Cheatsheet - Graphs

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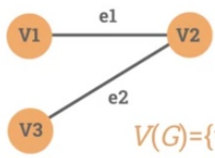
1. Intro

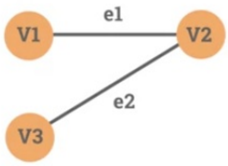
Graphs are **discrete** structures consisting of **vertices (nodes)** and **edges** connecting them. Graph theory is an area of discrete mathematics which studies these types of discrete structures.

The **graph** G can be represented as an ordered pair $G = (V, E)$, where V is a set of nodes/vertices and E is a set of edges, lines or connections. A **vertex** (singular of "vertices") is a basic element of a graph, usually drawn as a node or a dot. The set of vertices of G is usually denoted by $V(G)$ or V.



$V(G)=\{v_1, v_2, v_3\}$

An **edge** is a link between 2 vertices, usually drawn as a line connecting two vertices. The set of edges in a graph G is usually denoted by $E(G)$ or E.



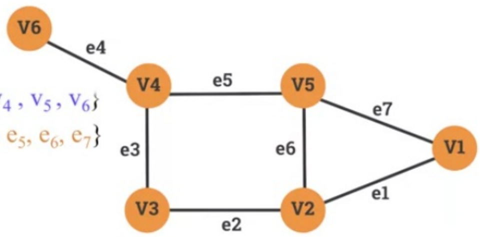
$E(G)=\{e_1, e_2\} = \{\{v_1, v_2\}, \{v_2, v_3\}\}$

Two vertices are said to be **adjacent** if they are endpoints of the same edge. Two edges are said to be **adjacent** if they share the same vertex. If a vertex v is an endpoint of an edge e, then we say that e and v are **incident**.

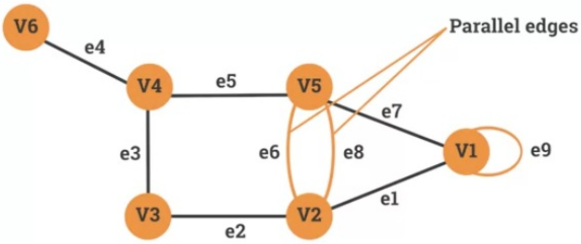
A **directed graph**, also called a **digraph**, is a graph in which the edges have a direction. This is usually indicated with an arrow on the edge.

1.1. Examples

- $V:=\{v_1, v_2, v_3, v_4, v_5, v_6\}$
- $E=\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$



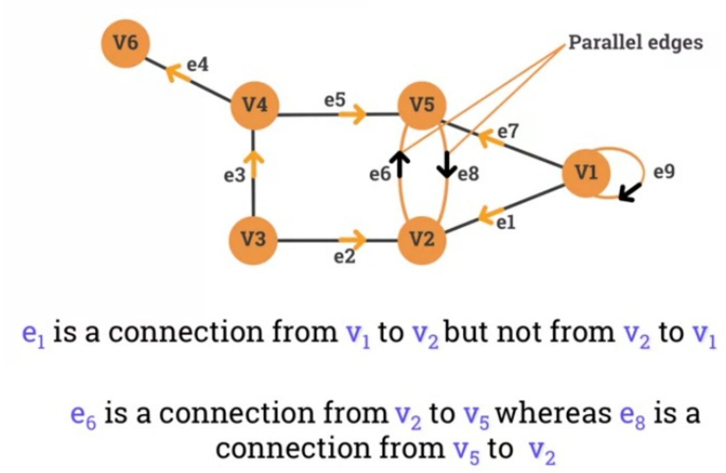
v_1 and v_2 are endpoints of the edge e_1. We say that v_1 and v_2 are **adjacent**.
The edges e_1 and e_7 share the same vertex v_1. We say that e_1 and e_7 are **adjacent**.
The vertex v2 is an endpoint of the edge e_1. We say that e_1 and v_2 are **incident**.



v_2 and v_5 are are linked with two edges (e_6 and e_8).
e_6 and e_8 are called parallel edges.

v_1 is linked to itself by e_9. The edge e_9 is called a loop.

