Static solution of a crack degenerated from dynamic solution of a propagating crack*

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Abstract

A two-dimensional planar crack which propagates along its self plane is taken as the fault model. The static solution of the classic (linear elastic fracture mechanics) model for this study for three types of two dimensional cracks are obtained by means of degeneration method based on the dynamic solution obtained by Kostrov (1975). The degeneration method used in this study has two points of convenience: (1) One can obtain the solutions of different types of cracks by using the unified method; (2) It is avoided to use displacement potential and stress function which physical meaning is not straight. The results obtained in this paper are just the same as that obtained by previous authors who solved the equilibrium equations by means of integral transform method. It is showed that: (1) The static solution cannot be separated from the dynamic one, because the static solution also has the meaning of duration time. (2) Both the static and the dynamic solutions of a critical crack must satisfy the same criteria, and the evolution from static to dynamic solution must be associated with some additional disturbance. Particularly, the quantity of disturbance in some form has to be imposed when a critical crack is ready to initiate.

Key words: dynamics of earthquake rupture, critical crack, static solution, degeneration method

1 Introduction

There are three models to imitate the initiation of a propagating fault in earthquake rupture dynamics: (1) The fault appears abruptly and then starts to propagate (Kostrov, 1966; Aki and Richards, 1980); (2) The fault starts to propagate from a point (Chen *et al.*, 1987); (3) The fault starts to propagate from a quasi-static length (Andrews, 1976; Chen and Wang, 1989). The physical meanings in these three models are different. It is obvious that the selection of the model depends on the scale of time duration of quasi-propagation of the crack and the scale of area that stress drop occurs.

In this paper, the third model is considered. It is supposed that the fault has formed long before the moment t_0, i. e., the fault has been under the condition of quasi-static extension after it formed. By quasi-static extension we mean the changing is so slow that the inertia term in wave equation can be neglected. We also suppose that from the moment $t=-\infty$ to $t=t_0$, the tectonic stress field of the earth medium does not change significantly. A fault in this stage is called an initial fault. According to this assumption, the solution of a static crack should be the degeneration of a dynamic crack with infinitesimal propagating velocity after infinite long time elapsed. In other words, one can directly obtain the solution of a static crack in critical state from the degeneration of the solution of a dynanic propagating crack.

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The basic dynamic solution of two dimensional crack have been given by Kostrov (1966; 1975) with the use of classic (linear elastic fracture mechanism) model. Kostrov assumed that the crack appears abruptly at the moment $t=0$. It is worthy to notice that the case which Kostrov (1975) discussed is that mode Ⅰ, Ⅱ and Ⅲ cracks exist alone respectively without any interaction among different modes. Otherwise, the boundary condition and the results will interfere each other.

2 Basic formulation of the problem

In this study a two-dimensional planar crack which propagates along its self-plane is still used as the fault model. This crack locates at the plane $x_1=0$, $l_-(t)<x_2<l_+(t)$, $-\infty<x_3<+\infty$, where x_1, x_2, x_3 are Cartesian coordinates, the x_1-axis being normal to the crack surface and the x_3-axis being parallel to its edges.

In this study we assume that the crack is in isotropic, homogeneous and elastic medium, the problem is of plane strain, and that all applied loads are independent of the x_3 coodinate. The solution can be treated as the superposition of the initial solution without crack and the perturbation with crack. Suppose the prestress at the moment $t=-\infty$ are $\sigma_{ij}^0(x_1, x_2, t)$, and the perturbation are $\sigma_{ij}(x_1, x_2, t)$, then the total stress are

$$\sigma_{ij}^T(x_1, x_2, t)=\sigma_{ij}(x_1, x_2, t)+\sigma_{ij}^0(x_1, x_2, t) \tag{1}$$

and the displacement also satisfy $u_i^T=u_i+u_i^0$. We also suppose the initial solution without crack may be known, or may be obtained easily. The action of body forces can also be treated as an initial solution. At the moment $t=-\infty$,

$$u_i(x_1, x_2, t)=0 \qquad \text{for } t=-\infty \tag{2}$$

We denote $\sigma_i(x_2, t)=\sigma_{1i}(x_2, t)$. The boundary conditions will take the form:

$$\sigma_i(x_2, t)\equiv\sigma_{i1}(0^+, x_2, t)=\sigma_{i1}(0^-, x_2, t)=-p_i(x_2, t)$$
$$i=1, 2, 3, \qquad l_-(t)<x_2<l_+(t) \tag{3}$$

