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***Highlights (for review)**

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- Optimal Control theory.
- Pontryagin's equations and Hamilton-Jacobi-Bellman equatic
- Dirac's constrained systems method.
- Quantum Mechanics.

The quantum dark side of the optimal control theory

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Abstract

In a recent article, a generic optimal control problem was studied from a physicist's point of view [43]. Through this optic, the Pontryagin equations are equivalent to the Hamilton equations of a classical constrained system. By quantizing this constrained system, using the right ordering of the operators, the corresponding quantum dynamics given by the Schrödinger equation is equivalent to that given by the Hamilton–Jacobi–Bellman equation of Bellman's theory. The conclusion drawn there were based on certain analogies between the equations of motion of both theories.

In this paper, a closer and more detailed examination of the quantization problem is carried out, by considering three possible quantization procedures: right quantization, left quantization, and Feynman's path integral approach. The Bellman theory turns out to be the classical limit $\hbar \to 0$ of these three different quantum theories. Also, the exact relation of the phase $S(x,t)$ of the wave function $\Psi(x,t) = e^{\frac{i}{\hbar}S(x,t)}$ of the quantum theory with Bellman's cost function $J_+(x,t)$ is obtained. In fact, $S(x,t)$ satisfies a 'conjugate' form of the Hamilton–Jacobi–Bellman equation, which implies that the cost functional $J_+(x,t)$ must necessarily satisfy the usual Hamilton–Jacobi–Bellman equation. Thus, the Bellman theory effectively corresponds to a quantum view of the optimal control problem.

Keywords: Optimal Control Theory; Pontryagin's equations; Hamilton–Jacobi–Bellman equation; Constrained systems; Dirac's method; Quantum mechanics.

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1 Introduction

In the last decades, physicists' methods and ideas have started to reach other areas not directly related to the physical world. An example is the area of Econophysics [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] and Sociophysics [11], [12].

Another example is path integrals, which from its introduction in physics by Feynman [13] has been widely used in physics. But now, its applications have extended to a different ambit, such as Econophysics, where its applications include, for example, path integral techniques applied to the study the Black–Scholes model in its different forms [3], [9] [14], [15], [16], [17], [18], stochastic processes [19], control theory [20], [21], time series analysis [22], data analysis [23], networks [24] and statistical mechanics [25], [26].

Quantum ideas have also permeated other areas, such as econophysics, in order to try to understand the Black–Scholes equation as a quantum mechanical Schrödinger system [3], [27], [28], [29].

In the recent few years, constrained systems techniques, which are the natural framework of the gauge theories of high energy physics, have been introduced into econophysics, through Dirac's method [30], [31], [32]. This kind of constrained analysis has been used to explain some features of stochastic volatility models [33], [34] and the multi-asset Black–Scholes equation [35] at the high correlation limit $\rho = \pm 1$.

On the other hand, dynamic optimization with its theory of optimal control is the foundation stone in economic analysis [36], [37], [38], [39], [40], [41]. Recently, these constrained methods, have been used to understand the structure of optimal control theory from a physicist perspective. As shown in [42], [43], a generic optimal control system associated to an economic model can be interpreted as a second class constrained physical system. At the classical level, the constrained dynamics given by Dirac's brackets is the same dynamics as given by the Pontryagin equations [44]. The right quantization of this second class constrained system resulted in a Schrödinger equation that by writing the wave function as $\Psi = e^{\frac{i}{\hbar}S}$, maps it to the Hamilton–Jacobi–Bellman equation of Bellman's theory. This is just amazing: the quantization of the Pontryagin theory gives Bellman's maximum principle. But, surprisingly, this quantum theory in terms of S does not depend on Planck's constant $\hbar$ at all, so it looks like more of a classical than a quantum theory. To understand this phenomenon we analyze other quantization proceduress, such as left quantization and the Feynman sum over histories description of Quantum Mechanics. In this last framework, the path integral that describes the quantum propagator can be computed, in some special cases, in a closed form. By taking the classical limit $\hbar \to 0$, this propagator reduces to a semi-classical one and this permits relating in a precise way the S function with Bellman's optimal cost function J_+. Thus, the Bellman theory is the classical limit of the fully quantum one.

The organization of this paper goes as follows: Section 2 summarizes the Pontryagin theory. Section 3 describes the optimal control problem from the physicist's perspective and Dirac's method is applied to understand the Pontryagin equations as a constrained problem in classical mechanics. In Section 4, the quantization of this classical constrained system is carried out by the operator

method of quantum mechanics. In Section 5, the quantum propagator is obtained as a constrained path integral and its semi-classical limit is taken. In Section 6, the relation between quantum mechanics and Bellman's theory is obtained, and the precise relation between the phase of the wave function $S(x,t)$ and Bellman's cost function $J_+(x,t)$ is derived. Lastly, in Section 7, the conclusions of this work are drawn.
There are three appendices which summarize, for the non-specialist reader, the basic background that is necessary to understand the paper. Appendix 1 review the Dirac's brackets. Appendix 2 takes the Feynman approach to quantum mechanics, and Appendix 3 review the semi-classical approximation and its related issues.

2 The control problem as a physical system

Consider an optimal control problem that is commonly used in financial applications (see, for example, [36]). We want to optimize the cost functional

$$A[x,u] = \int_{t_0}^{t_1} F(x,u,t)dt,$$

where u is a control variable and x represents a state variable subject to the equation

$$\dot{x} = f(x,u,t) \qquad x(t_0) = x_0. \tag{1}$$

The problem is to determine how to obtain the trajectory $x = x(t)$ and control path $u = u(t)$ that optimize the cost functional. Due to the restriction, one must use the Lagrange multipliers method to solve the problem, so instead one must optimize the functional

$$A[x,u,\lambda] = \int_{t_0}^{t_1} F(x,u,t) - \lambda\Big(\dot{x} - f(x,u,t)\Big)dt \tag{2}$$

with respect to the three variables $x(t), u(t), \lambda(t)$. The solution to this problem is given by Pontryagin's maximum principle, so that the optimal curves must satisfy the set of equations

$$\dot{x} = \frac{\partial H(x,u,\lambda,t)}{\partial \lambda} \tag{3}$$

$$\dot{\lambda} = -\frac{\partial H(x,u,\lambda,t)}{\partial x} \tag{4}$$

$$\frac{\partial H(x,u,\lambda,t)}{\partial u} = 0, \tag{5}$$

with

$$H(x,u,\lambda) = F(x,u,t) + \lambda f(x,u,t).$$

Equations (3), (4) and (5) are the well-known Pontryagin equations. Note that the first two, (3) and (4), are just Hamilton's equations of motion if one interprets the Lagrange multiplier λ as the canonical momentum p_x associated to the state variable x. In this way, by identifying λ with p_x, we can think of the Lagrangian (2) as the Lagrangian of a fictitious particle. With this identification, the action (2) written in terms of p_x is

$$A_1[x,u,\lambda] = \int_{t_0}^{t_1} \Big(H(x,p_x,t) - p_x\dot{x} \Big) dt \tag{6}$$

with

$$H(x,p_x,t) = F(x,u,t) + p_x f(x,u,t) \tag{7}$$

From the physicist's point of view, the Hamiltonian action in (6) has the wrong sign, so one must multiply the above action by -1 and consider instead the action

$$A_2[x,u,\lambda] = \int_{t_0}^{t_1} \Big(p_x\dot{x} - H(x,p_x,t) \Big) dt, \tag{8}$$

in fact this was the way taken in [43]. From the point of view of pure classical mechanics, A_1 and A_2 give the same classical equations of motions, so the Pontryagin equations emerge as a consequence of making A_1 or A_2 extremal. So, classically, the sign -1 is irrelevant. But quantum mechanically, the sign is important, because according to the Feynman scheme of quantum mechanics, the probability amplitude $\mathcal{A}$ of propagation between two points (x_0,t_0) and (x_1,t_1) is given by [13]

$$\mathcal{A} \sim \sum_{paths\ joining\ x_0,x_1} e^{\frac{i}{\hbar}A[x(t),p_x(t)]} \tag{9}$$

where A is the classical action evaluated over the path. Thus the relative sign of the action matters, since it could give different weights to the probability amplitude. So the signs are important in quantum mechanics. To fix the sign correctly to describe the control problem as a physical system, one must identify the momentum p_x with the Lagrangian multiplier λ not from the Pontryagin equations, but instead from the action A itself (which is the correct action for the control problem). Thus, if one identifies p_x with $-\lambda$ in (2), then A can be rewriten in terms of p_x as

$$A[x,u] = \int_{t_0}^{t_1} \Big(F + p_x\dot{x} - p_x f \Big) dt \tag{10}$$

or

$$A[x,u] = \int_{t_0}^{t_1} L(x,u,\dot{x},\dot{u},t) dt \tag{11}$$

with

$$L(x,u,\dot{x},\dot{u},t) = p_x\dot{x} - \Big(-F(x,u,t) + p_x f(x,u,t) \Big) \tag{12}$$

Note the sign change in the function F compared to the analysis given in [43]. Now, the starting point of this paper will be the action (11) with its Lagrangian (12). The object of this paper is to explore the quantum structure associated to this action in full detail and its relations with the Bellman equation of optimal control theory. In order to analyze the quantum theory, one must start with the classical model first. This will be done in the next section.

3 The optimization problem as a classical constrained system

Now, we analyze (12) from the phase space $(x, u, p_x, p_u,)$ point of view; that is, we formulate a Hamiltonian theory related to (12). To do this, we must first note that in the Lagrangian (12), the momentum definition for the variable u is

$$p_u = \frac{\partial L(x, u, \dot{x}, \dot{u}, t)}{\partial \dot{u}} = 0.$$

Note that in this case, the definition of the momentum variable does not allows us to write p_u in terms of its respective velocity $\dot{u}$, so the momentum definition gives rise to one constraint on the phase space. At this point we need to apply Dirac's method [30], [31], [32] for a constrained system, to study the problem in the right way. According to Dirac's classification, the above constraint is called a primary constraint, and we write it as

$$\Phi_1 = p_u = 0. \tag{13}$$

The Hamiltonian $H = H(x, u, p_x, p_u)$ is

$$H = p_x\dot{x} + p_u\dot{u} - L(x, u, \dot{x}, \dot{u}, t),$$

which expands to

$$H = p_x\dot{x} + p_u\dot{u} + (-F(x, u, t) + p_x f(x, u, t) - p_x\dot{x}),$$

or

$$H = p_u\dot{u} + H_0(x, u, p_x, p_u, t),$$

with

$$H_0(x, u, p_x, p_u, t) = -F(x, u, t) + p_x f(x, u, t). \tag{14}$$

For the constraint surface $\Phi_1 = 0$, we get

$$H(x, u, p_x, p_u, t) = H_0(x, u, p_x, p_u, t).$$

To incorporate these restrictions on the phase space, we define the extended Hamiltonian

$$\tilde{H}(x, u, p_x, p_u, t) = H_0 + \mu_1\Phi_1, \tag{15}$$

where μ_1 is a Lagrange multiplier. Now, we require the constraint Φ_1 to be preserved in time with the extended Hamiltonian (15) such that

$$\dot{\Phi}_1 = \{\Phi_1, \tilde{H}\} = 0,$$

where $\{A, B\}$ denotes the Poison bracket in the (x, u, p_x, p_u) phase space

$$\{A, B\} = \Big(\frac{\partial A}{\partial x}\frac{\partial B}{\partial p_x} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial p_x}\Big) + \Big(\frac{\partial A}{\partial u}\frac{\partial B}{\partial p_u} - \frac{\partial B}{\partial u}\frac{\partial A}{\partial p_u}\Big) \tag{16}$$

