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An Optimal Control Problem for the Schrödinger Equation with a Real-Valued Factor

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Abstract—We study an optimal control problem for the Schrödinger equation with a real-valued factor in its nonlinear part where the control function is square summable and the quality criterion is the Lions functional. First, we examine the correctness of the statement of the reduced problem and, second, we do that of the optimal control problem. We also study the differentiability of the Lions functional and obtain a necessary optimality condition in the form of a variational inequality.

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1. INTRODUCTION

Optimal control problems for the nonlinear Schrödinger equation often arise in quantum mechanics, nuclear physics, nonlinear optics, superconductivity theory, and other domains of the modern physics and engineering, where the coefficient of this equation [1, 2] plays the role of the control. Note that similar optimal control problems for the nonlinear Schrödinger equation were earlier investigated in detail, for example, in papers [3–8] and others, when control parameters were boundedly measurable functions.

In this paper we consider the optimal control problem for the nonlinear Schrödinger equation, where the control is a square summable function. Note that similar problems for the linear and nonlinear Schrödinger equations in another statement were investigated earlier, for example, in papers [9–12] and others. We pay an attention to the fact that in papers [10, 11] one studies optimal control problems for systems described by the Cauchy problems for the linear and nonlinear Schrödinger equations with a singular potential, considering weight functional spaces as the class of solutions to the Cauchy problems. We study the optimal control problem for systems described by the boundary-value problem for the nonlinear Schrödinger equation, using ordinary functional spaces as the class of solutions to the boundary-value problem. This paper essentially differs from previous works both in the problem definition and the obtained results.

2. PROBLEM DEFINITION

Let $l > 0$ and $T > 0$ be given numbers, $x \in (0,l)$, $t \in [0,T]$, $\Omega_t = (0,l) \times (0,t)$, and $\Omega = \Omega_T$. For $p > 1$ we denote by $L_p(\Omega)$ the Lebesgue space of measurable functions summable with the degree p. Denote by $C^k([0,T],B)$ the Banach space of all defined and $k > 0$ times continuously differentiable on $[0,T]$ functions whose values belong to the Banach space B. Let $W_p^k(0,l)$ and $W_p^{k,m}(\Omega)$ be Sobolev spaces [13, 14] of functions having generalized derivatives of order k with respect to x and of order $m \geq 0$ with respect to t, correspondingly, that are summable with the degree $p \geq 1$. Let the symbol $\overset{\circ}{W}{}_2^1(0,l)$ mean subspaces of the space $W_2^1(0,l)$, whose elements vanish at the endpoints of the segment $[0,l]$. We set $\overset{\circ}{W}{}_2^2(0,l) \equiv W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,l)$.

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Consider the minimization problem for the functional

$$J_\alpha(v) = \int_\Omega |\psi_1(x,t) - \psi_2(x,t)|^2 dx\,dt + \alpha\|v-\omega\|^2_{L_2(0,l)} \tag{1}$$

on the set $V \equiv \{v = v(x) : v \in L_2(0,l), \|v\|_{L_2(0,l)} \le b\}$ subject to

$$i\frac{\partial\psi_k}{\partial t} + a_0\frac{\partial^2\psi_k}{\partial x^2} - a(x)\psi_k - v(x)\psi_k + a_1|\psi_k|^2\psi_k = f_k(x,t),\ \ (x,t)\in\Omega,\ \ k=1,2, \tag{2}$$

$$\psi_k(x,0) = \varphi_k(0),\ \ x\in(0,l),\ \ k=1,2, \tag{3}$$

$$\psi_1(0,t) = \psi_1(l,t) = 0,\ \ t\in(0,T), \tag{4}$$

$$\frac{\partial\psi_2(0,t)}{\partial x} = \frac{\partial\psi_2(l,t)}{\partial x} = 0,\ \ t\in(0,T), \tag{5}$$

where $i^2=-1,\ a_0>0,\ b>0,\ \alpha\ge 0,$ and $-\infty<a_1<+\infty$ are given numbers, $a(x)$ is a bounded measurable function satisfying the condition

$$0<\mu_0\le a(x)\le\mu_1\ \ \forall x\in(0,l),\ \ \mu_0,\mu_1=\text{const}>0, \tag{6}$$

functions $\varphi_k=\varphi_k(x)$, $f_k=f_k(x,t)$, $k=1,2$, are given and satisfy the conditions

$$\varphi_1\in \overset{\circ}{W}{}^2_2(0,l),\ \ \varphi_2\in W^2_2(0,l),\ \ \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0, \tag{7}$$

$$f_k\in W_2^{0,1}(\Omega),\ \ k=1,2, \tag{8}$$

and $\omega\in L_2(0,l)$ is a given element.