Where $p_i(x_2, t)$ is the loading acting on the crack surface. Outside the crack, the dislocation must be zero which gives an additional boundary condition:

$$W_i(0, x_2, t)=0 \qquad x_2<l_-(t) \text{ or } x_2>l_+(t) \qquad i=1, 2, 3 \tag{4}$$

The stress intensity factors at the crack tips are defined as:

$$\sigma_i=\frac{\mathscr{K}_i(t)}{(2|x_2-l_\mp(t)|)^{\frac{1}{2}}}+O(1) \qquad x_2\to l_\pm(t)\pm 0 \tag{5}$$

We denote $x=x_2$, and α, β as elastic wave velocity, $\alpha=\sqrt{(\lambda+2\mu)/\rho}$, $\beta=\sqrt{\mu/\rho}$, where ρ is mass density of the medium, λ is Lamè parameter and μ, shear modulus. According to the assumption in this paper, the basic equations obtained by Kostrov (1975) are modified as follows:

① Boundary integral equations.

Denote G_i as generalized dislocation, $\nu_1=\nu_3=\beta$, $\nu_2=\alpha$, in the area Δ_i: $\nu_i(t_0-t)^2-(x_0-x)^2$

$\geqslant 0$, $-\infty \leqslant t_0 \leqslant t$, $i = 1, 2, 3$, we have:

$$-\frac{1}{\nu_i} G_i(x, t) = \frac{1}{\pi} \iint_{\Sigma_i} \frac{F_i(x_0, t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0}{[\nu_i^2(t_0 - t)^2 - (x_0 - x)^2]^{\frac{1}{2}}} \tag{6}$$

where F_i is generalized stress. The relationship between F_i and σ_i is $F_i(x, t) = L_0(\sigma_i)$ where L_0 is an integral operator (Kostrov, 1975). For the convenience of integration, the characteristic coordinates are introduced:

$$\xi = (\nu_i t - x)/\sqrt{2}, \quad \eta = (\nu_i t + x)/\sqrt{2}$$

It is not difficult to prove that the integral area that contributes to integral (6) is a strip (Aki and Richards, 1980):

$$G_i(\xi, \eta) = \frac{-1}{\sqrt{2}\pi} \int_{\eta^*(\xi)}^{\eta} \frac{\mathrm{d}\eta_0}{\sqrt{\eta - \eta_0}} \int^{\xi} \frac{F_i(\xi_0, \eta_0)}{\sqrt{\xi - \xi_0}} \mathrm{d}\xi_0 \tag{7}$$

where $\eta^*(\xi)$ is the η coordinate of one of the crack tips.

2) Equations for calculating generalized stress.

The generalized stress is

$$F_i(x, t) = \frac{1}{\pi[x - l_+(t^*)]^{\frac{1}{2}}} \int_{-\infty}^{l_+(t^*)} f_i\left(x_1, t - \frac{x - x_1}{\alpha}\right) \frac{[l_+(t^*) - x_1]^{\frac{1}{2}}}{x - x_1} \mathrm{d}x_1$$

$$x > l_+(t^*) \tag{8}$$

where t^* is the root of equation $x - l(t^*) = \nu_i(t - t^*)$, f_i is generalized loading, the relationship between f_i and p_i also satisfy $f_i(x, t) = L_0(p_i)$. This gives

$$f_i(x, t) = -F_i(x, t) \tag{9}$$

In this paper, we assume that the type of crack is independent. Thus, only u_i which corresponds to $F_i (i = 1, 2, 3)$ is symmetric about the plane $x_1 = 0$, outside the crack,

$$u_i(0, x_2, t) = 0 \qquad x_2 < l_-(t) \text{ or } x_2 > l_+(t) \qquad i = 1, 2, 3 \tag{10}$$

For example, for mode Ⅰ crack, we have $u_1(x_2, t) = 0$, $x < l_-(t)$ or $x > l_+(t)$, but in the same area, u_2 does not have to be zero. The results for mode Ⅱ or Ⅲ crack are in similar way.