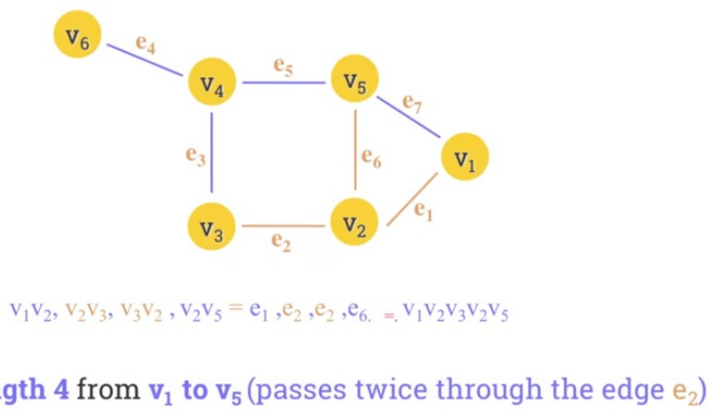
And an example of a directed graph:



2. Concepts

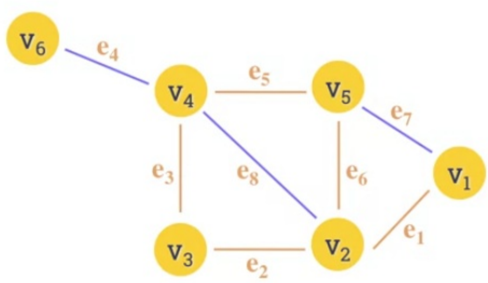
2.1. Walk

A **walk** is a sequence of vertices and edges of a graph were vertices and edges can be repeated. A **walk of length k** in a graph is a succession of k (not necessarily different) edges of the form uv, vw, wx, ..., yz.



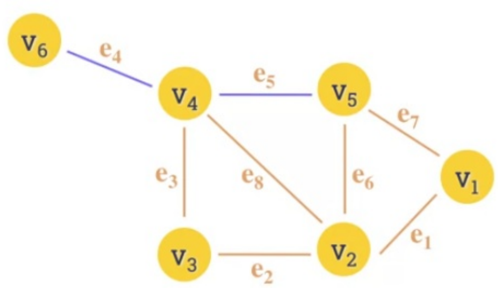
2.2. Trail

A **trail** is a walk in which no edge is repeated. In a trail, vertices can be repeated but no edge is ever repeated. For example, $e1$, $e2$, $e3$, $e5$, $e6$ is a trail:



2.3. Circuit

A **circuit** is a closed trail. Circuits can have repeated vertices only. For example, $e7$, $e6$, $e8$, $e3$, $e2$, $e1$ is a circuit:

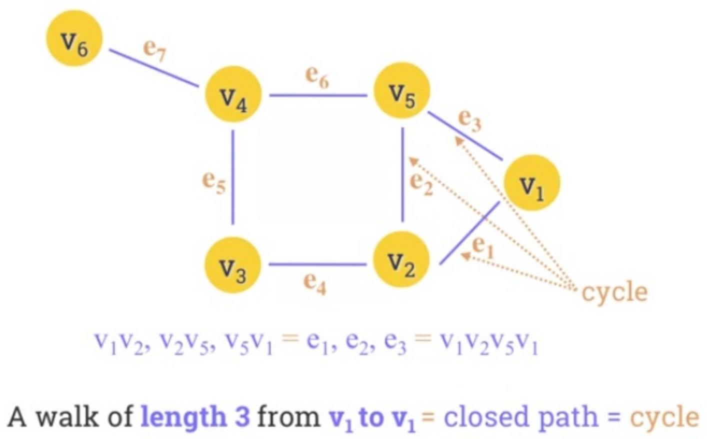


2.4. Path

A **path** is a trail in which neither vertices nor edges are repeated.

