)=x_0. \tag{5.12}$$

The solution to (5.12) can be expressed as:

$$x^*(t)=x_0+\int_{t_0}^{t} f\left[s,x^*(s),\psi_1^{(t_0)*}(s,x^*(s)),\psi_2^{(t_0)*}(s,x^*(s))\right]ds$$
$$+\int_{t_0}^{t}\sigma\left[s,x^*(s)\right]dz(s). \tag{5.13}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (5.13). The term x_t^* is used to denote an element in the set X_t^*.

The cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the time interval $[t_0, T]$ can be expressed more precisely as:

$$\left\{\left[\psi_1^{(t_0)*}(t, x^*(t)), \psi_2^{(t_0)*}(t, x^*(t))\right], \text{ for } t \in [t_0, T]\right\}. \tag{5.14}$$

Note that for group optimality to be achievable, the cooperative controls (5.14) must be exercised throughout time interval $[t_0, T]$.

To verify whether the players would find it optimal to adopt the coopeative controls (5.14) throughout the cooperative duration, we consider a cooperative game $\Gamma_c(x_\tau^*, T-\tau)$ with dynamics (5.1) and payoffs (5.2) which begins at time $\tau \in [t_0, T]$ and initial state $x_\tau^* \in X_t^*$. At time τ, the optimality principle ensuring group rationality requires the players to solve the problem $\Psi(x_\tau^*, T-\tau)$:

$$\begin{aligned}\max_{u_1,u_2} E_\tau \Bigg\{ & \int_\tau^T \sum_{j=1}^2 g^j[s, x(s), u_1(s), u_2(s)] \exp\left[-\int_\tau^s r(y)\,dy\right] ds \\ & + \exp\left[-\int_\tau^T r(y)\,dy\right] \sum_{j=1}^2 q^j(x(T)) \Bigg\},\end{aligned} \tag{5.15}$$

subject to

$$dx(s) = f[s, x(s), u_1(s), u_2(s)]\,ds + \sigma[s, x(s)]\,dz(s),$$
$$x(\tau) = x_\tau^* \in X_\tau^*.$$

Using Theorem 5.2.1 we can obtain the following result. A set of controls $\left\{\left[\psi_1^{(\tau)*}(t, x), \psi_2^{(\tau)*}(t, x)\right], \text{ for } t \in [\tau, T]\right\}$ provides an optimal solution to the control problem $\Psi(x_\tau^*, T-\tau)$ if there exists continuously differentiable function $W^{(\tau)}(t, x) : [\tau, T] \times R^m \to R$ satisfying the following equation:

$$\begin{aligned}-W_t^{(\tau)}(t, x) - \frac{1}{2} \sum_{h,\zeta=1}^m \Omega^{h\zeta}(t, x) W_{x^h x^\zeta}^{(\tau)}(t, x) = \\ \max_{u_1,u_2} \Bigg\{ \sum_{j=1}^2 g^j[t, x, u_1(t), u_2(t)] \exp\left[-\int_\tau^t r(y)\,dy\right] \\ + W_x^{(\tau)} f[s, x(s), u_1(s), u_2(s)] \Bigg\},\end{aligned} \tag{5.16}$$

with boundary condition

$$W^{(\tau)}(T, x) = \exp\left[-\int_\tau^T r(y)\,dy\right] \sum_{j=1}^2 q^j(x).$$

Remark 5.2.1. Note that the stochastic controls are Markovian in the sense that they depend on current time and current state. Comparing the stochastic control equations in (5.16) for different values of $\tau \in [t_0, T]$, one can readily observe that:

$$\psi_i^{(\tau)*}(t, x_t^*), \psi_i^{(t_0)*}(t, x_t^*) \text{ at the point } (t, x_t^*),$$
$$\text{for } t_0 \leq \tau \leq t \leq T \text{ and } x_t^* \in X_t^*.$$

Invoking Remark 5.2.1, one can show that the cooperative control for the game $\Gamma_c(x_\tau^*, T-\tau)$ over the time interval $[\tau, T]$ is identical to the cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the same period. Therefore, the optimal state trajectory of the game $\Gamma_c(x_\tau^*, T-\tau)$ is a continuation of the optimal state trajectory of the original game $\Gamma_c(x_0, T-t_0)$.

Moreover, along the optimal trajectory $\{x^*(s)\}_{s=t_0}^{T}$, one can use Remark 5.2.1 to show that

Remark 5.2.2.

$$\begin{aligned}
&W^{(t_0)}(t, x_t^*) = \\
&E_{t_0}\Bigg\{\int_t^T \sum_{j=1}^{2} g^j\left[s, x(s), \psi_1^{(t_0)*}(s, x^*(s)), \psi_2^{(t_0)*}(s, x^*(s))\right] \\
&\qquad\qquad \times \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds \\
&\qquad\qquad + \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] \sum_{j=1}^{2} q^j(x^*(T)) \Bigg|\, x(t) = x_t^* \Bigg\} \\
&= E_t\Bigg\{\int_t^T \sum_{j=1}^{2} g^j\left[s, x(s), \psi_1^{(\tau)*}(s, x^*(s)), \psi_2^{(\tau)*}(s, x^*(s))\right] \\
&\qquad\qquad \times \exp\left[-\int_{t}^{s} r(y)\,dy\right] ds \\
&\qquad + \exp\left[-\int_{t}^{T} r(y)\,dy\right] \sum_{j=1}^{2} q^j(x^*(T)) \Bigg|\, x(t) = x_t^* \Bigg\} \exp\left[-\int_{t_0}^{t} r(y)\,dy\right] \\
&= \exp\left[-\int_{t_0}^{t} r(y)\,dy\right] W^{(\tau)}(t, x_t^*), \qquad \text{for } i \in \{1, 2\}.
\end{aligned}$$

Example 5.2.1. Consider the stochastic control problem $\Psi(x_0, T-t_0)$ which involves the maximization of the sum of the expected payoffs of Player 1 and Player 2 in Example 5.1.1:

$$E_{t_0}\left\{\int_{t_0}^{T}\left(\left[u_1(s)^{1/2}-\frac{c_1}{x(s)^{1/2}}u_1(s)\right]\right.\right.$$
$$\left.+\left[u_2(s)^{1/2}-\frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\exp\left[-r\left(t-t_0\right)\right]ds$$
$$\left.+2\exp\left[-r\left(T-t_0\right)\right]qx(T)^{1/2}\right\}, \tag{5.17}$$

subject to (5.5).

Let $\left[\psi_1^{(t_0)*}(t,x),\psi_2^{(t_0)*}(t,x)\right]$ denote a set of controls that provides a solution to the stochastic control problem $\Psi\left(x_0,T-t_0\right)$, and $W^{(t_0)}(t,x):[t_0,T]\times R^n\to R$ denote the value function that satisfies the equations (see Theorem 5.2.1):

$$-W_t^{(t_0)}(t,x)-\frac{1}{2}\sigma^2x^2W_{xx}^{(t_0)}(t,x)=$$
$$\max_{u_1,u_2}\left\{\left(\left[u_1^{1/2}-\frac{c_1}{x^{1/2}}u_1\right]+\left[u_2^{1/2}-\frac{c_2}{x^{1/2}}u_2\right]\right)\exp\left[-r\left(t-t_0\right)\right]\right.$$
$$\left.+W_x^{(t_0)}(t,x)\left[ax^{1/2}-bx-u_1-u_2\right]\right\},\text{ and}$$
$$W^{(t_0)}(T,x)=2\exp\left[-r\left(T-t_0\right)\right]qx^{1/2}. \tag{5.18}$$

Performing the indicated maximization we obtain:

$$\psi_1^{(t_0)*}(t,x)=\frac{x}{4\left[c_1+W_x^{(t_0)}\exp\left[r\left(t-t_0\right)\right]x^{1/2}\right]^2},\text{ and}$$
$$\psi_2^{(t_0)*}(t,x)=\frac{x}{4\left[c_2+W_x^{(t_0)}\exp\left[r\left(t-t_0\right)\right]x^{1/2}\right]^2}.$$

Substituting $\psi_1^{(t_0)*}(t,x)$ and $\psi_2^{(t_0)*}(t,x)$ above into (5.18) yields the value function

$$W^{(t_0)}(t,x)=\exp\left[-r\left(t-t_0\right)\right]\left[\hat{A}(t)\,x^{1/2}+\hat{B}(t)\right],$$

where

$$\dot{\hat{A}}(t)=\left[r+\frac{\sigma^2}{8}+\frac{b}{2}\right]\hat{A}(t)-\frac{1}{2\left[c_1+\hat{A}(t)/2\right]}-\frac{1}{2\left[c_2+\hat{A}(t)/2\right]}$$
$$+\frac{c_1}{4\left[c_1+\hat{A}(t)/2\right]^2}+\frac{c_2}{4\left[c_2+\hat{A}(t)/2\right]^2}$$

$$+\frac{\hat{A}(t)}{8\left[c_1+\hat{A}(t)/2\right]^2}+\frac{\hat{A}(t)}{8\left[c_2+\hat{A}(t)/2\right]^2},$$

$$\dot{\hat{B}}(t)=r\hat{B}(t)-\frac{a}{2}\hat{A}(t),\ \hat{A}(T)=2q,\ \text{and } \hat{B}(T)=0.$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^{(t_0)*}(t,x)=\frac{x}{4\left[c_1+\hat{A}(t)/2\right]^2},\ \text{and}\ \psi_2^{(t_0)*}(t,x)=\frac{x}{4\left[c_2+\hat{A}(t)/2\right]^2}.$$

Substituting these control strategies into (5.5) yields the dynamics of the state trajectory under cooperation:

$$dx(s)=\left[ax(s)^{1/2}-bx(s)-\frac{x(s)}{4\left[c_1+\hat{A}(s)/2\right]^2}-\frac{x(s)}{4\left[c_2+\hat{A}(s)/2\right]^2}\right]ds$$
$$+\sigma x(s)\,dz(s),\qquad x(t_0)=x_0. \tag{5.19}$$

Solving (5.19) yields the optimal cooperative state trajectory for $\Gamma_c(x_0,T-t_0)$ as:

$$x^*(s)=\varpi(t_0,s)^2\left[x_0^{1/2}+\int_{t_0}^{s}\varpi^{-1}(t_0,t)\,H_1dt\right]^2,$$
$$\text{for}\ \ s\in[t_0,T], \tag{5.20}$$

where

$$\varpi(t_0,s)=\exp\left[\int_{t_0}^{s}\left[H_2(\tau)-\frac{\sigma^2}{8}\right]dv+\int_{t_0}^{s}\frac{\sigma}{2}dz(v)\right],$$

$$H_1=\frac{1}{2}a,\ \text{and}$$

$$H_2(s)=-\left[\frac{1}{2}b+\frac{1}{8\left[c_1+\hat{A}(s)/2\right]^2}+\frac{1}{8\left[c_2+\hat{A}(s)/2\right]^2}+\frac{\sigma^2}{8}\right].$$

The cooperative control for the game $\Gamma_c(x_0,T-t_0)$ over the time interval $[t_0,T]$ along the optimal trajectory can be expressed as:

$$\psi_1^{(t_0)*}(t,x_t^*)=\frac{x_t^*}{4\left[c_1+\hat{A}(t)/2\right]^2},\ \text{and}\ \psi_2^{(t_0)*}(t,x_t^*)=\frac{x_t^*}{4\left[c_2+\hat{A}(t)/2\right]^2},$$
$$\text{for}\ \ t\in[t_0,T]\ \ \text{and}\ \ x_t^*\in X_t^*. \tag{5.21}$$

Following the above analysis, in the subgame $\Gamma\left(x_\tau^*, T-\tau\right)$, for $x_\tau^* \in X_\tau^*$, the optimal controls are:

$$\psi_1^{(\tau)*}\left(t, x_t^*\right) = \frac{x_t^*}{4\left[c_1 + \hat{A}(t)/2\right]^2}, \text{ and } \psi_2^{(\tau)*}\left(t, x_t^*\right) = \frac{x_t^*}{4\left[c_2 + \hat{A}(t)/2\right]^2},$$
$$\text{for } t \in [\tau, T] \text{ and } x_t^* \in X_t^*. \tag{5.22}$$

Comparing the controls in (5.21) with those in (5.22), one can verify that the cooperative controls for the game $\Gamma_c\left(x_0, T-t_0\right)$ as in (5.21) remain optimal at any time instant $\tau \in [t_0, T]$ throughout the cooperative interval $[t_0, T]$.

Thus as shown in Remark 5.2.1.

5.2.2 Individual Rationality

Assume that at time t_0 when the initial state is x_0 the agreed upon optimality principle assigns an imputation vector $\xi\left(x_0, T-t_0\right) = \left[\xi^1\left(x_0, T-t_0\right), \xi^2\left(x_0, T-t_0\right)\right]$. This means that the players agree on an imputation of the gain in such a way that the expected share of the i^{th} player over the time interval $[t_0, T]$ is equal to $\xi^i\left(x_0, T-t_0\right)$.

Individual rationality requires that

$$\xi^i\left(x_0, T-t_0\right) \geq V^{(t_0)i}\left(t_0, x_0\right), \qquad \text{for } i \in \{1, 2\}.$$

Using the same optimality principle, at time τ when the state is $x_\tau^* \in X_\tau^*$ the same optimality principle assigns an imputation vector $\xi\left(x_\tau^*, T-\tau\right) = \left[\xi^1\left(x_\tau^*, T-\tau\right), \xi^2\left(x_\tau^*, T-\tau\right)\right]$. This means that the players agree on an imputation of the gain in such a way that (viewed at time τ) the expected share of the i^{th} player over the time interval $[\tau, T]$ is equal to $\xi^i\left(x_\tau^*, T-\tau\right)$. Individual rationality is satisfied if at time τ:

$$\xi^i\left(x_\tau^*, T-\tau\right) \geq V^{(\tau)i}\left(\tau, x_\tau^*\right), \qquad \text{for } i \in \{1, 2\}.$$

In a dynamic framework, individual rationality has to be maintained at every instant of time $\tau \in [t_0, T]$ along the optimal trajectory $\{x^*(t)\}_{t=t_0}^T$.

5.3 Dynamically Stable Cooperation and the Notion of Subgame Consistency

In Chapters 3 and 4, it has been shown that a stringent requirement for solutions of cooperative differential games is time consistency or dynamic stability. In particular, the dynamic stability of a solution of a cooperative differential game is the property that, when the game proceeds along an optimal trajectory, at each instant of time the players are guided by the same optimality principles, and hence do not have any ground for deviation from the previously adopted "optimal" behavior throughout the game.

In the presence of stochastic elements, a more stringent condition – *subgame consistency* – is required for a credible cooperative solution. A cooperative solution is subgame consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal.

Cooperative stochastic differential games represent one of the most complex form of optimization analysis. This complexity leads to great difficulties in the derivation of satisfactory solutions. In the field of cooperative stochastic differential games, little research has been published to date due to the inherent difficulties in deriving tractable subgame consistent solutions. Haurie et al. (1994) derived cooperative equilibria of a stochastic differential game of fishery with the use of monitoring and memory strategies. The recent work of Yeung and Petrosyan (2004) developed a generalized method for the derivation of analytically tractable subgame consistent solutions when payoffs are transferable.

Consider the cooperative game $\Gamma_c(x_0, T - t_0)$ in which the players agree to act to maximize their joint payoff and adopt a certain mechanism governing the sharing of the players' payoffs. To achieve group rationality, the players adopt the cooperative controls $\left[\psi_1^{(t_0)*}(t,x), \psi_2^{(t_0)*}(t,x)\right]$ characterized by Theorem 5.2.1. The optimal cooperative state trajectory follows the stochastic path $\{x^*(s)\}_{s=t_0}^{T}$ in (5.13).

At time t_0, with the state being x_0, the term $\xi^{(t_0)i}(t_0, x_0)$ denotes the expected share/imputation of total cooperative payoff (received over the time interval $[t_0, T]$) to Player i guided by the agreed-upon optimality principle.

Now, consider the cooperative game $\Gamma_c(x_\tau^*, T - \tau)$ in which the game starts at time $\tau \in [t_0, T]$ with initial state $x_\tau^* \in X_\tau^*$, and the same agreed-upon optimality principle as above is adopted. Let $\xi^{(\tau)i}(\tau, x_\tau^*)$ denote the expected share/imputation of total cooperative payoff given to Player i over the time interval $[\tau, T]$.

The vectors $\xi^{(\tau)}(\tau, x_\tau^*) = \left[\xi^{(\tau)1}(\tau, x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*)\right]$, for $\tau \in [t_0, T]$, are valid imputations if the following conditions are satisfied.

Definition 5.3.1. *The vector $\xi^{(\tau)}(\tau, x_\tau^*)$ is an imputation of the cooperative game $\Gamma_c(x_\tau^*, T - \tau)$, for $\tau \in [t_0, T]$ and $x_\tau^* \in X_\tau^*$, if it satisfies:*

(i) $\xi^{(\tau)}(\tau, x_\tau^*) = \left[\xi^{(\tau)1}(\tau, x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*)\right]$, is a Pareto optimal imputation vector,
(ii) $\xi^{(\tau)i}(\tau, x_\tau^*) \geq V^{(\tau)i}(\tau, x_\tau^*)$, for $i \in \{1, 2\}$.

In particular, part (i) of Definition 5.3.1 ensures Pareto optimality, while part (ii) guarantees individual rationality.

Following Yeung and Petrosyan (2004), we formulate a payoff distribution over time so that the agreed imputations can be realized. Let the vectors

$B^\tau(s) = [B_1^\tau(s), B_2^\tau(s)]$ denote the instantaneous payoff of the cooperative game at time $s \in [\tau, T]$ for the cooperative game $\Gamma_c(x_\tau^*, T-\tau)$. In other words, Player i, for $i \in \{1,2\}$, obtains a payoff equaling $B_i^\tau(s)$ at time instant s. A terminal value of $q^i(x^*(T))$ is received by Player i at time T.

In particular, $B_i^\tau(s)$ and $q^i(x^*(T))$ constitute a payoff distribution for the game $\Gamma_c(x_\tau^*, T-\tau)$ in the sense that $\xi^{(\tau)i}(\tau, x_\tau^*)$ equals:

$$E_\tau\left\{\left(\int_\tau^T B_i^\tau(s)\exp\left[-\int_\tau^s r(y)\,dy\right]ds\right.\right.$$
$$\left.\left.+q^i(x^*(T))\exp\left[-\int_\tau^T r(y)\,dy\right]\right)\middle|\; x(\tau)=x_\tau^*\right\}, \tag{5.23}$$
$$\text{for } \; i\in\{1,2\},\; \tau\in[t_0,T] \;\text{ and }\; x_\tau^*\in X_\tau^*.$$

Moreover, for $i \in \{1,2\}$, $t \in [\tau, T]$ and $x_t^* \in X_t^*$, we use the term $\xi^{(\tau)i}(t, x_t^*)$ which equals

$$E_\tau\left\{\left(\int_t^T B_i^\tau(s)\exp\left[-\int_\tau^s r(y)\,dy\right]ds\right.\right.$$
$$\left.\left.+q^i(x^*(T))\exp\left[-\int_\tau^T r(y)\,dy\right]\right)\middle|\; x(t)=x_t^*\right\}, \tag{5.24}$$

to denote the expected present value of Player i' cooperative payoff over the time interval $[t, T]$, given that the state is $x_t^* \in X_t^*$ at time $t \in [\tau, T]$, for the game which starts at time τ with state initial x_τ^*.

Definition 5.3.2. *The vector $\xi^{(\tau)}(\tau, x_\tau^*) = \left[\xi^{(\tau)1}(\tau, x_\tau^*), \xi^{(\tau)2}(\tau, x_\tau^*)\right]$, as defined by (5.23) and (5.24), is a subgame consistent imputation of $\Gamma_c(x_\tau^*, T-\tau)$, for $\tau \in [t_0, T]$ if it satisfies $\xi^{(\tau)}(\tau, x_\tau^*)$ is a Pareto Optimal imputation vector, for $t \in [\tau, T]$ and $\xi^{(\tau)i}(t, x_t^*) \geq V^{(t)i}(t, x_t^*)$, for $i \in \{1,2\}$, $t \in [\tau, T]$ and the condition that*

$$\xi^{(\tau)i}(t,x_t^*) = \exp\left[-\int_\tau^t r(y)\,dy\right]\xi^{(t)i}(t,x_t^*),$$
$$\textit{for } \; \tau\le t\le T,\;\; i\in\{1,2\}\;\; \textit{and any } \; x_t^*\in X_t^*.$$

Subgame consistency as defined in Definition 5.3.2 guarantees that the solution imputations throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal.

Conditions ensuring subgame consistency in cooperative solutions of stochastic differential games generally are more stringent and analytically intractable than those ensuring time consistency in cooperative solutions of differential games. The recent work of Yeung and Petrosyan (2004) developed a generalized theorem for the derivation of analytically tractable "payoff distribution procedure" of subgame consistent solution. Being capable of deriving analytical tractable solutions, the work is not only theoretically interesting but would enable the hitherto intractable problems in cooperative stochastic differential games to be fruitfully explored. A payment mechanism leading to the realization of the imputation scheme $\xi^{(\tau)}(\tau, x_\tau^*)$ satisfying Definition 5.3.2 has to be formulated. This will be done in the next section.

5.4 Transitory Compensation and Payoff Distribution Procedures

A payoff distribution procedure (PDP) of the cooperative game as specified in (5.23) and (5.24) must be formulated so that the agreed imputations can be realized.

For Definition 5.3.2 to hold, it is required that $B_i^\tau(s) = B_i^t(s)$, for $i \in \{1,2\}$ and $\tau \in [t_0, T]$ and $t \in [t_0, T]$ and $\tau \neq t$. Adopting the notation $B_i^\tau(s) = B_i^t(s) = B_i(s)$ and applying Definition 5.3.2, the PDP of a subgame consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau^*)$ has to satisfy the following condition.

Corollary 5.4.1. *The PDP with $B(s)$ and $q(x^*(T))$ corresponding to the subgame consistent imputation vectors $\xi^{(\tau)}(\tau, x_\tau^*)$ must satisfy the following conditions:*

(i)

$$\sum_{j=1}^{2} B_i(s) = \sum_{j=1}^{2} g^j\left[s, x_s^*, \psi_1^{(\tau)*}(s, x_s^*), \psi_2^{(\tau)*}(s, x_s^*)\right],$$
$$\text{for } s \in [t_0, T];$$

(ii)

$$E_\tau\left\{\int_\tau^T B_i(s)\exp\left[-\int_\tau^s r(y)\,dy\right]ds\right.$$
$$\left.+q^i(x^*(T))\exp\left[-\int_\tau^T r(y)\,dy\right]\middle|\, x(\tau) = x_\tau^*\right\} \geq V^{(\tau)i}(\tau, x_\tau^*),$$
$$\text{for } i \in \{1,2\},\ \ \tau \in [t_0, T] \ \text{ and all } x_\tau^* \in X_\tau^*; \text{ and}$$

(iii)

$$\xi^{(\tau)i}(\tau, x_\tau^*) = E_\tau\left\{\int_\tau^{\tau+\Delta t} B_i(s)\exp\left[-\int_\tau^s r(y)\,dy\right]ds\right.$$
$$\left.+\exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right]\xi^{(\tau+\Delta t)i}(\tau+\Delta t, x_\tau^*+\Delta x_\tau^*)\middle|\, x(\tau) = x_\tau^*\right\},$$

$$for \ \ \tau \in [t_0, T] \, , \ i \in \{1,2\} \ \ and \ all \ \ x_\tau^* \in X_\tau^*;$$

where

$$\Delta x_\tau^* = f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right] \Delta t + \sigma\left[\tau, x_\tau^*\right] \Delta z_\tau + o(\Delta t),$$
$$\Delta z_\tau = z(\tau + \Delta t) - z(\tau), \ \ and \ \ E_\tau\left[o(\Delta t)\right]/\Delta t \to 0 \ \ as \ \ \Delta t \to 0.$$

Consider the following condition concerning $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$:

Condition 5.4.1. *For $i \in \{1,2\}$ and $t \geq \tau$ and $\tau \in [t_0, T]$, the terms $\xi^{(\tau)i}(t, x_t^*)$ are functions that are continuously twice differentiable in t and x_t^*.*

If the imputations $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$ and $x_\tau^* \in X_\tau^*$, satisfy Condition 5.4.1, one can obtain the following relationship:

$$E_\tau\left\{\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds \,\middle|\, x(\tau) = x_\tau^*\right\}$$
$$= E_\tau\left\{\xi^{(\tau)i}(\tau, x_\tau^*) - \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right] \xi^{(\tau+\Delta t)i}(\tau + \Delta t, x_\tau^* + \Delta x_\tau^*)\right\}$$
$$= E_\tau\left\{\xi^{(\tau)i}(\tau, x_\tau^*) - \xi^{(\tau)i}(\tau + \Delta t, x_\tau^* + \Delta x_\tau^*)\right\},$$
$$\text{for all } \ \tau \in [t_0, T] \text{ and } i \in \{1,2\}. \tag{5.25}$$

With $\Delta t \to 0$, condition (5.25) can be expressed as:

$$E_\tau\left\{B_i(\tau)\,\Delta t + o(\Delta t)\right\} =$$
$$E_\tau\Bigg\{ -\left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] \Delta t$$
$$-\left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right] \Delta t$$
$$-\frac{1}{2} \sum_{h,\zeta=1}^m \Omega^{h\zeta}(\tau, x_\tau^*) \left[\xi_{x_t^h x_t^\zeta}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] \Delta t$$
$$-\left[\xi_{x_t}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] \sigma\left[\tau, x_\tau^*\right] \Delta z_\tau - o(\Delta t) \Bigg\}. \tag{5.26}$$

Dividing (5.26) throughout by Δt, with $\Delta t \to 0$, and taking expectation yield

$$B_i(\tau) =$$
$$-\left[\xi_t^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] - \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*)\Big|_{t=\tau}\right] f\left[\tau, x_\tau^*, \psi_1^{(\tau)*}(\tau, x_\tau^*), \psi_2^{(\tau)*}(\tau, x_\tau^*)\right]$$

$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(\tau,x_{\tau}^{*}\right)\left[\xi_{x_t^h x_t^\zeta}^{(\tau)i}\left(t,x_t^{*}\right)\Big|_{t=\tau}\right],$$

$$\text{for } i\in\{1,2\}. \tag{5.27}$$

Therefore, one can establish the following theorem.

Theorem 5.4.1. *If the solution imputations $\xi^{(\tau)i}\left(\tau,x_{\tau}^{*}\right)$, for $i \in \{1,2\}$ and $\tau \in [t_0,T]$, satisfy Definition 5.3.2 and Condition 5.4.1, a PDP with a terminal payment $q^i\left(x_T^{*}\right)$ at time T and an instantaneous payment at time $\tau \in [t_0,T]$ and $x_{\tau}^{*} \in X_{\tau}^{*}$:*

$$B_i(\tau) =$$

$$-\left[\xi_t^{(\tau)i}\left(t,x_t^{*}\right)\Big|_{t=\tau}\right]-\left[\xi_{x_t^*}^{(\tau)i}\left(t,x_t^{*}\right)\Big|_{t=\tau}\right]f\left[\tau,x_{\tau}^{*},\psi_1^{(\tau)*}\left(\tau,x_{\tau}^{*}\right),\psi_2^{(\tau)*}\left(\tau,x_{\tau}^{*}\right)\right],$$

$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(\tau,x_{\tau}^{*}\right)\left[\xi_{x_t^h x_t^\zeta}^{(\tau)i}\left(t,x_t^{*}\right)\Big|_{t=\tau}\right],$$

$$\textit{for } i\in\{1,2\},$$

yields a subgame consistent solution to the cooperative game $\Gamma_c\left(x_0,T-t_0\right)$.

5.5 Subgame Consistent Solutions under Specific Optimality Principles

In this section, we consider subgame consistent solutions under specific optimality principles.

5.5.1 The Nash Bargaining/Shapley Value Solution

Consider a cooperative game $\Gamma_c\left(x_0,T-t_0\right)$ in which the players agree to maximize the sum of their expected payoffs and divide the total cooperative payoff satisfying the Nash bargaining outcome – that is, they maximize the product of the expected individual gains in excess of the noncooperative payoffs. This scheme also coincides with the Shapley value in two-player games. The imputation scheme has to satisfy:

Proposition 5.5.1. *In the game $\Gamma_c\left(x_0,T-t_0\right)$, an imputation*

$$\xi^{(t_0)i}\left(t_0,x_0\right)=V^{(t_0)i}\left(t_0,x_0\right)+\frac{1}{2}\left[W^{(t_0)}\left(t_0,x_0\right)-\sum_{j=1}^{2}V^{(t_0)j}\left(t_0,x_0\right)\right],$$

is assigned to Player i, for $i\in\{1,2\}$;
and in the subgame $\Gamma_c\left(x_{\tau}^{*},T-\tau\right)$, for $\tau\in(t_0,T]$ and $x_{\tau}^{*}\in X_{\tau}^{*}$, an imputation

$$\xi^{(\tau)i}\left(\tau,x_\tau^*\right)=V^{(\tau)i}\left(\tau,x_\tau^*\right)+\frac{1}{2}\left[W^{(\tau)}\left(\tau,x_\tau^*\right)-\sum_{j=1}^{2}V^{(\tau)j}\left(\tau,x_\tau^*\right)\right],$$

is assigned to Player i, for $i\in\{1,2\}$.

Note that each player will receive a payoff equaling his expected noncooperative payoff plus half of the expected cooperative gains over the period $[\tau,T]$, for $\tau\in[t_0,T]$.

One can readily verify that $\xi^{(\tau)i}\left(\tau,x_\tau^*\right)$ in Proposition 5.5.1 satisfies Definition 5.3.2. Moreover, employing Remarks 5.1.1 and 5.2.1, one has:

$$\begin{aligned}
&\xi^{(t)i}\left(t,x_t^*\right)=\\
&\qquad\exp\left[\int_\tau^t r(y)\,dy\right]\left\{V^{(\tau)i}\left(t,x_t^*\right)+\frac{1}{2}\left[W^{(\tau)}\left(t,x_t^*\right)-\sum_{j=1}^{2}V^{(\tau)j}\left(t,x_t^*\right)\right]\right\}\\
&\quad=\exp\left[\int_\tau^t r(y)\,dy\right]\xi^{(\tau)i}\left(t,x_t^*\right),\\
&\qquad\qquad\text{for }\; t_0\le\tau\le t\;\text{ and }\; x_t^*\in X_t^*.
\end{aligned}\tag{5.28}$$

Hence, $\xi^{(\tau)i}\left(\tau,x_\tau^*\right)$ as in Proposition 5.5.1 is a subgame consistent imputations for the cooperative game $\Gamma_c\left(x_0,T-t_0\right)$. Using Theorem 5.4.1 one obtains:

Corollary 5.5.1. *A PDP with a terminal payment $q^i\left(x\left(T\right)\right)$ at time T and an instantaneous imputation rate at time $\tau\in[t_0,T]$:*

$$\begin{aligned}
&B_i(\tau)=\\
&\quad\frac{-1}{2}\left\{\left[V_t^{(\tau)i}\left(t,x_t^*\right)\Big|_{t=\tau}\right]+\left[W_t^{(\tau)}\left(t,x_t^*\right)\Big|_{t=\tau}\right]-\left[V_t^{(\tau)j}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right\}\\
&\quad-\frac{1}{2}\left\{\left[V_{x_t}^{(\tau)i}\left(t,x_t^*\right)\Big|_{t=\tau}\right]+\left[W_{x_t}^{(\tau)}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right.\\
&\qquad\left.-\left[V_{x_t}^{(\tau)j}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right\}f\left[\tau,x_\tau,\psi_1^{(\tau)*}\left(\tau,x_\tau^*\right),\psi_2^{(\tau)*}\left(\tau,x_\tau^*\right)\right]\\
&\quad-\frac{1}{4}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(\tau,x_\tau^*\right)\left\{\left[V_{x_t^h x_t^\zeta}^{(\tau)i}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right.\\
&\qquad\left.+\left[W_{x_t^h x_t^\zeta}^{(\tau)}\left(t,x_t^*\right)\Big|_{t=\tau}\right]-\left[V_{x_t^h x_t^\zeta}^{(\tau)j}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right\},\\
&\qquad\qquad\textit{for }\; i,j\in\{1,2\}\;\;\textit{and}\;\; i\neq j,
\end{aligned}\tag{5.29}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c\left(x_0,T-t_0\right)$, in which the players agree to divide their cooperative gains according to Proposition 5.5.1.

5.5.2 Proportional Distributions

Though one of the most commonly used allocation principles is the Shapley value, however, in the case when players may be asymmetric in their sizes of noncooperative payoffs. Equal imputation of cooperative gains may not be totally agreeable to asymmetric player. To overcome this, we consider the allocation principle in which the players' shares of the gain from cooperation are proportional to the relative sizes of their expected noncooperative profits. One can readily verify that this allocation principle allocates the cartel profit according to the relative sizes of the players' expected noncooperative payoffs. The imputation scheme has to satisfy:

Proposition 5.5.2. *In the game $\Gamma_c(x_0, T-t_0)$, an imputation*

$$\begin{aligned}&\xi^{(t_0)i}(t_0,x_0)=\\&V^{(t_0)i}(t_0,x_0)+\frac{V^{(t_0)i}(t_0,x_0)}{\sum_{j=1}^{2}V^{(t_0)j}(t_0,x_0)}\left[W^{(t_0)}(t_0,x_0)-\sum_{j=1}^{2}V^{(t_0)j}(t_0,x_0)\right]\\&=\frac{V^{(t_0)i}(t_0,x_0)}{\sum_{j=1}^{2}V^{(t_0)j}(t_0,x_0)}W^{(t_0)}(t_0,x_0)\end{aligned}$$

is assigned to Player i, for $i \in \{1,2\}$;
and in the subgame $\Gamma_c(x_\tau^*, T-\tau)$, for $\tau \in (t_0, T]$ and $x_\tau^* \in X_\tau^*$, an imputation

$$\xi^{(\tau)i}(\tau,x_\tau^*)=\frac{V^{(\tau)i}(\tau,x_\tau^*)}{\sum_{j=1}^{2}V^{(\tau)j}(\tau,x_\tau^*)}W^{(\tau)}(\tau,x_\tau^*),$$

is assigned to Player i, for $i \in \{1,2\}$.

Note that each player will receive a payoff equaling its expected noncooperative payoff plus a share of the cooperative gain proportional to the sizes of their expected noncooperative payoffs over the period $[\tau, T]$, for $\tau \in [t_0, T]$.

One can readily verify that $\xi^{(\tau)i}(\tau, x_\tau^*)$ in Proposition 5.5.2 satisfies Definition 5.3.2. Moreover, employing Remarks 5.1.1 and 5.2.1, one has:

$$\begin{aligned}\xi^{(t)i}(t,x_t^*)&=\exp\left[\int_\tau^t r(y)\,dy\right]\frac{V^{(\tau)i}(t,x_t^*)}{\sum_{j=1}^{2}V^{(\tau)j}(t,x_t^*)}W^{(\tau)}(t,x_t^*)\\&=\exp\left[\int_\tau^t r(y)\,dy\right]\xi^{(\tau)i}(t,x_t^*),\\&\quad\text{for } t_0\le\tau\le t \text{ and } x_t^*\in X_t^*.\end{aligned}\tag{5.30}$$

Hence, $\xi^{(\tau)i}(\tau, x_\tau^*)$ as in Proposition 5.5.2 is a subgame consistent imputations for the cooperative game $\Gamma_c(x_0, T-t_0)$.

Taking differentiation on $\xi^{(\tau)i}(\tau, x_\tau^*)$, we obtain:

$$\left[\xi_t^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]=$$

$$\frac{V^{(\tau)j}(\tau,x_\tau^*)\left[V_t^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-V^{(\tau)i}(\tau,x_\tau^*)\left[V_t^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]}{\left[V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)\right]^2}W^{(\tau)}(\tau,x_\tau^*)$$

$$+\frac{V^{(\tau)i}(\tau,x_\tau^*)}{V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)}\left[W_t^{(\tau)}(t,x_t^*)\Big|_{t=\tau}\right],$$

$$\left[\xi_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]=$$

$$\frac{V^{(\tau)j}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-V^{(\tau)i}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]}{\left[V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)\right]^2}W^{(\tau)}(\tau,x_\tau^*)$$

$$+\frac{V^{(\tau)i}(\tau,x_\tau^*)}{V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)}\left[W_{x_t^*}^{(\tau)}(t,x_t^*)\Big|_{t=\tau}\right], \text{ and}$$

$$\left[\xi_{x_t^*x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]=$$

$$\frac{V^{(\tau)j}(\tau,x_\tau^*)\left[V_{x_t^*x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-V^{(\tau)i}(\tau,x_\tau^*)\left[V_{x_t^*x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]}{\left[V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)\right]^2}W^{(\tau)}(\tau,x_\tau^*)$$

$$-2\left\{\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]+\left[V_{x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]\right\}\times$$

$$\frac{V^{(\tau)j}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-V^{(\tau)i}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]}{\left[V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)\right]^3}W^{(\tau)}(\tau,x_\tau^*)$$

$$+2\frac{V^{(\tau)j}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-V^{(\tau)i}(\tau,x_\tau^*)\left[V_{x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]}{\left[V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)\right]^2}$$

$$\times\left[W_{x_t^*}^{(\tau)}(t,x_t^*)\Big|_{t=\tau}\right]$$

$$+\frac{V^{(\tau)i}(\tau,x_\tau^*)}{V^{(\tau)i}(\tau,x_\tau^*)+V^{(\tau)j}(\tau,x_\tau^*)}\left[W_{x_t^*x_t^*}^{(\tau)}(t,x_t^*)\Big|_{t=\tau}\right].$$

Upon substituting these derivatives into Theorem 5.4.1 yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0,T-t_0)$, in which the players agree to divide their cooperative gains according to Proposition 5.5.2.

5.6 An Application in Cooperative Resource Extraction under Uncertainty

In this section, we illustrate the derivation of PDP of subgame consistent solutions in which the players agree to divide their cooperative gains according to Propositions 5.5.1 and 5.5.2 in a resource extraction game with stochastic elements. Consider the game in Example 5.1.1 in which the two extractors agree to maximize the sum of their payoffs and divide the total cooperative payoff according to Proposition 5.5.1.