Equation (16) yields

$$\dot{\Phi}_1 = \{p_u, \tilde{H}\} = \frac{\partial \tilde{H}}{\partial u} = \frac{\partial H_0}{\partial u} = 0. \tag{17}$$

Thus, (17) is a new secondary constraint:

$$\Phi_2 = \frac{\partial H_0}{\partial u} = -\frac{\partial F(x,u,t)}{\partial u} + p_x \frac{\partial f(x,u,t)}{\partial u} = 0 \tag{18}$$

In this way, the optimization of the Hamiltonian with respect to the control variable appears in the phase space as a secondary constraint. To incorporate the new constraint in the model, we must consider the Hamiltonian

$$\tilde{H}_2(x,u,p_x,p_u) = H_0 + \mu_1\Phi_1 + \mu_2\Phi_2$$

We start again and impose time preservation for the set of constraints $\{\Phi_1, \Phi_2\}$ with the new Hamiltonian $\tilde{H}_2$:

$$\begin{aligned} \dot{\Phi}_1 &= \{\Phi_1, \tilde{H}_2\} = 0, \\ \dot{\Phi}_2 &= \{\Phi_2, \tilde{H}_2\} = 0. \end{aligned}$$

The above set of two equations gives only restrictions for the Lagrange multipliers $\mu_1,\ \mu_2$ and no new constraint appears. In fact these equations are explicitly

$$\{\Phi_1, H_0\} + \mu_2\{\Phi_1, \Phi_2\} = 0 \tag{19}$$

$$\{\Phi_2, H_0\} + \mu_1\{\Phi_2, \Phi_1\} = 0, \tag{20}$$

or in matrix form

$$\begin{pmatrix} \{\Phi_1, H_0\} \\ \{\Phi_2, H_0\} \end{pmatrix} + \begin{pmatrix} 0 & \{\Phi_1, \Phi_2\} \\ -\{\Phi_1, \Phi_2\} & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The antisymmetric matrix

$$\Delta = \begin{pmatrix} 0 & \{\Phi_1, \Phi_2\} \\ -\{\Phi_1, \Phi_2\} & 0 \end{pmatrix},$$

is called the Dirac matrix. Now

$$\{\Phi_1, \Phi_2\} = \{p_u, \Phi_2\} = \frac{\partial \Phi_2}{\partial u} = \frac{\partial^2 H_0}{\partial u^2},$$

so

$$\Delta = \begin{pmatrix} 0 & \frac{\partial^2 H_0}{\partial u^2} \\ -\frac{\partial^2 H_0}{\partial u^2} & 0 \end{pmatrix}. \tag{21}$$

The determinant of the Dirac matrix is

$$\det(\Delta) = \Big(\frac{\partial^2 H_0}{\partial u^2}\Big)^2. \tag{22}$$

If

$$\frac{\partial^2 H_0}{\partial u^2} \neq 0, \tag{23}$$

on the constraint surface where $\Phi_1 = 0, \Phi_2 = 0$, then the Dirac matrix is invertible and the constraint set $\{\Phi_1,\ \Phi_2\}$ is second-class (see [30], [31], [35]).

For the rest of this paper, we will assume that (23) is valid (for example, in a typical control problem in economics, the function F is quadratic and the function f is linear in terms of the control variable, so $\frac{\partial^2 H_0}{\partial u^2} \neq 0$). Thus, the optimization problem defined by the Lagrangian (12), from a physical point of view, corresponds to a second-class constrained dynamic system in the phase space.

In the appendix 1, it is shown explicitly that the time evolution of this second class system in the Dirac's bracket (150), is completely equivalent to the dynamics given by the unconstrained Pontryagin's equations. Thus, the Pontryagin equations are then the classical equations of motion for our physical constrained system.

4 Operator quantization of the Pontryagin theory

Until now, the dynamic optimization problem has been studied from a classical point of view, and one has seen that it is equivalent to a classical physically constrained system. However, what happens at the quantum level? To explore that, we will quantize our classical system and study the consequences. First, the quantization will be done with the operator representation of quantum mechanics, and then it will be done in terms of the path integral representation of quantum mechanics due to Feynman.

4.1 Right quantization of the Pontryagin theory

Again, consider the classical Hamiltonian

$$H_0(x, u, \lambda, p_x, p_u) = -F(x, u, t) + f(x, u, t)\ p_x. \tag{24}$$

Now, we have to have to quantize the classical Hamiltonian (24). For this purpose, we replace p_x, p_u with the appropriate quantum momentum operators :

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$
$$\hat{p}_u = -i\hbar\frac{\partial}{\partial u},$$

thus, the Schrödinger equation

$$\hat{H}_0(x, u, \hat{p}_x, \hat{p}_u)\ \Psi(x, u, t) = i\hbar\frac{\partial}{\partial t}\ \Psi(x, u, t)$$

is in this case,

$$\left(-F(x, u, t) - i\hbar f(x, u, t)\frac{\partial}{\partial x}\right)\Psi = i\hbar\frac{\partial \Psi}{\partial t}\ , \tag{25}$$

with $\Psi = \Psi(x, u, t)$ and where right side order has been chosen for the momentum operator in the quantization process.

According to Dirac, not all solutions Ψ of the Schrödinger equation (25) are physically admissible. The physical solutions Ψ_P must satisfy the constraints (13) and (18). In the quantum case, these equations must be imposed as operator equations on the Hilbert space of states in the form

$$\hat{\Phi}_1 \ \Psi_P = 0,$$
$$\hat{\Phi}_2 \ \Psi_P = 0. \tag{26}$$

The physically admissible states are the solutions of (25) that satisfy the constraints in Equations (26). Explicitly, we can define these constraint operators as

$$\begin{aligned}\hat{\Phi}_1 &= \hat{p}_u = -i\hbar\frac{\partial}{\partial u}\\ \hat{\Phi}_2 &= -\frac{\partial F(x,u,t)}{\partial u} - i\hbar\frac{\partial f(x,u,t)}{\partial u}\frac{\partial}{\partial x}.\end{aligned}$$

Thus, physically admissible solutions $\Psi_P(x,u,t)$ must satisfy

$$-i\hbar\frac{\partial}{\partial u}\Psi_P = 0, \tag{27}$$

$$\left(-\frac{\partial F(x,u,t)}{\partial u} - i\hbar\frac{\partial f(x,u,t)}{\partial u}\frac{\partial}{\partial x}\right)\Psi_P = 0, \tag{28}$$

as well as the Schrödinger equation

$$\left(-F(x,u,t) - i\hbar f(x,u,t)\frac{\partial}{\partial x}\right)\ \Psi_P = i\hbar\frac{\partial}{\partial t}\ \Psi_P. \tag{29}$$

Equation (27) implies that the wave function is independent of u, thus $\Psi_P = \Psi_P(x,t)$. Now, if we define the function $S(x,t)$ as

$$\Psi_P(x,t) = e^{\frac{i}{\hbar}S(x,t)} \tag{30}$$

the quantum Schrödinger equations (28) and (29) can be written in terms of $S(x,t)$ as

$$-\frac{\partial F(x,u,t)}{\partial u} + \frac{\partial f(x,u,t)}{\partial u}\frac{\partial S(x,t)}{\partial x} = 0, \tag{31}$$

$$-F(x,u,t) + f(x,u,t)\frac{\partial S(x,t)}{\partial x} = -\frac{\partial S(x,t)}{\partial t}. \tag{32}$$

Note that the above equations do not depend on $\hbar$ due to the linear character of the Hamiltonian operator H_0 and the quantum constraint (28). The constraint (31) is an expression for the maximization of the quantity

$$-F(x,u,t) + f(x,u,t)\frac{\partial S(x,t)}{\partial x},$$

with respect to the control variable u, so Equations (31) and (32) can be written as one equation:

$$\max_u\Big(-F(x,u,t) + f(x,u,t)\frac{\partial S(x,t)}{\partial x}\Big) = -\frac{\partial S(x,t)}{\partial t}, \tag{33}$$

which has the form of a Hamilton–Jacobi–Bellman equation [46], [36]. Thus, $S(x,t)$ can be related by the optimal value function $J(x,t)$ of Bellman's theory. One can say, then, that the dynamic programming approach to the optimization problem can be related to a quantum view of the problem. In the next section, the precise relation will be shown explicitly.

Dirac's quantization method used has some problems. The commutator of the constraint operators $\hat{\Phi}_1$ and $\hat{\Phi}_2$ is, in general, different from zero, thus

$$[\hat{\Phi}_1, \hat{\Phi}_2] = \alpha(x,u)\hat{I} \neq 0, \tag{34}$$

for some function α, in such a way that, when applied to a physical state Ψ_P, we have

$$[\hat{\Phi}_1, \hat{\Phi}_2]\ \Psi_P = \alpha(x,u)\Psi_P.$$

This implies that $0 = \alpha(x,u)\Psi_P$ and $0 = \Psi_P$. Thus, there is no physical wave function at all.

This is related to the fact that the Poisson bracket of two second-class constraints $\{\Phi_1, \Phi_2\}$ is different from zero. After the quantization, the Poisson bracket becomes the commutator (34). Then, Dirac's quantization procedure is not well-defined for second-class constraints (for details related to this issue see [49]).

Note that we have quantized the model without solving the classical constraints. A more transparent procedure would be to solve the classical constraints first, and then to quantize (these procedures give, in general, different answers; see [49]).