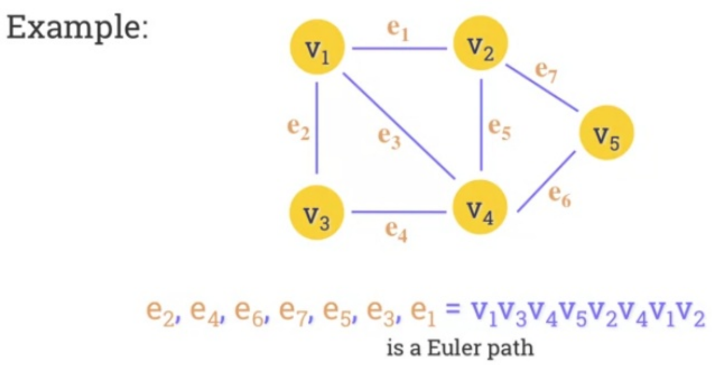
2.5. Cycle

A **cycle** is a closed path, consisting of edges and vertices where a vertex is reachable from itself.



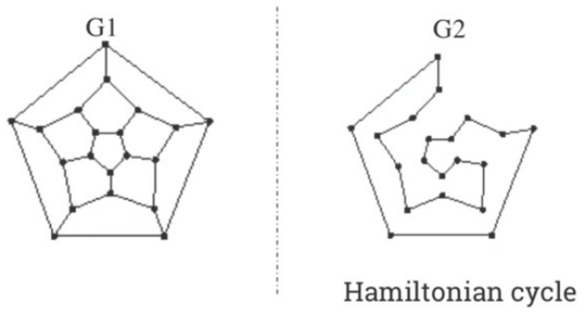
2.6. Eulerian Path

A **Eulerian path** in a graph is a path that uses each edge precisely once. If such a path exists, the graph is called **traversable**.



2.7. Hamiltonian Path, Cycle & Graph

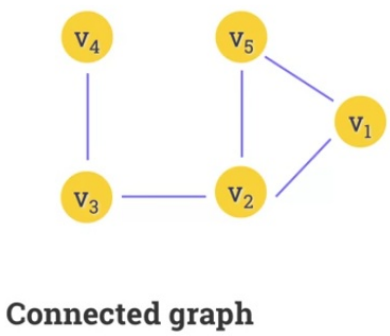
A **Hamiltonian path** (also called a *traceable path*) is a path that visits each vertex exactly once. A **Hamiltonian cycle** is a cycle that visits each vertex exactly once (except for the starting vertex, which is visited once at the start and once again at the end).



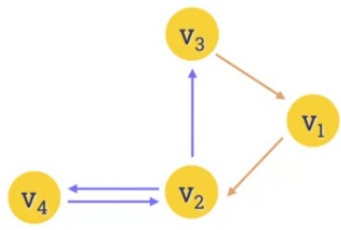
A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**. Any Hamiltonian cycle can be converted to a Hamiltonian path by removing one of its edges.

2.8. Connectivity

An **undirected** graph is **connected** if you can get from **any node to any other** by following a **sequence of edges**. Or, **any two nodes** are **connected** by a path.



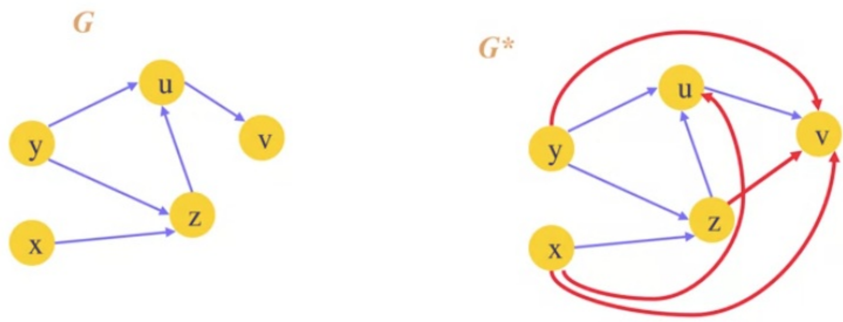
A directed graph is **strongly connected** if there is a **directed path** from any node to any other node.