Using the results in Example 5.1.1, Example 5.2.1 and Theorem 5.4.1 we obtain:

Corollary 5.6.1. *A PDP with a terminal payment $q^i\left(x^*\left(T\right)\right)$ at time T and an instantaneous imputation rate at time $\tau \in [t_0, T]$:*

$$B_i\left(\tau\right) = \frac{-1}{2}\Bigg\{\left(\left[\dot{A}_i\left(\tau\right)\left(x_\tau^*\right)^{1/2} + \dot{B}_i\left(\tau\right)\right] + r\left[A_i\left(\tau\right)\left(x_\tau^*\right)^{1/2} + B_i\left(\tau\right)\right]\right)$$

$$+\left[\frac{1}{2}A_i\left(\tau\right)\left(x_\tau^*\right)^{-1/2}\right]$$

$$\times\left[a\left(x_\tau^*\right)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}\left(\tau\right)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}\left(\tau\right)/2\right]^2}\right]$$

$$+\frac{1}{8}\sigma^2 A_i\left(\tau\right)\left(x_\tau^*\right)^{1/2}\Bigg\}$$

$$-\frac{1}{2}\Bigg\{\left(\left[\dot{\hat{A}}\left(\tau\right)\left(x_\tau^*\right)^{1/2} + \dot{\hat{B}}\left(\tau\right)\right] + r\left[\hat{A}\left(\tau\right)\left(x_\tau^*\right)^{1/2} + \hat{B}\left(\tau\right)\right]\right)$$

$$+\left[\frac{1}{2}\hat{A}\left(\tau\right)\left(x_\tau^*\right)^{-1/2}\right]$$

$$\times\left[a\left(x_\tau^*\right)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}\left(\tau\right)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}\left(\tau\right)/2\right]^2}\right]$$

$$
\left. +\frac{1}{8}\sigma^2 \hat{A}(\tau)(x_\tau^*)^{1/2} \right\}
$$

$$
+\frac{1}{2}\left\{ \left(\left[\dot{A}_j(\tau)(x_\tau^*)^{1/2} + \dot{B}_j(\tau) \right] + r\left[A_j(\tau)(x_\tau^*)^{1/2} + B_j(\tau) \right] \right) \right.
$$

$$
+ \left[\frac{1}{2} A_j(\tau)(x_\tau^*)^{-1/2} \right]
$$

$$
\times \left[a(x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_i + \hat{A}(\tau)/2\right]^2} - \frac{x_\tau^*}{4\left[c_j + \hat{A}(\tau)/2\right]^2} \right]
$$

$$
\left. +\frac{1}{8}\sigma^2 A_j(\tau)(x_\tau^*)^{1/2} \right\},
$$

$$
\text{for } i,j \in \{1,2\} \text{ and } i \neq j, \tag{5.31}
$$

yields a subgame consistent solution to the cooperative game $\overline{\Gamma}_c(x_0, T - t_0)$ in which the players agree to divide their cooperative gains according to Proposition 5.5.1.

5.7 An Exegesis of Transitory Compensation under Uncertainty

In this section, we examine the economic explanation of equilibrating transitory compensation in Theorem 5.4.1. Consider a cooperative scheme $\Gamma_c(x_0, T - t_0)$ in which the players agree to maximize the sum of their expected payoffs and divide the total cooperative payoff according to a certain imputation mechanism like those in Propositions 5.5.1 and 5.5.2. In a more general setting, $\xi^{(\tau)i}(\tau, x_\tau^*)$ may be expressed as a function of the expected cooperative payoff and the expected individual noncoopeative payoffs. In particular

$$
\begin{aligned}
&\xi^{(\tau)i}(\tau, x_\tau^*) = \omega^{(\tau)i}\left[W^{(\tau)}(\tau, x_\tau^*), V^{(\tau)i}(\tau, x_\tau^*), V^{(\tau)j}(\tau, x_\tau^*) \right], \text{ and} \\
&\xi^{(\tau)i}(t, x_t^*) = \omega^{(\tau)i}\left[W^{(\tau)}(t, x_t^*), V^{(\tau)i}(t, x_t^*), V^{(\tau)j}(t, x_t^*) \right], \\
&\qquad \text{for } i \in \{1,2\}.
\end{aligned} \tag{5.32}
$$

If $\omega^{(\tau)i}(t, x_t^*)$ is continuously differentiable in $W^{(\tau)}(t, x_t^*)$, $V^{(\tau)i}(t, x_t^*)$, and $V^{(\tau)j}(t, x_t^*)$, then Condition 5.4.1 is satisfied because the latter three expressions are continuously differentiable in t and x_t^*. Moreover, $\omega_W^{(\tau)i}(t, x_t^*) \geq 0$, $\omega_{V^i}^{(\tau)i}(t, x_t^*) \geq 0$ and $\omega_{V^j}^{(\tau)i}(t, x_t^*) \leq 0$.

Using Theorem 5.4.1 we obtain the equilibrating transition formula:

Formula 5.7.1.

$$
\begin{aligned}
B_i(\tau) = {} & \omega_W^{(\tau)i}(\tau, x_\tau^*) \sum_{j=1}^{2} g^j\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \\
& + \omega_{V^i}^{(\tau)i}(\tau, x_\tau^*) \left\{ g^i\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \right. \\
& + \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] \\
& \left. \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \right\} \\
& + \omega_{V^j}^{(\tau)i}(\tau, x_\tau^*) \left\{ g^j\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] \right. \\
& + \left[\xi_{x_t^*}^{(\tau)j}(t, x_t^*) \Big|_{t=\tau} \right] \\
& \left. \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right) \right\}.
\end{aligned}
\tag{5.33}
$$

Proof. See Appendix to Chapter 5.

Formula 5.7.1 provides the components of the equilibrating transitory compensation in economically interpretable terms. $\omega_W^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of total expected cooperative payoff that Player i is entitled to received according to the solution optimality principle. $\omega_{V^i}^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of his own expected payoff that Player i is entitled to received according to the optimality principle. $\omega_{V^j}^{(\tau)i}(\tau, x_\tau^*)$ is the marginal share of the other player's expected payoff that Player i is entitled to received according to agreed upon optimality principle.

The term $\sum_{j=1}^{2} g^j\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right]$ is the instantaneous cooperative payoff and $g^i\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right]$ is the instantaneous noncooperative payoffs of Player i given that the state at time τ is $x_\tau^* \in X_\tau^*$. The term

$$
\begin{aligned}
& \left[\xi_{x_t^*}^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] \\
& \quad \times \left(f\left[\tau, x_\tau^*, \phi_1^*(\tau, x_\tau^*), \phi_2^*(\tau, x_\tau^*)\right] - f\left[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)\right] \right)
\end{aligned}
$$

reflects the instantaneous effect on Player i's noncooperative expected payoff when the change in the state variable x_τ^* follows the cooperative trajectory governed by (5.13) instead of the noncooperative path (5.4).

Therefore the compensation $B_i(\tau)$ Player i receives at time τ given the state $x_\tau^* \in X_\tau^*$ is the sum of

(i) Player i's agreed upon marginal share of total expected cooperative profit,
(ii) Player i's agreed upon marginal share of his own expected nonccoperative profit plus the instantaneous effect on his noncooperative expected payoff when the change in the state variable x_τ^* follows the cooperative trajectory instead of the noncooperative path, and
(iii) Player i's agreed upon marginal share of Player j's nonccoperative profit plus the instantaneous effect on Player j's noncooperative payoff when the change in the state variable x follows the optimal trajectory instead of the noncooperative path.

5.8 Infinite-Horizon Cooperative Stochastic Differential Games

As discussed in Chapter 4, in many game situations, the terminal time of the game, T, is either very far in the future or unknown to the players. A way to resolve the problem, as suggested by Dockner et al (2000), is to set $T = \infty$. In this section, we examine cooperative stochastic differential games with infinite horizon.

Consider the two-person nonzero-sum stochastic differential game with objective

$$E_{t_0}\left\{\int_{t_0}^{\infty} g^i\left[x(s), u_1(s), u_2(s)\right] \exp\left[-r(s-t_0)\right] ds\right\},$$
$$\text{for } \; i \in \{1,2\} \tag{5.34}$$

and state dynamics

$$dx(s) = f\left[x(s), u_1(s), u_2(s)\right] ds + \sigma\left[x(s)\right] dz(s),$$
$$x(t_0) = x_0, \tag{5.35}$$

where $\sigma[x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[x(s)] = \sigma[x(s)]\,\sigma[x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[x(s)]$. $u_i \in U^i \subset compR^l$ is the control vector of Player i, for $i \in \{1,2\}$.

Since s does not appear in $g^i[x(s), u_1(s), u_2(s)]$ and the state dynamics, the game (5.34)–(5.35) is an autonomous problem. Consider the alternative game $\Gamma(x)$:

$$\max_{u_i} E_t \left\{ \int_t^{\infty} g^i \left[x(s), u_1(s), u_2(s) \right] \exp\left[-r(s-t) \right] ds \right\},$$

$$\text{for } i \in \{1, 2\}$$

subject to

$$dx(s) = f\left[x(s), u_1(s), u_2(s) \right] ds + \sigma\left[x(s) \right] dz(s), \quad x(t) = x.$$

The infinite-horizon autonomous problem $\Gamma(x)$ is independent of the choice of t and dependent only upon the state at the starting time x.

Invoking Theorem 2.7.1 a noncooperative feedback Nash equilibrium solution can be characterized as:

Theorem 5.8.1. *A set of strategies $\{\phi_1^*(x), \phi_2^*(x)\}$ constitutes a Nash equilibrium solution to the game $\Gamma(x)$, if there exist functionals $V^1(x) : R^m \to R$ and $V^2(x) : R^m \to R$, satisfying the following set of partial differential equations:*

$$\begin{aligned} r(t) V^i(x) - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x) V^i_{x^h x^\zeta}(x) = \\ \max_{u_i} \left\{ g^i \left[x, u_i, \phi_j^*(x) \right] + V^i_x(x) f\left[x, u_i, \phi_j^*(x) \right] \right\}, \\ i \in \{1, 2\} \quad \textit{and} \quad j \in \{1, 2\} \quad \textit{and} \quad j \neq i. \end{aligned}$$

In particular,

$$V^i(x) = E_t \left\{ \int_t^{\infty} g^i \left[x(s), \phi_1^*(s), \phi_2^*(s) \right] \exp\left[-r(s-t) \right] ds \middle| x(t) = x \right\}$$

represents the current-value payoffs of Player i at current time $t \in [t_0, \infty]$, given that the state is x at t.

Now consider the case when the players agree to cooperate. Let $\Gamma_c(x)$ denote a cooperative game with the game structure of $\Gamma(x)$ with the initial state being x. The players agree to act according to an agreed upon optimality principle.

To achieve group rationality the players agree to maximize the sum of their expected payoffs, that is

$$\max_{u_1, u_2} E_t \left\{ \int_t^{T} \sum_{j=1}^{2} g^j \left[x(s), u_1(s), u_2(s) \right] \exp\left[-r(s-t) \right] ds \right\}, \tag{5.36}$$

subject to $dx(s) = f\left[x(s), u_1(s), u_2(s) \right] ds + \sigma\left[x(s) \right] dz(s)$ and $x(t) = x$.

Following Theorem 2.1.6, we obtain:

Theorem 5.8.2. *A set of controls* $\{[\psi_1^*(x), \psi_2^*(x)]\}$ *provides a solution to the stochastic control problem associated with* $\Gamma_c(x)$ *if there exists continuously differentiable function* $W(x) : R^m \to R$ *satisfying the equation:*

$$rW(x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x) W_{x^h x^\zeta}(x) =$$
$$\max_{u_1,u_2}\left\{\sum_{j=1}^{2} g^j[x, u_1, u_2] + W_x f[x, u_1, u_2]\right\}.$$

Hence the players will adopt the cooperative control $[\psi_1^*(x), \psi_2^*(x)]$ characterized in Theorem 5.8.2. Note that these controls are functions of the current state x only. Substituting this set of control into (5.35) yields the dynamics of the optimal (cooperative) trajectory as;

$$dx(s) = f[x(s), \psi_1^*(x(s)), \psi_2^*(x(s))]\,ds + \sigma[x(s)]\,dz(s),$$
$$x(t) = x.$$

Consider the case at time t_0, where $x(t_0) = x_0$, we have

$$dx(s) = f[x(s), \psi_1^*(x(s)), \psi_2^*(x(s))]\,ds + \sigma[x(s)]\,dz(s),$$
$$x(t_0) = x_0. \tag{5.37}$$

The solution to (5.37) can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[x^*(s), \psi_1^{(t_0)*}(s, x^*(s)), \psi_2^{(t_0)*}(s, x^*(s))\right] ds$$
$$+ \int_{t_0}^{t} \sigma[x^*(s)]\,dz(s). \tag{5.38}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (5.38). The term x_t^* is used to denote an element in the set X_t^*.

Assume that at time $t\,(\geq t_0)$ when the initial state is $x_t^* \in X_t^*$ the agreed upon optimality principle assigns an imputation vector $\xi(x_t^*) = \left[\xi^1(x_t^*), \xi^2(x_t^*)\right]$. This means that the players agree on an imputation of the gains in such a way that the expected share of the i^{th} player is equal to $\xi^i(x_t^*)$.

Individual rationality requires that

$$\xi^i(x_t^*) \geq V^i(x_t^*), \qquad \text{for } i \in \{1, 2\}.$$

Following Petrosyan (1997) and Yeung and Petrosyan (2004), we use $B(s) = [B_1(s), B_2(s)]$ denote the instantaneous payoff of the cooperative game at time $s \in [t_0, \infty)$ for the cooperative game $\Gamma_c\left(x_{t_0}^*\right)$.

In particular, along the cooperative trajectory $\{x^*(t)\}_{t\geq t_0}$

$$\xi^i\left(x_\tau^*\right) = E_\tau\left\{\int_\tau^\infty B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right]ds \middle| x\left(\tau\right) = x_\tau^*\right\},$$

$$\text{for } i \in \{1,2\}, \text{ and } x_\tau^* \in X_\tau^*; \text{ and}$$

$$\xi^i\left(x_t^*\right) = E_t\left\{\int_t^\infty B_i\left(s\right)\exp\left[-r\left(s-t\right)\right]ds \middle| x\left(t\right) = x_t^*\right\},$$

$$\text{for } i \in \{1,2\}, \; x_t^* \in X_t^* \text{ and } t \geq \tau.$$

We further define

$$\gamma^i\left(\tau;\tau,x_\tau^*\right) = E_\tau\left\{\int_\tau^\infty B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right]ds \middle| x\left(\tau\right) = x_\tau^*\right\} = \xi^i\left(x_\tau^*\right),$$

and

$$\gamma^i\left(\tau;t,x_t^*\right) = E_\tau\left\{\int_t^\infty B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right]ds \middle| x\left(t\right) = x_t^*\right\},$$

$$\text{for } i \in \{1,2\} \text{ and } \tau \in [t_0,\infty).$$

Note that

$$\begin{aligned}
\gamma^i\left(\tau;t,x_t^*\right) &= \exp\left[-r\left(t-\tau\right)\right]E_t\left\{\int_t^\infty B_i\left(s\right)\exp\left[-r\left(s-t\right)\right]ds \middle| x\left(t\right) = x_t^*\right\} \\
&= \exp\left[-r\left(t-\tau\right)\right]\xi^i\left(x_t^*\right) \\
&= \exp\left[-r\left(t-\tau\right)\right]\gamma^i\left(t;t,x_t^*\right), \\
&\quad \text{for } i \in \{1,2\} \text{ and any } x_t^* \in X_t^*.
\end{aligned} \tag{5.39}$$

The condition in (5.39) guarantees subgame consistency of the solution imputations throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. Moreover, group and individual rationality are also required to be satisfied throughout the entire game interval.

Following the analysis in Section 5.4, we have

$$\begin{aligned}
\gamma^i\left(\tau;\tau,x_\tau^*\right) = E_\tau\Bigg\{&\int_\tau^{\tau+\Delta t} B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right]ds \\
&+ \exp\left[-r\left(\Delta t\right)\right]\gamma^i\left(\tau+\Delta t;\tau+\Delta t,x_\tau^*+\Delta x_\tau^*\right)\Bigg| x\left(\tau\right) = x_\tau^*\Bigg\}, \\
&\text{for } \tau \in [t_0,T], \; x_\tau^* \in X_\tau^* \text{ and } i \in \{1,2\};
\end{aligned} \tag{5.40}$$

where

$$\Delta x_\tau^* = f\left[x_\tau^*, \psi_1^*\left(x_\tau^*\right), \psi_2^*\left(x_\tau^*\right)\right]\Delta t + \sigma\left[x_\tau^*\right]\Delta z_\tau + o\left(\Delta t\right),$$
$$\Delta z_\tau = z\left(\tau + \Delta t\right) - z\left(\tau\right), \text{ and } E_\tau\left[o\left(\Delta t\right)\right]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

From (5.39), we have

$$\begin{aligned}\gamma^i\left(\tau; \tau + \Delta t, x_\tau^* + \Delta x_\tau^*\right) &= \exp\left[-r\left(\Delta t\right)\right]\xi^i\left(x_\tau^* + \Delta x_\tau^*\right) \\ &= \exp\left[-r\left(\Delta t\right)\right]\gamma^i\left(\tau + \Delta t; \tau + \Delta t, x_\tau^* + \Delta x_\tau^*\right). \quad (5.41)\end{aligned}$$

Therefore (5.40) becomes

$$\begin{aligned}\gamma^i\left(\tau; \tau, x_\tau^*\right) = E_\tau \Bigg\{ &\int_\tau^{\tau+\Delta t} B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right] ds \\ &+ \gamma^i\left(\tau; \tau + \Delta t, x_\tau^* + \Delta x_\tau^*\right) \Bigg| x\left(\tau\right) = x_\tau^* \Bigg\}, \\ &\text{for } \tau \in \left[t_0, T\right], \; x_\tau^* \in X_\tau^* \text{ and } i \in \{1, 2\}. \quad (5.42)\end{aligned}$$

One can obtain the following relationship:

$$\begin{aligned}E_\tau \Bigg\{ &\int_\tau^{\tau+\Delta t} B_i\left(s\right)\exp\left[-r\left(s-\tau\right)\right] ds \Bigg| x\left(\tau\right) = x_\tau^* \Bigg\} = \\ &\gamma^i\left(\tau; \tau, x_\tau^*\right) - \gamma^i\left(\tau; \tau + \Delta t, x_\tau^* + \Delta x_\tau^*\right), \\ &\text{for all } \tau \in \left[t_0, T\right], \; x_\tau^* \in X_\tau^* \text{ and } i \in \{1, 2\}. \quad (5.43)\end{aligned}$$

With $\Delta t \to 0$, condition (5.43) can be expressed as:

$$\begin{aligned}B_i\left(\tau\right)\Delta t = &-\left[\gamma_t^i\left(\tau; t, x_t^*\right)\big|_{t=\tau}\right]\Delta t - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x_\tau^*\right)\left[\gamma_{x_t^h x_t^\zeta}^i\left(\tau; t, x_t^*\right)\Big|_{t=\tau}\right]\Delta t \\ &-\left[\gamma_{x_t^*}^i\left(\tau; t, x_t^*\right)\Big|_{t=\tau}\right] f\left[x_\tau^*, \psi_1^*\left(x_\tau^*\right), \psi_2^*\left(x_\tau^*\right)\right]\Delta t - o\left(\Delta t\right). \quad (5.44)\end{aligned}$$

Dividing (5.44) throughout by Δt, with $\Delta t \to 0$, and taking expectation yield

$$\begin{aligned}B_i\left(\tau\right) = &-\left[\gamma_t^i\left(\tau; t, x_t^*\right)\big|_{t=\tau}\right] - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x_\tau^*\right)\left[\gamma_{x_t^h x_t^\zeta}^i\left(\tau; t, x_t^*\right)\Big|_{t=\tau}\right] \\ &-\left[\gamma_{x_t^*}^i\left(\tau; t, x_t^*\right)\Big|_{t=\tau}\right] f\left[x_\tau^*, \psi_1^*\left(x_\tau^*\right), \psi_2^*\left(x_\tau^*\right)\right]. \quad (5.45)\end{aligned}$$

Using (5.39) we have $\gamma^i\left(\tau; t, x_t^*\right) = \exp\left[-r\left(t-\tau\right)\right]\xi^i\left(x_t^*\right)$ and $\gamma^i\left(\tau; \tau, x_\tau^*\right) = \xi^i\left(x_\tau^*\right)$. Then (5.45) can be used to obtain:

Theorem 5.8.3. *An instantaneous payment at time* $\tau \in \left[t_0, T\right]$ *equaling*

$$B_i(\tau) =$$
$$r\xi^i(x_\tau^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x_\tau^*)\,\xi^i_{x_\tau^* x_\tau^*}(x_\tau^*) - \xi^i_{x_\tau^*}(x_\tau^*)\, f\left[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)\right],$$
$$\text{for } \tau \in [t_0, T],\ x_\tau^* \in X_\tau^* \text{ and } i \in \{1,2\}, \tag{5.46}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0)$.

$B_i(\tau)$ yields the transitory compensation that sustains a subgame consistent solution to the cooperative game $\Gamma_c(x_0)$. Since $B_i(\tau)$ is a function of the current state x_τ^* only, we can write $B_i(\tau)$ as $B_i(x_\tau^*)$.

Then, we consider subgame consistent solutions under specific optimality principles. Consider a cooperative game $\Gamma_c(x_0)$ in which the players agree to maximize the sum of their expected payoffs and divide the total cooperative payoff satisfying the Nash bargaining outcome. Hence the imputation scheme has to satisfy:

Proposition 5.8.1. *In the game* $\Gamma_c(x_0)$, *at time* t_0 *an imputation*

$$\xi^i(x_0) = V^i(x_0) + \frac{1}{2}\left[W(x_0) - \sum_{j=1}^{2} V^j(x_0)\right],$$

is assigned to Player i, *for* $i \in \{1,2\}$;
and at time $\tau \in (t_0, \infty)$, *an imputation*

$$\xi^i(x_\tau^*) = V^i(x_\tau^*) + \frac{1}{2}\left[W(x_\tau^*) - \sum_{j=1}^{2} V^j(x_\tau^*)\right],$$

is assigned to Player i, *for* $i \in \{1,2\}$ *and* $x_\tau^* \in X_\tau^*$.

Using Theorem 5.8.3, one can obtain a PDP with an instantaneous at time $\tau \in [t_0, \infty)$

$$B_i(\tau) = B_i(x_\tau^*) = \frac{1}{2}\Bigg\{ rV^i(x_\tau^*) - V^i_{x_\tau^*}(x_\tau^*)\, f\left[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)\right]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x_\tau^*)\, V^i_{x_\tau^* x_\tau^*}(x_\tau^*)\Bigg\}$$
$$+\frac{1}{2}\Bigg\{ rW(x_\tau^*) - W_{x_\tau^*}(x_\tau^*)\, f\left[x_\tau^*, \psi_1^*(x_\tau^*), \psi_2^*(x_\tau^*)\right]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x_\tau^*)\, W_{x_\tau^* x_\tau^*}(x_\tau^*)\Bigg\}$$

$$-\frac{1}{2}\left\{rV^j\left(x_\tau^*\right)-\xi_{x_\tau^*}^j\left(x_\tau^*\right)f\left[x_\tau^*,\psi_1^*\left(x_\tau^*\right),\psi_2^*\left(x_\tau^*\right)\right]\right.$$

$$\left.-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x_\tau^*\right)V_{x_\tau^* x_\tau^*}^j\left(x_\tau^*\right)\right\},$$

$$\text{for } i,j\in\{1,2\},\ i\neq j \text{ and } x_\tau^*\in X_\tau^*, \tag{5.47}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c\left(x_0\right)$, in which the players agree to divide their cooperative gains according to Proposition 5.8.1.

Example 5.8.1. Consider the resource extraction game in Example 5.1.1 in which the game horizon is infinity. At time t_0, the payoff function of Player 1 and Player 2 are respectively:

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_1(s)^{1/2}-\frac{c_1}{x(s)^{1/2}}u_1(s)\right]\exp\left[-r\left(t-t_0\right)\right]ds\right\},$$

and

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_2(s)^{1/2}-\frac{c_2}{x(s)^{1/2}}u_2(s)\right]\exp\left[-r\left(t-t_0\right)\right]ds\right\}. \tag{5.48}$$

The resource stock $x(s)\in X\subset R$ follows the dynamics in (5.5).

A Nash equilibrium solution of the game (5.5) and (5.48), can then be characterized as follows.

Invoking Theorem 5.8.1 a noncooperative feedback Nash equilibrium solution can be characterized by:

$$rV^i(x)-\frac{1}{2}\sigma^2x^2V_{xx}^i(x)=$$

$$\max_{u_i}\left\{u_i^{1/2}-\frac{c_i}{x^{1/2}}u_i+V_x^i(x)\left[ax^{1/2}-bx-u_i-\phi_j^*(x)\right]\right\},$$

$$\text{for } i,j\in\{1,2\} \text{ and } i\neq j. \tag{5.49}$$

Performing the indicated maximization in (5.49) yields:

$$\phi_i^*(x)=\frac{x}{4\left[c_i+V_x^i(x)\,x^{1/2}\right]^2},\ \text{for } i\in\{1,2\}.$$

Substituting $\phi_1^*(x)$ and $\phi_2^*(x)$ above into (5.49) and upon solving (5.49) one obtains the value function of player $i\in\{1,2\}$ as:

$$V^i(t,x)=\left[A_ix^{1/2}+B_i\right],$$

where for $i,j\in\{1,2\}$ and $i\neq j$, A_i, B_i, A_j and B_j satisfy:

$$\left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right] A_i - \frac{1}{2\left[c_i + A_i/2\right]} + \frac{c_i}{4\left[c_i + A_i/2\right]^2} + \frac{A_i}{8\left[c_i + A_i/2\right]^2} + \frac{A_i}{8\left[c_j + A_j/2\right]^2} = 0, \text{ and}$$

$$B_i = \frac{a}{2} A_i.$$

The game equilibrium strategies can be obtained as:

$$\phi_1^*(x) = \frac{x}{4\left[c_1 + A_1/2\right]^2}, \quad \text{and} \quad \phi_2^*(x) = \frac{x}{4\left[c_2 + A_2/2\right]^2}.$$

Consider the case when these two extractors agree to maximize the sum of their expected payoffs and divide the total cooperative payoff according to Proposition 5.8.1. The players have to solve the control problem of maximizing

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left(\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s)\right] + \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s)\right]\right)\exp\left[-r(t-t_0)\right] ds\right\} \tag{5.50}$$

subject to (5.5).

Invoking Theorem 5.8.2 we obtain:

$$rW(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}(x) = \max_{u_1,u_2}\left\{\left(\left[u_1^{1/2} - \frac{c_1}{x^{1/2}} u_1\right] + \left[u_2^{1/2} - \frac{c_2}{x^{1/2}} u_2\right]\right) + W_x(x)\left[a x^{1/2} - bx - u_1 - u_2\right]\right\}.$$

Following similar procedures, one can obtain:

$$W(x) = \left[\hat{A} x^{1/2} + \hat{B}\right],$$

where

$$\left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]\hat{A} - \frac{1}{2\left[c_1 + \hat{A}/2\right]} - \frac{1}{2\left[c_2 + \hat{A}/2\right]} + \frac{c_1}{4\left[c_1 + \hat{A}/2\right]^2} + \frac{c_2}{4\left[c_2 + \hat{A}/2\right]^2} + \frac{\hat{A}}{8\left[c_1 + \hat{A}/2\right]^2} + \frac{\hat{A}}{8\left[c_2 + \hat{A}/2\right]^2} = 0, \text{ and}$$

$$\hat{B} = \frac{a}{2r}\hat{A}.$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^* (x) = \frac{x}{4\left[c_1 + \hat{A}/2\right]^2} \quad \text{and} \quad \psi_2^* (x) = \frac{x}{4\left[c_2 + \hat{A}/2\right]^2}. \tag{5.51}$$

Substituting these control strategies into (5.5) yields the dynamics of the state trajectory under cooperation:

$$dx (s) = \left[ax (s)^{1/2} - bx (s) - \frac{x (s)}{4\left[c_1 + \hat{A}/2\right]^2} - \frac{x (s)}{4\left[c_2 + \hat{A}/2\right]^2}\right] ds$$
$$+\sigma [x (s)] dz (s), \qquad x (t_0) = x_0. \tag{5.52}$$

Solving (5.52) yields the optimal cooperative state trajectory for $\Gamma_c (x_0)$ as:

$$x^* (s) = \varpi (t_0, s)^2 \left[x_0^{1/2} + \int_{t_0}^{s} \varpi^{-1} (t_0, t) H_1 dt\right]^2,$$
$$\text{for } s \in [t_0, T], \tag{5.53}$$

where

$$\varpi (t_0, s) = \exp \left[\int_{t_0}^{s} \left[H_2 (\tau) - \frac{\sigma^2}{8}\right] dv + \int_{t_0}^{s} \frac{\sigma}{2} dz (v)\right],$$
$$H_1 = \frac{1}{2} a, \text{ and}$$
$$H_2 (s) = -\left[\frac{1}{2} b + \frac{1}{8\left[c_1 + \hat{A} (s) /2\right]^2} + \frac{1}{8\left[c_2 + \hat{A} (s) /2\right]^2} + \frac{\sigma^2}{8}\right].$$

Using (5.47) we obtain:

$$B_i (\tau) = \frac{1}{2} \left\{r \left[A_i (x_\tau^*)^{1/2} + B_i\right] + r \left[\hat{A} (x_\tau^*)^{1/2} + \hat{B}\right] - r \left[A_j (x_\tau^*)^{1/2} + B_j\right]\right\}$$
$$+\frac{\sigma^2}{8} \left\{A_i (x_\tau^*)^{1/2} + \hat{A} (x_\tau^*)^{1/2} + A_j (x_\tau^*)^{1/2}\right\}$$
$$-\frac{1}{4} \left\{A_i (x_\tau^*)^{-1/2} + \hat{A} (x_\tau^*)^{-1/2} - A_j (x_\tau^*)^{-1/2}\right\}$$
$$\times \left[a (x_\tau^*)^{1/2} - bx_\tau^* - \frac{x_\tau^*}{4\left[c_1 + \hat{A}/2\right]^2} - \frac{x_\tau^*}{4\left[c_2 + \hat{A}/2\right]^2}\right],$$
$$\text{for } i, j \in \{1, 2\} \text{ and } i \neq j. \tag{5.54}$$

5.9 Appendix to Chapter 5

Proof of Formula 5.7.1.

Using Theorem 5.4.1 and (5.32), we obtain:

$$\begin{aligned}
B_i(\tau) = -\omega_W^{(\tau)i}(\tau, x_\tau^*) \Bigg\{ & \left[W_t^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] \\
& + \left[W_{x_t^*}^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] f[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)] \\
& + \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(\tau, x_\tau^*) \left[W_{x_t^h x_t^\zeta}^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] \Bigg\} \\
-\omega_{V^i}^{(\tau)i}(\tau, x_\tau^*) \Bigg\{ & \left[V_t^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] \\
& + \left[V_{x_t^*}^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] f[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)] \\
& + \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(\tau, x_\tau^*) \left[V_{x_t^h x_t^\zeta}^{(\tau)i}(t, x_t^*) \Big|_{t=\tau} \right] \Bigg\} \\
-\omega_{V^j}^{(\tau)i}(\tau, x_\tau^*) \Bigg\{ & \left[V_t^{(\tau)j}(t, x_t^*) \Big|_{t=\tau} \right] \\
& + \left[V_{x_t^*}^{(\tau)j}(t, x_t^*) \Big|_{t=\tau} \right] f[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)] \\
& + \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(\tau, x_\tau^*) \left[V_{x_t^h x_t^\zeta}^{(\tau)j}(t, x_t^*) \Big|_{t=\tau} \right] \Bigg\}. \qquad (5.55)
\end{aligned}$$

Invoking Fleming's stochastic control equation, we obtain:

$$\begin{aligned}
& -\left[W_t^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(\tau, x_\tau^*) \left[W_{x_t^h x_t^\zeta}^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] = \\
& \sum_{j=1}^{2} g^j[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)] \\
& \quad + \left[W_{x_t^*}^{(\tau)}(t, x_t^*) \Big|_{t=\tau} \right] f[\tau, x_\tau^*, \psi_1^*(\tau, x_\tau^*), \psi_2^*(\tau, x_\tau^*)]. \qquad (5.56)
\end{aligned}$$

Invoking the Isaacs-Bellman-Fleming equation we obtain:

$$-\left[V_t^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(\tau,x_\tau^*)\left[V_{x_t^h x_t^\zeta}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]=$$
$$g^i\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right]$$
$$+\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]f\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right],$$
$$\text{for }\ i\in\{1,2\}. \tag{5.57}$$

Substituting (5.56) and (5.57) into (5.55), we obtain:

$$B_i(\tau)=\omega_W^{(\tau)i}(\tau,x_\tau^*)\sum_{j=1}^{2}g^j\left[\tau,x_\tau^*,\psi_1^*(\tau,x_\tau^*),\psi_2^*(\tau,x_\tau^*)\right]$$
$$+\omega_{V^i}^{(\tau)i}(\tau,x_\tau^*)\left\{g^i\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right]\right.$$
$$+\left[V_{x_t^*}^{(\tau)i}(t,x_t^*)\Big|_{t=\tau}\right]$$
$$\left.\times\left(f\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right]-f\left[\tau,x_\tau^*,\psi_1^*(\tau,x_\tau^*),\psi_2^*(\tau,x_\tau^*)\right]\right)\right\}$$
$$+\omega_{V^j}^{(\tau)i}(\tau,x_\tau^*)\left\{g^j\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right]\right.$$
$$+\left[V_{x_t^*}^{(\tau)j}(t,x_t^*)\Big|_{t=\tau}\right]$$
$$\left.\times\left(f\left[\tau,x_\tau^*,\phi_1^*(\tau,x_\tau^*),\phi_2^*(\tau,x_\tau^*)\right]-f\left[\tau,x_\tau^*,\psi_1^*(\tau,x_\tau^*),\psi_2^*(\tau,x_\tau^*)\right]\right)\right\}.$$

Hence Formula 5.7.1 follows. Q.E.D.

5.10 Problems

Problem 5.1. Consider a resource extraction game in which two extractors are awarded leases to extract a renewable resource over the time interval $[t_0,T]$. The resource stock $x(s)\in X\subset R$ follows the stochastic dynamics:

$$dx(s)=\left[a-bx(s)-u_1(s)-u_2(s)\right]ds+\sigma x(s)\,dz(s), \qquad x(t_0)=x_0\in X,$$

where a and b are constants, $z(s)$ is a Wiener process, $u_1(s)$ is the harvest rate of extractor 1, $u_2(s)$ and is the harvest rate of extractor 2.

At time t_0, the expected payoff of extractor 1 and extractor 2 are respectively:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[u_1(s)-\frac{c_1}{x(s)}u_1^2(s)\right]\exp\left[-r(t-t_0)\right]ds\right.$$
$$\left.+\exp\left[-r(T-t_0)\right]qx(T)\right\},$$

and

$$E_{t_0}\left\{\int_{t_0}^{T}\left[u_2(s)-\frac{c_2}{x(s)}u_2^2(s)\right]\exp\left[-r(t-t_0)\right]ds\right.$$
$$\left.+\exp\left[-r(T-t_0)\right]qx(T)\right\},$$

where q, c_1 and c_2 are constants and $c_1 \neq c_2$. Payoffs are transferable and r is the interest rate.

(a) Derive a Nash equilibrium solution for the game.
(b) If the extractors agree to cooperative, derive the optimal extraction strategies.
(c) Derive the optimal state trajectory under cooperation.
(d) Derive the expected cooperative profits along the cooperative path.

Problem 5.2. Consider the game in Problem 5.1, if the extractors agree to share the cooperative profit satisfying the Nash bargaining outcome – that is, they maximize the product of individual gains in excess of the expected noncooperative payoffs. The imputation scheme has to satisfy the condition that (see Proposition 5.5.1)

$$\xi^{(\tau)i}(\tau,x_\tau^*)=V^{(\tau)i}(\tau,x_\tau^*)+\frac{1}{2}\left[W^{(\tau)}(\tau,x_\tau^*)-\sum_{j=1}^{2}V^{(\tau)j}(\tau,x_\tau^*)\right],$$

is assigned to extractor i, for $i \in \{1,2\}$, and in the subgame $\Gamma_c(x_\tau^*, T-\tau)$, for $\tau \in [t_0, T]$.

(a) Formulate a payoff distribution procedure (PDP) of the cooperative game so that the agreed imputations can be realized.
(b) Show that Pareto optimality and individual rationality are satisfied.

Problem 5.3. Using the PDP obtained in part (a) of Problem 5.2, express the compensation $B_i(\tau)$ extractor i receives at time τ in terms of

(i) extractor i's agreed upon marginal share of total expected cooperative profit,
(ii) extractor i's agreed upon marginal share of his own expected nonccooperative profit plus the instantaneous effect on his noncooperative expected payoff when the change in the state variable x_τ^* follows the cooperative trajectory instead of the noncooperative path, and

(iii) extractor i's agreed upon marginal share of extractor j's nonccoperative profit plus the instantaneous effect on extractor j's noncooperative payoff when the change in the state variable x follows the optimal trajectory instead of the noncooperative path.

Problem 5.4. Consider the game in Problem 5.1, if the extractors agree to share the cooperative profit so that the players' shares of the gain from cooperation are proportional to the relative sizes of their expected noncooperative profits. The imputation scheme has to satisfy the condition that (see Proposition 5.5.2):

In the subgame $\Gamma_c\left(x_\tau^*, T-\tau\right)$, for $\tau \in (t_0, T]$ and $x_\tau^* \in X_\tau^*$, an imputation

$$\xi^{(\tau)i}\left(\tau, x_\tau^*\right) = \frac{V^{(\tau)i}\left(\tau, x_\tau^*\right)}{\sum_{j=1}^{2} V^{(\tau)j}\left(\tau, x_\tau^*\right)} W^{(\tau)}\left(\tau, x_\tau^*\right),$$

is assigned to Player i, for $i \in \{1,2\}$.