3 Solution for static crack by means of degeneration method

3.1 Solution for general case

In static case, i. e., $l_+(t) = c$, $l_-(t) = -c$,

$$F_i(x, t) = \sigma_i(x) \tag{11}$$

On the crack surface, the re-distributed stress is

$$\sigma_i(x) = -p_i(x) \qquad |x| < c \tag{12}$$

Where $p_i(x)$ is the loading on the crack surfaces. For mode Ⅱ or Ⅲ crack,

$$p_i(x) = \Delta\tau(x) \tag{13}$$

$$\Delta\tau = \tau_0(x) - \tau_d(x) \tag{14}$$

Where $\Delta\tau(x)$ is stress drop; $\tau_0(x)$ is prestress; and $\tau_d(x)$, sliding friction. According to the transform formula of G_i to W_i (Kostrov, 1975) and (10), for static case, the displacement is

$$u_i(x,0^+) - u_i(x,0^-) = \begin{cases} G_i(x)/(2C_i) & |x| < c \\ 0 & |x| \geqslant c \end{cases} \qquad i = 1,2,3 \tag{15}$$

where $C_1 = C_2 = \mu(1-\beta^2/\alpha^2) = 2\mu(\lambda+\mu)/(\lambda+2\mu)$, $C_3 = \mu/2$. In this paper, we assume that $u_i(x)$ on the crack surfaces and $\sigma(x)$ outside the crack are to be determined. Let the physical quantities in the dynamic solution of finite length crack be independent of t, and set the lower bound of integral being $t \to -\infty$, one can obtain the static solution degenerated from the dynamic solution. Degeneration of (8) to static case yields

$$\sigma_i(x) = \frac{-1}{\pi\sqrt{x-c}}\left[\int_{-\infty}^{-c} \sigma_i(\xi)\frac{\sqrt{c-\xi}}{x-\xi}\mathrm{d}\xi - \int_{-c}^{c} p_i(\xi)\frac{\sqrt{c-\xi}}{x-\xi}\mathrm{d}\xi\right] \qquad x > c \tag{16}$$

This is a second type of Fredholm integral equation. Suppose

$$\sigma_i(x) = \frac{\operatorname{sgn} x}{\pi\sqrt{x^2-c^2}}\int_{-c}^{c} p_i(\eta)\frac{\sqrt{c^2-\eta^2}}{x-\eta}\mathrm{d}\eta \qquad |x| > c \qquad i = 1,2,3 \tag{17}$$

where $\operatorname{sgn} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

It is not difficult to verify that (17) in the section $x < -c$ can be obtained by means of the coordinate transform from (17) in the section $x > c$, and *vice versa*. Substituting (17) into (16), it is not difficult to verify that (17) is just the solution of (16). Substituting (17) into (5), one obtains the static stress intensity factors at the crack tips.

$$\mathscr{K}_i^{\pm} = \lim_{x \to \pm c_-} (2)^{\frac{1}{2}}(|x \mp c|)^{\frac{1}{2}}\sigma_i(x) = \frac{1}{\pi\sqrt{c}}\int_{-c}^{c} p_i(\eta)\frac{\sqrt{c \pm \eta}}{\sqrt{c \mp \eta}}\mathrm{d}\eta \tag{18}$$

