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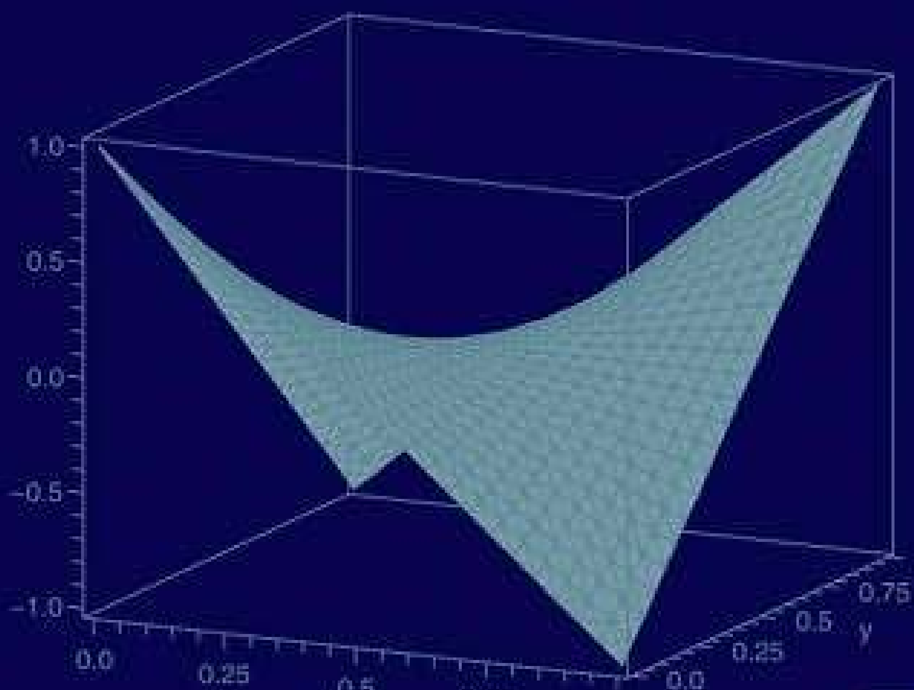
Game Theory

An Introduction

Emmanuel N. Barron



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GAME THEORY
AN INTRODUCTION

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GAME THEORY
AN INTRODUCTION

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To Christina, Michael,
and Anastasia; and Fotini
and Michael

CONTENTS

PREFACE

> Man is a gaming animal. He must always be trying to get the better in something or other.
>
> – Charles Lamb, *Essays of Elia, 1823*

Why do countries insist on obtaining nuclear weapons? What is a fair allocation of property taxes in a community? Should armies ever be divided, and in what way in order to attack more than one target? How should a rat run to avoid capture by a cat? Why do conspiracies almost always fail? What percentage of offensive plays in football should be passes, and what percentage of defensive plays should be blitzes? How should the assets of a bankrupt company be allocated to the debtors? These are the questions that game theory can answer. Game theory arises in almost every facet of human interaction (and inhuman interaction as well). Either every interaction involves objectives that are directly opposed, or the possibility of cooperation presents itself. Modern game theory is a rich area of mathematics for economics, political science, military science, finance, biological science (because of competing species and evolution), and so on.[1]

[1] In an ironic twist, game theory cannot help with most common games, like chess, because of the large number of strategies involved.

This book is intended as a mathematical introduction to the basic theory of games, including noncooperative and cooperative games. The topics build from zero sum matrix games, to nonzero sum, to cooperative games, to population games. Applications are presented to the basic models of competition in economics: Cournot, Bertrand, and Stackelberg models; von Neumann's economic growth model; and so on. The theory of auctions is introduced and the theory of duels is a theme example used in both matrix games, nonzero sum games, and games with continuous strategies.

The prerequisites for this course or book include a year of calculus, and very small parts of linear algebra and probability. For a more mathematical reading of the book, it would be helpful to have a class in advanced calculus, or real analysis. Chapter 6 uses ordinary differential equations. All of these courses are usually completed by the end of the sophomore year, and many can be taken concurrently with this course. Exercises are included at the end of almost every section, and almost all the problems have solutions at the end of the book. I have also included appendixes on the basics of linear algebra, probability, Maple,[2] and Mathematica,[3] commands for the code discussed in the book using Maple.

One of the unique features of this book is the use of Maple,[4] to find the values and strategies of games, both zero and nonzero sum, and noncooperative and cooperative. The major computational impediment to solving a game is the roadblock of solving a linear or nonlinear program. Maple gets rid of those problems and the theories of linear and nonlinear programming do not need to be presented to do the computations. To help present some insight into the basic simplex method which is used in solving matrix games and in finding the nucleolus, a section on the simplex method specialized to solving matrix games is included. If a reader does not have access to Maple or Mathematica, it is still possible to do most of the problems by hand, or using the free software Gambit,[5] or Gams.[6]

The approach I took in the Maple commands in this book is to not reduce the procedure to a canned program in which the student simply enters the matrix and Maple does the rest (Gambit does that). To use Maple and the commands to solve any of the games in this book, the student has to know the procedure, that is, what is going on with the game theory part of it, and then invoke Maple to do the computations.

My experience with game theory for undergraduates is that students greatly enjoy both the theory and applications, which are so obviously relevant and fun. I hope that instructors who offer this course as either a regular part of the curriculum, or as a topics course, will find that this is a very fun class to teach, and maybe to turn students on to a subject developed mostly in this century and still under hot pursuit. I also like to point out to students that they are studying the work of Nobel Prize

[2]Trademark of Maplesoft Corporation.

[3]Trademark of Wolfram Research Corp.

[4]version 10.0.

[5]available from www.gambit.sourceforge.net/.

[6]available from www.gams.com.

winners: Herbert Simon[7] in 1979, John Nash,[8] J. C. Harsanyi[9] and R. Selten[10] in 1994, William Vickrey[11] and James Mirrlees[12] in 1996, and Robert Aumann[13] and Thomas Schelling[14] in 2005. As this book was being written, on October 15, 2007 the Nobel Prize in economics was awarded to game theorists Roger Myerson,[15] Leonid Hurwicz,[16] and Erik Maskin.[17] In addition, game theory was pretty much invented by John von Neumann,[18] one of the true geniuses of the twentieth century.

E. N. BARRON

Chicago, Illinois
2007

[7] June 15, 1916–February 9, 2001, a political scientist who founded organizational decision making.

[8] See the short biography in the Appendix.

[9] May 29, 1920–August 9, 2000, Professor of Economics at University of California, Berkeley, instrumental in equilibrium selection.

[10] Born October 5, 1930, Professor Emeritus, University of Bonn, known for his work on bounded rationality.

[11] June 21, 1914–October 11, 1996, Professor of Economics at Columbia University, known for his work on auction theory.

[12] Born July 5,1936, Professor Emeritus at University of Cambridge.

[13] Born June 8, 1930, Professor at Hebrew University.

[14] Born April 14, 1921, Professor in School of Public Policy, University of Maryland.

[15] Born March 29, 1951, Professor at University of Chicago.

[16] Born August 21, 1917, Regents Professor of Economics Emeritus at the University of Minnesota.

[17] Born December 12, 1950, Professor of Social Science at Institute for Advanced Study, Princeton.

[18] See a short biography in the Appendix and Reference [14] for a full biography.

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E. N. B.

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INTRODUCTION

MOSTLY FOR THE INSTRUCTOR:

My experience is that the students who enroll in game theory are primarily mathematics students interested in applications, with about one-third to one-half of the class majoring in economics or other disciplines (such as biology or biochemistry or physics). The modern economics and operations research curriculum requires more and more mathematics, and game theory is typically a required course in those fields. For economics students with a more mathematical background, this course is set at an ideal level. For mathematics students interested in a graduate degree in something other than mathematics, this course exposes them to another discipline in which they might be interested and that will enable them to further their studies, or simply to learn some fun mathematics. Many students get the impression that applied mathematics is physics or engineering, and this class shows that there are other areas of applications that are very interesting and that opens up many other alternatives to a pure math or classical applied mathematics concentration.

Game theory can be divided into two broad classifications: noncooperative and cooperative. The sequence of main topics in this book is as follows:

1. Two-person zero sum matrix games.

2. Nonzero sum games, both bimatrix, and with a continuum of strategies.

3. Cooperative games, covering both the nucleolus concept and the Shapley value.

4. Bargaining with and without threats.

5. Evolution and population games and the merger with stability theory.

This is generally more than enough to fill one semester, but if time permits (which I doubt) or if the instructor would like to cover other topics (duels, auctions, economic growth, evolutionary stable strategies, population games) these are all presented at some level of depth in this book appropriate for the intended audience.

This book begins with the classical zero sum two-person matrix games, which is a very rich theory with many interesting results. I suggest that the first two chapters be covered in their entirety, although many of the examples can be chosen on the basis of time and the instructor's interest. For classes more mathematically oriented, one could cover the proofs given of the von Neumann minimax theorem. The use of linear programming as an algorithmic method for solving matrix games is essential, but one must be careful to avoid getting sucked into spending too much time on the simplex method. Linear programming is a course in itself, and as long as students understand the transformation of a game into a linear program, they get a flavor of the power of the method. It is a little magical when implemented in Maple, and I give two ways to do it, but there are reasons for preferring one over the other when doing it by hand.

The generalization to nonzero sum two-person games comes next with the foundational idea of a Nash equilibrium introduced. It is an easy extension from a saddle point of a zero sum game to a Nash equilibrium for nonzero sum. Several methods are used to find the Nash equilibria from the use of calculus to full-blown nonlinear programming. Again, Maple is an essential tool in the solution of these problems. Both linear and nonlinear programs are used in this course as a tool to study game theory, and not as a course to study the tools. I suggest that the entire chapter be covered.

It is essential that the instructor cover at least the main points in Chapters 1–5. Chapter 5 can be presented before Chapter 4. Otherwise, Chapter 4 is a generalization of two-person nonzero sum games with a finite number of strategies (basically matrix games) to games with a continuum of strategies. Calculus is the primary method used. The models included in Chapter 4 involve the standard economic models, the theory of duels, which are just games of timing, and the theory of auctions. An entire semester can be spent on this one chapter, so the instructor will probably want to select the applications for her or his own and the class's interest. Students find the economic problems particularly interesting and a very strong motivation for studying both mathematics and economics.

When cooperative games are covered, I present both the theory of the core, leading to the nucleolus, and the very popular Shapley value. Students find the nucleolus extremely computationally challenging because there are usually lots of inequalities to solve and one needs to find a special condition for which the constraints are nonempty. Doing this by hand is not trivial even though the algebra is easy. Once again, Maple can be used as a tool to assist in the solution for problems with four or more players, or even three players. In addition, the graphical abilities of Maple permit a demonstration of the actual shrinking or expanding of the core according to an adjustment of the dissatisfaction parameter. On the other hand, the use of Maple is not a procedure in which one simply inputs the characteristic function and out pops the answer. A student may use Maple to assist but not solve the problem. Chapter 5 ends with a presentation of the theory of bargaining in which nonlinear programming is used to solve the bargaining problem following Nash's ideas. In addition, the theory of bargaining with optimal threat strategies is included. This serves also as a review section because the concepts of matrix games are used for safety levels, saddle points, and so on.

The last chapter serves as a basic introduction to evolutionary stable strategies and population games. If you have a lot of biology or biochemistry majors in your class, you might want to make time for this chapter. The second half of the chapter does require an elementary class in ordinary differential equations. The connection between stability, Nash equilibria, and evolutionary stable strategies can be nicely illustrated with the assistance of the Maple differential equation packages, circumventing the need for finding the actual solution of the equations by hand. Testing for stability is a calculus method. One possible use of this chapter is for projects or extra credit.

My own experience is that I run out of time with a 14-week semester after Chapter 5. Too many topics need to be skipped, but adjusting the topics in different terms makes the course fresh from semester to semester. Of course, topics can be chosen according to your own and the class's interests. On the other hand, this book is not meant to be exhaustive of the theory of games in any way. In particular, I have omitted almost any mention of extensive form games. I consider that subject as a form of dynamic games. It should, or could, be presented in a second or more advanced course.

The prerequisites for this course have been kept to a minimum. This course is presented in our mathematics department, but I have had many economics, biology, biochemistry, business, finance, political science, and physics majors. The prerequisites are listed as a class in probability, and a class in linear algebra, but very little of those subjects are actually used in the class. I tell students that if they know how to multiply two matrices together, they should do fine; and the probability aspects are usually nothing more than the definitions. The important prerequisite is really not being afraid of the concepts. I have had many students with a background of only two semesters of calculus, no probability or linear algebra, or only high school mathematics courses. As a minor reference I include appendixes on linear algebra, probability, a more detailed explanation of some of the Maple commands, and a translation of the major procedures in the text to Mathematica.

The use of Maple in this class is also optional, but then it is like learning multivariable calculus without an ability to graph the surfaces. It can be done, but it is more difficult. Why do that when the tool is available? That may be one of the main features of this book, because before the technology was available, this subject had to be presented in a very mathematical way or a very nonmathematical way. I have tried to take the middle road, but it is not a soft class. My goal is to present the basic concepts of noncooperative and cooperative game theory and introduce students to a very important application of mathematics. In addition, this course introduces students to an understandable theory created by geniuses in many different fields that even today has a low cost of entry.

There are at least two important websites in game theory. The first is

gametheory.net,

which is a repository for all kinds of game theory stuff. I especially like the notes by T. Ferguson at UCLA, W. Bialis at SUNY Buffalo, and Y. Peres, at the University of California, Berkeley. The second site is

www.gambit.sourceforge.net,

which contains the extremely useful open source software **Gambit.exe,** which students may download and install on their own computers. Gambit is a game-solving program that will find the Nash equilibria of all N-person matrix games with any number of strategies. It may also be used to solve any zero sum game by entering the matrix appropriately. I use this software in my classes in addition to Maple, but it is entirely a blackbox. Students love it. Finally, if a user has Mathematica, there is a cooperative game solver available from the Wolfram website, known as **TuGames**, written by Holgar Meinhardt, that can be installed as a Mathematica package. TuGames can solve any characteristic function cooperative game, and much more. In addition, for those interested in the history of game theory see the wonderful resource for game theory due to Professor Paul Walker available at www.econ.canterbury.ac.nz/personal_pages/paul_walker/psw.htm.

Given the limits of my abilities, I expect errors. I would be grateful for any notification of errors, and I will post errata on my website (www.math.luc.edu/~enb). I will also make available for download a Maple worksheet containing the commands in the book.

I will end this introduction with an intriguing quote that Professor Avner Friedman included in his book *Differential Games* that he had me read for my graduate studies. The quote is from Lord Byron: "There are two pleasures for your choosing; the first is winning and the second is losing." Is losing really a pleasure?

CHAPTER 1

MATRIX TWO-PERSON GAMES

If you must play, decide upon three things at the start: the rules of the game, the stakes, and the quitting time.

–Chinese proverb

1.1 THE BASICS

What is a game? We need a mathematical description, but we will not get too technical. A game involves a number of players N, a set of **strategies** for each player, and a payoff that quantitatively describes the outcome of each play of the game in terms of the amount that each player wins or loses. A strategy for each player can be very complicated because it is a plan, determined at the start of the game, that describes what a player will do in every possible situation. In some games this is not too bad because the number of moves is small, but in other games, like chess, the number of moves is huge and so the number of possible strategies, although

Game Theory: An Introduction, First Edition. By E.N. Barron

finite, is gigantic. In this chapter we will consider two-person games and give several examples of exactly what is a strategy.

Let's call the two players I and II. Suppose that player I has a choice of n possible strategies and player II has a choice of m possible strategies. If player I chooses a strategy, say, strategy $i, i = 1, \ldots, n$, and player II chooses a strategy $j, j = 1, \ldots, m$, then they play the game and the payoff to each player is computed. In a **zero sum game**, whatever one player wins the other loses, so if a_{ij} is the amount player I receives, then II gets $-a_{ij}$. Now we have a collection of numbers $\{a_{ij}\}, i = 1, \ldots, n, j = 1, \ldots, m$, and we can arrange these in a matrix. These numbers are called the **payoffs to player I** and the matrix is called the **payoff** or **game matrix**.

	player II			
player I	Strategy 1	Strategy 2	$\cdots$	Strategy m
Strategy 1	a_{11}	a_{12}	$\cdots$	a_{1m}
Strategy 2	a_{21}	a_{22}		a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Strategy n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

By agreement we place player I as the row player and player II as the column player. We also agree that the numbers in the matrix represent the payoff to player I. In a zero sum game the payoffs to player II would be the negative of those in the matrix. Of course, if player I has some payoff which is negative, then player II would have a positive payoff.

Summarizing, a **two-person zero sum game** in matrix form means that there is a matrix $A = (a_{ij}), i = 1, \ldots, n, j = 1, \ldots, m$ of real numbers so that if player I, the row player, chooses to play row i and player II, the column player, chooses to play column j, then the payoff to player I is a_{ij} and the payoff to player II is $-a_{ij}$. Both players want to choose strategies that will maximize their individual payoffs.

Remark: Constant Sum Matrix Games. The discussion has assumed that whatever one player wins the other player loses, that is, that it is zero sum. A slightly larger class of games, the class of **constant sum games**, can also be reduced to this case. This means that if the payoff to player I is a_{ij} when player I uses row i and II uses column j, then the payoff to player II is $C - a_{ij}$, where C is a fixed constant, the same for all rows and columns. In a zero sum game $C = 0$. Now notice that even though this is nonzero sum, player II still gets C minus whatever player I gets. This means that from a game theory perspective, the optimal strategies for each player will not change even if we think of the game as zero sum. If we solve it as if the game were zero sum to get the optimal result for player I, then the optimal result for player II would be simply C minus the optimal result for I.

Now let's be concrete and work out some examples.

EXAMPLE 1.1

Pitching in Baseball. A pitcher has a collection of pitches that he has developed over the years. He can throw a fastball(F), curve(C), or slider(S). The batter he faces has also learned to expect one of these three pitches and to prepare for it. Let's call the batter player I and the pitcher player II. The strategies for each player in this case are simple; the batter looks for either F, C, or S, and the pitcher will decide to use F, C, or S. Here is a possible payoff, or game matrix,[1] to the batter:

I/II	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

For example, if the batter, player I, looks for a fastball and the pitcher actually pitches a fastball, then player I has probability 0.30 of getting a hit. This is a constant sum game because player II's payoff and player I's payoff actually add up to 1. The question for the batter is what pitch to expect and the question for the pitcher is what pitch to throw on each play.

EXAMPLE 1.2

Suppose that two companies are both thinking about introducing competing products into the marketplace. They choose the time to introduce the product, and their choices are 1 month, 2 months, or 3 months. The payoffs correspond to market share.

I/II	1	2	3
1	0.5	0.6	0.8
2	0.4	0.5	0.9
3	0.2	0.7	0.5

So, for instance, if player I introduces the product in 3 months and player II introduces it in 2 months, then it will turn out that player I will get 70% of the market. The companies want to introduce the product in order to maximize their market share. This is also a constant sum game.

EXAMPLE 1.3

Suppose an evader (called Rat) is forced to run a maze entering at point A. A pursuer (called Cat) will also enter the maze at point B. Rat and Cat will run

[1]The numbers in this matrix would be gathered by a statistical analysis of the history of this batter with this pitcher.

exactly four segments of the maze and the game ends. If Cat and Rat ever meet at an intersection point of segments at the same time, Cat wins +1 (Rat) and the Rat loses −1 because it is zero sum, while if they never meet during the run, both Cat and the Rat win 0. In other words, if Cat finds Rat, Cat gets +1 and otherwise, Cat gets 0. We are looking at the payoffs from Cat's point of view, who wants to maximize the payoffs, while Rat wants to minimize them. Figure 1.1 shows the setup.

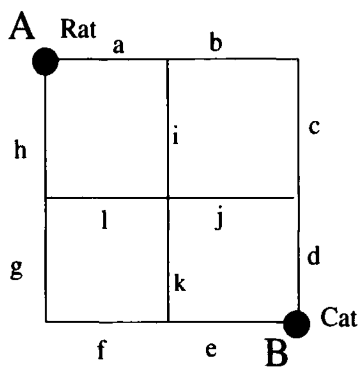


Figure 1.1 Maze for Cat vs. Rat

The strategies for Rat consist of all the choices of paths with four segments that Rat can run. Similarly, the strategies for Cat will be the possible paths it can take. With four segments it will turn out to be a 16×16 matrix. It would look like this:

Cat/Rat	abcd	abcj	⋯	hlkf
dcba	1	1	⋯	0
dcbi	1	1	⋯	0
⋮	⋮	⋮	⋮	⋮
eklg	0	0	⋯	1

The strategies in the preceding examples were fairly simple. The next example gives us a look at the fact that they can also be complicated, even in a simple game.

■ EXAMPLE 1.4

2×2 **Nim.** Four pennies are set out in two piles of two pennies each. Player I chooses a pile and then decides to remove one or two pennies from the pile chosen. Then player II chooses a pile with at least one penny and decides how many pennies to remove. Then player I starts the second round with the same rules. When both piles have no pennies, the game ends and the loser is the player who removed the last penny. The loser pays the winner one dollar.

Strategies for this game for each player must specify what each player will do depending on how many piles are left and how many pennies there are in each pile, at each stage. Let's draw a diagram of all the possibilities (see Figure 1.2).

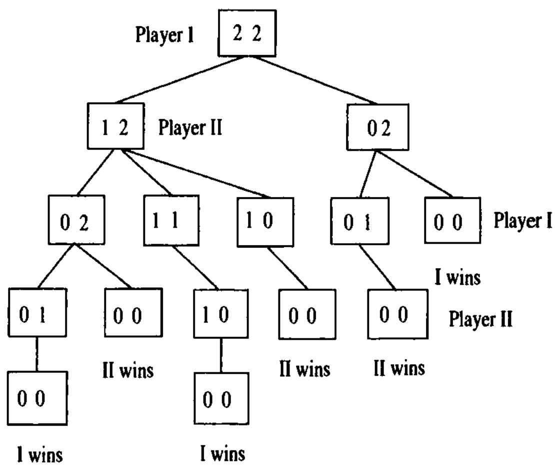


Figure 1.2 2×2 Nim tree

When the game is drawn as a tree representing the successive moves of a player, it is called a game in **extensive form**.

Next we need to write down the strategies for each player:

Strategies for player I
(1) Play (1,2) then, if at (0,2) play (0,1).
(2) Play (1,2) then if at (0,2) play (0,0).
(3) Play (0,2).

You can see that a strategy for I must specify what to do no matter what happens. Strategies for II are even more involved:

Strategies for player II
(1) If at (1,2) → (0,2); if at (0,2) → (0,1)
(2) If at (1,2)→ (1,1); if at (0,2)→ (0,1)
(3) If at (1,2)→ (1,0); if at (0,2)→ (0,1)
(4) If at (1,2)→ (0,2); if at (0,2)→ (0,0)
(5) If at (1,2)→ (1,1); if at (0,2)→ (0,0)
(6) If at (1,2)→ (1,0); if at (0,2)→ (0,0)

Playing strategies for player I against player II results in the payoff matrix for the game, with the entries representing the payoffs to player I.

player I/player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

Analysis of 2 × 2 Nim. Looking at this matrix, we see that player II would never play strategy 5 because player I then wins no matter which row I plays (the payoff is always $+1$). Any rational player in II's position would drop column 5 from consideration (column 5 is called a **dominated strategy**). By the same token, if you look at column 3, this game is finished. Why? Because no matter what player I does, player II by playing column 3 wins $+1$. Player I always loses as long as player II plays column 3; that is, if player I goes to $(1, 2)$, then player II should go to $(1, 0)$. If player I goes to $(0, 2)$, then player II should go to $(0, 1)$. This means that II can always win the game as long as I plays first.

We may also look at this matrix from I's perspective, but that is more difficult because there are times when player I wins and times when player I loses when I plays any fixed row. There is no row that player I can play in which the payoff is always the same and so, in contrast to the column player, no obvious strategy that player I should play.

We say that the **value of this game is** -1 and the strategies

$$(\mathrm{I}1, \mathrm{II}3), (\mathrm{I}2, \mathrm{II}3), (\mathrm{I}3, \mathrm{II}3)$$

are **saddle points**, or **optimal strategies** for the players. We will be more precise about what it means to be optimal in a little while, but for this example it means that player I can improve the payoff if player II deviates from column 3. Notice that there are three saddle points in this example, so saddles are not necessarily unique.

This game is not very interesting because there is always a winning strategy for player II and it is pretty clear what it is. Why would player I ever want to play this game? There are actually many games like this (tic-tac-toe is an obvious example) that are not very interesting because their outcome (the winner and the payoff) is determined as long as the players play optimally.

Chess is not so obvious an example because the number of strategies is so vast that the game cannot, or has not, been analyzed in this way.

One of the main points in this example is the complexity of the strategies. In a game even as simple as tic-tac-toe, the number of strategies is fairly large, and in a game like chess you can forget about writing them all down.

Here is one more example of a game in which we find the matrix by setting up a game tree. Technically, that means that we start with the game in extensive form.

EXAMPLE 1.5

In a version of the game of Russian roulette, made infamous in the movie *The Deer Hunter,* two players are faced with a six-shot pistol loaded with one bullet. The players ante \$1000, and player I goes first. At each play of the game, a player has the option of putting an additional \$1000 into the pot and **passing**, or spinning the chamber and **firing** (at his own head). If player I chooses the option of spinning and survives, then she passes the gun to player II, who has the same two options. If player I does not survive the shot, the game is over and II gets the pot. If player I has chosen to fire and survives, she passes the gun to player II; if player II chooses to fire and survives, the game is over and both players split the pot. If I fires and survives and then II passes, both will split the pot. The effect of this is that II will pay I \$500. On the other hand, if I chooses to pass and II chooses to fire, then, if II survives, he takes the pot. Remember that if either player passes, then that player will have to put

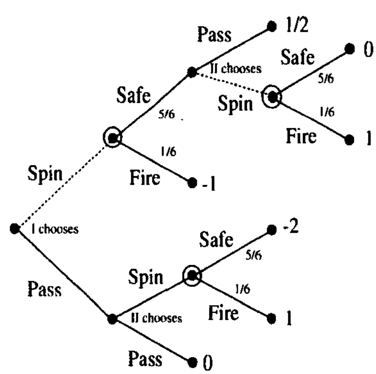


Figure 1.3 Russian roulette

an additional \$1000 in the pot. We begin by drawing in Figure 1.3 the game tree, which is nothing more than a picture of what happens at each stage of the game where a decision has to be made.

The numbers at the end of the branches are the payoffs to player I. The number $\frac{1}{2}$, for example, means that the net gain to player I is \$500 because player II had to pay \$1000 for the ability to pass and they split the pot in this case. The circled nodes are spots at which the next node is decided by chance. You could even consider Nature as another player. We analyze the game by first converting the tree to a game matrix which, in this example becomes

I/II	II1	II2	II3	II4
I1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{36}$
I2	$-\frac{3}{2}$	0	$-\frac{3}{2}$	0

To see how the numbers in the matrix are obtained, we first need to know what the pure strategies are for each player. For player I, this is easy because she makes only one choice and that is pass (I2) or spin (I1). For player II, II1 is the strategy; if I passes, then spin, but if I spins and survives, then pass. So, the expected payoff[2] to I is

$$\text{I1 against II1}: \frac{5}{6}\left(\frac{1}{2}\right) + \frac{1}{6}(-1) = \frac{1}{4}, \text{ and}$$
$$\text{I2 against II1}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

Strategy II3 says the following: If I spins and survives, then spin, but if I passes, then spin and fire. The expected payoff to I is

$$\text{I1 against II3}: \frac{5}{6}\left(\frac{5}{6}(0) + \frac{1}{6}(1)\right) + \frac{1}{6}(-1) = -\frac{1}{36}, \text{ and}$$
$$\text{I2 against II3}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

The remaining entries are left for the reader. The pure strategies for player II are summarized in the following table.

II1	If I2, then S;	If I1, then P.
II2	If I2, then P;	If I1, then P.
II3	If I1, then S;	If I2, then S.
II4	If I1, then S;	If I2, then P.

[2]This uses the fact that if X is a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then $EX = \sum_{i=1}^{n} x_i p_i$. In I1 against II1, X is $\frac{1}{2}$ with probability $\frac{5}{6}$ and -1 with probability $\frac{1}{6}$. See the appendix for more.

This is actually a simple game to analyze because we see that player II will never play II1, II2, or II4 because there is always a strategy for player II in which II can do better. This is strategy II3, which stipulates that if I spins and survives the shot, then II should spin, while if I passes, then II should spin and shoot. If I passes, II gets $\frac{1}{36}$ and I loses $-\frac{1}{36}$. If I spins and shoots, then II gets $\frac{3}{2}$ and I loses $-\frac{3}{2}$. The larger of these two numbers is $-\frac{1}{36}$, and so player I should always spin and shoot. Consequently, player II will also spin and shoot.

The dotted line in Figure 1.3 indicates the optimal strategies. The key to these strategies is that no significant value is placed on surviving.

Remark. Extensive form games can take into account information that is available to a player at each decision node. This is an important generalization. Extensive form games are a topic in sequential decision theory, a second course in game theory.

Finally, we present an example in which is it clear that randomization of strategies must be included as an essential element of games.

■ EXAMPLE 1.6

Evens or Odds. In this game, each player decides to show one, two, or three fingers. If the total number of fingers shown is even, player I wins +1 and player II loses −1. If the total number of fingers is odd, player I loses −1, and player II wins +1. The strategies in this game are simple: deciding how many fingers to show. We may represent the payoff matrix as follows:

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

The row player here and throughout this book will always want to maximize his payoff, while the column player wants to **minimize** the payoff to the row player, so that her own payoff is maximized (because it is a zero sum game). The rows are called the **pure strategies** for player I, and the columns are called the **pure strategies** for player II.

The following question arises: How should each player decide what number of fingers to show? If the row player **always** chooses the same row, say, one finger, then player II can **always** win by showing two fingers. No one would be stupid enough to play like that. So what do we do? In contrast to 2×2 Nim or Russian roulette, there is no obvious strategy that will always guarantee a win for either player.

Even in this simple game we have discovered a problem. If a player always plays the same strategy, the opposing player can win the game. It seems that the only alternative is for the players to mix up their strategies and play some rows and columns sometimes and other rows and columns at other times. Another way to put it is that the only way an opponent can be prevented from learning about your strategy is if you yourself do not know exactly what pure strategy you will use. That only can happen if you choose a strategy randomly. Determining exactly what this means will be studied shortly.

In order to determine what the players should do in any zero sum matrix game, we begin with figuring out a way of seeing if there is an obvious solution. The first step is to come up with a method so that if we have the matrix in front of us we have a systematic and mathematical way of finding a solution in pure strategies, if there is one.

We look at a game with matrix $A = (a_{ij})$ from player I's viewpoint. Player I assumes that player II is playing her best, so II chooses a column j so as to

$$\text{Minimize } a_{ij} \text{ over } j = 1, \ldots, m$$

for any given row i. Then player I can guarantee that he can choose the row i that will maximize this. So player I can **guarantee** that in the **worst possible situation** he can get at least

$$v^- \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij},$$

and we call this number v^- the **lower value of the game.**

Next, consider the game from II's perspective. Player II assumes that player I is playing his best, so that I will choose a row so as to

$$\text{Maximize } a_{ij} \text{ over } i = 1, \ldots, n$$

for any given column $j = 1, \ldots, m$. Player II can therefore choose her column j so as to **guarantee** a loss of no more than

$$v^+ \equiv \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij},$$

and we call this number v^+ the **upper value of the game.** So v^- represents the least amount that player I can be guaranteed to receive and v^+ represents the largest amount that player II can guarantee can be lost. This description makes it clear that we should have $v^- \leq v^+$.

Here is how to find the upper and lower values for any given matrix. In a two-person zero sum game with a finite number of strategies for each player, we write the game matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

For each row, find the minimum payoff in each column and write it in a new additional last column. Then the lower value is the largest number in that last column, that is, the maximum over rows of the minimum over columns. Similarly, in each column find the maximum of the payoffs (written in the last row). The upper value is the smallest of those numbers in the last row.

$$\begin{array}{cccc|l}
a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\
a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\
\vdots & \vdots & \cdots & \vdots & \\
a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\
\hline
\downarrow & \downarrow & \cdots & \downarrow & \\
\max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & \qquad v^- = \text{largest min} \\
 & & & & \qquad v^+ = \text{smallest max}
\end{array}$$

Here is the precise definition.

Definition 1.1.1 *A matrix game with matrix $A_{n\times m} = (a_{ij})$ has the lower value*

$$v^- \equiv \max_{i=1,\dots,n} \min_{j=1,\dots,m} a_{ij}.$$

and the upper value

$$v^+ \equiv \min_{j=1,\dots,m} \max_{i=1,\dots,n} a_{ij},$$

The lower value v^- is the smallest amount that player I is guaranteed to receive (v^- is player I's gain floor), and the upper value v^+ is the guaranteed greatest amount that player II can lose (v^+ is player II's loss ceiling). The **game has a value** *if $v^- = v^+$, and we write it as $v = v(A) = v^+ = v^-$. This means that the smallest max and the largest min must be equal and the row and column i^*, j^* giving the payoffs $a_{i^*,j^*} = v^+ = v^-$ are* **optimal**, *or a* **saddle point in pure strategies**.

One way to look at the value of a game is as a handicap. This means that if the value v is positive, player I should pay player II the amount v in order to make it a **fair game**, with $v = 0$. If $v < 0$, then player II should pay player I the amount $-v$ in order to even things out for player I before the game begins.

■ EXAMPLE 1.7

Let's work this out using 2×2 Nim.

$$\begin{array}{cccccc|cc}
1 & 1 & -1 & 1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & \longrightarrow & \min = -1 \\
\hline
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \boxed{v^- = -1} \\
\max = 1 & \max = 1 & \max = -1 & \max = 1 & \max = 1 & \max = 1 & \boxed{v^+ = -1} &
\end{array}$$

We see that $v^- =$ largest min $= -1$ and $v^+ =$ smallest max $= -1$. This says that $v^+ = v^- = -1$, and so 2×2 Nim has $v = -1$. The optimal strategies are located as the (row,column) where the smallest max is -1 and the largest min is also -1. This occurs at any row for player I, but player II must play column 3, so $i^* = 1, 2, 3,\ j^* = 3$. The optimal strategies are **not** at **any** row column combination giving -1 as the payoff. For instance, if II plays column 1, then II will play row 1 and receive $+1$. Column 1 is not part of an optimal strategy.

We have mentioned that the most that I can be guaranteed to win should be less than (or equal to) the most that II can be guaranteed to lose, (i.e., $v^- \le v^+$), Here is a quick verification of this fact.

For any column j we know that for any fixed row i, $\min_j a_{ij} \le a_{ij}$, and so taking the max of both sides over rows, we obtain

$$v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}.$$

This is true for any column $j = 1, \ldots, m$. The left side is just a number (i.e., v^-) independent of i as well as j, and it is smaller than the right side for any j. But this means that $v^- \le \min_j \max_i a_{ij} = v^+$, and we are done.

Now here is a precise definition of a (pure) saddle point involving only the payoffs, which basically tells the players what to do in order to obtain the value of the game when $v^+ = v^-$.

Definition 1.1.2 *We call a particular row i^* and column j^* a* **saddle point in pure strategies** *of the game if*

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \quad \textit{for all rows } i = 1, \ldots, n \textit{ and columns } j = 1, \ldots, m. \tag{1.1.1}$$

Lemma 1.1.3 *A game will have a saddle point in pure strategies if and only if*

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+. \tag{1.1.2}$$

Proof. If (1.1.1) is true, then

$$v^+ = \min_j \max_i a_{i,j} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$

But $v^- \le v^+$ always, and so we have equality throughout and $v = v^+ = v^- = a_{i^*,j^*}$.

On the other hand, if $v^+ = v^-$ then

$$\min_j \max_i a_{i,j} = \max_i \min_j a_{i,j}.$$

Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$. Then

$$a_{i^*,j} \ge v^- = v^+ \ge a_{i,j^*}, \text{ for any } i = 1, \ldots, n,\ j = 1, \ldots, m.$$

In addition, taking $j = j^*$ on the left, and $i = i^*$ on the right, gives $a_{i^*,j^*} = v^+ = v^-$. This satisfies the condition for (i^*, j^*) to be a saddle point. □

When a saddle point exists in pure strategies, (1.1.1) says that if any player deviates from playing her part of the saddle, then the other player can take advantage and improve his payoff. In this sense, each part of a saddle is a **best response** to the other. This will lead us a little later into considering a best response strategy. The question is that if we are given a strategy for a player, optimal or not, what is the best response on the part of the other player.

We now know that $v^+ \geq v^-$ is always true. We also know how to play if $v^+ = v^-$. The issue is what do we do if $v^+ > v^-$. Consider the following example.

■ **EXAMPLE 1.8**

In the baseball example player I, the batter, expects the pitcher (player II) to throw a fastball, a slider, or a curveball. This is the game matrix.

I/II	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

A quick calculation shows that $v^- = 0.28$ and $v^+ = 0.30$. So baseball does **not** have a saddle point in pure strategies. That shouldn't be a surprise because if there were such a saddle, baseball would be a very dull game, which nonfans say is true anyway. We will come back to this example below.

We end this section with the Maple commands to find the upper and lower values of a matrix game. Let's consider the matrix

$$A = \begin{bmatrix} 2 & -5 \\ -3 & 1 \\ 4 & -3 \end{bmatrix}.$$

A simple calculation shows that $v^- = -3$ and $v^+ = 1$. Here are the Maple commands to get this. These commands would be useful mostly for games with large matrices.

```
> with(LinearAlgebra):
> A:=Matrix([[2,-5],[-3,1],[4,-3]]);
> rows:=3 : cols:=2:
> vu:=min(seq(max(seq(A[i,j],i=1..rows)),j=1..cols));
> vl:=max(seq(min(seq(A[i,j],j=1..cols)),i=1..rows));
> print("the upper value is",vu);
> print("the lower value is",vl);
```

The number of rows and column could be obtained from Maple by using the statement

```
rows:=RowDimension(A); cols:=ColumnDimension(A).
```

PROBLEMS

1.1 In the game rock-paper-scissors both players select one of these objects simultaneously. The rules are as follows: paper beats rock, rock beats scissors, and scissors beats paper. The losing player pays the winner \$1 after each choice of object. What is the game matrix? Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.2 Each of two players must choose a number between 1 and 5. If a player's choice = opposing player's choice $+1$, she loses \$2; if a player's choice $\geq$ opposing player's choice $+2$, she wins \$1. If both players choose the same number the game is a draw. What is the game matrix? Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.3 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess either correctly or incorrectly, the game is a draw. If only one guesses correctly, he wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show, the number of fingers to guess. Find the game matrix, v^+, v^-, and optimal pure strategies if they exist.

1.4 In the Russian roulette example suppose that if player I spins and survives and player II decides to pass, then the net gain to I is \$1000 and so I gets all of the additional money that II had to put into the pot in order to pass. Draw the game tree and find the game matrix. What are the upper and lower values? Find the saddle point in pure strategies.

1.5 Let x be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & x \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ x & 0 \end{bmatrix}.$$

Show that no matter what x is, each matrix has a pure saddle point.

1.6 If we have a game with matrix A and we modify the game by adding a constant C to every element of A, call the new matrix $A + C$, is it true that $v^+(A + C) = v^+(A) + C$?

(a) If it happens that $v^-(A+C) = v^+(A+C)$, will it be true that $v^-(A) = v^+(A)$, and conversely?

(b) What can you say about the optimal pure strategies for $A + C$ compared to the game for just A?

1.7 Consider the square game matrix $A = (a_{ij})$ where $a_{ij} = i - j$ with $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Show that A has a saddle point in pure strategies. Find them and find $v(A)$.

1.8 Player I chooses 1, 2, or 3 and player II guesses which number I has chosen. The payoff to I is |I's number − II's guess|. Find the game matrix. Find v^- and v^+.

1.9 In the Cat versus Rat game determine v^+ and v^- without actually writing out the matrix (if you can; otherwise write it out). It is a 16×16 matrix.

1.10 In a football game the offense has two strategies: run or pass. The defense also has two strategies: defend against the run, or defend against the pass. A possible game matrix is

$$A = \begin{bmatrix} 3 & 6 \\ x & 0 \end{bmatrix}$$

This is the game matrix with the offense as the row player I. The numbers represent the number of yards gained on each play. The first row is run, the second is pass. The first column is defend the run and the second column is defend the pass. Assuming that $x > 0$, find the value of x so that this game has a saddle point in pure strategies.

1.2 THE VON NEUMANN MINIMAX THEOREM

Here now is the problem.[3] What do we do when $v^- < v^+$? If optimal pure strategies don't exist, then how do we play the game? If we use our own experience playing games, we know that it is rarely optimal and almost never interesting to always play the same moves. We know that if a poker player **always bluffs** when holding a weak hand, the power of bluffing disappears. We know that we have to bluff sometimes and hold a strong hand at others. If a pitcher always throws fastballs, it becomes much easier for a batter to get a hit. We know that no pitcher would do that (at least in the majors). We have to mix up the choices. John von Neumann figured out how to model **mixing** strategies in a game mathematically and then proved that if we allow **mixed strategies** in a matrix game, it will always have a value and optimal strategies. The rigorous verification of these statements is not elementary mathematics, but the theorem itself shows us how to make precise the concept of mixing pure strategies. So, you can skip all the proofs and try to understand the hypotheses of von Neumann's theorem. The assumptions of the von Neumann theorem will show us how to solve general matrix games.

We start by considering general functions of two variables $f = f(x, y)$, and give the definition of a saddle point for an arbitrary function f.

[3]This section may be skipped on first reading and you may go directly to the next section without loss of continuity.

Definition 1.2.1 *Let C and D be sets. A function $f : C \times D \to \mathbb{R}$ has at least one saddle point (x^*, y^*) with $x^* \in C$ and $y^* \in D$ if*

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y) \text{ for all } x \in C, y \in D.$$

Once again we could define the upper and lower values for the game defined using the function f, called a **continuous game**, by

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \quad \text{and} \quad v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

You can check as before that $v^- \le v^+$. If it turns out that $v^+ = v^-$ we say, as usual, that the **game has a value** $v = v^+ = v^-$. The next theorem, the most important in game theory and extremely useful in many branches of mathematics is called the **von Neumann minimax theorem**. It gives conditions on $f, C,$ and D so that the associated game has a value $v = v^+ = v^-$. It will be used to determine what we need to do in matrix games in order to get a value.

In order to state the theorem we need to introduce some definitions.

Definition 1.2.2 *A set $C \subset \mathbb{R}^n$ is* **convex** *if for any two points $a, b \in C$ and all scalars $\lambda \in [0, 1]$, the line segment connecting a and b is also in C, i.e., for all $a, b \in C$, $\lambda a + (1 - \lambda) b \in C, \forall\, 0 \le \lambda \le 1$.*

C is closed if it contains all limit points of sequences in C; C is bounded if it can be jammed inside a ball for some large enough radius. A closed and bounded subset of Euclidean space is compact.

A function $g : C \to \mathbb{R}$ is convex if

$$g(\lambda a + (1 - \lambda) b) \le \lambda g(a) + (1 - \lambda) g(b)$$

for any $a, b \in C, 0 \le \lambda \le 1$. This says that the line connecting $g(a)$ with $g(b)$, namely $\{\lambda g(a) + (1 - \lambda) g(b) : 0 \le \lambda \le 1\}$, must always lie above the function values $g(\lambda a + (1 - \lambda) b)$, $0 \le \lambda \le 1$.

The function is concave if $g(\lambda a + (1 - \lambda) b) \ge \lambda g(a) + (1 - \lambda) g(b)$ for any $a, b \in C, 0 \le \lambda \le 1$. A function is strictly convex or concave, if the inequalities are strict.

Figure 1.4 compares a convex set and a nonconvex set. Also, recall the common calculus test for twice differentiable functions of one variable. If $g = g(x)$ is a function of one variable and has at least two derivatives, then g is convex if $g'' \ge 0$ and g is concave if $g'' \le 0$.

Now the basic von Neumann minimax theorem.

Theorem 1.2.3 *Let $f : C \times D \to \mathbb{R}$ be a continuous function. Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be convex, closed, and bounded. Suppose that $x \mapsto f(x, y)$ is concave and $y \mapsto f(x, y)$ is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) = v^-.$$

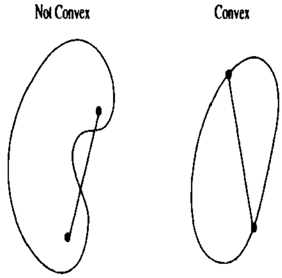


Figure 1.4 Convex and nonconvex sets

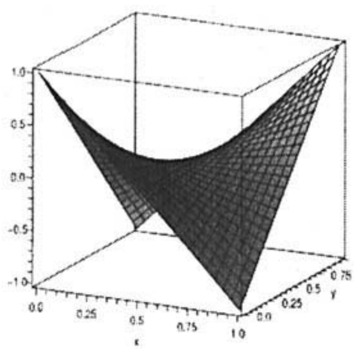
For an example, suppose we look at

$$f(x, y) = 4xy - 2x - 2y + 1 \text{ on } 0 \leq x, y \leq 1.$$

This function has $f_{xx} = 0 \geq 0, f_{yy} = 0 \leq 0$, so it is convex in y for each x and concave in x for each y. Since $(x, y) \in [0, 1] \times [0, 1]$, and the square is closed and bounded, von Neumann's theorem guarantees the existence of a saddle point for this function. To find it, solve $f_x = f_y = 0$ to get $x = y = \frac{1}{2}$. The Hessian for f, which is the matrix of second partial derivatives, is given by

$$H(f, [x, y]) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

Since $\det(H) = -16 < 0$ we are guaranteed by elementary calculus that $(x = \frac{1}{2}, y = \frac{1}{2})$ is an interior saddle for f. Here is a Maple generated picture of f :



Incidentally, another way to write our example function would be

$$f(x, y) = (x, 1 - x)A(y, 1 - y)^T, \text{ where } A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We will see that $f(x, y)$ is constructed from a matrix game in which player I uses the variable **mixed strategy** $X = (x, 1 - x)$, and player II uses the variable **mixed strategy** $Y = (y, 1 - y)$.

Obviously, not all functions will have saddle points. For instance, you will show in an exercise that $g(x, y) = (x - y)^2$ is not concave–convex and in fact does not have a saddle point in $[0, 1] \times [0, 1]$.

Many generalizations of von Neumann's theorem have been given. There are also lots of proofs. We will give a sketch of two [4] of them for your choice; one of them is hard, and the other is easy. You decide which is which. For the time being you can safely skip these proofs because they won't be used elsewhere.

Proof 1. Define the sets of points where the min or max is attained by

$$B_x := \{y^0 \in D : f(x, y^0) = \min_{y \in D} f(x, y)\} \quad \text{for each fixed } x \in C,$$

and

$$A_y := \{x^0 \in C : f(x^0, y) = \max_{x \in C} f(x, y)\} \quad \text{for each fixed } y \in D.$$

By the assumptions on f, C, D, these sets are nonempty, closed, and convex. For instance, here is why B_x is convex. Take $y_1^0, y_2^0 \in B_x$, and let $\lambda \in (0, 1)$. Then

$$f(x, \lambda y_1^0 + (1 - \lambda) y_2^0) \le \lambda f(x, y_1^0) + (1 - \lambda) f(x, y_2^0) = \min_{y \in D} f(x, y).$$

But $f(x, \lambda y_1^0 + (1 - \lambda) y_2^0) \ge \min_{y \in D} f(x, y)$ as well, and so they must be equal. This means that $\lambda y_1^0 + (1 - \lambda) y_2^0 \in B_x$.

Now define $g(x, y) \equiv A_y \times B_x$, which takes a point $(x, y) \in C \times D$ and gives the set $A_y \times B_x$. This function satisfies the continuity properties required by Kakutani's theorem , which is presented below. Furthermore, the sets $A_y \times B_x$ are nonempty, convex, and closed, and so Kakutani's theorem says that there is a point $(x^*, y^*) \in g(x^*, y^*) = A_{y^*} \times B_{x^*}$. Writing out what this says, we get

$$f(x^*, y^*) = \max_{x \in C} f(x, y^*) \quad \text{and} \quad f(x^*, y^*) = \min_{y \in D} f(x^*, y),$$

so that

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) \le f(x^*, y^*) \le \max_{x \in C} \min_{y \in D} f(x, y) = v^- \le v^+$$

and

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y), \ \forall\, x \in C, y \in D.$$

[4] These proofs follow the presentation by S. Karlin [11].

This says that (x^*, y^*) is a saddle point and $v = v^+ = v^- = f(x^*, y^*)$. $\square$

Here is the version of Kakutani's theorem that we are using.

Theorem 1.2.4 *(Kakutani) Let C be a closed, bounded, and convex subset of $\mathbb{R}^n$, and let g be a point (in C) to set (subsets of C) function. Assume that for each $x \in C$, the set $g(x)$ is nonempty and convex. Also assume that g is (upper semi)continuous*[5] *Then there is a point $x^* \in C$ satisfying $x^* \in g(x^*)$.*

This theorem makes the proof of the minimax theorem fairly simple. Kakutani's theorem is a **fixed-point theorem** with a very wide scope of uses and applications. A fixed-point theorem gives conditions under which a function has a point x^* that satisfies $f(x^*) = x^*$, so f fixes the point x^*. In fact, later we will use Kakutani's theorem to show that a generalized saddle point, called **Nash equilibrium**, is a fixed point.

Now for the second proof of von Neumann's theorem, we sketch a proof using only elementary properties of convex functions and some advanced calculus. You may refer to Devinatz [2] for all the calculus used in this book.

Proof 2. 1. Assume first that f is *strictly* concave–convex, meaning that

$$f(\lambda x + (1-\lambda)z, y) > \lambda f(x,y) + (1-\lambda) f(z,y), \; 0 < \lambda < 1,$$
$$f(x, \mu y + (1-\mu)w) < \mu f(x,y) + (1-\mu) f(x,w), \; 0 < \mu < 1.$$

The advantage of doing this is that for each $x \in C$ there is one and only one $y = y(x) \in D$ (y depends on the choice of x) so that

$$f(x, y(x)) = \min_{y \in D} f(x,y) := g(x).$$

This defines a function $g : C \to \mathbb{R}$ that is continuous (since f is continuous on the closed bounded sets $C \times D$ and thus is uniformly continuous). Furthermore, $g(x)$ is concave since

$$g(\lambda x + (1-\lambda)z) \geq \min_{y \in D}(\lambda f(x,y) + (1-\lambda) f(z,y)) \geq \lambda g(x) + (1-\lambda) g(z).$$

So, there is a point $x^* \in C$ at which g achieves its maximum:

$$g(x^*) = f(x^*, y(x^*)) = \max_{x \in C} \min_{y \in D} f(x,y).$$

2. Let $x \in C$ and $y \in D$ be arbitrary. Then, for any $0 < \lambda < 1$, we obtain

$$\begin{aligned} f(\lambda x + (1-\lambda)x^*, y) &> \lambda f(x,y) + (1-\lambda) f(x^*, y) \\ &\geq \lambda f(x,y) + (1-\lambda) f(x^*, y(x^*)) \\ &= \lambda f(x,y) + (1-\lambda) g(x^*). \end{aligned}$$

[5] That is, for any sequences $x_n \in C, y_n \in g(x_n)$, if $x_n \to x$ and $y_n \to y$, then $y \in g(x)$.

Now take $y = y(\lambda x + (1-\lambda)x^*) \in D$ to get

$$\begin{aligned} g(x^*) &\geq f(\lambda x + (1-\lambda)x^*, y(\lambda x + (1-\lambda)x^*)) = g(\lambda x + (1-\lambda)x^*) \\ &\geq g(x^*)(1-\lambda) + \lambda f(x, y(\lambda x + (1-\lambda)x^*)), \end{aligned}$$

where the first inequality follows from the fact that $g(x^*) \geq g(x),\ \forall x \in C$. As a result, we have,

$$g(x^*)[1-(1-\lambda)] = g(x^*)\lambda \geq \lambda f(x, y(\lambda x + (1-\lambda)x^*)),$$

or

$$f(x^*, y(x^*)) = g(x^*) \geq f(x, y(\lambda x + (1-\lambda)x^*)) \text{ for all } x \in C.$$

3. Sending $\lambda \to 0$, we see that $\lambda x + (1-\lambda)x^* \to x^*$ and $y(\lambda x + (1-\lambda)x^*) \to y(x^*)$. We obtain

$$f(x, y(x^*)) \leq f(x^*, y(x^*)) := v, \text{ for any } x \in C.$$

Consequently, with $y^* = y(x^*)$

$$f(x, y^*) \leq f(x^*, y^*) = v, \quad \forall\, x \in C.$$

In addition, since $f(x^*, y^*) = \min_y f(x^*, y) \leq f(x^*, y)$ for all $y \in D$, we get

$$f(x, y^*) \leq f(x^*, y^*) = v \leq f(x^*, y), \forall\, x \in C, y \in D.$$

This says that (x^*, y^*) is a saddle point and the minimax theorem holds, since

$$\min_y \max_x f(x,y) \leq \max_x f(x, y^*) \leq v \leq \min_y f(x^*, y) \leq \max_x \min_y f(x,y),$$

and so we have equality throughout because the right side is always less than the left side.

4. The last step would be to get rid of the assumption of strict concavity and convexity. Here is how it goes. For $\varepsilon > 0$, set

$$f_\varepsilon(x,y) \equiv f(x,y) - \varepsilon|x|^2 + \varepsilon|y|^2, \quad |x|^2 = \sum_{i=1}^n x_i^2, \quad |y|^2 = \sum_{j=1}^m y_j^2.$$

This function will be strictly concave–convex, so the previous steps apply to f_ε. Therefore, we get a point $(x_\varepsilon, y_\varepsilon) \in C \times D$ so that $v_\varepsilon = f_\varepsilon(x_\varepsilon, y_\varepsilon)$ and

$$f_\varepsilon(x, y_\varepsilon) \leq v_\varepsilon = f_\varepsilon(x_\varepsilon, y_\varepsilon) \leq f_\varepsilon(x_\varepsilon, y), \quad \forall\, x \in C, y \in D.$$

Since $f_\varepsilon(x, y_\varepsilon) \geq f(x, y_\varepsilon) - \varepsilon|x|^2$ and $f_\varepsilon(x_\varepsilon, y) \leq f(x_\varepsilon, y) + \varepsilon|y|^2$, we get

$$f(x, y_\varepsilon) - \varepsilon|x|^2 \leq v_\varepsilon \leq f(x_\varepsilon, y) + \varepsilon|y|^2, \quad \forall\, (x,y) \in C \times D.$$

Since the sets C, D are closed and bounded, we take a sequence $\varepsilon \to 0, x_\varepsilon \to x^* \in C, y_\varepsilon \to y^* \in D$ and also $v_\varepsilon \to v \in \mathbb{R}$. Sending $\varepsilon \to 0$, we get

$$f(x, y^*) \le v \le f(x^*, y) \;\; \forall \, (x, y) \in C \times D.$$

This says that $v^+ = v^- = v$ and (x^*, y^*) is a saddle point. □

So, von Neumann's theorem tells us what we need in order to guarantee that our game has a value. It is critical that we are dealing with a concave–convex function, and that the strategy sets be convex. Given a matrix game, how do we guarantee that? That is the subject of the next section.

PROBLEMS

1.11 Let $f(x, y) = x^2 + y^2, C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.12 Let $f(x, y) = x^2 - y^2, C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.13 Let $f(x, y) = (x - y)^2, C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.14 Show that for any matrix $A_{n \times m}$, the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by $f(\vec{x}, \vec{y}) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j = \vec{x} A \vec{y}^T$, is convex in $\vec{y} = (y_1, \ldots, y_m)$ and concave in $\vec{x} = (x_1, \ldots, x_n)$. In fact, it is **bilinear**.

1.15 Show that for any real-valued function $f = f(x, y), x \in C, y \in D$, where C and D are any old sets, it is always true that

$$\max_{x \in C} \min_{y \in D} f(x, y) \le \min_{y \in D} \max_{x \in C} f(x, y).$$

1.16 Verify that if there is $x^* \in C$ and $y^* \in D$ and a real number v so that

$$f(x^*, y) \ge v, \; \forall y \in D, \;\; \text{and} \;\; f(x, y^*) \le v, \; \forall x \in C,$$

then

$$v = f(x^*, y^*) = \max_{x \in C} \min_{y \in D} f(x, y) = \min_{y \in D} \max_{x \in C} f(x, y).$$

1.17 Suppose that $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is strictly concave in $x \in [0, 1]$ and strictly convex in $y \in [0, 1]$ and continuous. Then there is a point (x^*, y^*) so that

$$\min_{y \in [0,1]} \max_{x \in [0,1]} f(x, y) = f(x^*, y^*) = \max_{x \in [0,1]} \min_{y \in [0,1]} f(x, y).$$

In fact, define $y = \varphi(x)$ as the function so that $f(x, \varphi(x)) = \min_y f(x, y)$. This function is well defined and continuous by the assumptions. Also define the function $x = \psi(y)$ by $f(\psi(y), y) = \max_x f(x, y)$. The new function $g(x) = \psi(\varphi(x))$ is then a continuous function taking points in $[0, 1]$ and resulting in points in $[0, 1]$. There is a theorem, called the **Brouwer fixed-point theorem**, which now guarantees that there is a point $x^* \in [0, 1]$ so that $g(x^*) = x^*$. Set $y^* = \varphi(x^*)$. Verify that (x^*, y^*) satisfies the requirements of a saddle point for f.

1.3 MIXED STRATEGIES

Von Neumann's theorem suggests that if we expect to formulate a game model which will give us a saddle point, in some sense, we need convexity of the sets of strategies, whatever they may be, and convexity-concavity of the payoff function, whatever it may be.

Now let's review a bit. In most two-person zero sum games a saddle point in pure strategies will not exist because that would say that the players should **always** do the same thing. Such games, which include 2×2 Nim, tic-tac-toe, and many others, are not interesting when played over and over. It seems that if a player should not always play the same strategy, then there should be some randomness involved, because otherwise the opposing player will be able to figure out what the first player is doing and take advantage of it. A player who chooses a pure strategy randomly chooses a row or column according to some probability process that specifies the chance that each pure strategy will be played. These probability vectors are called **mixed strategies**, and will turn out to be the correct class of strategies for each of the players.

Definition 1.3.1 *A mixed strategy is a vector $X = (x_1, \ldots, x_n)$ for player I and $Y = (y_1, \ldots, y_m)$ for player II, where*

$$x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad y_j \geq 0, \sum_{j=1}^{m} y_j = 1.$$

The components x_i represent the probability that row i will be used by player I, so $x_i = Prob(I \text{ uses row } i)$, and y_j the probability column j will be used by player II, that is, $y_j = Prob(II \text{ uses row } j)$. Denote the set of mixed strategies with k components by

$$S_k \equiv \{(z_1, z_2, \ldots, z_k) \mid z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

In this terminology, a mixed strategy for player I is any element $X \in S_n$ and for player II any element $Y \in S_m$. A pure strategy $X \in S_n$ is an element of *the form* $X = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, *which represents always playing the row corresponding to the position of the* 1 *in* X.

If player I uses the mixed strategy $X = (x_1, \ldots, x_n) \in S_n$ then she will use row i on each play of the game with probability x_i. Every pure strategy is also a mixed strategy by choosing all the probability to be concentrated at the row or column that the player wants to always play. For example, if player I wants to always play row 3, then the mixed strategy she would choose is $X = (0, 0, 1, 0, \ldots, 0)$. Therefore, allowing the players to choose mixed strategies permits many more choices, and the mixed strategies make it possible to mix up the pure strategies used. The set of mixed strategies contains the set of all pure strategies in this sense, and it is a generalization of the idea of strategy.

Now, if the players use mixed strategies the payoff can be calculated only in the **expected** sense. That means the game payoff will represent what each player can expect to receive and will actually receive on average only if the game is played many, many times. More precisely, we calculate as follows.

Definition 1.3.2 *Given a choice of mixed strategy* $X \in S_n$ *for player I and* $Y \in S_m$ *for player II, chosen independently, the* **expected payoff** *to player I of the game is*

$$\begin{aligned} E(X, Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \mathit{Prob}(I \text{ uses } i \text{ and } II \text{ uses } j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \mathit{Prob}(I \text{ uses } i) P(II \text{ uses } j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = X\ A\ Y^T. \end{aligned}$$

In a zero sum two-person game the expected payoff to player II would be $-E(X, Y)$. *The independent choice of strategy by each player justifies the fact that*

$$\mathit{Prob}(I \text{ uses } i \text{ and } II \text{ uses } j) = \mathit{Prob}(I \text{ uses } i) P(II \text{ uses } j).$$

The expected payoff to player I, if I chooses $X \in S_n$ and II chooses $Y \in S_m$, will be

$$E(X, Y) = X\ A\ Y^T = (x_1 \ \cdots \ x_n) A_{n \times m} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

This is written in matrix form with Y^T denoting the transpose of Y, and where X is $1 \times n$, A is $n \times m$, and Y^T is $m \times 1$. If the game is played only once, player I receives exactly a_{ij}, for the pure strategies i and j for that play. Only when the game is played many times can player I **expect**[6] to receive approximately $E(X, Y)$.

In the mixed matrix zero sum game the goals now are that player I wants to maximize his expected payoff and player II wants to minimize the expected payoff to I.

We may define the upper and lower values of the mixed game as

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T, \quad \text{and } v^- = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

We will see shortly, however, that this is really not needed because it is always true that $v^+ = v^-$.

Now we can define what we mean by a saddle point in mixed strategies.

Definition 1.3.3 *A* **saddle point in mixed strategies** *is a pair (X^*, Y^*) of probability vectors $X^* \in S_n, Y^* \in S_m$, which satisfies*

$$E(X, Y^*) \leq E(X^*, Y^*) \leq E(X^*, Y), \ \forall (X \in S_n, Y \in S_m).$$

If player I decides to use a strategy other than X^* but player II still uses Y^*, then I receives an expected payoff smaller than that obtainable by sticking with X^*. A similar statement holds for player II. So (X^*, Y^*) is an **equilibrium** *in this sense.*

Does a game with matrix A have a saddle point in mixed strategies? von Neumann's minimax theorem, Theorem 1.2.3, tells us the answer is "Yes." All we need to do is define the function $f(X, Y) \equiv E(X, Y) = XAY^T$ and the sets S_n for X, and S_m for Y. For any $n \times m$ matrix A, this function is concave in X and convex in Y. Actually, it is even **linear** in each variable when the other variable is fixed. Recall that any linear function is both concave and convex, so our function f is concave–convex and certainly continuous. The second requirement of von Neumann's theorem is that the sets S_n and S_m be convex sets. This is very easy to check and we leave that as an exercise for the reader. These sets are also closed and bounded. Consequently, we may apply the general Theorem 1.2.3 to conclude the following.

Theorem 1.3.4 *For any $n \times m$ matrix A, we have*

$$\min_{Y \in S_m} \max_{X \in S_n} XAY^T = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

The common value is denoted $v(A)$, or $value(A)$, and that is the **value of the game.** *In addition, there is at least one saddle point $X^* \in S_n, Y^* \in S_m$ so that*

$$E(X, Y^*) \leq E(X^*, Y^*) = v(A) \leq E(X^*, Y), \textit{ for all } X \in S_n, Y \in S_m.$$

Remark. The theorem says there is always at least one saddle point in mixed strategies. There could be more than one. If the game happens to have a saddle point

[6] This is a statement in probability theory called the Law of Large Numbers.

in pure strategies, we should be able to discover that by calculating v^+ and v^- using the columns and rows as we did earlier. This is the first thing to check. The theorem is not used for finding the optimal pure strategies, and while it states that there is always a saddle point in mixed strategies, it does not give a way to find them. The next theorem is a step in that direction. First we need some notation that will be used throughout this book.

Notation 1.3.5 *For an $n \times m$ matrix $A = (a_{ij})$ we denote the jth column vector of A by A_j and the ith row vector of A by ${}_iA$. So*

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \quad \textit{and} \quad {}_iA = (a_{i1}, a_{i2}, \cdots, a_{im})$$

If player I decides to use the pure strategy $X = (0, \ldots, 0, 1, 0, \ldots, 0)$ with row i used 100% of the time and player II uses the mixed strategy Y, we denote the expected payoff by $E(i, Y) = {}_iA \cdot Y^T$. Similarly, if player II decides to use the pure strategy $Y = (0, \ldots, 0, 1, 0, \ldots, 0)$ with column j used 100% of the time, we denote the expected payoff by $E(X, j) = XA_j$. So we may also write

$$E(i, Y) = {}_iA \cdot Y^T = \sum_{j=1}^{m} a_{ij} y_j, \; E(X, j) = \sum_{i=1}^{n} x_i a_{ij}, \; \textit{and} \; E(i, j) = a_{ij}.$$

The next lemma says that **mixed against all pure is as good as mixed against mixed.** This lemma will be used in several places in the discussion following. It shows that if an inequality holds for a mixed strategy X for player I, no matter what column is used for player II, then the inequality holds even if player II uses a mixed strategy.

More precisely,

Lemma 1.3.6 *If $X \in S_n$ is any mixed strategy for player I and a is any number so that $E(X, j) \geq a$, $\forall j$, then for any $Y \in S_m$, it is also true that $E(X, Y) \geq a$.*

Here is why. The inequality $E(X, j) \geq a$ means that $\sum_i x_i a_{ij} \geq a$. Now multiply both sides by $y_j \geq 0$ and sum on j to see that

$$E(X, Y) = \sum_j \sum_i x_i a_{ij} y_j \geq \sum_j a y_j = a,$$

because $\sum_j y_j = 1$. Basically, this result says that if X is a good strategy for player I when player II uses any pure strategy, then it is still a good strategy for player I even if player II uses a mixed strategy. Seems obvious.

Our next theorem gives us a way of finding the value and the optimal mixed strategies. From now on, whenever we refer to the value of a game, we are assuming that the value is calculated using mixed strategies.

Theorem 1.3.7 *Let $A = (a_{ij})$ be an $n \times m$ game with value $v(A)$. Let w be a real number. Let $X^* \in S_n$ be a strategy for player I and $Y^* \in S_m$ be a strategy for player II.*

(a) *If $w \leq E(X^*, j) = X^* A_j = \sum_{i=1}^{n} x_i^* a_{ij}$, $j = 1, \ldots, m$, then $w \leq v(A)$.*

(b) *If $w \geq E(i, Y^*) = {}_iA\, Y^{*T} = \sum_{j=1}^{m} a_{ij} y_j^*$, $i = 1, 2, \ldots, n$, then $w \geq v(A)$.*

(c) *If $E(i, Y^*) = {}_iA\, Y^{*T} \leq w \leq E(X^*, j) = X^* A_j, i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then $w = v(A)$ and (X^*, Y^*) is a saddle point for the game.*

(d) *If $v(A) \leq E(X^*, j)$ for all columns $j = 1, 2, \ldots, m$, then X^* is optimal for player I. If $v(A) \geq E(i, Y^*)$ for all rows $i = 1, 2, \ldots, n$, then Y^* is optimal for player II.*

(e) *A strategy X^* for player I is optimal (i.e., part of a saddle point) if and only if $v(A) = \min_{1 \leq j \leq m} E(X^*, j)$. A strategy Y^* for player II is optimal if and only if $v(A) = \max_{1 \leq i \leq n} E(i, Y^*)$.*

Remarks

1. One important way to use the theorem is as a verification tool. If someone says that v is the value of a game and Y is optimal for player II, then you can check it by ensuring that $E(i, Y) \leq v$ for every row. If even one of those is not true, then either Y is not optimal for II, or v is not the value of the game. You can do the same thing given v and an X for player I. For example, let's verify that for the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the optimal strategies are $X^* = Y^* = (\frac{1}{2}, \frac{1}{2})$ and the value of the game is $v(A) = \frac{1}{2}$. All we have to do is check that

$$E(1, Y^*) = {}_1AY^{*T} = \frac{1}{2} \text{ and } E(2, Y^*) = \frac{1}{2},$$
$$E(X^*, 1) = \frac{1}{2} \text{ and } E(X^*, 2) = \frac{1}{2}.$$

Then the theorem guarantees that $v(A) = \frac{1}{2}$ and (X^*, Y^*) is a saddle and you can take it to the bank. Similarly, if we take $X = (\frac{3}{4}, \frac{1}{4})$, then $E(X, 2) = \frac{1}{4} < \frac{1}{2}$, and so, since $v = \frac{1}{2}$ is the value of the game, we know that X is not optimal for player I.

2. Part (c) of the theorem is particularly useful because it gives us a system of inequalities involving $v(A)$, X^*, and Y^*, which, if we can solve them, will give us the value of the game and the saddle points. We will illustrate that below.

Proof of Theorem 1.3.7.
(**a**) Suppose

$$w \le E(X^*, j) = X^* A_j = \sum_{i=1}^{n} x_i^* a_{ij}, j = 1, 2, \dots, m.$$

Let $Y^0 = (y_j) \in S_m$ be an optimal mixed strategy for player II. Multiply both sides by y_j and sum on j to see that

$$w = \sum_j y_j w \le \sum_{j=1}^{m} \sum_{i=1}^{n} x_i^* a_{ij} y_j = X^* A Y^{0T} = E(X^*, Y^0) \le v(A),$$

since $\sum_j y_j = 1$, and since $E(X, Y^0) \le v(A)$ for all $X \in S_n$.

Part (**b**) follows in the same way as (a). □

(**c**) If $\sum_j a_{ij} y_j^* \le w \le \sum_i a_{ij} x_i^*$, we have

$$E(X^*, Y^*) = \sum_i \sum_j x_i^* a_{ij} y_j^* \le \sum_i x_i^* w = w \le \sum_i a_{ij} x_i^*,$$

and

$$E(X^*, Y^*) = \sum_i \sum_j x_i^* a_{ij} y_j^* \ge \sum_j y_j^* w = w \ge E(X^*, Y^*).$$

This says that $w = E(X^*, Y^*)$. So now we have $E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for any row i and column j. Taking now any strategies $X \in S_n$ and $Y \in S_m$ and using Lemma 1.3.6, we get $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$ so that (X^*, Y^*) is a saddle point and $v(A) = E(X^*, Y^*) = w$. □

(**d**) Let $Y^0 \in S_m$ be optimal for player II. Then $E(i, Y^0) \le v(A) \le E(X^*, j)$, for all rows i and columns j, where the first inequality comes from the definition of optimal for player II. Now use part (c) of the theorem to see that X^* is optimal for player I. The second part of (d) is similar. □

(**e**) We begin by establishing that $\min_Y E(X, Y) = \min_j E(X, j)$ for any fixed $X \in S_n$. To see this, since every pure strategy is also a mixed strategy, it is clear that $\min_Y E(X, Y) \le \min_j E(X, j)$. Now set $a = \min_j E(X, j)$. Then

$$0 \le \min_{Y \in S_m} \sum_j (E(X, j) - a) y_j = \min_{Y \in S_m} E(X, Y) - a,$$

since $E(X,j) \geq a$ for each $j = 1,2,\ldots,m$. Consequently, $\min_Y E(X,Y) \geq a$, and putting the two inequalities together, we conclude that $\min_Y E(X,Y) = \min_j E(X,j)$.

Using the definition of $v(A)$, we then have

$$v(A) = \max_X \min_Y E(X,Y) = \max_X \min_j E(X,j).$$

In a similar way, we can also show that $v(A) = \min_Y \max_i E(i,Y)$. Consequently,

$$v(A) = \max_{X \in S_n} \min_{1 \leq j \leq m} E(X,j) = \min_{Y \in S_m} \max_{1 \leq i \leq n} E(i,Y).$$

If X^* is optimal for player I, then

$$v(A) = \max_X \min_Y E(X,Y) \leq \min_Y E(X^*,Y) = \min_j E(X^*,j).$$

On the other hand, if $v(A) \leq \min_j E(X^*,j)$, then $v(A) \leq E(X^*,j)$ for any column, and so $v(A) \leq E(X^*,Y)$ for any $Y \in S_m$, by Lemma 1.3.6, which implies that X^* is optimal for player I. □

The proof of part (e) contained a result important enough to separate into the following corollary.

Corollary 1.3.8 $v(A) = \min_{Y \in S_m} \max_{1 \leq i \leq n} E(i,Y) = \max_{X \in S_n} \min_{1 \leq j \leq m} E(X,j)$. *In addition,* $v^- = \max_i \min_j a_{ij} \leq v(A) \leq \min_j \max_i a_{ij} = v^+$.

Be aware of the fact that not only are the min and max in the corollary being switched but also the sets over which the min and max are taken are changing. The second part of the corollary is immediate from the first part.

Now let's use the theorem to see how we can compute the value and strategies for some games. Essentially, we consider the system of inequations

$$E(X,j) \geq v, j = 1,\ldots,m, \text{ for the unknowns } X = (x_1,\ldots,x_n),$$

along with the condition $x_1 + \cdots + x_n = 1$. We need the last equation because v is also an unknown. If we can solve these inequalities and the x_i variables turn out to be nonnegative, then that gives us a candidate for the optimal mixed strategy for player I, and our candidate for the value $v = v(A)$. Once we know, or think we know $v(A)$, then we can solve the system $E(i,Y) \leq v(A)$ for player II's Y strategy. If all the variables y_j are nonnegative and sum to one, then part (c) of Theorem 1.3.7 tells us that we have the solution in hand and we are done.

■ EXAMPLE 1.9

We start with a simple game with matrix

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix}.$$

Notice that $v^- = -1$ and $v^+ = 3$, so this game does not have a saddle in pure strategies. We will use parts (c) and (e) of Theorem 1.3.7 to find the mixed saddle. Suppose that $X = (x, 1 - x)$ is optimal and $v = v(A)$ is the value of the game. Then $v \le E(X, 1)$ and $v \le E(X, 2)$, which gives us $v \le 4x - 1$ and $v \le -10x + 9$. If we can find a solution of these, with equality instead of inequality, and it is valid (i.e., $0 \le x \le 1$), then we will have found the optimal strategy for player I and $v(A) = v$. So, first try replacing all inequalities with equalities. The equations become

$$v = 4x - 1 \text{ and } v = -10x + 9,$$

and can be solved for x and v to get $x = \frac{10}{14}$ and $v = \frac{26}{14} = v(A)$. Since $X = (\frac{10}{14}, \frac{4}{14})$ is a legitimate strategy and $v = \frac{26}{14}$ satisfies the conditions in Theorem 1.3.7, we know that X is optimal. Similarly $Y = (\frac{10}{14}, \frac{4}{14})$ is optimal for player II.

■ EXAMPLE 1.10

Evens and Odds Revisited. In the game of evens or odds we came up with the game matrix

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

We calculated that $v^- = -1$ and $v^+ = +1$, so this game does not have a saddle point using only pure strategies. But it does have a value and saddle point using mixed strategies. Suppose that v is the value of this game and $(X^* = (x_1, x_2, x_3), Y^* = (y_1, y_2, y_3))$ is a saddle point. According to Theorem 1.3.7 these quantities should satisfy

$$E(i, Y^*) = {}_iA\, Y^{*T} = \sum_{j=1}^{3} a_{ij} y_j \le v \le E(X^*, j) = X^* A_j = \sum_{i=1}^{3} x_i a_{ij},$$
$$i = 1, 2, 3,\ j = 1, 2, 3.$$

Using the values from the matrix, we have the system of inequalities

$$y_1 - y_2 + y_3 \le v, \quad -y_1 + y_2 - y_3 \le v, \quad \text{and } y_1 - y_2 + y_3 \le v,$$
$$x_1 - x_2 + x_3 \ge v, \quad -x_1 + x_2 - x_3 \ge v, \quad \text{and } x_1 - x_2 + x_3 \ge v.$$

Let's go through finding only the strategy X^* since finding Y^* is similar. We are looking for numbers x_1, x_2, x_3 and v satisfying $x_1 - x_2 + x_3 \ge v$, $-x_1 + x_2 - x_3 \ge v$, as well as $x_1 + x_2 + x_3 = 1$ and $x_i \ge 0, i = 1, 2, 3$. But then $x_1 = 1 - x_2 - x_3$, and so

$$1 - 2x_2 \ge v \quad \text{and} \quad -1 + 2x_2 \ge v \implies -v \ge 1 - 2x_2 \ge v.$$

If $v \ge 0$, this says that in fact $v = 0$ and then $x_2 = \frac{1}{2}$. Let's assume then that $v = 0$ and $x_2 = \frac{1}{2}$. This would force $x_1 + x_3 = \frac{1}{2}$ as well.

Instead of substituting for x_1, substitute $x_2 = 1 - x_1 - x_3$ hoping to be able to find x_1 or x_3. You would see that we would once again get $x_1 + x_3 = \frac{1}{2}$. Something is going on with x_1 and x_3, and we don't seem to have enough information to find them. But we can see from the matrix that it doesn't matter whether player I shows one or three fingers! The payoffs in all cases are the same. This means that row 3 (or row 1) is a redundant strategy and we might as well drop it. (We can say the same for column 1 or column 3.) If we drop row 3 we perform the same set of calculations but we quickly find that $x_2 = \frac{1}{2} = x_1$. Of course, we assumed that $v \ge 0$ to get this but now we have our candidates for the saddle points and value, namely, $v = 0$, $X^* = (\frac{1}{2}, \frac{1}{2}, 0)$ and also, in a similar way $Y^* = (\frac{1}{2}, \frac{1}{2}, 0)$. Check that with these candidates the inequalities of Theorem 1.3.7 are satisfied and so they are the actual value and saddle.

However, it is important to remember that with all three rows and columns, the theorem does not give a single characterization of the saddle point. Indeed, there are an infinite number of saddle points, $X^* = (x_1, \frac{1}{2}, \frac{1}{2} - x_1), 0 \le x_1 \le \frac{1}{2}$ and $Y^* = (y_1, \frac{1}{2}, \frac{1}{2} - y_1), 0 \le y_1 \le \frac{1}{2}$. Nevertheless, there is only one value for this, or any matrix game, and it is $v = 0$ in the game of odds and evens.

Later we will see that the theorem gives a method for solving any matrix game if we pair it up with another theory, namely, linear programming, which is a way to optimize a linear function over a set with linear constraints. Linear programming will accommodate the more difficult problem of solving a system of inequalities.

Here is a summary of the basic results in two-person zero sum game theory that we may use to find optimal strategies.

Properties of Optimal Strategies (1.3.1)

1. If w is any number such that $E(i, Y) \leq w \leq E(X, j), i = 1, \ldots, n, j = 1, \ldots, m$, where X is a strategy for player I and Y is a strategy for player II, then $w = value(A)$ and (X, Y) must be a saddle point. This is the way to check whether you have a solution to the game. This is part (c) of Theorem 1.3.7 but worth repeating.

2. If X is a strategy for player I and $value(A) \leq E(X, j), j = 1, \ldots, n$, then X is optimal for player I. If Y is a strategy for player II and $value(A) \geq E(i, Y), i = 1, \ldots, m$, then Y is optimal for player II.

3. If Y is optimal for II and $y_j > 0$, then $E(X, j) = value(A)$ for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$, then $E(i, Y) = value(A)$ for any optimal Y for II. Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value. This result is also called the **Equilibrium Theorem**.

4. If X is any optimal strategy for player I and $E(X, j) > value(A)$ for some column j, then for any optimal strategy Y for player II, we must have $y_j = 0$. Player II would never use column j in any optimal strategy for player II. Similarly, if Y is any optimal strategy for player II and $E(i, Y) < value(A)$, then any optimal strategy X for player I must have $x_i = 0$. If row i for player I gives a payoff when played against an optimal strategy for player II strictly below the value of the game, then player I would never use that row in any optimal strategy for player I.

5. If for any optimal strategy Y for player II, $y_j = 0$, then there is an optimal strategy X for player I so that $E(X, j) > value(A)$. If for any optimal strategy X for I, $x_i = 0$, then there is an optimal strategy Y for II so that $E(i, Y) < value(A)$. This is the converse statement to property 4.

6. If player I has more than one optimal strategy, then player I's set of optimal strategies is a convex, closed, and bounded set. Also, if player II has more than one optimal strategy, then player II's set of optimal strategies is a convex, closed, and bounded) set.

Remarks.

1. These properties and Theorem 1.3.7 give us a way of solving games algebraically without having to solve inequalities. The value of the game, $v(A)$, and the optimal strategies X^* for player I and Y^* for player II must satisfy the system of equations $E(i, Y^*) = v(A)$ for each row with $x_i^* > 0$ and $E(X^*, j) = v(A)$ for every column j with $y_j^* > 0$. Of course, we generally do not know $v(A)$, X^*, and Y^* ahead of time, but if we can solve these equations and then verify optimality using the properties, we have a plan for solving the game.

2. Property 4 is important enough that we should show why it is true.

Proof of Property 4. If it happens that (X^*, Y^*) are optimal and there is a component of $X^* = (x_1, \dots, x_k^*, \dots, x_n)$, say, $x_k^* > 0$ but $E(k, Y^*) < v(A)$, then multiplying both sides of $E(k, Y^*) < v(A)$ by x_k^* yields $x_k^* E(k, Y^*) < x_k^* v(A)$. Now, it is always true that for any row $i = 1, 2, \dots, n$,

$$E(i, Y^*) \leq v(A), \text{ which implies that } x_i E(i, Y^*) \leq x_i v(A).$$

But then, because $v(A) > E(k, Y^*)$ and $x_k^* > 0$, by adding, we get

$$\sum_{i=1, i\neq k}^{n} x_i E(i, Y^*) + x_k^* E(k, Y^*) = \sum_{i=1}^{n} x_i E(i, Y^*) < \sum_{i=1}^{n} x_i v(A) = v(A).$$

We see that, under the assumption $E(k, Y^*) < v(A)$, we have

$$v(A) = E(X^*, Y^*) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j = \sum_{i=1}^{n} x_i E(i, Y^*) < v(A),$$

which is a contradiction. But this means that if $x_k^* > 0$ we must have $E(k, Y^*) = v(A)$. $\square$

■ EXAMPLE 1.11

Let's consider the game with matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

We solve this game by using the properties. First, it appears that every row and column should be used in an optimal mixed strategy for the players, so we conjecture that if $X = (x_1, x_2, x_3)$ is optimal, then $x_i > 0$. This means, by property 3, that we have the following system of equations for $Y = (y_1, y_2, y_3)$:

$$\begin{aligned} E(1, Y) &= 1y_1 + 2y_2 + 3y_3 = v \\ E(2, Y) &= 3y_1 + 1y_2 + 2y_3 = v \\ E(3, Y) &= 2y_1 + 3y_2 + 1y_3 = v \\ y_1 + y_2 + y_3 &= 1. \end{aligned}$$

A small amount of algebra gives the solution $y_1 = y_2 = y_3 = \frac{1}{3}$, and $v = 2$. But, then, Theorem 1.3.7 guarantees that $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is indeed an optimal mixed strategy for player II and $v(A) = 2$ is the value of the game. A similar approach proves that $X = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is also optimal for player I. Incidentally, two very simple Maple commands gives the solution of the system of equations very quickly. Here they are:

```
> eqs:={y1+2*y2+3*y3-v=0,
        3*y1+y2+2*y3-v=0,
        2*y1+3*y2+y3-v=0,
        y1+y2+y3-1=0};
> solve(eqs,[y1,y2,y3,v]);
```

Maple gives us the same solution we got by hand.

■ EXAMPLE 1.12

Let's consider the game with matrix

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

This matrix has a saddle point at $X^* = (0, 1, 0), Y^* = (0, 0, 1)$, and $v(A) = 1$, as you should verify. We will show that player II has an optimal strategy $Y^* = (y_1, y_2, y_3)$, which has $y_j > 0$ for each j. But it is not true that player I has an optimal $X = (x_1, x_2, x_3)$ with $x_i > 0$. In fact, by the equilibrium theorem (1.3.1), property 3, if we assumed that X is optimal and $x_i > 0, i = 1, 2, 3$, then it would have to be true that

$$\begin{aligned} -2y_1 + 2y_2 - y_3 &= 1 \\ y_1 + y_2 + y_3 &= 1 \\ 3y_1 + y_3 &= 1 \end{aligned}$$

because we know that $v = 1$. But there is one and only one solution of this system, and it is given by $Y = (\frac{2}{5}, \frac{4}{5}, -\frac{1}{5})$, which is not a strategy. This means that our assumption about the existence of an optimal strategy for player I with

$x_i > 0, i = 1, 2, 3$, must be wrong. On the other hand, we know that player I has an optimal strategy of $X^* = (0, 1, 0)$, and so, by the equilibrium theorem (1.3.1), property 3, we know that $E(2, Y) = 1$, for an optimal strategy for player II, as well as $E(1, Y) < 1$, and $E(3, Y) < 1$. We need to look for y_1, y_2, y_3 so that

$$y_1 + y_2 + y_3 = 1, \; -2y_1 + 2y_2 - y_3 < 1, 3y_1 + y_3 < 1.$$

We may replace $y_3 = 1 - y_1 - y_2$ and then get a graph of the region of points satisfying all the inequalities in (y_1, y_2) space in Figure 1.5.

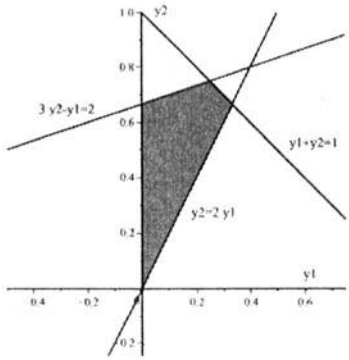


Figure 1.5 Optimal strategy set for Y.

There are lots of points which work. In particular, $Y = (0.15, 0.5, 0.35)$ will give an optimal strategy for player II in which all $y_j > 0$.

1.3.1 Dominated Strategies

Computationally, smaller game matrices are better than large matrices. Sometimes we can reduce the size of the matrix A by eliminating rows or columns (i.e., strategies) that will never be used because there is always a better row or column to use. This is elimination by **dominance.** We should check for dominance whenever we are trying to analyze a game before we begin because it can reduce the size of a matrix.

For example, if every number in row i is bigger than or equal to every corresponding number in row k, specifically $a_{ij} \geq a_{kj}, j = 1, \ldots, m$ (with strict inequality in at least one comparison), then the row player I would never play row k (since she wants the biggest possible payoff), and so we can drop it from the matrix. Similarly, if every number in column j is less than or equal to every corresponding number in column k (i.e., $a_{ij} \leq a_{ik}, i = 1, \ldots, n$), then the column player II would never play column k (since he wants player I to get the smallest possible payoff), and so we can drop it from the matrix. If we can reduce it to a $2 \times m$ or $n \times 2$ game, we can solve it by a graphical procedure that we will consider shortly. If we can reduce it to a 2×2 matrix, we can use the formulas that we derive in the next section. Here is the precise meaning of dominated strategies.

Definition 1.3.9 *Row i dominates row k if $a_{ij} \geq a_{kj}$ for all $j = 1, 2, \ldots, m$. This allows us to remove row k. Column j dominates column k if $a_{ij} \leq a_{ik}, i = 1, 2, \ldots, n$. This allows us to remove column k. Strict dominance means the inequalities are strict in at least one payoff pair in a row or a column.*

Remark. A row that is dropped because it is **strictly dominated** is played in a mixed strategy with probability 0. But a row that is dropped because it is equal to another row may not have probability 0 of being played. For example, suppose that we have a matrix with three rows and row 2 is the same as row 3. If we drop row 3, we now have two rows and the resulting optimal strategy will look like $X^* = (x_1, x_2)$ for the reduced game. Then for the original game the optimal strategy could be $X^* = (x_1, x_2, 0)$ or $X^* = (x_1, x_2/2, x_2/2)$, or in fact $X^* = (x_1, \lambda x_2, (1 - \lambda)x_2)$ for any $0 \leq \lambda \leq 1$. The set of all optimal strategies for player I would consist of all $X^* = (x_1, \lambda x_2, (1 - \lambda)x_2)$ for any $0 \leq \lambda \leq 1$, and this is the most general description. A duplicate row is a redundant row and may be dropped to reduce the size of the matrix. But you must account for redundant strategies.

Another way to reduce the size of a matrix, which is more subtle, is to drop rows or columns by dominance through a convex combination of other rows or columns. If a row (or column) is (strictly) dominated by a convex combination of other rows (or columns), then this row (column) can be dropped from the matrix. If, for example, row k is dominated by a convex combination or two other rows, say, p and q, then we can drop row k. This means that if there is a constant $\lambda \in [0, 1]$ so that

$$a_{kj} \leq \lambda a_{pj} + (1 - \lambda) a_{qj}, \quad j = 1, \ldots, m,$$

then row k is dominated and can be dropped. Of course, if the constant $\lambda = 1$, then row p dominates row k and we can drop row k. If $\lambda = 0$ then row q dominates row k. More than two rows can be involved in the combination.

For columns, the column player wants small numbers, so column k is dominated by a convex combination of columns p and q if

$$a_{ik} \geq \lambda a_{ip} + (1 - \lambda) a_{iq}, \quad i = 1, \ldots, n.$$

It may be hard to spot a combination of rows or columns that dominate, but if there are suspects, the next example shows how to verify it.

EXAMPLE 1.13

Consider the 3×4 game

$$A = \begin{bmatrix} 10 & 0 & 7 & 4 \\ 2 & 6 & 4 & 7 \\ 5 & 2 & 3 & 8 \end{bmatrix}$$

It seems that we may drop column 4 right away because every number in that column is larger than each corresponding number in column 2. So now we have

$$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 5 & 2 & 3 \end{bmatrix}$$

There is no obvious dominance of one row by another or one column by another. However, we suspect that row 3 is dominated by a convex combination of rows 1 and 2. If that is true, we must have, for some $0 \le \lambda \le 1$, the inequalities

$$5 \le \lambda(10) + (1-\lambda)(2), \ 2 \le 0(\lambda) + 6(1-\lambda), \ 3 \le 7(\lambda) + 4(1-\lambda).$$

Simplifying, $5 \le 8\lambda + 2, 2 \le 6 - 6\lambda, 3 \le 3\lambda + 4$. But this says any $\frac{3}{8} \le \lambda \le \frac{2}{3}$ will work. So, there is a λ that works to cause row 3 to be dominated by a convex combination of rows 1 and 2, and row 3 may be dropped from the matrix (i.e., an optimal mixed strategy will play row 3 with probability 0). Remember, to ensure dominance by a convex combination, all we have to show is that there are λs that satisfy all the inequalities. We don't actually have to find them. So now the new matrix is

$$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{bmatrix}$$

Again there is no obvious dominance, but it is a reasonable guess that column 3 is a bad column for player II and that it might be dominated by a combination of columns 1 and 2. To check, we need to have

$$7 \ge 10\lambda + 0(1-\lambda) = 10\lambda, \ \text{ and } \ 4 \ge 2\lambda + 6(1-\lambda) = -4\lambda + 6.$$

These inequalities require that $\frac{1}{2} \le \lambda \le \frac{7}{10}$, which is okay. So there are λs that work, and column 3 may be dropped. Finally, we are down to a 2×2 matrix

$$\begin{bmatrix} 10 & 0 \\ 2 & 6 \end{bmatrix}$$

We will see how to solve these small games graphically in the next section. They may also be solved by assuming that each row and column will be used with positive probability and then solving the system of equations. The answer is that the value of the game is $v(A) = \frac{30}{7}$ and the optimal strategies for the original game are $X^* = (\frac{2}{7}, \frac{5}{7}, 0)$ and $Y^* = (\frac{3}{7}, \frac{4}{7}, 0, 0)$. You should check that statement with part (c) of the Theorem 1.3.7 or property 1, of (1.3.1).

1.4 SOLVING 2 × 2 GAMES GRAPHICALLY

If we have a 2×2 matrix game, we can always find the optimal **mixed strategies** using a graph. It also reveals the basic concepts of how each player looks at the payoffs to determine their optimal plays. We will solve an example to illustrate this method.

Suppose that we have the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

The first step is that we must check whether there are pure optimal strategies because if there are, then we can't use the graphical method. It only finds the strategies that are mixed, but we don't need any fancy methods to find the pure ones. Since $v^- = 2$ and $v^+ = 3$, we know the optimal strategies must be mixed. Now we use Theorem 1.3.7 part (c) to find the optimal strategy $X = (x, 1-x), 0 < x < 1$, for player I, and the value of the game $v(A)$. Here is how it goes.

Playing X against each column for player II, we get

$$E(X,1) = XA_1 = x + 3(1-x) \text{ and } E(X,2) = XA_2 = 4x + 2(1-x).$$

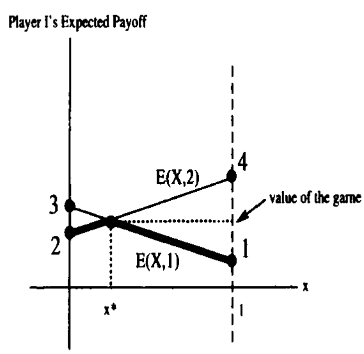


Figure 1.6 X against each column for player II.

We now plot each of these functions of x on the same graph in Figure 1.6. Each plot will be a straight line with $0 \le x \le 1$.

Now, here is how to analyze this graph from player I's perspective. First, the point at which the two lines intersect is $(x^* = \frac{1}{4}, \frac{10}{4})$. If player I chooses an $x < x^*$, then the best I can receive is on the highest line when x is on the left of x^*. In this case the line is $E(X,1) = x + 3(1-x) > \frac{10}{4}$. Player I will receive this higher payoff only if player II decides to play column 1. But player II will also have this graph and II will see that if I uses $x < x^*$ then II should definitely **not** use column 1 but should use column 2 so that I would receive a payoff on the **lower** line $E(X,2) < \frac{10}{4}$. In other words, I will see that II would switch to using column 2 if I chose to use any mixed strategy with $x < \frac{1}{4}$. What if I uses an $x > \frac{1}{4}$? The reasoning is similar; the best I could get would happen if player II chose to use column 2 and then I gets $E(X,2) > \frac{10}{4}$. But I cannot assume that II will play stupidly. Player II will see that if player I chooses to play an $x > \frac{1}{4}$, II will choose to play column 1 so that I receives some payoff on the line $E(X,1) < \frac{10}{4}$.

Conclusion: player I, assuming that player II will be doing her best, will choose to play $X = (x^*, 1-x^*) = (\frac{1}{4}, \frac{3}{4})$ and then receive exactly the payoff= $v(A) = \frac{10}{4}$. Player I will rationally choose the maximum minimum. The minimums are the bold lines and the maximum minimum is at the intersection, which is the highest point of the bold lines. Another way to put it is that player I will choose a mixed strategy so that she will get $\frac{10}{4}$ no matter what player II does, and if II does not play optimally, player I can get more than $\frac{10}{4}$.

What should player II do? Before you read the next section, try to figure it out. That will be an exercise.

PROBLEMS

1.18 Following the same procedure as that for player I, look at $E(i,Y), i = 1,2$. with $Y = (y, 1-y)$. Graph the lines $E(1,Y) = y + 4(1-y)$ and $E(2,Y) = 3y + 2(1-y), 0 \le y \le 1$. Now, how does player II analyze the graph to find Y^*?

1.19 Find the value and optimal X^* for the games with matrices

$$\textbf{(a)} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad \textbf{(b)} \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$$

What, if anything, goes wrong in part (b) if you use the graphical method?

1.20 Let z be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & z \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ z & 0 \end{bmatrix}$$

(a) Find $v(A)$ and $v(B)$ for any z.

(b) Now consider the game with matrix $A+B$. Find a value of z so that $v(A+B) < v(A) + v(B)$ and a value of z so that $v(A+B) > v(A) + v(B)$. Find the values of $A + B$ using the graphical method.

1.21 Suppose that we have the game matrix

$$A = \begin{bmatrix} 13 & 29 & 8 \\ 18 & 22 & 31 \\ 23 & 22 & 19 \end{bmatrix}.$$

Why can this be reduced to $B = \begin{bmatrix} 18 & 31 \\ 23 & 19 \end{bmatrix}$? Now solve the game graphically.

1.5 GRAPHICAL SOLUTION OF 2 $\times$ m AND n $\times$ 2 GAMES

Now we look at the general graphical method that applies whenever one or the other player has only two strategies. This generalizes the 2×2 case we looked at briefly.

Consider the game given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \end{bmatrix}$$

We denote, as usual, by A_j the jth column of the matrix and ${}_iA$ the ith row. We should always check at the outset whether A has a pure strategy saddle and if so, we don't need to apply the graphical method (in fact, it could lead to erroneous results). Therefore, **assuming that there is no pure saddle**, i.e., $v^+ > v^-$, we may approach this problem as follows.

Since player I has only two pure strategies we solve for I's optimal strategy first. Suppose that player I chooses a mixed strategy $X = (x, 1-x), 0 < x < 1$, and player II chooses column j. The payoff to player I is $E(X,j) = XA_j$ or, written out

$$E(X,j) = x\,a_{1j} + (1-x)a_{2j}.$$

Now, because there are only two rows, a mixed strategy is determined by the choice of the single variable $x \in [0,1]$. This is perfect for drawing a plot. On a graph (with x on the horizontal axis), $y = E(X,j)$ is a straight line through the two points $(0, a_{2j})$ and $(1, a_{1j})$. Now do this for each column j and look at

$$f(x) = \min_{1 \le j \le m} X\,A_j = \min_{1 \le j \le m} x\,a_{1j} + (1-x)a_{2j}.$$

This is called the **lower envelope** of all the straight lines associated to each strategy j for player II. Then let $0 \le x^* \le 1$ be the point where the maximum of f is achieved:

$$f(x^*) = \max_{0 \le x \le 1} f(x) = \max_{x} \min_{1 \le j \le m} x\,a_{1j} + (1-x)a_{2j} = \max_{x} \min_{j} E(X,j).$$

This is the maximum minimum of f. Then $X^* = (x^*, 1-x^*)$ is the optimal strategy for player I and $f(x^*)$ will be the value of the game $v(A)$. This is shown in Figure 1.7 for a 2×3 game.

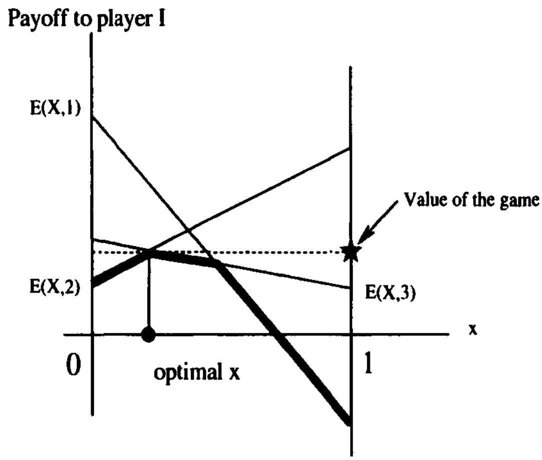


Figure 1.7 Graphical solution for 2 × 3 game.

Each line represents the payoff that player I would receive by playing the mixed strategy $X = (x, 1 - x)$, with player II always playing a fixed column. In the figure you can see that if player I decides to play the mixed strategy $X_1 = (x_1, 1 - x_1)$ where x_1 is to the left of the optimal value, then player II would choose to play column 2. If player I decides to play the mixed strategy $X_2 = (x_2, 1 - x_2)$, where x_2 is to the right of the optimal value, then player II would choose to play column 3, up to the point of intersection where $E(X, 1) = E(X, 3)$, and then switch to column 1. This shows how player I should choose $x, 0 \leq x \leq 1$; player I would choose the x that guarantees that she will receive the maximum of all the lower points of the lines. By choosing this optimal value, say, x^*, it will be the case that player II would play some combination of columns 2 and 3. It would be a mixture (a convex combination) of the columns because if player II always chose to play, say, column 2, then player I could do better by changing her mixed strategy to a point to the right of the optimal value.

Having found the optimal strategy for player I, we must find the optimal strategy for player II. Again, the graph reveals some information. The only two columns being used in an optimal strategy for player I are columns 2 and 3. This implies, by the properties of optimal strategies (1.3.1), that for this particular graph we can eliminate column 1 and reduce to a 2 × 2 matrix. Now let's work through a specific example.

EXAMPLE 1.14

The matrix payoff to player I is

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{bmatrix}$$

Consider the graph for player I first.

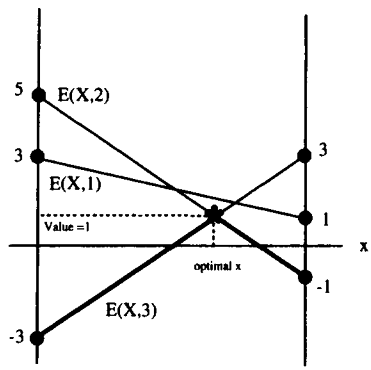


Figure 1.8 Mixed for player I versus player II's columns.

Looking at Figure 1.8, we see that the optimal strategy for I is the x value where the two lower lines intersect and yields $X^* = (\frac{2}{3}, \frac{1}{3})$. Also, $v(A) = E(X^*, 3) = E(X^*, 2) = 1$. To find the optimal strategy for player II, we see that II will never use the first column. The figure indicates that column 1 is dominated by columns 2 and 3 because it is always above the optimal point. In fact, $1 \geq -\lambda + 3(1-\lambda)$ and $3 \geq 5\lambda - 3(1-\lambda)$ implies that for any $\frac{1}{2} \leq \lambda \leq \frac{3}{4}$ the first column is dominated by a convex combination of columns 2 and 3 and may be dropped. So consider the subgame with the first column removed:

$$A1 = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}.$$

Now we will solve this graphically for player II assuming that II uses $Y = (y, 1-y)$. Consider the payoffs $E(1, Y)$ and $E(2, Y)$, which are the payoffs to I if I plays a row and II plays Y. Player II wants to choose y so that no matter what I does she is guaranteed the smallest maximum. This is now the lowest point of the highest part of the lines in Figure 1.9.

We see that the lines intersect with $y^* = \frac{1}{2}$. Hence the optimal strategy for II is $Y^* = (0, \frac{1}{2}, \frac{1}{2})$. Of course, we end up with the same $v(A) = 1$.

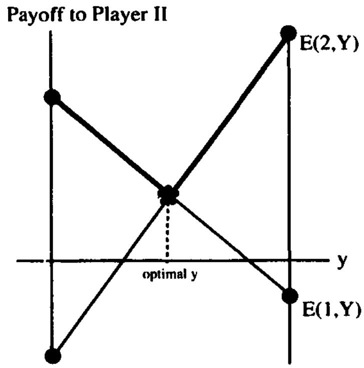


Figure 1.9 Mixed for player II versus I's rows.

The graphical solution of an $n \times 2$ game is similar to the preceding one except that we begin by finding the optimal strategy for player II first because now II has only two pure strategies. Here is an $n \times 2$ game matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}$$

Assume that player II uses the mixed strategy $Y = (y, 1-y), 0 < y < 1$. Then II wants to choose y to **minimize** the quantity

$$\max_{1 \le i \le n} E(i, Y) = \max_{1 \le i \le n} {}_iAY^T = \max_{1 \le i \le n} y(a_{i1}) + (1-y)(a_{i2}).$$

For each row $i = 1, 2, \ldots, n$, the graph of the payoffs (to player I) $E(i, Y)$ will again be a straight line. We will end up with n lines. Player I will want to go as high as possible, and there is not much player II can do about stopping him from doing that. Player II, playing conservatively, will play the mixed strategy Y, which will give the lowest maximum. The optimal y^* will be the point giving the minimum of the upper envelope. Notice that this is a guaranteed optimal strategy because if player II deviates from the lowest maximum, player I can do better. Let's work out another example.

■ **EXAMPLE 1.15**

Let's consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

This is a 4×2 game without a saddle point in pure strategies since $v^- = -1, v^+ = 6$. There is also no obvious dominance, so we try to solve the game graphically. Suppose that player II uses the strategy $Y = (y, 1-y)$, then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4$, as shown in Figure 1.10.

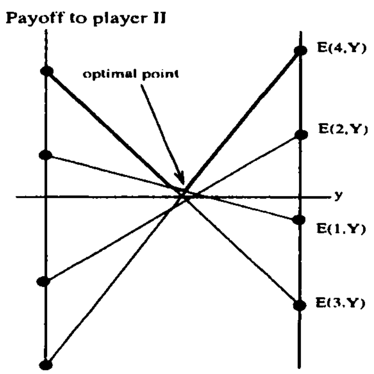


Figure 1.10 Mixed for player II versus 4 rows for player I.

You can see the difficulty with solving games graphically; you have to be very accurate with your graphs. Carefully reading the information, it appears that the optimal strategy for Y will be determined at the intersection point of $E(4, Y) = 7y - 8(1-y)$ and $E(1, Y) = -y + 2(1-y)$. This occurs at the point $y^* = \frac{5}{9}$ and the corresponding value of the game will be $v(A) = \frac{1}{3}$. The optimal strategy for player II is $Y^* = (\frac{5}{9}, \frac{4}{9})$.

Since this uses only rows 1 and 4, we may now drop rows 2 and 3 to find the optimal strategy for player I. In general, we may drop the rows (or columns) not used to get the optimal intersection point. Often that is true because the unused rows are dominated, but not always. To see that here, since $3 \le 7\frac{1}{2} - 1\frac{1}{2}$ and $-4 \le -8\frac{1}{2} + 2\frac{1}{2}$, we see that row 2 is dominated by a convex combination of rows 1 and 4; so row 2 may be dropped. On the other hand, there is no $\lambda \in [0, 1]$ so that $-5 \le 7\lambda - 1(1-\lambda)$ and $6 \le -8\lambda + 2(1-\lambda)$. Row 3 is not dominated by a convex combination of rows 1 and 4, but it is dropped because its payoff line $E(3, Y)$ does not pass through the optimal point.

Considering the matrix using only rows 1 and 4, we now calculate $E(X,1) = -x+7(1-x)$ and $E(X,2) = 2x-8(1-x)$ which intersect at $(x = \frac{5}{6}, v = \frac{1}{3})$. We obtain that row 1 should be used with probability $\frac{5}{6}$ and row 4 should be used with probability $\frac{1}{6}$, so $X^* = (\frac{5}{6}, 0, 0, \frac{1}{6})$. Again, $v(A) = \frac{1}{3}$.

A verification that these are indeed optimal uses Theorem 1.3.7(c). We check that $E(i, Y^*) \leq v(A) \leq E(X^*, j)$ for all rows and columns. This gives

$$\begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{bmatrix}.$$

Everything checks.

We end this section with a simple analysis of a version of poker, at least a small part of it.

■ EXAMPLE 1.16

This is a modified version of the endgame in poker. Here are the rules. Player I is dealt a card that may be an ace or a king. Player I sees the result but II does not. Player I may then choose to fold or bet. If I folds, he has to pay player II \$1. If I bets, player II may choose to fold or call. If II folds, she pays player I \$1. If player II calls and the card is a king, then player I pays player II \$2, but if the card comes up ace, then player II pays player I \$2.

Why wouldn't player I immediately fold when he gets dealt a king? It is the rule that I must pay II \$1 when I gets a king and he folds. Player I is hoping that player II will fold if I bets while holding a king. This is the element of bluffing, because if II calls while I is holding a king, then I must pay II \$2. Figure 1.11 is a graphical representation of the game.

Now player I has four strategies: FF =fold on ace and fold on king, FB=fold on ace and bet on King, BF =bet on ace and fold on king, and BB =bet on ace and bet on king. Player II has only two strategies, namely, F =fold or C =call.

Assuming that the probability of being dealt a king or an ace is $\frac{1}{2}$ we may calculate the expected reward to player I and get the matrix as follows:

I/II	C	F
FF	-1	-1
FB	$-\frac{3}{2}$	0
BF	$\frac{1}{2}$	0
BB	0	1

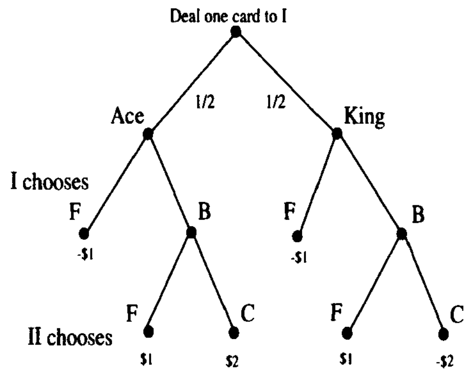


Figure 1.11 A simple poker game. F=fold, B=bet, C=call.

For example, if I plays BF and II plays C, this means that player I will bet if he got an ace, and fold if he got a king. Player II will call no matter what. We calculate the expected payoff to I as $\frac{1}{2} \cdot 2 + \frac{1}{2}(-1) = \frac{1}{2}$. Similarly,

$$E(FB, F) = \frac{1}{2}(-1) + \frac{1}{2} \cdot 1 = 0, \text{ and } E(FB, C) = \frac{1}{2}(-1) + \frac{1}{2}(-2) = -\frac{3}{2},$$

and so on. This is a 4×2, game which we can solve graphically.

1. The lower and upper values are $v^- = 0, v^+ = \frac{1}{2}$, so there is no saddle point in pure strategies, as we would expect. In addition, row 1, namely FF, is a strictly dominated strategy, so we may drop it. It is never worth it to player I to simply fold. Row 2 is also strictly dominated by row 4 and can be dropped. So we are left with considering the 2×2 matrix

$$A' = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

2. Suppose that II plays $Y = (y, 1 - y)$. Then

$$E(BF, Y) = A'Y^T = \frac{1}{2}\, y \text{ and } E(BB, Y) = (1 - y)$$

There are only two lines to consider in a graph for the game and these two lines intersect at $\frac{1}{2}y = 1 - y$, so that $y^* = \frac{2}{3}$. The optimal strategy for II is $Y^* = (\frac{2}{3}, \frac{1}{3})$, so II should call two-thirds of the time and bet one-third of the time. The value of the game is then at the point of intersection $v = \frac{1}{3}$. Notice that when there are only two lines, there is no need to actually draw the graph because we know that the optimal point must be at the point of intersection of the two lines, and no other lines are involved.

3. For player I, suppose that he plays $X = (x, 1 - x)$. Then

$$E(X, C) = XA' = \frac{1}{2}\,x \text{ and } E(X, F) = 1 - x.$$

Since there are only two lines, we again calculate the intersection point and obtain the optimal strategy for I as $X^* = (0, 0, \frac{2}{3}, \frac{1}{3})$.

Player II is at a distinct disadvantage since the value of this game is $v = \frac{1}{3}$. Player II in fact would never be induced to play the game unless player I pays II exactly $\frac{1}{3}$ before the game begins. That would make the value zero and hence a fair game.

Now we see a very interesting phenomenon that is well known to poker players. Namely, the optimal strategy for player I has him betting one-third of the time when he has a losing card (king). We expected some type of result like this because of the incentive for player I to bet even when he has a king. So bluffing with positive probability is a part of an optimal strategy when done in the right proportion.

PROBLEMS

1.22 In the 2×2 Nim game we saw that $v^+ = v^- = -1$. Reduce the game matrix using dominance.

1.23 Consider the matrix game

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) Find $v(A)$ and the optimal strategies.

(b) Show that $X^* = (\frac{1}{2}, \frac{1}{2}), Y^* = (1, 0)$ is not a saddle point for the game even though it does happen that $E(X^*, Y^*) = v(A)$.

1.24 Use the methods of this section to solve the games

$$\textbf{(a)} \begin{bmatrix} 4 & -3 \\ -9 & 6 \end{bmatrix}, \qquad \textbf{(b)} \begin{bmatrix} 4 & 9 \\ 6 & 2 \end{bmatrix}, \qquad \textbf{(c)} \begin{bmatrix} -3 & -4 \\ -7 & 2 \end{bmatrix}.$$

1.25 The third column of the matrix

$$A = \begin{bmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{bmatrix}$$

is dominated by a convex combination. Reduce the matrix and solve the game.

1.26 Show that for any strategy $X = (x_1, \ldots, x_n) \in S_n$ and any numbers $b_1, \ldots, b_n$, it must be that

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \max_{1 \le i \le n} b_i \text{ and } \min_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \min_{1 \le i \le n} b_i.$$

1.27 The properties of optimal strategies (1.3.1) show that $X^* \in S_n$ and $Y^* \in S_m$ are optimal if and only if $\min_j E(X^*, j) = \max_i E(i, Y^*)$. The common value will be the value of the game. Verify this.

1.28 Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*). In fact, (X_λ, Y_λ) where $X_\lambda = \lambda X^* + (1 - \lambda) X^0, Y_\lambda = \lambda Y^* + (1 - \lambda) Y^0$ and λ is any number in $[0, 1]$.

1.29 Consider the game with matrix

$$A = \begin{bmatrix} -2 & 3 & 5 & -2 \\ 3 & -4 & 1 & -6 \\ -5 & 3 & 2 & -1 \\ -1 & -3 & 2 & 2 \end{bmatrix}.$$

Someone claims that the strategies $X^* = (\frac{1}{9}, 0, \frac{8}{9}, 0)$ and $Y^* = (0, \frac{7}{9}, \frac{2}{9}, 0)$ are optimal.

(a) Is that correct? Why or why not? (**Hint**: Use a previous problem.)
(b) If $X^* = (\frac{13}{33}, \frac{5}{33}, 0, \frac{15}{33})$ is optimal and $v(A) = -\frac{26}{33}$, find Y^*.

1.30 In the baseball game Example 1.8 it turns out that an optimal strategy for player I, the batter, is given by $X^* = (x_1, x_2, x_3) = (\frac{2}{7}, 0, \frac{5}{7})$ and the value of the game is $v = \frac{2}{7}$. It is amazing that the batter should never expect a curveball with these payoffs under this optimal strategy. What is the pitcher's optimal strategy Y^*?