Formulate a payoff distribution procedure (PDP) of the cooperative game so that the agreed imputations can be realized.

Problem 5.5. Consider the infinite-horizon game in which extractor 1 and extractor 2 maximizes respectively:

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_1(s) - \frac{c_1}{x(s)}u_1^2(s)\right]\exp\left[-r(t-t_0)\right]ds\right\},$$

and

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_2(s) - \frac{c_2}{x(s)}u_2^2(s)\right]\exp\left[-r(t-t_0)\right]ds\right\}.$$

The extractors' payoffs are transferable and the resource stock $x(s) \in X \subset R$ follows the stochastic dynamics:

$$dx(s) = \left[a - bx(s) - u_1(s) - u_2(s)\right]ds + \sigma x(s)\,dz(s), \qquad x(t_0) = x_0 \in X.$$

(a) Derive a Nash equilibrium for the game.
(b) If the extractors agree to cooperative, derive the optimal extraction strategies.
(c) Derive the optimal state trajectory under cooperation.
(d) If the extractors agree to share the cooperative profit satisfying the Nash bargaining outcome, formulate a payoff distribution procedure (PDP) of the cooperative game so that the agreed imputations can be realized.

6

Multiplayer Cooperative Stochastic Differential Games

In Chapter 3 we consider multiplayer cooperative differential games in characteristic function form. However, as discussed in Section 3.6.3, in many game situations it is unlikely that a subset of players form a coalition to maximize the joint payoff and the remaining player would form an anti-coalition to harm their efforts. Hence the adoption of characteristic functions may not be realistic all the time. In this Chapter we present and analyze a class of multiplayer cooperative stochastic differential in technology development that does not require the use of characteristic functions.

6.1 A Class of Multiplayer Games in Cooperative Technology Development

As markets continue to become global and firms become more international, corporate joint ventures provide firms with opportunities to rapidly create economies of scale and learn new skills and technologies that would be very difficult for them to obtain on their own.

We first present the basic structure of a class of multiplayer cooperative stochastic differential in technology development.

Consider the scenario in which n players or firms, and firm i's objective is:

$$E_{t_0}\left\{\int_{t_0}^{T} g^i\left[s, x_i(s), u_i(s)\right]\exp\left[-\int_{t_0}^{s} r(y)\,dy\right]ds \right.$$
$$\left. +\exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i\left(x_i(T)\right)\right\},$$
$$\text{for } \ i \in [1, 2, \ldots, n] \equiv N, \tag{6.1}$$

where $x_i(s) \in X_i \subset R^{m_i}$ denotes the technology state of firm i, $u_i \in U^i \subset compR^l$ is the control vector of firm i, $\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]$ is the discount factor, and $q^i(x_i(T))$ the terminal payoff. In particular, $g^i[s,x_i,u_i]$ and $q^i(x_i)$ are positively related to x_i, reflecting the earning potent of the technology.

The state dynamics of the game is characterized by the set of vector-valued stochastic differential equations:

$$dx_i(s) = f^i[s,x_i(s),u_i(s)]\,ds + \sigma_i[s,x_i(s)]\,dz_i(s),$$
$$x_i(t_0) = x_i^0, \tag{6.2}$$

where $\sigma_i[s,x_i(s)]$ is a $m_i \times \Theta_i$ and $z_i(s)$ is a Θ_i-dimensional Wiener process and the initial state x_i^0 is given. Let $\Omega_i[s,x_i(s)] = \sigma_i[s,x_i(s)]\,\sigma_i[s,x_i(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega_i^{h\zeta}[s,x_i(s)]$. For $i \neq j$, $x_i \cap x_j = \emptyset$, and $z_i(s)$ and $z_j(s)$ are independent Wiener processes. We also used $x_N(s)$ to denote the vector $[x_1(s),x_2(s),\ldots,x_n(s)]$ and x_N^0 the vector $\left[x_1^0,x_2^0,\ldots,x_n^0\right]$.

Consider a coalition of a subset of firms $K \subseteq N$. There are k firms in the subset K. The participating firms can gain core skills and technology that would be very difficult for them to obtain on their own, and hence the state dynamics of firm i in the coalition K becomes

$$dx_i(s) = f^i[s,x_K(s),u_i(s)]\,ds + \sigma_i[s,x_i(s)]\,dz_i(s), \qquad x_i(t_0) = x_i^0,$$
$$\text{for } \; i \in K, \tag{6.3}$$

where $x_K(s)$ is the concatenation of the vectors $x_j(s)$ for $j \in K$. In particular, $\partial f_i^K[s,x_K,u_i]/\partial x_j \geq 0$, for $j \neq i$. Thus positive effects on the state of firm i could be derived from the technology of other firms within the coalition. Without much loss of generalization, the effect of x_j on $f_i^K[s,x_K,u_i]$ remains the same for all possible coalitions K containing firms i and j.

6.2 Theoretical Underpinning in a Deterministic Framework

To facilitate the readers in understanding the theoretic underpinning of the analysis we begin with a deterministic version of the stochastic differential games (6.1)–(6.3).

6.2.1 A Dynamic Model of Joint Venture

Consider a dynamic joint venture in which there are n firms. The state dynamics of the i^{th} firm is characterized by the set of vector-valued differential equations:

$$\dot{x}_i(s) = f_i^i[s,x_i(s),u_i(s)], \qquad x_i(t_0) = x_i^0,$$
$$\text{for } \; i \in [1,2,\ldots,n] \equiv N, \tag{6.4}$$

where $x_i(s) \in X_i \subset R^{m_i+}$ denotes the state variables of Player i, $u_i \in U^i \subset compR^{l_i+}$ is the control vector of firm i. The state of firm i include its capital stock, level of technology, special skills and productive resources. The objective of firm i is:

$$\int_{t_0}^{T} g^i[s, x_i(s), u_i(s)] \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds + \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i(x_i(T)),$$
$$\text{for } \ i \in [1, 2, \ldots, n] \equiv N, \tag{6.5}$$

where $\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]$ is the discount factor, $g^i[s, x_i(s), u_i(s)]$ the instantaneous profit, and $q^i(x_i(T))$ the terminal payment. In particular, $g^i[s, x_i, u_i]$ and $q^i(x_i)$ are positively related to the level of technology x_i.

Consider a joint venture consisting of a subset of companies $K \subseteq N$. There are k firms in the subset K. The participating firms can gain core skills and technology that would be very difficult for them to obtain on their own, and hence the state dynamics of firm i in the coalition K becomes

$$\dot{x}_i(s) = f_i^K[s, x_K(s), u_i(s)], \qquad x_i(t_0) = x_i^0, \qquad \text{for } \ i \in K, \tag{6.6}$$

where $x_K(s)$ is the concatenation of the vectors $x_j(s)$ for $j \in K$. In particular, $\partial f_i^K[s, x_i, u_K]/\partial u_j \geq 0$, for $j \neq i$. Thus positive effects on the state of firm i could be derived from the technology of other firms within the coalition. Again, without much loss of generalization, the effect of x_j on $f_i^K[s, x_K, u_i]$ remains the same for all possible coalitions K containing firms i and j.

6.2.2 Coalition Payoffs

At time t_0, the profit to the joint venture becomes:

$$\int_{t_0}^{T} \sum_{j \in K} g^j[s, x_j(s), u_j(s)] \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds + \sum_{j \in K} \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^j(x_j(T)),$$
$$\text{for } \ K \subseteq N. \tag{6.7}$$

To compute the profit of the joint venture K we have to consider the optimal control problem $\varpi\left[K; t_0, x_K^0\right]$ which maximizes (6.7) subject to (6.6).

For notational convenience, we express (6.6) as:

$$\dot{x}_K(s) = f^K[s, x_K(s), u_K(s)], \qquad x_K(t_0) = x_K^0, \tag{6.8}$$

where u_K is the set of u_j for $j \in K$, $f^K[t, x_K, u_K]$ is a column vector containing $f_j^K[t, x_K, u_K]$ for $j \in K$.

Using Bellman's technique of dynamic programming the solution of the problem $\varpi\left[K; t_0, x_K^0\right]$ can be characterized as follows.

Theorem 6.2.1. *A set of optimal controls* $\left\{u_K^*(t) = \psi_K^{(t_0)K*}(t, x_K)\right\}$ *provides an optimal solution to the problem* $\varpi\left[K; t_0, x_K^0\right]$ *if there exist continuously differentiable function* $W^{(t_0)K}(t, x_K) : [t_0, T] \times \Pi_{j \in K} R^{m_j} \to R$*, satisfying the Bellman equation:*

$$-W_t^{(t_0)K}(t, x_K) = \max_{u_K} \left\{ \sum_{j \in K} g^j [t, x_j, u_j] \exp\left[-\int_{t_0}^{t} r(y)\, dy\right] \right.$$
$$\left. + \sum_{j \in K} W_{x_j}^{(t_0)K}(t, x_K)\, f_j^K[t, x_K, u_j] \right\},$$
$$W^{(t_0)K}(T, x_K) = \sum_{j \in K} \exp\left[-\int_{t_0}^{T} r(y)\, dy\right] q^j(x_j).$$

Following the notation in Theorem 6.2.1, we use $\psi_j^{(t_0)K*}(t, x_k)$ to denote optimal controls of firm j in coalition K for the problem $\varpi\left[K; t_0, x_K^0\right]$.

In the case when all the n firms are in the joint venture, that is $K = N$, the optimal control

$$\psi_N^{(t_0)N*}(s, x_N(s)) = \left[\psi_1^{(t_0)N*}(s, x_N(s)), \psi_2^{(t_0)N*}(s, x_N(s)), \ldots, \psi_N^{(t_0)N*}(s, x_N(s))\right]$$

satisfying Theorem 6.2.1 will be adopted. The dynamics of the optimal state trajectory of the grand coalition can be obtained as:

$$\dot{x}_j(s) = f_j^N\left[s, x_N(s), \psi_j^{(t_0)N*}(s, x_N(s))\right],$$
$$x_j(t_0) = x_j^0, \qquad j \in N, \tag{6.9}$$

which can also be expressed as

$$\dot{x}_N(s) = f^N\left[s, x_N(s), \psi_N^{(t_0)N*}(s, x_N(s))\right],$$
$$\text{for} \quad x_N(t_0) = x_N^0.$$

Let $x_N^*(t) = [x_1^*(t), x_2^*(t), \ldots, x_n^*(t)]$ denote the solution to (6.9). The optimal trajectories $\{x_N^*(t)\}_{t=t_0}^{T}$ characterizes the states of the participating firms within the venture period. We use x_j^{t*} to denote the value of $x_j^*(t)$ at time $t \in [t_0, T]$.

Remark 6.2.1. Consider the problem $\varpi\left[K;\tau,x_K^\tau\right]$ which starts at time $\tau\in[t_0,T]$ with initial state x_K^τ. Using Theorem 6.2.1, one can readily show that:

$$\exp\left[\int_\tau^t r(y)\,dy\right]W^{(\tau)K}\left(t,x_K^t\right)=W^{(t)K}\left(t,x_K^t\right),\ \text{for}\ \ t_0\le\tau\le t\le T;\ \text{and}$$
$$\psi_K^{(\tau)K*}\left(t,x_K^t\right)=\psi_K^{(t)K*}\left(t,x_K^t\right),\qquad \text{for}\ \ t_0\le\tau\le t\le T.$$

Since (i) the level of technology of firm i enhances its profit as $g^i\left[s,x_i,u_i\right]$ and $q^i\left(x_i\right)$ are positively related to the level of technology x_i, and (ii) participating firms can gain core skills and technology from other firms in the coalition as $\partial f_i^K\left[s,x_i,u_K\right]/\partial u_j\ge 0$, for $j\ne i$; superadditivity of the coalition profit functions will result, that is,

$$W^{(\tau)K}\left(\tau,x_K^\tau\right)\ge W^{(\tau)L}\left(\tau,x_L^\tau\right)+W^{(\tau)K\backslash L}\left(\tau,x_{K\backslash L}^\tau\right),$$
$$\text{for}\ \ L\subset K\subseteq N,$$

where $K\backslash L$ is the relative complement of L in K.

6.2.3 The Dynamic Shapley Value Imputation

Consider the above joint venture involving n firms. The member firms would maximize their joint profit and share their cooperative profits according to the Shapley value (1953). The problem of profit sharing is inescapable in virtually every joint venture. The Shapley value is one of the most commonly used sharing mechanism in static cooperation games with transferable payoffs. Besides being individually rational and group rational, the Shapley value is also unique. The uniqueness property makes a more desirable cooperative solution relative to other solutions like the Core or the Stable Set. Specifically, the Shapley value gives an imputation rule for sharing the cooperative profit among the members in a coalition as:

$$\varphi^i(v)=\sum_{K\subseteq N}\frac{(k-1)!\,(n-k)!}{n!}\left[v(K)-v(K\backslash i)\right],$$
$$\text{for}\ \ i\in N, \tag{6.10}$$

where $K\backslash i$ is the relative complement of i in K, $v(K)$ is the profit of coalition K, and $\left[v(K)-v(K\backslash i)\right]$ is the marginal contribution of firm i to the coalition K.

Given the assumption that $v(K)$ is super-additive, the Shapley value yields the desirable properties of individual rationality and group optimality. Though the Shapley value is used as the profit allocation mechanism, there exist two features that do not conform with the standard Shapley value analysis. The first is that the present analysis is dynamic so that instead of a one-time allocation of the Shapley value, like the analysis in Chapter 3 we have to consider the maintenance of the Shapley value imputation over the joint venture horizon. The second is that the profit $v(K)$ is the maximized profit to coalition K, and is not a characteristic function (from the game in which coalition K is playing a zero-sum game against coalition $N \backslash K$)[1].

To maximize the joint venture's profits the firms would adopt the control vector $\left\{\psi_N^{(t_0)N*}(t, x_N^{t*})\right\}_{t=t_0}^{T}$ over the time $[t_0, T]$ interval , and the corresponding optimal state trajectory $\{x_N^*(t)\}_{t=t_0}^T$ in (6.9) would result. At time t_0 with state $x_N^{t_0}$, the firms agree that firm i's share of profits be:

$$v^{(t_0)i}\left(t_0, x_N^0\right) = \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(t_0)K}\left(t_0, x_K^0\right) - W^{(t_0)K\backslash i}\left(t_0, x_{K\backslash i}^0\right)\right], \quad \text{for } i \in N. \tag{6.11}$$

However, the Shapley value has to be maintained throughout the venture horizon $[t_0, T]$. In particular, at time $\tau \in [t_0, T]$ with the state being $x_N^{\tau*}$ the following imputation principle has to be maintained:

Condition 6.2.1. *At time τ, firm i's share of profits be:*

$$v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) = \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(\tau)K}\left(\tau, x_K^{\tau*}\right) - W^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{\tau*}\right)\right], \quad \textit{for } i \in N \textit{ and } \tau \in [t_0, T]. \tag{6.12}$$

Note that $v^{(\tau)}\left(\tau, x_N^{\tau*}\right) = \left[v^{(\tau)1}\left(\tau, x_N^{\tau*}\right), v^{(\tau)2}\left(\tau, x_N^{\tau*}\right), \ldots, v^{(\tau)n}\left(\tau, x_N^{\tau*}\right)\right]$, as specified in (6.12) satisfies the basic properties of an imputation vector:

(i)
$$\sum_{j=1}^{n} v^{(\tau)j}\left(\tau, x_N^{\tau*}\right) = W^{(\tau)N}\left(\tau, x_N^{\tau*}\right), \text{ and}$$

(ii)
$$v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) \geq W^{(\tau)i}\left(\tau, x_N^{\tau*}\right), \quad \text{for } i \in N \text{ and } \tau \in [t_0, T]. \tag{6.13}$$

Part (i) of (6.13) shows that $v^{(\tau)}\left(\tau, x_N^{\tau*}\right)$ satisfies the property of Pareto optimality throughout the game interval. Part (ii) demonstrates that $v^{(\tau)}\left(\tau, x_N^{\tau*}\right)$

[1] Application of the Shapley value in cost allocation usually do not follow the characteristics function approach. Moreover, since profit maximization by coalition K is not affected by firms outside the coalition, the analysis does not have to adopt arbitrary assumptions like that in Petrosyan and Zaccour (2003) in which the left-out players are assumed to stick with their feedback Nash strategies in computing a non-standard characteristic function.

guarantees individual rationality throughout the game interval. Pareto optimality and individual rationality are essential properties of imputation vectors. Moreover, if Condition 6.2.1 can be maintained, the solution optimality principle – sharing profits according to the Shapley value – is in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. Hence time consistency is satisfied and no firms would have any incentive to depart the joint venture. Therefore a dynamic imputation principle leading to (6.12) is dynamically stable or time consistent.

Crucial to the analysis is the formulation of a profit distribution mechanism that would lead to the realization of Condition 6.2.1. This will be done in the next section.

6.2.4 Transitory Compensation

In this section, a profit distribution mechanism will be developed to compensate transitory changes so that the Shapley value principle could be maintained throughout the venture horizon. First, an imputation distribution procedure (similar to those in Petrosyan and Zaccour (2003) and Yeung and Petrosyan (2004)) must be now formulated so that the imputation scheme in Condition 6.2.1 can be realized. Let $B_i(s)$ denote the payment received by firm $i \in N$ at time $s \in [t_0, T]$ dictated by $v^{(t_0)i}\left(t_0, x_N^0\right)$. In particular,

$$v^{(t_0)i}\left(t_0, x_N^0\right) =$$
$$\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!}\left[W^{(t_0)K}\left(t_0, x_K^0\right) - W^{(t_0)K\backslash i}\left(t_0, x_{K\backslash i}^0\right)\right] =$$
$$\int_{t_0}^{T} B_i(s)\exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds + q^i\left(x_i^*(T)\right)\exp\left[-\int_{t_0}^{T} r(y)\,dy\right], \quad (6.14)$$
$$\text{for } i \in N.$$

Moreover, for $i \in N$ and $t \in [t_0, T]$, we use

$$v^{(t_0)i}\left(t, x_N^{t*}\right) =$$
$$\int_{t}^{T} B_i(s)\exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds + q^i\left(x_i^*(T)\right)\exp\left[-\int_{t_0}^{T} r(y)\,dy\right], \quad (6.15)$$

to denote the present value of Player i's cooperative profit over the time interval $[t, T]$, given that the state x_N^{t*} is at time $t \in [t_0, T]$.

A necessary condition for $v^{(t_0)i}\left(t, x_N^{t*}\right)$ to follow Condition 6.2.1 is that:

$$v^{(t_0)i}\left(t, x_N^{t*}\right) = v^{(t)i}\left(t, x_N^{t*}\right)\exp\left[-\int_{t_0}^{t} r(y)\,dy\right],$$
$$\text{for } i \in N \text{ and } t \in [\tau, T]. \quad (6.16)$$

A candidate of $v^{(t_0)i}(t, x_N^{t*})$ satisfying (6.14)–(6.16) has to be found. A natural choice is

$$v^{(t_0)i}\left(t, x_N^{t*}\right) =$$
$$\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(t_0)K}\left(t, x_K^{t*}\right) - W^{(t_0)K\backslash i}\left(t, x_{K\backslash i}^{t*}\right)\right].$$

With Remark 6.2.1, one can readily that $v^{(t_0)i}(t, x_N^{t*})$ defined as above satisfies (6.14)–(6.16).

To fulfill the Pareto optimality property, the PDP $B(s) = [B_1(s), B_2(s), \ldots, B_n(s)]$ has to satisfy the following condition.

Condition 6.2.2.

$$\sum_{j=1}^{n} B_i(s) = \sum_{j=1}^{n} g^j\left[s, x_j^{S*}, \psi_N^{(t_0)N*}\left(s, x_N^{S*}\right)\right], \textit{ for } s \in [\tau, T].$$

Invoking Remark 6.2.1, we can write

$$v^{(\tau)i}\left(t, x_N^{t*}\right) = \exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] v^{(t_0)i}\left(t, x_N^{t*}\right) =$$
$$\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(\tau)K}\left(\tau, x_K^{t*}\right) - W^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{t*}\right)\right] =$$
$$\int_t^T B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds + q^i\left(x_i^*(T)\right) \exp\left[-\int_\tau^T r(y)\,dy\right],$$
$$\text{for } t_0 \le \tau \le t \le T. \tag{6.17}$$

Since the value function $W^{(\tau)K}(t, x_K^{t*})$ is twice continuously differentiable in t and x_K^{t*}, the term $v^{(\tau)i}(t, x_N^{t*})$ is twice continuously differentiable in t and x_N^{t*}.

Given the differentiability property of $v^{(\tau)i}(t, x_N^{t*})$, for $\Delta t \to 0$ one can use (6.16) and Remark 6.2.1 to obtain:

$$v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) =$$
$$\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds$$
$$+ \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right] v^{(\tau+\Delta t)i}\left(\tau+\Delta t, x_N^{\tau*} + \Delta x_N^{\tau*}\right)\Big| x_N(\tau) = x_N^{\tau*}, \tag{6.18}$$
$$\text{for } i \in N, \;\; t \in [\tau, T] \;\text{ and }\; \tau \in [t_0, T];$$

where

$$\Delta x_N^{\tau *} = [\Delta x_1^{\tau *}, \Delta x_2^{\tau *}, \ldots, \Delta x_n^{\tau *}],$$
$$\Delta x_j^{\tau *} = f_j^N \left[\tau, x_N^{\tau *}, \psi_j^{(\tau)N*}(\tau, x_N^{\tau *})\right] \Delta t + o(\Delta t),$$
$$\text{for } j \in N, \text{ and } [o(\Delta t)]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

Using (6.16) and (6.17), we express (6.18) as:

$$\begin{aligned}
&\int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\,dy\right] ds \\
&= v^{(\tau)i}(\tau, x_N^{\tau *}) - \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\,dy\right] v^{(\tau+\Delta t)i}(\tau+\Delta t, x_N^{\tau *} + \Delta x_N^{\tau *}) \\
&= v^{(\tau)i}(\tau, x_N^{\tau *}) - v^{(\tau)i}(\tau+\Delta t, x_N^{\tau *} + \Delta x_N^{\tau}), \\
&\qquad \text{for all } \tau \in [t_0, T].
\end{aligned} \tag{6.19}$$

When $\Delta t \to 0$, condition (6.19) can be expressed as:

$$\begin{aligned}
&B_i(\tau)\,\Delta t = \\
&\quad -\left[v_t^{(\tau)i}\left(t, x_N^{t*}\right)\Big|_{t=\tau}\right] \Delta t \\
&\quad -\sum_{j \in N} \left[v_{x_j^{t*}}^{(\tau)i}\left(t, x_N^{t*}\right)\Big|_{t=\tau}\right] f_j^N \left[\tau, x_N^{\tau *}, \psi_j^{(\tau)N}(\tau, x_N^{\tau *})\right] \Delta t + o(\Delta t),
\end{aligned} \tag{6.20}$$

where $\left[v_{x_j^{t*}}^{(\tau)i}(t, x_N^{t*})\Big|_{t=\tau}\right]$ is a row vector of partial derivatives with respect to x_j^{t*}.

Taking expectation and dividing (6.20) throughout by Δt, with $\Delta t \to 0$, yield

$$\begin{aligned}
B_i(\tau) = &-\left[v_t^{(\tau)i}(t, x_N^{\tau *})\Big|_{t=\tau}\right] \\
&-\sum_{j \in N} \left[v_{x_j^{t*}}^{(\tau)i}(t, x_N^{\tau *})\Big|_{t=\tau}\right] f_j^N \left[\tau, x_N^{\tau *}, \psi_j^{(\tau)N}(\tau, x_N^{\tau *})\right], \\
&\text{for } i \in N, \;\; t \in [\tau, T] \text{ and } \tau \in [t_0, T].
\end{aligned} \tag{6.21}$$

Using (6.17) and (6.21), we obtain:

Theorem 6.2.2. *A payment to player $i \in N$ at time $\tau \in [t_0, T]$ equaling*

$$\begin{aligned}
&B_i(\tau) = \\
&-\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left\{ \left[W_t^{(\tau)K}(t, x_K^{\tau *})\Big|_{t=\tau}\right] - \left[W_t^{(\tau)K\backslash i}\left(t, x_{K\backslash i}^{\tau *}\right)\Big|_{t=\tau}\right] \right. \\
&\quad + \left(\left[W_{x_N^{\tau *}}^{(\tau)K}(t, x_K^{\tau *})\Big|_{t=\tau}\right] - \left[W_{x_N^{\tau *}}^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{\tau *}\right)\Big|_{t=\tau}\right]\right) \\
&\qquad \left. \times f^N \left[\tau, x_N^{\tau *}, \psi_N^{(\tau)N}(\tau, x_N^{\tau *})\right] \right\},
\end{aligned} \tag{6.22}$$

will lead to the realization of the Condition 6.2.1.

Since the partial derivative of $W^{(\tau)K}\left(\tau, x_K^{\tau*}\right)$ with respect to x_j, where $j \notin K$, will vanish, a more concise form of Theorem 6.2.2 can be obtained as:

Theorem 6.2.3. *A payment to player $i \in N$ at time $\tau \in [t_0, T]$ leading to the realization of the Condition 6.2.1 can be expressed as:*

$$B_i(\tau) =$$

$$-\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left\{ \left[W_t^{(\tau)K}\left(t, x_K^{\tau*}\right)\Big|_{t=\tau} \right] - \left[W_t^{(\tau)K\backslash i}\left(t, x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau} \right] \right.$$

$$+ \sum_{j \in K} \left[W_{x_j^{\tau*}}^{(\tau)K}\left(t, x_K^{\tau*}\right)\Big|_{t=\tau} \right] f_j^N\left[\tau, x_N^{\tau*}, \psi_j^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right]$$

$$\left. - \sum_{h \in K\backslash i} \left[W_{x_h^{\tau*}}^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau} \right] f_h^N\left[\tau, x_N^{\tau*}, \psi_h^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right] \right\} =$$

$$-\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left\{ \left[W_t^{(\tau)K}\left(t, x_K^{\tau*}\right)\Big|_{t=\tau} \right] - \left[W_t^{(\tau)K\backslash i}\left(t, x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau} \right] \right.$$

$$+ \left[W_{x_K^{\tau*}}^{(\tau)K}\left(t, x_K^{\tau*}\right)\Big|_{t=\tau} \right] f_K^N\left[\tau, x_N^{\tau*}, \psi_K^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right]$$

$$\left. - \left[W_{x_{K\backslash i}^{\tau*}}^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau} \right] f_{K\backslash i}^N\left[\tau, x_N^{\tau*}, \psi_{K\backslash i}^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right] \right\},$$

where $f_K^N\left[\tau, x_N^{\tau}, \psi_K^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right]$ is a column vector containing $f_i^N\left[\tau, x_N^{\tau*}, \psi_i^{(\tau)N}\left(\tau, x_N^{\tau*}\right)\right]$ for $i \in K$.

The vector $B(\tau)$ serves as a form equilibrating transitory compensation that guarantees the realization of the Shapley value imputation throughout the game horizon. Note that the instantaneous profit $B_i(\tau)$ offered to Player i at time τ is conditional upon the current state $x_N^{\tau*}$ and current time τ. One can elect to express $B_i(\tau)$ as $B_i\left(\tau, x_N^{\tau*}\right)$. Hence an instantaneous payment $B_i\left(\tau, x_N^{\tau*}\right)$ to player $i \in N$ yields a dynamically stable solution to the joint venture.

6.3 An Application in Joint Venture

Consider the case when there are 3 companies involved in joint venture. The planning period is $[t_0, T]$. Company i profit is

$$\int_{t_0}^{T} \left[P_i\left[x_i(s)\right]^{1/2} - c_i u_i(s) \right] \exp\left[-r(s-t_0)\right] ds$$
$$+ \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}, \tag{6.23}$$
$$\text{for } i \in N = \{1, 2, 3\},$$

where P_i, c_i and q_i are positive constants, r is the discount rate, $x_i(s) \subset R^+$ is the level of technology of company i at time s, and $u_i(s) \subset R^+$ is its physical investment in technological advancement. The term $P_i [x_i(s)]^{1/2}$ reflects the net operating revenue of company i at technology level $x_i(s)$, and $c_i u_i$ is the cost of investment. $q_i [x_i(T)]^{1/2}$ gives the salvage value of company i's technology at time T.

The evolution of the technology level of company i follows the dynamics:

$$\dot{x}_i(s) = \left[\alpha_i [u_i(s) x_i(s)]^{1/2} - \delta x_i(s)\right],$$
$$x_i(t_0) = x_i^0 \in X_i, \qquad \text{for } i \in N = \{1,2,3\}, \tag{6.24}$$

where $\alpha_i [u_i(s) x_i(s)]^{1/2}$ is the addition to the technology brought about by $u_i(s)$ amount of physical investment, and δ is the rate of obsolescence.

In the case when each of these three firms acts independently. Using Theorem 2.1.1, we obtain the Bellman equation as:

$$-W_t^{(t_0)i}(t, x_i) =$$
$$\max_{u_i} \left\{ \left[P_i x_i^{1/2} - c_i u_i\right] \exp[-r(t - t_0)] + W_{x_i}^{(\tau)i}(t, x_i) \left[\alpha_i (u_i x_i)^{1/2} - \delta x_i\right] \right\},$$
$$W^{(t_0)i}(T, x_i) = \exp[-r(T - t_0)]\, q_i [x_i]^{1/2}, \qquad \text{for } i \in \{1,2,3\}.$$

Performing the indicated maximization yields

$$u_i = \frac{\alpha_i^2}{4(c_i)^2} \left[W_{x_i}^{(t_0)i}(t, x_i) \exp[r(t - t_0)]\right]^2 x_i, \qquad \text{for } i \in \{1,2,3\}.$$

Substituting u_i into the Bellman equation yields:

$$-W_t^{(t_0)i}(t, x_i) =$$
$$P_i x_i^{1/2} \exp[-r(t - t_0)] - \frac{\alpha_i^2}{4c_i} \left[W_{x_i}^{(t_0)i}(t, x_i)\right]^2 \exp[r(t - t_0)]\, x_i$$
$$+ \frac{\alpha_i^2}{2c_i} \left[W_{x_i}^{(t_0)i}(t, x_i)\right]^2 \exp[r(t - \tau)]\, x_i - \delta W_{x_i}^{(t_0)i}(t, x_i)\, x_i,$$
$$\text{for } i \in \{1,2,3\}.$$

Solving the above system of partial differential equations yields

$$W^{(t_0)i}(t, x_i) = \left[A_i^{\{i\}}(t)\, x_i^{1/2} + C_i^{\{i\}}(t)\right] \exp[-r(\tau - t_0)],$$
$$\text{for } i \in \{1,2,3\}, \tag{6.25}$$

where

$$\dot{A}_i^{\{i\}}(t) = \left(r + \frac{\delta}{2}\right) A_i^{\{i\}}(t) - P_i,$$
$$\dot{C}_i^{\{i\}}(t) = rC_i^{\{i\}}(t) - \frac{\alpha_i^2}{16c_i}\left[A_i^{\{i\}}(t)\right]^2,$$
$$A_i^{\{i\}}(T) = q_i \text{ and } C_i^{\{i\}}(T) = 0. \tag{6.26}$$

The first equation in the block-recursive system (6.26) is a first-order linear differential equation $inA_i^{\{i\}}(t)$ which can be solved independently by standard techniques. Upon substituting the solution of $A_i^{\{i\}}(t)$ into the second equation of (6.26) yields a first-order linear differential equation in $C_i^{\{i\}}(t)$. The solution of $C_i^{\{i\}}(t)$ can be readily obtained by standard techniques. The explicit solution is not stated here because of its lengthy expressions.

Moreover, as stated in Remark 6.2.1, one can easily derive for $\tau \in [t_0, T]$

$$W^{(\tau)i}(t, x_i) = \left[A_i^{\{i\}}(t)\, x_i + C_i^{\{i\}}(t)\right] \exp\left[-r(t-\tau)\right],$$
$$\text{for } i \in \{1,2,3\} \text{ and } \tau \in [t_0, T].$$

Consider the case when all these three firms agree to form a joint venture and share their joint profit according to the dynamic Shapley value in Condition 6.2.1. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology level of company i under joint venture becomes:

$$\dot{x}_i(s) = \Big[\alpha_i \left[u_i(s)\, x_i(s)\right]^{1/2} + b_j^{[j,i]}\left[x_j(s)\, x_i(s)\right]^{1/2}$$
$$+ b_k^{[k,i]}\left[x_k(s)\, x_i(s)\right]^{1/2} - \delta x_i(s)\Big],$$
$$x_i(t_0) = x_i^0 \in X_i, \qquad \text{for } i, j, k \in N = \{1,2,3\} \text{ and } i \neq j \neq k, \tag{6.27}$$

where $b_j^{[j,i]}$ and $b_k^{[k,i]}$ are non-negative constants. In particular, $b_j^{[j,i]} u_j(s)$ represents the technology transfer effect under joint venture on firm i brought about by firm j 's technology.

The profit of the joint venture is the sum of the participating firms' profits:

$$\int_{t_0}^{T} \sum_{j=1}^{3} \left[P_j\left[x_j(s)\right]^{1/2} - c_j u_j(s)\right] \exp\left[-r(s-t_0)\right] ds$$
$$+ \sum_{j=1}^{3} \exp\left[-r(T-t_0)\right] q_j \left[x_j(T)\right]^{1/2}. \tag{6.28}$$

The firms in the joint venture then act cooperatively to maximize (6.28) subject to (6.27). Using Theorem 2.1.1, we obtain the Bellman equation as:

$$
\begin{aligned}
&-W_t^{(t_0)\{1,2,3\}}\left(t, x_1, x_2, x_3\right)= \\
&\quad \max_{u_i}\left\{\sum_{i=1}^3\left[P_i\left[x_i\right]^{1/2}-c_i u_i\right] \exp \left[-r\left(t-t_0\right)\right]\right. \\
&\qquad +\sum_{i=1}^3 W_{x_i}^{(t_0)\{1,2,3\}}\left(t, x_1, x_2, x_3\right)\left[\alpha_i\left[u_i x_i\right]^{1/2}\right. \\
&\qquad\qquad \left.\left.+b_j^{[j,i]}\left[x_j x_i\right]^{1/2}+b_k^{[k,i]}\left[x_k x_i\right]^{1/2}-\delta x_i\right]\right\}, \\
&W^{(t_0)\{1,2,3\}}\left(T, x_1, x_2, x_3\right)=\sum_{j=1}^3 \exp \left[-r\left(T-t_0\right)\right] q_j\left[x_j\right]^{1/2}, \\
&\qquad \text{for } i, j, k \in N=\{1,2,3\} \text{ and } i \neq j \neq k.
\end{aligned} \tag{6.29}
$$

Performing the indicated maximization yields

$$
u_i=\frac{\alpha_i^2}{4\left(c_i\right)^2}\left[W_{x_i}^{(t_0)i}\left(t, x_1, x_2, x_3\right) \exp \left[r\left(t-t_0\right)\right]\right]^2 x_i, \quad \text{for } i \in\{1,2,3\}. \tag{6.30}
$$

Substituting (6.30) into (6.29) yields:

$$
\begin{aligned}
&-W_t^{(t_0)\{1,2,3\}}\left(t, x_1, x_2, x_3\right)= \\
&\quad \sum_{i=1}^3\left[P_i\left[x_i(s)\right]^{1/2} \exp \left[-r\left(t-t_0\right)\right]\right. \\
&\qquad \left.-\frac{\alpha_i^2 x_i}{4 c_i}\left[W_{x_i}^{(t_0)i}\left(t, x_1, x_2, x_3\right)\right]^2 \exp \left[r\left(t-t_0\right)\right]\right] \\
&+\sum_{i=1}^3 W_{x_i}^{(t_0)\{1,2,3\}}\left(t, x_1, x_2, x_3\right)\left[\frac{\alpha_i^2}{2 c_i^2}\left[W_{x_i}^{(t_0)i}\left(t, x_1, x_2, x_3\right) \exp \left[r\left(t-t_0\right)\right]\right] x_i\right. \\
&\qquad \left.+b_j^{[j,i]}\left[x_j x_i\right]^{1/2}+b_k^{[k,i]}\left[x_k x_i\right]^{1/2}-\delta x_i\right], \quad \text{and} \\
&W^{(t_0)\{1,2,3\}}\left(T, x_1, x_2, x_3\right)=\sum_{j=1}^3 \exp \left[-r\left(T-t_0\right)\right] q_j\left[x_j\right]^{1/2}, \\
&\qquad \text{for } i, j, k \in N=\{1,2,3\} \text{ and } i \neq j \neq k.
\end{aligned} \tag{6.31}
$$

Solving (6.31) yields

$$
\begin{aligned}
W^{(t_0)\{1,2,3\}}\left(t, x_1, x_2, x_3\right)= & {\left[A_1^{\{1,2,3\}}(t) x_1^{1/2}+A_2^{\{1,2,3\}}(t) x_2^{1/2}\right.} \\
& \left.+A_3^{\{1,2,3\}}(t) x_3^{1/2}+C^{\{1,2,3\}}(t)\right] \exp \left[-r\left(t-t_0\right)\right],
\end{aligned} \tag{6.32}
$$

where $A_1^{\{1,2,3\}}(t), A_2^{\{1,2,3\}}(t), A_3^{\{1,2,3\}}(t)$ and x_3, $C^{\{1,2,3\}}(t)$ satisfy

$$\dot{A}_i^{\{1,2,3\}}(t) = \left(r + \frac{\delta}{2}\right) A_i^{\{1,2,3\}}(t) - \frac{b_i^{[i,j]}}{2} A_j^{\{1,2,3\}}(t) - \frac{b_i^{[i,k]}}{2} A_k^{\{1,2,3\}}(t) - P_i$$
$$\text{for } i,j,k \in \{1,2,3\} \text{ and } i \neq j \neq k,$$
$$\dot{C}^{\{1,2,3\}}(t) = rC^{\{1,2,3\}}(t) - \sum_{i=1}^{3} \frac{\alpha_i^2}{16c_i}\left[A_i^{\{1,2,3\}}(t)\right]^2,$$
$$A_i^{\{1,2,3\}}(T) = q_i, \qquad \text{for } i \in \{1,2,3\}, \text{ and } C^{\{1,2,3\}}(T) = 0. \tag{6.33}$$

The first three equations in the block recursive system (6.33) is a system of three linear differential equations which can be solved explicitly by standard techniques. Upon solving $A_i^{\{1,2,3\}}(t)$ for $i \in \{1,2,3\}$, and substituting them into the fourth equation of (6.33), one has a linear differential equation in $C^{\{1,2,3\}}(t)$.

