Cheatsheet - Probability & Combinatorics

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1. Intro

Some events are impossible, while others are quite certain to happen. When something is impossible, we say that the probability is **0**. If something is very likely going to happen, we say the probability is **1**. We use the letter **P** to indicate a probability of something occuring.

$$P(\text{a human will live for 1'000 years}) = 0$$

$$P(\text{all of us will die someday}) = 1$$

No probabilities are outside the range of 0 to 1.

We say that two events are **complementary** if they exclude each other, for example something can either exist or not exist, but not exist and not exist at the same time. The sum of the probabilities of the two complementary events must always equal 1, also known as the **total probability**.

2. Calculating Theoretical Probabilities

If the odds of some event occuring is the same for all possible events (e.g. when rolling a dice), the probability is **unbiased**. This **theoretical probability** is calculated the following way:

$$P(\text{obtaining our chosen event}) = \left(\frac{\text{number of ways the chosen event can occur}}{\text{total number of possibilities}}\right)$$

For example, the probability of rolling *4* on a dice is:

$$P(\text{rolling } 4) = \frac{1}{6} = 0.1666...$$

Or, the probability of rolling above *4* (by rolling *5* or *6*):

$$P(\text{rolling above } 4) = \frac{2}{6} = \frac{1}{3} = 0.333...$$

3. Independent Events

If two events such as A and B are **independent** from each other, then the probability of obtaining A and B is given by:

$$P(A \quad \text{and} \quad B) = P(A) \times P(B)$$

For example, what is the probability that a coin toss and a dice throw result in a *head* and a *six*? We calculate:

$$\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

4. Combinatorics

We can calculate the permutations and combinations of elements. Permutations and combinations are similar, but differ due to the fact that combinations do not care about ordering.

4.1. Permutations

A **permutations** of *n* different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on. In order to calculate the number of different permutations of *n* elements we use the **factorial** of *n*, respectively $n!$:

$$n! = n \times (n-1) \times ...2 \times 1$$

For example:

$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$$

Additionally, there are $3! = 6$ permutations of the letters X, Y and Z:

$$XYZ$$

$$YXZ$$

$$XZY$$

$$ZYX$$

$$ZXY$$

$$YZX$$

NOTE $0! = 1.$

4.1.1. Permutations of n Elements Taken r at the Time

The number of permutations of n elements taken r at a time:

$$P_r^n = \frac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1)$$

For example, consider the question:

> *There are 8 horses running in a race. In how many different ways can these horses come in first, second and thrid (ties excluded)?*

We can calculate:

$$P_3^8 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

4.1.2. Distinguishable Permutations

When there are non-unique elements in a list, such as $\{A, B, B, C, C, C, D\}$, we might want to calculate the number of *distinguishable permutations*, not all possible permutation.

We the use formula:

$$\frac{n!}{n_1! \times n_2! \times ...n_k!}$$

Where n_k is the number of kinds of elements. As an example, for the set $\{A, B, B, C, C, C, D\}$, we have one *A*, two *B*'s, three *C*'s and one *D*:

$$\frac{7!}{1! \times 2! \times 3! \times 1!} = \frac{5040}{12} = 420$$

4.2. Combinations

Unlike permutations, combinations do not care about ordering. Respectively:

$$\{A, B, C\} = \{C, B, A\}$$

To calculate the number of possible combinations of $\{A, B, C\}$, we use the number of letters as r and use the following formula (where P_r^n is defined in the permutations section):

$$C_r^n = \frac{n!}{(n-r)!r!} = \frac{P_r^n}{r!}$$

respectively (note that $0! = 1$):

$$C_3^3 = \frac{3!}{(3-3)!3!} = \frac{6}{6} = 1$$

Unlike permutations, combinations always require an r (i.e. combinations of *n* elements taken at *r* at a time).

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