1.31 In a football game we use the matrix $A = \begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}$. The first row and column represent run, and the second row and column represent pass. The offense is the row player. Column pass means defend against the pass. Use the graphical method to solve this game.

1.6 BEST RESPONSE STRATEGIES

If you are playing a game and you determine, in one way or another, that your opponent is using a particular strategy, or is assumed to use a particular strategy, then what should you do? To be specific, suppose that you are player I and you know, or simply assume, that player II is using the mixed strategy Y, optimal or not for player II. In this case you should play the mixed strategy X that maximizes $E(X, Y)$. This strategy that you use would be a **best response** to the use of Y by player II. The best response strategy to Y may not be the same as what you would use if you knew that player II were playing optimally; that is, it may not be a part of a saddle point. Here is the precise definition.

Definition 1.6.1 *A mixed strategy X^* for player I is a* **best response strategy** *to the strategy Y for player II if it satisfies*

$$\max_{X \in S_n} E(X, Y) = \max_{X \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^* a_{ij} y_j = E(X^*, Y).$$

A mixed strategy Y^* for player II is a best response strategy to the strategy X for player I if it satisfies

$$\min_{Y \in S_m} E(X, Y) = \min_{Y \in S_m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j^* = E(X, Y^*).$$

Incidentally, if (X^*, Y^*) is a saddle point of the game, then X^* is the best response to Y^*, and vice versa. Unfortunately, knowing this doesn't provide a good way to calculate X^* and Y^* because they are **both** unknown at the start.

■ **EXAMPLE 1.17**

Consider the 3×3 game

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

The saddle point is $X^* = (0, \frac{1}{2}, \frac{1}{2}) = Y^*$ and $v(A) = 1$. Now suppose that player II, for some reason, thinks she can do better by playing $Y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. What is an optimal response strategy for player I?

Let $X = (x_1, x_2, 1 - x_1 - x_2)$. Calculate

$$E(X, Y) = XAY^T = -\frac{x_1}{4} - \frac{x_2}{2} + \frac{5}{4}.$$

We want to maximize this as a function of x_1, x_2 with the constraints $0 \leq x_1, x_2 \leq 1$. We see right away that $E(X, Y)$ is maximized by taking $x_1 = x_2 = 0$ and then necessarily $x_3 = 1$. Hence, the best response strategy for player I if player II uses $Y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is $X^* = (0, 0, 1)$. Then, using this strategy, the expected payoff to I is $E(X^*, Y) = \frac{5}{4}$, which is larger than the value of the game $v(A) = 1$. So that is how player I should play if player II decides to deviate from the optimal Y. This shows that any deviation from a saddle could result in a better payoff for the opposing player. However, if one player knows that the other player will not use her part of the saddle, then the best response may not be the strategy used in the saddle. In other words, if (X^*, Y^*) is a saddle point, the best response to $Y \neq Y^*$ may not be X^*, but some other X, even though it will be the case that $E(X^*, Y) \geq E(X^*, Y^*)$.

Because $E(X, Y)$ is linear in each strategy when the other strategy is fixed, the best response strategy for player I will usually be a pure strategy. For instance, if Y is given, then $E(X, Y) = ax_1 + bx_2 + cx_3$, for some values a, b, c that will depend on Y and the matrix. The maximum payoff is then achieved by looking at the largest of a, b, c, and taking $x_i = 1$ for the x multiplying the largest of a, b, c, and the remaining values of $x_j = 0$. How do we know that? Well, to show you what to do in general, we will show that

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\ = \max\{a, b, c\} \tag{1.6.1}$$

Suppose that $\max\{a, b, c\} = c$ for definiteness. Now, by taking $x_1 = 0, x_2 = 0, x_3 = 1$, we get

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\ \geq a \cdot 0 + b \cdot 0 + c \cdot 1 = c.$$

On the other hand, since $x_1 + x_2 + x_3 = 1$, we see that

$$ax_1 + bx_2 + c(1 - x_1 - x_2) = x_1(a - c) + x_2(b - c) + c \leq c,$$

since $a - c < 0, b - c < 0$ and $x_1, x_2 \geq 0$. So, we conclude that

$$c \geq \max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \geq c,$$

and this establishes (1.6.1). This shows that $X^* = (0, 0, 1)$ is a best response to Y.

It is possible to get a mixed strategy best response but only if some or all of the coefficients a, b, c are equal. For instance, if $b = c$, then

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} = \max\{a, c\}$$

To see this, suppose that $\max\{a, c\} = c$. We compute

$$\begin{aligned} &\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\ &= \max\{ax_1 + c(x_2 + x_3) \mid x_1 + x_2 + x_3 = 1\} \\ &= \max\{ax_1 + c(1 - x_1) \mid 0 \leq x_1 \leq 1\} \\ &= \max\{x_1(a - c) + c \mid 0 \leq x_1 \leq 1\} \\ &= c. \end{aligned}$$

This maximum is achieved at $X^* = (0, x_2, x_3)$ for any $x_2 + x_3 = 1, x_2 \geq 0, x_3 \geq 0$, and we see that we can get a mixed strategy as a best response.

In general, **if one of the strategies, say, Y is given and known**, then

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ij} y_j \right) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{m} a_{ij} y_j \right).$$

In other words,

$$\max_{X \in S_n} E(X, Y) = \max_{1 \leq i \leq n} E(i, Y).$$

We proved this in the proof of Theorem 1.3.7, part (e).

Best response strategies are frequently used when we assume that the opposing player is Nature or some nebulous player that we think may be trying to oppose us, like the market in an investment game, or the weather. The next example helps to make this more precise.

■ EXAMPLE 1.18

Suppose that player I has some money to invest with three options: stock(S), bonds(B), or CDs (certificates of deposit). The rate of return depends on the state of the market for each of these investments. Stock is considered risky, bonds have less risk than stock, and CDs are riskless. The market can be in one of three states: good(G), neutral(N), or bad(B), depending on factors such as the direction of interest rates, the state of the economy, prospects for future growth. Here is a possible game matrix in which the numbers represent the annual rate of return to the investor who is the row player:

I/II	G	N	B
S	12	8	-5
B	4	4	6
CD	5	5	5

The column player is the market. This game does not have a saddle in pure strategies. (Why?) If player I assumes that the market is the opponent with the goal of minimizing the investor's rate of return, then we may look at this as a two-person zero sum game. On the other hand, if the investor thinks that the market may be in any one of the three states with equal likelihood, then the market will play the strategy $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and then the investor must choose how to respond to that; that is, the investor seeks an X^* for which $E(X^*, Y) = \max_{X \in S_3} E(X, Y)$, where $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Of course, the investor uses the same procedure no matter which Y she thinks the market will use. The investor can use this game to compare what will happen under various Y strategies.

This problem may actually be solved fairly easily. If we assume that the market is an opponent in a game then the value of the game is $v(A) = 5$, and there are many optimal strategies, one of which is $X^* = (0, 0, 1), Y^* = (0, \frac{1}{2}, \frac{1}{2})$. If instead $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then the best response for player I is $X = (0, 0, 1)$, with payoff to I equal to 5.If $Y = (\frac{2}{3}, 0, \frac{1}{3})$, the best response is $X = (1, 0, 0)$, that is, invest in the stock if there is a $\frac{2}{3}$ probability of a good market. The payoff then is $\frac{19}{3} > 5$.

It may seem odd that the best response strategy in a zero sum two person game is usually a pure strategy, but it can be explained easily with a simple example. Suppose that someone is flipping a coin that is not fair–say heads comes up 75% of the time. On each toss you have to guess whether it will come up heads or tails. How often should you announce heads in order to maximize the percentage of time you guess correctly? If you think it is 75% of the time, then you will be correct $75 \times 75 = 56.25\%$ of the time! If you say heads **all the time**, you will be correct 75% of the time, and that is the best you can do.

Here is a simple game with an application of game theory to theology!

EXAMPLE 1.19

Blaise Pascal constructed a game to show that belief in God is the only rational strategy. The model assumes that you are player I, and you have two possible strategies: believe or don't believe. The opponent is taken to be God, who either plays God exists, or God doesn't exist. (Remember, God can do anything without contradiction.) Here is the matrix, assuming $\alpha, \beta, \gamma > 0$:

You/God	God exists	God doesn't exist
Believe	α	$-\beta$
Don't believe	$-\gamma$	0

If you believe and God doesn't exist, then you receive $-\beta$ because you have foregone evil pleasures in the belief God exists. If you don't believe and God exists, then you pay a price $-\gamma$. Pascal argued that this would be a big price. If you believe and God exists, then you receive the amount α, from God, and Pascal argued this would be a very large amount of spiritual currency.

To solve the game, we first calculate v^+ and v^-. Clearly $v^+ = 0$ and $v^- = \max(-\beta, -\gamma) < 0$, so this game does not have a saddle point in pure strategies unless $\beta = 0$ or $\gamma = 0$. If there is no loss or gain to you if you play don't believe, then that is what you should do, and God should play **not exist**. In this case the value of the game is zero.

Assuming that none of α, β, or γ is zero, we will have a mixed strategy saddle. Let $Y = (y, 1 - y)$ be an optimal strategy for God. Then it must be true that

$$E(1,Y) = \alpha y - \beta(1-y) = v(A) = -\gamma y = E(2,Y).$$

These are the two equations for a mixed strategy from the graphical method. Solving, we get that $y = \beta/(\alpha+\beta+\gamma)$. The optimal strategy for God is

$$Y = \left(\frac{\beta}{\alpha+\beta+\gamma}, \frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)$$

and the value of the game to you is

$$v(A) = \frac{-\gamma\beta}{\alpha+\beta+\gamma} < 0.$$

Your optimal strategy $X = (x, 1-x)$ must satisfy

$$E(X,1) = \alpha x - \gamma(1-x) = -\beta x = E(X,2) \implies$$
$$x = \frac{\gamma}{\alpha+\beta+\gamma}, \text{ and } X = \left(\frac{\gamma}{\alpha+\beta+\gamma}, \frac{\alpha+\beta}{\alpha+\beta+\gamma}\right).$$

Pascal argued that if γ, the penalty to you if you don't believe and God exists is loss of eternal life, represented by a very large number. In this case, the percent of time you play believe, $x = \gamma/(\alpha+\beta+\gamma)$ should be fairly close to 1, so you should play believe with high probability. For example, if $\alpha = 10$, $\beta = 5, \gamma = 100$, then $x = 0.87$.

From God's point of view, if γ is a very large number and this is a zero sum game, God would then play doesn't exist with high probability!! It may not make much sense to think of this as a zero sum game, at least not theologically. Maybe we should just look at this like a best response for you, rather than as a zero sum game.

So, let's suppose that God plays the strategy $Y^0 = (\frac{1}{2}, \frac{1}{2})$. What this really means is that you think that God's existence is as likely as not. What is your best response strategy? For that, we calculate $f(x) = E(X, Y^0)$, where $X = (x, 1-x), 0 \le x \le 1$. We get

$$f(x) = x\,\frac{\alpha+\gamma-\beta}{2} - \frac{\gamma}{2}.$$

The maximum of $f(x)$ over $x \in [0,1]$ is

$$f(x^*) = \begin{cases} \dfrac{\alpha-\beta}{2}, & \text{at } x^* = 1 \text{ if } \alpha+\gamma > \beta; \\ -\dfrac{\gamma}{2}, & \text{at } x^* = 0 \text{ if } \alpha+\gamma < \beta; \\ -\dfrac{\gamma}{2}, & \text{at any } 0 \le x \le 1 \text{ if } \alpha+\gamma = \beta. \end{cases}$$

Pascal argued that γ would be a very large number (and so would α) compared to β. Consequently, the best response strategy to Y^0 would be $X^* = (1, 0)$. Any rational person who thinks that God's existence is as likely as not would choose to play believe.

PROBLEMS

1.32 Solve the game with matrix

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ -2 & 8 & 4 & 0 \end{bmatrix}.$$

Find the best response for player I to the strategy $Y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8})$. What is II's best response to I's best response?

1.33 Suppose that the batter in the baseball game Example 1.8 hasn't done his homework to learn the percentages in the game matrix. So, he uses the strategy $X^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. What is the pitcher's best response strategy?

1.34 In general, if we have two payoff functions $f(x, y)$ for player I and $g(x, y)$ for player II, suppose that both players want to maximize their own payoff functions with the variables that they control. Then $y^* = y^*(x)$ is a best response of player II to x if $g(x, y^*(x)) = \max_y g(x, y)$, and $x^* = x^*(y)$ is a best response of player I to y if $f(x^*(y), y) = \max_x f(x, y)$. Find the best responses if $f(x, y) = (C - x - y)x$ and $g(x, y) = (D - x - y)y$, where C and D are constants. Solve the best responses and show that the solution x^*, y^* satisfies $f(x^*, y^*) \geq f(x, y^*)$ for all x, and $g(x^*, y^*) \geq g(x^*, y)$ for all y.

1.35 Given a strategy for player II, Y_0, let X_1 be a best response for player I. Next, let Y_1 be a best response to X_1 for player II and then X_2 a best response for player I to Y_1, and so on. Define the sequences of strategies $\{X_n\}$ and $\{Y_n\}$ precisely using mathematical symbols. Assuming $X_n \to X'$ and $Y_n \to Y'$ as $n \to \infty$, what should be true about X', Y'? Justify your answer. Now try the idea on the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. What goes wrong? How can you get the saddle point $X^* = (\frac{1}{3}, \frac{2}{3}) = Y^*$ from the procedure of this problem?

BIBLIOGRAPHIC NOTES

The classic reference for the material in this chapter is the wonderful two-volume set of books by S. Karlin[7] [11], one of the pioneers of game theory when he was a member of the Rand Institute. The Cat versus Rat game is a problem in Karlin's book [11], and I am sure Karlin has a clever way of solving it. The proofs of the von Neumann minimax theorem are modifications of those in Karlin's book. Proof 2, which uses only advanced calculus, is original to Karlin. On the other hand, the birth of game theory can be dated to the appearance of the seminal book by von Neumann and Morgenstern, reference [26].

The idea of using mixed strategies in order to establish the existence of a saddle point in this wider class is called **relaxation**. This idea is due to von Neumann and extends to games with continuous strategies, as well as the modern theory of optimal control, differential games, and the calculus of variations. It has turned out to be one of the most far-reaching ideas of the twentieth century.

The graphical solution of matrix games is extremely instructive as to the objectives for each player. Apparently, this method is present at the founding of game theory, but its origin is unknown to the author of this book.

The Russian roulette Example 1.5 is an adaptation of a similar example in the book by Jones [7]. Poker models have been considered from the birth of game theory. Any gambler interested in poker should consult the book by Karlin [11] for much more information and results on poker models.

The ideas of a best response strategy and viewing a saddle point as a pair of strategies, each of which is a best response to the other, leads naturally in later chapters to a Nash equilibrium for more general games.

[7] Professor Emeritus of Mathematics, Stanford University.

CHAPTER 2

SOLUTION METHODS FOR MATRIX GAMES

I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, ...; but time and chance happeneth to them all.

–Ecclesiates 9:11

2.1 SOLUTION OF SOME SPECIAL GAMES

Graphical methods reveal a lot about exactly how a player reasons her way to a solution, but it is not a very practical method. Now we will consider some special types of games for which we actually have a formula giving the value and the mixed strategy saddle points. Let's start with the easiest possible class of games that can always be solved explicitly and without using a graphical method.

2.1.1 2×2 Games Revisited

We have seen that any 2×2 matrix game can be solved graphically, and many times that is the fastest and best way to do it. But there are also explicit formulas giving the

Game Theory: An Introduction, First Edition. By E.N. Barron

value and optimal strategies with the advantage that they can be run on a calculator or computer. Also the method we use to get the formulas is instructive because it uses calculus.

Each player has exactly two strategies, so the matrix and strategies look like

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{player I} : X = (x, 1-x); \quad \text{player II} : Y = (y, 1-y).$$

For any mixed strategies we have $E(X,Y) = XAY^T$, which, written out, is

$$E(X,Y) = xy(a_{11} - a_{12} - a_{21} + a_{22}) + x(a_{12} - a_{22}) + y(a_{21} - a_{22}) + a_{22}.$$

Now here is the theorem giving the solution of this game.

Theorem 2.1.1 *In the* 2×2 *game with matrix* A, **assume that there are no pure optimal strategies.** *If we set*

$$x^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \qquad y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

then $X^* = (x^*, 1 - x^*), Y^* = (y^*, 1 - y^*)$ *are optimal mixed strategies for players I and II, respectively. The value of the game is*

$$v(A) = E(X^*, Y^*) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

Remarks.

1. The main assumption you need before you can use the formulas is that the game does **not** have a pure saddle point. If it does, you find it by checking $v^+ = v^-$, and then finding it directly. You don't need to use any formulas. Also, when we write down these formulas, it had better be true that $a_{11} - a_{12} - a_{21} + a_{22} \neq 0$, but if we assume that there is no pure optimal strategy, then this must be true. In other words, it isn't difficult to check that when $a_{11} - a_{12} - a_{21} + a_{22} = 0$, then $v^+ = v^-$ and that violates the assumption of the theorem.

2. A more compact way to write the formulas and easier to remember is

$$X^* = \frac{(1\ 1)A^*}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \text{and} \quad Y^* = \frac{A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$value(A) = \frac{\det(A)}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

where

$$A^* = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad \text{and} \quad \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Recall that the inverse of a 2×2 matrix is found by swapping the main diagonal numbers, and putting a minus sign in front of the other diagonal numbers, and finally dividing by the determinant of the matrix. A^* is exactly the first two steps, but we don't divide by the determinant. The matrix we get is defined even if the matrix A doesn't have an inverse. Remember, however, that we need to make sure that the matrix doesn't have optimal pure strategies first.

Notice too, that if $\det(A) = 0$, the value of the game is zero.

Here is why the formulas hold. Write $f(x, y) = E(X, Y)$, where $X = (x, 1 - x), Y = (y, 1 - y), 0 \le x, y \le 1$. Then

$$
\begin{aligned}
f(x, y) &= (x, 1 - x) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix} \qquad (2.1.1) \\
&= x[y(a_{11} - a_{21}) + (1 - y)(a_{12} - a_{22})] + a_{12} - a_{22}.
\end{aligned}
$$

By assumption there are no optimal pure strategies, and so the extreme points of f will be found inside the intervals $0 < x, y < 1$. But that means that we may take the partial derivatives of f with respect to x and y and set them equal to zero to find all the possible critical points. The function f has to look like the function depicted by Maple in Figure 2.1.

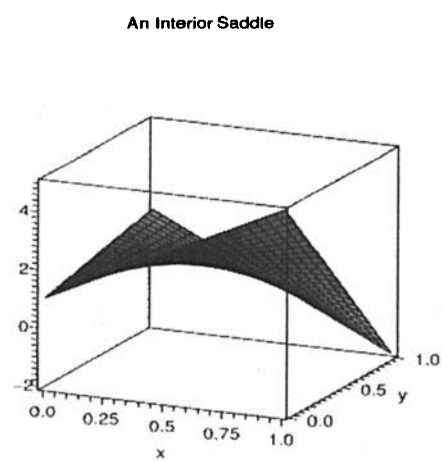


Figure 2.1 Concave in x, convex in y, saddle at $(\frac{1}{8}, \frac{1}{2})$.

Figure 2.1 is the graph of $f(x,y) = XAY^T$ taking $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$. It is a concave–convex function which has a saddle point at $x = \frac{1}{8}, y = \frac{1}{2}$. You can see now why it is called a **saddle**.

Returning to our general function $f(x,y)$ in (2.1.1), take the partial derivatives and set to zero:

$$\frac{\partial f}{\partial x} = y\alpha + \beta = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x\alpha + \gamma = 0,$$

where we have set

$$\alpha = (a_{11} - a_{12} - a_{21} + a_{22}), \quad \beta = (a_{12} - a_{22}), \quad \gamma = (a_{21} - a_{22}).$$

Notice that if $\alpha = 0$, the partials are never zero (assuming $\beta, \gamma \neq 0$), and that would imply that there are pure optimal strategies (in other words, the min and max must be on the boundary). The existence of a pure saddle is ruled out by assumption. We solve where the partial derivatives are zero to get

$$x^* = -\frac{\gamma}{\alpha} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} \quad \text{and} \quad y^* = -\frac{\beta}{\alpha} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

which is the same as what the theorem says. How do we know this is a saddle and not a min or max of f? The reason is because if we take the second derivatives, we get the matrix of second partials (called the **Hessian**):

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

Since $\det(H) = -\alpha^2 < 0$ (unless $\alpha = 0$, which is ruled out) a theorem in elementary calculus says that an interior critical point with this condition must be a saddle.[1] □

So the procedure is as follows. Look for pure strategy solutions first (calculate v^- and v^+ and see if they're equal), but if there aren't any, use the formulas to find the mixed strategy solutions.

■ EXAMPLE 2.1

In the game with $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$ we see that $v^- = 1 < v^+ = 2$, so there is no pure saddle for the game. If we apply the formulas, we get the mixed strategies

[1]The calculus definition of a saddle point of a function $f(x,y)$ is a point so that in every neighborhood of the point there are x and y values that make f bigger and smaller than f at the candidate saddle point.

$X^* = (\frac{1}{8}, \frac{7}{8})$ and $Y^* = (\frac{1}{2}, \frac{1}{2})$ and the value of the game is $v(A) = \frac{3}{2}$. Notice here that

$$A^* = \begin{bmatrix} 1 & -5 \\ -2 & -2 \end{bmatrix}$$

and $\det(A) = -12$. The matrix formula for player I gives

$$X = \frac{(1\ 1)A^*}{((1\ 1)A^*(1\ 1)^T)} = \left(\frac{1}{8}, \frac{7}{8}\right).$$

PROBLEMS

2.1 In a simplified analysis of a football game suppose that the offense can only choose a pass or run, and the defense can choose only to defend a pass or run. Here is the matrix in which the payoffs are the average yards gained:

	Defense	
Offense	Run	Pass
Run	1	8
Pass	10	0

The offense's goal is to maximize the average yards gained per play. Find $v(A)$ and the optimal strategies using the explicit formulas. Check your answers by solving graphically as well.

2.2 Suppose that an offensive pass against a run defense now gives 12 yards per play on average to the offense (so the 10 in the previous matrix changes to a 12). Believe it or not, the offense should pass less, not more. Verify that and give a game theory (not math) explanation of why this is so.

2.3 Does the same phenomenon occur for the defense? To answer this, compare the original game in the first problem to the new game in which the defense reduces the number of yards per run to 6 instead of 8 when defending against the pass. What happens to the optimal strategies?

2.4 Show that the formulas in the theorem are exactly what you would get if you solved the 2×2 game graphically under the assumptions of the theorem.

2.5 Solve the 2×2 games using the formulas and check by solving graphically:

$$\textbf{(a)} \begin{bmatrix} 4 & -3 \\ -9 & 6 \end{bmatrix}; \quad \textbf{(b)} \begin{bmatrix} 8 & 99 \\ 29 & 6 \end{bmatrix}; \quad \textbf{(c)} \begin{bmatrix} -32 & 4 \\ 74 & -27 \end{bmatrix}.$$

2.6 Give an example to show that when optimal pure strategies exist, the formulas in the theorem won't work.

2.7 Show that if $a_{11} + a_{22} = a_{12} + a_{21}$, then $v^+ = v^-$ or, equivalently, there are optimal pure strategies for both players. This means that if you end up with a zero denominator in the formula for $v(A)$, it turns out that there had to be a saddle in pure strategies for the game and hence the formulas don't apply from the outset.

2.2 INVERTIBLE MATRIX GAMES

In this section we will solve the class of games in which the matrix A is square, say, $n \times n$ and invertible, so $\det(A) \neq 0$ and A^{-1} exists and satisfies $A^{-1}A = AA^{-1} = I_{n\times n}$. The matrix $I_{n\times n}$ is the $n \times n$ identity matrix consisting of all zeros except for ones along the diagonal. For the present let us suppose that

> Player I has an optimal strategy that is completely mixed, specifically, $X = (x_1, \ldots, x_n)$, and $x_i > 0, i = 1, 2, \ldots, n$. So player I plays every row with positive probability.

By the Properties of Strategies (1.3.1), property 4, we know that this implies that every optimal Y strategy for player II, must satisfy

$$E(i, Y) = {}_iAY^T = value(A), \text{ for every row } i = 1, 2, \ldots, n.$$

Y played against any row will give the value of the game. If we write $J_n = (1\ 1\ \cdots 1)$ for the row vector consisting of all 1s, we can write

$$AY^T = v(A)J_n^T = \begin{bmatrix} v(A) \\ \vdots \\ v(A) \end{bmatrix}. \tag{2.2.1}$$

Now, if $v(A) = 0$, then $AY^T = 0J_n^T = 0$, and this is a system of n equations in the n unknowns $Y = (y_1, \ldots, y_n)$. It is a homogeneous linear system. Since A is invertible this would have the one and only solution $Y = A^{-1}0 = 0$. But that is impossible if Y is a strategy (the components must add to 1). So, if this is going to work, the value of the game cannot be zero, and we get, by multiplying both sides of (2.2.1) by A^{-1}, the following equation:

$$A^{-1}AY^T = Y^T = v(A)A^{-1}J_n^T.$$

This gives us Y if we knew $v(A)$. How do we get that? The extra piece of information we have is that the components of Y add to 1 (i.e., $\sum_{j=1}^n y_j = J_nY^T = 1$). So

$$J_nY^T = 1 = v(A)J_nA^{-1}J_n^T,$$

and therefore

$$v(A) = \frac{1}{J_nA^{-1}J_n^T} \text{ and then } Y^T = \frac{A^{-1}J_n^T}{J_nA^{-1}J_n^T}.$$

We have found the only candidate for the optimal strategy for player II assuming that every component of X is greater than 0. However, if it turns out that this formula for Y has at least one $y_j < 0$, something would have to be wrong; specifically, it would **not** be true that X was completely mixed because that was our hypothesis. But, if it turns out that $y_j \geq 0$ for every component, we could try to find an optimal X for player I by the exact same method. This would give us

$$X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

This method will work if the formulas we get for X and Y end up satisfying the condition that they are strategies. If either X or Y has a negative component, then it fails. But notice that the strategies do not have to be completely mixed as we assumed from the beginning, only bona fide strategies.

Here is a summary of what we know.

Theorem 2.2.1 *Assume that*

1. $A_{n\times n}$ has an inverse A^{-1}.

2. $J_n A^{-1} J_n^T \neq 0$.

3. $v(A) \neq 0$.

Set $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_m)$, and

$$v \equiv \frac{1}{J_n A^{-1} J_n^T}, \qquad Y^T = \frac{A^{-1} J_n^T}{J_n A^{-1} J_n^T}, \qquad X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

If $x_i \geq 0, i = 1, \ldots, n$ and $y_j \geq 0, j = 1, \ldots, n$, we have that $v = v(A)$ is the value of the game with matrix A and (X, Y) is a saddle point in mixed strategies.

Now the point is that when we have an invertible game matrix, we can always use the formulas in the theorem to calculate the number v and the vectors X and Y. If the result gives vectors with nonnegative components, then, by the properties (1.3.1) of games, v must be the value, and (X, Y) is a saddle point. Notice that, directly from the formulas, $J_n Y^T = 1$ and $X J_n^T = 1$, so the components given in the formulas will automatically sum to 1; it is only the sign of the components that must be checked.

Here is a simple direct verification that (X, Y) is a saddle and v is the value assuming $x_i \geq 0, y_j \geq 0$. Let $Y' \in S_n$ be any mixed strategy and let X be given by the formula $X = \dfrac{J_n A^{-1}}{J_n A^{-1} J_n^T}$. Then, since $J_n Y'^T = 1$, we have

$$\begin{aligned} E(X, Y') = X A Y'^T &= \frac{1}{J_n A^{-1} J_n^T} J_n A^{-1} A Y'^T \\ &= \frac{1}{J_n A^{-1} J_n^T} J_n Y'^T \\ &= \frac{1}{J_n A^{-1} J_n^T} = v. \end{aligned}$$

Similarly, for any $X' \in S_n$, $E(X', Y) = v$, and so (X, Y) is a saddle and v is the value of the game by the Theorem 1.3.7 or property 1 of (1.3.1).

Incidentally, these formulas match the formulas when we have a 2×2 game with an invertible matrix because then $A^{-1} = (1/\det(A))A^*$.

In order to guarantee that the value of a game is not zero, we may add a constant to every element of A that is large enough to make all the numbers of the matrix positive. In this case the value of the new game could not be zero. Since $v(A+b) = v(A) + b$, where b is the constant added to every element, we can find the original $v(A)$ by subtracting b. Adding the constant to all the elements of A will not change the probabilities of using any particular row or column; that is, the optimal mixed strategies are not affected by doing that.

Even if our original matrix A does not have an inverse, if we add a constant to all the elements of A, we get a new matrix $A + b$ and the new matrix **may** have an inverse (of course, it may not as well). We may have to try different values of b. Here is an example.

EXAMPLE 2.2

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}$$

This matrix has negative and positive entries, so it's possible that the value of the game is zero. The matrix does not have an inverse because the determinant of A is 0. So, let's try to add a constant to all the entries to see if we can make the new matrix invertible. Since the largest negative entry is -4, let's add 5 to everything to get

$$A + 5 = \begin{bmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{bmatrix}$$

This matrix does have an inverse given by

$$B = \frac{1}{80}\begin{bmatrix} 61 & -18 & -39 \\ -18 & 4 & 22 \\ -39 & 22 & 21 \end{bmatrix}$$

Next we calculate using the formulas $v = 1/(J_3 B J_3^T) = 5$, and

$$X = v(J_3 B) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \quad \text{and } Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

Since both X and Y are strategies (they have nonnegative components), the theorem tells us that they are optimal and the value of our original game is $v(A) = v - 5 = 0$.

The next example shows what can go wrong.

EXAMPLE 2.3

Let

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 2 & 2 \\ 5 & 2 & 8 \end{bmatrix}$$

Then it is immediate that $v^- = v^+ = 2$ and there is a pure saddle $X^* = (0,0,1), Y^* = (0,1,0)$. If we try to use Theorem 2.2.1, we have $\det(A) = 10$, so A^{-1} exists and is given by

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 6 & -2 & -1 \\ -23 & 11 & 3 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

If we use the formulas of the theorem, we get

$$v = -1, \ X = v(A)(J_3 A^{-1}) = \left(3, -\frac{3}{2}, -\frac{1}{2}\right),$$

and

$$Y = v(A^{-1} J_3^T)^T = \left(-\frac{3}{5}, \frac{9}{5}, -\frac{1}{5}\right).$$

Obviously these are completely messed up (i.e., wrong). The problem is that the components of X and Y are not nonnegative even though they do sum to 1.

Here are the Maple commands to work out any invertible matrix game. You can try many different constants to add to the matrix to see if you get an inverse if the original matrix does not have an inverse.

```
> restart:with(LinearAlgebra):
> A:=Matrix([[0 ,1 ,-2 ],[-1 ,-2 ,3],[2 ,-3 ,-4 ]]);
> Determinant(A);
> A:=MatrixAdd( ConstantMatrix(-1,3,3), A );
> Determinant(A);
> B:=A^(-1);
> J:=Vector[row]([1 ,1 ,1 ]);
> J.B.Transpose(J);
> v:=1/(J.B.Transpose(J));
> X:=v*(J.B);
> Y:=v*(B.Transpose(J));
```

The first line loads the `LinearAlgebra` package. The next line enters the matrix. The `Determinant` command finds the determinant of the matrix so you can see if it has an inverse. The `MatrixAdd` command adds the constant 3×3 matrix which consists of all -1s to A. The `ConstantMatrix(-1,3,3)` says the matrix is 3×3 and has all -1s. If the determinant of the original matrix A is not zero, you change the `MatrixAdd` command to `A:=MatrixAdd(ConstantMatrix(0,3,3),A)`. In general you keep changing the constant until you get tired or you get an invertible matrix.

The inverse of A is put into the matrix B, and the row vector consisting of 1s is put into J. Finally we calculate $v = value(A)$ and the optimal mixed strategies X, Y.

In this particular example the determinant of the original matrix is -12, but we add -1 to all entries anyway for illustration. This gives us $v = -\frac{14}{11}, X = (\frac{13}{22}, \frac{4}{11}, \frac{1}{22})$ and $Y = (\frac{13}{22}, \frac{2}{11}, \frac{5}{22})$, and these are legitimate strategies. So X and Y are optimal and the value of the original game is $v(A) = -\frac{14}{11} + 1 = -\frac{3}{11}$.

Completely Mixed Games. We just dealt with the case that the game matrix is invertible, or can be made invertible by adding a constant. A related class of games that are also easy to solve is the class of completely mixed games. We have already mentioned these earlier, but here is a precise definition.

Definition 2.2.2 *A game is* **completely mixed** *if every saddle point consisting of strategies $X = (x_1, \ldots, x_n) \in S_n$, $Y = (y_1, \ldots, y_m) \in S_m$ satisfies the property $x_i > 0, i = 1, 2, \ldots, n$ and $y_j > 0, j = 1, 2, \ldots, m$. So, every row and every column is used with positive probability.*

If you know, or have reason to believe, that the game is completely mixed ahead of time, then there is only one saddle point! It can also be shown that for completely mixed games with a square game matrix if you know that $v(A) \neq 0$, then the game matrix A must have an inverse, A^{-1}. In this case the formulas for the value and the saddle

$$v(A) = \frac{1}{J_n A^{-1} J_n^T}, \quad X^* = v(A)\ J_n A^{-1}, \quad \text{and} \quad Y^{*T} = v(A) A^{-1} J_n^T$$

from Theorem 2.2.1 will give the completely mixed saddle. The procedure is that if you are faced with a game and you think that it is completely mixed, then solve it using the formulas and verify.

■ **EXAMPLE 2.4**

Hide and Seek. Suppose that we have $a_1 > a_2 > \cdots > a_n > 0$. The game matrix is

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

Because $a_i > 0$ for every $i = 1, 2, \ldots, n$, we know $\det(A) = a_1 a_2 \cdots a_n > 0$, so A^{-1} exists. It seems pretty clear that a mixed strategy for both players should use each row or column with positive probability; that is, we think that the game is completely mixed. In addition, since $v^- = 0$ and $v^+ = a_n$, the value of the game satisfies $0 \le v(A) \le a_n$. Choosing $X = (1/n, \ldots, 1/n)$ we see that $\min_Y XAY^T = a_n/n > 0$ so that $v(A) > 0$. This isn't a fair game for player II. It is also easy to see that

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}.$$

Then, we may calculate from Theorem 2.2.1 that

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}},$$

$$X^* = v(A)\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right) = Y^*.$$

Notice that for any $i = 1, 2, \ldots, n$, we obtain

$$1 < a_i\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right),$$

so that $v(A) < \min(a_1, a_2, \ldots, a_n) = a_n$.

PROBLEMS

2.8 Consider the matrix game

$$A = \begin{bmatrix} 3 & 5 & 3 \\ 4 & -3 & 2 \\ 3 & 2 & 3 \end{bmatrix}.$$

Show that there is a saddle in pure strategies at $(1,3)$ and find the value. Verify that $X^* = (\frac{1}{3}, 0, \frac{2}{3}), Y^* = (\frac{1}{2}, 0, \frac{1}{2})$ is also an optimal saddle point. Does A have an inverse? Find it and use the formulas in the theorem to find the optimal strategies and value.

2.9 Solve the following games:

$$\textbf{(a)} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix}; \quad \textbf{(b)} \begin{bmatrix} 5 & 4 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 5 \end{bmatrix}; \quad \textbf{(c)} \begin{bmatrix} 4 & 2 & -1 \\ -4 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}.$$

Finally, use dominance to solve this game even though it has no inverse:

$$A = \begin{bmatrix} -4 & 2 & -1 \\ -4 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}.$$

2.10 To underscore that the formulas can be used only if you end up with legitimate strategies, consider the game with matrix

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 4 & 4 & 4 \\ 6 & 3 & 4 \end{bmatrix}.$$

(a) Does this matrix have a saddle in pure strategies? If so find it.
(b) Find A^{-1}.
(c) Without using the formulas, find an optimal **mixed** strategy for player II.
(d) Use the formula to calculate Y^*. Why isn't this optimal? What's wrong?

2.11 Show that $value(A + b) = value(A) + b$ for any constant b, where, by $A + b = (a_{ij} + b)$, is meant A plus the matrix with all entries $= b$. Show also that (X, Y) is a saddle for the matrix $A + b$ if and only if it is a saddle for A.

2.12 Derive the formula for $X = \dfrac{J_n A^{-1}}{J_n A^{-1} J_n^T}$, assuming the game matrix has an inverse. Follow the same procedure as that in obtaining the formula for Y.

2.13 A magic square game has a matrix in which each row has a row sum that is the same as each of the column sums. For instance, consider the matrix

$$A = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix}.$$

This is a magic square of order 5 and sum 65. Find the value and optimal strategies of this game and show how to solve any magic square game.

2.14 Why is the hide-and-seek game called that? Determine what happens in the hide-and-seek game if there is at least one $a_k = 0$.

2.15 For the game with matrix

$$\begin{bmatrix} -1 & 0 & 3 & 3 \\ 1 & 1 & 0 & 2 \\ 2 & -2 & 0 & 1 \\ 2 & 3 & 3 & 0 \end{bmatrix},$$

we determine that the optimal strategy for player II is $Y = (\frac{3}{7}, 0, \frac{1}{7}, \frac{3}{7})$. We are also told that player I has an optimal strategy X which is completely mixed. Given that the value of the game is $\frac{9}{7}$, find X.

2.16 A triangular game is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Find conditions under which this matrix has an inverse. Now consider

$$A = \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 4 & 4 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & 50 \end{bmatrix}.$$

Solve the game by finding $v(A)$ and the optimal strategies.

2.17 Another method that can be used to solve a game uses calculus to find the interior saddle points. For example, consider

$$A = \begin{bmatrix} 4 & -3 & -2 \\ -3 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

A strategy for each player is of the form $X = (x_1, x_2, 1 - x_1 - x_2), Y = (y_1, y_2, 1 - y_1 - y_2)$, so we consider the function $f(x_1, x_2, y_1, y_2) = XAY^T$. Now solve the system of equations

$$f_{x_1} = f_{x_2} = f_{y_1} = f_{y_2} = 0$$

to get X^* and Y^*. If these are legitimate completely mixed strategies, then you can verify that they are optimal and then find $v(A)$. Carry out these calculations for A and verify that they give optimal strategies.

2.18 Consider the Cat versus Rat game (Example 1.3). The game matrix is 16×16 and consists of 1s and 0s, but the matrix can be considerably reduced by eliminating dominated rows and columns. Show that you can reduce the game to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Now solve the game.

2.19 In tennis two players can choose to hit a ball left, center, or right of where the opposing player is standing. Name the two players I and II and suppose that I hits the ball, while II anticipates where the ball will be hit. Suppose that II can return a ball hit right 90% of the time, a ball hit left 60% of the time, and a ball hit center 70% of the time. If II anticipates incorrectly, she can return the ball only 20% of the time. Score a return as $+1$ and not return as -1. Find the game matrix and the optimal strategies.

2.3 SYMMETRIC GAMES

Symmetric games are important classes of two-person games in which the players can use the exact same set of strategies and any payoff that player I can obtain using strategy X can be obtained by player II using the same strategy $Y = X$. The two players can switch roles. Such games can be quickly identified by the rule that $A = -A^T$. Any matrix satisfying this is said to be **skew symmetric**. If we want the roles of the players to be symmetric, then we need the matrix to be skew symmetric.

Why is skew symmetry the correct thing? Well, if A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or $-A$ represent the payoffs to player II. So player II wants to maximize the column entries in $-A$. This means that from player II's perspective, the game matrix must be $(-A)^T$ because it is always the row player by convention who is the maximizer; that is, A is the payoff matrix to player I and $-A^T$ is the payoff to player II. So, if we want the payoffs to player II to be the same as the payoffs to player I, then we must have the same game matrices for each player and so $A = -A^T$. If this is the case, the matrix must be square, $a_{ij} = -a_{ji}$, and the diagonal elements of A, namely, a_{ii}, must be 0. We can say more. In what follows it is helpful to keep in mind that for any appropriate size matrices $(AB)^T = B^T A^T$.

Theorem 2.3.1 *For any skew symmetric game* $v(A) = 0$ *and if* X^* *is optimal for player I, then it is also optimal for player II.*

Proof. Let X be any strategy for I. Then

$$E(X,X) = X\ A\ X^T = -X\ A^T\ X^T = -(X\ A^T\ X^T)^T = -X\ A\ X^T = -E(X,X).$$

Therefore $E(X,X) = 0$ and any strategy played against itself has zero payoff.

Let (X^*, Y^*) be a saddle point for the game so that $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$, for all strategies (X, Y). Then for any (X, Y), we have

$$E(X,Y) = X\ A\ Y^T = -X\ A^T\ Y^T = -(X\ A^T\ Y^T)^T = -Y\ A\ X^T = -E(Y,X).$$

Hence, from the saddle point definition, we obtain

$$E(X,Y^*) = -E(Y^*,X) \le E(X^*,Y^*) = -E(Y^*,X^*) \le E(X^*,Y) = -E(Y,X^*).$$

Then

$$-E(Y^*,X) \le -E(Y^*,X^*) \le -E(Y,X^*) \Longrightarrow$$
$$E(Y^*,X) \ge E(Y^*,X^*) \ge E(Y,X^*).$$

But this says that Y^* is optimal for player I and X^* is optimal for player II and also that $E(X^*, Y^*) = -E(Y^*, X^*) \Longrightarrow v(A) = 0$. □

EXAMPLE 2.5

General Solution of 3×3 **Symmetric Games.** For any 3×3 symmetric game we must have

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Any of the following conditions gives a pure saddle point:

1. $a \geq 0, b \geq 0 \Longrightarrow$ saddle at $(1, 1)$ position,
2. $a \leq 0, c \geq 0 \Longrightarrow$ saddle at $(2, 2)$ position,
3. $b \leq 0, c \leq 0 \Longrightarrow$ saddle at $(3, 3)$ position.

Here's why. Let's assume that $a \leq 0, c \geq 0$. In this case if $b \leq 0$ we get $v^- = \max\{\min\{a, b\}, 0, -c\} = 0$ and $v^+ = \min\{\max\{-a, -b\}, 0, c\} = 0$, so there is a saddle in pure strategies at $(2, 2)$. All cases are treated similarly. To have a mixed strategy, all three of these must fail.

We next solve the case $a > 0, b < 0, c > 0$ so there is no pure saddle and we look for the mixed strategies.

Let player I's optimal strategy be $X^* = (x_1, x_2, x_3)$. Then

$$\begin{aligned} E(X^*, 1) &= -ax_2 - bx_3 \geq 0 = v(A) \\ E(X^*, 2) &= ax_1 - cx_3 \geq 0 \\ E(X^*, 3) &= bx_1 + cx_2 \geq 0 \end{aligned}$$

Each one is nonnegative since $E(X^*, Y) \geq 0 = v(A)$, for all Y. Now, since $a > 0, b < 0, c > 0$ we get

$$\frac{x_3}{a} \geq \frac{x_2}{-b}, \qquad \frac{x_1}{c} \geq \frac{x_3}{a}, \qquad \frac{x_2}{-b} \geq \frac{x_1}{c}$$

so

$$\frac{x_3}{a} \geq \frac{x_2}{-b} \geq \frac{x_1}{c} \geq \frac{x_3}{a},$$

and we must have equality throughout. Thus, each fraction must be some scalar λ, and so $x_3 = a\,\lambda, x_2 = -b\,\lambda, x_1 = c\,\lambda$. Since they must sum to one, $\lambda = 1/(a - b + c)$. We have found the optimal strategies $X^* = Y^* = (c\,\lambda, -b\,\lambda, a\,\lambda)$. The value of the game, of course is zero.

For example, the matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

is skew symmetric and does not have a saddle point in pure strategies. Using the formulas in the case $a > 0, b < 0, c > 0$, we get $X^* = (\frac{3}{8}, \frac{3}{8}, \frac{2}{8}) = Y^*$, and $v(A) = 0$. It is recommended, however, that you derive the result from scratch rather than memorize formulas.

EXAMPLE 2.6

Two companies will introduce a number of new products that are essentially equivalent. They will **introduce** one or two products but they each must also **guess** how many products their opponent will introduce. If they introduce the same number of products and guess the correct number the opponent will introduce, the payoff is zero. Otherwise the payoff is determined by whoever introduces more products and guesses the correct introduction of new products by the opponent. This accounts for the fact that new products result in more sales and guessing correctly results in accurate advertising, and so on. This is the payoff matrix to player I.

player I / player II	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$	0	1	-1	-1
$(1,2)$	-1	0	-2	-1
$(2,1)$	1	2	0	1
$(2,2)$	1	1	-1	0

This game is symmetric. We can drop the first row and the first column by dominance and are left with the following symmetric game (which could be further reduced):

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Since $a \leq 0, c \geq 0$, we have a saddle point at position $(2,2)$. We have $v = 0, X^* = (0,0,1,0) = Y^*$. Each company should introduce two new products and guess that the opponent will introduce one.

EXAMPLE 2.7

Alexander Hamilton was challenged to a duel with pistols by Aaron Burr after Hamilton wrote a defamatory article about Burr in New York. We will analyze one version of such a duel by the two players as a game with the object of trying to find the optimal point at which to shoot. First, here are the rules.

Each pistol has exactly one bullet. They will face each other starting at 10 paces apart and walk toward each other, each deciding when to shoot. In a silent duel a player does not know whether the opponent has taken the shot. In a noisy duel, the players know when a shot is taken. This is important because if a player shoots and misses, he is certain to be shot. The Hamilton–Burr duel will be considered as a silent duel because it is more interesting. We leave as an exercise the problem of the noisy duel.

We assume that each player's accuracy will increase the closer the players are. In a simplified version, suppose that they can choose to fire at 10 paces, 6 paces, or 2 paces. Suppose also that the probability that a shot hits and kills the opponent is 0.2 at 10 paces, 0.4 at 6 paces, and 1.0 at 2 paces. An opponent who is hit is assumed killed.

If we look at this as a zero sum two-person game, the player strategies consist of the pace distance at which to take the shot. The row player is Burr(B), and the column player is Hamilton (H). Incidentally, it is worth pointing out that the game analysis should be done before the duel is actually carried out.

We assume that the payoff to both players is $+1$ if they kill their opponent, -1 if they are killed, and 0 if they both survive.

So here is the matrix setup for this game.

B/H	10	6	2
10	0	−0.12	−0.6
6	0.12	0	−0.2
2	0.6	0.2	0

To see where these numbers come from, let's consider the pure strategy $(6, 10)$, so Burr waits until 6 to shoot (assuming that he survived the shot by Hamilton at 10) and Hamilton chooses to shoot at 10 paces. Then the expected payoff to Burr is

$$(+1) \cdot Prob(\text{H misses at } 10) \cdot Prob(\text{Kill H at } 6) \\ + (-1) \cdot Prob(\text{Killed by H at } 10) = 0.8 \cdot 0.4 - 0.2 = 0.12.$$

The rest of the entries are derived in the same way.

This is a symmetric game with skew symmetric matrix so the value is zero and the optimal strategies are the same for both Burr and Hamilton, as we would expect since they have the same accuracy functions. In this example, there is a pure saddle at position $(3, 3)$ in the matrix, so that $X^* = (0, 0, 1)$ and $Y^* = (0, 0, 1)$. Both players should wait until the probability of a kill is certain.

To make this game a little more interesting, suppose that the players will be penalized if they wait until 2 paces to shoot. In this case we may use the matrix

B/H	10	6	2
10	0	−0.12	1
6	0.12	0	−0.2
2	−1	0.2	0

This is still a symmetric game with skew symmetric matrix, so the value is still zero and the optimal strategies are the same for both Burr and Hamilton. To find the optimal strategy for Burr, we can remember the formulas or, even better, the procedure. So here is what we get knowing that $E(X^*, j) \geq 0, j = 1, 2, 3$:

$$E(X^*, 1) = 0.12\, x_2 - 1 \cdot x_3 \geq 0,$$
$$E(X^*, 2) = -0.12\, x_1 + 0.2\, x_3 \geq 0 \text{ and } E(X^*, 3) = x_1 - 0.2\, x_2 \geq 0.$$

These give us

$$x_2 \geq \frac{x_3}{0.12}, \frac{x_3}{0.12} \geq \frac{x_1}{0.2}, \text{ and } \frac{x_1}{0.2} \geq x_2,$$

which means equality all the way. Consequently

$$x_1 = \frac{0.2}{0.12 + 1 + 0.2} = \frac{0.2}{1.32}, \, x_2 = \frac{1}{1.32}, \, x_3 = \frac{0.12}{1.32}$$

or, $x_1 = 0.15, x_2 = 0.76, x_3 = 0.09$, so each player will shoot, with probability 0.76 at 6 paces.

In the real duel, that took place on July 11, 1804, Alexander Hamilton, who was the first US Secretary of the Treasury and widely considered to be a future president and a genius, was shot by Aaron Burr, who was John Adams' vice president of the United States and who also wanted to be president. Hamilton died of his wounds the next day. Aaron Burr was charged with murder but was later either acquitted or the charge was dropped (dueling was in the process of being outlawed). The duel was the end of the ambitious Burr's political career, and he died an ignominious death in exile.

In the next section we will consider some duels that are not symmetric and in which the payoff for survival is not zero. We can't treat such games with the methods of symmetry.

PROBLEMS

2.20 Find the matrix for a noisy Hamilton–Burr duel and solve the game. Notice that there is a pure saddle in this case also.

2.21 Assume that we have a silent duel but the choice of a shot may be taken at 10,8,6,4, or 2 paces. The accuracy, or probability of a kill is 0.2, 0.4, 0.6, 0.8, and 1, respectively, at the paces. Set up and solve the game.

2.22 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess correctly or both incorrectly, the game is a draw. If only one guesses correctly, that player wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show and the number of fingers to guess. Find the game matrix and the optimal strategies.

2.23 This exercise shows that symmetric games are more general than they seem at first and in fact this is the main reason they are important. Assuming that $A_{n\times m}$ is **any** payoff matrix with $value(A) > 0$, define the matrix B that will be of size $(n+m+1) \times (n+m+1)$, by

$$B = \begin{bmatrix} 0 & A & -\vec{1} \\ -A^T & 0 & \vec{1} \\ \vec{1} & -\vec{1} & 0 \end{bmatrix}.$$

The notation $\vec{1}$, for example in the third row and first column, is the $1 \times n$ matrix consisting of all 1s. B is a skew symmetric matrix and it can be shown that if $P = (p_1, \ldots, p_n, q_1, \ldots, q_m, \gamma)$ is an optimal strategy for matrix B, then, setting

$$b = \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j > 0, \quad x_i = \frac{p_i}{b}, \quad y_j = \frac{q_j}{b},$$

we have $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_m)$ as a saddle point for the game with matrix A. In addition, $value(A) = \gamma/b$. The converse is also true. Verify all these points with the matrix

$$A = \begin{bmatrix} 5 & 2 & 6 \\ 1 & \frac{7}{2} & 2 \end{bmatrix}.$$

2.4 MATRIX GAMES AND LINEAR PROGRAMMING

Linear programming is an area of optimization theory developed since World War II that is used to find the minimum (or maximum) of a linear function of many variables subject to a collection of linear constraints on the variables. It is extremely important to any modern economy to be able to solve such problems that are used to model many fundamental problems that a company may encounter. For example, the best routing of oil tankers from all of the various terminals around the world to the unloading points is a linear programming problem in which the oil company wants to minimize the total cost of transportation subject to the consumption constraints at each unloading point. But there are millions of applications of linear programming, which can range from the problems just mentioned to modeling the entire US economy. One can imagine the importance of having a very efficient way to find the optimal variables involved. Fortunately, George Dantzig,[2] in the 1940s, because of the necessities of the war effort, developed such an algorithm, called the **simplex method** that will quickly solve very large problems formulated as linear programs.

[2]George Bemard Dantzig was born on November 8, 1914 and died May 13, 2005. He is considered the "father of linear programming." He was the recipient of many awards, including the National Medal of Science in 1975, and the John von Neumann Theory Prize in 1974.

Mathematicians and economists working on game theory (including von Neumann), once they became aware of the simplex algorithm, recognized the connection.[3] After all, a game consists in minimizing and maximizing linear functions with linear things all over the place. So a method was developed to formulate a matrix game as a linear program (actually two of them) so that the simplex algorithm could be applied.

This means that using linear programming, we can find the value and optimal strategies for a matrix game of any size without any special theorems or techniques. In many respects, this approach makes it unnecessary to know any other computational approach. The downside is that in general one needs a computer capable of running the simplex algorithm to solve a game by the method of linear programming. We will show how to set up the game in two different ways to make it amenable to the linear programming method and also the Maple commands to solve the problem. We show both ways to set up a game as a linear program because one method is easier to do by hand (the first) since it is in **standard form**. The second method is easier to do using Maple and involves no conceptual transformations. Let's get on with it.

A linear programming problem is a problem of the **standard form** (called the **primal program**):

$$\begin{aligned}&\text{Minimize } z = \mathbf{c}\cdot\mathbf{x}\\ &\text{subject to } \mathbf{x}\,A \geq \mathbf{b}, \mathbf{x} \geq 0,\end{aligned}$$

where $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $A_{n\times m}$ is an $n \times m$ matrix, and $\mathbf{b} = (b_1, \ldots, b_m)$.

The primal problem seeks to minimize a **linear objective function**, $z(\mathbf{x}) = \mathbf{c}\cdot\mathbf{x}$, over a set of constraints (viz., $\mathbf{x}\cdot A \geq \mathbf{b}$) that are also linear. You can visualize what happens if you try to minimize or maximize a linear function of one variable over a closed interval on the real line. The minimum and maximum must occur at an endpoint. In more than one dimension this idea says that the minimum and maximum of a linear function over a variable that is in a convex set must occur on the boundary of the convex set. If the set is created by linear inequalities, even more can be said, namely, that the minimum or maximum must occur at an extreme point, or corner point, of the constraint set. The method for solving a linear program is to efficiently go through the extreme points to find the best one. That is essentially the simplex method.

[3]In 1947 Dantzig met with von Neumann and began to explain the linear programming model "as I would to an ordinary mortal." After a while von Neumann ordered Dantzig to "get to the point." Dantzig relates that in "less than a minute I slapped the geometric and algebraic versions..on the blackboard." Von Neumann stood up and said "Oh, that." Dantzig relates that von Neumann then "proceeded to give me a lecture on the mathematical theory of linear programs." This story is related in the excellent article by Cottle, Johnson, and Wets in the Notices of the A.M.S., March 2007.

For any primal there is a related linear program called the **dual program**:

$$\text{Maximize } w = \mathbf{y}\, \mathbf{b}^T$$
$$\text{subject to } A\, \mathbf{y}^T \leq \mathbf{c}^T, \mathbf{y} \geq 0.$$

A very important result of linear programming, which is called the **duality theorem**, states that if we solve the primal problem and obtain the optimal objective $z = z^*$, and solve the dual obtaining the optimal $w = w^*$, then $z^* = w^*$. In our game theory formulation to be given next, this theorem will tell us that the two objectives in the primal and the dual will give us the value of the game.

We will now show how to formulate any matrix game as a linear program. We need the primal and the dual to find the optimal strategies for each player.

Setting up the Linear Program; First Method. Let A be the game matrix. We may assume $a_{ij} > 0$ by adding a large enough constant to A if that isn't true. As we have seen earlier, adding a constant won't change the strategies and will only add a constant to the value of the game.

Hence we assume that $v(A) > 0$. Now consider the properties of optimal strategies (1.3.1). player I looks for a mixed strategy $X = (x_1, \ldots, x_n)$ so that

$$E(X, j) = X A_j = x_1\, a_{1j} + \cdots + x_n\, a_{nj} \geq v,\ 1 \leq j \leq m, \tag{2.4.1}$$

where $\sum x_i = 1, x_i \geq 0$, and $v > 0$ is as **large as possible**, because that is player I's reward. It is player I's objective to get the largest value possible. Notice that we don't have any connection with player II here except through the inequalities that come from player II using each column.

We will make this problem look like a regulation linear program. We change variables by setting

$$p_i = \frac{x_i}{v},\ \ 1 \leq i \leq n,\ \ \mathbf{p} = (p_1, \ldots, p_n).$$

This is where we need $v > 0$. Then $\sum x_i = 1$ implies that

$$\sum_{i=1}^{n} p_i = \frac{1}{v}.$$

Thus maximizing v is the same as **minimizing** $\sum p_i = 1/v$. This gives us our objective. For the constraints, if we divide the inequalities (2.4.1) by v and switch to the new variables, we get the set of constraints

$$\frac{x_1}{v}\, a_{1j} + \cdots + \frac{x_n}{v}\, a_{nj} = p_1 a_{1j} + \cdots + p_n a_{nj} \geq 1,\ 1 \leq j \leq m.$$

Now we summarize this as a linear program.

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_{\text{I}} = \ \mathbf{p}\, J_n^T = \sum_{i=1}^{n} p_i, & J_n = (1, 1, \ldots, 1) \\ \text{subject to: } \mathbf{p}\, A \geq J_m, & \mathbf{p} \geq 0. \end{cases}$$

Notice that the constraint of the game $\sum_i x_i = 1$ is used to get the objective function! It is not one of the constraints of the linear program. The set of constraints is

$$\mathbf{p}\, A \geq J_m \iff \mathbf{p} \cdot A_j \geq 1, j = 1, \ldots, m.$$

Also $\mathbf{p} \geq 0$ means $p_i \geq 0, i = 1, \ldots, n$.

Once we solve player I's program, we will have in our hands the optimal $\mathbf{p} = (p_1, \ldots, p_n)$ that minimizes the objective $z_{\text{I}} = \mathbf{p}\, J_n^T$. The solution will also give us the minimum objective z_{I}, labeled z_{I}^*.

Unwinding the formulation back to our original variables, we find the optimal strategy X for player I and the value of the game as follows:

$$\boxed{value(A) = \frac{1}{\sum_{i=1}^{n} p_i} = \frac{1}{z_{\text{I}}^*} \quad \text{and} \quad x_i = p_i \; value(A).}$$

Remember, too, that if you had to add a constant to the matrix to ensure $v(A) > 0$, then you have to subtract that same constant to get the value of the original game.

Now we look at the problem for player II. Player II wants to find a mixed strategy $Y = (y_1, \ldots, y_m), y_j \geq 0, \sum_j y_j = 1$, so that

$$y_1\, a_{i1} + \cdots + y_m\, a_{im} \leq u, \; i = 1, \ldots, n$$

with $u > 0$ as **small as possible**. Setting

$$q_j = \frac{y_j}{u}, \; j = 1, 2, \ldots, m, \quad \mathbf{q} = (q_1, q_2, \ldots, q_m),$$

we can restate player II's problem as the standard linear programming problem

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{\text{II}} = \mathbf{q}\, J_m^T, & J_m = (1, 1, \ldots, 1), \\ \text{subject to: } A\, \mathbf{q}^T \leq J_n^T & , \mathbf{q} \geq 0. \end{cases}$$

Player II's problem is the **dual** of player I's. At the conclusion of solving this program we are left with the optimal maximizing vector $\mathbf{q} = (q_1, \ldots, q_m)$ and the optimal objective value z_{II}^*. We obtain the optimal mixed strategy for player II and the value of the game from

$$\boxed{value(A) = \frac{1}{\sum_{j=1}^{m} q_j} = \frac{1}{z_{\text{II}}^*} \quad \text{and} \quad y_j = q_j \; value(A).}$$

However, how do we know that the value of the game using player I's program will be the same as that given by player II's program? The important **duality theorem** mentioned above and given explicitly below tells us that is exactly true, and so $z_{\text{I}}^* = z_{\text{II}}^*$.

Remember again that if you had to add a number to the matrix to guarantee that $v > 0$, then you have to subtract that number from z_I^*, and z_{II}^*, in order to get the value of the original game with the starting matrix A.

Theorem 2.4.1 (Duality Theorem) *If one of the pair of linear programs (primal and dual) has a solution, then so does the other. If there is at least one feasible solution (i.e., a vector that solves all the constraints so the constraint set is nonempty), then there is an optimal feasible solution for both, and their values, i.e. the objectives, are equal.*

This means that in a game we are guaranteed that $z_I^* = z_{II}^*$ and so the values given by player I's program will be the same as that given by player II's program.

■ EXAMPLE 2.8

Use the linear programming method to find a solution of the game with matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -3 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

Since it is possible that the value of this game is zero, begin by making all entries positive by adding 4 (other numbers could also be used) to everything:

$$A' = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 1 & 3 \\ 4 & 6 & 1 \end{bmatrix}.$$

Here is player I's program. We are looking for $X = (x_1, x_2, x_3), x_i \geq 0, \sum_{i=1}^{3} x_i = 1$, which will be found from the linear program. Setting $p_i = \dfrac{x_i}{v}$, player I's problem is

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_I = p_1 + p_2 + p_3 \quad \left(= \dfrac{1}{v}\right) \\ \text{subject to} \\ 2p_1 + 6p_2 + 4p_3 \geq 1 \\ 5p_1 + p_2 + 6p_3 \geq 1 \\ 4p_1 + 3p_2 + p_3 \geq 1 \\ p_i \geq 0 \qquad\qquad\qquad\qquad i = 1, 2, 3. \end{cases}$$

After finding the p_i's, we will set

$$v = \frac{1}{z_I^*} = \frac{1}{p_1 + p_2 + p_3}$$

and then $v(A) = v - 4$ is the value of the original game with matrix A, and v is the value of the game with matrix A'. Then $x_i = v\, p_i$ will give the optimal strategy for player I.

We next set up player II's program. We are looking for $y = (y_1, y_2, y_3), y_j \geq 0, \sum_{j=1}^{3} y_j = 1$. Setting $q_j = (y_j/v)$, player II's problem is

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{\text{II}} = q_1 + q_2 + q_3 \left(= \dfrac{1}{v}\right) \\ \text{subject to} \\ 2q_1 + 5q_2 + 4q_3 \leq 1 \\ 6q_1 + q_2 + 3q_3 \leq 1 \\ 4q_1 + 6q_2 + q_3 \leq 1 \\ q_j \geq 0 \qquad\qquad j = 1, 2, 3. \end{cases}$$

Having set up each player's linear program, we now are faced with the formidable task of having to solve them. By hand this would be very computationally intensive, and it would be very easy to make a mistake. Fortunately, the simplex method is part of all standard Maple and Mathematica software so we will solve the linear programs using Maple.

For player I we use the Maple commands

```
> with(simplex):
> cnsts:={2*p[1]+6*p[2]+4*p[3] >=1,
            5*p[1]+p[2]+6*p[3] >=1,4*p[1]+3*p[2]+p[3] >=1};
> obj:=p[1]+p[2]+p[3];
> minimize(obj,cnsts,NONNEGATIVE);
```

The `minimize` command incorporates the constraint that the variables be nonnegative by the use of the third argument. Maple gives the following solution to this program:

$$p_1 = \frac{21}{124},\ p_2 = \frac{13}{124},\ p_3 = \frac{1}{124} \quad \text{and} \quad p_1 + p_2 + p_3 = \frac{35}{124}.$$

Unwinding this to the original game, we have

$$\frac{35}{124} = \frac{1}{v} \implies v(A') = \frac{124}{35}.$$

So, the optimal mixed strategy for player I, using $x_i = p_i\, v$, is $X^* = (\frac{21}{35}, \frac{13}{35}, \frac{1}{35})$, and the value of our original game is $v(A) = \frac{124}{35} - 4 = -\frac{16}{35}$.

We may also use the `Optimization` package in Maple to solve player I's program, but this solves the problem numerically rather than symbolically. To use this package and solve the problem, use the following commands.

```
> with(Optimization):
> cnsts:={2*p[1]+6*p[2]+4*p[3] >=1,
          5*p[1]+p[2]+6*p[3] >=1,
          4*p[1]+3*p[2]+p[3] >=1};
>obj:=p[1]+p[2]+p[3];
>Minimize(obj,cnsts,assume=nonnegative);
```

This will give the solution in floating-point form: $p[1] = 0.169$, $p[2] = 0.1048$, $p[3] = 0.00806$. The minimum objective is $0.169 + 0.1048 + 0.00806 = 0.28186$ and so the value of the game is $v = 1/0.28186 - 4 = -0.457$ which agrees with our earlier result.

Next we turn to solving the program for player II. We use the Maple commands

```
> with(simplex):
> cnsts:={2*q[1]+5*q[2]+4*q[3]<=1,
          6*q[1]+q[2]+3*q[3]<=1,
          4*q[1]+6*q[2]+q[3]<=1};
> obj:=q[1]+q[2]+q[3];
> maximize(obj,cnsts,NONNEGATIVE);
```

Maple gives the solution

$$q_1 = \frac{13}{124}, q_2 = \frac{10}{124}, q_3 = \frac{12}{124},$$

so again $q_1 + q_2 + q_3 = 1/v = \frac{35}{124}$, or $v(A') = \frac{124}{35}$. Hence the optimal strategy for player II is

$$Y^* = \frac{124}{35}\left(\frac{13}{124}, \frac{10}{124}, \frac{12}{124}\right) = \left(\frac{13}{35}, \frac{10}{35}, \frac{12}{35}\right).$$

The value of the original game is then $\frac{124}{35} - 4 = -\frac{16}{35}$.

Remark. As mentioned earlier, the linear programs for each player are the **duals** of each other. Precisely, for player I the problem is

$$\text{Minimize } \mathbf{c} \cdot \mathbf{p}, \quad \mathbf{c} = (1,1,1) \text{ subject to } A^T\mathbf{p} \geq \mathbf{b}, \mathbf{p} \geq 0,$$

where $\mathbf{b} = (1,1,1)$. The dual of this is the linear programming problem for player II:

$$\text{Maximize } \mathbf{b} \cdot \mathbf{q} \quad \text{subject to } A\mathbf{q} \leq \mathbf{c}, \ \mathbf{q} \geq 0.$$

The duality theorem of linear programming guarantees that the minimum in player I's program will be equal to the maximum in player II's program, as we have seen in this example.

EXAMPLE 2.9

A Nonsymmetric Noisy Duel. We consider a nonsymmetric duel at which the two players may shoot at paces $(10, 6, 2)$ with accuracies $(0.2, 0.4, 1.0)$ each. This is the same as our previous duel, but now we have the following payoffs to player I at the end of the duel:

- If only player I survives, then player I receives payoff a.
- If only player II survives, player I gets payoff $b < a$. This assumes that the survival of player II is less important than the survival of player I.
- If both players survive, they each receive payoff zero.
- If neither player survives, player I receives payoff g.

We will take $a = 1, b = \frac{1}{2}, g = 0$. Then here is the expected payoff matrix for player I:

I/II	$(0.2, 10)$	$(0.4, 6)$	$(1, 2)$
$(0.2, 10)$	0.24	0.6	0.6
$(0.4, 6)$	0.9	0.36	0.70
$(1.0, 2)$	0.9	0.8	0

The pure strategies are labeled with the two components (accuracy,paces). The elements of the matrix are obtained from the general formula

$$E((x,i),(y,j)) = \begin{cases} ax + b(1-x) & \text{if } x < y; \\ ax + bx + (g - a - b)x^2 & \text{if } x = y; \\ a(1-y) + by & \text{if } x > y. \end{cases}$$

For example, if we look at

$$\begin{aligned} E((0.4, 6), (0.2, 10)) &= a Prob(\text{II misses at } 10) + b Prob(\text{II kills I at } 10) \\ &= a(1 - 0.2) + b(0.2). \end{aligned}$$

because if player II kills I at 10, then he survives and the payoff to I is b. If player II shoots, but misses player I at 10, then player I is certain to kill player II later and will receive a payoff of a. Similarly,

$$\begin{aligned} E((x,i),(x,i)) &= a Prob(\text{II misses}) Prob(\text{I hits}) \\ &\quad + b Prob(\text{I misses}) Prob(\text{II hits}) + g Prob(\text{I hits}) Prob(\text{II hits}) \\ &= a(1-x)x + b(1-x)x + g(x \cdot x) \end{aligned}$$

To solve the game, we use the Maple commands that will help us calculate the constraints:

```
> with(LinearAlgebra):
> R:=Matrix([[0.24,0.6,0.6],[0.9,0.36,0.70],[0.9,0.8,0]]);
> with(Optimization): P:=Vector(3,symbol=p);
> PC:=Transpose(P).R;
> Xcnst:={seq(PC[i]>=1,i=1..3)};
> Xobj:=add(p[i],i=1..3);
> Z:=Minimize(Xobj,Xcnst,assume=nonnegative);
> v:=evalf(1/Z[1]); for i from 1 to 3 do evalf(v*Z[2,i]) end do;
```

Alternatively, use the simplex package:

```
> with(simplex):Z:=minimize(Xobj,Xcnst,NONNEGATIVE);
```

Maple produces the output

$$Z = [1.821, [p_1 = 0.968, p_2 = 0.599, p_3 = 0.254]].$$

This says that the value of the game is $v(A) = 1/1.821 = 0.549$, and the optimal strategy for player I is $X^* = (0.532, 0.329, 0.140)$. By a similar analysis of player II's linear programming problem, we get $Y^* = (0.141, 0.527, 0.331)$. So player I should fire at 10 paces more than half the time, even though they have the same accuracy functions, and a miss is certain death. Player II should fire at 10 paces only about 14% of the time.

An asymmetric duel could arise also if the players have differing accuracy functions. That analysis is left as an exercise.

2.4.1 A Direct Formulation Without Transforming: Method 2

It is not necessary to make the transformations we made in order to turn a game into a linear programming problem. In this section we give a simpler and more direct way. We will start from the beginning and recapitulate the problem.

Recall that player I wants to choose a mixed strategy $X^* = (x_1^*, \ldots, x_n^*)$ so as to

$$\text{Maximize } v$$

subject to the constraints

$$\sum_{i=1}^{n} a_{ij} x_i^* = X^* A_j = E(X^*, j) \geq v, \; j = 1, \ldots, m,$$

and

$$\sum_{i=1}^{n} x_i^* = 1, \; x_i \geq 0, \; i = 1, \ldots, n.$$

This is a direct translation of the properties (1.3.1) that say that X^* is optimal and v is the value of the game if and only if, when X^* is played against any column for player II, the expected payoff must be at least v. Player I wants to get the largest possible v so that the expected payoff against any column for player II is at least v. Thus, if we can find a solution of the program subject to the constraints, it must give the optimal strategy for I as well as the value of the game. Similarly, player II wants to choose a strategy $Y^* = (y_j{}^*)$ so as to

$$\text{Minimize } v$$

subject to the constraints

$$\sum_{j=1}^{m} a_{ij} y_j^* = {}_iAY^{*T} = E(i, Y^*) \leq v, \; i = 1, \ldots, n,$$

and

$$\sum_{j=1}^{m} y_j^* = 1, \; y_j \geq 0, \; j = 1, \ldots, m.$$

The solution of this dual linear programming problem will give the optimal strategy for player II and the same value of the game as that for player I. We can solve these programs directly without changing to new variables. Since we don't have to divide by v in the conversion, we don't need to ensure that $v > 0$ (although it is usually a good idea to do that anyway), so we can avoid having to add a constant to A. This formulation is much easier to set up in Maple, but if you ever have to solve a game by hand using the simplex method, the first method is much easier.

Let's work an example and give the Maple commands to solve the game.

■ EXAMPLE 2.10

In this example we want to solve by the linear programming method with the second formulation the game with the skew symmetric matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Here is the setup for solving this using Maple. We begin by entering the matrix A. For the row player we define his strategy as `X:=Vector(3,symbol=x)`, which defines a vector of size 3 and uses the symbol x for the components. We could do this with Y as well, but a more direct way is to use `Y:=<y[1],y[2],y[3]>`. In the Maple commands which follow anything to the right of the symbol # is a comment.

```
> with(LinearAlgebra):with(simplex):
>#Enter the matrix of the game here,row by row:
> A:=Matrix([[0,-1,1],[1,0,-1],[-1,1,0]]);

>#The row player's Linear Programming problem:

> X:=Vector(3,symbol= x);
#Defines X as a column vector with 3 components
> B:=Transpose(X).A;
# Used to calculate the constraints; B is a vector.
> cnstx:={seq(B[i] >=v,i=1..3),add(x[i],i=1..3)=1};
#The components of B must be >=v and the
#components of X must sum to 1.
> maximize(v,cnstx,NONNEGATIVE);
#player I wants v as large as possible

#Hitting enter will give X=(1/3,1/3,1/3) and v=0.

>#Column players Linear programming problem:
> Y:=<y[1],y[2],y[3]>;#Another way to set up the vector for Y.
> B:=A.Y;
> cnsty:={seq(B[j]<=w,j=1..3),add(y[j],j=1..3)=1};
>minimize(w,cnsty,NONNEGATIVE);

#Again, hitting enter gives Y=(1/3,1/3,1/3) and w=0.
```

Since A is skew symmetric, we know ahead of time that the value of this game is 0. Maple gives us the optimal strategies

$$X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = Y^*,$$

which checks with the fact that for a symmetric matrix the strategies are the same for both players.

Remark. One point that you have to be careful about is the fact that in the Maple statement `maximize(v,cnstx,NONNEGATIVE)` the term `NONNEGATIVE` means that Maple is trying to solve this problem by looking for **all variables** ≥ 0. If it happens that the actual value of the game is < 0, then Maple will not give you the solution. You can do either of two things to fix this:

1. Drop the `NONNEGATIVE` word and change `cnstx` to

```
> cnstx:={seq(B[i] >=v,i=1..3),seq(x[i] >= 0,i=1..3),
                                    add(x[i],i=1..3)=1};
```

which puts the nonnegativity constraints of the strategy variables directly into `cnstx`. You have to do the same for `cnsty`.

2. Add a large enough constant to the game matrix A to make sure that $v(A) > 0$.

Now let's look at a much more interesting example.

EXAMPLE 2.11

Colonel Blotto Games. This is a simplified form of a military game in which the leaders must decide how many regiments to send to attack or defend two or more targets. It is an optimal allocation of forces game. In one formulation from reference [11], suppose that there are two opponents (players), which we call Red and Blue. Blue controls four regiments, and Red controls three. There are two targets of interest, say, A and B. The rules of the game say that the player who sends the most regiments to a target will win one point for the win and one point for every regiment captured at that target. A tie, in which Red and Blue send the same number of regiments to a target, gives a zero payoff. The possible strategies for each player consist of the number of regiments to send to A and the number of regiments to send to B, and so they are pairs of numbers. The payoff matrix to Blue is

Blue/Red	$(3,0)$	$(0,3)$	$(2,1)$	$(1,2)$
$(4,0)$	4	0	2	1
$(0,4)$	0	4	1	2
$(3,1)$	1	-1	3	0
$(1,3)$	-1	1	0	3
$(2,2)$	-2	-2	2	2

For example, if Blue plays $(3,1)$ against Red's play of $(2,1)$, then Blue sends three regiments to A while Red sends two. So Blue will win A, which gives $+1$ and then capture the two Red regiments for a payoff of $+3$ for target A. But Blue sends one regiment to B and Red also sends one to B, so that is considered a tie, or standoff, with a payoff to Blue of 0. So the net payoff to Blue is $+3$.

This game can be easily solved using Maple as a linear program, but here we will utilize the fact that the strategies have a form of symmetry that will simplify the calculations and are an instructive technique used in game theory.

It seems clear from the matrix that the Blue strategies $(4,0)$ and $(0,4)$ should be played with the same probability. The same should be true for $(3,1)$ and $(1,3)$. Hence, we may consider that we really only have three numbers to determine for a strategy for Blue:

$$X = (x_1, x_1, x_2, x_2, x_3),\ 2x_1 + 2x_2 + x_3 = 1.$$

Similarly, for Red we need to determine only two numbers:

$$Y = (y_1, y_1, y_2, y_2),\ 2y_1 + 2y_2 = 1.$$

But don't confuse this with reducing the matrix by dominance because that isn't what's going on here. Dominance would say that a dominated row would be played with probability zero, and that is not the case here. We are saying that for Blue rows 1 and 2 would be played with the same probability, and rows 3 and 4 would be played with the same probability. Similarly, columns 1 and 2, and columns 3 and 4 would be played with the same probability for Red. The net result is that we only have to find (x_1, x_2, x_3) for Blue, and (y_1, y_2) for Red.

Red wants to choose $y_1 \geq 0, y_2 \geq 0$ to make $value(A) = v$ as small as possible but subject to $E(i, Y) \leq v,\ i = 1, 3, 5$. That is what it means to play optimally. This says

$$
\begin{aligned}
E(1, Y) &= 4y_1 + 3y_2 \leq v, \\
E(3, Y) &= 3y_2 \leq v, \quad \text{and} \\
E(5, Y) &= -4y_1 + 4y_2 \leq v.
\end{aligned}
$$

Now, $3y_2 \leq 4y_1 + 3y_2 \leq v$ implies $3y_2 \leq v$ automatically, so we can drop the second inequality. Since $2y_1 + 2y_2 = 1$, substitute $y_2 = \frac{1}{2} - y_1$ to eliminate y_2 and get

$$4y_1 + 3y_2 = y_1 + \frac{3}{2} \leq v, \quad -4y_1 + 4y_2 = -8y_1 + 2 \leq v.$$

The two lines $v = y_1 + \frac{3}{2}$, and $v = -8y_1 + 2$ intersect at $y_1 = \frac{1}{18}, v = \frac{28}{18}$. Since the inequalities require that v be above the two lines, the smallest v satisfying the inequalities is at the point of intersection. Observe the connection with the graphical method for solving games. Thus,

$$y_1 = \frac{1}{18},\ y_2 = \frac{8}{18},\ v = \frac{28}{18},$$

and $Y^* = (\frac{1}{18}, \frac{1}{18}, \frac{8}{18}, \frac{8}{18})$. To check that it is indeed optimal, we have

$$(E(i, Y^*), i = 1, 2, 3, 4, 5) = \left(\frac{14}{9}, \frac{14}{9}, \frac{12}{9}, \frac{12}{9}, \frac{14}{9}\right),$$

so that $E(i, Y^*) \leq \frac{14}{9},\ i = 1, 2, \ldots, 5$. This verifies that Y^* is optimal for Red.

Next, we find the optimal strategy for Blue. We have $E(3, Y^*) = 3y_2 = \frac{24}{18} < \frac{28}{18}$ and, because it is a strict inequality, property 4 of the Properties (1.3.1), tells us that any optimal strategy for Blue would have 0 probability of using row 3, that is, $x_2 = 0$. (Recall that $X = (x_1, x_1, x_2, x_2, x_3)$ is the strategy we are looking for.) With that simplification we obtain the inequalities for Blue as

$$E(X, 1) = 4x_1 - 2x_3 \geq v = \frac{28}{18},\ E(X, 3) = 3x_1 + 2x_3 \geq \frac{28}{18}.$$

In addition, we must have $2x_1 + x_3 = 1$. The maximum minimum occurs at $x_1 = \frac{4}{9}, x_3 = \frac{1}{9}$ where the lines $2x_1 + x_3 = 1$ and $3x_1 + 2x_3 = \frac{14}{9}$ intersect. The solution yields $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$.

Naturally, Blue, having more regiments, will come out ahead, and a rational opponent (Red) would capitulate before the game even began. Observe also that with two equally valued targets, it is optimal for the superior force (Blue) to not divide its regiments, but for the inferior force to split its regiments, except for a small probability of doing the opposite.

Finally, if we use Maple to solve this problem, we use the commands

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>X:=Vector(5,symbol=x);
>B:=Transpose(X).A;
>cnst:={seq(B[i]>=v,i=1..4),add(x[i],i=1..5)=1};
>with(simplex):
>maximize(v,cnst,NONNEGATIVE,value);
```

The outcome of these commands is $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$ and $value = 14/9$. Similarly, using the commands

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>Y:=Vector(4,symbol=y);
>B:=A.Y;
>cnst:={seq(B[j]<=v,j=1..5),add(y[i],j=1..4)=1};
>with(simplex):
>minimize(v,cnst,NONNEGATIVE,value);
```

results in $Y^* = (\frac{7}{90}, \frac{3}{90}, \frac{32}{90}, \frac{48}{90})$ and again $v = \frac{14}{9}$. The optimal strategy for Red is not unique, and Red may play one of many optimal strategies but all resulting in the same expected outcome. Any convex combination of the two Y^*s we found will be optimal for player II.

Suppose, for example, we take the Red strategy

$$
\begin{aligned}
Y &= \frac{1}{2}\left(\frac{1}{18}, \frac{1}{18}, \frac{8}{18}, \frac{8}{18}\right) + \frac{1}{2}\left(\frac{7}{90}, \frac{3}{90}, \frac{32}{90}, \frac{48}{90}\right) \\
&= \left(\frac{1}{15}, \frac{2}{45}, \frac{2}{5}, \frac{22}{45}\right)
\end{aligned}
$$

we get $(E(i, Y), i = 1, 2, 3, 4, 5) = (\frac{14}{9}, \frac{14}{9}, \frac{11}{9}, \frac{13}{9}, \frac{14}{9})$. Suppose also that Blue deviates from his optimal strategy $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$ by using $X = (\frac{3}{9}, \frac{3}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$. Then $E(X, Y) = \frac{122}{81} < v = \frac{14}{9}$, and Red decreases the payoff to Blue.

Remark. The Maple commands for solving a game may be automated somewhat by using a Maple procedure:

```
>restart:with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>value:=proc(A,rows,cols)
    local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
    X:=Vector(rows,symbol=x): Y:=Vector(cols,symbol=y):
    B:=Transpose(X).A; C:=A.Y;
    cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
    cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
    with(simplex):
    vu:=maximize(vI,cnstx,NONNEGATIVE);
    vl:=minimize(vII,cnsty,NONNEGATIVE);
    print(vu,vl);
    end:
>value(A,5,4);
```

You may now enter the matrix A and run the procedure through the statement `value(A,rows,columns)`. The procedure will return the value of the game and the optimal strategies for each player. Remember, it never hurts to add a sufficiently large constant to the matrix to make sure that the value is positive. In all cases you should verify that you actually have the value of the game and the optimal strategies by using the verification: (X^*, Y^*) is a saddle point and v is the value if and only if

$$_iAY^{*T} = E(i, Y^*) \le v \le E(X^*, j) = X^* A_j,$$

for every $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

PROBLEMS

2.24 Use any method to solve the games with the following matrices.

(a) $\begin{bmatrix} 0 & 3 & 3 & 2 & 4 \\ 4 & 4 & 3 & 1 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & 4 & -4 & -1 \\ -4 & -2 & 4 & 4 \\ 2 & -4 & -1 & -5 \\ -3 & 1 & 0 & -4 \end{bmatrix}$; (c) $\begin{bmatrix} 2 & -5 & 3 & 0 \\ -4 & -5 & -5 & -6 \\ 3 & -4 & -1 & -2 \\ 0 & 4 & 1 & 3 \end{bmatrix}$

2.25 Solve this game using the two methods of setting it up as a linear program:

$$\begin{bmatrix} -2 & 3 & 3 & 4 & 1 \\ 3 & -2 & -5 & 2 & 4 \\ 4 & -5 & -1 & 4 & -1 \\ 2 & -4 & 3 & 4 & -3 \end{bmatrix}$$

In each of the following problems, set up the payoff matrices and solve the games using the linear programming method with both formulations, that is, both with and without transforming variables. You may use Maple to solve these linear programs.

2.26 Drug runners can use three possible methods for running drugs through Florida: small plane, main highway, or backroads. The cops know this, but they can only patrol one of these methods at a time. The street value of the drugs is \$100,000 if the drugs reach New York using the main highway. If they use the backroads, they have to use smaller-capacity cars so the street value drops to \$80,000. If they fly, the street value is \$150,000. If they get stopped, the drugs and the vehicles are impounded, they get fined, and they go to prison. This represents a loss to the drug kingpins of \$90,000, by highway, \$70,000 by backroads, and \$130,000 if by small plane. On the main highway they have a 40% chance of getting caught if the highway is being patrolled, a 30% chance on the backroads, and a 60% chance if they fly a small plane (all assuming that the cops are patrolling that method). Set this up as a zero sum game assuming that the cops want to minimize the drug kingpins gains, and solve the game to find the best strategies the cops and drug lords should use.

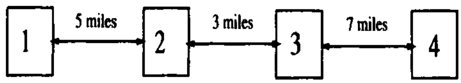
2.27 Consider the asymmetric silent duel at which the two players may shoot at paces $(10, 6, 2)$ with accuracies $(0.2, 0.4, 1.0)$ for player I, but, since II is a better shot, $(0.6, 0.8, 1.0)$ for player II. Given that the payoffs are $+1$ to a survivor and -1 otherwise (but 0 if a draw), set up the matrix game and solve it.

2.28 Solve the noisy duel game assuming that each player may fire at $10, 8, 6, 4, 2$ paces. The accuracy for player I is given by $(0.1, 0.3, 0.4, 0.8, 1.0)$ and for player II by $(0.2, 0.4, 0.6, 0.8, 1.0)$. We have the following payoffs to player I at the end of the duel:

- If only player I survives, then player I receives payoff a.
- If only player II survives, player I gets payoff $b < a$.
- If both players survive, they each receive payoff zero.
- If neither player survives, player I receives payoff g

Assuming that $a = 1, b = \frac{1}{2}, g = \frac{1}{2}$, set up the payoff matrix with player I as the row player and solve the game.

2.29 Suppose that there are four towns connected by a highway as in the following diagram: Assume that 15% of the total populations of the four towns are nearest to town 1, 30% nearest to town two, 20% nearest to town three, and 35% nearest to town four. There are two superstores, say, I and II, thinking about building a store to serve these four towns. If both stores are in the same town or in two different towns but with the same distance to a town, then I will get a 65% market share of business.



Each store gets 90% of the business of the town in which they put the store if they are in two different towns. If store I is closer to a town than II is, then I gets 90% of the business of that town. If store I is farther than store II from a town, store I still gets 40% of that town's business, except for the town II is in. Find the payoff matrix and solve the game.

2.30 Two farmers are having a fight over a disputed 6-yard-wide strip of land between their farms. Both farmers think that the strip is theirs. A lawsuit is filed and the judge orders them to submit a confidential settlement offer to settle the case fairly. The judge has decided to accept the settlement offer that concedes the most land to the other farmer. In case both farmers make no concession or they concede equal amounts, the judge will confiscate all the land for himself. Formulate this as a zero sum game assuming that both farmers' pure strategies must be the yards that they concede: $0, 1, 2, 3, 4, 5, 6$. Solve the game. What if the judge awards 3 yards if equal concessions?

2.31 Two football teams, B and P, meet for the Superbowl. Each offense can play run right (RR), run left (RL), short pass (SP), deep pass (DP), or screen pass (ScP). Each defense can choose to blitz (Bl), defend a short pass (DSP), or defend a long pass (DLP), or defend a run (DR). Suppose that team B does a statistical analysis and determines the following payoff when they are on defense:

B/P	RR	RL	SP	DP	ScP
Bl	-5	-7	-7	5	4
DSP	-6	-5	8	6	3
DLP	-2	-3	-8	6	-5
DR	3	3	-5	-15	-7

A negative payoff represents the number of yards gained by the offense, so a positive number is the number of yards lost by the offense on that play of the game. Solve this game and find the optimal mix of plays by the defense and offense. (**Caution**: This is a matrix in which you might want to add a constant to ensure $v(A) > 0$. Then subtract the constant at the end to get the real value. You do not need to do that with the strategies.)

2.32 Let $a > 0$. Use the graphical method to solve the game in which player II has an infinite number of strategies with matrix

$$\begin{bmatrix} a & 2a & \frac{1}{2} & 2a & \frac{1}{4} & 2a & \frac{1}{6} & \cdots \\ a & 1 & 2a & \frac{1}{3} & 2a & \frac{1}{5} & 2a & \cdots \end{bmatrix}$$

2.33 A Latin square game is a square game in which the matrix A is a Latin square. A Latin square of size n has every integer from 1 to n in each row and column. Solve the game of size 5

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}$$

Prove that a Latin square game of size n has $v(A) = (n + 1)/2$.

2.34 Two cards, an ace and a jack, are face down on the table. Ace beats jack. Two players start by putting \$1 each into a pot. Player I picks a card at random and looks at it. Player I then bets either \$2 or \$4. Player II can either quit and yield the pot to player I, or **call** the bet (meaning that she may bet the \$2 or \$4 that player I has chosen the jack). If II calls, she matches I's bet, and then if I holds the ace, player I wins the pot, while if player I has the jack, then player II wins the pot.

(a) Find the matrix. (**Hint**: Each player has four pure strategies. Draw the game tree.)

(b) Assuming that the linear programming method by transforming variables gives the result $p_1 = p_2 = 0, p_3 = \frac{1}{10}, p_4 = \frac{1}{5}$ as the solution by the simplex method for player I, find the value of the game, and the optimal strategies for players I and II.

2.35 Two players, Curly and Shemp, are betting on the toss of a fair coin. Shemp tosses the coin, hiding the result from Curly. Shemp looks at the coin. If the coin is heads, Shemp says that he has heads and demands \$1 from Curly. If the coin is tails, then Shemp may tell the truth and pay Curly \$1, or he may lie and say that he got a head and demands \$1 from Curly. Curly can challenge Shemp whenever Shemp demands \$1 to see whether Shemp is telling the truth, but it will cost him \$2 if it turns out that Shemp was telling the truth. If Curly challenges the call and it turns out that Shemp was lying, then Shemp must pay Curly \$2. Find the matrix and solve the game.

2.5 LINEAR PROGRAMMING AND THE SIMPLEX METHOD (OPTIONAL)

Linear programming is one of the major success stories of mathematics for applications. Matrix games of arbitrary size are solved using linear programming, but so is finding the best route through a network for every single telephone call or computer network connection. Oil tankers are routed from port to port in order to minimize costs by a linear programming method. Linear programs have been found useful in virtually every branch of business and economics and many branches of physics and engineering–even in the theory of nonlinear partial differential equations. If you are curious about how the simplex method works to find an optimal solution, this is the section for you. We will have to be brief and only give a bare outline, but rest assured that this topic can easily consume a whole year or two of study, and it is necessary to know linear programming before one can deal adequately with nonlinear programming.

The standard linear programming problem consists of maximizing (or minimizing) a **linear function** $f : \mathbb{R}^m \to \mathbb{R}$ over a special type of convex set called a **polyhedral set** $S \subset \mathbb{R}^n$, which is a set given by a collection of linear constraints

$$S = \{\mathbf{x} \in \mathbb{R}^m \mid A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge 0\},$$

where $A_{n \times m}$ is a matrix, $\mathbf{x} \in \mathbb{R}^m$ is considered as an $m \times 1$ matrix, or vector, and $\mathbf{b} \in \mathbb{R}^n$ is considered as an $n \times 1$ column matrix. The inequalities are meant to be taken componentwise, so $\mathbf{x} \ge 0$ means $x_j \ge 0,\ j = 1, 2, \ldots, m$, and $A\mathbf{x} \le \mathbf{b}$ means

$$\sum_{j=1}^{m} a_{ij} x_j \le b_i, \text{ for each } i = 1, 2, \ldots, n.$$

The extreme points of S are the key. Formally, an extreme point, as depicted in

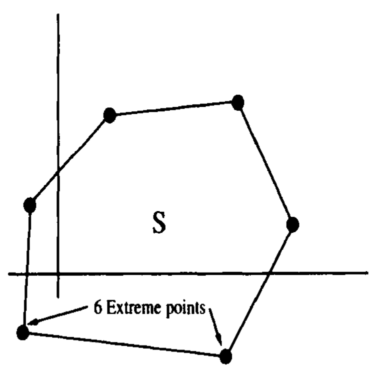


Figure 2.2 Extreme points of a convex set.

Figure 2.2, is a point of S that cannot be written as a convex combination of two other points of S; in other words, if $x = \lambda x_1 + (1 - \lambda) x_2$, for some $0 < \lambda < 1, x_1 \in S,\ x_2 \in S$, then $x = x_1 = x_2$.

Now here are the standard linear programming problems:

$$\begin{array}{lcl} \text{Maximize } \mathbf{c}\cdot\mathbf{x} & & \text{Minimize } \mathbf{c}\cdot\mathbf{x} \\ \text{subject to} & \text{or} & \text{subject to} \\ A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge 0 & & A\mathbf{x} \ge \mathbf{b}, \mathbf{x} \ge 0 \end{array}$$

The matrix A is $n \times m$, $\mathbf{b}$ is $n \times 1$, and $\mathbf{c}$ is $1 \times m$. To get an idea of what is happening, we will solve a two dimensional problem in the next example by graphing it.

EXAMPLE 2.12

The linear programming problem is

$$\begin{array}{l} \text{Minimize } z = 2x - 4y \\ \text{subject to} \\ x + y \ge \frac{1}{2},\ x - y \le 1,\ y + 2x \le 3, \quad x \ge 0,\ y \ge 0 \end{array}$$

So $c = (2, -4), b = (\frac{1}{2}, -1, -3)$ and

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & -1 \end{bmatrix}.$$

We graph (using Maple) the constraints and we also graph the objective $z = 2x - 4y = k$ on the same graph with various values of k as in Figure 2.3 The values of k that we have chosen are $k = -6, -8, -10$. The graph of the objective is, of course, a straight line for each k. The important thing is that as k decreases, the lines go up. From the direction of decrease, we see that the furthest we can go in decreasing k before we leave the constraint set is at the top extreme point. That point is $x = 0, y = 3$, and so the minimum objective subject to the constraints is $z = -12$.

The simplex method of linear programming basically automates this procedure for any number of variables or constraints. The idea is that the maximum and minimum must occur at extreme points of the region of constraints, and so we check the objective at those points only. At each extreme point, if it does not turn out to be optimal, then there is a direction that moves us to the next extreme point and that does improve the objective. The simplex method then does not have to check all the extreme points, just the ones that improve our goal.

The first step in using the simplex method is to change the inequality constraints into equality constraints $A\mathbf{x} = \mathbf{b}$, which we may always do by adding variables to x, called **slack variables**. So we may consider

$$S = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}.$$

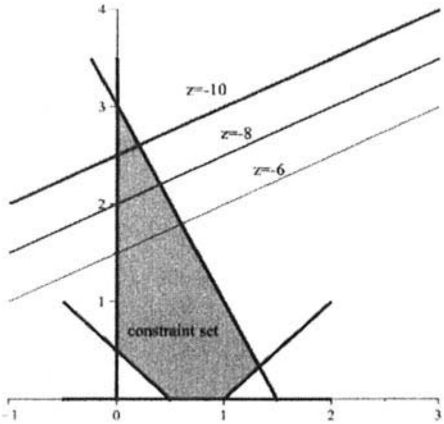


Figure 2.3 Objective Plotted over the Constraint Set

We assume that $S \neq \emptyset$. We also need the idea of an extreme direction for S. A vector $\mathbf{d} \neq 0$ is an **extreme direction** of S if and only if $A\,\mathbf{d} = 0$ and $\mathbf{d} \geq 0$. The simplex method idea then formalizes the approach that we know the optimal solution must be at an extreme point, so we begin there. If we are at an extreme point which is not our solution, then move to the next extreme point along an extreme direction. The extreme directions show how to move from extreme point to extreme point in the quickest possible way, improving the objective the most. Reference [30] is a comprehensive resource for all of these concepts.

In the next section we will show how to use the simplex method by hand. We will not go into any more theoretical details as to why this method works, and simply refer you to Winston's book [30].

2.5.1 The Simplex Method Step by Step

We will present the simplex method specialized for solving matrix games, starting with the standard problem setup for player II. The solution for player I can be obtained directly from the solution for player II, so we only have to do the computations once. Here are the steps. We list them first and then go through them so don't worry about unknown words for the moment.

1. Convert the linear programming problem to a system of linear equations using **slack variables**.

2. Set up the **initial tableau**.

3. Choose a **pivot column**.

   Look at all the numbers in the bottom row, excluding the answer column. From these, choose the largest number in absolute value. The column it is in is the pivot column. If there are two possible choices, choose either one. If all the numbers in the bottom row are zero or positive, then you are done. The basic solution is the optimal solution.

4. Select the **pivot** in the pivot column according to the following rules:

   (a) The pivot must always be a positive number. This rules out zeros and negative numbers.

   (b) For each positive entry b in the pivot column, excluding the bottom row, compute the ratio $\frac{a}{b}$, where a is the number in the rightmost column in that row.

   (c) Choose the smallest ratio. The corresponding number b which gave you that ratio is the **pivot**.

5. Use the pivot to clear the pivot column by row reduction. This means making the pivot element 1 and every other number in the pivot column a zero. Replace the x variable in the pivot row and column 1 by the x variable in the first row and pivot column. This results in the next tableau.

6. Repeat Steps 3–5 until there are no more negative numbers in the bottom row except possibly in the answer column. Once you have done that and there are no more positive numbers in the bottom row, you can find the optimal solution easily.

7. The solution is as follows. Each variable in the first column has the value appearing in the last column. All other variables are zero. The optimal objective value is the number in the last row, last column.

Here is a worked example following the steps. It comes from the game with matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix}$$

We will consider the linear programming problem for player II. **Player II's problem is always in standard form when we transform the game to a linear program using the first method of section 2.4. It is easiest to start with player II rather than player I.**

1. The problem is

$$\text{Maximize } q := x + y + z + w$$

subject to

$$4x + y + z + 3w \leq 1, \;\; 2x + 4y - 2z - w \leq 1, \;\; x, y, z, w \geq 0.$$

In matrix form this is

$$\text{Maximize } q := (1,1,1,1) \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

subject to

$$\begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \end{bmatrix}, \;\; x, y, z, w \geq 0.$$

At the end of our procedure we will find the value as $v(A) = \frac{1}{q}$, and the optimal strategy for player II as $Y^* = v(A)(x, y, z, w)$. We are not modifying the matrix because we know ahead of time that $v(A) \neq 0$. (Why?)

We need two slack variables to convert the two inequalities to equalities. Let's call them $s, t \geq 0$, so we have the equivalent problem

$$\text{Maximize } q = x + y + z + w + 0\,s + 0\,t$$

subject to

$$4x + y + z + 3w + s = 1, \;\; 2x + 4y - 2z - w + t = 1, \;\; x, y, z, w, s, t \geq 0.$$

The coefficients of the objective give us the vector $\mathbf{c} = (1, 1, 1, 1, 0, 0)$. Now we set up the initial tableau.

2. Here is where we start:

Variable	x	y	z	w	s	t	Answer
s	$\boxed{4}$	1	1	3	1	0	1
t	2	4	-2	-1	0	1	1
$-\mathbf{c}$	-1	-1	-1	-1	0	0	0

This is the initial tableau.

3. The pivot column is the column with the smallest number in the last row. Since we have several choices for the pivot column (because there are four -1s) we choose the first column arbitrarily. Notice that the last row uses $-\mathbf{c}$, not $\mathbf{c}$, because the method is designed to minimize, but we are actually maximizing here.

4. We choose the pivot in the first column by looking at the ratios $\frac{a}{b}$, where a is the number in the answer column and b is the number in the pivot column, for each row not including the bottom row. This gives $\frac{1}{4}$ and $\frac{1}{2}$ with the smaller ratio $\frac{1}{4}$. This means that the 4 in the second column is the pivot element.

5. Now we **row reduce** the tableau using the 4 in the first column until all the other numbers in that column are zero. Here is the next tableau after carrying out those row operations:

Variable	x	y	z	w	s	t	Answer
x	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$
t	0	$\boxed{\frac{7}{2}}$	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$
$-\mathbf{c}$	0	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$

Notice that on the leftmost (first) column we have replaced the variable s with the variable x to indicate that the s variable is leaving (has left) and the x variable is entering.

6. We repeat the procedure of looking for a pivot column. We have a choice of choosing either column with the $\frac{3}{4}$ on the bottom. We choose the first such column because if we calculate the ratios we get $(\frac{1}{4})/(\frac{1}{4}) = 1$ and $(\frac{1}{2})/(\frac{7}{2}) = \frac{1}{7}$, and that is the smallest ratio. So the pivot element is the $\frac{7}{2}$ in the third column.

After row reducing on the pivot $\frac{7}{2}$, we get the next tableau.

Variable	x	y	z	w	s	t	Answer
x	1	0	$\boxed{\frac{3}{7}}$	$\frac{13}{14}$	$\frac{2}{7}$	$-\frac{1}{14}$	$\frac{3}{14}$
y	0	1	$-\frac{5}{7}$	$-\frac{5}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
$-\mathbf{c}$	0	0	$-\frac{9}{7}$	$-\frac{11}{14}$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{5}{14}$

The slack variable t has left and the variable y has entered.

7. The largest element in the bottom row is $-\frac{9}{7}$ so that is the pivot column. The pivot element in that column must be $\frac{3}{7}$ because it is the only positive number left in the column. So we row reduce on the $\frac{3}{7}$ pivot and finally end up with the tableau

Variable	x	y	z	w	s	t	Answer
z	$\frac{7}{3}$	0	1	$\frac{13}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$	$\frac{1}{2}$
y	$\frac{5}{3}$	1	0	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
$-\mathbf{c}$	3	0	0	2	1	0	1

We have reached the end. All of the numbers on the bottom row are nonnegative so we are done. Now we read off the solution.

The **maximum objective is in the lower right corner**, $q = 1$.

The **maximum is achieved** at the variables $z = \frac{1}{2}, y = \frac{1}{2}, x = 0, w = 0$, because the z and y variables are on the left column and equal the corresponding value in the answer column. The remaining variables are zero because they are not in the variable column.

We conclude that $v(A) = \dfrac{1}{q} = 1$, and $Y^* = (0, \frac{1}{2}, \frac{1}{2}, 0)$ is optimal for player II.

To determine X^* we write down the dual of the problem for player II:

$$\text{Minimize } p = \mathbf{b}^T\mathbf{x}$$

subject to

$$A^T\mathbf{x} \geq \mathbf{c} \quad \text{and} \quad \mathbf{x} \geq 0.$$

For convenience let's match this up with player II's problem:

$$\text{Maximize } q = \mathbf{c} \cdot \mathbf{x} = (1, 1, 1, 1) \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

subject to

$$A\mathbf{x} = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \leq \mathbf{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad x, y, z, w \geq 0.$$

Notice that to find the dual, the cost vector is replaced by the right-hand vector of the inequality constraints for player II, we replace A by A^T, and the original cost vector $\mathbf{c}$ becomes the inequality constraints.

We could set up the simplex method for player I and solve it, but the solution is already present in the solution for player II; that is, **the solution of the dual is already present in the final tableau of the primal**. Ain't math wonderful! For the example we computed, when we get to the final tableau, the optimal objective for the dual is the same as the primal (that is the duality theorem), but the optimal variables are in the bottom row corresponding to the columns headed by the slack variables! That means that $v(A) = 1/q = 1$ and $X^* = (1, 0)$, the 1 coming from s and the 0 coming from t. That's it. We are done.

Now let's check our answer by solving the game directly. If you look at the game matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix}$$

it is obvious that column 3 dominates all the other columns and so there is a saddle point at row 1, column 3. The saddle is $X^* = (1, 0)$ and $Y^* = (0, 0, 1, 0)$. In fact there are an infinite number of optimal strategies for player II,

$$Y^* = (0, \frac{1}{2}(1 - \lambda), \frac{1}{2}(1 + \lambda), 0),$$

for any $0 \leq \lambda \leq 1$. The simplex method gave us one of them.

Here is one last example.

■ EXAMPLE 2.13

There are two presidential candidates, Harry and Tom, who will choose which states they will visit to garner votes. Their pollsters estimate that if, for example, Tom goes to state 2 and Harry goes to state 1, then Tom will lose 8 percentage points to Harry in that state. Suppose that there are 3 states that each candidate can select. Here is the matrix with Tom as the row player:

Tom/Harry	State 1	State 2	State 3
State 1	12	−9	14
State 2	−8	7	12
State 3	11	−10	10

Each player wants to know which visits to which states he should make and what will be the net result. This is a game. There is an obvious dominance that allows us to drop state 3 for both players, but we will ignore this to see if we end up with that in our solution. That would be a quick check to see if everything works.

So, we have two methods to solve this problem. We could use the invertible matrix formulas and see if that works, or we could use linear programming, which is our choice here.

Step one is to set up the linear program for player II.

$$\text{Maximize } z_{II} = q_1 + q_2 + q_3$$

subject to

$$12q_1 - 9q_2 + 14q_3 \leq 1, \quad -8q_1 + 7q_2 + 12q_3 \leq 1, \quad 11q_1 - 10q_2 + 10q_3 \leq 1,$$

and $q_1, q_2, q_3 \geq 0$.

Next we set up the initial tableau, which will have three slack variables because we have three inequality constraints:

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
s_1	$\boxed{12}$	-9	14	1	0	0	1
s_2	-8	7	12	0	1	0	1
s_3	11	-10	10	0	0	1	1
z_{II}	-1	-1	-1	0	0	0	0

Instead of labeling the last row $-\mathbf{c}$ we are labeling it with the objective value z_{II} to keep track of the player number.

After pivoting on the 12 in the second column, we replace s_1 by q_1 in the first column:

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
q_1	1	$-\frac{3}{4}$	$\frac{7}{6}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$
s_2	0	$\boxed{1}$	$\frac{64}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{3}$
s_3	0	$-\frac{7}{4}$	$-\frac{17}{6}$	$-\frac{11}{12}$	0	1	$\frac{1}{12}$
z_{II}	0	$-\frac{7}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$

Finally, we pivot on the 1 in the third column and arrive at the final tableau:

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
q_1	1	0	$\frac{103}{6}$	$\frac{7}{12}$	$\frac{3}{4}$	0	$\frac{4}{3}$
q_2	0	1	$\frac{64}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{3}$
s_3	0	0	$\frac{69}{2}$	$\frac{1}{4}$	$\frac{7}{4}$	1	3
z_{II}	0	0	$\frac{75}{2}$	$\frac{5}{4}$	$\frac{7}{4}$	0	3

This is the final tableau because all numbers on the bottom are nonnegative. Notice, too, that s_2 in the first column was replaced by q_2 because that was the pivot column.

Now we read off the information:

$$z_{\mathrm{II}} = 3 = \frac{1}{v(A)}, \; q = \left(\frac{4}{3}, \frac{5}{3}, 0\right) \implies v(A) = \frac{1}{3} \text{ and } Y^* = \left(\frac{4}{9}, \frac{5}{9}, 0\right).$$

For player I we look at the numbers under the slack variables in the final tableau, namely, $\frac{5}{4}, \frac{7}{4}, 0$. This is the solution to the dual problem. In other words, the optimal solution for player I is $p = (\frac{5}{4}, \frac{7}{4}, 0)$, which gives us

$$X^* = v(A)\, p = \left(\frac{5}{12}, \frac{7}{12}, 0\right).$$

Sure enough, state 3 is never to be visited by either Tom or Harry. Tom should visit state 1, 5 out of 12 times and state 2, 7 out of 12 times. Interestingly, Harry should visit state 1, 4 out of 9 times and state 2, 5 out of 9 times. Tom ends up with a net gain of $v(A) = 0.33\%$. It hardly seems worth the effort (unless the opponent doesn't play optimally).

Of course, you can check these results with the Maple commands

```
> with (LinearAlgebra):
> value:=proc(A,rows,cols)
    local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
    X:=Vector(rows,symbol=x):
    Y:=Vector(cols,symbol=y):
    B:=Transpose(X).A; C:=A.Y;
    cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
    cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
    with(simplex):
    vu:=maximize(vI,cnstx,NONNEGATIVE);
    vl:=minimize(vII,cnsty,NONNEGATIVE);
    print(vu,vl);
 end:

> City:=Matrix([[12,-9,14],[-8,7,12],[11,-10,10]]);
> value(City,3,3);
```

These Maple commands produce the output

$$vI = \frac{1}{3}, \quad X^* = (x_1, x_2, x_3) = \left(\frac{5}{12}, \frac{7}{12}, 0\right),$$

and

$$vII = \frac{1}{3}, \quad Y^* = (y_1, y_2, y_3) = \left(\frac{4}{9}, \frac{5}{9}, 0\right).$$

PROBLEMS

2.36 Use the simplex method to solve the games with the following matrices. Check your answers using Maple and verify that the strategies obtained are indeed optimal.

$$\text{(a)} \begin{bmatrix} 0 & 3 & 3 & 2 & 4 \\ 4 & 4 & 3 & 1 & 4 \end{bmatrix}; \ \text{(b)} \begin{bmatrix} 4 & 4 & -4 & -1 \\ -4 & -2 & 4 & 4 \\ 2 & -4 & -1 & -5 \\ -3 & 1 & 0 & -4 \end{bmatrix}; \ \text{(c)} \begin{bmatrix} 2 & -5 & 3 & 0 \\ -4 & -5 & -5 & -6 \\ 3 & -4 & -1 & -2 \\ 0 & 4 & 1 & 3 \end{bmatrix}.$$

2.37 Solve this problem using Maple in two ways:

$$\begin{bmatrix} -2 & 3 & 3 & 4 & 1 \\ 3 & -2 & -5 & 2 & 4 \\ 4 & -5 & -1 & 4 & -1 \\ 2 & -4 & 3 & 4 & -3 \end{bmatrix}.$$

2.6 A GAME THEORY MODEL OF ECONOMIC GROWTH (OPTIONAL)

In this section we present an application of game theory (due to von Neumann) to an input/output model of economic growth.[4]

An economy has many goods (or goods and services), each of which has a price. Suppose that there are $i = 1, 2, \ldots, n$ goods with prices $p_1, p_2, \ldots, p_n$, respectively. Set the

$$\text{Price vector } = \mathbf{p} = (p_1, p_2, \ldots, p_n).$$

There are also $j = 1, 2, \ldots, m$ activities to produce the goods and services. Let

$$y_j = \text{production intensity of process } j, \quad \text{and } \mathbf{y} = (y_1, \ldots, y_m).$$

We consider two processes, an input process and an output process. The input process is:

$$\text{Amount of good } i = 1, 2, \ldots, n \text{ used by process } j = 1, 2, \ldots, m, \text{ is } = a_{ij} y_j \geq 0.$$

The output process is:

$$\text{Amount of good } i = 1, 2, \ldots, n, \text{ produced by process } j \text{ is } = b_{ij} y_j \geq 0.$$

It is assumed that the constants $a_{ij} \geq 0, b_{ij} \geq 0$, for all i, j. Set the matrices

$$A_{n \times m} = (a_{ij}), \quad \text{and } B = (b_{ij}).$$

[4]In this section we follow the formulation in Reference [10]; but see also Reference [11] for this model and much more.

For the time being we also consider two constants:

$$\rho_g = \text{ rate of growth of goods and services }, \tag{2.6.1}$$

$$\rho_r = \text{ rate of growth of money } = \text{ risk-free interest rate.} \tag{2.6.2}$$

Since all prices must be nonnegative and all intensities must be nonnegative

$$p_i \geq 0, \quad i = 1, 2, \ldots, n, \quad y_j \geq 0, \quad j = 1, 2, \ldots, m.$$

Assume that every row and every column of the matrices A and B has at least one positive element. This implies that

$$\sum_{i=1}^{n} a_{ij} > 0, j = 1, 2, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} b_{ij} > 0, i = 1, 2, \ldots, n.$$

The economic meaning is that every process requires at least one good, and every good is produced by at least one process. Of course, we also assume that $a_{ij} \geq 0, b_{ij} \geq 0$ for all elements of the matrices A, B.

The summary of the input/output model in matrix form is

1. $B\mathbf{y} \geq \rho_g A\mathbf{y}$.
2. $\mathbf{p}(\rho_g A - B)\mathbf{y} = 0$.
3. $\rho_r \mathbf{p}A \geq \mathbf{p}B$.
4. $\mathbf{p}(\rho_m A - B)\mathbf{y} = 0$.
5. $\mathbf{p}B\mathbf{y} > 0$.

Condition 1 says that the output of goods must grow by a factor of ρ_g. Condition 2 says that if any component of $(\rho_g A - B)\mathbf{y} < 0$, say, the ith element, then $p_i = 0$. If $(\rho_g a_{ij} - b_{ij})y_j < 0$ that would mean that the output of good i exceeds the consumption of that good. We assume in this model that when demand is exceeded by supply, the price of that good will be zero, so $p_i = 0$. Conversely, if $p_i > 0$, so the price of good i is positive, then $[(\rho_g A - B)y]_i = 0$. But then

$$\sum_{j=1}^{m} (\rho_g a_{ij} - b_{ij}) y_j = 0,$$

and so the output of good i is exactly balanced by the input of good i. Conditions 3 and 4 have a similar economic interpretation but for prices. Condition 5 is that $p_i b_{ij} y_j > 0$ for at least one i and j since the economy must produce at least one good with a positive price and with positive intensity. (See the texts by Karlin [11] and Kaplan [10] for a full derivation of the model.)

The question is whether there are vectors $\mathbf{p}$ and $\mathbf{y}$ and growth factors ρ_r, ρ_g so that these five conditions are satisfied. We can easily reach some conclusions under the assumptions of the model.

Conclusion 1. For an economy satisfying the assumptions of the input/output model, it must be true that the growth factor for goods must be the same as the growth rate of money $\rho_g = \rho_r$.