4.2 Left quantization of the Pontryagin theory

Due to the non-commutativity of the operators in quantum mechanics, right quantization is not the only possibility for the quantization procedure. So, what about other operator orderings. For example, if left quantization is employed, that is, putting all momentum operators on the left side, the resulting Schrödinger equations (27), (28) and (29) are

$$-i\hbar\frac{\partial}{\partial u}\Psi_P = 0, \tag{35}$$

$$\left(-\frac{\partial F(x,u,t)}{\partial u} - i\hbar\frac{\partial f(x,u,t)}{\partial u}\frac{\partial}{\partial x} - i\hbar\frac{\partial^2 f(x,u,t)}{\partial x \partial u}\right)\Psi_P = 0, \tag{36}$$

$$\left(-F(x,u,t) - i\hbar f(x,u,t)\frac{\partial}{\partial x} - i\hbar\frac{\partial f(x,u,t)}{\partial x}\right)\ \Psi_P = i\hbar\frac{\partial}{\partial t}\ \Psi_P. \tag{37}$$

By substituting $\Psi_P(x,u,t) = e^{\frac{i}{\hbar}S(x,u,t)}$ in the last three equations one has that

$$\frac{\partial}{\partial u}S(x,u,t) = 0, \tag{38}$$

$$\left(-\frac{\partial F(x,u,t)}{\partial u} + \frac{\partial f(x,u,t)}{\partial u}\frac{\partial S(x,u,t)}{\partial x} - i\hbar\frac{\partial^2 f(x,u,t)}{\partial x \partial u}\right)S(x,u,t) = 0, \tag{39}$$

$$\Big(-F(x,u,t)+f(x,u,t)\frac{\partial S(x,u,t)}{\partial x}-i\hbar\frac{\partial f(x,u,t)}{\partial x}\Big)=-\frac{\partial S(x,u,t)}{\partial t}. \tag{40}$$

Hence, (38) implies that $S=S(x,t)$ and

$$\Bigg(-\frac{\partial F(x,u,t)}{\partial u}+\frac{\partial f(x,u,t)}{\partial u}\frac{\partial S(x,t)}{\partial x}-i\hbar\frac{\partial^2 f(x,u,t)}{\partial x\partial u}\Bigg)=0, \tag{41}$$

$$\Big(-F(x,u,t)+f(x,u,t)\frac{\partial S(x,t)}{\partial x}-i\hbar\frac{\partial f(x,u,t)}{\partial x}\Big)=-\frac{\partial S(x,t)}{\partial t}. \tag{42}$$

The last two equations have again a similar form to that of Hamilton–Jacobi–Bellman equations, but in this case these equations depend explicitly on Planck's constant $\hbar$. One can call these last equations the full Quantum Hamilton–Jacobi–Bellman equations.

4.3 Bellman's theory as the classical limit of the quantum Pontryagin theory

Note that the Hamilton–Jacobi–Bellman type equations (31), (32) or (33) do not depend on $\hbar$, so these look more like classical equations than quantum ones. But these equations came from a quantization process, so what really happened here?

To understand this phenomenon, consider the usual Schrödinger equation for a non-relativistic particle of mass m:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{43}$$

If we write the wave function in the form (30), by replacing it in the the Schrödinger equation (43) one obtain the following fully quantum Hamilton–Jacobi equation for $S(x,t)$

$$\frac{1}{2m}\Big(\frac{\partial S(x,t)}{\partial x}\Big)^2+U(x)-\frac{i\hbar}{2m}\frac{\partial^2 S(x,t)}{\partial x^2}=-\frac{\partial S(x,t)}{\partial t}. \tag{44}$$

This equation is completely equivalent to the Schrödinger equation, but here the classical and quantum realms can be clearly identified. In fact by taking the limit $\hbar\to 0$ in (44), one gets

$$\frac{1}{2m}\Big(\frac{\partial S(x,t)}{\partial x}\Big)^2+U(x)=-\frac{\partial S(x,t)}{\partial t}. \tag{45}$$

which corresponds to the classical limit of the quantum theory. Equation (45) is the well known Hamilton–Jacobi equation of classical mechanics.
Note that the momentum operator acting on the wave function gives an independent $\hbar$ term:

$$\hat{p}_x\Psi=\hat{p}_x e^{\frac{i}{\hbar}S(x,t)}=-i\hbar\frac{\partial}{\partial x}e^{\frac{i}{\hbar}S(x,t)}=-i\hbar\frac{i}{\hbar}\frac{\partial S(x,t)}{\partial x}e^{\frac{i}{\hbar}S(x,t)}=\frac{\partial S(x,t)}{\partial x}\Psi. \tag{46}$$

whereas $\hat{p}_x^2$ acting on the wave function gives

$$\hat{p}_x^2\ \Psi=-\hbar^2\frac{\partial^2}{\partial x^2}\ e^{\frac{i}{\hbar}S(x,t)}=-i\hbar\frac{\partial^2 S(x,t)}{\partial x^2}\Psi+\Big(\frac{\partial S(x,t)}{\partial x}\Big)^2\Psi \tag{47}$$

which has a term dependent on $\hbar$. Thus, for a Hamiltonian that is linear in the momentum, when acting on $\Psi = e^{\frac{i}{\hbar}S(x,t)}$, the equation for $S(x,t)$ gives an equation independent of $\hbar$. But, for a quadratic Hamiltonian such as for the non-relativistic particle,

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + U(x), \tag{48}$$

equation (44) for $S(x,t)$ necessarily has $\hbar$ dependent terms. Also note that for the non-relativistic Hamiltonian, different orderings of the operators give the same Schrödinger equation (43). Thus right quantization is equivalent to left quantization in that case.

But the Hamiltonian associated to the optimal control problem is linear in the momentum operator so right quantization produces the $\hbar$ independent equations (31) and (32) for $S(x,t)$, but left quantization produced the $\hbar$ dependent equations (41) and (42). Now by taking the classical limit $\hbar \to 0$ in the full quantum Hamilton–Jacobi–Bellman equations (41) and (42), one recovers the Hamilton–Jacobi–Bellman equations (31) and (32). Thus these last equation can be seen has the classical limit of the full quantum Hamilton–Jacobi–Bellman equations.

In this way, one can say that due to the special linear character of the optimal control Hamiltonian, the right-quantization of the classical constrained systems associated to the optimal control problem, gives just the classical limit of the full quantum theory of the Pontryagin equations.

5 The Feynman approach to the Quantum Mechanics

In spit of the inherent problems of the quantization in terms of an operator algebra, one can try surpass this difficulties if one do quantum mechanics by using the Feynman sum over stories approach of the quantum mechanics [13], [50], [51]. For the nonspecialist reader, in the appendix 2 a little review of the Feynman Hamiltonian path integrals will be done. And in the next subsection, these will be applied to the Pontryagin theory to obtain the quantum propagator of the optimal control theory.

5.1 The Feynman propagator of the Pontryagin theory

The exact path integral quantization of our classical constrained system will now be carried out. Due to the presence of a constraint, the naive path integral (164) is incorrect for our control system. The true propagator is expressed in terms of a constrained path integral (for technical details see [32], [49]). In our specific optimal control case, that constrained path integral looks like

$$K = \int \mathcal{D}x\mathcal{D}u \frac{\mathcal{D}p_x}{2\pi\hbar}\frac{\mathcal{D}p_u}{2\pi\hbar}\sqrt{\det(\Delta)}\delta(\Phi_1)\delta(\Phi_2)\exp(\frac{i}{\hbar}A[x,u,p_x,p_u]) \tag{49}$$

where the Hamiltonian action A is

$$A[x,u,p_x,p_u] = \int [p_x\dot{x} + p_u\dot{u} - H_0(x,p_x,u,p_u)]dt. \tag{50}$$

The term

$$\mathcal{D}x\mathcal{D}u\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{\mathcal{D}p_u}{2\pi\hbar}\sqrt{\det(\Delta)}\delta(\Phi_1)\delta(\Phi_2) \tag{51}$$

generates an invariant measure on the constraint surface $\Phi_1 = 0$, $\Phi_2 = 0$ and $\sqrt{\det(\Delta)}$ is the reminder that our model has second class constraints. In fact, in (49) $\det(\Delta) = (\frac{\partial^2 H_0}{\partial u^2})^2$ is the determinant of the Dirac matrix (22), and Φ_1 and Φ_2 are the second-class constraints (13), (18).

An explicit form of the propagator is then

$$K = \int \mathcal{D}x\mathcal{D}u\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{\mathcal{D}p_u}{2\pi\hbar}\ \frac{\partial^2 H_0}{\partial u^2}\ \delta(p_u)\ \delta(\frac{\partial H_0}{\partial u})\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1}[p_x\,\dot{x} + p_u\dot{u} - H_0]\ dt) \tag{52}$$

Integration with respect to p_u yields

$$K = \int \mathcal{D}x\mathcal{D}u\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{1}{2\pi\hbar}\ \frac{\partial^2 H_0}{\partial u^2}\ \delta(\frac{\partial H_0}{\partial u})\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1}[p_x\dot{x} - H_0]\ dt) \tag{53}$$

Now, $\delta(\frac{\partial H_0}{\partial u})$ can be written as

$$\delta(\frac{\partial H_0}{\partial u}) = \frac{1}{\left(\frac{\partial^2 H_0}{\partial u^2}\right)}\delta(u - u_P) \tag{54}$$

where $u_P = u_P(x,t)$ is the optimal control path, given by the maximization of the Hamiltonian H_0 with respect to u, that is, the Pontryagin equation for the control variable:

$$\frac{\partial H_0(x, u_P, p_x, t)}{\partial u} = -\frac{\partial F(x, u_P, t)}{\partial u} + p_x\frac{\partial f(x, u_P, t)}{\partial u} = 0. \tag{55}$$

The fact that u_P is the solution of the above equation, implies that $u_P = u_P(x, p_x, t)$.
Substituting (54) in (53) gives

$$K = \int \mathcal{D}x\mathcal{D}u\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{1}{2\pi\hbar}\ \delta(u - u_P)\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1}[p_x\dot{x} - H_0]dt). \tag{56}$$

Integration with respect to u yields

$$K = \int \mathcal{D}x\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{1}{2\pi\hbar}\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1}[p_x\dot{x} - H_0(x, u_P(x, p_x, t), p_x, t)]dt) \tag{57}$$

That is, an explicit form for the full quantum propagator of the Pontryagin theory is

$$K = \int \mathcal{D}x\frac{\mathcal{D}p_x}{2\pi\hbar}\frac{1}{2\pi\hbar}\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1}[F(x, u_P(x, p_x, t), t) + p_x(\dot{x} - f(x, u_P(x, p_x, t), t)\ )\ ]dt) \tag{58}$$

Note that the effective Hamiltonian

$$H(x, p_x, t) = H_0(x, u_P(x, p_x, t), p_x, t) = -F(x, u_P(x, p_x, t), t) + p_x f(x, u_P(x, p_x, t), t) \tag{59}$$

generated after the integration with respect to u in general will be non-linear in p_x. Equation (58) represents the most general form of the propagator for the optimal control problem. Note that in (58), the summation is over

1) arbitrary quantum paths $p_x(t)$ that ended at 0 at time t_1, and
2) arbitrary quantum paths $x(t)$ that start at x_0 at time t_0, so the term $p_x(\dot{x} - f(x, u_P(x, p_x, t), t))$ gives non-vanishing contributions for all non-classical trajectories (see Figure 4 in appendix 3).

5.2 The exact propagator for the Pontryagin linear case

There is an interesting case where this propagator can be integrated exactly, so as to obtain a closed form expression for it. If u_P does not depend on p_x, that is, $u_P = u_P(x, t)$ (in which case the effective Hamiltonian is linear in p_x), then one can integrate with respect to p_x to obtain

$$K = \int \mathcal{D}x \frac{1}{2\pi\hbar}\ \delta(\dot{x} - f(x, u_P, t))\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1} F(x, u_P, t)dt) \tag{60}$$

Finally, by integrating with respect to x, one arrives at the exact closed form

$$K(x_1 t_1|x_0 t_0) = \frac{1}{2\pi\hbar}\ \delta(x_1 - x_P(t_1, x_0))\ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1} F(x_P, u_P(x_P, t), t)dt), \tag{61}$$

where $x_P = x_P(t, x_0)$ denotes the solution $x(t)$ of the Pontryagin equation

$$\dot{x} = f(x, u_P(x, t), t) \tag{62}$$

with the initial condition $x(t_0) = x_0$. Note that for this special case in which the Hamiltonian is linear in the momentum, we hope to find a Dirac delta function as a factor in the propagator, because this calculation is analogous to the linear case discussed in the appendix 3.