Strongly connected directed graph

2.9. Transitive Close

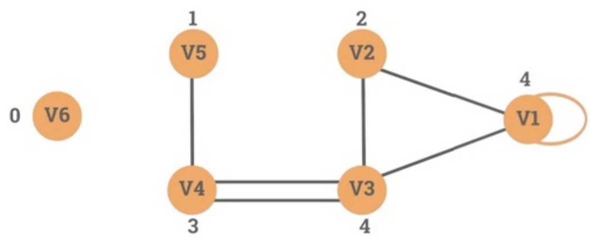
Given a digraph G, the transitive closure of G is the digraph G^* such that G^* has the same vertices as G. If G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v.



The transitive closure provides reachability information about a digraph.

3. Degree of a Vertex

The degree of a vertex is the number of edges incident on v. A loop contributes **twice** to the degree. An **isolated vertex** has a degree of 0.

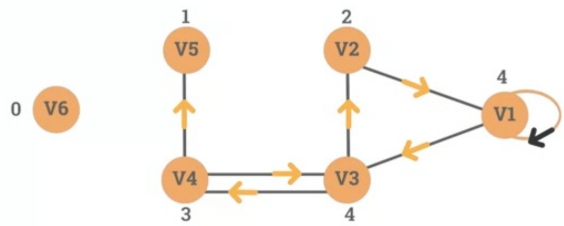


In the case of directed graphs, $\text{In-deg}(v)$ is the number of edges for which v is the terminal vertex. $\text{Out-deg}(v)$ is the number of edges for which v is the initial vertex.

And the degree $\deg(v)$ is:

$$\deg(v) = \text{Out-deg}(v) + \text{In-deg}(v)$$

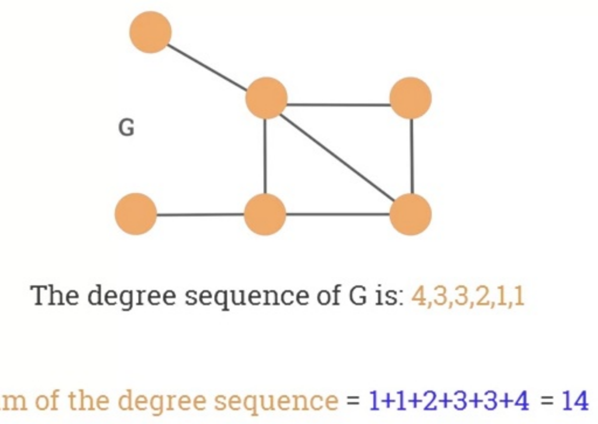
A loop contributes **twice** to the degree as it contributes 1 to both in-degree and out-degree.



$\deg(v_1) = \text{in-deg}(v_1) + \text{out-deg}(v_1) = 2 + 2 = 4$
$\deg(v_2) = \text{in-deg}(v_2) + \text{out-deg}(v_2) = 1 + 1 = 2$
$\deg(v_3) = \text{in-deg}(v_3) + \text{out-deg}(v_3) = 2 + 2 = 4$
$\deg(v_4) = \text{in-deg}(v_4) + \text{out-deg}(v_4) = 1 + 2 = 3$
$\deg(v_5) = \text{in-deg}(v_5) + \text{out-deg}(v_5) = 1 + 0 = 1$
$\deg(v_6) = \text{in-deg}(v_6) + \text{out-deg}(v_6) = 0 + 0 = 0$

3.1. Degree Sequence

Given an undirected graph G, a **degree sequence** is a **monotonic non-increasing** sequence of the vertex degrees of all the vertices of G. The **sum of the degree sequence** of a graph is always **even**.



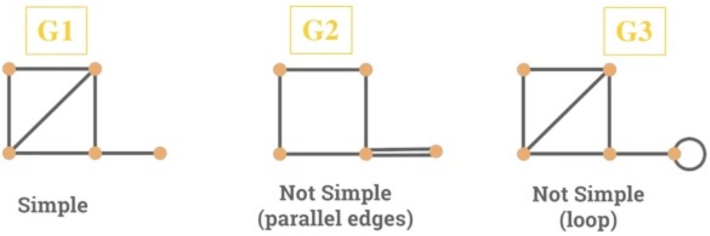
Given a graph G, the sum of the degree sequence of G is twice the number of edges in G.

$$\text{Number of edges of } \; G = \frac{\text{sum of degree sequences of } \; G}{2}$$