Proof. Using Conditions 1-5, we first have

$$\mathbf{p}(\rho_g A - B)\mathbf{y} = 0 \implies \rho_g \mathbf{p} A \mathbf{y} = \mathbf{p} B \mathbf{y} > 0$$

and

$$\mathbf{p}(\rho_m A - B)\mathbf{y} = 0 \implies \rho_r \mathbf{p} A \mathbf{y} = \mathbf{p} B \mathbf{y},$$

so that

$$\rho_g \mathbf{p} A \mathbf{y} = \rho_r \mathbf{p} A \mathbf{y} > 0.$$

Because this result is strictly positive, by dividing by $\mathbf{p}A\mathbf{y}$ we may conclude that $\rho_g = \rho_r$. $\square$

Considering our Conclusion 1, from now on we just write $\rho = \rho_g = \rho_r$. Then, from the conditions 1-5, we have the inequalities

$$\mathbf{p}(\rho A - B) \geq 0 \geq (\rho A - B)\mathbf{y}. \tag{2.6.3}$$

In addition

$$\mathbf{p}(\rho A - B)\mathbf{y} = 0 = \mathbf{p}(\rho A - B)\mathbf{y}. \tag{2.6.4}$$

Can we find $\mathbf{p}$ and $\mathbf{y}$ and a scalar ρ so that these inequalities will be satisfied? This is where game theory enters.

Consider the two-person zero sum game with matrix $\rho A - B$. This game has a value using mixed strategies denoted by $v(\rho A - B)$, as well as a saddle point in mixed strategies, say, (X^*, Y^*), satisfying the saddle point condition

$$X(\rho A - B)Y^* \leq v(\rho A - B) \leq X^*(\rho A - B)Y, \text{ for all } X \in S_n, Y \in S_m.$$

These strategies will also depend on ρ. The focus now is on the constant $\rho > 0$. It would be nice if there were a constant $\rho = \rho_0$ so that $v(\rho_0 A - B) = 0$, because then the saddle condition becomes

$$X(\rho_0 A - B)Y^* \leq 0 \leq X^*(\rho_0 A - B)Y, \text{ for all } X \in S_n, Y \in S_m.$$

In particular, for every row and column $E(i, Y^*) \leq 0 \leq E(X^*, j)$, or in matrix form

$$(\rho_0 A - B)Y^* \leq 0 \leq X^*(\rho_0 A - B),$$

which is exactly the same as (2.6.3) with $\mathbf{p}$ replaced by X^* and $\mathbf{y}$ replaced by Y^*. In addition, $v(\rho_0 A - B) = 0 = X^*(\rho_0 A - B)Y^*$, which is the same as (2.6.4) with $\mathbf{p}$ replaced by X^* and $\mathbf{y}$ replaced by Y^*. Prices and intensities do not have to be strategies, but the strategies $\mathbf{p} = X^*$ and $\mathbf{y} = Y^*$ can be considered as normalized prices and intensities, noting that the inequalities still hold if we divide by constants. Thus we may assume without loss of generality from the beginning that $\sum p_i = \sum y_j = 1$. We are now ready to state our second conclusion.

Conclusion 2. There is a constant $\rho_0 > 0$ (which is unique if $a_{ij} + b_{ij} > 0$) and price and intensity vectors $\mathbf{p}, \mathbf{y}$ so that $\mathbf{p}B\mathbf{y} > 0$,

$$\mathbf{p}(\rho A - B) \geq 0 \geq (\rho A - B)\mathbf{y},$$

and

$$\mathbf{p}(\rho A - B)\mathbf{y} = 0 = \mathbf{p}(\rho A - B)\mathbf{y}.$$

In other words, there is a $\rho_0 > 0$ such that $value(\rho_0 A - B) = 0$, there is a saddle point $(\mathbf{y}_{\rho_0}, \mathbf{p}_{\rho_0})$, and the saddle point satisfies $\mathbf{p}_{\rho_0} B \mathbf{y}_{\rho_0} > 0$.

Proof. We will show only that in fact there is ρ_0 satisfying $v(\rho_0 A - B) = 0$. If we set $f(\rho) = value(\rho A - B)$, we have $f(0) = value(-B)$. Let (X, Y) be optimal strategies for the game with matrix $-B$. Then

$$value(-B) = \min_j E_B(X, j) = \min_j \sum_{i=1}^{n} (-b_{ij}) x_i < 0$$

because we are assuming that at least one $b_{ij} > 0$ and at least one x_i must be positive. Since every row of A has at least one strictly positive element a_{ij}, it is always possible to find a large enough $\rho > 0$ so that $f(\rho) > 0$. Since $f(\rho)$ is a continuous function, the intermediate value theorem of calculus says that there must be at least one $\rho_0 > 0$ for which $f(\rho_0) = 0$. $\square$

The conclusion of all this is that, under the assumptions of the model, there is a set of equilibrium prices, intensity levels, and growth rate of money that permits the expansion of the economy. We have used game theory to prove it.

■ EXAMPLE 2.14

Let's consider the input/output matrices

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

We will find the $\rho > 0$ so that $value(\rho A - B) = 0$, as well as the optimal strategies. We will use Maple to find ρ. Notice that $a_{ij} + b_{ij} > 0$ so there is only one $\rho_0 > 0$ which will work. The commands we use are as follows:

```
>with(LinearAlgebra):
>value:=proc(A,rows,cols)
       local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
       X:=Vector(rows,symbol=x): Y:=Vector(cols,symbol=y):
       B:=Transpose(X).A;
       C:=A.Y;
       cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
       cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
       vu:=maximize(vI,cnstx,NONNEGATIVE);
       vl:=minimize(vII,cnsty,NONNEGATIVE);
       print(vu,vl);
 end:

> A:=a->Matrix([[-2,2*a-1,a-3],[3*a-2,a-3,2*a-1]]);
> # This is the matrix (a A-B) as a function of a>0;
> B1:=a->A(a)+ConstantMatrix(5,2,3);
> # This command adds a constant (5) to each element of A(a) so
> # that the value is not close to zero. We subtract 5 at the end to
> # get the actual value.
> value(B1(1.335),2,3);
```

By plugging in various values of a, we get

$$
\begin{aligned}
&value(B1(0),2,3) = 3 \implies value(0A - B) = 3 - 5 = -2\\
&value(B1(2),2,3) = 6 \implies value(2A - B) = 6 - 5 = 1\\
&value(B1(1.5),2,3) = 5.25 \implies value(1.5A - B) = 5.25 - 5 = 0.25\\
&\vdots\\
&value(B1(1.335),2,3) = 5.0025\\
&\qquad\qquad\qquad \implies value(1.335A - B) = 5.0025 - 5 = 0.0025.
\end{aligned}
$$

We eventually arrive at the conclusion that when $a = 1.335$, we have

$$value(B1(1.335), 2, 3) = value(aA - B + 5) = 5.0025.$$

Subtracting 5, we get $value(1.335A - B) = 0$, and so $\rho_0 = 1.335$. The optimal strategies are $\mathbf{p} = X = (\frac{1}{2}, \frac{1}{2})$, and $\mathbf{y} = Y = (0, \frac{1}{2}, \frac{1}{2})$.

We end this section with a sketch of the proof of a useful result if you want to use Newton's method to calculate ρ_0. To do that, you need the derivative of $f(\rho) = value(\rho A - B)$, which is where we will stop with this. Here is the derivative from the right:

$$\lim_{h\to 0+} \frac{value((\rho + h)A - B) - value(\rho A - B)}{h} = \max_{X\in S_n(A)} \min_{Y\in S_m(A)} X(\rho A - B)Y^T,$$

where $S_n(A)$ denotes the set of strategies that are **optimal for the game with matrix** A. Similarly, $S_m(A)$ is the set of strategies for player II that are optimal for the game with matrix A.

Proof. Suppose that (X^h, Y^h) are optimal for $A_h \equiv (\rho + h)A - B$ and (X^ρ, Y^ρ) are optimal for $A_\rho \equiv \rho A - B$. Then, if we play X^h against Y^ρ, we get

$$\begin{aligned} value((\rho + h)A - B) &\leq X^h((\rho + h)A - B)Y^{\rho T} \\ &= X^h(\rho A - B)Y^{\rho T} + hX^h AY^{\rho T} \\ &\leq value(\rho A - B) + hX^h AY^{\rho T}. \end{aligned}$$

The last inequality follows from the fact that $value(\rho A - B) \geq X^h(\rho A - B)Y^{\rho T}$ because Y^ρ is optimal for $\rho A - B$. In a similar way, we can see that

$$value((\rho + h)A - B) \geq (value(\rho A - B) + hX^\rho AY^{hT}).$$

Putting them together, we have

$$value(\rho A - B) + hX^\rho AY^{hT} \leq value((\rho + h)A - B) \leq value(\rho A - B) + hX^h AY^{\rho T}.$$

Now, divide these inequalities by $h > 0$ to get

$$X^\rho AY^{hT} \leq \frac{value((\rho + h)A - B) - value(\rho A - B)}{h} \leq X^h(\rho A - B)Y^{\rho T}.$$

Let $h \to 0+$. Since $X^h \in S_n, Y^h \in S_m$, these strategies, as a function of h, are bounded uniformly in h. Consequently, as $h \to 0$, it can be shown that $X^h \to X^* \in S_n(\rho A - B)$ and $Y^h \to Y^* \in S_m(\rho A - B)$. We conclude that

$$X^\rho AY^{*T} \leq \lim_{h \to 0+} \frac{value((\rho + h)A - B) - value(\rho A - B)}{h} \leq X^* AY^{\rho T},$$

or

$$X^\rho AY^{*T} \leq f'(\rho)_+ \leq X^* AY^{\rho T}.$$

Consequently, we obtain

$$\begin{aligned} \min_{Y \in S_n(\rho A - B)} \max_{X \in S_n(\rho A - B)} XAY^T &\leq f'(\rho)_+ \\ &\leq \min_{Y \in S_n(\rho A - B)} X^* AY^{\rho T} \\ &\leq \max_{X \in S_n(\rho A - B)} \min_{Y \in S_n(\rho A - B)} XAY^T. \end{aligned}$$

Since

$$\max_{X \in S_n(\rho A - B)} \min_{Y \in S_n(\rho A - B)} XAY^T \leq \min_{Y \in S_n(\rho A - B)} \max_{X \in S_n(\rho A - B)} XAY^T,$$

we have shown that

$$f'(\rho)_+ = \lim_{h\to 0+} \frac{value((\rho+h)A - B) - value(\rho A - B)}{h}$$
$$= \max_{X\in S_n(\rho A - B)} \min_{Y\in S_n(\rho A-B)} XAY^T.$$

□

A special case of this result is that for any matrix $D_{n\times m}$, we have a sort of formula for the directional derivative of $v(A)$ in the *direction* D:

$$\lim_{h\to 0+} \frac{v(A+hD) - v(A)}{h} = \max_{X\in S_n(A)} \min_{Y\in S_n(A)} XDY^T.$$

In particular, if we fix any payoff entry of A, say, a_{ij}, and take D to be the matrix consisting of all zeros except for $d_{ij} = 1$, we get a formula for the partial derivative of $v(A)$ with respect to the components of A:

$$\left(\frac{\partial v(A)}{\partial a_{ij}}\right)_+ = \max_{X\in S_n(A)} \min_{Y\in S_n(A)} XDY^T = (\max_{X\in S_n(A)} x_i)(\min_{Y\in S_n(A)} y_j). \quad (2.6.5)$$

PROBLEMS

2.38 Consider the input/output matrices

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 2 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

Find the $\rho > 0$ so that $value(\rho A - B) = 0$.

2.39 Verify equation (2.6.5) for the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ by calculating $v(A + hD)$ as a function of h, calculating the limit, and calculating the right side of (2.6.5).

BIBLIOGRAPHIC NOTES

Invertible matrices are discussed by Owen [20] and Ferguson [3] and in many other references. Symmetric games were studied early in the development of the theory because it was shown (see problem 2.23, from Karlin [11]) that any matrix game could be symmetrized so that techniques developed for symmetric games could be used to yield information about general games. Before the advent of high-speed computers, this was mostly of theoretical interest because the symmetrization made the game matrix much larger. Today, this is not much of a problem.

The introduction of the Burr–Hamilton duel is an example of a game of timing that is properly considered in the continuous-time setting discussed in later chapters. For the full generality see Karlin's book [11].

The fundamental idea of using linear programming to solve games, and the use of the duality theorem to even prove the existence of value of a matrix game, might have been known to several mathematicians at the same time. The exact originator of this idea is unknown to the present author but possibly may be due to Dantzig, von Neumann, Karlin, or others.

Problem 2.29 is a classic example of the use of game theory in a business problem. Here it is used to determine the location of a business depending on marketing conditions and the location of a competitor's store. Versions of this problem appear throughout the literature, but this version is due to R. Bronson. Problem 2.30 is an example of a dispute resolution problem (refer to Winston's book [30]) presaging the idea of cooperative games considered in chapter 5. Problem 2.32 is an instructive problem on the use of the graphical method and appears in at least the books by [11] and [7]. The Colonel Blotto game is an early application of game theory to military theory. There is an entire class of games known as Blotto games. The analysis presented in Example 2.11 is due to Karlin and shows the techniques involved to solve an important game without the use of linear programming.

The use of the simplex method for solving matrix games by hand or by computer is standard and appears in almost all books on game theory. Here we follow Ferguson's approach in reference [3] because, as is pointed out in those notes, by starting with the program for player II, the solution for player I can be read off of the final tableau. This approach requires the transformation of the game into a linear programming problem by the first method.

The game theory application to an equilibrium model of economic growth follows the approach by Kaplan [10] and Karlin [11], where many more models of economic growth appear. The determination of the directional derivative of $\rho \mapsto v(\rho A - B)$ is a modification of an exercise by Karlin [11].

CHAPTER 3

TWO-PERSON NONZERO SUM GAMES

But war's a game, which, were their subjects wise, Kings would not play at.

–William Cowper, *The Winter Morning Walk*

3.1 THE BASICS

The previous chapter considered two person games in which whatever one player gains, the other loses. This is far too restrictive for many games, especially games in economics or politics, where both players can win something or both players can lose something. We no longer assume that the game is zero sum, or even constant sum. All players will have their own individual payoff matrix and the goal of maximizing their own individual payoff. We will have to reconsider what we mean by a solution, how to get optimal strategies, and exactly what is a strategy.

Game Theory: An Introduction, First Edition. By E.N. Barron

In a two-person nonzero sum game, we simply assume that each player has her or his own payoff matrix. Suppose that the payoff matrices are

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

For example, if player I plays row 1 and player II plays column 2, then the payoff to player I is a_{12} and the payoff to player II is b_{12}. In a zero sum game we always had $a_{ij} + b_{ij} = 0$, or more generally $a_{ij} + b_{ij} = k$, where k is a fixed constant. In a nonzero sum game we do not assume that. Instead, the payoff when player I plays row i and player II plays column j is now a pair of numbers (a_{ij}, b_{ij}) where the first component is the payoff to player I and the second number is the payoff to player II. The individual rows and columns are called **pure strategies** for the players. Finally, every zero sum game can be put into the bimatrix framework by taking $B = -A$, so this is a true generalization of the theory in the first chapter. Let's start with a simple example.

■ **EXAMPLE 3.1**

Two students have an exam tomorrow. They can choose to study, or go to a party. The payoff matrices, written together as a bimatrix, are given by

I/II	Study	Party
Study	$(2,2)$	$(3,1)$
Party	$(1,3)$	$(4,-1)$

If they both study, they each receive a payoff of 2, perhaps in grade point average (GPA) points. If player I studies and player II parties, then player I receives a better grade because the curve is lower. But player II also receives a payoff from going to the party (in good time units). If they both go to the party, player I has a really good time, but player II flunks the exam the next day and his girlfriend is stolen by player I, so his payoff is -1. What should they do?

Players can choose to play pure strategies, but we know that greatly limits their options. So we allow mixed strategies. A mixed strategy for player I is again $X = (x_1, \ldots, x_n) \in S_n$ with $x_i \geq 0$ representing the probability that player I uses row i, so again $x_1 + x_2 + \cdots + x_n = 1$. Similarly, a mixed strategy for player II is $Y = (y_1, \ldots, y_m) \in S_m$, with $y_j \geq 0$ and $y_1 + \cdots + y_m = 1$. Now given the player's choice of mixed strategies, each player will have their own expected payoffs given by

$$E_{\mathrm{I}}(X, Y) = XAY^T \text{ for player I,}$$
$$E_{\mathrm{II}}(X, Y) = XBY^T \text{ for player II.}$$

We need to define a concept of optimal play that should reduce to a saddle point in mixed strategies in the case $B = -A$. It is a fundamental and far-reaching definition due to another genius of mathematics who turned his attention to game theory in the middle twentieth century, John Nash.

Definition 3.1.1 *A pair of mixed strategies $(X^* \in S_n, Y^* \in S_m)$ is a Nash equilibrium if $E_I(X, Y^*) \le E_I(X^*, Y^*)$ for every mixed $X \in S_n$ and $E_{II}(X^*, Y) \le E_{II}(X^*, Y^*)$ for every mixed $Y \in S_m$. If (X^*, Y^*) is a Nash equilibrium we denote by $v_A = E_I(X^*, Y^*)$ and $v_B = E_{II}(X^*, Y^*)$ as the optimal payoff to each player. Written out with the matrices, (X^*, Y^*) is a Nash equilibrium if*

$$E_I(X^*, Y^*) = X^* A Y^{*T} \ge X A Y^{*T} = E_I(X, Y^*), \text{ for every } X \in S_n,$$

$$E_{II}(X^*, Y^*) = X^* B Y^{*T} \ge X^* B Y^T = E_{II}(X^*, Y), \text{ for every } Y \in S_m.$$

This says that neither player can gain any expected payoff if either one chooses to deviate from playing the Nash equilibrium, **assuming that the other player is implementing his or her piece of the Nash equilibrium.** On the other hand, if it is known that one player will not be using his piece of the Nash equilibrium, then the other player may be able to increase her payoff by using some strategy other than that in the Nash equilibrium. The player then uses a **best response strategy**. In fact, the definition of a Nash equilibrium says that each strategy in a Nash equilibrium is a best response strategy against the opponent's Nash strategy. Here is a precise definition for two players.

Definition 3.1.2 *A strategy $X^0 \in S_n$ is a* **best response strategy** *to a given strategy $Y^0 \in S_m$ for player II. if*

$$E_I(X^0, Y^0) = \max_{X \in S_n} E_I(X, Y^0).$$

Similarly, a strategy $Y^0 \in S_m$ is a **best response strategy** *to a given strategy $X^0 \in S_n$ for player I, if*

$$E_{II}(X^0, Y^0) = \max_{Y \in S_m} E_{II}(X^0, Y).$$

In particular, another way to define a Nash equilibrium (X^*, Y^*) is that X^* maximizes $E_I(X, Y^*)$ over all $X \in S_n$ and Y^* maximizes $E_{II}(X^*, Y)$ over all $Y \in S_m$. X^* is a best response to Y^* and Y^* is a best response to X^*.

If $B = -A$, a bimatrix game is a zero sum two-person game and a Nash equilibrium is the same as a saddle point in mixed strategies. It is easy to check that from the definitions because $E_I(X, Y) = XAY^T = -E_{II}(X, Y)$.

Note that a Nash equilibrium in pure strategies will be a row i^* and column j^* satisfying

$$a_{ij^*} \le a_{i^*j^*} \text{ and } b_{i^*j} \le b_{i^*j^*}, i = 1, \dots, n, j = 1, \dots, m.$$

So $a_{i^*j^*}$ is the largest number in column j^* and $b_{i^*j^*}$ is the largest number in row i^*. In the bimatrix game a Nash equilibrium in pure strategies must be the **pair that is, at the same time, the largest first component in the column and the largest second component in the row**.

Just as in the zero sum case, a pure strategy can always be considered as a mixed strategy by concentrating all the probability at the row or column, which should always be played.

As in earlier sections we will use the notation that if player I uses the pure strategy row i, and player II uses mixed strategy $Y \in S_m$, then the expected payoffs to each player are

$$E_{\mathrm{I}}(i,Y) = {}_iAY^T \quad \text{and} \quad E_{\mathrm{II}}(i,Y) = {}_iBY^T.$$

Similarly, if player II uses column j and player I uses the mixed strategy X, then

$$E_{\mathrm{I}}(X,j) = XA_j \quad \text{and} \quad E_{\mathrm{II}}(X,j) = XB_j.$$

The questions we ask for a given bimatrix game are

1. Is there a Nash equilibrium using pure strategies?
2. Is there a Nash equilibrium using mixed strategies? Maybe more than one?
3. How do we compute these?

To start, we consider the classic example.

Prisoner's Dilemma. Two criminals have just been caught after committing a crime. The police interrogate the prisoners by placing them in separate rooms so that they cannot communicate and coordinate their stories. The goal of the police is to try to get one or both of them to confess to having committed the crime. We consider the two prisoners as the players in a game in which they have two pure strategies: confess, or don't confess. Their prison sentences, if any, will depend on whether they confess and agree to testify against each other. But if they both confess, no benefit will be gained by testimony that is no longer needed. If neither confesses, there may not be enough evidence to convict either of them of the crime. The following matrices represent the possible payoffs remembering that they are set up to maximize the payoff.

Prisoner I/II	Confess	Don't confess
Confess	$(-5,-5)$	$(0,-20)$
Don't confess	$(-20,0)$	$(-1,-1)$

The individual matrices for the two prisoners are

$$A = \begin{bmatrix} -5 & 0 \\ -20 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & -20 \\ 0 & -1 \end{bmatrix}$$

The numbers are negative because they represent the number of years in a prison sentence and each player wants to maximize the payoff, that is, minimize their own sentences.

To see whether there is a Nash equilibrium in pure strategies we are looking for a payoff pair (a, b) in which a is the largest number in a column and b is the largest number in a row simultaneously. There may be more than one such pair. Looking at the bimatrix is the easiest way to find them. The **systematic way** is to put a bar over the first number that is the **largest in each column and put a bar over the second number that is the largest in each row.** Any pair of numbers that both have bars is a Nash equilibrium in pure strategies. In the prisoner's dilemma problem we have

Prisoner I/II	Confess	Don't confess
Confess	$\boxed{(-\bar{5}, -\bar{5})}$	$(\bar{0}, -20)$
Don't confess	$(-20, \bar{0})$	$(-1, -1)$

We see that there is exactly one pure Nash equilibrium at (confess, confess), they should both confess and settle for 5 years in prison each. If either player deviates from *confess*, while the other player still plays *confess*, then the payoff to the player who deviates goes from -5 to -20.

Wait a minute: clearly both players can do better if they both choose *don't confess* because then there is not enough evidence to put them in jail for more than one year. But there is an incentive for each player to **not** play *don't confess*. If one player chooses *don't confess* and the other chooses *confess*, the payoff to the confessing player is 0–he won't go to jail at all! The players are rewarded for a **betrayal of the other prisoner**, and so that is exactly what will happen. This reveals a major reason why conspiracies almost always fail. As soon as one member of the conspiracy senses an advantage by confessing, that is exactly what they will do, and then the game is over.

The payoff pair $(-1, -1)$ is **unstable** in the sense that a player can do better by deviating, assuming that the other player does not, whereas the payoff pair $(-5, -5)$ is **stable** because neither player can improve their own individual payoff if they both play it. Even if they agree before they are caught to not confess, it would take extraordinary will power for both players to stick with that agreement in the face of the numbers. In this sense, the Nash equilibrium is self-enforcing.

Notice that in matrix A in the prisoner's dilemma, the first row is always better than the second row for player I. This says that for player I, row 1, (i.e., confess), **strictly dominates** row 2 and hence row 2 can be eliminated from consideration by player I, no matter what player II does because player I will never play row 2. Similarly, in matrix B for the other player, column 1 strictly dominates column 2, so player II, who chooses the columns, would never play column 2. Once again, player II would always confess. This problem can be solved by domination.

EXAMPLE 3.2

Can a bimatrix game have more than one Nash equilibrium? Absolutely. If we go back to the study–party game and change one number, we will see that it has two Nash equilibria:

I/II	Study	Party
Study	$\boxed{(2,2)}$	$(3,1)$
Party	$(1,3)$	$\boxed{(4,4)}$

There is a Nash equilibrium at payoff $(2,2)$ and at $(4,4)$. Which one is better? In this example, since both students get a payoff of 4 by going to the party, that Nash point is clearly better for both. But if player I decides to study instead, then what is best for player II? That is not so obvious.

In the next example we get an insight into why countries want to have nuclear bombs.

EXAMPLE 3.3

The Arms Race. Suppose that two countries have the choice of developing or not developing nuclear weapons. There is a cost of the development of the weapons in the price that the country might have to pay in sanctions, and so forth. But there is also a benefit in having nuclear weapons in prestige, defense, deterrence, and so on. Of course, the benefits of nuclear weapons disappear if a country is actually going to use the weapons. If one considers the outcome of attacking an enemy country with nuclear weapons and the risk of having your own country vaporized in retaliation, a rational person would certainly consider the cold war acronym MAD, Mutually Assured Destruction, as completely accurate.

Suppose that we quantify the game using a bimatrix in which each player wants to maximize the payoff.

Country I/ II	Nuclear	Conventional
Nuclear	$\boxed{(1,1)}$	$(10,-5)$
Conventional	$(-5,10)$	$(1,1)$

To explain these numbers, if country I goes nuclear and country II does not, then country I can dictate terms to country II to some extent because a war with I, who holds nuclear weapons and will credibly use them, would result in destruction of country II, which has only conventional weapons. The result is a representative payoff of 10 to country I and -5 to country II, as I's lackey now. On the other hand, if both countries go nuclear, neither country has an advantage or can dictate terms to the other because war would result in mutual destruction, assuming that the weapons are actually used. Consequently, there is only a minor benefit to both countries going nuclear, represented by a payoff of 1 to each. That is the same as if they remain conventional because then they do not have to spend money to develop the bomb, dispose of nuclear waste, and so on.

We see from the bimatrix that we have a Nash equilibrium at the pair $(1, 1)$ corresponding to the strategy (nuclear, nuclear). The pair $(1, 1)$ when both countries maintain conventional weapons is **not** a Nash equilibrium because each player can improve its own payoff by unilaterally deviating from this. Observe, too, that if one country decides to go nuclear, the other country clearly has no choice but to do likewise. The only way that the situation could change would be to make the benefits of going nuclear much less, perhaps by third-party sanctions or in other ways.

This simple matrix game captures the theoretical basis of the MAD policy of the United States and the former Soviet Union during the cold war. Once the United States possessed nuclear weapons, the payoff matrix showed the Soviet Union that it was in their best interest to also own them and to match the US nuclear arsenal to maintain the MAD option.

It also explains why Pakistan had almost no choice except to obtain nuclear weapons once their historical enemy, India, obtained them. It explains why Iran wants nuclear weapons, knowing that Israel has nuclear weapons.[1] Finally, it explains why the international community attempts to change the payoff matrix for governments attempting to obtain nuclear weapons by increasing their costs, or decreasing their payoffs in several ways. North Korea has learned this lesson and reaped the benefits of concessions from the international community by making a credible threat of obtaining nuclear weapons, as has Libya garnered the benefits of foregoing the nuclear option.

The lesson to learn here is that once one government obtains nuclear weapons, it is a Nash equilibrium–and self-enforcing equilibrium–for opposing countries to also obtain the weapons.

■ EXAMPLE 3.4

Do all bimatrix games have Nash equilibrium points in pure strategies? Game theory would be pretty boring if that were true. For example, if we look at the

[1] Israel has never admitted to having nuclear weapons, but it is widely accepted as true and reported as a fact. The secrecy on the part of the Israelis also indicates an implicit understanding of the bimatrix game here.

game

$$\begin{bmatrix} (2,0) & (1,3) \\ (0,1) & (3,0) \end{bmatrix}$$

there is no pair (a, b) in which a is the largest in the column and b is the largest in the row. In a case like this it seems reasonable that we use mixed strategies. In addition, even though a game might have pure strategy Nash equilibria, it could also have a mixed strategy Nash equilibrium.

In the next section we will see how to solve such games and find the mixed Nash equilibria.

Finally, we end this section with a concept that is a starting point for solving bimatrix games and that we will use extensively when we discuss cooperation. Each player asks, what is the worst that can happen to me in this game?

The amount that player I can be guaranteed to receive is obtained by assuming that player II is actually trying to minimize player I's payoff. In the bimatrix game with two players with matrices $(A_{n\times m}, B_{n\times m})$, we consider separately the two games arising from each matrix. Matrix A is considered as the matrix for a **zero sum** game with player I against player II (player I is the row player=maximizer and player II is the column player=minimizer). The value of the game with matrix A is the guaranteed amount for player I. Similarly, the amount that player II can be guaranteed to receive is obtained by assuming player I is actively trying to minimize the amount that II gets. For player II, the zero sum game is B^T because the row player is always the maximizer. Consequently, player II can guarantee that she will receive the value of the game with matrix B^T. The formal definition is summarized below.

Definition 3.1.3 *Consider the bimatrix game with matrices (A, B). The* **safety value** *for player I is $value(A)$. The* **safety value** *for player II in the bimatrix game is $value(B^T)$.*

If A has the saddle point (X^A, Y^A), then X^A is called the **maxmin strategy for player I.**

If B^T has saddle point (X^{B^T}, Y^{B^T}), then X^{B^T} is the **maxmin strategy for player II.**.

In the prisoner's dilemma game, the safety values are both -5 to each player, as you can quickly verify.

In the game with matrix

$$\begin{bmatrix} (2,0) & (1,3) \\ (0,1) & (3,0) \end{bmatrix},$$

we have

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}.$$

Then $v(A) = \frac{3}{2}$ is the safety value for player I and $v(B^T) = \frac{3}{4}$ is the safety value for player II.

The maxmin strategy for player I is $X = (\frac{3}{4}, \frac{1}{4})$, and the implementation of this strategy guarantees that player I can get at least her safety level. In other words, if I uses $X = (\frac{3}{4}, \frac{1}{4})$, then $E_I(X, Y) \geq v(A) = \frac{3}{2}$ no matter what Y strategy is used by II. In fact

$$E_I\left(\left(\frac{3}{4}, \frac{1}{4}\right), Y\right) = \frac{3}{2}(y_1 + y_2) = \frac{3}{2}, \text{ for any strategy } Y = (y_1, y_2).$$

The maxmin strategy for player II is $Y = X^{B^T} = (\frac{3}{4}, \frac{1}{4})$, which she can use to get at least her safety value of $\frac{3}{4}$.

Is their a connection between the safety levels and a Nash equilibrium? The safety levels are the guaranteed amounts each player can get by using their own individual maxmin strategies, so any rational player must get at least the safety level in a bimatrix game. In other words, it has to be true that if (X^*, Y^*) is a Nash equilibrium for the bimatrix game (A, B), then

$$E_I(X^*, Y^*) = X^*AY^{*T} \geq value(A) \text{ and } E_{II}(X^*, Y^*) = X^*BY^{*T} \geq value(B^T).$$

This would say that in the bimatrix game, if players use their Nash points, they get at least their safety levels. That's what it means to be **individually rational**.

Here's why that's true.

Proof. It's really just from the definitions. The definition of Nash equilibrium says

$$E_I(X^*, Y^*) = X^*AY^{*T} \geq E_I(X, Y^*) = XAY^{*T}, \text{ for all } X \in S_n.$$

But if that is true for all mixed X, then

$$E_I(X^*, Y^*) \geq \max_{X \in S_n} XAY^{*T} \geq \min_{Y \in S_m} \max_{X \in S_n} XAY^T = value(A).$$

The other part of a Nash definition gives us

$$\begin{aligned} E_{II}(X^*, Y^*) = X^*BY^{*T} &\geq \max_{Y \in S_m} X^*BY^T \\ &= \max_{Y \in S_m} YB^TX^{*T} \qquad (\text{since } X^*BY^T = YB^TX^{*T}) \\ &\geq \min_{X \in S_n} \max_{Y \in S_m} YB^TX^T = value(B^T). \end{aligned}$$

Each player does at least as well as assuming the worst. □

PROBLEMS

3.1 Show that (X^*, Y^*) is a saddle point of the game with matrix A if and only if (X^*, Y^*) is a Nash equilibrium of the bimatrix game $(A, -A)$.

3.2 Suppose that a married couple, both of whom have just finished medical school, now have choices regarding their residencies. One of the new doctors has three choices of programs, while the other has two choices. They value their prospects numerically on the basis of the program itself, the city, staying together, and other factors, and arrive at the bimatrix

$$\begin{bmatrix} (5.2, 5.0) & (4.4, 4.4) & (4.4, 4.1) \\ (4.2, 4.2) & (4.6, 4.9) & (3.9, 4.3) \end{bmatrix}$$

Find all the pure Nash equilibria. Which one should be played? Find the safety levels for each player.

3.3 Consider the bimatrix game that models the game of chicken:

I/II	Turn	Straight
Turn	$(19, 19)$	$(-42, 68)$
Straight	$(68, -42)$	$(-45, -45)$

Two cars are headed toward each other at a high rate of speed. Each player has two options: turn off, or continue straight ahead. This game is a macho game for reputation, but leads to mutual destruction if both play straight ahead. There are two pure Nash equilibria. Find them. Verify by using the definition of mixed Nash equilibrium that the mixed strategy pair $X^* = (\frac{3}{52}, \frac{49}{52}), Y^* = (\frac{3}{52}, \frac{49}{52})$ is a Nash equilibrium, and find the expected payoffs to each player. Find the safety levels for each player.

3.4 We may eliminate a row or a column by dominance. If $a_{ij} \geq a_{i'j}$ for every column j, then we may eliminate row i'. If player I drops row i', then the entire pair of numbers in that row are dropped. Similarly, if $b_{ij} \geq b_{ij'}$ for every row i, then we may drop column j', and all the pairs of payoffs in that column. By this method, solve the game

$$\begin{bmatrix} (3, -1) & (2, 1) \\ (-1, 7) & (1, 3) \\ (4, -3) & (-2, 9) \end{bmatrix}.$$

Find the safety levels for each player.

3.2 2 × 2 BIMATRIX GAMES

Now we will analyze all two-person 2×2 nonzero sum games. We are after a method to find all Nash equilibria for a bimatrix game, mixed and pure. Let $X = (x, 1 - x), Y = (y, 1 - y), 0 \leq x \leq 1, 0 \leq y \leq 1$ be mixed strategies for players I and II, respectively. As in the zero sum case, X represents the mixture of the rows that player I has to play; specifically, player I plays row 1 with probability x and row

2 with probability $1 - x$. Similarly for player II and the mixed strategy Y. Now we may calculate the expected payoff to each player. As usual,

$$E_{\text{I}}(X,Y) = XAY^T = (x,\ 1-x)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix},$$
$$E_{\text{II}}(X,Y) = XBY^T = (x,\ 1-x)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix},$$

are the expected payoffs to I and II, respectively. It is the goal of each player to maximize her own expected payoff **assuming that the other player is doing her best to maximize her own payoff** with the strategies she controls.

The first theorem shows us that we may check that mixed strategies form a Nash equilibrium against only pure strategies. This is similar to the result in zero sum games that (X^*, Y^*) is a saddle and v is the value if and only if $E(X^*, j) \geq v$ and $E(i, Y^*) \leq v$, for all rows and columns.

Proposition 3.2.1 *A necessary and sufficient condition for $X^* = (x^*, 1-x^*), Y^* = (y^*, 1-y^*)$ to be a Nash equilibrium point of the game with matrices (A, B) is*

$$\begin{array}{lll} (1) & E_I(1,Y^*) & \leq E_I(X^*,Y^*) \\ (2) & E_I(2,Y^*) & \leq E_I(X^*,Y^*) \\ (3) & E_{II}(X^*,1) & \leq E_{II}(X^*,Y^*) \\ (4) & E_{II}(X^*,2) & \leq E_{II}(X^*,Y^*) \end{array}$$

Proof. To see why this is true, we first note that if (X^*, Y^*) is a Nash equilibrium, then the inequalities must hold by definition (simply choose pure strategies for comparison). So we need only show that the inequalities are sufficient.

Suppose that the inequalities hold for (X^*, Y^*). Let $X = (x, 1-x)$ and $Y = (y, 1-y)$ be any mixed strategies. Successively multiply (1) by $x \geq 0$ and $1 - x \geq 0$ to get

$$xE_{\text{I}}(1, Y^*) = x(1\ 0)AY^{*T} = x(a_{11}y^* + a_{12}(1-y^*)) \leq xE_{\text{I}}(X^*, Y^*)$$

and

$$\begin{aligned}(1-x)E_{\text{I}}(2, Y^*) &= (1-x)(0\ 1)AY^{*T} \\ &= (1-x)(a_{21}y^* + a_{22}(1-y^*)) \leq (1-x)E_{\text{I}}(X^*, Y^*).\end{aligned}$$

Add these up to get

$$\begin{aligned} xE_{\text{I}}(1,Y^*) + (1-x)E_{\text{I}}(2,Y^*) &= x(a_{11}y^* + a_{12}(1-y^*)) \\ &\quad + (1-x)(a_{21}y^* + a_{22}(1-y^*)) \\ &\leq xE_{\text{I}}(X^*,Y^*) + (1-x)E_{\text{I}}(X^*,Y^*) \\ &= E_{\text{I}}(X^*,Y^*) \end{aligned}$$

But then, since

$$x(a_{11}y^* + a_{12}(1 - y^*)) + (1 - x)(a_{21}y^* + a_{22}(1 - y^*)) = XAY^{*^T},$$

we see that

$$(x, (1 - x))AY^{*T} = XAY^{*T} = E_I(X, Y^*) \leq E_I(X^*, Y^*).$$

Since $X \in S_2$ is any old mixed strategy for player I, this gives the first part of the definition that (X^*, Y^*) is a Nash equilibrium. The rest of the proof is similar and left as an exercise. □

Consequently, to find the Nash equilibria we need to find all solutions of the inequalities (1)–(4). This will return us to first principles for solving games.

Let's proceed to see what happens. First we need a definition of a certain set.

Definition 3.2.2 *Let* $X = (x, 1-x), Y = (y, 1-y)$ *be strategies, and set* $f(x, y) = E_I(X, Y)$, *and* $g(x, y) = E_{II}(X, Y)$. *The* **rational reaction set** *for player I is the set of points*

$$R_I = \{(x, y) \mid 0 \leq x, y \leq 1, \max_{0 \leq z \leq 1} f(z, y) = f(x, y)\},$$

and the rational reaction set for player II is the set

$$R_{II} = \{(x, y) \mid 0 \leq x, y \leq 1, \max_{0 \leq w \leq 1} g(x, w) = g(x, y)\}.$$

A point $(x^*, y^*) \in R_I$ means that $X^* = (x^*, 1 - x^*)$ is a best response to $Y^* = (y^*, 1 - y^*)$. Similarly, if $(x^*, y^*) \in R_{II}$, then $Y^* = (y^*, 1 - y^*)$ is a best response to $X^* = (x^*, 1 - x^*)$. Consequently, a point (x^*, y^*) in both R_I and R_{II} says that $X^* = (x^*, 1 - x^*)$ and $Y^* = (y^*, 1 - y^*)$ is a Nash equilibrium.

To simplify notation, let's drop the star on $X^* = (x^*, 1-x^*)$ and $Y^* = (y^*, 1-y^*)$ so that they will be simply $(x, 1-x)$, $(y, 1-y)$, and assumed to be a Nash equilibrium. Since $E_I(1, Y) \leq E_I(X, Y)$ and $E_I(2, Y) \leq E_I(X, Y)$, we have the inequalities

$$\begin{aligned}(a_{11} - a_{12} - a_{21} + a_{22})y &+ (a_{12} - a_{22}) + (a_{21} - a_{22})y + a_{22} \\ &\leq (a_{11} - a_{12} - a_{21} + a_{22})xy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}\end{aligned}$$

and

$$\begin{aligned}(a_{11} - a_{12} - a_{21} + a_{22})xy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}& \\ \geq (a_{21} - a_{22})y + a_{22}.&\end{aligned}$$

Simplifying these two, we get

$$M(1 - x)y - m(1 - x) \leq 0 \quad \text{and} \quad Mxy - mx \geq 0, \tag{3.2.1}$$

where we have set

$$M = a_{11} - a_{12} - a_{21} + a_{22} \text{ and } m = a_{22} - a_{12}.$$

So, this means that any (x, y) that will give us a Nash point $X = (x, 1 - x), Y = (y, 1 - y)$ must satisfy (3.2.1), and, of course, we must also have $0 \le x, y \le 1$. The inequalities are nonlinear and not that easy to solve, and certainly not without some assumptions on M, m.

We consider the following cases:

1. $M = m = 0$. In this case $M(1 - x)y - m(1 - x) = 0$, and $Mxy - mx = 0$, for any $x, y \in [0, 1]$. This is the trivial case because if $M = m = 0$, then $a_{22} = a_{12}$ and $a_{11} = a_{21}$. So it doesn't matter what player I does.

2. $M = 0, m > 0$. Then $-m(1 - x) \le 0$ and $-mx \ge 0$, implying that $x = 0$ and y is anything in $[0, 1]$.

3. $M = 0, m < 0$. Then $(1 - x) \le 0$, and $x \ge 0$. Solutions are $x = 1, 0 \le y \le 1$.

4. $M > 0$. Then $M(1 - x)y - m(1 - x) \le 0$, and $Mxy - mx \ge 0$, and there are many solutions of these:

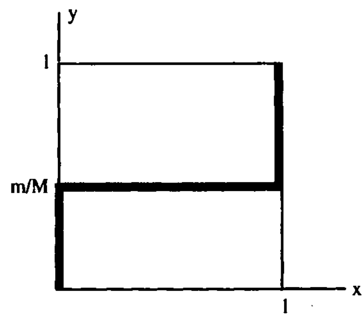
$$\begin{aligned} &\text{if } x = 0 \implies 0 \le y \le \frac{m}{M}, \\ &\text{if } 0 < x < 1 \implies y = \frac{m}{M}, \\ &\text{if } x = 1 \implies 1 \ge y \ge \frac{m}{M}. \end{aligned}$$

   To see this, if $x = 1$, then $M(1-x)y-m(1-x) = 0 \le 0$, and $My-m \ge 0$. So $y \ge m/M$. If $x = 0$, then $M(1 - x)y - m(1 - x) = My - m \le 0$, and $Mxy - mx = 0 \ge 0$, so that $y \le m/M$. If $0 < x < 1$, then $M(1 - x)y - m(1 - x) \le 0 \implies My - m \le 0$ and $Mxy - mx \ge 0 \implies My - m \ge 0$. Consequently $y = m/M$.

5. $M < 0$. Then again $M(1 - x)y - m(1 - x) \le 0$, and $Mxy - mx \ge 0$, and we have multiple solutions:

$$\begin{aligned} &\text{if } x = 0 \implies 1 \ge y \ge \frac{m}{M}, \\ &\text{if } 0 < x < 1 \implies y = \frac{m}{M}, \\ &\text{if } x = 1 \implies 0 \le y \le \frac{m}{M}. \end{aligned}$$

Figure 3.1 below is a graph of the **set of the possible solutions** of (3.2.1) in the case $M > 0$. The axes are the x and y variables, which must be between zero and one. The bold line is the set of points (x, y) so that $X = (x, 1-x), Y = (y, 1-y)$ solve the inequalities $E_I(1, Y) \leq E_I(X, Y)$ and $E_I(2, Y) \leq E_I(X, Y)$, the sufficient inequalities for a Nash point. For example, let's suppose that player II decides to use the strategy $Y = (\frac{1}{4}, \frac{3}{4})$ (so that $y = \frac{1}{4}$) and we have $M = 2, m = 1$. We are in case 4 and, since $y = \frac{1}{4} < m/M = \frac{1}{2}$, case 4 would tell us that the best strategy for player I is to play $x = 0$, that is, $X = (0, 1)$. This is the bold portion of the y axis in Figure 3.1.



Looking for Nash: the case M>0

Figure 3.1 Rational reaction set for player I.

So, for a given $0 \leq y \leq 1$, the best response for player I is the corresponding $X = (x, 1-x)$, where x lies on the bold zigzag line for the given y. The bold zigzag line is the rational reaction set for player I for a given Y.

The rational reaction set for player I has the explicit expression in the case $M > 0$ given by

$$R_I = \left\{(0, y) \mid 0 \leq y \leq \frac{m}{M}\right\} \cup \left\{\left(x, \frac{m}{M}\right) \mid 0 < x < 1\right\} \cup \left\{(1, y) \mid \frac{m}{M} \leq y \leq 1\right\}.$$

You can draw a similar figure and a similar representation for each case that we have considered so far.

Next, we have a similar result if we consider the inequalities in Proposition 3.2.1(3)-(4). Let

$$R = b_{11} - b_{12} - b_{21} + b_{22}, \quad r = b_{22} - b_{21}.$$

Then the inequalities we have to solve become

$$Rx(1-y) - r(1-y) \leq 0, \quad Rxy - ry \geq 0.$$

Again, we consider cases and do an analysis similar to the preceding one:

1. $R = 0, r = 0$. Solutions are all $0 \leq x \leq 1, 0 \leq y \leq 1$.

2. $R = 0, r > 0$. Solutions are $0 \leq x \leq 1, y = 0$.

3. $R = 0, r < 0$. Solutions are $0 \leq x \leq 1, y = 1$.

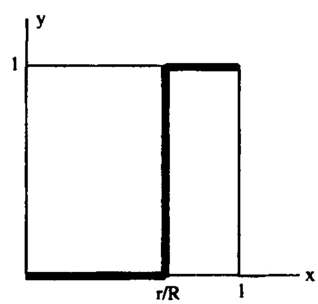
4. $R > 0$. Solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 0 \leq x \leq \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 1 \geq x \geq \frac{r}{R}. \end{aligned}$$

5. $R < 0$. In this final case the set of all possible solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 1 \geq x \geq \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 0 \leq x \leq \frac{r}{R}. \end{aligned}$$

We may also draw a zigzag line for player II that will be a graph of the rational reaction set for player II to a given strategy X for player I. For example, if $R > 0$, the graph would look like Figure 3.2. This bold line would be a graph of the rational



Looking for Nash: the case R>0

Figure 3.2 Rational reaction set for player II.

reaction set for player II against a given X. It has the explicit representation in the case $R > 0$ given by

$$R_{\text{II}} = \left\{(x, 0) \mid 0 \leq x \leq \frac{r}{R}\right\} \cup \left\{\left(\frac{r}{R}, y\right) \mid 0 < y < 1\right\} \cup \left\{(x, 1) \mid \frac{r}{R} \leq x \leq 1\right\}.$$

If you lay the graph in Figure 3.2 of R_{II} on top of that in Figure 3.1 of the set R_{I}, the point of intersection (or points of intersection) in $R_{\text{I}} \cap R_{\text{II}}$ is the Nash equilibrium. In other words, the mixed Nash equilibrium is at the point which is in both rational reaction sets for each player. The next example will illustrate that (see Figure 3.3 below).

Finally, in the case $M \neq 0, R \neq 0$, we have a mixed Nash equilibrium $(X^* = (x, 1-x), Y^* = (y, 1-y)$ at

$$x = \frac{r}{R} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} \quad \text{and} \quad y = \frac{m}{M} = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}}.$$

The pure Nash equilibria will be the intersection points of the rational reaction sets at the corners. (draw them!) In all cases, the expected payoffs to each player are calculated after determination of the Nash equilibria by calculating X^*AY^{*T} and X^*BY^{*T}.

■ EXAMPLE 3.5

The bimatrix game with the two matrices

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

will have multiple Nash equilibria (see Figure 3.3 below). Two of them are obvious; the pair $(2,1)$ and the pair $(1,2)$ in (A, B) have the property that the first number is the largest in the first column and at the same time the second number is the largest in the first row. So we know that $X^* = (1,0), Y^* = (1,0)$ is a Nash point (with payoff 2 for player I and 1 for player II) as is $X^* = (0,1), Y^* = (0,1)$, (with payoff 1 for player I and 2 for player II).

If we apply the solution results we obtained in the theorem, then

$$M = 2 - (-1) - (-1) + 1 = 5 > 0, m = 1 - (-1) = 2, \frac{m}{M} = \frac{2}{5},$$

and

$$R = 5, r = 3, \frac{r}{R} = \frac{3}{5},$$

so that we have three equilibria associated with the points $(x,y) = (0,0)$, $(x,y) = (\frac{3}{5}, \frac{2}{5})$, and $(x,y) = (1,1)$. In Figure 3.3 we can see that the three equilibria are exactly where the two zigzag lines cross corresponding to the cases $M > 0, R > 0$. Here they cross at $(x,y) = (0,0)$, and $(1,1)$ on the boundary of the square, but also at $(x,y) = (\frac{3}{5}, \frac{2}{5})$ in the interior, corresponding to the unique mixed Nash equilibrium $X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{2}{5}, \frac{3}{5})$.The expected payoffs are

$$E_{\text{I}}(X^*, Y^*) = \frac{1}{5} \quad \text{and} \quad E_{\text{II}}(X^*, Y^*) = \frac{1}{5}.$$

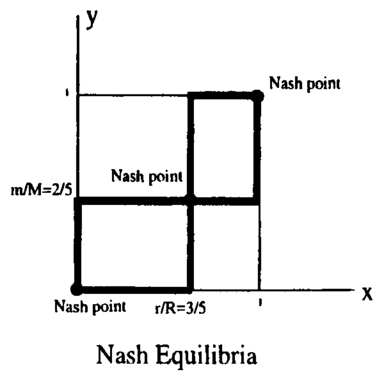


Figure 3.3 Rational reaction sets for both players

This is curious because the expected payoffs to each player are **much less** than they could get at the other Nash points.

We will see pictures like Figure 3.3 again in the next section when we consider an easier way to get Nash equilibria.

Remark: A direct way to calculate the rational reaction sets for 2×2 games. This is a straightforward derivation of the rational reaction sets for the bimatrix game with matrices (A, B). Let $X = (x, 1-x)$, $Y = (y, 1-y)$ be any strategies and define

$$f(x, y) = E_I(X, Y) \text{ and } g(x, y) = E_{II}(X, Y).$$

The idea is to find for a fixed $0 \le y \le 1$, the best response to y. Accordingly,

$$\begin{aligned}\max_{0 \le x \le 1} f(x, y) &= \max_{0 \le x \le 1} xE_I(1, Y) + (1-x)E_I(2, Y) \\ &= x[E_I(1, Y) - E_I(2, Y)] + E_I(2, Y) \\ &= \begin{cases} E_I(2, Y) & \text{at } x = 0 \text{ if } E_I(1, Y) < E_I(2, Y); \\ E_I(1, Y) & \text{at } x = 1 \text{ if } E_I(1, Y) > E_I(2, Y); \\ E_I(2, Y) & \text{at any } 0 < x < 1 \text{ if } E_I(1, Y) = E_I(2, Y). \end{cases}\end{aligned}$$

Now we have to consider the inequalities in the conditions. For example,

$$E_I(1, Y) < E_I(2, Y) \Leftrightarrow My < m, \ M = a_{11} - a_{12} - a_{21} + a_{22}, \ m = a_{22} - a_{12}.$$

If $M > 0$ this is equivalent to the condition $0 \le y < m/M$. Consequently, in the case $M > 0$, the best response to any $0 \le y < M/m$ is $x = 0$. All remaining cases are considered in the same way. See the examples in the next section for a calculation of the rational reaction sets using this approach.

In general, the requirement of the inequalities of this section that we solved in the 2×2 case can also be expressed for the more general cases. That is our next proposition.

Proposition 3.2.3 *(X^*, Y^*) is a Nash equilibrium if and only if*

$$E_I(i, Y^*) = {}_iAY^{*T} \leq X^*AY^{*T} = E_I(X^*, Y^*), \quad i = 1, \dots, n,$$
$$E_{II}(X^*, j) = X^*B_j \leq X^*BY^{*T} = E_{II}(X^*, Y^*), \quad j = 1, \dots, m.$$

In other words, we need to check the inequalities only against pure strategies for the opponent.

Proof. If $E_{\mathrm{I}}(i, Y^*) = {}_iAY^{*T} \leq X^*AY^{*T} \equiv v_A$, for all rows, then we take any $X = (x_1, \dots, x_n) \in S_n$. By multiplying and adding, we obtain

$$\sum_{i=1}^{n} x_i E_{\mathrm{I}}(i, Y^*) \leq \sum_{i=1}^{n} x_i v_A = v_A.$$

But the left side of this inequality is $E_{\mathrm{I}}(X, Y^*) = XAY^{*T}$ and so $E_{\mathrm{I}}(X, Y^*) \leq E_{\mathrm{I}}(X^*, Y^*)$, for any $X \in S_n$. The remaining parts of this claim follow in the same way. The converse is immediate by playing pure strategies in the definition of Nash equilibrium. □

One important use of this result is as a check to make sure that we actually have a Nash equilibrium. The next example illustrates that.

■ EXAMPLE 3.6

Someone says that the bimatrix game

$$\begin{bmatrix} (2,1) & (-1,-1) \\ (-1,-1) & (1,2) \end{bmatrix}$$

has a Nash equilibrium at $X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{2}{5}, \frac{3}{5})$. To check that, first compute $E_{\mathrm{I}}(X^*, Y^*) = E_{\mathrm{II}}(X^*, Y^*) = \frac{1}{5}$. Now check to make sure that this number is at least as good as what could be gained if the other player plays a pure strategy. You can readily verify that, in fact, $E_{\mathrm{I}}(1, Y^*) = \frac{1}{5} = E_{\mathrm{I}}(2, Y^*)$ and also $E_{\mathrm{II}}(X^*, 1) = E_{\mathrm{II}}(X^*, 2) = \frac{1}{5}$, so we do indeed have a Nash point.

Finally, we present a way to find Nash equilibria of any two-person bimatrix game. It is a necessary condition that may be used for computation and is very similar to the equilibrium theorem for zero sum games. If you recall in property 3 of (1.3.1), we had the result that in a zero sum game whenever a row, say, row k, is played with positive probability against Y^*, then the expected payoff $E(k, Y^*)$ must give the value of the game. The next result says the same thing, but now for each of the two players. This is called the **equality of payoffs theorem.**

Theorem 3.2.4 ***(Equality of Payoffs Theorem)*** *Suppose that*

$$X^* = (x_1, x_2, \dots, x_n), \quad Y^* = (y_1, y_2, \dots, y_m)$$

is a Nash equilibrium for the bimatrix game (A, B).

For any row k that has a positive probability of being used, $x_k > 0$, we have $E_I(k, Y^*) = E_I(X^*, Y^*) \equiv v_I$.

For any column j that has a positive probability of being used, $y_j > 0$, we have $E_{II}(X^*, j) = E_{II}(X^*, Y^*) \equiv v_{II}$. That is,

$$x_k > 0 \Longrightarrow E_I(k, Y^*) = v_I$$
$$y_j > 0 \Longrightarrow E_{II}(X^*, j) = v_{II}.$$

Proof. We know that since we have a Nash point, $E_{\mathrm{I}}(X^*, Y^*) = v_{\mathrm{I}} \geq E_{\mathrm{I}}(i, Y^*)$ for any row i. Now, suppose that row k has positive probability of being played against Y^* and that it gives player I a strictly smaller expected payoff $v_{\mathrm{I}} > E_{\mathrm{I}}(k, Y^*)$. Then $v_{\mathrm{I}} \geq E_{\mathrm{I}}(i, Y^*)$ for all the rows $i = 1, 2, \dots, n, i \neq k$, and $v_{\mathrm{I}} > E_{\mathrm{I}}(k, Y^*)$ together imply that

$$x_i v_{\mathrm{I}} \geq x_i E_{\mathrm{I}}(i, Y^*), i \neq k, \quad \text{and} \quad x_k v_{\mathrm{I}} > x_k E_{\mathrm{I}}(k, Y^*).$$

Adding up all these inequalities, we get

$$\sum_{i=1}^{n} x_i v_{\mathrm{I}} = v_{\mathrm{I}} > \sum_{i=1}^{n} x_i E_{\mathrm{I}}(i, Y^*) = E_{\mathrm{I}}(X^*, Y^*) = v_{\mathrm{I}}.$$

This contradiction says it must be true that $v_{\mathrm{I}} = E_{\mathrm{I}}(k, Y^*)$. The only thing that could have gone wrong with this argument is $x_k = 0$. (Why?) Now you can argue in the same way for the assertion about player II. Observe, too, that the argument we just gave is basically the identical one we gave for zero sum games. □

The idea now is that we can find the (completely) mixed Nash equilibria by solving a system of equations rather than inequalities for player II:

$${}_kAY^{*T} = E_{\mathrm{I}}(k, Y^*) = E_{\mathrm{I}}(s, Y^*) = {}_sAY^{*T}, \text{ assuming that } x_k > 0, x_s > 0,$$

and

$$X^*B_j = E_{\mathrm{II}}(X^*, j) = E_{\mathrm{II}}(X^*, r) = X^*B_r, \text{ assuming that } y_j > 0, y_r > 0.$$

This won't be enough to solve the equations, however. You need the additional condition that the components of the strategies must sum to one:

$$x_1 + x_2 + \cdots + x_n = 1, \quad y_1 + y_2 + \cdots + y_m = 1.$$

■ **EXAMPLE 3.7**

As a simple example of this theorem, suppose that we take the matrices

$$A = \begin{bmatrix} -4 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

Suppose that $X = (x_1, x_2), Y = (y_1, y_2)$ is a mixed Nash equilibrium. If $0 < x_1 < 1$, then, since both rows are played with positive probability, by the equality of payoffs Theorem 3.2.4, $v_{\mathrm{I}} = E_{\mathrm{I}}(1, Y) = E_{\mathrm{I}}(2, Y)$. So the equations

$$2y_1 + y_2 = -4y_1 + 2y_2, \quad \text{and } y_1 + y_2 = 1$$

have solution $y_1 = 0.143,\ y_2 = 0.857$, and $v_{\mathrm{I}} = 1.143$. Similarly, $E_{\mathrm{II}}(X, 1) = E_{\mathrm{II}}(X, 2)$ gives

$$x_1 + 2x_2 = 3x_2 \quad \text{and } x_1 + x_2 = 1 \implies x_1 = x_2 = 0.5, v_{\mathrm{II}} = 1.5.$$

Notice that we can find the Nash point without actually knowing v_{I} or v_{II}. Also, assuming that $E_{\mathrm{I}}(1, Y) = v_{\mathrm{I}} = E_{\mathrm{I}}(2, Y)$ gives us the optimal Nash point for player II, and assuming $E_{\mathrm{II}}(X, 1) = E_{\mathrm{II}}(X, 2)$ gives us the Nash point for player I. In other words, the Nash point for II is found from the payoff function for player I and vice versa.

PROBLEMS

3.5 Complete the verification that the inequalities in Proposition 3.2.1 are sufficient for a Nash equilibrium in a 2×2 game.

3.6 Apply the method of this section to analyze the modified study–party game:

I/II	Study	Party
Study	(2, 2)	(3, 1)
Party	(1, 3)	(4, 4)

Find all Nash equilibria and graph the rational reaction sets.

3.7 Determine all Nash equilibria and graph the rational reaction sets for the game

$$\begin{bmatrix} (-10, 5) & (2, -2) \\ (1, -1) & (-1, 1) \end{bmatrix}$$

3.8 Verify by checking against pure strategies that the mixed strategies $X^* = (\frac{3}{4}, 0, \frac{1}{4})$ and $Y^* = (0, \frac{1}{3}, \frac{2}{3})$ is a Nash equilibrium for the game with matrix

$$\begin{bmatrix} (x,2) & (3,3) & (1,1) \\ (y,y) & (0,z) & (2,w) \\ (a,4) & (5,1) & (0,7) \end{bmatrix}$$

where x, y, z, w, a are arbitrary.

3.9 In a modified story of the prodigal son, a man had two sons, the prodigal and the one who stayed home. The man gave the prodigal son his share of the estate, which he squandered, and told the son who stayed home that all that he (the father) has is his (the son's). When the man died, the prodigal son again wanted his share of the estate. They each tell the judge (it ends up in court) a share amount they would be willing to take, either $\frac{1}{4}, \frac{1}{2}$, or $\frac{3}{4}$. Call the shares for each player $\mathrm{I}_i, \mathrm{II}_i, i = 1, 2, 3$. If $\mathrm{I}_i + \mathrm{II}_j > 1$, all the money goes to the game theory society. If $\mathrm{I}_i + \mathrm{II}_j \leq 1$, then each gets the share they asked for and the rest goes to an antismoking group. Find all pure Nash equilibria. Can you find the mixed Nash equilibria using the equality of payoffs Theorem 3.2.4?

3.10 Use the equality of payoffs Theorem 3.2.4 to solve the welfare game. In the welfare game the state, or government, wants to aid a pauper if he is looking for work and not otherwise. The pauper looks for work only if he cannot depend on welfare, but he may not be able to find a job even if he looks. The game matrix is

G/P	Look for work	Be a bum
Welfare	$(3, 2)$	$(-1, 3)$
No welfare	$(-1, 1)$	$(0, 0)$

Find all Nash equilibria and graph the rational reaction sets.

3.3 INTERIOR MIXED NASH POINTS BY CALCULUS

Whenever we are faced with a problem of maximizing or minimizing a function, we are taught in calculus that we can find them by finding critical points and then trying to verify that they are minima or maxima or saddle points. Of course, a critical point doesn't have to be any of these special points. When we look for Nash equilibria that simply supply the maximum expected payoff, assuming that the other players are doing the same for their payoffs, why not apply calculus? That's exactly what we can do, and it will give us all the interior, that is, completely mixed Nash points. The reason it works here is because of the nature of functions like $f(x, y) = XAY^T$. Calculus cannot give us the pure Nash equilibria because those are achieved on the boundary of the strategy region.

The easy part in applying calculus is to find the partial derivatives, set them equal to zero, and see what happens. Here is the procedure.

Calculus Method for Interior Nash. (3.3.1)

1. The payoff matrices are $A_{n\times m}$ for player I and $B_{n\times m}$ for player II. The expected payoff to I is $E_I(X,Y) = XAY^T$, and the expected payoff to II is $E_{II}(X,Y) = XBY^T$.

2. Let $x_n = 1 - (x_1 + \cdots + x_{n-1}) = x_n - \sum_{i=1}^{n} x_i$, $y_m = 1 - \sum_{j=1}^{m-1} y_j$ so each expected payoff is a function only of $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}$. We can write

$$E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = E_I(X,Y),$$
$$E_2(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = E_{II}(X,Y).$$

3. Take the partial derivatives and solve the system of equations $\partial E_1/\partial x_i = 0$, $\partial E_2/\partial y_j = 0, i = 1, \ldots, n-1,\ j = 1, \ldots, m-1$.

4. If there is a solution of this system of equations which satisfies the constraints $x_i \geq 0, y_j \geq 0$ and $\sum_{i=1}^{n-1} x_i \leq 1, \sum_{j=1}^{m-1} y_j \leq 1$, then this is the mixed strategy Nash equilibrium.

It is important to observe that we do **not** maximize $E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$ over **all** variables x and y, but only over the x variables. Similarly, we do **not** maximize $E_2(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$ over **all** variables x and y, but only over the y variables. A Nash equilibrium for a player is a maximum of the player's payoff over those variables that player controls, assuming that the other player's variables are held fixed.

■ **EXAMPLE 3.8**

The bimatrix game we considered in the preceding section had the two matrices

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

What happens if we use the calculus method on this game? First, set up the functions (using $X = (x, 1-x), Y = (y, 1-y)$)

$$E_1(x,y) = [2x - (1-x)]y + [-x + (1-x)](1-y),$$
$$E_2(x,y) = [x - (1-x)]y + [-x + 2(1-x)](1-y).$$

Player I wants to maximize E_1 for each fixed y, so we take

$$\frac{\partial E_1(x,y)}{\partial x} = 3y + 2y - 2 = 5y - 2 = 0 \implies y = \frac{2}{5}.$$

Similarly, player II wants to maximize $E_2(x, y)$ for each fixed x, so

$$\frac{\partial E_2(x,y)}{\partial y} = 5x - 3 = 0 \implies x = \frac{3}{5}.$$

Everything works to give us $X^* = (\frac{3}{5}, \frac{2}{5})$ and $Y^* = (\frac{2}{5}, \frac{3}{5})$ is a Nash equilibrium for the game, just as we had before. We do not get the pure Nash points for this problem. But those are easy to get by determining the pairs of payoffs that are simultaneously the largest in the column and the largest in the row, just as we did before. We don't need calculus for that.

■ EXAMPLE 3.9

Two partners have two choices for where to invest their money, say, O_1, O_2 where the letter O stands for opportunity, but they have to come to an agreement. If they do not agree on joint investment, there is no benefit to either player. We model this using the bimatrix

	O_1	O_2
O_1	$(1,2)$	$(0,0)$
O_2	$(0,0)$	$(2,1)$

If player I chooses O_1 and II chooses O_1 the payoff to I is 1 and the payoff to II is 2 units because II prefers to invest the money into O_1 more than into O_2. If the players do not agree on how to invest, then each receives 0.

To solve this game, first notice that there are two pure Nash points at (O_1, O_1) and (O_2, O_2), so total cooperation will be a Nash equilibrium. We want to know if there are any mixed Nash points. We will start the analysis from the beginning rather than using the formulas from section 3.2. The reader should verify that our result will match if we do use the formulas. We will derive the rational reaction sets for each player directly.

Set

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then player I's expected payoff (using $X = (x, 1-x), Y = (y, 1-y)$) is

$$\begin{aligned} E_1(x,y) &= (x\ 2(1-x)) \cdot (y, 1-y)^T \\ &= xy + 2(1-x)(1-y) = 3xy - 2x - 2y + 2. \end{aligned}$$

Keep in mind that player I wants to make this as large as possible for any fixed $0 \leq y \leq 1$. We need to find the maximum of E_1 as a function of $x \in [0, 1]$ for each fixed $y \in [0, 1]$.

Write E_1 as

$$E_1(x,y) = x(3y - 2) - 2y + 2.$$

If $3y - 2 > 0$, then $E_1(x, y)$ is maximized as a function of x at $x = 1$. If $3y - 2 < 0$, then the maximum of $E_1(x, y)$ will occur at $x = 0$. If $3y - 2 = 0$, then $y = \frac{2}{3}$ and $E_1(x, \frac{2}{3}) = \frac{2}{3}$; that is, we have shown that

$$\max_{0 \le x \le 1} E_1(x, y)$$
$$= \begin{cases} y & \text{if } 3y - 2 > 0 \implies y > \frac{2}{3}, \text{ achieved at } x = 1; \\ \frac{2}{3} & \text{if } y = \frac{2}{3}, \text{ achieved at any } x \in [0, 1]; \\ -2y + 2 & \text{if } 3y - 2 < 0 \implies y < \frac{2}{3}, \text{ achieved at } x = 0. \end{cases}$$

Recall that the set of points where the maximum is achieved by player I for each fixed y for player II is the rational reaction set for player I:

$$R_{\text{I}} = \{(x^*, y) \in [0, 1] \times [0, 1] \mid \max_{0 \le x \le 1} E_1(x, y) = E_1(x^*, y)\}.$$

In this example we have shown that

$$R_{\text{I}} = \left\{(1, y), \frac{2}{3} < y \le 1\right\} \cup \left\{\left(x, \frac{2}{3}\right), 0 \le x \le 1\right\} \cup \left\{(0, y), 0 \le y < \frac{2}{3}\right\}.$$

This is the rational reaction set for player I because no matter what II plays, player I should use an $(x, y) \in R_{\text{I}}$. For example, if $y = \frac{1}{2}$, then player I should use $x = 0$ and I receives payoff $E(0, \frac{1}{2}) = 1$; if $y = \frac{15}{16}$, then I should choose $x = 1$ and I receives $E_1(1, \frac{15}{16}) = \frac{15}{16}$; and when $y = \frac{2}{3}$, it doesn't matter what player I chooses because the payoff to I will be exactly $\frac{2}{3}$ for any $x \in [0, 1]$.

Next, we consider $E_2(x, y)$ in a similar way. Write

$$E_2(x, y) = 2xy + (1 - x)(1 - y) = 3xy - x - y + 1 = y(3x - 1) - x + 1.$$

Player II wants to choose $y \in [0, 1]$ to maximize this, and that will depend on the coefficient of y, namely, $3x - 1$. We see as before that

$$\max_{y \in [0,1]} E_2(x, y) = \begin{cases} -x + 1 & \text{if } 0 \le x < \frac{1}{3} \text{ achieved at } y = 0; \\ \frac{2}{3} & \text{if } x = \frac{1}{3} \text{ achieved at any } y \in [0, 1] \\ 2x & \text{if } \frac{1}{3} < x \le 1 \text{ achieved at } y = 1. \end{cases}$$

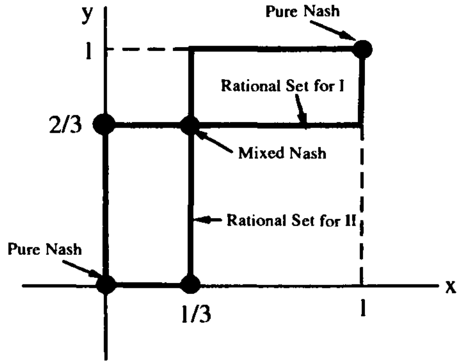
The rational reaction set for player II is the set of points where the maximum is achieved by player II for each fixed x for player II:

$$R_{\text{II}} = \{(x, y^*) \in [0, 1] \times [0, 1] \mid \max_{0 \le y \le 1} E_2(x, y) = E_2(x, y^*)\}.$$

We see that in this example

$$R_{\text{II}} = \left\{(x, 0), 0 \le x < \frac{1}{3}\right\} \cup \left\{\left(\frac{1}{3}, y\right), 0 \le y \le 1\right\} \cup \left\{(x, 1), \frac{1}{3} < x \le 1\right\}.$$

Here is the graph of R_{I} and R_{II} on the same set of axes:



The zigzag lines form the rational reaction sets of the players. For example, if player I decides for some reason to play O_1 with probability $x = \frac{1}{2}$, then player II would rationally play $y = 1$. Where the zigzag lines cross (which is the set of points $R_{\text{I}} \cap R_{\text{II}}$) are all the Nash points; that is, the Nash points are at $(x, y) = (0,0), (1,1)$ and $(\frac{1}{3}, \frac{2}{3})$. So either they should always cooperate to the advantage of one player or the other, or player I should play O_1 one-third of the time and player II should play O_1 two-thirds of the time. The associated expected payoffs are

$$E_1(0,0) = 2, \quad E_2(0,0) = 1,$$

$$E_1(1,1) = 1, \quad E_2(1,1) = 2,$$

and

$$E_1\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3} = E_2\left(\frac{1}{3}, \frac{2}{3}\right).$$

Only the mixed strategy Nash point $(X^*, Y^*) = ((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}))$ gives the same expected payoffs to the two players. This seems to be a problem. Only the mixed strategies gives the same payoff to each player, but it will result in **less** for each player than they could get if they play the pure Nash points. Permitting the other player the advantage results in a **bigger** payoff to both players! If one player decides to cave, they both can do better, but if both players insist that the outcome be **fair**, whatever that means, then they both do worse.

Calculus will give us the interior mixed Nash very easily:

$$\frac{\partial E_1(x,y)}{\partial x} = 3y - 2 = 0 \implies y = \frac{2}{3}, \text{ and}$$
$$\frac{\partial E_2(x,y)}{\partial y} = 3x - 1 = 0 \implies x = \frac{1}{3}.$$

The method of solution we used in this example gives us the entire rational reaction set for each player. It is essentially the way Proposition 3.2.1 was proved.

Another point to notice is that the rational reaction sets and their graphs do **not** indicate what the **payoffs** are to the individual players, only what their **strategies** should be.

Finally, we record here the definition of the rational reaction sets in the general case with arbitrary size matrices.

Definition 3.3.1 *The rational reaction sets for each player are defined as follows:*

$$R_I = \{(X,Y) \in S_n \times S_m \mid E_I(X,Y) = \max_{p \in S_n} E_I(p,Y)\},$$
$$R_{II} = \{(X,Y) \in S_n \times S_m \mid E_{II}(X,Y) = \max_{t \in S_m} E_{II}(X,t)\}.$$

The set of all Nash equilibria is then the set of all common points $R_I \cap R_{II}$.

We can write down the system of equations that we get using calculus in the general case, after taking the partial derivatives and setting to zero. We start with

$$E_I(X,Y) = XAY^T = \sum_{j=1}^{m}\sum_{i=1}^{n} x_i a_{ij} y_j.$$

Following the calculus method (3.3.1), we replace [1] $x_n = 1 - \sum_{k=1}^{n-1} x_k$ and do some algebra:

$$\begin{aligned} XAY^T &= \sum_{j=1}^{m}\sum_{i=1}^{n} x_i a_{ij} y_j \\ &= \sum_{j=1}^{m}\left(\sum_{i=1}^{n-1} x_i a_{ij} y_j + \left(1 - \sum_{k=1}^{n-1} x_k\right) a_{nj} y_j\right) \\ &= \sum_{j=1}^{m}\left(a_{nj} y_j + \sum_{i=1}^{n-1} x_i a_{ij} y_j - \sum_{k=1}^{n-1} x_k (a_{nj} y_j)\right) \\ &= \sum_{j=1}^{m}\left(a_{nj} y_j + \sum_{i=1}^{n-1} x_i [a_{ij} - a_{nj}] y_j\right) \\ &= E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m). \end{aligned}$$

But then, for each $k = 1, 2, \ldots, n-1$, we obtain

$$\frac{\partial E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m)}{\partial x_k} = \sum_{j=1}^{m} y_j [a_{kj} - a_{nj}].$$

[1] Alternatively we may take the partial derivative of the function with a Lagrange multiplier $E_1(\vec{x}, \vec{y}) - \lambda(\sum_i x_i - 1)$. Taking a partial with respect to x_k, shows that $\partial E_1/\partial x_k = \partial E_1/\partial x_n$ for all $k = 1, 2, \ldots, n-1$. This gives us the same system as (3.3.2).

Similarly, for each $s = 1, 2, \ldots, m-1$, we get the partials

$$\frac{\partial E_2(x_1, \ldots, x_n, y_1, \ldots, y_{m-1})}{\partial y_s} = \sum_{i=1}^{n} x_i[b_{is} - b_{im}].$$

So, the system of equations we need to solve to get an interior Nash equilibrium is

$$\left.\begin{array}{ll} \displaystyle\sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0, & k = 1, 2, \ldots, n-1, \\ \displaystyle\sum_{i=1}^{n} x_i[b_{is} - b_{im}] = 0, & s = 1, 2, \ldots, m-1, \\ \displaystyle x_n = 1 - \sum_{i=1}^{n-1} x_i, \quad y_m = 1 - \sum_{j=1}^{m-1} y_j. & \end{array}\right\} \tag{3.3.2}$$

Once these are solved, we check that $x_i \geq 0, y_j \geq 0$. If these all check out, we have found a Nash equilibrium $X^* = (x_1, \ldots, x_n)$ and $Y^* = (y_1, \ldots, y_m)$. Notice also that the equations are really two separate systems of linear equations and can be solved separately because the variables x_i and y_j appear only in their own system. Also notice that these equations are really nothing more than the equality of payoffs Theorem 3.2.4, because, for example

$$\sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0 \Longrightarrow \sum_{j=1}^{m} y_j a_{kj} = \sum_{j=1}^{m} y_j a_{nj},$$

which is the same as saying that for $k = 1, 2, \ldots, n-1$, we have

$$E_{\mathrm{I}}(k, Y^*) = \sum_{j=1}^{m} y_j a_{kj} = \sum_{j=1}^{m} y_j a_{nj} = E_{\mathrm{I}}(n, Y^*).$$

All the payoffs are equal. This, of course, assumes that each row of A is used by player I with positive probability in a Nash equilibrium, but that is our assumption about an **interior** Nash equilibrium. These equations won't necessarily work for the pure Nash or the ones with zero components.

It is not recommended that you memorize these equations but rather that you start from scratch on each problem.

EXAMPLE 3.10

We are going to use the equations (3.3.2) to find interior Nash points for the following bimatrix game:

$$A = \begin{bmatrix} -2 & 5 & 1 \\ -3 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & -2 & 4 \\ -3 & 1 & 4 \\ 3 & 1 & -1 \end{bmatrix}.$$

The system of equations (3.3.2) for an interior Nash point become

$$-2y_1 + 6y_2 - 2 = 0, \qquad -5y_1 + y_2 = 0$$

and

$$-12x_1 - 11x_2 + 4 = 0, \qquad -8x_1 - 5x_2 + 2 = 0.$$

There is one and only one solution given by $y_1 = \frac{1}{14}, y_2 = \frac{5}{14}$ and $x_1 = \frac{1}{14}, x_2 = \frac{4}{14}$, so we have $y_3 = \frac{8}{14}, x_3 = \frac{9}{14}$, and our interior Nash point is

$$X^* = \left(\frac{1}{14}, \frac{4}{14}, \frac{9}{14}\right) \text{ and } Y^* = \left(\frac{1}{14}, \frac{5}{14}, \frac{8}{14}\right).$$

The expected payoffs to each player are $E_{\text{I}}(X^*, Y^*) = X^* A Y^{*T} = \frac{31}{14}$ and $E_{\text{II}}(X^*, Y^*) = X^* B Y^{*T} = \frac{11}{14}$. It appears that player I does a lot better in this game with this Nash.

We have found the interior, or mixed Nash point. There are also two pure Nash points and they are $X^* = (0,0,1), Y^* = (1,0,0)$ with payoffs $(2,3)$ and $X^* = (0,1,0), Y^* = (0,0,1)$ with payoffs $(3,4)$. With multiple Nash points, the game can take on one of many forms.

Maple can be used to solve the system of equations giving an interior Nash point. The commands for solving the preceding example are shown here:

```
> restart:with(LinearAlgebra):
> A:=Matrix([[-2,5,1],[-3,2,3],[2,1,3]]);
> B:=Matrix([[-4,-2,4],[-3,1,4],[3,1,-1]]);
> Y:=Vector(3,symbol=y):
> X:=Vector(3,symbol=x):
> yeq:=seq(add(y[j]*(A[i,j]-A[3,j]),j=1..3),i=1..2);
> xeq:=seq(add(x[i]*(B[i,s]-B[i,3]),i=1..3),s=1..2);
> xsols:=solve({xeq[1]=0,xeq[2]=0,add(x[i],i=1..3)=1},
                          [x[1],x[2],x[3]]);
> assign(xsols);
> ysols:=solve({yeq[1]=0,yeq[2]=0,add(y[j],j=1..3)=1},
                          [y[1],y[2],y[3]]);
> assign(ysols);
> Transpose(X).A.Y; Transpose(X).B.Y;
```

The last line gives the expected payoffs to each player. The `assign` command assigns the solutions to the vectors. Observe too that when we calculate in Maple `Transpose(X).A.Y` the expected payoff to player I, the transpose appears to be on the wrong vector. But in Maple, vectors are defined as **column** matrices, so a correct multiplication is as is shown in the Maple commands, even though in the book we use XAY^T.

If you change some of the numbers in the matrices A, B and rerun the Maple code, you will see that frequently the solutions will have negative components or the components will be greater than one. You have to be careful in trying to do this with Maple because solving the system of equations doesn't always work, especially if there is more than one interior Nash equilibrium (which could occur for matrices that have more than two pure strategies).

■ EXAMPLE 3.11

Here is a last example in which the equations do not work (see problem 3.11) because it turns out that one of the columns should never be played by player II. That means that the mixed Nash is not in the interior, but on the boundary of $S_n \times S_m$.

Let's consider the game with payoff matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

You can calculate that the safety levels are $value(A) = 2$, with pure saddle $X_A = (1,0), Y_A = (1,0,0)$, and $value(B^T) = 1$, with saddle $X_B = (0,0,1), Y_B = (\frac{1}{2}, \frac{1}{2})$. These are the amounts that each player can get assuming that both are playing in a zero sum game with the two matrices.

Now, let $X = (x, 1-x), Y = (y_1, y_2, 1-y_1-y_2)$ be a Nash point for the bimatrix game. Calculate

$$E_1(x, y_1, y_2) = XAY^T = x[y_1 - 2y_2 + 1] + y_2 - 3y_1 + 3.$$

Player I wants to maximize $E_1(x, y_1, y_2)$ for given fixed $y_1, y_2 \in [0,1], y_1 + y_2 \leq 1$, using $x \in [0,1]$. So we look for $\max_x E_1(x, y_1, y_2)$. For fixed y_1, y_2, we see that $E_1(x, y_1, y_2)$ is a straight line with slope $y_1 - 2y_2 + 1$. The maximum of that line will occur at an endpoint depending on the sign of the slope. Here is what we get:

$$\begin{aligned} \max_{0 \leq x \leq 1} x[y_1 - 2y_2 + 1] & + y_2 - 3y_1 + 3 \\ &= \begin{cases} -2y_1 - y_2 + 4 & \text{if } y_1 > 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 = 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 < 2y_2 - 1. \end{cases} \\ &= \begin{cases} E_1(1, y_1, y_2) & \text{if } y_1 > 2y_2 - 1; \\ E_1(x, 2y_2 - 1, y_2) & \text{if } y_1 = 2y_2 - 1; \\ E_1(0, y_1, y_2) & \text{if } y_1 < 2y_2 - 1. \end{cases} \end{aligned}$$

Along any point of the straight line $y_1 = 2y_2 - 1$ the maximum of $E_1(x, y_1, y_2)$ is achieved at any point $0 \leq x \leq 1$. We end up with the following set of maximums for $E_1(x, y_1, y_2)$:

$$R_I = \{(x, y_1, y_2) \mid [(1, y_1, y_2), y_1 > 2y_2 - 1], \text{or} \\ [(x, 2y_2 - 1, y_2), 0 \le x \le 1], \text{or } [(0, y_1, y_2), y_1 < 2y_2 - 1]\},$$

which is exactly the rational reaction set for player I. This is a set in three dimensions.

Now we go through the same procedure for player II, for whom we calculate,

$$E_2(x, y_1, y_2) = XBY^T = y_1 + y_2(3x - 1) + (-2x + 1).$$

Player II wants to maximize $E_2(x, y_1, y_2)$ for given fixed $x \in [0, 1]$ using $y_1, y_2 \in [0, 1], y_1 + y_2 \le 1$. We look for $\max_{y_1, y_2} E_2(x, y_1, y_2)$.

Here is what we get:

$$\begin{aligned} \max_{\substack{y_1+y_2\le 1,\\ y_1,y_2\ge 0}} \quad & y_1 + y_2(3x-1) + (-2x+1) \\ &= \begin{cases} -2x+2 & \text{if } 3x - 1 < 1; \\ \frac{2}{3} & \text{if } 3x - 1 = 1; \\ x & \text{if } 3x - 1 > 1. \end{cases} \\ &= \begin{cases} E_2(x, 1, 0) & \text{if } 0 \le x < \frac{2}{3}; \\ E_2\left(\frac{2}{3}, y_1, y_2\right) = \frac{2}{3} & \text{if } x = \frac{2}{3},\ y_1 + y_2 = 1; \\ E_2(x, 0, 1) & \text{if } \frac{2}{3} < x \le 1. \end{cases} \end{aligned}$$

To explain where this came from, let's consider the case $3x - 1 < 1$. In this case, the coefficient of y_2 is less than the coefficient of y_1 (which is 1), so the maximum will be achieved by taking $y_2 = 0$ and $y_1 = 1$ because that gives the biggest weight to the largest coefficient. Then plugging in $y_1 = 1, y_2 = 0$ gives payoff $-2x + 2$. If $3x - 1 = 1$ the coefficients of y_1 and y_2 are the same, then we can take any y_1 and y_2 as long as $y_1 + y_2 = 1$. Then the payoff becomes $(y_1 + y_2)(3x - 1) + (-2x + 1) = x$, but $3x - 1 = 1$ requires that $x = \frac{2}{3}$, so the payoff is $\frac{2}{3}$. The case $3x - 1 > 1$ is similar.

We end up with the following set of maximums for $E_2(x, y_1, y_2)$:

$$R_{II} = \left\{(x, y_1, y_2) \mid \left[(x, 1, 0), 0 \le x < \frac{2}{3}\right], \text{ or} \\ \left[\left(\frac{2}{3}, y_1, y_2\right), y_1 + y_2 = 1\right], \text{ or } \left[(x, 0, 1), \frac{2}{3} < x \le 1\right]\right\},$$

which is the rational reaction set for player II.

The graph of $R_{\rm I}$ and $R_{\rm II}$ on the same graph (in three dimensions) will intersect at the mixed Nash equilibrium points. In this example the Nash equilibrium is given by

$$X^* = \left(\frac{2}{3}, \frac{1}{3}\right), \quad Y^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right).$$

Then, $E_1(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}) = \frac{8}{3}$ and $E_2(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$ are the payoffs to each player. We could have simplified the calculations significantly if we had noticed from the beginning that we could have eliminated the third column from the bimatrix because column 3 for player II is dominated by column 1, and so may be dropped.

PROBLEMS

3.11 Write down the equations (3.3.2) for the game

$$\begin{bmatrix} (2,0) & (3,2) & (4,1) \\ (0,2) & (4,0) & (3,1) \end{bmatrix}.$$

Try to solve the equations. What, if anything, goes wrong?

3.12 The game matrix in the welfare problem is

G/P	Look for work	Be a bum
Welfare	$(3,2)$	$(-1,3)$
No welfare	$(-1,1)$	$(0,0)$

Write the system of equations for an interior Nash and solve them.

3.13 Find all possible Nash equilibria for the game and the rational reaction sets:

$$\begin{bmatrix} (a,a) & (0,0) \\ (0,0) & (b,b) \end{bmatrix}$$

Consider all cases $(a > 0, b > 0)$, $(a > 0, b < 0)$, and so on.

3.14 Solve directly by finding the sets $R_{\rm I}$ and $R_{\rm II}$ the game with matrix

$$\begin{bmatrix} (2,0) & (3,1) & (4,-1) \\ (0,2) & (4,0) & (3,1) \end{bmatrix}.$$

3.15 A game called the **battle of the sexes** is a game between a husband and wife trying to decide about cooperation. On a given evening, the husband wants to see wrestling at the stadium, while the wife wants to attend a concert at orchestra hall. Neither the husband nor the wife wants to go to what the other has chosen, but neither do they want to go alone to their preferred choice. They view this as a two-person nonzero sum game with matrix

H/W	Wr	Co
Wr	$(2,1)$	$(-1,-1)$
Co	$(-1,-1)$	$(1,2)$

If they decide to cooperate and both go to wrestling, the husband receives 2 and the wife receives 1, because the husband gets what he wants and the wife partially gets what she wants. The rest are explained similarly, so this is a model of compromise and cooperation. Use the method of this section to find all Nash equilibria. Graph the rational reaction sets.

3.16 **(Hawk–Dove game)** Two companies both want to take over a sales territory. They have the choices of defending the territory and fighting if necessary, or act as if willing to fight but if the opponent fights (F), then backing off (Bo). They look at this as a two-person nonzero sum game with matrix

I/II	F	Bo
F	$(-1,-1)$	$(2,0)$
Bo	$(0,2)$	$(0,0)$

Solve this game and graph the rational reaction sets.

3.17 **(Stag–Hare game)** Two hunters are pursuing a stag. Each hunter has the choice of going after the stag (S), which will be caught if they both go after it and it will be shared equally, or peeling off and going after a rabbit (R). Only one hunter is needed to catch the rabbit and it will not be shared. Look at this as a nonzero sum two-person game with matrix

I/II	S	R
S	$(2,2)$	$(0,1)$
R	$(1,0)$	$(1,1)$

This assumes that they each prefer stag meat to rabbit meat, and they will each catch a rabbit if they decide to peel off. Solve this game and graph the rational reaction sets.

3.18 Find all Nash equilibria and expected payoffs for the game with bimatrix

$$\begin{bmatrix} (0,9) & (8,2) & (7,3) \\ (4,7) & (2,8) & (9,1) \\ (10,-1) & (5,5) & (1,9) \end{bmatrix}.$$

3.19 Consider the game with bimatrix

$$\begin{bmatrix} (5,4) & (3,6) \\ (6,3) & (1,1) \end{bmatrix}.$$

(**a**) Find the safety levels for each player.

(**b**) Find the maxmin strategies.

(**c**) Find all Nash equilibria using the equality of payoffs theorem for the mixed strategy.

(**d**) Verify that the mixed Nash equilibrium is individually rational.

(**e**) Verify that $X = (\frac{1}{4}, \frac{3}{4}), Y = (\frac{5}{8}, \frac{3}{8})$ is not a Nash equilibrium.

3.3.1 Proof that There Is a Nash Equilibrium for Bimatrix Games (Optional)

The last section contained the sentence "The set of all Nash equilibria is then the set of all common points $R_{\text{I}} \cap R_{\text{II}}$." How do we know whether this intersection has any points at all? It might have occurred to you that we have no guarantee that looking for a Nash equilibrium in a bimatrix game with matrices (A, B) would ever be successful. So maybe we need a guarantee that what we are looking for actually exists. That is what Nash's theorem gives us.

Theorem 3.3.2 *There exists $X^* \in S_n$ and $Y^* \in S_m$ so that*

$$E_I(X^*, Y^*) = X^* A Y^{*T} \geq E_I(X, Y^*),$$
$$E_{II}(X^*, Y^*) = X^* B Y^{*T} \geq E_{II}(X^*, Y),$$

for any other mixed strategies $X \in S_n, Y \in S_m$.

The theorem guarantees at least one Nash equilibrium if we are willing to use mixed strategies. In the zero sum case, this theorem reduces to von Neumann's minimax theorem.

Proof. We will give a proof that is very similar to that of von Neumann's theorem using the Kakutani fixed-point theorem for point to set maps, but there are many other proofs that have been developed, as is true of many theorems that are important. As we go through this, note the similarities with the proof of von Neumann's theorem.

First $S_n \times S_m$ is a closed, bounded and convex set. Now for each given pair of strategies (X, Y), we could consider the best response of player II to X and the best response of player I to Y. In general there may be more than one best response, so we consider the best response sets:

Definition 3.3.3 *The* **best response sets** *for each player are defined as*

$$BR_I(Y) = \{X \in S_n \mid E_I(X, Y) = \max_{p \in S_n} E_I(p, Y)\},$$
$$BR_{II}(X) = \{Y \in S_m \mid E_{II}(X, Y) = \max_{t \in S_m} E_I(X, t)\}.$$

The difference between the best response set and the rational reaction set is that the rational reaction set R_{I} consists of the **pairs of strategies** (X, Y) for which

$E_I(X,Y) = \max_p E_I(p,Y)$, whereas the set $BR_I(Y)$ consists of the strategy (or collection of strategies) X for player I that is the best response to a **fixed** Y. Whenever you are maximizing a continuous function, which is true of $E_I(X,Y)$, over a closed and bounded set (which is true of S_n), you always have a point at which the maximum is achieved. So we know that $BR_I(Y) \neq \emptyset$. Similarly, the same is true of $BR_{II}(X) \neq \emptyset$.

We define the point to set mapping

$$\varphi : (X,Y) \in S_n \times S_m \to BR_I(Y) \times BR_{II}(X) \subset S_n \times S_m,$$

which gives, for each pair (X,Y) of mixed strategies, the best response strategies $(X',Y') \in \varphi(X,Y)$ with $X' \in BR_I(Y)$ and $Y' \in BR_{II}(X)$.

It seems natural that our Nash equilibrium should be among the best response strategies to the opponent. Translated, this means that a Nash equilibrium (X^*,Y^*) should satisfy $(X^*,Y^*) \in \varphi(X^*,Y^*)$. But that is exactly what it means to be a fixed point of φ. If φ satisfies the required properties to apply Kakutani's fixed-point theorem, we have the existence of a Nash equilibrium. This is relatively easy to check because $X \mapsto E_I(X,Y)$ and $X \mapsto E_{II}(X,Y)$ are linear maps, as are $Y \mapsto E_I(X,Y)$ and $Y \mapsto E_{II}(X,Y)$. Hence it is easy to show that $\varphi(X,Y)$ is a convex, closed, and bounded subset of $S_n \times S_m$. It is also not hard to show that φ will be an (upper) semicontinuous map, and so Kakutani's theorem applies.

This gives us a pair $(X^*,Y^*) \in \varphi(X^*,Y^*)$. Written out, this means $X^* \in BR_I(Y^*)$ so that

$$E_I(X^*,Y^*) = \max_{p \in S_n} E_I(p,Y^*) \geq E_I(X,Y^*) \text{ for all } X \in S_n$$

and $Y^* \in BR_{II}(X^*)$ so that

$$E_{II}(X^*,Y^*) = \max_{t \in S_m} E_{II}(X^*,t) \geq E_{II}(X^*,Y) \text{ for all } Y \in S_m.$$

That's it. (X^*,Y^*) is a Nash equilibrium. □

Finally, it is important to understand the difficulty in obtaining the existence of a Nash equilibrium. If our problem was

$$\max_{X \in S_n, Y \in S_m} E_I(X,Y) \quad \text{and} \quad \max_{X \in S_n, Y \in S_m} E_{II}(X,Y),$$

then the existence of an (X_I, Y_I) providing the maximum of E_I is immediate from the fact that $E_I(X,Y)$ is a continuous function over a closed and bounded set. The same is true for the existence of an (X_{II}, Y_{II}) providing the maximum of $E_{II}(X,Y)$. The problem is that we don't know whether $X_I = X_{II}$ and $Y_I = Y_{II}$. In fact, that is very unlikely and it is a stronger condition than what is required of a Nash equilibrium.

3.4 NONLINEAR PROGRAMMING METHOD FOR NONZERO SUM TWO-PERSON GAMES

We have shown how to calculate the pure Nash equilibria in the 2×2 case, and the system of equations that will give a mixed Nash equilibrium in more general cases. In this section we present a method (introduced by Lemke and Howson [12] in the 1960s) of finding all Nash equilibria for arbitrary two-person nonzero sum games with any number of strategies. We will formulate the problem of finding a Nash equilibrium as a **nonlinear program**, in contrast to the formulation of solution of a zero sum game as a **linear program**.

In general, a nonlinear program is simply an optimization problem involving nonlinear functions and nonlinear constraints. For example, if we have an objective function f and constraint functions $h_1, \ldots, h_k$, the problem

$$\text{Minimize } f(x_1, \ldots, x_n) \quad \text{subject to } h_j(x_1, \ldots, x_n) \leq 0, j = 1, \ldots, k,$$

is a fairly general formulation of a nonlinear programming problem. In general, the functions f, h_j are not linear, but they could be. In that case, of course, we have a linear program, which is solved by the simplex algorithm. If the function f is quadratic and the constraint functions are linear, then this is called a **quadratic programming problem** and special techniques are available for those. Nonlinear programs are more general and the techniques to solve them are more involved, but fortunately there are several methods available, both theoretical and numerical. Nonlinear programming is a major branch of operations research and is under very active development. In our case, once we formulate the game as a nonlinear program, we will use the packages developed in Maple to solve them numerically. So the first step is to set it up.

Theorem 3.4.1 *Consider the two-person game with matrices (A, B) for players I and II. Then, $(X^* \in S_n, Y^* \in S_m)$ is a Nash equilibrium if and only if they satisfy, along with scalars p^*, q^* the nonlinear program*

$$\begin{aligned}
&\max_{X,Y,p,q} XAY^T + XBY^T - p - q \\
&\textit{subject to} \\
&AY^T \leq pJ_n^T \\
&B^T X^T \leq qJ_m^T \qquad (\textit{equivalently } XB \leq qJ_m) \\
&x_i \geq 0, y_j \geq 0, \quad XJ_n = 1 = YJ_m^T
\end{aligned}$$

where $J_k = (1\ 1\ 1\ \cdots 1)$ is the $1 \times k$ row vector consisting of all 1s. In addition, $p^* = E_I(X^*, Y^*)$, and $q^* = E_{II}(X^*, Y^*)$.

Remark. Expanded, this program reads as

$$\max_{X,Y,p,q} \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i b_{ij} y_j - p - q$$

subject to

$$\sum_{j=1}^{m} a_{ij} y_j \le p, \; i = 1, 2, \dots, n,$$

$$\sum_{i=1}^{n} x_i b_{ij} \le q, \; j = 1, 2, \dots, m,$$

$$x_i \ge 0, y_j \ge 0, \; \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 1.$$

You can see that this is a nonlinear program because of the presence of the terms $x_i y_j$. That is why we need a nonlinear programming method, or a quadratic programming method because it falls into that category.

Proof. Here is how the proof of this useful result goes. Recall that a strategy pair (X^*, Y^*) is a Nash equilibrium if and only if

$$\begin{aligned} E_{\mathrm{I}}(X^*, Y^*) &= X^* A Y^{*T} \ge X A Y^{*T}, \text{ for every } X \in S_n, \\ E_{\mathrm{II}}(X^*, Y^*) &= X^* B Y^{*T} \ge X^* B Y^T, \text{ for every } Y \in S_m. \end{aligned} \tag{3.4.1}$$

Keep in mind that the quantities $E_{\mathrm{I}}(X^*, Y^*)$ and $E_{\mathrm{II}}(X^*, Y^*)$ are scalars. In the first inequality of (3.4.1), successively choose $X = (0, \dots, 1, \dots, 0)$ with 1 in each of the n spots, and in the second inequality of (3.4.1) choose $Y = (0, \dots, 1, \dots, 0)$ with 1 in each of the m spots, and we see that $E_{\mathrm{I}}(X^*, Y^*) \ge E_{\mathrm{I}}(i, Y^*) = {}_iAY^{*T}$ for each i, and $E_{\mathrm{II}}(X^*, Y^*) \ge E_{\mathrm{II}}(X^*, j) = X^* B_j$, for each j. In matrix form, this is

$$\begin{aligned} E_{\mathrm{I}}(X^*, Y^*) J_n^T &= X^* A Y^{*T} J_n^T \ge A Y^{*T}, \\ E_{\mathrm{II}}(X^*, Y^*) J_m &= (X^* B Y^{*T}) J_m \ge X^* B. \end{aligned} \tag{3.4.2}$$

However, it is also true that if (3.4.2) holds for a pair (X^*, Y^*) of strategies, then these strategies must be a Nash point, that is, (3.4.1) must be true. Why? Well, if (3.4.2) is true, we choose any $X \in S_n$ and $Y \in S_m$ and multiply

$$\begin{aligned} E_{\mathrm{I}}(X^*, Y^*) X J_n^T &= E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} X J_n^T \ge X A Y^{*T}, \\ E_{\mathrm{II}}(X^*, Y^*) J_m Y^T &= E_{\mathrm{II}}(X^*, Y^*) = (X^* B Y^{*T}) J_m Y^T \ge X^* B Y^T, \end{aligned}$$

because $XJ_n^T = J_m Y^T = 1$. But this is exactly what it means to be a Nash point. This means that (X^*, Y^*) is a Nash point if and only if

$$X^* A Y^{*T} J_n^T \geq AY^{*T}, \quad (X^* B Y^{*T}) J_m \geq X^* B.$$

We have already seen this in Proposition 3.2.3.

Now suppose that (X^*, Y^*) is a Nash point. We will see that if we choose the scalars

$$p^* = E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} \text{ and } q^* = E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T},$$

then (X^*, Y^*, p^*, q^*) is a solution of the nonlinear program. To see this, we first show that all the constraints are satisfied. In fact, by the equivalent characterization of a Nash point we just derived, we get

$$X^* A Y^{*T} J_n^T = p^* J_n^T \geq AY^{*T} \quad \text{and} \quad (X^* B Y^{*T}) J_m = q^* J_m \geq X^* B.$$

The rest of the constraints are satisfied because $X^* \in S_n$ and $Y^* \in S_m$. In the language of nonlinear programming, we have shown that (X^*, Y^*, p^*, q^*) is a **feasible point**. The **feasible set** is the set of all points that satisfy the constraints in the nonlinear programming problem.

We have left to show that (X^*, Y^*, p^*, q^*) maximizes the objective function

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q$$

over the set of the possible feasible points.

Since every feasible solution (meaning it maximizes the objective over the feasible set) to the nonlinear programming problem must satisfy the constraints $AY^T \leq pJ_n^T$ and $XB \leq qJ_m$, multiply the first on the left by X and the second on the right by Y^T to get

$$XAY^T \leq pXJ_n^T = p, \qquad XBY^T \leq qJ_m Y^T = q.$$

Hence, any **possible** solution gives the objective

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q \leq 0.$$

So $f(X, Y, p, q) \leq 0$ for any feasible point. But with $p^* = X^* A Y^{*T}, q^* = X^* B Y^{*T}$, we have seen that (X^*, Y^*, p^*, q^*) is a feasible solution of the nonlinear programming problem and

$$f(X^*, Y^*, p^*, q^*) = X^* A Y^{*T} + X^* B Y^{*T} - p^* - q^* = 0$$

by definition of p^* and q^*. Hence this point (X^*, Y^*, p^*, q^*) both is feasible and gives the maximum objective (which we know is zero) over any possible feasible solution and so is a solution of the nonlinear programming problem. This shows that if we have a Nash point, it must solve the nonlinear programming problem.

Now we have to show the reverse, namely, that any solution of the nonlinear programming problem must be a Nash point (and we get the optimal expected payoffs as well).

For the opposite direction, let X_1, Y_1, p_1, q_1 be **any** solution of the nonlinear programming problem, let (X^*, Y^*) be a Nash point for the game, and set $p^* = X^* A Y^{*T}$, $q^* = X^* B Y^{*T}$. We will show that (X_1, Y_1) must be a Nash equilibrium of the game.

Since X_1, Y_1 satisfy the constraints of the nonlinear program $A{Y_1}^T \leq p_1 J_n^T$ and $X_1 B \leq q_1 J_m$, we get, by multiplying the constraints appropriately

$$X_1 A Y_1^T \leq p_1 X_1 J_n^T = p_1 \quad \text{and} \quad X_1 B Y_1^T \leq q_1 Y_1^T J_m = q_1.$$

Now, we know that if we use the Nash point (X^*, Y^*) and $p^* = X^* A Y^{*T}, q^* = X^* B Y^{*T}$, then $f(X^*, Y^*, p^*, q^*) = 0$, so zero is the maximum objective. But we have just shown that our **solution** to the program (X_1, Y_1, p_1, q_1) satisfies $f(X_1, Y_1, p_1, q_1) \leq 0$. Consequently, it must in fact be equal to zero:

$$f(X_1, Y_1, p_1, q_1) = (X_1 A Y_1^T - p_1) + (X_1 B Y_1^T - q_1) = 0.$$

The terms in parentheses are nonpositive, and the two terms add up to zero. That can happen only if they are each zero. Hence

$$X_1 A Y_1^T = p_1 \text{ and } X_1 B Y_1^T = q_1.$$

Then we write the constraints as

$$A Y_1^T \leq (X_1 A Y_1^T) J_n^T, \qquad X_1 B \leq (X_1 B Y_1^T) J_m.$$

However, we have shown at the beginning of this proof that this condition is exactly the same as the condition that (X_1, Y_1) is a Nash point. So that's it; we have shown that any solution of the nonlinear program must give a Nash point, and the scalars must be the expected payoffs using that Nash point. $\square$

Remark. It is **not** necessarily true that $E_{\text{I}}(X_1, Y_1) = p_1 = p^* = E_{\text{I}}(X^*, Y^*)$ and $E_{\text{II}}(X_1, Y_1) = q_1 = q^* = E_{\text{II}}(X^*, Y^*)$. Different Nash points can, and usually do, give different expected payoffs, as we have seen many times.

Using this theorem and some nonlinear programming implemented in Maple or Mathematica, we can numerically solve any two-person nonzero sum game. For an example, suppose that we have the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Now, before you get started looking for mixed Nash points, you should first find the pure Nash points. For this game we have two Nash points at $(X_1 = (0,1,0), Y_1 = (1,0,0))$ with expected payoff $E_{\rm I}^1 = 2, E_{\rm II}^1 = 1$, and $(X_2 = (0,0,1) = Y_2)$ with expected payoffs $E_{\rm I}^2 = 1, E_{\rm II}^2 = 2$.

On the other hand, we may solve the nonlinear programming problem with these matrices and obtain a mixed Nash point

$$X_3 = \left(0, \frac{2}{3}, \frac{1}{3}\right), \; Y_3 = \left(\frac{1}{3}, 0, \frac{2}{3}\right), \; p = E_{\rm I}(X_3, Y_3) = \frac{2}{3}, \; q = E_{\rm II}(X_3, Y_3) = \frac{2}{3}.$$

Here are the Maple commands to get the solutions:

```
> with(LinearAlgebra):
> A:=Matrix([[-1,0,0],[2,1,0],[0,1,1]]);
> B:=Matrix([[1,2,2],[1,-1,0],[0,1,2]]);
> X:=<x[1],x[2],x[3]>; #Or X:=Vector(3,symbol=x):
> Y:=<y[1],y[2],y[3]>; #Or Y:=Vector(3,symbol=y):
> Cnst:={seq((A.Y)[i]<=p,i=1..3),
         seq((Transpose(X).B)[i]<=q,i=1..3),
         add(x[i],i=1..3)=1,add(y[i],i=1..3)=1};
> with(Optimization);
> objective:=expand(Transpose(X).A.Y+Transpose(X).B.Y-p-q);
> QPSolve(objective,Cnst,assume=nonnegative,maximize);
> QPSolve(objective,Cnst,assume=nonnegative,maximize,
                                  initialpoint=({q=1,p=2}));
> NLPSolve(objective,Cnst,assume=nonnegative,maximize);
```

This gives us the result $p = 0.66, q = 0.66, x[1] = 0, x[2] = 0.66, x[3] = 0.33$ and $y[1] = 0.33, y[2] = 0, y[3] = 0.66$. The commands also tell us that the value of the objective function at the optimal points is zero, which is what the theorem guarantees, viz., $\max f(X, Y, p, q) = 0$. By changing the initial point, which is indicated with the option `initialpoint=({q=1,p=2})`, we may also find the pure Nash solution $X = (0,1,0)$ and $Y = (1,0,0)$. This is rather a waste, however, because there is no need to use a computer to find the pure Nash points unless it is a very large game. Nevertheless, if you see a solution that seems to be pure, you could check it directly.

The commands indicate there are two ways Maple can solve this problem. First, recognizing the payoff objective function as a quadratic function, we can use the command `QPSolve` which specializes to quadratic programming problems and is faster. Second, in general for any nonlinear objective function we use `NLPSolve`.

You do have to make one change to the Maple commands if the expected payoffs of the game can be negative. The commands

```
>QPSolve(objective,Cnst,assume=nonnegative,maximize,
                                  initialpoint=({q=1,p=2}));
>NLPSolve(objective,Cnst,assume=nonnegative,maximize);
```

are set up assuming that all variables including p and q are nonnegative. If they can possibly be negative, then these commands will not find the Nash equilibria associated with negative payoffs. It's not a big deal, but we have to drop the `assume=nonnegative` part of these commands and add the nonnegativity of the strategy variables to the constraints. Here is what you need to do:

```
> with(LinearAlgebra):
> A:=Matrix([[-1,0,0],[2,1,0],[0,1,1]]);
> B:=Matrix([[1,2,2],[1,-1,0],[0,1,2]]);
> X:=<x[1],x[2],x[3]>;
> Y:=<y[1],y[2],y[3]>;
> Cnst:={seq((A.Transpose(Y))[i]<=p,i=1..3),seq((X.B)[i]<=q,i=1..3),
        add(x[i],i=1..3)=1,add(y[i],i=1..3)=1,
        seq(y[i]>=0,i=1..3),seq(x[i]>=0,i=1..3)};
> with(Optimization);
> objective:=expand(Transpose(X).A.Y+Transpose(X).B.Y-p-q);
> QPSolve(objective,Cnst,maximize);
> QPSolve(objective,Cnst,maximize,initialpoint=({q=1,p=2}));

> NLPSolve(objective,Cnst,maximize);
```

■ EXAMPLE 3.12

Suppose that two countries are involved in an arms control negotiation. Each country can decide to either cooperate or not cooperate (don't). For this game, one possible bimatrix payoff situation may be

	Cooperate	Don't
Cooperate	(1,1)	(0,3)
Don't	(3,0)	(2,2)

This game has a pure Nash equilibrium at $(2,2)$, so these countries will not actually negotiate in good faith. This would lead to what we might call **deadlock** because the two players will decide not to cooperate. If a third party managed to intervene to change the payoffs, you might get the following payoff matrix:

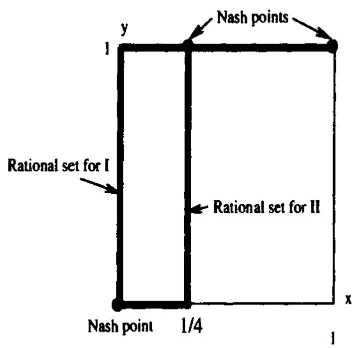
	Cooperate	Don't
Cooperate	$(3,3)$	$(-1,-3)$
Don't	$(3,-1)$	$(1,1)$

.

What's happened is that both countries will receive a greater reward if they cooperate and the benefits of not cooperating on the part of both of them have decreased. In addition, we have lost symmetry. If the row player chooses to cooperate but the column player doesn't, they both lose, but player I will lose less than player II. On the other hand, if player I doesn't cooperate but player II does cooperate, then player I will gain and player II will lose, although not as much. Will that guarantee that they both cooperate? Not necessarily. We now have pure Nash equilibria at **both** $(3,3)$ and $(1,1)$. Is there also a mixed Nash equilibrium? If we apply the Maple commands, or use the calculus method, or the formulas, we obtain

$$X_1 = \left(\frac{1}{4}, \frac{3}{4}\right), Y_1 = (1,0), p_1 = 3, q_1 = 0,$$
$$X_2 = (0,1), Y_2 = (0,1), p_2 = q_2 = 1,$$
$$X_3 = (1,0) = Y_3, p_3 = q_3 = 3.$$

Actually, by graphing the rational reaction sets we see that any mixed strategy $X = (x, 1-x), Y = (1,0), \frac{1}{4} \leq x \leq 1$, is a Nash point.



Each player receives the most if both cooperate, but how does one move them from *Don't Cooperate* to *Cooperate*?

Here is one last example applied to dueling.

EXAMPLE 3.13

A Discrete Silent Duel. Consider a gun duel between two persons, Pierre (player I) and Bill (player II). They each have a gun with exactly one bullet. They face each other initially 10 paces apart. They will walk toward each other, one pace at a time. At each step, including the initial one, that is, at 10 paces, they each may choose to either fire or hold. If they fire, the probability of a hit depends on how far apart they are according to the following distribution:

$$Prob(\text{Hit}|\text{Paces apart} = k) = \begin{cases} 0.2 & \text{if } k = 10; \\ 0.6 & \text{if } k = 6; \\ 0.8 & \text{if } k = 4; \\ 1 & \text{if } k = 2. \end{cases}$$

Assume in this special case that they choose to fire only at $k = 10, 6, 4, 2$ paces apart. It is possible for both to fire and miss, or both fire and kill.

The hardest part of the analysis is coming up with the matrices without making a computational mistake. Maple can be a big help with that. First we define the accuracy functions

$$p_1(x) = p_2(x) = \begin{cases} 0.2 & \text{if } x = 0; \\ 0.6 & \text{if } x = 0.4; \\ 0.8 & \text{if } x = 0.6; \\ 1 & \text{if } x = 0.8. \end{cases}$$

Think of $0 \leq x \leq 1$ as the **time** to shoot. It is related to k by $x = 1 - k/10$. Now define the payoff to player I, Pierre, as

$$u_1(x,y) = \begin{cases} a_1 p_1(x) + b_1(1 - p_1(x))p_2(y) \\ \qquad\qquad +c_1(1 - p_1(x))(1 - p_2(y)) & \text{if } x < y; \\ d_1 p_2(y) + e_1(1 - p_2(y))p_1(x) \\ \qquad\qquad +f_1(1 - p_2(y))(1 - p_1(x)) & \text{if } x > y; \\ g_1 p_1(x)p_2(x) + h_1 p_1(x)(1 - p_2(x)) \\ \quad +k_1(1 - p_1(x))p_2(x) + \ell_1(1 - p_1(x))(1 - p_2(x)) & \text{if } x = y. \end{cases}$$

For example, if $x < y$, then Pierre is choosing to fire before Bill and the expected payoff is calculated as

$$\begin{aligned} u_1(x,y) &= a_1 Prob(\text{II killed at } x) + b_1 Prob(\text{I misses at } x) Prob(\text{I killed at } y) \\ &\qquad + c_1 Prob(\text{I misses at } x) Prob(\text{II misses at } y) \\ &= a_1 p_1(x) + b_1(1 - p_1(x))(1) + c_1(1 - p_1(x))(1 - p_2(y)). \end{aligned}$$

Notice that the **silent** part appears in the case that I misses at x and I is killed at y because the probability I is killed by II is not necessarily 1 if I misses. The remaining cases are similar.

The constants multiplying the accuracy functions are the payoffs. For example, if $x < y$, Pierre's payoffs are a_1 if I kills II at x, b_1 if I misses at x and II kills I at y, and c_1 if they both miss. We have derived the payoff for player I using general payoff constants so that you may simply change the constants to see what happens with different rewards (or penalties).