We say that the problem on finding functions $\psi_k=\psi_k(x,t)$, $k=1,2$, from conditions (2)–(5) with given $v\in V$ is reduced. We understand its solution as functions $\psi_k=\psi_k(x,t)$, $k=1,2$, that belong to $B_1\equiv C^0([0,T],\overset{\circ}{W}{}^2_2(0,l))\cap C^1([0,T],L_2(0,l))$ and $B_2\equiv C^0([0,T],W_2^2(0,l))\cap C^0([0,T],L_2(0,l))$, respectively, and satisfy conditions (2)–(5) for almost all $x\in(0,l)$ and $\forall t\in[0,T]$.

The reduced problem (2)–(5) consists of two boundary-value problems for the nonlinear Schrödinger equation (2). For the function $\psi_1=\psi_1(x,t)$ this problem is the first boundary-value problem, and for $\psi_2=\psi_2(x,t)$ it is the second boundary-value problem for the Schrödinger equation. Boundary value problems for the linear and nonlinear Schrödinger equations were studied earlier, for example, in papers [3–5, 15–17], and others. However, in these papers the widest class of feasible controls is the set of measurable bounded functions. In this case the set of feasible controls belongs to the class of measurable square summable functions. Therefore it is necessary to study boundary-value problems for the nonlinear Schrödinger equation with given controls from the class of square summable functions. Taking into account this fact, in the paper [18] with the help of the Galerkin method we prove the following theorem.

Theorem 1. *Let functions $a(x)$, $\varphi_k(x)$, and $f_k(x,t)$ satisfy conditions* (6)–(8). *Then the reduced problem* (2)–(5) *with $v\in V$ has a unique solution $\psi_1\in B_1$ and $\psi_2\in B_2$, and the following estimates are valid:*

$$\begin{aligned}\|\psi_1(\cdot,t)\|_{\overset{\circ}{W}{}^2_2(0,l)} + \left\|\frac{\partial\psi_1(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} &\le c_1\Big(\|\varphi_1\|_{\overset{\circ}{W}{}^2_2(0,l)} + \|f_1\|_{W_2^{0,1}(\Omega)} + \|\varphi_1\|^3_{\overset{\circ}{W}{}^1_2(0,l)}\\ &+\|f_1\|^3_{W_2^{0,1}(\Omega)} + \|\varphi_1\|^9_{L_2(0,l)} + \|f_1\|^9_{L_2(\Omega)}\Big)\ \ \forall t\in[0,T],\end{aligned} \tag{9}$$

$$\begin{aligned}\|\psi_2(\cdot,t)\|_{W^2_2(0,l)} + \left\|\frac{\partial\psi_2(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} &\le c_2\Big(\|\varphi_2\|_{W^2_2(0,l)} + \|f_2\|_{W_2^{0,1}(\Omega)} + \|\varphi_2\|^3_{W^1_2(0,l)}\\ &+\|f_2\|^3_{W_2^{0,1}(\Omega)} + \|\varphi_2\|^9_{L_2(0,l)} + \|f_2\|^9_{L_2(\Omega)}\Big)\ \ \forall t\in[0,T],\end{aligned} \tag{10}$$

where $c_1>0$ and $c_2>0$ are some constants.

Note that estimates (9) and (10) correspond to the case $a_1\ne 0$. If $a_1=0$, then these estimates contain no nonlinear terms with respect to norms.

3. THE UNIQUE SOLVABILITY OF THE OPTIMAL CONTROL PROBLEM

Further we need the following theorem.

Theorem 2 (a corollary of the Goebel theorem [19]). *Assume that $\widetilde{X}$ is a uniformly convex space, U is a closed bounded set on $\widetilde{X}$, a functional $I(v)$ is lower semicontinuous and bounded from below on U, and $\alpha > 0$ is a given number. Then there exists a dense subset G of the space $\widetilde{X}$ such that for any $\omega \in G$ the functional*

$$J_\alpha(v) = I(v) + \alpha\|v - \omega\|^2_{\widetilde{X}}$$

attains its minimal value on U at a unique element.