The investment strategies of the grand coalition joint venture can be derived as:

$$\psi_i^{\{1,2,3\}}(t,x) = \frac{\alpha_i^2}{16(c_i)^2}\left[A_i^{\{1,2,3\}}(t)\right]^2,$$
$$\text{for } i \in \{1,2,3\}. \tag{6.34}$$

The dynamics of the state trajectories of the joint venture over the time interval $s \in [t_0, T]$ can be expressed as:

$$\dot{x}_i(s) = \frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}}(t)\, x_i(s)^{1/2} + b_j^{[j,i]}\left[x_j(s)\, x_i(s)\right]^{1/2}$$
$$+ b_k^{[k,i]}\left[x_k(s)\, x_i(s)\right]^{1/2} - \delta x_i(s),$$
$$x_i(t_0) = x_0, \tag{6.35}$$
$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

Taking the transforming $y_i(s) = x_i(s)^{1/2}$, for $i \in \{1,2,3\}$, equation system (6.35) can be expressed as:

$$\dot{y}_i(s) = \frac{\alpha_i^2}{8c_i} A_i^{\{1,2,3\}}(t) + \frac{1}{2} b_j^{[j,i]} y_j(s)^{1/2} + \frac{1}{2} b_k^{[k,i]} y_k(s)^{1/2} - \frac{\delta}{2} y_i(s),$$
$$y_i(t_0) = x_0^{1/2}, \tag{6.36}$$
$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

(6.36) is a system of linear differential equations which can be solved by standard techniques. Solving (6.36) yields the joint venture's state trajectory. Let $\{y_1^*(t), y_2^*(t), y_3^*(t)\}$ denote the solution to (6.36). Transforming $x_i = y_i^2$, we obtain the state trajectories of the joint venture over the time interval $s \in [t_0, T]$ as

$$\{x_1^*(t), x_2^*(t), x_3^*(t)\}_{t=t_0}^{T} = \left\{[y_1^*(t)]^2, [y_2^*(t)]^2, [y_3^*(t)]^2\right\}_{t=t_0}^{T}. \tag{6.37}$$

Once again, we use the terms $x_i^*(t)$ and x_i^{t*} interchangeably.

Remark 6.3.1. One can readily verify that:

$$W^{(t_0)\{1,2,3\}}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) = W^{(t)\{1,2,3\}}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) \exp[-r(t-t_0)]$$
$$\text{and } \psi_i^{(t_0)\{1,2,3\}*}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) = \psi_i^{(t)\{1,2,3\}*}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right).$$

For computation of the dynamic the Shapley value, we consider cases when two of the firms form a coalition $\{i, j\} \subset \{1, 2, 3\}$ to maximize joint profit:

$$\int_{t_0}^{T} \left[P_i [x_i(s)]^{1/2} - c_i u_i(s) + P_j [x_j(s)]^{1/2} - c_j u_j(s)\right] \exp[-r(s-t_0)]\, ds$$
$$+ \exp[-r(T-t_0)] \left\{q_i [x_i(T)]^{1/2} + q_j [x_j(T)]^{1/2}\right\} \tag{6.38}$$

subject to

$$\dot{x}_i(s) = \left[\alpha_i [u_i(s) x_i(s)]^{1/2} + b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s)\right],$$
$$x_i(t_0) = x_i^0 \in X_i, \qquad \text{for } i, j \in \{1, 2, 3\} \text{ and } i \neq j. \tag{6.39}$$

Following the above analysis, we obtain the following value functions:

$$W^{(t_0)\{i,j\}}(t, x_i, x_j) =$$
$$\left[A_i^{\{i,j\}}(t)\, x_i^{1/2} + A_j^{\{i,j\}}(t)\, x_j^{1/2} + C^{\{i,j\}}(t)\right] \exp[-r(t-t_0)], \tag{6.40}$$
$$\text{for } i, j \in \{1, 2, 3\} \text{ and } i \neq j,$$

where $A_i^{\{i,j\}}(t)$, $A_j^{\{i,j\}}(t)$ and $C^{\{i,j\}}(t)$ satisfy

$$\dot{A}_i^{\{1,2\}}(t) = \left(r + \frac{\delta}{2}\right) A_i^{\{1,2\}}(t) - \frac{b_i^{[i,j]}}{2} A_j^{\{1,2\}}(t) - P_i, \text{ and } A_i^{\{1,2\}}(T) = q_i,$$
$$\text{for } i, j \in \{1, 2, 3\} \text{ and } i \neq j;$$
$$\dot{C}^{\{1,2,3\}}(t) = r C^{\{1,2,3\}}(t) - \sum_{i=1}^{2} \frac{\alpha_i^2}{16 c_i} \left[A_i^{\{1,2\}}(t)\right]^2,$$
$$C^{\{1,2\}}(T) = 0. \tag{6.41}$$

The block-recursive system (6.41) can be solved readily by standard techniques. Moreover, as stated in Remark 6.2.1, one can easily derive for $\tau \in [t_0, T]$

$$W^{(t_0)\{i,j\}}(t,x_i,x_j)=\exp\left[-r(\tau-t_0)\right]W^{(\tau)\{i,j\}}(t,x_i,x_j),$$
$$\text{for } i,j\in\{1,2,3\} \text{ and } i\neq j.$$

Using the results derived above, one can readily obtain:

$$f_i^{\{1,2,3\}}\left[\tau,x_1^{\tau*},x_2^{\tau*},x_3^{\tau*},\psi_i^{(\tau)\{1,2,3\}}(\tau,x_1^{\tau*},x_2^{\tau*},x_3^{\tau*})\right]=$$
$$\frac{\alpha_i^2}{4c_i}A_i^{\{1,2,3\}}(\tau)(x_i^{\tau*})^{1/2}+b_j^{[j,i]}\left[x_j^{\tau*}x_i^{\tau*}\right]^{1/2}+b_k^{[k,i]}\left[x_k^{\tau*}x_i^{\tau*}\right]^{1/2}-\delta x_i^{\tau*},$$
$$\text{for } i\in\{1,2,3\}. \tag{6.42}$$

Denoting $[x_1^{\tau*},x_2^{\tau*},x_3^{\tau*}]$ by $x_{\{1,2,3\}}^{\tau*}$, we can write

$$f_{\{i,j\}}^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_i^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right),\psi_j^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]$$
$$=\begin{bmatrix} f_i^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_i^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]\\ f_j^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_j^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]\end{bmatrix},$$
$$\text{for } i,j\in\{1,2,3\} \text{ and } i\neq j;$$
$$f_{\{1,2,3\}}^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_1^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right),\psi_2^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right),\right.$$
$$\left.\psi_3^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]$$
$$=\begin{bmatrix} f_1^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_1^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]\\ f_2^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_2^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]\\ f_3^{\{1,2,3\}}\left[\tau,x_{\{1,2,3\}}^{\tau*},\psi_3^{(\tau)\{1,2,3\}}\left(\tau,x_{\{1,2,3\}}^{\tau*}\right)\right]\end{bmatrix}. \tag{6.43}$$

Using (6.26), (6.32) and (6.41), we obtain:

$$W_t^{(\tau)\{1,2,3\}}\left(t,x_{\{1,2,3\}}^{\tau*}\right)\Big|_{t=\tau}=\left[\dot{A}_1^{\{1,2,3\}}(\tau)(x_1^{\tau*})^{1/2}+\dot{A}_2^{\{1,2,3\}}(\tau)(x_2^{\tau*})^{1/2}\right.$$
$$\left.+\dot{A}_3^{\{1,2,3\}}(\tau)(x_3^{\tau*})^{1/2}+\dot{C}^{\{1,2,3\}}(\tau)\right]$$
$$-r\left[A_1^{\{1,2,3\}}(\tau)(x_1^{\tau*})^{1/2}+A_2^{\{1,2,3\}}(\tau)(x_2^{\tau*})^{1/2}\right.$$
$$\left.+A_3^{\{1,2,3\}}(\tau)(x_3^{\tau*})^{1/2}+C^{\{1,2,3\}}(\tau)\right];$$

$$W_t^{(\tau)\{i,j\}}\left(t,x_{\{i,j\}}^{\tau*}\right)\Big|_{t=\tau}=$$
$$\left[\dot{A}_i^{\{i,j\}}(\tau)(x_i^{\tau*})^{1/2}+\dot{A}_j^{\{i,j\}}(\tau)\left(x_j^{\tau*}\right)^{1/2}+\dot{C}^{\{i,j\}}(\tau)\right]$$
$$-r\left[A_i^{\{i,j\}}(\tau)(x_i^{\tau*})^{1/2}+A_j^{\{i,j\}}(\tau)\left(x_j^{\tau*}\right)^{1/2}+C^{\{i,j\}}(\tau)\right],$$
$$\text{for } i,j\in\{1,2,3\} \text{ and } i\neq j;$$

$$
\begin{aligned}
& W_t^{(\tau)i}\left(t, x_i^{\tau *}\right)\Big|_{t=\tau} = \\
& \qquad \left[\dot{A}_i^{\{i\}}(\tau)\, x_i^{\tau *} + \dot{C}^{\{i\}}(\tau)\right] - r\left[A_i^{\{i\}}(\tau)\, x_i^{\tau *} + C^{\{i\}}(\tau)\right], \\
& \qquad\qquad \text{for } \ i \in \{1,2,3\}; \\
& W_{x_i^{\tau *}}^{(\tau)K}\left(t, x_K^{\tau *}\right)\Big|_{t=\tau} = \frac{1}{2} A_i^K(\tau)\left(x_i^{\tau *}\right)^{-1/2}, \qquad \text{for } \ i \in K \subseteq \{1,2,3\}. \qquad (6.44)
\end{aligned}
$$

Upon substituting the results from (6.42) to (6.44) into Theorem 6.2.3, we obtain $B_i(\tau)$. A payment $B_i(\tau)$ offered to player $i \in \{1,2,3\}$ at time $\tau \in [t_0, T]$ will lead to the realization of the dynamic Shapley value in Condition 6.2.1. Hence a dynamically stable solution to the joint venture will result.

6.4 The Stochastic Version

Now we return to the multiplayer cooperative stochastic differential game (6.1)–(6.3).

6.4.1 Expected Coalition Payoffs

The expected profit to coalition K can be obtained by maximizing:

$$
\begin{aligned}
E_{t_0} \Bigg\{ & \int_{t_0}^{T} \sum_{j \in K} g^j\left[s, x_j(s), u_j(s)\right] \exp\left[-\int_{t_0}^{s} r(y)\, dy\right] ds \\
& + \sum_{j \in K} \exp\left[-\int_{t_0}^{T} r(y)\, dy\right] q^j\left(x_j(T)\right) \Bigg\}, \qquad \text{for } \ K \subseteq N, \quad (6.45)
\end{aligned}
$$

subject to

$$
\begin{aligned}
& dx_i(s) = f^i\left[s, x_K(s), u_i(s)\right] ds + \sigma_i\left[s, x_i(s)\right] dz_i(s), \qquad x_i(t_0) = x_i^0, \\
& \qquad \text{for } \ i \in K. \qquad (6.46)
\end{aligned}
$$

We use $\Gamma[K; t_0, x_0]$ to denote the stochastic control problem (6.45)–(6.46). Using Fleming's technique of stochastic control the solution of the problem $\Gamma[K; t_0, x_0]$ can be characterized as follows.

Theorem 6.4.1. *A set of controls* $\left\{u_K^*(t) = \psi_K^{(t_0)K*}(t, x_K)\right\}$ *provides an optimal solution to the stochastic control problem* $\Gamma[K; t_0, x_0]$ *if there exist continuously differentiable function* $W^{(t_0)K}(t, x_K) : [t_0, T] \times \Pi_{j \in K} R^{m_j} \to R$, *satisfying the following partial differential equation:*

$$-W_t^{(t_0)K}(t, x_K) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega_K^{h\zeta}(t, x_K)\, W_{x^h x^\zeta}^{(t_0)K}(t, x_K) =$$

$$\max_{u_K}\left\{\sum_{j\in K} g^j\left[t, x_j, u_j\right]\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right.$$

$$\left. + \sum_{j\in K} W_{x_j}^{(t_0)K}(t, x_K)\, f_j^K\left[t, x_K, u_j\right]\right\},$$

$$W^{(t_0)K}(T, x_K) = \sum_{j\in K}\exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^j(x_j),$$

where $\Omega_K(t, x_K)$ *is a matrix with* $\Omega_i\left[s, x_i(s)\right]$, *for* $i \in K \subseteq N$, *lying in the diagonal and zeros elsewhere.* $\Omega_K^{h\zeta}(t, x_K)$ *is the element in row* h *and column* ζ *of* $\Omega_K(t, x_K)$.

In the case when all the firms are in the joint venture, that is $K = N$, the optimal control

$$\psi_N^{(t_0)N*}(s, x_N(s)) =$$
$$\left[\psi_1^{(t_0)N*}(s, x_N(s)), \psi_2^{(t_0)N*}(s, x_N(s)), \ldots, \psi_n^{(t_0)N*}(s, x_N(s))\right]$$

satisfying Theorem 6.4.1 will be adopted. The dynamics of the optimal trajectory of joint venture be obtained as:

$$dx_j(s) = f_j^N\left[s, x_N(s), \psi_j^{(t_0)N*}(s, x_N(s))\right]ds + \sigma_j\left[s, x_j(s)\right]dz_j(s),$$
$$x_j(t_0) = x_j^0, \qquad \text{for } j \in N. \tag{6.47}$$

System (6.47) can also be expressed as

$$dx_N(s) = f^N\left[s, x_N(s), \psi_N^{(t_0)N*}(s, x_N(s))\right]ds + \sigma_N\left[s, x_N(s)\right]dz_N(s),$$

where $dx_N(s)$ is a column vector of $\left[dx_1(s), dx_2(s), \ldots, dx_n(s)\right]$, $dz_N(s)$ is a column vector of $\left[dz_1(s), dz_2(s), \ldots, dz_n(s)\right]$,

$$f^N\left[s, x_N(s), \psi_N^{(t_0)N*}(s, x_N(s))\right] = \begin{bmatrix} f_1^N\left[s, x_N(s), \psi_1^{(t_0)N*}(s, x_N(s))\right] \\ f_2^N\left[s, x_N(s), \psi_2^{(t_0)N*}(s, x_N(s))\right] \\ \vdots \\ f_n^N\left[s, x_N(s), \psi_n^{(t_0)N*}(s, x_N(s))\right] \end{bmatrix},$$

$$\text{and } \sigma_N\left[s, x_N(s)\right] = \begin{bmatrix} \sigma_1\left[s, x_1(s)\right] \\ \sigma_2\left[s, x_2(s)\right] \\ \vdots \\ \sigma_n\left[s, x_n(s)\right] \end{bmatrix}.$$

The solution to (6.47) yields the optimal trajectories:

$$x_j^*(t) = x_j^0 + \int_{t_0}^{t} f^j\left[s, x_N^*(s), \psi_j^{(t_0)N*}(s, x_N^*(s))\right] ds$$
$$+ \int_{t_0}^{t} \sigma_j\left[s, x_j^*(s)\right] dz_j(s),$$
$$\text{for } j \in N. \tag{6.48}$$

We use X_j^{t*} to denote the set of realizable values of $x_j^*(t)$ at time t generated by (6.48). We denote the stochastic optimal trajectory by $\{x_N^*(t)\}_{t=t_0}^T = \{x_1^*(t), x_2^*(t), \ldots, x_n^*(t)\}_{t=t_0}^T$ and the set of realizable values of $x_N^*(t)$ at time t by X_N^{t*}. We use x_j^{t*} to denote an element in X_t^* and x_N^{t*} to denote an element in X_N^*.

Note that the stochastic version of Remark 6.2.1 can be obtained as follow.

Remark 6.4.1. Consider the stochastic control problem $\Gamma[K; \tau, x_K^{\tau*}]$ which starts at time $\tau \in [t_0, T]$ with initial state x_K^τ. Using Theorem 6.4.1, one can readily show that:

$$\exp\left[\int_\tau^t r(y)\,dy\right] W^{(\tau)K}\left(t, x_K^t\right) = W^{(t)K}\left(t, x_K^t\right),$$
$$\text{for } t_0 \le \tau \le t \le T; \text{ and}$$
$$\psi_K^{(\tau)K*}\left(t, x_K^t\right) = \psi_K^{(t)K*}\left(t, x_K^t\right), \qquad \text{for } t_0 \le \tau \le t \le T.$$

Similar to the deterministic case, (i) the level of technology of firm i enhances its profit as $g^i[s, x_i, u_i]$ and $q^i(x_i)$ are positively related to the level of technology x_i, and (ii) participating firms can gain core skills and technology from other firms in the coalition as $\partial f_i^K[s, x_i, u_K]/\partial u_j \ge 0$, for $j \ne i$; hence superadditivity of the coalition profit functions will result, that is,

$$W^{(\tau)K}(\tau, x_K^\tau) \ge W^{(\tau)L}(\tau, x_L^\tau) + W^{(\tau)K\backslash L}\left(\tau, x_{K\backslash L}^\tau\right), \qquad \text{for } L \subset K \subseteq N,$$

where $K \backslash L$ is the relative complement of L in K,

6.4.2 Stochastic Dynamic Shapley Value

Consider the above joint venture involving n firms. The member firms would maximize their joint expected profit and share their cooperative profits according to the Shapley value (1953).

To maximize the joint venture's profits the firms would adopt the control vector $\left\{\psi_N^{(t_0)N*}(t, x_N^{t*})\right\}_{t=t_0}^T$ over the time interval $[t_0, T]$, and the corresponding optimal state trajectory $\{x_N^*(t)\}_{t=t_0}^T$ in (6.47) would result. At time t_0 with state $x_N^{t_0}$, the firms agree that firm i's share of profits be:

$$v^{(t_0)i}\left(t_0, x_N^0\right) = \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!}\left[W^{(t_0)K}\left(t_0, x_K^0\right) - W^{(t_0)K\backslash i}\left(t_0, x_{K\backslash i}^0\right)\right], \quad \text{for } i \in N. \tag{6.49}$$

However, the Shapley value has to be maintained throughout the venture horizon $[t_0, T]$. In particular, at time $\tau \in [t_0, T]$ with the state being $x_N^{\tau *} \in X_N^{\tau *}$ the following imputation principle has to be maintained:

Condition 6.4.1. *At time τ, firm i's share of profits be:*

$$v^{(\tau)i}\left(\tau, x_N^{\tau *}\right) = \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!}\left[W^{(\tau)K}\left(\tau, x_K^{\tau *}\right) - W^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{\tau *}\right)\right],$$

$$\textit{for } \; i \in N \;\; \textit{and} \;\; \tau \in [t_0, T] \;\; \textit{and} \;\; x_N^{\tau *} \in X_N^{\tau *}.$$

Note that $v^{(\tau)}\left(\tau, x_N^{\tau *}\right) = \left[v^{(\tau)1}\left(\tau, x_N^{\tau *}\right), v^{(\tau)2}\left(\tau, x_N^{\tau *}\right), \ldots, v^{(\tau)n}\left(\tau, x_N^{\tau *}\right)\right]$, as specified in Condition 6.4.1 satisfies the basic properties of an imputation vector:

(i)
$$\sum_{j=1}^{n} v^{(\tau)j}\left(\tau, x_N^{\tau *}\right) = W^{(\tau)N}\left(\tau, x_N^{\tau *}\right), \quad \text{and}$$

(ii)
$$v^{(\tau)i}\left(\tau, x_N^{\tau *}\right) \geq W^{(\tau)i}\left(\tau, x_N^{\tau *}\right), \quad \text{for } i \in N \text{ and } \tau \in [t_0, T]. \tag{6.50}$$

Part (i) of (6.50) shows that $v^{(\tau)}\left(\tau, x_N^{\tau *}\right)$ satisfies the property of Pareto optimality throughout the game interval. Part (ii) demonstrates that $v^{(\tau)}\left(\tau, x_N^{\tau *}\right)$ guarantees individual rationality throughout the game interval. Pareto optimality and individual rationality are essential properties of imputation vectors. Moreover, if Condition 6.4.1 can be maintained, the solution optimality principle – sharing profits according to the Shapley value – is in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. Hence time consistency is satisfied and no firms would have any incentive to depart the joint venture. Therefore a dynamic imputation principle leading to Condition 6.4.1 is dynamically stable or time consistent.

Crucial to the analysis is the formulation of a profit distribution mechanism that would lead to the realization of Condition 6.4.1. This will be done in the next section.

6.4.3 Transitory Compensation

In this section, a profit distribution mechanism will be developed to compensate transitory changes so that the Shapley value principle in Condition 6.4.1

could be maintained throughout the venture horizon. First, an imputation distribution procedure (Yeung and Petrosyan (2004)) must be now formulated so that the imputation scheme in Condition 6.4.1 can be realized. Let the $B_i(s)$ denote the payment received by firm $i \in N$ at time $s \in [t_0, T]$ dictated by $v^{(t_0)i}\left(t_0, x_N^0\right)$. In particular,

$$\begin{aligned} &v^{(t_0)i}\left(t_0, x_N^0\right) = \\ &\quad \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(t_0)K}\left(t_0, x_K^0\right) - W^{(t_0)K\backslash i}\left(t_0, x_{K\backslash i}^0\right)\right] = \\ &\qquad E_{t_0}\left\{\int_{t_0}^{T} B_i(s) \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds \right. \\ &\qquad\qquad \left. + q^i\left(x_i^*(T)\right) \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] \right| x_N(t_0) = x_N^0 \Bigg\}, \\ &\qquad \text{for } \ i \in N. \end{aligned} \tag{6.51}$$

Moreover, for $i \in N$ and $t \in [t_0, T]$, we use

$$\begin{aligned} &v^{(t_0)i}\left(t, x_N^{t*}\right) = \\ &\quad E_{t_0}\left\{\int_{t_0}^{T} B_i(s) \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds \right. \\ &\qquad \left. + q^i\left(x_i^*(T)\right) \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] \right| x_N(t) = x_N^{t*} \Bigg\}, \end{aligned} \tag{6.52}$$

to denote the present value of Player i's cooperative profit over the time interval $[t, T]$, given that the state is $x_N^{t*} \in X_N^{t*}$ at time $t \in [t_0, T]$.

A necessary condition for $v^{(t_0)i}(t, x_N^{t*})$ to follow Condition 6.4.1 is that:

$$\begin{aligned} v^{(t_0)i}\left(t, x_N^{t*}\right) &= v^{(t)i}\left(t, x_N^{t*}\right) \exp\left[-\int_{t_0}^{t} r(y)\,dy\right], \\ &\text{for } \ i \in N \ \text{ and } \ t \in [\tau, T] \ \text{ and } \ x_N^{t*} \in X_N^{t*}. \end{aligned} \tag{6.53}$$

A candidate of $v^{(t_0)i}(t, x_N^{t*})$ satisfying (6.51)–(6.53) has to be found. A natural choice is

$$\begin{aligned} v^{(t_0)i}\left(t, x_N^{t*}\right) &= \sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(t_0)K}\left(t, x_K^{t*}\right) - W^{(t_0)K\backslash i}\left(t, x_{K\backslash i}^{t*}\right)\right], \\ &\text{for } \ x_N^{t*} \in X_N^{t*}. \end{aligned} \tag{6.54}$$

With Remark 6.4.1, one can readily that $v^{(t_0)i}(t, x_N^{t*})$ as defined in (6.54) satisfies (6.51)–(6.53).

To fulfill the Pareto optimality property, the PDP $B(s) = [B_1(s), B_2(s), \ldots, B_n(s)]$ has to satisfy the following condition.

Condition 6.4.2.

$$\sum_{j=1}^{n} B_i(s) = \sum_{j=1}^{n} g^j \left[s, x_j^{S*}, \psi_N^{(t_0)N*}\left(s, x_N^{S*}\right)\right],$$
$$\textit{for } \; s \in [\tau, T] \quad \textit{and } \; x_N^{s*} \in X_N^{s*}.$$

Invoking Remark 6.4.1, we can write

$$v^{(\tau)i}\left(t, x_N^{t*}\right) = \exp\left[\int_{t_0}^{\tau} r(y)\, dy\right] v^{(t_0)i}\left(t, x_N^{t*}\right) =$$
$$\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \left[W^{(\tau)K}\left(\tau, x_K^{t*}\right) - W^{(\tau)K\backslash i}\left(\tau, x_{K\backslash i}^{t*}\right)\right] =$$
$$E_\tau \left\{ \int_t^T B_i(s) \exp\left[-\int_\tau^s r(y)\, dy\right] ds \right.$$
$$\left. + q^i\left(x_i^*(T)\right) \exp\left[-\int_\tau^T r(y)\, dy\right] \right| x_N(t) = x_N^{t*} \Bigg\}, \tag{6.55}$$
$$\text{for } \; t_0 \le \tau \le t \le T \; \text{ and } \; x_N^{t*} \in X_N^{t*}.$$

Since the value function $W^{(\tau)K}\left(t, x_K^{t*}\right)$ is twice continuously differentiable in t and x_K^{t*}, the term $v^{(\tau)i}\left(t, x_N^{t*}\right)$ is twice continuously differentiable in t and x_N^{t*}.

Given the differentiability property of $v^{(\tau)i}\left(t, x_N^{t*}\right)$, for $\Delta t \to 0$ one can use (6.53) and Remark 6.4.1 to obtain:

$$v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) =$$
$$E_\tau \left\{ \int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\, dy\right] ds \right.$$
$$\left. + \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\, dy\right] v^{(\tau+\Delta t)i}\left(\tau + \Delta t, x_N^{\tau*} + \Delta x_N^{\tau*}\right) \right| x_N(\tau) = x_N^{\tau*} \Bigg\}, \tag{6.56}$$
$$\text{for } \; i \in N, \; t \in [\tau, T], \; \tau \in [t_0, T] \; \text{ and } \; x_N^{\tau*} \in X_N^{\tau*};$$

where

$$\Delta x_N^{\tau*} = \left[\Delta x_1^{\tau*}, \Delta x_2^{\tau*}, \ldots, \Delta x_n^{\tau*}\right],$$
$$\Delta x_j^{\tau*} = f_j^N\left[\tau, x_N^{\tau*}, \psi_j^{(\tau)N*}\left(\tau, x_N^{\tau*}\right)\right] \Delta t + \sigma_j\left[\tau, x_j^{\tau*}\right] \Delta z_j^\tau + o(\Delta t),$$
$$\text{for } \; j \in N,$$
$$\Delta z_j^\tau = z_j(\tau + \Delta t) - z_j(\tau), \; \text{ and } \; E_\tau\left[o(\Delta t)\right]/\Delta t \to 0 \; \text{ as } \; \Delta t \to 0.$$

Using (6.53) and (6.55), we express (6.56) as:

$$E_\tau \left\{ \int_\tau^{\tau+\Delta t} B_i(s) \exp\left[-\int_\tau^s r(y)\, dy \right] ds \middle| x_N(t) = x_N^{t*} \right\}$$
$$= v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) - \exp\left[-\int_\tau^{\tau+\Delta t} r(y)\, dy \right] v^{(\tau+\Delta t)i}\left(\tau+\Delta t, x_N^{\tau*} + \Delta x_N^{\tau*}\right)$$
$$= v^{(\tau)i}\left(\tau, x_N^{\tau*}\right) - v^{(\tau)i}\left(\tau+\Delta t, x_N^{\tau*} + \Delta x_N^{\tau}\right),$$
$$\text{for all } \tau \in [t_0, T]. \tag{6.57}$$

When $\Delta t \to 0$, condition (6.57) can be expressed as:

$$\begin{aligned} B_i(\tau)\,\Delta t = E_\tau \Bigg\{ & -\left[v_t^{(\tau)i}\left(t, x_N^{t*}\right)\Big|_{t=\tau} \right] \Delta t \\ & - \sum_{j \in N} \left[v_{x_j^{t*}}^{(\tau)i}\left(t, x_N^{t*}\right)\Big|_{t=\tau} \right] f_j^N\left[\tau, x_N^{\tau*}, \psi_j^{(\tau)N}\left(\tau, x_N^{\tau*}\right) \right] \Delta t \\ & - \frac{1}{2} \sum_{h,\zeta=1}^{n} \Omega_N^{h\zeta}\left(\tau, x_\tau^*\right) \left[v_{x_t^h x_t^\zeta}^{(\tau)i}\left(t, x_t^*\right)\Big|_{t=\tau} \right] \Delta t \\ & - \sum_{j \in N} \left[v_{x_j^{t*}}^{(\tau)i}\left(t, x_N^{t*}\right)\Big|_{t=\tau} \right] \sigma_j\left[\tau, x_j^{\tau*} \right] \Delta z_j^\tau - o(\Delta t) \Bigg\}. \end{aligned}$$

Taking expectation and dividing the above equation throughout by Δt, with $\Delta t \to 0$, yield

$$\begin{aligned} B_i(\tau) = & -\left[v_t^{(\tau)i}\left(t, x_N^{\tau*}\right)\Big|_{t=\tau} \right] \\ & - \sum_{j \in N} \left[v_{x_j^{t*}}^{(\tau)i}\left(t, x_N^{\tau*}\right)\Big|_{t=\tau} \right] f_j^N\left[\tau, x_N^{\tau*}, \psi_j^{(\tau)N}\left(\tau, x_N^{\tau*}\right) \right] \\ & - \frac{1}{2} \sum_{h,\zeta=1}^{n} \Omega_N^{h\zeta}\left(\tau, x_\tau^*\right) \left[v_{x_t^h x_t^\zeta}^{(\tau)i}\left(t, x_t^*\right)\Big|_{t=\tau} \right], \end{aligned}$$
$$\text{for } i \in N \text{ and } t \in [\tau, T],\ \tau \in [t_0, T] \text{ and } x_N^{\tau*} \in X_N^{\tau*}. \tag{6.58}$$

Using (6.55) and (6.58), we obtain:

Theorem 6.4.2. *A payment to player $i \in N$ at time $\tau \in [t_0, T]$ equaling*

$$B_i(\tau) =$$
$$-\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \Bigg\{ \left[W_t^{(\tau)K}\left(t, x_K^{\tau*}\right)\Big|_{t=\tau} \right] - \left[W_t^{(\tau)K \setminus i}\left(t, x_{K\setminus i}^{\tau*}\right)\Big|_{t=\tau} \right]$$

$$+\left(\left[W_{x_N^{\tau*}}^{(\tau)K}\left(t,x_K^{\tau*}\right)\Big|_{t=\tau}\right]\right.$$
$$\left.-\left[W_{x_N^{\tau*}}^{(\tau)K\backslash i}\left(\tau,x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau}\right]\right)f^N\left[\tau,x_N^{\tau*},\psi_N^{(\tau)N}\left(\tau,x_N^{\tau*}\right)\right]$$
$$+\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega_K^{h\zeta}\left(\tau,x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{(\tau)K}\left(t,x_t^*\right)\Big|_{t=\tau}\right]$$
$$\left.-\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega_{K\backslash i}^{h\zeta}\left(\tau,x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{(\tau)K\backslash i}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right\}$$

when $x_N(\tau)=x_N^{\tau*}\in X_N^{\tau*}$ will lead to the realization of the Condition 6.4.1.

Since the partial derivative of $W^{(\tau)K}\left(\tau,x_K^{\tau*}\right)$ with respect to x_j, where $j\notin K$, will vanish, a more concise form of Theorem 6.4.2 can be obtained as:

Theorem 6.4.3. *A payment to player $i\in N$ at time $\tau\in[t_0,T]$ leading to the realization of the Condition 6.4.1 can be expressed as:*

$$B_i(\tau)=$$
$$-\sum_{K\subseteq N}\frac{(k-1)!\,(n-k)!}{n!}\left\{\left[W_t^{(\tau)K}\left(t,x_K^{\tau*}\right)\Big|_{t=\tau}\right]-\left[W_t^{(\tau)K\backslash i}\left(t,x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau}\right]\right.$$
$$+\sum_{j\in K}\left[W_{x_j^{\tau*}}^{(\tau)K}\left(t,x_K^{\tau*}\right)\Big|_{t=\tau}\right]f_j^N\left[\tau,x_N^{\tau*},\psi_j^{(\tau)N}\left(\tau,x_N^{\tau*}\right)\right]$$
$$-\sum_{h\in K\backslash i}\left[W_{x_h^{\tau*}}^{(\tau)K\backslash i}\left(\tau,x_{K\backslash i}^{\tau*}\right)\Big|_{t=\tau}\right]f_h^N\left[\tau,x_N^{\tau*},\psi_h^{(\tau)N}\left(\tau,x_N^{\tau*}\right)\right]$$
$$+\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega_K^{h\zeta}\left(\tau,x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{(\tau)K}\left(t,x_t^*\right)\Big|_{t=\tau}\right]$$
$$\left.-\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega_{K\backslash i}^{h\zeta}\left(\tau,x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{(\tau)K\backslash i}\left(t,x_t^*\right)\Big|_{t=\tau}\right]\right\}$$

when $x_N(\tau)=x_N^{\tau*}\in X_N^{\tau*}$.

The vector $B(\tau)$ serves as a form equilibrating transitory compensation that guarantees the realization of the Shapley value imputation throughout the game horizon. Note that the instantaneous profit $B_i(\tau)$ offered to Player i at time τ is conditional upon the current state $x_N^{\tau*}$ and current time τ. One can elect to express $B_i(\tau)$ as $B_i\left(\tau,x_N^{\tau*}\right)$. Hence an instantaneous payment $B_i\left(\tau,x_N^{\tau*}\right)$ to player $i\in N$ yields a dynamically stable solution to the joint venture.

6.5 An Application in Cooperative R&D under Uncertainty

Consider the case when there are 3 companies. The planning period is $[t_0, T]$. Company i expected profit is

$$E_{t_0}\left\{\int_{t_0}^{T}\left[P_i\left[x_i\left(s\right)\right]^{1/2}-c_iu_i\left(s\right)\right]\exp\left[-r\left(s-t_0\right)\right]ds\right.$$
$$\left.+\exp\left[-r\left(T-t_0\right)\right]q_i\left[x_i\left(T\right)\right]^{1/2}\right\},$$
$$\text{for } i\in N=\{1,2,3\}, \tag{6.59}$$

where P_i, c_i and q_i are positive constants, r is the discount rate, $x_i(s) \subset R^+$ is the level of technology of company i at time s, and $u_i(s) \subset R^+$ is its physical investment in technological advancement. The term $P_i[x_i(s)]^{1/2}$ reflects the net operating revenue of company i at technology level $x_i(s)$, and c_iu_i is the cost of investment. $q_i[x_i(T)]^{1/2}$ gives the salvage value of company i's technology at time T.

The evolution of the technology level of company i follows the dynamics:

$$dx_i\left(s\right)=\left[\alpha_i\left[u_i\left(s\right)x_i\left(s\right)\right]^{1/2}-\delta x_i\left(s\right)\right]ds+\sigma_ix_i\left(s\right)dz_i\left(s\right),$$
$$x_i\left(t_0\right)=x_i^0\in X_i,\qquad \text{for } i\in N=\{1,2,3\}, \tag{6.60}$$

where $z_i(s)$ is a Wiener process. For $i \neq j$, $x_i \cap x_j = \emptyset$, and $z_i(s)$ and $z_j(s)$ are independent Wiener processes. the term $\alpha_i[u_i(s)x_i(s)]^{1/2}$ is the expected addition to technology brought about by $u_i(s)$ amount of physical investment, and δ is the rate of obsolescence. There is a noise term in technology advancement reflected by $\sigma_ix_i(s)dz_i(s)$.

In the case when each of these three firms acts independently. Using Theorem 2.1.5, we obtain the Fleming stochastic control equation as:

$$-W_t^{(t_0)i}\left(t,x_i\right)-\frac{1}{2}W_{x_ix_i}^{(t_0)i}\left(t,x_i\right)\sigma_i^2x_i^2=$$
$$\max_{u_i}\left\{\left[P_ix_i^{1/2}-c_iu_i\right]\exp\left[-r\left(t-t_0\right)\right]+W_{x_i}^{(\tau)i}\left(t,x_i\right)\left[\alpha_i\left(u_ix_i\right)^{1/2}-\delta x_i\right]\right\},$$
$$W^{(t_0)i}\left(T,x_i\right)=\exp\left[-r\left(T-t_0\right)\right]q_i\left[x_i\right]^{1/2},\qquad \text{for } i\in\{1,2,3\}. \tag{6.61}$$

Performing the indicated maximization yields

$$u_i=\frac{\alpha_i^2}{4\left(c_i\right)^2}\left[W_{x_i}^{(t_0)i}\left(t,x_i\right)\exp\left[r\left(t-t_0\right)\right]\right]^2x_i,$$
$$\text{for } i\in\{1,2,3\}. \tag{6.62}$$

Substituting (6.62) into (6.61) yields:

$$
\begin{aligned}
&-W_t^{(t_0)i}(t,x_i) - \frac{1}{2} W_{x_i x_i}^{(t_0)i}(t,x_i)\,\sigma_i^2 x_i^2 = \\
&\qquad P_i x_i^{1/2} \exp\left[-r(t-t_0)\right] - \frac{\alpha_i^2}{4c_i}\left[W_{x_i}^{(t_0)i}(t,x_i)\right]^2 \exp\left[r(t-t_0)\right] x_i \\
&\qquad + \frac{\alpha_i^2}{2c_i}\left[W_{x_i}^{(t_0)i}(t,x_i)\right]^2 \exp\left[r(t-\tau)\right] x_i - \delta W_{x_i}^{(t_0)i}(t,x_i)\, x_i, \\
&\qquad\qquad \text{for } \ i \in \{1,2,3\}. \qquad (6.63)
\end{aligned}
$$

Solving (6.63) yields

$$
\begin{aligned}
&W^{(t_0)i}(t,x_i) = \left[A_i^{\{i\}}(t)\, x_i^{1/2} + C_i^{\{i\}}(t)\right] \exp\left[-r(\tau-t_0)\right], \\
&\qquad \text{for } \ i \in \{1,2,3\}, \qquad (6.64)
\end{aligned}
$$

where

$$
\begin{aligned}
&\dot{A}_i^{\{i\}}(t) = \left(r + \frac{\sigma^2}{8} + \frac{\delta}{2}\right) A_i^{\{i\}}(t) - P_i, \\
&\dot{C}_i^{\{i\}}(t) = r C_i^{\{i\}}(t) - \frac{\alpha_i^2}{16 c_i}\left[A_i^{\{i\}}(t)\right]^2, \\
&A_i^{\{i\}}(T) = q_i \ \text{ and } \ C_i^{\{i\}}(T) = 0. \qquad (6.65)
\end{aligned}
$$

The first equation in the block-recursive system (6.65) is a first-order linear differential equation in $A_i^{\{i\}}(t)$ which can be solved independently by standard techniques. Upon substituting the solution of $A_i^{\{i\}}(t)$ into the second equation of (6.65) yields a first-order linear differential equation in $C_i^{\{i\}}(t)$. The solution of $C_i^{\{i\}}(t)$ can be readily obtained by standard techniques. The explicit solution is not stated here because of its lengthy expressions.