For Pierre we will use the payoff values

$$\begin{aligned} &a_1 = -2, b_1 = -1, c_1 = 2, d_1 = -1, \\ &e_1 = 1, f_1 = 2, g_1 = -2, h_1 = 1, k_1 = -1, \ell_1 = 2. \end{aligned}$$

The expected payoff to Bill is similarly

$$u_2(x,y) = \begin{cases} a_2 p_1(x) + b_2(1-p_1(x))p_2(y) \\ \qquad\qquad + c_2(1-p_1(x))(1-p_2(y)) & \text{if } x < y; \\ d_2 p_2(y) + e_2(1-p_2(y))p_1(x) \\ \qquad\qquad + f_2(1-p_2(y))(1-p_1(x)) & \text{if } x > y; \\ g_2 p_1(x)p_2(x) + h_2 p_1(x)(1-p_2(x)) \\ \quad + k_2(1-p_1(x))p_2(x) + \ell_2(1-p_1(x))(1-p_2(x)) & \text{if } x = y. \end{cases}$$

For Bill we will take the payoff values

$$a_2 = -1, b_2 = 1, c_2 = 1, d_2 = 1,$$
$$e_2 = -1, f_2 = 1, g_2 = 0, h_2 = -1, k_2 = 1, \ell_2 = 1.$$

The payoff matrix then for player I is

$$A = (u_1(x,y) : x = 0, 0.4, 0.6, 0.8, y = 0, 0.4, 0.6, 0.8),$$

or

$$A = \begin{bmatrix} 1.20 & -0.24 & -0.72 & -1.2 \\ 0.92 & -0.40 & -1.36 & -1.6 \\ 0.76 & -0.12 & -1.20 & -1.8 \\ 0.6 & -0.2 & -0.6 & -2 \end{bmatrix}.$$

Similarly, player II's matrix is

$$B = \begin{bmatrix} 0.64 & 0.60 & 0.60 & 0.6 \\ 0.04 & 0.16 & -0.20 & -0.2 \\ -0.28 & 0.36 & 0.04 & -0.6 \\ -0.6 & 0.2 & 0.6 & 0 \end{bmatrix}.$$

To solve this game, you may use Maple and adjust the initial point to obtain multiple equilibria. Here is the result:

X	Y	E_{I}	E_{II}
$(0.97, 0, 0, 0.03)$	$(0.17, 0, 0.83, 0)$	-0.4	0.6
$(0.95, 0, 0.03, 0.02)$	$(0.08, 0.8, 0.12)$	-0.19	0.58
$(0.94, 0, 0.06, 0)$	$(0.21, 0.79, 0, 0)$	0.07	0.59
$(0, 0, 0.55, 0.45)$	$(0, 0.88, 0.12, 0)$	-0.25	0.29
$(0, 0, 1, 0)$	$(0, 1, 0, 0)$	-0.12	0.36
$(1, 0, 0, 0)$	$(1, 0, 0, 0)$	1.2	0.64
$(0, 0, 0, 1)$	$(0, 0, 1, 0)$	-0.6	0.6

It looks like the best Nash for each player is to shoot at 10 paces.

SUMMARY OF METHODS FOR FINDING MIXED NASH EQUILIBRIA

The methods we have for finding the mixed strategies for nonzero sum games are recapped here.

1. Equality of payoffs. Suppose that we have mixed strategies $X^* = (x_1, \ldots, x_n)$ and $Y^* = (y_1, \ldots, y_m)$. For any rows $k_1, k_2, \ldots$ that have a positive probability of being used, the expected payoffs to player I for using any of those rows must be equal: $E_{\mathrm{I}}(k_r, Y^*) = E_{\mathrm{I}}(k_s, Y^*) = E_{\mathrm{I}}(X^*, Y^*)$. You can find Y^* from these equations. Similarly, for any columns j that have a positive probability of being used, we have $E_{\mathrm{II}}(X^*, j_r) = E_{\mathrm{II}}(X^*, j_s) = E_{\mathrm{II}}(X^*, Y^*)$. You can find X^* from these equations.

2. You can use the calculus method directly by computing

$$f(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = \left(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i\right) A \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ 1 - \sum_{j=1}^{m-1} y_j \end{bmatrix}$$

and then

$$\frac{\partial f}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n-1.$$

This will let you find Y^*. Next, compute

$$g(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = \left(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i\right) B \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ 1 - \sum_{j=1}^{m-1} y_j \end{bmatrix}$$

and then

$$\frac{\partial g}{\partial y_j} = 0, \quad j = 1, 2, \ldots, m-1.$$

From these you will find X^*.

3. You can use the system of equations to find interior Nash points given by

$$\sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0, \qquad k = 1, 2, \dots, n-1$$

$$\sum_{i=1}^{n} x_i[b_{is} - b_{im}] = 0, \qquad s = 1, 2, \dots, m-1.$$

$$x_n = 1 - \sum_{i=1}^{n-1} x_i, \quad y_m = 1 - \sum_{j=1}^{m-1} y_j.$$

4. In the 2×2 case you can find the rational reaction sets for each player and see where they intersect. This gives all the Nash equilibria including the pure ones.

5. Use the nonlinear programming method: set up the objective, the constraints, and solve. Use the option `initialpoint` to modify the starting point the algorithm uses to find additional Nash points.

PROBLEMS

3.20 Suppose you are told that the following is the nonlinear program for solving a game with matrices (A, B):

$$\begin{aligned}
\text{obj} &= (90 - 40x)y + (60x + 20)(1 - y) \\
&\qquad + (40x + 10)y + (80 - 60x)(1 - y) - p - q \\
\text{cnst1} &= 80 - 30y \le p \\
\text{cnst2} &= 20 + 70y \le p \\
\text{cnst3} &= 10 + 40x \le q \\
\text{cnst4} &= 80 - 60x \le q \\
0 &\le x \le 1, 0 \le y \le 1.
\end{aligned}$$

Find the associated matrices A and B and then solve the problem to find the Nash equilibrium.

3.21 Suppose that the wife in a battle of the sexes game has an additional strategy she can use to try to get the husband to go along with her to the concert, instead of wrestling. Call it strategy Z. The payoff matrix then becomes

H/W	Wr	Co	Z
Wr	$(2, 1)$	$(0, 0)$	$(-1, 1)$
Co	$(0, 0)$	$(1, 2)$	$(1, 3)$

Find all Nash equilibria.

3.22 Since every two-person **zero sum** game can be formulated as a bimatrix game, show how to modify the Lemke–Howson algorithm to be able to calculate saddle points of zero sum two-person games. Then use that to find the value and saddle point for the game with matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -3 \\ 4 & -3 & 5 \\ -3 & \frac{1}{2} & -9 \end{bmatrix}.$$

Check your answer by using the linear programming method to solve this game.

3.23 Consider the following bimatrix for a version of the game of chicken (see Problem 3.3):

I/II	Straight	Avoid
Straight	$(1,1)$	$(-1,2)$
Avoid	$(2,-1)$	$(-3,-3)$

(**a**) What is the explicit objective function for use in the Lemke–Howson algorithm?
(**b**) What are the explicit constraints?
(**c**) Solve the game.

3.24 Use nonlinear programming to find all Nash equilibria for the game and expected payoffs for the game with bimatrix

$$\begin{bmatrix} (1,2) & (0,0) & (2,0) \\ (0,0) & (2,3) & (0,0) \\ (2,0) & (0,0) & (-1,6) \end{bmatrix}$$

3.25 Consider the game with bimatrix

$$\begin{bmatrix} (-3,-4) & (2,-1) & (0,6) & (1,1) \\ (2,0) & (2,2) & (-3,0) & (1,-2) \\ (2,-3) & (-5,1) & (-1,-1) & (1,-3) \\ (-4,3) & (2,-5) & (1,2) & (-3,1) \end{bmatrix}$$

There are six Nash equilibria for this game. Find as many as you can by adjusting `initialpoint=` in the Maple commands.

3.26 Consider the gun duel between Pierre and Bill. Modify the payoff functions so that it becomes a noisy duel with $a_1 = -2, b_1 = -1, c_1 = 2, d_1 = -1, e_1 = 1, f_1 = 2, g_1 = -2, h_1 = 1, k_1 = -1, \ell_1 = 2$, for player I, and $a_2 = -1, b_2 = 1, c_2 = 1, d_2 = 1, e_2 = -1, f_2 = 1, g_2 = 0, h_2 = -1, k_2 = 1, \ell_2 = 1$, for player II. Then solve the game and obtain at least one mixed Nash equilibrium.

3.5 CHOOSING AMONG SEVERAL NASH EQUILIBRIA (OPTIONAL)

The concept of a Nash equilibrium is the most widely used idea of equilibrium in most fields, certainly in economics. But there is obviously a problem we have to deal with. How do we choose among the Nash equilibria in games where there are more than one? This must be addressed if we ever want to be able to predict the outcome of a game, which is, after all, the reason why we are studying this to begin with. But we have to warn the reader that many different criteria are used to choose a Nash equilibrium and there is no definitive way to make a choice.

The first idea is to use some sort of stability. Intuitively that means that we start with any strategy, say, for player II. Then we calculate the best response strategy for player I to this first strategy, then we calculate the best response strategy for player II to the best response for player I, and so on. If you have done Problem 1.35, you have already considered what could happen. Here is our example.

EXAMPLE 3.14

Let's carry out the repeated best response idea for the two-person zero sum game

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & -2 \\ 0 & -1 & 4 \end{bmatrix}.$$

Notice that $v^+ = v^- = 1$ and we have a saddle point at $X^* = (1,0,0), Y^* = (0,1,0)$. Suppose that we start with a strategy for player II, $Y^0 = (0,0,1)$, so player II starts by playing column 3. The table summarizes the sequence of best responses:

	Best response Strategy for II	Best response Strategy for I	Payoff to I
Step 0	Column 3 (Start)	Row 3	4
Step 1	Column 2	Row 1	1
Step 2	Column 2	Row 1	1
⋮	⋮	⋮	⋮

The best response to player II by player I is to play row 3 (and receive 4). The best response to that by player II is to play column 2 (and II receives -1). The best response to II's choice of column 2 is now to play row 1 (and I receives 1). The best response of player II to player I's choice of row 1 is to play column 2, and that is the end. We have arrived at the one and only saddle point of the matrix, namely, I plays row 1 and II plays column 2.

Similarly, you can check that this convergence to the saddle point will happen no matter where we start with a strategy, and no matter who chooses first. This is a really stable saddle point. Maybe it is because it is the only saddle point?

Nope. That isn't the thing going on, because here is a matrix with only one saddle but the best response sequence doesn't converge to it:

$$A = \begin{bmatrix} 6 & 3 & 2 \\ 5 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}$$

The value of this game is $v(A) = 4$, and there is a unique saddle at row 2 column 2. Now suppose that the players play as in the following table:

	Best response Strategy for II	Best response Strategy for I
Step 0	Column 1	Row 1
Step 1	Column 3	Row 3
Step 2	Column 1	Row 1
⋮	⋮	⋮

This just cycles from corner payoff to corner payoff and doesn't get to the saddle. Only starting with row 2 or column 2 would bring us to the saddle, and that after only one step.

Stability seems to be a desirable characteristic for saddles and also for Nash points, especially with games that will be played many times. The next example considers a sort of stability to determine the choice of a Nash equilibrium among several candidates.

■ EXAMPLE 3.15

Let's consider the bimatrix game

$$\begin{bmatrix} (-a,-a) & (0,-b) \\ (-b,0) & (-c,-c) \end{bmatrix}.$$

Assume $a > b, c > 0$. Because the game is symmetric with respect to payoffs, it doesn't matter whether we talk about the row or column player. Then we have two pure Nash equilibria at $(-b, 0)$ and at $(0, -b)$. By calculating

$$E_1(x,y) = (x\ 1-x) \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = -ayx - (1-x)[(b-c)y + c],$$

we see that setting

$$\frac{\partial E_1(x,y)}{\partial x} = -(a-b+c)y + c = 0 \implies y = \frac{c}{a-b+c}.$$

Similarly, $x = \dfrac{c}{a-b+c}$. Defining $h = \dfrac{c}{a-b+c}$, we see that we also have a mixed Nash equilibrium at $X = (h, 1-h) = Y$. Here is the table of payoffs for each of the three equilibria:

	$E_1(x,y)$	$E_2(x,y)$
$x=0, y=1$	$-b$	0
$x=1, y=0$	0	$-b$
$x=h, y=h$	z	z

where $z = E_1(h,h) = -h^2(a-b+c) - h(b-2c) - c$. To be concrete, suppose $a = 3, b = 1, c = 2$. Then $h = \frac{1}{2}$ gives the mixed Nash equilibrium $X = Y = (\frac{1}{2}, \frac{1}{2})$. These are the payoffs:

Equilibrium	Payoff to I	Payoff to II
$x = y = \frac{1}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$
$x=1, y=0$	0	-1
$x=0, y=1$	-1	0

Now suppose that we go through a thought experiment. Both players are trying to maximize their own payoffs without knowing what the opponent will do. Player I sees that choosing $x = 1$ will do that, but only if II chooses $y = 0$. player II sees that choosing $y = 1$ will maximize her payoff, but only if I chooses $x = 0$. Without knowing the opponent's choice, they will end up playing $(x = 1, y = 1)$, resulting in the nonoptimal payoff $E_1(1,1) = -3 = E_2(1,1)$.

If there are **many players playing this game whenever two players encounter each other** and they each play nonoptimally, namely, $x = 1, y = 1$, they will all receive less than they could otherwise get. Eventually, one of the players will realize that everyone else is playing nonoptimally $(x = 1)$ and decide to switch to playing $x = 0$ for player I, or $y = 0$ for player II. Knowing, or believing, that in the next game a player will be using $y = 1$, then player I can switch and use $x = 0$, receiving -1, instead of -3.

Now we are at an equilibrium $x = 0, y = 1$. But there is nothing to prevent other players from reaching this conclusion as well. Consequently others also start playing $x = 0$ or $y = 0$, and now we move again to the nonoptimal play $x = 0, y = 0$, resulting in payoffs -2 to each player.

If this reasoning is correct, then we could cycle forever between $(x = 1, y = 1)$ and $(x = 0, y = 0)$, until someone stumbles on trying $x = h = \frac{1}{2}$. If player I uses $x = \frac{1}{2}$ and everyone else is playing $y = 0$, then player I gets $E_1(\frac{1}{2}, 0) = -1$ and player II gets $E_2(\frac{1}{2}, 0) = -\frac{3}{2}$. Eventually, everyone will see that $\frac{1}{2}$ is a better response to 0 and everyone will switch to $h = \frac{1}{2}$, that is, half the time playing the first row (or column) and half the time playing the second row (or column), When that happens, everyone receives $-\frac{3}{2}$.

In addition, notice that since $E_1(x, \frac{1}{2}) = E_2(\frac{1}{2}, y) = -\frac{3}{2}$ for any x, y, **no strategy chosen by either player can get a higher payoff if the opposing player chooses** $h = \frac{1}{2}$. That means that once a player hits on using $x = \frac{1}{2}$ or

$y = \frac{1}{2}$ the cycling is over. No one can do better with a unilateral switch to something else, and there is no incentive to move to another equilibrium.

This Nash equilibrium $x = y = \frac{1}{2}$ is the only one that allows the players to choose without knowing the other's choice and then have no incentive to do something else. It is stable in that sense.

This strategy is called **uninvadable**, or an **evolutionary stable strategy**, and shows us, sometimes, one way to pick the **right** Nash equilibrium when there are more than one. We will discuss this in much more depth when we discuss evolutionary stable strategies and population games in Chapter 6.

Stability is one criterion for picking a good Nash equilibrium, but there are others. Another criterion taking into account the idea of risk is discussed with an example.

■ EXAMPLE 3.16

Entry Deterrence. There are two players producing gadgets. Firm (player) A is already producing and selling the gadgets, while firm (player) B is thinking of producing and selling the gadgets and competing with firm A. Firm A has two strategies: (1) join with firm B to control the total market (perhaps by setting prices), or (2) resist firm B and make it less profitable or unprofitable for firm B to enter the market (perhaps by lowering prices unilaterally). Firm B has the two strategies to (1) enter the market and compete with firm A or (2) move on to something else. Here is the bimatrix:

A/B	Enter	Move on
Resist	$(0, -1)$	$(10, 0)$
Join	$(5, 4)$	$(10, 0)$

The idea is that a war hurts both sides, but sharing the market means less for firm A. There are two pure Nash equilibria:

$$X_1 = (1, 0), Y_1 = (0, 1),$$
$$X_2 = (0, 1), Y_2 = (1, 0).$$

The associated payoffs for these Nash equilibria are

$$(A_1, B_1) = (10, 0), \ (A_2, B_2) = (5, 4).$$

Assuming that firm B actually will make the first move, if firm B decides to enter the market, firm A will get 0 if A resists, but 5 if firm A joins with firm B. If firm B decides to not enter the market, then firm A gets 10. So, we consider the two Nash points (resist, move on) and (join, enter) in which the second Nash point gives firm A significantly less (5 vs. 10). So long as firm B does not enter the market, firm A can use either resist or join, but with any chance that firm B enters the market, firm A definitely prefers to join.

If firm B moves first, the best choice is Y_2, which will yield it B_2 (if A uses X_2). Since B doesn't know what A will do, suppose that B just chooses Y_2 in the hope that A will use X_2. Now, firm A looks over the payoffs and sees that X_1 gives A a payoff of 10 (if B plays Y_1). So the best for firm A is X_1, which is resist. So, if each player plays the best for themselves without regard to what the other player will do, they will play $X = (1,0), Y = (1,0)$ with the result that A gets 0 and B gets -1. This is the worst outcome for both players. Now, what?

In the previous example we did not account for the fact that if there is any positive probability that player B will enter the market, then firm A must take this into account in order to reduce the risk. From this perspective, firm A would definitely not play resist because if $Y^* = (\varepsilon, 1-\varepsilon)$, with $\varepsilon > 0$ a very small number, then firm A is better off playing join. Economists say that equilibrium X_2, Y_2 **risk dominates** the other equilibrium and so that is the **correct one**. A risk-dominant Nash equilibrium will be **correct** the more uncertainty exists on the part of the players as to which strategy an opponent will choose; that is, the more risk and uncertainty, the more likely the risk-dominant Nash equilibrium will be played.

Finally, we end this section with a definition and discussion of Pareto-optimality. Pareto optimality is yet another criterion used to choose among several Nash equilibria. Here is the definition and we will use this again later in the book.

Definition 3.5.1 *Given a collection of payoff functions*

$$(u_1(q_1,\dots,q_n),\dots,u_n(q_1,\dots,q_n))$$

for an n–person nonzero sum game, where the q_i is a pure or mixed strategy for player $i = 1,2,\dots,n$, we say that $(q_1^*,\dots,q_n^*)$ is Pareto-optimal if there does not exist any other strategy for any of the players that makes that player better off, that is, increases her or his payoff, without making other players worse off, namely, decreasing at least one other player's payoff.

From this perspective, it is clear that $(5,4)$ is the Pareto-optimal payoff point for the firms in entry deterrence because if either player deviates from using X_2, Y_2, then at least one of the two players does worse. On the other hand, if we look back at the prisoner's dilemma problem at the beginning of this chapter we showed that $(-5,-5)$ is a Nash equilibrium, but it is **not** Pareto-optimal because $(-1,-1)$ simultaneously improves both their payoffs.

Closely related to Pareto-optimality is the concept of a payoff-dominant Nash equilibrium, which was introduced by Nobel Prize winners Harsanyi and Selten.

Definition 3.5.2 *A Nash equilibrium is* **payoff-dominant** *if it is Pareto-optimal compared to all other Nash equilibria in the game.*

Naturally, **risk-dominant** and **payoff-dominant** are two different things. Here is an example, commonly known as the **stag hunt game**:

	Hunt	Gather
Hunt	$(5,5)$	$(0,4)$
Gather	$(4,0)$	$(2,2)$

This is an example of a **coordination** game. The idea is that if the players can coordinate their actions and hunt, then they can both do better. Gathering alone is preferred to gathering together, but hunting alone is much worse than gathering alone.

The pure Nash equilibria are (hunt,hunt) and (gather, gather). There is also a mixed Nash equilibrium at $X_1 = (\frac{2}{3}, \frac{1}{3}), Y_1 = (\frac{2}{3}, \frac{1}{3})$. The following table summarizes the Nash points and their payoffs:

$X_1 = (\frac{2}{3}, \frac{1}{3}) = Y_1$	$E_I = \frac{10}{3}$	$E_{II} = \frac{10}{3}$
$X_2 = (0,1) = Y_2$	$E_I = 2$	$E_{II} = 2$
$X_3 = (1,0) = Y_3$	$E_I = 5$	$E_{II} = 5$

The Nash equilibrium X_3, Y_3 is payoff-dominant because no player can do better no matter what. But the Nash equilibrium X_2, Y_2 risk dominates X_3, Y_3 (i.e., (gather,gather) risk dominates (hunt,hunt), The intuitive reasoning is that if either player is not absolutely certain that the other player will join the hunt, then the player who was going to hunt sees that she can do better by gathering. The result is that both players end up playing gather in order to minimize the risk of getting zero. Even though they both do worse, (gather, gather) is risk-dominant. Notice the resemblance to the prisoner's dilemma game in which both players choose to confess because that is the risk-dominant strategy.

PROBLEMS

3.27 In the following games solved earlier, determine which, if any, Nash equilibrium is payoff dominant, risk-dominant, and Pareto-optimal.

(a) (The Game of Chicken) :

I/II	Straight	Avoid
Straight	$(1,1)$	$(-1,2)$
Avoid	$(2,-1)$	$(-3,-3)$

(b) (Arms Control)

	Cooperate	Don't
Cooperate	$(3,3)$	$(-1,-3)$
Don't	$(3,-1)$	$(1,1)$

(c)

$$\begin{bmatrix} (1,2) & (0,0) & (2,0) \\ (0,0) & (2,3) & (0,0) \\ (2,0) & (0,0) & (-1,6) \end{bmatrix}$$

BIBLIOGRAPHIC NOTES

The example games considered in this chapter (prisoner's dilemma, chicken, battle of sexes, hawk–dove, stag–hare, etc.) are representative examples of the general class of games of pure competition, or cooperation, and so on. The general approaches to solving 2×2 games using inequalities (Proposition 3.2.1) also appears in Wang's book [28]. Rational reaction is a standard concept appearing in all game theory texts (see References [28], [15], [3], for example).

The proof of existence of a Nash equilibrium given here is due to Nash, but there are many proofs of this result that do not use the Kakutani fixed-point theorem and are considered more elementary. The nonlinear programming approach to the calculation of a Nash equilibrium originates with Lemke and Howson [12]. The main advantage of their approach is the fact that it combines two separate optimization problems over two sets of variables into one optimization problem over one set of variables; that is, instead of maximizing $E_{\text{I}}(X,Y)$ over X for fixed Y and $E_{\text{II}}(X,Y)$ over Y for fixed X, we maximize $E_{\text{I}}(X,Y) + E_{\text{II}}(X,Y) - p - q$ over all variables (X, Y, p, q) at the same time. Computationally, this is much easier to carry out on a computer, which is the main advantage of the Lemke Howson algorithm.

All the games of this chapter can be solved using the software package **Gambit** as a blackbox in which the matrices are entered and all the Nash equilibria as well as the expected payoffs are calculated.

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CHAPTER 4

N-PERSON NONZERO SUM GAMES WITH A CONTINUUM OF STRATEGIES

The race is not always to the swift nor the battle to the strong, but that's the way to bet.

–Damon Runyon, *More than Somewhat*

4.1 THE BASICS

In previous sections the games all assumed that the players each had a finite or countable number of pure strategies they could use. A major generalization of this is to consider games in which the players have many more strategies. For example, in bidding games the amount of money a player could bid for an item tells us that the players can choose any strategy that is a positive real number. The dueling games we considered earlier are discrete versions of games of timing in which the time to act is the strategy. In this chapter, we will consider some games of this type with N players and various strategy sets.

If there are N players in a game, we assume that each player has her or his own payoff function depending on her or his choice of strategy and the choices of the other players. Suppose that the strategies must take values in sets $Q_i, i = 1, \ldots, N$ and the payoffs are real-valued functions

$$u_i : Q_1 \times \cdots \times Q_N \to \mathbb{R}, \qquad i = 1, 2, \ldots, N.$$

Here is a formal definition of a Nash equilibrium point, keeping in mind that each player wants to maximize their own payoff.

Definition 4.1.1 *A collection of strategies $q^* = (q_1^*, \ldots, q_n^*) \in Q_1 \times \cdots \times Q_N$ is a Nash equilibrium for the game with payoff functions $\{u_i(q)\}, i = 1, \ldots, N$, if for each player $i = 1, \ldots, N$, we have*

$$u_i(q_1^*, \ldots, q_{i-1}^*, q_i^*, q_{i+1}^*, \ldots, q_N^*) \\ \geq u_i(q_1^*, \ldots, q_{i-1}^*, q_i, q_{i+1}^*, \ldots, q_N^*), \text{ for all } q_i \in Q_i.$$

A Nash equilibrium consists of strategies that are all best responses to each other. The point is that no player can do better by deviating from a Nash point, assuming that no one else deviates. It doesn't mean that a **group** of players couldn't do better by playing something else.

Remarks.

1. Notice that if there are two players and $u_1 = -u_2$, then a point (q_1^*, q_2^*) is a Nash point if

$$u_1(q_1^*, q_2^*) \geq u_1(q_1, q_2^*) \text{ and } u_2(q_1^*, q_2^*) \geq u_2(q_1^*, q_2), \forall (q_1, q_2).$$

But then

$$-u_1(q_1^*, q_2^*) \geq -u_1(q_1^*, q_2) \Longrightarrow u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2),$$

and putting these together we see that

$$u_1(q_1, q_2^*) \leq u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2), \forall (q_1, q_2).$$

This, of course, says that (q_1^*, q_2^*) is a saddle point of the two-person zero sum game. This would be a pure saddle point if the Q sets were pure strategies and a mixed saddle point if the Q sets were the set of mixed strategies.

2. In many cases, the problem of finding a Nash point can be reduced to a simple calculus problem. To do this we need to have the strategy sets Q_i to be open intervals and the payoff functions to have at least two continuous derivatives because we are going to apply the second derivative test. The steps involved in determining $(q_1^*, q_2^*, \ldots, q_n^*)$ as a Nash equilibrium are the following:

(a) Solve

$$\frac{\partial u_i(q_1, \ldots, q_n)}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n.$$

(b) Verify that q_i^* is the only stationary point of the function

$$q \mapsto u_i(q_1^*, \ldots, q_{i-1}^*, q, q_{i+1}^*, \ldots, q_n^*) \text{ for } q \in Q_i.$$

(c) Verify

$$\frac{\partial^2 u_i(q_1, \ldots, q_n)}{\partial q_i^2} < 0, \quad i = 1, 2, \ldots, n,$$

evaluated at $q_1^*, \ldots, q_n^*$.

If these three points hold for $(q_1^*, q_2^*, \ldots, q_n^*)$, then it must be a Nash equilibrium. The last condition guarantees that the function is concave down in each variable when the other variables are fixed. This guarantees that the critical point (in that variable) is a maximum point. There are many problems where this is all we have to do to find a Nash equilibrium, but remember that this is a sufficient but not necessary set of conditions because many problems have Nash equilibria that do not satisfy any of the three conditions.

3. Carefully read the definition of Nash equilibrium. For the calculus approach we take the partial of u_i with respect to q_i, **not** the partial of each payoff function with respect to all variables. We are not trying to maximize each payoff function over **all** the variables, but each payoff function to each player as a function only of the variable they control, namely, q_i. That is the difference between a Nash equilibrium and simply the old calculus problem of finding the maximum of a function over a set of variables.

Let's start with a straightforward example.

■ EXAMPLE 4.1

We have a two-person game with pure strategy sets $Q_1 = Q_2 = \mathbb{R}$ and payoff functions

$$u_1(q_1, q_2) = -q_1 q_2 - q_1^2 + q_1 + q_2 \quad \text{and} \quad u_2(q_1, q_2) = -3q_2^2 - 3q_1 + 7q_2.$$

Then

$$\frac{\partial u_1}{\partial q_1} = -q_2 - 2q_1 + 1 \quad \text{and} \quad \frac{\partial u_2}{\partial q_2} = -6q_2 + 7.$$

There is one and only one solution of these (so only one stationary point), and it is given by $q_1 = -\frac{1}{12}, q_2 = \frac{7}{6}$. Finally, we have

$$\frac{\partial^2 u_1}{\partial q_1^2} = -2 < 0 \quad \text{and} \quad \frac{\partial^2 u_2}{\partial q_2^2} = -6 < 0,$$

and so $(q_1, q_2) = (-\frac{1}{12}, \frac{7}{6})$ is indeed a Nash equilibrium.

For a three-person example take

$$u_1(q_1, q_2, q_3) = -q_1^2 + 2q_1q_2 - 3q_2q_3 - q_1$$
$$u_2(q_1, q_2, q_3) = q_2^2 - q_1^2 + 4q_1q_3 - 5q_1q_2 + 2q_2$$
$$u_3(q_1, q_2, q_3) = 4q_3^2 - (q_1 + q_3)^2 + q_1q_3 - 3q_2q_3 + q_1q_2q_3.$$

Taking the partials and finding the stationary point gives the unique solution $q_1^* = \frac{1}{7}, q_2^* = \frac{9}{14}$, and $q_3^* = -\frac{97}{490}$. Since $\partial^2 u_1/\partial q_1^2 = -2 < 0$, $\partial^2 u_2/\partial q_2^2 = -2 < 0$, and $\partial^2 u_3/\partial q_3^2 = -10 < 0$, we know that our stationary point is a Nash equilibrium.

It is not true that Nash equilibria can be found only for payoff functions that have derivatives but if there are no derivatives then we have to do a direct analysis.

EXAMPLE 4.2

Do politicians pick a position on issues to maximize their votes?[1] Suppose that is exactly what they do (after all, there are plenty of real examples). Suppose that voter preferences on the issue are distributed from $[0, 1]$ according to a continuous probability density function $f(x) > 0$ and $\int_0^1 f(x)\,dx = 1$. The density $f(x)$ approximately represents the percentage of voters who have preference $x \in [0, 1]$ over the issue. The midpoint $x = \frac{1}{2}$ is taken to be middle of the road, while x values in $[0, \frac{1}{2})$ are **leftist, or liberal,** and x values in $(\frac{1}{2}, 1]$ are **rightist, or conservative.** The question a politician might ask is: "Given f, what position in $[0, 1]$ should I take in order to maximize the votes that I get in an election against my opponent?" The opponent also asks the same question. **We assume that voters will always vote for the candidate nearest to their own positions.**

Let's call the two candidates I and II, and let's take the position of player I to be $q_1 \in [0, 1]$ and for player II, $q_2 \in [0, 1]$. Let V be the random variable that is the position of a randomly chosen voter so that V has continuous density function f.

The payoff functions for player I and II is given by

$$u_1(q_1, q_2) \equiv \begin{cases} Prob(V \le \dfrac{q_1 + q_2}{2}) & \text{if } q_1 < q_2; \\ \frac{1}{2} & \text{if } q_1 = q_2; \\ Prob(V > \dfrac{q_1 + q_2}{2}) & \text{if } q_1 > q_2. \end{cases}$$
$$u_2(q_1, q_2) \equiv 1 - u_1(q_1, q_2).$$

[1] This example is adapted from a similar example in Reference [1], commonly known as the median voter problem.

Here is the reasoning behind these payoffs. Suppose that $q_1 < q_2$, so that candidate I takes a position to the left of candidate II. The midpoint of

$$[q_1, q_2] \quad \text{is} \quad \gamma \equiv \frac{q_1 + q_2}{2}.$$

Now, the voters below q_1 will certainly vote for candidate I because q_2 is farther away. The voters above q_2 will vote for candidate II. The voters with positions in the interval $[q_1, \gamma]$ are closer to q_1 than to q_2, and so they vote for candidate I. Consequently, candidate I receives the total percentage of votes $Prob(V \leq \gamma = (q_1 + q_2)/2)$ if $q_1 < q_2$. Similarly, if $q_2 < q_1$, candidate I receives the percentage of votes $Prob(V > \gamma)$. Finally, if $q_1 = q_2$, there is no distinction between the candidates' positions and they evenly split the vote. Recall from elementary probability theory (see Appendix B) that the probabilities for a given density are given by (see Figure 4.1)

$$Prob(V \leq v) = \int_0^v f(x)\, dx \text{ and } Prob(V > v) = 1 - Prob(V \leq v).$$

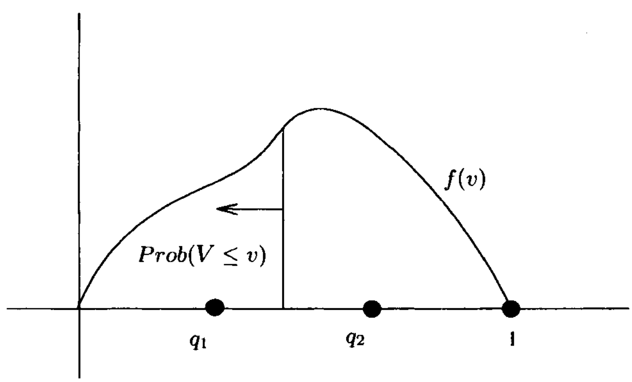


Figure 4.1 Area to the left of v is candidate I's percentage of the vote.

This is a problem with a discontinuous payoff pair, and we cannot simply take derivatives and set them to zero to find the equilibrium. But we can take an educated guess as to what the equilibrium should be by the following reasoning. Suppose that I takes a position to the left of II, $q_1 < q_2$. Then she receives $Prob(V \leq \gamma)$ percent of the vote. Because this is the cumulative distribution function of V, we know that it increases as γ (which is the midpoint of q_1 and q_2) increases. Therefore, player I wants γ as large as possible. Once q_1 increases past q_2, then candidate II starts to gain because $u_1(q_1, q_2) = 1 - Prob(V \leq \gamma)$

if $q_1 > q_2$. It seems that player I should not go further to the right than q_2, but should equal q_2. In addition, we should have

$$Prob(V \leq \gamma) = Prob(V \geq \gamma) = 1 - Prob(V \leq \gamma) \implies Prob(V \leq \gamma) = \frac{1}{2}.$$

In probability theory we know that this defines γ as the **median** of the random variable V. Therefore, the equilibrium γ should be the solution of

$$F_V(\gamma) \equiv P(V \leq \gamma) = \int_0^\gamma f(x)\,dx = \frac{1}{2}.$$

Because $F'_V(\gamma) = f(\gamma) > 0$, $F_V(\gamma)$ is strictly increasing, and there can only be one such $\gamma = \gamma^*$ that solves the equation; that is, a random variable has only one median when the density is strictly positive.

On the basis of this reasoning we now verify that

If γ^* is the median of the voter positions

then

(γ^*, γ^*) is a Nash equilibrium for the candidates.

If this is the case, then $u_i(\gamma^*, \gamma^*) = \frac{1}{2}$ for each candidate and both candidates split the vote.

How do we check this? We need to verify directly from the definition of equilibrium that

$$u_1(\gamma^*, \gamma^*) = \frac{1}{2} \geq u_1(q_1, \gamma^*) \text{ and } u_2(\gamma^*, \gamma^*) = \frac{1}{2} \geq u_2(\gamma^*, q_2)$$

for every $q_1, q_2 \in [0, 1]$. If we assume $q_1 > \gamma^*$, then

$$u_1(q_1, \gamma^*) = P\left(V \geq \frac{q_1 + \gamma^*}{2}\right) \leq P\left(V \geq \frac{\gamma^* + \gamma^*}{2}\right) = P(V \geq \gamma^*) = \frac{1}{2}.$$

If, on the other hand, $q_1 < \gamma^*$, then

$$u_1(q_1, \gamma^*) = P\left(V \leq \frac{q_1 + \gamma^*}{2}\right) \leq P\left(V \geq \frac{\gamma^* + \gamma^*}{2}\right) = P(V \geq \gamma^*) = \frac{1}{2}.$$

and we are done.

In the special case V has a uniform distribution on $[0, 1]$, so that $Prob(V \leq v) = v, 0 \leq v \leq 1$, we have $\gamma^* = \frac{1}{2}$. In that case, each candidate should be in the center. That will also be true if voter positions follow a bell curve (or, more generally, is symmetric). It seems reasonable that in the United States if we account for all regions of the country, national candidates should be in the center. Naturally, that is where the winner usually is, but not always. For instance, when James Earl Carter was president, the country swung to the right, and so did γ^*, with Ronald Reagan elected as president in 1980. Carter should have moved to the right.

It is not true that every collection of payoff functions will have a Nash equilibrium, but there is at least one result guaranteeing that one exists. It should remind you of von Neumann's minimax theorem, but it is more general than that because it doesn't have to be zero sum. It also generalizes our previous statement of Nash's theorem for bimatrix games.

Theorem 4.1.2 *Let $Q_1 \subset \mathbb{R}^n$ and $Q_2 \subset \mathbb{R}^m$ be compact and convex sets. Suppose that the payoff functions $u_i : Q_1 \times Q_2 \to \mathbb{R}^n \times \mathbb{R}^m$, $i = 1, 2$, satisfy*

1. *u_1 and u_2 are continuous.*
2. *$q_1 \mapsto u_1(q_1, q_2)$ is concave for each fixed q_2.*
3. *$q_2 \mapsto u_2(q_1, q_2)$ is concave for each fixed q_1.*

Then, there is a Nash equilibrium for (u_1, u_2).

We will not prove this theorem, which may be generalized to N payoff functions, but it again uses a fixed-point theorem. It indicates that if we do not have convexity of the strategy sets and concavity of the payoff functions, we may not have an equilibrium. It also says more than that. It gives us a way to set up mixed strategies for games that do not have pure Nash equilibria; namely, we have to convexify the strategy sets, and then make the payoffs concave in each variable. We do that by using **continuous probability distributions** instead of the discrete ones corresponding to mixed strategies for matrix games. That is a topic for a more advanced course (and, in addition, computing them is next to impossible), and so we skip it here. Later we will present an example in a silent duel to give a flavor of what needs to be done.

Getting back to the similarities with von Neumann's minimax theorem, von Neumann's theorem says roughly that a function $f(x, y)$ that is concave in x and convex in y will have a saddle point. The connection with the Nash theorem is made by noticing that $f(x, y)$ is the payoff for player I and $-f(x, y)$ is the payoff for player II, and so if $y \mapsto f(x, y)$ is convex, then $y \mapsto -f(x, y)$ is concave. So Nash's result is a true generalization of the von Neumann minimax theorem. The Nash theorem is only a sufficient condition, not a necessary condition for a Nash equilibrium. In fact, many of the conditions in the theorem may be weakened, but we will not go into that here.

4.2 ECONOMICS APPLICATIONS OF NASH EQUILIBRIA

The majority of game theorists are economists (or some combination of economics and mathematics), and in recent years many winners of the Nobel Prize in economics have been game theorists (Harsanyi, Selten, Aumann, Schelling, Myerson and, of course, Nash, to mention a only a few. Nash, however, would probably consider himself primarily a pure mathematician). In this section we will discuss some of the basic applications of game theory to economics problems.

Cournot Duopoly. Cournot[1] developed one of the earliest economic models of the competition between two firms. Suppose that there are two companies producing the same gadget. Firm $i = 1,2$ chooses to produce the quantity $q_i \geq 0$, so the total quantity produced by both companies is $q = q_1 + q_2$.

We assume in this simplified model that the price of a gadget is a decreasing function of the total quantity produced by the two firms. Let's take it to be

$$P(q) = (\Gamma - q)^+ = \begin{cases} \Gamma - q & \text{if } 0 \leq q \leq \Gamma; \\ 0 & \text{if } q > \Gamma. \end{cases}$$

Γ represents the price of gadgets beyond which the quantity to produce is essentially zero, and the price a consumer is willing to pay for a gadget if there are no gadgets on the market. Suppose also that to make one gadget costs firm $i = 1, 2$, c_i dollars per unit so the total cost to produce q_i units is $c_i q_i, i = 1, 2$. Assume that

$$\Gamma > c_1 + c_2,$$

because if $\Gamma < c_1 + c_2$, the costs to produce gadgets by the two firms exceeds the price a consumer would pay if there were no gadgets on the market.

The total quantity of gadgets produced by the two firms together is $q_1 + q_2$, so that the revenue to firm i for producing q_1 units of the gadget is $q_i P(q_1 + q_2)$. The cost of production to firm i is $c_i q_i$. Each firm wants to maximize its own profit function, which is total revenue minus total costs and is given by

$$u_1(q_1, q_2) = P(q_1 + q_2)q_1 - c_1 q_1 \text{ and } u_2(q_1, q_2) = P(q_1 + q_2)q_2 - c_2 q_2. \quad (4.2.1)$$

Let's begin by taking the partials and setting to zero. We assume that the optimal production quantities are in the interval $(0, \Gamma)$:

$$\begin{aligned} \frac{\partial u_1(q_1, q_2)}{\partial q_1} = 0 \quad &\Longrightarrow -2q_1 - q_2 + \Gamma - c_1 = 0, \\ \frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0 \quad &\Longrightarrow -2q_2 - q_1 + \Gamma - c_2 = 0. \end{aligned}$$

Notice that we take the partial of u_i with respect to q_i, not the partial of each payoff function with respect to both variables. We are not trying to maximize each profit

[1] Antoine Augustin Cournot (August 28, 1801–March 31, 1877) was a French philosopher and mathematician. One of his students was Auguste Walras, who was the father of Leon Walras. Cournot and Auguste Walras convinced Leon Walras to give economics a try. Walras then came up with his famous equilibrium theory in economics.

function over both variables, but each profit function to each firm as a function only of the variable they control, namely, q_i. That is a Nash equilibrium. Now solving the resulting equations gives the optimal production quantities for each firm at

$$q_1^* = \frac{\Gamma + c_2 - 2c_1}{3} \text{ and } q_2^* = \frac{\Gamma + c_1 - 2c_2}{3}.$$

If $\Gamma > c_1 + c_2$ then both $\Gamma > q_1^* > 0$ and $\Gamma > q_2^* > 0$. At these points we have

$$\frac{\partial^2 u_1(q_1^*, q_2^*)}{\partial^2 q_1} = -2 < 0 \text{ and } \frac{\partial^2 u_2(q_1^*, q_2^*)}{\partial q_2^2} = -2 < 0,$$

and so (q_1^*, q_2^*) are values that maximize the profit functions, when the other variable is fixed. The total amount the two firms should produce is

$$q^* = q_1^* + q_2^* = \frac{2\Gamma - c_1 - c_2}{3}$$

and $\Gamma > q^* > 0$ if $\Gamma > c_1 + c_2$. We see that our assumption about where the optimal production quantities would be found was correct.

The price function at the quantity q^* is then

$$P(q_1^* + q_2^*) = \Gamma - q_1^* - q_2^* = \Gamma - \frac{2\Gamma - c_1 - c_2}{3} = \frac{\Gamma + c_1 + c_2}{3}.$$

That is the market price of the gadgets produced by both firms when producing optimally.

Turn it around now and suppose that the price of gadgets is set at

$$P(q_1 + q_2) = p = \frac{\Gamma + c_1 + c_2}{3}.$$

If this is the market price of gadgets how many gadgets should each firm produce? The total quantity that both firms should produce (and will be sold) at this price is $q = P^{-1}(p)$, or

$$\begin{aligned} q = P^{-1}(p) = \Gamma - p = \frac{2\Gamma - c_1 - c_2}{3} \\ = \frac{\Gamma + c_2 - 2c_1}{3} + \frac{\Gamma + c_1 - 2c_2}{3} = q_1^* + q_2^*. \end{aligned}$$

We conclude that the quantity of gadgets sold (demanded) will be exactly the total amount that each firm **should** produce at this price. This is called a **market equilibrium** and it turns out to be given by the Nash point equilibrium quantity to produce. In other words, in economics, a market equilibrium exists when the quantity demanded at a price p is $q_1^* + q_2^*$ and the firms will optimally produce the quantities q_1^*, q_2^* at price p. That is exactly what happens when we use a Nash equilibrium to determine q_1^*, q_2^*.

Finally, substituting the Nash equilibrium point into the profit functions gives the equilibrium profits

$$u_1(q_1^*, q_2^*) = \frac{(\Gamma + c_2 - 2c_1)^2}{9} \quad \text{and} \quad u_2(q_1^*, q_2^*) = \frac{(\Gamma + c_1 - 2c_2)^2}{9}.$$

Notice that the profit of each firm depends on the costs of the other firm. That's a problem because how is a firm supposed to know the costs of a competing firm? The costs can be estimated, but known for sure...? This example is only a first-cut approximation, and we will have a lot more to say about this later.

A Slight Generalization of Cournot. Suppose that price is a function of the demand, which is a function of the total supply q of gadgets, so $P = D(q)$ and D is the demand function. We assume that if q gadgets are made, they will be sold at price $P = D(q)$. Suppose also that $C(z)$ is the cost to the two firms if z units of the gadget are produced. Again, each firm is trying to maximize its own profit, and we want to know how many gadgets each firm should produce. Another famous economist (A. Wald[2]) solved this problem (as presented in Reference [21]). Figure 4.2 shows the relationship of of the demand and cost functions.

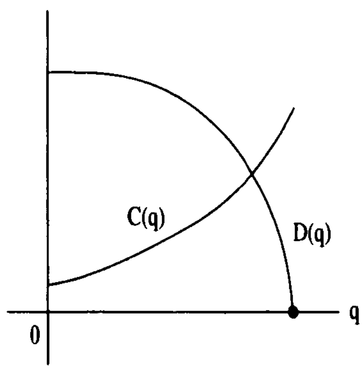


Figure 4.2 Demand and cost functions.

The next theorem generalizes the Cournot duopoly example to cases in which the price function and the cost functions are not necessarily linear. But we have to assume that they have the **same** cost function.

[2]October 31, 1902–December 13, 1950. Wald was a mathematician born in Hungary who made fundamental contributions to statistics, probability, differential geometry, and econometrics. He was a professor at Columbia University at the time of his death in a plane crash in India.

Theorem 4.2.1 *Suppose that $P = D(q)$ has two continuous derivatives, is nonincreasing, and is concave in the interval $0 \le q \le \Gamma$, and suppose that*

$$D(0) > 0 \text{ and } D(q) = 0, q \ge \Gamma.$$

So there is a positive demand if there are no gadgets, but the demand (or price) shrinks to zero if too many gadgets are on the market. Also, $P = D(q)$, the price per gadget, decreases as the total available quantity of gadgets increases. Suppose that firm $i = 1, 2$ has $M_i \ge \Gamma$ gadgets available for sale.

Suppose that the cost function C has two continuous derivatives, is strictly increasing, nonnegative, and convex, and that $C'(0) < D(0)$. The payoff functions are again the total profit to each firm:

$$u_1(q_1, q_2) = q_1 D(q_1 + q_2) - C(q_1) \text{ and } u_2(q_1, q_2) = q_2 D(q_1 + q_2) - C(q_2).$$

Then, there is one and only one Nash equilibrium given by (q^*, q^*), where $q^* \in [0, \Gamma]$ is the unique solution of the equation

$$D(2q) + qD'(2q) - C'(q) = 0 \text{ in the interval } 0 < q < \frac{\Gamma}{2}.$$

Under our assumptions, both firms produce exactly the same amount $q^* \in (0, \frac{\Gamma}{2})$ and each receives the same profit $u_1(q^*, q^*) = u_2(q^*, q^*)$. If they had differing cost functions, this would not be true.

Sketch of the Proof. By the assumptions we put on D and C, we may apply the theorem that guarantees that there is a Nash equilibrium to know that we are looking for something that exists. Call it (q_1^*, q_2^*). We assume that this will happen with $0 < q_1^* + q_2^* < \Gamma$. By taking the partial derivatives and setting equal to zero, we see that

$$\frac{\partial u_1(q_1^*, q_2^*)}{\partial q_1} = D(q_1^* + q_2^*) + q_1^* D'(q_1^* + q_2^*) - C'(q_1^*) = 0$$

and

$$\frac{\partial u_2(q_1^*, q_2^*)}{\partial q_2} = D(q_1^* + q_2^*) + q_2^* D'(q_1^* + q_2^*) - C'(q_2^*) = 0.$$

We solve these equations by subtracting to get

$$(q_1^* - q_2^*)D'(q_1^* + q_2^*) - (C'(q_1^*) - C'(q_2^*)) = 0. \tag{4.2.2}$$

Remember that $C''(q) \ge 0$ (so C' is increasing) and $D' < 0$. This means that if $q_1^* < q_2^*$, then we have the sum of two positive quantities in (4.2.2) adding to zero, which is impossible. So, it must be true that $q_1^* \ge q_2^*$. However, by a similar argument, strict inequality would be impossible, and so we conclude that $q_1^* = q_2^* \equiv q^*$. Actually, this should be obvious because the firms are symmetric and have the same costs.

So now we have

$$D(2q^*) + q^* D'(q^*) - C'(q^*) = 0 \text{ and } 0 < q^* < \frac{\Gamma}{2}.$$

In addition, it is not too difficult to show that q^* is the unique root of this equation (and therefore the only stationary point). That is the place where the assumption that $C'(0) < D(0)$ is used. Finally, by taking second derivatives, we see that

$$\frac{\partial^2 u_i(q_1^*, q_2^*)}{\partial q_i^2} = 3D'(2q^*) + q^* D''(2q^*) - C''(q^*) < 0, \quad i = 1, 2,$$

for each payoff function, and hence the unique root of the equation is indeed a Nash equilibrium. □

■ EXAMPLE 4.3

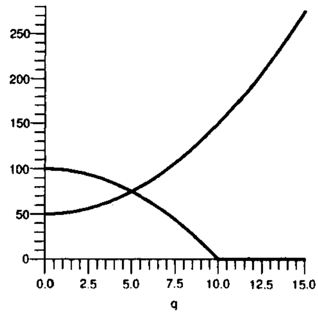
Let's take $D(q) = 100 - q^2, 0 \leq q \leq 10$, and cost function $C(q) = 50 + q^2$. Then $C'(0) = 0 < D(0) = 100$, and the functions satisfy the appropriate convexity assumptions. Then q is the unique solution of

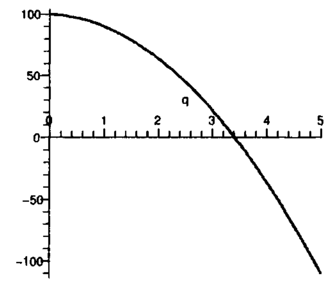
$$D(2q) + qD'(2q) - C'(q) = 100 - 8q^2 - 2q = 0$$

in the interval $0 < q < 10$. The unique solution is given by $q^* = 3.413$. The unit price at this quantity should be $D(2q^*) = 88.353$, and the profit to each firm will be $u_i(q^*, q^*) = q^* D(2q^*) - C(q^*) = 120.637$. These numbers, as well as some pictures, can be obtained using the Maple commands

```
> restart:
> De:=q->piecewise(q<10,100-q^2,q>=10,0);
> C:=q->50+q^2;
> plot({De(q),C(q)},q=0..15,color=[red,green]);
> diff(De(q),q);
> a:=eval( %, [q = 2*q] );
> qstar:=fsolve(De(2*q)+q*a-diff(C(q),q)=0,q);
> plot(De(2*q)+q*a-diff(C(q),q),q=0..5);
> De(qstar);
> qstar*De(2*qstar)-C(qstar);
```

This will give you a plot of the demand function and cost function on the same set of axes, and then a plot of the function giving you the root where it crosses the q axis. You may modify the demand and cost functions to solve the exercises. Here are the Maple generated figures for our problem.





Cournot model with Uncertain Costs. Now here is a generalization of the Cournot model that is more realistic and also more difficult to solve because it involves a lack of information on the part of at least one player. It is assumed that one firm has no information regarding the other firm's cost function. Here is the model setup.

Assume that both firms produce gadgets at constant unit cost. Both firms know that firm 1's cost is c_1, but firm 1 does not know firm 2's cost of c_2, which is known only to firm 2. Suppose that the cost for firm 2 is considered as a random variable to firm 1, say, C_2. Now firm 1 has reason to believe that

$$Prob(C_2 = c^+) = p \text{ and } Prob(C_2 = c^-) = 1 - p$$

for some $0 < p < 1$ that is known by firm 1.

Again, the payoffs to each firm are its profits. For firm 1 we have the payoff function, assuming that firm 1 makes q_1 gadgets and firm 2 makes q_2 gadgets

$$u_1(q_1, q_2) = q_1[P(q_1 + q_2) - c_1],$$

where $P(q_1 + q_2)$ is the market price for the total production of $q_1 + q_2$ gadgets. Firm 2's payoff function is

$$u_2(q_1, q_2) = q_2[P(q_1 + q_2) - C_2].$$

From firm 1's perspective this is a random variable because of the unknown cost. The way to find an equilibrium now is the following:

1. Find the optimal production level for firm 2 using the costs c^+ and c^- giving the two numbers q_2^+ and q_2^-.

2. Firm 1 now finds the optimal production levels for the two firm 2 quantities from step 1 using the expected payoff

$$\begin{aligned} E(u_1(q_1, q_2(C_2))) &= [u_1(q_1, q_2^-)]Prob(C_2 = c^-) \\ &\quad + [u_1(q_1, q_2^+)]Prob(C_2 = c^+) \\ &= q_1[P(q_1 + q_2^-) - c_1](1 - p) + q_1[P(q_1 + q_2^+) - c_1]p. \end{aligned}$$

3. From the previous two steps you end up with three equations involving q_1, q_2^+, q_2^-. Treat these as three equations in three unknowns and solve.

For example, let's take the price function $P(q) = \Gamma - q, 0 \le q \le \Gamma$.

1. Firm 2 has the cost of production $c^+ q_2$ with probability p and the cost of production $c^- q_2$ with probability $1-p$. Firm 2 will solve the problem for each cost c^+, c^- assuming that q_1 is known:

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^+) \implies q_2^+ = \frac{1}{2}(\Gamma - q_1 - c^+)$$

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^-) \implies q_2^- = \frac{1}{2}(\Gamma - q_1 - c^-)$$

2. Next, firm 1 will maximize the expected profit using the two quantities q_2^+, q_2^-. Firm 1 seeks the production quantity q_1, which solves

$$\max_{q_1} q_1[\Gamma - (q_1 + q_2^+) - c_1]p + q_1[\Gamma - (q_1 + q_2^-) - c_1](1-p).$$

This is maximized at

$$q_1 = \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1-p)(\Gamma - q_2^- - c_1)].$$

3. Summarizing, we now have the following system of equations for the variables q_1, q_2^-, q_2^+:

$$\begin{aligned}
q_2^+ &= \frac{1}{2}(\Gamma - q_1 - c^+),\\
q_2^- &= \frac{1}{2}(\Gamma - q_1 - c^-),\\
q_1 &= \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1-p)(\Gamma - q_2^- - c_1)].
\end{aligned}$$

Solving these, we finally arrive at the optimal production levels:

$$\begin{aligned}
q_1^* &= \frac{1}{3}[\Gamma - 2c_1 + pc^+ + (1-p)c^-],\\
q_2^{+*} &= \frac{1}{3}[\Gamma + c_1] - \frac{1}{6}[(1-p)c^- + pc^+] - \frac{1}{2}c^+,\\
q_2^{-*} &= \frac{1}{3}[\Gamma - 2c^- + c_1] + \frac{1}{6}p(c^- - c^+).
\end{aligned}$$

Notice that if we require that the production levels be nonnegative, we need to put some conditions on the costs and Γ.

The Bertrand Model. Joseph Bertrand[3] didn't like Cournot's model. He thought firms should set prices to accommodate demand. Here is the setup. We again have two companies making identical gadgets. In this model they can set prices, not quantities, and they will only produce the quantity demanded at the given price. So the quantity sold is a function of the price set by each firm, say, $q = \Gamma - p$. This is better referred to as the **demand function** for a given price:

$$D(p) = \Gamma - p, \quad 0 \le p \le \Gamma \text{ and } D(p) = 0 \text{ when } p > \Gamma.$$

In a classic problem the model says that if both firms charge the same price, they will split the market evenly, with each selling exactly half of the total sold. But the company that charges a lower price will capture the entire market. We have to assume that each company has enough capacity to make the entire quantity if it captures the whole market. The cost to make gadgets is still $c_i,\ i = 1, 2$, dollars per unit gadget. We first assume:

$$c_1 \neq c_2 \text{ and } \max\{c_1, c_2\} < \Gamma + \min\{c_1, c_2\}.$$

The profit function for firm $i = 1, 2$, assuming that firm 1 sets the price as p_1 and firm 2 sets the price at p_2, is

$$u_1(p_1, p_2) = \begin{cases} p_1(\Gamma - p_1) - c_1(\Gamma - p_1) & \text{if } p_1 < p_2; \\ \dfrac{(p - c_1)(\Gamma - p)}{2} & \text{if } p_1 = p_2 = p \ge c_1; \\ 0, & \text{if } p_1 > p_2. \end{cases}$$

This says that if firm 1 sets the price lower than firm 2, firm 1's profit will be (price − cost) × quantity demanded; if the prices are the same, firm 1's profits will be $(\frac{1}{2})$(price − cost) × quantity demanded; and zero if firm 1's price of a gadget is greater than firm 2's. This assumes that the **lower price captures the entire market.** Similarly, firm 2's profit function is

$$u_2(p_1, p_2) = \begin{cases} p_2(\Gamma - p_2) - c_2(\Gamma - p_2) & \text{if } p_2 < p_1; \\ \dfrac{(p - c_2)(\Gamma - p)}{2} & \text{if } p_1 = p_2 = p \ge c_2; \\ 0 & \text{if } p_2 > p_1. \end{cases}$$

Figure 4.3 is a Maple generated plot of $u_1(p_1, p_2)$ with $\Gamma = 10, c_1 = 3$.

So now we have to find a Nash equilibrium. Let's suppose that there is a Nash equilibrium point at (p_1^*, p_2^*). By definition, we have

$$u_1(p_1^*, p_2^*) \ge u_1(p_1, p_2^*) \text{ and } u_2(p_1^*, p_2^*) \ge u_2(p_1^*, p_2), \text{ for all } (p_1, p_2).$$

[3]Joseph Louis François Bertrand, March 11, 1822–April 5, 1900, a French mathematician who contributed to number theory, differential geometry, probability theory, thermodynamics, and economics. He reviewed the Cournot competition model, arguing that Cournot had reached a misleading conclusion.

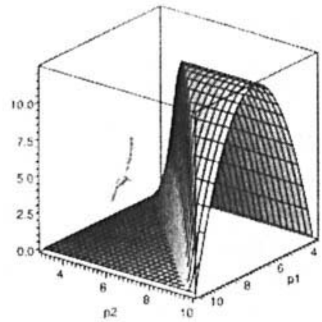


Figure 4.3 Discontinuous payoff function for firm 1.

Let's break this down by considering three cases:

Case 1. $p_1^* > p_2^*$. Then it should be true that firm 2, having a lower price, captures the entire market so that for firm 1

$$u_1(p_1^*, p_2^*) = 0 \geq u_1(p_1, p_2^*), \text{ for every } p_1.$$

But if we take any price $c_1 < p_1 < p_2^*$, the right side will be

$$u_1(p_1, p_2^*) = (p_1 - c_1)(\Gamma - p_1) > 0,$$

so $p_1^* > p_2^*$ cannot hold and still have (p_1^*, p_2^*) a Nash equilibrium.

Case 2. $p_1^* < p_2^*$. Then it should be true that firm 1 captures the entire market and so for firm 2

$$u_2(p_1^*, p_2^*) = 0 \geq u_2(p_1^*, p_2), \text{ for every } p_2.$$

But if we take any price for firm 2 with $c_2 < p_2 < p_1^*$, the right side will be

$$u_2(p_1^*, p_2) = (p_2 - c_2)(\Gamma - p_2) > 0,$$

that is, strictly positive, and again it cannot be that $p_1^* < p_2^*$ and fulfill the requirements of a Nash equilibrium. So the only case left is the following.

Case 3. $p_1^* = p_2^*$. But then the two firms split the market and we must have for firm 1

$$u_1(p_1^*, p_2^*) = \frac{(p_1^* - c_1)(\Gamma - p_1^*)}{2} \geq u_1(p_1, p_2^*), \text{ for all } p_1 \geq c_1.$$

If we take firm 1's price to be $p_1 = p_1^* - \varepsilon < p_2^*$ with really small $\varepsilon > 0$, then firm 1 drops the price ever so slightly below firm 2's price. Under the Bertrand model, firm 1 will capture the entire market at price p_1 so that in this case we have

$$u_1(p_1^*, p_2^*) = \frac{(p_1^* - c_1)(\Gamma - p_1^*)}{2} < u_1(p_1^* - \varepsilon, p_2^*) = (p_1^* - \varepsilon - c_1)(\Gamma - p_1^* + \varepsilon).$$

This inequality won't be true for every ε, but it will be true for small enough $\varepsilon > 0$, (say, $0 < \varepsilon < (p_1^* - c_1)/2$).

So, in all cases, we can find prices so that the condition that (p_1^*, p_2^*) be a Nash point is violated and so there is no Nash equilibrium in pure strategies. But there is one case when there is a Nash equilibrium. In the analysis above we assumed in several places that prices would have to be above costs. What if we drop that assumption? The first thing that happens is that in case 3 we won't be able to find a positive ε to drop p_1^*.

What's the problem here? It is that neither player has a continuous profit function (as you can see in Figure 4.3). By lowering the price just below the competitor's price, the firm with the lower price can capture the entire market. So the incentive is to continue lowering the price of a gadget. In fact, we are led to believe that maybe $p_1^* = c_1, p_2^* = c_2$ is a Nash equilibrium. Let's check that, and let's just assume that $c_1 < c_2$, because a similar argument would apply if $c_1 > c_2$.

In this case, $u_1(c_1, c_2) = 0$, and if this is a Nash equilibrium, then it must be true that $u_1(c_1, c_2) = 0 \geq u_1(p_1, c_2)$ for all p_1. But if we take any price $c_1 < p_1 < c_2$, then

$$u_1(p_1, c_2) = (p_1 - c_1)(\Gamma - p_1) > 0,$$

and we conclude that (c_1, c_2) also is not a Nash equilibrium. The only way that this could work is if $c_1 = c_2 = c$, so the costs to each firm are the same. In this case we leave it as an exercise to show that $p_1^* = c, p_2^* = c$ is a Nash equilibrium and optimal profits are zero for each firm. So, what good is this if the firms make no money, and even that is true only when their costs are the same? This leads us to examine assumptions about exactly how profits arise in competing firms. Is it strictly prices and costs, or are there other factors involved?

Here is another example where a pure Nash equilibrium exists but seems to be completely unrealistic. It is connected to the Bertrand model in that it also shows that when there is an incentive to drop to the lowest price, it leads to unrealistic expectations.

■ **EXAMPLE 4.4**

The Traveler's Paradox.[4] Two airline passengers who have luggage with identical contents are informed by the airline that their luggage has been lost.

[4] Adapted from an example in Reference [5].

The airline offers to compensate them if they make a claim in some range acceptable to the airline. Here are the payoff functions for each player

$$u_1(q_1, q_2) = \begin{cases} q_1 & \text{if } q_1 = q_2; \\ q_1 + R & \text{if } q_2 > q_1; \\ q_2 - R & \text{if } q_1 > q_2. \end{cases} \quad \text{and} \quad u_2(q_1, q_2) = \begin{cases} q_2 & \text{if } q_1 = q_2; \\ q_1 - R & \text{if } q_2 > q_1; \\ q_2 + R & \text{if } q_1 > q_2. \end{cases}$$

It is assumed that the acceptable range is $[a, b]$ and $q_i \in [a, b], i = 1, 2$. The idea behind these payoffs is that if the passengers' claims are equal the airline will pay the amount claimed. If passenger I claims less than passenger II, $q_1 < q_2$, then passenger II will be penalized an amount R and passenger I will receive the amount she claimed plus R. Passenger II will receive the lower amount claimed minus R. Similarly, if passenger I claims more than does passenger II, $q_1 > q_2$, then passenger I will receive $q_2 - R$, and passenger II will receive the amount claimed q_2 plus R.

Suppose, to be specific, that $a = 80, b = 200$, so the airline acceptable range is from \$80 to \$200. We take $R = 2$, so the penalty is only 2 dollars for claiming high. Believe it or not, we will show that the Nash equilibrium is $(q_1 = 80, q_2 = 80)$ so both players should claim the low end of the airline range (under the Nash equilibrium concept). To see that, we have to show that

$$u_1(80, 80) = 80 \geq u_1(q_1, 80) \quad \text{and} \quad u_2(80, 80) = 80 \geq u_2(80, q_2)$$

for all $80 \leq q_1, q_2 \leq 200$. Now

$$u_1(q_1, 80) = \begin{cases} 80 & \text{if } q_1 = 80; \\ 80 - 2 = 78 & \text{if } q_1 > 80. \end{cases}$$

and

$$u_2(80, q_2) = \begin{cases} 80 & \text{if } q_2 = 80; \\ 80 - 2 = 78 & \text{if } q_2 > 80; \end{cases}$$

So indeed $(80, 80)$ is a Nash equilibrium with payoff 80 to each passenger. But clearly, they can do better if they both claim $q_1 = q_2 = \$200$. Why don't they do that? The problem is that there is an incentive to undercut the other traveler. If R is \$2, then, if one of the passengers drops her claim to \$199, this passenger will actually receive \$201. This cascades downward, and the undercutting disappears only at the lowest range of the acceptable claims. Do you think that the passengers would, in reality, make the lowest claim?

The Stackelberg Model. What happens if two competing firms don't choose the production quantities at the same time, but one after the other? Stackelberg[5] gave an answer to this question. In this model we will assume that there is a dominant

[5]Heinrich Freiherr von Stackelberg (1905–1946) was a German economist who contributed to game theory and oligopoly theory.

firm, say, firm 1, who will announce its production quantity publicly. Then firm 2 will decide how much to produce. In other words, given that one firm knows the production quantity of the other, determine how much each will or should produce.

Suppose that firm 1 announces that it will produce q_1 gadgets at cost c_1 dollars per unit. It is then up to firm 2 to decide how many gadgets, say, q_2 at cost c_2, it will produce. We again assume that the unit costs are constant. The price per unit will then be considered a function of the total quantity produced so that $p = p(q_1, q_2) = (\Gamma - q_1 - q_2)^+ = \max\{\Gamma - q_1 - q_2, 0\}$. The profit functions will be

$$u_1(q_1, q_2) = (\Gamma - q_1 - q_2)q_1 - c_1 q_1,$$
$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - c_2 q_2.$$

These are the same as in the simplest Cournot model, but now q_1 is fixed as given. It is not variable when firm 1 announces it. So what we are really looking for is the best response of firm 2 to the production announcement q_1 by firm 1. In other words, firm 2 wants to know how to choose $q_2 = q_2(q_1)$ so as to

Maximize over q_2, given q_1, the function $u_2(q_1, q_2(q_1))$.

This is given by calculus as

$$q_2(q_1) = \frac{\Gamma - q_1 - c_2}{2}.$$

This is the amount that firm 2 should produce when firm 1 announces the quantity of production q_1.

Now, firm 1 has some clever employees who know calculus and game theory and can perform this calculation as well as we can. Firm 1 knows what firm 2's optimal production quantity should be, given its own announcement of q_1. Therefore, firm 1 should choose q_1 to maximize its own profit function knowing that firm 2 will use production quantity $q_2(q_1)$:

$$\begin{aligned} u_1(q_1, q_2(q_1)) &= q_1(\Gamma - q_1 - q_2(q_1)) - c_1 q_1 \\ &= q_1 \left(\Gamma - q_1 - \frac{\Gamma - q_1 - c_2}{2} \right) - c_1 q_1 \\ &= q_1 \frac{\Gamma - q_1}{2} + q_1 \left(\frac{c_2}{2} - c_1 \right). \end{aligned}$$

Firm 1 wants to choose q_1 to make this as large as possible. By calculus, we find that

$$q_1^* = \frac{\Gamma - 2c_1 + c_2}{2}, \quad \text{and} \quad u_1(q_1^*, q_2^*) = \frac{(\Gamma - 2c_1 + c_2)^2}{8}.$$

Then the optimal production quantity for firm 2 will be

$$q_2^* = q_2(q_1^*) = \frac{\Gamma + 2c_1 - 3c_2}{4}.$$

The equilibrium profit function for firm 2 is then

$$u_2(q_1^*, q_2^*) = \frac{(\Gamma + 2c_1 - 3c_2)^2}{16},$$

and for firm 1, it is

$$u_1(q_1^*, q_2^*) = \left(\frac{\Gamma - 2c_1 + c_2}{2}\right)^2.$$

For comparison, we will set $c_1 = c_2 = c$ and then recall the optimal production quantities for the Cournot model:

$$q_1^c = \frac{\Gamma - 2c_1 + c_2}{3} = \frac{\Gamma - c}{3}, \qquad q_2^c = \frac{\Gamma + c_1 - 2c_2}{3} = \frac{\Gamma - c}{3}.$$

The equilibrium profit functions were

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma + c_2 - 2c_1)^2}{9} = \frac{(\Gamma - c)^2}{9},$$
$$u_2(q_1^c, q_2^c) = \frac{(\Gamma + c_1 - 2c_2)^2}{9} = \frac{(\Gamma - c)^2}{9}.$$

In the Stackelberg model we have

$$q_1^* = \frac{\Gamma - c}{2} > q_1^c, \quad q_2^* = \frac{\Gamma - c}{4} < q_2^c.$$

So firm 1 produces more and firm 2 produces less in the Stackelberg model than if firm 2 did not have the information announced by firm 1. For the firm's profits, we have

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} < u_1(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{4},$$

$$u_2(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} > u_2(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{16}.$$

Firm 1 makes more money by announcing the production level, and firm 2 makes less with the information.

One last comparison is the total quantity produced

$$q_1^c + q_2^c = \frac{2}{3}(\Gamma - c) < q_1^* + q_2^* = \frac{3\Gamma - 2c_1 - c_2}{4} = \frac{3}{4}(\Gamma - c)$$

and the price at equilibrium (recall that $\Gamma > c$):

$$P(q_1^* + q_2^*) = \frac{\Gamma + 3c}{4} < P(q_1^c + q_2^c) = \frac{\Gamma + 2c}{3}.$$

Entry Deterrence. In this example we ask the following question. If there is currently only one firm producing a gadget, what should be the price of the gadget in order to make it unprofitable for another firm to enter the market and compete with firm 1? This is a famous problem in economics called the **entry deterrence problem**.

Of course, a monopolist may charge any price at all as long as there is a demand for gadgets at that price. But it should also be true that competition should lower prices, implying that the price to prevent entry by a competitor should be lower than what the firm would otherwise set.

We call the existing company firm 1 and the potential challenger firm 2. The demand function is $p = D(q) = (\Gamma - q)^+$.

Now, before the challenger enters the market the profit function to firm 1 is

$$u_1(q_1) = (\Gamma - q_1)q_1 - (aq_1 + b),$$

where we assume that the cost function $C(q) = aq + b$, with $\Gamma > a, b > 0$. This cost function includes a fixed cost of $b > 0$ because even if the firm produces nothing, it still has expenses.

Now firm 1 is acting as a monopolist in our model because it has no competition. So firm 1 wants to maximize profit, which gives a production quantity of

$$q_1^* = \frac{\Gamma - a}{2}$$

and maximum profit for a monopolist of

$$u_1(q_1^*) = \frac{(\Gamma - a)^2}{4} - b.$$

In addition, the price of a gadget at this quantity of production will be

$$p = D(q_1^*) = \frac{\Gamma + a}{2}.$$

Now firm 2 enters the picture and calculates firm 2's profit function knowing that firm 1 will or should produce $q_1^* = (\Gamma - a)/2$ to get firm 2's payoff function

$$u_2(q_2) = \left(\Gamma - \frac{\Gamma - a}{2} - q_2\right) q_2 - (aq_2 + b).$$

So firm 2 calculates its maximum possible profit and optimal production quantity as

$$u_2(q_2^*) = \frac{(\Gamma - a)^2}{16} - b, \quad q_2^* = \frac{\Gamma - a}{4}.$$

The price of gadgets will now drop (recall $\Gamma > a$) to

$$p = D(q_1^* + q_2^*) = \Gamma - q_1^* - q_2^* = \frac{\Gamma + 3a}{4} < D(q_1^*) = \frac{\Gamma + a}{2}.$$

As long as $u_2(q_2^*) \geq 0$, firm 2 has an incentive to enter the market. If we interpret the constant b as a fixed cost to enter the market, this will require that

$$\frac{(\Gamma - a)^2}{16} > b,$$

or else firm 2 cannot make a profit.

Now here is a more serious analysis because firm 1 is not about to sit by idly and let another firm enter the market. Therefore, firm 1 will now analyze the Cournot model assuming that there is a firm 2 against which firm 1 is competing. Firm 1 looks at the profit function for firm 2:

$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - (aq_2 + b),$$

and maximizes this as a function of q_2 to get

$$q_2^m = \frac{\Gamma - q_1 - a}{2} \text{ and } u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b$$

as the maximum profit to firm 2 if firm 1 produces q_1 gadgets. Firm 1 reasons that it can set q_1 so that firm 2's profit is zero:

$$u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b = 0 \implies q_1^0 = \Gamma - 2\sqrt{b} - a.$$

This gives a zero profit to firm 2. Consequently, if firm 1 decides to produce q_1^0 gadgets, firm 2 has no incentive to enter the market. The price at this quantity will be

$$D(q_1^0) = \Gamma - (\Gamma - 2\sqrt{b} - a) = 2\sqrt{b} + a,$$

and the profit for firm 1 at this level of production will be

$$u_1(q_1^0) = (\Gamma - q_1^0)q_1^0 - (aq_1^0 + b) = 2\sqrt{b}(\Gamma - a) - 5b.$$

This puts a requirement on Γ that $\Gamma > a + \frac{5}{2}\sqrt{b}$, or else firm 1 will also make a zero profit.

PROBLEMS

4.1 Suppose instead of two firms in the Cournot model with payoff functions (4.2.1), that there are N firms. Formulate this model and find the optimal quantities each of the N firms should produce. Instead of a duopoly, this is an oligopoly. What happens when the firms all have the same costs and $N \to \infty$?

4.2 Two firms produce identical products. The market price for total production quantity q is $P(q) = 100 - 2\sqrt{q}$. Firm 1's production cost is $C_1(q_1) = q_1 + 10$, and firm 2's production cost is $C_2(q_2) = 2q_2 + 5$. Find the profit functions and the Nash equilibrium quantities of production and profits.

4.3 Compare profits for firm 1 in the model with uncertain costs and the standard Cournot model. Assume $\Gamma = 15, c_1 = 4, c^+ = 5, c^- = 1$ and $p = 0.5$.

4.4 Suppose that we consider the Cournot model with uncertain costs but with three possible costs, $Prob(C_2 = c^i) = r_i$, $i = 1, 2, 3$, where $r_i \geq 0, r_1 + r_2 + r_3 = 1$. Solve for the optimal production quantities. Find the explicit production quantities when $r_1 = \frac{1}{2}, r_2 = \frac{1}{8}, r_3 = \frac{3}{8}, \Gamma = 100$, and $c_1 = 2, c^1 = 1, c^2 = 2, c^3 = 5$.

4.5 In the Stackelberg model compare the quantity produced, the profit, and the prices for firm 1 assuming that firm 2 did not exist so that firm 1 is a monopolist.

4.6 Suppose that two firms have constant unit costs $c_1 = 2, c_2 = 1$ and $\Gamma = 19$ in the Stackelberg model.

(a) What are the profit functions?

(b) How much should firm 2 produce as a function of q_1?

(c) How much should firm 1 produce? **(d)** How much, then, should firm 2 produce?

4.7 Set up and solve a Stackelberg model given three firms with constant unit costs c_1, c_2, c_3 and firm 1 announcing production quantity q_1.

4.8 In the Bertrand model show that if $c_1 = c_2 = c$, then $(p_1^*, p_2^*) = (c, c)$ is a Nash equilibrium.

4.9 Determine the entry deterrence level of production for firm 1 given $\Gamma = 100, a = 2, b = 10$. How much profit is lost by setting the price to deter a competitor?

4.10 We could make one more adjustment in the Bertrand model and see what effect it has on the model. What if we put a limit on the total quantity that a firm can produce? This limits the supply and possibly will put a floor on prices. Let $K \geq \frac{\Gamma}{2}$ denote the maximum quantity of gadgets that each firm can produce and recall that $D(p) = \Gamma - p$ is the quantity of gadgets demanded at price p. Find the profit functions for each firm.

4.11 Suppose that the demand functions in the Bertrand model are given by

$$q_1 = D_1(p_1, p_2) = (a - p_1 + bp_2)^+ \text{ and } q_2 = D_2(p_1, p_2) = (a - p_2 + bp_1)^+,$$

where $1 \geq b > 0$. This says that the quantity of gadgets sold by a firm will increase if the price set by the opposing firm is too high. Assume that both firms have a cost of production $c \leq \min\{p_1, p_2\}$.

(a) Show that the profit functions will be given by

$$u_i(p_1, p_2) = D_i(p_1, p_2)(p_i - c), \quad i = 1, 2.$$

(**b**) Using calculus, show that there is a unique Nash equilibrium at

$$p_1^* = p_2^* = \frac{a+c}{2-b}.$$

(**c**) Find the profit functions at equilibrium.

4.12 Suppose that firm 1 announces a price in the Bertrand model with demand functions

$$q_1 = D_1(p_1, p_2) = (a - p_1 + bp_2)^+ \text{ and } q_2 = D_2(p_1, p_2) = (a - p_2 + bp_1)^+,$$

where $1 \geq b > 0$.

(**a**) Construct the Stackelberg model; firm 2 should announce $p_2 = p_2(p_1)$, to maximize $u_2(p_1, p_2)$ and then firm 1 should choose p_1 to maximize $u_1(p_1, p_2(p_1))$.

(**b**) Find the equilibrium prices and profits.

4.3 DUELS (OPTIONAL)

Duels are used to model not only the actual dueling situation but also many problems in other fields. For example, a battle between two companies for control of a third company or asset can be regarded as a duel in which the accuracy functions could represent the probability of success. Duels can be used to model competitive auctions between two bidders. So there is ample motivation to study a theory of duels.

In earlier chapters we considered discrete versions of a duel in which the players were allowed to fire only at certain distances. In reality, a player can shoot at any distance (or time) once the duel begins. That was only one of our simplifications. The theory of duels includes multiple bullets, machine gun duels, silent and noisy, and so on.[6]

Here are the precise rules that we use here. There are two participants, I and II, each with a gun, and each has exactly one bullet. They will fire their guns at the opponent at a moment of their own choosing. The players each have functions representing their accuracy or probability of killing the opponent, say, $p_{\mathrm{I}}(x)$ for player I and $p_{\mathrm{II}}(y)$ for player II, with $x, y \in [0, 1]$. The choice of strategies is a **time** in $[0, 1]$ at which to shoot. Assume that $p_{\mathrm{I}}(0) = p_{\mathrm{II}}(0) = 0$ and $p_{\mathrm{I}}(1) = p_{\mathrm{II}}(1) = 1$. So, in the setup here you may assume that they are farthest apart at time 0 or $x = y = 0$ and closest together when $x = y = 1$. It is realistic to assume also that both p_{I} and p_{II} are continuous, are strictly increasing, and have continuous derivatives up to any order needed.

Finally, if I hits II, player I receives +1 and player II receives -1, and conversely. If both players miss, the payoff is 0 to both. The payoff functions will be the expected payoff depending on the accuracy functions and the choice of the $x \in [0, 1]$

[6]Refer to Karlin's book [11] for a very nice exposition of the general theory of duels.

or $y \in [0, 1]$ at which the player will take the shot. We could take more general payoffs to make the game asymmetric as we did in Example 3.13, for instance, but we will consider only the symmetric case with continuous strategies.

We break our problem down into the cases where player I shoots before player II, player II shoots before player I, or they shoot at the same time.

If player I shoots before player II, then $x < y$, and

$$\begin{aligned} u_1(x, y) &= (+1)p_{\mathrm{I}}(x) + (-1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(y), \\ u_2(x, y) &= (-1)p_{\mathrm{I}}(x) + (+1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(y). \end{aligned}$$

If player II shoots before player I, then $y < x$ and we have similar expected payoffs:

$$\begin{aligned} u_1(x, y) &= (-1)p_{\mathrm{II}}(y) + (+1)(1 - p_{\mathrm{II}}(y))p_{\mathrm{I}}(x), \\ u_2(x, y) &= (+1)p_{\mathrm{II}}(y) + (-1)(1 - p_{\mathrm{II}}(y))p_{\mathrm{I}}(x). \end{aligned}$$

Finally, if they choose to shoot at the same time, then $x = y$ and we have

$$u_1(x, x) = (+1)p_{\mathrm{I}}(x)(1 - p_{\mathrm{II}}(x)) + (-1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(x) = -u_2(x, x).$$

In this simplest setup, this is a zero sum game, but, as mentioned earlier, it is easily changed to nonzero sum.

We have set up this duel without consideration yet that the duel is noisy. Each player will hear (or see, or feel) the shot by the other player, so that if a player shoots and misses, the surviving player will know that all she has to do is wait until her accuracy reaches 1. With certainty that occurs at $x = y = 1$, and she will then take her shot. In a **silent** duel the players would not know that a shot was taken (unless they didn't survive). Silent duels are more difficult to analyze, and we will consider a special case later.

Let's simplify the payoffs in the case of a noisy duel. In that case, when a player takes a shot and misses, the other player (if she survives) waits until time 1 to kill the opponent with certainty. So, the payoffs become

$$u_1(x, y) = \begin{cases} (+1)p_{\mathrm{I}}(x) + (-1)(1 - p_{\mathrm{I}}(x)) = 2p_{\mathrm{I}}(x) - 1, & x < y; \\ p_{\mathrm{I}}(x) - p_{\mathrm{II}}(x), & x = y; \\ 1 - p_{\mathrm{II}}(y), & x > y. \end{cases}$$

For player II, $u_2(x, y) = -u_1(x, y)$.

Now, to solve this, we cannot use the procedure outlined using derivatives, because this function has no derivatives exactly at the places where the optimal things happen.

Figure 4.4 below shows a graph of $u_1(x, y)$ in the case when the players have the distinct accuracy functions given by $p_{\mathrm{I}}(x) = x^3$ and $p_{\mathrm{II}}(x) = x^2$. Player I's accuracy function increases at a slower rate than that for player II. Nevertheless, we will see that both players will fire at the same time. That conclusion seems reasonable when it is a noisy duel. If one player fires before the opponent, the accuracy suffers, and, if it is a miss, death is certain.

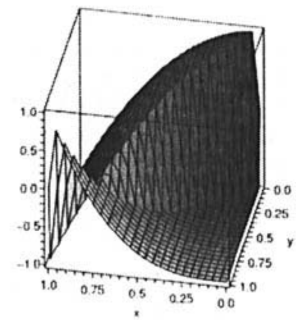


Figure 4.4 $u_1(x, y)$ with accuracy functions $p_{\mathrm{I}} = x^3, p_{\mathrm{II}} = x^2$.

In fact, we will show that there is a unique point $x^* \in [0, 1]$ that is the unique solution of

$$p_{\mathrm{I}}(x^*) + p_{\mathrm{II}}(x^*) = 1, \tag{4.3.1}$$

so that

$$u_1(x^*, x^*) \geq u_1(x, x^*) \text{ for all } x \in [0, 1] \tag{4.3.2}$$

and

$$u_2(x^*, x^*) \geq u_2(x^*, y) \text{ for all } y \in [0, 1]. \tag{4.3.3}$$

This says that (x^*, x^*) is a Nash equilibrium for the noisy duel. Of course, since $u_2 = -u_1$, the inequalities reduce to

$$u_1(x, x^*) \leq u_1(x^*, x^*) \leq u_1(x^*, y), \text{ for all } x, y \in [0, 1],$$

so that (x^*, x^*) is a saddle point for u_1.

To verify that the inequalities (4.3.2) and (4.3.3) hold for x^* defined in (4.3.1), we have, from the definition of u_1, that

$$u_1(x^*, x^*) = p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*).$$

Using the fact that both accuracy functions are increasing, we have, by (4.3.1)

$$\begin{aligned}
u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) = 1 - 2p_{\mathrm{II}}(x^*) \\
&\leq 1 - 2p_{\mathrm{II}}(y) = u_1(x^*, y) \text{ if } x^* > y, \\
u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) \\
&= 2p_{\mathrm{I}}(x^*) - 1 = u_1(x^*, y) \text{ if } x^* < y, \text{ and} \\
u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) \\
&= u_1(x^*, y) \text{ if } x^* = y.
\end{aligned}$$

So, in all cases $u_1(x^*, x^*) \leq u_1(x^*, y)$ for all $y \in [0, 1]$. We verify (4.3.3) in a similar way and leave that as an exercise. We have shown that (x^*, x^*) is indeed a Nash point.

That x^* exists and is unique is shown by considering the function $f(x) = p_{\mathrm{I}}(x) + p_{\mathrm{II}}(x)$. We have $f(0) = 0, f(1) = 2$, and $f'(x) = p_{\mathrm{I}}'(x) + p_{\mathrm{II}}'(x) > 0$. By the **intermediate value theorem** of calculus we conclude that there is an x^* satisfying $f(x^*) = 1$. The uniqueness of x^* follows from the fact that p_{I} and p_{II} are strictly increasing.

In the example shown in Figure 4.4 with $p_{\mathrm{I}}(x) = x^3$ and $p_{\mathrm{II}}(x) = x^2$, we have the condition $x^{*3} + x^{*2} = 1$, which has solution at $x^* = 0.754877$. With these accuracy functions, the duelists should wait until less than 25% of the time is left until they **both** fire. The expected payoff to player I is $u_1(x^*, x^*) = -0.1397$, and the expected payoff to player II is $u_2(x^*, x^*) = 0.1397$. It appears that player I is going down. We expect that result in view of the fact that player II has greater accuracy at x^*, namely, $p_{\mathrm{II}}(x^*) = 0.5698 > p_{\mathrm{I}}(x^*) = 0.4301$.

The following Maple commands are used to get all of these results and see some great pictures:

```
> restart:with(plots):
> p1:=x->x^3;p2:=x->x^2;
> v1:=(x,y)->piecewise(x<y,2*p1(x)-1,x=y,p1(x)-p2(x),x>y,1-2*p2(x));
> plot3d(v1(x,y),x=0..1,y=0..1,axes=boxed);
> xstar:=fsolve(p1(x)+p2(x)-1=0,x);
> v1(xstar,xstar);
```

For a more dramatic example, suppose that $p_{\mathrm{I}}(x) = x^3$, $p_{\mathrm{II}}(x) = x$. This says that player I is at a severe disadvantage. His accuracy does not improve until a lot of time has passed (and so the duelists are closer together). In this case $x^* = 0.68232$ and they both fire at time 0.68232 and $u_1(0.68232, 0.68232) = -0.365$. Player I would be stupid to play this game with real bullets. That is why game theory is so important.

Silent Duel on* $[0, 1]$. *(Optional). In case you are curious as to what happens when we have a silent duel, we will present this example to show that things get considerably more complicated. We take the simplest possible accuracy functions $p_{\mathrm{I}}(x) = p_{\mathrm{II}}(x) = x \in [0, 1]$ because this case is already much more difficult than the noisy duel. The payoff of this game to player I is

$$u_1(x, y) = \begin{cases} x - (1 - x)y, & x < y; \\ 0, & x = y; \\ -y + (1 - y)x, & x > y. \end{cases}$$

For player II, since this is zero sum, $u_2(x, y) = -u_1(x, y)$. Now, in the problem with a silent duel, intuitively it seems that there cannot be a pure Nash equilibrium because silence would dictate that an opponent could always take advantage of a pure strategy. But how do we allow mixed strategies in a game with continuous strategies?

In a discrete matrix game a mixed strategy is a probability distribution over the pure strategies. Why not allow the players to choose continuous probability distributions? No reason at all. So we consider the mixed strategy choice for each player

$$X(x) = \int_0^x f(a)\,da,$$
$$Y(y) = \int_0^y g(b)\,db,$$
$$\int_0^1 f(a)\,da = \int_0^1 g(b)\,db = 1.$$

The cumulative distribution function $X(x)$ represents the probability that player I will choose to fire at a point $\leq x$. The expected payoff to player I if he chooses X and his opponent chooses Y is

$$E(u_1(X,Y)) = \int_0^1 \int_0^1 u_1(x,y) f(x) g(y)\,dx\,dy.$$

As in the discrete-game case, we define the value of the game as

$$v \equiv \min_Y \max_X E(u_1(X,Y)) = \max_X \min_Y E(u_1(X,Y)).$$

The equality follows from the existence theorem of a Nash equilibrium (actually a saddle point in this case) because the expected payoff is not only concave–convex, but actually linear in each of the probability distributions X, Y. It is completely analogous to the existence of a mixed strategy saddle point for matrix games. The value of games with a continuum of strategies exists if the players choose from within the class of probability distributions. (Actually, the probability distributions should include the possibility of point masses, but we do not go into this generality here.) A saddle point in mixed strategies has the same definition as before: (X^*, Y^*) is a saddle if

$$E(X, Y^*) \leq E(X^*, Y^*) = v \leq E(X^*, Y), \forall X, Y \text{ probability distributions.}$$

Now, the fact that both players are symmetric and have the same accuracy functions allows us to guess that $v = 0$ for the silent duel. To find the optimal strategies, namely, the density functions $f(x), g(y)$, we use the necessary condition that if X^*, Y^* are optimal, then

$$E(X, y) = \int_0^1 u_1(x,y) f(x)\,dx = v = 0,$$
$$E(x, Y) = \int_0^1 u_1(x,y) g(y)\,dy = v = 0, \quad \forall x, y \in [0,1].$$

This is completely analogous to the equality of payoffs Theorem 3.2.4 to find mixed strategies in bimatrix games, or to the geometric solution of two person 2×2 games in which the value occurs where the two payoff lines cross. We replace $u_1(x, y)$ to work with the following equation:

$$\int_0^y [x - (1 - x)y]f(x)\, dx + \int_y^1 [-y + (1 - y)x]f(x)\, dx = 0,\ \forall\, y \in [0, 1].$$

If we expand this, we get

$$\begin{aligned}
0 &= \int_0^y [x - (1 - x)y]f(x)\, dx + \int_y^1 [-y + (1 - y)x]f(x)\, dx \\
&= \int_0^y xf(x)\, dx - y\int_0^y (1 - x)f(x)\, dx - y\int_y^1 f(x)dx + (1 - y)\int_y^1 xf(x)\, dx \\
&= \int_0^1 xf(x)\, dx - y\int_0^1 f(x)\, dx + y\int_0^y xf(x)\, dx - y\int_y^1 xf(x)\, dx \\
&= \int_0^1 xf(x)\, dx - y + y\int_0^y xf(x)\, dx - y\int_y^1 xf(x)\, dx.
\end{aligned}$$

The first term in this last line is actually a constant, and the constant is $E[X] = \int_0^1 xf(x)\, dx$, which is the **mean of the strategy** X.

Now a key observation is that the equation we have should be looked at, not in the unknown function $f(x)$, but in the unknown function $xf(x)$. Let's call it $\varphi(x) \equiv xf(x)$, and we see that

$$E[X] - y + y\int_0^y \varphi(x)\, dx - y\int_y^1 \varphi(x)\, dx = 0.$$

Consider the left side as a function of $y \in [0, 1]$. Call it

$$F(y) \equiv E[X] - y + y\int_0^y \varphi(x)\, dx - y\int_y^1 \varphi(x)\, dx.$$

Then $F(y) = 0, 0 \leq y \leq 1$. We take a derivative using the fundamental theorem of calculus in an attempt to get rid of the integrals:

$$F'(y) = -1 + \int_0^y \varphi(x)\, dx + y[\varphi(y)] - \int_y^1 \varphi(x)\, dx + y[\varphi(y)] = 0$$

and then another derivative

$$\begin{aligned}
F''(y) &= \varphi(y) + \varphi(y) + y\varphi'(y) + \varphi(y) + \varphi(y) + y\varphi'(y) \\
&= 4\varphi(y) + 2y\varphi'(y) = 0.
\end{aligned}$$

So we are led to the differential equation for $\varphi(y)$, which is

$$4\varphi(y) + 2y\varphi'(y) = 0, \quad 0 \le y \le 1.$$

This is a first-order **ordinary differential equation** that will have general solution

$$\varphi(y) = C\frac{1}{y^2},$$

as you can easily check by plugging in. Then $\varphi(y) = yf(y) = C/y^2$ implies that $f(y) = C/y^3$, or $f(x) = C/x^3$ returning to the x variable. We have to determine the constant C.

You might think that the way to find C is to apply the fact that $\int_0^1 f(x)\, dx = 1$. That would normally be correct, but it also points out a problem with our formulation. Look at $\int_0^1 x^{-3}\, dx$. This integral **diverges** (that is, it is infinite), because x^{-3} is not integrable on $(0, 1)$. This would stop us dead in our tracks because there would be no way to fix that with a constant C unless the constant were zero. That can't be, because then $f = 0$, and it is not a probability density. The way to fix this is to assume that the function $f(x)$ is zero on the starting subinterval $[0, a]$ for some $0 < a < 1$. In other words, we are assuming that the players will not shoot on the interval $[0, a]$ for some unknown time $a > 0$. The lucky thing is that the procedure we used at first, but now repeated with this assumption, is the same and leads to the equation

$$E[X] - y + y\int_a^y \varphi(x)\, dx - y\int_y^1 \varphi(x)\, dx = 0,$$

which is the same as where we were before except that 0 is replaced by a. So, we get the same function $\varphi(y)$ and eventually the same $f(x) = C/x^3$, except we are now on the interval $0 < a \le x \le 1$. This idea does not come for free, however, because now we have two constants to determine, C and a. C is easy to find because we must have $\int_a^1 C/x^3\, dx = 1$. This says, $C = 2a^2/(1 - a^2) > 0$. To find $a > 0$, we substitute $f(x) = C/x^3$ into (recall that $\varphi(x) = x\,f(x)$)

$$\begin{aligned} 0 &= E[X] - y + y\int_0^y \varphi(x)\, dx - y\int_y^1 \varphi(x)\, dx \\ &= y\left(C + \frac{C}{a} - 1\right) + C\left(-3 + \frac{1}{a}\right). \end{aligned}$$

This must hold for all $a \le y \le 1$ which implies that $C + C/a - 1 = 0$. Therefore, $C = a/(a+1)$. But then we must have

$$C = \frac{2a^2}{1 - a^2} = \frac{a}{a+1} \implies a = \frac{1}{3}, \quad C = \frac{1}{4}.$$

So, we have found $X(x)$. It is the cumulative distribution function of the strategy for player I and has density

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{3}; \\ \dfrac{1}{4}x^3 & \text{if } \frac{1}{3} \le x \le 1. \end{cases}$$

We know that $\int_a^1 u_1(x, y) f(x)\, dx = 0$ for $y \ge a$, but we have to check that with this $C = \frac{1}{4}$ and $a = \frac{1}{3}$ to make sure that

$$\int_a^1 u_1(x, y) f(x)\, dx > 0 = v, \text{ when } y < a. \tag{4.3.4}$$

That is, we need to check that X played against any pure strategy in $[0, a]$ must give at least the value $v = 0$ if X is optimal. Let's take a derivative of the function $G(y) = \int_a^1 u_1(x, y) f(x)\, dx$, $0 \le a \le 1$. We have,

$$G(y) = \int_a^1 u_1(x, y) f(x)\, dx = \int_a^1 (-y + (1-y)x) f(x)\, dx,$$

which implies that

$$G'(y) = \int_a^1 [-1 - x f(x)]\, dx = -\frac{3}{2} < 0.$$

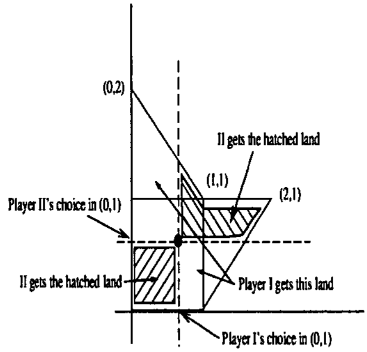
So, $G(y)$ is decreasing on $[0, a]$. Since $G(a) = \int_a^1 (-a + (1-a)x) f(x)\, dx = 0$, it must be true that $G(y) > 0$ on $[0, a)$, so the condition (4.3.4) checks out. Finally, since this is a symmetric game, it will be true that $Y(y)$ will have the same density as player I.

PROBLEMS

4.13 Determine the optimal time to fire for each player in the noisy duel with accuracy functions $p_{\mathrm{I}}(x) = \sin(\frac{\pi}{2}x)$ and $p_{\mathrm{II}}(x) = x^2$, $0 \le x \le 1$.

4.14 Consider the zero sum game with payoff function for player 1 given by $u(x, y) = -2x^2 + y^2 + 3xy - x - 2y, 0 \le x, y \le 1$. Show that this function is concave in x and convex in y. Find the saddle point and the value.

4.15 This is known as the **division of land game**. Suppose that there is a parcel of land as in the following figure:



Player I chooses a vertical line between $(0, 1)$ on the x axis and player II chooses a horizontal line between $(0, 1)$ on the y axis. Player I gets the land below II's choice and right of I's choice as well as the land above II's choice and left of I's line. Player II gets the rest of the land. Both players want to choose their line so as to maximize the amount of land they get. Formulate this as a game with continuous strategies and solve it.

4.16 Two countries are at war over a piece of land. Country 1 places value v_1 on the land, and country 2 values it at v_2. The players choose the time at which to concede the land to the other player, but there is a cost for letting time pass. Suppose that the cost to each country is $C_i, i = 1, 2$ per unit of time. The first player to concede yields the land to the other player at that time. If they concede at the same time, each player gets half the land. Determine the payoffs to each player and determine the pure Nash equilibria. Consider the case $C_1 = C_2$.

4.17 Two countries are at war over an asset with a total value of $V > 0$. Country 1 allocates an effort $x > 0$, and country 2 allocates an effort $y > 0$ to acquire all or a portion of V. The portion of V won by country 1 if they allocate effort x is $(x/(x + y))V$ at cost $C_1 x$, where $C_1 > 0$ is a constant. Similarly, the portion of V won by country 2 if they allocate effort y is $(y/(x+y))V$ at cost $C_2 y$, where $C_2 > 0$ is a constant. The total reward to each country is then

$$u_1(x, y) = V\frac{x}{x+y} - C_1 x \quad \text{and} \quad u_2(x, y) = V\frac{y}{x+y} - C_2 y, \quad x > 0, y > 0.$$

Show that these payoff functions are concave in the variable they control and then find the Nash equilibrium using calculus.

4.18 Corn is a food product in high demand but also enjoys a government price subsidy. Assume that the demand for corn (in bushels) is given by $D(p) = 150000(15 - p)^+$, where p is the price per bushel. The government program guarantees that $p \geq 2$. Suppose that there are three corn producers who have reaped 1 million bushels each. They each have the choice of how much to send to market and how much to use for feed (at no profit). Find the Nash equilibrium. What happens if one farmer sends the entire crop to market?

4.19 **(The Samaritan's Dilemma** [4]) A citizen works in period 1 at a job giving a current annual net income of p. Out of her income she saves s for her retirement and earns $r\%$ interest per period. When she retires, considered to occur at period 2, she will receive an amount g from the government. The payoff to the citizen is

$$u_C(s,g) \equiv u_1(p-s) + u_2(s(1+r)+g)\delta,$$

where u_1 is the first-period utility function, and is increasing but concave down, and u_2 is the second-period utility function, that is also increasing and concave down. The constant $\delta > 0$, called a discount rate, finds the present value of dollars that are not delivered until the second period. The government has a payoff function

$$u_G(s,g) = u(t-g) + \alpha u_C(s,g)$$

where $u(t)$ is the government's utility function for income level t such as tax receipts and $\alpha > 0$ represents the factor of benefits received by the government for a happy citizen, called an **altruism factor**. Assume that utilities u_1, u_2, u are all logarithmic functions of their variables. (**a**) Find a Nash equilibrium. (**b**) Given $\alpha = 0.25, p = 22000, t = 100000, r = 0.15$ and $\delta = 0.03$, find the optimal amount for the citizen to save and the optimal amount for the government to transfer to the citizen in period 2.

4.4 AUCTIONS (OPTIONAL)

There were probably auctions by cavemen for clubs, tools, skins, and so on, but we can be sure (because there is a written record) that there was bidding by Babylonians for both men and women slaves and wives. Auctions today are ubiquitous with many internet auction houses led by eBay, which does more than 6 billion dollars of business a year and earns almost 2 billion dollars as basically an auctioneer. This pales in comparison with United States treasury bills, notes, and bond auctions, which have total dollar values each year in the trillions of dollars. Rights to the airwaves, oil leases, pollution rights, and tobacco, all the way down to auctions for a Mickey Mantle-signed baseball are common occurrences.

There are different types of auctions we study. Their definitions are summarized here.

Definition 4.4.1 *The different types of auctions are:*

- **English Auction.** *Bids are announced publicly, and the bids rise until only one bidder is left. That bidder wins the object at the highest bid.*
- **Sealed Bid, First Price.** *Bids are private and are made simultaneously. The highest sealed bid wins and the winner pays that bid.*

- **Sealed Bid, Second Price.** *Bids are private and made simultaneously. The high bid wins, but the winner pays the second highest bid. This is also called a Vickrey auction after the Nobel Prize winner who studied them.*

- **Dutch Auction.** *The auctioneer (which could be a machine) publicly announces a high bid. Bidders may accept the bid or not. The announced prices are gradually lowered until someone accepts that price. The first bidder who accepts the announced price wins the object and pays that price.*

- **Private Value Auction.** *Bidders are certain of their own valuation of the object up for auction and these valuations (which may be random variables) are independent.*

- **Common Value Auction.** *The object for sale has the same value (that is not known for certain to the bidders) to all the bidders. Each bidder has their own estimate of this value.*

In this section we will present a game theory approach to the theory of auctions and will be more specific about the type of auction as we cover it. Common value auctions require the use of more advanced probability theory and will not be considered further.

Let's start with a simple nonzero sum game that shows why auction firms like eBay even exist (or need to exist).

■ EXAMPLE 4.5

In an online auction with no middleman the seller of the object and the buyer of the object may choose to renege on the deal dishonestly or go through with the deal honestly. How is that carried out? The buyer could choose to wait for the item and then not pay for it. The seller could simply receive payment but not send the item. Here is a possible payoff matrix for the buyer and the seller:

Buyer/Seller	Send	Keep
Pay	$(1,1)$	$(-2,2)$
Don't Pay	$(2,-2)$	$(-1,-1)$

There is only one Nash equilibrium in this problem and it is at (don't pay,keep); neither player should be honest! Amazing, the transaction will never happen and it is all due to either lack of trust on the part of the buyer and seller, or total dishonesty on both their parts. If a buyer can't trust the seller to send the item and the seller can't depend on the buyer to pay for the item, there won't be a deal.

Now let's introduce an auction house that serves two purposes: (1) it guarantees payment to the seller and (2) it guarantees delivery of the item for the buyer. Of course, the auction house will not do that out of kindness but because it is paid by the seller (or the buyer) in the form of a commission. This introduces a third strategy for the buyer and seller to use: Auctioneer. This changes the payoff matrix as follows:

Buyer/Seller	Send	Keep	Auctioneer
Pay	$(1,1)$	$(-2,2)$	$(1,1-c)$
Don't pay	$(2,-2)$	$(-1,-1)$	$(0,-c)$
Auctioneer	$(1-c,1)$	$(-c,0)$	$(1-c,1-c)$

The idea is that each player has the **option, but not the obligation,** of using an auctioneer. If somehow they should agree to both be honest, they both get $+1$. If they both use an auctioneer, the auctioneer will charge a fee of $0 < c < 1$ and the payoff to each player will be $1 - c$.

Observe that $(-1,-1)$ is no longer a pure Nash equilibrium. We use a calculus procedure to find the mixed Nash equilibrium for this symmetric game as a function of c. The result of this calculation is

$$X_c = \left(\frac{1}{2}(1-c), \frac{1}{2}c, \frac{1}{2}\right) = Y_c$$

and (X_c, Y_c) is the unique Nash equilibrium. The expected payoffs to each player are

$$E_{\mathrm{I}}(X_c, Y_c) = 1 - \frac{3}{2}c = E_{\mathrm{II}}(X_c, Y_c).$$

As long as $\frac{2}{3} > c > 0$, both players receive a positive expected payoff. Because we want the payoffs to be close to $(1,1)$, which is what they each get if they are both honest and don't use a auctioneer, it will be in the interest of the auctioneer to make $c > 0$ as small as possible because at some point the transaction will not be worth the cost to the buyer or seller.

The Nash equilibrium tells the buyer and seller to use the auctioneer half the time, no matter what value c is. Each player should be dishonest with probability $c/2$, which will increase as c increases. The players should be honest with probability only $(1-c)/2$. If $c = 1$ they should never play honestly and either play dishonestly or use an auctioneer half the time.

You can see that the existence and only the existence of an auctioneer will permit the transaction to go through. From an economics perspective, this implies that auctioneers **will** come into existence as an economic necessity for online auctions and it can be a very profitable business (which it is). But this is true not only for online auctions. Auction houses for all kinds of specialty and nonspecialty items (like horses, tobacco, diamonds, gold, houses, etc.) have been in existence for decades, if not centuries, because they serve the economic function of guaranteeing the transaction.

A common feature of auctions is that the seller of the object may set a price, called the **reserve price,** so that if none of the bids for the object are above the reserve price the seller will not sell the object. One question that arises is whether sellers should use a reserve price. Here is an example to illustrate.

EXAMPLE 4.6

Assume that the auction has two possible buyers. The seller must have some information, or estimates, about how the buyers value the object. Perhaps these are the seller's own valuations projected onto the buyers. Suppose that the seller feels that each buyer values the object at either $\$s$ (small amount) or $\$L$ (large amount) with probability $\frac{1}{2}$ each. But, assuming bids may go up by a minimum of $1, the winning bids with no reserve price set are $\$s, \$(s+1), \$(s+1)$, or $\$L$ each with probability $\frac{1}{4}$. Without a reserve price, the expected payoff to the seller is

$$\frac{s+2(s+1)+L}{4} = \frac{3s+2+L}{4}.$$

Suppose next that the seller sets a reserve price at the higher valuation $\$L$, and this is the lowest acceptable price to the seller. Let $B_i, i = 1, 2$ denote the random variable that is the bid for buyer i. Without collusion or passing of information we may assume that the B_i values are independent. The seller is assuming that the valuations of the bidders are $(s, s), (s, L), (L, s)$ and (L, L), each with probability $\frac{1}{4}$. If the reserve price is set at $L, the sale will not go through 25% of the time, but the expected payoff to the seller will be

$$\left(\frac{3}{4}\right)L + \left(\frac{1}{4}\right)0 = \frac{3L}{4}.$$

The question is whether it can happen that there are valuations s and L, so that

$$\frac{3L}{4} > \frac{3s+2+L}{4}.$$

Solving for L, the requirement is that $L > (3s+2)/2$, and this certainly has solutions. For example, if $L = 100$, any lower valuation $s < 66$ will lead to a higher expected payoff to the seller with a reserve price set at $100. Of course, this result depends on the valuations of the seller.

EXAMPLE 4.7

If you need more justification for a reserve price, consider that if only one bidder shows up for your auction and you must sell to the high bidder, then you do not make any money at all unless you set a reserve price. Of course, the question is how to set the reserve price. Assume that a buyer has a valuation of your gizmo at $\$V$, where V is a random variable with cumulative distribution function $F(v)$. Then, if your reserve price is set at \$$p$, the expected payoff (assuming one bidder) will be

$$\begin{aligned} E[\text{Payoff}] &= pProb(V > p) + 0Prob(V \le p) \\ &= pProb(V > p) = p(1 - F(p)). \end{aligned}$$

You want to choose p to maximize this function $g(p) \equiv p(1 - F(p))$. Let's find the critical points assuming that $f(p) = F'(p)$ is the probability density function of V :

$$g'(p) = (1 - F(p)) - pf(p) = 0.$$

Assuming that $g''(p) = 2f(p)+pf'(p) \ge 0$, we will have a maximum. Observe that the maximum is not achieved at $p = 0$ because $g'(0) = 1 - F(0) = 1 > 0$. So, we need to find the solution, p^*, of

$$1 - F(p^*) - p^* f(p^*) = 0.$$

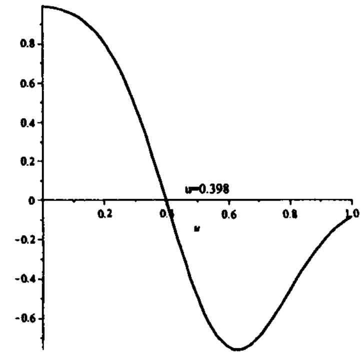
For a concrete example, we assume that the random variable V has a normal distribution with mean 0.5 and standard deviation 0.2 The density is

$$f(p) = \frac{2.5}{\sqrt{2\pi}} e^{-12.5(p-0.5)^2}.$$

We need to solve $1 - F(p) - pf(p) = 0$, where $F(p)$ is the cumulative normal distribution with mean 0.5 and standard deviation 0.2. This is easy to do using Maple. The Maple commands to get our result are

```
> restart:
> with(Statistics):
> X := RandomVariable(Normal(0.5,0.2)):
> plot(1-CDF(X,u)-u*PDF(X,u),u=0..1);
> fsolve(1-CDF(X,u)-u*PDF(X,u),u);
```

The built-in functions PDF and CDF give the density and cumulative normal functions. We have also included a command to give us a plot of the function giving the root. The plot of $b(u) = 1 - F(u) - p * f(u)$ is shown in the following figure.



We see that the function $b(u)$ crosses the axis at $u = p^* = 0.398$. The reserve price should be set at 39.8% of the maximum valuation.

Having seen why auction houses exist, let's get into the theory from the bidders' perspective. There are N bidders (=players) in this game. There is one item up for bid and each player **values the object** at $v_1 \geq v_2 \geq \cdots \geq v_N > 0$ dollars. One question we have to deal with is whether the bidders know this ranking of values.

4.4.1 Complete Information

In the simplest and almost totally unrealistic model, all the bidders have complete information about the valuations of all the bidders. Now we define the rules of the auctions considered in this section.

Definition 4.4.2 *A first-price, sealed-bid auction is an auction in which each bidder submits a bid $b_i, i = 1, 2, \ldots, N$ in a sealed envelope. After all the bids are received the envelopes are opened by the auctioneer and the person with the highest bid wins the object and pays the bid b_i. If there are identical bids, the winner is chosen at random from the identical bidders.*

So what is the payoff to player i if the bids are $b_1, \ldots, b_N$? Well, if bidder i doesn't win the object, she pays nothing and gets nothing. That will occur if she is not a high bidder:

$$b_i < \max\{b_1, \ldots, b_N\} \equiv M.$$

On the other hand, if she is a high bidder, so that $b_i = M$, then the payoff is the difference between what she bid and what she thinks it's worth (i.e., $v_i - b_i$). If she bids less than her valuation of the object, and wins the object, then she gets a positive payoff, but she gets a negative payoff if she bids more than it's worth to her. To take into account the case when there are k ties in the high bids, she would get the average payoff. Let's use the notation that $\{k\}$ is the set of high bidders. So, in symbols

$$u_i(b_1, \ldots, b_N) = \begin{cases} 0 & \text{if } b_i < M, \text{ she is not a high bidder;} \\ v_i - b_i & \text{if } b_i = M, \text{ she is the sole high bidder;} \\ \dfrac{v_i - b_i}{k} & \text{if } i \in \{k\}, \text{ she is one of } k \text{ high bidders.} \end{cases}$$

Naturally, bidder i wants to know the amount to bid. That is determined by finding the maximum payoff of player i, assuming that the other players are fixed. We want a Nash equilibrium for this game. This doesn't really seem too complicated. Why not just bid v_i? That would guarantee a payoff of zero to each player, but is that the maximum? Should a player ever bid more than her valuation? These questions are answered in the following rules.

With complete information and a sealed-bid, first-price auction:

1. Each bidder should bid $b_i \leq v_i, i = 1, 2, \ldots, N$. Never bid more than the valuation. To see this, just consider the following cases. If player i bids $b_i > v_i$ and wins the auction, then $u_i < 0$, even if there are ties. If player i bids $b_i > v_i$ and does not win the auction, then $u_i = 0$. But if player i bids $b_i \leq v_i$, in all cases $u_i \geq 0$.

2. In the case when the highest valuation is strictly bigger than the second highest valuation, $v_1 > v_2$, player 1 bids $b_1 \approx v_2, v_1 > b_1 > v_2$; that is, player 1 wins the object with any bid greater than v_2 and so should bid very close to but higher than v_2. Notice that this is an open interval and the maximum is not actually achieved by a bid. If bidding is in whole dollars, then $b_1 = v_2 + 1$ is the optimal bid. There is no Nash equilibrium achieved in the case where the winning bid is in $(v_2, v_1]$ because it is an open interval at v_2.

3. In the case when $v_1 = v_2 = \cdots = v_k$, so there are k players with the highest, but equal, value of the object, then player i should bid v_i (i.e, $b_i = v_i, 1 \leq i \leq N$). So, the bid $(b_1, \ldots, b_N) = (v_1, \ldots, v_N)$ will be a Nash equilibrium in this case. We leave it as an exercise to verify that.

PROBLEMS

4.20 Verify that in the first-price auction $(b_1, \ldots, b_N) = (v_1, \ldots, v_N)$ is a Nash equilibrium assuming $v_1 = v_2$.

4.21 In a second-price sealed-bid auction with complete information the winner is the high bidder but she pays, not the price she bid, but the second highest bid. If there are ties, then the winner is drawn at random from among the high bidders and she pays the highest bid. Formulate the payoff functions and show that the following rules are optimal:

(a) Each player bids $b_i \leq v_i$.

(b) If $v_1 > v_2$, then player 1 wins by bidding any amount $v_2 < b_1 < v_1$.
(c) If $v_1 = v_2 = \cdots = v_k$, then $(v_1, v_2, \ldots, v_N)$ is a Nash equilibrium.