Theorem 3. *Let conditions of Theorem 1 be fulfilled and let $\omega \in L_2(0,l)$ be a given element. Then there exists a dense subset G of the space $L_2(0,l)$ such that for any $\omega \in G$ with $\alpha > 0$ the optimal control problem (1)–(5) has a unique solution.*

Proof. Let us prove that the following functional is continuous

$$J_0(v) = \int_\Omega |\psi_1(x,t) - \psi_2(x,t)|^2 dx\, dt. \tag{11}$$

Let $\delta v \in L_2(0,l)$ be an increment of an arbitrary element $v \in V$ such that $v + \delta v \in V$. Then with $v \in V$ the solution $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$, $k = 1,2$, to the reduced problem (2)–(5) obtains the increment $\delta\psi_k = \delta\psi_k(x,t) \equiv \psi_k(x,t;v+\delta v) - \psi_k(x,t;v)$; here $\psi_k(x,t;v+\delta v)$ is a solution to the reduced problem with $v + \delta v \in V$. From conditions (2)–(5) it follows that functions $\delta\psi_k = \delta\psi_k(x,t)$, $k = 1,2$, present a solution to the boundary-value problem

$$i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - (v+\delta v)\delta\psi_k + a_1(|\psi_{k\delta}|^2 + |\psi_k|^2)\delta\psi_k$$
$$+ a_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k = \delta v\psi_k(x,t;v),\ \ (x,t) \in \Omega, \tag{12}$$

$$\delta\psi_k(x,0) = 0,\ \ x \in (0,l),\ \ k = 1,2, \tag{13}$$

$$\delta\psi_1(0,t) = \delta\psi_1(l,t) = 0,\ \ t \in (0,T), \tag{14}$$

$$\frac{\partial\delta\psi_2(0,t)}{\partial x} = \frac{\partial\delta\psi_2(l,t)}{\partial x} = 0, \quad t \in (0,T), \tag{15}$$

where $\psi_{k\delta} = \psi_{k\delta}(x,t) \equiv \psi_k(x,t;v+\delta v)$, $k = 1,2$.

Let us estimate the solution to this problem. We multiply both parts of Eq. (2) by the function $\delta\overline{\psi}_k(x,t)$, $k = 1,2$, and integrate the obtained correlations over the area Ω_t. As a result we obtain

$$\int_{\Omega_t}\Bigg[i\frac{\partial\delta\psi_k}{\partial\tau}\cdot\delta\overline{\psi}_k - a_0\left|\frac{\partial\delta\psi_k}{\partial x}\right|^2 - a(x)|\delta\psi_k|^2 - (v+\Delta v)|\delta\psi_k|^2$$
$$+a_1(|\psi_{k\delta}|^2 + |\psi_k|^2)|\delta\psi_k|^2 + a_1\psi_{k\delta}\psi_k(\delta\overline{\psi}_k)^2\Bigg]dx\,d\tau = \int_{\Omega_t}\delta v(x)\psi_k\delta\overline{\psi}_k dx\,d\tau,\ \ t \in [0,T],\ \ k = 1,2.$$

We subtract from this equality its complex conjugate one and get

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)} = 2\int_{\Omega_t}\mathrm{Im}[a_1\psi_{k\delta}\psi_k(\delta\overline{\psi}_k)^2]dx\,d\tau + 2\int_{\Omega_t}\mathrm{Im}[\delta v(x)\psi_k\delta\overline{\psi}_k]dx\,d\tau,\ \ t\in[0,T],\ \ k = 1,2.$$

Hence it follows that

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)} \le 2|a_1|\int_{\Omega_t}|\psi_k|\,|\psi_{k\delta}|\,|\delta\psi_k|^2 dx\,d\tau$$
$$+ 2\int_{\Omega_t}|\delta v(x)|\,|\psi_k|\,|\delta\psi_k|dx\,d\tau, \quad t \in [0,T], \quad k = 1,2. \tag{16}$$

Using estimates (9) and (10), the known inequality from [13]

$$\|\Phi(\cdot,t)\|_{L_\infty(0,l)} \le c_3\left\|\frac{\partial\Phi(\cdot,t)}{\partial x}\right\|^{1/2}_{L_2(0,L)}\|\Phi(\cdot,t)\|^{1/2}_{L_2(0,l)},\ \ t\in[0,T], \tag{17}$$

and its analog for functions which do not vanish at the endpoints of the segment $[0,l]$, it is not difficult to establish the validity of inequalities

$$\|\psi_k\|_{L_\infty(\Omega)} \le c_4, \quad \|\psi_{k\delta}\|_{L_\infty(\Omega)} \le c_4, \quad k=1,2, \tag{18}$$

where $c_4 > 0$ is some constant. Then, taking into account these inequalities and the Cauchy–Bunyakovsky inequality, from (16) we obtain

$$\|\delta\psi_k(\cdot,t)\|^2_{L_2(0,l)} \le c_5\int_0^t \|\delta\psi_k(\cdot,\tau)\|^2_{L_2(0,l)}d\tau + c_6\|\delta v\|^2_{L_2(0,l)} \ \ \forall t\in[0,T],\ \ k=1,2.$$

Applying the Gronwall lemma, we obtain the estimates $\forall t \in [0,T]$,

$$\|\delta\psi_k(\cdot,t)\|_{L_2(0,l)} \le c_7\|\delta v\|_{L_2(0,l)}, \quad k=1,2. \tag{19}$$