Moreover, as stated in Remark 6.4.1, one can easily derive for $\tau \in [t_0, T]$

$$
\begin{aligned}
&W^{(\tau)i}(t,x_i) = \left[A_i^{\{i\}}(t)\, x_i + C_i^{\{i\}}(t)\right] \exp\left[-r(t-\tau)\right], \\
&\qquad \text{for } \ i \in \{1,2,3\} \ \text{ and } \ \tau \in [t_0, T], \qquad (6.66)
\end{aligned}
$$

Consider the case when all these three firms agree to cooperate in research and development and share their joint profit according to the dynamic Shapley value in Condition 6.4.1. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology level of company i under joint venture becomes:

$$
dx_i(s) = \left[\alpha_i \left[u_i(s)\, x_i(s)\right]^{1/2} + b_j^{[j,i]} \left[x_j(s)\, x_i(s)\right]^{1/2}\right.
$$

$$+b_k^{[k,i]}\left[x_k(s)\,x_i(s)\right]^{1/2}-\delta x_i(s)\Big]\,ds$$
$$+\sigma_i x_i(s)\,dz_i(s)\,,\qquad x_i(t_0)=x_i^0\in X_i,$$
$$\text{for }\; i,j,k\in N=\{1,2,3\}\;\text{ and }\; i\neq j\neq k,\tag{6.67}$$

where $b_j^{[j,i]}$ and $b_k^{[k,i]}$ are non-negative constants. In particular, $b_j^{[j,i]}u_j(s)$ represents the technology transfer effect under joint venture on firm i brought about by firm j 's technology.

The expected profit of the joint venture is the sum of the participating firms' expected profits:

$$E_{t_0}\left\{\int_{t_0}^{T}\sum_{j=1}^{3}\Big[P_j\left[x_j(s)\right]^{1/2}-c_ju_j(s)\Big]\exp\left[-r(s-t_0)\right]ds\right.$$
$$\left.+\sum_{j=1}^{3}\exp\left[-r(T-t_0)\right]q_j\left[x_j(T)\right]^{1/2}\right\}.\tag{6.68}$$

The firms in the joint venture then act cooperatively to maximize (6.68) subject to (6.67). Using Theorem 2.1.5, we obtain the stochastic control equation as:

$$-W_t^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3)-\frac{1}{2}\sum_{i=1}^{3}W_{x_ix_i}^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3)\,\sigma_i^2x_i^2=$$
$$\max_{u_i}\left\{\sum_{i=1}^{3}\Big[P_i\left[x_i\right]^{1/2}-c_iu_i\Big]\exp\left[-r(t-t_0)\right]\right.$$
$$+\sum_{i=1}^{3}W_{x_i}^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3)\Big[\alpha_i\left(u_ix_i\right)^{1/2}$$
$$\left.+b_j^{[j,i]}\left[x_jx_i\right]^{1/2}+b_k^{[k,i]}\left[x_kx_i\right]^{1/2}-\delta x_i\Big]\right\},$$

$$W^{(t_0)\{1,2,3\}}(T,x_1,x_2,x_3)=\sum_{j=1}^{3}\exp\left[-r(T-t_0)\right]q_j\left[x_j\right]^{1/2},$$
$$\text{for }\; i,j,k\in N=\{1,2,3\}\;\text{ and }\; i\neq j\neq k.\tag{6.69}$$

Performing the indicated maximization yields

$$u_i=\frac{\alpha_i^2}{4\left(c_i\right)^2}\left[W_{x_i}^{(t_0)i}(t,x_1,x_2,x_3)\exp\left[r(t-t_0)\right]\right]^2x_i,\qquad \text{for }\; i\in\{1,2,3\}\,.\tag{6.70}$$

Substituting (6.70) into (6.69) yields:

$$-W_t^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3) - \frac{1}{2}\sum_{i=1}^{3} W_{x_i x_i}^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3)\,\sigma_i^2 x_i^2 =$$

$$\sum_{i=1}^{3}\left[P_i\,[x_i]^{1/2}\exp[-r(t-t_0)] - \frac{\alpha_i^2 x_i}{4c_i}\left[W_{x_i}^{(t_0)i}(t,x_1,x_2,x_3)\right]^2\exp[r(t-t_0)]\right]$$

$$+\sum_{i=1}^{3} W_{x_i}^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3)\left[\frac{\alpha_i^2}{2c_i^2}\left[W_{x_i}^{(t_0)i}(t,x_1,x_2,x_3)\exp[r(t-t_0)]\right]x_i\right.$$

$$\left.+\,b_j^{[j,i]}[x_j x_i]^{1/2} + b_k^{[k,i]}[x_k x_i]^{1/2} - \delta x_i\right],\ \text{and}$$

$$W^{(t_0)\{1,2,3\}}(T,x_1,x_2,x_3) = \sum_{j=1}^{3}\exp[-r(T-t_0)]\,q_j\,[x_j]^{1/2}, \tag{6.71}$$

$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

Solving (6.71) yields

$$W^{(t_0)\{1,2,3\}}(t,x_1,x_2,x_3) = \left[A_1^{\{1,2,3\}}(t)\,x_1^{1/2} + A_2^{\{1,2,3\}}(t)\,x_2^{1/2}\right.$$
$$\left.+A_3^{\{1,2,3\}}(t)\,x_3^{1/2} + C^{\{1,2,3\}}(t)\right]\exp[-r(t-t_0)], \tag{6.72}$$

where $A_1^{\{1,2,3\}}(t), A_2^{\{1,2,3\}}(t),\ A_3^{\{1,2,3\}}(t)$ and $x_3, C^{\{1,2,3\}}(t)$ satisfy

$$\dot{A}_i^{\{1,2,3\}}(t) =$$
$$\left(r+\frac{\sigma^2}{8}+\frac{\delta}{2}\right)A_i^{\{1,2,3\}}(t) - \frac{b_i^{[i,j]}}{2}A_j^{\{1,2,3\}}(t) - \frac{b_i^{[i,k]}}{2}A_k^{\{1,2,3\}}(t) - P_i,$$
$$\text{for } i,j,k \in \{1,2,3\} \text{ and } i \neq j \neq k,$$

$$\dot{C}^{\{1,2,3\}}(t) = rC^{\{1,2,3\}}(t) - \sum_{i=1}^{3}\frac{\alpha_i^2}{16c_i}\left[A_i^{\{1,2,3\}}(t)\right]^2,$$

$$A_i^{\{1,2,3\}}(T) = q_i \text{ for } i \in \{1,2,3\}, \text{ and } C^{\{1,2,3\}}(T) = 0. \tag{6.73}$$

The first three equations in the block recursive system (6.73) is a system of three linear differential equations which can be solved explicitly by standard techniques. Upon solving $A_i^{\{1,2,3\}}(t)$ for $i \in \{1,2,3\}$, and substituting them into the fourth equation of (6.73), one has a linear differential equation in $C^{\{1,2,3\}}(t)$.

The investment strategies of the grand coalition joint venture can be derived as:

$$\psi_i^{\{1,2,3\}}(t,x) = \frac{\alpha_i^2}{16(c_i)^2}\left[A_i^{\{1,2,3\}}(t)\right]^2, \qquad \text{for } i \in \{1,2,3\}. \tag{6.74}$$

The dynamics of the state trajectories of the joint venture over the time interval $s \in [t_0, T]$ can be expressed as:

$$dx_i(s) = \left[\frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}}(t)\, x_i(s)^{1/2} + b_j^{[j,i]} \left[x_j(s)\, x_i(s)\right]^{1/2} + b_k^{[k,i]} \left[x_k(s)\, x_i(s)\right]^{1/2} - \delta x_i(s)\right] ds$$
$$+ \sigma_i x_i(s)\, dz_i(s), \qquad x_i(t_0) = x_0, \tag{6.75}$$
$$\text{for } i, j, k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

Taking the transforming $y_i(s) = x_i(s)^{1/2}$, for $i \in \{1,2,3\}$, and using Ito's lemma equation system (6.75) can be expressed as:

$$dy_i(s) = \left[\frac{\alpha_i^2}{8c_i} A_i^{\{1,2,3\}}(s) + \frac{1}{2} b_j^{[j,i]} y_j(s)^{1/2} + \frac{1}{2} b_k^{[k,i]} y_k(s)^{1/2} - \frac{\delta}{2} y_i(s) - \frac{\sigma^2}{8} A_i^{\{1,2,3\}}(s)\, y_i(s)\right] ds$$
$$+ \frac{1}{2}\sigma_i y_i(s)\, dz_i(s), \qquad y_i(t_0) = x_0^{1/2}, \tag{6.76}$$
$$\text{for } i, j, k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

(6.76) is a system of linear stochastic differential equations which can be solved by standard techniques. Solving (6.76) yields the joint venture's state trajectory. Let $\{y_1^*(t), y_2^*(t), y_3^*(t)\}$ denote the solution to (6.76). Transforming $x_i = y_i^2$, we obtain the state trajectories of the joint venture over the time interval $s \in [t_0, T]$ as

$$\{x_1^*(t), x_2^*(t), x_3^*(t)\}_{t=t_0}^T = \left\{[y_1^*(t)]^2, [y_2^*(t)]^2, [y_3^*(t)]^2\right\}_{t=t_0}^T. \tag{6.77}$$

We use X_i^{t*} to denote the set of realizable values of $x_i^*(t)$ at time t generated by (6.77). The term x_i^{t*} is used to denote an element in the set X_i^{t*}.

Remark 6.5.1. One can readily verify that:

$W^{(t_0)\{1,2,3\}}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) = W^{(t)\{1,2,3\}}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) \exp[-r(t-t_0)]$ and $\psi_i^{(t_0)\{1,2,3\}*}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right) = \psi_i^{(t)\{1,2,3\}*}\left(t, x_1^{t*}, x_2^{t*}, x_3^{t*}\right)$.

For computation of the dynamic the Shapley value, we consider cases when two of the firms form a coalition $\{i, j\} \subset \{1,2,3\}$ to maximize joint profit:

$$E_{t_0}\left\{\int_{t_0}^T \left[P_i [x_i(s)]^{1/2} - c_i u_i(s) + P_j [x_j(s)]^{1/2} - c_j u_j(s)\right] \times \exp[-r(s-t_0)]\, ds + \exp[-r(T-t_0)] \left(q_i [x_i(T)]^{1/2} q_j [x_j(T)]^{1/2}\right)\right\} \tag{6.78}$$

subject to

$$dx_i(s) = \left[\alpha_i \left[u_i(s)\, x_i(s)\right]^{1/2} + b_j^{[j,i]} \left[x_j(s)\, x_i(s)\right]^{1/2} - \delta x_i(s)\right] ds$$
$$+\sigma_i x_i(s)\, dz_i(s), \qquad x_i(t_0) = x_i^0 \in X_i,$$
$$\text{for } i, j \in \{1,2,3\} \text{ and } i \neq j. \tag{6.79}$$

Following the above analysis, we obtain the following value functions:

$$W^{(t_0)\{i,j\}}(t, x_i, x_j) =$$
$$\left[A_i^{\{i,j\}}(t)\, x_i^{1/2} + A_j^{\{i,j\}}(t)\, x_j^{1/2} + C^{\{i,j\}}(t)\right] \exp\left[-r(t-t_0)\right], \tag{6.80}$$

where $A_i^{\{i,j\}}(t), A_j^{\{i,j\}}(t)$ and $C^{\{i,j\}}(t)$ satisfy

$$\dot{A}_i^{\{1,2\}}(t) = \left(r + \frac{\sigma^2}{8} + \frac{\delta}{2}\right) A_i^{\{1,2\}}(t) - \frac{b_i^{[i,j]}}{2} A_j^{\{1,2\}}(t) - P_i, \text{ and}$$
$$A_i^{\{1,2\}}(T) = q_i,$$
$$\dot{C}^{\{i,j\}}(t) = rC^{\{i,j\}}(t) - \sum_{i=1}^{2} \frac{\alpha_i^2}{16c_i} \left[A_i^{\{i,j\}}(t)\right]^2,$$
$$C^{\{i,j\}}(T) = 0, \qquad \text{for } i, j \in \{1,2,3\} \text{ and } i \neq j. \tag{6.81}$$

The block-recursive system (6.81) can be solved readily by standard techniques. Moreover, as stated in Remark 6.4.1, one can easily derive for $\tau \in [t_0, T]$

$$W^{(t_0)\{i,j\}}(t, x_i, x_j) = \exp\left[-r(\tau - t_0)\right] W^{(\tau)\{i,j\}}(t, x_i, x_j),$$
$$\text{for } i, j \in \{1,2,3\} \text{ and } i \neq j.$$

Using the results derived above, one can readily obtain:

$$f_i^{\{1,2,3\}}\left[\tau, x_1^{\tau *}, x_2^{\tau *}, x_3^{\tau *}, \psi_i^{(\tau)\{1,2,3\}}(\tau, x_1^{\tau *}, x_2^{\tau *}, x_3^{\tau *})\right] =$$
$$\frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}}(\tau)\,(x_i^{\tau *})^{1/2} + b_j^{[j,i]}\left[x_j^{\tau *} x_i^{\tau *}\right]^{1/2} + b_k^{[k,i]}\left[x_k^{\tau *} x_i^{\tau *}\right]^{1/2} - \delta x_i^{\tau *},$$
$$\text{for } i \in \{1,2,3\}. \tag{6.82}$$

Using (6.66), (6.73) and (6.81), we obtain:

$$W_t^{(\tau)\{1,2,3\}}\left(t, x_{\{1,2,3\}}^{\tau *}\right)\Big|_{t=\tau} = \left[\dot{A}_1^{\{1,2,3\}}(\tau)\left(x_1^{\tau *}\right)^{1/2} + \dot{A}_2^{\{1,2,3\}}(\tau)\left(x_2^{\tau *}\right)^{1/2} + \dot{A}_3^{\{1,2,3\}}(\tau)\left(x_3^{\tau *}\right)^{1/2} + \dot{C}^{\{1,2,3\}}(\tau)\right]$$
$$-r\left[A_1^{\{1,2,3\}}(\tau)\left(x_1^{\tau *}\right)^{1/2} + A_2^{\{1,2,3\}}(\tau)\left(x_2^{\tau *}\right)^{1/2} + A_3^{\{1,2,3\}}(\tau)\left(x_3^{\tau *}\right)^{1/2} + C^{\{1,2,3\}}(\tau)\right];$$

$$W_t^{(\tau)\{i,j\}}\left(t, x_{\{i,j\}}^{\tau *}\right)\Big|_{t=\tau} =$$
$$\left[\dot{A}_i^{\{i,j\}}(\tau)\left(x_i^{\tau *}\right)^{1/2} + \dot{A}_j^{\{i,j\}}(\tau)\left(x_j^{\tau *}\right)^{1/2} + \dot{C}^{\{i,j\}}(\tau)\right]$$
$$-r\left[A_i^{\{i,j\}}(\tau)\left(x_i^{\tau *}\right)^{1/2} + A_j^{\{i,j\}}(\tau)\left(x_j^{\tau *}\right)^{1/2} + C^{\{i,j\}}(\tau)\right],$$
$$\text{for } i,j \in \{1,2,3\} \text{ and } i \neq j.$$

$$W_t^{(\tau)i}\left(t, x_i^{\tau *}\right)\Big|_{t=\tau} = \left[\dot{A}_i^{\{i\}}(\tau)\, x_i^{\tau *} + \dot{C}^{\{i\}}(\tau)\right] - r\left[A_i^{\{i\}}(\tau)\, x_i^{\tau *} + C^{\{i\}}(\tau)\right],$$
$$\text{for } i \in \{1,2,3\};$$

$$W_{x_i^{\tau *}}^{(\tau)K}\left(t, x_K^{\tau *}\right)\Big|_{t=\tau} = \frac{1}{2} A_i^K(\tau)\left(x_i^{\tau *}\right)^{-1/2}, \qquad \text{for } i \in K \subseteq \{1,2,3\};$$

$$W_{x_i x_i}^{(\tau)K}\left(t, x_K^{\tau *}\right)\Big|_{t=\tau} = \frac{-1}{4} A_i^K(\tau)\left(x_i^{\tau *}\right)^{-3/2}, \qquad \text{for } i \in K \subseteq \{1,2,3\}. \tag{6.83}$$

Moreover

$$\Omega_N(t, x_N) = \begin{bmatrix} \sigma_1^2 x_1^2 & 0 & 0 \\ 0 & \sigma_2^2 x_2^2 & 0 \\ 0 & 0 & \sigma_3^2 x_3^2 \end{bmatrix}. \tag{6.84}$$

Upon substituting the results from (6.82) to (6.84) into Theorem 6.4.3, we obtain $B_i(\tau)$. A payment $B_i(\tau)$ offered to player $i \in \{1,2,3\}$ at time $\tau \in [t_0, T]$ will lead to the realization of the dynamic Shapley value in Condition 6.4.1. Hence a dynamically stable solution to a cooperative R&D joint venture will result.

6.6 Infinite-Horizon Analysis

As discussed in Chapter 4, in many game situations, the terminal time of the game, T, is either very far in the future or unknown to the players. A way to resolve the problem is to set $T = \infty$. In this section, we examine a multiplayer cooperative stochastic differential games with infinite horizon.

Consider the infinite-horizon version of the game (6.1)–(6.2) in which n players or firms, and firm i's objective is:

$$E_{t_0}\left\{\int_{t_0}^{\infty} g^i\left[x_i(s), u_i(s)\right]\exp\left[-r(s-t_0)\right] ds\right\},$$
$$\text{for } i \in [1,2,\ldots,n] \equiv N, \tag{6.85}$$

subject to the state dynamics

$$dx_i(s) = f^i[x_i(s), u_i(s)]\,ds + \sigma_i[x_i(s)]\,dz_i(s), \qquad x_i(t_0) = x_i^0, \tag{6.86}$$

where $\sigma_i[x_i(s)]$ is a $m_i \times \Theta_i$ and $z_i(s)$ is a Θ_i-dimensional Wiener process and the initial state x_i^0 is given. Let $\Omega_i[x_i(s)] = \sigma_i[x_i(s)]\sigma_i[x_i(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega_i^{h\zeta}[x_i(s)]$. For $i \neq j$, $x_i \cap x_j = \emptyset$, and $z_i(s)$ and $z_j(s)$ are independent Wiener processes. We also used $x_N(s)$ to denote the vector $[x_1(s), x_2(s), \ldots, x_n(s)]$ and x_N^0 the vector $[x_1^0, x_2^0, \ldots, x_n^0]$.

The expected profit to coalition K can be obtained by maximizing:

$$E_{t_0}\left\{\int_{t_0}^{\infty}\sum_{j\in K} g^j[x_j(s), u_j(s)]\exp[-r(s-t_0)]\,ds\right\}, \qquad \text{for } K \subseteq N, \tag{6.87}$$

subject to

$$dx_i(s) = f^i[x_K(s), u_i(s)]\,ds + \sigma_i[x_i(s)]\,dz_i(s), \qquad x_i(t_0) = x_i^0, \\ \text{for } i \in K. \tag{6.88}$$

Since s does not appear in $g^i[x(s), u_i(s)]$ and the state dynamics (6.88), the problem (6.87)–(6.88) is autonomous. Consider the alternative problem $\Gamma(x_K^t)$:

$$\max_{u_K} E_t\left\{\int_t^{\infty}\sum_{j\in K} g^j[x_j(s), u_j(s)]\exp[-r(s-t)]\,ds \,\middle|\, x_K(t) = x_K^t\right\}, \\ \text{for } K \subseteq N, \tag{6.89}$$

subject to

$$dx_i(s) = f^i[x_K(s), u_i(s)]\,ds + \sigma_i[x_i(s)]\,dz_i(s), \qquad x_i(t) = x_i^t, \\ \text{for } i \in K. \tag{6.90}$$

The infinite-horizon autonomous problem $\Gamma(x_K^t)$ is independent of the choice of t and dependent only upon the state at the starting time x_K^t.

Invoking Theorem 2.1.6 a solution to the problem $\Gamma(x_K^t)$ can be characterized as:

Theorem 6.6.1. *A set of controls $\left\{\psi_K^{K*}(x_K^t)\right\}$ provides an optimal solution to the stochastic control problem $\Gamma(x_K^t)$ if there exist continuously differentiable function $W^K(x_K^t) : \Pi_{j\in K} R^{m_j} \to R$, satisfying the following partial differential equation:*

$$rW^K\left(x_K^t\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega_K^{h\zeta}\left(x_K^t\right) W_{x^h x^\xi}^K\left(x_K^t\right) =$$
$$\max_{u_K}\left\{\sum_{j\in K} g^j\left[x_j, u_j\right] + \sum_{j\in K} W_{x_j}^K\left(x_K^t\right) f_j^K\left[x_k, u_j\right]\right\}.$$

In the case when all the n firms are in the joint venture, that is $K = N$, the optimal control

$$\psi_N^{N*}\left(x_N^t\right) = \left[\psi_1^{N*}\left(x_N^t\right), \psi_2^{N*}\left(x_N^t\right), \ldots, \psi_n^{N*}\left(x_N^t\right)\right]$$

satisfying Theorem 6.6.1 will be adopted. The dynamics of the optimal trajectory of joint venture be obtained as:

$$dx_j(s) = f_j^N\left[x_N(s), \psi_j^{N*}\left(x_N(s)\right)\right] ds + \sigma_j\left[x_j(s)\right] dz_j(s),$$
$$x_j(t_0) = x_j^0, \qquad \text{for } j \in N. \tag{6.91}$$

The solution to (6.91) yields the optimal trajectories:

$$x_j^*(t) = x_j^0 + \int_{t_0}^{t} f^j\left[x_N^*(s), \psi_j^{N*}\left(x_N^*(s)\right)\right] ds + \int_{t_0}^{t} \sigma_j\left[x_j^*(s)\right] dz_j(s),$$
$$\text{for } j \in N. \tag{6.92}$$

We use X_j^{t*} to denote the set of realizable values of $x_j^*(t)$ at time t generated by (6.92). We denote the stochastic optimal trajectory by $\{x_N^*(t)\}_{t\geq t_0} = \{x_1^*(t), x_2^*(t), \ldots, x_n^*(t)\}_{t\geq t_0}$ and the set of realizable values of $x_N^*(t)$ at time t by X_N^{t*}. We use x_j^{t*} to denote an element in X_t^* and x_N^{t*} to denote an element in X_N^*.

The member firms would maximize their joint expected profit and share their cooperative profits according to the Shapley value.

To maximize the joint venture's profits the firms would adopt the control vector $\psi_N^{N*}(x_N^t) = \left[\psi_1^{N*}(x_N^t), \psi_2^{N*}(x_N^t), \ldots, \psi_n^{N*}(x_N^t)\right]$ satisfying Theorem 6.6.1, and the corresponding optimal state trajectory in (6.92) would result. At time t_0 with state $x_N^{t_0}$, the firms agree that firm i's share of profits be:

$$v^i\left(x_N^0\right) = \sum_{K\subseteq N} \frac{(k-1)!\,(n-k)!}{n!}\left[W^K\left(x_K^0\right) - W^{K\backslash i}\left(x_{K\backslash i}^0\right)\right],$$
$$\text{for } i \in N. \tag{6.93}$$

In subsequent time $t \geq t_0$, if the state is $x_N^{t*} \in X_N^{t*}$, the firms agree that firm i's share of profits be:

$$v^i\left(x_N^{t*}\right) = \sum_{K\subseteq N} \frac{(k-1)!\,(n-k)!}{n!}\left[W^K\left(x_K^{t*}\right) - W^{K\backslash i}\left(x_{K\backslash i}^{t*}\right)\right],$$
$$\text{for } i \in N. \tag{6.94}$$

Following the analysis leading to Theorem 5.8.3 a profit distribution mechanism that would lead to the realization of Condition (6.93) and (6.94) can be obtained

Theorem 6.6.2. *A payment to player $i \in N$ at time $\tau \in [t_0, T]$ leading to the realization of the (6.93)–(6.94) can be expressed as:*

$$B_i(\tau) = -\sum_{K \subseteq N} \frac{(k-1)!\,(n-k)!}{n!} \Bigg\{ -\left[r W^K\left(x_K^{\tau *}\right)\right] + \left[r W_t^{K\backslash i}\left(x_{K\backslash i}^{\tau *}\right)\right]$$

$$+ \sum_{j \in K} \left[W_{x_j^{\tau *}}^{K}\left(x_K^{\tau *}\right)\right] f_j^N\left[x_N^{\tau *}, \psi_j^{N*}\left(x_N^{\tau *}\right)\right]$$

$$- \sum_{h \in K\backslash i} \left[W_{x_h^{\tau *}}^{K\backslash i}\left(x_{K\backslash i}^{\tau *}\right)\right] f_h^N\left[x_N^{\tau *}, \psi_h^{N*}\left(x_N^{\tau *}\right)\right]$$

$$+\frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega_K^{h\zeta}\left(x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{K}\left(x_t^*\right)\right] - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega_{K\backslash i}^{h\zeta}\left(x_\tau^*\right)\left[W_{x_t^h x_t^\zeta}^{K\backslash i}\left(x_t^*\right)\right] \Bigg\}$$

when $x_N(\tau) = x_N^{\tau *} \in X_N^{\tau *}$.

6.7 An Example of Infinite-Horizon Joint Venture

Consider an infinite-horizon version of the problem in Section 6.5. Company i expected profit is

$$E_{t_0}\left\{ \int_{t_0}^{\infty} \left[P_i \left[x_i(s)\right]^{1/2} - c_i u_i(s)\right] \exp\left[-r(s-t_0)\right] ds \right\},$$
$$\text{for } \ i \in N = \{1,2,3\}. \tag{6.95}$$

The evolution of the technology level of company i follows the dynamics:

$$dx_i(s) = \left[\alpha_i \left[u_i(s)\,x_i(s)\right]^{1/2} - \delta x_i(s)\right] ds + \sigma_i x_i(s)\, dz_i(s),$$
$$x_i(t_0) = x_i^0 \in X_i, \qquad \text{for } \ i \in N = \{1,2,3\}. \tag{6.96}$$

Consider the alternative problem:

$$E_{t}\left\{ \int_{t}^{\infty} \left[P_i \left[x_i(s)\right]^{1/2} - c_i u_i(s)\right] \exp\left[-r(s-t)\right] ds \right\},$$
$$\text{for } \ i \in N = \{1,2,3\}. \tag{6.97}$$

The evolution of the technology level of company i follows the dynamics:

$$dx_i(s) = \left[\alpha_i \left[u_i(s)\, x_i(s)\right]^{1/2} - \delta x_i(s)\right] ds + \sigma_i x_i(s)\, dz_i(s),$$
$$x_i(t) = x_i^t, \qquad \text{for } \ i \in N = \{1,2,3\}. \tag{6.98}$$

In the case when each of these three firms acts independently. Using Theorem 2.1.6, we obtain the Fleming stochastic control equation for the problem (6.97)–(6.98) as:

$$rW_t^i(x_i) - \frac{1}{2} W_{x_i x_i}^i(x_i)\, \sigma_i^2 x_i^2 =$$
$$\max_{u_i} \left\{ \left[P_i x_i^{1/2} - c_i u_i\right] + W_{x_i}^i(x_i) \left[\alpha_i (u_i x_i)^{1/2} - \delta x_i\right] \right\},$$
$$\text{for } \ i \in \{1,2,3\}. \tag{6.99}$$

Performing the indicated maximization yields

$$u_i = \frac{\alpha_i^2}{4(c_i)^2} \left[W_{x_i}^i(x_i)\right]^2 x_i, \qquad \text{for } \ i \in \{1,2,3\}. \tag{6.100}$$

Substituting (6.100) into (6.99) yields:

$$rW_t^i(x_i) - \frac{1}{2} W_{x_i x_i}^i(x_i)\, \sigma_i^2 x_i^2 =$$
$$P_i x_i^{1/2} - \frac{\alpha_i^2}{4c_i} \left[W_{x_i}^i(x_i)\right]^2 x_i + \frac{\alpha_i^2}{2c_i} \left[W_{x_i}^i(x_i)\right]^2 x_i - \delta W_{x_i}^i(x_i)\, x_i,$$
$$\text{for } \ i \in \{1,2,3\}. \tag{6.101}$$

Solving (6.101) yields

$$W^i(x_i) = \left[A_i^{\{i\}} x_i^{1/2} + C_i^{\{i\}}\right], \qquad \text{for } \ i \in \{1,2,3\}, \tag{6.102}$$

where

$$A_i^{\{i\}} = \frac{P_i}{\left(r + \frac{\sigma^2}{8} + \frac{\delta}{2}\right)}, \text{ and } \ C_i^{\{i\}} = -\frac{\alpha_i^2}{16 r c_i} \left[A_i^{\{i\}}\right]^2.$$

Consider the case when all these three firms agree to cooperate in research and development and share their joint profit according to the dynamic Shapley Value in (6.93)–(6.94). Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology level of company i under joint venture becomes:

$$dx_i(s) = \left[\alpha_i \left[u_i(s)\, x_i(s)\right]^{1/2} + b_j^{[j,i]} \left[x_j(s)\, x_i(s)\right]^{1/2}\right.$$
$$\left. + b_k^{[k,i]} \left[x_k(s)\, x_i(s)\right]^{1/2} - \delta x_i(s)\right] ds$$
$$+ \sigma_i x_i(s)\, dz_i(s), \qquad x_i(t_0) = x_i^0 \in X_i,$$
$$\text{for } \ i,j,k \in N = \{1,2,3\} \ \text{ and } \ i \neq j \neq k. \tag{6.103}$$

where $b_j^{[j,i]}$ and $b_k^{[k,i]}$ are non-negative constants. In particular, $b_j^{[j,i]} u_j(s)$ represents the technology transfer effect under joint venture on firm i brought about by firm j 's technology.

The expected profit of the joint venture is the sum of the participating firms' expected profits:

$$E_{t_0}\left\{\int_{t_0}^{\infty}\sum_{j=1}^{3}\left[P_j\left[x_j(s)\right]^{1/2}-c_j u_j(s)\right]\exp\left[-r\left(s-t_0\right)\right]ds\right\}. \tag{6.104}$$

The firms in the joint venture then act cooperatively to maximize (6.104) subject to (6.103). Once again consider the problem

$$\max_{u_1,u_2,u_3} E_t\left\{\int_{t}^{\infty}\sum_{j=1}^{3}\left[P_j\left[x_j(s)\right]^{1/2}-c_j u_j(s)\right]\exp\left[-r\left(s-t\right)\right]ds\right\} \tag{6.105}$$

subject to dynamics (6.103).

Using Theorem 2.1.6, we obtain the stochastic control equation for the problem (6.103) and (6.105) as:

$$\begin{aligned}
& rW^{\{1,2,3\}}(x_1,x_2,x_3)-\frac{1}{2}\sum_{i=1}^{3}W^{\{1,2,3\}}_{x_i x_i}(x_1,x_2,x_3)\,\sigma_i^2 x_i^2 = \\
& \qquad \max_{u_i}\left\{\sum_{i=1}^{3}\left[P_i\left[x_i\right]^{1/2}-c_i u_i\right]\right. \\
& \qquad\qquad +\sum_{i=1}^{3}W^{\{1,2,3\}}_{x_i}(x_1,x_2,x_3)\left[\alpha_i\left[u_i x_i\right]^{1/2}\right. \\
& \qquad\qquad\qquad \left.\left.+b_j^{[j,i]}\left[x_j x_i\right]^{1/2}+b_k^{[k,i]}\left[x_k x_i\right]^{1/2}-\delta x_i\right]\right\}, \\
& \qquad\qquad \text{for } \ i,j,k\in N=\{1,2,3\} \ \text{ and } \ i\neq j\neq k.
\end{aligned} \tag{6.106}$$

Performing the indicated maximization yields

$$u_i=\frac{\alpha_i^2}{4\left(c_i\right)^2}\left[W^i_{x_i}(x_1,x_2,x_3)\right]^2 x_i, \qquad \text{for } \ i\in\{1,2,3\}. \tag{6.107}$$

Substituting (6.107) into (6.106) yields:

$$
\begin{aligned}
& rW^{\{1,2,3\}}(x_1,x_2,x_3) - \frac{1}{2}\sum_{i=1}^{3} W_{x_i x_i}^{\{1,2,3\}}(x_1,x_2,x_3)\,\sigma_i^2 x_i^2 = \\
& \qquad \sum_{i=1}^{3}\left[P_i\,[x_i]^{1/2} - \frac{\alpha_i^2 x_i}{4c_i}\left[W_{x_i}^{(t_0)i}(x_1,x_2,x_3)\right]^2\right] \\
& \qquad + \sum_{i=1}^{3} W_{x_i}^{\{1,2,3\}}(t,x_1,x_2,x_3)\left[\frac{\alpha_i^2}{2c_i^2}\left[W_{x_i}^{i}(x_1,x_2,x_3)\right]x_i\right. \\
& \qquad \qquad \left. + b_j^{[j,i]}\,[x_j x_i]^{1/2} + b_k^{[k,i]}\,[x_k x_i]^{1/2} - \delta x_i\right], \qquad (6.108)
\end{aligned}
$$

$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

Solving (6.108) yields

$$
\begin{aligned}
& W^{\{1,2,3\}}(x_1,x_2,x_3) = \\
& \qquad \left[A_1^{\{1,2,3\}} x_1^{1/2} + A_2^{\{1,2,3\}} x_2^{1/2} + A_3^{\{1,2,3\}} x_3^{1/2} + C^{\{1,2,3\}}\right], \qquad (6.109)
\end{aligned}
$$

where $A_1^{\{1,2,3\}}, A_2^{\{1,2,3\}},\ A_3^{\{1,2,3\}}$ and $x_3, C^{\{1,2,3\}}$ satisfy

$$
0 = \left(r + \frac{\sigma^2}{8} + \frac{\delta}{2}\right) A_i^{\{1,2,3\}} - \frac{b_i^{[i,j]}}{2} A_j^{\{1,2,3\}} - \frac{b_i^{[i,k]}}{2} A_k^{\{1,2,3\}} - P_i,
$$

$$\text{for } i,j,k \in \{1,2,3\} \text{ and } i \neq j \neq k,$$

$$
C^{\{1,2,3\}} = \sum_{i=1}^{3} \frac{\alpha_i^2}{16 c_i r}\left[A_i^{\{1,2,3\}}\right]^2. \qquad (6.110)
$$

The first three equations in the block recursive system (6.110) is a system of three linear equations which can be solved explicitly by standard techniques. Upon solving $A_i^{\{1,2,3\}}$ for $i \in \{1,2,3\}$, and substituting them into the fourth equation of (6.110), one can obtain $C^{\{1,2,3\}}$.

The investment strategies of the grand coalition joint venture can be derived as:

$$
\psi_i^{\{1,2,3\}}(x) = \frac{\alpha_i^2}{16\,(c_i)^2}\left[A_i^{\{1,2,3\}}\right]^2, \qquad \text{for } i \in \{1,2,3\}. \qquad (6.111)
$$

The dynamics of the state trajectories of the joint venture can be expressed as:

$$
\begin{aligned}
dx_i(s) = & \left[\frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}} x_i(s)^{1/2} + b_j^{[j,i]}\,[x_j(s)\,x_i(s)]^{1/2}\right. \\
& \qquad \left. + b_k^{[k,i]}\,[x_k(s)\,x_i(s)]^{1/2} - \delta x_i(s)\right] ds \\
& + \sigma_i x_i(s)\,dz_i(s), \qquad x_i(t_0) = x_0, \qquad (6.112)
\end{aligned}
$$

$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

Taking the transforming $y_i(s) = x_i(s)^{1/2}$, for $i \in \{1,2,3\}$, and using Ito's lemma equation system (6.112) can be expressed as:

$$dy_i(s) = \left[\frac{\alpha_i^2}{8c_i} A_i^{\{1,2,3\}} + \frac{1}{2} b_j^{[j,i]} y_j(s)^{1/2} + \frac{1}{2} b_k^{[k,i]} y_k(s)^{1/2} \right.$$
$$\left. - \frac{\delta}{2} y_i(s) - \frac{\sigma^2}{8} A_i^{\{1,2,3\}} y_i(s) \right] ds$$
$$+ \frac{1}{2} \sigma_i y_i(s) \, dz_i(s), \qquad y_i(t_0) = x_0^{1/2}, \tag{6.113}$$
$$\text{for } i,j,k \in N = \{1,2,3\} \text{ and } i \neq j \neq k.$$

(6.113) is a system of linear stochastic differential equations which can be solved by standard techniques. Solving (6.113) yields the joint venture's state trajectory. Let $\{y_1^*(t), y_2^*(t), y_3^*(t)\}$ denote the solution to (6.113). Transforming $x_i = y_i^2$, we obtain the state trajectories of the joint venture over the time interval $s \in [t_0, T]$ as

$$\{x_1^*(t), x_2^*(t), x_3^*(t)\}_{t=t_0}^T = \left\{ [y_1^*(t)]^2, [y_2^*(t)]^2, [y_3^*(t)]^2 \right\}_{t=t_0}^T. \tag{6.114}$$

We use X_i^{t*} to denote the set of realizable values of $x_i^*(t)$ at time t generated by (6.114). The term x_i^{t*} is used to denote an element in the set X_i^{t*}.