4.22 A homeowner is selling her house by auction. Two bidders place the same value on the house at \$100,000, while the next bidder values the house at \$80,000. Should the homeowner use a first-price or second-price auction to sell the house, or does it matter? What if the highest valuation is \$100,000, the next is \$95,000 and the rest are no more than \$90,000?

4.23 Find the optimal reserve price to set in an auction assuming that the density of the value random variable V is $f(p) = 6p(1-p), 0 \leq p \leq 1$.

4.4.2 Incomplete Information

The setup now is that we have $N > 1$ bidders with valuations of the object for sale $V_i, i = 1, 2, \ldots, N$. The problem is that the valuations are not known to either the buyers or the seller, except for their own valuation. Thus, since we must have some place to start we assume that the seller and buyers think of the valuations as random variables. The information that we assume known to the seller is the joint cumulative distribution function

$$F(v_1, v_2, \ldots, v_N) = Prob(V_1 \leq v_1, \ldots, V_n \leq v_N),$$

and each buyer i has knowledge of his or her own distribution function $F_i(v_i) = P(V_i \leq v_i)$.

Take-It-or-Leave-It Rule. This is the simplest possible problem that may still be considered an auction. But it is not really a game. It is a problem of the seller of an object as to how to set the **buy-it-now** price.

In an auction you may set a **reserve price** r, which, as we have seen, is a nonnegotiable lowest price you must get to consider selling the object. You may also declare a price $p \geq r$, which is your **take-it-or-leave-it price** or **buy-it-now price** and wait for some buyer, who hopefully has a valuation greater than or equal to p to buy the object. The problem is to determine p.

The solution involves calculating the expected payoff from the trade and then maximizing the expected payoff over $p \geq r$. The payoff is the function $U(p)$, which is $p - r$, if there is a buyer with a valuation at least p, and 0 otherwise:

$$U(p) = \begin{cases} p - r & \text{if } \max\{V_1, \ldots, V_N\} \geq p; \\ 0 & \text{if } \max\{V_1, \ldots, V_N\} < p. \end{cases}$$

This is a random variable because it depends on $V_1, \ldots, V_N$, which are random. The expected payoff is

$$\begin{aligned} u(p) = E[U(p)] &= (p-r)P(\max\{V_1, \ldots, V_N\} \geq p) + 0 \cdot P(\max\{V_1, \ldots, V_N\} < p) \\ &= (p-r)[1 - P(\max\{V_1, \ldots, V_N\} < p)] \\ &= (p-r)[1 - P(V_1 < p, \cdots, V_N < p)] \\ &= (p-r)[1 - F(p, p, \ldots, p)] \\ &= (p-r)f(p), \end{aligned}$$

where $f(p) = 1 - F(p, p, \ldots, p)$. The seller wants to find p^* such that

$$\max_{p \geq r}(p-r)f(p) = (p^* - r)f(p^*).$$

If there is a maximum $p^* > r$, we could find it by calculus. It is the solution of

$$(p^* - r)f'(p^*) + f(p^*) = 0.$$

This will be a maximum as long as $(p^* - r)f''(p^*) + 2f'(p^*) \leq 0$. To proceed further, we need to know something about the valuation distribution. Let's take the simplest case that $\{V_i\}$ is a collection of N independent and identically distributed random variables. In this case

$$F(v_1, \ldots, v_N) = G(v_1) \cdots G(v_N), \text{where } G(v) = F_i(v), 1 \leq i \leq N.$$

Then

$$f(p) = 1 - G(p)^N, \quad f'(p) = -NG(p)^{N-1}G'(p), \quad \text{and} \quad G'(p) = g(p)$$

is the density function of V_i, if it is a continuous random variable. So the condition for a maximum at p^* becomes

$$-(p^* - r)NG(p^*)^{N-1}g(p^*) + 1 - G(p^*)^N = 0.$$

Now we take a particular distribution for the valuations that is still realistic. In the absence of any other information, we might as well assume that the valuations are uniformly distributed over the interval $[r, R]$, remembering that r is the reserve price. For this uniform distribution. We have

$$g(p) = \begin{cases} \dfrac{1}{R-r} & r < p < R; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad G(p) = \begin{cases} 0 & \text{if } p < r; \\ \dfrac{p-r}{R-r} & \text{if } r < p < R; \\ 1 & \text{if } p > R. \end{cases}$$

If we assume that $r < p^* < R$, then we may solve

$$-(p^* - r)N\left(\frac{p^* - r}{R-r}\right)^{N-1}\left(\frac{1}{R-r}\right) + 1 - \left(\frac{p^* - r}{R-r}\right)^N = 0$$

for p^* to get the **take-it-or-leave-it price**

$$p^* = r + (R - r)\left(\frac{1}{N+1}\right)^{1/N}.$$

For this p^* we have the expected payoff

$$u(p^*) = (p^* - r)f(p^*) = (R - r)N\left(\frac{1}{N+1}\right)^{N+1}$$

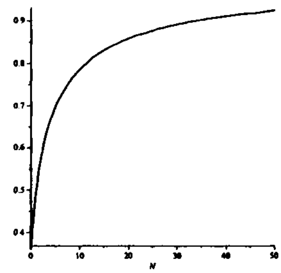
Of particular interest are the cases $N = 1, N = 2$, and $N \to \infty$. We label the take-it-or-leave-it price as $p^* = p^*(N)$. Here are the results:

1. When there is only one potential buyer the **take-it-or-leave-it price** should be set at
$$p^*(1) = r + \frac{R-r}{2} = \frac{r+R}{2}$$
the midpoint of the range $[r, R]$. The expected payoff to the seller is
$$u(p^*(1)) = \frac{R-r}{4}.$$

2. When there are two potential buyers, the take-it-or-leave-it price should be set at
$$p^*(2) = r + \frac{R-r}{\sqrt{3}}, \text{ and then } u(p^*(2)) = (R-r)\frac{2\sqrt{3}}{9}.$$

3. As $N \to \infty$, we have the take-it-or-leave-it price should be set at
$$p^*(\infty) = \lim_{N\to\infty} p^*(N) = \lim_{N\to\infty} r + (R-r)\left(\frac{1}{N+1}\right)^{1/N} = R,$$
and then the expected payoff is $u(p^*(\infty)) = R - r$. Notice that we may calculate $\lim_{N\to\infty}(1/N+1)^{1/N} = 1$ using L'Hôpital's rule. We conclude that as the number of potential buyers increases, the price should be set at the upper range of valuations. A totally reasonable result.

You can see from the following figure, which plots $p^*(N)$ as a function of N (with $R = 1, r = 0$), that as the number of buyers increases, there is a rapid increase in the take-it-or-leave-it price before it starts to level off:



4.4.3 Symmetric Independent Private Value Auctions

In this section [7] we take it a step further and model the auction as a game. Again, there is one object up for auction by a seller and the bidders know their own valuations of the object but not the valuations of the other bidders. It is assumed that the unknown valuations $V_1, \ldots, V_N$ are independent and identically distributed continuous random variables. The **symmetric** part of the title of this section comes from assuming that the bidders all have the same valuation distribution.

We will consider two types of auction:

1. English auction, where bids increase until everyone except the highest one or two bidders are gone. In a first–price auction the high bidder gets the object.

2. Dutch auction, where the auctioneer asks a price and lowers the price continuously until one or more bidders decide to buy the item at the latest announced price. In a tie the winner is chosen randomly.

Remarks.

A Dutch auction is equivalent to the first-price sealed-bid auction. Why? A bidder in each type of auction has to decide how much to bid to place in the sealed envelope or when to yell "Buy" in the Dutch auction. That means that the strategies and payoffs are the same for these types of auction. In both cases the bidder must decide the highest price she is willing to pay and submit that bid. In a Dutch auction the object will be awarded to the highest bidder at a price equal to her bid. That is exactly what happens in a first-price sealed-bid auction. The strategies for making a bid are identical in each of the two seemingly different types of auction.

An English auction can also be shown to be equivalent to a **second-price sealed-bid auction** (as long as we are in the private values set up). Why? As long as bidders do not change their valuations based on the other bidders' bids, which is assumed, then bidders should accept to pay any price up to their own valuations. A player will continue to bid until the current announced price is greater than how much the bidder is willing to pay. This means that the item will be won by the bidder who has the

[7] Refer to the article by Milgrom [16] for an excellent essay on auctions from an economic perspective.

highest valuation and she will win the object at a price equal to the **second highest valuation**.

The equivalence of the various types of auctions was first observed by William Vickrey.[8] The result of an English auction can be achieved by using a sealed-bid auction in which the item is won by the second highest submitted bid. A bidder should submit a bid that is equal to her valuation of the object since she is willing to pay an amount less than her valuation but not willing to pay a price greater than her valuation. Assuming that each player does that, the result of the auction is that the object will be won by the bidder with the highest valuation and the price paid will be the second highest valuation.

For simplicity it will be assumed that the reserve price is normalized to $r = 0$. As above, we assume that the joint distribution function of the valuations is given by

$$F_J(v_1, \ldots, v_N) = Prob(V_1 \leq v_1, \ldots, V_N \leq v_N) = F(v_1)F(v_2)\cdots F(v_N),$$

which holds because we assume independence and identical distribution of the bidder's valuations. Suppose that the maximum possible valuation of all the players is a fixed constant $w > 0$. Then, a bidder gets to choose a bidding function $b_i(v)$ that takes points in $[0, w]$ and gives a positive real bid. Because of the symmetry property, it should be the case that all players have the same payoff function and that all optimal strategies for each player are the same for all other players.

Now suppose that we have a player with bidding function $b = b(v)$ for a valuation $v \in [0, w]$. The payoff to the player is given by

$$u(b(v), v) = Prob(b(v) \text{ is high bid})v - E(\text{payment for bid } b(v)).$$

We have to use the expected payment for the bid $b(v)$ because we don't know whether the bid $b(v)$ will be a winning bid. The probability that $b(v)$ is the high bid is

$$Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\}).$$

We will simplify notation a bit by setting

$$f(b(v)) = Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\}),$$

and then write the payoff as

$$u(b(v), v) = f(b(v))v - E(b(v)).$$

We want the bidder to maximize this payoff by choosing a bidding strategy $\beta(v)$.

[8]William Vickrey 1914–1996, won the 1996 Nobel Prize in economics primarily for his foundational work on auctions. He earned a BS in mathematics from Yale in 1935 and a PhD in economics from Columbia University.

One property of $\beta(v)$ should be obvious, namely, as the valuation increases, the bid must increase. The fact that $\beta(v)$ is strictly increasing as a function of v can be proved using some **convex analysis** but we will skip the proof.

Once we know the strict increasing property, the fact that all bidders are essentially the same leads us to the conclusion that the bidder with the highest valuation wins the auction.

Let's take the specific example that bidder's valuations are uniform on $[0, w] = [0, 1]$. Then for each player, we have

$$F(v) = \begin{cases} 0 & \text{if } v < 0; \\ v & \text{if } 0 \le v \le 1; \\ 1 & \text{if } v > 1. \end{cases}$$

Remark. Whenever we say in the following that we are normalizing to the interval $[0, 1]$, this is not a restriction because we may always transform from an interval $[r, R]$ to $[0, 1]$ and the reverse by the linear transformation $t = (s - r)/(R - r),\ r \le s \le R$, or $s = r + (R - r)t, 0 \le t \le 1$.

We can now establish the following theorem.[9]

Theorem 4.4.3 *Suppose that valuations $V_1, \ldots, V_N$, are uniformly distributed on $[0, 1]$, and the expected payoff function for each bidder is*

$$u(b(v), v) = f(b(v))v - E(b(v)), \text{ where} \tag{4.4.1}$$
$$f(b(v)) = Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\}).$$

Then there is a unique Nash equilibrium $(\beta, \ldots, \beta)$ given by

$$\beta(v_i) = v_i, \quad i = 1, 2, \ldots, N$$

in the case when we have an English auction, and

$$\beta(v_i) = \left(1 - \frac{1}{N}\right) v_i, \quad i = 1, 2, \ldots, N,$$

in the case when we have a Dutch auction. In either case the expected payment price for the object is

$$p^* = \frac{N - 1}{N + 1}.$$

Proof. To see why this is true, we start with the Dutch auction result. Since all players are indistinguishable, we might as well say our that guy is player 1. Suppose that player 1 bids $b = \beta(v)$. Then the probability that she wins the object is given by

$$Prob(\beta(\max\{V_2, \ldots, V_N\}) < b).$$

[9] See the article by Wolfstetter [31] for a very nice presentation and many more results on auctions, including common value auctions.

Now here is where we use the fact that β is strictly increasing, because then it has an inverse, $v = \beta^{-1}(b)$, and so we can say

$$\begin{aligned} f(b) &= Prob(\beta(\max\{V_2, \dots, V_N\}) < b) \\ &= Prob(\max\{V_2, \dots, V_N\} < \beta^{-1}(b)) = Prob(V_i < \beta^{-1}(b), i = 2, \dots, N) \\ &= F(\beta^{-1}(b))^{N-1} = [\beta^{-1}(b)]^{N-1} = v^{N-1}, \end{aligned}$$

because all valuations are independent and identically distributed. The next-to-last-equality is because we are assuming a uniform distribution here. The function $f(b)$ is the probability of winning the object with a bid of b.

Now, for the given bid $b = \beta(v)$, player 1 wants to maximize her expected payoff. The expected payoff (4.4.1) becomes

$$\begin{aligned} u(\beta(v), v) &= f(\beta(v))v - E(\beta(v)) \\ &= f(b)v - (bProb(\text{win}) + 0 \cdot Prob(\text{lose})) \\ &= f(b)v - bf(b) = (v - b)f(b). \end{aligned}$$

Taking a derivative of $u(b, v)$ with respect to b, evaluating at $b = \beta(v)$ and setting to zero, we get the condition

$$f'(\beta)(v - \beta) - f(\beta) = 0. \tag{4.4.2}$$

Since $f(b) = [\beta^{-1}(b)]^{N-1} = v^{N-1}, v = \beta^{-1}(b)$, we have

$$\frac{df(b)}{db} = (N-1)[\beta^{-1}(b)]^{N-2}\frac{d\beta^{-1}(b)}{db}.$$

Therefore, after dividing out the term $[\beta^{-1}(b)]^{N-2}$, the condition (4.4.2) becomes

$$(N-1)[\beta^{-1}(b) - b]\frac{d\beta^{-1}(b)}{db} - \beta^{-1}(b) = 0.$$

Let's set $y(b) = \beta^{-1}(b)$ to see that this equation becomes

$$(N-1)[y(b) - b]y'(b) - y(b) = 0. \tag{4.4.3}$$

This is a **first-order ordinary differential equation** for $y(b)$, that we may solve to get

$$y(b) = \beta^{-1}(b) = v = \frac{N}{N-1}b.$$

Actually, we may find the general solution of (4.4.3) using Maple fairly easily. The Maple commands to do this are

```
> ode:=a*(y(x)-x)*diff(y(x),x)=y(x);
> dsolve(ode,y(x));
```

That's it, just two commands without any packages to load. We are setting $a = N-1$. Here is what Maple gives you:

$$x - \frac{a}{a+1}y(x) - (y(x))^{-a}\ \_C1 = 0.$$

The problem with this is that we are looking for the solution with $y(0) = 0$ because when the valuation of the item is zero, the bid must be zero. Notice that the term $y(x)^{-a}$ is undefined if $y(0) = 0$. Consequently, we must take $C1 = 0$, leaving us with $x - (a/(a+1))y(x) = 0$ which gives $y(x) = ((a+1)/a)x$. Now we go back to our original variables to get $y(b) = v = \beta^{-1}(b) = (N/(N-1))b$. Solving for b in terms of v we get

$$b = \beta(v) = \left(1 - \frac{1}{N}\right)v,$$

which is the claimed optimal bidding function in a Dutch auction.

Next we calculate the expected payment. In a Dutch auction, we know that the payment will be the highest bid. We know that is going to be the random variable $\beta(\max\{V_1, \ldots, V_N\})$, which is the optimal bidding function evaluated at the largest of the random valuations. Then

$$\begin{aligned} E(\beta(\max\{V_1,\ldots,V_N\})) &= E\left[\left(1 - \frac{1}{N}\right)\max\{V_1,\ldots,V_N\}\right] \\ &= \left(1 - \frac{1}{N}\right)E[\max\{V_1,\ldots,V_N\}] \\ &= \frac{N-1}{N+1} \end{aligned}$$

because $E[\max\{V_1, \ldots, V_N\}] = N/(N+1)$, as we will see next, when the V_i values are uniform on $[0, 1]$.

Here is why $E[\max\{V_1, \ldots, V_N\}] = N/(N+1)$. The cumulative distribution function of $Y = \max\{V_1, \ldots, V_N\}$ is derived as follows. Since the valuations are independent and all have the same distribution,

$$F_Y(x) = Prob(\max\{V_1,\ldots,V_N\} \le x) = P(V_i \le x)^N = F_V(x)^N.$$

Then the density of Y is

$$f_Y(x) = F_Y'(x) = N(F_V(x))^{N-1} f_V(x).$$

In case V has a uniform distribution on $[0, 1]$, $f_V(x) = 1, F_V(x) = x, 0 < x < 1$, and so

$$f_Y(x) = Nx^{N-1}, 0 < x < 1 \implies E[Y] = \int_0^1 x f_Y(x)\,dx = \frac{N}{N+1}.$$

So we have verified everything for the Dutch auction case.

Next we solve the English auction game. We have already discussed informally that in an English auction each bidder should bid his or her true valuation so that $\beta(v) = v$. Here is a formal statement and proof.

Theorem 4.4.4 *In an English auction with valuations $V_1, \dots, V_N$ and $V_i = v_i$ known to player i, then player i's optimal bid is v_i.*

Proof.

1. Suppose that player i bids $b_i > v_i$, more than her valuation. Recall that an English auction is equivalent to a second-price sealed-bid auction. Using that we calculate her payoff as

$$u_i(b_1, \dots, b_i, \dots, b_N) \tag{4.4.4}$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k \geq 0 & \text{if } v_i \geq \max_{k \neq i} b_k; \\ v_i - \max_{k \neq i} b_k < 0 & \text{if } v_i < \max_{k \neq i} b_k < b_i; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

(We are ignoring possible ties.) To see why this is her payoff, if $v_i \geq \max_{k \neq i} b_k$, then player i's valuation is more than the bids of all the other players, so she wins the auction (since $b_i > v_i$) and pays the highest bid of the other players (which is $\max_{k \neq i} b_k$), giving her a payoff of $v_i - \max_{k \neq i} b_k \geq 0$.

If $v_i < \max_{k \neq i} b_k < b_i$, she values the object as less than at least one other player but bids more than this other player. So she pays $\max_{k \neq i} b_k$, and her payoff is $v_i - \max_{k \neq i} b_k < 0$. In the last case, she does not win the object, and her payoff is zero.

2. If player i's bid is $b_i \leq v_i$, then her payoff is

$$u_i(b_1, \dots, b_i, \dots, b_N) \tag{4.4.5}$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k > 0 & \text{if } v_i \geq b_i > \max_{k \neq i} b_k; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

So, in all cases the payoff function in (4.4.5) is at least as good as the payoff in (4.4.4) and better in some cases. Therefore, player i should bid $b_i \leq v_i$.

3. Now what happens if player i bids $b_i < v_i$? In that case,

$$u_i(b_1, \dots, b_i, \dots, b_N)$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k > 0 & \text{if } v_i > b_i > \max_{k \neq i} b_k; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

Player i will get a strictly positive payoff if $v_i > b_i > \max\{b_k : k \neq i\}$.

Putting steps (3) and (2) together, we therefore want to maximize the payoff for player i using bids b_i subject to the constraint $v_i \geq b_i > \max_{k \neq i} b_k$. Looking at the payoffs, the biggest that u_i can get is when $b_i = v_i$. In other words, since $b_i < v_i$ and the probability increases,

$$Prob(b_i > \max_{k \neq i} b_k(V_k)) \leq Prob(v_i > \max_{k \neq i} b_k(V_k)).$$

The largest probability, and therefore the largest payoff, will occur when $b_i = v_i$. $\square$

Having proved that the optimal bid in an English auction is $b_i = \beta(v_i) = v_i$, we next calculate the expected payment and complete the proof of Theorem 4.4.3. The winner of the English auction with uniform valuations makes the payment of the second highest bid, which is given by

$$\text{If } V_j = \max\{V_1, \ldots, V_N\}, \text{ then } E[\max_{i \neq j}\{V_i\}] = \frac{N-1}{N+1}.$$

This follows from knowing the density of the ranaom variable $Y = \max_{i \neq j}\{V_i\}$, the second highest valuation. In the case when V is uniform on $[0, 1]$ the density of Y is

$$f_Y(x) = N(N-1)x(1-x)^{N-1}, 0 < x < 1 \implies E[Y] = \frac{N-1}{N+1}.$$

You can refer to the Appendix B for a derivation of this using order statistics. Therefore we have proved all parts of Theorem 4.4.3. $\square$

One major difference between English and Dutch auctions is the risk characteristics as measured by the variance of the selling price (see Wolfstetter [31] for the derivation).

1. In an English auction, the selling price random variable is the second highest valuation, that we write as $P_E = \max_2\{V_1, \ldots, V_N\}$. In probability theory this is an **order statistic** (see Appendix B), and it is shown that if the valuations are all uniformly distributed on $[0, 1]$, then

$$Var(P_E) = \frac{2(N-1)}{(N+1)^2(N+2)}.$$

2. In a Dutch auction, equivalent to a first-price sealed-bid auction, the selling price is $P_D = \beta(\max\{V_1, \ldots, V_N\})$, and we have seen that with uniform valuations

$$\beta(\max\{V_1, \ldots, V_N\}) = \frac{N-1}{N} \max\{V_1, \ldots, V_N\}.$$

Consequently

$$\begin{aligned} Var(P_D) &= Var(\beta(\max\{V_1, \dots, V_N\})) \\ &= \left(\frac{N-1}{N}\right)^2 Var(\max\{V_1, \dots, V_N\}) \\ &= \frac{(N-1)^2}{N(N+1)^2(N+2)}. \end{aligned}$$

We claim that $Var(P_D) < Var(P_E)$. That will be true if

$$\frac{2(N-1)}{(N+1)^2(N+2)} > \frac{(N-1)^2}{N(N+1)^2(N+2)}.$$

After using some algebra, this inequality reduces to the condition $2 > \frac{(N-1)}{N}$, which is absolutely true for any $N \geq 1$. We conclude that Dutch auctions are less risky for the seller than are English auctions, as measured by the variance of the payment.

If the valuations are not uniformly distributed, the problem will be much harder to solve explicitly. But there is a general formula for the Nash equilibrium still assuming independence and that each valuation has distribution function $F(v)$. If the distribution is continuous, the Dutch auction will have a unique Nash equilibrium given by

$$\beta(v) = v - \frac{1}{F(v)^{N-1}} \int_r^v F(y)^{N-1}\, dy.$$

The proof of this formula comes basically from having to solve the differential equation that we derived earlier for the Nash equilibrium

$$(N-1)f(y(b))(y(b)-b)y'(b) - F(y(b)) = 0,$$

where we have set $y(b) = \beta^{-1}(b)$, and $F(y)$ is the cumulative distribution function and $f = F'$ is the density function.

The expected payment in a Dutch auction with uniformly distributed valuations was shown to be $E[P_D] = (N-1)/(N+1)$. The expected payment in an English auction was also shown to be $E[P_E] = (N-1)/(N+1)$. You will see in the problems that the expected payment in an **all pay** auction, in which all bidders will have to pay their bid, is also $(N-1)/(N+1)$. What's going on? Is the expected payment for an auction always $(N-1)/(N+1)$, at least for valuations that are uniformly distributed? The answer is "Yes," and not just for uniform distributions:

Theorem 4.4.5 *Any symmetric private value auction with identically distributed valuations, satisfying the following conditions, always has the same expected payment to the seller of the object:*

1. *They have the same number of bidders (who are risk-neutral).*

2. *The object at auction always goes to the bidder with the highest valuation.*

3. *The bidder with the lowest valuation has a zero expected payoff.*

This is known as the **revenue equivalence theorem.**

In the remainder of this section we will verify directly as sort of a review that **linear trading rules** are the way to bid when the valuations are uniformly distributed in the interval $[r, R]$, where r is the reserve price. For simplicity we consider only two bidders who will have payoff functions

$$u_1((b_1, v_1), (b_2, v_2)) = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2; \\ \dfrac{v_1 - b_1}{2} & \text{if } b_1 = b_2; \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

and

$$u_2((b_1, v_1), (b_2, v_2)) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1; \\ \dfrac{v_2 - b_2}{2} & \text{if } b_1 = b_2; \\ 0 & \text{if } b_2 < b_1. \end{cases}$$

We have explicitly indicated that each player has two variables to work with, namely, the bid and the valuation. Of course the bid will depend on the valuation eventually. The independent valuations of each player are random variables V_1, V_2 with identical cumulative distribution function $F_V(v)$. Each bidder knows his or her own valuation but not the opponent's. So the expected payoff to player 1 is

$$U_1(b_1, b_2) \equiv Eu_1(b_1, v_1, b_2(V_2), V_2) = Prob(b_1 > b_2(V_2))(v_1 - b_1)$$

because in all other cases the expected value is zero. In the case $b_1 = b_2(V_2)$ it is zero since we have continuous random variables. In addition, we write player 1's bid and valuation with lower case letters because player 1 knows her own bid and valuation with certainty. Similarly

$$U_2(b_1, b_2) \equiv Eu_2(b_1(V_1), V_1, b_2, v_2) = Prob(b_2 > b_1(V_1))(v_2 - b_2).$$

A Nash equilibrium must satisfy

$$U_1(b_1^*, b_2^*) \geq U_1(b_1, b_2^*) \text{ and } U_2(b_1^*, b_2^*) \geq U_1(b_1^*, b_2).$$

In the case that the valuations are uniform on $[r, R]$, we will verify that the bidding rules

$$\beta_1^*(v_1) = \frac{r + v_1}{2} \text{ and } \beta_2^*(v_2) = \frac{r + v_2}{2}$$

constitute a Nash equilibrium. We only need to show it is true for β_1^* because it will be the same procedure for β_2^*.

So, by the assumptions, we have

$$\begin{aligned} Prob(b_1 > \beta_2^*(V_2)) &= Prob\left(b_1 > \frac{r + V_2}{2}\right) \\ &= Prob(2b_1 - r > V_2) \\ &= \frac{(2b_1 - r) - r}{R - r}, \end{aligned}$$

if $r/2 < b_1 < (r + R)/2$, and the expected payoff

$$\begin{aligned} U_1(b_1, \beta_2^*(V_2)) &= (v_1 - b_1)Prob(b_1 > \beta_2^*(V_2)) \\ &= (v_1 - b_1)Prob(V_2 < 2b_1 - r) \\ &= \begin{cases} 0 & \text{if } b_1 < \frac{r}{2}; \\ (v_1 - b_1)\dfrac{2b_1 - 2r}{R - r} & \text{if } \frac{r}{2} < b_1 < \frac{(r+R)}{2}; \\ v_1 - b_1 & \text{if } \frac{(r+R)}{2} < b_1. \end{cases} \end{aligned}$$

We want to maximize this as a function of b_1. To do so, let's consider the case $r/2 < b_1 < (r + R)/2$ and set

$$g(b_1) = (v_1 - b_1)\frac{2b_1 - 2r}{R - r}.$$

The function g is strictly concave down as a function of b_1 and has a unique maximum at $\beta_1 = r/2 + v_1/2$, as the reader can readily verify by calculus. We conclude that $\beta_1^*(v_1) = (r+v_1)/2$ maximizes $U_1(b_1, \beta_2^*)$. This shows that $\beta_1^*(v_1)$ is a best response to $\beta_2^*(v_2)$. We have verified the claim.

Remark. A similar argument will work if there are more than two bidders. Here are the details for three bidders. Start with the payoff functions (we give only the one for player 1 since the others are similar):

$$u_1(b_1, b_2, b_3) = \begin{cases} v_1 - b_1 & \text{if } b_1 > \max\{b_2, b_3\}; \\ \dfrac{v_1 - b_1}{k} & \text{if there are } k = 2, 3 \text{ high bids}; \\ 0 & \text{otherwise.} \end{cases}$$

The expected payoff if player 1 knows her own bid and valuation, but V_2, V_3 are random variables is

$$\begin{aligned} U_1(b_1, b_2, b_3) &= Eu_1(b_1, b_2(V_2), b_3(V_3)) \\ &= Prob(b_1 > \max\{b_2(V_2), b_3(V_3)\})(v_1 - b_1). \end{aligned}$$

Assuming independence of valuations, we obtain

$$U_1(b_1, b_2, b_3) = Prob(b_1 > b_2(V_2))Prob(b_1 > b_3(V_3))(v_1 - b_1).$$

In the exercises you will find the optimal bidding functions.

EXAMPLE 4.8

Two players are bidding in a first-price sealed-bid auction for a 1901s United States penny, a very valuable coin for collectors. Each player values it at somewhere between \$750K (where K= 1000) and \$1000K dollars with a uniform distribution (so $r = 750, R = 1000$). In this case, each player should bid $\beta_i(v_i) = \frac{1}{2}(750 + v_i)$. So, if player 1 values the penny at \$800K, she should optimally bid $\beta_1(800)$ =775K. Of course, if bidder 2 has a higher valuation, say, at \$850K, then player 2 will bid $\beta_2(850) = \frac{1}{2}(750 + 850)$ =800K and win the penny. It will sell for \$800K and player 2 will benefit by $850 - 800 = 50$K.

On the other hand, if this were a second-price sealed-bid auction, equivalent to an English auction, then each bidder would bid their own valuations. In this case $b_1 =$ \$800K and $b_2 =$ \$850K. Bidder 2 still gets the penny, but the selling price is the same. On the other hand, if player 1 valued the penny at \$775K, then $b_1 =$ \$775K, and that would be the selling price. It would sell for \$25K less than in a first-price auction.

We conclude with a very nice application of auction theory to economic models appearing in the survey article by Wolfstetter [31].

EXAMPLE 4.9

Application of Auctions to Bertrand's Model of Economic Competition. If we look at the payoff for a bidder in a Dutch auction we have seen in Theorem 4.4.3

$$u(b(v), v) = f(b(v))v - E(b(v)) = f(b)v - bf(b) = (v - b)f(b),$$

where

$$f(b(v)) = Prob(b(v) > \max\{b(V_1), \dots, b(V_N)\}).$$

Compare u with the profit function in a Bertrand model of economic competition. This looks an awful lot like a profit function in an economic model by identifying v as price per gadget, b as a cost, and $f(b)$ as the probability the firm sets the highest price. That is almost correct but not exactly. Here is the exact analogy.

We consider Bertrand's model of competition between $N \geq 2$ identical firms. Assume that each firm has a constant and identical unit cost of production

$c > 0$, and that each firm knows its own cost but not those of the other firms. For simplicity, we assume that each firm considers the costs of the other firms as random variables with a uniform distribution on $[0, 1]$.

A basic assumption in the Bertrand model of competition is that the firm offering the **lowest price** at which the gadgets are sold will win the market. With that assumption, we may make the Bertrand model to be a Dutch auction with a little twist, namely, the buyers are bidding to win the **market** at the lowest selling price of the gadgets. So, if we think of the **market as the object up for auction**, each firm wants to bid for the object and obtain it at the **lowest possible selling price**, namely, the lowest possible price for the gadgets.

We take a strategy to be a function that converts a unit cost for the gadget (valuation) into a price for the gadget. Then we equate the price to set for a gadget to a bid for the object (market).

The Bertrand model is really an inverse Dutch auction, so if we invert variables, we should be able to apply the Dutch auction results. Here is how to do that. Call a valuation $v = 1 - c$. Then call the price to be set for a gadget as one minus the bid, $p = 1 - b$, or $b = 1 - p$. In this way, low costs lead to high valuations, and low prices imply higher bids, and conversely. So the highest bid (=lowest price) wins the market. The expected payoff function in the Dutch auction is then

$$\begin{aligned} u(b(v), v) &= (v - b)f(b) \\ &= (1 - c - ((1 - p))f(1 - p) \\ &= (p - c)f(1 - p), \end{aligned}$$

where

$$\begin{aligned} f(1 - p) &= Prob(1 - p(c) > \max\{1 - p(C_1), \dots, 1 - p(C_N)\}) \\ &= Prob(p(c) < \min\{p(C_1), \dots, p(C_N)\}). \end{aligned}$$

Putting them together, we have

$$w(p, c) = (p - c)Prob(p(c) < \min\{p(C_1), \dots, p(C_N)\})$$

as the profit function to each firm, because $w(p, c)$ is the expected net revenue to the firm if the firm sets price p for a gadget and has cost of production c per gadget, but the firm has to set the lowest price for a gadget to capture the market.

Now in a Dutch auction we know that the optimal bidding function is $b(v) = ((N - 1)/N)v$, so that

$$p^*(c) = 1 - b(c) = 1 - \frac{N-1}{N}v = 1 - \frac{N-1}{N}(1 - c) = \frac{c(N-1)+1}{N}.$$

This is the optimal expected price to set for gadgets as a function of the production cost. Next, the expected payment in a Dutch auction is

$$E(b(\max\{V_1, \dots, V_N\})) = \frac{N-1}{N+1}.$$

So

$$\begin{aligned}E[p^*(C_1, \dots, C_N)] &= 1 - E[b(\max\{V_1, \dots, V_N\})] \\ &= 1 - \frac{N-1}{N+1} \\ &= \frac{2}{N+1}.\end{aligned}$$

We conclude that the equilibrium expected price of gadgets in this model for each firm as a function of the unit cost of production will be

$$p^*(c) = \frac{c(N-1)+1}{N}.$$

PROBLEMS

4.24 Show directly that $\beta_i^* = \frac{1}{3}(r + 2v_i), i = 1, 2, 3$ is a Nash equilibrium for the problem with three bidders.

4.25 In an **all-pay auction** all the bidders must actually pay their bids to the seller, but only the high bidder gets the object up for sale. This type of auction is also called a **charity auction**. By following the procedure for a Dutch auction, show that the equilibrium bidding function for all players is $\beta(v) = ((N-1)/N)v^N$, assuming that bidders' valuations are uniformly distributed on the normalized interval $[0, 1]$. Find the expected total amount collected by the seller.

BIBLIOGRAPHIC NOTES

The voter model in Example 4.2 is a standard application of game theory to voting. More than two candidates are allowed for a model to get an $N > 2$ person game. Here we follow the presentation in the reference by Aliprantis and Chakrabarti [1] in the formulation of the payoff functions.

The economics models presented in Section 4.2 have been studied for many years, and these models appear in both the game theory and economic theory literature. The generalization of the Cournot model in Theorem 4.2.1 due to Wald is proved in full generality in the book by Parthasarathy and Raghavan [21]. The Bertrand model and its ramifications (Problems 4.11 and 4.12) follows Ferguson [3] and Aliprantis and Chakrabarti [1]. The traveler's paradox Example 4.4 (from reference [5]) is also well known as a critique of both the Nash equilibrium concept and a Bertrand-type model of markets. Entry deterrence is a modification of the same model in Ferguson's notes [3]. The theory of duels as examples of games of timing is due to Karlin [11]. Our derivation of the payoff functions and the solution of the silent continuous duel also follows Karlin [11].

The land division game in Problem 4.15 is a wonderful calculus and game theory problem. For this particular geometry of land we take the formulation in Jones [7]. Problem 4.17 is formulated as an advertising problem in Ferguson [3]. H. Gintis [4] is an outstanding source of problems in many areas of game theory. Problems 4.18 and 4.19 are modifications of the tobacco farmer problem and Social Security problem in Gintis [4].

The theory of auctions has received considerable interest since the original analysis by W. Vickrey beginning in the 1960s. In this section we follow the survey by Wolfstetter [31], where many more results are included. In particular, we have not discussed the theory of auctions with a common value because it uses conditional expectations, which is beyond our level of prerequisites. The application of auction theory to solving an otherwise complex Bertrand economic model in Example 4.9 is presented by Wolfstetter [31]. The direct verification that linear trading rules form a Nash equilibrium is from Aliprantis and Chakrabarti [1].

CHAPTER 5

COOPERATIVE GAMES

Government and cooperation are in all things the laws of life; anarchy and competition the laws of death.

–John Ruskin, *Unto this Last*

We must all hang together, or assuredly we will all hang separately.

–Benjamin Franklin, at the signing of the Declaration of Independence

5.1 COALITIONS AND CHARACTERISTIC FUNCTIONS

There are $n > 1$ players numbered $1, 2, \ldots, n$. **In this chapter we use the letter n to denote the number of players and N to denote the set of all the players** $N = \{1, 2, \ldots, n\}$. We consider a game in which the players may choose to cooperate by forming coalitions. A **coalition** is any subset $S \subset N$, or numbered collection of the players. Since there are 2^n possible subsets of N, there are 2^n possible coalitions. Coalitions form in order to benefit every member of the coalition so that all members might receive more than they could individually on their own. In this section we

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try to determine a **fair allocation** of the benefits of cooperation among the players to each member of a coalition. A major problem in cooperative game theory is to precisely define what **fair** means. The definition of **fair** of course determines how the allocations to members of a coalition are made.

First we need to quantify the benefits of a coalition through the use of a real-valued function, called the **characteristic function**. The characteristic function of a coalition $S \subset N$ is the **largest** guaranteed payoff to the coalition.

Definition 5.1.1 *Let 2^N denote the set of all possible coalitions for the players N. If $S = \{i\}$ is a coalition containing the single member i, we simply denote S by i.*

Any function $v : 2^N \to \mathbb{R}$ satisfying

$$v(\emptyset) = 0 \text{ and } v(N) \geq \sum_{i=1}^{n} v(i)$$

is a characteristic function (of an n-person cooperative game).

In other words, the only condition placed on a characteristic function is that the benefit of the empty coalition be zero and the benefit of the **grand coalition** N, consisting of all the players, be at least the sum of the benefits of the individual players if no coalitions form. This means that every one pulling together should do better than each player on his or her own. With that much flexibility, games may have more than one characteristic function.Let's start with some simple examples.

EXAMPLE 5.1

1. Suppose that there is a factory with n workers each doing the same task. If each worker earns the same amount b dollars, then we can take the characteristic function to be $v(S) = b|S|$, where $|S|$ is the number of workers in S. Clearly, $v(\emptyset) = b|\emptyset| = 0$, and $v(N) = b|N| = bn = b\sum_{i=1}^{n} v(i)$.

2. Suppose that the owner of a car, labeled player 1, offers it for sale for \$M. There are two customers interested in the car. Customer C, labeled player 2, values the car at c and customer D, labeled player 3, values it at d. Assume that the price is nonnegotiable. This means that if $M > c$ and $M > d$, then no deal will be made. We will assume then that $M < \min\{c, d\}$, and, for definiteness we may assume $M < c \leq d$. The set of possible coalitions are $2^N \equiv \{123,\ 12,\ 13,\ 23,\ 1,\ 2,\ 3,\ \emptyset\}$. For simplicity we are dropping the braces in the notation for any individual coalition.

It requires a seller and a buyer to reach a deal. Therefore, we may define the characteristic function as follows:

$$v(123) = d, \quad v(1) = M, \quad v(\emptyset) = 0$$
$$v(13) = d, \quad v(12) = c, \quad v(23) = 0,$$
$$v(2) = v(3) = 0.$$

Why? Well, $v(123) = d$ because the car will be sold for d, $v(1) = M$ because the car is worth M to player 1, $v(13) = d$ because player 1 will sell the car to player 3 for $d > M$, $v(12) = c$ because the car will be sold to player 2 for $c > M$, and so on. The reader can easily check that v is a characteristic function.

3. A customer wants to buy a bolt and a nut for the bolt. There are three players but player 1 owns the bolt and players 2 and 3 each own a nut. A bolt together with a nut is worth 5. We could define a characteristic function for this game as

$$v(123) = 5,\ v(12) = v(13) = 5,\ \ v(1) = v(2) = v(3) = 0, \quad \text{and} \quad v(\emptyset) = 0.$$

In contrast to the car problem $v(1) = 0$ because a bolt without a nut is worthless to player 1.

4. A small research drug company, labeled 1, has developed a drug. It does not have the resources to get FDA (Food and Drug Administration) approval or to market the drug, so it considers selling the rights to the drug to a big drug company. Drug companies 2 and 3 are interested in buying the rights but only if both companies are involved in order to spread the risks. Suppose that the research drug company wants \$1 billion, but will take \$100 million if only one of the two big drug companies are involved. The profit to a participating drug company 2 or 3 is \$5 billion, which they split. Here is a possible characteristic function with units in billions:

$$v(1) = v(2) = v(3) = 0, v(12) = 0.1, v(13) = 0.1, v(23) = 0, v(123) = 5,$$

because any coalition which doesn't include player 1 will be worth nothing.

5. A **simple game** is one in which $v(S) = 1$ or $v(S) = 0$ for all coalitions S. A coalition with $v(S) = 1$ is called a **winning coalition** and one with $v(S) = 0$ is a **losing coalition**. For example, if we take $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ otherwise, we have a simple game that is a model of majority voting. If a coalition contains more than half of the players, it has the majority of votes and is a winning coalition.

6. In any bimatrix (A, B) nonzero sum game we may obtain a characteristic function by taking $v(1) = value(A)$, $v(2) = value(B^T)$, and $v(12)$ =sum of largest payoff pair in (A, B). Checking that this is a characteristic function is skipped. The next example works one out.

EXAMPLE 5.2

In this example we will construct a characteristic function for a version of the prisoner's dilemma game in which we assumed that there was no cooperation. Now we will assume that the players may cooperate and negotiate. One form of the prisoner's dilemma is with the bimatrix

$$\begin{bmatrix} (8,8) & (0,10) \\ (10,0) & (2,2) \end{bmatrix}.$$

Here $N = \{1,2\}$ and the possible coalitions are $2^N = \{\emptyset, 1, 2, 12\}$. If the players do not form a coalition, they are playing the nonzero sum noncooperative game. Each player can guarantee only that they receive their **safety level**. For player I that is the value of the zero sum game with matrix $A = \begin{bmatrix} 8 & 0 \\ 10 & 2 \end{bmatrix}$, which is $value(A) = 2$. For player II the safety level is the value of the game with matrix $B^T = \begin{bmatrix} 8 & 0 \\ 10 & 2 \end{bmatrix}$. Again $value(B^T) = 2$.

Thus we could define $v(1) = v(2) = 2$ as the characteristic function for single member coalitions. Now, if the players cooperate and form the coalition $S = \{12\}$, can they do better? Figure 5.1, which is generated by Maple, shows what is going on. The parallelogram is the boundary of the set of all possible payoffs to the two players when they use all possible mixed strategies. The

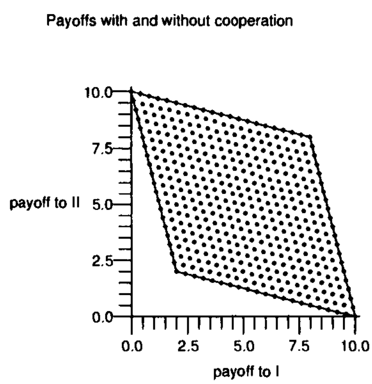


Figure 5.1 Payoff to player I versus payoff to player II.

vertices are the pure payoff pairs in the bimatrix. You can see that without cooperation they are each at the lower left vertex point $(2,2)$. Any point in the parallelogram is attainable with some suitable selection of mixed strategies if the players cooperate. Consequently, the maximum benefit to cooperation for both players results in the payoff pair at vertex point $(8,8)$, and so we set

$v(12) = 16$ as the **maximum sum of the benefits** awarded to each player. With $v(\emptyset) = 0$, the specification is complete.

As an aside, notice that $(8,8)$ is Pareto-optimal as is any point on the two lines connecting $(0,10)$ and $(8,8)$ and $(10,0)$ with $(8,8)$. This is the Pareto-optimal boundary of the payoff set. This is clear from Figure 5.1 because if you take any point on the lines, you cannot simultaneously move up and right and remain in the set.

EXAMPLE 5.3

Here is a much more complicated but systematic way to create a characteristic function given any n-person, noncooperative, nonzero sum game. The idea is to create a two-person zero sum game in which any given coalition is played against a pure opposing coalition consisting of everybody else. The two players are the coalition S versus all the other players, which is also a coalition $N - S$. The characteristic function will be the value of the game associated with each coalition S. The way to set this up will become clear if we go through an example.

Let's work out a specific example using a three-player game. Suppose that we have a three-player nonzero sum game with the following matrices:

$\boxed{3}$ plays A		player $\boxed{2}$	
		A	B
player $\boxed{1}$	A	$(1,1,0)$	$(4,-2,2)$
	B	$(1,2,-1)$	$(3,1,-1)$

$\boxed{3}$ plays B		player $\boxed{2}$	
		A	B
player $\boxed{1}$	A	$(-3,1,2)$	$(0,1,1)$
	B	$(2,0,-1)$	$(2,1,-1)$

Each player has the two pure strategies A and B. Because there are three players, in matrix form this could be represented in three dimensions (a cube $3 \times 3 \times 3$ matrix). That is a little hard to write down, so instead we have broken this into two 2×2 matrices. Each matrix assumes that player 3 plays one of the two strategies that is fixed. Now we want to find the characteristic function of this game.

We need to consider all of the zero sum games which would consist of the two-player coalitions versus each player, and the converse, which will switch the roles from maximizer to minimizer and vice versa. The possible two-player coalitions are $\{12\}, \{13\}, \{23\}$ versus the single-player coalitions, $\{1,2,3\}$, and conversely. For example, one such possible game is $S = \{12\}$ versus $N - S = 3$, in which player $S = \{12\}$ is the row player and player 3 is the column player We also have to consider the game 3 versus $\{12\}$, in which player 3 is the row player and coalition $\{12\}$ is the column player. So now we go through the construction.

1. **Play** $S = \{12\}$ **versus** $\{3\}$**.** players 1 and 2 team up against player 3. We first write down the associated matrix game.

$\boxed{12}$ versus $\boxed{3}$		player $\boxed{3}$	
		A	B
player $\boxed{12}$	AA	2	−2
	AB	2	1
	BA	3	2
	BB	4	3

For example, if 1 plays A and 2 plays A and 3 plays B, the payoffs in the nonzero sum game are $(-3, 1, 2)$ and so the payoff to player 12 is $-3 + 1 = -2$, the **sum of the payoff to player 1 and player 2**, which is our coalition. Now we calculate the value of the zero sum two-person game with this matrix to get the $value(12 \text{ vs. } 3) = 3$ and we write $v(12) = 3$. This is the maximum possible guaranteed benefit to coalition $\{12\}$ because it even assumes that player 3 is actively working against the coalition.

In the game $\{3\}$ versus $\{12\}$ we have $\boxed{3}$ as the row player and players $\{12\}$ as the column player. We now want to know the maximum possible payoff to player 3 assuming that the coalition $\{12\}$ is actively working against player 3. The matrix is

$\boxed{3}$ versus $\boxed{12}$		$\boxed{12}$			
		AA	AB	BA	BB
$\boxed{3}$	A	0	2	−1	−1
	B	2	1	−1	−1

The value of this game is -1. Consequently, in the game $\{3\}$ versus $\{12\}$ we would get $v(3) = -1$. Observe that the game matrix for 3 versus 12 is **not** the transpose of the game matrix for 12 versus 3.

2. **Play** $S = \{13\}$ **versus** $\{2\}$. The game matrix is

$\boxed{13}$ versus $\boxed{2}$		player $\boxed{2}$	
		A	B
player $\boxed{13}$	AA	1	6
	AB	−1	1
	BA	0	2
	BB	1	1

We see that the value of this game is 1 so that $v(13) = 1$. In the game $\{2\}$ versus $\{13\}$ we have $\boxed{2}$ as the row player and the matrix

$\boxed{2}$ versus $\boxed{13}$		$\boxed{13}$			
		AA	AB	BA	BB
$\boxed{2}$	A	1	1	2	0
	B	-2	1	1	1

The value of this game is $\frac{1}{4}$, and so $v(2) = \frac{1}{4}$.

Continuing in this way, we summarize that the characteristic function for this three–person game is

$$\begin{aligned}
&v(1) = 1,\ v(2) = \frac{1}{4},\ v(3) = -1,\\
&v(12) = 3,\ v(13) = 1,\ v(23) = 1,\\
&v(123) = 4,\ v(\emptyset) = 0.
\end{aligned}$$

The value $v(123) = 4$ is obtained by taking the largest sum of the payoffs that they would achieve if they all cooperated. This number is obtained from the pure strategies: 3 plays A, 1 plays A, and 2 plays B with payoffs $(4, -2, 2)$. Summing these payoffs for all the players gives $v(123) = 4$. This is the most the players can get if they form a grand coalition, and they can get this only if all the players cooperate. The central question in cooperative game theory is how to allocate the reward of 4 to the three players. In this example, player 2 contributes a payoff of -2 to the grand coalition, so should player 2 get an equal share of the 4? On the other hand, the 4 can only be obtained if player 2 agrees to play strategy B, so player 2 does have to be induced to do this. What would be a **fair allocation**?

One more observation is that player 3 seems to be in a bad position. On her own she can be guaranteed to get only $v(3) = -1$, but the assistance of player 1 does help since $v(13) = 1$. Player 2 doesn't do that well on her own but does do better with player 3.

Remark. There is a general formula for the characteristic function obtained by converting an n-person nonzero sum game to a cooperative game. Given any coalition $S \subset N$, the characteristic function is

$$v(S) = \max_{X \in X_S} \min_{Y \in Y_{N-S}} \sum_{i \in S} E_i(X, Y) = \min_{Y \in Y_{N-S}} \max_{X \in X_S} \sum_{i \in S} E_i(X, Y),$$

where X_S is the set of mixed strategies for the coalition S, Y_{N-S} is the set of mixed strategies for the coalition $N - S$, $E_i(X, Y)$ is the expected payoff to player $i \in S$, and $\sum_{i\in S} E_i(X, Y)$ is the total payoff for each player in $i \in S$ and represents the payoff to the coalition S. The set of pure strategies for coalition S is the set of all combinations of pure strategies for the members of S. This definition of characteristic function satisfies the requirements to be a characteristic function and the property of superadditivity discussed below.

It is important to not be confused about the definition of characteristic function. A characteristic function is **any** function that satisfies $v(\emptyset) = 0, v(N) \geq \sum v(i)$. It does not have to be defined as we did in the examples with the matrices but that is a convenient way of obtaining one that will work.

Here are some additional observations and definitions.

Remarks on Characteristic Functions.

1. A very desirable property of a characteristic function is that it satisfy

$$v(S \cup T) \geq v(S) + v(T) \quad \text{for all } S, T \subset N, S \cap T = \emptyset.$$

This is called **superadditivity**. It says that the benefits of the larger consolidated coalition $S \cup T$ of the two separate coalitions S, T must be at least the total benefits of the individual coalitions S and T. Many results on cooperative games do not need superadditivity, but we will take it as an **axiom that our characteristic functions in all that follows must be superadditive**. With the assumption of superadditivity, the players have the incentive to form and join the grand coalition N.

2. A game is **inessential** if and only if $v(N) = \sum_{i=1}^{n} v(i)$. An **essential** game therefore is one with $v(N) > \sum_{i=1}^{n} v(i)$.

3. Any game with $v(S \cup T) = v(S) + v(T)$, for all $S, T \subset N, S \cap T = \emptyset$, is called an **additive** game. A game is inessential if and only if it is additive.

The word **inessential** implies that these games are not important. That turns out to be true. They turn out to be easy to analyze, as we will see.

To see why a characteristic function for an inessential game must be additive, we simply write down the definitions. In fact let $S, T \subset N$, $S \cap T = \emptyset$. Then

$$\begin{aligned}
v(N) &= \sum_{i=1}^{n} v(i) && \text{(inessential game)} \\
&= \sum_{i\in S} v(i) + \sum_{i\in T} v(i) + \sum_{i\in N-(S\cup T)} v(i) \\
&\leq v(S) + v(T) + v(N - (S \cup T)) && \text{(superadditivity)} \\
&\leq v(S \cup T) + v(N - (S \cup T)) && \text{(superadditivity)} \\
&\leq v(N) && \text{(superadditivity again).}
\end{aligned}$$

Since we now have equality throughout

$$v(S) + v(T) + v(N - (S \cup T)) = v(S \cup T) + v(N - (S \cup T)),$$

and so $v(S) + v(T) = v(S \cup T)$.

We need a basic definition regarding the allocation of rewards to each player. Recall that $v(N)$ represents the reward available if all players cooperate.

Definition 5.1.2 *Let x_i be a real number for each $i = 1, 2, \ldots, n$. A vector $\vec{x} = (x_1, \ldots, x_n)$ is an* **imputation** *if*

- $x_i \geq v(i)$ **(individual rationality)**
- $\displaystyle\sum_{i=1}^{n} x_i = v(N)$ **(group rationality)**

Each x_i represents the share of the value of $v(N)$ received by player i. The imputation $\vec{x}$ is also called a **payoff vector** *or an* **allocation**, *and we will use these words interchangeably.*

Remarks.

1. It is possible for x_i to be a negative number! That allows us to model coalition members that do not benefit and may be a detriment to a coalition.
2. Individual rationality means that the share received by player i should be at least what he could get on his own. Each player must be individually rational, or else why join the grand coalition?
3. Group rationality means that the total rewards allocated to each individual in the grand coalition should **equal** the total rewards available by cooperation. We know that $v(N) \geq \sum_i v(i) \geq \sum_i x_i$, just by definition. If in fact $\sum_i x_i < v(N)$, then each player could actually receive a bigger share than simply x_i; in fact, one possibility is an additional amount $(v(N) - \sum_i x_i)/n$. This says that the allocation x_i would be rejected by each player, so it must be true that $\sum_i x_i = v(N)$ for any reasonable allocation. Nothing should be left over.
4. Any inessential game, i.e., $v(N) = \sum_{i=1}^{n} v(i)$, has one and only one imputation and it is $\vec{x} = (v(1), \ldots, v(n))$. The verification is a simple exercise (see the problems). These games are uninteresting because there is no incentive for any of the players to form any sort of coalition and there is no wiggle room in finding a better allocation.

The main objective in cooperative game theory is to determine the imputation that results in a **fair** allocation of the total rewards. Of course, this will depend on the definition of **fair**, as we mentioned earlier. That word is not at all precise. If you change the meaning of **fair** you will change the imputation.

We begin by presenting a way to transform a given characteristic function for a cooperative game to one which is frequently easier to work with. It is called the **(0,1) normalization of the original game**. This is not strictly necessary, but it does simplify the computations in many problems. The normalized game will result in a characteristic function with $v(i) = 0, v(N) = 1$. In addition, any two games may be compared by comparing their normalized characteristic functions. If they are the same, the two games are said to be **strategically equivalent**.

The proof of the lemma will show how to make the conversion to $(0,1)$ normalized.

Lemma 5.1.3 *Any essential game with characteristic function v has a $(0,1)$ normalization with characteristic function v; that is, given the characteristic function $v(\cdot)$ there is a unique characteristic function $v'(\cdot)$ that satisfies $v'(N) = 1, v'(i) = 0, 1 \le i \le n$, and $v'(S) = cv(S) + \sum_{i \in S} a_i$ for some constants $c > 0, a_1, \ldots, a_n$. The constants are given by*

$$c \equiv \frac{1}{v(N) - \sum_{i=1}^{n} v(i)} \quad \textit{and} \quad a_i \equiv -c\, v(i), \quad i = 1, 2, \ldots, n.$$

Proof. Consider the $n + 1$ system of equations for constants $c, a_i, 1 \le i \le n$, given by

$$cv(i) + a_i = 0, \quad 1 \le i \le n,$$
$$cv(N) + \sum_{i=1}^{n} a_i = 1.$$

If we add up the first n equations, we get $c \sum v(i) + \sum a_i = 0$. Subtracting this from the second equation results in

$$c[v(N) - \sum v(i)] = 1,$$

and we can solve for $c > 0$ because the game is essential (so $v(N) > \sum v(i)$). Now that we have c, we set $a_i = -c\, v(i)$. Solving this system, we get

$$c = \frac{1}{v(N) - \sum v(i)} > 0, \qquad a_i = -\frac{v(i)}{v(N) - \sum_i v(i)}.$$

Then, for any coalition $S \subset N$ define the characteristic function

$$v'(S) = cv(S) + \sum_{i \in S} a_i \quad \text{for all } S \subset N.$$

The equations we started with give immediately that $v'(N) = 1$ and $v'(i) = 0, i = 1, 2, \ldots, n$. $\square$

Remark: How Does Normalizing Affect Imputations? If we have an imputation for an unnormalized game, what does it become for the normalized game? Conversely, if we have an imputation for the normalized game, how do we get the imputation for the original game?

The set of imputations for the original game is

$$X = \{\vec{x} = (x_1, \ldots, x_n) \mid x_i \geq v(i), \sum_{i=1}^{n} x_i = v(N)\}.$$

For the normalized game, indeed, for any game with $v(i) = 0, v(N) = 1$, the set of all possible imputations is given by

$$X' = \{\vec{x}' = (x'_1, \ldots, x'_n) \mid x'_i \geq 0, \sum_{i=1}^{n} x'_i = 1\}.$$

That is because any imputation $\vec{x}' = (x'_1, \ldots, x'_n)$ must satisfy $x'_i \geq v'(i) = 0$ and $\sum_i x'_i = v'(N) = 1$.

If $\vec{x}' = (x'_1, \ldots, x'_n) \in X'$ is an imputation for v' then the imputation for v becomes

$$\vec{x} = (x_1, \ldots, x_n) \in X \text{ where } x_i = \frac{(x'_i - a_i)}{c},$$

$$c = \frac{1}{v(N) - \sum_{i=1}^{n} v(i)} \quad \text{and} \quad a_i = -c\, v(i), \quad i = 1, 2, \ldots, n.$$

Conversely, if $\vec{x} = (x_1, \ldots, x_n) \in X$ is an imputation for the original game, then $\vec{x}' = (x'_1, \ldots, x'_n)$ is the imputation for the normalized game, where $x'_i = cx_i + a_i, i = 1, 2, \ldots, n$.

■ EXAMPLE 5.4

In the three–person nonzero sum game considered above we found the (unnormalized) characteristic function to be

$$\begin{aligned}
&v(1) = 1,\ v(2) = \frac{1}{4},\ v(3) = -1 \\
&v(12) = 3,\ v(13) = 1,\ v(23) = 1 \\
&v(123) = 4.
\end{aligned}$$

To normalize this game we compute

$$c = \frac{1}{v(N) - \sum_{i=1}^{3} v(i)} = \frac{1}{4 - \frac{1}{4}} = \frac{4}{15}, \text{ and } a_i = -\frac{4}{15} v(i).$$

So

$$a_1 = \frac{4}{15}, a_2 = -\frac{1}{15}, \text{ and } a_3 = \frac{4}{15}.$$

Then the normalized characteristic function by v' is calculated as

$$\begin{aligned}
v'(i) &= \frac{4}{15}\, v(i) + a_i = 0 \\
v'(12) &= \frac{4}{15}\, v(12) + a_1 + a_2 = \frac{7}{15} \\
v'(13) &= \frac{4}{15}\, v(13) + a_1 + a_3 = \frac{4}{15} \\
v'(23) &= \frac{4}{15}\, v(23) + a_2 + a_3 = \frac{7}{15} \\
v'(123) &= \frac{4}{15}\, v(123) + a_1 + a_2 + a_3 = 1.
\end{aligned}$$

In the rest of this section we let X denote the set of imputations $\vec{x}$. We look for an allocation $\vec{x} \in X$ as a **solution** to the game. The problem is the definition of the word **solution**. It is as vague as the word **fair**. We are seeking an imputation that, in some sense, is fair and allocates a fair share of the payoff to each player. To get a handle on the idea of **fair** we introduce the following subset of X.

Definition 5.1.4 *The* **reasonable allocation set** *of a cooperative game is a set of imputations $R \subset X$ given by*

$$R \equiv \{\vec{x} \in X \mid x_i \leq \max_{T \in \Pi^i}\{v(T) - v(T - i)\}, i = 1, 2, \ldots, n\},$$

where Π^i is the set of all coalitions for which player i is a member. So, if $T \in \Pi^i$, then $i \in T \subset N$, and $T - i$ denotes the coalition T without the player i.

In other words, the reasonable set is the set of imputations so that the amount allocated to each player is no greater than the maximum benefit that the player brings to any coalition of which the player is a member. The difference $v(T) - v(T - i)$ is the measure of the rewards for coalition T due to player i. The reasonable set gives us a first way to reduce the size of X and try to focus in on a solution.

If the reasonable set has only one element, which is extremely unlikely for most games, then that is our solution. If there are many elements in R, we need to cut it down further. In fact, we need to cut it down to the **core** imputations, or even further. Here is the definition.

Definition 5.1.5 *Let $S \subset N$ be a coalition and let $\vec{x} \in X$. The* **excess** *of coalition $S \subset N$ for imputation $\vec{x} \in R$ is defined by*

$$e(S, x) = v(S) - \sum_{i \in S} x_i.$$

The **core of the game** *is*

$$C(0) = \{\vec{x} \in X \mid e(S,\vec{x}) \le 0,\ \forall S \subset N\} = \{\vec{x} \in X \mid v(S) \le \sum_{i\in S} x_i,\ \forall S \subset N\}.$$

The ε**-core**, *for* $-\infty < \varepsilon < +\infty$, *is*

$$C(\varepsilon) = \{\vec{x} \in X \mid e(S,\vec{x}) \le \varepsilon,\ \forall S \subset N, S \neq N, S \neq \emptyset\}$$

Let $\varepsilon^1 \in (-\infty, \infty)$ *be the smallest* ε *for which* $C(\varepsilon) \neq \emptyset$. *The* **least core** , *labeled* X^1, *is* $C(\varepsilon^1)$. *It is possible for* ε^1 *to be positive, negative, or zero.*

In the definition of the excess function with the grand coalition $e(N,\vec{x}) = v(N) - \sum_{i=1}^{n} x_i = 0$ for any imputation. The grand coalition is excluded in the requirements for $C(\varepsilon)$ because if N were an eligible coalition, then $e(N,\vec{x}) = 0 \le \varepsilon$, and it would force ε to be nonnegative. That would put too strict a requirement on ε in order for $C(\varepsilon)$ to be nonempty. That is why N is excluded. In fact, we could exclude it in the definition of $C(0)$ because any imputation must satisfy $e(N,\vec{x}) = 0$ and it automatically satisfies the requirement $e(N,\vec{x}) \le 0$.

We will use the notation that for a given imputation $\vec{x} = (x_1, \ldots, x_n)$ and a given coalition $S \subset N$

$$\vec{x}(S) = \sum_{i\in S} x_i,$$

the total amount allocated to coalition S.

Remarks.

1. The idea behind the definition of the core is that an imputation $\vec{x}$ is a member of the core if no matter which coalition S is formed, the total payoff given to the members of S, namely, $\vec{x}(S) = \sum_{i\in S} x_i$, must be at least as large as $v(S)$, the maximum possible benefit of forming the coalition. If $e(S,\vec{x}) > 0$, this would say that the maximum possible benefits of joining the coalition S are greater than the total allocation to the members of S using the imputation $\vec{x}$. But then the members of S would not be very happy with $\vec{x}$ and would want to change to a better allocation. In that sense, if $\vec{x} \in C(0)$, then $e(S,\vec{x}) \le 0$ for every coalition S, and there would be no incentive for any coalition to try to use a different imputation. An imputation is in the core of a game if it is acceptable to all coalitions. The excess function $e(S,\vec{x})$ is a **measure of dissatisfaction** of a particular coalition S with the allocation $\vec{x}$. Consequently, $\vec{x}$ is in the core if all coalitions are satisfied with $\vec{x}$. If the core has only one allocation, that is our solution.

2. Likewise, if $\vec{x} \in C(\varepsilon)$, then the measure of dissatisfaction of a coalition with $\vec{x}$ is limited to ε. The size of ε determines the measure of dissatisfaction because $e(S,\vec{x}) \le \varepsilon$.

3. It is possible for the core of the game $C(0)$ to be empty, but there will always be some $\varepsilon \in (-\infty, \infty)$ so that $C(\varepsilon) \neq \emptyset$. The least core uses the smallest such ε. If the smallest $\varepsilon > 0$, then $C(0) = \emptyset$.

4. It should be clear, since $C(\varepsilon)$ is just a set of inequalities, that as ε increases, $C(\varepsilon)$ gets bigger, and as ε decreases, $C(\varepsilon)$ gets smaller. In other words, $\varepsilon < \varepsilon' \implies C(\varepsilon) \subset C(\varepsilon')$. So, the idea is that we should shrink (or expand if necessary) $C(\varepsilon)$ by adjusting ε until we get one and only one imputation in it, if possible.

5. We will see shortly that $C(0) \subset R$, every allocation in the core is always in the reasonable set.

6. The definition of **solution** for a cooperative game we are going to use in this section is that an imputation should be a fair allocation if it is the allocation which minimizes the maximum dissatisfaction for all coalitions.

■ **EXAMPLE 5.5**

Let's give an example of a calculation of $C(0)$. Take the three-person game $N = \{1, 2, 3\}$, with characteristic function

$$v(1) = 1,\ v(2) = 2,\ v(3) = 3,$$
$$v(23) = 6, v(13) = 5, v(12) = 4, v(\emptyset) = 0, v(N) = 8.$$

The excess functions for a given imputation $\vec{x} = (x_1, x_2, x_3) \in C(0)$ must satisfy

$$e(1, \vec{x}) = 1 - x_1 \leq 0,\ e(2, \vec{x}) = 2 - x_2 \leq 0,\ e(3, \vec{x}) = 3 - x_3 \leq 0$$
$$e(12, \vec{x}) = 4 - x_1 - x_2 \leq 0,\ e(13, \vec{x}) = 5 - x_1 - x_3 \leq 0,$$
$$e(23, \vec{x}) = 6 - x_2 - x_3 \leq 0,$$

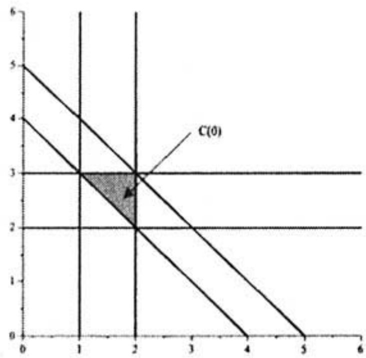
and we must have $x_1 + x_2 + x_3 = 8$. These inequalities imply that $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$, and

$$x_1 + x_2 \geq 4, x_1 + x_3 \geq 5, x_2 + x_3 \geq 6.$$

If we use some algebra and the substitution $x_3 = 8 - x_1 - x_2$ to solve these inequalities, we see that

$$C(0) = \{(x_1, x_2, 8 - x_1 - x_2) \mid 1 \leq x_1 \leq 2, 2 \leq x_2 \leq 3, 4 \leq x_1 + x_2 \leq 5\}.$$

If we plot this region in the (x_1, x_2) plane, we get the following Maple–generated diagram



In a similar way it can be shown that the smallest ε for which $C(\varepsilon) \neq \emptyset$ is $\varepsilon = \varepsilon^1 = -\frac{1}{3}$. In fact, the least core is the single imputation $C\left(-\frac{1}{3}\right) = \left\{\left(\frac{5}{3}, \frac{8}{3}, \frac{11}{3}\right)\right\}$. Indeed, the imputations in $C(\varepsilon)$ must satisfy $e(S, \vec{x}) \leq \varepsilon$ for all coalitions $S \subsetneq N$. Written out, these inequalities become

$$1 - \varepsilon \leq x_1 \leq 2 + \varepsilon, \; 2 - \varepsilon \leq x_2 \leq 3 + \varepsilon,$$
$$4 - \varepsilon \leq x_1 + x_2 \leq 5 + \varepsilon,$$

where we have eliminated $x_3 = 8 - x_1 - x_2$. Adding the inequalities involving only x_1, x_2 we see that $4 - \varepsilon \leq x_1 + x_2 \leq 5 + 2\varepsilon$, which implies that $\varepsilon \geq -\frac{1}{3}$. You can check that this is the smallest ε for which $C(\varepsilon) \neq \emptyset$. With $\varepsilon = -\frac{1}{3}$, it follows that $x_1 + x_2 = \frac{13}{3}$ and $x_1 \leq \frac{5}{3}$, $x_2 \leq \frac{8}{3}$. Then,

$$\left(\frac{5}{3} - x_1\right) + \left(\frac{8}{3} - x_2\right) = 0,$$

which implies that $x_1 = \frac{5}{3}$, $x_2 = \frac{8}{3}$ because two nonnegative terms adding to zero must each be zero. This is one technique for finding ε^1 and $C(\varepsilon^1)$.

Now we will formalize some properties of the core. We begin by showing that the core must be a subset of the reasonable set.

Lemma 5.1.6 $C(0) \subset R$.

Proof. We may assume the game is in normalized form because we can always transform it to one that is and then work with that one. So $v(N) = 1, v(i) = 0, i = 1, \ldots, n$. Let $\vec{x} \in C(0)$. If $\vec{x} \notin R$ there is some player j such that

$$x_j > \max_{T \in \Pi^j} v(T) - v(T - j).$$

This means that for every $T \subset N$ with $j \in T$, $x_j > v(T) - v(T - j)$, and so the amount allocated to player j is larger than the amount of her benefit to any coalition.

Take $T = N$. Then

$$x_j > v(N) - v(N - j) = 1 - v(N - j).$$

But then, $v(N - j) > 1 - x_j = \sum_{i \neq j} x_i$, and so $e(N - j, \vec{x}) > 0$, which means $\vec{x} \notin C(0)$. □

EXAMPLE 5.6

In this example we will normalize the given characteristic function, find the reasonable set, and find the core of the game. Finally, we will find the least core and then find the unnormalized imputation.

We have the characteristic function in the three-player game from Example 5.3:

$$\bar{v}(1) = 1, \bar{v}(2) = \frac{1}{4}, \bar{v}(3) = -1, \bar{v}(12) = 3, \bar{v}(13) = 1, \bar{v}(23) = 1, \bar{v}(123) = 4.$$

This is an essential game that we normalized in Example 5.4 to obtain the characteristic function that we will use:

$$v(i) = 0,\ v(123) = 1,\ v(12) = \frac{7}{15},\ v(13) = \frac{4}{15},\ v(23) = \frac{7}{15}.$$

The normalization constants are $c = \frac{4}{15}$, and $a_1 = -\frac{4}{15}, a_2 = -\frac{1}{15}$, and $a_3 = \frac{4}{15}$. Since this is now in normalized form, the set of imputations is

$$X = \{\vec{x} = (x_1, x_2, x_3) \mid x_i \geq 0, \sum_{i=1}^{3} x_i = 1\}.$$

The reasonable set is easy to find:

$$\begin{aligned} R &= \{\vec{x} = (x_1, x_2, x_3) \in X \mid x_i \leq \max_{T \in \Pi^i}\{v(T) - v(T - i)\}, i = 1, 2, 3\} \\ &= \{(x_1, x_2, 1 - x_1 - x_2) \mid x_1 \leq \frac{8}{15}, x_2 \leq \frac{11}{15}, \frac{7}{15} \leq x_1 + x_2 \leq 1\}. \end{aligned}$$

For example, let's consider

$$x_1 \leq \max_{T \in \Pi^1} v(T) - v(T - 1).$$

The coalitions containing player 1 are $\{1, 12, 13, 123\}$, so we are calculating the maximum of

$$\begin{aligned} &v(1) - v(\emptyset) = 0,\ \ v(12) - v(2) = \frac{7}{15},\ \ v(13) - v(3) = \frac{4}{15}, \\ &v(123) - v(23) = 1 - \frac{7}{15} = \frac{8}{15}. \end{aligned}$$

Hence $0 \le x_1 \le \frac{8}{15}$. Similarly, $0 \le x_2 \le \frac{11}{15}$. We could also show $0 \le x_3 \le \frac{8}{15}$, but this isn't good enough because we can't ignore $x_1 + x_2 + x_3 = 1$. That is where we use

$$0 \le 1 - x_1 - x_2 = x_3 \le \frac{8}{15} \implies \frac{7}{15} \le x_1 + x_2 \le 1.$$

Another benefit of replacing x_3 is that now we can draw the reasonable set in (x_1, x_2) space. Figure 5.2 below is a plot of R.

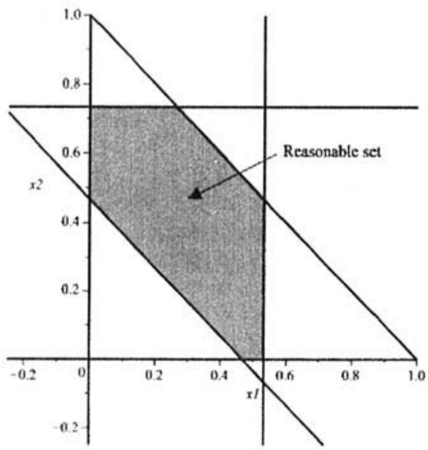


Figure 5.2 The set of reasonable imputations.

Figure 5.2 was generated with the simple Maple commands

```
> with(plots):with(plottools):
> inequal( { x<=8/15,y<=11/15,x+y >=7/15, x+y<=1,x>=0,y>=0},
            x=-.25..1, y=-.25..1,
     optionsfeasible=(color=red),
     optionsopen=(color=blue,thickness=2),
     optionsclosed=(color=gray, thickness=2),
     optionsexcluded=(color=white),labels=['x[1]','x[2]'] );
```

You can see from Figure 5.2 that there are lots of reasonable imputations. This is a starting point. We would like to find next the point (or points) in the reasonable set which is acceptable to all coalitions. That is the core of the game:

$$\begin{aligned}C(0) &= \{\vec{x} \in X \mid e(S,x) \le 0,\ \forall S \subset N, S \neq N, S \neq \emptyset\}\\ &= \{x_1 \ge 0, x_2 \ge 0, \frac{7}{15} - x_1 - x_2 \le 0, -\frac{11}{15} + x_2 \le 0, -\frac{8}{15} + x_1 \le 0,\\ &\qquad x_1 + x_2 \le 1\}.\end{aligned}$$

Unfortunately, this gives us exactly the same set as the reasonable set, $C(0) = R$ in this example, and that is too big a set.

Now let's calculate the ε-core for any $\varepsilon \in (-\infty, \infty)$. The ε-core is, by definition

$$\begin{aligned}C(\varepsilon) &= \{\vec{x} \in X \mid e(S,\vec{x}) \le \varepsilon,\ \forall S \subset N, S \neq N, S \neq \emptyset\}\\ &= \{x_1 \ge 0, x_2 \ge 0, \tfrac{7}{15} - x_1 - x_2 \le \varepsilon, -\tfrac{11}{15} + x_2 \le \varepsilon, -\tfrac{8}{15} + x_1 \le \varepsilon,\\ &\qquad -x_1 \le \varepsilon, -x_2 \le \varepsilon, -1 + x_1 + x_2 \le \varepsilon\}.\end{aligned}$$

We have used the fact that $x_1 + x_2 + x_3 = 1$ to substitute $x_3 = 1 - x_1 - x_2$.

By working with the inequalities in $C(\varepsilon)$, we can find the least core X^1. We verify that the smallest ε so that $C(\varepsilon) \neq \emptyset$ is $\varepsilon^1 = -\frac{4}{15}$. The procedure is to add the inequality for x_2 with the one for x_1 and then use the first inequality:

$$-\frac{11}{15} + x_2 - \frac{8}{15} + x_1 = -\frac{19}{15} + x_1 + x_2 \le 2\varepsilon,$$

but $x_1 + x_2 \ge \frac{7}{15} - \varepsilon$, so that

$$-\frac{12}{15} - \varepsilon = -\frac{19}{15} + \frac{7}{15} - \varepsilon \le -\frac{19}{15} + x_1 + x_2 \le 2\varepsilon$$

which can be satisfied if and only if $\varepsilon \ge -\frac{4}{15} = \varepsilon^1$.

If we replace ε by $\varepsilon^1 = -\frac{4}{15}$, the **least core** is the set

$$\begin{aligned}C(\varepsilon^1) = X^1 &= \{(x_1, x_2, 1 - x_1 - x_2) \mid\\ &\qquad \frac{4}{15} \ge x_1 \ge 0, \frac{7}{15} \ge x_2 \ge 0, x_1 + x_2 = \frac{11}{15}\}\\ &= \left\{\left(\frac{4}{15}, \frac{7}{15}, \frac{4}{15}\right)\right\}.\end{aligned}$$

The least core is exactly the single point $(x_1 = \frac{4}{15}, x_2 = \frac{7}{15}, x_3 = \frac{4}{15})$ because if $x_1 + x_2 = \frac{11}{15}$, then $(\frac{4}{15} - x_1) + (\frac{7}{15} - x_2) = 0$ and each of the two terms is nonnegative, and therefore both zero or they couldn't add to zero. This says that there is one and only one allocation that gives all coalitions the smallest dissatisfaction.

If we want the imputation of the original unnormalized game, we use $\bar{x}_i = (x_i - a_i)/c$ and obtain

$$\bar{x}_1 = \frac{\frac{4}{15} + \frac{4}{15}}{\frac{4}{15}} = 2, \ \bar{x}_2 = \frac{\frac{7}{15} + \frac{1}{15}}{\frac{4}{15}} = 2, \ \bar{x}_3 = \frac{\frac{4}{15} - \frac{4}{15}}{\frac{4}{15}} = 0.$$

It is an exercise to directly get this imputation for the original game without going through the normalization and solving the inequalities from scratch.

Now here is a surprising (?) conclusion. Remember from Example 5.3 that the payoff of 4 for the grand coalition $\{123\}$ is obtained only by cooperation of the three players in the game in which the payoffs to each player was $(4, -2, 2)$. But remember also that player 3 was in a very weak position according to the characteristic function we derived. The conclusion of the imputation we derived is that players 1 and 2 will split the 4 units available to the grand coalition and player 3 gets nothing. That is still better than what she could get on her own because $\bar{v}(3) = -1$. So that is our fair allocation.

We have already argued that the core $C(0)$ should consist of the good imputations and so would be considered the solution of our game. If in fact $C(0)$ contained exactly one point, then that would be true. Unfortunately, the core may contain many points, as in the last example, or may even be empty. Here is an example of a game with an empty core.

■ EXAMPLE 5.7

Suppose that the characteristic function of a three-player game is given by

$$v(123) = 1 = v(12) = v(13) = v(23) \quad \text{and} \quad v(1) = v(2) = v(3) = 0.$$

Since this is already in normalized form, the set of imputations is

$$X = \{\vec{x} = (x_1, x_2, x_3) \mid x_i \geq 0, \sum_{i=1}^{3} x_i = 1\}.$$

To calculate the reasonable set R, we need to find

$$x_i \leq \max_{T \in \Pi^i} \{v(T) - v(T - i)\}, i = 1, 2, 3.$$

Starting with $\Pi^1 = \{1, 12, 13, 123\}$, we calculate

$$v(1) - v(\emptyset) = 0, \ v(12) - v(2) = 1, v(13) - v(3) = 1, v(123) - v(23) = 0,$$

so $x_1 \leq \max\{0, 1, 1, 0\} = 1$. This is true for x_2 as well as x_3. So all we get from this is $R = X$, all the imputations are reasonable.

Next we have

$$C(0) = \{\vec{x} \in X \mid v(S) \leq \sum_{i \in S} x_i, \ \forall S \subsetneq N\}.$$

If $\vec{x} \in C(0)$, we calculate

$$e(i, \vec{x}) = v(i) - x_i = -x_i \leq 0, \ e(12, \vec{x}) = 1 - (x_1 + x_2) \leq 0$$

and, in likewise fashion

$$e(13, \vec{x}) = 1 - (x_1 + x_3) \leq 0, \ e(23, \vec{x}) = 1 - (x_2 + x_3) \leq 0.$$

The set of inequalities we have to solve are

$$x_1 + x_2 \geq 1, \ x_1 + x_3 \geq 1, \ x_2 + x_3 \geq 1, \ x_1 + x_2 + x_3 = 1, \ x_i \geq 0.$$

But clearly there is no $\vec{x} \in X$ that can satisfy these inequalities, because it is impossible to have three positive numbers, any two of which have sum at least 1, which can add up to 1, so $C(0) = \emptyset$.

In the next example we will determine a necessary and sufficient condition for any cooperative game with three players to have a nonempty core.

■ EXAMPLE 5.8

We take $N = \{1, 2, 3\}$ and a characteristic function in normalized form

$$v(i) = v(\emptyset) = 0, \ i = 1, 2, 3, \ v(123) = 1,$$
$$v(12) = a_{12}, \ v(13) = a_{13}, \ v(23) = a_{23}.$$

Of course, we have $0 \leq a_{ij} \leq 1$. We can state the proposition.

Proposition 5.1.7 *For the three-person cooperative game with normalized characteristic function v we have $C(0) \neq \emptyset$ if and only if*

$$a_{12} + a_{13} + a_{23} \leq 2.$$

Proof. We have

$$C(0) = \{(x_1, x_2, 1 - x_1 - x_2) \mid x_i \geq 0, a_{12} \leq x_1 + x_2,$$
$$a_{13} \leq x_1 + (1 - x_1 - x_2) = 1 - x_2, \ \text{and} \ a_{23} \leq 1 - x_1\}.$$

So, $x_1 + x_2 \geq a_{12}$, $x_2 \leq 1 - a_{13}$, and $x_1 \leq 1 - a_{23}$. Adding the last two inequalities says $x_1 + x_2 \leq 2 - a_{23} - a_{13}$ so that with the first inequality

$a_{12} \le 2 - a_{13} - a_{23}$. Consequently, if $C(0) \neq \emptyset$, it must be true that $a_{12} + a_{13} + a_{23} \le 2$.

For the other side, if $a_{12} + a_{13} + a_{23} \le 2$, we define the imputation

$$\begin{aligned}\vec{x} &= (x_1, x_2, x_3)\\ &= \left(\frac{1 - 2a_{23} + a_{13} + a_{12}}{3}, \frac{1 + a_{23} - 2a_{13} + a_{12}}{3}, \frac{1 + a_{23} + a_{13} - 2a_{12}}{3}\right).\end{aligned}$$

Then $x_1 + x_2 + x_3 = 1 = v(123)$. Furthermore

$$\begin{aligned}v(23) - x_2 - x_3 &= a_{23} - x_2 - x_3 = a_{23} - x_2 - (1 - x_1 - x_2)\\ &= a_{23} - 1 + x_1\\ &= a_{23} - 1 + \frac{1 - 2a_{23} + a_{13} + a_{12}}{3}\\ &= \frac{a_{23} + a_{13} + a_{12} - 2}{3} \le 0.\end{aligned}$$

Similarly, $v(12) - x_1 - x_2 \le 0$ and $v(13) - x_1 - x_3 \le 0$. Hence $\vec{x} \in C(0)$ and so $C(0) \neq \emptyset$. □

Remark: An Automated Way to Determine Whether $C(0) = \emptyset$.

Maple can give us a simple way of determining whether the core is empty. Consider the linear program:

$$\begin{aligned}&\text{Minimize } z = x_1 + \cdots + x_n\\ &\text{subject to } v(S) \le \sum_{i \in S} x_i \text{ for every } S \subsetneq N.\end{aligned}$$

It is not hard to check that $C(0)$ is not empty if and only if the linear program has a minimum, say, z^*, and $z^* \le v(N)$. If the game is normalized, then we need $z^* \le 1$. When this condition is not satisfied, $C(0) = \emptyset$. For instance, in the last example the commands would be

```
> with(simplex):
> obj:=x+y+z;
> cnsts:={1-x-z<=0,1-y-z<=0,1-x-y<=0};
> minimize(obj,cnsts,NONNEGATIVE);
> assign(%);
> obj;
```

Maple gives the output $\{x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{2}\}$ as the allocation and $obj = \frac{3}{2}$ as the sum of the allocation components. Since this is a game in which the allocation components must sum to 1, because $v(N) = 1$, we see that the core must be empty.

PROBLEMS

5.1 Look back at Example 5.5. Find the normalized characteristic function and the normalized element in the least core.

5.2 Consider the bimatrix game with

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}.$$

(a) Find the characteristic function of this game.
(b) Find the core of the game $C(0)$.
(c) Find the least core.

5.3 Given the characteristic function

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13,$$

find the normalized characteristic function. Given the fair allocation

$$\vec{x} = \left(\frac{1}{4}, \frac{33}{104}, \frac{33}{104}, \frac{3}{26}\right)$$

for the normalized game find the unnormalized allocation.

5.4 Find the characteristic function for the following three-player game. Each player has two strategies, A, B. If player 1 plays A the matrix is

$$\begin{bmatrix} (1,2,1) & (3,0,1) \\ (-1,6,-3) & (3,2,1) \end{bmatrix},$$

while if player 1 plays B the matrix is

$$\begin{bmatrix} (-1,2,4) & (1,0,3) \\ (7,5,4) & (3,2,1) \end{bmatrix}.$$

In each matrix player 2 is the row player and player 3 is the column player. Next find the normalized characteristic function.

5.5 Derive the least core for the game with

$$v(123) = 1 = v(12) = v(13) = v(23) \quad \text{and} \quad v(1) = v(2) = v(3) = 0.$$

5.6 Given the characteristic function

$$v(1) = 1, v(2) = \frac{1}{4}, v(3) = -1, v(12) = 3, v(13) = -1, v(23) = 1, v(123) = 4,$$

find the least core without normalizing.

5.7 A **constant sum game** is one in which $v(S) + v(N - S) = v(N)$ for all coalitions $S \subset N$. Show that any essential constant sum game must have empty core $C(0) = \emptyset$.

5.8 In this problem you will see why inessential games are of no interest. Show that an **inessential** game has one and only one imputation and is given by

$$\vec{x} = (x_1, \ldots, x_n) = (v(1), v(2), \ldots, v(n));$$

that is, each player is allocated exactly the benefit of the one-player coalition.

5.9 A player i is a **dummy** if $v(S) = v(S \cup i)$, for every $S \subset N$. It looks like a dummy contributes nothing. Show that if i is a dummy, $v(i) = 0$, and $\vec{x} \in C(0)$, then $x_i = 0$.

5.10 Show that a vector $\vec{x} = (x_1, x - 2, \ldots, x_n)$ is an imputation if and only if there are nonnegative constants $a_i \geq 0, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n a_i = v(N) - \sum_{i=1}^n v(i)$, and $x_i = v(i) + a_i$ for each $i = 1, 2, \ldots, n$.

5.11 Let $\delta_i = v(N) - v(N - i)$. Show that $C(0) = \emptyset$ if $\sum_{i=1}^n \delta_i < v(N)$.

5.12 Verify the statement: $C(0) \neq \emptyset$ if and only if the linear program

$$\begin{aligned}
&\text{Minimize } z = x_1 + \cdots + x_n \\
&\text{subject to } v(S) \leq \sum_{i \in S} x_i \text{ for every } S \subsetneq N
\end{aligned}$$

has a finite minimum, say z^*, and $z^* \leq v(N)$.

5.1.1 Finding the Least Core

The next theorem formalizes the idea above that when $e(S, \vec{x}) \leq 0$ for all coalitions, then the player should be happy with the imputation $\vec{x}$ and would not want to switch to another one.

One way to describe the fact that one imputation is better than another is the concept of domination.

Definition 5.1.8 *If we have two imputations $\vec{x} \in X, \vec{y} \in X$, and a nonempty coalition $S \subset N$, then $\vec{x}$* **dominates** *$\vec{y}$ (for the coalition S) if $x_i > y_i$ for all members $i \in S$, and $\vec{x}(S) = \sum_{i \in S} x_i \leq v(S)$.*

If $\vec{x}$ dominates $\vec{y}$ for the coalition S, then members of S prefer the allocation $\vec{x}$ to the allocation $\vec{y}$, because they get more $x_i > y_i$, for each $i \in S$, and the coalition S can actually achieve the allocation because $v(S) \geq \sum_{i \in S} x_i$. The next result is another characterization of the core of the game.

Theorem 5.1.9 *The core of a game is the set of all undominated imputations for the game; that is,*

$$C(0) = \{\vec{x} \in X \mid \text{there is no } \vec{z} \in X \text{ and } S \subset N \text{ such that}$$
$$z_i > x_i, \forall\, i \in S, \text{ and } \sum_{i \in S} z_i \leq v(S)\}.$$

Proof. Call the right hand side the set B. We have to show $C(0) \subset B$ and $B \subset C(0)$.

We may assume that the game is in $(0, 1)$ normalized form.

Let $\vec{x} \in C(0)$ and suppose $\vec{x} \notin B$. Since $\vec{x} \notin B$ that means that $\vec{x}$ must be dominated by another imputation for at least one nonempty coalition $S \subset N$; that is, there is $\vec{y} \in X$ and $S \subset N$ such that $y_i > x_i$ for all $i \in S$ and $v(S) \geq \sum_{i \in S} y_i$. Summing on $i \in S$ this shows

$$v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \Longrightarrow e(S, \vec{x}) > 0,$$

contradicting the fact that $\vec{x} \in C(0)$. Therefore $C(0) \subset B$.

Now let $\vec{x} \in B$. If $\vec{x} \notin C(0)$, there is a nonempty coalition $S \subset N$ so that $e(S, \vec{x}) = v(S) - \sum_{i \in S} x_i > 0$. Let

$$\varepsilon = v(S) - \sum_{i \in S} x_i > 0 \text{ and } \alpha = 1 - v(S) \geq 0.$$

Let $s = |S|$, the number of players in S, and

$$z_i = \begin{cases} x_i + \dfrac{\varepsilon}{s} & \text{if } i \in S; \\ \dfrac{\alpha}{n - s} & \text{if } i \notin S. \end{cases}$$

We will show that $\vec{z} = (z_1, \ldots, z_n)$ is an imputation and $\vec{z}$ dominates $\vec{x}$ for the coalition S; that is, that $\vec{z}$ is a better allocation for the members of S than is $\vec{x}$.

First $z_i \geq 0$ and

$$\sum_{i=1}^{n} z_i = \sum_{S} x_i + \sum_{S} \frac{\varepsilon}{s} + \sum_{N-S} \frac{\alpha}{n - s} = \sum_{S} x_i + \varepsilon + \alpha = v(S) + 1 - v(S) = 1.$$

Therefore $\vec{z}$ is an imputation.

Next we show $\vec{z}$ is a better imputation than is $\vec{x}$ for the coalition S.If $i \in S$ $z_i = x_i + \varepsilon/s > x_i$ and $\sum_{i \in S} z_i = \sum_{i \in S} x_i + \varepsilon = v(S)$. Therefore $\vec{z}$ dominates $\vec{x}$. But this says $\vec{x} \notin B$ and that is a contradiction. Hence $B \subset C(0)$. □

EXAMPLE 5.9

This example[1] will present a game with an empty core. We will see that when we calculate the least core $X^1 = C(\varepsilon^1)$, where ε^1 is the smallest value for which $C(\varepsilon^1) \neq \emptyset$, we will obtain a reasonable fair allocation (and hopefully only one). The fact that $C(0) = \emptyset$ means that when we calculate ε^1 it must be the case that $\varepsilon^1 > 0$ because if $\varepsilon^1 < 0$, by the definition of ε^1 as the smallest ε making $C(\varepsilon) \neq \emptyset$, we know immediately that $C(0) \neq \emptyset$ because $C(\varepsilon)$ increases as ε gets bigger.

Suppose that Bill has 150 sinks to give away to whomever shows up to take them away. Amy(1), Agnes(2), and Agatha(3) simultaneously show up with their trucks to take as many of the sinks as their trucks can haul. Amy can haul 45, Agnes 60, and Agatha 75, for a total of 180, 30 more than the maximum of 150. The wrinkle in this problem is that the sinks are too heavy for any one person to load onto the trucks so they must cooperate in loading the sinks. The question is: How many sinks should be allocated to each person?

Define the characteristic function $v(S)$ as the number of sinks the coalition $S \subset N = \{1, 2, 3\}$ can load. We have $v(i) = 0, i = 1, 2, 3$, since they must cooperate to receive any sinks at all, and

$$v(12) = 105,\ v(13) = 120,\ v(23) = 135,\ v(123) = 150.$$

It will be easier to not normalize this problem, so the set of imputations will be $X = \{(x_1, x_2, x_3) | x_i \geq 0, \sum x_i = 150\}$. First let's use Maple to see if the core is nonempty:

```
> with(simplex):
> obj:=x1+x2+x3:
> cnsts:={105-x1-x2<=0,120-x1-x3<=0,135-x2-x3<=0};
> minimize(obj,cnsts,NONNEGATIVE);
> assign(%);
> obj;
```

Maple gives the output $x_1 = 45, x_2 = 60, x_3 = 75$, and $obj = x_1 + x_2 + x_3 = 180 > v(123) = 150$. So the core of this game is empty. A direct way to get this is to note that the inequalities

$$x_1 + x_2 \geq 105, x_1 + x_3 \geq 120 \text{ and } x_2 + x_3 \geq 135$$

imply that $2(x_1 + x_2 + x_3) = 2(150) = 300 \geq 360$, which is impossible.

[1] due to Mesterton-Gibbons [15].

The next step is to calculate the least core. Begin with the definition:

$$\begin{aligned}C(\varepsilon) &= \{\vec{x} \in X \mid e(S, \vec{x}) \le \varepsilon, \forall S \subsetneq N\} \\ &= \{\vec{x} \in X \mid v(S) - \sum_{i \in S} x_i \le \varepsilon\} \\ &= \{\vec{x} \mid 105 \le x_1 + x_2 + \varepsilon, 120 \le x_1 + x_3 + \varepsilon, \\ &\qquad 135 \le x_2 + x_3 + \varepsilon, -x_i \le \varepsilon\}.\end{aligned}$$

We know that $x_1 + x_2 + x_3 = 150$ so by replacing $x_3 = 150 - x_1 - x_2$ we obtain as conditions on ε that

$$120 \le 150 - x_2 + \varepsilon,\ 135 \le 150 - x_1 + \varepsilon,\ 105 \le x_1 + x_2 + \varepsilon.$$

We see that $45 \ge x_1 + x_2 - 2\varepsilon \ge 105 - 3\varepsilon$, implying that $\varepsilon \ge 20$. This is in fact the smallest $\varepsilon^1 = 20$, making $C(\varepsilon) \ne \emptyset$. Using $\varepsilon^1 = 20$, we calculate

$$C(20) = \{(x_1 = 35, x_2 = 50, x_3 = 65)\}.$$

Hence the fair allocation is to let Amy have 35 sinks, Agnes 50, and Agatha 65 sinks, and they all cooperate.

We conclude that our fair allocation of sinks is as follows:

player	Truck capacity	Allocation
Amy	45	35
Agnes	60	50
Agatha	75	65
Total	180	150

Observe that each player in the fair allocation gets 10 less than the capacity of her truck. It seems that this is certainly a reasonably fair way to allocate the sinks; that is, there is an undersupply of 30 sinks so each player will receive $\frac{30}{3} = 10$ less than her truck can haul. You might think of other ways in which you would allocate the sinks (e.g, maybe it would be better to fill the large trucks first), but the solution here minimizes the maximum dissatisfaction over any other allocation for all coalitions.

The least core plays a critical role in solving the problem when $C(0) = \emptyset$ or there are more than one allocations in $C(0)$.

Lemma 5.1.10 *Let*

$$\varepsilon^1 = \min_{\vec{x} \in X} \max_{S \subsetneq N} e(S, \vec{x}).$$

Then the least core $X^1 = C(\varepsilon^1) \ne \emptyset$ *and if* $\varepsilon > \varepsilon^1$, *then* $C(\varepsilon^1) \subsetneq C(\varepsilon)$.

Proof. Since the set of imputations is compact (=closed and bounded) and $\vec{x} \mapsto \max_S e(S, \vec{x})$ is at least lower semicontinuous, there is an allocation $\vec{x}_0$ so that the minimum in the definition of ε^1 is achieved, namely, $\varepsilon^1 = \max_S e(S, \vec{x}_0) \geq e(S, \vec{x}_0), \forall S \subsetneq N$. This is the very definition of $\vec{x}_0 \in C(\varepsilon^1)$ and so $C(\varepsilon^1) \neq \emptyset$.

On the other hand, if we have a smaller $\varepsilon < \varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N}$, then for every allocation $\vec{x} \in X$, we have $\varepsilon < \max_S e(S, \vec{x})$. So, for any allocation there is at least one coalition $S \subsetneq N$ for which $\varepsilon < e(S, \vec{x})$. This means that for this ε, no matter which allocation is given, $\vec{x} \notin C(\varepsilon)$. Thus, $C(\varepsilon) = \emptyset$. As a result, ε^1 is the smallest ε so that $C(\varepsilon) \neq \emptyset$. □

Remarks.
These remarks summarize the ideas behind the use of the least core.

1. For a given grand allocation $\vec{x}$, the coalition S_0 that most objects to $\vec{x}$ is the coalition giving the largest excess and so satisfies

$$e(S_0, \vec{x}) = \max_{S \subsetneq N} e(S, \vec{x}).$$

For each fixed coalition S, the allocation giving the minimum dissatisfaction is

$$e(S, \vec{x}_0) = \min_{\vec{x} \in X} e(S, \vec{x}).$$

2. The value of ε giving the least ε-core is

$$\varepsilon^1 \equiv \min_{\vec{x} \in X} \max_{S \subsetneq N} e(S, \vec{x}),$$

and this is the smallest level of dissatisfaction.

3. If $\varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N} < 0$, then some allocation $\vec{x}^*$ satisfies $\max_S e(S, \vec{x}^*) < 0$. That means that $e(S, \vec{x}^*) < 0$ for every coalition $S \subsetneq N$. Every coalition is satisfied with $\vec{x}^*$ because $v(S) < \vec{x}^*(S)$, so that every coalition is allocated at least its maximum value.

If $\varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N} > 0$, then for every allocation $\vec{x} \in X$, $\max_S e(S, \vec{x}) > 0$. Consequently, there is at least one coalition S so that $e(S, \vec{x}) = v(S) - \vec{x}(S) > 0$. For any allocation, there is at least one coalition that will not be happy with it.

4. The excess function $e(S, \vec{x})$ is a measure of dissatisfaction of S with the imputation $\vec{x}$. It makes sense that the best imputation would minimize the largest dissatisfaction over all the coalitions. This leads us to one possible definition of a solution for the n-person cooperative game. An allocation $\vec{x}^* \in X$ is a solution to the cooperative game if

$$\varepsilon^1 = \min_{\vec{x} \in X} \max_S e(S, x) = \max_S e(S, x^*),$$

so that $\vec{x}^*$ minimizes the maximum excess for any coalition S. When there is only one such allocation $\vec{x}^*$, it is the fair allocation. The problem is that there may be more than one element in the least core, then we still have a problem as to how to choose among them.

Remark: Maple Calculation of the Least Core. The point of calculating the ε-core is that the core is not a sufficient set to ultimately solve the problem in the case when the core $C(0)$ is (1) empty or (2) consists of more than one point. In case (2) the issue, of course, is which point should be chosen as the fair allocation. The ε-core seeks to address this issue by shrinking the core at the same rate from each side of the boundary until we reach a single point. We can use Maple to do this.

The calculation of the least core is equivalent to the linear programming problem

$$\begin{aligned}
&\text{Minimize } z \\
&\text{subject to} \\
&v(S) - \vec{x}(S) = v(S) - \sum_{i \in S} x_i \leq z, \ \text{for all } S \subsetneq N.
\end{aligned}$$

The characteristic function need not be normalized. So all we really need to do is to formulate the game using characteristic functions, write down the constraints, and plug them into Maple. The result will be the smallest $z = \varepsilon^1$ that makes $C(\varepsilon^1) \neq \emptyset$, as well as an imputation which provides the minimum.

For example, let's suppose we start with the characteristic function

$$v(i) = 0,\ i = 1, 2, 3,\ v(12) = 2, v(23) = 1, v(13) = 0,\ \ v(123) = \frac{5}{2}.$$

The constraint set is the ε-core

$$\begin{aligned}
C(\varepsilon) &= \{\vec{x} = (x_1, x_2, x_3) \mid v(S) - x(S) \leq \varepsilon, S \subsetneq N\} \\
&= \{-x_i \leq \varepsilon, i = 1, 2, 3, 2 - x_1 - x_2 \leq \varepsilon, 1 - x_2 - x_3 \leq \varepsilon, \\
&\qquad\qquad 0 - x_1 - x_3 \leq \varepsilon, x_1 + x_2 + x_3 = \frac{5}{2}\}
\end{aligned}$$

The Maple commands used to solve this are very simple:

```
> with(simplex):
> cnsts:={-x1<=z,-x2<=z,-x3<=z,2-x1-x2<=z,1-x2-x3<=z,-x1-x3<=z,
        x1+x2+x3=5/2};
> minimize(z,cnsts);
```

Maple produces the output

$$x_1 = \frac{5}{4},\ x_2 = 1,\ x_3 = \frac{1}{4},\ z = -\frac{1}{4}.$$

Hence the smallest $\varepsilon^1 = z$ for which the ε-core is nonempty is $\varepsilon^1 = -\frac{1}{4}$. Now, Maple also gives us the allocation $\vec{x} = \left(\frac{5}{4}, 1, \frac{1}{4}\right)$ which will be in $C\left(-\frac{1}{4}\right)$, but we don't know if that is the **only point** in $C\left(-\frac{1}{4}\right)$. With Maple we can graph the core and the set $C\left(-\frac{1}{4}\right)$ with the following commands:

```
> cnsts:={-x1<=z,-x2<=z,-(5/2-x1-x2)<=z,2-x1-x2<=z,
            1-x2-(5/2-x1-x2)<=z,-x1-(5/2-x1-x2)<=z};
> Core:=subs(z=0,cnsts);
> with(plots):
> inequal(Core,x1=0..2,x2=0..3,optionsfeasible=(color=red),
>     optionsopen=(color=blue,thickness=2),
>     optionsclosed=(color=green, thickness=3),
>     optionsexcluded=(color=yellow));
> ECore:=subs(z=-1/4,cnsts);
> inequal(ECore,x1=0..2,x2=0..3,optionsfeasible=(color=red),
>     optionsopen=(color=blue,thickness=2),
>     optionsclosed=(color=green, thickness=3),
>     optionsexcluded=(color=yellow));
```

Figure 5.3 shows the core $C(0)$.

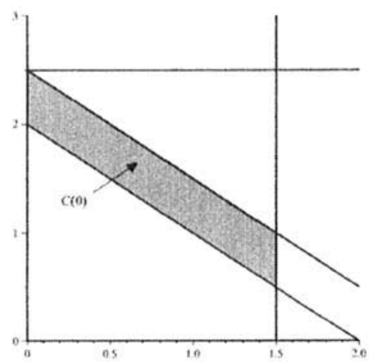


Figure 5.3 Graph of $C(0)$.

You can even see how the core shrinks to the ε-core using an animation:

```
> animate(inequal,[cnsts, x1=0..2,x2=0..3,
                    optionsfeasible=(color=red),
                    optionsopen=(color=blue,thickness=2),
                    optionsclosed=(color=green,thickness=3),
                    optionsexcluded=(color=white)],
                    z=-1..0,frames=50);
```

Figure 5.4 results from the animation at $z = -0.18367$ with the dark region constituting the core $C(-0.18367)$. You will see that at $z = -\frac{1}{4}$ the dark region becomes the line segment. Hence $C\left(-\frac{1}{4}\right)$ is certainly not empty, but it is also not just one point.

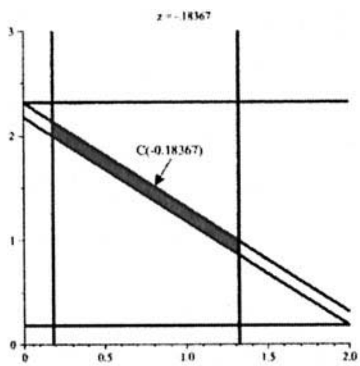


Figure 5.4 The shrinking of the core frozen at $C(-0.18367)$.

So we have solved any cooperative game if the least core contains exactly one point. But when $C(\varepsilon^1) = X^1$ has more than one point, we still have a problem, and that leads us in the next section to the **nucleolus**.

PROBLEMS

5.13 In a three–player game each player has two strategies A, B. If player 1 plays A, the matrix is

$$\begin{bmatrix} (1,2,1) & (3,0,1) \\ (-1,6,-3) & (3,2,1) \end{bmatrix},$$

while if player 1 plays B, the matrix is

$$\begin{bmatrix} (-1,2,4) & (1,0,3) \\ (7,5,4) & (3,2,1) \end{bmatrix}.$$

In each matrix player 2 is the row player and player 3 is the column player. The characteristic function is $v(\emptyset) = 0, v(1) = \frac{3}{5}, v(2) = 2, v(3) = 1, v(12) = 5, v(13) = 4, v(23) = 3, v(123) = 16$. Verify that and then find the core and the least core.

5.14 A weighted majority game has a characteristic function of the form $v(S) = 1$ if $\sum_{i \in S} w_i > q$, and $v(S) = 0$; otherwise, where $w_i \geq 0$ are weights and $q > 0$ is called a quota. Take $q = \frac{1}{2} \sum_{i \in N} w_i$. Suppose that there is one large group with two-fifths of the votes and two equal-sized groups with three-tenths of the vote each. **(a)** Find the characteristic function. **(b)** Find the core and the least core of the game.

5.15 A classic game is the **garbage game**. Suppose that there are four property owners, each with one bag of garbage that needs to be dumped on somebody's property (one of the four). If n bags of garbage are dumped on a coalition S of property owners, the coalition receives a reward of $-n$. The characteristic function is taken to be the best that the members of a coalition S can do, which is to dump all their garbage on the property of the owners not in S.

(a) Explain why the characteristic function should be $v(S) = -(4 - |S|)$, where $|S|$ is the number of members in S.

(b) Show that the core of the game is empty.

(c) Recall that an imputation $\vec{y}$ dominates an imputation $\vec{x}$ through the coalition S if $e(S, \vec{y}) \geq 0$ and $y_i > x_i$ for each component i. Find a coalition S so that $\vec{y} = (-1.5, -0.5, -1, -1)$ dominates $\vec{x} = (-2, -1, -1, 0)$.

5.2 THE NUCLEOLUS

The core $C(0)$ might be empty, but we can find an ε so that $C(\varepsilon)$ is not empty. We can fix the **empty** problem. Even if $C(0)$ is not empty, it may contain more than one point and again we can use $C(\varepsilon)$ to **maybe** shrink the core down to one point or, if $C(0) = \emptyset$, to expand the core until we get it nonempty. The problem is what happens when the least core $C(\varepsilon)$ itself has too many points.

In this section we will address the issue of what to do when the least core $C(\varepsilon)$ contains more than one point. Remember that $e(S, \vec{x}) = v(S) - \sum_{i \in S} x_i = v(S) - \vec{x}(S)$ and the larger the excess, the more unhappy the coalition S is with the allocation $\vec{x}$. So, no matter what, we want the excess to be as small as possible for all coalitions and we want the imputation which achieves that.

In the previous section we saw that we should shrink $C(0)$ to $C(\varepsilon^1)$, so if $C(\varepsilon^1)$ has more than one allocation, why not shrink that also? No reason at all.

Let's begin by working through an example to see how to shrink the ε^1-core.

■ EXAMPLE 5.10

Let us take the normalized characteristic function for the three-player game

$$v(12) = \frac{4}{5}, v(13) = \frac{2}{5}, v(23) = \frac{1}{5} \text{ and } v(123) = 1, v(i) = 0, \quad i = 1, 2, 3.$$

Step 1: Calculate the least core. We have the ε-core

$$\begin{aligned} C(\varepsilon) &= \{(x_1, x_2, x_3) \in X \mid e(S, x) \leq \varepsilon, \forall S \subsetneq N\} \\ &= \{(x_1, x_2, 1 - x_1 - x_2) \mid -\varepsilon \leq x_1 \leq \frac{4}{5} + \varepsilon, \\ &\qquad -\varepsilon \leq x_2 \leq \frac{3}{5} + \varepsilon, \ \frac{4}{5} - \varepsilon \leq x_1 + x_2 \leq 1 + \varepsilon\}. \end{aligned}$$

We then calculate that the smallest ε for which $C(\varepsilon) \neq \emptyset$ is $\varepsilon^1 = -\dfrac{1}{10}$, and then

$$C\left(\varepsilon^1 = -\frac{1}{10}\right) = \{(x_1, x_2, 1 - x_1 - x_2) \mid x_1 \in \left[\frac{2}{5}, \frac{7}{10}\right], \\ x_2 \in \left[\frac{1}{5}, \frac{1}{2}\right], \quad x_1 + x_2 = \frac{9}{10}\}.$$

This is a line segment in the (x_1, x_2) plane as we see in Figure 5.5, which is obtained from the Maple animation shrinking the core down to the line frozen at $z = -0.07$.

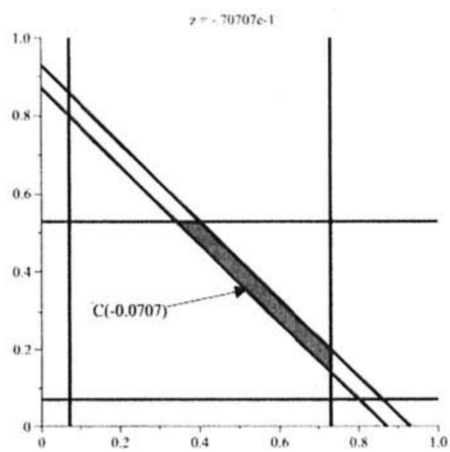


Figure 5.5 $C(-0.07)$: shrinking down to a line segment.

So we have the problem that the least core does not have only one imputation that we would be able to call our solution. What is the fair allocation now? We must shrink the line down somehow. Here is what to do.

Step 2: Calculate the next least core. The idea is, **restricted to the allocations in the first least core**, minimize the maximum excesses over all the allocations in the least core. So we must take allocations with $\vec{x} = (x_1, x_2, x_3)$, with $\frac{2}{5} \leq x_1 \leq \frac{7}{10}$, $\frac{1}{5} \leq x_2 \leq \frac{1}{2}$, and $x_1 + x_2 = \frac{9}{10}$. This last equality then requires that $x_3 = 1 - x_1 - x_2 = \frac{1}{10}$.

If we take any allocation $\vec{x} \in C(\varepsilon^1)$, we want to calculate the excesses for each coalition:

$$\begin{array}{ll}
e(1,\vec{x}) = -x_1 & e(2,\vec{x}) = -x_2 \\
e(13,\vec{x}) = x_2 - \frac{3}{5} & e(23,\vec{x}) = x_1 - \frac{4}{5} \\
e(12,\vec{x}) = \frac{4}{5} - x_1 - x_2 = -\frac{1}{10} & e(3,\vec{x}) = -x_3 = -\frac{1}{10}
\end{array}$$

Since $\vec{x} \in C(\varepsilon^1)$, we know that these are all $\leq -\frac{1}{10}$. Observe that the excesses $e(12,\vec{x}) = e(3,\vec{x}) = -\frac{1}{10}$ do not depend on the allocation $\vec{x}$ as long as it is in $C(\varepsilon^1)$. But then, there is nothing we can do about those coalitions by changing the allocation. Those coalitions will always have an excess of $-\frac{1}{10}$ as long as the imputations are in $C(\varepsilon^1)$, and they cannot be reduced. Therefore, we may eliminate those coalitions from further consideration.

Now we set

$$\Sigma^1 \equiv \{S \subsetneq N \mid e(S,\vec{x}) < \varepsilon^1, \text{ for some } \vec{x} \in C(\varepsilon^1)\}.$$

This is a set of coalitions with excesses for some imputation smaller than ε^1. These coalitions can use some imputation that gives a **better** allocation for them, as long as the allocations used are also in $C\left(-\frac{1}{10}\right)$. For our example, we get

$$\Sigma^1 = \{1, 2, 13, 23\}.$$

The coalitions $\{12\}$ and $\{3\}$ are out because the excesses of those coalitions cannot be dropped below $-\frac{1}{10}$ no matter what allocation we use in $C(-\frac{1}{10})$. Their level of dissatisfaction cannot be dropped any further.

Now pick any allocation in $C(-\frac{1}{10})$ and calculate the smallest level of dissatisfaction for the coalitions in Σ^1:

$$\varepsilon^2 \equiv \min_{\vec{x} \in X^1} \max_{S \in \Sigma^1} e(S,\vec{x}).$$

The number ε^2 is then the smallest maximum excess over all allocations in $C(-\frac{1}{10})$. It is defined just as is ε^1 except we restrict to the coalitions that can have their dissatisfaction reduced. Finally, set

$$X^2 \equiv \{\vec{x} \in X^1 = C(\varepsilon^1) \mid e(S,\vec{x}) \leq \varepsilon^2, \ \forall S \in \Sigma^1\}.$$

The set X^2 is the subset of allocations from X^1 that are preferred by the coalitions in Σ^1. It plays exactly the same role as the least core $C(\varepsilon^1)$, but now we can only use the coalitions in Σ^1. If X^2 contains exactly one imputation, then that point is our fair allocation, namely, the solution to our problem.

