1. Binomial Theorem

An expression consisting of two terms, connected by a + or - sign, is called a **binomial expression**. As we increase the power of binomials, expanding them becomes more and more complicated:

$$(x+y)^{1} = x + y$$
$$(x+y)^{2} = x^{2} + 2xy + y^{2}$$
$$(x+y)^{3} = x^{3} + 3x^{2}y + 3xy^{2} + y^{3}$$
...

The **binomial theorem** helps us to simplify this expansion. Let x and y be variables, and n a non-negative integer. The expansion of $(x+y)^n$ can be formalized as:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

The **binomial coefficients** are the coefficients in the binomial theorem and denoted as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here we say "n choose k".

For example:

What is the coefficient of x^8y^7 in the expansion of $(3x-y)^{15}$.

We can view the expression as $(3x+(-y))^{15}$. By the binomial theorem:

$$(3x+(-y))^{15} = \sum_{k=0}^{15} inom{15}{k} (3x)^k (-y)^{15-k}$$

The coefficient of x^8y^7 in the expansion is obtained when k=8:

$$\binom{15}{8}(3)^8(-1)^7 = -3^8 \frac{15!}{8!7!}$$

TODO: Expand on this.

1.1. Pascal's Identity

If n and k are integers with $n \geq k \geq 1$, then:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

1.2. Pascal's Triangle

Pascals' triangle is a number triangle with numbers arranged in staggered rows such that $a_{n,r}$ is the binomial coefficient $\binom{n}{r}$.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & 1 & 1 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} & 1 & 2 & 1 \\ \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} & 1 & 3 & 3 & 1 \\ \begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} & 4 & 4 & 4 & 4 \\ \end{pmatrix}$$

$$\binom{4}{3} = \binom{3}{2} + \binom{3}{3} = 3 + 1 = 4$$

Using Pascal's identity, we can show that the result of adding two **adjacent** coefficients in this triangle is **equal** to the binomial coefficient in the **next** row between these two coefficients.