4. Special Graphs

4.1. Simple Graphs

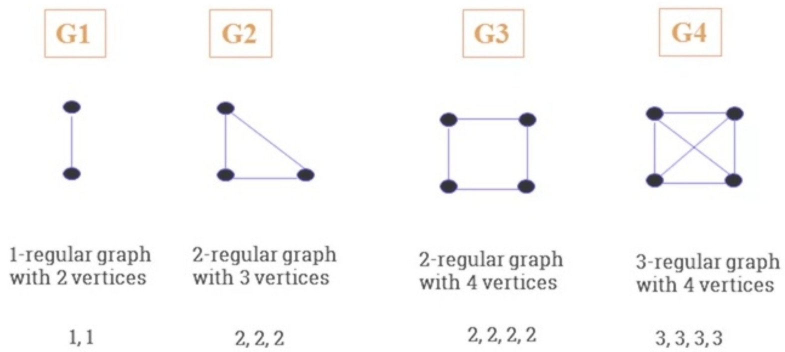
A **simple graph** is a graph without **loops** and **parallel** edges.



Given a **simple** graph G with n vertices, then the degree of each vertex of G is at most equal to $n - 1$.

4.2. Regular Graphs

A graph is said to be **regular** of degree if all local degrees are the same number. A graph G where all the vertices are of the same degree, r, is called an **r-regular** graph.

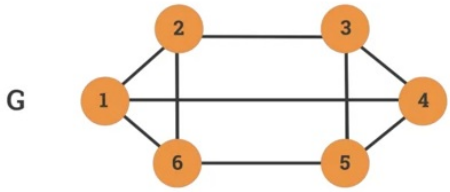


Given a **r-regular** graph G with n vertices, then the following is true:

$$\text{Degree sequence of } \; G = r, r, r, ..., r \quad (n \;\; \text{times})$$

$$\text{Sum of degree sequence of } \; G = r \times n$$

$$\text{Number of edges in } \; G = r \times \frac{n}{2}$$

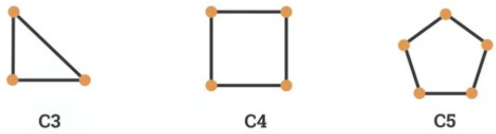


Degree Sequence = 3,3,3,3,3,3

Sum of degree sequence = 3x6=18

Number of edges = 18/2=9

4.3. Special Regular Graphs: Cycles

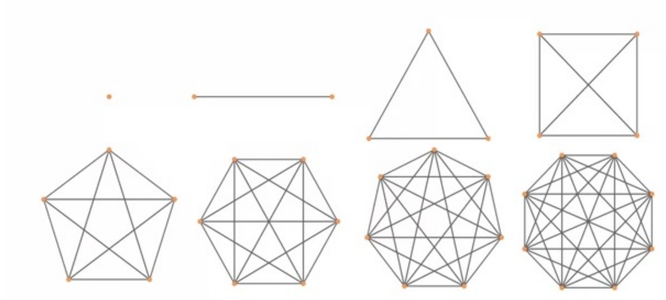


C_3 is 2-regular graph with 3 vertices
C_4 is 2-regular graph with 4 vertices
C_5 is 2-regular graph with 5 vertices

deg seq. of C_3 = 2,2,2
deg seq. of C_4 = 2,2,2,2
deg seq. of C_5 = 2,2,2,2,2

4.4. Complete Graphs

A **complete graph** is a **simple** graph where **every pair of vertices** are **adjacent** (linked with an edge). We represent a complete graph with n vertices using the symbol K_n.

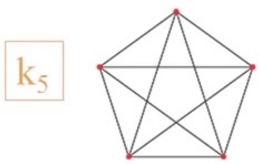


A complete graph with n vertices, k_n, has the following properties:

$$\text{Every vertex has a degree } \ (n-1)$$

$$\text{Sum of degree sequence } = n(n-1)$$

$$\text{Number of edges } = \frac{n(n-1)}{n}$$



There are 5 vertices
Degree of each vertex = (5-1) = 4
Sum of deg. Seq. = 5(5-1) = 20
Number of edges = 5(5-1)/2=20/2=10