5.3 The semi-classical propagator for the Pontryagin theory

Now, a semi-classical expression for the exact Feynman propagator of the Pontryagin theory will be determined (a review of the semi-classical ideas are given in the appendix 3). The semi-classical propagator K_{SC} is important because it corresponds to the limit $\hbar \to 0$ of the full quantum theory in the Feynman representation of quantum mechanics:

$$K_{SC} = \lim_{\hbar \to 0} K, \tag{63}$$

and this limit just corresponds to the Bellman theory, so one can say that the Bellman theory has as propagator the semi-classical one.

In the limit $\hbar \to 0$, the only paths that contribute to the path integral are the classical ones, that is, the paths that make the action in the exponent of (57) or (58), extremal, so the classical path must fulfill the Hamilton equations of motion

$$\dot{x} = \frac{\partial H(x,p_x,t)}{\partial p_x} = \frac{\partial H_0(x,u_P,p_x,t)}{\partial p_x} + \frac{\partial H_0(x,u_P,p_x,t)}{\partial u_P}\frac{\partial u_P}{\partial p_x} = f(x,u_P(x,p_x,t),t) \tag{64}$$

$$\dot{p_x} = -\frac{\partial H(x,p_x,t)}{\partial x} = -\frac{\partial H_0(x,u_P,p_x,t)}{\partial x} - \frac{\partial H_0(x,u_P,p_x,t)}{\partial u_P}\frac{\partial u_P}{\partial x} = -\frac{\partial H_0(x,u_P,p_x,t)}{\partial x} \tag{65}$$

because $\frac{\partial H_0(x,u_P,p_x,t)}{\partial u_P} = 0$ for the optimal control u_P. Thus, the classical equations are precisely the Pontryagin equations. Then, the semi-classical propagator is, according to [1] (183)

$$K_{SC}(x_1t_1|x_0t_0) = \ \delta(x_1 - x_P(t_1,x_0)) \ \exp(\frac{i}{\hbar}\int_{t_0}^{t_1} F(x_P,u_P(x_P,p_x,t),t)dt), \tag{66}$$

where the delta factor is due to the fact that the classical equations of motions are of first order, and satisfy the initial and final conditions of Figure 4 in appendix 3, so there is no possible propagation for $x_1 \neq x_P(t_1,x_0)$.

With this semi-classical propagator K_{SC}, one has the possibility of determining the time evolution of the wave function in the classical limit $\hbar \to 0$ that corresponds to the right quantization of the Pontryagin theory. In fact, in view of Equations (152) and (29) one can write the wave function in (29) as (the P subindex has been omitted for simplicity)

$$\Psi(x,t) = \int K_{SC}(xt|x't_0)\Psi(x',t_0)dx' \tag{67}$$

where $\Psi(x,t_0) = \Phi_0(x)$ is some fixed initial state. By substituting (66) in (67) one has

$$\Psi(x,t) = \int \delta(x - x_P(t,x')) \ e^{\frac{i}{\hbar}\int_{t_0}^{t} F(x_P,u_P,\tau)d\tau} \ \Phi_0(x')dx' \tag{68}$$

For some fixed value x, the integral in x' contributes only when the initial condition $x_0 = x'$ permits the Pontryagin trajectory to reach the point x at a future time t, see Figure 1. So, the initial point x' is a function of x and t through the solution of the Pontryagin equation, and it will be denoted by $x' = \sigma(x,t)$, that is,

$$x - x_P(t,x') = 0 \quad \Leftrightarrow \quad x' = \sigma(x,t). \tag{69}$$

Then, one can write

$$\delta(x - x_P(t,x')) = \frac{\delta(x' - \sigma(x,t))}{\left(\frac{\partial x_P(t,x')}{\partial x'}\right)_{x'=\sigma(x,t)}}. \tag{70}$$

[1] K_{SC} would include a constant N in front of the Dirac delta function, but it has been chosen equal to one. In fact, all multiplicative constants that appear in the propagator and in the wave function would finally be determined by the normalization of the wave function $\int_{-\infty}^{\infty}\Psi\Psi^* dx = 1$, so one always can take $N = 1$.

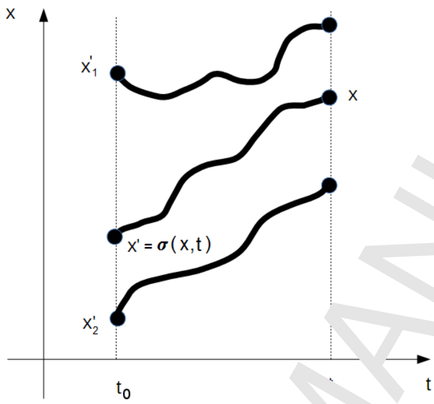

Figure 1: Path from $x_0 = x'$ to x

Thus the wave function is

$$\Psi(x,t) = \int \frac{\delta(x' - \sigma(x,t))}{\left(\frac{\partial x_P(t,x')}{\partial x'}\right)_{x'=\sigma(x,t)}} e^{\frac{i}{\hbar}\int_{t_0}^{t} F(x_P,u_P,\tau)d\tau}\ \Phi_0(x')dx' \tag{71}$$

that is

$$\Psi(x,t) = \frac{1}{\left(\frac{\partial x_P(t,x')}{\partial x'}\right)_{x'=\sigma(x,t)}} e^{\frac{i}{\hbar}\int_{t_0}^{t} F(x_P,u_P(x_P,\tau),\tau)d\tau}\ \Phi_0(\sigma(x,t)) \tag{72}$$

where x_P in F in the above equation is the solution of the Pontryagin equation $x_P(\tau, \sigma_0 = \sigma(x))$ for $t_0 \leq \tau \leq t$.

Note that if one fixes x_0 and varies x, then the curves $x_P(\tau, x_0)$ for $t_0 \leq \tau \leq T$ and $x_P(\tau, x)$ for $t \leq \tau \leq T$ are in general distinct, but if (x,t) belongs to $x_P(\tau, x_0)$, then

$$x_P(\tau, x_0) = x_P(\tau, x) \qquad t \leq \tau \leq T \tag{73}$$

as illustrated in Figure 2.
This implies that

$$x_0 = \sigma(x,t) \quad \forall \ \ (x,t) \in x_P(\tau, x_0) \qquad t_0 \leq t \leq T, \quad t \leq \tau \leq T \tag{74}$$

Thus, $\sigma(x,t)$ is constant when (x,t) moves along the Pontryagin curve $x_P(t)$ and its value is the initial value x_0.

Now, by defining

$$N(x,t) = \left(\frac{\partial x_P}{\partial x'}\right)_{x'=\sigma(x,t)} \quad , \tag{75}$$

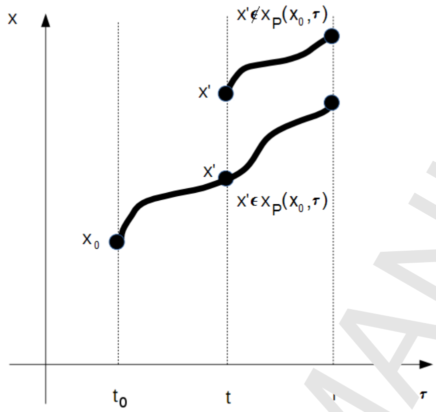

Figure 2: When $(x,t) \in x_P(\tau, x_0) \Rightarrow x_P(\tau, x_0) = x_P(\tau, x)$ for $t \leq \tau \leq T$

the wave function can be written as

$$\Psi(x,t) = e^{\frac{i}{\hbar}\left(\int_{t_0}^{t} F(x_P, u_P(x_P,\tau),\tau)d\tau + i\hbar \ln(N(x,t))\right)} \; \Phi_0(\sigma(x,t)) \tag{76}$$

Equation (76) can be written in terms of $S(x,t)$ as

$$e^{\frac{i}{\hbar}S(x,t)} \;=\; e^{\frac{i}{\hbar}\left(\int_{t_0}^{t} F(x_P, u_P(x_P,\tau),\tau)d\tau + i\hbar \ln(N(x,t))\right)} \; e^{\frac{i}{\hbar}S_0(\sigma(x,t),t_0)}, \tag{77}$$

which implies

$$S(x,t) = \int_{t_0}^{t} F(x_P, u_P(x_P,\tau),\tau)d\tau + i\hbar \ln(N(x,t)) + S_0(\sigma(x,t),t_0) \tag{78}$$

Thus, S is essentially the value of the cost functional between t_0 and t plus quantum terms that depend on $\hbar$. Taking now the classical limit $\hbar \to 0$ one then finds

$$S(x,t) = \int_{t_0}^{t} F(x_P, u_P(x_P,\tau),\tau)d\tau + S_0(\sigma(x,t),t_0) \tag{79}$$

Note that in this limit, the function $S(x,t)$ obtained through the path integral formalism has no dependence on $\hbar$, so that $S(x,t)$ must be a solution of the Hamilton–Jacobi–Bellman type equations (31) and (32) for the right quantization procedure with initial condition $S(x,t_0) = S_0(x,t_0)$.

6 Quantum mechanics and the Bellman theory

To make contact with the Bellman theory, consider a point (x,t) and the functions $J_+(x,t)$ and $J_-(x,t)$ defined by

$$J_+(x,t) = \int_{t}^{T} F(x_P, u_P, \tau)d\tau \tag{80}$$

$$J_-(x,t) = \int_{t_0}^{t} F(x_P, u_P, \tau)d\tau \tag{81}$$

where $J_+(x,t)$ is the line integral along the Pontryagin curve that starts at x at time t, and $J_-(x,t)$ is the line integral along the curve that start at $x_0 = \sigma(x,t)$ at time t_0. Of course

$$\int_{t_0}^{T} F(x_P, u_P, \tau)d\tau = \int_{t_0}^{t} F(x_P, u_P, \tau) + \int_{t}^{T} F(x_P, u_P, \tau)d\tau. \tag{82}$$

The integral on the left side of this equation must ultimately depend on $x_0 = \sigma(x,t)$, t_0, and T, and one can denote it by $I(\sigma(x,t), t_0, T)$, so this equation is just

$$I(\sigma(x,t), t_0, T) = J_-(x,t) + J_+(x,t) \tag{83}$$

Note that when (x,t) moves along the Pontryagin trajectory $I(\sigma(x,t), t_0, T)$ remains constant. By taking partial derivatives with respect to x and t one has that

$$\Big(\frac{\partial I(\sigma, t_0, T)}{\partial \sigma}\Big)_{\sigma=\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial x} = \frac{\partial J_-(x,t)}{\partial x} + \frac{\partial J_+(x,t)}{\partial x} \tag{84}$$

$$\Big(\frac{\partial I(\sigma, t_0, T)}{\partial \sigma}\Big)_{\sigma=\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial t} = \frac{\partial J_-(x,t)}{\partial t} + \frac{\partial J_+(x,t)}{\partial t} \tag{85}$$

