Cheatsheet - Propositional & First-order Logic

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1. Intro

Predicates describe properties of objects.

For example:

$$\text{odd}(3)$$

$\text{odd}(3)$ means 3 is an odd number. odd is a predicate, 3 is an object. Predicates take arguments and become **propositions**. A proposition is a statement that can be either *true* or *false*. It must be one or the other, and it cannot be both.

Connectives can be applied:

$$\text{odd}(3) \land \text{prime}(3)$$

This means that 3 is odd but also prime.

2. Syntax

Propositions are denoted by capital letters, such as $P, Q, ...$. General statements are denoted by lowercase letters, such as $p, q, ...$

3. Connectives

Logical NOT: $\neg p$ is true if and only if p is false (also called *negation*).

Logical OR: $p \lor q$ is true if and only if at least one of p or q is true or if both p and q are true (also called *disjunction*).

Logical AND: $p \land q$ is true if and only if both p and q are true (also called *conjunction*).

Logical IF...THEN: $p \rightarrow q$ is true if and only if either p is false or q is true (also called *conditional* or *implication*). p is the premise, q is the conclusion.

Logical IF and only IF: $p \leftrightarrow q$ is true if and only if both p and q are true (also called *bi-conditional*).

Exclusive OR: XOR: $p \oplus q$ is true if p or q is true but not both.

3.1. Translation to Connectives

As an example, lets consider the propositions:

- $P = \text{I study 20 hours a week}$
- $R = \text{I will pass the exam}$
- $S = \text{I will be happy}$
- $Q = \text{I attend all the lectures}$

And the following connectives:

$$(P \lor Q) \rightarrow (R \land S)$$

which is a translation of: "**If** I study 20 hours a week **or** attend all the lectures, **then** I will pass the exam **and** I will be happy."

4. Truth Tables

4.1. Negation: $\neg$

$$true = \neg false$$
$$false = \neg true$$

4.2. Conjunction: $\land$

$$true \land true = true$$
$$true \land false = false$$
$$false \land true = false$$
$$false \land false = false$$

4.3. Disjunction: $\lor$

$$true \lor true = true$$

$$true \lor false = true$$

$$false \lor true = true$$

$$false \lor false = false$$

4.4. Implication: $\rightarrow$

$$true \rightarrow true = true$$

$$true \rightarrow false = false$$

$$false \rightarrow true = true$$

$$false \rightarrow false = true$$

NOTE This can seem weird at first, this answer helps: https://math.stackexchange.com/a/100288

> *If you start out with a false premise, then, as far as implication is concerned, you are free to conclude anything. (This corresponds to the fact that, when p is false, the implication $p \rightarrow q$ is true no matter what q is.)*

> *If you start out with a true premise, then the implication should be true only when the conclusion is also true. (This corresponds to the fact that, when p is true, the truth of the implication is the same as the truth of q.)*

Additionally, let p and q be propositions and A the conditional statement:

$$p \rightarrow q$$

then:

- p is called the **hypothesis** (or antecedent or premise) and q is called the **conclusion** (or consequence).
- The proposition $q \rightarrow p$ is the **converse** of A.
- The proposition $\neg q \rightarrow \neg p$ is the **contrapositive** of A.

4.5. Bi-conditional: $\leftrightarrow$

$$1 \leftrightarrow 1 = 1$$

$$1 \leftrightarrow 0 = 0$$

$$0 \leftrightarrow 1 = 0$$

$$0 \leftrightarrow 0 = 1$$

4.6. Exclusive or: XOR, $\oplus$

$$1 \oplus 1 = 0$$

$$1 \oplus 0 = 1$$

$$0 \oplus 1 = 1$$

$$0 \oplus 0 = 0$$

5. Operator Precedence

Operators are applied in the following order (ascending):

1. $\neg$
2. $\land$
3. $\lor$
4. $\rightarrow$
5. $\leftrightarrow$

For example:

$$p \rightarrow p \land \neg q \lor s \equiv (p \rightarrow ((p \land (\neg q)) \lor s))$$

6. Descriptors

A formula that's always true is called a **tautology**. A formula that is true for at least one scenario is **consistent**. A formula that's never true is **inconsistent**. A formula can also result in a **contradiction**.

7. Equivalances

Formulas are equivalanent if they result in the same logical outcomes.

For example (*De Morgan's Laws*):

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

For example:

$$\neg(true \land true) \equiv false \lor false \equiv false$$
$$\neg true \lor \neg true \equiv \neg(true \land true) = \neg true = false$$

8. Quantifiers

We use the symbol $\exists$ to indicate the existence of something (**existential quantifier**).

$$\exists x \quad \text{odd}(x)$$

This means that there exists some x that is odd.

We denote the **universal quantifier** as $\forall$.

$$\forall x(\text{odd}(x) \lor \text{even}(x))$$

This meant that **for all** x the number is either even or odd.

Other examples, "All Ps are Qs":

$$\forall x(P(x) \to Q(x))$$

And "No Ps are Qs":

$$\forall x(P(x) \to \neg Q(x))$$

8.1. Quantifiers to Connectives

$\exists x, P(x)$ where $x \in \{x_1, x_2, ..., x_n\}$ means that there exists some x for which $P(x)$ is true.

Denoted alternatively:

$$\exists x, P(x) \equiv P(x_1) \lor P(x_2) \lor ... \lor P(x_3)$$

We can also conclude:

$$\neg \exists x, P(x) \equiv \neg(P(x_1) \lor P(x_2) \lor ... \lor P(x_3))$$
$$\neg \exists x, P(x) \equiv \neg P(x_1) \land \neg P(x_2) \land ... \land \neg P(x_3)$$
$$\neg \exists x, P(x) \equiv \forall x, \neg P(x)$$