Cheatsheet - Binomial Coefficients & Identities

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1. Binomial Theorem

An expression consisting of two terms, connected by a $+$ or $-$ sign, is called a **binomial expression**. As we increase the power of binomials, expanding them becomes more and more complicated:

$$\begin{aligned}(x+y)^1 &= x+y\\(x+y)^2 &= x^2+2xy+y^2\\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\&\dots\end{aligned}$$

The **binomial theorem** helps us to simplify this expansion. Let x and y be variables, and n a non-negative integer. The expansion of $(x+y)^n$ can be formalized as:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

The **binomial coefficients** are the coefficients in the binomial theorem and denoted as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here we say **"n choose k"**.

For example:

> *What is the coefficient of* x^8y^7 *in the expansion of* $(3x-y)^{15}$.

We can view the expression as $(3x+(-y))^{15}$. By the binomial theorem:

$$(3x+(-y))^{15} = \sum_{k=0}^{15}\binom{15}{k}(3x)^k(-y)^{15-k}$$

The coefficient of x^8y^7 in the expansion is obtained when $k=8$:

$$\binom{15}{8}(3)^8(-1)^7 = -3^8\frac{15!}{8!7!}$$

TODO: Expand on this.

1.1. Pascal's Identity

If n and k are integers with $n \geq k \geq 1$, then:

$$\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$$

1.2. Pascal's Triangle

Pascals' triangle is a number triangle with numbers arranged in staggered rows such that $a_{n,r}$ **is the binomial coefficient** $\binom{n}{r}$.