For computation of the dynamic the Shapley value, we consider cases when two of the firms form a coalition $\{i,j\} \subset \{1,2,3\}$ to maximize joint profit:

$$E_{t_0} \left\{ \int_{t_0}^T \left[P_i [x_i(s)]^{1/2} - c_i u_i(s) + P_j [x_j(s)]^{1/2} - c_j u_j(s) \right] \right.$$
$$\left. \times \exp[-r(s - t_0)] \, ds \right\} \tag{6.115}$$

subject to

$$dx_i(s) = \left[\alpha_i [u_i(s) x_i(s)]^{1/2} + b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s) \right] ds$$
$$+ \sigma_i x_i(s) \, dz_i(s), \qquad x_i(t_0) = x_i^0 \in X_i,$$
$$\text{for } i,j \in \{1,2,3\} \text{ and } i \neq j. \tag{6.116}$$

Following the above analysis, we obtain the following value functions:

$$W^{\{i,j\}}(x_i, x_j) = \left[A_i^{\{i,j\}} x_i^{1/2} + A_j^{\{i,j\}} x_j^{1/2} + C^{\{i,j\}} \right], \tag{6.117}$$
$$\text{for } i,j \in \{1,2,3\} \text{ and } i \neq j,$$

where $A_i^{\{i,j\}}$, $A_j^{\{i,j\}}$ and $C^{\{i,j\}}$ satisfy

$$0 = \left(r + \frac{\sigma^2}{8} + \frac{\delta}{2} \right) A_i^{\{1,2\}} - \frac{b_i^{[i,j]}}{2} A_j^{\{1,2\}} - P_i,$$

$$rC^{\{i,j\}} = \sum_{i=1}^{2} \frac{\alpha_i^2}{16c_i} \left[A_i^{\{i,j\}} \right]^2,$$

$$\text{for } \ i, j \in \{1,2,3\} \ \text{ and } \ i \neq j; \tag{6.118}$$

Using the results derived above, one can readily obtain:

$$f_i^{\{1,2,3\}} \left[x_1^{\tau *}, x_2^{\tau *}, x_3^{\tau *}, \psi_i^{\{1,2,3\}*} \left(x_1^{\tau *}, x_2^{\tau *}, x_3^{\tau *} \right) \right] =$$

$$\frac{\alpha_i^2}{4c_i} A_i^{\{1,2,3\}} \left(x_i^{\tau *} \right)^{1/2} + b_j^{[j,i]} \left[x_j^{\tau *}, x_i^{\tau *} \right]^{1/2} + b_k^{[k,i]} \left[x_k^{\tau *}, x_i^{\tau *} \right]^{1/2} - \delta x_i^{\tau *},$$

$$\text{for } \ i \in \{1,2,3\}. \tag{6.119}$$

Using (6.102), (6.109) and (6.117), we obtain:

$$W_{x_i^{\tau *}}^{(\tau)K} \left(x_K^{\tau *} \right) \Big|_{t=\tau} = \frac{1}{2} A_i^K \left(x_i^{\tau *} \right)^{-1/2}, \ \text{for } \ i \in K \subseteq \{1,2,3\};$$

$$W_{x_i x_i}^{(\tau)K} \left(x_K^{\tau *} \right) \Big|_{t=\tau} = \frac{-1}{4} A_i^K \left(x_i^{\tau *} \right)^{-3/2}, \ \text{for } \ i \in K \subseteq \{1,2,3\}. \tag{6.120}$$

Moreover

$$\Omega_N \left(t, x_N \right) = \begin{bmatrix} \sigma_1^2 x_1^2 & 0 & 0 \\ 0 & \sigma_2^2 x_2^2 & 0 \\ 0 & 0 & \sigma_3^2 x_3^2 \end{bmatrix}. \tag{6.121}$$

Upon substituting the results from (6.119) to (6.121) into Theorem 6.6.2, we obtain $B_i(\tau)$. A payment $B_i(\tau)$ offered to player $i \in \{1,2,3\}$ at time $\tau \in [t_0, T]$ will lead to the realization of the dynamic Shapley value in (6.93)–(6.94).

6.8 Problems

Problem 6.1. If three firms act independently, the profits of firms 1, 2 and 3 are respectively $i \in \{1,2,3\}$,

$$\int_0^{10} \left[10 \left[x_1(s) \right]^{1/2} - 2u_1(s) \right] \exp\left[-0.01s \right] ds + \exp\left[-0.1 \right] 4 \left[x_1(T) \right]^{1/2},$$

$$\int_0^{10} \left[8 \left[x_2(s) \right]^{1/2} - 2u_2(s) \right] \exp\left[-0.01s \right] ds + \exp\left[-0.1 \right] 3 \left[x_2(T) \right]^{1/2},$$

$$\int_0^{10} \left[7 \left[x_3(s) \right]^{1/2} - u_3(s) \right] \exp\left[-0.01s \right] ds + \exp\left[-0.1 \right] 2 \left[x_3(T) \right]^{1/2},$$

where $x_i(s)$ is the level of technology and $u_i(s)$ is the level of investment of firm i at time s.

If they act independently the evolution of the technology levels of the companies follow the dynamics:

$$\begin{aligned}
\dot{x}_1(s) &= 2\left[u_1(s)\,x_1(s)\right]^{1/2} - 0.01x_1(s), \qquad x_1(0) = 15,\\
\dot{x}_2(s) &= \left[u_2(s)\,x_2(s)\right]^{1/2} - 0.02x_2(s), \qquad x_2(0) = 10,\\
\dot{x}_3(s) &= 2.5\left[u_3(s)\,x_3(s)\right]^{1/2} - 0.01x_3(s), \qquad x_3(0) = 20.
\end{aligned}$$

Consider the case when all these three firms agree to form a joint venture. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology levels of company i under joint venture become:

$$\begin{aligned}
\dot{x}_1(s) &= 2\left[u_1(s)\,x_1(s)\right]^{1/2} + 0.1\left[x_2(s)\,x_1(s)\right]^{1/2}\\
&\qquad + 0.1\left[x_3(s)\,x_1(s)\right]^{1/2} - 0.01x_1(s),\\
\dot{x}_2(s) &= \left[u_2(s)\,x_2(s)\right]^{1/2} + 0.15\left[x_1(s)\,x_2(s)\right]^{1/2} - 0.02x_2(s),\\
\dot{x}_3(s) &= 2.5\left[u_3(s)\,x_3(s)\right]^{1/2} + 0.05\left[x_1(s)\,x_3(s)\right]^{1/2}\\
&\qquad + 0.1\left[x_2(s)\,x_3(s)\right]^{1/2} - 0.01x_3(s).
\end{aligned}$$

(a) Compute the optimal state trajectory of the three-firm joint venture.
(b) Obtain the profit of the joint venture over the time interval $[t, 10]$, where $t \in [0, 10]$.

Problem 6.2. Consider the joint venture in Problem 6.1. In particular, all the three firms agree to form a joint venture and share their joint profit according to the dynamic Shapley value.

For computation of the dynamic Shapley value, we have to consider cases when two of the firms form a coalition $\{i, j\} \subset \{1, 2, 3\}$.

The evolution of the technology levels of companies 1 and 2 in a joint venture of these two companies become:

$$\begin{aligned}
\dot{x}_1(s) &= 2\left[u_1(s)\,x_1(s)\right]^{1/2} + 0.1\left[x_2(s)\,x_1(s)\right]^{1/2} - 0.01x_1(s), \text{ and}\\
\dot{x}_2(s) &= \left[u_2(s)\,x_2(s)\right]^{1/2} + 0.15\left[x_1(s)\,x_2(s)\right]^{1/2} - 0.02x_2(s).
\end{aligned}$$

The evolution of the technology levels of companies 1 and 3 in a joint venture of these two companies become:

$$\begin{aligned}
\dot{x}_1(s) &= 2\left[u_1(s)\,x_1(s)\right]^{1/2} + 0.1\left[x_3(s)\,x_1(s)\right]^{1/2} - 0.01x_1(s), \text{ and}\\
\dot{x}_3(s) &= 2.5\left[u_3(s)\,x_3(s)\right]^{1/2} + 0.05\left[x_1(s)\,x_3(s)\right]^{1/2} - 0.01x_3(s).
\end{aligned}$$

The evolution of the technology levels of companies 2 and 3 in a joint venture of these two companies become:

$$\begin{aligned}
\dot{x}_2(s) &= \left[u_2(s)\,x_2(s)\right]^{1/2} - 0.02x_2(s), \text{ and}\\
\dot{x}_3(s) &= 2.5\left[u_3(s)\,x_3(s)\right]^{1/2} + 0.1\left[x_2(s)\,x_3(s)\right]^{1/2} - 0.01x_3(s).
\end{aligned}$$

Compute transitory compensation leading to the realization of the dynamic Shapley value for the joint venture.

Problem 6.3. Consider a stochastic version of the deterministic joint venture in Problems 6.1 with the following specifications:

If all the firms act independently, the expected profits of firms 1, 2 and 3 are respectively $i \in \{1, 2, 3\}$,

$$E_0 \left\{ \int_0^{10} \left[15 \left[x_1(s) \right]^{1/2} - 2u_1(s) \right] \exp\left[-0.01s\right] ds + \exp\left[-0.1\right] 2 \left[x_1(T) \right]^{1/2} \right\},$$
$$E_0 \left\{ \int_0^{10} \left[10 \left[x_2(s) \right]^{1/2} - 2u_2(s) \right] \exp\left[-0.01s\right] ds + \exp\left[-0.1\right] 3 \left[x_2(T) \right]^{1/2} \right\},$$
$$E_0 \left\{ \int_0^{10} \left[6 \left[x_3(s) \right]^{1/2} - u_3(s) \right] \exp\left[-0.01s\right] ds + \exp\left[-0.1\right] 4 \left[x_3(T) \right]^{1/2} \right\},$$

where $x_i(s)$ is the level of technology and $u_i(s)$ is the level of investment of firm i at time s.

If they act independently the evolution of the technology levels of the companies follow the following stochastic dynamics

$$\begin{aligned}
dx_1(s) &= \left[2 \left[u_1(s) x_1(s) \right]^{1/2} - 0.01 x_1(s) \right] ds + 0.01 x_1(s) dz_1(s), \\
&\quad x_1(0) = 12, \\
dx_2(s) &= 3 \left[\left[u_2(s) x_2(s) \right]^{1/2} - 0.05 x_2(s) \right] ds + 0.02 x_2(s) dz_2(s), \\
&\quad x_2(0) = 10, \\
dx_3(s) &= \left[3 \left[u_3(s) x_3(s) \right]^{1/2} - 0.02 x_3(s) \right] ds + 0.01 x_3(s) dz_3(s), \\
&\quad x_3(0) = 10,
\end{aligned}$$

where $z_i(s)$ is a Wiener process, for $i \in N = \{1, 2, 3\}$. For $i \neq j$, $x_i \cap x_j = \emptyset$, $z_i(s)$ and $z_j(s)$ are independent Wiener processes.

Consider the case when all these three firms agree to form a joint venture. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own. The evolution of the technology levels of company i under joint venture become:

$$\begin{aligned}
dx_1(s) &= \left[2 \left[u_1(s) x_1(s) \right]^{1/2} + 0.15 \left[x_2(s) x_1(s) \right]^{1/2} + 0.1 \left[x_3(s) x_1(s) \right]^{1/2} \right. \\
&\qquad \left. - 0.01 x_1(s) \right] ds + 0.01 x_1(s) dz_1(s), \\
dx_2(s) &= 3 \left[\left[u_2(s) x_2(s) \right]^{1/2} + 0.1 \left[x_1(s) x_2(s) \right]^{1/2} + 0.3 \left[x_3(s) x_2(s) \right]^{1/2} \right. \\
&\qquad \left. - 0.05 x_2(s) \right] ds + 0.02 x_2(s) dz_2(s), \text{ and} \\
dx_3(s) &= \left[3 \left[u_3(s) x_3(s) \right]^{1/2} + 0.1 \left[x_1(s) x_3(s) \right]^{1/2} + 0.1 \left[x_2(s) x_3(s) \right]^{1/2} \right.
\end{aligned}$$

$$- 0.02x_3(s)\Big]\, ds + 0.01x_3(s)\, dz_3(s)\,.$$

Obtain the expected profit of the joint venture over the time interval $[t, 10]$, where $t \in [0, 10]$.

Problem 6.4. Characterize the expected profit of the joint venture in Problem 6.3 over the time interval $[t, 10]$, where $t \in [0, 10]$.

Problem 6.5. Consider the joint venture in Problem 6.3. In particular, all the three firms agree to form a joint venture and share their joint profit according to the dynamic Shapley value.

For computation of the dynamic Shapley value, we have to consider cases when two of the firms form a coalition $\{i, j\} \subset \{1, 2, 3\}$.

The evolution of the technology levels of companies 1 and 2 in a joint venture of these two companies become:

$$\begin{aligned} dx_1(s) &= \Big[2\left[u_1(s)\, x_1(s)\right]^{1/2} + 0.15\left[x_2(s)\, x_1(s)\right]^{1/2} - 0.01x_1(s)\Big]\, ds \\ &\qquad +0.01x_1(s)\, dz_1(s)\,, \text{ and} \\ dx_2(s) &= 3\Big[\left[u_2(s)\, x_2(s)\right]^{1/2} + 0.1\left[x_1(s)\, x_2(s)\right]^{1/2} - 0.05x_2(s)\Big]\, ds. \end{aligned}$$

The evolution of the technology levels of companies 1 and 3 in a joint venture of these two companies become:

$$\begin{aligned} dx_1(s) &= \Big[2\left[u_1(s)\, x_1(s)\right]^{1/2} + 0.1\left[x_3(s)\, x_1(s)\right]^{1/2} - 0.01x_1(s)\Big]\, ds \\ &\qquad +0.01x_1(s)\, dz_1(s)\,, \text{ and} \\ dx_3(s) &= \Big[3\left[u_3(s)\, x_3(s)\right]^{1/2} + 0.1\left[x_1(s)\, x_3(s)\right]^{1/2} - 0.02x_3(s)\Big]\, ds \\ &\qquad +0.01x_3(s)\, dz_3(s)\,. \end{aligned}$$

The evolution of the technology levels of companies 2 and 3 in a joint venture of these two companies become:

$$\begin{aligned} dx_2(s) &= 3\Big[\left[u_2(s)\, x_2(s)\right]^{1/2} + 0.3\left[x_3(s)\, x_2(s)\right]^{1/2} - 0.05x_2(s)\Big]\, ds \\ &\qquad +0.02x_2(s)\, dz_2(s)\,, \text{ and} \\ dx_3(s) &= \Big[3\left[u_3(s)\, x_3(s)\right]^{1/2} + 0.1\left[x_2(s)\, x_3(s)\right]^{1/2} - 0.02x_3(s)\Big]\, ds \\ &\qquad +0.01x_3(s)\, dz_3(s)\,. \end{aligned}$$

Compute transitory compensation leading to the realization of the dynamic Shapley value for the joint venture.

7

Cooperative Stochastic Differential Games with Nontransferable Payoffs

In this chapter we consider cooperative stochastic differential games with nontransferable payoffs. This class of games is probably one the most, if not the most complex form dynamic game theory analysis. The solution mechanism becomes extremely complicated and intractable. So far, subgame consistent solutions are developed for very few classes of cooperative stochastic differential games with nontransferable payoffs (see Yeung and Petrosyan (2005) and Yeung et al. (2004)).

7.1 Game Formulation and Noncooperative Outcome

Consider the two-person nonzero-sum stochastic differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \in R^m$, with permissible state trajectories $\{x(s), t_0 \le s \le T\}$. The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$dx(s) = f[s, x(s), u_1(s), u_2(s)]\, ds + \sigma[s, x(s)]\, dz(s), \quad x(t_0) = x_0, \quad (7.1)$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\, \sigma[s, x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. $u_i \in U^i \subset compR^l$ is the control vector of Player i, for $i \in \{1, 2\}$.

At time instant $s \in [t_0, T]$, the instantaneous payoff of Player i, for $i \in \{1, 2\}$, is denoted by $g^i[s, x(s), u_1(s), u_2(s)]$, and, when the game terminates at time T, Player i receives a terminal payment of $q^i(x(T))$. Payoffs are nontransferable across players. Given a time-varying instantaneous discount rate $r(s)$, for $s \in [t_0, T]$, values received t time after t_0 have to be discounted by the factor $\exp\left[-\int_{t_0}^{t} r(y)\, dy\right]$. Hence at time t_0, the expected payoff of Player i can be expressed as:

$$E_{t_0}\left\{\int_{t_0}^{T} g^i\left[s, x(s), u_1(s), u_2(s)\right] \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds \right.$$
$$\left. + \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i(x(T))\right\}, \tag{7.2}$$
$$\text{for } i \in \{1,2\},$$

where E_{t_0} denotes the expectation operator performed at time t_0.

We use $\Gamma(x_0, T-t_0)$ to denote the stochastic differential game (7.1)–(7.2).
Invoking Theorem 2.5.1 a non-cooperative Nash equilibrium solution of the game $\Gamma(x_0, T-t_0)$ can be characterized as follows.

Theorem 7.1.1. *A set of feedback strategies* $\left[\phi_1^{(t_0)*}(t,x), \phi_2^{(t_0)*}(t,x)\right]$ *provides a Nash equilibrium solution to the game* $\Gamma(x_0, T-t_0)$*, if there exist continuously differentiable functions* $V^{(t_0)i}(t,x) : [t_0, T]\times R^n \to R$, $i \in \{1,2\}$*, satisfying the following (Fleming-Bellman-Isaacs) partial differential equations:*

$$-V_t^{(t_0)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, V_{x^h x^\zeta}^{(t_0)i}(t,x) =$$
$$\max_{u_i}\left\{g^i\left[t, x, u_i, \phi_j^{(t_0)*}(t,x)\right] \exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right.$$
$$\left. + V_x^{(t_0)i}(t,x)\, f\left[t, x, u_i, \phi_j^{(t_0)*}(t,x)\right]\right\},$$

and

$$V^{(t_0)i}(T,x) = \exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i(x),$$
$$i \in \{1,2\} \quad \textit{and} \quad j \in \{1,2\} \quad \textit{and} \quad j \neq i.$$

Consider the alternative game $\Gamma(x_\tau, T-\tau)$ with payoff structure (7.1) and dynamics (7.2) starting at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$. Let $\left\{u_i^{(\tau)*}(t) = \phi_i^{(\tau)*}(t,x) \in \Phi^i, \text{ for } i \in \{1,2\} \text{ and } t \in [\tau, T]\right\}$ denote a set of feedback strategies that constitutes a Nash equilibrium solution to the game $\Gamma(x_\tau, T-\tau)$, and $V^{(\tau)i}(t, x_t) : [\tau, T] \times R^n \to R$ denote the value function of player $i \in \{1,2\}$ that satisfies the corresponding Bellman-Isaacs-Fleming equations. Analogous to Remark 5.1.1, we have

Remark 7.1.1.

$$\phi_i^{(\tau)*}(s, x(s)) = \phi_i^{(t_0)*}(s, x(s)), \; s \in [\tau, T],$$
$$V^{(\tau)i}(\tau, x_\tau) = \exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] V^{(t_0)i}(\tau, x_\tau), \text{ and}$$

$$V^{(t)i}(t, x_t) = \exp\left[\int_\tau^t r(y)\,dy\right] V^{(\tau)i}(t, x_t),$$
$$\text{for } t_0 \le \tau \le t \le T \text{ and } i \in \{1, 2\}.$$

The game equilibrium dynamics of $\Gamma(x_\tau, T-\tau)$ can be obtained as:

$$dx(s) = f\left[s, x(s), \phi_1^{(\tau)*}(s, x(s)), \phi_2^{(\tau)*}(s, x(s))\right] ds + \sigma[s, x(s)]\,dz(s),$$
$$x(\tau) = x_\tau. \tag{7.3}$$

Example 7.1.1. Consider a resource extraction game with nontransferable payoffs. The state space of the game is $X \subset R$, with permissible state trajectories $\{x(s), t_0 \le s \le T\}$. The state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s)\right] ds + \sigma x(s)\,dz(s),$$
$$x(t_0) = x_0 \in X, \tag{7.4}$$

where $u_i \in U^i$ is the control vector of Player i, for $i \in \{1, 2\}$, a, b and σ are positive constants, and $z(s)$ is a Wiener process.

At time t_0, the expected payoff of player $i \in \{1, 2\}$ is:

$$J^i(t_0, x_0) = E_{t_0}\left\{\int_{t_0}^T \left[[k_i u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s)\right] \exp[-r(s-t_0)]\,ds\right.$$
$$\left. + \exp[-r(T-t_0)]\, q_i x(T)^{1/2} \,\middle|\, x(t_0) = x_0\right\}, \tag{7.5}$$

where k_1, k_2, c_1 and c_2 are positive constants.

The term $[k_i u_i(s)]^{1/2}$ reflects Player i's satisfaction level obtained from the consumption of the resource extracted, and $c_i u_i(s)\, x(s)^{-1/2}$ measures the (dis)satisfaction level brought about by the cost of extraction. Total utility of Player i is the aggregate level of satisfaction. Payoffs in the form of utility are not transferable between players. There exists a discount rate r, and utility received at time t has to be discounted by the factor $\exp[-r(t-t_0)]$. At time T, Player i will receive a termination bonus with satisfaction $q_i x(T)^{1/2}$, where q_i is nonnegative.

Once again we use $\Gamma(x_0, T-t_0)$ to denote the game (7.4)–(7.5), and $\Gamma(x_\tau, T-\tau)$ to denote a game with payoff structure (7.5) and dynamics (7.4) starting at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$.

A set of feedback strategies $\left[\phi_1^{(\tau)*}(t, x), \phi_2^{(\tau)*}(t, x)\right]$ provides a Nash equilibrium solution to the game $\Gamma(x_\tau, T-\tau)$, if there exist twice continuously differentiable functions $V^{(\tau)i}(t, x) : [\tau, T] \times R \to R$, $i \in \{1, 2\}$, satisfying the following partial differential equations:

$$
\begin{aligned}
&-V_t^{(\tau)i}(t,x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^{(\tau)i}(t,x) = \\
&\qquad \max_{u_i}\left\{\left[[k_i u_i]^{1/2} - \frac{c_i}{x^{1/2}} u_i\right]\exp[-r(t-\tau)] \right.\\
&\qquad\qquad \left. + V_x^{(\tau)i}(t,x)\left[a x^{1/2} - bx - u_i - \phi_j^{(\tau)*}(t,x)\right]\right\}, \text{ and} \\
&V^{(\tau)i}(T,x) = \exp[-r(T-\tau)]\, q_i x^{1/2}, \\
&\qquad \text{for } i \in \{1,2\} \text{ and } j \in \{1,2\} \text{ and } j \neq i. \qquad (7.6)
\end{aligned}
$$

Performing the indicated maximization yields:

$$
\begin{aligned}
&\phi_i^{(\tau)*}(t,x) = \frac{k_i x}{4\left[c_i + x^{1/2} V_x^{(\tau)i}(t,x)\exp[r(t-\tau)]\right]^2}, \\
&\qquad \text{for } i \in \{1,2\},\ t \in [\tau, T],\ \tau \in [t_0, T] \text{ and } x \in X. \qquad (7.7)
\end{aligned}
$$

Proposition 7.1.1. *The value function of Player i in the game $\Gamma(x_\tau, T-\tau)$ is:*

$$
\begin{aligned}
&V^{(\tau)i}(t,x) = \exp[-r(t-\tau)]\left[A_i(t)\, x^{1/2} + B_i(t)\right], \\
&\qquad \textit{for } i \in \{1,2\},\ t \in [\tau, T], \qquad (7.8)
\end{aligned}
$$

where $A_i(t)$, $B_i(t)$, $A_j(t)$ and $B_j(t)$, for $i \in \{1,2\}$, $j \in \{1,2\}$ and $j \neq i$, satisfy:

$$
\begin{aligned}
&\dot{A}_i(t) = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] A_i(t) - \frac{k_i}{4[c_i + A_i(t)/2]} + \frac{A_i(t)\, k_j}{8[c_j + A_j(t)/2]^2}, \\
&\dot{B}_i(t) = r B_i(t) - \frac{a}{2} A_i(t), \\
&A_i(T) = q_i, \quad \textit{and} \quad B_i(T) = 0.
\end{aligned}
$$

Proof. *Upon substitution of $\phi_i^{(\tau)*}(t,x)$ from (7.7) into (7.6) yields a set of partial differential equations. One can readily verify that (7.8) is a solution to this set of equations.*

Upon substituting $\left[\phi_1^{(\tau)*}(t,x), \phi_2^{(\tau)*}(t,x)\right]$ into (7.4) yields the game equilibrium trajectory.

7.2 Cooperative Arrangement under Uncertainty and Nontransferable Payoffs

Now consider the case when the players agree to cooperatively. Let $\Gamma_c(x_0, T-t_0)$ denote a cooperative stochastic differential game with the game structure of $\Gamma(x_0, T-t_0)$ in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game $\Gamma_c(x_0, T-t_0)$ with nontransferable payoffs involves an agreement on a set of cooperative strategies/controls and the payoff of individual will be determined by the cooperative controls adopted.

The solution optimality principle will remain in effect along the cooperative state trajectory path $\{x_s^*\}_{s=t_0}^{T}$. Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

7.2.1 Pareto Optimal Trajectories

Under cooperation with nontransferable payoffs, the players negotiate to establish an agreement (optimality principle) on how to play the cooperative game and how to distribute the resulting payoff. A necessary condition is that this optimality principle must satisfy group rationality and individual rationality. To achieve group rationality, the Pareto optimality of outcomes must be validated.

Consider the cooperative game $\Gamma_c(x_0, T-t_0)$ in which the payoffs are nontransferable. Pareto optimal outcomes for $\Gamma_c(x_0, T-t_0)$ can be identified by choosing a weight $\alpha_1 \in (0,\infty)$ that solves the following control problem (see Leitmann (1974)):

$$
\begin{aligned}
&\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1 J^2(t_0,x_0)\right\} \equiv \\
&\quad \max_{u_1,u_2} E_{t_0}\left\{\int_{t_0}^{T}\left(g^1[s,x(s),u_1(s),u_2(s)]\right.\right. \\
&\qquad \left.+\alpha_1 g^2[s,x(s),u_1(s),u_2(s)]\right)\exp\left[-\int_{t_0}^{s} r(y)\,dy\right]ds \\
&\qquad \left.+\left[q^1(x(T))+\alpha_1 q^2(x(T))\right]\exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\right| x(t_0)=x_0\Bigg\}, \qquad (7.9)
\end{aligned}
$$

subject to the dynamics (7.1). Note that the optimal control strategies for the problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1 J^2(t_0,x_0)\right\}$ is identical to those for the problem $\max_{u_1,u_2}\left\{J^2(t_0,x_0)+\alpha_2 J^1(t_0,x_0)\right\}$ when $\alpha_1 = 1/\alpha_2$.

In $\Gamma_c(x_0, T-t_0)$, let α_1^0 be the selected weight according an agreed upon optimality principle. Using Theorem 2.1.5, we obtain:

Theorem 7.2.1. *A set of controls* $\left\{\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right],\right.$ *for* $\left. t \in [t_0, T]\right\}$ *provides an optimal solution to the stochastic control problem* $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\}$ *if there exists continuously differentiable function* $W^{\alpha_1^0(t_0)}(t,x) : [t_0,T] \times R^m \to R$ *satisfying the following partial differential equation:*

$$
\begin{aligned}
&-W_t^{\alpha_1^0(t_0)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, W_{x^h x^\zeta}^{\alpha_1^0(t_0)}(t,x) = \\
&\qquad \max_{u_1,u_2}\left\{\left(g^1[t,x,u_1,u_2] + \alpha_1^0 g^2[t,x,u_1,u_2]\right)\exp\left[-\int_{t_0}^{t} r(y)\,dy\right]\right. \\
&\qquad\qquad \left. + W_x^{\alpha_1^0(t_0)} f[t,x,u_1,u_2]\right\}, \text{ and} \\
&W^{\alpha_1^0(t_0)}(T,x) = \\
&\qquad \exp\left[-\int_{t_0}^{T} r(y)\,dy\right]\sum_{j=1}^{2} q^j(x)\left[q^1(x) + \alpha_1 q^2(x)\right]\exp\left[-\int_{t_0}^{T} r(y)\,dy\right].
\end{aligned}
$$

Substituting $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ into (7.1) yields the dynamics of the Pareto optimal trajectory associated with weight α_1^0:

$$
\begin{aligned}
&dx(s) = f\left[s, x(s), \psi_1^{\alpha_1^0(t_0)}(s,x(s)), \psi_2^{\alpha_1^0(t_0)}(s,x(s))\right] ds + \sigma[s,x(s)]\,dz(s), \\
&\qquad x(t_0) = x_0.
\end{aligned}
\tag{7.10}
$$

The solution to (7.10), denoted by $x^{\alpha_1^0}(t)$, can be expressed as:

$$
\begin{aligned}
x^{\alpha_1^0}(t) = x_0 &+ \int_{t_0}^{t} f\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right] ds \\
&+ \int_{t_0}^{t} \sigma\left[s, x^{\alpha_1^0}(s)\right] dz(s), \qquad \text{for } t \in [t_0, T].
\end{aligned}
\tag{7.11}
$$

The path $\left\{x^{\alpha_1^0}(t)\right\}_{t=t_0}^{T}$ yields the stochastic cooperative trajectory of the problem

$$
\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\}.
$$

We use $X_t^{\alpha_1^0}$ to denote the set of realizable values of $x^{\alpha_1^0}(t)$ at time t generated by (7.11). The term $x_t^{\alpha_1^0}$ is used to denote an element in $X_t^{\alpha_1^0}$.

For group optimality to be achievable, the cooperative controls $\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$ must be adopted throughout time interval $[t_0, T]$.

Now, consider the cooperative game $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ for $\tau \in [t_0, T]$. Let α_1^τ be the weight selected according to the originally agreed upon optimality principle. We use

$$\left\{\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right],\ t \in [\tau, T]\right\}$$

to denote a set of optimal controls, and $W^{\alpha_1^\tau(\tau)}(t,x) : [\tau, T] \times R^n \to R$ the corresponding value function in Theorem 7.2.1.

One can readily verify that

$$\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right] = \left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right] \text{ when } \alpha_1^\tau = \alpha_1^0.$$

Remark 7.2.1. Group optimality can only be maintained if $\alpha_1^\tau = \alpha_1^0$ is the chosen weight in $\Gamma_c(x_\tau, T-\tau)$ for all $\tau \in [t_0, T]$.

Example 7.2.1. Consider the game in Example 7.1.1. The players agree to cooperative and negotiate to establish an agreement (optimality principle) on how to play the cooperative game and hence how to distribute the resulting payoff. A necessary condition is that this optimality principle must satisfy group rationality and individual rationality.

Let $\Gamma_c(x_0, T-t_0)$ denote a cooperative game with payoff structure (7.5) and dynamics (7.4). Pareto optimal trajectories for $\Gamma_c(x_0, T-t_0)$ can be identified by choosing specific weights $\alpha_1^0 \in (0, \infty)$ that solves the stochastic control problem:

$$\begin{aligned}
&\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\} \equiv \\
&\quad \max_{u_1,u_2} E_{t_0}\left\{\int_{t_0}^T \left(\left[[k_1u_1(s)]^{1/2} - \frac{c_1}{x(s)^{1/2}}u_1(s)\right]\right.\right. \\
&\qquad \left. + \alpha_1^0\left[[k_2u_2(s)]^{1/2} - \frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\exp[-r(s-t_0)]\,ds \\
&\qquad \left.\left. + \exp[-r(T-t_0)]\left(q_1x(T)^{1/2}\,\alpha_1q_2x(T)^{1/2}\right)\right|\, x(t_0) = x_0\right\}, \quad (7.12)
\end{aligned}$$

subject to dynamics (7.4).

Let $\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$, for $t \in [t_0, T]$ denote a set of controls that provides a solution to the stochastic control problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0)\right.$ $\left.+\alpha_1^0 J^2(t_0,x_0)\right\}$, and $W^{(t_0)}(t,x) : [t_0,T] \times R^n \to R$ denote the value function that satisfies the equations (see Theorem 7.2.1):

$$-W_t^{\alpha_1^0(t_0)}(t,x) - \frac{1}{2}\sigma^2x^2W_{xx}^{\alpha_1^0(t_0)}(t,x) =$$

$$\max_{u_1,u_2}\left\{\left(\left[\left[k_1u_1\right]^{1/2}-\frac{c_1}{x^{1/2}}u_1\right]+\alpha_1^0\left[\left[k_2u_2\right]^{1/2}-\frac{c_2}{x^{1/2}}u_2\right]\right)\exp\left[-r\left(t-t_0\right)\right]\right.$$
$$\left.+W_x^{\alpha_1^0(t_0)}\left(t,x\right)\left[ax^{1/2}-bx-u_1-u_2\right]\right\},\text{ and}$$
$$W^{\alpha_1^0(t_0)}\left(T,x\right)=\exp\left[-r\left(T-t_0\right)\right]\left[q_1x^{1/2}+\alpha_1^0q_2x^{1/2}\right]. \tag{7.13}$$

Performing the indicated maximization in (7.13) yields:

$$\psi_1^{\alpha_1^0(t_0)}\left(t,x\right)=\frac{k_1x}{4\left[c_1+x^{1/2}W_x^{\alpha_1^0(t_0)}\left(t,x\right)\exp\left[r\left(t-t_0\right)\right]\right]^2}\text{ and}$$
$$\psi_2^{\alpha_1^0(t_0)}\left(t,x\right)=\frac{k_2x}{4\left[c_2+x^{1/2}W_x^{\alpha_1^0(t_0)}\left(t,x\right)\exp\left[r\left(t-t_0\right)\right]/\alpha_1^0\right]^2},$$
$$\text{for }\ t\in\left[t_0,T\right]. \tag{7.14}$$

Substituting $\psi_1^{\alpha_1^0(t_0)}\left(t,x\right)$ and $\psi_2^{\alpha_1^0(t_0)}\left(t,x\right)$ from (7.14) into (7.13) yields the value function

$$W^{\alpha_1^0(t_0)}\left(t,x\right)=\exp\left[-r\left(t-t_0\right)\right]\left[A^{\alpha_1^0}\left(t\right)x^{1/2}+B^{\alpha_1^0}\left(t\right)\right],$$
$$\text{for }\ t\in\left[t_0,T\right], \tag{7.15}$$

where $A^{\alpha_1^0}\left(t\right)$ and $B^{\alpha_1^0}\left(t\right)$ satisfy:

$$\dot{A}^{\alpha_1^0}\left(t\right)=\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]A^{\alpha_1^0}\left(t\right)-\frac{k_1}{4\left[c_1+A^{\alpha_1^0}\left(t\right)/2\right]}$$
$$-\frac{\alpha_1^0k_2}{4\left[c_2+A^{\alpha_1^0}\left(t\right)/2\alpha_1^0\right]},$$
$$\dot{B}^{\alpha_1^0}\left(t\right)=rB^{\alpha_1^0}\left(t\right)-\frac{a}{2}A^{\alpha_1^0}\left(t\right),$$
$$A^{\alpha_1^0}\left(T\right)=q_1+\alpha_1^0q_2,\text{ and }\ B^{\alpha_1^0}\left(T\right)=0. \tag{7.16}$$

Substituting the partial derivative $W_x^{\alpha_1^0(t_0)}\left(t,x\right)$ from (7.16) into $\psi_1^{\alpha_1^0(t_0)}\left(t,x\right)$ and $\psi_2^{\alpha_1^0(t_0)}\left(t,x\right)$ one obtains the controls of the problem $\max_{u_1,u_2}\left\{J^1\left(t_0,x_0\right)+\alpha_1^0J^2\left(t_0,x_0\right)\right\}$. Substituting these controls into (7.4) yields the dynamics of the Pareto optimal trajectory associated with a weight α_1^0 as:

$$dx\left(s\right)=$$
$$\left[ax\left(s\right)^{1/2}-\left(b+\frac{k_1}{4\left[c_1+A^{\alpha_1^0}\left(s\right)/2\right]^2}+\frac{k_2}{4\left[c_2+A^{\alpha_1^0}\left(s\right)/2\alpha_1^0\right]^2}\right)x\left(s\right)\right]ds,$$
$$+\sigma x\left(s\right)dz\left(s\right),\quad x\left(t_0\right)=x_0\in X. \tag{7.17}$$

Solving (7.17) yields the Pareto optimal trajectory associated with weight α_1^0 as:

$$x^{\alpha_1^0}(t) = \left\{ \Phi(t,t_0) \left[x_0^{1/2} + \int_{t_0}^{t} \Phi^{-1}(s,t_0) \frac{a}{2} ds \right] \right\}^2 , \qquad (7.18)$$

where

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} \left(\frac{-b}{2} - \frac{k_1}{8\left[c_1 + A^{\alpha_1^0}(s)/2\right]^2} - \frac{k_2}{8\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]^2} - \frac{3\sigma^2}{8} \right) ds + \int_{t_0}^{t} \frac{\sigma}{2} dz(s) \right] .$$

We use $X_t^{\alpha_1^0}$ to denote the set of realizable values of $x^{\alpha_1^0}(t)$ at time t generated by (7.18). We denote an element in $X_t^{\alpha_1^0}$ by $x_t^{\alpha_1^0}$.