In characteristic co-ordinates, (7) is degenerated as

$$u_i(\xi,\eta) = -\frac{e_i}{\sqrt{2}}\int_{\eta_0^*(\xi)}^{\eta}\frac{d\eta_0}{\sqrt{\eta-\eta_0}}\int_{-\infty}^{\xi}\frac{\sigma_i(\xi_0,\eta_0)}{\sqrt{\xi-\xi_0}}\mathrm{d}\xi_0 \tag{19}$$

where $e_i = 1/(2\pi C_i)$. In terms of (x, t) variables, denoting $t_0 - t = t_1$, then (19) takes the form:

$$u_i(x) = -e_i\nu_i \iint_{S_2} \frac{\sigma_i(x_1)\mathrm{d}x_1\mathrm{d}t_1}{[\nu_i^2 t_1^2 + (x - x_1)^2]^{\frac{1}{2}}}$$

$$= -e_i\nu_i\left[\int_{-c}^{c} p_i(x_1) - \int_{|x_1|>c} \sigma_i(x_1)\right]\int_{S_2} \frac{\mathrm{d}t_1\mathrm{d}x_1}{[\nu_i^2 t_1^2 + (x - x_1)^2]^{\frac{1}{2}}}$$

$$= e_i\left[\int_{-c}^{c} p_i(x_1) - \int_{|x_1|>c} \sigma_i(x_1)\right]\ln\left|\frac{\sqrt{x_1 \pm c} + \sqrt{x \pm c}}{\sqrt{x_1 + c} - \sqrt{x + c}}\right| dx_1 \tag{20}$$

we treat (20) as an integral with variable x, the differentiation of u_i with respect to x is

$$\frac{\partial u_i}{\partial x} = \frac{e_i}{\sqrt{x + c}}\left[\int_{-c}^{c} p_i(\xi) - \int_{|\xi|>c} \sigma_i(\xi)\right]\frac{\sqrt{\xi + c}}{\xi - x}\mathrm{d}\xi \tag{21}$$

Substituting (17) into (21), integrating the both sides over x, exchanging the integration order, and determining the integral constant according to the condition $u_i(\pm c) = 0$, one obtains

$$u_i(x) = \frac{1}{2C_i\pi}\int_{-c}^{c} p_i(\xi)\left\{\ln\left|\frac{\xi\sqrt{c^2 - x^2} - x\sqrt{c^2 - \xi^2} + c(\xi - x)}{\xi\sqrt{c^2 - x^2} + x\sqrt{c^2 - \xi^2} + c(\xi - x)}\right|\right.$$

$$\left. - \ln\left|\frac{\sqrt{c + \xi} + \sqrt{c - \xi}}{\sqrt{c + \xi} - \sqrt{c - \xi}}\right|\right\}\mathrm{d}\xi \qquad |x| < c \qquad i = 1, 2, 3 \tag{22}$$

(17), (18) and (22) are just the static solution of stress increment, stress intensity factor and displacement for general case, respectively.

Particularly, for the case of constant loading, $p_i(x) = p_i$ $(|x| < c)$, if we denote

$$p_2 = p_3 = \Delta\tau = \tau_b - \tau_d \qquad |x| < c \tag{23}$$

where $\Delta\tau$ is stress drop; τ_b is pre-stress; and τ_d is sliding friction, then the corresponding results are

$$\sigma_i(x) = \begin{cases} -p_i & |x| < c \\ p_i\left(\dfrac{|x|}{\sqrt{x^2 - c^2}} - 1\right) & |x| > c \end{cases} \tag{24}$$

$$\mathscr{K}_i^{\pm} = p_i\sqrt{c} \tag{25}$$

$$u_i(x, 0^{\pm}) = \begin{cases} p_i\sqrt{c^2 - x^2}/2C_i & |x| < c \\ 0 & |x| \geqslant c \end{cases} \qquad i = 1, 2, 3 \tag{26}$$

The prestress in the crack plane is σ_i^0, where

$$\sigma_1^0 = p, \qquad \sigma_2^0 = \sigma_3^0 = \tau_b \tag{27}$$

The total stress is

$$\sigma_i^*(x) = \sigma_i(x) + \sigma_i^0 \tag{28}$$

In the case of $i=1$, the results shown in (23)—(28) are just that obtained by Inglis (1913), Griffith (1920) and Irwin (1957) in their pioneer work for the case of uniaxial tension (Sneddon and Lowengrub, 1969). In the case of $i=2$, the results are just that obtained by Starr (1928). In the case of $i=3$, the results are just that obtained by Knopoff (1958).

