Cheatsheet - Sequences and Series

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1. Intro

Sequence: A set of (arbitrary) numbers ("terms") written down in a specific order.

$$S = (1, 3, 5, 7, 9)$$

A sequence can be *finite* or *infinite*:

$$S = (0, 1, 2, 3, ...)$$

2. Notations

Given $S = (0, 1, 2, 3)$, then $S_2 \in S$ equals to 1 and $S_3 \in S$ equals to 2 (sometimes the index starts at *0*, common in Computer Science).

Terms of a sequence can often be found by using a formula. For example, given:

$$x_k = 2k + 3$$

then $x_3 = 2 \times 3 + 3 = 9$ and $x_4 = 11$.

3. Arithmetic Progression

A arithmetic progression (or sequence) *adds* a fixed amount to the previous term.

$$S = (a, a + d, a + 2d, a + 3d, ...)$$

where a is the *first term* and d is the *common difference* of the arithmetic progression S. For example:

$$S = (1, 7, 13, 19, ...)$$

where a is *1* and d is *6*. The n-th term of an arithmetic progression is given by:

$$a + (n - 1)d$$

For example, the third term in S is:

$$S_3 = 1 + (3 - 1)6$$
$$S_3 = 1 + 2 \times 6$$
$$S_3 = 13$$

4. Geometric Progression

A geometric progression (or sequence) *multiplies* the previous terms by a fixed amount.

$$S = (a, ar, ar^2, ar^3, ...)$$

where a is the *first term* and r is the *common ratio*. For example:

$$S = (2, 10, 50, 250, ...)$$

where a is *2* and r is *5*. The n-th term of an geometric progression is given by:

$$ar^{n-1}$$

For example, the third term in S is:

$$S_3 = 2 \times 5^{3-1}$$
$$S_3 = 2 \times 5^2$$
$$S_3 = 50$$

5. Infinite Sequences

An infinite sequences continues indefinitely. As the sequences progresses, it gets closer and closer to a fixed value. For example, the following sequences and smaller

$$S = \left(s, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right)$$

which can be written as $x_k = \frac{1}{k}$ for $k = (1, 2, 3, ...)$. We say that "$\frac{1}{k}$ tends to zero as k tends to infinity" or "as k tends to infinity, the *limit* of the sequence is zero":

$$\lim_{k \to \infty} \frac{1}{k} = 0$$

When a sequence possesses a limit it is said to **converge**. A sequence such as $x_k = 3k - 2$ which is $(1, 4, 7, 10, ...)$ does have a limit, which is said to **diverge**.

6. Series & Sigma Notation

If the terms of a sequence are added, the result is known as a *series*.

$$\sum_{k=1}^{k=5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Note that notations can be abbreviated:

$$\sum_{k=1}^{k=5} k = \sum_{k=1}^{5} k = \sum_{1}^{5} k$$

7. Arithmetic Series

If the terms of an arithmetic sequence are added, the result is known as an arithmetic series. The sum of the first n terms of an arithmetic series with first term a and common difference d is denoted by S_n and given by:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Alternatively, this can be written as: the sum of the first n terms of an arithmetic series with first term a_1 and last term a_2 is denoted by S_n and given by:

$$S_n = \frac{n}{2}(a_1 + a_2)$$

For example, the sum of of the first 3 items in $\sum_{k=1}^{k=50} k$ is:

$$S_3 = \frac{3(1+3)}{2}$$

$$S_3 = \frac{3 \times 4}{2} = \frac{12}{2} = 6$$

Additional, the sum of $\sum_{k=1}^{k=50} (2k+1)$ is:

$$S_{50} = \frac{50}{2} \times ((2 \times 1 + 1) + (2 \times 50 + 1))$$

$$S_{50} = \frac{50}{2} \times (3 + 101) = \frac{5200}{2} = 2600$$

8. Geometric Series

If the terms of a geometric sequence are added, the result is known as a geometric series. The sum of the first n terms of a geometric series with first term a and common ratio r is denoted by S_n and given by:

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{provided} \quad r \quad \text{is not equal to} \quad 1$$

For example, the sum of the 3 first terms of $(2, 10, 50, 250, ..)$ is:

$$S_3 = \frac{2(1-5^3)}{1-5}$$

$$S_3 = \frac{2 \times 1 - 2 \times 5^3}{-4}$$

$$S_3 = \frac{-248}{-4} = 62$$

9. Infinite Geometric Series

If the terms of an infinite sequence are added, the result is known as an infinite series. The sum of an infinite number of terms of a geometric series with first term a and common ratio r is denoted by S_∞ and given by:

$$S_\infty = \frac{a}{1-r} \quad \text{provided} \quad -1 < r < 1$$

For example, a first term of 2 and a common ration of $\frac{1}{3}$ is:

$$S_\infty = \frac{2}{1 - \left(\frac{1}{3}\right)} = \frac{2}{\frac{2}{3}} = 3$$

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