Now, by evaluating (84) and (85) along the Pontryagin curve (where $\sigma(x,t)$ remains constant), then $\frac{\partial I}{\partial \sigma} = 0$. Equations (84) and (85) thus imply that

$$0 = \frac{\partial J_-(x,t)}{\partial x} + \frac{\partial J_+(x,t)}{\partial x} \tag{86}$$

$$0 = \frac{\partial J_-(x,t)}{\partial t} + \frac{\partial J_+(x,t)}{\partial t} \tag{87}$$

or

$$\frac{\partial J_+(x,t)}{\partial x} = -\frac{\partial J_-(x,t)}{\partial x} \tag{88}$$

$$\frac{\partial J_+(x,t)}{\partial t} = -\frac{\partial J_-(x,t)}{\partial t} \tag{89}$$

along the Pontryagin curve. But note that $J_+(x,t)$ is just Bellman's cost functional, so if $J_+(x,t)$ satisfies the Hamilton–Jacobi–Bellman equation (via Bellman's maximum principle [46], [36]), then

$$\frac{\partial J_+(x,t)}{\partial t} + max_u \left(F(x,u,t) + \frac{\partial J_+(x,t)}{\partial x} f(x,u,t) \right) = 0 \tag{90}$$

which is equivalent to

$$\frac{\partial J_+(x,t)}{\partial t} + \left(F(x,u,t) + \frac{\partial J_+(x,t)}{\partial x} f(x,u,t) \right) = 0 \tag{91}$$

$$\frac{\partial H_0(x,u,t)}{\partial u} = \frac{\partial F(x,u,t)}{\partial u} + \frac{\partial J_+(x,t)}{\partial x}\frac{\partial f(x,u,t)}{\partial x} = 0 \tag{92}$$

By substituting (88) and (89) into the last two equations, one obtains the equations that J_- must satisfy:

$$\frac{\partial J_-(x,t)}{\partial t} + \left(-F(x,u,t) + \frac{\partial J_-(x,t)}{\partial x} f(x,u,t)\right) = 0 \tag{93}$$

$$\frac{\partial H_0(x,u,t)}{\partial u} = \frac{\partial - F(x,u,t)}{\partial u} + \frac{\partial J_-(x,t)}{\partial x}\frac{\partial f(x,u,t)}{\partial x} = 0 \tag{94}$$

that is,

$$\frac{\partial J_-(x,t)}{\partial t} + max_u\left(-F(x,u,t) + \frac{\partial J_-(x,t)}{\partial x} f(x,u,t)\right) = 0. \tag{95}$$

Thus, if $J_+(x,t)$ satisfies the Hamilton–Jacobi–Bellman equation (90), then $J_-(x,t)$ satisfies the "conjugate" Hamilton–Jacobi–Bellman equation (95), and vice versa.

But what does all this have to do with the quantum model? Well, from (79), one has that

$$S(x,t) - S_0(\sigma(x,t),t_0) = J_-(x,t) \tag{96}$$

Upon taking partial derivatives with respect to x and t, one obtains

$$\frac{\partial S(x,t)}{\partial x} - \Big(\frac{\partial S_0(\sigma,t)}{\partial \sigma}\Big)_{\sigma=\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial x} = \frac{\partial J_-(x,t)}{\partial x} \tag{97}$$

$$\frac{\partial S(x,t)}{\partial t} - \Big(\frac{\partial S_0(\sigma,t)}{\partial \sigma}\Big)_{\sigma=\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial t} = \frac{\partial J_-(x,t)}{\partial t} \tag{98}$$

Now evaluating this along the Pontryagin curve, where $\frac{\partial S_0(\sigma,t)}{\partial \sigma} = 0$, one gets

$$\frac{\partial S(x,t)}{\partial x} = \frac{\partial J_-(x,t)}{\partial x} = -\frac{\partial J_+(x,t)}{\partial x}, \tag{99}$$

$$\frac{\partial S(x,t)}{\partial t} = \frac{\partial J_-(x,t)}{\partial t} = -\frac{\partial J_+(x,t)}{\partial t}. \tag{100}$$

Substituting these two last equations in the conjugate Hamilton–Jacobi–Bellman equation (33) that $S(x,t)$ satisfies, one obtains finally that

$$\frac{\partial J_-(x,t)}{\partial t} + max_u\left(-F(x,u,t) + \frac{\partial J_-(x,t)}{\partial x} f(x,u,t)\right) = 0 \tag{101}$$

or

$$\frac{\partial J_+(x,t)}{\partial t} + max_u\left(F(x,u,t) + \frac{\partial J_+(x,t)}{\partial x} f(x,u,t)\right) = 0. \tag{102}$$

The last equation is just the Hamilton–Jacobi–Bellman equation for J_+. Thus, the fact that the phase $S(x,t)$ of the wave function $\Psi(x,t)$ satisfies the conjugate Hamilton–Jacobi–Bellman equation (33) implies that $J_-(x,t)$ satisfy the same conjugate equation and that $J_+(x,t)$ satisfy then the Hamilton–Jacobi–Bellman equation!. Then, the Bellman theory is a consequence of the quantization of the classical Pontryagin theory!

7 An example

Consider a typical linear-quadratic optimization problem of the form

$$A[x,u] = \int_{t_0}^{T} \Big(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\Big)\,dt, \tag{103}$$

where u is a control variable and x represents a state variable subject to the equation

$$\dot{x} = \frac{1}{2}\sigma x^2 + \beta u, \qquad x(t_0) = x_0. \tag{104}$$

For this system

$$F(x,u,t) = \frac{1}{2}Qx^2 - \frac{1}{2}Bu^2 \quad , \quad f(x,u,t) = \frac{1}{2}\sigma x^2 + \beta u \ , \tag{105}$$

and we assume that $B \neq 0$ so that the cost functional A will depend on the control variable.

For a open-loop strategy (that is, if the system has no feedback), the optimal solution of the optimization problem is given by the Pontryagin equations (3), (4), (5), where the Pontryagin Hamiltonian is

$$H = \frac{1}{2}Qx^2 - \frac{1}{2}Bu^2 + \lambda\Big(\frac{1}{2}\sigma x^2 + \beta u\Big)\ . \tag{106}$$

and the Pontryagin equations are in this case respectively

$$\dot{x} = \frac{1}{2}\sigma x^2 + \beta u\ , \tag{107}$$

$$\dot{\lambda} = -\Big(Qx + \lambda\sigma x\Big)\ , \tag{108}$$

$$0 = -Bu + \lambda\beta\ . \tag{109}$$

From (109) one has that

$$u = \frac{\beta}{B}\ \lambda. \tag{110}$$

and by replacing this last u in (107) and (108) one obtains finally

$$\dot{x} = \frac{1}{2}\sigma x^2 + \frac{\beta^2}{B}\lambda, \tag{111}$$

$$\dot{\lambda} = -\Big(Qx + \lambda\sigma x\Big). \tag{112}$$

These equations are non-linear differential equations for x and λ. These must be solved numerically to obtain the optimal solution for the state variable $x(t)$.

For a closed-loop strategy (that is, if the system has explicit feedback) the optimal solution of the optimization problem is given by the Hamilton–Jacobi–Bellman (HJB) equation (102), which for our example is

$$\frac{\partial J_+(x,t)}{\partial t} + max_u \left(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2 + \frac{\partial J_+(x,t)}{\partial x}\Big(\frac{1}{2}\sigma x^2 + \beta u\Big) \right) = 0. \tag{113}$$

The maximization with respect to u gives

$$-Bu + \frac{\partial J_+(x,t)}{\partial x}\beta = 0\,. \tag{114}$$

so the control variable is

$$u = \frac{\beta}{B}\frac{\partial J_+(x,t)}{\partial x}. \tag{115}$$

By putting this in (113) one obtains the Hamilton–Jacobi–Bellman equation for our example:

$$\frac{\partial J_+(x,t)}{\partial t} + \frac{1}{2}Qx^2 + \frac{1}{2}\frac{\beta^2}{B}\Big(\frac{\partial J_+(x,t)}{\partial x}\Big)^2 + \frac{1}{2}\sigma x^2\frac{\partial J_+(x,t)}{\partial x} = 0. \tag{116}$$

7.1 The physical point of view

Now we now study our particular example from the phase space point of view. The physical action (11) in this case has the Lagrangian

$$L = p_x\dot{x} - H_0 \tag{117}$$

with

$$H_0 = -\Big(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\Big) + p_x\Big(\frac{1}{2}\sigma x^2 + \beta u\Big) \tag{118}$$

Using this Lagrangian, the definition of the canonical momentum of u is

$$p_u = \frac{\partial L}{\partial \dot{u}} = 0. \tag{119}$$

which generates the first constraint (13) of the model in the phase space:

$$\Phi_1 = p_u = 0. \tag{120}$$

Time consistency of this first constraint gives the second one (18), which is

$$\Phi_2 = Bu + \beta p_x = 0. \tag{121}$$

Thus, for our example the dynamics of the system must lie on the intersection of the planes given by equations (120) and (121). The Dirac matrix (21) is in this case given by

$$\Delta = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}. \tag{122}$$

For our example, it is invertible since $B \neq 0$. As is shown in Appendix 1 the dynamics of the canonical variables $x, p_x, u.p_u$ in the Dirac brackets (150) give the same equations (111) and (112) of the Pontryagin theory after identifying the Lagrangian multiplier λ with $-p_x$.

7.2 Right quantization

For right quantization, the physical wave function $\Psi_P(x,u,t)$ must satisfy the constraints (27), (28)

$$-i\hbar\frac{\partial}{\partial u}\Psi_P = 0, \tag{123}$$

$$\left(Bu - i\hbar\beta\frac{\partial}{\partial x}\right)\Psi_P = 0, \tag{124}$$

and the Schrödinger equation

$$\left[-\left(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\right) - i\hbar\left(\frac{1}{2}\sigma x^2 + \beta u\right)\frac{\partial}{\partial x}\right]\ \Psi_P = i\hbar\frac{\partial}{\partial t}\ \Psi_P.$$

By writing

$$\Psi_P(x,u,t) = e^{\frac{i}{\hbar}S(x,u,t)} \tag{125}$$

one gets for $S(x,u,t)$

$$\frac{\partial S(x,u,t)}{\partial u} = 0, \tag{126}$$

$$Bu + \beta\frac{\partial S(x,u,t)}{\partial x} = 0, \tag{127}$$

$$-\left(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\right) + \left(\frac{1}{2}\sigma x^2 + \beta u\right)\frac{\partial S(x,u,t)}{\partial x} = -\frac{\partial S(x,u,t)}{\partial t}\ . \tag{128}$$

Equation (126) implies that $S = S(x,t)$, so $S(x,t)$ satisfies

$$Bu + \beta\frac{\partial S(x,t)}{\partial x} = 0, \tag{129}$$

$$-\left(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\right) + \left(\frac{1}{2}\sigma x^2 + \beta u\right)\frac{\partial S(x,t)}{\partial x} = -\frac{\partial S(x,t)}{\partial t}\ . \tag{130}$$

Note that u becomes an auxiliary variable. In fact, u can be determined from equation (129) as

$$u = -\frac{\beta}{B}\left(\frac{\partial S(x,t)}{\partial x}\right) \tag{131}$$

and by replacing it in (130) one found

$$-\frac{1}{2}Qx^2 + \frac{1}{2}\sigma x^2 \frac{\partial S(x,t)}{\partial x} - \frac{1}{2}\frac{\beta^2}{B}\Big(\frac{\partial S(x,t)}{\partial x}\Big)^2 = -\frac{\partial S(x,t)}{\partial t}. \tag{132}$$

The last Schrödinger equation for $S(x,t)$ is very similar to the Hamilton–Jacobi–Bellman equation (116). Now by using the fact that

$$\frac{\partial S}{\partial x} = -\frac{\partial J_+}{\partial x} \tag{133}$$

and

$$\frac{\partial S}{\partial t} = -\frac{\partial J_+}{\partial t}\ , \tag{134}$$

the Schrödinger equation (132) becomes the Hamilton–Jacobi–Bellman equation (116) for $J_+(x,t)$. Note that for right quantization, the Schrödinger equation (132) for $S(x,t)$ does not have any terms that depend on $\hbar$.

