

Cheatsheet - (First-order) Predicate & Propositional Logic

Fabio Lama – fabio.lama@pm.me

1. Intro

Predicates describe properties of objects.

For example:

$$\text{odd}(3)$$

$\text{odd}(3)$ means 3 is an odd number. odd is a predicate, 3 is an object. Predicates take arguments and become **propositions**. A proposition is a statement that can be either *true* or *false*. It must be one or the other, and it cannot be both.

Connectives can be applied:

$$\text{odd}(3) \wedge \text{prime}(3)$$

This means that 3 is odd but also prime.

2. Syntax

Propositions are denoted by capital letters, such as P, Q, \dots . General statements are denoted by lowercase letters, such as p, q, \dots

3. Connectives

Logical NOT: $\neg p$ is true if and only if p is false (also called *negation*).

Logical OR: $p \vee q$ is true if and only if at least one of p or q is true or if both p and q are true (also called *disjunction*).

Logical AND: $p \wedge q$ is true if and only if both p and q are true (also called *conjunction*).

Logical IF...THEN: $p \rightarrow q$ is true if and only if either p is false or q is true (also called *conditional* or *implication*). p is the premise, q is the conclusion.

Logical IF and only IF: $p \leftrightarrow q$ is true if and only if both p and q are true (also called *bi-conditional*).

Exclusive OR: XOR: $p \oplus q$ is true if p or q is true but not both.

3.1. Translation to Connectives

As an example, let's consider the propositions:

- P = I study 20 hours a week
- R = I will pass the exam
- S = I will be happy
- Q = I attend all the lectures

And the following connectives:

$$(P \vee Q) \rightarrow (R \wedge S)$$

which is a translation of: "If I study 20 hours a week **or** attend all the lectures, **then** I will pass the exam **and** I will be happy."

4. Truth Tables

4.1. Negation: \neg

$$\text{true} = \neg \text{false}$$

$$\text{false} = \neg \text{true}$$

4.2. Conjunction: \wedge

$$\text{true} \wedge \text{true} = \text{true}$$

$$\text{true} \wedge \text{false} = \text{false}$$

$$\text{false} \wedge \text{true} = \text{false}$$

$$\text{false} \wedge \text{false} = \text{false}$$

4.3. Disjunction: \vee

$$\text{true} \vee \text{true} = \text{true}$$

$$true \vee false = true$$

$$false \vee true = true$$

$$false \vee false = false$$

4.4. Implication: \rightarrow

$$true \rightarrow true = true$$

$$true \rightarrow false = false$$

$$false \rightarrow true = true$$

$$false \rightarrow false = true$$

NOTE

This can seem weird at first, this answer helps: <https://math.stackexchange.com/a/100288>

“If you start out with a false premise, then, as far as implication is concerned, you are free to conclude anything. (This corresponds to the fact that, when p is false, the implication $p \rightarrow q$ is true no matter what q is.)

“If you start out with a true premise, then the implication should be true only when the conclusion is also true. (This corresponds to the fact that, when p is true, the truth of the implication is the same as the truth of q .)

Additionally, let p and q be propositions and A the conditional statement:

$$p \rightarrow q$$

then:

- p is called the **hypothesis** (or antecedent or premise) and q is called the **conclusion** (or consequence).
- The proposition $q \rightarrow p$ is the **converse** of A .
- The proposition $\neg q \rightarrow \neg p$ is the **contrapositive** of A .

4.5. Bi-conditional: \leftrightarrow

$$1 \leftrightarrow 1 = 1$$

$$1 \leftrightarrow 0 = 0$$

$$0 \leftrightarrow 1 = 0$$

$$0 \leftrightarrow 0 = 1$$

4.6. Exclusive or: XOR, \oplus

$$1 \oplus 1 = 0$$

$$1 \oplus 0 = 1$$

$$0 \oplus 1 = 1$$

$$0 \oplus 0 = 0$$

5. Operator Precedence

Operators are applied in the following order (ascending):

1. \neg
2. \wedge
3. \vee
4. \rightarrow
5. \leftrightarrow

For example:

$$p \rightarrow p \wedge \neg q \vee s \equiv (p \rightarrow ((p \wedge (\neg q)) \vee s))$$

6. Descriptors

A formula that's always true is called a **tautology**. A formula that is true for at least one scenario is **consistent**. A formula that's never true is **inconsistent**. A formula can also result in a **contradiction**.

7. Equivalances

Formulas are equivalent if they result in the same logical outcomes.

For example (*De Morgan's Laws*):

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

For example:

$$\neg(true \wedge true) \equiv false \vee false \equiv false$$
$$\neg true \vee \neg true \equiv \neg(true \wedge true) = \neg true = false$$

8. Quantifiers

We use the symbol \exists to indicate the existence of something (**existential quantifier**).

$$\exists x \text{ odd}(x)$$

This means that there exists some x that is odd.

We denote the **universal quantifier** as \forall .

$$\forall x(\text{odd}(x) \vee \text{even}(x))$$

This meant that **for all** x the number is either even or odd.

Other examples, "All Ps are Qs":

$$\forall x(P(x) \rightarrow Q(x))$$

And "No Ps are Qs":

$$\forall x(P(x) \rightarrow \neg Q(x))$$

8.1. Quantifiers to Connectives

$\exists x, P(x)$ where $x \in \{x_1, x_2, ..., x_n\}$ means that there exists some x for which $P(x)$ is true.

Denoted alternatively:

$$\exists x, P(x) \equiv P(x_1) \vee P(x_2) \vee ... \vee P(x_3)$$

We can also conclude:

$$\neg \exists x, P(x) \equiv \neg(P(x_1) \vee P(x_2) \vee ... \vee P(x_3))$$
$$\neg \exists x, P(x) \equiv \neg P(x_1) \wedge \neg P(x_2) \wedge ... \wedge \neg P(x_3)$$
$$\neg \exists x, P(x) \equiv \forall x, \neg P(x)$$

8.1.1. De Morgan's Law for Negation

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists P(x) \equiv \forall x \neg P(x)$$

9. Laws of Propositional Logic

9.1. Logic 1

| | Disjunction | Conjunction |
|-------------------|---|---|
| idempotent laws | $p \vee p \equiv p$ | $p \wedge p \equiv p$ |
| commutative laws | $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ |
| associative laws | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ |
| distributive laws | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| identity laws | $p \vee F \equiv p$ | $p \wedge T \equiv p$ |
| domination laws | $p \vee T \equiv T$ | $p \wedge F \equiv F$ |

9.2. Logic 2

| | | |
|--|--|--|
| | | |
|--|--|--|

| | Disjunction | Conjunction |
|---------------------|--|--|
| De Morgan's laws | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ |
| absorption laws | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| negation laws | $p \vee \neg p \equiv T$ | $p \wedge \neg p \equiv F$ |
| double negation law | $\neg \neg p \equiv p$ | |