3.2 General solution as superposition of symmetric and asymmetric solution

In general, the loadings on the crack surfaces can always be expressed as the superposition of symmetric and asymmetric loadings.

$$\begin{aligned} p_i(x) &= \frac{1}{2}[p_i(x) + p_i(-x)] + \frac{1}{2}[p_i(x) - p_i(-x)] \\ &= p_{is}(x) + p_{ia}(x) \end{aligned} \tag{29}$$

where $p_{is}(-x) = p_{is}(x)$ represents the symmetric loading, and $p_{ia}(-x) = -p_{ia}(x)$ represents the asymmetric loading. The corresponding results are:

$$\sigma_i(x) = \frac{2x\,\mathrm{sgn}\,x}{\pi\sqrt{x^2 - c^2}} \int_0^c [\xi p_{is}(\xi) + x p_{ia}(\xi)] \frac{\sqrt{c^2 - \xi^2}}{x^2 - \xi^2} d\xi \qquad |x| > c \tag{30}$$

$$\mathscr{K}_i^{\pm} = \frac{\pm 2}{\pi\sqrt{c}} \int_0^c [c p_{is}(\xi) \mp \xi p_{ia}(\xi)] \frac{d\xi}{\sqrt{c^2 - \xi^2}} \tag{31}$$

$$\begin{aligned} u_i(x) = \frac{1}{2C_i\pi} \int_0^c \Bigg\{ & p_{is}(\xi) \ln \left| \frac{\sqrt{c^2 - x^2} + \sqrt{c^2 - \xi^2}}{\sqrt{c^2 - x^2} - \sqrt{c^2 - \xi^2}} \right| \\ & + p_{ia}(\xi) \ln \left| \frac{\xi\sqrt{c^2 - x^2} + x\sqrt{c^2 - \xi^2}}{\xi\sqrt{c^2 - x^2} - x\sqrt{c^2 - \xi^2}} \right| \Bigg\} d\xi \qquad |x| < c \end{aligned} \tag{32}$$

In the case of $i=3$, (30), (32) gives just the same results as that given by Chen and Chen (1990).

If the loading or stress drop distributes symmetrically, i.e., $p_{ia}(x)=0$, in the case of $i=3$, the solution (30) and (32) are just the same as that given by Chen and Knopoff (1986).

If the loading or stress drop distributs asymmetrically, i.e., $p_{is}(x)=0$, in the case of $i=1$, multipling $\mathscr{K}_i$ in (31) by $\sqrt{\pi}$ (cosidering the difference of the definition of $\mathscr{K}_i$ in different papers), we obtain the solution which is just the same as Chen (1989).

The static solution of a two-dimensional crack have been obtained by many authors previously. In general, the way to obtain the static solution is to drop the inertial terms in the wave equations, thus degenerate into equilibrium equations which can be solved by using the methods of integral transform (Sneddon and Lowengrub, 1969). In these papers, C_1 *and* C_2 are often written as $C_1 = C_2 = \mu/(1-\nu)$. This is because, according to the relationship among the elastic constants, we have $(\lambda+2\mu)/[2(\lambda+\mu)] = 1-\nu$, where ν is Poisson ratio. The uniqueness theorem of the solution assures that whatever degenerating step one starts from, the results obtained should be the same (Li, 1991).

4 Conclusions

We have presented the deduction of the solution for a static crack by means of the degenerating method from the solution for the corresponding dynamic crack. We have showed that the degenerate method presented in this paper is convenient. The solution of three types of crack can be derived by using the unified method and the displacement potential and stress function of which the physical meaning are not straight are avoided.

It is clear that as a special stage of the dynamic propagation, the static solution also has the meaning of duration time. In this paper, it is showed that a static crack can remain sub-critical under static loading in infinite long time unless an additional disturbance exists. Specifically, the quantity of disturbance in some form has to be given when the initiation problem of a critical crack is calculated.

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