Now we consider the increment of the functional $J_0(v)$ on any element $v\in V$. Evidently,

$$\delta J_0(v) = J_0(v+\delta v) - J_0(v) = 2\int_\Omega \mathrm{Re}[(\psi_1(x,t)-\psi_2(x,t))\cdot(\delta\overline{\psi}_1(x,t)-\delta\overline{\psi}_2(x,t))]dx\,dt$$
$$+\|\delta\psi_1\|^2_{L_2(\Omega)} + \|\delta\psi_2\|^2_{L_2(\Omega)} - 2\int_\Omega \mathrm{Re}(\delta\psi_1\delta\overline{\psi}_2)dx\,dt. \tag{20}$$

By applying the Cauchy–Bunyakovsky inequality and estimates (9), (10), and (19), we obtain the estimate

$$|\delta J_0(v)| \le c_7\big(\|\delta v\|_{L_2(0,l)} + \|\delta v\|^2_{L_2(0,l)}\big), \tag{21}$$

where the constant $c_7 > 0$ is independent of δv. Hence we get the continuity of the functional $J_0(v)$ on any element $v\in V$, i.e., on the set V. The latter set is a closed bounded convex subset of the uniformly convex space $L_2(0,l)$ [20]. Then due to Theorem 2 and the boundedness and continuity of the functional $J_0(v)$ on the set V, there exists an everywhere dense subset G of the space $L_2(0,l)$ such that $\forall\omega\in G$ with $\alpha>0$ problem (1)–(5) has a unique solution. □

From Theorem 3 it follows that with $\alpha>0$ the optimal control problem (1)–(5) does not necessarily have a solution for any $\omega\in L_2(0,l)$.

Let us show that with $\alpha\ge 0$ problem (1)–(5) has at least one solution for any element $\omega\in L_2(0,l)$.

Theorem 4. *Let conditions of Theorem* 3 *be fulfilled and* $\alpha\ge 0$ *be a given number. Then the optimal control problem* (1)–(5) *has at least one solution for any* $\omega\in L_2(0,l)$.

Proof. Choose a minimizing sequence $\{v^m\}\in V$ for the functional $J_\alpha(v)$

$$\lim_{m\to\infty} J_\alpha(v^m) = \inf_{v\in V} J_\alpha(v) = J_{\alpha*}.$$

Put $\psi_{km}(x,t)=\psi_k(x,t;v^m)$, $k=1,2$, $m=1,2,\dots$ Since $\{v^m\}\subset V$, under conditions of Theorem 1 the reduced problem (2)–(5) has a unique solution $\psi_{1m}\in B_1$, $\psi_{2m}\in B_2$, $m=1,2,\dots$, and the following estimates are valid:

$$\|\psi_{1m}(\cdot,t)\|_{\overset{\circ}{W}{}^2_2(0,l)} + \left\|\frac{\partial\psi_{1m}(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le c_8, \quad m=1,2,\dots; \tag{22}$$

$$\|\psi_{2m}(\cdot,t)\|_{W^2_2(0,l)} + \left\|\frac{\partial\psi_{2m}(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le c_9, \quad m=1,2,\dots, \tag{23}$$

$\forall t\in[0,T]$, where c_8 and c_9 stand for the right-hand sides of estimates (9) and (10); they are independent of m.

Since V is a closed bounded convex subset of the reflexive Banach space $L_2(0,l)$, it is weakly compact and weakly closed in this space. Therefore from the sequence $\{v^m\} \subset V$ we can extract a subsequence (for simplicity we denote it by $\{v^m\}$) such that $v^m \to v$ weakly in $L_2(0,l)$ as $m \to \infty$ and $v \in V$. Then $\forall q \in L_2(0,l)$,

$$\lim_{m\to\infty} \int_0^l v^m(x)q(x)dx = \int_0^l v(x)q(x)dx. \tag{24}$$

From estimates (22) and (23) it follows that sequences $\{\psi_{km}(x,t)\}$, $k=1,2$, are uniformly bounded in the norm of spaces B_1 and B_2, respectively. From these sequences we can extract subsequences (for simplicity we again denote them by $\{\psi_{km}(x,t)\}$) such that $\{\psi_{km}(x,t)\}$, $\{\frac{\partial\psi_{km}(x,t)}{\partial x}\}$, $\{\frac{\partial\psi_{km}(x,t)}{\partial t}\}$, $\{\frac{\partial^2\psi_{km}(x,t)}{\partial x^2}\}$, $k=1,2$, converge weakly in $L_2(0,l)$ as $m\to\infty$ to functions $\psi_k(x,t)$, $\frac{\partial\psi_k(x,t)}{\partial x}$, $\frac{\partial\psi_k(x,t)}{\partial t}$, $\frac{\partial^2\psi_k(x,t)}{\partial x^2}$, $k=1,2$, respectively, with each $t\in[0,T]$. Let us make sure that the limit functions $\psi_k = \psi_k(x,t)$, $k=1,2$, satisfy Eqs. (2) for almost all $x\in(0,l)$ and each $t\in[0,T]$. To this end we consider the integral identities

