Cheatsheet - Proofs

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1. Intro

A **proof** is a sequence of logical statements that explain why a statement is true (or not true). However, some statements are true even if they cannot be logically proven.

2. Direct Proof

Direct proofs are easy because no particular technique is used. But it can be difficult to find a starting point.

Lets consider the example:

$$\text{even}(n) \land \text{even}(m) \rightarrow \text{even}(n+m)$$

I.e. if both numbers are even, then the sum of both numbers is even, too.

We can prove this: knowing that when an integer is even, it is twice another integer. We conclude:

$$n = 2k, m = 2l$$

$$m + n = 2k + 2l = 2(k + l) \quad \text{(factored)}$$

$$k + l = t \quad \text{(arbitrary name)}$$

$$m + n = 2t$$

3. Proof by Contradiction (Indirect Proof)

Lets say we want to prove that a statement is true. We then assume that that statement is false. Proof by contradiction means we now have to arrive at a statement that contradicts our assumption (implying that the original statement is true).

> **NOTE** The following example is from Quora: https://qr.ae/pvqpcu

For example, consider the statement *"there is no largest number in the world"*. We now assume the opposite: *"N is the largest number in the world"*. What happens if we multiply that number by itself?

$$N \times N = N$$

... because we assume N is the largest number in the world, meaning $N \times N$ cannot be bigger than N.

We know that when we divide a number by itself, the result is 1:

$$\frac{N}{N} = 1$$

Respectively:

$$\frac{N \times N}{N} = \frac{N \times \cancel{N}}{\cancel{N}} = N = 1$$

This implies that the largest number in the world is 1, but we can prove by contradiction that:

$$2 > 1$$

Hence the original statement *"there is not largest number in the world"* is correct.

4. Proof by Contrapositive (Indirect Proof)

Lets say we want to prove the following statement:

$$A \rightarrow B \quad \text{is true}$$

The **contrapositive** being:

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

Now prove that $\neg B \rightarrow \neg A$ is true. Sometimes it's easier to prove the contrapositive of a statement.

> **NOTE** The following example is from StackExchange: https://math.stackexchange.com/q/88565

For example:

$$x^2 \neq x \rightarrow x \neq 1$$

is equivalent to:

$$x = 1 \rightarrow x^2 = x$$

Respectively, we can prove:

$$1^2 = 1$$

Hence, the original statement is true.

5. Proof by Induction

If $P(0)$, $P(k)$ and $P(k+1)$ is true, then $P(n)$ is always true. In other words, if we can prove that the proposition is true for the first step, the next step and the one after that, then we can conclude that proposition is always true.

For example, lets prove that for any positive integer n, the function $n^3 + 2n$ results in a number divisible by 3.

$$1^3 + 2 \times 1 = 1 + 2 = 3 \rightarrow \frac{3}{3} = 1$$
$$2^3 + 2 \times 2 = 8 + 4 = 12 \rightarrow \frac{12}{3} = 4$$
$$3^3 + 2 \times 3 = 27 + 6 = 33 \rightarrow \frac{33}{3} = 11$$
$$\ldots$$

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