Cheatsheet - Postulates of Boolean Algebra

Fabio Lama – fabio.lama@pm.me

1. Intro

Boolean algebra describes the field of study where the values of variables are *true* or *false*. The following operators are used:

- **AND**: represented as $x.\, y$, $x \cap y$ or $x \wedge y$
- **OR**: represented as $x + y$, $x \cup y$ or $x \vee y$
- **NOT**: represented as $x\,'$, $\bar{x}$ or $\neg x$

Order of precedence: $\text{NOT} > \text{AND} > \text{OR}$

2. Axioms

The following axioms must be satisfied by any boolean algebra:

- **closure**: any result of logical operations belongs to the set $\{0, 1\}$ (*true* or *false*).
- **identity**: elements have an identity for the applied operator:
  - $x + 0 = x$
  - $x.\, 1 = x$
- **commutativity**: the order of the applied operators does not matter.
  - $x + y = y + x$
  - $x.\, y = y.\, x$
- **distributivity**:
  - $x(y + z) = (x.\, y) + (x.\, z)$
  - $x + (y.\, z) = (x + y).\, (x + z)$
- **complements**: exist for all the elements
  - $x + x\,' = 1$
  - $x.\, x\,' = 0$
- **distinct elements**:
  - $0 \neq 1$

2.1. Basic Theorems

Based on those axioms, we can establish basic theorems:

Theorem 1: Idempotent Laws

$$x + x = x$$
$$x.\, x = x$$

Theorem 2: Tautology and Contradiction

$$x + 1 = 1$$
$$x.\, 0 = 0$$

Theorem 3: Involution

$$(x\,')\,' = x$$

Theorem 4: Associative Laws

$$(x + y) + z = x + (y + x)$$
$$(x.\, y).\, z = x.\, (y.\, z)$$

Theorem 5: Absorption Laws

$$x + (x.\, y) = x$$
$$x.\, (x + y) = x$$

Theorem 6: Uniqueness of Complement

$$\text{if} \quad y + x = 1 \quad \text{and} \quad y.\, x = 0, \quad \text{then} \quad x = y'$$

Theorem 7: Inversion Law

$$0' = 1, 1' = 0$$

2.2. De Morgans' Theorems

$$\overline{x.y} = \bar{x} + \bar{y}$$

$$\overline{x+y} = \bar{x}.\bar{y}$$

2.3. Principles of Duality

Starting with a boolean relation, we can build another equivalent boolean relation by:

- changing each $\text{OR}(+)$ to an $\text{AND}(.)$
- changing each $\text{AND}(.)$ to an $\text{OR}(+)$
- changing each *0* to *1* and each *1* to *0*.

For example:

$$(a.1).(0+\bar{a}) = 0$$

$$\equiv$$

$$(a+0)+(1.\bar{a}) = 1$$

3. Boolean Functions

3.1. Standardized Forms of a Functions

The two most common standardized forms are **sum-of-products** and **product-of-sums**.

The **sum-of-products form**: $f(x, y, z) = xy + xz + yz$ The **product-of-sums form**: $f(x, y, z) = (x+y)(x+z)(y+z)$

For example:

$$f(x, y) = x'y + xy' + xy$$

3.2. Useful Functions

The **exclusive-or** function:

$$x \oplus y = x'y + xy'$$

The **implies** function:

$$x \to y = x' + y$$

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