$$\int_0^l \Big[i\frac{\partial\psi_{km}(x,t)}{\partial t} + a_0\frac{\partial^2\psi_{km}(x,t)}{\partial x^2} - a(x)\psi_{km}(x,t) \\ -v^m(x)\psi_{km}(x,t) + a_1|\psi_{km}(x,t)|^2\psi_{km}(x,t) - f_k(x,t)\Big]\overline{g}_k(x)dx = 0, \quad k=1,2, \tag{25}$$

for any functions $g_k \in L_2(0,l)$, $k=1,2$, and for $t\in[0,T]$. At first we prove the validity of the following limit correlations for $m\to\infty$:

$$\int_0^l v^m(x)\psi_{km}(x,t)\overline{g}_k(x)dx \to \int_0^l v(x)\psi_k(x,t)\overline{g}_k(x)dx, \tag{26}$$

$k=1,2$, for any functions $g_k\in L_2(0,l)$ and $t\in[0,T]$. It is clear that the following equalities are true:

$$\int_0^l v^m(x)\psi_{km}(x,t)\overline{g}_k(x)dx = \int_0^l v^m(x)(\psi_{km}(x,t)-\psi_k(x,t))\overline{g}_k(x)dx \\ + \int_0^l (v^m(x)-v(x))\psi_k(x,t)\overline{g}_k(x)dx + \int_0^l v(x)\psi_k(x,t)\overline{g}_k(x)dx, \quad k=1,2, \quad t\in[0,T]. \tag{27}$$

Let us estimate the first term in the right-hand side of (27)

$$\left|\int_0^l v^m(x)(\psi_{km}(x,t)-\psi_k(x,t))\overline{g}_k(x)dx\right| \le \|v^m\|_{L_2(0,l)}\|\overline{g}_k\|_{L_2(0,l)}\|\psi_{km}(\cdot,t)-\psi_k(\cdot,t)\|_{L_\infty(0,l)} \tag{28}$$

for $t\in[0,T]$, $k=1,2,\dots$, $m=1,2,\dots$ According to embedding theorems, spaces B_1 and B_2 are compactly embedded in $C([0,T],L_\infty(0,l))$ (see [21, 12]). This means that

$$\|\psi_{km}(\cdot,0)-\psi_k(\cdot,t)\|_{L_\infty(0,l)} \to 0, \quad k=1,2, \quad m\to\infty, \tag{29}$$

uniformly with respect to $t\in[0,T]$. Hence and from inequality (28) it follows that the first term in the right-hand side of (27) converges to zero as $m\to\infty$ for each $t\in[0,T]$. In addition, for each $t\in[0,T]$ functions $\psi_k(x,t)$, $k=1,2$, belong to the space $L_\infty(0,l)$, and $\psi_k(\cdot,t)\overline{g}_k \in L_2(0,l)$, $k=1,2$. Therefore, taking into account the limit correlation (24), we obtain that the second term in the right-hand side of (27) also tends to zero as $m\to\infty$ for each $t\in[0,T]$. Proceeding to the limit in (27) as $m\to\infty$, we obtain that formula (26) is valid.

Using the limit correlation (29), we can validate the formula

$$\lim_{m\to\infty}\int_o^l a_1|\psi_{km}(x,t)|^2\psi_{km}(x,t)\overline{g}_k(x)dx = \int_o^l a_1|\psi_k(x,t)|^2\psi_k(x,t)\overline{g}_k(x)dx, \quad k=1,2, \tag{30}$$

$\forall g_k\in L_2(0,l)$, $k=1,2$, and for each $t\in[0,T]$.

Therefore, using correlations (26) and (30) and the weak convergence of the sequence $\{\psi_{km}(x,t)\}$, $k=1,2$, in B_1 and B_2, respectively, proceeding to the limit in integral identities (25) as $m\to\infty$, we obtain that functions $\psi_k(x,t)$, $k=1,2$, satisfy Eqs. (2) for almost all $x\in(0,l)$ and for each $t\in[0,T]$. Moreover, due to the properties of convergence, sequences $\{\psi_{km}(x,t)\}$, $k=1,2$, satisfy initial conditions (3) and boundary conditions (4) and (5). Thus, we have proved that the limit functions $\psi_k=\psi_k(x,t)$, $k=1,2$, which correspond to the limit control function $v=v(x)$ from V present a solution to the reduced problem (2)–(5), i.e., $\psi_k=\psi_k(x,t)\equiv\psi_k(x,t;v)$, $k=1,2$. It is clear that functions $\psi_k(x,t)$, $k=1,2$, belong to spaces B_1 and B_2, respectively, and satisfy estimates (9) and (10) which directly follow from those (22) and (23) by proceeding to the limit as $m\to\infty$.