7.3 Left quantization

For left quantization, the physical wave function $\Psi_P(x,u,t)$ satisfies the same constraints (123), (124), and the Schrödinger equation is in this case

$$\Big[-\Big(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\Big) - i\hbar\Big(\frac{1}{2}\sigma x^2 + \beta u\Big)\frac{\partial}{\partial x} - i\hbar\sigma x\Big]\ \Psi_P = i\hbar\frac{\partial}{\partial t}\ \Psi_P.$$

Again, by writing

$$\Psi_P(x,u,t) = e^{\frac{i}{\hbar}S(x,u,t)} \tag{135}$$

one gets for $S(x,u,t)$

$$\frac{\partial S(x,u,t)}{\partial u} = 0, \tag{136}$$

$$Bu + \beta\frac{\partial S(x,u,t)}{\partial x} = 0, \tag{137}$$

$$-\Big(\frac{1}{2}Qx^2 - \frac{1}{2}Bu^2\Big) - i\hbar\sigma x + \Big(\frac{1}{2}\sigma x^2 + \beta u\Big)\frac{\partial S(x,u,t)}{\partial x} = -\frac{\partial S(x,u,t)}{\partial t}\ . \tag{138}$$

And again (136) implies that $S = S(x,t)$, so for left quantization case one has that

$$-\frac{1}{2}Qx^2 + \frac{1}{2}\sigma x^2\Big(\frac{\partial S(x,t)}{\partial x}\Big) - i\hbar\sigma x - \frac{1}{2}\frac{\beta^2}{B}\Big(\frac{\partial S(x,t)}{\partial x}\Big)^2 = -\frac{\partial S(x,t)}{\partial t}\ . \tag{139}$$

By using (133) and (134) in the above equation, one arrives at a full quantum Hamilton–Jacobi–Bellman equation

$$\frac{1}{2}Qx^2 + \frac{1}{2}\sigma x^2\Big(\frac{\partial J_+(x,t)}{\partial x}\Big) + i\hbar\sigma x + \frac{1}{2}\frac{\beta^2}{B}\Big(\frac{\partial J_+(x,t)}{\partial x}\Big)^2 = -\frac{\partial J_+(x,t)}{\partial t}\ . \tag{140}$$

Thus, left quantization generates a Hamilton–Jacobi–Bellman equation that contains a term that explicitly depends on $\hbar$. In the classical limit $\hbar \to 0$, the above quantum equation reduces to the usual Hamilton–Jacobi–Bellman equation (116).

7.4 The Feynman approach and its classical limit

As is shown in (58), the full quantum propagator in the Feynman approach is

$$K = \int \frac{\mathcal{D}x}{2\pi\hbar} \frac{\mathcal{D}p_x}{2\pi\hbar} \; \exp\left(\frac{i}{\hbar} \int_{t_0}^{t_1} [\, p_x \dot{x} - H(x, p_x, t)]\, dt \right) \tag{141}$$

where the effective Hamiltonian $H(x, p_x, t)$ is

$$H(x, p_x, t) = H_0(x, u_P(x, p_x, t), p_x, t) = -F(x, u_P(x, p_x, t), t) + p_x f(x, u_P(x, p_x, t), t) \tag{142}$$

Note the above propagator is given by an unconstrained path integral which depends on the (x, p_x) phase space. Thus, the classical theory underlying the quantum one can be obtained by taking the classical limit of the propagator (141). In this limit, the most important contribution comes from the classical path, which makes the Hamiltonian action in the exponent of (141) an extremal. Then, the classical path in the (x, p_x) phase space must satisfy the Hamiltonian equations of motion for the effective Hamiltonian (142). The control $u_P(x, p_x, t)$ is given by equation (55), which for our example is the solution of the constraint equation (121), that is

$$u_P(x, p_x, t) = -\frac{\beta}{B}\, p_x$$

By replacing this u_P in (118), one finds that the effective Hamiltonian $H(x, p_x, t)$ is given by

$$H(x, p_x, t) = -\frac{1}{2} Q x^2 - \frac{1}{2} \frac{\beta^2}{B} p_x^2 + \frac{1}{2} \sigma x^2 p_x. \tag{143}$$

So the Hamiltonian equations of motions are

$$\dot{x} = \frac{\partial H(x, p_x, t)}{\partial p_x} = -\frac{\beta^2}{B} p_x + \frac{1}{2} \sigma x^2 \tag{144}$$

$$\dot{p}_x = -\frac{\partial H(x, p_x, t)}{\partial x} = Qx - \sigma x p_x. \tag{145}$$

By identifying p_x with $-\lambda$, the above Hamiltonian equations can be written as

$$\dot{x} = \frac{\beta^2}{B} \lambda + \frac{1}{2} \sigma x^2 \tag{146}$$

$$-\dot{\lambda} = Qx + \sigma x \lambda, \tag{147}$$

which are the Pontryagin equations (111) and (112). Thus, the constrained quantum theory defined by the propagator (49) has, as its classical limit, the Pontryagin theory.

From the historical point of view [52], [53], the theory of optimal control was developed in the 1950s, in the middle of the Cold War, so any tool that would give some kind of advantage in the race for supremacy was very valuable. That is why the first applications of the theory of optimal control were in aeronautics and aerospace travel, in which the control variable is associated with the propulsion of a rocket (acceleration) and where you want to minimize either the cost of the fuel, or to minimize the flight time.

As expected, the two main actors in the development of the theory belonged to the countries that led the conflict. On the one hand, there was Richard Bellman (USA) who developed his approach to the theory in what is known as 'dynamic programming'.
On the opposite side was the mathematician Lev Pontryagin (USSR), who defined the principle of the maximum. He had made several contributions to algebraic topology, but decided to devote the last years of his life to applied mathematics. At the moment of publishing his work on optimal control theory he was criticized, to the point of its being considered simply an engineering tool rather than a new way of optimization. Later his work was valued and recognized as a new mathematical discipline.

On the other hand, the theory of P. A. M. Dirac on systems with constraints, was developed in the 1950s. In his article, 'Generalized Hamiltonian Dynamics' published in 1950 in the *Canadian Journal of Mathematics*, Dirac built his theory on constrained systems. Its applications in Physics have continued until today, pervading the theory of general relativity, electromagnetism, the gauge models of particle physics, loop quantum gravity, and string theory.

To our knowledge, the first attempt to understand the theory of control as a constrained system from Dirac's point of view, is in Teturo [42]. In this article, the theory of non-linear control given by the Hamilton–Jacobi–Bellman equation, is treated as a linear Schrödinger equation. Here, there is no mention of the fact that there are different possible quantizations in the operator scheme. The role of Planck's constant in the quantization process is not analysed, nor is quantization discussed in terms of path integrals.

Other attempts to understand control theory as a Schrödinger equation, include Kappen [20], in which the stochastic control theory described by the stochastic version of the Hamilton–Jacobi–Bellman equation (for a particular choice of the covariance matrix) is transformed into a linear Schrödinger equation.

Thus, this opens the possibility of seeing whether stochastic control theory is also equivalent to a quantum model with constraints. In future articles we will explore these ideas in detail.

8 Conclusions

In this paper, the classical constrained Pontryagin system has been studied from a quantum point of view. To do that, three different quantizations of the Pontryagin theory have been carried out: the right, left, and Feynman quantizations has been analyzed in detail. As a result, the Bellman theory has been found to correspond to the classical limit $\hbar \to 0$ of these quantum theories. In fact, the phase S of the wave function $\Psi = e^{\frac{i}{\hbar}S}$ satisfies in this classical limit, a conjugate form of the Hamilton–Jacobi–Bellman equation, and this implies that Bellman's cost functional must satisfy the Hamilton–Jacobi–Bellman equation. Thus, the Bellman theory can be viewed as the result of taking the low-energy limit of the quantum behavior associated to the Pontryagin theory.

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Appendix 1: Dirac brackets

In this appendix we show that the time evolution in the phase space of the physical second class system (associated to the optimization problem), is completely equivalent to the dynamics given by the unconstrained Pontryagin's equations.

To do that one must consider the Dirac's bracket $\{A, B\}_{DB}$. In fact, equations (19) and (20) can be used to obtain the Lagrange multipliers μ_1 and μ_2 as

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = -\Delta^{-1} \begin{pmatrix} \{\Phi_1, H_0\} \\ \{\Phi_2, H_0\} \end{pmatrix},$$

that is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\{\Phi_1,\Phi_2\}}\{\Phi_2, H_0\} \\ -\frac{1}{\{\Phi_1,\Phi_2\}}\{\Phi_1, H_0\} \end{pmatrix}. \tag{148}$$

The time evolution of any function $C = C(x, u, p_x, p_u)$ in the phase space generated by the Hamiltonian $\tilde{H}_2$ is

$$\dot{C} = \{C, \tilde{H}_2\} = \{C, H_0\} + \mu_1\{C, \Phi_1\} + \mu_2\{C, \Phi_2\}.$$

Substituting the Lagrangian multipliers (148) in the above equation we obtain

$$\begin{aligned} \dot{C} = \quad & \{C, H_0\} + \{C, \Phi_1\}\frac{1}{\{\Phi_1,\Phi_2\}}\{\Phi_2, H_0\} + \\ & \{C, \Phi_2\}\frac{1}{\{\Phi_2,\Phi_1\}}\{\Phi_1, H_0\}. \end{aligned} \tag{149}$$

The whole expression in the right hand side of (149) is called the Dirac bracket, defined by

$$\begin{aligned} \{A, B\}_{DB} = \quad & \{A, B\} + \{A, \Phi_1\}\Delta_{12}^{-1}\{\Phi_2, B\} + \\ & \{A, \Phi_2\}\Delta_{21}^{-1}\{\Phi_1, B\}, \end{aligned} \tag{150}$$

so we can write (149) as

$$\dot{C} = \{C, H_0\}_{DB}.$$

The dynamical evolution of the variables x, p_x, u, p_u in the phase space, in the presence of the second-class constraints Φ_1, Φ_2, is then given by

$$\begin{aligned} \dot{x} &= \{x, H_0\}_{DB} \\ \dot{p_x} &= \{p_x, H_0\}_{DB} \\ \dot{u} &= \{u, H_0\}_{DB} \\ \dot{p_u} &= \{p_u, H_0\}_{DB}. \end{aligned}$$

The last equation is

$$\dot{p_u} = \{\Phi_1, H_0\}_{DB} = \{\Phi_1, \tilde{H}_2\} = 0, \tag{151}$$

which is consistent with the time preservation of $\Phi_1 = 0$.

