Cheatsheet - Relations

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1. Intro

Relations between **elements** of sets occur in many contexts. In mathematics, we study relationships such as:

- a relation between a **positive integer** and one that it **divides**.
- a relation between a **real number** and **one** that is **larger** than it.
- a relation between a **real number** x and the **value f** (x) where f is a function, and so on.

2. Definition

A relation can be defined between elements of a set A and elements of another set B. It can also be defined between elements of the same set. We always use the letter R to refer to a relation.

For example, we say that x **is related** to y with respect to the relation R and we write:

$$xRy \quad \text{where} \quad x \in A, y \in B$$

> NOTE: Relations are **defined arbitrarily**.

2.1. Cartesian Product & Binary Relation

The **Cartesian product** $A \times B$ is defined by a **set of pairs** (x, y) such that $x \in A$ and $y \in B$.

$$A \times B = \{(x, y) : x \in A \quad \text{and} \quad y \in B\}$$

For example:

$$A = \{a_1, a_2\} \quad \text{and} \quad B = \{b_1, b_2, b_3\}$$
$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}$$

Note that:

$$A \times A = A^2$$

A **binary relation** from A to B is a **subset** of a **Cartesian product** $A \times B$:

$$R \subseteq A \times B$$

which means that R is a set of ordered pairs of the form (x, y) where $x \in A$ and $y \in B$.

$$(x, y) \in R \quad \text{means} \quad xRy \quad (\text{... is related to ...})$$

2.1.1. Relation on a Set

When $A = B$, a relation R **on the set** A is a relation from A to A:

$$R \subseteq A \times A$$

For example:

$$A = \{1, 2, 3, 4\}$$
$$R = \{x, y \in A, xRy \mid \text{if and only if} \quad x < y\}$$
$$\text{We have} \quad 1R2, 1R3, 1R4, 2R3, 2R4, 3R4 \quad \text{respectively:}$$
$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

3. Representation

3.1. Matrices

Given a **relation** R from a set A to set B. We can list the elements of sets A and B in a particular order.

Let $n_a = |A|$ and $n_b = |B|$. The **matrix of** R is $n_a \times n_b$:

$$M_r = [m_{ij}]n_a \times n_b$$

$$m_{ij} = \begin{cases} 1 & \text{if} \quad (a_i, bj) \in R \\ 0 & \text{if} \quad (a_i, bj) \notin R \end{cases}$$

Example:

$$A = \{a, b, c\}$$

$$B = \{1, 2, 3\}$$

$$R = \{(a, 1), (a, 2), (b, 2), (b, 3), (c, 1), (c, 3)\}$$

$$M_r = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

3.1.1. Combining Relations

The **union** of two relations is a new set that contains all of the pairs of elements that are in at least one of the two relations. The union is written as $R \cup S$ or **"R or S"**.

$$R \cup S = \{(a, b) \colon (a, b) \in R \quad \text{or} \quad (a, b) \in S\}$$

The **intersection** of two relations is a new set that contains all of the pairs that are in both sets. The intersection is written as $R \cap S$ or **"R and S"**.

$$R \cap S = \{(a, b) \colon (a, b) \in R \quad \text{and} \quad (a, b) \in S\}$$

For example, given:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

To **join** (union):

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To **meet** (intersection):

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.2. Graphs

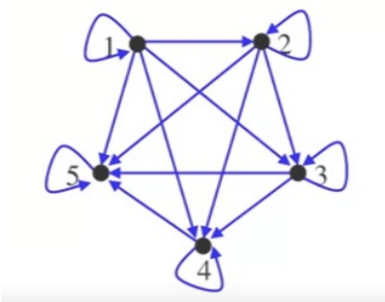
When a relation is defined on a set, it can be represented by a digraph. First, the elements of A are written down, then $(a, b) \in R$ arrows are drawn from a to b.

For example:

$$A = \{1, 2, 3, 4, 5\}$$

$$R = \{x, y \in A, xRy \mid \text{if and only if} \quad x \leq y\}$$

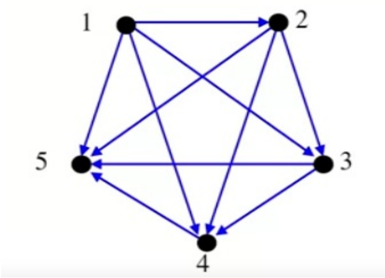
Then the graph is:



And if (*strictly less*):

$$R = \{x, y \in A, xRy \mid \text{if and only if} \quad x < y\}$$

Then the graph is:



4. Properties

4.1. Reflexivity

A relation R in a set S is said to be **reflexive** if and only if:

$$xRx, \forall x \in S$$
$$\equiv$$
$$(x, x) \in R, \forall x \in S$$

For example, the following is reflexive:

$$R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$$
$$1 \leq 1 \quad (\text{i.e.} \quad a \leq a)$$
$$1 \leq 2$$
$$\ldots$$

While this example is **not** reflexive:

$$R = \{(a, b) \in \mathbb{Z}^2 \mid a < b\}$$
$$1 < 2$$
$$\text{but not} \quad 1 < 1 \quad (\text{i.e.} \quad a < a)$$
$$\ldots$$

4.2. Symmetry

A relation R on a set S is said to be **symmetric** if and only if:

$$\forall (a, b) \in S, \quad \text{if} \quad aRb \quad \text{then} \quad bRa$$

For example, the following is symmetric:

$$R = \{(a, b) \in \mathbb{Z}^2 \mid a + b = b + a\}$$
$$1 + 2 = 2 + 1$$
$$2 + 1 = 1 + 2$$
$$\ldots$$

While this example is **not** symmetric:

$$R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$$
$$1 \leq 2$$
$$\text{but not} \quad 2 \leq 1$$
$$\ldots$$

4.3. Anti-Symmetry

NOTE — The symmetric and anti-symmetric properties are not necessarily mutually exclusive, meaning a relation can be both.

A relation R on a set S is said to be **anti-symmetric** if and only if:

$$\forall (a, b) \in S, \quad \text{if} \quad (aRb \quad \text{and} \quad bRa) \quad \text{then} \quad a = b$$

For example, the following is anti-symmetric:

$$R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$$
$$1 \leq 1 \quad \text{and} \quad 1 \leq 1$$

$$\text{implies} \quad 1 = 1$$

$$\ldots$$

4.4. Transitivity

A relation R on set S is called **transitive** if and only if:

$$\forall (a, b, c) \in S, \quad \text{if} \quad (aRb \quad \text{and} \quad bRc) \quad \text{then} \quad aRc$$

For example, the following is transitive:

$$R = \left\{(a, b) \in \mathbb{Z}^2 \mid a \leq b\right\}$$

$$2 \leq 3 \quad \text{and} \quad 3 \leq 4$$

$$\text{implies} \quad 2 \leq 4$$

$$\ldots$$