In our example we now use allocations $\vec{x} \in X^1$ so that $x_1 + x_2 = \frac{9}{10}, \frac{2}{5} \le x_1 \le \frac{7}{10}$, and $\frac{1}{5} \le x_2 \le \frac{1}{2}$. The next least core is

$$\begin{aligned} C(\varepsilon^2) \equiv X^2 &= \{\vec{x} \in X^1 \mid e(1,\vec{x}) \le \varepsilon^2, e(2,\vec{x}) \le \varepsilon^2, \\ &\qquad\qquad e(13,\vec{x}) \le \varepsilon^2, e(23,\vec{x}) \le \varepsilon^2\} \\ &= \{\vec{x} \in X^1 \mid -x_1 \le \varepsilon^2, -x_2 \le \varepsilon^2, \\ &\qquad\qquad x_2 - \frac{3}{5} \le \varepsilon^2, x_1 - \frac{4}{5} \le \varepsilon^2\}. \end{aligned}$$

We need to find the smallest ε^2 for which X^2 is nonempty. We do this by hand as follows. Since $x_1 + x_2 = \frac{9}{10}$, we get rid of $x_2 = \frac{9}{10} - x_1$. Then

$$\begin{aligned} &-x_1 \le \varepsilon^2 \quad -x_2 = x_1 - \frac{9}{10} \le \varepsilon^2 \implies -\varepsilon^2 - \frac{9}{10} \le \varepsilon^2 \implies \varepsilon^2 \ge -\frac{9}{20} \\ &\frac{9}{10} - x_1 - \frac{3}{5} \le \varepsilon^2, \ x_1 - \frac{4}{5} \le \varepsilon^2 \implies -\frac{5}{10} - \varepsilon^2 \le \varepsilon^2 \implies \varepsilon^2 \ge -\frac{5}{20}, \end{aligned}$$

and so on.

The smallest ε^2 satisfying all the requirements is then $\varepsilon^2 = -\frac{5}{20} = -\frac{1}{4}$. Next, we replace ε^2 by $-\frac{1}{4}$ in the definition of X^2 to get

$$\begin{aligned} C\left(-\frac{1}{4}\right) \equiv X^2 &= \{\vec{x} \in X^1 \mid -x_1 \le -\frac{1}{4}, -x_2 \le -\frac{1}{4}, \\ &\qquad\qquad x_2 - \frac{3}{5} \le -\frac{1}{4}, x_1 - \frac{4}{5} \le -\frac{1}{4}\} \\ &= \{\vec{x} \in X^1 \mid \frac{1}{4} \le x_1 \le \frac{11}{20}, \ \frac{1}{4} \le x_2 \le \frac{7}{20} \\ &\qquad\qquad x_1 + x_2 = 18/20.\} \end{aligned}$$

The last equality gives us

$$0 = \left(\frac{11}{20} - x_1\right) + \left(\frac{7}{20} - x_2\right) \implies x_1 = \frac{11}{20}, \ x_2 = \frac{7}{20},$$

since both terms are nonnegative and cannot add up to zero unless they are each zero. So we have found $x_1 = \frac{11}{20}, x_2 = \frac{7}{20}$, and, finally, $x_3 = \frac{2}{20}$. We have our second least core

$$X^2 = \left\{\left(\frac{11}{20}, \frac{7}{20}, \frac{2}{20}\right)\right\},$$

and X^2 consists of exactly one point. That is our solution to the problem. Notice that for this allocation

$$e(13,\vec{x}) = x_2 - \frac{3}{5} = \frac{7}{20} - 12/20 = -\frac{1}{4}$$
$$e(23,\vec{x}) = x_1 - \frac{4}{5} = 11/20 - \frac{4}{5} = -\frac{1}{4}$$
$$e(1,\vec{x}) = -11/20, \quad \text{and} \quad e(2,x) = -\frac{7}{20},$$

and each of these is a constant smaller than $-\frac{1}{10}$. Because they are all independent of any specific allocation, we know that they cannot be reduced any further by adjusting the imputation. Since X^2 contains only one allocation, no further adjustments are possible in any case. This is the allocation that minimizes the maximum dissatisfaction of all coalitions.

The most difficult part of this procedure is finding ε^1, ε^2, and so on. This is where Maple is a great help. For instance, we can find $\varepsilon^2 = -\frac{1}{4}$ very easily if we use the commands

```
> with(simplex):
> cnsts:={-x1<=z,-x2<=z,x2-3/5<=z,x1-4/5<=z,x1+x2=9/10};
> minimize(z,cnsts);
```

Maple informs us that z=-1/4,x1=11/20,x2=7/20, but you have to be careful of the x1 and x2 because these are points providing the minimum but you don't know whether they are the only such points. That is what you must verify.

In general, we would need to continue this procedure if X^2 also contained more than one point. Here are the sequence of steps to take in general until we get down to one point:

1. **Step 0: Initialize.** We start with the set of all possible imputations X and the coalitions excluding N and $\emptyset$:

$$X^0 \equiv X, \quad \Sigma^0 \equiv \{S \subsetneq N, S \neq \emptyset\}$$

2. **Step $k \geq 1$: Successively calculate**

   (a) The minimum of the maximum dissatisfaction

$$\varepsilon^k \equiv \min_{\vec{x} \in X^k} \max_{S \in \Sigma^{k-1}} e(S,\vec{x}).$$

   (b) The set of allocations achieving the minimax dissatisfaction

$$X^k \equiv \{\vec{x} \in X^{k-1} \mid \varepsilon^k = \min_{\vec{x} \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S,\vec{x}) = \max_{S \in \Sigma^{k-1}} e(S,\vec{x})\}$$
$$= \{\vec{x} \in X^{k-1} \mid e(S,\vec{x}) \leq \varepsilon^k, \ \forall S \subsetneq \Sigma^{k-1}\}.$$

(c) The set of coalitions achieving the minimax dissatisfaction

$$\Sigma_k = \{S \in \Sigma^{k-1} \mid e(S,\vec{x}) = \varepsilon^k, \forall\, \vec{x} \in X^k\}.$$

(d) Delete these coalitions from the previous set

$$\Sigma^k \equiv \Sigma^{k-1} - \Sigma_k.$$

3. **Step: Test if Done.** If $\Sigma^k = \emptyset$ we are done; otherwise set $k = k+1$ and go to step (2) with the new k.

When this algorithm stops at, say, $k = m$, then X^m is the **nucleolus** of the core and will satisfy the relationships

$$X^m \subset X^{m-1} \subset \cdots \subset X^1 = C(\varepsilon^1) \subset X^0 = X.$$

Also, $\Sigma^0 \supset \Sigma^1 \supset \Sigma^2 \cdots \Sigma^{m-1} \supset \Sigma^m - \emptyset$. The allocation sets decrease down to a single point, **the nucleolus**, and the unhappiest coalitions decrease down to the empty set. The nucleolus is guaranteed to contain only one allocation $\vec{x}$, and this is the solution of the game. In fact, the following theorem can be proved.[1]

Theorem 5.2.1 *The nucleolus algorithm stops in a finite number of steps $m < \infty$ and for each $k = 1, 2, \ldots, m$ we have*

1. $-\infty < \varepsilon_k < \infty$.
2. $X^k \neq \emptyset$ *are convex, closed, and bounded.*
3. $\Sigma_k \neq \emptyset$ *for* $k = 1, 2, \ldots, m-1$.
4. $\varepsilon_{k+1} < \varepsilon_k$.

In addition, X^m is a single point, called the **nucleolus of the game**:

$$Nucleolus = X^m = \bigcap_{k=1}^{m} X^k.$$

The nucleolus algorithm stops when all coalitions have been eliminated, but when working this out by hand you don't have to go that far. When you see that X^k is a single point you may stop.

The procedure to find the nucleolus can be formulated as a sequence of linear programs that can be solved using Maple.

[1] See, for example, the book by Wang [28]

To begin, set $k = 1$ and calculate the constraint set

$$X^1 = \{\vec{x} \in X \mid e(S, \vec{x}) \le \varepsilon, \forall S \subsetneq N\}.$$

The smallest ε that makes this nonempty is ε^1, given by

$$\varepsilon^1 = \min_{\vec{x} \in X} \max_{S \in \Sigma^0} e(S, \vec{x}), \quad \Sigma^0 = \{S \mid S \subsetneq N, \emptyset\}.$$

The first linear programming problem that will yield $\varepsilon^1, X^1, \Sigma^1$ is

$$\begin{aligned}&\text{Minimize } \varepsilon\\ &\text{subject to } v(S) - \vec{x}(S) \le \varepsilon, \ \vec{x} \in X^0 = X.\end{aligned}$$

The set of $\vec{x}$ values that provide the minimum in this problem is labeled X^1 (this is the least core). Now we take

$$\Sigma_1 = \{S \in \Sigma^0 \mid e(S, \vec{x}) = \varepsilon^1, \forall \vec{x} \in X^1\},$$

which is the set of coalitions that give excess ε^1 for any allocation in X^1. Getting rid of those gives us the next set of coalitions that we have to deal with, $\Sigma^1 = \Sigma^0 - \Sigma_1$.

The next linear programming problem can now be formulated:

$$\begin{aligned}&\text{Minimize } \varepsilon\\ &\text{subject to } v(S) - \vec{x}(S) \le \varepsilon, \ \vec{x} \in X^1, S \in \Sigma^1 = \Sigma^0 - \Sigma_1.\end{aligned}$$

The minimum such ε is ε^2, and we set X^2 to be the set of allocations in X^1 at which $\varepsilon^2 = \max_{S \in \Sigma^1} e(S, \vec{x})$. Then

$$\Sigma_2 = \{S \in \Sigma^1 | e(S, \vec{x}) = \varepsilon^2, \forall \vec{x} \in X^2\}.$$

Set $\Sigma^2 = \Sigma^1 - \Sigma_2$ and see if this is empty. If so, we are done; if not, we continue until we get our solution.

■ EXAMPLE 5.11

Three hospitals, A,B,C, want to have a proton therapy accelerator (PTA) to provide precise radiological cancer therapy. These are very expensive devices because they are subatomic particle accelarators. The hospitals can choose to build their own or build one, centrally located, PTA to which they may refer their patients. The costs for building their own PTA are estimated at 50, 30, 50, for A,B,C, respectively. The units for these numbers are million-dollars. If A and B cooperate to build a PTA, the total cost will be 60 because of land costs for the location, coordination, and so on. If B and C cooperate, the cost will be 70; if A and C cooperate, the cost will be 110. Because the cost for cooperation between A and C is greater than what it would cost if they built their own, they would decide to build their own, so the cost is still 100 for AC cooperation. Finally, the cost to build one PTA for all three hospitals A,B,C is 105.

In this setup the players want to minimize their costs, but since our previous theory is based on maximizing benefits instead of minimizing, we reformulate the problem by looking at the **amount saved** by each player and for each coalition. The characteristic function is then

$$v(S) = \text{total cost if each } i \in S \text{ builds its own} - \text{cost if they cooperate.}$$

With A=player 1, B=player 2, C=player 3, we get

$$v(1) = v(2) = v(3) = v(13) = 0, v(12) = 20, v(23) = 10, v(123) = 25.$$

For instance, $v(123) = 50 + 30 + 50 - 105 = 25$. We are looking for the fair allocation of the savings to each hospital that we can then translate back to costs. This game is not in normalized form and need not be.

The first linear program finds the least core:

$$\begin{aligned}&\text{Minimize } \varepsilon\\&\text{subject to}\\&\{-x_i \le \varepsilon, i = 1,2,3,\ -(x_1 + x_3) \le \varepsilon,\ 20 - (x_1 + x_2) \le \varepsilon,\\&\qquad\qquad\qquad 10 - (x_2 + x_3) \le \varepsilon,\ x_1 + x_2 + x_3 = 25\}.\end{aligned}$$

This gives us $\varepsilon^1 = -\frac{5}{2}$, which you should check without using Maple.Replacing ε by $\varepsilon^1 = \frac{5}{2}$ and simplifying, we see that the least core will be the set

$$X^1 = \left\{(x_1, x_2, 25 - x_1 - x_2) \mid \frac{5}{2} \le x_1 \le \frac{25}{2}, x_1 + x_2 = \frac{45}{2}\right\}.$$

Notice that $x_3 = 25 - \frac{45}{2} = \frac{5}{2}$. Next we have to calculate

$$\Sigma_1 = \{S \in \Sigma^0 \mid e(S, \vec{x}) = \varepsilon^1, \forall \vec{x} \in X^1.\}$$

Calculate the excesses for all the coalitions except $N, \emptyset$, assuming that the allocations are in X^1 :

$$\begin{aligned}&e(1,\vec{x}) = v(1) - x_1 = -x_1,\ \ e(2,\vec{x}) = -x_2,\ \ e(3,\vec{x}) = -x_3 = -\frac{5}{2}\\&e(12,\vec{x}) = v(12) - x_1 - x_2 = 20 - \frac{45}{2} = -\frac{5}{2}\\&e(13,\vec{x}) = v(13) - x_1 - x_3 = 0 - x_1 - \frac{5}{2} = -x_1 - \frac{5}{2}\\&e(23,\vec{x}) = v(23) - x_2 - x_3 = 10 - x_2 - \frac{5}{2} = \frac{15}{2} - x_2\end{aligned}$$

Thus the coalitions that give $\varepsilon^1 = -\frac{5}{2}$ are $\Sigma_1 = \{12, 3\}$, and so these two coalitions are dropped from consideration in the next step. The ones left are

$$\Sigma^1 = \Sigma^0 - \{12, 3\} = \{1, 2, 13, 23\}.$$

This is going to lead to the constraint set for the next linear program:

$$\begin{aligned}
&\text{Minimize } \varepsilon,\\
&\text{subject to } \vec{x} \in X^2,\\
&X^2 = \{\vec{x} \in X^1 \mid v(S) - \vec{x}(S) \le \varepsilon,\ \forall S \in \Sigma^1\}\\
&\quad = \left\{\left(x_1, x_2, \frac{5}{2}\right) \mid x_1 + x_2 = \frac{45}{2},\ -x_1 \le \varepsilon, -x_2 \le \varepsilon\right.\\
&\qquad\qquad \left. -\left(x_1 + \frac{5}{2}\right) \le \varepsilon,\ 10 - \left(x_2 + \frac{5}{2}\right) \le \varepsilon\right\}.
\end{aligned}$$

We get the solution of this linear program as

$$\varepsilon^2 = -\frac{15}{2},\ x_1 = \frac{15}{2}, x_2 = 15, \text{ and } x_3 = \frac{5}{2}.$$

Furthermore, it is the one and only solution, so we should be done, but we will continue with the algorithm until we end up with an empty set of coalitions.

Calculate the excesses for all the coalitions excluding $N, \emptyset$, assuming that the allocations are in X^2 :

$$\begin{aligned}
&e(1, \vec{x}) = -x_1 = -\frac{15}{2},\quad e(2, \vec{x}) = -x_2 = -15,\quad e(3, \vec{x}) = -x_3 = -\frac{5}{2},\\
&e(12, \vec{x}) = v(12) - x_1 - x_2 = 20 - \frac{45}{2} = -\frac{5}{2},\\
&e(13, \vec{x}) = v(13) - x_1 - x_3 = 0 - x_1 - \frac{5}{2} = -\frac{15}{2} - \frac{5}{2} = -\frac{20}{2},\\
&e(23, \vec{x}) = v(23) - x_2 - x_3 = 10 - x_2 - \frac{5}{2} = \frac{15}{2} - x_2 = \frac{15}{2} - 15 = -\frac{15}{2},
\end{aligned}$$

so that $e(23, \vec{x}) = e(1, \vec{x}) = -\frac{15}{2}$. Now we can get rid of coalitions $\Sigma_2 = \{1, 3, 12, 13, 23\}$ because none of the excesses for those coalitions can be further reduced by changing the allocations. Then

$$\Sigma^2 = \Sigma^1 - \Sigma_2 = \emptyset.$$

We are now done, having followed the algorithm all the way through. We conclude that

$$\text{Nucleolus} = \left\{\left(\frac{15}{2}, 15, \frac{5}{2}\right)\right\}.$$

Therefore, the savings to hospital A is $\frac{15}{2}$, the savings to B is 15, and the savings to C is $\frac{5}{2}$. Consequently the costs allocated to each player if all the hospitals cooperate is as follows: A pays $50 - 7.5 = 42.5$, B pays $30 - 15 = 15$, and C pays $50 - 2.5 - 47.5$. Hospital B saves the most and pays the least.

■ EXAMPLE 5.12

In this example we will give the Maple commands at each stage to find the nucleolus. This entire procedure can be automated but that is a programming problem.

We take the characteristic function

$$v(i) = 0, i = 1,2,3,\ v(12) = \frac{1}{3},\ v(13) = \frac{1}{6},\ v(23) = \frac{5}{6}, v(123) = 1,$$

and this is in normalized form. We see that $\frac{1}{3} + \frac{1}{6} + \frac{5}{6} < 2$, and so the core of the game $C(0)$ is not empty by Proposition 5.1.7. We need to find the allocation within the core which solves our problem.[2]

1. First linear programming problem. We start with the full set of possible coalitions excluding the grand coalition N and $\emptyset : \Sigma^0 = \{1,2,3,12,13,23\}$. In addition, with the given characteristic function, we get the excesses

$$\begin{aligned} e(1,\vec{x}) &= -x_1,\ e(2,x) = -x_2,\ e(3,\vec{x}) = -x_3 \\ e(12,\vec{x}) &= \tfrac{1}{3} - x_1 - x_2, e(13,\ \vec{x}) = \tfrac{1}{6} - x_1 - x_3 \\ e(23,\vec{x}) &= \tfrac{5}{6} - x_2 - x_3 \end{aligned}$$

The Maple commands that give the solution are

```
> with(simplex):v1:=0:v2:=0:v3:=0:v12:=1/3:v13:=1/6:
              v23:=5/6:v123:=1;                              1
> cnsts:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-(x1+x2)<=z,
            v13-(x1+x3)<=z,v23-(x2+x3)<=z,x1+x2+x3=v123};
> minimize(z,cnsts);
```

Maple gives the solution $\varepsilon^1 = z = -\frac{1}{12}, x_1 = \frac{1}{12}, x_2 = \frac{3}{4}, x_3 = \frac{1}{6}$.So this gives the allocation $\vec{x} = \left(\frac{1}{12}, \frac{3}{4}, \frac{1}{6}\right)$.But this is not necessarily the unique allocation and therefore the solution to our game.

To see if there are more allocations in X^1, substitute $z = -\frac{1}{12}$ as well as $x_3 = 1 - x_1 - x_2$ in the constraint set. To do that in Maple use the `substitute` command

```
> fcnsts:=subs(z=-1/12,x1=1-x2-x3,cnsts);
```

This will put the new constraint set into the variable `fcnsts` and gives us the output

```
fcnsts={1=1,x3 <= 7/12,-x2 <= -1/12, -x3 <= -1/12,
            -x2-x3 <= -11/12, x2+x3 <= 11/12, x2 <= 3/4}.
```

[2]To make things simpler, we will not use subscripts to denote the variables in the use of Maple. For instance, we use in Maple $x1$ instead of x_1, $z2$ instead of z_2, and so on. In our discussion we will use normal subscripts.

To get rid of the first equality so that we can continue, use

```
> gcnsts:=fcnsts[2..7];
```

This puts the second through seventh elements of `fcnsts` into `gcnsts`. Now, to see if there are other solutions, we need to solve the system of inequalities in `gcnsts` for x_1, x_2. Maple does that as follows:

```
> with(SolveTools:-Inequality):
> glc:=LinearMultivariateSystem(gcnsts,[x2,x3]);
```

Maple solves the system of inequalities in the sense that it reduces the inequalities to simplest form and gives the following output:

```
{x2 <=3/4,1/3<=x2} {x3<=-x2+11/12, -x2+11/12<=x3}.
```

We see that $\frac{1}{3} \le x_2 \le \frac{3}{4}$, $x_2 + x_3 = \frac{11}{12}$ and $x_1 = \frac{1}{12}$.

2. To get the second linear program we first have to see which coalitions are dropped. First we assign the variables that are known from the first linear program and recalculate the constraints:

```
> assign(x1=1/12,z=-1/12);
> cnsts1:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-(x1+x2)<=z,
          v13-(x1+x3)<=z,v23-(x2+x3)<=z,x1+x2+x3=v123};
```

Maple gives the output:

```
cnsts1:={-x3 <= -1/12, -x2-x3<=-11/12,-x2 <=-1/12,-x2 <= -1/3,
 -1/12 <= -1/12,-x3 <=-1/6, 1/12+x2+x3=1}.
```

Getting rid of the coalitions that have excess$= -\frac{1}{12}$ (indicated by the output without any x variables), we have the new constraint set

```
> cnsts2:={v2-x2<=z2,v3-x3<=z2,v12-(x1+x2)<=z2,
          v13-(x1+x3)<=z2,x1+x2+x3=v123};
```

Now we solve the second linear program

```
> minimize(z2,cnsts2);
```

which gives

$$x_2 = \frac{13}{24}, x_3 = \frac{3}{8}, z_2 = -\frac{7}{24}.$$

At each stage we need to determine whether there is more than one solution of the linear programming problem. To do that, we have to substitute our solution for z_2 into the constraints and solve the inequalities:

```
> fcnsts2:=subs(z2=-7/24, 11/12=x2+x3,cnsts2);
> gcnsts2:=fcnsts2[2..5] union
                   {x2+x3<=11/12,x2+x3>=11/12};
> glc2:=LinearMultivariateSystem(gcnsts2,[x2,x3]);
```

We get

```
glc2:={x2=13/24,x3 <= x2+x3=11/12},
```

and we know now that $x_1 = \frac{1}{12}, x_2 = \frac{13}{24}$, and $x_3 = \frac{3}{8}$ because $x_2 + x_3 = \frac{11}{12}$.

We could continue setting up linear programs until we get the set of empty coalitions, but there is no point to that when we are doing it by hand (or with a Maple assist), because we are now at the point when we have one and only one allocation.

So we are finally done, and we conclude that

$$\text{Nucleolus} = X^2 = \left\{ \left(\frac{1}{12}, \frac{13}{24}, \frac{3}{8} \right) \right\}.$$

The first $\varepsilon^1 = -\frac{1}{12}$ and the second $\varepsilon^2 = -\frac{7}{24}$.

Now for a another practical application.

EXAMPLE 5.13

Three cities are to be connected to a water tower at a central location. Label the three cities 1, 2, 3 and the water tower as 0. The cost to lay pipe connecting location i with location j is denoted as $c_{ij}, i \neq j$. Figure 5.6 contains the data for our problem.

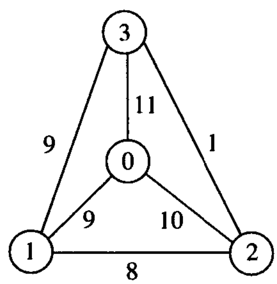


Figure 5.6 Three cities and a water tower.

Coalitions among cities can form for pipe to be laid to the water tower. For example, it is possible for city 1 and city 3 to join up so that the cost to the coalition $\{13\}$ would be the sum of the cost of going from 1 to 3 and then 3 to 0. It may be possible to connect from 1 to 3 to 0 but not from 3 to 1 to 0 depending on land conditions. We have the following costs in which we do not treat the water tower as a player:

$$c_1 = 9, c_2 = 10, c_3 = 11, c_{123} = 18,$$
$$c_{12} = 17, c_{13} = 18, c_{23} = 11.$$

The single-player coalitions correspond to hooking up that city directly to location 0. Converting this to a savings game, we let $c(S)$ be the total cost for coalition S and

$$v(S) = \sum_{i \in S} c_i - c(S) = \text{ amount saved by coalition } S.$$

This is a three-player game with the set of possible coalitions

$$2^N = \{\emptyset, N, 1, 2, 3, 12, 13, 23, 123\},$$

and we calculate the characteristic function

$$v(i) = 0,\ i = 1, 2, 3,\ \ v(123) = 30 - 18 = 12,$$
$$v(12) = 19 - 17 = 2,\ v(13) = 2,\ v(23) = 10.$$

We will find the nucleolus of this game. First, the core of the game is illustrated in Figure 5.7.

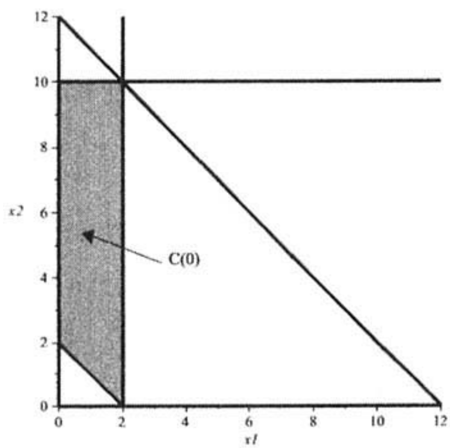


Figure 5.7 Core of three-city problem.

This is clearly a nonempty set with many points, so we need to find the nucleolus. This is the complete set of Maple commands needed to do this:

```
> restart:with(simplex):v1:=0:v2:=0:v3:=0:
                          v12:=2:v13:=2:v23:=10:v123:=12;
> with(SolveTools:-Inequality):
> cnsts:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-(x1+x2)<=z,
           v13-(x1+x3)<=z,v23-(x2+x3)<=z,x1+x2+x3=v123};
> minimize(z,cnsts);
```

```
> fcnsts:=subs(z=-1,x3=12-x1-x2,cnsts);
> gcnsts:=fcnsts[1..7] minus {fcnsts[2]};
> Core1:=subs(z=0,x3=12-x1-x2,cnsts);
> Core:=Core1 minus {Core1[1]};#This is needed to get rid of all
                               #equalities in Core1

> with(plots):#The next command plots the core.
> inequal( Core,x1=0..12, x2=0..12,
    optionsfeasible=(color=red),
    optionsopen=(color=blue,thickness=2),
    optionsclosed=(color=green, thickness=3),
    optionsexcluded=(color=yellow),labels=[x1,x2] );
    > # Now we set up for the next least core.
> glc:=LinearMultivariateSystem(gcnsts,[x1,x2]);
> assign(x1=1,z=-1);
> cnsts1:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-(x1+x2)<=z,
             v13-(x1+x3)<=z,v23-(x2+x3)<=z,x1+x2+x3=v123};

> cnsts2:={v2-x2<=z2,v3-x3<=z2,v13-(x1+x3)<=z2,
             v12-(x1+x2)<=z2,x1+x2+x3=v123};
> minimize(z2,cnsts2); #This command results in z2=-9/2
                         for the second least core.
> fcnsts2:=subs(z2=-9/2,cnsts2);
> gcnsts2:=fcnsts2[2..4] union {x2+x3<=11,x2+x3>=11};#Needed to
                     #convert equality to inequality
> # We now see if the second least core has more than one point.
> glc2:=LinearMultivariateSystem(gcnsts2,[x2,x3]);
> assign(x2=11-x3,z2=-9/2);
> cnsts3:={v2-x2<=z2,v3-x3<=z2,v12-(x1+x2)<=z2,
           v13-(x1+x3)<=z2,x1+x2+x3=v123};
```

When we get to the last execution group, we have already determined that $x_1 = 1, x_2 + x_3 = 11$, and the last constraint set gives

```
cnst3={12 = 12, -x3 <= -9/2, -x3 <= -11/2, x3 <= 13/2, x3 <= 11/2},
```

which tells us that $x_3 = \frac{11}{2}$ and $x_2 = 11 - \frac{11}{2} = \frac{11}{2}$.We have found that the nucleolus consists of the single allocation

$$\vec{x} = \left(1, \frac{11}{2}, \frac{11}{2}\right).$$

Our conclusion is that of the 12 units of savings possible for all the cities to cooperate, city 1 only gets 1 unit, while cities 2 and 3 get $\frac{11}{2}$ units each. It makes sense that city 1 would get the least because city 1 brings the least benefit to any coalition compared with 2 and 3.

PROBLEMS

5.16 Consider the normalized characteristic function for a three-person game $v(12) = \frac{4}{5}, v(13) = \frac{2}{5}, v(23) = \frac{1}{5}$. Without using Maple, find the core, the least core X^1, and the next least core X^2.

5.17 Find the fair allocation in the nucleolus for the three-person characteristic function game with

$$\begin{aligned}&v(i) = 0, i = 1, 2, 3,\\&v(12) = v(13) = 2, v(23) = 10,\\&v(123) = 12.\end{aligned}$$

5.18 Use Maple to solve the four-person game with unnormalized characteristic function

$$\begin{aligned}&v(i) = 0, i = 1, 2, 3, 4,\\&v(12) = v(34) = v(14) = v(23) = 1,\\&v(13) = \frac{3}{4}, v(24) = 0,\\&v(123) = v(124) = v(134) = v(234) = 1,\\&v(1234) = 3.\end{aligned}$$

5.19 Four doctors, Moe, Larry, Curly, and Shemp,[3] are in partnership and cover for each other. At most one doctor needs to be in the office to see patients at any one time. They advertise their office hours as follows:

(1)	Shemp	12:00-5:00
(2)	Curly	9:00-4:00
(3)	Larry	10:00-4:00
(4)	Moe	2:00-5:00

A coalition is an agreement by one or more doctors as to the times they will really be in the office to see all of their patients. The characteristic function should be the amount of time saved by a given coalition. Note that $v(i) = 0$, and $v(1234) = 13$ hours. This problem is an example of the use of cooperative game theory in an allocation of resources.

(a) Find the characteristic function for all coalitions.

(b) Find X^1, X^2. When you get to X^2, you should have the fair allocation in terms of hours saved.

[3]My favorite stooges, Moe Howard, Larry Fine, Curly Howard, and Shemp Howard. This scheduling problem is adapted from an example in Mesterton-Gibbons [15].

(c) Find the exact times of the day that each doctor will be in the office according to the allocation you found in X^2.

5.20 We have four players involved in a game to minimize their costs. We have transformed such games to savings games by defining $v(S) = \sum_{i \in S} c(i) - C(S)$, the total cost if each player is in a one-player coalition, minus the cost involved if players form the coalition S. In this problem we want a direct solution without going through savings but a fair allocation of costs directly. Find the nucleolus allocation of costs for the four-player game with costs

$$\begin{aligned}
&c(1) = 7,\ c(2) = 6,\ c(3) = 4,\ c(4) = 5,\\
&c(12) = 7.5,\ c(13) = 7,\ c(14) = 7.5,\ c(23) = 6.5,\ c(24) = 6.5,\ c(34) = 5.5,\\
&c(123) = 7.5,\ c(124) = 8,\ c(134) = 7.5,\ c(234) = 7,\\
&c(1234) = 8.5,\ c(\emptyset) = 0.
\end{aligned}$$

5.3 THE SHAPLEY VALUE

In an entirely different approach to deciding a fair allocation in a cooperative game, we change the definition of **fair** from minimizing the maximum dissatisfaction to **allocating an amount proportional to the benefit each coalition derives from having a specific player as a member**. The fair allocation would be the one recognizing the amount that each member adds to a coalition. Players who add nothing should receive nothing, and players who are indispensable should be allocated a lot. The question is how do we figure out how much benefit each player adds to a coalition. Lloyd Shapley[4] came up with a way.

Definition 5.3.1 *An allocation $\vec{x} = (x_1, \ldots, x_n)$ is called the* **Shapley value** *if*

$$x_i = \sum_{\{S \in \Pi^i\}} [v(S) - v(S - i)] \frac{(|S| - 1)!(|N| - |S|)!}{|N|!}, \quad i = 1, 2, \ldots, n,$$

where Π^i is the set of all coalitions $S \subset N$ containing i as a member (i.e., $i \in S$), $|S|$ =number of members in S, and $|N| = n$.

To see where this definition comes from, fix a player, say, i, and consider the random variable Z_i, which takes its values in the set of all possible coalitions 2^N. Z_i is the coalition S in which i is the last player to join S and $n - |S|$ players join the grand coalition after player i. Diagrammatically, if i joins the coalition S on the way to the

[4]Born June 2, 1923, is Emeritus Professor of Mathematics at UCLA and a member of the greatest generation. He and John Nash had the same PhD advisor, A. W. Tucker at Princeton. He has received many honors, not the least of which is a Bronze Star for Valor from the US Army for his service in World War II.

formation of the grand coalition, we have

$$\underbrace{(1)(2)\cdots(|S|-2)(|S|-1)}_{|S|-1 \text{ arrive}} \quad \underbrace{(i)}_{i \text{ arrives}} \quad \underbrace{(n-|S|)(n-|S|-1)\cdots(2)(1)}_{\text{remaining arrive}}$$

Remember that because a characteristic function is superadditive, the players have the incentive to form the grand coalition.

For a given coalition S, by elementary probability, there are $(|S|-1)!(n-|S|)!$ ways i can join the grand coalition N, joining S first. With this reasoning, we assume that Z_i has the probability distribution

$$Prob(Z_i = S) = \frac{(|S|-1)!(n-|S|)!}{n!}.$$

We choose this distribution because $|S|-1$ players have joined before player i, and this can happen in $(|S|-1)!$ ways; and $n-|S|$ players join after player i, and this can happen in $(n-|S|)!$ ways. The denominator is the total number of ways that the grand coalition can form among n players. Any of the $n!$ permutations has probability $\frac{1}{n!}$ of actually being the way the players join. This distribution assumes that they are **all equally likely**. One could debate this choice of distribution, but this one certainly seems reasonable. Also, see Example 5.17 below for a direct example of the calculation of the arrival of a player to a coalition and the consequent benefits.

Therefore, for the fixed player i, the benefit player i brings to the coalition Z_i is $v(Z_i) - v(Z_i - i)$. It seems reasonable that the amount of the total grand coalition benefits that should be allocated to player i should be the expected value of $v(Z_i) - v(Z_i - i)$. This gives,

$$\begin{aligned} x_i \equiv E[v(Z_i) - v(Z_i - i)] &= \sum_{\{S \in \Pi_i\}} [v(S) - v(S-i)] Prob(Z_i = S) \\ &= \sum_{\{S \in \Pi^i\}} [v(S) - v(S-i)] \frac{(|S|-1)!(n-|S|)!}{n!}. \end{aligned}$$

The **Shapley value (or vector)** is then the allocation $\vec{x} = (x_1, \ldots, x_n)$. At the end of this chapter you can find the Maple code to find the Shapley value.

■ EXAMPLE 5.14

Two players have to divide \$M, but they each get zero if they can't reach an agreement as to how to divde it. What is the fair division? Obviously, without regard to the benefit derived from the money the allocation should be $M/2$ to each player. Let's see if Shapley gives that.

Define $v(1) = v(2) = 0, v(12) = M$. Then

$$x_1 = [v(1) - v(\emptyset)]\frac{0!1!}{1!}2! + [v(12) - v(2)]\frac{1!0!}{2!} = \frac{M}{2}.$$

Note that if we solve this problem using the least core approach, we get

$$\begin{aligned} C(\varepsilon) &= \{(x_1, x_2) \mid e(S, x) \le \varepsilon, \forall S \subsetneq N\} \\ &= \{(x_1, x_2) \mid -x_1 \le \varepsilon, -x_2 \le \varepsilon,\ x_1 + x_2 = M\} \\ &= \left\{(x_1, x_2) \mid x_1 = x_2 = \frac{M}{2}\right\}. \end{aligned}$$

The reason for this is that if $x_1 \ge -\varepsilon$, $x_2 \ge -\varepsilon$, then adding, we have $-2\varepsilon \le x_1 + x_2 = M$. This implies that $\varepsilon \ge -M/2$, is the restriction on ε. The smallest ε that makes $C(\varepsilon) \ne \emptyset$ is then $\varepsilon^1 = -M/2$. Then $x_1 \ge M/2$, $x_2 \ge M/2$ and, since they add to M, it must be that $x_1 = x_2 = M/2$. So the least core allocation and the Shapley value coincide in the problem.

■ EXAMPLE 5.15

Let's go back to the sink allocation (Example 5.9) with Amy, Agnes, and Agatha. Using the core concept, we obtained

player	Truck Capacity	Allocation
Amy	45	35
Agnes	60	50
Agatha	75	65
Total	180	150

Let's see what we get for the Shapley value. Recall that the characteristic function was $v(i) = 0, v(13) = 120, v(12) = 105, v(23) = 135, v(123) = 150$. In this case $n = 3, n! = 6$ and for player $i = 1, \Pi^1 = \{1, 12, 13, 123\}$, so

$$\begin{aligned} x_1 &= \sum_{\{S \in \Pi^1\}} [v(S) - v(S - 1)] \frac{(|S| - 1)!(3 - |S|)!}{3!} \\ &= [v(1) - v(\emptyset)] \frac{2!0!}{3!} + [v(12) - v(2)] \frac{1!1!}{3!} \\ &+ [v(13) - v(3)] \frac{1!1!}{3!} + [v(123) - v(23)] \frac{2!0!}{3!} \\ &= 0 + 105 \frac{1}{6} + 120 \frac{1}{6} + [150 - 135] \frac{2}{6} \\ &= 42.5 \end{aligned}$$

Similarly, with more details,

$$\begin{aligned}
x_2 &= \sum_{\{S\in\Pi^2\}} [v(S) - v(S-2)]\frac{(|S|-1)!(3-|S|)!}{6} \\
&= [v(2) - v(\emptyset)]Prob(Z_1 = 2) + [v(12) - v(1)]Prob(Z_1 = 12) \\
&+ [v(23) - v(3)]Prob(Z_1 = 23) + [v(123) - v(13)]Prob(Z_1 = 123) \\
&= 0 + 105\frac{1}{6} + 135\frac{1}{6} + [150 - 135]\frac{2}{6} \\
&= 50,
\end{aligned}$$

$$\begin{aligned}
x_3 &= \sum_{\{S\in\Pi^3\}} [v(S) - v(S-3)]\frac{(|S|-1)!(3-|S|)!}{6} \\
&= [v(3) - v(\emptyset)]Prob(Z_3 = 3) + [v(13) - v(1)]Prob(Z_3 = 13) \\
&+ [v(23) - v(2)]Prob(Z_3 = 23) + [v(123) - v(12)]Prob(Z_3 = 123) \\
&= 0 + 120\frac{1}{6} + 135\frac{1}{6} + [150 - 105]\frac{2}{6} \\
&= 57.5.
\end{aligned}$$

Consequently, the Shapley vector is $\vec{x} = (42.5, 50, 57.5)$, or, since we can't split sinks $\vec{x} = (43, 50, 57)$, an allocation quite different from the nucleolus solution of $(35, 50, 65)$.

■ EXAMPLE 5.16

A typical and interesting fair allocation problem involves a debtor who owes money to more than one creditor. The problem is that the debtor does not have enough money to pay off the entire amount owed to all the creditors. Consequently, the debtor must negotiate with the creditors to reach an agreement about what portion of the assets of the debtor will be paid to each creditor. Usually, but not always, these agreements are imposed by a bankruptcy court.

Let's take a specific problem. Suppose that debtor D has exactly $100,000 to pay off three creditors A,B,C. Debtor D owes A $50,000; D owes B $65,000, and D owes C $10,000.

Now it is possible for D to split up the $100K (K=1000) on the basis of percentages; that is, the total owed is $125,000 and the amount owed to A is 40% of that, to B is 52%, and to C about 8%, so A would get $40K, B would get $52K and C would get $8K. What if the players could form coalitions to try to get more?

Let's take the characteristic function as follows. The three players are A,B,C and (with amounts in thousands of dollars)

$$v(A) = 25,\ v(B) = 40,\ v(C) = 0,$$
$$v(AB) = 90,\ v(AC) = 35,\ v(BC) = 50,\ v(ABC) = 100.$$

To explain this choice of characteristic function, consider the coalition consisting of just A, If we look at the worst that could happen to A, it would be that B and C get paid off completely and A gets what's left, if anything. If B gets \$65K and C gets \$10K then \$25K is left for A, and so we take $v(A) = 25$. Similarly, if A and B get the entire \$100K, then C gets \$0. If we consider the coalition AC they look at the fact that in the worst case B gets paid \$65K and they have \$35K left as the value of their coalition. This characteristic function is a little pessimistic since it is also possible to consider that AC would be paid \$75K and then $v(AC) = 75$. So other characteristic functions are certainly possible. On the other hand, if two creditors can form a coalition to freeze out the third creditor not in the coalition, then the characteristic function we use here is exactly the result.

Now we compute the Shapley values. For player A, we have

$$\begin{aligned} x_A &= [v(A) - v(\emptyset)]\frac{1}{3} + [v(AB) - v(B)]\frac{1}{6} \\ &\qquad + [v(AC) - v(C)]\frac{1}{6} + [v(ABC) - v(BC)]\frac{1}{3} \\ &= \frac{25}{3} + \frac{50}{6} + \frac{35}{6} + \frac{50}{3} = \frac{235}{6} = 39.17K. \end{aligned}$$

Similarly, for players B and C

$$\begin{aligned} x_B &= [v(B) - v(\emptyset)]\frac{1}{3} + [v(AB) - v(A)]\frac{1}{6} \\ &\qquad + [v(BC) - v(C)]\frac{1}{6} + [v(ABC) - v(AC)]\frac{1}{3} \\ &= 40\frac{1}{3} + 65\frac{1}{6} + 50\frac{1}{6} + 65\frac{1}{3} = \frac{325}{6} = 54.17K \end{aligned}$$

$$\begin{aligned} x_C &= [v(C) - v(\emptyset)]\frac{1}{3} + [v(BC) - v(B)]\frac{1}{6} \\ &\qquad + [v(AC) - v(A)]\frac{1}{6} + [v(ABC) - v(AB)]\frac{1}{3} \\ &= 0\frac{1}{3} + 10\frac{1}{6} + 10\frac{1}{6} + 10\frac{1}{3} = \frac{40}{6} = 6.67K, \end{aligned}$$

where again K=1000. The Shapley allocation is $\vec{x} = (39.17, 54.17, 6.67)$ compared to the allocation by percentages of $(40, 52, 8)$. Player B will receive more under the Shapley allocation at the expense of players A and C, who are owed the least.

A reasonable question to ask is why is the Shapley allocation any better than the percentage allocation? After all, the percentage allocation gives a perfectly reasonable answer, or does it? Actually, it ignores the basic fact that players can combine to freeze out another player. The players do not all have the same negotiating power. The Shapley allocation takes that into account through the characteristic function, while the percentage allocation does not.

EXAMPLE 5.17

In the beginning of this chapter we presented a typical problem involving small biotech companies. They can discover a new drug but they don't have the resources to manufacture and market it so they have to team up with a large partner. Let's say that A is the biotech firm and B and C are the candidate big pharmaceutical companies. If B or C teams up with A, the big firm will split \$1 billion with A. Here is a possible characteristic function:

$$v(A) = v(B) = v(C) = v(BC) = 0, \quad v(AB) = v(AC) = v(ABC) = 1$$

We will indicate a quicker way to calculate the Shapley allocation when there are a small number of players. We make a table indicating the value brought to a coalition by each player on the way to formation of the grand coalition:

Order of arrival	Player A	Player B	Player C
ABC	0	1	0
ACB	0	0	1
BAC	1	0	0
BCA	1	0	0
CAB	1	0	0
CBA	1	0	0

The numbers in the table are the amount of value added to a coalition when that player arrives. For example, if A arrives first, no benefit is added; then, if B arrives and joins A, player B has added 1 to the coalition AB; finally, when C arrives (so we have the coalition ABC), C adds no additional value. Since it is assumed in the derivation of the Shapley value that each arrival sequence is **equally likely** we calculate the average benefit brought by each player as the total benefit brought by each player (the sum of each column), divided by the total number of possible orders of arrival. We get

$$x_A = \frac{4}{6}, \; x_B = \frac{1}{6}, \; \text{and} \; x_C = \frac{1}{6}.$$

So company A, the discoverer of the drug should be allocated two-thirds of the \$1 billion and the big companies split the remaining third.

It is interesting to compare this with the nucleolus. The core, which will be the nucleolus for this example, is

$$\begin{aligned} C(0) = \{\vec{x} = (x_A, x_B, 1 - x_A - x_B) \mid & -x_A \le 0, -x_B \le 0, \\ & -(x_B + 1 - x_A - x_B) \le 0, 1 - x_A - x_B \le 0, \\ & 1 - x_A - (1 - x_A - x_B) \le 0, x_A + x_B \le 1\} \\ = \{(x_A, x_B, x_C) = (1,0,0)\}. & \end{aligned}$$

This says that A gets the entire \$1 billion and the other companies get nothing. The Shapley value is definitely more realistic.

Shapley vectors can also quickly analyze the winning coalitions in games where winning or losing is all we care about: who do we team up with to win. Here are the definitions.

Definition 5.3.2 *Suppose that we are given a normalized characteristic function $v(S)$ that satisfies that for every $S \subset N$, either $v(S) = 0$ or $v(S) = 1$. This is called a* **simple game**. *If $v(S) = 1$, the coalition S is said to be a* **winning coalition**. *If $v(S) = 0$, the coalition S is said to be a* **losing coalition**. *Let*

$$W^i = \{S \in \Pi^i \mid v(S) = 1, v(S - i) = 0\},$$

denote the set of coalitions who win with player i and lose without player i.

Simple games are very important in voting systems. For example, a game in which the coalition with a majority of members wins has $v(S) = 1$, if $|S| > n/2$, as the winning coalitions. Losing coalitions have $|S| \le n/2$ and $v(S) = 0$. If only unanimous votes win, then $v(N) = 1$ is the only winning coalition. Finally, if there is a certain player who has dictatorial power, say, player 1, then $v(S) = 1$ if $1 \in S$ and $v(S) = 0$ if $1 \notin S$.

In the case of a simple game for player i we need only consider coalitions $S \in \Pi^i$ for which S is a winning coalition, but $S - i$, that is, S without i, is a losing coalition. We have denoted that set by W^i. We need only consider $S \in W^i$ because $v(S) - v(S - i) = 1$ only when $v(S) = 1$, and $v(S - i) = 0$. In all other cases $v(S) - v(S - i) = 0$. Hence, the Shapley value for a simple game is

$$\begin{aligned} x_i &= \sum_{\{S \in \Pi^i\}} [v(S) - v(S - i)] \frac{(|S| - 1)!(n - |S|)!}{n!} \\ &= \sum_{\{S \in W^i\}} \frac{(|S| - 1)!(n - |S|)!}{n!} \end{aligned}$$

The Shapley allocation for player i represents the power that player i holds in a game. It is also called the **Shapley–Shubik index**.

EXAMPLE 5.18

A corporation has four stockholders (with 100 total shares) who all vote their own individual shares on any major decision. The majority of shares voted decides an issue. A majority consists of more than 50 shares. Suppose that the holdings of each stockholder are as follows:

player	1	2	3	4
shares	10	20	30	40

The winning coalitions, that is, with $v(S) = 1$ are

$$W = \{24, 34, 123, 124, 234, 1234\}.$$

We find the Shapley allocation. For x_1, it follows that $W^1 = \{123\}$ because $S = \{123\}$ is winning but $S - 1 = \{23\}$ is losing. Hence

$$x_1 = \frac{(4-3)!(3-1)!}{4!} = \frac{1}{12}.$$

Similarly, $W^2 = \{24, 123, 234\}$, and so

$$x_2 = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

Also, $x_3 = \frac{1}{4}$ and $x_4 = \frac{5}{12}$. We conclude that the Shapley allocation for this game is $\vec{x} = (\frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{5}{12})$.Notice that player 1 has the least power, but players 2 and 3 have the same power even though player 3 controls 10 more shares than does player 2. Player 4 has the most power, but a coalition is still necessary to constitute a majority.

Continue this example, but change the shares as follows

player	1	2	3	4
shares	10	30	30	40

Computing the Shapley value as $x_1 = 0, x_2 = x_3 = x_4 = \frac{1}{3}$, we see that player 1 is completely marginalized as she contributes nothing to any coalition. She has no power. In addition, player 4's additional shares over players 2 and 3 provide no advantage over those players since a coalition is essential to carry a majority in any case.

EXAMPLE 5.19

The United Nations Security Council has 15 members, five of whom are permanent (Russia, Great Britain, France, China, and the United States). These five players have veto power over any resolution. To pass a resolution requires all five permanent member's votes and four of the remaining 10 nonpermanent member's votes. This is a game with fifteen players, and we want to determine the Shapley–Shubik index of their power. We label players 1,2,3,4,5 as the permanent members.

Instead of the natural definition of a winning coalition as one that can pass a resolution, it is easier to use the definition that a winning coalition is one that can **defeat** a resolution. So, for player 1 the winning coalitions are those for which $S \in \Pi^1$, and $v(S) = 1$, $v(S - 1) = 0$; that is, player 1, or player 1 and any number up to six nonpermanent members can defeat a resolution, so that the winning coalitions for player 1 is the set

$$W^1 = \{1, 1a, 1ab, 1abc, 1abcd, 1abcde, 1abcdef\},$$

where the letters denote distinct nonpermanent members. The number of distinct two-player winning coalitions is $10 = \binom{10}{1}$,[5] three-player coalitions is $\binom{10}{2}$, four-player coalitions is $\binom{10}{3}$, and so on, and each of these coalitions will have the same coefficients in the Shapley value. So we get

$$x_1 = \frac{0!14!}{15!} + \binom{10}{1}\frac{1!13!}{15!} + \binom{10}{2}\frac{2!12!}{15!} + \cdots + \binom{10}{6}\frac{6!8!}{15!}$$

We can use Maple to give us the result with this command:

```
> tot:=0;
> for k from 0 to 6 do
    tot:=tot+binomial(10,k)*k!*(14-k)!/15!
    end do:
> print(tot);
```

We get $x_1 = \frac{421}{2145} = 0.1963$. Obviously, it must also be true that $x_2 = x_3 = x_4 = x_5 = 0.19623$. The five permanent members have a total power of $5 \times 0.19623 = 0.9812$ or 98.12% of the power, while the nonpermanent members have $x_6 = \cdots = x_{15} = 0.0019$ or 0.19% each, or a total power of 1.88%.

■ EXAMPLE 5.20

In this example[6] we show how cooperative game theory can determine a fair allocation of taxes to a community. For simplicity, assume that there are only four households and that the community requires expenditures of \$100,000. The question is how to allocate the cost of the \$100,000 among the four households.

[5] Recall that the binomial coefficient is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

[6] Adapted from Aliprantis and Chakrabarti [1, p.232].

As in most communities, we consider the wealth of the households as represented by the value of their property. Suppose the wealth of household i is w_i. Our four households have specific wealth values

$$w_1 = 75, \ w_2 = 175, \ w_3 = 200, \ w_4 = 300,$$

again with units in thousands of dollars. In addition, suppose that there is a cap on the amount that each household will have to pay (on the basis of age, income, or some other factors) that is independent of the value of their property value. In our case we take the maximum amount each of the four households will be required to pay as

$$u_1 = 25, \ u_2 = 30, \ u_3 = 20, \ u_4 = 80.$$

What is the fair allocation of expenses to each household?

Let's consider the general problem first. Define the variables

T	Total costs of community
u_i	Maximum amount i will have to pay
w_i	Net worth of player i
z_i	Amount player i will have to pay
$u_i - z_i$	Surplus of the cap over the assessment

The quantity $u_i - z_i$ is the difference between the maximum amount that household i would ever have to pay and the amount household i actually pays. It represents the amount household i does not have to pay.

We will assume that the total wealth of all the players is greater than T, and that the total amount that the players are willing (or are required) to pay is greater than T, but the total actual amount that the players will have to pay is exactly T :

$$\sum_{i=1}^{n} w_i > T, \ \sum_{i=1}^{n} u_i > T, \ \text{ and } \ \sum_{i=1}^{n} z_i = T. \tag{5.3.1}$$

This makes sense because "you can't squeeze blood out of a turnip." Here is the characteristic function we will use:

$$v(S) = \begin{cases} \max\left(\sum_{i \in S} u_i - T, 0\right) & \text{if } \sum_{i \in S} w_i \geq T; \\ 0 & \text{if } \sum_{i \in S} w_i < T. \end{cases}$$

In other words, $v(S) = 0$ in two cases: (1) if the total wealth of the members of coalition S is less than the total cost, $\sum_{i \in S} w_i < T$, or (2) if the total maximum amount coalition S is required to pay is less than T, $\sum_{i \in S} u_i < T$. If a coalition S cannot afford the expenditure T, then the characteristic function of that coalition is zero.

The Shapley value involves the expression $v(S) - v(S - j)$ in each term. Only the terms with $v(S) - v(S - j) > 0$ need to be considered.

Suppose first that the coalition S and player $j \in S$ **satisfies** $v(S) > 0$ **and** $v(S - j) > 0$. That means the coalition S and the coalition S without player j can finance the community. We compute

$$v(S) - v(S - j) = \sum_{i \in S} u_i - T - \left(\sum_{i \in S, i \neq j} u_i - T \right) = u_j.$$

Next, suppose that the coalition S can finance the community, but not without $j : v(S) > 0, v(S - j) = 0$. Then

$$v(S) - v(S - j) = \sum_{i \in S} u_i - T.$$

Summarizing the cases, we have

$$v(S) - v(S - j) = \begin{cases} u_j & \text{if } v(S) > 0, v(S - j) > 0; \\ \sum_{i \in S} u_i - T & \text{if } v(S) > 0, v(S - j) = 0; \\ 0 & \text{if } v(S) = v(S - j) = 0. \end{cases}$$

Notice that if $j \in S$ and $v(S - j) > 0$, then automatically $v(S) > 0$. We are ready to compute the Shapley allocation. For player $j = 1, \ldots, n$, we have,

$$\begin{aligned} x_j &= \sum_{\{S \in \Pi^j\}} [v(S) - v(S - j)] \frac{(|S| - 1)!(n - |S|)!}{n!} \\ &= \sum_{\{S | j \in S, v(S - j) > 0\}} u_j \frac{(|S| - 1)!(n - |S|)!}{n!} \\ &+ \sum_{\{S | j \in S, v(S) > 0, v(S - j) = 0\}} \left(\sum_{i \in S} u_i - T \right) \frac{(|S| - 1)!(n - |S|)!}{n!} \end{aligned}$$

By our definition of the characteristic function for this problem, the allocation x_j is the portion of the surplus $\sum_i u_i - T > 0$ that will be assessed to household j. Consequently, the amount player j will be billed is actually $z_j = u_j - x_j$.

For the four-person problem data above we have $T = 100, \sum w_i = 750 > 100, \sum_i u_i = 155 > 100$, so all our assumptions in (5.3.1) are verified. Remember that the units are in thousands of dollars. Then we have

$$v(i) = 0,\ v(12) = v(13) = v(23) = 0,\ v(14) = 5, v(24) = 10, v(34) = 0,$$
$$v(123) = 0, v(134) = 25, v(234) = 30, v(124) = 35, v(1234) = 55.$$

For example, $v(134) = \max(u_1 + u_3 + u_4 - 100, 0) = 125 - 100 = 25$. We compute

$$
\begin{aligned}
x_1 &= \sum_{\{S \mid 1 \in S, v(S-1) > 0\}} u_1 \frac{(|S|-1)!(4-|S|)!}{4!} \\
&+ \sum_{\{S \mid 1 \in S, v(S)>0, v(S-1)=0\}} \left(\sum_{i \in S} u_i - T\right) \frac{(|S|-1)!(n-|S|)!}{n!} \\
&= \frac{2!1!}{4!} \cdot u_1 + \frac{3!0!}{4!} u_1 \\
&+ \frac{1!2!}{4!}\left([u_1 + u_4 - 100]\right) + \frac{2!1!}{4!}[u_1 + u_3 + u_4 - 100] \\
&= \frac{65}{6}.
\end{aligned}
$$

The first term comes from coalition $S = 124$; the second term, from coalition $S = 1234$; the third term comes from coalition $S = 14$; and the last term from coalition $S = 134$.

As a result, the amount player 1 will be billed will be $z_1 = u_1 - x_1 = 25 - \frac{65}{6} = \frac{85}{6}$ thousand dollars. In a similar way we calculate

$$x_2 = \frac{40}{3}, \quad x_3 = \frac{25}{3}, \quad \text{and } x_4 = \frac{45}{2},$$

so that the actual bill to each player will be

$$
\begin{aligned}
z_1 &= 25 - \tfrac{65}{6} = 14.167, \\
z_2 &= 30 - \tfrac{40}{3} = 16.667, \\
z_3 &= 20 - \tfrac{25}{3} = 11.667, \\
z_4 &= 80 - \tfrac{45}{2} = 57.5.
\end{aligned}
$$

For comparison purposes it is not too difficult to calculate the nucleolus for this game to be $\left(\frac{25}{2}, 15, 10, \frac{35}{2}\right)$, so that the payments using the nucleolus will be

$$
\begin{aligned}
z_1 &= 25 - \tfrac{25}{2} = \tfrac{25}{2} = 12.5, \\
z_2 &= 30 - 15 = 15, \\
z_3 &= 20 - 10 = 10, \\
z_4 &= 80 - \tfrac{35}{2} = \tfrac{125}{2} = 62.5.
\end{aligned}
$$

There is yet a third solution, the straightforward solution that assesses the amount to each player in proportion to each household's maximum payment to the total assessment. For example, $u_1/(\sum_i u_i) = 25/155 = 0.1613$ and so player 1 could be assessed the amount $0.1613 \times 100 = 16.13$.

We end this section by explaining how Shapley actually came up with his fair allocation, because it is very interesting in its own right.

First we separate players who don't really matter. A player i is a **dummy** if for any coalition S in which $i \notin S$, we have

$$v(S \cup i) = v(S).$$

So dummy player i contributes nothing to any coalition. The players who are not dummies are called the **carriers** of the game. Let's define C = set of carriers.

Shapley now looked at things this way. Given a characteristic function v, we should get an allocation as a function of v, $\varphi(v) = (\varphi_1(v), \ldots, \varphi_n(v))$, where $\varphi_i(v)$ will be the allocation or worth or value of player i in the game, and this function φ should satisfy the following properties:

1. $v(N) = \sum_{i=1}^{n} \varphi_i(v)$. (Group rationality).

2. If players i and j satisfy $v(S \cup i) = v(S \cup j)$ for any coalition with $i \notin S, j \notin S$, then $\varphi_i(v) = \varphi_j(v)$. If i and j provide the same benefit, they should have the same worth.

3. If i is a dummy player, $\varphi_i(v) = 0$. Dummies should be worth nothing.

4. If v_1 and v_2 are two characteristic functions, then $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$.

The last property is the strongest and most controversial. It essentially says that the allocation to a player using the sum of characteristic functions should be the sum of the allocations corresponding to each characteristic function.

Now these properties, if you agree that they are reasonable, leads to a surprising conclusion. There is one and only one function φ that satisfies them! It is given by $\varphi(v) = (\varphi_1(v), \ldots, \varphi_n(v))$, where

$$\varphi_i(v) = \sum_{\{S \in \Pi^i\}} [v(S) - v(S - i)] \frac{(|S| - 1)!(|N| - |S|)!}{|N|!}, i = 1, 2, \ldots, n.$$

This is the **only** function satisfying the properties, and, sure enough, it is the Shapley value.

PROBLEMS

5.21 Show that the Shapley value is in fact an imputation but may not be in the core. For example, suppose that the seller of an object (which is worthless to the seller) has two potential buyers who are willing to pay \$100 or \$130, respectively. Find the characteristic function and then the core and the Shapley value.

5.22 Consider the characteristic function in Example 5.16 for the creditor–debtor problem. By writing out the table of the order of arrival of each player versus the benefit the player brings to a coalition when the player arrives as in Example 5.17, calculate the Shapley value.

5.23 Find the Shapley allocation for the three-person characteristic function game with

$$\begin{aligned}
&v(i) = 0, i = 1, 2, 3,\\
&v(12) = v(13) = 2, v(23) = 10,\\
&v(123) = 12.
\end{aligned}$$

5.24 Once again we consider the four doctors, Moe, Larry, Curly, and Shemp, and their problem to minimize the amount of hours they work. The characteristic function is the number of hours saved by a coalition. We have $v(i) = 0$ and

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2,$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13.$$

Find the Shapley allocation.

5.25 **Modified Garbage Game.** Suppose that there are four property owners each with one bag of garbage that needs to be dumped on somebody's property (one of the four).

(**a**) Take the characteristic function to be $v(N) = 4$ and $v(S) = (4 - |S|)$, where $|S|$ is the number of members in S. Is the core empty?

(**b**) Find the least core.

(**c**) Now find the Shapley allocation.

(**d**) Find the Shapley value for the original garbage game with $v(N) = -4$ and $v(S) = -(4 - |S|)$.

5.26 A farmer (player 1) owns some land which he values at \$100K. A speculator (player 2) feels that if she buys the land, she can subdivide it into lots and sell the lots for a total of \$150K. A home developer (player 3) thinks that he can develop the land and build homes that he can sell. So the land to the developer is worth \$160K.

(**a**) Find the characteristic function and the Shapley allocation.

(**b**) Compare the Shapley allocation with the nucleolus allocation.

5.27 Find the Shapley allocation for the cost game in Problem 5.20.

5.4 BARGAINING

In this section we will introduce a new type of cooperative game in which the players bargain to improve both of their payoffs. Let us start with a simple example to illustrate the benefits of bargaining and cooperation. Consider the prisoner's dilemma two-player nonzero sum game with bimatrix

	II_1	II_2
I_1	$(2,1)$	$(-1,-1)$
I_2	$(-1,-1)$	$(1,2)$

You can easily check that there are three Nash equilibria given by $X_1 = (0,1) = Y_1, X_2 = (1,0) = Y_2$, and $X_3 = (\frac{3}{5}, \frac{2}{5}), Y_3 = (\frac{2}{5}, \frac{3}{5})$. Now consider Figure 5.8.

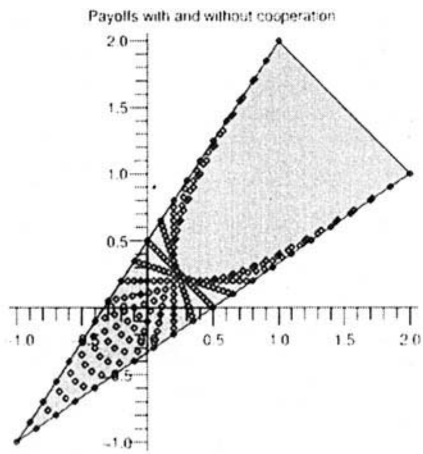


Figure 5.8 Payoff I versus payoff II for the prisoner's dilemma.

The points represent the possible pairs of payoffs to each player $(E_1(x,y), E_2(x,y))$ given by

$$E_1(x,y) = (x, 1-x)A\begin{pmatrix} y \\ 1-y \end{pmatrix}, \qquad E_2(x,y) = (x, 1-x)B\begin{pmatrix} y \\ 1-y \end{pmatrix}.$$

It was generated with the following Maple commands:

```
> with(plots):with(plottools):with(LinearAlgebra):
> A:=Matrix([[2,-1],[-1,1]]);B:=Matrix([[1,-1],[-1,2]]);
> f:=(x,y)->expand(Transpose(<x,1-x>).A.<y,1-y>);
> g:=(x,y)->expand(Transpose(<x,1-x>).B.<y,1-y>);
> points:={seq(seq([f(x,y),g(x,y)],x=0..1,0.05),y=0..1,0.05)}:
> pure:=([[2,1],[-1,-1],[-1,-1],[1,2]]);
> pp:=pointplot(points);
> pq:=polygon(pure,color=yellow);
> display(pq,pp,title="Payoffs with and without cooperation");
```

The horizontal axis (abscissa) is the payoff to player I, and the vertical axis (ordinate) is the payoff to player II. Any point in the parabolic region is achievable for a some $0 \le x \le 1, 0 \le y \le 1$.

The parabola is given by the implicit equation $5(E_1 - E_2)^2 - 2(E_1 + E_2) + 1 = 0$. If the players play pure strategies, the payoff to each player will be at one of the vertices. The pure Nash equilibria yield the payoff pairs $(E_1 = 1, E_2 = 2)$ and $(E_1 = 2, E_2 = 1)$. The mixed Nash point gives the payoff pair $(E_1 = \frac{1}{5}, E_2 = \frac{1}{5})$, which is strictly inside the region of points, called the **noncooperative payoff set.**

Now, if the players do not cooperate, they will achieve one of two possibilities: (1) The vertices of the figure, if they play pure strategies; or (2) any point in the region of points bounded by the two lines and the parabola, if they play mixed strategies. The portion of the triangle outside the parabolic region is **not** achievable simply by the players using mixed strategies. However, if the players agree to cooperate, then any point on the boundary of the triangle, the entire shaded region,[7] including the boundary of the region, are achievable payoffs, which we will see shortly. Cooperation here means an agreement as to which combination of strategies each player will use and the proportion of time that the strategies will be used.

Player I wants a payoff as large as possible and thus as far to the right on the triangle as possible. Player II wants to go as high on the triangle as possible. So player I wants to get the payoff at $(2, 1)$, and player II wants the payoff at $(1, 2)$, but this is possible if and only if the opposing player agrees to play the correct strategy. In addition, it seems that nobody wants to play the mixed Nash equilibrium because they can both do better, but they have to cooperate to achieve a higher payoff.

Here is another example illustrating the achievable payoffs.

■ **EXAMPLE 5.21**

	II_1	II_2	II_3
I_1	$(1, 4)$	$(-2, 1)$	$(1, 2)$
I_2	$(0, -2)$	$(3, 1)$	$(\frac{1}{2}, \frac{1}{2})$

We will draw the pure payoff points of the game as the vertices of the graph and connect the pure payoffs with straight lines, as in Figure 5.9. The vertices of the polygon are the payoffs from the matrix. The solid lines connect the pure payoffs. The dotted lines extend the region of payoffs to those payoffs that **could** be achieved if both players cooperate. For example, suppose that player I always chooses row 2, I_2, and player II plays the mixed strategy $Y = (y_1, y_2, y_3)$, where $y_i \ge 0, y_1 + y_2 + y_3 = 1$. The expected payoff to I is then

$$E_1(2, Y) = 0\, y_1 + 3y_2 + \frac{1}{2}\, y_3,$$

[7]This region is called the **convex hull** of the pure payoff pairs. The convex hull of a set of points is the smallest convex set containing all the points.

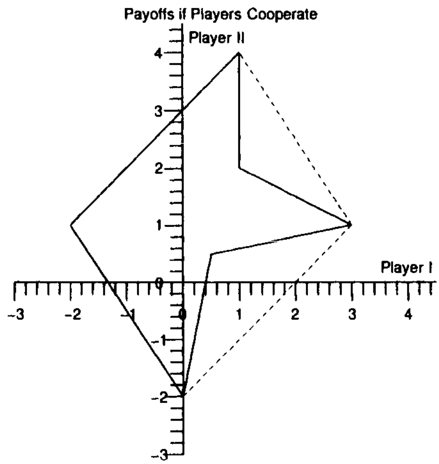


Figure 5.9 Achievable payoffs with cooperation.

and the expected payoff to II is

$$E_2(2, Y) = -2y_1 + 1y_2 + \frac{1}{2}\, y_3.$$

Hence

$$(E_1, E_2) = y_1(0, -2) + y_2(3, 1) + y_3\left(\frac{1}{2}, \frac{1}{2}\right),$$

which, as a linear combination of the three points $(0, -2)$, $(3, 1)$, and $(\frac{1}{2}, \frac{1}{2})$, is in the convex hull of these three points. This means that if players I and II can agree that player I will always play row 2, then player II can choose a (y_1, y_2, y_3) so that the payoff pair to each player will be in the triangle bounded by the lower dotted line in Figure 5.9 and the lines connecting $(0, -2)$ with $(\frac{1}{2}, \frac{1}{2})$ with $(3, 1)$. The conclusion is that any point in the convex hull of all the payoff points is achievable if the players agree to cooperate.

One thing to be mindful of is that the Figure 5.9 does not show the actual payoff pairs that are achievable in the noncooperative game as we did for the 2×2 prisoner's dilemma game (Figure 5.8) because it is too involved. The boundaries of that region may not be straight lines or parabolas.

The entire triangle in Figure 5.9 is called the **feasible set** for the problem. The precise definition in general is as follows.

Definition 5.4.1 *The* **feasible set** *is the convex hull of all the payoff points corresponding to pure strategies of the players.*

The objective of player I in Example 5.21 is to obtain a payoff as far to the right as possible in Figure 5.9, and the objective of player II is to obtain a payoff as far up as possible in Figure 5.9. Player I's ideal payoff is at the point $(3, 1)$, but that is attainable only if II agrees to play II_2. Why would he do that? Similarly, II would do best at $(1, 4)$, which will happen only if I plays I_1, and why would she do that? There is an incentive for the players to reach a compromise agreement in which they would agree to play in such a way so as to obtain a payoff along the line connecting $(1, 4)$ and $(3, 1)$. That portion of the boundary is known as the **Pareto-optimal boundary** because it is the edge of the set and has the property that if either player tries to do better (say, player I tries to move further right), then the other player will do worse (player II must move down to remain feasible). That is the definition. We have already defined what it means to be Pareto-optimal, but it is repeated here for convenience.

Definition 5.4.2 *The Pareto-optimal boundary of the feasible set is the set of payoff points in which no player can improve his payoff without at least one other player decreasing her payoff.*

The point of this discussion is that there is an incentive for the players to cooperate and try to reach an agreement that will benefit both players. The result will always be a payoff pair occuring on the Pareto-optimal boundary of the feasible set.

In any bargaining problem there is always the possibility that negotiations will fail. Hence, each player must know what the payoff would be if there were no bargaining. This leads us to the next definition.

Definition 5.4.3 *The* **status quo payoff point**, *or* **safety point**, *or* **security point** *in a two-person game is the pair of payoffs* (u^*, v^*) *that each player can achieve if there is no cooperation between the players.*

The safety point usually is, but does not have to be, the same as the safety levels defined earlier. Recall that the safety levels we used in previous sections were defined by the pair $(value(A), value(B^T))$. In the context of bargaining it is simply a noncooperative payoff to each player if no cooperation takes place. For most problems considered in this section, the status quo point **will** be taken to be the values of the zero sum games associated with each player, because those values can be guaranteed to be achievable, no matter what the other player does.

■ **EXAMPLE 5.22**

We will determine the security point for each player in Example 5.21. In this example we take the security point to be the values that each player can guarantee receiving no matter what. This means that we take it to be the value of the zero sum games for each player.

Consider the payoff matrix for player I:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & \frac{1}{2} \end{bmatrix}$$

We want the value of the game with matrix A. By the methods of Chapter 1 we find that $v(A) = \frac{1}{2}$ and the optimal strategies are $Y = \left(\frac{5}{6}, \frac{1}{6}, 0\right)$ for player II and $X = \left(\frac{1}{2}, \frac{1}{2}\right)$ for player I.

Next we consider the payoff matrix for player II. We call this matrix B but since we want to find the value of the game from player II's perspective, we actually need to work with B^T since it is always the row player who is the maximizer (and II is trying to maximize his payoff). Now

$$B^T = \begin{bmatrix} 4 & -2 \\ 1 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}.$$

For this matrix $v(B^T) = 1$, and we have a saddle point at row 1 column 2.

We conclude that the status quo point for this game is $(E_1, E_2) = (\frac{1}{2}, 1)$ since that is the guaranteed payoff to each player without cooperation or negotiation. This means that any bargaining must begin with the guaranteed payoff pair $(\frac{1}{2}, 1)$. This cuts off the feasible set as in Figure 5.10. The new feasible

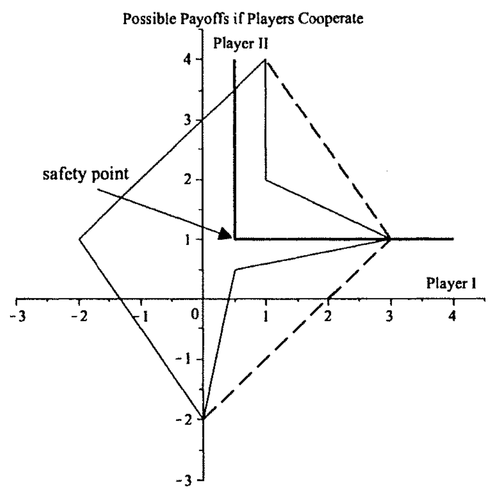


Figure 5.10 The reduced feasible set; safety at $(\frac{1}{2}, 1)$.

set consists of the points in Figure 5.10 above and to the right of the lines emanating from the security point $(\frac{1}{2}, 1)$. It is like moving the origin to the new point $(\frac{1}{2}, 1)$.

Notice that in this problem the Pareto-optimal boundary is the line connecting $(1,4)$ and $(3,1)$ because no player can get a bigger payoff on this line without forcing the other player to get a smaller payoff. A point in the set can't go to the right and stay in the set without also going down; a point in the set can't go up and stay in the set without also going to the left.

The question now is to find the cooperative, negotiated best payoff for each player. How does cooperation help? Well, suppose, for example, that the players agree to play as follows: I will play row I_1 half the time and row I_2 half the time as long as II plays column II_1 half the time and column II_2 half the time. This is not optimal for player II in terms of his safety level. But, if they agree to play this way, they will get $\frac{1}{2}(1,4) + \frac{1}{2}(3,1) = (2, \frac{5}{2})$. So player I gets $2 > \frac{1}{2}$ and player II gets $\frac{5}{2} > 1$, a big improvement for each player over his or her own individual safety level. So, they definitely have an incentive to cooperate.

EXAMPLE 5.23

Here is another example. The bimatrix is

	II_1	II_2
I_1	$(2,17)$	$(-10,-22)$
I_2	$(-19,-7)$	$(17,2)$

The reader can calculate that the safety level is given by the point

$$(value(A), value(B^T)) = \left(-\frac{13}{4}, -\frac{5}{2}\right),$$

and the optimal strategies that will give these values are $X_A = (\frac{3}{4}, \frac{1}{4}), Y_A = (\frac{9}{16}, \frac{7}{16})$, and $X_B = (\frac{1}{2}, \frac{1}{2}), Y_B = (\frac{3}{16}, \frac{13}{16})$. Negotiations start from the safety point. Figure 5.11 shows the safety point and the associated feasible payoff pairs above and to the right of the dark lines. The shaded region in Figure 5.11 is the convex hull of the pure payoffs, namely, the feasible set, and is the set of all possible negotiated payoffs. The region of dot points is the set of noncooperative payoff pairs if we consider the use of all possible mixed strategies. The set we consider is the shaded region above and to the right of the safety point. It appears that a negotiated set of payoffs will benefit both players and will be on the line farthest to the right, which is the Pareto-optimal boundary. Player I would love to get $(17,2)$, while player II would love to get $(2,17)$. That probably won't occur but they could negotiate a point along the line connecting these two points and compromise on obtaining, say, the

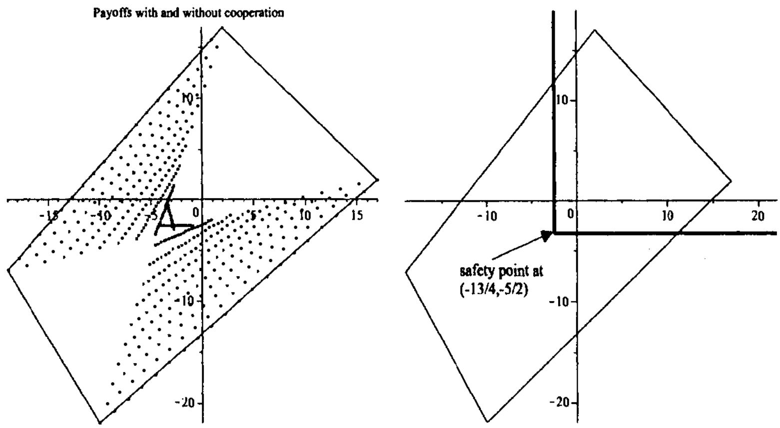


Figure 5.11 Achievable payoff pairs with cooperation; safety point $= \left(\frac{13}{4}, -\frac{5}{2}\right)$.

midpoint

$$\frac{1}{2}(2,17) + \frac{1}{2}(17,2) = (9.5, 9.5).$$

So they could negotiate to get 9.5 each if they agree that each player would use the pure strategies $X = (1,0) = Y$ half the time and play pure strategies $X = (0,1) = Y$ exactly half the time. They have an incentive to cooperate.

Now suppose that player II threatens player I by saying that she will always play strategy II_1 unless I cooperates. Player II's goal is to get the 17 if and when I plays I_1, so I would receive 2. Of course, I does not have to play I_1, but if he doesn't, then I will get -19, and II will get -7. So, if I does not cooperate and II carries out her threat, they will both lose, but I will lose much more than II. Therefore, II is in a much stronger position than I in this game and can essentially **force** I to cooperate. This implies that the safety level of $\left(-\frac{13}{4}, -\frac{5}{2}\right)$ loses its effect here because II has a credible threat that she can use to force player I to cooperate. This also seems to imply that maybe player II should expect to get more than 9.5 to reflect her stronger bargaining position from the start.

The preceding example indicates that there may be a more realistic choice for a safety level than the values of the associated games, taking into account various threat possibilities. We will see later that this is indeed the case.

5.4.1 The Nash Model with Security Point

We start with any old security status quo point (u^*, v^*) for a two-player cooperative game with matrices A, B. This leads to a feasible set of possible negotiated outcomes depending on the point we start from (u^*, v^*). This may be the safety point $u^* = value(A), v^* = value(B^T)$, or not. For any given such point and feasible set S, we are looking for a negotiated outcome, call it $(\overline{u}, \overline{v})$. This point will depend on (u^*, v^*) and the set S, so we may write

$$(\overline{u}, \overline{v}) = f(S, u^*, v^*).$$

The question is how to determine the point $(\overline{u}, \overline{v})$? John Nash proposed the following requirements for the point to be a negotiated solution:

- **Axiom 1.** We must have $\overline{u} \geq u^*$ and $\overline{v} \geq v^*$. Each player must get at least the status quo point.
- **Axiom 2.** The point $(\overline{u}, \overline{v}) \in S$, that is, it must be a feasible point.
- **Axiom 3.** If (u, v) is any point in S, so that $u \geq \overline{u}$ and $v \geq \overline{v}$, then it must be the case that $u = \overline{u}, v = \overline{v}$. In other words, there is no other point in S, where **both** players receive more. This is **Pareto-optimality**.
- **Axiom 4.** If $(\overline{u}, \overline{v}) \in T \subset S$ and $(\overline{u}, \overline{v}) = f(T, u^*, v^*)$ is the solution to the bargaining problem with feasible set T, then for the larger feasible set S, either $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ is the bargaining solution for S, or the actual bargaining solution for S is in $S - T$. We are assuming that the security point is the same for T and S. So, if we have more alternatives, the new negotiated position can't be one of the old possibilities.
- **Axiom 5.** If T is an affine transformation of $S, T = aS + b = \varphi(S)$ and $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ is the bargaining solution of S with security point (u^*, v^*), then $(a\overline{u} + b, a\overline{v} + b) = f(T, au^* + b, av^* + b)$ is the bargaining solution associated with T and security point $(au^* + b, av^* + b)$. This says that the solution will not depend on the scale or units used in measuring payoffs.
- **Axiom 6.** If the game is symmetric with respect to the players, then so is the bargaining solution. In other words, if $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ and (i) $u^* = v^*$, and (ii) $(u, v) \in S \Rightarrow (v, u) \in S$, then $\overline{u} = \overline{v}$. So, if the players are essentially interchangeable they should get the same negotiated payoff.

The amazing thing that Nash proved is that if we assume these axioms, there is one and only one solution of the bargaining problem. In addition, the theorem gives a constructive way of finding the bargaining solution.

Theorem 5.4.4 *Let the set of feasible points for a bargaining game be nonempty and convex, and let $(u^*, v^*) \in S$ be the security point. Consider the nonlinear programming problem*

$$\begin{aligned} &\textit{Maximize } g(u,v) := (u - u^*)(v - v^*) \\ &\textit{subject to } (u,v) \in S, u \geq u^*, v \geq v^*. \end{aligned}$$

Assume that there is at least one point $(u, v) \in S$ with $u > u^*, v > v^*$. Then there exists one and only one point $(\overline{u}, \overline{v}) \in S$ that solves this problem, and this point is the unique solution of the bargaining problem $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ that satisfies the axioms $1 - 6$. If, in addition, the game satisfies the symmetry assumption, then the conclusion of axiom 6 tells us that $\overline{u} = \overline{v}$

Proof. We will only sketch a part of the proof and skip the rest.

1. **Existence.** Define the function $g(u,v) \equiv (u - u^*)(v - v^*)$. The set

$$S^* = \{(u,v) \in S \mid u \geq u^*, v \geq v^*\}$$

is convex, closed, and bounded. Since $g : S^* \to \mathbb{R}$ is continuous, a theorem of analysis (any continuous function on a closed and bounded set achieves a maximum and a minimum on the set) guarantees that g has a maximum at some point $(\overline{u}, \overline{v}) \in S^*$. By assumption there is at least one feasible point with $u > u^*, v > v^*$. For this point $g(u, v) > 0$ and so the maximum of g over S^* must be > 0 and therefore does not occur at the safety points $u = u^*$ or $v = v^*$.

2. **Uniqueness.** Suppose the maximum of g occurs at two points $0 < M = g(u', v') = g(u'', v'')$. If $u' = u''$, then

$$g(u', v') = (u' - u^*)(v' - v^*) = g(u'', v'') = (u'' - u^*)(v'' - v^*),$$

so that dividing out $u' - u^*$, implies that $v' = v''$ also. So we may as well assume that $u' < u''$ and that implies $v' > v''$ because $(u' - u^*)(v' - v^*) = (u'' - u^*)(v'' - v^*) = M > 0$ so that

$$\frac{u'' - u^*}{u' - u^*} = \frac{v' - v^*}{v'' - v^*} > 1 \implies v' - v^* > v'' - v^* \implies v' > v''.$$

Set $(u, v) = \frac{1}{2}(u', v') + \frac{1}{2}(u'', v'')$. Since S is convex, $(u, v) \in S$ and $u > u^*, v > v^*$. So $(u, v) \in S^*$. Some simple algebra shows that

$$g(u,v) = M + \frac{(u' - u'')(v'' - v')}{4} > M, \text{ since } u'' > u', v'' < v'.$$

This contradicts the fact that (u', v') provides a maximum for g over S^* and so the maximum point must be unique.

3. **Pareto-optimality.** We show that the solution of the nonlinear program, say, $(\overline{u}, \overline{v})$, is Pareto-optimal. If it is not Pareto-optimal, then there must be another

feasible point $(u', v') \in S$ for which either $u' > \overline{u}$ and $v' \geq \overline{v}$, or $v' > \overline{v}$ and $u' \geq \overline{u}$. We may as well assume the first possibility. Since $\overline{u} > u^*, \overline{v} > v^*$, we then have $u' > u^*$ and $v' > v^*$ and so $g(u', v') > 0$. Next, we have

$$g(u', v') = (u' - u^*)(v' - v^*) > (\overline{u} - u^*)(\overline{v} - v^*) = g(\overline{u}, \overline{v}).$$

But this contradicts the fact that $(\overline{u}, \overline{v})$ maximizes g over the feasible set. Hence $(\overline{u}, \overline{v})$ is Pareto-optimal.

The rest of the proof will be skipped, and the interested reader can refer to the references, e.g., the book by Jones [7], for a complete proof.

■ EXAMPLE 5.24

In an earlier example we considered the game with bimatrix

	II_1	II_2
I_1	$(2, 17)$	$(-10, -22)$
I_2	$(-19, -7)$	$(17, 2)$

The safety levels are $u^* = value(A) = -\frac{13}{4}, v^* = value(B^T) = -\frac{5}{2}$,

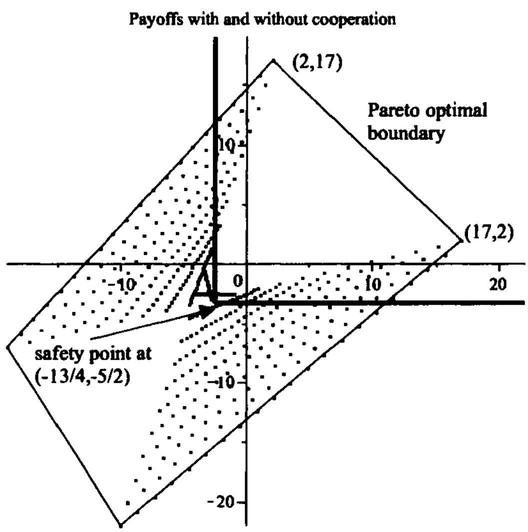


Figure 5.12 Pareto-optimal boundary is line connecting $(2, 17)$ and $(17, 2)$.

Figure 5.12 for this problem shows the safety point and the associated feasible payoff pairs above and to the right. Now we need to describe the Pareto-optimal boundary of the feasible set. We need the equation of the lines forming the Pareto-optimal boundary. In this example it is simply $v = -u + 19$, which is the line with negative slope to the right of the safety point. It is the only place where both players cannot simultaneously improve their payoffs. (If player I moves right, to stay in the feasible set player II must go down.)

To find the bargaining solution for this problem, we have to solve the nonlinear programming problem

$$\text{Maximize } \left(u + \frac{13}{4}\right)\left(v + \frac{5}{2}\right)$$
$$\text{subject to } u \geq -\frac{13}{4}, \quad v \geq -\frac{5}{2}, \quad v \leq -u + 19$$

The Maple commands used to solve this are

```
> with(Optimization):
> NLPSolve((u+13/4)*(v+5/2),
    {u>=-13/4,v>=-5/2,v<=-u+19},maximize);
```

This gives the optimal bargained payoff pair ($\overline{u} = \frac{73}{8} = 9.125, \overline{v} = \frac{79}{8} = 9.875$). The maximum of g is $g(\overline{u}, \overline{v}) = 153.14$, which we do not really use or need.

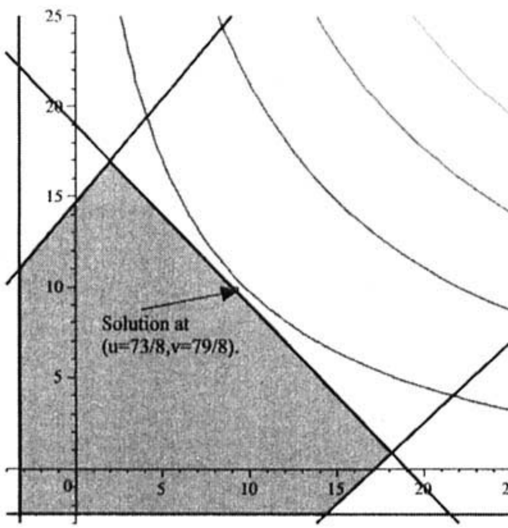


Figure 5.13 Bargaining solution where curves just touch Pareto boundary at (9.125, 9.875).

The bargained payoff to player I is $\overline{u} = 9.125$ and the bargained payoff to player II is $\overline{v} = 9.875$. We do **not** get the point we expected, namely, $(9.5, 9.5)$; that is due to the fact that the security point is not symmetric. Player II has a small advantage.

You can see in the Maple generated Figure 5.13 that the solution of the problem occurs just where the level curves, or contours of g are tangent to the boundary of the feasible set. Since the function g has concave up contours and the feasible set is convex, this must occur at exactly one point.

The Maple commands used to get Figure 5.13 are as follows.

```
> f:=(x,y)->(x+13/4)*(y+5/2);
> gcnst:={x >=-13/4,y>=-5/2,y<=-x+19,
                  y<=24/21*x+14.71,y>=24/27*x-15.22};
> with(plots):with(plottools):
> cp:=contourplot(f(x,y),x=0..25,y=0..25,
                  axes=normal,thickness=2,contours=4):
> ineq:=inequal( gcnst,x=-4..25, y=-3..25,
    optionsfeasible=(color=yellow),
    optionsopen=(color=blue,thickness=2),
    optionsclosed=(color=green, thickness=2),
    optionsexcluded=(color=white),labels=[x,y] ):
> pointp:=pointplot([73/8,79/8],thickness=5,symbol=circle):
> t1:=textplot([16,13,"(73/8,79/8)"],align={BELOW,LEFT});
> display3d(cp,ineq,t1,pointp,title="Bargaining Solution",
                  labels=['u','v'] );
```

Finally, knowing that the optimal point must occur on the Pareto-optimal boundary means we could solve the nonlinear programming problem by calculus. We want to maximize

$$f(u) = g(u, -u + 19) = (u + \frac{13}{4})(-u + 19 + \frac{5}{2}), \text{ on the interval } 2 \leq u \leq 17.$$

This is an elementary calculus maximization problem.

■ EXAMPLE 5.25

We will work through another example from scratch. We start with the following bimatrix:

	II_1	II_2
I_1	$(1, 3)$	$(-4, -2)$
I_2	$(-1, -3)$	$(2, 1)$

1. **Find the security point.** To begin we find the values of the associated matrices

$$A = \begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix}.$$

Then, $value(A) = -\frac{1}{4}$ and $value(B^T) = -\frac{1}{3}$. Hence the security point is $(u^*, v^*) = \left(-\frac{1}{4}, -\frac{1}{3}\right)$.

2. **Find the feasible set.** The feasible set, taking into account the security point, is

$$S^* = \{(u,v) \mid u \geq -\frac{1}{4}, v \geq -\frac{1}{3}, 0 \leq 10 + 5u - 5v, 0 \leq 10 + u + 3v,$$
$$0 \leq 5 - 4u + 3v, 0 \leq 5 - 2u - v\}.$$

3. **Set up and solve the nonlinear programming problem.** The nonlinear programming problem is then

$$\text{Maximize } g(u,v) \equiv \left(u + \frac{1}{4}\right)\left(v + \frac{1}{3}\right)$$
$$\text{subject to } (u,v) \in S^*.$$

Maple gives us the solution $\overline{u} = \frac{29}{24} = 1.208, \overline{v} = \frac{31}{12} = 2.583$. If we look at Figure 5.14 for S^*, we see that the Pareto-optimal boundary is the line $v = -2u + 5, 1 \leq u \leq 2$.

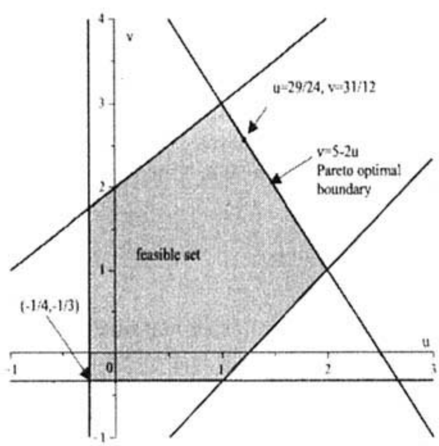


Figure 5.14 Security point $\left(-\frac{1}{4}, -\frac{1}{3}\right)$, Pareto boundary $v = -2u + 5$, solution $(1.208, 2.583)$.

The solution with the safety point given by the values of the zero sum games is at point $(\overline{u}, \overline{v}) = (1.208, 2.583)$. The conclusion is that with this security point, player I receives the negotiated solution $\overline{u} = 1.208$ and player II the amount $\overline{v} = 2.583$. Again, we do not need Maple to solve this problem if we know the line where the maximum occurs, which here is $v = -2u + 5$, because then we may substitute into g and use calculus:

$$\begin{aligned} f(u) &= g(u, -2u+5) = (u + \frac{1}{4})(-2u + \frac{16}{3}) \\ \implies f'(u) &= -4u + \frac{29}{6} = 0 \\ \implies u &= \frac{29}{24}. \end{aligned}$$

So this gives us the solution as well.

4. **Find the strategies giving the negotiated solution.** How should the players cooperate in order to achieve the bargained solutions we just obtained? To find out, the only points in the bimatrix that are of interest are the endpoints of the Pareto-optimal boundary, namely, $(1,3)$ and $(2,1)$. So the cooperation must be a linear combination of the strategies yielding these payoffs. Solve

$$\left(\frac{29}{24}, \frac{31}{12}\right) = \lambda(1,3) + (1-\lambda)(2,1),$$

to get $\lambda = \frac{19}{24}$. This says that (I,II) must agree to play (row 1,col 1) with probability $\frac{19}{24}$ and (row 2, col 2) with probability $\frac{5}{24}$.

The Nash bargaining theorem also applies to games in which the players have payoff functions $u_1(x,y), u_2(x,y)$, where x, y are in some interval and the players have a continuum of strategies. As long as the feasible set contains some security point u_1^*, u_2^*, we may apply Nash's theorem. Here is an example.

■ EXAMPLE 5.26

Suppose that two persons are given \$1000, which they can split if they can agree on how to split it. If they cannot agree they each get nothing. One player is rich, so her payoff function is

$$u_1(x) = \frac{x}{2}, \quad 0 \le x \le 1000$$

because the receipt of more money will not mean that much. The other player is poor, so his utility function is

$$u_2(y) = \ln(y+1) \quad ,0 \le y \le 1000,$$

because small amounts of money mean a lot but the money has less and less impact as he gets more but no more than \$1000. We want to find the bargained solution. The safety points are taken as $(0,0)$ because that is what they get if they can't agree on a split. The feasible set is $S = \{(x, y) \mid 0 \leq x, y \leq 1000, x + y \leq 1000\}$.

Figure 5.15 illustrates the feasible set and the contours of the objective function.

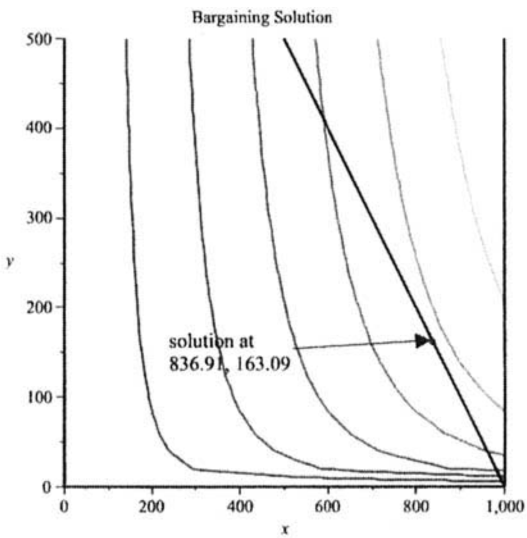


Figure 5.15 Rich and poor split \$1000: solution at $(836.91, 163.09)$.

The solution is obtained using Maple as follows.

```
> f:=(x,y)->(x/2)*(ln(1+y));
> cnst:={0<=x, x<=1000, 0 <=y, y<=1000, x+y <=1000};
> with(Optimization):
> NLPSolve(f(x,y),cnst,assume=nonnegative,maximize);
```

Maple tells us that the maximum is achieved at $x = 836.91, y = 163.09$, so the poor man gets \$163 while the rich woman gets \$837. Figure 5.15 shows the feasible set as well as several level curves of $f(x, y) = k$. The optimal solution is obtained by increasing k until the curve is tangent to the Pareto-optimal boundary. That occurs here at the point $(836.91, 163.09)$. The actual value of the maximum is of no interest to us.

5.4.2 Threats

Negotiations of the type considered in the previous section do not take into account the relative strength of the positions of the players in the negotiations. As mentioned earlier, a player may be able to force the opposing player to play a certain strategy by threatening to use a strategy that will be very detrimental for the opponent. These types of threats will change the bargaining solution. Let's start with an example.

■ EXAMPLE 5.27

We will consider the two-person game with bimatrix

	II_1	II_2
I_1	$(2,4)$	$(-3,-10)$
I_2	$(-8,-2)$	$(10,1)$

Player I's payoff matrix is

$$A = \begin{bmatrix} 2 & -3 \\ -8 & 10 \end{bmatrix}$$

and for II

$$B = \begin{bmatrix} 4 & -10 \\ -2 & 1 \end{bmatrix}, \text{ so we look at } B^T = \begin{bmatrix} 4 & -2 \\ -10 & 1 \end{bmatrix}.$$

It is left to the reader to verify that $value(A) = -\frac{4}{23}, value(B^T) = -\frac{16}{17}$ so the security point is $(u^*, v^*) = (-\frac{4}{23}, -\frac{16}{17})$.

With this security point we solve the problem

$$\text{Maximize } g(u,v) = \left(u + \frac{4}{23}\right)\left(v + \frac{16}{17}\right)$$
$$\text{subject to } u \geq -\frac{4}{23}, v \geq -\frac{16}{17}, v \geq \frac{11}{13}u - \frac{97}{13},$$
$$v \leq -\frac{3}{8}u + \frac{38}{8}, v \leq \frac{6}{10}u + \frac{28}{10}.$$

In the usual way we get the solution $\overline{u} = 7.501, \overline{v} = 1.937$. This is achieved by players I and II agreeing to play the pure strategies (I_1,II_1) 31.2% of the time and pure strategies (I_2,II_2) 68.8% of the time. So with the safety levels as the value of the games, we get the bargained payoffs to each player as 7.501 to player I and 1.937 to player II. Figure 5.16 below is a three-dimensional diagram of the contours of $g(u, v)$ over the shaded feasible set. The dot shown on the Pareto boundary is the solution to our problem.

The problem is that this solution is not realistic for this game. Why? The answer is that player II is actually in a much stronger position than player I. In fact, player II can threaten player I with always playing II_1. If player II does that and player I plays I_1, then I gets $2 < 7.501$, but II gets $4 > 1.937$. So, why would player I do that? Well, if player I instead plays I_2, in order to avoid

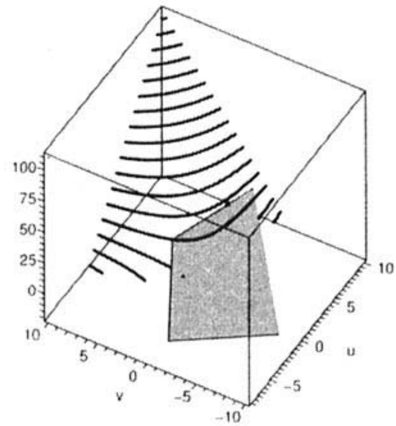


Figure 5.16 The feasible set and level curves in three dimensions. Solution is at $(7.5, 1.93)$ for security point $\left(-\frac{4}{23}, -\frac{16}{17}\right)$.

getting less, then player I actually gets -8, while player II gets -2. So both will lose, but I loses much more. So player II's threat to always play II_1, if player I doesn't cooperate on II's terms, is a credible and serious threat that player I cannot ignore. Next we consider how to deal with this problem.

Finding the Threat Strategies. In a threat game we replace the security levels (u^*, v^*), which we have so far taken to be the value of the associated games $u^* = value(A), v^* = value(B^T)$, with the expected payoffs to each player if threat strategies are used. In the Example 5.27 player II seemed to have a pure threat strategy. In general, both players will have a mixed threat strategy, and we have to find a way to determine them. For now, suppose that in the bimatrix game player I has a threat strategy X_t and player II has a threat strategy Y_t. The new status quo or security point will be the expected payoffs to the players if they both use their threat strategies:

$$u^* = E_A(X_t, Y_t) = X_t A Y_t^T \text{ and } v^* = E_B(X_t, Y_t) = X_t B Y_t^T.$$

Then we return to the cooperative bargaining game and apply the same procedure as before but with the new threat security point; that is, we seek to

$$\begin{aligned}&\text{Maximize } g(u, v) := (u - X_t A Y_t^T)(v - X_t B Y_t^T)\\&\text{subject to } (u, v) \in S, u \geq X_t A Y_t^T, v \geq X_t B Y_t^T.\end{aligned}$$

Notice that we are not using B^T for player II's security point but the matrix B because we only need to use B^T when we consider player II as the row maximizer and player I as the column minimizer.

In the Example 5.27 let's suppose that the threat strategies are $X_t = (0, 1)$ and $Y_t = (1, 0)$. Then the expected payoffs give us the safety point $u^* = X_t{}^T A Y_t = -8$ and $v^* = X_t B Y_t{}^T = -2$ (see Figure 5.17). Changing to this security point increases the size of the feasible set and changes the objective function to $g(u, v) = (u+8)(v+2)$. When we solved this example with the security point $(-\frac{4}{23}, -\frac{16}{17})$ we obtained the

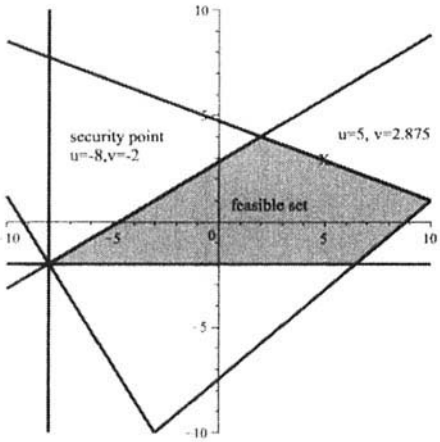


Figure 5.17 Feasible set with security point $(-8, -2)$ using threat strategies.

payoffs 7.501 for player I, and 1.937 for player II. The solution of the threat problem is $\overline{u} = 5 < 7.501, \overline{v} = 2.875 > 1.937$. This reflects the fact that player II has a credible threat and therefore should get more than if we ignore the threat.

The question now is how to pick the threat strategies? How do we know in the previous example that the threat strategies we chose were the best ones? We continue our example to see how to solve this problem. This method follows the procedure as presented in the book by A. J. Jones [7].

We look for a different security point associated with threats that we call the **optimal threat security point**..

The Pareto-optimal boundary for our problem is the line segment $v = -\frac{3}{8}u + \frac{38}{8}, 2 \leq u \leq 10$. This line has slope $m_p = -\frac{3}{8}$. Consider now a line with slope $-m_p = \frac{3}{8}$ through **any possible threat security point in the feasible set** (u^t, v^t). Referring to Figure 5.18, the line will intersect the Pareto-optimal boundary line segment at some possible negotiated solution $(\overline{u}, \overline{v})$. The line with slope $-m_p$ through

(u^t, v^t), whatever the point is, has the equation

$$v - v^t = -m_p(u - u^t).$$

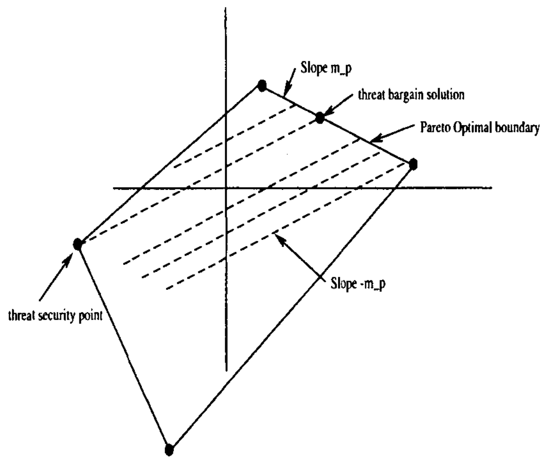


Figure 5.18 Lines through possible threat security points.

The equation of the Pareto-optimal boundary line is

$$v = m_p u + b = -\frac{3}{8}u + \frac{38}{8},$$

so the intersection point of the two lines will be at the coordinates

$$\overline{u} = \frac{m_p\, u^t + v^t - b}{2m_p} = \frac{3u^t - 8v^t + 38}{6},$$

$$\overline{v} = \frac{1}{2}(m_p u^t + v^t + b) = \frac{-(3u^t - 8v^t) + 38}{16}.$$

Now, remember that we are trying to find the best threat strategies to use, but the primary objective of the players is to maximize their payoffs $\overline{u}, \overline{v}$. This tells us exactly what to do to find the optimal threat security point.

- Player I will **maximize** $\overline{u}$ if she chooses threat strategies to maximize the quantity $m_p\, u^t + v^t = -\frac{3}{8}\, u^t + v^t$.

- Player II will maximize $\overline{v}$ if he chooses threat strategies to **minimize** the same quantity $m_p\, u^t + v^t$ because the Pareto-optimal boundary will have $m_p < 0$, so the sign of the term multiplying u^t will be opposite in $\overline{u}$ and $\overline{v}$.

Putting these two goals together, it seems that we need to solve a game with some matrix. The rules following will show exactly what we need to do.

Summary Approach for Bargaining with Threat Strategies.

Here is the general procedure for finding u^t, v^t and the optimal threat strategies as well as the solution of the bargaining game:

1. Identify the Pareto-optimal boundary of the feasible payoff set and find the slope of that line, call it m_p. This slope should be < 0.

2. Construct the new matrix for a zero sum game

$$-m_p u^t - v^t = -m_p(X_t A Y_t^T) - X_t B Y_t^T = X_t(-m_p A - B)Y_t^T$$

with matrix $-m_p A - B$.

3. Find the optimal strategies X_t, Y_t for that game and compute $u^t = X_t A Y_t^T$ and $v^t = X_t B Y_t^T$. This (u^t, v^t) is the threat security point to be used to solve the bargaining problem.

4. Once we know (u^t, v^t), we may use the following formulas for $(\overline{u}, \overline{v})$:

$$\overline{u} = \frac{m_p\, u^t + v^t - b}{2m_p}, \quad \overline{v} = \frac{1}{2}(m_p u^t + v^t + b). \tag{5.4.1}$$

Alternatively, we may apply the nonlinear programming method with security point (u^t, v^t) to find $(\overline{u}, \overline{v})$.

EXAMPLE 5.27, continued.

Carrying out these steps for our example, $m_p = -\frac{3}{8}, b = \frac{38}{8}$, we find

$$\frac{3}{8}A - B = \begin{bmatrix} -\frac{26}{8} & \frac{71}{8} \\ -1 & \frac{22}{8} \end{bmatrix}$$

We find $value(\frac{3}{8}A - B) = -1$ and, because there is a saddle point at the second row and first column, optimal threat strategies $X_t = (0, 1), Y_t = (1, 0)$. Then $u^t = X_t A Y_t^T = -8$, and $v^t = X_t B Y_t^T = -2$. Once we know that, we can use the formulas above to get

$$\overline{u} = \frac{-\frac{3}{8}(-8) + (-2) - \frac{38}{8}}{2(-\frac{3}{8})} = 5,$$

$$\overline{v} = \frac{1}{2}(-\frac{3}{8}(-8) + (-2) + \frac{38}{8}) = 2.875.$$

This matches with our previous solution in which we simply took the threat security point to be $(-8, -2)$. Now we see that $(-8, -2)$ is indeed the **optimal threat** security point.

The preceding discussion gives us a general procedure to solve for the threat strategies. Notice, however, that several things can make this procedure more complicated. First, the determination of the Pareto-optimal boundary of S is of critical importance. In Example 5.18 it consisted of only one line segment, but in practice there may be many such line segments and we have to work separately with each segment. That is because we need the slopes of the segments. This means that the threat strategies and the threat point u^t, v^t could change from segment to segment. An example below will illustrate this.

Another problem is that the Pareto-optimal boundary could be a point of intersection of two segments, so there is no slope for the point. Then, what do we do? The answer is that when we calculate the threat point (u^t, v^t) for each of the two line segments that intersect at a vertex, if this threat point is in the cone emanating from this vertex with the slopes shown in the Figure 5.19, then the threat solution of our problem is in fact at the vertex.

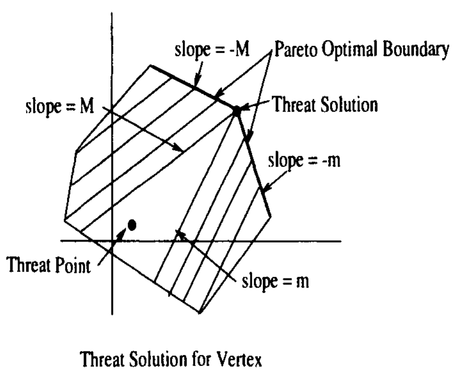


Figure 5.19 Bargaining solution for threats when threat point is in the cone is the vertex.

■ EXAMPLE 5.28

Consider the cooperative game with bimatrix

	II_1	II_2
I_1	$(-1, -1)$	$(1, 1)$
I_2	$(2, -2)$	$(-2, 2)$

The individual matrices are

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

It is easy to calculate that $value(A) = 0, value(B^T) = 1$ and so the status quo security point for this game is at $(u^*, v^*) = (0, 1)$. The problem we then need to solve is

$$\text{Maximize } u(v-1),$$
$$\text{subject to } (u, v) \in S^*,$$

where

$$S^* = \{(u, v) \mid v \leq (-\frac{1}{3})u + \frac{4}{3}, v \leq -3u + 4, u \geq 0, v \geq 1\}.$$

The solution of this problem is at the unique point $(\overline{u}, \overline{v}) = (\frac{1}{2}, \frac{7}{6})$, which you can see in the Figure 5.20.

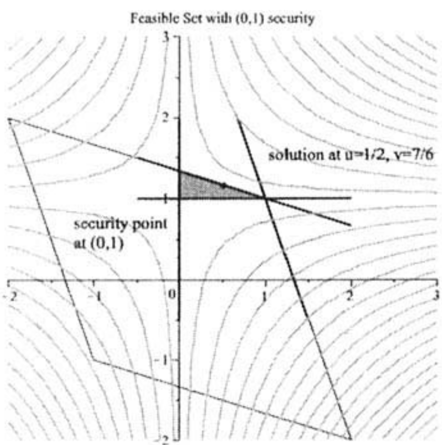


Figure 5.20 Security point $(0, 1)$; cooperative solution $(\overline{u} = \frac{1}{2}, \overline{v} = \frac{7}{6})$.

Figure 5.20 was created with the following Maple commands

```
> mypoints:=[[-1, -1], [-2, 2], [1, 1], [2, -2], [-1, -1]];
> constr:={0 <=x,-3*x-y<=4, x+3*y<=4, 3*x+y<=4, -x-3*y<=4, 1<=y};
> z:=(x+0)*(y-1);
```

```
> iplot2:=plots[inequal](constr,x=-0.5..2, y=-0.5..2,
    optionsfeasible=(color=white),
    optionsclosed=(color=black, thickness=2),
    optionsexcluded=(color=white),title="Feasible Set
                                    with (0,1) security"):
> pol:=plots[polygonplot](mypoints, color=yellow):
> cp:=plots[contourplot](z, x=-2..3,y=-2..3,
                          contours=40, axes=boxed,thickness=2):
> plots[display](iplot2, pol,cp);
```

The solution of the problem is given by the Maple commands:

```
> with(Optimization):
> NLPSolve(z,constr,maximize);
```

We get from these commands that $z = 0.083, x = \overline{u} = 0.5, y = \overline{v} = 1.167$. As mentioned earlier, you may also get this by hand using calculus. You need to find the maximum of $g(u, v) = u(v - 1)$ subject to $u \geq 0, v \geq 1$, and $v = -\frac{u}{3} + \frac{4}{3}$. So, $f(u) = g(u, v) = u(-\frac{u}{3} + \frac{4}{3})$ is the function to maximize. Since $f'(u) = -\frac{2u}{3} + \frac{1}{3} = 0$ at $u = \frac{1}{2} \geq 0$, we have that $\overline{u} = \frac{1}{2} > 0, \overline{v} = \frac{7}{6} > 1$ as our interior feasible maximum.

Next, to find the threat strategies we note that we have two possibilities because we have two line segments in Figure 5.20 as the Pareto-optimal boundary. We have to consider both $m_p = -\frac{1}{3}, b = \frac{4}{3}$ and $m_p = -3, b = 4$.

Let's use $m_p = -3, b = 4$. We look for the value of the game with matrix $3A - B$:

$$3A - B = \begin{bmatrix} -2 & 2 \\ 8 & -8 \end{bmatrix}$$

Then $value(3A - B) = 0$, and the optimal threat strategies are $X_t = (\frac{1}{2}, \frac{1}{2}) = Y_t$. Then the security threat points are

$$u^t = X_t A Y_t^T = 0 \text{ and } v^t = X_t B Y_t^T = 0.$$

This means that each player threatens to use (X_t, Y_t) and receive 0 rather than cooperate and receive more.

Now the maximization problem becomes

$$\text{Maximize } uv,$$
$$\text{subject to } (u, v) \in S^t,$$

where

$$S^t = \{(u, v) \mid v \leq (-\frac{1}{3})u + \frac{4}{3}, v \leq -3u + 4, u \geq 0, v \geq 0\}.$$

The solution of this problem is at the unique point $(\overline{u}, \overline{v}) = (1, 1)$. You can see in Figure 5.21 how the level curves have bent over to touch at the vertex.

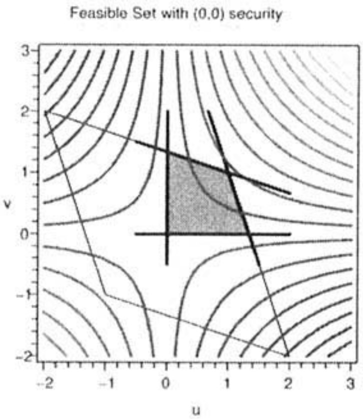


Figure 5.21 Security point $(0,0)$; cooperative solution $(\overline{u}=1,\overline{v}=1)$.

What would have happened if we used the slope of the other line of the Pareto-optimal boundary? Let's look at $m_p = -\frac{1}{3}, b = \frac{4}{3}$. The matrix is

$$\frac{1}{3}A - B = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{8}{3} & -\frac{8}{3} \end{bmatrix}$$

Then $value(\frac{1}{3}A - B) = -\frac{2}{3}$, and the optimal threat strategies are $X_t = (1,0), Y_t = (0,1)$. The security threat points are

$$u^t = X_t A {Y_t}^T = 1 \text{ and } v^t = X_t B {Y_t}^T = 1.$$

This point is exactly the vertex of the feasible set.

Now the maximization problem becomes

$$\begin{aligned} &\text{Maximize } (u-1)(v-1), \\ &\text{subject to } (u,v) \in S^t, \end{aligned}$$

where

$$S^t = \{(u,v) | v \le (-\frac{1}{3})u + \frac{4}{3},\ v \le -3u+4,\ u \ge 1,\ v \ge 1\}.$$

But this set has exactly one point (as you should verify), and it is $(1,1)$, so we immediately get the solution $(\overline{u}=1,\overline{v}=1)$, the same as what we got earlier.