The cooperative control associated with weight α_1^0 over the time interval $[t_0, T]$ can be expressed precisely as:

$$\psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right) = \frac{k_1 x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \quad \text{and}$$

$$\psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right) = \frac{k_2 x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2},$$

$$\text{for } t \in [t_0, T] \text{ and } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}. \qquad (7.19)$$

7.3 Individual Player's Payoffs under Cooperation

In order to verify individual rationality in a cooperative scheme, we have to derive individual players' expected payoff functions under cooperation. To do this, we first substitute the optimal controls $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ into the objective functions (7.2) to derive the players' expected payoff in $\Gamma_c(x_0, T-t_0)$ with α_1^0 being chosen as the cooperative weight. We follow Yeung (2004) and define:

Definition 7.3.1. *We define Player i's expected cooperative payoff over the interval $[t,T]$ as:*

$$\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) = E_{t_0}\left\{ \int_t^T g^i \left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right) \right] \right.$$

$$\times \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds$$
$$+\exp\left[-\int_{t_0}^{T} r(y)\,dy\right] q^i\left(x^{\alpha_1^0}(T)\right)\bigg| x(t) = x_t^{\alpha_1^0}\Bigg\},$$
$$\textit{for } i \in \{1,2\} \quad \textit{and } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0},$$

where

$$dx^{\alpha_1^0}(s) = f\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right] ds$$
$$+\sigma\left[s, x^{\alpha_1^0}(s)\right] dz(s),$$
$$x^{\alpha_1^0}(t) = x_t^{\alpha_1^0}.$$

Note that for $\Delta t \to 0$, we can express $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right)$ as:

$$\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) =$$
$$E_{t_0}\left\{\int_t^{t+\Delta t} g^i\left[s, x^{\alpha_1^0}(s), \psi_1^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right), \psi_2^{\alpha_1^0(t_0)}\left(s, x^{\alpha_1^0}(s)\right)\right]\right.$$
$$\times \exp\left[-\int_{t_0}^{s} r(y)\,dy\right] ds$$
$$\left.+\hat{W}^{\alpha_1^0(t_0)i}\left(t+\Delta t, x^{\alpha_1^0} + \Delta x^{\alpha_1^0}\right)\bigg| x(t) = x_t^{\alpha_1^0}\right\}, \tag{7.20}$$

where

$$\Delta x^{\alpha_1^0} = f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \Delta t$$
$$+\sigma\left(t, x_t^{\alpha_1^0}\right) \Delta z + o(\Delta t),$$
$$\Delta z = z(t+\Delta t) - z(t), \text{ and } E_{t_0}\left[o(\Delta t)\right]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

Applying Taylor's Theorem, we have

$$\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) =$$
$$g^i\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \exp\left[-\int_{t_0}^{t} r(y)\,dy\right] \Delta t$$
$$+\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) + \hat{W}_t^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) \Delta t$$
$$+\hat{W}_x^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) f\left[t, x_t^{\alpha_1^0}, \psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right] \Delta t$$

$$+\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(t,x_t^{\alpha_1^0}\right)\hat{W}_{x^h x^\zeta}^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)\Delta t$$

$$+\hat{W}_x^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)\sigma\left(t,x_t^{\alpha_1^0}\right)\Delta z+o\left(\Delta t\right),$$

$$\text{for } i\in\{1,2\}. \tag{7.21}$$

Canceling terms, performing the expectation operator, dividing throughout by Δt and taking $\Delta t \to 0$, we obtain:

$$-\hat{W}_t^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(t,x_t^{\alpha_1^0}\right)\hat{W}_{x^h x^\zeta}^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)=$$

$$g^i\left[t,x_t^{\alpha_1^0},\psi_1^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right),\psi_2^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right)\right]\exp\left[-\int_{t_0}^{t}r\left(y\right)dy\right]$$

$$+\hat{W}_x^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)f\left[t,x_t^{\alpha_1^0},\psi_1^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right),\psi_2^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right)\right],$$

$$\text{for } i\in\{1,2\}. \tag{7.22}$$

Boundary conditions require:

$$\hat{W}^{\alpha_1^0(t_0)i}\left(T,x_T^{\alpha_1^0}\right)=\exp\left[-\int_{t_0}^{T}r\left(y\right)dy\right]q^i\left(x_T^{\alpha_1^0}\right), \qquad \textit{for } i\in\{1,2\}. \tag{7.23}$$

Theorem 7.3.1. ***(Yeung's (2004))*** *If there exist continuously differentiable functions* $\hat{W}^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right):[t_0,T]\times R^m\to R$, $i\in\{1,2\}$, *satisfying*

$$-\hat{W}_t^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(t,x_t^{\alpha_1^0}\right)\hat{W}_{x^h x^\zeta}^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)=$$

$$g^i\left[t,x_t^{\alpha_1^0},\psi_1^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right),\psi_2^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right)\right]\exp\left[-\int_{t_0}^{t}r\left(y\right)dy\right]$$

$$+\hat{W}_x^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)f\left[t,x_t^{\alpha_1^0},\psi_1^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right),\psi_2^{\alpha_1^0(t_0)}\left(t,x_t^{\alpha_1^0}\right)\right],$$

and

$$\hat{W}^{\alpha_1^0(t_0)i}\left(T,x_T^{\alpha_1^0}\right)=\exp\left[-\int_{t_0}^{T}r\left(y\right)dy\right]q^i\left(x_T^{\alpha_1^0}\right),$$

$$\textit{for } i\in\{1,2\},$$

then $\hat{W}^{\alpha_1^0(t_0)i}\left(t,x_t^{\alpha_1^0}\right)$ *gives Player i's expected cooperative payoff over the interval* $[t,T]$ *with* α_1^0 *being the cooperative weight and* $x(t)=x_t^{\alpha_1}$.

For the sake of subsequent comparison, we repeat the above analysis for the cooperative game $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ which starts at time τ with initial state $x_\tau^{\alpha_1^0}$ and chosen cooperative weight $\alpha_1^\tau = \alpha_1^0$. One can readily verify that

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right)\exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1,2\} \text{ and } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}. \tag{7.24}$$

Remark 7.3.1. To maintain individual rationality throughout the game, the chosen α_1^0 has to satisfy

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right)\exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1,2\}.$$

Example 7.3.1. Consider the cooperative game in Example 7.2.1 in which a cooperative weight α_1^0 is chosen. By substituting the cooperative controls $\left[\psi_1^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right), \psi_2^{\alpha_1^0(t_0)}\left(t, x_t^{\alpha_1^0}\right)\right]$ in Example 7.2.1 into (7.5) we obtain the expected payoff of Player 1 over the interval $[t,T]$ as:

$$\hat{W}^{t_0\left(\alpha_1^0\right)1}\left(t, x_t^{\alpha_1^0}\right) =$$
$$E_{t_0}\left\{\int_t^T \left[\frac{k_1 x^{\alpha_1^0}(s)^{1/2}}{2\left[c_1 + A^{\alpha_1^0}(s)/2\right]} - \frac{c_1 k_1 x^{\alpha_1^0}(s)^{1/2}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2}\right]\exp\left[-r(s-t_0)\right]ds\right.$$
$$\left. + \exp\left[-r(T-t_0)\right] q_1 x^{\alpha_1^0}(T)^{1/2} \,\middle|\, x(t) = x_t^{\alpha_1^0}\right\};$$

and the corresponding expected payoff of Player 2 over the interval $[t,T]$ as:

$$\hat{W}^{t_0\left(\alpha_1^0\right)2}\left(t, x_t^{\alpha_1^0}\right) =$$
$$E_{t_0}\left\{\int_t^T \left[\frac{k_2 x^{\alpha_1^0}(s)^{1/2}}{2\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]} - \frac{c_2 k_2 x^{\alpha_1^0}(s)^{1/2}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}\right]\right.$$
$$\times \exp\left[-r(s-t_0)\right]ds$$
$$\left. + \exp\left[-r(T-t_0)\right] q_2 x^{\alpha_1^0}(T)^{1/2} \,\middle|\, x(t) = x_t^{\alpha_1^0}\right\};$$

where

$$dx^{\alpha_1^0}(s) = \left[ax^{\alpha_1^0}(s)^{1/2} - \left(b + \frac{k_1}{4\left[c_1 + A^{\alpha_1^0}(s)/2\right]^2} + \frac{k_2}{4\left[c_2 + A^{\alpha_1^0}(s)/2\alpha_1^0\right]^2}\right) x^{\alpha_1^0}(s)\right] ds$$
$$+\sigma x^{\alpha_1^0}(s)\, dz(s),$$
$$x(t) = x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}.$$

Invoking Theorem 7.3.1, we know if there exist continuously differentiable functions $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right) : [t_0, T] \times R^m \to R$, $i \in \{1, 2\}$, satisfying

$$-\hat{W}_t^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right) - \frac{1}{2}\hat{W}_{xx}^{t_0(\alpha_1^0)1}\left(t, x_t^{\alpha_1^0}\right)\sigma^2\left(x_t^{\alpha_1^0}\right)^2 =$$
$$\left[\frac{k_1\left(x_t^{\alpha_1^0}\right)^{1/2}}{2\left[c_1 + A^{\alpha_1^0}(t)/2\right]} - \frac{c_1 k_1\left(x_t^{\alpha_1^0}\right)^{1/2}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2}\right]\exp\left[-r(t - t_0)\right]$$
$$+\hat{W}_x^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right)\left[a\left(x_t^{\alpha_1^0}\right)^{1/2} - bx_t^{\alpha_1^0} - \frac{k_1 x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} - \frac{k_2 x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}\right],$$

$$\hat{W}^{\alpha_1^0(t_0)1}\left(T, x_T^{\alpha_1^0}\right)\exp\left[-r(T - \tau)\right] q_1\left(x_T^{\alpha_1^0}\right)^{1/2};$$

and

$$-\hat{W}_t^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) - \frac{1}{2}\hat{W}_{xx}^{t_0(\alpha_1^0)2}\left(t, x_t^{\alpha_1^0}\right)\sigma^2\left(x_t^{\alpha_1^0}\right)^2 =$$
$$\left[\frac{k_2\left(x_t^{\alpha_1^0}\right)^{1/2}}{2\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]} - \frac{c_2 k_2\left(x_t^{\alpha_1^0}\right)^{1/2}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}\right]\exp\left[-r(t - t_0)\right]$$
$$+\hat{W}_x^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)\left[a\left(x_t^{\alpha_1^0}\right)^{1/2} - bx_t^{\alpha_1^0} - \frac{k_1 x_t^{\alpha_1^0}}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} - \frac{k_2 x_t^{\alpha_1^0}}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}\right],$$

$$\hat{W}^{\alpha_1^0(t_0)2}\left(T, x_T^{\alpha_1^0}\right)\exp\left[-r(T - \tau)\right] q_2\left(x_T^{\alpha_1^0}\right)^{1/2};$$

$$\text{for } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0} \text{ and } x_T^{\alpha_1^0} \in X_T^{\alpha_1^0}, \tag{7.25}$$

then $\hat{W}^{\alpha_1^0(t_0)i}\left(t, x_t^{\alpha_1^0}\right)$ gives Player i's expected cooperative payoff over the interval $[t, T]$ with α_1^0 being the cooperative weight.

Proposition 7.3.1. *The function* $\hat{W}^{\alpha_1^0(t_0)1}(t, x) : [\tau, T] \times R \to R$ *satisfying (7.25) can be solved as:*

$$\hat{W}^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right) = \exp\left[-r\left(t - t_0\right)\right]\left[\hat{A}_1^{\alpha_1^0}(t)\, x^{1/2} + \hat{B}_1^{\alpha_1^0}(t)\right], \tag{7.26}$$

where

$$\begin{aligned}
\dot{\hat{A}}_1^{\alpha_1^0}(t) = {} & \left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]\hat{A}_1^{\alpha_1^0}(t) - \frac{k_1}{2\left[c_1 + A^{\alpha_1^0}(t)/2\right]} + \frac{c_1 k_1}{4\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} \\
& + \frac{\hat{A}_1^{\alpha_1^0}(t)\, k_1}{8\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} + \frac{\hat{A}_1^{\alpha_1^0}(t)\, k_2}{8\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}, \\
\dot{\hat{B}}_1^{\alpha_1^0}(t) = {} & r\hat{B}_1^{\alpha_1^0}(t) - \frac{a}{2}\hat{A}_1^{\alpha_1^0}(t), \text{ and} \\
\hat{A}_1^{\alpha_1^0}(T) = {} & q_1, \text{ and } \hat{B}_1^{\alpha_1^0}(T) = 0.
\end{aligned}$$

Proof. *Upon calculating the derivatives* $\hat{W}_t^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right)$, $\hat{W}_{x_t}^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right)$ *and* $\hat{W}_{x_t x_t}^{\alpha_1^0(t_0)1}\left(t, x_t^{\alpha_1^0}\right)$ *from (7.26) and then substituting them into (7.25) yield Proposition 7.3.1.*

Proposition 7.3.2. *The function* $\hat{W}^{\alpha_1^0(t_0)2}(t, x) : [\tau, T] \times R \to R$ *satisfying (7.25) can be solved as:*

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right) = \exp\left[-r\left(t - t_0\right)\right]\left[\hat{A}_2^{\alpha_1^0}(t)\, x^{1/2} + \hat{B}_2^{\alpha_1^0}(t)\right], \tag{7.27}$$

where

$$\begin{aligned}
\dot{\hat{A}}_2^{\alpha_1^0}(t) = {} & \\
& \left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]\hat{A}_2^{\alpha_1^0}(t) - \frac{k_2}{2\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]} + \frac{c_2 k_2}{4\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2} \\
& + \frac{\hat{A}_2^{\alpha_1^0}(t)\, k_1}{8\left[c_1 + A^{\alpha_1^0}(t)/2\right]^2} + \frac{\hat{A}_2^{\alpha_1^0}(t)\, k_2}{8\left[c_2 + A^{\alpha_1^0}(t)/2\alpha_1^0\right]^2}, \\
\dot{\hat{B}}_2^{\alpha_1^0}(t) = {} & r\hat{B}_2^{\alpha_1^0}(t) - \frac{a}{2}\hat{A}_2^{\alpha_1^0}(t), \\
\hat{A}_2^{\alpha_1^0}(T) = {} & q_2, \text{ and } \hat{B}_2^{\alpha_1^0}(T) = 0.
\end{aligned}$$

Proof. Upon calculating the derivatives $\hat{W}_t^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)$, $\hat{W}_{x_t}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)$ *and* $\hat{W}_{x_t x_t}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)$ *from (7.27) and then substituting them into (7.25) yield Proposition 7.3.2.*

Repeating the above, one can obtain:

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)\exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)1}\left(t, x_t^{\alpha_1^0}\right)$$

$$= \exp\left[-r\left(t-\tau\right)\right]\left[\hat{A}_1^{\alpha_1^0}(t)\left(x_t^{\alpha_1^0}\right)^{1/2} + \hat{B}_1^{\alpha_1^0}(t)\right]$$

and

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)\exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)2}\left(t, x_t^{\alpha_1^0}\right)$$

$$= \exp\left[-r\left(t-\tau\right)\right]\left[\hat{A}_2^{\alpha_1^0}(t)\left(x_t^{\alpha_1^0}\right)^{1/2} + \hat{B}_2^{\alpha_1^0}(t)\right];$$

$$\text{for } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}. \tag{7.28}$$

In order to fulfill individual rationality, the choice of α_1^0 must satisfy the condition

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)$$
$$\text{for any } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}.$$

7.4 Subgame Consistent Solutions

In Chapter 5, the notion of subgame consistency is introduced and applied to cooperative stochastic differential games with transferable payoffs. The principle of subgame consistency ensures that an extension of a cooperative solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. In addition, the two essential properties of cooperation – Pareto optimality and individual rationality – are required to be present. In the present framework, a subgame consistent solution to the nontransferable payoffs game $\Gamma_c\left(x_0, T - t_0\right)$ requires:

Condition 7.4.1.

(i) The imputation vector

$$\left[\hat{W}^{\tau(\alpha_1^\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\tau(\alpha_1^\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right], \textit{ for } \tau \in [t_0, T],$$

be Pareto optimal;

(ii)

$$\hat{W}^{\alpha_1^\tau(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$

for $i \in \{1,2\}$, $\tau \in [t_0, T]$ *and any* $x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}$; *and*

(iii)

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \exp\left[\int_{t_0}^{\tau} r(y)\, dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$

for $i \in \{1,2\}$, $\tau \in [t_0, T]$ *and any* $x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}$.

Part (i) of Condition 7.4.1 guarantees Pareto optimality. Part (ii) ensures that individual rationality is satisfied at a later starting time and any feasible state brought about by prior optimal behaviors. Part (iii) guarantees the subgame consistency of the solution imputations throughout the game interval.

Consider that at time t_0 the players reach an agreed-upon optimality principle for $\Gamma_c(x_0, T - t_0)$ governing

(i) the choice a weight α_1^0 leading to a set of cooperative controls

$$\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right], \text{ and}$$

(ii) an imputation

$$\left[\hat{W}^{\alpha_1^0(t_0)1}(t_0, x_0), \hat{W}^{\alpha_1^0(t_0)2}(t_0, x_0)\right]$$

will then follow.

At a subsequent time $\tau \in (t_0, T]$, using the same optimality principle for $\Gamma_c\left(x_\tau^{\alpha_1^0}, T - \tau\right)$, the players agree to adopt

(i) a weight α_1^τ leading to a set of cooperative controls

$$\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right], \text{ and}$$

(ii) an imputations

$$\left[\hat{W}^{\alpha_1^\tau(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\alpha_1^\tau(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right]$$

will then follow.

Theorem 7.4.1. *A solution optimality principle under which the players agree to choose the same weight* α_1^0 *in all the games* $\Gamma_c\left(x_\tau^{\alpha_1^0}, T - \tau\right)$ *such that*

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right),$$

for $\tau \in [t_0, T]$ *and* $x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}$,

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

Proof. Given that the same weight α_1^0 will be chosen for all the subgames $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$, the cooperative control $\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$ will be adopted throughout time interval $[t_0, T]$. Group optimality is assured and the imputation vector

$$\xi^{(\tau)}(x_\tau, T-\tau) = \left[\hat{W}^{\tau\left(\alpha_1^0\right)1}\left(\tau, x_\tau^{\alpha_1^0}\right), \hat{W}^{\tau\left(\alpha_1^0\right)2}\left(\tau, x_\tau^{\alpha_1^0}\right)\right],$$
$$\text{for } \tau \in [t_0, T],$$

is indeed Pareto optimal. Hence part (i) of Condition 7.4.1 is satisfied.

With

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } \tau \in [t_0, T] \text{ and any } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0},$$

individual rationality is fulfilled throughout the game horizon given any feasible state brought about by prior optimal behaviors. Part (ii) of Condition 7.4.1 is satisfied.

Moreover, from (7.24), we have

$$\hat{W}^{\alpha_1^0(t_0)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \exp\left[\int_{t_0}^{\tau} r(y)\,dy\right] = \hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1,2\},\ \tau \in [t_0, T] \text{ and } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}.$$

Part (iii) of Condition 7.4.1 is satisfied.

7.5 A Proposed Solution

In this section, we propose a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$ for Example 7.2.1. Invoking Theorem 7.4.1, a solution optimality principle under which the players agree to choose the same weight α_1^0 in all the games $\Gamma_c\left(x_\tau^{\alpha_1^0}, T-\tau\right)$ and

$$\hat{W}^{\alpha_1^0(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)1}\left(\tau, x_\tau^{\alpha_1^0}\right) \text{ and } \hat{W}^{\alpha_1^0(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)2}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } \tau \in [t_0, T] \text{ and } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0},$$

would yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$.

The condition

$$\hat{W}^{\alpha_1^0(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right) \geq V^{(\tau)i}\left(\tau, x_\tau^{\alpha_1^0}\right),$$
$$\text{for } i \in \{1,2\} \text{ and } \tau \in [t_0, T] \text{ and } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0},$$

implies the restriction on the choice of α_1^0 such that:

$$\left[\hat{A}_i^{\alpha_1^0}(\tau)\left(x_\tau^{\alpha_1^0}\right)^{1/2}+\hat{B}_i^{\alpha_1^0}(\tau)\right] \geq \left[A_i(\tau)\left(x_\tau^{\alpha_1^0}\right)^{1/2}+B_i(\tau)\right], \quad (7.29)$$

$$\text{for } i \in \{1,2\}, \ \tau \in [t_0, T] \text{ and any } x_\tau^{\alpha_1^0} \in X_\tau^{\alpha_1^0}.$$

Note that from Propositions 7.3.1 and 7.3.2 one can obtain

$$\hat{B}_i^{\alpha_1^0}(\tau) = (a/2r)\,\hat{A}_i^{\alpha_1^0}(\tau),$$

and from Proposition 7.1.1 one can obtain $B_i(\tau) = (a/2r)\,A_i(\tau)$. Therefore if $\hat{A}_i^{\alpha_1^0}(\tau) \geq A_i(\tau)$ then

$$\hat{B}_i^{\alpha_1^0}(\tau) \geq B_i(\tau), \quad \text{for } i \in \{1,2\} \text{ and } \tau \in [t_0, T].$$

For (7.29) to hold it is necessary that

$$\hat{A}_i^{\alpha_1^0}(\tau) \geq A_i(\tau), \ \text{ for } i \in \{1,2\} \text{ and } \tau \in [t_0, T]. \quad (7.30)$$

We define:

Definition 7.5.1. *We denote the set of α_1^t that satisfies*

$$\hat{A}_i^{\alpha_1^t}(t) \geq A_i(t), \ \textit{for } i \in \{1,2\}$$

at time $t \in [t_0, T)$ by S_t. We use $\underline{\alpha}_1^t$ to denote the lowest value of α_1 in S_t, and $\overline{\alpha}_1^t$ the highest value. In the case when t tends to T, we use $\overline{\alpha}_1^{T-}$ to stand for $\lim_{t\to T^-} \overline{\alpha}_1^t$, and $\overline{\alpha}_1^{T-}$ for $\lim_{t\to T^-} \underline{\alpha}_1^t$.

Definition 7.5.2. *We define the set $S_\tau^T = \bigcap_{\tau \leq t < T} S_t$, for $\tau \in [t_0, T)$.*

S_t is a set of α_1 that satisfies individual rationality at time $t \in [t_0, T)$ and S_τ^T is a set of α_1 that satisfies individual rationality throughout the interval $[\tau, T)$. In general $S_\tau^T \neq S_t^T$ for $\tau, t \in [t_0, T)$ where $\tau \neq t$.

7.5.1 Typical Configurations of S_t

To find out typical configurations of the set S_t for $t \in [t_0, T)$ of the game $\Gamma_c(x_0, T - t_0)$, we perform extensive numerical simulations with a wide range of parameter specifications for a, b, σ, c_1, c_2, k_1, k_2, q_1, q_2, T, r, x_0. We calculate the time paths of $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ for $t \in [t_0, T]$ from the Example 7.1.1. Then we select weights α_1^t and calculate the time paths of $\hat{A}_1^{\alpha_1^t}(t)$ and $\hat{A}_2^{\alpha_1^t}(t)$, $\hat{B}_1^{\alpha_1^t}(t)$ and $\hat{B}_2^{\alpha_1^t}(t)$ in Propositions 7.3.1 and 7.3.2, for $t \in [t_0, T]$. At each time instant $t \in [t_0, T]$, we derive the set of α_1^t that yields $\hat{A}_i^{\alpha_1^t}(t) \geq A_i(t)$, for $i \in \{1,2\}$, to construct the set S_t, for $t \in [t_0, T)$.

We denote the locus of the values of $\underline{\alpha}_1^t$ along $t \in [t_0, T)$ as curve $\underline{\alpha}_1$ and the locus of the values of $\overline{\alpha}_1^t$ as curve $\overline{\alpha}_1$. In particular, three typical characteristics prevail:

(i) The curves $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are continuous and always move in the same direction over the entire game duration: either both increase monotonically or both decrease monotonically.

(ii) The set $S_t = \left[\underline{\alpha}_1^t, \overline{\alpha}_1^t\right]$, and

$$\hat{W}^{t\left(\overline{\alpha}_1^t\right)1}(t,x) = V^{(t)1}(t,x) \quad \text{and} \quad \hat{W}^{t\left(\underline{\alpha}_1^t\right)2}(t,x) = V^{(t)2}(t,x),$$
$$\text{for} \quad t \in [t_0, T).$$

(iii) The set $S_{t_0}^T$ can be nonempty or empty.

These typical configurations are very similar to those appearing in Figures 4.1 – 4.3 in Chapter 4.

Remark 7.5.1. Consider the case when $S_{t_0}^T \neq \emptyset$. If both $\underline{\alpha}_1$ and $\overline{\alpha}_1$ decrease monotonically, the condition $\underline{\alpha}_1^{T-} \notin S_\tau^T$ and $\overline{\alpha}_1^{T-} \in S_\tau^T$ for $\tau \in [t_0, T)$ prevails. If both $\underline{\alpha}_1$ and $\overline{\alpha}_1$ increase monotonically, the condition $\underline{\alpha}_1^{T-} \in S_\tau^T$ and $\overline{\alpha}_1^{T-} \notin S_\tau^T$ for $\tau \in [t_0, T)$ prevails.

7.5.2 A Subgame Consistent Solution

A subgame consistent solution to $\Gamma_c(x_0, T - t_0)$ is presented below.

Theorem 7.5.1. *If $S_{t_0}^T \neq \emptyset$, an optimality principle under which the players agree to choose the weight*

$$\alpha_1^* = \begin{cases} \underline{\alpha}_1^{T-}, & \text{if } \underline{\alpha}_1^{T-} \in S_\tau^T \text{ and } \overline{\alpha}_1^{T-} \notin S_\tau^T, \text{ for } \tau \in [t_0, T], \\ \overline{\alpha}_1^{T-}, & \text{if } \overline{\alpha}_1^{T-} \in S_\tau^T \text{ and } \underline{\alpha}_1^{T-} \notin S_\tau^T, \text{ for } \tau \in [t_0, T], \end{cases} \tag{7.31}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

Proof. *If any one of the two mutually exclusive conditions governing the choice of α_1^* in (7.31) prevails, according to the optimality principle in Theorem 7.5.1, a unique α_1^* will be chosen for all the subgames $\Gamma_c\left(x_\tau^{\alpha_1^*}, T - \tau\right)$, for $t_0 \leq \tau \leq t < T$. The vector*

$$\xi^{(\tau)}\left(x_\tau^{\alpha_1^*}, T - \tau\right) = \left[\hat{W}^{\tau(\alpha_1^*)1}(\tau, x_\tau), \hat{W}^{\tau(\alpha_1^*)2}(\tau, x_\tau)\right],$$
$$\textit{for} \quad \tau \in [t_0, T],$$

yields a Pareto optimal pair of imputations. Hence part (i) of Condition 7.4.1 is satisfied.

Since $\alpha_1^* \in S_{t_0}^T$,

$$\left[\hat{A}_i^{\alpha_1^*}(t)\left(x_t^{\alpha_1^*}\right)^{1/2} + \hat{B}_i^{\alpha_1^*}(t)\right] \geq \left[A_i(t)\left(x_t^{\alpha_1^*}\right)^{1/2} + B_i(t)\right],$$
$$\textit{for} \quad i \in \{1, 2\}.$$

Hence part (ii) of Condition 7.4.1 is satisfied.

Finally from (7.24),

$$\hat{W}^{\alpha_1^0(t_0)2}\left(t, x_t^{\alpha_1^0}\right)\exp\left[-r\left(\tau - t_0\right)\right] = \hat{W}^{\alpha_1^0(\tau)1}\left(t, x_t^{\alpha_1^0}\right),$$
$$\textit{for } i \in \{1,2\}, \quad \tau \in [t_0, T].$$

Part (iii) of Condition 7.4.1 is satisfied.
Hence Theorem 7.4.1 follows.

7.5.3 Numerical Illustrations

Three numerical examples are provided below.

Example 7.5.1. Consider the cooperative game $\Gamma_c(x_0, T-t)$ in Example 7.2.1. with the following parameter specifications: $a = 0.4$, $b = 0.1$, $\sigma = 0.05$, $k_1 = 4$, $k_2 = 1$, $c_1 = 1$, $c_2 = 1.5$, $q_1 = 2$, $q_2 = 1.2$, $T = 6$, $r = 0.04$, and $x_0 = 5$.

The numerical results are displayed in the upper panel of Fig. 7.1. The curve $\underline{\alpha}_1$ is the locus of the values of $\underline{\alpha}_1^t$ along $t \in [t_0, T)$. The curve $\overline{\alpha}_1$ is the locus of the values of $\overline{\alpha}_1^t$ along $t \in [t_0, T)$. In particular, the set $S_{t_0}^T = \bigcap_{t_0 \le t < T} S_t = \left[\underline{\alpha}_1^{t_0}, \overline{\alpha}_1^{T-}\right] = [0.764, 1.07]$. Note that $\overline{\alpha}_1^{T-} \in S_\tau^T$ and $\underline{\alpha}_1^{T-} \notin S_\tau^T$, for $\tau \in [t_0, T)$. According to Theorem 7.5.1, the players would agree to the optimality principle of choosing a weight $\alpha_1^* = \overline{\alpha}_1^{T-} = 1.07$ throughout the game interval, and a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$ would result.

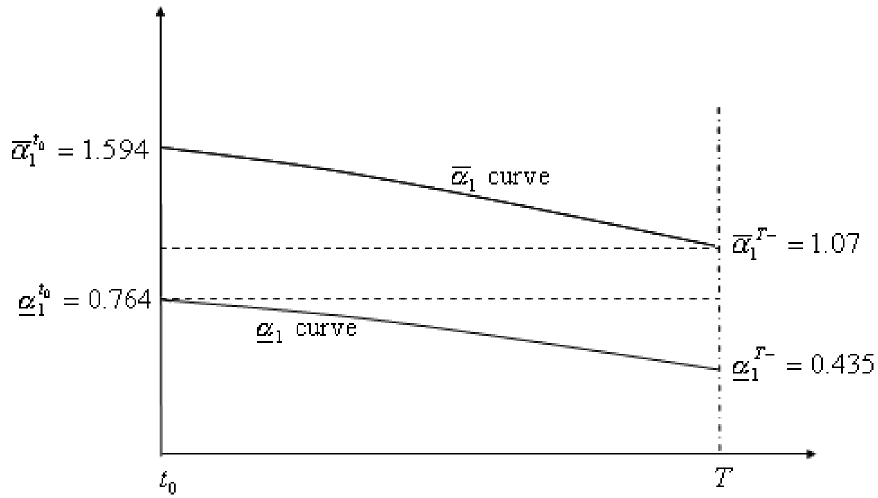

Fig. 7.1. Bounds of the values of α_1 in Example 7.5.1.

Example 7.5.2. Consider again the cooperative game $\Gamma_c(x_0, T-t)$ in Example 7.2.1 with the following parameter specifications: $a = 0.4$, $b = 0.1$, $\sigma = 0.1$, $k_1 = 1$, $k_2 = 5$, $c_1 = 1$, $c_2 = 1.5$, $q_1 = 2$, $q_2 = 1.2$, $T = 6$, $r = 0.04$, and $x_0 = 10$.

The numerical results are displayed in the upper panel of Fig. 7.2. The curve $\underline{\alpha}_1$ is the locus of the values of $\underline{\alpha}_1^t$ along $t \in [t_0, T)$. The curve $\overline{\alpha}_1$ is the locus of the values of $\overline{\alpha}_1^t$ along $t \in [t_0, T)$. In particular, the set $S_{t_0}^T = \bigcap_{t_0 \le t < T} S_t = \left[\underline{\alpha}_1^{t_0}, \overline{\alpha}_1^{T-}\right] = [1.964, 2.561]$. Note that $\overline{\alpha}_1^{T-} \notin S_\tau^T$ and $\underline{\alpha}_1^{T-} \in S_\tau^T$, for $\tau \in [t_0, T)$. According to Theorem 7.5.1, the players would agree to the optimality principle of choosing a weight $\alpha_1^* = \underline{\alpha}_1^{T-} = 1.964$ throughout the game interval, and a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$ would result.

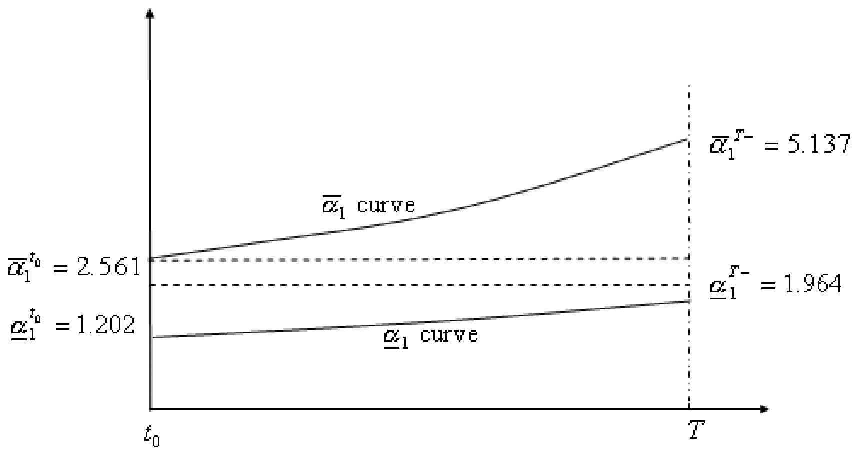

Fig. 7.2. Bounds of the values of α_1 in Example 7.5.2.

Example 7.5.3. Consider the cooperative game $\Gamma_c(x_0, T-t)$ with parameters: $a = 2$, $b = 0.5$, $\sigma = 0.01$, $k_1 = 5$, $k_2 = 1$, $c_1 = 0.5$, $c_2 = 3$, $q_1 = 1$, $q_2 = 0.5$, $T = 6$, $r = 0.05$, and $x_0 = 10$.

In Fig. 7.3, the curve $\overline{\alpha}_1$ and the curve $\underline{\alpha}_1$ are displayed. In particular, the set $S_{t_0}^T = \bigcap_{t_0 \le t < T} S_t$ is an empty set. Hence there does not exist any candidate for a subgame consistent solution for the game $\Gamma_c(x_0, T-t_0)$.

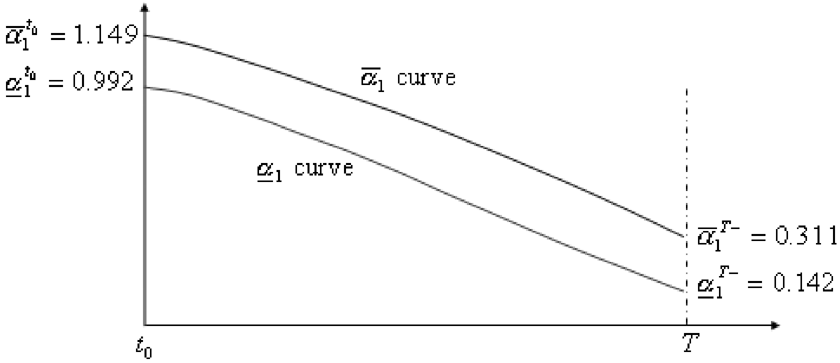

Fig. 7.3. Bounds of the values of α_1 in Example 7.5.3.

7.6 Infinite-Horizon Problems

In this section, we consider infinite-horizon cooperative stochastic differential games with nontransferable payoffs.

7.6.1 Noncooperative Equilibria and Pareto Optimal Trajectories

Consider the two-person nonzero-sum differential game with objective

$$E_{t_0}\left\{\int_{t_0}^{\infty} g^i\left[x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]\exp\left[-r\left(s-t_0\right)\right]ds\right\},$$
$$\text{for}\;\; i\in\{1,2\} \tag{7.32}$$

and state dynamics

$$dx\left(s\right)=f\left[x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]ds+\sigma\left[x\left(s\right)\right]dz\left(s\right),\quad x\left(t_0\right)=x_0, \tag{7.33}$$

where $\sigma\left[x\left(s\right)\right]$ is a $m\times\Theta$ matrix and $z\left(s\right)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega\left[x\left(s\right)\right]=\sigma\left[x\left(s\right)\right]\sigma\left[x\left(s\right)\right]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}\left[x\left(s\right)\right]$.

Since s does not appear in $g^i\left[x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]$ and the state dynamics, the game (7.32) – (7.33) is an autonomous problem. Consider the alternative game $\Gamma\left(x\right)$:

$$\max_{u_i} E_t\left\{\int_{t}^{\infty} g^i\left[x\left(s\right),u_1\left(s\right),u_2\left(s\right)\right]\exp\left[-r\left(s-t\right)\right]ds\right\},$$
$$\text{for}\;\; i\in\{1,2\} \tag{7.34}$$

subject to

$$dx(s) = f[x(s), u_1(s), u_2(s)]\,ds + \sigma[x(s)]\,dz(s), \quad x(t) = x, \tag{7.35}$$

The infinite-horizon autonomous problem $\Gamma(x)$ is independent of the choice of t and dependent only upon the state at the starting time x. Invoking Theorem 2.7.1, we obtain

Theorem 7.6.1. *A set of strategies $\{\phi_1^*(x), \phi_2^*(x)\}$ constitutes a feedback Nash equilibrium solution to the game $\Gamma(x)$, if there exist functionals $V^1(x) : R^m \to R$ and $V^2(x) : R^m \to R$, satisfying the following set of partial differential equations:*

$$rV^i(x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x)\, V^i_{x^h x^\zeta}(x) =$$
$$\max_{u_i}\left\{g^i\left[x, u_i, \phi_j^*(x)\right] + V^i_x(x)\, f\left[x, u_i, \phi_j^*(x)\right]\right\},$$
$$i \in \{1,2\} \quad \text{and} \quad j \in \{1,2\} \quad \text{and} \quad j \neq i.$$

In particular,

$$V^i(x) = E_t\left\{\int_t^\infty g^i[x(s), \phi_1^*(s), \phi_2^*(s)]\exp[-r(s-t)]\,ds \,\middle|\, x(t) = x\right\}$$

represents the current-value payoffs of Player i at current time $t \in [t_0, \infty]$, given that the state is x at t.

Now consider the case when the players agree to cooperatively. Let $\Gamma_c(x)$ denote a cooperative game with the game structure of $\Gamma(x)$ with the initial state being x. The players agree to act according to an agreed upon optimality principle. Pareto optimal outcomes for $\Gamma_c(x)$ can be identified by choosing a weight $\alpha_1 \in (0, \infty)$ that solves the following control problem (See Yeung and Petrosyan (2005)):

$$\max_{u_1,u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\} \equiv$$
$$\max_{u_1,u_2} E_t\left\{\int_t^T \left(g^1[x(s), u_1(s), u_2(s)]\right.\right.$$
$$\left.\left. + \alpha_1 g^2[x(s), u_1(s), u_2(s)]\right)\exp[-r(s-t)]\,ds\right\}, \tag{7.36}$$

subject to (7.35).