Due to the compactness of the embedding of spaces B_1 and B_2 in $L_2(\Omega)$ we get

$$\psi_{km}(x,t)\to\psi_k(x,t) \ \text{ strongly in } \ L_2(\Omega)$$

as $m\to\infty$, $k=1,2$, a fortiori weakly in $L_2(\Omega)$. Then, using the weak lower semicontinuity of norms of spaces $L_2(\Omega)$ and $L_2(0,l)$ and the condition $\alpha\geq 0$, from the form of the functional $J_\alpha(v)$ we obtain that the functional $J_\alpha(v)$ is weakly lower semicontinuous on an element $v\in V$ and for any $\omega\in L_2(0,l)$. Therefore the following correlation takes place:

$$J_{\alpha*}\leq J_\alpha(v)\leq \varliminf_{m\to\infty} J_\alpha(v^m)=J_{\alpha*}.$$

Thus, $v\in V$ is the minimum of the functional $J_\alpha(v)$ on the set V, i.e., $v\in V$ is a solution to the optimal control problem (1)–(5). □

4. THE DIFFERENTIABIITY OF THE FUNCTIONAL AND THE NECESSARY OPTIMALITY CONDITION

Now let us study the differentiability of the functional and establish the necessary optimality condition in problem (1)–(5). We introduce the conjugate problem that implies the definition of functions $\eta_k=\eta_k(x,t)$, $k=1,2$, from conditions

$$i\frac{\partial\eta_k}{\partial t}+a_0\frac{\partial^2\eta_k}{\partial x^2}-a(x)\eta_k-v(x)\eta_k+a_1(2|\psi_k|^2\eta_k+\psi_k^2\overline{\eta}_k)=2(-1)^k(\psi_1(x,t)-\psi_2(x,t)),\ \ k=1,2, \tag{31}$$

$$\eta_k(x,T)=0,\ \ k=1,2,\ \ x\in(0,l), \tag{32}$$

$$\eta_1(0,t)=\eta_1(l,t)=0,\ \ t\in(0,T), \tag{33}$$

$$\frac{\partial\eta_2(0,t)}{\partial x}=\frac{\partial\eta_2(l,t)}{\partial x}=0,\ \ t\in(0,T), \tag{34}$$

where $\psi_k=\psi_k(x,t)$, $k=1,2$, is a solution to the reduced problem (2)–(5) with $v\in V$. We understand a solution to this problem as functions $\eta_k=\eta_k(x,t)$, $k=1,2$, from $C^0([0,T],\overset{\circ}{W}{}_2^1(0,l))$ and $C^0([0,T],W_2^1(0,l))$, respectively, that satisfy the integral identities

$$\int_\Omega\Big[-i\eta_k\frac{\partial\overline{\Phi}_k}{\partial t}-a_0\frac{\partial\eta_k}{\partial x}\frac{\partial\overline{\Phi}_k}{\partial x}-a(x)\eta_k\overline{\Phi}_k-v(x)\eta_k\overline{\Phi}_k+2a_1|\psi_k|^2\eta_k\overline{\Phi}_k$$
$$+a_1\psi_k^2\overline{\eta}_k\overline{\Phi}_k\Big]dx\,dt=2(-1)^k\int_\Omega(\psi_1(x,t)-\psi_2(x,t))\overline{\Phi}_k(x,t)dx\,dt,\ \ k=1,2, \tag{35}$$

for any functions $\Phi_1\in\overset{\circ}{W}{}_2^{1,1}(\Omega)$ and $\Phi_2\in W_2^{1,1}(\Omega)$, such that $\Phi_k(x,0)=0$, $k=1,2$.

Using the same technique as that applied for the reduced problem (2)–(5), we can analogously prove the unique solvability of the conjugate boundary-value problem (31)–(34) and the following estimates for its solution:

$$\|\eta_1(\cdot,t)\|_{\overset{\circ}{W}{}_2^1(0,l)}\leq c_{10}\|\psi_1-\psi_2\|_{W_2^{0,1}(\Omega)}, \tag{36}$$

$$\|\eta_2(\cdot,t)\|_{W_2^1(0,l)}\leq c_{11}\|\psi_1-\psi_2\|_{W_2^{0,1}(\Omega)} \tag{37}$$

$\forall t \in [0, T]$, where $c_{10} > 0$ and $c_{11} > 0$ are some constants.