The Dirac brackets between the second-class constraints Φ_1, Φ_2 are

$$\begin{aligned}
\{\Phi_1, \Phi_2\}_{DB} = &\ \{\Phi_1, \Phi_2\} + \{\Phi_1, \Phi_1\} \tfrac{1}{\{\Phi_1, \Phi_2\}} \{\Phi_2, \Phi_2\} \\
+ &\ \{\Phi_1, \Phi_2\} \tfrac{1}{\{\Phi_2, \Phi_1\}} \{\Phi_1, \Phi_2\} \\
= &\ 0.
\end{aligned}$$

Thus, the use of the Dirac bracket is equivalent to eliminating the second-class constraints from the theory or, which is the same, to setting all second constraints to zero.

Now we compute explicitly the dynamic behavior of x and p_x for the constrained system:

$$\begin{aligned}
\dot{x} &= \{x, H_0\}_{DB} \\
&= \{x, H_0\} + \{x, \Phi_1\} \tfrac{1}{\{\Phi_1, \Phi_2\}} \{\Phi_2, H_0\} \\
&+ \{x, \Phi_2\} \tfrac{1}{\{\Phi_2, \Phi_1\}} \{\Phi_1, H_0\} \\
&= \tfrac{\partial H_0}{\partial p_x} + \{x, p_u\} \tfrac{1}{\{\Phi_1, \Phi_2\}} \{\Phi_2, H_0\} \\
&+ \{x, \Phi_2\} \tfrac{1}{\{\Phi_2, \Phi_1\}} \{p_u, H_0\},
\end{aligned}$$

but $\{x, p_u\} = 0$ and $\{p_u, H_0\} = -\frac{\partial H_0}{\partial u} = -\Phi_2 = 0$, so

$$\dot{x} = \{x, H_0\}_{DB} = \frac{\partial H_0}{\partial p_x}.$$

$$\dot{x} = f(x, u, t)$$

which is the first Pontryagin equation (3) . In the same way, one obtain, for the momentum,

$$\begin{aligned}
\dot{p}_x &= \{p_x, H_0\}_{DB} \\
&= \{p_x, H_0\} + \{p_x, \Phi_1\} \tfrac{1}{\{\Phi_1, \Phi_2\}} \{\Phi_2, H_0\} \\
&+ \{p_x, \Phi_2\} \tfrac{1}{\{\Phi_2, \Phi_1\}} \{\Phi_1, H_0\} \\
&= -\tfrac{\partial H_0}{\partial x} + \{p_x, p_u\} \tfrac{1}{\{\Phi_1, \Phi_2\}} \{\Phi_2, H_0\} \\
&+ \{p_x, \Phi_2\} \tfrac{1}{\{\Phi_2, \Phi_1\}} \{p_u, H_0\},
\end{aligned}$$

but $\{p_x, p_u\} = 0$, and so

$$\dot{p_x} = \{p_x, H_0\}_{DB} = -\frac{\partial H_0}{\partial x},$$

that is,

$$\dot{p_x} = -\Big(-\frac{\partial F}{\partial x} + p_x\frac{\partial f}{\partial x}\Big)$$

but $p_x = -\lambda$, so

$$-\dot{\lambda} = -\Big(-\frac{\partial F}{\partial x} - \lambda\frac{\partial f}{\partial x}\Big)$$

or

$$\dot{\lambda} = -\Big(\frac{\partial F}{\partial x} + \lambda\frac{\partial f}{\partial x}\Big) = -\frac{\partial H(x,u,\lambda,t)}{\partial x}$$

which is the second Pontryagin equation (4). Thus, the constrained dynamic given by the Dirac bracket is the same unconstrained dynamic system given by the Pontryagin equations.

Appendix 2: Feynman Hamiltonian path integrals

In physics, quantum dynamical behavior is defined by the Hamiltonian operator. For the simple case of a non-relativistic one-dimensional particle subjected to an external potential $U(x)$, the Hamiltonian operator is given by (48), where $p_x = -i\hbar\frac{\partial}{\partial x}$ is the momentum operator. The wave function satisfies the Schrödinger equation (43) and this equation can be integrated to give the wave function at time t if it is known at the initial time $t = 0$, according to [13], [50], [51]

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi_0(x).$$

By inserting the Dirac delta function one can write the above equations as

$$\Psi(x,t) = \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\ \delta(x-x')\Psi_0(x')dx'.$$

But this is just a convolution

$$\Psi(x,t) = \int K(x,t|x't_0)\Psi_0(x')dx', \tag{152}$$

where the propagator K is defined by

$$K(x,t|x't_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\ \delta(x-x') \tag{153}$$

Now, dividing the interval $t - t_0$ into n equal time segments we can write

$$K(x,t|x't_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_n)}e^{-\frac{i}{\hbar}\hat{H}(t-t_{n-1})}...\ e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})}...\ e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}\ \delta(x-x') \tag{154}$$

and by inserting n Dirac delta functions we can write it as

$$\begin{aligned}K(x,t|x't_0) &= \int dx_n \int dx_{n-1}...\ \int dx_k \int dx_{k-1}...\ \int dx_1\quad e^{-\frac{i}{\hbar}\hat{H}(t-t_n)}\delta(x-x_n)\\ &\quad e^{-\frac{i}{\hbar}\hat{H}(t_n-t_{n-1})}\delta(x_n-x_{n-1})\ ...\ e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})}\delta(x_k-x_{k-1})\ ...\\ &\quad e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}\delta(x_2-x_1)\ e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}\ \delta(x_1-x'),\end{aligned}$$

that is,

$$K(x,t|x't_0) = \int ... \int dx_1 \; ... \; dx_n \;\; \Pi_{k=1}^{n} \; e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})}\delta(x_k - x_{k-1}) \tag{155}$$

with $x_n = x$ and $x_0 = x'$. By using the representation of the Dirac delta function as

$$\delta(x_k - x_{k-1}) = \frac{1}{2\pi\hbar}\int dp_k \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} \tag{156}$$

one has that

$$K(x,t|x't_0) = \int ... \int dx_1 \frac{dp_1}{2\pi\hbar} \; ... \; dx_n \frac{dp_n}{2\pi\hbar} \;\; \Pi_{k=1}^{n} \; e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} \tag{157}$$

But note that each $e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})}$ is an eigenstate of the momentum operator $\hat{p}_{x_k}$ with eigenvalue p_k:

$$\hat{p}_{x_k} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} = p_k \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} \tag{158}$$

so

$$\hat{H} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} = \Big(\frac{1}{2m}\hat{p}_{x_k}^2 + U(x)\Big)e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} = H_0(x_k,p_k) \;\; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} \tag{159}$$

then

$$e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} = e^{-\frac{i}{\hbar}H_0(x_k,p_k)(t_k-t_{k-1})} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} \tag{160}$$

wich can be written as

$$e^{-\frac{i}{\hbar}\hat{H}(t_k-t_{k-1})} \; e^{\frac{i}{\hbar}p_k(x_k-x_{k-1})} = e^{\frac{i}{\hbar}\left(p_k\frac{(x_k-x_{k-1})}{(t_k-t_{k-1})} - H_0(x_k,p_k)\right) \; (t_k-t_{k-1})} \tag{161}$$

By substituting in (157) one obtains

$$K(x,t|x't_0) = \int ... \int dx_1 \frac{dp_1}{2\pi\hbar} \; ... \; dx_n \frac{dp_n}{2\pi\hbar} \;\; \Pi_{k=1}^{n} \; e^{\frac{i}{\hbar}\left(p_k\frac{\Delta X_k}{\Delta t_k} - H_0(x_k,p_k)\right) \; \Delta t_k} \tag{162}$$

where $\Delta x_k = (x_k - x_{k-1})$ and $\Delta t_k = (t_k - t_{k-1})$, so

$$K(x,t|x't_0) = \int ... \int dx_1 \frac{dp_1}{2\pi\hbar} \; ... \; dx_n \frac{dp_n}{2\pi\hbar} \;\; e^{\frac{i}{\hbar} \; \Sigma_{k=1}^{n}\left(p_k\frac{\Delta X_k}{\Delta t_k} - H_0(x_k,p_k)\right) \; \Delta t_k} \tag{163}$$

In the limit $n \to \infty$ and $\Delta t_k \to 0$, one finds that the propagator admits the Hamiltonian Feynman path integral representation:

$$K(x,t|x't_0) = \int \mathcal{D}x \frac{\mathcal{D}p_x}{2\pi\hbar} \exp(\frac{i}{\hbar}A[x,p_x]), \tag{164}$$

where $A[x,p_x]$ is the classical Hamiltonian action functional, given by

$$A[x,p_x] = \int_{t_0}^{t} p_x\dot{x} - H(x,p_x)dt, \tag{165}$$

where H is the classical Hamiltonian function.

The symbol $\int \mathcal{D}x\frac{\mathcal{D}p_x}{2\pi\hbar}$ denotes the sum over the set of all trajectories in the phase space. Thus, the quantum propagator is summed over an infinite set of paths.

Appendix 3: Semi-classical approximation

In this appendix we review some important issues related to the Semi-classical approximation, that will be applied to the Feynman quantization of the second class Pontryagin theory.

