

# Cheatsheet - Functions

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## 1. Intro

A function maps one element from a set to **exactly one** element of another (or the same) set.

For example, we have sets:

$$A = \{1, 2, 3\}$$
$$B = \{2, 4, 6, 7, 8\}$$

and the function:

$$f: A \rightarrow B$$

defined as:

$$f(x) = 2x$$

Now lets apply each element from  $A$  to  $f$ :

$$f(1) = 2$$
$$f(2) = 4$$
$$f(3) = 6$$

We call set  $A$  the **domain** and set  $B$  the **co-domain**. The set of all possible values when mapping elements from set  $A$  to set  $B$ , respectively set  $R = \{2, 4, 6\}$ , is called the **range**. Hence  $R \subseteq B$ .

Additionally, we say that 1 is the **pre-image** of 2, which in return is the **image** of 1. The element 2 is the pre-image of 4, which in return is the image of 2. And so on.

## 2. Injective, Surjective & Bijective Functions

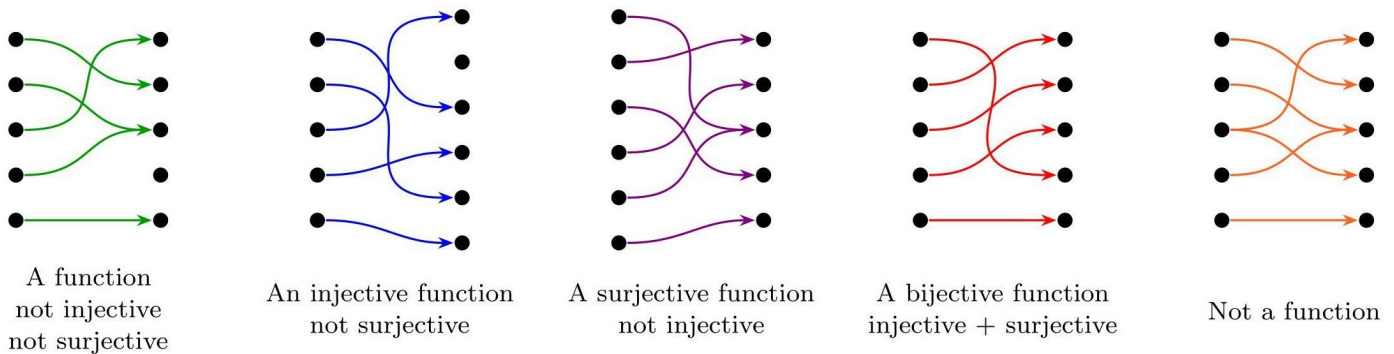


Figure 1. Source: <https://twitter.com/JDHamkins/status/841318019397779456>

- **General function:**  $A$  has at most one  $B$  (not injective, not surjective).
- **Injective:**  $A$  has exactly one  $B$  (not surjective).
- **Surjective:** Each and every  $B$  has one or many  $A$  (not injective).
- **Bijective:** Each and every  $B$  has exactly one  $A$  (injective and surjective).
- **NOT a function:**  $A$  has many  $B$ .

### 2.1. Proofs

We can prove whether a function is injective, surjective or bijective by solving an equation. Lets use the following function as an example:

$$f(x) = 2x + 3$$

#### Injective Proof

Let  $a, b \in R$ , show that if  $a \neq b$  then  $f(a) \neq f(b)$ :

$$\begin{aligned} a &\neq b && \times 2 \\ 2a &\neq 2b && + 3 \\ 2a + 3 &\neq 2b + 3 \\ \Rightarrow f(a) &\neq f(b) \end{aligned}$$

**Or:** Let  $a, b \in R$ , show that if  $f(a) = f(b)$  then  $a = b$ :

$$\begin{aligned}f(a) &= f(b) \Rightarrow \\2a + 3 &= 2b + 3 \quad - 3 \\2a &= 2b \quad \div 2 \\a &= b\end{aligned}$$

Hence, function  $f$  is injective.

#### Surjective Proof

Let  $y \in R$ , show that there exists  $x \in R$  such that  $f(x) = y$ :

$$\begin{aligned}f(x) &= y \Rightarrow \\2x + 3 &= y \quad - 3 \\2x &= y - 3 \quad \div 2 \\x &= \frac{y - 3}{2}\end{aligned}$$

Hence, function  $f$  is surjective (TODO: clarify this).

### 3. Composition

Function composition means we apply one function to the result of another.

For example:

$$(f \circ g)(x) = f(g(x))$$

which means that the result of  $g()$  is passed on to  $f()$ . If we define  $f(x) = 2x$  and  $g(x) = x^2$ , then:

$$(f \circ g)(5) = f(g(5)) = 2 \times (5^2) = 50$$

Do note that function composition is not commutative, meaning  $f \circ g \neq g \circ f$ :

$$(f \circ g)(5) = f(g(5)) = 2 \times (5^2) = 50$$

$$(g \circ f)(5) = g(f(5)) = (5 \times 2)^2 = 100$$

### 4. Inverse Function

If function  $f$  is bijective, then there exists an inverse function  $f^{-1}$ .

$$\begin{aligned}f: A &\rightarrow B \\f^{-1}: B &\rightarrow A\end{aligned}$$

For example, given  $f(x) = 2x$ , then  $f^{-1}(x) = \frac{x}{2}$ .

$$\begin{aligned}f(2) &= 4 \\f^{-1}(4) &= 2\end{aligned}$$

Additionally:

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

### 5. Exponential Functions

Properties of exponential the function:

$$y = f(x) = b^x \quad (b > 0 \text{ and } b \neq 1)$$

- The domain is  $(-\infty, \infty)$
- The range is  $(0, \infty)$
- It passes through the point  $(0, 1)$
- If  $b > 1$  then it's increasing on  $(-\infty, \infty)$  ("exponential growth")
- If  $b < 1$  then it's decreasing on  $(-\infty, \infty)$  ("exponential decay")

### 6. Logarithmic Functions

The logarithmic function with base  $b$  where  $b > 0$  and  $b \neq 1$  is defined as:

$$\log_b x = y \text{ if and only if } x = b^y$$

Respectively:

$$x = b^y \Leftrightarrow \log_b(x) = y$$

For example:

$$81 = 3^4 \Leftrightarrow \log_3(81) = 4$$

## 6.1. Laws

$$\log_b(m \times n) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b(m^n) = n \times \log_b(m)$$

$$\log_b(1) = 0$$

$$\log_b(b) = 1$$

Conventionally, we also define **natural logarithms** as:

$$\log = \log_{10}$$

$$\ln = \log_e$$

Where  $e$  is the "Euler number" ([https://en.wikipedia.org/wiki/E\\_\(mathematical\\_constant\)](https://en.wikipedia.org/wiki/E_(mathematical_constant))) ( $e = 2.71828$ ).

## 7. Floor and Ceiling Functions

We define the **floor** of the real number  $x$  as (round **down** to the previous integer **or equal**):

$$x = 3.6$$

$$\lfloor x \rfloor = 3$$

We define the **ceiling** of real number  $x$  as (round **up** to the next integer **or equal**):

$$x = 3.6$$

$$\lceil x \rceil = 4$$

Additionally (equal):

$$y = 5$$

$$\lfloor y \rfloor = 5$$

$$\lceil y \rceil = 5$$

and (negative numbers)

$$x = -3.5$$

$$\lfloor x \rfloor = -4$$

$$\lceil x \rceil = -3$$

Both the floor and the ceiling function convert a real number to an integer, respectively  $\mathbb{R} \rightarrow \mathbb{Z}$ .

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