What happens if we try to use the formulas (5.4.1) for the threat problem? This question arises now because the contours of g are hitting the feasible set right at the point of intersection of two lines. The two lines have the equations

$$v = -3u + 4 \text{ and } v = -\frac{1}{3}u + \frac{4}{3}.$$

So, do we use $m_p = -3, b = 4$, or $m_p = -\frac{1}{3}, b = \frac{4}{3}$? Let's calculate for both. For $m_p = -3, b = 4, u^t = v^t = 0$, we have

$$\overline{u} = \frac{m_p\, u^t + v^t - b}{2m_p} = \frac{-3(0) + (0) - 4}{2(-3)} = \frac{2}{3},$$
$$\overline{v} = \frac{1}{2}(m_p u^t + v^t + b) = \frac{1}{2}(-3(0) + (0) + 4) = 2.$$

The point $(\frac{2}{3}, 2)$ is not in S^t because $(-\frac{1}{3})(\frac{2}{3}) + \frac{4}{3} = \frac{10}{9} < 2$. So we no longer consider this point. However, because the point $(u^t, v^t) = (0, 0)$ is inside the cone region formed by the lines through $(1, 1)$ with slopes $\frac{1}{3}$ and 3, we know that the threat solution should be $(1, 1)$.

For $m_p = -\frac{1}{3}, b = \frac{4}{3}, u^t = v^t = 1$,

$$\overline{u} = \frac{m_p\, u^t + v^t - b}{2m_p} = \frac{-\frac{1}{3}(1) + (1) - \frac{4}{3}}{2(-\frac{1}{3})} = 1,$$
$$\overline{v} = \frac{1}{2}(m_p u^t + v^t + b) = \frac{1}{2}(-\frac{1}{3}(1) + (1) + \frac{4}{3}) = 1.$$

This gives $(\overline{u} = 1, \overline{v} = 1)$, which is the correct solution.

■ EXAMPLE 5.29

At the risk of undermining your confidence, this example will show that the Nash bargaining solution can be totally unrealistic, and in an important problem. Suppose that there is a person, Moe, who has owes money to two creditors, Larry and Curly. He owes more than he can pay. Let's say that he can pay at most \$100 but he owes a total of \$150 > \$100 dollars, \$90 to Curly and \$60 to Larry. The question is how to divide the \$100 among the two creditors. We set this up as a bargaining game and use Nash's method to solve it.

First, the feasible set is

$$S = \{(u, v) \mid u \leq 60, v \leq 90, u + v \leq 100\},$$

where u is the amount Larry gets, and v is the amount Curly will get.

The objective function we want to maximize at first is $g(u, v) = uv$ because if Larry and Curly can't agree on the split, then we assume that they each get nothing (because they have to sue and pay lawyers, etc.).

For the solution, we want to maximize $g(u, v)$ subject to $(u, v) \in S, u \geq 0, v \geq 0$. It is straightforward to show that the maximum occurs at $\overline{u} = \overline{v} = 50$, as shown in Figure 5.22.

In fact, if we take any safety point of the form $u^* = a = v^*$, we would get the exact same solution. This says that even though Moe owes Curly \$90 and Larry \$60, they both get the same amount as a settlement. That doesn't seem reasonable, and I'm sure Curly would be very upset.

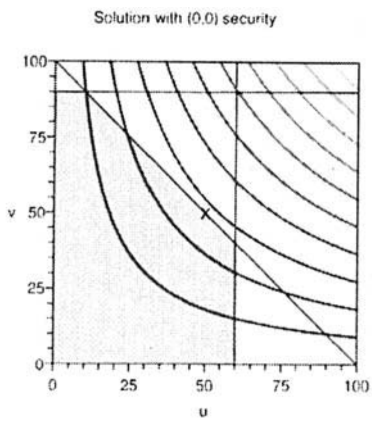


Figure 5.22 Moe pays both Curly and Larry \$50 each.

Now let's modify the safety point to $u^* = -60$ and $v^* = -90$, which is still feasible and reflects the fact that the players actually lose the amount owed in the worst case, that is, when they are left holding the bag. This case is illustrated in Figure 5.23. The solution is now obtained from maximizing $g(u, v) = (u + 60)(v + 90)$ subject to $(u, v) \in S, u \geq -60, v \geq -90$, and results in $\overline{u} = 60$ and $\overline{v} = 40$. This is ridiculous because it says that Larry should be paid off in full while Curly, who is owed more, gets less than half of what he is owed.

The problem occurs whenever the players are not even close to being symmetric as in this example. In this case, there is an idea for solution due to Kalai and Smorodinsky [9] that gives an alternate approach to solving the bargaining problem.

Following the discussion in Aliprantis and Chakrabarti [1], we give a brief description of Kalai and Smorodinsky's idea.

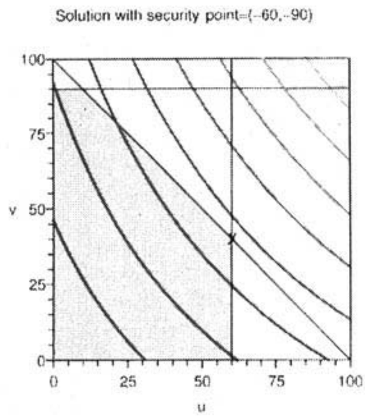


Figure 5.23 Moe pays Larry \$60 and pays Curly \$40.

1. Given the feasible set S, find

$$a = \max_{(u,v)\in S} u \text{ and } b = \max_{(u,v)\in S} v.$$

   Essentially, a is the maximum possible payoff for player I and b is the maximum possible payoff for player II.

2. Given any status quo security point (u^*, v^*), consider the line that has equation

$$v - v^* = k(u - u^*), \quad k = \frac{b - v^*}{a - u^*}. \tag{5.4.2}$$

   This line passes through (u^*, v^*) and (a, b), and is called the **KS line**, after Kalai and Smorodinsky.

3. The KS solution of the bargaining problem is the highest exit point on line KS from the feasible set. It roughly allocates to each player an amount proportional to the ratio of their maximum possible payoffs.

For the Moe-Larry-Curly problem, with $u^* = -60, v^* = -90$, we calculate

$$a = 60, b = 90 \implies k = \frac{180}{120} = \frac{3}{2}, \text{ and } v + 90 = \frac{3}{2}(u + 60).$$

This KS line will exit the feasible set where it crosses the line $u + v = 100$. This occurs at $\overline{u} = 40$, and so $\overline{v} = 60$. Consequently, now Larry gets \$40 and Curly gets

\$60. This is much more reasonable because almost everyone would agree that each creditor should be paid the same percentage of the amount Moe has as his percentage of the total owed. In this case $90/(90+60) = \frac{3}{5}$, so that Curly should get three-fifths of the \$100 or \$60. That is the KS solution as well.

The KS solution gives another alternative for solution of the bargaining problem. However, it should be noted that the KS solution does not satisfy all the axioms for a desirable solution. In particular it does not satisfy an economic axiom called the **independence of irrelevant alternatives axiom.**[8]

PROBLEMS

5.28 Find the Nash bargaining solution, the threat solution, and the KS solution to the battle of the sexes game with matrix

$$\begin{bmatrix} (4,2) & (2,-1) \\ (-1,2) & (2,4) \end{bmatrix}.$$

Compare the solutions with the solution obtained using the characteristic function approach.

5.29 Find the Nash bargaining solution and the threat solution to the game with bimatrix

$$\begin{bmatrix} (-2,5) & (-7,3) & (3,4) \\ (4,-3) & (6,1) & (-6,-6) \end{bmatrix}.$$

Find the KS line and solution.

5.30 The Nash solution also applies to payoff functions with a continuum of strategies. For example, suppose that two investors are bargaining over a piece of real estate and they have payoff functions $u(x,y) = x + y$, while $v(x,y) = x + \sqrt{y}$, with $x, y \geq 0$, and $x + y \leq 1$. Both investors want to maximize their own payoffs. The bargaining solution with safety point $u^* = 0, v^* = 0$ (because both players get zero if negotiations break down) is given by the solution of the problem

$$\begin{aligned} &\text{Maximize } (u(x,y) - u^*)(v(x,y) - v^*) = (x+y)(x+\sqrt{y}) \\ &\text{subject to } x, y \geq 0, x + y \leq 1. \end{aligned}$$

Solve this problem to find the Nash bargaining solution.

5.31 A classic bargaining problem involves a union and management in contract negotiations. If management hires $w \geq 0$ workers, the company produces $f(w)$ revenue units, where f is a continuous, increasing function. The maximum number of workers who are represented by the union is W. A person who is not employed by the company gets a payoff $p_0 \geq 0$, which is either unemployment benefits or the pay

[8] See, for example, Aliprantis and Chakrabarti [1] for the proof and further discussion.

at another job. In negotiations with the union, the firm agrees to the pay level p and to employ $0 \leq w \leq W$ workers. We may consider the payoff functions as

$$u(p, w) = f(w) - pw \quad \text{to the company}$$

and

$$v(p, w) = pw + (W - w)p_0 \quad \text{to the union.}$$

Assume the safety security point is $u^* = 0$ for the company and $v^* = Wp_0$ for the union.

(a) What is the nonlinear program to find the Nash bargaining solution?

(b) Assuming an interior solution, show that the solution (p^*, w^*) of the Nash bargaining solution satisfies

$$w^* f'(w^*) = p_0 \text{ and } p^* = \frac{p_0 + f(w^*)}{2w^*}.$$

(c) Find the Nash bargaining solution for $f(w) = aw + b, a > 0$.

THE SHAPLEY VALUE WITH MAPLE

The following Maple commands can be used to calculate the Shapley value of a cooperative game. All you need to do is to let S be the set of numbered players, and define the characteristic function as v. The list $M = [M[k]]$ consists all the possible coalitions.

```
>restart:with(combinat):S:={1,2,3,4};
>L:=powerset(S):M:=convert(L,list):M:=sort(M,length);K:=nops(L);
># Define the characteristic function
>for k from 1 to K do if nops(M[k])<=1 then v(M[k]):=0; end if;end do;
v({1,2}):=0:v({1,3}):=0:v({2,3}):=0:v({1,4}):=5:v({2,4}):=10:v({3,4}):=0:
v({1,2,3}):=0:v({1,3,4}):=25:v({2,3,4}):=30:v({1,2,4}):=35:v({1,2,3,4}):=55:
># Calculate Shapley
> for i from 1 to nops(S) do
  x[i]:=0:
     for k from 1 to K do
        if member(i,M[k]) and nops(M[k])>=1 then
           x[i]:=x[i]+(v(M[k])-v(M[k] minus {i}))*
                ((nops(M[k])-1)!*(nops(S)-nops(M[k]))!)/nops(S)!
        end if;
     end do;
  end do:

> for i from 1 to nops(S) do lprint(shapley[i]=x[i]); end do;
```

BIBLIOGRAPHIC NOTES

The pioneers of the theory of cooperative games include L. Shapley, W. F. Lucas, M. Shubik, and many others, but may go back to Francis Edgeworth in the 1880s. It received a huge boost in the publication in 1944 of the seminal work by von Neumann and Morgenstern [26] and then again in a 1953 paper by L. Shapley in which he introduced the Shapley value of a cooperative game.

There are many very good discussions on cooperative game theory, and they are listed in the references. The conversion of any N-person non-zero sum game to characteristic form is due to von Neumann and Morgenstern, which we follow, as presented in references by Wang [28] and Jones [7]. Example 5.9 (used here with permission of Mesterton-Gibbons) is called the "log hauling problem" by Mesterton-Gibbons [15] as a realistic example of a game with empty core. It is a good candidate to illustrate how the least core with a positive ε^1 results in a fair allocation in which all the players are dissatisfied with the allocation. The use of Maple to plot and animate $C(\varepsilon)$ as ε varies is a great way to show what is happening with the level of dissatisfaction and the resulting allocations. For the concept of the nucleolus we follow the sequence in Wang's book [28], but this is fairly standard. The allocation of costs and savings games can be found in the early collection of survey papers in reference [13]. Problem 5.19 is a modification of a scheduling problem known as the "antique dealer's problem" in Mesterton-Gibbon's fine book [15], in which we may consider savings games in **time** units rather than monetary units.

The Shapley value is popular because it is relatively easy to compute but also because, for the most part, it is based on a commonly accepted set of economic principles. The United Nations Security Council example (Example 5.19) has been widely used as an illustration of quantifying the power of members of a group. The solution given here follows the computation by Jones [7]. Example 5.20 is adapted from an example due to Aliprantis and Chakrabarti [1] and gives an efficient way to compute the Shapley allocation of expenses to multiple users of a resource, and taking into account the ability to pay and requirement to meet the expenditures.

The theory of bargaining presented in Section 5.4 has two primary developers: Nash and Shapley. Our presentation for finding the optimal threat strategies in section 5.4.2 follows that in Jones' book [7]. The alternative method of bargaining using the KS solution is from Aliprantis and Chakrabarti [1], where more examples and much more discussion can be found. Our union versus management problem (Problem 5.31) is a modification of an example due to Aliprantis and Chakrabarti [1].

We have only scratched the surface of the theory of cooperative games. Refer to the previously mentioned references and the books by Gintis [4], Rasmussen [22], and especially the book by Osborne [19], for many more examples and further study of cooperative games.

CHAPTER 6

EVOLUTIONARY STABLE STRATEGIES AND POPULATION GAMES

I was a young man with uninformed ideas. I threw out queries, suggestions, wondering all the time over everything; and to my astonishment the ideas took like wildfire. People made a religion of them.

– *Charles Darwin*

All the ills from which America suffers can be traced to the teaching of evolution.

– *William Jennings Bryan*

If automobiles had followed the same development cycle as the computer, a Rolls-Royce would today cost $100, get a million miles per gallon, and explode once a year, killing everyone inside.

– *Robert Cringely*

Game Theory: An Introduction, First Edition. By E.N. Barron

6.1 EVOLUTION

A major application and extension of game theory to evolutionary theory was initiated by Maynard-Smith and Price.[1] They had the idea that if you looked at interactions among players as a game, then better strategies would eventually evolve and dominate among the players. They introduced the concept of an **evolutionary stable strategy (ESS)** as a good strategy that would not be invaded by any mutants so that bad mutations would not overtake the population. These concepts naturally apply to biology but can be used to explain and predict many phenomena in economics, finance, and other areas in social and political arenas. In this chapter we will present a brief introduction to this important concept.

Consider a population with many members. Whenever two players encounter each other, they play a symmetric bimatrix game with matrices (A, B). Symmetry in this set up means that $B = A^T$.

To make things simple, we assume for now that A is a 2×2 matrix. Then, for a strategy $X = (x, 1-x)$ and $Y = (y, 1-y)$, we will work with the expected payoffs

$$E_{\mathrm{I}}(X,Y) = XAY^T = u(x,y) \quad \text{and} \quad E_{\mathrm{II}}(X,Y) = XA^TY^T = v(x,y).$$

Because of symmetry, we have $u(x,y) = v(y,x)$, so we really only need to talk about $u(x,y)$ and focus on the payoff to player I.

Suppose that there is a strategy $X^* = (x^*, 1-x^*)$ that is used by most members of the population and s fixed strategies $X_1, \ldots, X_s$, which will be used by **deviants,** a small proportion of the population. Again, we will refer only to the first component of each strategy, $x_1, x_2, \ldots, x_s$.

Suppose that we define a random variable Z, which will represent the strategy played by player I's next opponent. The discrete random variable Z takes on the possible values $x^*, x_1, x_2, \ldots, x_s$ depending on whether the next opponent uses the **usual** strategy, x^*, or a deviant (or mutant) strategy, $x_1, \ldots, x_s$. We assume that the distribution of Z is given by

$$Prob(Z = x^*) = 1 - p, \ Prob(Z = x_j) = p_j, \ \ j = 1, 2, \ldots, s, \ \sum_{j=1}^{s} p_j = p.$$

It is assumed that X^* will be used by most opponents, so we will take $0 < p \approx 0$, to be a small, but positive number. If player I uses X^* and player II uses Z, the expected payoff, also called the **fitness of the strategy** X^*, is given by

$$F(x^*) \equiv E(u(x^*, Z)) = u(x^*, x^*)(1-p) + \sum_{j=1}^{s} u(x^*, x_j)p_j. \tag{6.1.1}$$

[1]John Maynard-Smith, F. R. S. (January 6, 1920–April 19, 2004) was a British biologist and geneticist. George R. Price (1922 - January 6, 1975) was an American population geneticist. He and Maynard-Smith introduced the concept of the evolutionary stable strategy (ESS).

Similarly, the expected payoff to player I if she uses one of the deviant strategies $x_k, \ k = 1, 2, \ldots, s$, is

$$F(x_k) = E(u(x_k, Z)) = u(x_k, x^*)(1-p) + \sum_{j=1}^{s} u(x_k, x_j)p_j. \tag{6.1.2}$$

Subtracting (6.1.2) from (6.1.1), we get

$$F(x^*) - F(x_k) = (u(x^*, x^*) - u(x_k, x^*))(1-p) + \sum_{j=1}^{s}[u(x^*, x_j) - u(x_k, x_j)]p_j. \tag{6.1.3}$$

We want to know when it happens that $F(x^*) > F(x_k)$ so that x^* is a strictly better, or more **fit**, strategy against any deviant strategy. If

$u(x^*, x^*) > u(x_k, x^*), k = 1, 2, \ldots, s,$ and p is a **small enough** positive number,

then, from (6.1.3) we will have $F(x^*) > F(x_k)$, and so no player can do better by using any deviant strategy. This defines x^* as an **uninvadable,** or an **evolutionary stable strategy.** In other words, the strategy X^* resists being overtaken by one of the mutant strategies $X_1, \ldots, X_s$, in case X^* gives a higher payoff when it is used by both players than when any one player decides to use a mutant strategy, and the proportion of deviants in the population p is small enough. We need to include the requirement that p is small enough because $p = \sum_j p_j$ and we need

$$u(x^*, x^*) - u(x_k, x^*) \geq \frac{1}{1-p}\sum_{j=1}^{s}[u(x^*, x_j) - u(x_k, x_j)]p_j > 0.$$

We can arrange this to be true if p is small but not for all $0 < p < 1$.

Now, the other possibility in (6.1.3) is that the first term could be zero for even one x_k. In that case $u(x^*, x^*) - u(x_k, x^*) = 0$, and the first term in (6.1.3) drops out for that k. In order for $F(x^*) > F(x_k)$ we would now need $u(x^*, x_j) > u(x_k, x_j)$ for all $j = 1, 2, \ldots, s$. This has to hold for any x_k such that $u(x^*, x^*) = u(x_k, x^*)$. In other words, if there is even one deviant strategy that is as good as x^* when played against x^*, then, in order for x^* to result in a higher average fitness, we must have a bigger payoff when x^* is played against x_j than any payoff with x_k played against x_j. Thus, x^* played against deviant strategy must be better than x_k against any other deviant strategy x_j.

We summarize these conditions as a definition.

Definition 6.1.1 *A strategy X^* is an ESS against (deviant strategy) strategies $X_1, \ldots, X_s$ if either of (1) or (2) hold:*

$$(1)\ u(x^*, x^*) > u(x_k, x^*), \quad \textit{for each } k = 1, 2, \ldots, s,$$

$$(2)\ \textit{for any } x_k \textit{ such that } u(x^*, x^*) = u(x_k, x^*),$$
$$\textit{we must have } u(x^*, x_j) > u(x_k, x_j), \textit{ for all } j = 1, 2, \ldots, s.$$

In the case when there is only one deviant strategy, $s = 1$, and we label $X_1 = X = (x, 1-x), 0 < x < 1$. That means that every player in the population must use X^* or X with X as the deviant strategy. The proportion of the population which uses the deviant strategy is p. In this case, the definition reduces to: X^* is an ESS if and only if either (1) or (2) holds:

$$(1)\ u(x^*, x^*) > u(x, x^*), \ \forall\, 0 \le x \le 1$$

$$(2)\ u(x^*, x^*) = u(x, x^*) \implies u(x^*, x) > u(x, x), \quad \forall\, x \ne x^*.$$

In the rest of this chapter we consider only the case $s = 1$.

Notice that if X^* and X are any mixed strategies and $0 < p < 1$, then $(1-p)X + pX^*$ is also a mixed strategy and can be used in an encounter between two players. Here, then, is another definition of an ESS that we will show shortly is equivalent to the first definition.

Definition 6.1.2 *A strategy $X^* = (x^*, 1-x^*)$ is an evolutionary stable strategy if for every strategy $X = (x, 1-x)$, with $x \ne x^*$, there is some $p_x \in (0,1)$, which depends on the particular choice x, such that*

$$u(x^*, px + (1-p)x^*) > u(x, px + (1-p)x^*), \textit{ for all } 0 < p < p_x. \tag{6.1.4}$$

This definition says that X^* should be a good strategy if and only if this strategy played against the mixed strategy $Y_p \equiv pX + (1-p)X^*$ is better than any deviant strategy X played against $Y_p = pX + (1-p)X^*$, given that the probability p, that a member of the population will use a deviant strategy is sufficiently small.

The left side of (6.1.4) is

$$\begin{aligned} u(x^*, px + (1-p)x^*) &= (x^*, 1-x^*)A \begin{bmatrix} px + (1-p)x^* \\ p(1-x) + (1-p)(1-x^*) \end{bmatrix} \\ &= X^* A[pX + (1-p)X^*]^T \\ &= pX^* A X^T + (1-p) X^* A X^{*T} \\ &= pu(x^*, x) + (1-p)u(x^*, x^*). \end{aligned}$$

The right side of (6.1.4) is

$$u(x, px + (1-p)x^*) = (x, 1-x)A \begin{bmatrix} px + (1-p)x^* \\ p(1-x) + (1-p)(1-x^*) \end{bmatrix}$$
$$= XA[pX + (1-p)X^*]^T$$
$$= pXAX^T + (1-p)XAX{*}^T$$
$$= pu(x,x) + (1-p)u(x,x^*).$$

Putting them together yields that X^* is an ESS according to this definition if and only if

$$pu(x^*,x) + (1-p)u(x^*,x^*) > pu(x,x) + (1-p)u(x,x^*) \text{ for } 0 < p < p_x.$$

But this condition is equivalent to

$$p[u(x^*,x) - u(x,x)] + (1-p)[u(x^*,x^*) - u(x,x^*)] > 0 \text{ for } 0 < p < p_x. \quad (6.1.5)$$

Now we can show the two definitions of ESS are equivalent.

Proposition 6.1.3 *X^* is an ESS according to definition* (6.1.1) *if and only if X^* is an ESS according to definition* (6.1.2).

Proof. Suppose X^* satisfies definition (6.1.1). We will see that inequality (6.1.5) will be true. Now either

$$u(x^*,x^*) > u(x,x^*), \text{ for all } x \neq x^*, \text{ or} \quad (6.1.6)$$

$$u(x^*,x^*) = u(x,x^*) \implies u(x^*,x) > u(x,x) \text{ for all } x \neq x^*. \quad (6.1.7)$$

If we suppose that $u(x^*,x^*) > u(x,x^*)$, for all $x \neq x^*$, then for each x there is a small $\gamma_x > 0$ so that

$$u(x^*,x^*) - u(x,x^*) > \gamma_x > 0.$$

For a fixed $x \neq x^*$, since $\gamma_x > 0$, we can find a small enough $p_x > 0$ so that for $0 < p < p_x$ we have

$$u(x^*,x^*) - u(x,x^*) > \frac{1}{1-p}[\gamma_x - p(u(x^*,x) - u(x,x))] > 0.$$

This says that for all $0 < p < p_x$, we have

$$p(u(x^*,x) - u(x,x)) + (1-p)(u(x^*,x^*) - u(x,x^*)) > \gamma_x > 0,$$

which means that (6.1.5) is true. A similar argument shows that if we assume $u(x^*,x^*) = u(x,x^*) \implies u(x^*,x) > u(x,x)$ for all $x \neq x^*$, then (6.1.5) holds. So, if X^* is an ESS in the sense of definition (6.1.1), then it is an ESS in the sense of definition (6.1.2).

Conversely, if X^* is an ESS in the sense of definition (6.1.2), then for each x there is a $p_x > 0$ so that

$$p(u(x^*, x) - u(x, x)) + (1 - p)(u(x^*, x^*) - u(x, x^*)) > 0, \quad \forall\, 0 < p < p_x.$$

If $u(x^*, x^*) - u(x, x^*) = 0$, then since $p > 0$, it must be the case that $u(x^*, x) - u(x, x) > 0$. In case $u(x^*, x^*) - u(x', x^*) \neq 0$ for some $x' \neq x^*$, sending $p \to 0$ in

$$p(u(x^*, x') - u(x', x')) + (1 - p)(u(x^*, x^*) - u(x', x^*)) > 0, \quad \forall\, 0 < p < p_{x'},$$

we conclude that $u(x^*, x^*) - u(x', x^*) > 0$, and this is true for every $x' \neq x^*$. But that says that X^* is an ESS in the sense of definition (6.1.1). □

Properties of an ESS (6.1.8)

1. **If X^* is an ESS, then (X^*, X^*) is a Nash equilibrium.** Why? Because if it isn't a Nash equilibrium, then there is a player who can find a strategy $Y = (y, 1 - y)$ such that $u(y, x^*) > u(x^*, x^*)$. Then, for all small enough $p = p_y$, we have

$$p(u(x^*, y) - u(y, y)) + (1 - p)(u(x^*, x^*) - u(y, x^*)) < 0, \quad \forall\, 0 < p < p_y,$$

This is a contradiction of the definition (6.1.2) of ESS. One consequence of this is that **only the symmetric Nash equilibria** of a game are candidates for ESSs.

2. **If (X^*, X^*) is a strict Nash equilibrium, then X^* is an ESS.** Why? Because if (X^*, X^*) is a strict Nash equilibrium, then $u(x^*, x^*) > u(y, x^*)$ for any $y \neq x^*$. But then, for every small enough p, we would obtain

$$pu(x^*, y) + (1 - p)u(x^*, x^*) > pu(y, y) + (1 - p)u(y, x^*), \quad \text{for } 0 < p < p_y,$$

for some $p_y > 0$ and all $0 < p < p_y$. But this defines X^* as an ESS according to (6.1.2).

3. A symmetric Nash equilibrium X^* is an ESS for a symmetric game if and only if $u(x^*, y) > u(y, y)$ for every strategy $y \neq x^*$ that is a **best response strategy** to x^*. Recall that $Y = (y, 1 - y)$ is a best response to $X^* = (x^*, 1 - x^*)$ if

$$u(y, x^*) = YAX^{*T} = \max_Z u(z, x^*) = \max_Z ZAX^{*T}.$$

We now present a series of examples illustrating the concepts and calculations.

■ EXAMPLE 6.1

In a simplified model of the evolution of currency, suppose that members of a population have currency in either euros or dollars. When they want to trade for some goods, the transaction must take place in the same currency. Here is a possible matrix representation

I/II	Euros	Dollars
Euros	(1,1)	(0,0)
Dollars	(0,0)	(1,1)

Naturally, there are three symmetric Nash equilibria $X_1 = (1,0) = Y_1, X_2 = (0,1) = Y_2$ and one mixed Nash equilibrium at $X_3 = (\frac{1}{2}, \frac{1}{2}) = Y_3$. These correspond to everyone using euros, everyone using dollars, or half the population using euros and the other half dollars. We want to know which, if any, of these are ESSs. We have for X_1:

$$u(1,1) = 1 \text{ and } u(x,1) = x, \text{ so that } u(1,1) > u(x,1),\ x \neq 1.$$

This says X_1 is ESS.

Next, for X_2 we have

$$u(0,0) = 1 \text{ and } u(x,0) = x, \text{ so that } u(0,0) > u(x,0),\ x \neq 1.$$

Again, X_2 is an ESS. Note that both X_1 and X_2 are strict Nash equilibria and so ESSs according to properties (6.1.8), (2).

Finally for X_3, we have

$$u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \text{ and } u\left(x, \frac{1}{2}\right) = \frac{1}{2},$$

so $u(x^*, x^*) = u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} = u(x, x^*) = u(x, \frac{1}{2})$, for all $x \neq x^* = \frac{1}{2}$. We now have to check the second possibility in definition (6.1.1):

$$u(x^*, x^*) = u(x, x^*) = \frac{1}{2} \implies$$
$$u(x^*, x) = \frac{1}{2} > u(x,x) = x^2 + (1-x)^2 \text{ for all } x \neq \frac{1}{2}.$$

But that is false because there are plenty of x values for which this is false. We can take, for instance $x = \frac{1}{3}$, to get $\frac{1}{9} + \frac{4}{9} = \frac{5}{9} > \frac{1}{2}$. Consequently, the mixed Nash equilibrium X_3 in which a player uses dollars and euros with equal likelihood is not an ESS. The conclusion is that eventually, the entire population will evolve to either all euros or all dollars, and the other currency will disappear. This seems to predict the future eventual use of one currency worldwide, just as the euro has become the currency for all of Europe.

EXAMPLE 6.2

Hawk–Dove Game. Each player can choose to act like a hawk or act like a dove, can either fight or yield, when they meet over some roadkill. The payoff matrix is

H/D	Fight	Yield
Fight	$\left(\frac{v-c}{2}, \frac{v-c}{2}\right)$	$(v, 0)$
Yield	$(0, v)$	$\left(\frac{v}{2}, \frac{v}{2}\right)$

, so $A = B^T = \begin{bmatrix} (v-c)/2 & v \\ 0 & v/2 \end{bmatrix}$

The reward for winning a fight is $v > 0$, and the cost of losing a fight is $c > 0$, and each player has an equal chance of winning a fight. The payoff to each player if they both fight is thus $v\frac{1}{2} + (-c)\frac{1}{2}$. If hawk fights, and dove yields, hawk gets v while dove gets 0. If they both yield they both receive $v/2$. This is a symmetric two person game.

Consider the following cases:

Case 1: $v > c$. When the reward for winning a fight is greater than the cost of losing a fight, there is a unique symmetric Nash equilibrium at (fight,fight). In addition, it is strict, because $(v - c)/2 > 0$, and so it is the one and only ESS. Fighting is evolutionary stable, and you will end up with a population of fighters.

Case 2: $v = c$. When the cost of losing a fight is the same as the reward of winning a fight, there are two nonstrict nonsymmetric Nash equilibria at (yield, fight) and (fight,yield). But there is only one symmetric, nonstrict Nash equilibrium at (fight,fight). Since $u(\text{fight}, \text{yield}) = v > u(\text{yield}, \text{yield}) = v/2$, the Nash equilibrium (fight,fight) satisfies the conditions in the definition of an ESS.

Case 3: $c > v$. Under the assumption $c > v$, we have two pure nonsymmetric Nash equilibria at $X_1 = (0, 1), Y_1 = (1, 0)$ with payoff $u(0, 1) = v/2$ and $X_2 = (1, 0), Y_2 = (0, 1)$, with payoff $u(1, 0) = (v - c)/2$. It is easy to calculate that $X_3 = \left(\frac{v}{c}, 1 - \frac{v}{c}\right) = Y_3$ is the symmetric mixed strategy Nash equilibrium. This is a symmetric game, and so we want to know which, if any, of the three Nash points are evolutionary stable in the sense of definition (6.1.1). But we can immediately eliminate the nonsymmetric equilibria.

We take now specifically $v = 4, c = 6$. Then $v/c = \frac{2}{3}, (v - c)/2 = -1$, so

$$u(1, 1) = -1, \quad u(0, 0) = 2, \quad \text{and } u\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3}.$$

Let's consider the mixed strategy $X_3 = Y_3 = (\frac{2}{3}, \frac{1}{3})$. We have

$$u\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} = u\left(x, \frac{2}{3}\right),$$
$$u\left(\frac{2}{3}, x\right) = -4x + \frac{10}{3} \text{ and } u(x, x) = -3x^2 + 2.$$

Since $u\left(\frac{2}{3}, \frac{2}{3}\right) = u\left(x, \frac{2}{3}\right)$ we need to show that the second case in definition (6.1.1) holds, namely, $u\left(\frac{2}{3}, x\right) = -4x + \frac{10}{3} > -3x^2 + 2 = u(x, x)$ for all $0 \le x \le 1, x \ne \frac{2}{3}$. By algebra, this is the same as showing that $\left(x - \frac{2}{3}\right)^2 > 0$, for $x \ne \frac{2}{3}$, which is obvious. We conclude that $X_3 = Y_3 = (\frac{2}{3}, \frac{1}{3})$ is evolutionary stable.

For comparison purposes, let's try to show that $X_1 = (0, 1)$ is not an ESS directly from the definition. Of course, we already know it isn't because (X_1, X_1) is not a Nash equilibrium. So if we try the definition for $X_1 = (0, 1)$, we would need to have that

$$u(0, 0) = 2 \ge u(x, 0) = 2x + 2,$$

which is clearly false for any $0 < x \le 1$. So X_1 is not evolutionary stable.

From a biological perspective, this says that only the mixed strategy Nash equilibrium is evolutionary stable, and so always fighting, or always yielding is not evolutionary stable. Hawks and doves should fight two-thirds of the time when they meet.

In the general case with $v < c$ we have the mixed symmetric Nash equilibrium $X^* = (\frac{v}{c}, 1 - \frac{v}{c}) = Y^*$, and since

$$u\left(\frac{v}{c}, \frac{v}{c}\right) = u\left(x, \frac{v}{c}\right), \quad x \ne \frac{v}{c},$$

we need to check whether $u(x^*, x) > u(x, x)$. An algebra calculation shows that

$$u\left(\frac{v}{c}, x\right) - u(x, x) = \frac{c}{2}\left(\frac{v}{c} - x\right)\left(\frac{v}{c} - x\right) > 0, \text{ if } x \ne \frac{v}{c}.$$

Consequently, X^* is an ESS.

It is not true that every symmetric game will have an ESS. The game in the next example will illustrate that.

■ EXAMPLE 6.3

We return to consideration of the rock-paper-scissors game with a variation on what happens when there is a tie. Consider the matrix

I/II	Rock	Paper	Scissors
Rock	$(\frac{1}{2},\frac{1}{2})$	$(-1,1)$	$(1,-1)$
Paper	$(1,-1)$	$(\frac{1}{2},\frac{1}{2})$	$(-1,1)$
Scissors	$(-1,1)$	$(1,-1)$	$(\frac{1}{2},\frac{1}{2})$

A tie in this game gives each player a payoff of $\frac{1}{2}$, so this is not a zero sum game, but it is symmetric. You can easily verify that there is one Nash equilibrium mixed strategy, and it is $X^* = (\frac{1}{3},\frac{1}{3},\frac{1}{3}) = Y^*$, with expected payoff $\frac{1}{6}$ to each player. (If the diagonal terms gave a payoff to each player ≥ 1, there would be many more Nash equilibria, both mixed and pure.) We claim that (X^*, X^*) is not an ESS.

If X^* is an ESS, it would have to be true that either

$$u(X^*,X^*) > u(X,X^*) \text{ for all } X \neq X^* \tag{6.1.9}$$

or

$$u(X^*,X^*) = u(X,X^*) \implies u(X^*,X) > u(X,X) \text{ for all } X \neq X^*. \tag{6.1.10}$$

Notice that this is a 3×3 game, and so we have to use strategies $X = (x_1, x_2, x_3) \in S_3$. Now $u(X^*,X^*) = X^*AX^{*T} = \frac{1}{6}$, where

$$A = \begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 1 & \frac{1}{2} & -1 \\ -1 & 1 & \frac{1}{2} \end{bmatrix}$$

and

$$u(X,X^*) = \frac{x_1}{6} + \frac{x_2}{6} + \frac{x_3}{6} = \frac{1}{6}, \quad u(X,X) = \frac{(x_1^2 + x_2^2 + x_3^2)}{2}.$$

The first possibility (6.1.9) does not hold.

For the second possibility condition (6.1.10), we have

$$u(X^*,X^*) = \frac{1}{6} = u(X,X^*),$$

but

$$u(X^*,X) = \frac{x_1 + x_2 + x_3}{6} = \frac{1}{6}$$

is **not** greater than $u(X,X) = (x_1^2 + x_2^2 + x_3^2)/2$ for all mixed $X \neq X^*$. For example, take the pure strategy $X = (1,0,0)$ to see that it fails. Therefore, neither possibility holds, and X^* is not an ESS.

Our conclusion can be phrased in this way. Suppose that one player decides to use $X^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The best response to X^* is the strategy $\overline{X}$ so that

$$u(\overline{X}, X^*) = \max_{Y \in S_3} u(Y, X^*) = \max_{Y \in S_3} \frac{y_1 + y_2 + y_3}{6} = \frac{1}{6},$$

and any pure strategy, for instance $\overline{X} = (1, 0, 0)$, will give that. But then $u(\overline{X}, \overline{X}) = \frac{1}{2} > \frac{1}{6}$, so that any deviant using any of the pure strategies can do better and will eventually invade the population. There is no **uninvadable** strategy.

Remark. There is much more to the theory of evolutionary stability and many extensions of the theory, as you will find in the references. In the next section we will present one of these extensions because it shows how the stability theory of ordinary differential equations enters game theory.

PROBLEMS

6.1 In the currency game (Example 6.1), derive the same result we obtained but using the equivalent definition of ESS: X^* is an evolutionary stable strategy if for every strategy $X = (x, 1 - x)$, with $x \neq x^*$, there is some $p_x \in (0, 1)$, which depends on the particular choice x, such that

$$u(x^*, px + (1 - p)x^*) > u(x, px + (1 - p)x^*) \text{ for all } 0 < p < p_x.$$

Find the value of p_x in each case an ESS exists.

6.2 It is possible that there is an economy that uses a dominant currency in the sense that the matrix becomes

I/II	Euros	Dollars
Euros	(1,1)	(0,0)
Dollars	(0,0)	(2,2)

Find all Nash equilibria and determine which are ESSs.

6.3 Analyze the Nash equilibria for a version of the prisoner's dilemma game:

I/II	Confess	Deny
Confess	(4,4)	(1,6)
Deny	(6,1)	(1,1)

6.4 Determine the Nash equilibria for rock-paper-scissors with matrix

I/II	Rock	Paper	Scissors
Rock	$(2,2)$	$(-1,1)$	$(1,-1)$
Paper	$(1,-1)$	$(2,2)$	$(-1,1)$
Scissors	$(-1,1)$	$(1,-1)$	$(2,2)$

There are three pure Nash equilibria and four mixed equilibria (all symmetric). Determine which are evolutionary stable strategies, if any, and if an equilibrium is not an ESS, show how the requirements fail.

6.5 Consider a game with matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Suppose that $ab \neq 0$.

(**a**) Show that if $ab < 0$, then there is exactly one ESS. Find it.

(**b**) Suppose that if $a > 0, b > 0$, then there are three symmetric Nash equilibria. Show that the Nash equilibria which are evolutionary stable are the pure ones and that the mixed Nash is not an ESS.

(**c**) Suppose that $a < 0, b < 0$. Show that this game has two pure nonsymmetric Nash equilibria and one symmetric mixed Nash equilibrium. Show that the mixed Nash is an ESS.

6.6 Verify that $X^* = (x^*, 1 - x^*)$ is an ESS if and only if (X^*, X^*) is a Nash equilibrium and $u(x^*, x) > u(x, x)$ for every $X = (x, 1 - x) \neq X^*$ that is a best response to X^*.

6.2 POPULATION GAMES

One important idea introduced by considering evolutionary stable strategies is the idea that eventually, players will choose strategies that produce a better-than-average payoff. The clues for this section are the words **eventually** and **better-than-average**. The word **eventually** implies a time dependence and a limit as time passes, so it is natural to try to model what is happening by introducing a time-dependent equation and let $t \to \infty$. That is exactly what we study in this section.

Here is the setup. There are N members of a population. At random two members of the population are chosen to play a certain game against each other. It is assumed that N is a really big number, so that the probability of a faceoff between two of the same members is virtually zero.

We assume that the game they play is a symmetric two-person bimatrix game. **Symmetry** here again means that $B = A^T$. Practically, it means that the players can switch roles without change. In a symmetric game it doesn't matter who is player I and who is player II.

Assume that there are many players in the population. Any particular player is chosen from the population, chooses a mixed strategy, and plays in the bimatrix game with matrix A against any other player chosen from the population. This is called a **random contest**. Notice that all players will be using the same payoff matrix.

The players in the population may use pure strategies $1, 2, \ldots, n$. Suppose that the **percentage of players in the population using strategy** j is

$$P(\text{player uses } j) = p_j, \quad p_j \geq 0, \sum_{j=1}^{n} p_j = 1.$$

Set $\pi = (p_1, \ldots, p_n)$. These p_i components of π are called the **frequencies** and represent the probability a randomly chosen individual in the population will use the strategy i. Denote by

$$\Pi = \{\pi = (p_1, p_2, \ldots, p_n) \mid p_j \geq 0, j = 1, 2, \ldots, n, \sum_{j=1}^{n} p_j = 1\}$$

the **set of all possible frequencies**.

If two players are chosen from the population and player I chooses strategy i and player II chooses strategy j, we calculate the payoffs from the matrix A.

We define the **fitness** of a player playing strategy $i = 1, 2, \ldots, n$ as

$$E(i, \pi) = \sum_{k=1}^{n} a_{i,k} p_k = {}_iA\pi.$$

This is the expected payoff of a random contest to player I who uses strategy i against the other possible strategies $1, 2, \ldots n$, played with probabilities $p_1, \ldots, p_n$. It measures the worth of strategy i in the population. You can see that the π looks just like a mixed strategy and we may identify $E(i, \pi)$ as the expected payoff to player I if player I uses the pure strategy i and the opponent (player II) uses the mixed strategy π.

Next we calculate the expected fitness of the entire population as

$$E(\pi, \pi) := \sum_{i=1}^{n} p_i \left[\sum_{k=1}^{n} a_{i,k} p_k \right] = \pi A \pi^T.$$

Now suppose that the frequencies $\pi = (p_1, \ldots, p_n) = \pi(t) \in \Pi$ can change with time. This is where the evolutionary characteristics are introduced. We need a model describing how the frequencies can change in time and we use the **frequency dynamics** as the following system of differential equations:

$$\begin{aligned} \frac{dp_i(t)}{dt} &= p_i(t) \left[E(i, \pi(t)) - E(\pi(t), \pi(t)) \right] \\ &= p_i(t) \left[\sum_{k=1}^{n} a_{i,k} p_k(t) - \pi(t) A \pi(t)^T \right], \quad i = 1, 2, \ldots, n, \end{aligned} \tag{6.2.1}$$

or, equivalently,

$$\frac{dp_i(t)}{p_i(t)} = \left[\sum_{k=1}^{n} a_{i,k} p_k(t) - \pi(t) A \pi(t)^T \right] dt.$$

This is also called the **replicator dynamics**. The idea is that the growth rate at which the population percentage using strategy i changes is measured by how much greater (or less) the expected payoff (or fitness) using i is compared with the expected fitness using all strategies in the population. Better strategies should be used with increasing frequency and worse strategies with decreasing frequency. In the one-dimensional case the right side will be positive if the fitness using pure strategy i is better than average and negative if worse than average. That makes the derivative $dp_i(t)/dt$ positive if better than average, which causes $p_i(t)$ to increase as time progresses, and strategy i will be used with increasing frequency.

We are not specifying at this point the initial conditions $\pi(0) = (p_1(0), \ldots, p_n(0))$, but we know that $\sum_i p_i(0) = 1$. We also note that any realistic solution of equations (6.2.1) must have $0 \leq p_i(t) \leq 1$ as well as $\sum_i p_i(t) = 1$ for all $t > 0$. In other words, we must have $\pi(t) \in \Pi$ for all $t \geq 0$. Here is one way to check that: add up the equations in (6.2.1) to get

$$\begin{aligned}\sum_{i=1}^{n} \frac{dp_i(t)}{dt} &= \sum_{i=1}^{n}\sum_{k=1}^{n} p_i(t)a_{i,k}p_k(t) - \sum_{i=1}^{n} p_i(t)\pi(t)A\pi(t)^T \\ &= \pi(t)A\pi(t)^T[1 - \sum_{i=1}^{n} p_i(t)]\end{aligned}$$

or, setting $\gamma(t) = \sum_{i=1}^{n} p_i(t)$,

$$\frac{d\gamma}{dt} = \pi(t)A\pi(t)^T[1 - \gamma(t)].$$

If $\gamma(0) = 1$, the **unique solution** of this equation is $\gamma(t) \equiv 1$, as you can see by plugging in $\gamma(t) = 1$. Consequently, $\sum_{i=1}^{n} p_i(t) = 1$ for all $t \geq 0$. By uniqueness it is the one and only solution. In addition, assuming that $p_i(0) > 0$, if it ever happens that $p_i(t_0) = 0$ for some $t_0 > 0$, then, by considering the trajectory with that new initial condition, we see that $p_i(t) \equiv 0$ for all $t \geq t_0$. This conclusion also follows from the fact that (6.2.1) has one and only one solution through any initial point, and zero will be a solution if the initial condition is zero. Similar reasoning, which we skip, shows that for each $i, 0 \leq p_i(t) \leq 1$. Therefore $\pi(t) \in \Pi$, for all $t \geq 0$, if $\pi(0) \in \Pi$.

Notice that when the right side of (6.2.1)

$$p_i(t)\left[\sum_{k=1}^{n} a_{i,k}p_k(t) - \pi(t)A\pi(t)^T\right] = 0,$$

then $dp_i(t)/dt = 0$ and $p_i(t)$ is not changing in time. In differential equations a solution that doesn't change in time will be a **steady-state, equilibrium,** or **stationary** solution. So, if there is a constant solution $\pi^* = (p_1^*, \ldots, p_n^*)$ of

$$p_i^* \left[\sum_{k=1}^{n} a_{i,k} p_k^* - \pi^* A \pi^{*T} \right] = 0,$$

then if we start at $\pi(0) = \pi^*$, we will stay there for all time and $\lim_{t \to \infty} \pi(t) = \pi^*$ is a steady state solution of (6.2.1).

Remark. If the symmetric game has a completely mixed Nash equilibrium $X^* = (x_1^*, \ldots, x_n^*), x_i^* > 0$, then $\pi^* = X^*$ is a stationary solution of (6.2.1). The reason is provided by the equality of payoffs Theorem 3.2.4, which guarantees that $E(i, X^*) = E(j, X^*) = E(X^*, X^*)$ for any pure strategies i, j played with positive probability. So, if the strategy is completely mixed, then $x_i^* > 0$ and the payoff using row i must be the average payoff to the player. But, then if $E(i, X^*) = E(i, \pi^*) = E(\pi^*, \pi^*) = E(X^*, X^*)$, we have the right side of (6.2.1) is zero and then that $\pi^* = X^*$ is a stationary solution.

■ EXAMPLE 6.4

Consider the symmetric game with

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, B = A^T.$$

Suppose that the players in the population may use the pure strategies $k \in \{1, 2\}$. The frequency of using k is $p_k(t)$ at time $t \geq 0$, and it is the case that $\pi(t) = (p_1(t), p_2(t)) \in \Pi$. Then the fitness of a player using $k = 1, 2$ becomes

$$E(1, \pi) = {}_1A \cdot \pi = \sum_{j=1}^{2} a_{1,j} p_j = p_1 + 3p_2,$$

$$E(2, \pi) = {}_2A \cdot \pi = \sum_{j=1}^{2} a_{2,j} p_j = 2p_1,$$

and the average fitness in the population is

$$E(\pi, \pi) = \pi A \pi^T = (p_1, p_2) A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p_1^2 + 5p_1p_2.$$

We end up with the following system of equations:

$$\frac{dp_1(t)}{dt} = p_1(t) \left[p_1 + 3p_2 - (p_1^2 + 5p_1p_2) \right]$$

$$\frac{dp_2(t)}{dt} = p_2(t) \left[2p_1 - (p_1^2 + 5p_1p_2) \right].$$

We are interested in the long-term behavior $\lim_{t\to\infty} p_i(t)$ of these equations because the limit is the eventual evolution of the strategies. The steady state solution of these equations occurs when $dp_i(t)/dt = 0$, which, in this case implies that

$$p_1\left[p_1 + 3p_2 - (p_1^2 + 5p_1p_2)\right] = 0 \text{ and } p_2\left[2p_1 - (p_1^2 + 5p_1p_2)\right] = 0.$$

Since both p_1 and p_2 cannot be zero (because $p_1 + p_2 = 1$), you can check that we have steady–state solutions

$$(p_1 = 0, p_2 = 1), (p_1 = 1, p_2 = 0), \text{ and } \left(p_1 = \frac{3}{4}, p_2 = \frac{1}{4}\right).$$

The arrows in Figure 6.1 show the direction a trajectory (=solution of the frequency dynamics) will take as time progresses depending on the starting condition. You can see in the figure that no matter where the initial condition is **inside** the square $(p_1, p_2) \in (0,1) \times (0,1)$, the trajectories will be sucked into the solution at the point $(p_1 = \frac{3}{4}, p_2 = \frac{1}{4})$ as the equilibrium solution. If, however, we start exactly at the edges $p_1(0) = 0, p_2(0) = 1$ or $p_1(0) =$

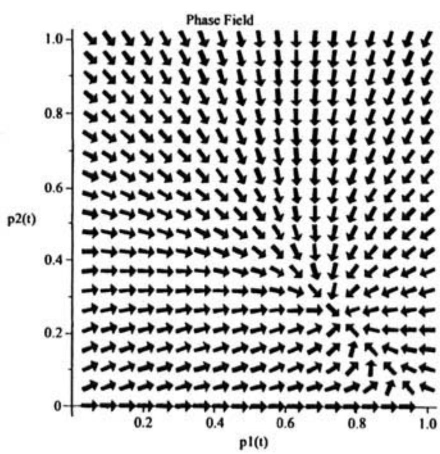


Figure 6.1 Stationary solution at $\left(\frac{3}{4}, \frac{1}{4}\right)$.

$1, p_2(0) = 0$, then we stay on the edge forever. Any deviation off the edge gets the trajectory sucked into $(\frac{3}{4}, \frac{1}{4})$, where it stays forever.

Whenever there are only two pure strategies used in the population, we can simplify down to one equation for $p(t)$ using the substitutions $p_1(t) = p(t), p_2(t) = 1 - p(t)$.

Equation (6.2.1) then becomes

$$\frac{dp(t)}{dt} = p(t)(1 - p(t))(E(1, \pi) - E(2, \pi)), \quad \pi = (p, 1 - p), \tag{6.2.2}$$

and we must have $0 \leq p(t) \leq 1$. For Example 6.4 the equation reduces to

$$\frac{dp(t)}{dt} = p(t)(1 - p(t))(-4p(t) + 3),$$

which is no easier to solve exactly. This equation can be solved implicitly using integration by parts to give the implicitly defined solution

$$\ell n \left[p^{1/3}|p - 1|(4p - 3)^{-4/3}\right] = t + C.$$

This is valid only away from the stationary points $p = 0, 1, \frac{3}{4}$, because, as you can see, the logarithm is a problem at those points. Figure 6.2 shows the direction field of $p(t)$ versus time on the left and the direction field superimposed with the graphs of four trajectories starting from four different initial conditions, two on the edges, and two in the interior of $(0, 1)$. The interior trajectories get pulled quickly as $t \to \infty$ to the steady state solution $p(t) = \frac{3}{4}$, while the trajectories that start on the edges stay there forever. In the long run, as long as the population is not using strategies on

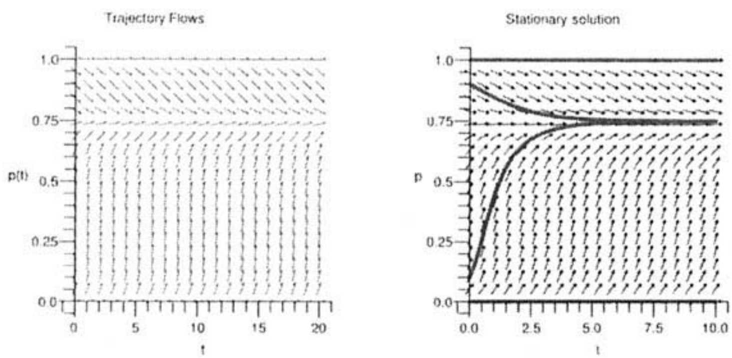


Figure 6.2 Direction field and trajectories for $dp(t)/dt = p(t)(1 - p(t))(-4p(t) + 3)$, $p(t)$ versus time with four initial conditions.

the edges (i.e., pure strategies), the population will eventually use the pure strategy $k = 1$ exactly 75% of the time and strategy $k = 2$ exactly 25% of the time.

Now, the idea is that the limit behavior of $\pi(t)$ will result in conclusions about how the population will evolve regarding the use of strategies. But that is what we studied in the previous section on evolutionary stable strategies. There must be a connection.

EXAMPLE 6.5

In an earlier problem (6.5) we looked at the game with matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, and assumed $ab \neq 0$. You showed that

1. If $ab < 0$, then there is exactly one ESS. It is $X^* = (0,1) = Y^*$ if $a < 0, b > 0$, and $X^* = (1,0) = Y^*$ if $b < 0, a > 0$.

2. If $a > 0, b > 0$, then there are three symmetric Nash equilibria. The Nash equilibria that are evolutionary stable are $X_1^* = (1,0), X_2^* = (0,1)$, and the mixed Nash $X_3^* = (b/(a+b), a(a+b))$ is not an ESS.

3. If $a < 0, b < 0$ this game has two pure nonsymmetric Nash equilibria and one symmetric mixed Nash equilibrium. The mixed Nash $X^* = (b/(a+b), a/(a+b))$ is an ESS.

The system of equations for this population game becomes

$$
\begin{aligned}
\frac{dp_1(t)}{dt} &= p_1(t)\left[\sum_{k=1}^{2} a_{1,k}p_k(t) \quad (p_1(t), p_2(t))A\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}\right] \\
&= p_1(t)[ap_1(t) - ap_1^2(t) - bp_2^2(t)], \\
\frac{dp_2(t)}{dt} &= p_2(t)\left[\sum_{k=1}^{2} a_{2,k}p_k(t) - (p_1(t), p_2(t))A\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}\right] \\
&= p_2(t)[bp_2(t) - ap_1^2(t) - bp_2^2(t)].
\end{aligned}
$$

We can simplify using the fact that $p_2 = 1 - p_1$ to get

$$
\frac{dp_1(t)}{dt} = p_1(t)[ap_1(t)(1 - p_1(t)) - bp_2^2(t)]. \tag{6.2.3}
$$

Now we can see that if $a > 0$ and $b < 0$, the right side is always > 0, and so $p_1(t)$ increases while $p_2(t)$ decreases. Similarly, if $a < 0, b > 0$, the right side of (6.2.3) is always < 0 and so $p_1(t)$ decreases while $p_2(t)$ increases. In either case, as $t \to \infty$ we converge to either $(p_1, p_2) = (1,0)$ or $(0,1)$, which is at the unique ESS for the game. When $p_1(t)$ must always increase, for example, but cannot get above 1, we know it must converge to 1.

In the case when $a > 0, b > 0$ there is only one mixed Nash (which is not an ESS) occurring with the mixture $X^* = (b/(a+b), a/(a+b))$. It is not hard to see that in this case, the trajectory $(p_1(t), p_2(t))$ will converge to one of the two pure Nash equilibria that are the ESSs of this game. In fact, if we integrate the differential equation, which is (6.2.3), replacing $p_2 = 1 - p_1$ and $p = p_1$, we have using integration by partial fractions

$$\frac{dp}{p(1-p)[ap - b(1-p)]} = dt \implies$$
$$\ell n\left[|1-p|^{-1/a}p^{-1/b}(ap - b(1-p))^{1/a+1/b}\right] = t + C.$$

or

$$\frac{(ap - b(1-p))^{1/a+1/b}}{|1-p|^{1/a}p^{1/b}} = Ce^t.$$

As $t \to \infty$, assuming $C > 0$, the right side goes to ∞. There is no way that could happen on the left side unless $p \to 0$ or $p \to 1$. It would be impossible for the left side to become infinite if $\lim_{t\to\infty} p(t)$ is strictly between 0 and 1.

We illustrate the case $a > 0, b > 0$ in Figure 6.3 on the left for choices $a = 1, b = 2$ and several distinct initial conditions. You can see that $\lim_{t\to\infty} p_2(t) = 0$ or $= 1$, so it is converging to a pure ESS as long as we do not start at $p_1(0) = \frac{2}{3}, p_2(0) = \frac{1}{3}$. In the case $a < 0, b < 0$ the tra-

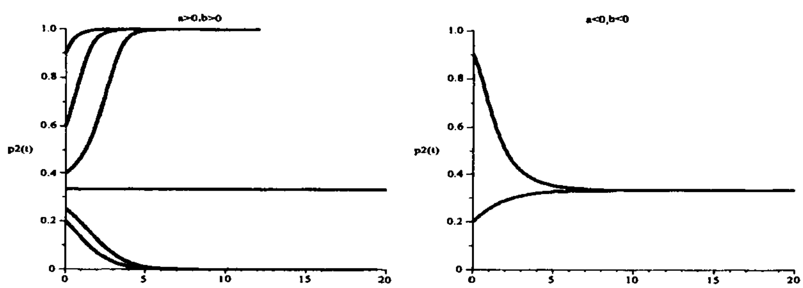


Figure 6.3 Left: $a > 0, b > 0$; Right: $a < 0, b < 0$.

jectories converges to the mixed Nash equilibrium, which is the unique ESS. This is illustrated in Figure 6.3 on the right with $a = -1, b = -2$, and you can see that $\lim_{t\to\infty} p_2(t) = \frac{1}{3}$ and so $\lim_{t\to\infty} p_1(t) = \frac{2}{3}$. The phase portraits in Figure 6.4 show clearly what is happening if you follow the arrows.

The graphs in Figure 6.3 and Figure 6.4 were created using the following Maple commands:

```
>restart:a:=1:b:=2:
> ode:=diff(p1(t),t)=p1(t)*(a*p1(t)-a*p1(t)^2-b*p2(t)^2),
```

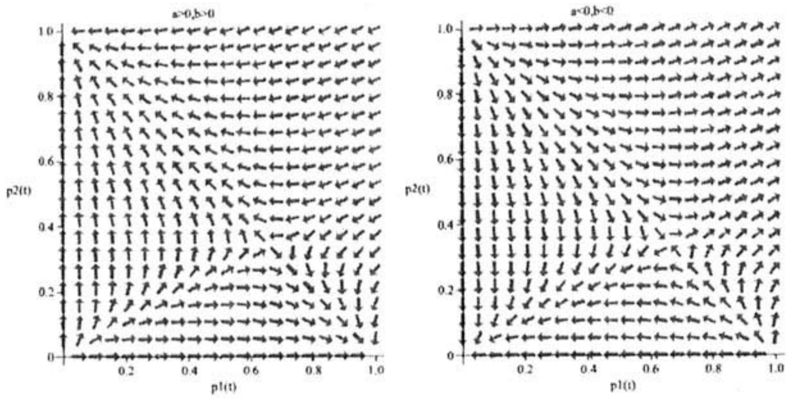


Figure 6.4 Left:Convergence to ESS $(1,0)$ or $(0,1)$; Right: Convergence to the mixed ESS.

```
        diff(p2(t),t)=p2(t)*(b*p2(t)-a*p1(t)^2-b*p2(t)^2);
> with(DEtools):
> DEplot([ode],[p1(t),p2(t)],t=0..20,
        [[p1(0)=.8,p2(0)=.2],
              [p1(0)=.1,p2(0)=.9],[p1(0)=2/3,p2(0)=1/3]],
        stepsize=.05,p1=0..1,p2=0..1,
        scene=[p1(t),p2(t)],arrows=large,linecolor=black,
        title="a>0,b>0");
> DEplot([ode],[p1(t),p2(t)],t=0..20,
        [[p1(0)=.8,p2(0)=.2],[p1(0)=.4,p2(0)=.6],
        [p1(0)=.6,p2(0)=.4],[p1(0)=.75,p2(0)=.25],
        [p1(0)=.1,p2(0)=.9],[p1(0)=2/3,p2(0)=1/3]],
        stepsize=.01,p1=0..1,p2=0..1,
        scene=[t,p2(t)],arrows=large,linecolor=black,
        title="a>0,b>0");
```

Remark. Example 6.5 is not as special as it looks because if we are given any two-player symmetric game with matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{bmatrix} \quad B = A^T,$$

then the game is equivalent to the symmetric game with matrix

$$A = \begin{bmatrix} a_{11} - a & a_{12} - b \\ a_{21} - a & a_{22} - b, \end{bmatrix} \quad B = A^T$$

for any a, b, in the sense that they have the same set of Nash equilibria. (Verify that!) So, in particular, if we take $a = a_{21}$ and $b = a_{12}$, we have the equivalent matrix game

$$\bar{A} = \begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{21}, \end{bmatrix} = \begin{bmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{21}, \end{bmatrix} \quad \bar{B} = \bar{A}^T$$

and so we are in the case discussed in the example.

Let's step aside and write down some results we need from ordinary differential equations. Here is a theorem guaranteeing existence and uniqueness of solutions of differential equations.

Theorem 6.2.1 *Suppose that you have a system of differential equations*

$$\frac{d\pi}{dt} = f(\pi(t)), \pi = (p_1, \ldots, p_n). \tag{6.2.4}$$

Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\partial f / \partial p_i$ are continuous. Then for any initial condition $\pi(0) = \pi_0$, there is a unique solution up to some time $T > 0$.

Definition 6.2.2 *A steady state (or stationary, or equilibrium, or fixed-point) solution of the system of ordinary differential equations* (6.2.4) *is a constant vector π^* that satisfies $f(\pi^*) = 0$. It is (locally)* **stable** *if for any $\varepsilon > 0$ there is a $\delta > 0$ so that every solution of the system with initial condition π_0 satisfies*

$$|\pi_0 - \pi^*| < \delta \implies |\pi(t) - \pi^*| < \varepsilon, \forall t > 0.$$

A stationary solution of (6.2.4) *is (locally)* **asymptotically stable** *if it is locally stable and if there is $\rho > 0$ so that*

$$|\pi_0 - \pi^*| < \rho \implies \lim_{t \to \infty} |\pi(t) - \pi^*| = 0.$$

The set

$$B_{\pi^*} = \{\pi_0 \mid \lim_{t \to \infty} \pi(t) = \pi^*\}$$

is called the **basin of attraction** *of the steady state π^*. Here $\pi(t)$ is a trajectory through the initial point $\pi(0) = \pi_0$. If every initial point that is possible is in the basin of attraction of π^*, we say that the point π^* is* **globally asymptotically stable**.

Stability means that if you have a solution of the equations (6.2.4) that starts **near** the stationary solution, then it will stay near the stationary solution as time progresses. Asymptotic stability means that if a trajectory starts near enough to π^*, then it must eventually converge to π^*. For a system of two equations, such as those that arise with 2×2 games, or to which a 3×3 game may be reduced, we have the criterion given in the following theorem.

Theorem 6.2.3 *A steady-state solution* (p_1^*, p_2^*) *of the system*

$$\frac{dp_1(t)}{dt} = f(p_1(t), p_2(t)),$$
$$\frac{dp_2(t)}{dt} = g(p_1(t), p_2(t))$$

is asymptotically stable if

$$f_{p_1}(p_1^*, p_2^*) + g_{p_2}(p_1^*, p_2^*) < 0$$

and

$$\det J(p_1^*, p_2^*) = \det \begin{bmatrix} f_{p_1}(p_1^*, p_2^*) & f_{p_2}(p_1^*, p_2^*) \\ g_{p_1}(p_1^*, p_2^*) & g_{p_2}(p_1^*, p_2^*) \end{bmatrix} > 0.$$

If either $f_{p_1} + g_{p_1} > 0$ *or* $\det J(p_1^*, p_2^*) < 0$, *the steady-state solution* $\pi^* = (p_1^*, p_2^*)$ *is unstable (i.e., not stable).*

The J matrix in the proposition is known as the **Jacobian matrix** of the system.

■ EXAMPLE 6.6

If there is only one equation $p' = f(p)$, (where $' = d/dt$) then a steady state p^* is asymptotically stable if $f'(p^*) < 0$ and unstable if $f'(p^*) > 0$. It is easy to see why that is true in the one-dimensional case. Set $x(t) = p(t) - p^*$, so now we are looking at $\lim_{t\to\infty} x(t)$ and testing whether that limit is zero. By Taylor's theorem

$$\frac{dx}{dt} = f(x + p^*) = f(p^*) + f'(p^*)x(t) = f'(p^*)x(t)$$

should be true up to first-order terms. The solution of this linear equation is $x(t) = C\exp[f'(p^*)t]$. If $f'(p^*) < 0$, we see that $\lim_{t\to\infty} x(t) = 0$, and if $f'(p^*) > 0$, then $\lim_{t\to\infty} x(t)$ does not exist. So that is why stability requires $f'(p^*) < 0$.

Consider the differential equation

$$\frac{dx}{dt} = -x(1-x)(1-2x) = f(x).$$

The stationary states are $x = 0, 1, \frac{1}{2}$, but only two of these are asymptotically stable. These are the pure states $x = 0, 1$, while the state $x = \frac{1}{2}$ is unstable because no matter how small you perturb the initial condition, the solution will eventually be drawn to one of the other asymptotically stable solutions. As $t \to \infty$, the trajectory moves away from $\frac{1}{2}$ unless you start at exactly that point. Figure 6.5 shows this and shows how the arrows lead away from $\frac{1}{2}$.

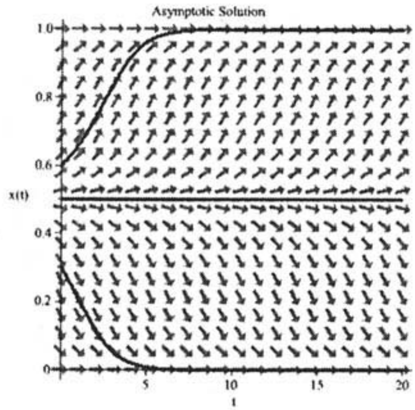


Figure 6.5 Trajectories of $dx/dt = -x(1-x)(1-2x)$; $x = \frac{1}{2}$ is unstable.

Checking the stability condition for $f'(0)$, $f'(1)$, $f'(\frac{1}{2})$ we have $f'(x) = -1 + 6x - 6x^2$, and $f'(0) = f'(1) = -1 < 0$, while $f'(\frac{1}{2}) = \frac{1}{2} > 0$, and so $x = 0, 1$ are asymptotically stable, while $x = \frac{1}{2}$ is not.

Now here is the connection for evolutionary game theory.

Theorem 6.2.4 *In any* 2×2 *game, a strategy* $X^* = (x_1^*, x_2^*)$ *is an ESS if and only if the system* (6.2.1) *has* $p_1^* = x_1^*, p_2^* = x_2^*$ *as an asymptotically stable steady state.*

Before we indicate the proof of this, let's recall the definition of what it means to be an ESS. X^* is an ESS if and only if either $E(X^*, X^*) > E(Y, X^*)$, for all strategies $Y \neq X^*$ or $E(X^*, X^*) = E(Y, X^*) \implies E(X^*, Y) > E(Y, Y), \forall\, Y \neq X^*$.

Proof. Now we will show that any ESS of a 2×2 game must be asymptotically stable. We will use the stability criterion in Theorem 6.2.3 to do that. Because there are only two pure strategies, we have the equation for $\pi = (p, 1-p)$:

$$\frac{dp(t)}{dt} = p(t)(1-p(t))(E(1,\pi) - E(2,\pi)) \equiv f(p(t)).$$

Because of an earlier Problem 6.5 we will consider only the case where

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad E(1,\pi) = ap, \quad E(2,\pi) = b(1-p),$$

and, in general, $a = a_{11} - a_{21}, b = a_{22} - a_{12}$. Then

$$f(p) = p(1-p)(E(1,\pi) - E(2,\pi)) = p(1-p)[ap - b(1-p)]$$

and

$$f'(p) = p(2a + 4b) + p^2(-3a - 3b) - b.$$

The three steady-state solutions where $f(p) = 0$ are $p^* = 0, 1, b/(a+b)$. Now consider the following cases:

Case 1: $ab < 0$. In this case there is a unique strict symmetric Nash equilibrium and so the ESSs are either $X = (1,0)$ (when $a > 0, b < 0$), or $X = (0,1)$ (when $a < 0, b > 0$). We look at $a > 0, b < 0$ and the steady-state solution $p^* = 1$. Then, $f'(1) = 2a + 4b - 3a - 3b - b = -a < 0$, so that $p^* = 1$ is asymptotically stable. Similarly, if $a < 0, b > 0$, the ESS is $X = (0,1)$ and $p^* = 0$ is asymptotically stable. For an example, if we take $a = 1, b = -2$, the following diagram shows convergence to $p^* = 1$ for trajectories from fours different initial conditions:

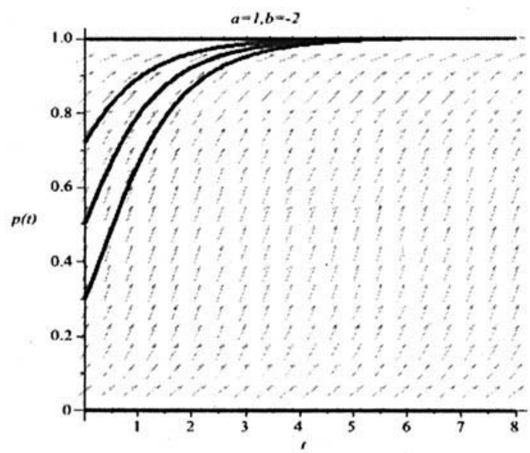


Figure 6.6 Convergence to $p^* = 1$.

Case 2: $a > 0, b > 0$. In this case there are three symmetric Nash equilibria: $X_1 = (1,0), X_2 = (0,1)$, and the mixed Nash $X_3 = (\gamma, 1-\gamma), \gamma = b/(a+b)$. The two pure Nash X_1, X_2 are strict and thus are ESSs by the properties 6.1.8. The mixed Nash is **not** an ESS because $E(X_3, X_3) = a\gamma$ is not larger than $E(Y, X_3) = a\gamma, \forall Y \neq X_3$ (they are equal), and taking $Y = (1,0)$, $E(Y,Y) = E(1,1) = a > a\gamma = E(X_3, 1)$. Consequently, X_3 does not satisfy the criteria to be an ESS. Hence we only need consider the stationary solutions $p^* = 0, 1$. But then, in the case $a > 0, b > 0$, we have $f'(0) = -b < 0$ and $f'(1) = -a < 0$, so they are both asymptotically stable. This is illustrated in the Figure 6.7 with $a = 2, b = 1$.

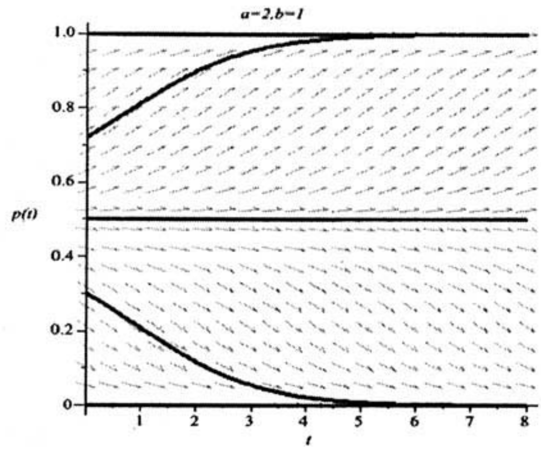


Figure 6.7 Mixed Nash not stable.

You can see that from any initial condition $x_0 < \frac{1}{2}$, the trajectory will converge to the stationary solution $p^* = 0$. For any initial condition $x_0 > \frac{1}{2}$, the trajectory will converge to the stationary solution $p^* = 1$, and only for $x_0 = \frac{1}{2}$, will the trajectory stay at $\frac{1}{2}$; $p^* = \frac{1}{2}$ is unstable.

Case 3: $a < 0, b < 0$. In this case, there are two strict but asymmetric Nash equilibria and one symmetric Nash equilibrium $X = (\gamma, 1 - \gamma)$. This symmetric X is an ESS because

$$E(Y, X) = ay_1\gamma + by_2(1 - \gamma) = \frac{ab}{a+b} = E(X, X),$$

and for every $Y \neq X$

$$E(Y, Y) = ay_1^2 + by_2^2 < \frac{ab}{a+b} = E(X, X),$$

since $a < 0, b < 0$. Consequently the only ESS is $X = (\gamma, 1-\gamma)$, and so we consider the steady state $p^* = \gamma$. Then

$$f'(\gamma) = \frac{ab}{a+b} < 0,$$

and so $p^* = \gamma$ is asymptotically stable. For an example, if we take $a = -2, b = -1$, we get Figure 6.8, in which all interior initial conditions lead to convergence to $\frac{1}{3}$.

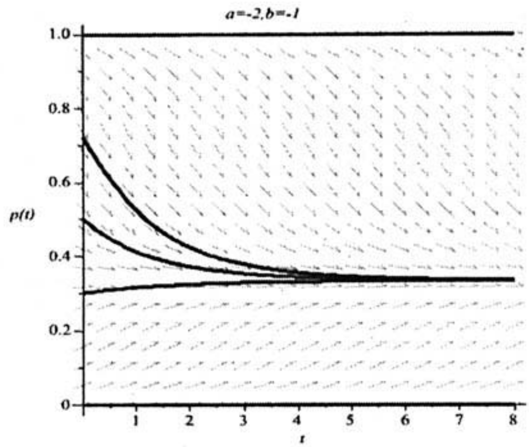


Figure 6.8 Convergence to the mixed ESS $X^* = (b/(a+b), a/(a+b))$, in case $a < 0, b < 0$.

Figure 6.8 was created with the Maple commands

```
> with(plots): with(DEtools):
> DEplot(D(x)(t)=(x(t)*(1-x(t))*(-2*x(t)+1*(1-x(t)))),
        x(t),t=0..8,
        [[x(0)=0],[x(0)=1],[x(0)=.5],[x(0)=.3],[x(0)=0.72]],
        title='a=-2,b=-1',colour=magenta,linecolor=[gold,yellow,
        black,red,blue],stepsize=0.05,labels=['t','p(t)']);
```

It is important to remember that if the trajectory starts exactly at one of the stationary solutions, then it stays there forever. It is starting nearby and seeing where it goes that determines stability.

We have proved that any ESS of a 2×2 game must be asymptotically stable. We skip the opposite direction of the proof and end this section by reviewing the remaining connections between Nash equilibria, ESSs, and stability. The first is a summary of the connections with Nash equilibria.

1. If X^* is a Nash equilibrium for a symmetric game with matrix A, then it is a stationary solution of (6.2.1).

2. If X^* is a strict Nash equilibrium, then it is locally asymptotically stable.

3. If X^* is a stationary solution and if $\lim_{t \to \infty} p(t) = X^*$, where $p(t)$ is a solution of (6.2.1) such that each component $p_i(t)$ of $p(t) = (p_1(t), \ldots, p_n(t))$ satisfies $0 < p_i(t) < 1$, then X^* is a Nash equilibrium.

4. If X^* is a locally asymptotically stable stationary solution of (6.2.1), then it is a Nash equilibrium.

The converse statements do not necessarily hold. The verification of all these statements, and much more, can be found in the book by Weibull [29]. Now here is the main result for ESSs and stability.

Theorem 6.2.5 *If X^* is an ESS, then X^* is an asymptotically stable stationary solution of* (6.2.1). *In addition, if X^* is completely mixed, then it is globally asymptotically stable.*

Again, the converse does not necessarily hold.

EXAMPLE 6.7

In this example we consider a 3×3 symmetric game so that the frequency dynamics is a system in the three variables (p_1, p_2, p_3). Let's look at the game with matrix

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}.$$

This game has Nash equilibria

X	Y
$(1,0,0)$	$(0,0,1)$
$(0,1,0)$	$(0,0,1)$
$(0,0,1)$	$(1,0,0)$
$(0,0,1)$	$(0,1,0)$
$(0,\frac{1}{4},\frac{3}{4})$	$(0,\frac{1}{4},\frac{3}{4})$

There is only one symmetric Nash equilibrium $X^* = (0, \frac{1}{4}, \frac{3}{4}) = Y^*$. It will be an ESS because we will show that $u(Y,Y) < u(X^*,Y)$ for every best response strategy $Y \neq X^*$ to the strategy X^*. By properties 6.1.8(3), we then know that X^* is an ESS.

First we find the set of best response strategies Y. To do that, calculate

$$u(Y, X^*) = y_1\left(\frac{1}{4}\right) + (y_2 + y_3)\frac{3}{4}.$$

It is clear that u is maximized for $y_1 = 0, y_2 + y_3 = 1$. This means that any best response strategy must be of the form $Y = (0, y, 1-y)$, where $0 \leq y \leq 1$.

Next, taking any best response strategy, we have

$$u(Y,Y) = 4y - 4y^2 \text{ and } u(X^*,Y) = X^*AY^T = 2y + \frac{1}{4}.$$

We have to determine whether $4y - 4y^2 < 2y + \frac{1}{4}$ for all $0 \le y \le 1, y \ne \frac{1}{4}$. This is easy to do by calculus since $f(y) = 4y^2 - (2y + \frac{1}{4})$ has a minimum at $y = \frac{1}{4}$ and $f(\frac{1}{4}) = 0$. Here are the Maple commands to get all of this:

```
> restart:with(LinearAlgebra):
> A:=Matrix([[0,-2,1],[1,0,1],[1,3,0]]);
> X:=<0,1/4,3/4>; Y:=<y1,y2,y3>;
> uYY:=expand(Transpose(Y).A.Y); uYX:=expand(Transpose(Y).A.X);
> with(Optimization):
> Maximize(uYX,{y1+y2+y3=1,y1>=0,y2>=0,y3>=0});
> uXY:=Transpose(X).A.Y;
> w:=subs(y1=0,y2=y,y3=1-y,uYY); v:=subs(y1=0,y2=y,y3=1-y,uXY);
> plot(w-v,y=0..1);
```

The last plot, exhibited in Figure 6.9, shows that $u(Y, Y) < u(X^*, Y)$ for best response strategies, Y.

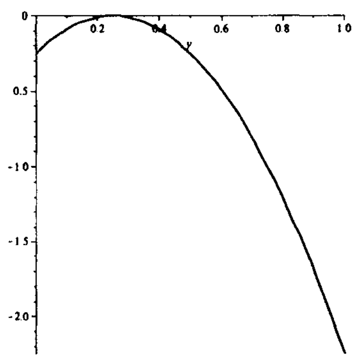


Figure 6.9 Plot of $f(y) = u(Y, Y) - u(X^*, Y) = 4y^2 - 2y - \frac{1}{4}$.

The replicator dynamics (6.2.1) for this game are

$$\frac{dp_1(t)}{dt} = p_1 [-2p_2 + p_3 - (-p_1p_2 + 2p_1p_3 + 4p_2p_3)],$$
$$\frac{dp_2(t)}{dt} = p_2 [p_1 + p_3 - (-p_1p_2 + 2p_1p_3 + 4p_2p_3)],$$
$$\frac{dp_3(t)}{dt} = p_3 [p_1 + 3p_2 - (-p_1p_2 + 2p_1p_3 + 4p_2p_3)].$$

Since $p_1 + p_2 + p_3 = 1$, they can be reduced to two equations to which we may apply the stability theorem. The equations become

$$\frac{dp_1(t)}{dt} = f(p_1,p_2) \equiv p_1\left[-7p_2 + 1 - 3p_1 + 7p_1p_2 + 2p_1^2 + 4p_2^2\right],$$
$$\frac{dp_2(t)}{dt} = g(p_1,p_2) \equiv p_2\left[1 - 5p_2 + 7p_1p_2 - 2p_1 + 2p_1^2 + 4p_2^2\right].$$

The steady-state solutions are given by the solutions of the pair of equations $f(p_1,p_2) = 0, g(p_1,p_2) = 0$, which make the derivatives zero and are given by

$$a = [p_1 = 0, p_2 = 0], b = [p_1 = 0, p_2 = \frac{1}{4}],$$
$$c = [p_1 = 0, p_2 = 1], d = [p_1 = \frac{1}{2}, p_2 = 0], e = [p_1 = 1, p_2 = 0],$$

and we need to analyze each of these. We start with the condition from Theorem 6.2.3 that $f_{p_1} + g_{p_2} < 0$. Calculation yields,

$$\begin{aligned} k(p_1,p_2) &= f_{p_1}(p_1,p_2) + g_{p_2}(p_1,p_2) \\ &= -17p_2 + 2 - 8p_1 + 28p_1p_2 + 8p_1^2 + 16p_2^2. \end{aligned}$$

Directly plugging our points into $k(p_1,p_2)$, we see that

$$k(0,0) = 2 > 0,\ k(0,\frac{1}{4}) = -\frac{5}{4} < 0,\ k(0,1) = 1 > 0,$$
$$k(\frac{1}{2},0) = 0,\ k(1,0) = 2 > 0,$$

and since we only need to consider the negative terms as possibly stable, we are left with the one possible asymptotically stable value $b = (0, \frac{1}{4})$.

Next we check the Jacobian at that point and get

$$\det\begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{16} & -\frac{3}{4} \end{bmatrix} = \frac{3}{8} > 0$$

By the stability Theorem 6.2.3 $p_1^* = 0, p_2^* = \frac{1}{4}, p_3^* = \frac{3}{4}$, is indeed an asymptotically stable solution, and hence a Nash equilibrium and an ESS. Figure 6.10 shows three trajectories $(p_1(t), p_2(t))$ starting at time $t = 0$ from the initial conditions $(p_1(0), p_2(0)) = (0.1, 0.2),\ (0.6, 0.2),\ (0.33, 0.33)$.

We see the asymptotic convergence of $(p_1(t), p_2(t))$ to $(0, \frac{1}{4})$. Since $p_1 + p_2 + p_3 = 1$, this means $p_3(t) \to \frac{3}{4}$. It also shows a trajectory starting from $p_1 = 0, p_2 = 0$ and shows that the trajectory stays there for all time.

Figure 6.10 was obtained using the following Maple commands.

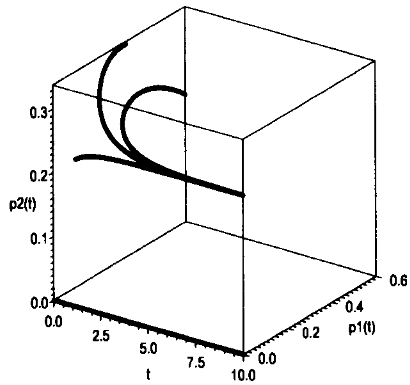


Figure 6.10 Convergence to $p_1^* = 0, p_2^* = \frac{1}{4}, p_3^* = \frac{3}{4}$ from three different Initial conditions.

```
> restart:with(DEtools):with(plots):with(LinearAlgebra):
> A:=Matrix([[0,-2,1],[1,0,1],[1,3,0]]);
> X:=<x1,x2,x3>;
> Transpose(X).A.X;
> s:=expand(%);
> L:=A.X;
> L[1]-s;L[2]-s;L[3]-s;
> DEplot3d({D(p1)(t)=p1(t)*(-2*p2(t)+p3(t)+p1(t)*p2(t)
                       -2*p1(t)*p3(t)-4*p2(t)*p3(t)),
          D(p2)(t)=p2(t)*(p1(t)+p3(t)+p1(t)*p2(t)
                       -2*p1(t)*p3(t)-4*p2(t)*p3(t)),
          D(p3)(t)=p3(t)*(p1(t)+3*p2(t)+p1(t)*p2(t)
                       -2*p1(t)*p3(t)-4*p2(t)*p3(t))},
          {p1(t),p2(t),p3(t)},
          t=0..10,[[p1(0)=0.1,p2(0)=0.2,p3(0)=.7],
                   [p1(0)=0.6,p2(0)=.2,p3(0)=.2],
                   [p1(0)=1/3,p2(0)=1/3,p3(0)=1/3],
                   [p1(0)=0,p2(0)=0,p3(0)=1]],
          scene=[t,p1(t),p2(t)],
          stepsize=.1,linecolor=t);
```

The right sides of the frequency equations are calculated in $L[i] - s, i = 1, 2, 3$ and then explicitly entered in the `DEplot3d` command. Notice the command `scene=[t,p1(t),p2(t)]` gives the plot viewing $p_1(t), p_2(t)$ versus time. If you want to see p_1 and p_3 instead, then this is the command to change. These commands result in Figure 6.10

EXAMPLE 6.8

This example also illustrates instability with cycling around the Nash mixed strategy as shown in Figure 6.11. The game matrix is

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

The system of frequency equations, after making the substitution $p_3 = 1 - p_1 - p_2$, becomes

$$\frac{dp_1(t)}{dt} = f(p_1, p_2) \equiv p_1 [2p_2 - 1 + p_1],$$
$$\frac{dp_2(t)}{dt} = g(p_1, p_2) \equiv -p_2 [2p_1 - 1 + p_2].$$

The steady state solutions are $(0,0), (0,1), (1,0)$, and $(\frac{1}{3}, \frac{1}{3})$. Then

$$k(p_1, p_2) \equiv f_{p_1}(p_1, p_2) + g_{p_2}(p_1, p_2) = 4p_2 - 2 + 4p_1,$$

and, evaluating at the stationary points,

$$k(0,0) = -2 < 0, \ k(0,1) = 2 > 0, \ k(1,0) = 2 > 0, \ k\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3} > 0.$$

The steady-state solution $(0,0)$ is indeed asymptotically stable since the determinant of the Jacobian at that point is readily calculated to be $J = 1 > 0$, and so Theorem 6.2.3 tells us that $(p_1, p_2, p_3) = (0,0,1)$ is asymptotically stable. On the other hand, the unique mixed strategy corresponding to $(p_1, p_2, p_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ completely violates the conditions of the theorem because $k(\frac{1}{3}, \frac{1}{3}) > 0$ as well as $J = -\frac{1}{3} < 0$.

It is unstable, as is shown in Figure 6.11.

Three trajectories are shown starting from the points $(0.1, 0.2), (0.6, 0.2)$ and $(0.33, 0.33)$. You can see that unless the starting position is exactly at $(0.33, 0.33)$, the trajectories will cycle around and not converge to the mixed strategy.

Finally, a somewhat automated set of Maple commands to help with the algebra are given:

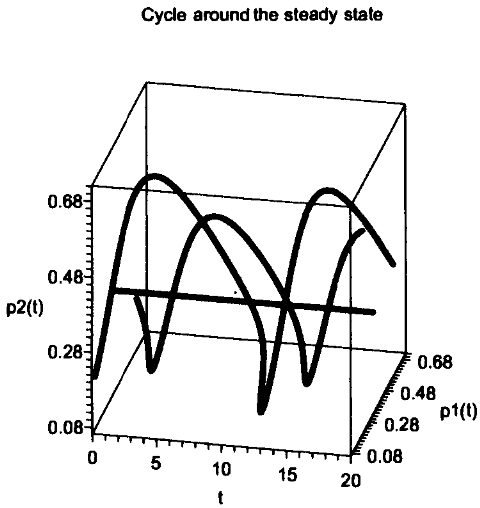


Figure 6.11 $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is unstable.

```
> restart:with(DEtools):with(plots):with(LinearAlgebra):
> A:=Matrix([[0,1,-1],[-1,0,1],[1,-1,0]]);
> X:=<x1,x2,x3>;
> Transpose(X).A.X;
> s:=expand(%);
> L:=A.X; L[1]-s;L[2]-s;L[3]-s;
> ode:={D(p1)(t)=p1(t)*(p2(t)-p3(t)),
        D(p2)(t)=p2(t)*(-p1(t)+p3(t)),
        D(p3)(t)=p3(t)*(p1(t)-p2(t))};
> subs(p3(t)=1-p1(t)-p2(t),ode);
> G:=simplify(%);
> DEplot3d({G[1],G[2]},{p1(t),p2(t)}, t=0..20,
           [[p1(0)=0.1,p2(0)=0.2],
           [p1(0)=0.6,p2(0)=.2],
           [p1(0)=1/3,p2(0)=1/3]],
           scene=[t,p1(t),p2(t)],
           stepsize=.1,linecolor=t,
           title="Cycle around the steady state");
```

Remember that Maple makes `X:=<x1,x2,x3>` a column vector, and that is why we need `Transpose(X).A.X` to calculate $E(X, X)$.

PROBLEMS

6.7 Show that a 2×2 symmetric game

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{bmatrix}, \quad B = A^T$$

has exactly the same set of Nash equilibria as does the symmetric game with matrix

$$A' = \begin{bmatrix} a_{11} - a & a_{12} - b \\ a_{21} - a & a_{22} - b, \end{bmatrix}, \quad B = A'^T$$

for any a, b.

6.8 Consider a game in which a seller can be either honest or dishonest and a buyer can either inspect or trust (the seller). One game model of this is the matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, where the rows are inspect and trust, and the columns correspond to dishonest and honest.

(**a**) Find the replicator dynamics for this game.

(**b**) Find the Nash equilibria and determine which, if any, are ESSs.

(**c**) Analyze the stationary solutions for stability.

6.9 Analyze all stationary solutions for the game with matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$.

6.10 Consider the symmetric game with matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 5 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix}.$$

Find the one and only Nash equilibrium. Determine whether the Nash equilibrium is an ESS. Reduce the replicator equations to two equations and find the stationary solutions. Check for stability using the stability Theorem 6.2.3.

6.11 Consider the symmetric game with matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Show that $X = (1, 0, 0)$ is an ESS that is asymptotically stable for (6.2.1).

6.12 The simplest version of the rock-paper-scissors game has matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Show that there is one and only one completely mixed Nash equilibrium but it is not an ESS. Show that this statement is still true if you replace 0 in each row of the matrix by $a > 0$. Analyze the stability of the stationary points for the replicator dynamics (6.2.1) by reducing to two equations and using the stability Theorem 6.2.3.

6.13 Find the frequency dynamics (6.2.1) for the game

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the steady-state solutions and investigate their stability, then reach a conclusion about the Nash equilibria and the ESSs.

Remark. There is an apocryphal story [1] appearing on the Internet as an eRumor that is an interesting counterexample to evolution and implies that populations may not always choose a better strategy in the long run. The story starts with a measurement. In the United States the distance between railroad rails is 4 feet 8.5 inches. Isn't that strange? Why isn't it 5 feet, or 4 feet? The distance between rails determines how rail cars are built, so wouldn't it be easier to build a car with exact measurements, that determine all the parts in the drive train?

> Why is that the measurement chosen?
>
> Well, that's the way they built them in Great Britain, and it was immigrants from Britain who built the US railroads.
>
> Why did the English choose that measurement for the railroads in England?
>
> Because, before there were railroads there were tramways, and that's the measurement they chose for the tramways. (A tramway is a light-rail system for passenger trams.)
>
> Why?
>
> Because they used the same measurements as the people who built wagons for the wheel spacing.
>
> And why did the wagon builders use that measurement?
>
> Because the spacing of the wheel ruts in the old roads had that spacing, and if they used another spacing, the wagon wheels would break apart.
>
> How did the road ruts get that spacing?
>
> The Romans' chariots for the legions made the ruts, and since Rome conquered most of the known world, the ruts ended up being the same almost everywhere they traveled because they were made by the same version of General Motors then, known as Imperial Chariots, Inc.
>
> And why did they choose the spacing that started this story?
>
> Well, that is exactly the width the chariots need to allow two horse's rear ends to fit. And that is how the world's most advanced transportation system is based on 4 feet 8.5 inches.

[1] It turns out that this is a fictional story. The real reason why the measurement is 4 feet 8.5 inches can be found at http://www.truthorfiction.com/rumors/r/railwidth.htm.

BIBLIOGRAPHIC NOTES

Evolutionary games can be approached with two distinct goals in mind. The first is the matter of determining a way to choose a **correct** Nash equilibrium in games with multiple equilibria. The second is to model biological processes in order to determine the eventual evolution of a population. This was the approach of Maynard-Smith and Price (see Vincent and Brown's book [25] for references and biologic applications) and has led to a significant impact in biology, both experimental and theoretical. Our motivation for the definition of evolutionary stable strategy is the very nice derivation in the treatise by Mesterton-Gibbons [15], which is used here with permission. The equivalent definitions, some of which are easier to apply, are standard and appear in the literature (for example in references [25] and [29]) as are many more results. The hawk–dove game is a classic example appearing in all books dealing with this topic. The rock-paper-scissors game is an illustrative example of a wide class of games (see the book by Weibull [29] for a lengthy discussion of rock-paper-scissors).

Population games are a natural extension of the idea of evolutionary stability by allowing the strategies to change with time and letting time become infinite. The stability theory of ordinary differential equations can now be brought to bear on the problem (see the book by Scheinermann [24] for the basic theory of stability). The first derivation of the replicator equations seems to have been due to Taylor and Jonker in the article in Reference [8]. Hofbauer and Sigmund, who have made important contributions to evolutionary game theory, have published recently an advanced survey of evolutionary dynamics in Reference [6]. Refer to the book by Gintis [4] for an exercise-based approach to population games and extensions.

APPENDIX A
THE ESSENTIALS OF MATRIX ANALYSIS

A matrix is a rectangular collection of numbers. If there are n rows and m columns, we write the matrix as $A_{n\times m}$, and the numbers of the matrix are a_{ij}, where i gives the row number and j gives the column number. These are also called the **dimensions** of the matrix. We compactly write $A = (a_{ij})_{i=1,j=1}^{i=n,j=m}$. In rectangular form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

Game Theory: An Introduction, First Edition. By E.N. Barron

If $n = m$, we say that the matrix is square. The square matrix in which there are all 1s along the diagonal and 0s everywhere else is called the **identity matrix**:

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Here are some facts about algebra with matrices:

1. We add two matrices that have the same dimensions, $A + B$, by adding the respective components $A + B = (a_{ij} + b_{ij})$, or

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix}$$

2. A matrix may be multiplied by a scalar c by multiplying every element of A by c; that is, $cA = (ca_{ij})$ or

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & \cdots & ca_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nm} \end{bmatrix}.$$

3. We may multiply two matrices $A_{n\times m}$ and $B_{m\times k}$ only if the number of columns of A is exactly the same as the number of rows of B. You have to be careful because not only is $A \cdot B \neq B \cdot A$; in general it is not even defined if the rows and columns don't match up. So, if $A = A_{n\times m}$ and $B = B_{m\times k}$, then $C = A \cdot B$ is defined and $C = C_{n\times k}$, and is given by

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix}$$

We multiply each row $i = 1, 2, \ldots, n$, of A by each column $j = 1, 2, \ldots, k$, of B in this way

$${}_iA \cdot B_j = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = c_{ij}.$$

This gives the $(i,j$th element of the matrix C. The matrix $C_{n\times k} = (c_{ij})$ has elements written compactly as

$$c_{ij} = \sum_{r=1}^{m} a_{ir}b_{rj}, \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,k.$$

4. As special cases of multiplication that we use throughout this book

$$\begin{aligned} X_{1\times n} \cdot A_{n\times m} &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} . \\ &= \begin{bmatrix} \sum_{i=1}^{n} x_i a_{i1} & \sum_{i=1}^{n} x_i a_{i2} & \cdots & \sum_{i=1}^{n} x_i a_{im} \end{bmatrix} \end{aligned}$$

Each element of the result is $E(X,j), \quad j = 1,2,\ldots,m$.

5. If we have any matrix $A_{n\times m}$, the **transpose** of A is written as A^T and is the $m \times n$ matrix, which is A with the rows and columns switched:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix} .$$

If $Y_{1\times m}$ is a row matrix, then Y^T is an $m \times 1$ column matrix, so we may multiply $A_{n\times m}$ by Y^T on the right to get

$$A_{n\times m} Y^T_{m\times 1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m} a_{1j} y_j \\ \sum_{j=1}^{m} a_{2j} y_j \\ \vdots \\ \sum_{j=1}^{m} a_{nj} y_j \end{bmatrix} .$$

Each element of the result is $E(i,Y), \quad i = 1,2,\ldots,n$.

6. A square matrix $A_{n\times n}$ has an **inverse** A^{-1} if there is a matrix $B_{n\times n}$ that satisfies $A \cdot B = B \cdot A = I$, and then B is written as A^{-1}. Finding the inverse

is computationally tough, but luckily you can determine whether there is an inverse by finding the determinant of A. The linear algebra theorem says that A^{-1} exists if and only if $\det(A) \neq 0$. The determinant of a 2×2 matrix is easy to calculate by hand:

$$\det(A_{2\times 2}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

One way to calculate the determinant of a larger matrix is **expansion by minors** which we illustrate for a 3×3 matrix:

$$\begin{aligned} \det(A_{3\times 3}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

This reduces the calculation of the determinant of a 3×3 matrix to the calculation of the determinants of three 2×2 matrices, which are called the **minors** of A. They are obtained by crossing out the row and column of the element in the first row (other rows may also be used). The determinant of the minor is multiplied by the element and the sign + or − alternates starting with a + for the first element. Here is the determinant for a 4×4 reduced to four 3×3 determinants:

$$\begin{aligned} \det(A_{4\times 4}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \\ &\quad + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \end{aligned}$$

7. A system of linear equations for the unknowns $\vec{y} = (y_1, \ldots, y_m)$ may be written in matrix form as $A_{n\times m}\vec{y} = \vec{b}$, where $\vec{b} = (b_1, b_2, \ldots, b_n)$. This is called an **inhomogeneous system** if $\vec{b} \neq \vec{0}$ and a **homogeneous system** if $\vec{b} = \vec{0}$. If A is a square matrix and is invertible, then $\vec{y} = A^{-1}\vec{b}$ is the one and only solution. In particular, if $\vec{b} = \vec{0}$, then $\vec{y} = \vec{0}$ is the only solution.

8. A matrix $A_{n\times m}$ has associated with it a number called the **rank** of A, which is the largest square submatrix of A that has a nonzero determinant. So, if A is a

4×4 invertible matrix, then $rank(A) = 4$. Another way to calculate the rank of A is to **row-reduce** the matrix to row-reduced echelon form. The number of nonzero rows is $rank(A)$.

9. The rank of a matrix is intimately connected to the solution of equations $A_{n\times m}y_{m\times 1} = b_{n\times 1}$. This system will have a unique solution if and only if $rank(A) = m$. If $rank(A) < m$, then $Ay_{m\times 1} = b_{n\times 1}$ has an infinite number of solutions. For the homogeneous system $Ay_{m\times 1} = 0_{n\times 1}$, this has a unique solution, namely, $y_{m\times 1} = 0_{m\times 1}$, if and only if $rank(A) = m$. In any other case $Ay_{m\times 1} = 0_{n\times 1}$ has an infinite number of solutions.

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APPENDIX B
THE ESSENTIALS OF PROBABILITY

In this section we give a brief review of the basic probability concepts and definitions used or alluded to in this book. For further information, there are many excellent books on probability (e.g., see the book by Ross [23]).

The space of all possible outcomes of an experiment is labeled Ω. Events are subsets of the space of all possible outcomes, $A \subset \Omega$. Given two events A, B

1. The event $A \cup B$ is the event **either A occurs or B occurs, or they both occur**.

2. The event $A \cap B$ is the event **both A and B occur**.

3. The event $\emptyset$ is called the **impossible event**, while the event Ω itself is called the **sure event**.

4. The event $A^c = \Omega \setminus A$ is called the **complement of** A, so either A occurs or, if A does not occur, then A^c occurs.

The probability of an event A is written as $P(A)$ or $Prob(A)$. It must be true that for any event A, $0 \leq P(A) \leq 1$. In addition

1. $P(\emptyset) = 0, P(\Omega) = 1$.
2. $P(A^c) = 1 - P(A)$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. $P(A \cap B) = P(A)P(B)$ if A and B are **independent.**

Given two events A, B with $P(B) > 0$, the **conditional probability of A given the event B has occurred**, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Two events are therefore independent if and only if $P(A|B) = P(A)$ so that the knowledge that B has occurred does not affect the probability of A. The very important **laws of total probability** are frequently used:

1. $P(A) = P(A \cap B) + P(A \cap B^c)$
2. $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

These give us a way to calculate the probability of A by breaking down cases. Using the formulas, if we want to calculate $P(A|B)$ but know $P(B|A)$ and $P(B|A^c)$, we can use **Bayes' rule**:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

A random variable is a function $X : \Omega \to \mathbb{R}$, that is, a real-valued function of outcomes of an experiment. As such, this random variable takes on its values by chance. We consider events of the form $\{\omega \in \Omega : X(\omega) \leq x\}$ for values of $x \in \mathbb{R}$. For simplicity, this event is written as $\{X \leq x\}$ and the function

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}$$

is called the **cumulative distribution function** of X. We have

1. $P(X > a) = 1 - F_X(a)$.
2. $P(a < X \leq b) = F_X(b) - F_X(a)$.

A random variable X is said to be **discrete** if there is a finite or countable set of numbers $x_1, x_2, \ldots$, so that X takes on only these values with $P(X = x_i) = p_i$, where $0 \leq p_i \leq 1$, and $\sum_i p_i = 1$. The cumulative distribution function is a step function with a jump of size p_i at each x_i, $F_X(x) = \sum_{x_i \leq x} p_i$.

A random variable is said to be **continuous** if $P(X = x) = 0$ for every $x \in \mathbb{R}$, and the cumulative distribution function $F_X(x) = P(X \le x)$ is a continuous function. The probability that a continuous random variable is any particular value is always zero. A probability density function for X is a function

$$f_X(x) \ge 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} f_X(x)\, dx = 1.$$

In addition, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy \quad \text{and} \quad f_X(x) = \frac{d}{dx} F_X(x).$$

The cumulative distribution up to x is the area under the density from $-\infty$ to x. With an abuse of notation, we often see $P(X = x) = f_X(x)$, which is clearly nonsense, but it gets the idea across that the density at x is roughly the probability that $X = x$.

Two random variables X and Y are **independent** if $P(X \le x \text{ and } Y \le y) = P(X \le x)P(X \le y)$ for all $x, y \in \mathbb{R}$. If the densities exist, this is equivalent to $f(x, y) = f_X(x) f_Y(y)$, where $f(x, y)$ is the joint density of (X, Y).

The **mean** or **expected value** of a random variable X is

$$E[X] = \sum_i x_i P(X = x_i) \quad \text{if } X \text{ is discrete}$$

and

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx \quad \text{if } X \text{ is continuous.}$$

In general, a much more useful measure of X is the **median** of X, which is any number satisfying

$$P(X \ge m) = P(X \le m) = \frac{1}{2}.$$

Half the area under the density is to the left of m and half is to the right.

The mean of a function of X, say $g(X)$, is given by

$$E[g(X)] = \sum_i g(x_i) P(X = x_i) \quad \text{if } X \text{ is discrete}$$

and

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx \quad \text{if } X \text{ is continuous.}$$

With the special case $g(x) = x^2$, we may get the **variance of** X defined by

$$Var(X) = E[X^2] - (E[X])^2 = E[X - E[X]]^2.$$

This gives a measure of the spread of the values of X around the mean defined by the **standard deviation of** X

$$\sigma(X) = \sqrt{Var(X)}.$$

We end this appendix with a list of properties of the main discrete and continuous random variables.

Discrete Random Variables

1. **Bernoulli.** Consider an experiment in which the outcome is either success, with probability $p > 0$ or failure, with probability $1 - p$. The random variable

$$X = \begin{cases} 1 & \text{if success;} \\ 0 & \text{if failure.} \end{cases}$$

is Bernoulli with parameter p. Then $E[X] = p, Var(X) = p(1-p)$.

2. Suppose that we perform an independent sequence of Bernoulli trials, each of which has probability p of success. Let X be a count of the number of successes in n trials. X is said to be a **binomial random variable** with distribution

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, 1, 2, \ldots, n, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then $E[X] = np, Var(X) = np(1-p)$.

3. In a sequence of independent Bernoulli trials the number of trials until the first success is called a **geometric random variable**. If X is geometric, it has distribution

$$P(X = k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

with mean $E[X] = \frac{1}{p}$, and variance $Var(X) = (1-p)/p^2$.

4. A random variable has a **Poisson distribution with parameter** λ if X takes on the values $k = 0, 1, 2, \ldots,$ with probability

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2 \ldots.$$

It has $E[X] = \lambda$, and $Var(X) = \lambda$. It arises in many situations and is a limit of binomial random variables. In other words, if we take a large number n of Bernoulli trials with p as the probability of success on any trial, then as $n \to \infty$ and $p \to 0$ but np remaining the constant λ, the total number of successes will follow a Poisson distribution with parameter λ.

Continuous Distributions

1. X is **uniformly distributed on the interval** $[a, b]$ if it has the density

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b; \\ 0 & \text{otherwise.} \end{cases}$$

   It is a model of picking a number at random from the interval $[a, b]$ in which every number has equal likelihood. The cdf is $F_X(x) = (x-a)/(b-a), a \leq x \leq b$. The mean is $E[X] = (a+b)/2$, the midpoint, and variance is $Var(X) = (b-a)^2/12$.

2. X has a **normal distribution** with mean μ and standard deviation σ if it has the density

$$f_X(x) = \frac{1}{\sqrt{\sigma 2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

   Then the mean is $E[X] = \mu$, and the variance is $Var(X) = \sigma^2$. The graph of f_X is the classic bell-shaped curve centered at μ. The central limit theorem makes this the most important distribution because it roughly says that sums of independent random variables normalized by $\sigma\sqrt{n}$, converge to a normal distribution, no matter what the distribution of the random variables in the sum. More precisely

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

   Here $E[X_i] = \mu, Var(X_i) = \sigma^2$ are the mean and variance of the arbitrary members of the sum.

3. The random variable $X \geq 0$ is said to have an **exponential distribution** if

$$P(X \leq x) = F_X(x) = 1 - e^{-\lambda x}, x \geq 0.$$

   The density is $f_X(x) = F'_X(x) = \lambda e^{-\lambda x}, x > 0$. Then $E[X] = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Order Statistics In the theory of auctions one needs to put in order the valuations of each bidder and then calculate things such as the mean of the highest valuation, the second highest valuation, and so on. This is an example of the use of **order statistics**, which is an important topic in probability theory. We review a basic situation.

Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables with common density function $f(x)$ and common cumulative distribution function $F(x)$. The order statistics are the random variables

$$\begin{aligned} X_{(1)} &= \min\{X_1, \ldots, X_n\} = \text{ smallest of } X_i \\ X_{(2)} &= \min\{X_i : i \neq (1)\} = \text{ second smallest of } X_i \\ &\vdots \\ X_{(n-1)} &= \text{ second largest of } X_i \\ X_{(n)} &= \max\{X_1, \ldots, X_n\}, \end{aligned}$$

and automatically $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. This simply reorders the random variables from smallest to largest.

To get the density function of $X_{(k)}$ we may argue formally that $X_{(k)} \leq x$ if and only if there $k-1$ of the random variables are $\leq x$ and $n-k$ of the random variables are $> x$. In symbols,

$$X_{(k)} \leq x \Leftrightarrow k-1 \;\; X_i's \leq x, \text{ and } n-k \quad X_i's > x, \text{ and } 1 \quad X_i = x.$$

This leads to the cumulative distribution function of $X_{(k)}$. Consequently, it is not too difficult to show that the density of $X_{(k)}$ is

$$f_{X_{(k)}}(x) = \binom{n}{k-1, n-k, 1} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x),$$

where

$$\binom{n}{k-1, n-k, 1} = \frac{n!}{(n-k)!(k-1)!},$$

because there are

$$\binom{n}{k-1, n-k, 1}$$

ways of splitting up n things into three groups of size $k-1, n-k$, and 1.

If we start with a uniform distribution $f(x) = 1, 0 < x < 1$, we have

$$f_{X_{(k)}}(x) = \binom{n}{k-1, n-k, 1} [x]^{k-1} [1-x]^{n-k}, \quad 0 < x < 1.$$

If $k = n-1$, then

$$f_{X_{(n-1)}}(x) = \frac{n!}{(n-2)!} x^{n-2} (1-x), \quad 0 < x < 1.$$

Consequently, for $X_1, \ldots, X_n$ independent uniform random variables, we have $E(X_{(n-1)}) = (n-1)/(n+1)$. The means and variances of all the order statistics can be found by integration.

APPENDIX C
THE ESSENTIALS OF MAPLE

In this appendix we will review some of the basic Maple operations used in this book. Maple has a very helpful feature of providing context-sensitive help. That means that if you forget how a command is used, or even what the command is, simply type the word and click on it. Go to the Help menu, and you will see "Help on *your word*" which will give you the help available for that command.

Throughout this book we have used the **worksheet** mode of Maple version 10.0.

Features The major features of Maple are listed as follows:

Operators, Signs, Symbols, and Commands. The basic arithmetic operations are $+, -, *, /, ^\wedge$ for addition, subtraction, multiplication, division, and exponentiation, respectively. Any command must end with a semicolon ";" if you use a colon ":" instead, the statement will be executed but no output will appear. To clear all variables and definitions in a worksheet use `restart:`.

Parentheses (...) are used for algebraic grouping and arguments of functions just as they are in algebra. You cannot use braces or brackets for that! Braces {...} are used for sets, and brackets [...] for lists.

Multiplication uses an asterisk (e.g., `x*y`, not `xy`); exponents by a caret (e.g., `x^2`= x^2).

To make an assignment like `a:=3*x+4`, use `:=` to assign $3 * x + 2$ to the symbol a. The **equation** `a=3*x+4` is made with the equal sign and not colon equal.

The percent sign (%) refers to the immediately preceding output; %% gives the next-to-last output, and so on.

The command `unapply` converts a mathematical expression into a function. For example, if you type the command `g:=unapply(3*x,4*z,[x,z])` this converts $3x + 4z$ into the function $g(x, z) = 3x + 4z$.

Functions. A direct way of defining a function is by using

```
f:=variables-> expression.
```

For example,

```
h:=(u,v)->u*sin(v^2+Pi)-sqrt(u+v),
```

defines $h(u, v) = u\sin(v^2 + \pi) - \sqrt{(u + v)}$. A piecewise defined function is entered using the command `piecewise`. For example

```
f:=x->piecewise(x<1,x^2,x<3,2*x-sin(x),x=3,7,x>3,-x^3)
```

defines

$$f(x) = \begin{cases} x^2 & \text{if } x < 1; \\ 2x - \sin(x) & \text{if } x < 3; \\ 7 & \text{if } x = 3; \\ -x^3 & \text{if } x > 3. \end{cases}$$

Derivatives and Integrals of Functions. The derivative of a function with respect to a variable is obtained by

```
diff(f(x,y),y)
```

to get $\partial f(x, y)/\partial y$. Second derivatives simply repeat the variable:

```
diff(f(x,y),y,y)
```

is $\partial^2 f/\partial y^2$.

To get an integral, use

```
int(f(x), x=a..b)
```

to obtain $\int_a^b f(x)dx$. Use

```
int(g(x,y),x=a..b, y=c..d)
```

for $\int_a^b \int_c^d g(x, y)\, dy\, dx$. If Maple cannot get an exact answer, you may use

```
evalf(int(f(x),x=a..b))
```

to evaluate it numerically.

Whenever you get an expression that is complicated, use `simplify(%)` to simplify it. If you want the expression expanded, use `expand(%)`, or factored `factor(%)`.

Plotting Functions. Maple has a very powerful graphing capability. The full range of plotting capabilities is accessed by using the command `with(plots):` and `with(plottools):`. To get a straightforward plot of the function $f(x)$, use

```
plot(f(x),x=a..b)
```

or, for a three-dimensional (3d) plot

```
plot3d(f(x,y),x=a..b,y=c..d).
```

There are many options for the display of the plots which can be accessed by typing `?plot/options`.

To plot **more than one curve on the same graph**, use

```
fplot:=plot(f(x),x=a..b):
gplot:=plot(g(x),x=c..d):
display(fplot,gplot);
```

Matrices and Vectors. A matrix is entered by first loading the set of Maple commands dealing with linear algebra: `with(LinearAlgebra):` and then using

```
A:=Matrix([[row 1],[row 2],...,[row n]]);
```

It is a list of lists. Individual elements of A are obtained using $A[i,j]$.

The commands to manipulate matrices are

- `A+B` or `MatrixAdd(A,B)` adds element by element $a_{ij} + b_{ij}$.
- `c*A` gives ca_{ij}.
- `A.B` is the product of A and B assuming that the number of rows of B is the same as the number of columns of A.
- `Transpose(A)` gives A^T.
- Either `Inverse(A)` or simply `A^(-1)` gives A^{-1} for a square matrix which has an inverse, and can be checked by calculating `Determinant(A)`.
- `ConstantMatrix(-1,3,3)` defines a 3×3 matrix in which each element is -1.

Suppose now that A is a 3×3 matrix. If you define a vector in Maple by `X:=<x1,x2,x3>`, Maple writes this as a 3×1 column matrix. So both `Transpose(X).A` and `A.X` are defined. You can always find the dimensions of a matrix using `RowDimension(A)` and `ColDimension(A)`.

Some Commands Used in This Book. We will briefly describe the use of some specialized commands used throughout the book.

1. Given a game with matrix A, the upper value v^+ and lower value v^- are calculated with the respective commands

```
vu:=min(seq(max(seq(A[i,j],i=1..rows)),j=1..cols));
```

and

```
vl:=max(seq(min(seq(A[i,j],j=1..cols)),i=1..rows));
```

The command `seq(A[i,j],i=1..rows)` considers for each fixed jth column each element of the ith row of the matrix A. Then `max(seq)` finds the largest of those.

2. To solve a system of equations, we use

```
eqs:={y1+2*y2+3*y3-v=0,3*y1+y2+2*y3-v=0,
                2*y1+3*y2+y3-v=0,y1+y2+y3-1=0};
solve(eqs,[y1,y2,y3,v]);
```

Define the set of equations using $\{...\}$ and solve for the list of variables using $[...]$.