The most likely quantum path

The most probable path in quantum mechanics is the classical path, which correspond to the trajectory for which the action (165) is extremal, that is

$$\delta A[x,p_x] = A[x+\delta x, p_x+\delta p_x] - A[x,p_x] = 0. \tag{166}$$

Now

$$A[x+\delta x, p_x+\delta p_x] = \tag{167}$$

$$\int_{t_0}^{t_1} -[p_x+\delta p_x][\dot{x}+(\dot{\delta x})] + \ H(x+\delta x, p_x+\delta p_x, t)dt, \tag{168}$$

where δx and δp_x are the corresponding functional variations of the initial variables. Now, expanding the Hamiltonian in a Taylor series and keeping only the first-order terms, we have

$$\delta A = \int_{t_0}^{t_1} \Big[(\frac{\partial H}{\partial p_x} - \dot{x})\delta p_x - p_x \dot{\delta x} + \frac{\partial H}{\partial x}\delta x \Big] dt.$$

Finally, integrating by parts, we get

$$\delta A = \int_{t_0}^{t_1} \Big[(\frac{\partial H}{\partial p_x} - \dot{x})\delta p_x + (\frac{\partial H}{\partial x} + \dot{p_x})\delta x \Big] dt + \big[p_x(t_1)\delta x(t_1) - p_x(t_0)\delta x(t_0) \big] \tag{169}$$

For the action to have an extremal, all the first-order terms of δA in δx and δp_x must vanish. Since the $\delta x(t)$ are independent for all $t \in [t_0, t_1]$, to achieve an extremum one needs

i) $p_x(t_0)\delta x(t_0) = 0$,
ii) $p_x(t_1)\delta x(t_1) = 0$ and
iii) the Hamiltonian equations of motion:

$$\dot{x} = \frac{\partial H(x,p_x)}{\partial p_x}, \tag{170}$$

$$\dot{p_x} = -\frac{\partial H(x,p_x)}{\partial x}. \tag{171}$$

Note that the Hamiltonian equations are first order equations, so to integrate them one need give two initial conditions. For example, for a non-relativistic one-dimensional particle under the action of a conservative force, the classical Hamiltonian is

$$H_0 = \frac{1}{2m} p_x^2 + U(x), \tag{172}$$

so the Hamiltonian equations of motions are

$$\dot{x} = \frac{\partial H(x, p_x)}{\partial p_x} = m p_x \tag{173}$$

$$\dot{p}_x = -\frac{\partial H(x, p_x)}{\partial x} = -\frac{\partial U(x)}{\partial x} \tag{174}$$

By solving p_x from the (173) and substituting in (174), one obtains Newton's law of motion:

$$m\ddot{x} = -\frac{\partial U(x)}{\partial x}. \tag{175}$$

Since this is a second order equation, one needs two initial conditions for $x(t)$ to completely fix the dynamics. For example, one can fix the positions at t_0 and t_1 (see Figure 3a).

The initial conditions needed to determinate a trajectory of the Hamiltonian system must satisfy the transversality conditions (i) and (ii). This restricts the possibles choices and implies that there are only four possibilities for fixing the initial conditions in the Hamiltonian framework. Figures 3, 4, 5 and 6 illustrate these conditions:

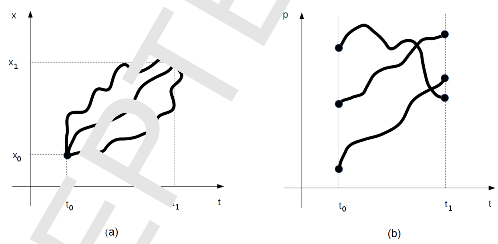

Figure 3: Fixed extremal points for x at t_0, t_1 and unrestricted momentum conditions at t_0, t_1.

Figure 3 show the case in which the extremal points of the trajectory $x(t)$ are fixed (Figure 3a). In this case $\delta x(t_0) = 0$ and $\delta x(t_1) = 0$, so (i) and (ii) are satisfied automatically. Note that the extremal point of the momentum curve $p(t)$ is arbitrary (Figure 3b).

The second possibility is illustrated in Figure 4: in this case the first extremum $t = t_0$ of $x(t)$ is fixed, so $\delta x(t_0) = 0$ and the first transversality condition (i) is satisfied. Instead, the second extremum is free, so $\delta x(t_1)$ is arbitrary, and the second condition (ii) implies that $p(t_1) = 0$. Thus, the second transversality condition fixed the momentum at t_1 to be zero.

The third possibility is given in Figure 5: here, the second extremum $x(t_1)$ is fixed, so $\delta x(t_1) = 0$ and condition (ii) is satisfied, but the first extremum is free, so $\delta x(t_0)$ is arbitrary and (i) implies

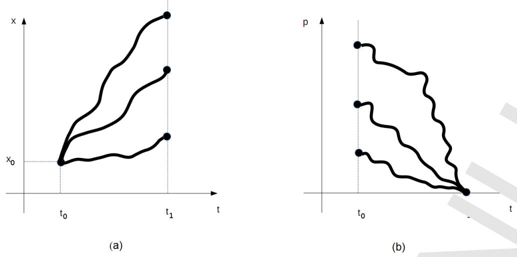

Figure 4: Fixed extremal point for x at t_0, and fixed extremal momentum conditions at t_1.

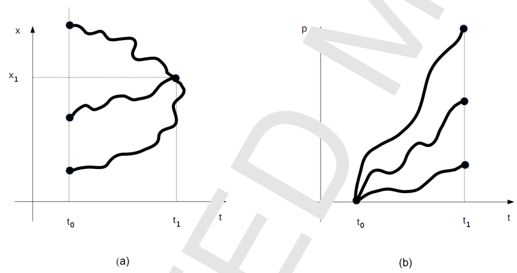

Figure 5: Fixed extremal point for x at t_1, and fixed extremal momentum conditions at t_0.

that $p(t_0) = 0$. Thus, the first transversality condition fixed the momentum at t_0 to be zero.

The last possibility is seen in Figure 6 and corresponds to the case that the extremal points of $x(t)$ are free, so both $\delta x(t_0)$ and $\delta x(t_1)$ are arbitrary and then conditions (i) and (ii) imply that $p(t_0) = 0$ and $p(t_1) = 0$, as shown in Figure 6b.

These four conditions essentially correspond to the Heisenberg uncertainty conditions, that diffuse to the classical world.

For a quadratic Hamiltonian as in (172), the natural possibility is the first one (because the equations of motion for x are second order so one can fix both extrema). Note that the usual initial conditions of Newton's equation (175) $x(t_0) = x_0$ and $v(t_0) = v_0$ satisfy the Hamiltonian equation, but do not provide an extremal for the Hamiltonian action.

But, what about for a linear Hamiltonian? For example, for

$$H(x, p_x) = -F(x, t) + p_x f(x, t) \tag{176}$$

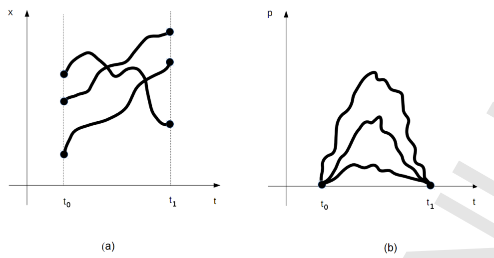

Figure 6: Unrestricted conditions for x at t_0, t_1 and fixed extremal points for p at t_0, t_1.

(this analog to the Hamiltonian (14) appears in optimal control theory). For a such a linear Hamiltonian, Hamilton's equations are

$$\dot{x} = \frac{\partial H(x, p_x)}{\partial p_x} = f(x,t) \tag{177}$$

$$\dot{p}_x = -\frac{\partial H(x, p_x)}{\partial x} = +\frac{\partial F(x,t)}{\partial x} - p_x \frac{\partial f(x,t)}{\partial x} \tag{178}$$

In this case, the equation for $x(t)$ is independent of the equation for $p_x(t)$, so we can not construct a second order equation for $x(t)$ using both Hamiltonian equations. Thus, only one initial condition must be given to integrate the first Hamiltonian equation for $x(t)$. Then, the possible trajectories that satisfy conditions (i) and (ii), in order to provide an extremal of the action in this linear case, are those of Figure 4. This implies that there is only one point $x_P(t_1)$ at which the classical trajectory can end at time t_1. So, the propagation to an arbitrary point $x_1 \neq x_C(t_1)$ at time t_1 is forbidden at the classical level.

The semi-classical approximation

In general, the evaluation of the propagator (164) is a complicated task; one can consider the alternative of obtaining an approximate expression for the quantum propagator. In this section one of these approximations, called the semi-classical approximation [50], [51] will be applied to the Pontryagin theory. In fact, the heart of the semi-classical approximation is to replace the full sum over the set of histories in (164) by the most likely trajectory, that is, the classical one, so in this approximation the propagator K would take the form

$$K = \sum_{path\ joining\ x_0, x_1} e^{\frac{i}{\hbar}A[x(t), p_x(t)]} \sim e^{\frac{i}{\hbar}A[x_C(t), p_{x_C}(t)]} \tag{179}$$

that is,

$$K_{SC} \sim \exp\Big(\frac{i}{\hbar}\int_{t_0}^{t_1} \dot{x}_C p_{x_C} - H(x_C, p_{x_C}) dt\Big), \tag{180}$$

where $x_C(t)$ and $p_{x_C}(t)$ are the solutions of the Hamiltonian equations that makes the action an extremal.

For a quadratic Hamiltonian such as (172), the classical equations of motion are of second order, so the possible classical trajectories $x(t)$ for which the action is an extremum are those of the form in Figure (3), so one always finds a classical trajectory that joins the point $(x_1.t_1)$ and (x_0, t_0). Then, the propagator K_{SC} depends finally on both extremal points $K_{SC} = K_{SC}(x_1t_1|x_0t_0)$, just as in (164). The propagator in fact is interpreted as the quantum transition probability amplitude that the particle goes from $x_0 = x'$ to $x_1 = x$.

In contrast, for the linear Hamiltonian (176), the propagation to an arbitrary point $x_1 \neq x_C(t_1)$ at time t_1 is forbidden. So the propagator would be zero if $x_1 \neq x_C(t_1)$, which means that it must be proportional to $\delta(x_1 - x_C(t_1))$. Thus, for the linear Hamiltonian (176), the semi-classical propagator would be

$$K_{SC}(x_1t_1|x_0t_0) \sim \delta(x_1 - x_C(t_1))\ \exp\Big(\frac{i}{\hbar}\int_{t_0}^{t_1} \dot{x}_C p_{x_C} - H(x_C, p_{x_C})\ dt\Big), \tag{181}$$

or

$$K_{SC}(x_1t_1|x_0t_0) \sim \delta(x_1 - x_C(t_1))\ \exp\Big(\frac{i}{\hbar}\int_{t_0}^{t_1} F(x_C, t) + p_{x_C}(\dot{x}_C - f(x_C, t))\ dt\Big), \tag{182}$$

By the first Hamiltonian equation, $\dot{x}_C - f(x_C, t) = 0$, so [2]

$$K_{SC}(x_1t_1|x_0t_0) \sim \delta(x_1 - x_C(t_1))\ \exp\Big(\frac{i}{\hbar}\int_{t_0}^{t_1} F(x_C, t)\ dt\Big). \tag{183}$$

In the Feynman quantization of the classical constrained Pontryagin theory, the form of the propagator (183) appears again.

[2]Note that in the linear case the exact path integral (164) is

$$K(x_1t_1|x_0t_0) = \int \mathcal{D}x \frac{\mathcal{D}p_x}{2\pi\hbar}\ \ e^{\frac{i}{\hbar}(\int_{t_0}^{t_1} F(x,t) + p_x[\dot{x} - f(x,t)]dt)}.$$

Integrating with respect to p_x gives

$$K(x_1t_1|x_0t_0) = \int \mathcal{D}x\ \ \delta(\dot{x} - f(x,t)) e^{\frac{i}{\hbar}(\int_{t_0}^{t_1} F(x,t)dt)}$$

and integrating with respect to x yields

$$K(x_1t_1, t_1|x_0t_0) = \delta(x_1 - x_C(t_1)) e^{\frac{i}{\hbar}(\int_{t_0}^{t_1} F(x_C,t)dt)}.$$

Thus, for the linear theory, the semi-classical approximation gives the exact result!

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