Consider the increment of the functional $J_\alpha(v)$ on any element $v \in V$. Let $\delta v \in L_2(0,l)$ be the increment of $v \in V$ such that $v + \delta v \in V$. In view of (1) and (20) we have

$$\delta J_\alpha(v) = J_\alpha(v+\delta v) - J_\alpha(v) = 2\int_\Omega \mathrm{Re}[(\psi_1(x,t) - \psi_2(x,t)) \cdot (\delta\overline{\psi}_1(x,t) - \delta\overline{\psi}_2(x,t))]dx\,dt$$
$$+ 2\alpha \int_0^l (v(x) - \omega(x))\delta v(x)dx + \|\delta\psi_1\|^2_{L_2(\Omega)} + \|\delta\psi_2\|^2_{L_2(\Omega)}$$
$$+ \alpha\|\delta v\|^2_{L_2(0,l)} - 2\int_\Omega \mathrm{Re}(\delta\psi_1(x,t)\delta\overline{\psi}_2(x,t))dx\,dt, \quad (38)$$

where $\delta\psi_k = \delta\psi_k(x,t)$, $k = 1, 2$, are solutions to the boundary-value problem (12)–(15).

Theorem 5. *Let conditions of Theorem 2 be fulfilled. Then the functional $J_\alpha(v)$ is Freshet differentiable on the set V and its gradient admits the representation*

$$J'_\alpha(v) = -\int_0^T \mathrm{Re}(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t))dt + 2\alpha(v(x) - \omega(x)), \quad (39)$$

where $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$ and $\eta_k = \eta_k(x,t) \equiv \eta_k(x,t;v)$ are solutions to the reduced problem (2)–(5) and the conjugate problem (31)–(34), respectively.

Proof. It is clear that $\delta\psi_k = \delta\psi_k(x,t) \equiv \psi_k(x,t;v+\delta v) - \psi_k(x,t;v)$ satisfy the following integral identities for any functions $\Phi_{1k} = \Phi_{1k}(x,t)$ from $L_2(\Omega)$, $k = 1, 2$:

$$\int_\Omega \Big[i\frac{\partial\delta\psi_k}{\partial t} + a_0\frac{\partial^2\delta\psi_k}{\partial x^2} - a(x)\delta\psi_k - (v+\delta v)\delta\psi_k + a_1(|\psi_{k\delta}|^2 + |\psi_k|^2)\delta\psi_k$$
$$+ a_1\psi_{k\delta}\psi_k\delta\overline{\psi}_k\Big]\overline{\Phi}_{1k}(x,t)dx\,dt = \int_\Omega \delta v\psi_k(x,t)\overline{\Phi}_{1k}(x,t)dx\,dt, \quad k = 1, 2.$$

Taking into account these identities and integral identities (35), as well as the conditions $\delta\psi_1 \in \overset{\circ}{W}{}^{2,1}_2(\Omega)$, $\delta\psi_2 \in W^{2,1}_2(\Omega)$, and $\delta\psi_k(x,0) = 0$, $k = 1, 2$, we get the equality

$$2\int_\Omega \mathrm{Re}[(\psi_1(x,t) - \psi_2(x,t))(\delta\overline{\psi}_1(x,t) - \delta\overline{\psi}_2(x,t))]dx\,dt$$
$$= -\int_\Omega \mathrm{Re}(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t))\delta v(x)dx\,dt$$
$$- \int_\Omega \mathrm{Re}(\delta\psi_1(x,t)\overline{\eta}_1(x,t) + \delta\psi_2(x,t)\overline{\eta}_2(x,t))\delta v(x)dx\,dt$$
$$+ \int_\Omega \mathrm{Re}[a_1(|\psi_{1\delta}|^2\delta\psi_1\overline{\eta}_1 - |\psi_1|^2\delta\psi_1\overline{\eta}_1)]dx\,dt + \int_\Omega \mathrm{Re}[a_1(|\psi_{2\delta}|^2\delta\psi_2\overline{\eta}_2 - |\psi_2|^2\delta\psi_2\overline{\eta}_2)]dx\,dt$$
$$+ \int_\Omega \mathrm{Re}[a_1(\psi_{1\delta}\psi_1\delta\overline{\psi}_1\overline{\eta}_1 - \psi_1^2\delta\overline{\psi}_1\overline{\eta}_1)]dx\,dt + \int_\Omega \mathrm{Re}[a_1(\psi_{2\delta}\psi_2\delta\overline{\psi}_2\overline{\eta}_2 - \psi_2^2\delta\overline{\psi}_2\overline{\eta}_2)]dx\,dt.$$