3. To invoke the simplex method and solve a linear programming problem load, the package `with(simplex)`: and then the command

```
minimize(objective,constraints,NONNEGATIVE)
```

minimizes the objective function, subject to the constraints, along with the constraint that the variables must be nonnegative. This returns exact solutions if there are any. You may also use the `Optimization` package, which picks the best method to solve: `with(Optimization)` loads the package, and then the command `Minimize(obj,cnsts,assume=nonnegative);` solves the problem.

4. The `Optimization` package is used to solve nonlinear programming problems as well. In game theory the problems can be solved using either `QPSolve`, which solves quadratic programming problems, or `NLPSolve`, which numerically solves general nonlinear programs. For example, we have used either the command

```
QPSolve(objective,Cnst,assume=nonnegative,maximize,
                                initialpoint=({q=1,p=2}));
```

or the command

```
NLPSolve(objective,Cnst,assume=nonnegative,maximize);
```

to solve problems. The search for additional solutions can be performed by setting the variable `initialpoint`.

5. To substitute a specific value into a variable use the command

```
subs(variable=value,expression).
```

We use this in several places, one of which is in evaluating the core of a cooperative game. For example

```
cnsts:={-x1<=z,-x2<=z,-(5/2-x1-x2)<=z,2-x1-x2<=z,
              1-x2-(5/2-x1-x2)<=z,-x1-(5/2-x1-x2)<=z};
Core:=subs(z=0,cnsts);
```

gives

$$Core = \{-x_1 \leq 0, -x_2 \leq 0, -\left(\frac{5}{2} - x_1 - x_2\right) \leq 0, 2 - x_1 - x_2 \leq 0, ...\}.$$

6. A region in the plane defined by inequalities are plotted using the command

```
> with(plots):
> inequal(Core,x1=0..2,x2=0..3,
             optionsfeasible=(color=red),
             optionsopen=(color=blue,thickness=2),
             optionsclosed=(color=green, thickness=3),
             optionsexcluded=(color=yellow));
```

The region Core is plotted. The set of points that satisfy the inequalities are in red, and the set of points that violate at least one inequality are colored yellow. The boundary of the feasible set is drawn in different colors depending on whether the inequality is strict or not. Strict inequalities are blue, and closed inequalities are drawn green.

7. A system of linear inequalities may be solved, actually reduced, to a minimal set of inequalities using the package with(SolveTools:-Inequality). For example

```
> with(SolveTools:-Inequality):
> cnsts:={x+y<=1,y>=3*x-20,x<=3/4,y>=-1,x>=0};
> glc:=LinearMultivariateSystem(cnsts,[x,y]);
```

gives the output $glc := \{0 \leq x \leq \frac{3}{4}, x + y \leq 1, y \geq -1\}$.

8. To plot a polygon connecting a list of points, we use

```
> with(plottools):
> pure:=([[2,1],[-1,-1],[-1,-1],[1,2]]);
> pp:=pointplot(pure);
> pq:=polygon(pure,color=yellow);
> display(pp,pq)
```

This plots the points pure and the polygon with the points as the vertices, and then colors the interior of the polygon yellow. To plot a sequence of points generated from two functions first define the points using

```
> points:={seq(seq([f(x,y),g(x,y)],x=0..1,0.05),y=0..1,0.05)}:
```

This is on the square $(x, y) \in [0,1] \times [0,1]$ with step size 0.05 for each variable. This produces 400 points (20 x values and 20 y values). Notice the colon at the end of this command, which suppresses the actual display of all 400 points. Then `pointplot(points);` produces the plot.

9. In order to plot the solution of a system of ordinary differential equations we need the package `with(DEtools)` and `with(plots)`. For instance consider the commands

```
> ode:={D(p1)(t)=p1(t)*(p2(t)-p3(t)),
        D(p2)(t)=p2(t)*(-p1(t)+p3(t)),
        D(p3)(t)=p3(t)*(p1(t)-p2(t))};
> subs(p3=1-p1-p2,ode);
> G:=simplify(%);

> DEplot3d({G[1],G[2]},{p1(t),p2(t)},t=0..20,
            [[p1(0)=0.1,p2(0)=0.2],[p1(0)=0.6,
                        p2(0)=.2],[p1(0)=1/3,p2(0)=1/3]],
          scene=[t,p1(t),p2(t)],
          stepsize=.1,linecolor=t,
          title="Cycle around the steady state");
```

The first statement assigns the system of differential equations to the set named ode. The `subs` command substitutes p3=1-p1-p2 in the set ode. `G:=simplfy(%)` puts the result of simplifying the previous substitution into G. There are now exactly two equations into the set G and they are assigned to $G[1]$ and $G[2]$. The `DEplot3d` command numerically solves the system $\{G[1], G[2]\}$ for the functions $[p_1(t), p_2(t)]$ on the interval $0 \le t \le 20$ and plots the resulting curves starting from the different initial points listed in three dimensions.

APPENDIX D
THE MATHEMATICA COMMANDS

In this appendix we will translate the primary Maple commands we used throughout the book to Mathematica version 5.2.

The Upper and Lower Values of a Game. We begin by showing how to use Mathematica to calculate the lower value

$$v^- = \max_{1 \le i \le n} \min_{1 \le j \le m} a_{i,j}$$

and the upper value

$$v^+ = \min_{1 \le j \le m} \max_{1 \le i \le n} a_{i,j}.$$

of the matrix A.

Enter the matrix

```
A={{1,4,7},{-1,3,5},{2,-6,1.4}}
```

```
rows = Dimensions[A][[1]]
```

```
cols = Dimensions[A][[2]]
  a = Table[Min[A[[i]]], {i, rows}]
    b = Max[a[[]]]
      Print["The lower value of the game is= ", b]
      c = Table[Max[A[[All, j]]], {j, cols}]
        d = Min[c[[]]]
        Print["The upper value is=", d]
```

These commands will give the upper and lower values of A. Observe that we do not need to load any packages and we do not need to end a statement with a semicolon. On the other hand, to execute a statement we need to push Shift-Enter at the same time.

The Value of an Invertible Matrix Game with Mixed Strategies. The value and optimal strategies of a game with an invertible matrix (or one which can be made invertible by adding a constant) are calculated in the following. The formulas we use are

$$v(A) = \frac{1}{J_n A^{-1} J_n^T},$$

and

$$X = v(A)(J_n A^{-1}), \text{ and } Y = v(A)(A^{-1} J_n^T).$$

If these are legitimate strategies, then the whole procedure works; that is, the procedure is actually calculating the saddle point and the value:

```
A = {{4, 2, -1}, {-4, 1, 4}, {0, -1, 5}}
         {{4, 2, -1}, {-4, 1, 4}, {0, -1, 5}}
Det[A]
     72
A1 = A + 3
     {{7, 5, 2}, {-1, 4, 7}, {3, 2, 8}}
B=Inverse[A1]
     {{2/27, -4/27,1/9}, {29/243, 50/243, -17/81},
      {-14/243,1/243,11/81}}
J= {1, 1, 1}
     {1, 1, 1}
v = 1/J.B.J
     81/19
X =v*(J.B)
     {11/19, 5/19, 3/19}
Y = v*(B.J)
     {3/19, 28/57,20/57}
```

At the end of each statement, having pushed shift+enter, we are showing the Mathematica result. Remember that you have to subtract the constant you added to the matrix at the beginning in order to get the value for the original matrix. So the actual value of the game is $v(A) = \frac{81}{19} - 3 = \frac{24}{19}$.

Solving Matrix Games by Linear Programming In this subsection we will present the use of Mathematica to solve a game using the two methods of using linear programming.

Method 1. The Mathematica command

```
LinearProgramming[c, m, b]
```

finds a vector x that minimizes the quantity $c.x$ subject to the constraints $m.x \geq b, x \geq 0$, where m is a matrix and b is a vector. This is the easiest way to solve the game.

Recall that for the matrix

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 1 & 3 \\ 4 & 6 & 1 \end{bmatrix}$$

player I's problem becomes

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_{\mathrm{I}} = p_1 + p_2 + p_3 \ (= 1/v) & \\ \text{subject to} & \\ 2p_1 + 6p_2 + 4p_3 & \geq 1 \\ 5p_1 + p_2 + 6p_3 & \geq 1 \\ 4p_1 + 3p_2 + p_3 & \geq 1 \\ p_i \geq 0 & i = 1, 2, 3. \end{cases}$$

After finding the p_i values, we will set

$$v = \frac{1}{p_1 + p_2 + p_3}.$$

Then $x_i = vp_i$ will give the optimal strategy for player I. Player II's problem is

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{\mathrm{II}} = q_1 + q_2 + q_3 \ (= 1/v) & \\ \text{subject to} & \\ 2q_1 + 5q_2 + 4q_3 & \leq 1 \\ 6q_1 + q_2 + 3q_3 & \leq 1 \\ 4q_1 + 6q_2 + q_3 & \leq 1 \\ q_j \geq 0 & j = 1, 2, 3. \end{cases}$$

The Mathematica solution of these is given as follows:

```
The matrix is:
A={{2,5,4},{6,1,3},{4,6,1}}
c={1,1,1}
b={1,1,1}

PlayI=LinearProgramming[c,Transpose[A],b]
      {21/124, 13/124, 1/124}
PlayII=LinearProgramming[-b,A,{{1,-1},{1,-1},{1,-1}}]
      \frac{13/124, 10/124, 12/124}

v=1/(21/124+13/124+1/124)
124/35

Y=v*PlayII
{13/35, 10/35, 12/35}

X=v*PlayI

{21/35, 13/35, 1/35}
```

The value of the original game is $v - 4 = -\frac{16}{35}$. In `PlayII` the constraints must be of the form $\leq$. The way to get that in Mathematica is to make each constraint a pair where the second number is either 0, if the constraint is $=$; -1, if the constraint is $\leq$; and 1, if the constraint is $\geq$. The default is 1 and that is why we don't need the pair in `PlayI`.

It is also possible and sometimes preferable to use the Mathematica function `Minimize`, or `Maximize`. For this problem the commands become:

For player I,

```
Minimize[{x + y + z, 2x + 6y + 4z >= 1, 5x + y + 6z >= 1,
            4x + 3y + z >= 1,x>=0,y>=0,z>=0}, {x, y, z}]

  {35/124,{x->21/124,y->13/124,z->1/124}}
```

For player II,

```
Maximize[{x + y + z, 2x + 5y + 4z <= 1,
                    6x + y + 3z <= 1, 4x + 6y + z <= 1,
                    x >= 0, y >= 0, z >= 0}, {x, y, z}]

  {35/124,{x->13/124,y->5/62,z->3/31}}
```

You may also use these commands with matrices and vectors.

Method 2. Player II's problem is

$$\text{Minimize } v \text{ subject to } {}_iAY^T \leq v, 1 \leq i \leq n, \sum_{j=1}^{m} y_j = 1,\ y_j \geq 0.$$

player I's problem is

$$\text{Maximize } v \text{ subject to } XA_j \geq v, 1 \leq j \leq m, \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0.$$

For example, if we start with the game matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$

we may solve player I's problem using the Mathematica command

```
Maximize[{v, y - z >= v, -x + z >= v, x - y >= v, x + y + z == 1,
          x >= 0,y >= 0, z >=0}, {x, y, z, v}]
```

This gives the output $\{x = \frac{1}{3}, y = \frac{1}{3}, z = \frac{1}{3}, v = 0\}$. Similarly, you can use `Minimize` to solve player II's program.

We may also use the `LinearProgramming` function in matrix form, but now it becomes more complicated because we have an equality constraint as well as the additional variable v.

Here is an example. Take the Colonel Blotto game

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{bmatrix}.$$

The Mathematica command

```
LinearProgramming[c,m,b]
```

solves the following program: Minimize $c \cdot x$ subject to $mx \geq b, x \geq 0$. To use this command for player II, our coefficient variables are $c = (0, 0, 0, 0, 1)$ for the objective function $z_{II} = 0y_1 + 0y_2 + 0y_3 + 0y_4 + 1v$. The matrix m is

$$m = \begin{bmatrix} 4 & 0 & 2 & 1 & -1 \\ 0 & 4 & 1 & 2 & -1 \\ 1 & -1 & 3 & 0 & -1 \\ -1 & 1 & 0 & 3 & -1 \\ -2 & -2 & 2 & 2 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The last column is added for the variable v. The last row is added for the constraint $y_1 + y_2 + y_3 + y_4 = 1$. Finally, the vector b becomes

$$b = \{\{0,-1\},\{0,-1\},\{0,-1\},\{0,-1\},\{0,-1\},\{1,0\}\}.$$

The first number, 0, is the actual constraint, and the second number makes the inequality $\leq$ instead of the default $\geq$. The pair $\{1,0\}$ in b comes from the equality constraint $y_1 + y_2 + y_3 + y_4 = 1$ and the second zero converts the inequality to equality. So here it is:

```
m = {{4, 0, 2, 1, -1}, {0, 4, 1, 2, -1}, {1, -1, 3, 0, -1},
    {-1, 1, 0, 3, -1}, {-2, -2, 2, 2, -1}, {1, 1, 1, 1, 0}}
c = {0, 0, 0, 0, 1}
b = {{0, -1}, {0, -1}, {0, -1}, {0, -1}, {0, -1}, {1, 0}}
LinearProgramming[c, m, b]
```

Mathematica gives the output

$$\{y_1 = \frac{7}{90}, y_2 = \frac{1}{30}, y_3 = \frac{16}{45}, y_4 = \frac{8}{15}, v = \frac{14}{9}\}.$$

The solution for player I is found similarly, but we have to be careful because we have to put it into the form for the `LinearProgramming` command in Mathematica, which always minimizes and the constraints in Mathematica form are $\geq$. Player I's problem is a maximization problem so the cost vector is

$$c = (0,0,0,0,0,-1)$$

corresponding to the variables $x_1, x_2, \ldots, x_5$, and $-v$. The constraint matrix is the transpose of the Blotto matrix but with a row added for the constraint $x_1 + \cdots + x_5 = 1$. It becomes

$$q = \begin{bmatrix} 4 & 0 & 1 & -1 & -2 & -1 \\ 0 & 4 & -1 & 1 & -2 & -1 \\ 2 & 1 & 3 & 0 & 2 & -1 \\ 1 & 2 & 0 & 3 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, the b vector becomes

$$b = \{\{0,1\},\{0,1\},\{0,1\},\{0,1\},\{1,0\}\}.$$

In each pair, the second number 1 means that the inequalities are of the form $\geq$. In the last pair, the second 0 means the inequality is actually $=$. For instance, the first and last constraints are, respectively

$$4x_1 + 0x_2 + x_3 - x_4 - 2x_5 - v \geq 0 \text{ and } x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

With this setup the Mathematica command to solve is simply

```
LinearProgramming[c, q, b]
```

with output

$$\{x_1 = \frac{4}{9}, x_2 = \frac{4}{9}, x_3 = 0, x_4 = 0, x_5 = \frac{1}{9}, v = \frac{14}{9}\}$$

Interior Nash Points. The system of equations for an interior mixed Nash equilibrium are given by

$$\sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0, \qquad k = 1, 2, \ldots, n-1$$

$$\sum_{i=1}^{n} x_i[b_{is} - b_{im}] = 0, \qquad s = 1, 2, \ldots, m-1.$$

$$x_n = 1 - \sum_{i=1}^{n-1} x_i \text{ and } y_m = 1 - \sum_{j=1}^{m-1} y_j.$$

The game has matrices (A, B) and the equations, if they have a solution that are strategies, will yield $X^* = (x_1, \ldots, x_n)$ and $Y^* = (y_1, \ldots, y_m)$.

The following example shows how to set up and solve using Mathematica:

```
A = {{-2, 5, 1}, {-3, 2, 3}, {2, 1, 3}}
B = {{-4, -2, 4}, {-3, 1, 4}, {3, 1, -1}}
rowdim = Dimensions[A][[1]]
coldim = Dimensions[B][[2]]
Y = Array[y, coldim]
X= Array[x, rowdim]
EQ1 - Table[Sum[y[j](A[[k, j]]-A[[rowdim, j]]),
              {j, coldim}], {k, rowdim-1}]
EQ2 = Table[Sum[x[i](B[[i, s]]-B[[i, coldim]]),
              {i, rowdim}], {s, coldim-1}]
Solve[{EQ1[[1]] == 0, EQ1[[2]] == 0, y[1] + y[2] + y[3] == 1},
          {y[1],y[2],y[3]}]
Solve[{EQ2[[1]] == 0, EQ2[[2]] == 0, x[1] + x[2] + x[3] == 1},
          {x[1],x[2],x[3]}]
```

Mathematica gives the output

$$y[1] = \frac{1}{14}, y[2] = \frac{5}{14}, y[3] = \frac{8}{14}, \text{ and } x[1] = \frac{1}{14}, x[2] = \frac{2}{7}, x[3] = \frac{9}{14}.$$

Lemke–Howson Algorithm for Nash Equilibrium A Nash equilibrium of a bimatrix game with matrices (A, B) is found by solving the nonlinear programming problem

$$\text{Maximize } f(X, Y, p, q) = XAY^T + XBY^T - p - q$$

subject to

$$AY^T \le p, XB \le q, \sum_i x_i = 1, \sum_j y_j = 1, X \ge 0, Y \ge 0.$$

Here is the use of Mathematica to solve for a particular example:

```
A = {{-1, 0, 0}, {2, 1, 0}, {0, 1, 1}}
B = {{1, 2, 2}, {1, -1, 0}, {0, 1, 2}}
f[X_] = X.B
g[Y_] = A.Y
NMaximize[{f[{x, y, z}].{a, b, c} + {x, y, z}.g[{a, b, c}] - p - q,
 f[{x, y, z}][[1]] - q <= 0,
 f[{x, y, z}][[2]] - q <= 0, f[{x, y, z}][[3]] - q <= 0,
 g[{a, b, c}][[1]] - p <= 0, g[{a, b, c}][[2]] - p <= 0,
 g[{a, b, c}][[3]] - p <= 0, x + y + z == 1, x >= 0, y >= 0, z >= 0,
 a >= 0, b >= 0, c >= 0, a + b + c == 1}, {x, y, z, a, b, c, p, q}]
```

The command `NMaximize` will seek a maximum numerically rather than symbolically. This is the preferred method for solving using the Lemke–Howson algorithm because of the difficulty of solving optimization problems with constraints. This command produces the Mathematica output

$$\begin{aligned}
&4.1389 \times 10^{-10}, \{a = 0.333, b = 0.0, c = 0.666\},\\
&p = 0.666, q = 0.666,\\
&\{x = -2.328 \times 10^{-10}, y = 0.666, z = 0.333\}.
\end{aligned}$$

The first number verifies that the maximum of f is indeed zero. The optimal strategies for each player are $Y^* = (\frac{1}{3}, 0, \frac{2}{3}), X^* = (0, \frac{2}{3}, \frac{1}{3})$. The expected payoff to player I is $p = \frac{2}{3}$, and the expected payoff to player II is $q = \frac{2}{3}$.

Is the Core Empty? A simple check to determine if the core is empty uses a simple linear program. The core is

$$C(0) = \{\vec{x} = (x_1, \dots, x_n) \mid \sum_{i=1}^{n} x_i = 1, x_i \ge 0, v(S) - \sum_{i \in S} x_i \le 0, \forall S \subsetneq N\}.$$

Convert to a linear program

$$\begin{aligned}
&\text{Minimize } x_1 + \cdots + x_n\\
&\text{subject to } v(S) - \sum_{i \in S} x_i \le 0, \ \forall S \subsetneq N.
\end{aligned}$$

If the solution of this program produces a minimum of $x_1 + \cdots + x_n > v(N)$, we know that the core is empty; otherwise it is not. For example, if $N = \{1, 2, 3\}$, and

$$v(i) = 0, v(12) = 105, v(13) = 120, v(23) = 135, v(123) = 150,$$

we may use the Mathematica command

```
Minimize[{x1+x2+x3,x1+x2>=105,x1+x3>=120,
         x2+x3>=135,x1>=0,x2>=2},{x1,x2,x3}].
```

Mathematica tells us that the minimum of $x_1 + x_2 + x_3$ is $180 > v(123) = 150$. So $C(0) = \emptyset$. The other way to see this is with the Mathematica command

```
LinearProgramming[{1, 1, 1}, {{1, 1, 0}, {1, 0, 1},
                              {0, 1, 1}}, {105, 120,135}]
```

which gives the output $\{x_1 = 45, x_2 = 60, x_3 = 75\}$ and minimum 180. The vector $c = (1, 1, 1)$ is the coefficient vector of the objective $c \cdot (x_1, x_2, x_3)$; the matrix

$$m = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is the matrix of constraints of the form $\geq b$; and $b = (105, 120, 135)$ is the right hand side of the constraints.

Find and Plot the Least Core. The ε-core is

$$C(\varepsilon) = \{\vec{x} = (x_1, \ldots, x_n) \mid v(S) - \sum_{i \in S} x_i \leq \varepsilon, \ \forall\, S \subsetneq N\}.$$

You need to find the smallest $\varepsilon^1 \in \mathbb{R}$ for which $C(\varepsilon^1) \neq \emptyset$.

For example, we use the characteristic function

```
v1 = 0; v2 = 0; v3 = 0;
v12 = 2; v13 = 0; v23 = 1; v123 = 5/2;
```

and then the Mathematica command

```
Minimize[{z, v1 - x1 <= z, v2 - x2 <= z, v3 - x3 <= z,
         v12 - x1 - x2 <= z, v13 - x1 - x3 <= z, v23 - x2 - x3 <= z
         x1 + x2 + x3 == v123}, {z, x1, x2, x3}]
```

This gives the output

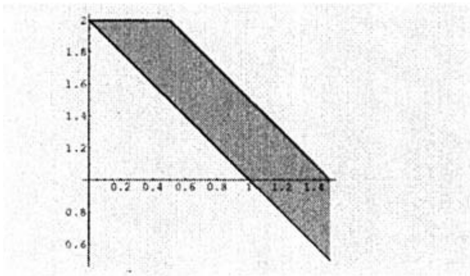
$$z = -\frac{1}{4},\ x_1 = \frac{5}{4},\ x_2 = 1, x_3 = \frac{1}{4}.$$

The important quantity is $z = \varepsilon^1 = -\frac{1}{4}$ because we do not know yet whether $C(\varepsilon^1)$ contains only one point. At this stage, we know that $C(-\frac{1}{4}) \neq \emptyset$, and if we take any $\varepsilon < -\frac{1}{4}$, then $C(\varepsilon) = \emptyset$.

Here is how you find the least core and get a plot of $C(0)$ in Mathematica:

```
Core[z]:= {v1 - x1 <= z, v2 - x2 <= z, v3 - x3 <= z,
           v12 - x1 - x2 <= z, v13 - x1 - x3 <= z,
           v23 - x2 - x3 <= z, x1 + x2 + x3 == v123}

A[z]:= {z, Core[z]} Minimize[A[z], {z, x1, x2, x3}]

Output:-1/4, {z -> -1/4, x1 -> 5/4, x2 -> 1, x3 -> 1/4}

S = Simplify[Core[z] /. {z -> 0, x3 -> v123 - x1 - x2}]

      Substitute z=0, x3=v123-x1-x2 in C[z]

<< Algebra'InequalitySolve'
      Load the InequalitySolve package

<< Graphics'InequalityGraphics'
      Load the InequalityGraphics package.

InequalitySolve[S, {x1, x2}]
      Output:  0 <= x1 <= 3\/2 && 2 - x1 <= x2 <= 1/2(5 - 2 x1)
      Solve inequalities in C[0] to see if one point.

InequalityPlot[S, {x1, -2, 2}, {x2, -2, 2}]
      Plot the core C[0] (see below)

F = Simplify[Core[z] /. {z -> -1/4, x3 -> v123 - x1 - x2}]
      Substitute z=-1/4 for least core.

InequalitySolve[F, {x1, x2}]

      Output:  1/4 <= x1 <= 5/4 && x2 == 1/4(9 - 4 x1)
      Solve inequalities in C[-1/4] to see if one point.

Plot[(9 - 4 x1)/4, {x1, 1/4, 5/4}]
      C[-1/4] is a line segment.
```

Here is $C(0)$ for the example, generated with Mathematica:



Nucleolus Procedure. This section will give the Mathematica procedure to find the nucleolus. This is presented as a three-player example. The Mathematica commands are interspersed with output, and is self documenting.

```
Clear the variables:
Clear[x1,x2,x3,v1,v2,v3,v12,v13,v23,v123,z,w,S,S2,A,Core]
Define the characteristic function:
v1=0;v2=0;v3=0;v12=1/3;v13=1/6;v23=5/6;v123=1;

Core[z]:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-x1-x2<=z,
          v13-x1-x3<=z,v23-x2-x3<=z, x1+x2+x3==v123}

A[z]:={z,Core[z]}

Minimize[A[z],{z,x1,x2,x3}]

Output:-1/12, {z=-1/12, x1=1/12, x2=1/3, x3=7/12}

S=Simplify[Core[z]/.{z->-1/12,x3->v123-x1-x2}]

<<Algebra'InequalitySolve'

<<Graphics'InequalityGraphics'

InequalitySolve[S,{x1,x2}]

Output:
x1=1/12 && 1/3<= x2 <= 3/4

Assign the known variables.

x1=1/12;z=-1/12;x3=v123-x1-x2
```

```
Now check the excesses to see which coalitions can be dropped:

v1-x1=-1/2
v2-x2=-x2
v3-x3=-11/12+x2
v12-x1-x2=1/4-x2
v13-x1-x3=-5/6+x2
v23-x2-x3=-1/12

We may drop coalitions {1} and {23} and then recalculate

B[w]={v2-x2<=w,v3-x3<=w,v12-x1-x2<=w,
                     v13-x1-x3<=w,x1+x2+x3==v123}

Find the smallest w which makes B[w] nonempty
Minimize[{w,B[w]},{x2,w}]

Output:-7/24, {x2 = 13/24, w = -7/24}

Check to see if we are done:
S2=Simplify[B[w]/.w->-7/24]

InequalitySolve[S2,{x2}]

Output:x2= 13/24

This is the end because we now know x1=1/12,x2=13/24,x3=9/24.
```

The final result gives the nucleolus as $(x_1 = \frac{1}{12}, x_2 = \frac{13}{24}, x_3 = \frac{9}{24})$.

Plotting the Payoff Pairs. Given a bimatrix game with matrices (A, B) that are each 2×2 matrices, the expected payoff to player I is

$$E_{\mathrm{I}}(x,y) = (x, 1-x)A \begin{bmatrix} y \\ 1-y \end{bmatrix},$$

and for player II

$$E_{\mathrm{II}}(x,y) = (x, 1-x)B \begin{bmatrix} y \\ 1-y \end{bmatrix}.$$

We want to get an idea of the shape of the set of payoff pairs $(E_{\mathrm{I}}(x,y), E_{\mathrm{II}}(x,y))$. In Mathematica we can do that with the following commands:

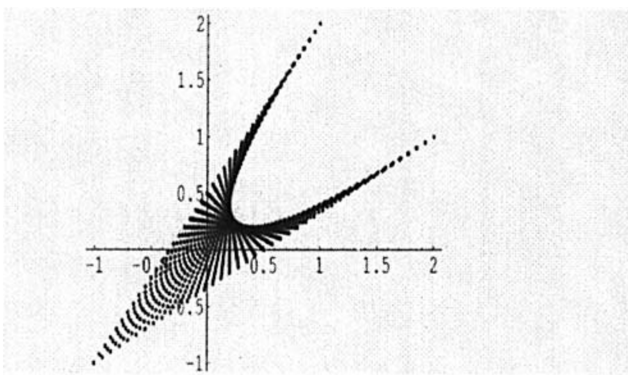
```
A = {{2, -1}, {-1, 1}}
B = {{1, -1}, {-1, 2}}
```

```
f[x_, y_] = {x, 1 - x}.A.{y, 1 - y}
g[x_, y_] = {x, 1 - x}.B.{y, 1 - y}

h[x_, y_] = {f[x, y], g[x, y]}

values = Table[Table[h[x, y], {x, 0, 1, .025}], {y, 0, 1, 0.025}];

s = Flatten[values, 1];

ListPlot[s]
```

Observe that a function is defined in Mathematica as, for example,

```
f[x_, y_] = {x, 1 - x}.A.{y, 1 - y},
```

which gives $E_I(x, y)$. Mathematica uses the symbol x_ to indicate that x is a variable. The result of these commands is the following graph:



Bargaining Solutions. We will use Mathematica to solve the following bargaining problem with bimatrix:

$$(A, B) = \begin{bmatrix} (1,4) & (-\frac{4}{3}, -4) \\ (-3, -1) & (4, 1) \end{bmatrix}$$

First we use the safety point $u^* = value(A), v^* = value(B^T)$. Here are the commands to find (u^*, v^*):

```
The matrix is:
A={{1,-4/3},{-3,4}}
```

```
player I's problem is:
Maximize[{v, {x, y}.A[[All, 1]] >= v, {x, y}.A[[All, 2]] >= v,
    x + y == 1, x >= 0, y >= 0}, {x, y, v}]

player II's problem is:
Minimize[{v, A[[1]].{x, y} <= v, A[[2]].{x, y} <= v,
    x + y == 1, x >= 0, y >= 0}, {x, y, v}]
```

This finds $u^* = value(A) = 0$, and, even though we don't need it, $X^* = (\frac{3}{4}, \frac{1}{4}), Y^* = (\frac{4}{7}, \frac{3}{7})$. For v^* we use

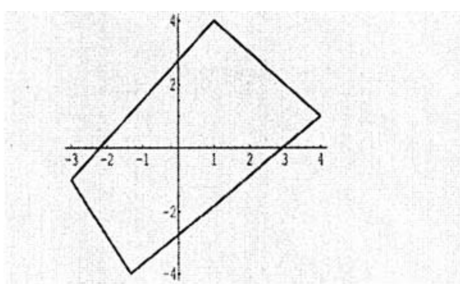
```
B = {{4, -4}, {-1, 1}}
BT = Transpose[B]
Minimize[{v, BT[[1]].{x, y} <= v, BT[[2]].{x, y} <= v,
   x + y == 1, x >= 0, y >= 0}, {x, y, v}]
```

which tells us that $v^* = value(B^T) = 0$ and $Y^* = (\frac{1}{5}, \frac{4}{5})$.

So our safety point is $(u^*, v^*) = (0, 0)$. A plot of the feasible set is found from

```
ListPlot[{{1, 4}, {-3, -1}, {-4/3, -4}, {4, 1}, {1, 4}},
                                          PlotJoined -> True]
```

which produces



You can see that the Pareto-optimal boundary is the line through $(4, 1)$ and $(1, 4)$ which has the equation given by $v - 1 = -(u - 4)$. So now the bargaining problem with safety point $(0, 0)$ is

$$\text{Maximize } g(u, v) = (u - u^*)(v - v^*) = uv$$

subject to the constraints

$$(u, v) \in S = \{v \leq -u + 5, v \leq \frac{5}{4}\,u - 4, v \geq -\frac{9}{5}\,u - \frac{32}{5}, v \geq \frac{15}{16}\,u - \frac{11}{16}\}.$$

The Mathematica command is

```
Maximize[{u v, v <= -u + 5, v <= 5/4 u + 11/4, v >= -9/5 u - 32/5,
    v >= 15/16 u - 11/16, u >= 0, v >= 0}, {u, v}]
```

and gives the output $\overline{u} = \frac{5}{2}, \overline{v} = \frac{5}{2}$ and the maximum is $g = \frac{25}{4}$. This is achieved if the two players agree to play and receive $(1, 4)$ and $(4, 1)$ exactly half the time. Then they each get $\frac{5}{2}$.

Next we find the optimal threat strategies.

First we have seen that the Pareto-optimal boundary is the line $v = -u + 5$, which has slope $m_p = -1$. So we look at the game with matrix $-m_pA - B$, or

$$A - B = \begin{bmatrix} -3 & \frac{8}{3} \\ -2 & 3 \end{bmatrix}$$

It is the optimal strategies we need for this game. However, in this case it is easy to see that there is a pure saddle point $X^t = (0, 1), Y^t = (1, 0)$. Consequently, the threat safety point is

$$u^t = X_tAY_t{}^T = -3, v^t = X_tBY_t{}^T = -1.$$

The Mathematica command to solve the bargaining problem with this safety point is

```
Maximize[{(u + 3) (v + 1), v <= -u + 5, v <= 5/4 u + 11/4,
              v >= -9/5 u - 32/5,
              v >= 15/16 u - 11/16, u >= -3, v >= -1}, {u, v}]
```

which gives the output $\overline{u} = \frac{3}{2}, \overline{v} = \frac{7}{2}$ and maximum $g = \frac{81}{4}$. In this solution player I gets less and player II gets more to reflect the fact that player II has a credible and powerful threat. So the payoffs are obtained by each agreeing to receive $(1, 4)$ exactly five-sixths of the time and payoff $(4, 1)$ exactly one-sixth of the time (player I throws player II a bone once every 6 plays).

Mathematica for Replicator Dynamics. Naturally, the solution of the replicator dynamics equations (6.2.1) depends on the matrices involved so we will only give some sample Mathematica commands to produce graphs of the vector fields and trajectories.

For example

```
Needs["Graphics'PlotField'"];
a = 2
b = 1
Here is the system we consider:
e1 = p1'[t] == p1[t] (a p1[t] - a p1[t]^2 - b p2[t]^2)
e2 = p2'[t] == p2[t] (b p2[t] - a p1[t]^2 - b p2[t]^2)
This command gives a plot of the vector field in the figure below:
```

```
field = PlotVectorField[{x (a x - a x^2 - b y^2),
        y (b y - a x^2 - b y^2)}, {x, 0, 1, 0.25}, {y, 0, 1, 0.2},
        PlotPoints -> 20, Frame -> True];
```

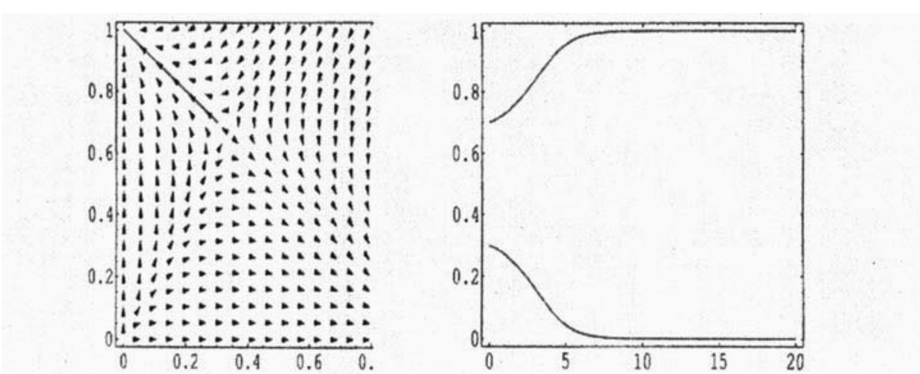
```
This command solves the system for the initial point p1[0],p2[0]:
nsoln = NDSolve[{e1, e2, p1[0] == .3,
                         p2[0] == .7}, {p1, p2}, {t, 0, 20}];
```

```
This plots the trajectory (p1[t],p2[t]) with p1 versus p2:
trajectory =
    ParametricPlot[{p1[t], p2[t]} /. nsoln[[1]], {t, 0, 20},
              PlotRange -> All,
              PlotStyle -> {Hue[1]}, Frame -> True];
```

```
This command plots the trajectory superimposed on the vector field:

Show[trajectory, field];
```

```
If you want a graph of the individual functions use:
trajectory1 = ParametricPlot[{t, p1[t]} /. nsoln[[1]],
                             {t, 0, 20}, PlotRange -> All,
                             PlotStyle -> {Hue[1]}, Frame -> True];

trajectory2 = ParametricPlot[{t, p2[t]} /. nsoln[[1]],
                             {t, 0, 20}, PlotRange -> All,
                             PlotStyle -> {Hue[1]}, Frame -> True];
Show[trajectory1, trajectory2];
```

This last command produces the following plot of both functions on one graph:



APPENDIX E
BIOGRAPHIES

JOHN VON NEUMANN. John von Neumann was born to an aristocratic family on December 28, 1903 in Budapest, Austria-Hungary and died February 8, 1957 in Washington, DC, of brain cancer. Von Neumann earned two doctorates at the same time, a PhD in mathematics from the University of Budapest, and a PhD in chemistry from the University of Zurich. He joined the faculty of the prestigious University of Berlin in 1927. Von Neumann made fundamental contributions to quantum physics, functional analysis, economics, computer science, numerical analysis, and many other fields. There was hardly any subject that came to his attention that he didn't revolutionize. Some areas, especially in mathematics, he invented. At age 19, he applied abstract operator theory, much of which he developed, to the brand new field of quantum mechanics. He was an integral member of the Manhattan Project and made fundamental contributions to the development of the hydrogen bomb and the development of the Mutual Assured Destruction policy of the United States. In 1932 he was appointed one of the original and youngest permanent members of the

Institute for Advanced Study at Princeton New Jersey (along with A. Einstein) and helped to make it the most prestigious research institute in the world.

The von Neumann minimax theorem was proved in 1928 and was a major milestone in the theory of games. Von Neumann continued to think about games and wrote the classic *Theory of Games and Economic Behavior* [26] (written with economist Oskar Morgenstern [1]) in 1944. It was a breakthrough in the development of economics using mathematics and in the mathematical theory of games. His contributions to pure mathematics fills volumes. The cleverness and ingenuity of his arguments amaze mathematicians to this day.

Von Neumann was one of the most creative mathematicians of the twentieth century. In a century in which there were many geniuses and many breakthroughs, von Neumann contributed more than his fair share. He ranks among the greatest mathematicians of all time for his depth, breadth, and scope of contributions. In addition, and perhaps more importantly, von Neumann was famous for the parties he hosted throughout his lifetime and in the many places he lived. He was an aristocratic *bon vivant* who managed several careers even among the political sharks of the cold war era without amassing enemies. He was well liked by all of his colleagues and lived a contributory life.

If you want to read a very nice biography of John von Neumann read MacRae's excellent book [14].

JOHN FORBES NASH. John Forbes Nash, Jr. was born June 13, 1928 in Bluefield, West Virginia. He was awarded the Nobel Prize in Economics (formally the 1994 Bank of Sweden Prize in Economic Sciences), which he shared with the mathematical economists and game theorists Reinhard Selten and John Harsanyi. This is the most prestigious prize in economics but certainly not the only prize won by Nash. In 1978 Nash was awarded the John Von Neumann Theory Prize for his invention of noncooperative equilibria, now called **Nash equilibria** and in 1999 Nash was awarded the Leroy P. Steele Prize by the American Mathematical Society.

Nash continues to work on game theory, and his contributions to the theory of games has certainly been as profound as that of von Neumann and others. Like von Neumann, Nash is truly a pure mathematician with a very creative, penetrating, and inquisitive mind, with fundamental contributions to differential geometry, global analysis, and partial differential equations. Despite the power and depth of his thinking and mathematical ability, between 1945 and 1996, he published only 23 papers (most of which contain major and fundamental results). The primary reason for the unproductive period in his life is the illness that he suffered, and suffers to

[1] Born January 24, 1902, in Germany and died July 26, 1977, in Princeton, NJ. Morgenstern's mother was the daughter of the German emperor Frederick III. He was a professor at the University of Vienna when he came to the United States on à Rockafellar Foundation fellowship. In 1938, while in the US, he was dismissed from his post in Vienna by the Nazis and became a professor of economics at Princeton University where he remained until his death.

this day, described in his autobiography[2] written for the Nobel award. John Nash's life was dramatized in the movie *A Beautiful Mind* in which Russell Crowe played the lead, based on the book by Sylvia Nasar [18].

Nash is currently a Senior Research Mathematician at Princeton.

For further information about Nash, read the book by Nasar [18] and see the movie.

[2] *Les Prix Nobel. The Nobel Prizes 1994*, edited by Tore Frängsmyr (Nobel Foundation), Stockholm, 1995. You can read it and the biographies of R. Selten and J. Harsanyi online at www.nobelprize.org/nobel_prizes/economics/laureates/1994/nash-autobio.html.

PROBLEM SOLUTIONS

SOLUTIONS FOR CHAPTER 1

1.1 $v^+ = 1, v^- = -1$. No saddle point in pure strategies.

1.2 $v^+ = 1,\ v^- = -1$, no pure saddle point.

1.3 $A = \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$ $v^+ = 2, v^- = -2$, no pure optimal strategies.

1.4

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3}, & -\frac{1}{36} & -\frac{1}{36} \\ -\frac{3}{2} & 0 & -\frac{3}{2} & 0 \end{bmatrix}, \quad v^- = -\frac{1}{36} = v^+$$

Saddle at row 1, column 3. Even if player I takes the entire pot there will be a saddle at row 1, column 3, both players should spin.

1.5 Saddle for both at payoff 1 for any value of x.

1.7 $v = 0$ at (n, n).

1.8

	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

1.9 $v^+ = 1, v^- = 0$ because there is always at least one 1 and one 0 in each row and column.

1.10 Any $0 < x \le 3$.

1.11 $v^+ = 1, v^- = 1$.

1.12 $v^+ = 0, v^- = 0$.

1.13 $v^+ = 1, v^- = 0$. Here's why. For $v^- = \max_x \min_y (x-y)^2$, y can be chosen to be $y = x$ to get a minimum of zero. For $v^+ = \min_y \max_x (x-y)^2$, x wants to be as far away from y as possible. So, if $y < 0$, then $x = 1$, and if $y > 0$, then $x = -1$, so

$$\max_x (x-y)^2 = \begin{cases} (1+y)^2 & \text{if } y > 0; \\ (1-y)^2 & \text{if } y \le 0. \end{cases}$$

The minimum of this over $y \in [-1, 1]$ is 1, so $v^+ = 1$. You can see this with the Maple commands

```
> f:=y->piecewise(y<0,(1-y)^2,y>=0,(1+y)^2);
> plot(f(y),y=-1..1,view=[-1..1,0..3]);
```

1.17 Use the definitions of $y^* = \varphi(x^*)$ and $x^* = \psi(y^*)$.

1.18 Since player II wants to guarantee that player I gets the smallest maximum, look at the line that is the highest and then choose the y^* that gives the smallest payoff. The point of intersection of the two lines is where the y^* will be and the corresponding vertical coordinate will be the value of the game.

1.19 **(b)** The matrix has a saddle at 5, so it won't be interior. The method gives $x = -\frac{1}{2}$, which is bogus.

1.20 $v(A) = v(B) = 1$ and $v(A) + v(B) = 2$. Now pick $z = 3$ and $z = -1$ to show that $v(A + B) \ne v(A) + v(B)$. This shows that the value of a sum of matrices is not necessarily the value of the games with the individual matrices.

1.21 Column 2 may be eliminated by dominance: Any $\frac{9}{13} \le \lambda \le \frac{3}{4}$ will make

$$\begin{aligned} 13\lambda + 8(1-\lambda) &\le 29 \\ 18\lambda + 31(1-\lambda) &\le 22 \\ 23\lambda + 19(1-\lambda) &\le 22. \end{aligned}$$

Once column 2 is gone, row 1 may be dropped. Then $X^* = (0, \frac{4}{17}, \frac{13}{17})$ and $Y^* = (\frac{12}{17}, 0, \frac{5}{17})$.

1.23 $v(A) = 1 = E(X^*, Y^*)$, but $E(X, Y^*) = 2x$, where $X = (x, 1-x), 0 \le x \le 1$, and it is not true that $2x < v(A)$ for all x in that range.

1.24 (a) $X^* = (\frac{15}{22}, \frac{7}{22})$;(b) $Y^* = (\frac{7}{9}, \frac{2}{9})$;(c) $Y^* = (\frac{6}{10}, \frac{4}{10})$.

1.25 Any $\frac{3}{8} \le \lambda \le \frac{7}{16}$ will work for a convex combination of columns 2 and 1.

1.26 Let $\max_i b_i = b_k$. Then $\sum_i x_i b_i - b_k = \sum_i x_i(b_i - b_k) = z$ since $\sum_i x_i = 1$. Now $b_i \le b_k$ for each i, so $z \le 0$. Its maximum value is achieved by taking $x_k = 1$ and $x_i = 0, i \ne k$. Hence $\max_X \sum_i x_i b_i - b_k = 0$, which says $\max_X \sum_i x_i b_i = b_k = \max_i b_i$.

1.27 This uses $v = \min_Y \max_i E(i, Y) = \max_X \min_j E(X, j)$.

1.28 By definition of saddle

$$E(X^0, Y^*) \le E(X^*, Y^*) \le E(X^*, Y^0)$$

and

$$E(X^*, Y^0) \le E(X^0, Y^0) \le E(X^0, Y^*).$$

Now put them together.

1.29 (a) The given strategies in the first part are not optimal because $\max_i E(i, Y) = \frac{31}{9}$ and $\min_j E(X, j) = -\frac{42}{9}$.
(b) The optimal Y^* is $Y^* = (\frac{52}{99}, \frac{8}{33}, 0, \frac{23}{99})$.

1.30 $Y^* = (\frac{5}{7}, \frac{2}{7}, 0)$.

1.31 $X^* = (\frac{8}{11}, \frac{3}{11}), Y^* = (\frac{6}{11}, \frac{5}{11}), v(A) = \frac{48}{11}$.

1.32 $X^* = (\frac{2}{3}, \frac{1}{3}), Y^* = (\frac{2}{3}, \frac{1}{3}, 0, 0), v(A) = \frac{4}{3}$.The best response for player I to $Y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8})$ is $X = (0, 1)$.

1.33 Since $XA = (0.28, 0.2933, 0.27)$, the smallest of these is 0.27, so the best response is $Y = (0, 0, 1)$.

1.34 Best responses are $x(y) = (C - y)/2, y(x) = (D - x)/2$, which can be solved to give $x^* = (2C - D)/3, y^* = (2D - C)/3$.

1.35 $\max_X E(X, Y_n) = E(X_n, Y_n), \min_Y E(X_n, Y) = E(X_n, Y_{n+1}), n = 0, 1, 2 \ldots$. Then

$$E(X, Y_n) \le E(X_n, Y_n) \text{ and } E(X_n, Y_{n+1}) \le E(X_n, Y),\ \forall, X, Y.$$

Now let $n \to \infty$. Then (X', Y') should be a saddle point.

Unfortunately, since best response strategies are generally pure, it will not be the case that $X_n \to X'$. You can see that in the matrix A, regardless of what Y is initially, the sequence of best responses will not converge. For example, if $Y_0 = (0, 1)$, then $X_0 = (0, 1)$, and $Y_1 = (1, 0), X_1 = (1, 0)$, and so on. Look at the following table:

Y	Payoff	X	Payoff
$(0,1)$	0	$(0,1)$	1
$(1,0)$	0	$(1,0)$	2
$(0,1)$	0	$(0,1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

It seems that the **percentage of the time** that player I will get a payoff of 2 is $\frac{1}{3}$. Calculate $X_n = \frac{1}{n}\sum_{k=0}^{n} X_k$, and $Y_n = \frac{1}{n}\sum_{k=0}^{n} Y_k$, for various values of n and see if those sequences converge to X^*, Y^*.

SOLUTIONS FOR CHAPTER 2

2.1 $X^* = (\frac{10}{17}, \frac{7}{17}), Y^* = (\frac{8}{17}, \frac{9}{17})$ and $v(A) = \frac{80}{17} = 4.7059$.

2.2 $X^* = (\frac{12}{19}, \frac{7}{19}), Y^* = (\frac{8}{19}, \frac{11}{19}), v(A) = \frac{96}{19} = 5.0526$. Since $\frac{7}{19} = 0.3684 < \frac{7}{17} = 0.4118$, we see that the passing percentage goes down even though the payoff goes up.

2.3 $X^* = (\frac{2}{3}, \frac{1}{3}), Y^* = (\frac{2}{5}, \frac{3}{5}), v(A) = 4$. The defense will guard against the pass more.

2.5 **(a)** $X^* = (\frac{15}{22}, \frac{7}{22}), Y^* = (\frac{9}{22}, \frac{13}{22}), v = -\frac{3}{22}$.

(b) $X^* = (\frac{23}{114}, \frac{91}{114}), Y^* = (\frac{31}{38}, \frac{7}{38}), v = \frac{941}{38}$.

(c) $X^* = (\frac{101}{137}, \frac{36}{137}), Y^* = (\frac{31}{137}, \frac{106}{137}), v = -\frac{568}{137}$.

2.6 One possible example is $A = \begin{bmatrix} 8 & 9 \\ 6 & 2 \end{bmatrix}$, which obviously has a saddle at payoff 8, so that the actual optimal strategies are $X^* = (1, 0), Y^* = (1, 0), v(A) = 8$. However, if we simply apply the formulas, we get the nonoptimal alleged solution $X^* = (\frac{4}{5}, \frac{1}{5}), Y^* = (\frac{7}{5}, -\frac{2}{5})$ which is completely incorrect.

2.7 Consider cases. If $a_{12} > a_{22}$, then because $a_{11} + a_{22} = a_{12} + a_{21}$, it must be that $a_{11} > a_{21}$ so there is a saddle point in the first row, and so on.

2.9 **(a)** Matrix has a saddle at row 1, column 2; formulas do not apply.

(b) $X^* = (\frac{1}{2}, 0, \frac{1}{2}), Y^* = (\frac{1}{5}, \frac{9}{20}, \frac{7}{20}), v = \frac{7}{2}$. There is another optimal $Y^* = (\frac{1}{2}, 0, \frac{1}{2})$ but we don't get that one using the formula.

(c) $X^* = (\frac{11}{19}, \frac{5}{19}, \frac{3}{19}), Y^* = (\frac{3}{19}, \frac{28}{57}, \frac{20}{57}), v = \frac{24}{19}$. The last matrix has $X^* = (\frac{1}{7}, 0, \frac{6}{7}), Y^* = (\frac{3}{7}, \frac{4}{7}, 0), v(A) = -\frac{4}{7}$.

2.10 The inverse is

$$B = \frac{1}{10}\begin{bmatrix} 2 & -7 & 6 \\ 4 & -4 & 2 \\ -6 & \frac{27}{2} & -8 \end{bmatrix}.$$

The pure saddle is at $X^* = (0,1,0), Y^* = (0,0,1)$. The formula gives $Y^* = (\frac{2}{5}, \frac{4}{5}, -\frac{1}{5})$, which is not optimal but the mixed strategy $Y^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is, as you should verify.

2.13 Let S denote the common sum of the rows and columns in an $n \times n$ magic square. Then $v(A) = \frac{S}{n}$ and $X^* = Y^* = (\frac{1}{n}, \ldots, \frac{1}{n})$.

2.15 $X = \dfrac{9}{7} J_n A^{-1}$.

2.16 $X^* = (\frac{40}{91}, \frac{40}{91}, \frac{10}{91}, \frac{1}{91}),\ Y^* = (\frac{57}{65}, \frac{57}{910}, \frac{47}{910}, \frac{4}{455}), v(A) = \frac{40}{91}$.

2.17 $X^* = (\frac{1}{7}, \frac{1}{7}, \frac{5}{7}), Y^* = (\frac{3}{7}, \frac{3}{7}, \frac{1}{7}), v(A) = \frac{1}{7}$.

2.18 *value* $= \frac{2}{3}$, and the cat and rat play each of the rows and columns of the reduced game with probability $\frac{1}{3}$. This corresponds to *dcbi, djke, ekjd* for the cat, each with probability $\frac{1}{3}$ and *abcj, aike, hlia* each with probability $\frac{1}{3}$ for the rat. Equivalent paths exist for eliminated rows and columns.

2.19 The game matrix with player I as the row player is $A = \begin{bmatrix} -0.8 & 0.6 & 0.6 \\ 0.6 & -0.2 & 0.6 \\ 0.6 & 0.6 & -0.4 \end{bmatrix}$.

$X^* = (20, 35, 28)/83 = Y^*$.

2.20 $\begin{bmatrix} 0 & -0.6 & -0.6 \\ 0.6 & 0 & -0.2 \\ 0.6 & 0.2 & 0 \end{bmatrix}$.

2.21 The game matrix is

$$\begin{bmatrix} 0.0 & -0.12 & -0.28 & -0.44 & -0.6 \\ 0.12 & 0.0 & 0.04 & -0.08 & -0.2 \\ 0.28 & -0.04 & 0.0 & 0.28 & 0.2 \\ 0.44 & 0.08 & -0.28 & 0.0 & 0.6 \\ 0.6 & 0.2 & -0.2 & -0.6 & 0 \end{bmatrix}$$

The game is symmetric and has solution $X^* = (0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}) = Y^*, v = 0$.

2.22 $A = \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$, $X^* = Y^* = (0, \frac{4}{7}, \frac{3}{7}, 0)$, for example.

2.23 The matrix B is given by

$$B = \begin{bmatrix} 0 & 0 & 5 & 2 & 6 & -1 \\ 0 & 0 & 1 & \frac{7}{2} & 2 & -1 \\ -5 & -1 & 0 & 0 & 0 & 1 \\ -2 & -\frac{7}{2} & 0 & 0 & 0 & 1 \\ -6 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{bmatrix},$$

and will have $v(B) = 0$, and $P = Q = \left(\frac{5}{53}, \frac{6}{53}, \frac{3}{53}, \frac{8}{53}, 0, \frac{31}{53}\right)$. Then $b = (5 + 6)/53 = \frac{11}{53}$ and $b = (3 + 8 + 0)/53 = \frac{11}{53}$, so $x_1 = \frac{5}{11}, x_2 = \frac{6}{11}$ and $y_1 = \frac{3}{11}, y_2 = \frac{8}{11}, y_3 = 0$. Also, $v(A) = \gamma/b = \frac{31}{53}/\frac{11}{53} = \frac{31}{11}$.

2.24 **(a)**$X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{1}{5}, 0, 0, \frac{4}{5}, 0), v = \frac{8}{5}$.
(b) $X^* = (\frac{21}{53}, \frac{24}{53}, \frac{8}{53}, 0), Y^* = (\frac{23}{53}, \frac{4}{53}, \frac{26}{53}, 0), v = \frac{4}{53}$;
(c) $X^* = (\frac{9}{55}, 0, \frac{1}{5}, \frac{7}{11}), Y^* = (\frac{34}{55}, \frac{2}{11}, \frac{1}{5}, 0), v = \frac{51}{55}$.

2.25 $X^* = (\frac{47}{82}, \frac{10}{41}, \frac{15}{82}, 0), Y^* = (\frac{22}{41}, \frac{14}{41}, \frac{5}{41}, 0, 0), v = \frac{13}{41}$.

2.26 The pure strategies are labeled plane(P), highway(H), roads(R), for each player. The drug runner chooses one of those to try to get to New York, and the cops choose one of those to patrol. The game matrix in which the drug runner is the row player, becomes

$$A = \begin{bmatrix} -18 & 150 & 150 \\ 100 & 24 & 100 \\ 80 & 80 & 35 \end{bmatrix}.$$

For example, if drug runner plays H and cops patrol H, the drug runner's expected payoff is $(-90)(0.4) + (100)(0.6) = 24$. The saddle point is

$$X^* = (0.144, 0.3183, 0.5376) \text{ and } Y^* = (0.4628, 0.3651, 0.1721).$$

The drug runners should use the back roads more than half the time, but the cops should patrol the back roads only about 17% of the time.

2.27 The game matrix is

$$\begin{bmatrix} -0.40 & -0.44 & -0.6 \\ -0.28 & -0.40 & -0.2 \\ -0.2 & -0.6 & 0 \end{bmatrix}$$

To use Maple to get these matrices:

```
Accuracy functions:
>p1:=x->piecewise(x=0,.2,x=.2,.4,x=.4,.4,x=.6,.4,x=.8,1);
>p2:=x->piecewise(x=0,0.6,x=.2,.8,x=.4,.8,x=.6,.8,x=.8,1);
Payoff function
>u1:=(x,y)->
    piecewise(x<y,1*p1(x)+
                   (-1)*(1-p1(x))*p2(y)+
                   (0)*(1-p1(x))*(1-p2(y)),
    x>y,(-1)*p2(y)+(1)*(1-p2(y))*p2(x)+0*(1-p2(y))*(1-p1(x)),
    x=y,0*p1(x)*p2(x)+(1)*p1(x)*(1-p2(x))+(-1)*(1-p1(x))*p2(x)
             +0*(1-p1(x))*(1-p2(x)));
>with(LinearAlgebra):
>A:=Matrix([[u1(0,0),u1(0,.4),u1(0,.8)],
           [u1(.4,0),u1(.4,.4),u1(.4,.8)],
           [u1(.8,0),u1(.8,.4),u1(.8,.8)]]);
```

2.28 The game matrix is

$$\begin{bmatrix} 0.18 & 0.28 & 0.37 & 0.46 & 0.55 \\ 0.34 & 0.38 & 0.51 & 0.58 & 0.65 \\ 0.42 & 0.44 & 0.46 & 0.64 & 0.70 \\ 0.74 & 0.68 & 0.62 & 0.56 & 0.90 \\ 0.90 & 0.80 & 0.70 & 0.60 & 0.50 \end{bmatrix}$$

The value of the game is $v = 0.61$ with saddle $X^* = (0,0,0.34,0.10,0.56)$ and $Y^* = (0,0,0.21,0.11,0.68)$.

2.29 To find the game matrix with player I as the row player, if I locates the store in town 1, and II locates the store in town 4,for example, then the payoff to I, in terms of market share, is

$$(0.9)(15) + (0.9)(30) + (0.4)(20) + (0.9)(35) = 52\%.$$

Similarly, if store I is in town 1 and store II is in town 2, then the expected payoff to I is

$$(0.9)(15) + (0.1)(30) + ((0.4)(20) + (0.4)(35) = 38.5\%.$$

The game matrix becomes

$$A = \begin{bmatrix} 65 & 38.5 & 41.5 & 52 \\ 78 & 65 & 56.5 & 62 \\ 78 & 58.5 & 65 & 62 \\ 63 & 48.5 & 51.5 & 65 \end{bmatrix}$$

The saddle point is $X^* = (0, 0.43, 0.57, 0)$ and $Y^* = (0, 0.57, 0.43, 0)$.

2.30 The matrix is

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	5	0	2	3	4	5	6
2	4	4	0	3	4	5	6
3	3	3	3	0	4	5	6
4	2	2	2	2	0	5	6
5	1	1	1	1	1	0	6
6	0	0	0	0	0	0	0

Then, $X^* = (20, 4, 7, 15, 0, 0, 0)/46, Y^* = (2, 10, 19, 15, 0, 0, 0)/46, v = \frac{93}{46}$. In the second case, each farmer concedes 2 yards.

2.31 The saddle point is

$$X^* = \left(\frac{23}{226}, \frac{165}{452}, \frac{73}{452}, \frac{42}{113}\right),$$
$$Y^* = \left(\frac{169}{452}, \frac{55}{226}, 0, \frac{61}{452}, \frac{28}{113}\right).$$

The value is $v = -\frac{431}{226}$. According to this, team P should never play a short pass but team B should defend against SP $\frac{165}{452}$ fraction of the time.

2.32 $X = (\frac{1}{2}, \frac{1}{2}), Y = (1, 0, 0...), v = a$. If you graph the lines for player I versus the columns, you will see that the lines that determine the optimal strategy for I will correspond to column 1 and the column with $(2a, 1/n)$, or $(1/n, 2a)$. This gives $x^* = a/(2a - 1/n)$. Letting $n \to \infty$ gives $x^* = \frac{1}{2}$. For player II you can see from the graph that the last column and the first column are the only ones being used, which leads to $ay + a(1 - y) = ay + (1/n)(1 - y)$, and so for all large enough n so that $a > 1/n$, $y^* = 1$ and then $Y^* = (1, 0, 0, \ldots)$.

2.35 Shemp has the two strategies to (1) tell the truth or (2) call H no matter what. Curly also has two strategies: (1) believe a call of tails or heads, or (2) believe a call of tails and challenge a call of heads. The game matrix to Shemp as the row player is $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$. Shemp should call the actual toss two-thirds of the time and lie one-third of the time.

2.36 (a) $X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{1}{5}, 0, 0, \frac{4}{5}, 0)$

(b) $X^* = (\frac{21}{53}, \frac{24}{53}, \frac{8}{53}, 0), Y^* = (\frac{23}{53}, \frac{4}{53}, \frac{26}{53}, 0), v = \frac{4}{53}$.

(c) $X^* = (\frac{9}{55}, 0, \frac{1}{5}, \frac{7}{11}), Y^* = (\frac{34}{55}, \frac{2}{11}, \frac{1}{5}, 0), v = \frac{51}{55}$.

2.37 $X^* = \left(\frac{47}{82}, \frac{10}{41}, \frac{15}{82}, 0\right), Y^* = \left(\frac{22}{41}, \frac{14}{41}, \frac{5}{41}, 0, 0\right), v = \frac{13}{41}$.

SOLUTIONS FOR CHAPTER 3

3.1 Use the definitions. For example, if X^*, Y^* is a Nash equilibrium, then $X^* A Y^{*T} \geq X A Y^{*T}$, $\forall X$, and $X^*(-A)Y^{*T} \geq X^*(-A)Y^T$, $\forall Y$.

3.2 Pure Nash equilibria at $X = (0,1), Y = (0,1,0)$, and $X = (1,0), Y = (1,0,0)$. There is also a mixed Nash at $X = (\frac{7}{13}, \frac{6}{13}), Y = (\frac{1}{6}, \frac{5}{6}, 0), E_{\mathrm{I}} = \frac{68}{15}, E_{\mathrm{II}} = \frac{301}{65}$. Notice that column 3 is dominated and may be dropped. Safety levels: $v(A) = \frac{22}{5}, X^* = (1,0), Y^* = (0,0,1)$. $X^* = (1,0)$ is the maxmin strategy for player I. $v(B^T) = 4.9, X^* = (0,1,0), Y^* = (0,1)$, and $X^* = (0,1,0)$ is the maxmin strategy for player II.

3.3 Pure Nash at $(Turn, Straight)$ and $(Straight, Turn)$. Expected payoffs for the mixed Nash are $E_{\mathrm{I}} = E_{\mathrm{II}} = -\frac{2001}{52}$. Safety for I is -42; safety for II is also -42.

3.4 First, row 2 is dominated for player I, so get rid of it. Second, column 1 is dominated for player II, so get rid of it. Finally, the payoff at 2 is a solution; that is, $X^* = (1,0,0), Y^* = (0,1), E_{\mathrm{I}} = 2, E_{\mathrm{II}} = 1$. Safety for I is 2, and safety for II is 1.

3.6 $X_1 = (\frac{1}{2}, \frac{1}{2}), Y_1 = (\frac{1}{2}, \frac{1}{2}), X_2 = (0,1) = Y_2, X_3 = (1,0) = Y_3$.

3.7 The only Nash is $X = (\frac{2}{9}, \frac{7}{9}), Y = (\frac{3}{14}, \frac{11}{14})$, with payoffs $-\frac{4}{7}, \frac{1}{3}$. The rational reaction sets intersect at only this Nash.

3.9 The matrix is

$$\begin{bmatrix} (0.25, 0.25) & (0.25, 0.50) & (0.25, 0.75) \\ (0.50, 0.25) & (0.50, 0.50) & (0, 0) \\ (0.75, 0.25) & (0, 0) & (0, 0) \end{bmatrix}.$$

The table has the pure and mixed Nash points as well as the payoffs.

X	Y	E_{I}	E_{II}
$(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$	$\frac{1}{4}$	$\frac{1}{4}$
$(\frac{2}{3}, \frac{1}{3}, 0)$	$(0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{4}$	$\frac{1}{2}$
$(1,0,0)$	$(0,0,1)$	$\frac{1}{4}$	$\frac{3}{4}$
$(0,1,0)$	$(0,1,0)$	$\frac{1}{2}$	$\frac{1}{2}$
$(\frac{1}{3}, 0, \frac{2}{3})$	$(\frac{1}{3}, 0, \frac{2}{3})$	$\frac{1}{4}$	$\frac{1}{4}$
$(0,0,1)$	$(1,0,0)$	$\frac{3}{4}$	$\frac{1}{4}$
$(0, \frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{1}{3}, 0)$	$\frac{1}{2}$	$\frac{1}{4}$

To use the equality of payoffs theorem to find these you must consider all possibilities in which at least two rows (or columns) are played with positive probability.

3.10 $X = (\frac{1}{2}, \frac{1}{2}), Y = (\frac{1}{5}, \frac{4}{5})$ with payoffs $E_I = -\frac{1}{5}, E_{II} = \frac{3}{2}$. The government should aid half the paupers, but 80% of paupers should be bums. The rational reaction sets intersect only at the mixed Nash point.

3.11 $X^* = (\frac{1}{2}, \frac{1}{2}), Y^* = (\frac{1}{3}, \frac{2}{3}, 0)$, or $Y^* = (0, \frac{1}{2}, \frac{1}{2})$, but you can't get this from the equations $2y_1 - y_2 + y_3 = 0$ and $y_1 + y_2 + y_3 = 1$ because it is underdetermined. It reduces to $y_1 = 2y_2 - 1$, and you can see that $y_1 = \frac{1}{3}, y_2 = \frac{2}{3}$ is one possible solution and so is $y_1 = 0, y_2 = \frac{1}{2}$. The equations require that all components be > 0.

3.12 $0 = 4y_1 - y_2, y_1 + y_2 = 1$ gives $y_1 = \frac{1}{5}, y_2 = \frac{4}{5}$.

3.13 If $ab < 0$, there is exactly one strict Nash equilibrium: (i) $a > 0, b < 0 \implies$ the Nash point is $X^* = Y^* = (1,0)$; (ii) $a < 0, b > 0 \implies$ the Nash point is $X^* = Y^* = (0,1)$.

If $a > 0, b > 0$ there are three Nash equilibria $X_1 = Y_1 = (1,0)$, $X_2 = Y_2 = (0,1)$, and the mixed Nash $X_3 = Y_3 = (b/(a+b), a/(a+b))$.

If $a < 0, b < 0$, there are three Nash equilibria $X_1 = (1,0), Y_1 = (0,1)$, $X_2 = (0,1), Y_2 = (1,0)$, and the mixed Nash $X_3 = Y_3 = (b/(a+b), a/(a+b))$.

3.14 $X^* = (\frac{2}{3}, \frac{1}{3}), Y^* = (\frac{1}{3}, \frac{2}{3}, 0)$.

3.15 Three Nash: $X_1 = (0,1) = Y_1$, $X_2 = (1,0) = Y_2$, which represents one player always giving in to the other so the one who wins gets 2 and the other gets 1, and $X_3 = (\frac{3}{5}, \frac{2}{5}), Y_3 = (\frac{2}{5}, \frac{3}{5})$ in which they each get payoff $\frac{1}{5}$. Notice that the mixed Nash leads to a lower payoff to *both players*.

3.16 Three Nash equilibria: $X_1 = (1,0), Y_1 = (0,1)$, $X_2 = (0,1), Y_2 = (1,0)$, $X_3 = (\frac{2}{3}, \frac{1}{3}) = Y_3$. The last Nash gives payoffs $(0,0)$.

3.17 The pure Nash $X = Y = (1,0)$ in which they both go after the stag gives a payoff of 2 to each player. The mixed Nash $X = Y = (\frac{1}{2}, \frac{1}{2})$ and the pure Nash $X = Y = (0,1)$, in which they go after the rabbit either all the time or half the time, gives a payoff of 1 to each player.

3.18 One Nash equilibrium, $X^* = (\frac{23}{64}, \frac{17}{64}, \frac{3}{8}), Y^* = (\frac{21}{67}, \frac{22}{67}, \frac{24}{67})$. The expected payoffs are $E_I = \frac{344}{67}, E_{II} = \frac{151}{32}$.

3.19 The mixed Nash is $X = (\frac{1}{2}, \frac{1}{2}), Y = (\frac{2}{3}, \frac{1}{3})$. For player I the safety level is 3 and the maxmin strategy is $X = (1,0)$.

3.20 $A = \begin{bmatrix} 50 & 80 \\ 90 & 20 \end{bmatrix}, B = \begin{bmatrix} 50 & 20 \\ 10 & 80 \end{bmatrix}$. Then $X^* = (0.7, 0.3), Y^* = (0.6, 0.4), p = 62, q = 38$.

3.21

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (1,0,0) & E_I = 2, E_{II} = 1 \\ X_2 = (1,0) & Y_2 = (\frac{1}{2},0,\frac{1}{2}) & E_I = \frac{1}{2}, E_{II} = 1 \\ X_3 = (0,1) & Y_3 = (0,0,1) & E_I = 1, E_{II} = 3 \end{array}$$

3.22 Take $B = -A$. The Nash equilibrium is $X^* = (\frac{5}{8},\frac{3}{8},0), Y^* = (0,\frac{5}{8},\frac{3}{8})$, and the value of the game is $v(A) = \frac{1}{8}$.

3.23 The objective function is $f(x,y,p,q) = 7x + 7y - 6xy - 6 - p - q$ with constraints $2y - 1 \le p, 5y - 3 \le p, 2x - 1 \le q, 5x - 3 \le q$, and $0 \le x, y \le 1$.

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_I = -1, E_{II} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_I = 2, E_{II} = -1 \\ X_3 = (\frac{2}{3},\frac{1}{3}) & Y_3 = (\frac{2}{3},\frac{1}{3}) & E_I = \frac{1}{3}, E_{II} = \frac{1}{3} \end{array}$$

3.24 $X_1 = \left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right), Y_1 = \left(\frac{5}{13},\frac{5}{13},\frac{2}{13}\right), E_I = \frac{10}{13}$, $E_{II} = 1$. $X_2 = \left(\frac{3}{4},0,\frac{1}{4}\right) = Y_2$ with payoffs $E_I = \frac{5}{4}, E_{II} = \frac{3}{2}$. $X_3 = Y_3 = (0,1,0)$.

3.26 The matrices are

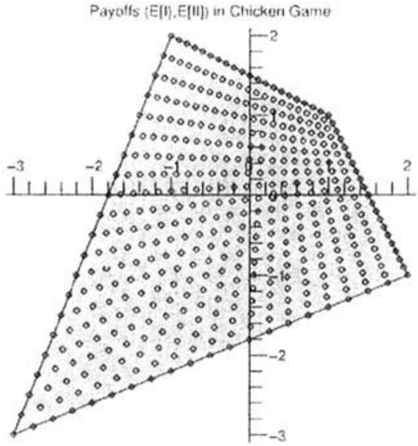
$$A = \begin{bmatrix} 1.20 & -0.56 & -0.88 & -1.2 \\ 1.24 & -0.40 & -1.44 & -1.6 \\ 0.92 & -0.04 & -1.20 & -1.8 \\ 0.6 & -0.2 & -0.6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0.64 & 0.92 & 0.76 & 0.6 \\ -0.28 & 0.16 & -0.12 & -0.2 \\ -0.44 & 0.28 & 0.04 & -0.6 \\ -0.6 & 0.2 & 0.6 & 0 \end{bmatrix}.$$

One Nash equilibrium is $X = (0.71,0,0,0.29), Y = (0,0,0.74,0.26)$. So Pierre fires at 10 paces about 75% of the time and waits until 2 paces about 25% of the time. Bill, on the other hand, waits until 4 paces before he takes a shot but 1 out of 4 times waits until 2 paces.

3.27 (a) The Nash equilibria are

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_I = -1, E_{II} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_I = 2, E_{II} = -1 \\ X_3 = (\frac{2}{3},\frac{1}{3}) & Y_3 = (\frac{2}{3},\frac{1}{3}) & E_I = \frac{1}{3}, E_{II} = \frac{1}{3} \end{array}$$

They are all Pareto-optimal because it is impossible for either player to improve their payoff without simultaneously decreasing the other player's payoff, as you can see from the figure:



None of the Nash equilibria are payoff-dominant. The mixed Nash (X_3, Y_3) risk dominates the other two.

(b) The Nash equilibria are

$$X_1 = \left(\frac{1}{4}, \frac{3}{4}\right), Y_1 = (1,0), E_1 = 3, E_2 = 0,$$
$$X_2 = (0,1), Y_2 = (0,1), E_1 = E_2 = 1,$$
$$X_3 = (1,0) = Y_3, E_1 = E_2 = 3.$$

(X_3, Y_3) is payoff-dominant and Pareto-optimal.

(c)

$$X_1 = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right), Y_1 = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right), E_I = \frac{10}{13}, E_{II} = 1.$$
$$X_2 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) = Y_2, E_I = \frac{5}{4}, E_{II} = \frac{3}{2}.$$
$$X_3 = Y_3 = (0,1,0), E_I = 2, E_{II} = 3.$$

Clearly X_3, Y_3 is payoff-dominant and Pareto-optimal. Neither (X_1, Y_1) nor (X_2, Y_2) are Pareto-optimal relative to the other Nash equilibria, but they each risk dominate (X_3, Y_3).

SOLUTIONS FOR CHAPTER 4

4.1 $u_i(q_1, \ldots, q_i, \ldots, q_N) = q_i(\sum_{j=1}^N q_j - c_i)$. The optimal quantities that each firm should produce is

$$q_i = \frac{1}{N+1}\left(NC_i - \sum_{j=1, j\neq i}^{N} c_j\right).$$

If $C_i = c, \quad i = 1, 2, \ldots, N$, then $q_i = c/(N+1) \to 0, N \to \infty$.

4.2 With $u_1(q_1, q_2) = q_1(100 - 2\sqrt{(q_1 + q_2)}) - q_1 - 10$, and $u_2(q_1, q_2) = q_2(100 - 2\sqrt{(q_1 + q_2)}) - 2q_2 - 5$, we get $q_1^* = 795.88, q_2^* = 756.48$, and profits $u_1^* = 16066.77$, and $u_2^* = 14519.42$.

4.3 Profit for firm 1 is 10, compared with 16 or 7.11 if $c_2 = 5$ or $c_2 = 1$, resp.

4.4 We have to solve the system

$$\begin{aligned}
q_1 &= \frac{1}{2}[(\Gamma - q_2^1 - c_1)r_1 + (\Gamma - q_2^2 - c_1)r_2 + (\Gamma - q_2^3 - c_1)r_3],\\
q_2^i &= \frac{1}{2}[\Gamma - q_1 - c^i], i = 1, 2, 3,
\end{aligned}$$

which has solution

$$\begin{aligned}
q_1 &= \frac{1}{3}\left[\Gamma - 2c_1 + r_1(c^1 - c^3) + r_2(c^2 - c^3)\right]\\
q_2^1 &= \frac{1}{3}\left[\Gamma + c_1 - \frac{c^3}{2} + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{c^1}{2}\\
q_2^2 &= \frac{1}{3}\left[\Gamma + c_1 - \frac{c^3}{2} + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{c^2}{2}\\
q_2^3 &= \frac{1}{3}\left[\Gamma + c_1 + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{2c^3}{3}
\end{aligned}$$

The quantities with the information given are $q_1 = \frac{263}{8}$ for firm 1, and $q_2^1 = \frac{529}{16}, q_2^2 = \frac{521}{16}$, and $q_2^3 = \frac{497}{16}$.

4.6 **(b)** $q_2 = \frac{1}{2}(18 - q_1)$; **(c)** $q_1 = 8$; **(d)** $q_2 = 5$.

4.7 $q_2(q_1) = (\Gamma - q_1 - 2c_2 + c_3)/3, q_3 = (\Gamma - q_1 - 2c_3 + c_2)/3$. Then $q_1 = (\Gamma + c_2 + c_3 - 3c_1)/2$, and $u_1(q_1, q_2, q_3) = (\Gamma + c_2 + c_3 - 3c_1)^2/12$.

4.9 $q_1^0 = 91.67, p = 8.32, u_1 = 569.8$.

4.10

$$u_1(p_1,p_2) = \begin{cases} (p_1 - c_1)\min\{(\Gamma - p_1), K\} & \text{if } p_1 < p_2; \\ ((p - c_1)(\Gamma - p))/2 & \text{if } p_1 = p_2 = p \geq c_1; \\ 0 & \text{if } p_1 > p_2, p_2 \geq \Gamma - K. \\ (p_1 - c_1)(\Gamma - p_1 - K) & \text{if } p_1 > p_2, p_2 < \Gamma - K. \end{cases}$$

and

$$u_2(p_1,p_2) = \begin{cases} (p_2 - c_2)\min\{(\Gamma - p_2), K\} & \text{if } p_2 < p_1; \\ ((p - c_2)(\Gamma - p))/2 & \text{if } p_1 = p_2 = p \geq c_2; \\ 0 & \text{if } p_2 > p_1, p_1 \geq \Gamma - K. \\ (p_2 - c_2)(\Gamma - p_2 - K) & \text{if } p_2 > p_1, p_2 < \Gamma - K. \end{cases}$$

To explain the profit function for firm 1, if $p_1 < p_2$, firm 1 will be able to sell the quantity of gadgets $q = \min\{\Gamma - p_1, K\}$, the smaller of the demand quantity at price p_1 and the capacity of production for firm 1. If $p_1 = p_2$, so that both firms are charging the same price, they split the market. If $p_1 > p_2$ in the previous model, this is enough for firm 1 to lose all the business, but now firm 1 loses all the business only if in addition $p_2 \geq \Gamma - K$; that is, if $K \geq \Gamma - p_2$, which is the amount that firm 2 will be able to sell at price p_2, and this quantity is less than the capacity of production. Finally, if $p_1 > p_2$, and $K < \Gamma - p_2$, firm 1 will be able to sell the amount of gadgets that exceed the capacity of firm 2. That is, if $K < \Gamma - p_2$, then the quantity demanded from firm 2 at price p_2 is greater than the production capacity of firm 2 so the residual amount of gadgets can be sold to consumers by firm 1 at price p_1. But notice that in this case, the number of gadgets that are demanded at price p_1 is $\Gamma - p_1$, so firm 1 can sell at most $\Gamma - p_1 - K < \Gamma - p_2 - K$. Even in the case $c_1 = c_2 = 0$ there is no pure Nash equilibrium even at $p_1^* = p_2^* = 0$. This is known as the Edgeworth paradox.

4.12 Start with $u_1(p_1,p_2) = (a - p_1 - bp_2)(p_1 - c_1)$, and $u_2(p_1,p_2) = (a - p_2 - bp_1)(p_2 - c_2)$. Solve $\partial u_2/\partial p_2 = 0$ to get $p_2(p_1) = \frac{1}{2}(c_2 + a - bp_1)$.

Next solve $\partial u_1(p_1, p_2(p_1))/\partial p_1 = 0$, to get

$$p_1^* = \frac{1}{2}\frac{2c_1 + 2a - c_1b^2 - bc_2 - ba}{2 - b^2}$$

and then

$$p_2^* = p_2(p_1^*) = \frac{1}{4}\frac{4c_2 + 4a + c_1b^3 - b^2(a + c_2) - 2bc_1 - 2ba}{2 - b^2}.$$

The profit functions become

$$u_1(p_1^*, p_2^*) = \frac{1}{8}\frac{(2c_1 - b^2c_1 - 2a + bc_2 + ba)^2}{2 - b^2}$$

and

$$u_2(p_1^*, p_2^*) = \frac{1}{16}\frac{\left(-4\,a + ab^2 + 4c_2 - 3c_2b^2 + 2\,bc_1 - c_1b^3 + 2\,ba\right)^2}{(-2+b^2)^2}$$

Notice that with the Stackelberg formulation and the formulation in the preceding problem the Nash equilibrium has positive prices and profits.

The Maple commands to do these calculations are:

```
> u1:=(p1,p2)->(a-p1-b*p2)*(p1-c1);
> u2:=(p1,p2)->(a-p2-b*p1)*(p2-c2);
> eq1:=diff(u2(p1,p2),p2)=0;
> solve(eq1,p2);
> assign(p2,%);
> f:=p1->u1(p1,p2);
> eq2:=diff(f(p1),p1)=0;
> solve(eq2,p1);
> simplify(%);
> assign(p1,%);
> p2;simplify(%);
> factor(%);
> assign(a2,%);
> u1(p1,a2);
> simplify(%);
> u2(p1,a2);
> simplify(%);
```

4.13 $x = 0.52$.

4.14 $x = \frac{4}{17}, y = \frac{11}{17}, u = -\frac{13}{17}$.

4.15 This is actually a constant sum game since the portions of land that each player gets must add to the total, which is 2. The area for player I is

$$A(x, y) = x - 2xy + y + x - x^2/2 + y^2/2,$$

and the area for player II is

$$B(x, y) = 2xy + 2 - 2x - y + x^2/2 - y^2/2.$$

The Nash equilibrium is found by taking partial derivatives of A (with respect to x) and B (with respect to y), setting to zero, and solving. The result is $x^* = \frac{4}{5}, y^* = \frac{3}{5}$, and $A(x^*, y^*) = \frac{11}{10}, B(x^*, y^*) = \frac{9}{10}$. Surprisingly, they do not occur with equal areas.

4.16 $$u_1(q_1, q_2) = \begin{cases} v_1 - C_1q_2 & \text{if } q_1 > q_2; \\ -C_1q_1 & \text{if } q_1 < q_2; \\ v_1/2 - C_1q_1 & \text{if } q_1 = q_2. \end{cases}$$

First, calculate $\max_{q_1 \geq 0} u_1(q_1, q_2)$, and $\max_{q_2 \geq 0} u_2(q_1, q_2)$. For example

$$\max_{q_1 \geq 0} u_1(q_1, q_2) = \begin{cases} v_1 - c_1 q_2 & \text{if } q_2 < v_1/C_1; \\ 0 & \text{if } q_2 > v_1/C_1; \\ 0 & \text{if } q_2 = v_1/C_1. \end{cases}$$

and the maximum is achieved at the set valued function

$$q_1^*(q_2) = \begin{cases} (q_2, \infty) & \text{if } q_2 < v_1/C_1; \\ 0 & \text{if } q_2 > v_1/C_1; \\ 0 \cup (q_2, \infty), & \text{if } q_2 = v_1/C_1. \end{cases}$$

This is the best response to q_2. Next calculate the best response to q_1 for country 2, $q_2^*(q_1)$:

$$q_2^*(q_1) = \begin{cases} (q_1, \infty) & \text{if } q_1 < v_2/C_2; \\ 0 & \text{if } q_1 > v_2/C_2; \\ 0 \cup (q_1, \infty) & \text{if } q_1 = v_2/C_2. \end{cases}$$

If you now graph these sets on the same set of axes, the Nash equilibrium are points of intersection of the sets. The result is

$$(q_1^*, q_2^*) = \begin{cases} \text{either } q_1^* = 0 \text{ and } q_2^* \geq v_1/C_1 \\ \text{or } q_2^* = 0 \text{ and } q_1^* \geq v_2/C_2 \end{cases}$$

For example, this says that either (1) country 1 should concede immediately and country 2 should wait until time v_1/C_1, or (2) country 2 should concede immediately and country 1 should wait until time v_2/C_2.

4.17 We have

$$x^* = \frac{C_2 V}{(C_1 + C_2)^2}, y^* = \frac{C_1 V}{(C_1 + C_2)^2}.$$

Then

$$u_1(x, y) = \frac{C_2^2 V}{(C_1 + C_2)^2} \text{ and } u_2(x^*, y^*) = \frac{C_1^2 V}{(C_1 + C_2)^2}$$

4.18 Each farmer has the payoff function

$$u_i(q_1, q_2, q_3) = pq_i = (15 - \frac{1}{150,000}(q_1 + q_2 + q_3))q_i, \quad i = 1, 2, 3.$$

Take a partial derivative and set to zero to get $q_i = 562500$ bushels each. So an interior pure Nash equilibrium consists of each farmer sending 562500 bushels each to market and using 437500 bushels for feed. The price per bushel will be \$3.75, which is greater than the government-guaranteed price.

4.19 (**a**) The Nash equilibrium is

$$s^* = \frac{\alpha t - p(1 + r)(1 + \alpha\delta)}{(1 + r)(1 + \alpha(1 + \delta t))}, \quad g^* = \frac{\alpha t(1 + \delta) - p(1 + r)}{\alpha(1 + \delta) + 1}.$$

(b) For the data given in the problem $s^* = 338.66, g^* = 357.85$.

4.23 The cumulative distribution function is $F(p) = -2p^3 + 3p^2, 0 < p < 1$. The interior solution of $1 - F(p) - pf(p) = 0$ is $p^* = 0.422$, so the reserve price should be set at 42.2% of the normalized range of prices. Notice that even though the density is symmetric around $p = \frac{1}{2}$, the optimal reserve price is not 0.5. The Maple commands to solve are

```
> restart: f:=x->6*x*(1-x);
> F:=x->int(f(y),y=0..x);
> fsolve(1-F(x)-x*f(x)=0,x);
```

4.25 The expected payoff of a bidder with valuation v who makes a bid of b is given by

$$u(b) = vProb(b \text{ is high bid}) - b = vF(\beta^{-1}(b))^{N-1} - b = v\beta^{-1}(b)^{N-1} - b.$$

Differentiate, set to zero, and solve to get $\beta(v) = ((N-1)/N)v^N$.

Since all bidders will actually pay their own bids and each bid is $\beta(v) = (N - 1/N)v^N$, the expected payment from each bidder is

$$E[\beta(V)] = \frac{N-1}{N}\int_0^1 v^N dv = \frac{N-1}{N(N+1)}.$$

Since there are N bidders, the total expected payment to the seller will be $(N-1)/(N+1$

SOLUTIONS FOR CHAPTER 5

5.2 **(a)** $v(1) = value(A) = \frac{8}{5}, v(2) = value(B^T) = \frac{8}{5}, v(12) = 6, v(\emptyset) = 0$.

(b) The core is $C(0) = \{(6 - x_2, x_2) \mid \frac{8}{5} \le x_2 \le \frac{22}{5}\}$.

(c) The least core is $C\left(-\frac{7}{5}\right) = \{(3,3)\}$.

5.3 In normalized form simply divide each number by 13 : $\vec{x}$ unnormalized$=\{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8},$

5.4 $v(\emptyset) = 0, v(1) = \frac{3}{5}, v(2) = 2, v(3) = 1, v(12) = 5, v(13) = 4, v(23) = 3, v(123) = 16$.

5.5 $\varepsilon^1 = \frac{1}{3}$ and $C\left(\frac{1}{3}\right) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$.

5.6 $\varepsilon^1 = -1$ least core is $C(-1) = \{\vec{x} = (2,2,0)\}$.

5.7 Suppose $\vec{x} \in C(0)$ so that $e(S, \vec{x}) \le 0, \forall S \subsetneq N$. Take the single player coalition $S = \{i\}$ so $v(i) + v(N-i) = v(N)$. Since the game is essential, $v(N) > \sum_{i=1}^{n} v(i)$.

Since $\vec{x}$ is in the core, we have

$$v(N) > \sum_{i=1}^{n} v(i) = \sum_{i=1}^{n} v(N) - v(N-i) = nv(N) - \sum_{i=1}^{n} v(N-i),$$

$$\Longrightarrow$$

$$v(N)(n-1) < \sum_{i=1}^{n} v(N-i) \le \sum_{i=1}^{n}\sum_{j\neq i} x_j = \sum_{i=1}^{n} v(N) - x_i$$
$$= nv(N) - \sum_{i=1}^{n} x_i = (n-1)v(N) \Rightarrow\Leftarrow .$$

5.8 Since the game is inessential, $v(N) = \sum_{i=1}^{n} v(i)$. It is obvious that $\vec{x} = (v(1), \dots, v(n)) \in C(0)$. If there is another $\vec{y} \in C(0), \vec{y} \neq \vec{x}$, there must be one component $y_i < v(i)$ or $y_i > v(i)$. Since $\vec{y} \in C(0)$, the first possibility cannot hold and so $y_i > v(i)$. This is true at any j component of $\vec{y}$ not equal to $v(j)$. But then, adding them up gives $\sum_{i=1}^{n} y_i > \sum_{i=1}^{n} v(i) = v(N)$, which contradicts the fact that $\vec{y} \in C(0)$.

5.9 Suppose $i = 1$. Then

$$x_1 + \sum_{j\neq 1} x_j = v(N) = v(N-1) \le \sum_{j\neq 1} x_j,$$

and so $x_1 \le 0$. But since $-x_1 = v(1) - x_1 \le 0$, we have $x_1 = 0$.

5.11 Let $\vec{x} \in C(0)$. Since $v(N-1) \le x_2 + \cdots + x_n = v(N) - x_1$, we have $x_1 \le v(N) - v(N-1)$. In general, $x_i \le v(N) - v(N-i), 1 \le i \le n$. Now add these up to get $v(N) = \sum_i x_i \le \sum_i \delta_i < v(N)$, which says $C(0) = \emptyset$.

5.13 The core is

$$C(0) = \{(x_1, x_2, 16 - x_1 - x_2) : \frac{3}{5} \le x_1 \le 13, 2 \le x_2 \le 12, 5 \le x_1 + x_2 \le 15\}.$$

The least core: $\varepsilon^1 = -\frac{62}{15}, C(\varepsilon^1) = \left\{\left(\frac{71}{15}, \frac{92}{15}, \frac{77}{15}\right)\right\}$.

5.14 $q = \frac{1}{2}\left(\frac{2}{5} + \frac{3}{10} + \frac{3}{10}\right) = \frac{1}{2}$. The characteristic function is $v(i) = 0, v(12) = v(13) = v(23) = 1, v(123) = 1$.

5.15 **(b)** To see why the core is empty, show first that it must be true $x_1 + x_2 = -2$, and $x_3 + x_4 = -2$. Then, since $-1 \le x_1 + x_2 + x_3 = -2 + x_3$, we have $x_3 \ge 1$. Similarly $x_4 \ge 1$. But then $x_3 + x_4 \ge 2$ and that is a contradiction.

(c) A coalition that works is $S = \{12\}$.

5.16 $X^1 = C(-\frac{1}{10}) = \{x_1 + x_2 = \frac{9}{10}, \frac{4}{10} \le x_1, \frac{2}{10} \le x_2\}$. The next least core is $X^2 = C(-\frac{1}{4}) = \{(\frac{11}{20}, \frac{7}{20}, \frac{2}{20})\}$.

5.17 The least core is the set $C(-1) = \{x_1 = 1, x_2 + x_3 = 11, x_2 \ge 1, x_3 \ge 2\}$. The nucleolus is the single point $\{(1, \frac{11}{2}, \frac{11}{2})\}$

5.18 For the least core $\varepsilon^1 = -\frac{1}{2}$:

$$\begin{aligned}\text{Least core} = X^1 = C(-\frac{1}{2}) = \{&x_1 + x_2 = \frac{3}{2}, x_3 + x_4 = \frac{3}{2}, x_i \ge \frac{1}{2}, i = 1,2,3,4,\\ &x_2 + x_3 \ge \frac{3}{2}, x_1 + x_4 \ge \frac{3}{2}, x_1 + x_3 \ge \frac{5}{4}, x_2 + x_4 \ge \frac{1}{2},\\ &x_1 + x_2 + x_3 \ge \frac{3}{2}, x_1 + x_2 + x_4 \ge \frac{3}{2}, x_1 + x_3 + x_4 \ge \frac{3}{2},\\ &x_2 + x_3 + x_4 \ge \frac{3}{2}, x_1 + x_2 + x_3 + x_4 = 3\}.\end{aligned}$$

Next X^2 has $\varepsilon^2 = 1$. X^3 has $\varepsilon^3 = 3$, and nucleolus=$\{(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})\}$.

5.19 **(a)** The characteristic function is the number of hours saved by a coalition. $v(i) = 0$, and

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2,$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13.$$

(b) Nucleolus=$\{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with units in hours. The least core is

$$\begin{aligned}X^1 = C(-\frac{3}{2}) = \{&x_1 + x_2 + x_3 = \frac{23}{2}, x_4 = \frac{3}{2},\\ &x_1 + x_2 + x_3 + x_4 = 13, x_1 + x_2 + x_4 \ge \frac{17}{2},\\ &x_2 + x_3 + x_4 \ge \frac{19}{2}, x_1 \ge \frac{3}{2}, x_2 \ge \frac{3}{2},\\ &x_1 + x_2 \ge \frac{11}{2}, x_3 \ge \frac{3}{2}, x_1 + x_3 \ge \frac{11}{2},\\ &x_2 + x_3 \ge \frac{15}{2}, x_1 + x_4 \ge \frac{9}{2}, x_2 + x_4 \ge \frac{7}{2},\\ &x_3 + x_4 \ge \frac{7}{2}, x_1 + x_3 + x_4 \ge \frac{17}{2}\}\end{aligned}$$

The next least core, which will be the nucleolus, is $X^2 = \{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with $\varepsilon^2 = 10$.

(c) The schedule is set up as follows: (i) Curly works from 9:00 to 11:52.5, (ii) Larry works from 11:52.5 to 1:45, (iii) Shemp works from 1:45 to 3;30, and (iv) Moe works from 3:30 to 5:00.

5.20 The characteristic function for the **savings game** is $v(\emptyset) = 0, v(i) = 0, v(1234) = 22 - 8.5$, and

$$\begin{aligned} v(12) &= 13 - 7.5, \quad v(13) = 11 - 7, \quad v(14) = 12 - 7.5, \\ v(23) &= 10 - 6.5, \quad v(24) = 11 - 6.5, \quad v(34) = 9 - 5.5, \\ v(123) &= 17 - 7.5, \quad v(124) = 18 - 8, \quad v(134) = 16 - 7.5, \\ v(234) &= 15 - 7. \end{aligned}$$

The least core is

$$\begin{aligned} X^1 = C(-1.125) = \{ & x_1 + x_2 + x_3 + x_4 = 13.5, x_1 + x_2 + x_3 \geq 10.625, \\ & x_1' + x_2 + x_4 \geq 11.125, x_1 + x_3 + x_4 \geq 9.625, \\ & x_2 + x_3 + x_4 \geq 9.125, x_1 \geq 1.125, x_2 \geq 1.125, x_1 + x_2 \geq 6.625, \\ & x_3 \geq 1.125, x_1 + x_3 \geq 5.125, x_2 + x_3 \geq 4.625, \\ & x_4 \geq 1.125, x_1 + x_4 \geq 5.625, x_2 + x_4 \geq 5.625, x_3 + x_4 \geq 4.625\} \end{aligned}$$

$X^2 = \{(4.375, 3.875, 2.375, 2.875)\}$, so this is the nucleolus.

5.21 The characteristic function is $v(i) = 0, v(12) = 100, v(13) = 130, v(23) = 0, v(123) = 130$.The Shapley value is $\{(\frac{245}{3}, \frac{50}{3}, \frac{95}{3})\}$, and this point is not in $C(0)$. The nucleolus of this game is $\{(115, 0, 15)\}$.

5.22

	player A	player B	player C
ABC	25	65	10
ACB	25	65	10
BAC	50	40	10
BCA	50	40	10
CAB	35	65	0
CBA	50	50	0
Total	235	325	40

5.23 Shapley value=$(\frac{4}{3}, \frac{16}{3}, \frac{16}{3})$.

5.24 Shapley value=$\{\frac{10}{3}, \frac{23}{6}, \frac{23}{6}, 2\}$. The hours of work for each player using the Shapley value are as follows: (i) Curly 9:00 to 12:10, (ii) Shemp from 12:10 to 1:50, (iii) Larry from 1:50 to 4:00, and (iv) Moe from 4:00 to 5:00.

5.25 **(a)** $C(0) = \emptyset$.
(b) $X^1 = C(2) = \{(1, 1, 1, 1)\}$.
(c) Shapley value=$\{(1, 1, 1, 1)\}$.
(d) Shapley value for original game is $(-1, -1, -1, -1)$.

5.26 **(a)** $v(1) = 100, v(2) = v(3) = 0, v(12) = 150, v(13) = 160, v(23) = 0, v(123) = 160$.

Shapley allocation is $\{(\frac{415}{3}, \frac{25}{3}, \frac{40}{3})\}$.

(b) The nucleolus is $\{(155, 0, 5)\}$. This reflects the fact that player 2 has no power at all because she is not the high bidder. The Shapley allocation grants player 2 the amount $\frac{25}{3}$ to reflect the fact that without player 2, player 1 has less negotiating power.

5.27 Shapley value=$\{(3.91667, 3.66667, 2.75, 3.16667)\}$.

5.28 With $(2, 2)$ security point, the bargaining solution is $(3, 3)$. The threat security point is $(u^t, v^t) = (4, 2)$ with threat strategies $X_t = (1, 0) = Y_t$. The threat solution is $(\overline{u}, \overline{v}) = (4, 2)$. The KS line for $(u^*, v^*) = (2, 2)$ intersects the Pareto-optimal boundary at $(3, 3)$. There is no KS line for $(u^*, v^*) = (4, 2)$ because it is on the edge of the Pareto line. The characteristic function is $v(1) = v(2) = 2, v(12) = 6$, which gives nucleolus and Shapley value $(3, 3)$ and matches with the solution for the $(2, 2)$ security point.

5.29 With security point $(u^*, v^*) = (-\frac{12}{11}, 1)$ we get the bargaining solution $(\overline{u}, \overline{v}) = (3, 4)$. This occurs at the point of intersection of the 2 lines forming the Pareto-optimal boundary $v = -u + 7, v = (-\frac{1}{5})u + \frac{23}{5}$.

The threat solution uses $m_p = -1$ or $m_p = -\frac{1}{5}$. In both cases $u^t = 6, v^t = 1$ and player I has a credible and serious threat to always play the second row, in which case player II has no choice but to play column 2. Their payoffs are $(\overline{u}, \overline{v}) = (6, 1)$.

For the KS line we take $u^* = -\frac{12}{11}, v^* = 1$. The maximum possible feasible payoffs for each player is $a = 6$ for player I and $b = 4.818$ for player II, and so $k = 0.538$. The equation of the KS line is then $v - 1 = k(u + \frac{12}{11})$, and this intersects the Pareto-optimal boundary at the line $v = -u + 7$ and at the point $\overline{u} = 3.518, \overline{v} = 3.482$.

5.30 Use calculus or Maple to get $x^* = \frac{3}{4}, y^* = \frac{1}{4}$, and then $\overline{u} = 1, \overline{v} = \frac{5}{4}$.

5.31 **(a)** The problem to solve is

$$\max_{(p,w):f(w)-pw\geq 0\ 0\leq w\leq W} (f(w) - pw)w(p - p_0).$$

SOLUTIONS FOR CHAPTER 6

6.1 Let's look at the equilibrium $X_1 = (1, 0)$. We need to show that for $x \neq 1$, $u(1, px + (1 - p)) > u(x, px + (1 - p))$ for some p_x, and for all $0 < p < p_x$. Now $u(1, px + (1 - p)) = 1 - p + px$, and $u(x, px + (1 - p)) = p + x - 3px + 2px^2$. In order for X_1 to be an ESS, we need $1 > 2p(x - 1)^2$, which implies $0 < p < 1/(2(x - 1)^2)$. So, for $0 \leq x < 1$, we can take $p_x = 1/(2(x - 1)^2)$ and the ESS requirement will be satisfied. Similarly, the equilibrium $X_2 = (0, 1)$, can be shown to be an ESS. For $X_3 = (\frac{1}{2}, \frac{1}{2})$, we have

$$u(\frac{1}{2}, px + (1 - p)/2) = \frac{1}{2} \text{ and } u(x, px + (1 - p)/2) = \frac{1}{2} + \frac{p}{2} - 2px + 2px^2.$$

In order for X_3 to be an ESS, we need

$$\frac{1}{2} > \frac{1}{2} + \frac{p}{2} - 2px + 2px^2,$$

which becomes $0 > 2p(x - \frac{1}{2})^2$, for $0 < p < p_x$. This is clearly impossible, so X_3 is not an ESS.

6.2 There are three Nash equilibria $X_1 = Y_1 = (1,0)$, $X_2 = Y_2 = (0,1)$, and the mixed $X_3 = Y_3 = (\frac{2}{3}, \frac{1}{3})$. The first two are ESSs. For X_3, $u(\frac{2}{3}, \frac{2}{3}) = \frac{2}{3}$, $u(x, \frac{2}{3}) = \frac{2}{3}$. Is $u\left(\frac{2}{3}, x\right) = \frac{2}{3} > u(x,x)$? No, because $\frac{2}{3} > x^2 + 2(1-x)^2$ is false for all $0 < x < 1$.

6.3 The only symmetric (nonstrict) Nash is $(X^* = (0,1), X^*)$. Then $u(0,0) = 1, u(x,0) = 1, u(x,x) = -2x^2 + 5x + 1$, and $u(0,x) = 5x + 1$. Hence, $u(0,0) = 1 = u(x,0)$ and $u(x,x) < u(0,x)$, for any $0 < x \le 1$. This means that $X^* = (0,1)$ is an ESS.

6.4 The Nash equilibria and their payoffs are shown in the following table; they are all symmetric.

X^*	$u(X^*, X^*)$
$(1,0,0)$	2
$(0,1,0)$	2
$(0,0,1)$	2
$(\frac{3}{4}, \frac{1}{4}, 0)$	$\frac{5}{4}$
$(\frac{1}{4}, 0, \frac{3}{4})$	$\frac{5}{4}$
$(0, \frac{3}{4}, \frac{1}{4})$	$\frac{5}{4}$
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{2}{3}$

For $X^* = (1,0,0)$ you can see this is an ESS because it is strict. Consider next $X^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Since $u(Y, X^*) = \frac{5}{4}(y_1 + y_2) - y_3/2$, the set of best response strategies is $Y = (y, 1-y, 0)$. Then $u(Y,Y) = 4y^2 - 4y + 2$,and $u(X^*, Y) = -\frac{1}{4} + 2y$. Since it is **not** true that $u(Y,Y) < u(X^*, Y)$, for all best responses $Y \neq X^*$, X^* is not an ESS.

6.5 **(a)** There is a unique Nash, strict and symmetric ESS$= (0,1)$ if $a < 0, b > 0, = (1,0)$ if $b < 0, a > 0$.

(b) Three Nash equilibria, all symmetric, NE$= (1,0), (0,1), X, X = (b/(a+b), a/(a+b))$. Both $(1,0), (0,1)$ are strict, so $(1,0), (0,1) \in ESS$. The mixed X is not an ESS since $E(1,1) = a > ab/(a+b) = E(X,1)$ so $ESS = \{(0,1), (1,0)\}$.

(c) Two strict asymmetric Nash Equilibria, one symmetric Nash Equilibrium $X = (c = b/(a+b), a/(a+b))$, but now X is ESS since

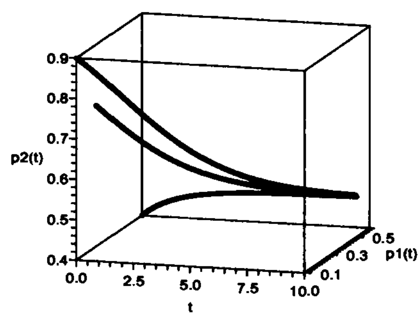
$$E(X,Y) = cay_1 + (1-c)by_2 = ab/(a+b)$$

and for every strategy $Y \neq X$, $E(Y,Y) = ay_1^2 + by_2^2 < ab/(a+b) = E(X,Y)$, so X is the ESS.

6.7 The pure Nash equilibria are clearly equivalent. For the interior mixed Nash, the calculus method shows that the partial in the appropriate variables of the payoff functions lead to equations for the Nash equilibrium independent of a, b. You may also calculate directly that $E'(X,Y) = XA'Y^T = XAY^T - (a \quad b)Y^T = E(X,Y) - (a \quad b)Y^T$. Therefore, $E'(X^*,Y^*) \geq E'(X,Y^*)$ for all X, if and only if $E(X^*,Y^*) \geq E(X,Y^*)$, for all X.

6.8 (b) The three Nash equilibria are $X_1 = (\frac{1}{2}, \frac{1}{2}) = Y_1$, and the two nonsymmetric Nash points $((0,1),(1,0))$ and $((1,0),(0,1))$. So only X_1 is a possible ESS.

(c) From the following figure you can see that $(p_1(t), p_2(t)) \to (\frac{1}{2}, \frac{1}{2})$ as $t \to \infty$ and conclude that (X_1, X_1) is an ESS. Verify directly using the stability theorem that it is asymptotically stable.



The figure shows trajectories starting from three different initial points. In the three-dimensional figure you can see that the trajectories remain in the plane $p_1 + p_2 = 1$. The Maple commands used to find the stationary solutions, check their stability, and produce the graph are

```
> restart:with(DEtools):with(plots):with(LinearAlgebra):
> A:=Matrix([[3,2],[4,1]]); X:=<x1,x2>;
> Transpose(X).A.X;
> s:=expand(%);
> L:=A.X; f:=(x1,x2)->L[1]-s;g:=(x1,x2)->L[2]-s;
> solve({f(x1,x2)=0,g(x1,x2)=0},[x1,x2]);
```

```
> q:=diff(f(x1,x2),x1)+diff(g(x1,x2),x2);
> with(VectorCalculus):Jacobian(<f(x1,x2),g(x1,x2)>,[x1,x2]);
> j:=simplify(%);
> a1:=subs(x1=1/2,x2=1/2,j);b1:=subs(x1=1/2,x2=1/2,q);
> Determinant(a1);
> DEplot3d({D(p1)(t)=p1(t)*(3*p1(t)+2*p2(t)
                    -3*p1(t)^2-6*p1(t)*p2(t)-p2(t)^2),
           D(p2)(t)=p2(t)*(4*p1(t)+p2(t)
                    -3*p1(t)^2-6*p1(t)*p2(t)-p2(t)^2) },
           {p1(t),p2(t)}, t=0..10,
           [[p1(0)=0.1,p2(0)=0.9],[p1(0)=0.6,p2(0)=.4],
           [p1(0)=1/4,p2(0)=3/4]],scene=[t,p1(t),p2(t)],
           stepsize=.1,linecolor=t);
```

For the problem, `Determinant(a1)`=5>0, and `b1`=-6<0, so the stability Theorem 6.2.3 allows us to conclude that $(\frac{1}{2}, \frac{1}{2})$ is asymptotically stable.

6.9 The replicator equation becomes

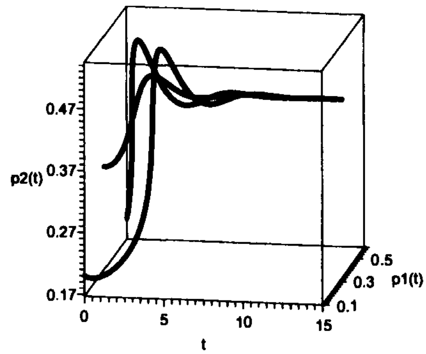
$$\frac{dp_1}{dt} = -2p_1(1-p_1)^2 \equiv f(p_1).$$

The stationary solutions are $p_1 = 0, 1$. The Nash equilibria of this game are $X_1 = Y_1 = (0,1)$ and $X_2 = Y_2 = (1,0)$. Now $f'(p_1) = -2(1-p_1)^2 + 4(1-p_1)$ and $f'(0) = -2 < 0$, but $f'(1) = 0$. The asymptotically stable solution is $X_1 = (0,1)$, so that X_1 is an ESS, which you may verify directly.

6.10 The unique Nash is $X = (\frac{15}{44}, \frac{20}{44}, \frac{9}{44}) = Y$, and it is symmetric. The replicator equations for p_1, p_2 are

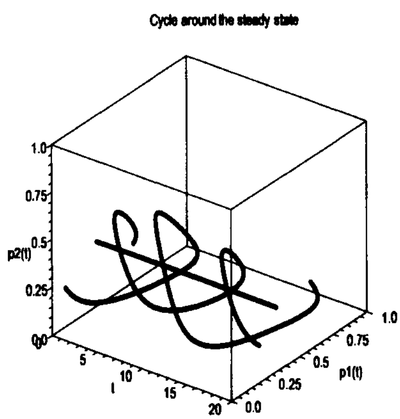
$$\frac{dp_1(t)}{dt} = p_1(t)(-9p_1(t) - 8p_2(t) + 5 + 4p_1(t)^2 + 4p_1(t)p_2(t) + 3p_2(t))^2$$
$$\frac{dp_2(t)}{dt} = p_2(t)(-p_1(t) - 3p_2(t) + 4p_1(t)^2 + 4p_1(t)p_2(t) + 3p_2(t))^2$$

Then $p_1 = \frac{15}{44}, p_2 = \frac{20}{44}$ is a stationary solution, and is asymptotically stable (check the stability theorem). The remaining stationary solutions are $(p_1 = 0, p_2 = 1)$, $(p_1 = 0, p_2 = 0)$, $(p_1 = 1, p_2 = 0)$. The convergence to $(\frac{15}{44}, \frac{20}{44}, \frac{9}{44})$ is shown in the figure.



6.11 The set of best response strategies to X^* consists of all strategies in which $Y = (y, 1-y, 0), 0 \leq y \leq 1$. Then $u(Y,Y) = -y^2 + 2y, u(X^*,Y) = 1$, and $-y^2 + 2y < 1$ for all $0 \leq y < 1$.

6.12 $X^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique symmetric Nash. But since $u(Y,Y) = 0, u(Y,X^*) = 0$, and $u(X^*,Y) = 0$, it will **not** be true that $u(Y,Y) < u(X^*,Y)$ for all Y that is a best response to X^*. Next, we see that the stationary solution is not asymptotically stable, as is illustrated in the figure.



6.13 The frequency dynamics become

$$\frac{dp_1(t)}{dt} = p_1(2p_2 - p_1 - 4p_1p_2 + p_1^2 + p_2^2)$$
$$\frac{dp_2(t)}{dt} = p_2(2p_1 - p_2 - 4p_1p_2 + p_1^2 + p_2^2)$$

The stationary solution $p_1 = \frac{1}{2}, p_2 = \frac{1}{2}$ is asymptotically stable because $f_{p_1}(\frac{1}{2}, \frac{1}{2}) + g_{p_2}(\frac{1}{2}, \frac{1}{2}) = -2 < 0$ and $\det J(\frac{1}{2}, \frac{1}{2}) = \frac{3}{4} > 0$.

REFERENCES

1. C. Aliprantis and S. Chakrabarti, *Games and Decision Making*, Oxford University Press, New York, 2000.
2. A. Devinatz, *Advanced Calculus*, Holt, Rinehart & Winston, New York, 1968.
3. T. Ferguson, *Game Theory*, Notes for a Course in Game Theory, available at www.gametheory.net.
4. H. Gintis, *Game Theory Evolving*, Princeton University Press, Princeton, NJ, 2000.
5. J. Goeree and C. Holt, *Stochastic game theory: For playing games, not just doing theory*, Proc. Natl. Acad. Sci. **96** (1999), 10564–10567.
6. J. Hofbauer and K. Sigmund, *Evolutionary game dynamics*, Bull. Amer. Math. Soc. **40** (2003), no. 4, 479–519.
7. A. J. Jones, *Game THEORY: Mathematical Models of Conflict*, Horwood Publishing, West Sussex, UK, 2000.
8. L. Jonker and P. D. Taylor, *Evolutionarily stable strategies and game dynamics*, Math.Biosci. **40** (1978), 145–156.
9. E. Kalai and M. Smorodinsky, *Other solutions to the Nash's bargaining problem*, Econometrica **43** (1975), 513–518.
10. E. L. Kaplan, *Mathematical Programming and Games*, Wiley, New York, 1982.

Game Theory: An Introduction, First Edition. By E.N. Barron

11. S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics, Vols. I, II*, Dover, New York, 1992.
12. C. Lemke and J. Howson, *Equilibrium points of bimatrix games*, Soc. Indust. Appl. Math. **12** (1964).
13. W. Lucas (ed.), *Game Theory and its Applications*, Proceedings of Symposia in Applied Mathematics, vol. 24, Providence, R. I., American Mathematical Society, 1981.
14. N. MacRae, *John Von Neumann: The Scientific Genius who Pioneered the Modern Computer, Game Theory, Nuclear Deterrence, and Much More*, American Mathematical Society, Providence, R. I., 1999.
15. M. Mesterton-Gibbons, *Introduction to Game Theoretic Modelling*, 2nd ed., American Mathematical Society, 2000.
16. P. Milgrom, *Auctions and bidding: A primer*, J. of Econ. Perspect. **3** (1989), no. 3, 3–22.
17. R. Myerson, *Game Theory, Analysis of Conflict*, Harvard University Press, Cambridge, MA, 1991.
18. S. Nasar, *A Beautiful Mind: The Life of Mathematical Genius and Nobel Laureate John Nash*, Touchstone, New York, 1998.
19. M. Osborne, *An Introduction to Game Theory*, Oxford University Press, New York, 2004.
20. G. Owen, *Game Theory*, 3rd ed., Academic Press, San Diego, CA, 2001.
21. T. Parthasarathy and T. E. S. Raghavan, *Some Topics in Two-Person Games*, vol. 22, Elsevier, New York, 1971.
22. E. Rasmussen, *Games and Information*, Basil Blackwell, Cambridge, UK, 1989.
23. S. Ross, *A First Course in Probability*, Prentice-Hall, Englewood Cliffs, NJ, 2002.
24. E. Scheinerman, *Introduction to Dynamical Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1996.
25. T. Vincent and J. Brown, *Evolutionary Game Theory, Natural Selection, and Darwinian Dynamics*, Cambridge University Press, Cambridge, UK, 2005.
26. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 3rd ed., Princeton University Press, Princeton, NJ, 1953 (1st ed., 1944).
27. N. Vorobev, *Game Theory, Lectures for Economists and Systems Scientists*, Springer Verlag, New York, 1977.
28. J. Wang, *The Theory of Games*, Oxford University Press, New York, 1988.
29. J. Weibull, *Evolutionary Game Theory*, MIT University Press, Cambridge, MA, 1995.
30. W. Winston, *Operations Research, Applications and Algorithms*, 2nd ed., PWS-Kent, Boston, 1991.
31. E. Wolfstetter, *Auctions: An introduction*, J. Econ. Surv. **10** (1996), 367–420.

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