Using Theorem 2.1.6, we obtain:

Theorem 7.6.2. *A set of controls $[\psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)]$ provides an optimal solution to the stochastic control problem $\max_{u_1,u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ if there exists continuously differentiable function $W^{\alpha_1}(x) : R^m \to R$ satisfying the following partial differential equation:*

$$-W_t^{\alpha_1}(x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x)\, W_{x^h x^\zeta}^{\alpha_1}(x) =$$

$$\max_{u_1,u_2}\left\{\left(g^1[t,x,u_1,u_2] + \alpha_1 g^2[t,x,u_1,u_2]\right) + W_x^{\alpha_1} f[t,x,u_1,u_2]\right\}.$$

Substituting $\psi_1^{\alpha_1}(x)$ and $\psi_2^{\alpha_1}(x)$ into (7.33) yields the dynamics of the Pareto optimal trajectory for the game (7.32)–(7.33) associated with weight α_1:

$$dx(s) = f\left[x(s), \psi_1^{\alpha_1}(x(s)), \psi_2^{\alpha_1}(x(s))\right] ds + \sigma[x(s)]\, dz(s),$$
$$x(t_0) = x_0. \tag{7.37}$$

The solution to (7.37), denoted by $x^{\alpha_1^0}(t)$, can be expressed as:

$$x^{\alpha_1}(t) = x_0 + \int_{t_0}^{t} f\left[x^{\alpha_1}(s), \psi_1^{\alpha_1}(x^{\alpha_1}(s)), \psi_2^{\alpha_1}(x^{\alpha_1}(s))\right] ds$$
$$+ \int_{t_0}^{t} \sigma[x^{\alpha_1}]\, dz(s). \tag{7.38}$$

We use $X_t^{\alpha_1}$ to denote the set of realizable values of $x^{\alpha_1}(t)$ at time t generated by (7.38). The term $x_t^{\alpha_1}$ is used to denote an element in $X_t^{\alpha_1}$.

For group optimality to be achievable, the cooperative controls $[\psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)]$ must be adopted throughout the game interval.

Assume that at time $t\,(\geq t_0)$ when the initial state is $x_t^{\alpha_1} \in X_t^{\alpha_1}$, the expected payoff to Player i in current value can be expressed as:

$$\xi(x_t^{\alpha_1}) = E_t\left\{\int_t^{\infty} g^i\left[x(s), \psi_1^{\alpha_1}(x^{\alpha_1}(s)), \psi_2^{\alpha_1}(x^{\alpha_1}(s))\right]\right.$$
$$\left.\times \exp[-r(s-t)]\, ds \,\middle|\, x^{\alpha_1}(t) = x_t^{\alpha_1}\right\}$$
$$\text{for } i \in \{1,2\}. \tag{7.39}$$

Individual rationality requires that

$$\xi^i(x_t^{\alpha_1}) \geq V^i(x_t^{\alpha_1}), \qquad \text{for } i \in \{1,2\}. \tag{7.40}$$

The choice of cooperative weights α_1 must also satisfy (7.40).

Example 7.6.1. Consider the infinite-horizon game in which the state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = \left[a x(s)^{1/2} - b x(s) - u_1(s) - u_2(s)\right] ds + \sigma x(s)\, dz(s),$$
$$x(t_0) = x_0 \in X, \tag{7.41}$$

where $u_i \in U^i$ is the control vector of Player i, for $i \in \{1,2\}$,
a, b and σ are positive constants, and $z(s)$ is a Wiener process.

At time t_0, the expected payoff of player $i \in \{1,2\}$ is:

$$E_{t_0}\left\{ \int_{t_0}^{\infty} \left[[k_i u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s-t_0)]\, ds \,\middle|\, x(t_0) = x_0 \right\}, \tag{7.42}$$

where k_1, k_2, c_1 and c_2 are positive constants.

Consider the alternative game $\Gamma(x)$ in which the Player i maximizes

$$E_t\left\{ \int_t^{\infty} \left[[k_i u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s-t)]\, ds \,\middle|\, x(t) = x \right\},$$

subject to

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s) \right] ds + \sigma x(s)\, dz(s), \qquad x(t) = x.$$

Invoking Theorem 7.6.1, a set of feedback strategies $[\phi_1^*(x), \phi_2^*(x)]$ provides a Nash equilibrium solution to the game $\Gamma(x)$, if there exist twice continuously differentiable functions $V^i(x) : R \to R$, $i \in \{1,2\}$, satisfying the following partial differential equations:

$$\begin{aligned} &rV_t^i(x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^i(x) = \\ &\quad \max_{u_i}\left\{ \left[[k_i u_i]^{1/2} - \frac{c_i}{x^{1/2}} u_i \right] + V_x^i(x) \left[ax^{1/2} - bx - u_i - \phi_j^*(x) \right] \right\}, \\ &\qquad \text{for } i \in \{1,2\},\ j \in \{1,2\} \text{ and } j \neq i. \end{aligned} \tag{7.43}$$

Performing the indicated maximization yields:

$$\phi_i^*(x) = \frac{k_i x}{4\left[c_i + x^{1/2} V_x^i(x) \right]^2}, \qquad \text{for } i \in \{1,2\}. \tag{7.44}$$

Upon substitution of $\phi_i^*(x)$ from (7.44) into (7.43), one can obtain the value function of Player i in the game $\Gamma(x)$ as:

$$V^i(x) = \left[A_i x^{1/2} + B_i \right], \quad \text{for } i \in \{1,2\}, \tag{7.45}$$

where A_i, B_i, A_j and B_j, for $i \in \{1,2\}$, $j \in \{1,2\}$ and $j \neq i$, satisfy:

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right] A_i - \frac{k_i}{4[c_i + A_i/2]} + \frac{A_i k_j}{8[c_j + A_j/2]^2}, \text{ and}$$

$$B_i = \frac{a}{2r} A_i.$$

Consider the case when the players agree to cooperative and negotiate to establish an agreement (optimality principle) on how to play the cooperative game and hence how to distribute the resulting payoff. A necessary condition is that this optimality principle must satisfy group rationality and individual rationality.

Let $\Gamma_c(x)$ denote a cooperative game with game structure $\Gamma_c(x)$. Pareto optimal trajectories for $\Gamma_c(x)$ can be identified by choosing specific weights $\alpha_1 \in (0,\infty)$ that solves the infinite-horizon stochastic control problem:

$$\max_{u_1,u_2} \left\{ J^1(x) + \alpha_1 J^2(x) \right\} \equiv$$
$$\max_{u_1,u_2} E_t \left\{ \int_t^T \left(\left[[k_1 u_1(s)]^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] \right.\right.$$
$$\left.\left. + \alpha_1 \left[[k_2 u_2(s)]^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \right) \exp[-r(s-t)]\, ds \,\middle|\, x(t_0) = x \right\} \tag{7.46}$$

subject to dynamics (7.41).

Let $[\psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)]$ denote a set of controls that provides a solution to the stochastic control problem $\max_{u_1,u_2} \left\{ J^1(x) + \alpha_1 J^2(x) \right\}$. Invoking Theorem 7.6.2, the value function $W(x) \times R^n \to R$ has to satisfy the equations:

$$-W_t^{\alpha_1}(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}^{\alpha_1}(x) =$$
$$\max_{u_1,u_2} \left\{ \left(\left[[k_1 u_1]^{1/2} - \frac{c_1}{x^{1/2}} u_1 \right] + \alpha_1 \left[[k_2 u_2]^{1/2} - \frac{c_2}{x^{1/2}} u_2 \right] \right) \right.$$
$$\left. + W_x^{\alpha_1}(x) \left[a x^{1/2} - bx - u_1 - u_2 \right] \right\}. \tag{7.47}$$

Performing the indicated maximization in (7.47) yields:

$$\psi_1^{\alpha_1}(x) = \frac{k_1 x}{4\left[c_1 + x^{1/2} W_x^{\alpha_1}(x)\right]^2} \quad \text{and}$$
$$\psi_2^{\alpha_1}(x) = \frac{k_2 x}{4\left[c_2 + x^{1/2} W_x^{\alpha_1}(x)/\alpha_1\right]^2}. \tag{7.48}$$

Substituting $\psi_1^{\alpha_1}(x)$ and $\psi_2^{\alpha_1}(x)$ from (7.48) into (7.47) yields the value function

$$W^{\alpha_1}(x) = \left[A^{\alpha_1} x^{1/2} + B^{\alpha_1} \right], \tag{7.49}$$

where A^{α_1} and B^{α_1} satisfy:

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2} \right] A^{\alpha_1} - \frac{k_1}{4\left[c_1 + A^{\alpha_1}/2\right]} - \frac{\alpha_1 k_2}{4\left[c_2 + A^{\alpha_1}(t)/2\alpha_1\right]}, \text{ and}$$
$$B^{\alpha_1} = \frac{a}{2r} A^{\alpha_1}.$$

Substituting the partial derivative $W_x^{\alpha_1}(x)$ from (7.49) into $\psi_1^{\alpha_1}(x)$ and $\psi_2^{\alpha_1}(x)$ one obtains the controls of the problem $\max_{u_1,u_2} \left\{ J^1(x) + \alpha_1 J^2(x) \right\}$.

Substituting these controls into (7.41) yields the dynamics of the Pareto optimal trajectory of the game $\Gamma_c(x)$ associated with a weight α_1 as:

$$dx(s) = \left[ax(s)^{1/2} - \left(b + \frac{k_1}{4\left[c_1 + A^{\alpha_1}/2\right]^2} + \frac{k_2}{4\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}\right)x(s)\right]ds$$
$$+\sigma x(s)\,dz(s), \quad x(t_0) = x_0 \in X. \tag{7.50}$$

Solving (7.50) yields the Pareto optimal trajectory associated with weight α_1 as:

$$x^{\alpha_1}(t) = \left\{\Phi(t,t_0)\left[x_0^{1/2} + \int_{t_0}^{t} \Phi^{-1}(s,t_0)\,\frac{a}{2}ds\right]\right\}^2, \tag{7.51}$$

where

$$\Phi(t,t_0) =$$
$$\exp\left[\int_{t_0}^{t}\left(\frac{-b}{2} - \frac{k_1}{8\left[c_1 + A^{\alpha_1}(s)/2\right]^2} - \frac{k_2}{8\left[c_2 + A^{\alpha_1}(s)/2\alpha_1\right]^2} - \frac{3\sigma^2}{8}\right)ds\right.$$
$$\left.+\int_{t_0}^{t}\frac{\sigma}{2}dz(s)\right].$$

We use $X_t^{\alpha_1}$ to denote the set of realizable values of $x^{\alpha_1}(t)$ at time t generated by (7.51). We denote an element in $X_t^{\alpha_1}$ by $x_t^{\alpha_1}$.

7.6.2 Individual Expected Payoffs

In order to verify individual rationality in an infinite-horizon cooperative scheme, we have to derive the players' expected payoff functions under cooperation. To do this, we substitute the controls $\psi_1^{\alpha_1}(x)$ and $\psi_2^{\alpha_1}(x)$ satisfying Theorem 7.6.2 along the optimal trajectory $\{x^{\alpha_1}(t)\}_{t=t_0}^{\infty}$ from (7.38) into (7.32) to derive the players' expected payoff in $\Gamma_c(x_0)$ with α_1 being chosen as the cooperative weight. We then define:

$$\hat{W}^{\alpha_1(t_0)i}(t,x_t^{\alpha_1}) =$$
$$E_{t_0}\left\{\int_t^{\infty} g^i\left[s, x^{\alpha_1}(s), \psi_1^{\alpha_1}(x^{\alpha_1}(s)), \psi_2^{\alpha_1}(x^{\alpha_1}(s))\right]\right.$$
$$\left.\times\ \exp\left[-r(s-t_0)\right]ds\right| x(t) = x_t^{\alpha_1}\Big\},$$
$$\text{for } i \in \{1,2\} \text{ and } x_t^{\alpha_1} \in X_t^{\alpha_1},$$

where

$$dx^{\alpha_1}(s) = f\left[x^{\alpha_1}(s), \psi_1^{\alpha_1}(x^{\alpha_1}(s)), \psi_2^{\alpha_1}(x^{\alpha_1}(s))\right]ds + \sigma\left[x^{\alpha_1}(s)\right]dz(s),$$
$$x^{\alpha_1}(t) = x_t^{\alpha_1}.$$

Then we define

$$\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) = E_t\left\{\int_t^{\infty} g^i\left[s, x^{\alpha_1}(s), \psi_1^{\alpha_1}\left(x^{\alpha_1}(s)\right), \psi_2^{\alpha_1}\left(x^{\alpha_1}(s)\right)\right] \times \exp\left[-r(s-t)\right] ds \middle| x(t) = x_t^{\alpha_1}\right\},$$

$$\text{for } i \in \{1,2\} \text{ and } x_t^{\alpha_1} \in X_t^{\alpha_1}.$$

Note that $\hat{W}^{\alpha_1(t_0)i}\left(t, x_t^{\alpha_1}\right) = \exp\left[-r(t-t_0)\right]\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right)$, and hence

$$\begin{aligned}
&\hat{W}_t^{\alpha_1(t_0)i}\left(t, x_t^{\alpha_1}\right) = -r\exp\left[-r(t-t_0)\right]\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right),\\
&\hat{W}_x^{\alpha_1(t_0)i}\left(t, x_t^{\alpha_1}\right) = \exp\left[-r(t-t_0)\right]\hat{W}_x^{(\alpha_1)i}\left(x_t^{\alpha_1}\right), \text{ and}\\
&\hat{W}_{xx}^{\alpha_1(t_0)i}\left(t, x_t^{\alpha_1}\right) = \exp\left[-r(t-t_0)\right]\hat{W}_{xx}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right).
\end{aligned} \tag{7.52}$$

Substituting the results from (7.52) into Theorem 7.3.1 yields

Theorem 7.6.3. *If there exist continuously differentiable functions $\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) : R^m \to R$, $i \in \{1,2\}$, satisfying*

$$r\hat{W}_t^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}\left(x_t^{\alpha_1}\right)\hat{W}_{x^h x^\zeta}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) = g^i\left[x_t^{\alpha_1}, \psi_1^{\alpha_1}\left(x_t^{\alpha_1}\right), \psi_2^{\alpha_1}\left(x_t^{\alpha_1}\right)\right] + \hat{W}_x^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) f\left[x_t^{\alpha_1}, \psi_1^{\alpha_1}\left(x_t^{\alpha_1}\right), \psi_2^{\alpha_1}\left(x_t^{\alpha_1}\right)\right],$$

$$\textit{for } i \in \{1,2\},$$

then $\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right)$ gives Player i's expected cooperative payoff over the interval $[t,\infty)$ with α_1 being the cooperative weight and $x(t) = x_t^{\alpha_1}$.

To maintain individual rationality throughout the game, the chosen α_1 has to satisfy

$$\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) \geq V^i\left(x_t^{\alpha_1}\right), \quad \text{for } i \in \{1,2\} \text{ and all } t \in [t_0,\infty). \tag{7.53}$$

Example 7.6.2. Consider the infinite-horizon game in Example 7.6.1 in which a cooperative weight α_1 is chosen. By substituting cooperative controls $\left[\psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)\right]$ and the optimal trajectory in Example 7.6.1 into (7.42) we obtain the expected payoff of Player 1 under cooperation as:

$$\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) = E_t\left\{\int_t^T \left[\frac{k_1 x^{\alpha_1}(s)^{1/2}}{2\left[c_1 + A^{\alpha_1}/2\right]} - \frac{c_1 k_1 x^{\alpha_1}(s)^{1/2}}{4\left[c_1 + A^{\alpha_1}/2\right]^2}\right] \times \exp\left[-r(s-t)\right] ds \middle| x(t) = x_t^{\alpha_1}\right\};$$

and the corresponding expected payoff of Player 2 under cooperation as:

$$\hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) = E_t\left\{\int_t^T \left[\frac{k_2 x^{\alpha_1}(s)^{1/2}}{2\left[c_2 + A^{\alpha_1}/2\alpha_1\right]} - \frac{c_2 k_2 x^{\alpha_1}(s)^{1/2}}{4\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}\right] \times \exp\left[-r(s-t)\right] ds \,\middle|\, x(t) = x_t^{\alpha_1}\right\}$$

where

$$dx^{\alpha_1}(s) = \left[a x^{\alpha_1}(s)^{1/2} - \left(b + \frac{k_1}{4\left[c_1 + A^{\alpha_1}/2\right]^2} + \frac{k_2}{4\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}\right) x^{\alpha_1^0}(s)\right] ds + \sigma x^{\alpha_1}(s)\, dz(s), \qquad x(t) = x_t^{\alpha_1} \in X_t^{\alpha_1}.$$

Invoking Theorem 7.6.3, we know if there exist continuously differentiable functions $\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) : R^m \to R$, $i \in \{1, 2\}$, satisfying

$$r\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) - \frac{1}{2}\hat{W}_{xx}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right)\sigma^2\left(x_t^{\alpha_1}\right)^2 = \left[\frac{k_1\left(x_t^{\alpha_1}\right)^{1/2}}{2\left[c_1 + A^{\alpha_1}/2\right]} - \frac{c_1 k_1\left(x_t^{\alpha_1}\right)^{1/2}}{4\left[c_1 + A^{\alpha_1}/2\right]^2}\right] + \hat{W}_x^{(\alpha_1)1}\left(x_t^{\alpha_1}\right)\left[a\left(x_t^{\alpha_1}\right)^{1/2} - b x_t^{\alpha_1} - \frac{k_1 x_t^{\alpha_1}}{4\left[c_1 + A^{\alpha_1}(t)/2\right]^2} - \frac{k_2 x_t^{\alpha_1}}{4\left[c_2 + A^{\alpha_1}(t)/2\alpha_1\right]^2}\right],$$

and

$$r\hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) - \frac{1}{2}\hat{W}_{xx}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)\sigma^2\left(x_t^{\alpha_1}\right)^2 = \left[\frac{k_2\left(x_t^{\alpha_1}\right)^{1/2}}{2\left[c_2 + A^{\alpha_1}/2\alpha_1^0\right]} - \frac{c_2 k_2\left(x_t^{\alpha_1}\right)^{1/2}}{4\left[c_2 + A^{\alpha_1}/2\alpha_1^0\right]^2}\right] + \hat{W}_x^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)\left[a\left(x_t^{\alpha_1}\right)^{1/2} - b x_t^{\alpha_1} - \frac{k_1 x_t^{\alpha_1}}{4\left[c_1 + A^{\alpha_1}(t)/2\right]^2} - \frac{k_2 x_t^{\alpha_1}}{4\left[c_2 + A^{\alpha_1}(t)/2\alpha_1\right]^2}\right],$$

$$\text{for } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}, \tag{7.54}$$

then $\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right)$ gives Player i's expected cooperative payoff with α_1 being the cooperative weight.

Proposition 7.6.1. *The function $\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right): R \to R$ satisfying (7.54) can be solved as:*

$$\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) = \left[\hat{A}_1^{\alpha_1}\left(x_t^{\alpha_1}\right)^{1/2} + \hat{B}_1^{\alpha_1}\right], \tag{7.55}$$

where

$$0 = \left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]\hat{A}_1^{\alpha_1} - \frac{k_1}{2\left[c_1 + A^{\alpha_1}/2\right]} + \frac{c_1 k_1}{4\left[c_1 + A^{\alpha_1}/2\right]^2}$$
$$+ \frac{\hat{A}_1^{\alpha_1} k_1}{8\left[c_1 + A^{\alpha_1}/2\right]^2} + \frac{\hat{A}_1^{\alpha_1} k_2}{8\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}, \text{ and}$$
$$\hat{B}_1^{\alpha_1} = \frac{a}{2r}\hat{A}_1^{\alpha_1^0}.$$

Proof. *Upon calculating the derivatives $\hat{W}_t^{(\alpha_1)1}\left(x_t^{\alpha_1}\right)$, $\hat{W}_{x_t}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right)$ and $\hat{W}_{x_t x_t}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right)$ from (7.55) and then substituting them into (7.54) yield Proposition 7.6.1.*

Proposition 7.6.2. *The function $\hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right): R \to R$ satisfying (7.54) can be solved as:*

$$\hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) = \left[\hat{A}_2^{\alpha_1}\left(x_t^{\alpha_1}\right)^{1/2} + \hat{B}_2^{\alpha_1}\right], \tag{7.56}$$

where

$$0 = \left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]\hat{A}_2^{\alpha_1} - \frac{k_2}{2\left[c_2 + A^{\alpha_1}/2\alpha_1\right]} + \frac{c_2 k_2}{4\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}$$
$$+ \frac{\hat{A}_2^{\alpha_1} k_1}{8\left[c_1 + A^{\alpha_1}/2\right]^2} + \frac{\hat{A}_2^{\alpha_1} k_2}{8\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2},$$
$$\hat{B}_2^{\alpha_1} = \frac{a}{2r}\hat{A}_2^{\alpha_1}.$$

Proof. *Upon calculating the derivatives $\hat{W}_t^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)$, $\hat{W}_{x_t}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)$ and $\hat{W}_{x_t x_t}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)$ from (7.56) and then substituting them into (7.54) yield Proposition 7.6.2.*

7.6.3 Subgame Consistent Solutions

The principle of subgame consistency ensures that an extension of a cooperative solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. In addition, the two essential properties of cooperation – Pareto optimality and individual rationality – are required to be present. In the present framework, a subgame consistent solution to the infinite-horizon nontransferable payoffs game (7.34) – (7.35) requires:

Condition 7.6.1.

(i) The imputation vector

$$\left[\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right), \hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)\right]$$

is Pareto optimal;

(ii)

$$\hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right) \geq V^i\left(x_t^{\alpha_1}\right),$$
$$\textit{for } i \in \{1,2\},\ t \in [t_0, \infty) \ \ \textit{and any } x_t^{\alpha_1} \in X_t^{\alpha_1};\ \textit{and}$$

(iii)

$$\hat{W}^{\alpha_1(t_0)i}\left(t_0, x_t^{\alpha_1}\right) \exp\left[r\left(t - t_0\right)\right] = \hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right),$$
$$\textit{for } i \in \{1,2\},\ t \in [t_0, \infty) \ \ \textit{and any } x_t^{\alpha_1} \in X_t^{\alpha_1}.$$

Part (i) of Condition 7.6.1 guarantees Pareto optimality. Part (ii) ensures that individual rationality is satisfied at a later starting time and any feasible state brought about by prior optimal behaviors. Part (iii) guarantees the subgame consistency of the solution imputations throughout the game interval.

Theorem 7.6.4. *A solution optimality principle under which the players agree to choose the same weight α_1 in all the games $\Gamma_c\left(x_t^{\alpha_1}\right)$ such that*

$$\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) \geq V^1\left(x_t^{\alpha_1}\right) \ \ \textit{and } \hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) \geq V^2\left(x_t^{\alpha_1}\right),$$
$$\textit{for any } x_t^{\alpha_1} \in X_t^{\alpha_1},$$

yields a subgame consistent solution to the cooperative game (7.34) – (7.35).

Proof. Given that the same weight α_1 will be chosen for all the subgames $\Gamma_c\left(x_t^{\alpha_1}\right)$, the cooperative control $\left[\psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)\right]$ will be adopted throughout time interval. Group optimality is assured and the imputation vector $\left[\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right), \hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right)\right]$ is indeed Pareto optimal. Hence part (i) of Condition 7.6.1 is satisfied.
With

$$\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) \geq V^1\left(x_t^{\alpha_1}\right) \ \ \textit{and } \hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) \geq V^2\left(x_t^{\alpha_1}\right),$$
$$\textit{for any } x_t^{\alpha_1} \in X_t^{\alpha_1},$$

individual rationality is fulfilled throughout the game horizon given any feasible state brought about by prior optimal behaviors. Part (ii) of Condition 7.6.1 is satisfied.

Moreover, from (7.52), we have

$$\hat{W}^{\alpha_1(t_0)i}\left(t_0, x_t^{\alpha_1}\right) \exp\left[r\left(t - t_0\right)\right] = \hat{W}^{(\alpha_1)i}\left(x_t^{\alpha_1}\right),$$
$$\textit{for } i \in \{1,2\},\ t \in [t_0, \infty) \ \ \textit{and any } x_t^{\alpha_1} \in X_t^{\alpha_1}.$$

Part (iii) of Condition 7.4.1 is satisfied.

7.7 An Infinite-Horizon Application

In this section, we propose subgame consistent solutions to the infinite-horizon cooperative game in Example 7.6.1.

Invoking Theorem 7.6.1, a subgame consistent solution requires

$$\hat{W}^{(\alpha_1)1}\left(x_t^{\alpha_1}\right) \geq V^1\left(x_t^{\alpha_1}\right) \text{ and } \hat{W}^{(\alpha_1)2}\left(x_t^{\alpha_1}\right) \geq V^2\left(x_t^{\alpha_1}\right), \quad \text{for any } x_t^{\alpha_1} \in X_t^{\alpha_1}. \tag{7.57}$$

Condition (7.57) implies that the choice of α_1 has to satisfy:

$$\left[\hat{A}_i^{\alpha_1}\left(x_t^{\alpha_1}\right)^{1/2} + \hat{B}_i^{\alpha_1}\right] \geq \left[A_i\left(x_t^{\alpha_1}\right)^{1/2} + B_i\right] \quad \text{for } i \in \{1,2\} \text{ and any } x_t^{\alpha_1^0} \in X_t^{\alpha_1^0}. \tag{7.58}$$

Note that from Proposition 7.6.2 one can obtain $\hat{B}_2^{\alpha_1} = (a/2r)\hat{A}_2^{\alpha_1}$, and from (7.45) one obtains $B_i = (a/2r)A_i$. Therefore if $\hat{A}_i^{\alpha_1} \geq A_i$ then $\hat{B}_i^{\alpha_1} \geq B_i$, for $i \in \{1,2\}$.

Therefore for (7.58) to hold it is necessary that

$$\hat{A}_i^{\alpha_1} \geq A_i, \qquad \text{for } i \in \{1,2\}. \tag{7.59}$$

Since the solution to the control problem $\max_{u_1,u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ yields a Pareto optimal outcome there exist some α_1 such that $\hat{A}_i^{\alpha_1} \geq A_i$, for $i \in \{1,2\}$.

Proposition 7.7.1. *$d\hat{A}_1^{\alpha_1}/d\alpha_1 < 0$ and $d\hat{A}_2^{\alpha_1}/d\alpha_1 > 0$.*

Proof. See Appendix.

Let S^∞ denote the set of α_1 such that $\hat{A}_i^{\alpha_1} \geq \overline{A}_i$, for $i \in \{1,2\}$. Using Proposition 7.7.1, one can readily show that

Condition 7.7.1. *There exist a continuous and bounded set $S^\infty = [\underline{\alpha}_1, \overline{\alpha}_1]$ where $\underline{\alpha}_1$ is the lowest value of α_1 in S^∞, and $\overline{\alpha}_1$ the highest. Moreover, $\hat{A}_1^{\overline{\alpha}_1} = A_1$ and $\hat{A}_2^{\underline{\alpha}_1} = A_2$.*

Theorem 7.7.1. *An optimality principle under which the players agree to choose the weight*

$$\alpha_1^* = \left(\underline{\alpha}_1\right)^{0.5}\left(\overline{\alpha}_1\right)^{0.5} \tag{7.60}$$

yields a subgame consistent solution to the cooperative game (7.34) – (7.35).

Proof. According to the optimality principle in Theorem 7.7.1 a unique weight $\alpha_1^* = \left(\underline{\alpha}_1\right)^{1/2}\left(\overline{\alpha}_1\right)^{1/2}$ is chosen for any game $\Gamma\left(x_t^{\alpha_1^*}\right)$, the imputation vector $\left[\hat{W}^{(\alpha_1^*)1}\left(x_t^{\alpha_1^*}\right), \hat{W}^{(\alpha_1^*)2}\left(x_t^{\alpha_1^*}\right)\right]$ yields a Pareto optimal pair. Hence part (i) of Condition 7.6.1 is satisfied.

Since $(\underline{\alpha}_1)^{1/2}(\overline{\alpha}_1)^{1/2} \in S^\infty$, $\hat{A}_i^{\alpha_1} \geq \overline{A}_i$, *for* $i \in \{1,2\}$. *Hence individual rationality as required by part (ii) of Condition 7.6.1 is satisfied.*

Finally, from (7.52) one obtains

$$\hat{W}^{\alpha_1(t_0)i}(t, x_t^{\alpha_1}) = \exp[-r(t-t_0)]\,\hat{W}^{(\alpha_1)i}(x_t^{\alpha_1}),$$

and hence part (iii) of Condition 7.6.1 holds.

Therefore Theorem 7.7.1 follows.

Remark 7.7.1. The Pareto optimal cooperative solution proposed in Theorem 7.7.1 invoking the geometric mean $(\underline{\alpha}_1)^{1/2}(\overline{\alpha}_1)^{1/2}$ satisfies the axioms of symmetry. In particular, the symmetry of the geometric mean gives rise to a symmetrical solution in the sense that the switching of the labels of the players (that is labeling Player 1 as Player 2, and vice versa) leaves the solution unchanged.

Example 7.7.1. Consider the infinite-horizon stochastic differential game in Example 7.6.1 with the following parameter specifications: $a = 0.4$, $b = 0.1$, $\sigma = 0.05$, $k_1 = 4$, $k_2 = 1$, $c_1 = 1$, $c_2 = 1.5$, $q_1 = 2$, $q_2 = 1.2$, and $r = 0.04$.

We obtain $S^\infty = [\underline{\alpha}_1, \overline{\alpha}_1] = [1.286, 2.831]$ and $(\underline{\alpha}_1)^{1/2}(\overline{\alpha}_1)^{1/2} = 1.908$.

If significant difference in bargaining powers exists between the players, asymmetric solutions like the following may arise.

Theorem 7.7.2. *An optimality principle under which the players agree to choose the bargaining weight equaling the weighted geometric mean of* $\overline{\alpha}_1$ *and* $\underline{\alpha}_1$, *that is*

$$\alpha_1^* = (\underline{\alpha}_1)^\theta (\overline{\alpha}_1)^{1-\theta}, \quad \text{for } 0 \leq \theta \leq 1, \tag{7.61}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x, \infty)$.

Proof. Follow the proof of Theorem 7.7.1.

7.8 Appendix to Chapter 7

Proof of Proposition 7.7.1

Note that $W^{\alpha_1}(x_t^{\alpha_1}) = \hat{W}^{\alpha_1(1)}(x_t^{\alpha_1}) + \alpha_1 \hat{W}^{\alpha_1(2)}(x_t^{\alpha_1})$, therefore from (7.49), (7.55) and (7.56), we have $A^{\alpha_1} = \hat{A}_1^{\alpha_1} + \alpha_1 \hat{A}_2^{\alpha_1}$. Since when $u_i = 0$, the integrand

$$\left[[k_i u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s) \right]$$

equal zero, the maximized value functions will never be negative. Hence

$$\hat{W}^{\alpha_1(1)}(x_t^{\alpha_1}) \geq 0 \text{ and } \hat{W}^{\alpha_1(2)}(x_t^{\alpha_1}) \geq 0.$$

This implies A^{α_1}, $\hat{A}_1^{\alpha_1}$ and $\hat{A}_2^{\alpha_1}$ are nonnegative.

Define the equation

$$\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]A^{\alpha_1}-\frac{k_1}{4\left[c_1+A^{\alpha_1}/2\right]}-\frac{\alpha_1 k_2}{4\left[c_2+A^{\alpha_1}/2\alpha_1\right]}=0.$$

associated with (7.49) as $\Psi\left(A^{\alpha_1},\alpha_1\right)=0$.

Implicitly differentiating $\Psi\left(A^{\alpha_1},\alpha_1\right)=0$ yields the effect of a change in α_1 on A^{α_1} as:

$$\begin{aligned}\frac{dA^{\alpha_1}}{d\alpha_1}&=\frac{k_2\left[c_2+A^{\alpha_1}/\alpha_1\right]}{4\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}\\&\quad\div\left\{\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+A^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}\right\}>0.\end{aligned} \tag{7.62}$$

Then we define the equation

$$\begin{aligned}&\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]\hat{A}_1^{\alpha_1}-\frac{k_1}{2\left[c_1+A^{\alpha_1}/2\right]}+\frac{c_1 k_1}{4\left[c_1+A^{\alpha_1}/2\right]^2}\\&\quad+\frac{\hat{A}_1^{\alpha_1}k_1}{8\left[c_1+A^{\alpha_1}/2\right]^2}+\frac{\hat{A}_1^{\alpha_1}k_2}{8\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}=0\end{aligned}$$

in Proposition 7.6.1 as $\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)=0$.

The effect of a change in α_1 on $\hat{A}_1^{\alpha_1}$ can be obtained as:

$$\frac{d\hat{A}_1^{\alpha_1}}{d\alpha_1}=-\frac{\partial\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)/\partial\alpha_1}{\partial\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)/\partial\hat{A}_1^{\alpha_1}}, \tag{7.63}$$

where

$$\frac{\partial\Psi^1}{\partial\hat{A}_1^{\alpha_1}}=\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+A^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}>0, \tag{7.64}$$

and

$$\frac{\partial\Psi^1}{\partial\alpha_1}=\frac{k_1\left[A^{\alpha_1}-\hat{A}_1^{\alpha_1}\right]}{8\left[c_1+A^{\alpha_1}/2\right]^3}\frac{dA^{\alpha_1}}{d\alpha_1}+\frac{\hat{A}_1^{\alpha_1}k_2/\alpha_1^2}{8\left[c_2+A^{\alpha_1}/2\alpha_1\right]^3}\left[A^{\alpha_1}-\alpha_1\frac{dA^{\alpha_1}}{d\alpha_1}\right]. \tag{7.65}$$

From Proposition 7.6.2, we can express:

$$\begin{aligned}\hat{A}_2^{\alpha_1}&=\frac{k_2\left[c_2+A^{\alpha_1}/\alpha_1\right]}{4\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}\\&\quad\div\left\{\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+A^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+A^{\alpha_1}/2\alpha_1\right]^2}\right\}.\end{aligned} \tag{7.66}$$

Comparing (7.66) with (7.62) shows that $dA^{\alpha_1}/d\alpha_1 = \hat{A}_2^{\alpha_1}$. Upon substituting $dA^{\alpha_1}/d\alpha_1$ by $\hat{A}_2^{\alpha_1}$ into (7.65), and invoking the relation $A^{\alpha_1} = \hat{A}_1^{\alpha_1} + \alpha_1 \hat{A}_2^{\alpha_1}$, we can express (7.65) as

$$\frac{\partial \Psi^1}{\partial \alpha_1} = \frac{k_1 \alpha_1 \left(\hat{A}_2^{\alpha_1}\right)^2}{8\left[c_1 + A^{\alpha_1}/2\right]^3} + \frac{\left(\hat{A}_1^{\alpha_1}\right)^2 k_2/\alpha_1^2}{8\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^3} > 0. \tag{7.67}$$

Using (7.64) and (7.67), we have

$$\frac{d\hat{A}_1^{\alpha_1}}{d\alpha_1} = -\frac{\partial \Psi^1\left(\hat{A}_1^{\alpha_1}, \alpha_1\right)/\partial \alpha_1}{\partial \Psi^1\left(\hat{A}_1^{\alpha_1}, \alpha_1\right)/\partial \hat{A}_1^{\alpha_1}} < 0.$$

Following the above analysis, we have:

$$\begin{aligned} \frac{d\hat{A}_2^{\alpha_1}}{d\alpha_1} = &\left\{\frac{k_2\left(\hat{A}_1^{\alpha_1}\right)^2/\alpha_1^3}{8\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^3} + \frac{\left(\hat{A}_2^{\alpha_1}\right)^2 k_1}{8\left[c_1 + A^{\alpha_1}/2\right]^3}\right\} \\ &\div \left\{\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] + \frac{k_1}{8\left[c_1 + A^{\alpha_1}/2\right]^2} + \frac{k_2}{8\left[c_2 + A^{\alpha_1}/2\alpha_1\right]^2}\right\} > 0. \end{aligned} \tag{7.68}$$

Hence Proposition 7.7.1 follows.

7.9 Problems

Problem 7.1. Consider a 2-person game in which the expected payoffs of the players at time t_0 are:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[u_1(s) - \frac{c_1}{x(s)}u_1^2(s)\right]\exp\left[-r(t-t_0)\right]ds \right.$$
$$\left. + \exp\left[-r(T-t_0)\right]qx(T)\right\},$$

and

$$E_{t_0}\left\{\int_{t_0}^{T}\left[u_2(s) - \frac{c_2}{x(s)}u_2^2(s)\right]\exp\left[-r(t-t_0)\right]ds \right.$$
$$\left. + \exp\left[-r(T-t_0)\right]qx(T)\right\},$$

where q, c_1 and c_2 are constants and r is the interest rate. $u_1(s)$ is the control of Player 1 and $u_2(s)$ is the control of Player 2. Payoffs are not transferable. The state dynamics $x(s) \in X \subset R$ follows the stochastic dynamics:

$$dx(s) = [a - bx(s) - u_1(s) - u_2(s)]\,ds + \sigma x(s)\,dz(s), \qquad x(t_0) = x_0 \in X,$$

where a and b are constant, $z(s)$ is a Wiener process.

If the players agree to cooperate, describe the Pareto optimal outcomes and the corresponding optimal trajectories.

Problem 7.2. Derive individual player's payoff functions under cooperation along the optimal trajectory for the game in Problem 7.1.

Problem 7.3. Consider the infinite-horizon game in which Player 1 and Player 2 maximizes respectively:

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_1(s) - \frac{c_1}{x(s)}u_1^2(s)\right]\exp[-r(t-t_0)]\,ds\right\},$$

and

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u_2(s) - \frac{c_2}{x(s)}u_2^2(s)\right]\exp[-r(t-t_0)]\,ds\right\}.$$

The players' payoffs are transferable and the state variable $x(s) \in X \subset R$ follows the stochastic dynamics:

$$dx(s) = [a - bx(s) - u_1(s) - u_2(s)]\,ds + \sigma x(s)\,dz(s), \qquad x(t_0) = x_0 \in X.$$

(a) If the players agree to cooperate, describe the Pareto optimal outcomes and the corresponding optimal trajectories.

Problem 7.4. Derive individual player's expected payoff functions under cooperation along the optimal trajectory for the game in Problem 7.3.

Problem 7.5. Consider the game in Problem 7.3.

(a) Present a subgame consistent solution which satisfies the axiom of symmetry
(b) Compute the solution numerically if $a = 2$, $b = 0.1$, $r = 0.04$, $t_0 = 0$, $x_0 = 15$, $c_1 = 0.5$, $c_2 = 0.2$, $\sigma = 0.05$.

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