By applying it to the right-hand side of correlation (38), we obtain the following formula for the increment of the functional:

$$\delta J_\alpha(v) = \int_0^l \Big[-\int_0^T \mathrm{Re}(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t))dt + 2\alpha(v(x) - \omega(x))\Big]\delta v(x)dx + R,$$

$$R = \|\delta\psi_1\|^2_{L_2(\Omega)} + \|\delta\psi_2\|^2_{L_2(\Omega)} + \alpha\|\delta v\|^2_{L_2(0,l)} - 2\int_\Omega \mathrm{Re}(\delta\psi_1\delta\overline{\psi}_2)dx\,dt$$

$$- \int_\Omega \mathrm{Re}(\delta\psi_1 \overline{\eta}_1 + \delta\psi_2 \overline{\eta}_2)\delta v(x) dx\, dt + a_1 \int_\Omega \mathrm{Re}[|\psi_{1\delta}|^2 \delta\psi_1 \overline{\eta}_1 - |\psi_1|^2 \delta\psi_1 \overline{\eta}_1] dx\, dt$$
$$+ a_1 \int_\Omega \mathrm{Re}[|\psi_{2\delta}|^2 \delta\psi_2 \overline{\eta}_2 - |\psi_2|^2 \delta\psi_2 \overline{\eta}_2] dx\, dt + a_1 \int_\Omega \mathrm{Re}[\psi_{1\delta}\psi_1 \delta\overline{\psi}_1 \overline{\eta}_1 - \psi_1^2 \delta\overline{\psi}_1 \overline{\eta}_1] dx\, dt$$
$$+ a_1 \int_\Omega \mathrm{Re}[\psi_{2\delta}\psi_2 \delta\overline{\psi}_2 \overline{\eta}_2 - \psi_2^2 \delta\overline{\psi}_2 \overline{\eta}_2] dx\, dt.$$

Hence due to the Cauchy–Bunyakovsky inequality and estimates (18) and (19) we obtain

$$|R| \le c_{12} \|\delta v\|^2_{L_2(0,l)} + c_{13} \|\delta v\|^2_{L_2(0,l)} (\|\eta_1\|_{L_\infty(\Omega)} + \|\eta_2\|_{L_\infty(\Omega)}), \tag{40}$$

where constants $c_{12} > 0$ and $c_{13} > 0$ are independent of δv.

Let us estimate $\|\eta_k\|_{L_\infty(\Omega)}$, $k = 1, 2\ldots$ Using an inequality of the type (17) for the function $\eta_1(x,t)$ and its analog for the function $\eta_2(x,t) \in C^0([0,T], W_2^1(0,l))$, as well as estimates (9), (10), (36), and (37), we obtain $\|\eta_k\|_{L_\infty(\Omega)} \le c_{14}$, $k = 1, 2$. Hence, taking into account formula (40), we get $|R| \le c_{15} \|\delta v\|^2_{L_2(0,l)}$, where the constant $c_{15} > 0$ is independent of δv. This means that

$$R = o(\|\delta v\|_{L_2(0,l)}). \tag{41}$$

Then, taking into account this correlation, we can represent the increment of the functional $J_\alpha(v)$ in the form

$$\Delta J_\alpha(v) = \int_0^l \Big[-\int_0^T \mathrm{Re}(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t)) dt$$
$$+ 2\alpha(v(x) - \omega(x)) \Big] \delta v(x) dx + o\big(\|\delta v\|_{L_2(0,l)}\big). \tag{42}$$

In accordance with (42), the functional $J_\alpha(v)$ is Freshet differentiable and its gradient admits representation (39). □

Using Theorem 5 and the known theorem from [22] (P. 28), we can establish the necessary optimality condition in problem (1)–(5).

Theorem 6. *Let conditions of Theorem* 4 *be fulfilled and let* $v^* = v^*(x)$ *be an optimal control from* V *in problem* (1)–(5). *Then* $\forall v \in V$ *the following inequality takes place*:

$$\int_0^l \Big[\int_0^T \mathrm{Re}(\psi_1^*(x,t)\overline{\eta}_1^*(x,t) + \psi_2^*(x,t)\overline{\eta}_2^*(x,t)) dt - 2\alpha(v^*(x) - \omega(x)) \Big] (v(x) - \omega^*(x)) dx \le 0,$$

where $\psi_k^*(x,t) \equiv \psi_k^*(x,t;v^*)$, $k = 1, 2$, *is a solution to the reduced problem* (2)–(5), *and* $\eta_k^*(x,t) \equiv \eta_k^*(x,t;v^*)$, $k = 1, 2$, *is a solution to the defined problem* (34)–(37) *with* $v^* \in V$.

One can prove this theorem with the help of the well-known theorem from the paper [22] (P. 28).

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