

Cheatsheet - Postulates of Boolean Algebra

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1. Intro

Boolean algebra describes the field of study where the values of variables are *true* or *false*. The following operators are used:

- **AND**: represented as $x \cdot y$, $x \cap y$ or $x \wedge y$
- **OR**: represented as $x + y$, $x \cup y$ or $x \vee y$
- **NOT**: represented as x' , \bar{x} or $\neg x$

Order of precedence: NOT > AND > OR

2. Axioms

The following axioms must be satisfied by any boolean algebra:

- **closure**: any result of logical operations belongs to the set $\{0, 1\}$ (*true* or *false*).
- **identity**: elements have an identity for the applied operator:
 - $x + 0 = x$
 - $x \cdot 1 = x$
- **commutativity**: the order of the applied operators does not matter.
 - $x + y = y + x$
 - $x \cdot y = y \cdot x$
- **distributivity**:
 - $x(y + z) = (x \cdot y) + (x \cdot z)$
 - $x + (y \cdot z) = (x + y) \cdot (x + z)$
- **complements**: exist for all the elements
 - $x + x' = 1$
 - $x \cdot x' = 0$
- **distinct elements**:
 - $0 \neq 1$

2.1. Basic Theorems

Based on those axioms, we can establish basic theorems:

Theorem 1: Idempotent Laws

$$x + x = x$$

$$x \cdot x = x$$

Theorem 2: Tautology and Contradiction

$$x + 1 = 1$$

$$x \cdot 0 = 0$$

Theorem 3: Involution

$$(x')' = x$$

Theorem 4: Associative Laws

$$(x + y) + z = x + (y + z)$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Theorem 5: Absorption Laws

$$x + (x \cdot y) = x$$

$$x \cdot (x + y) = x$$

Theorem 6: Uniqueness of Complement

$$\text{if } y + x = 1 \text{ and } y \cdot x = 0, \text{ then } x = y'$$

$$0' = 1, 1' = 0$$

2.2. De Morgans' Theorems

$$\overline{\bar{x} \cdot \bar{y}} = \bar{x} + \bar{y}$$

$$\overline{\bar{x} + \bar{y}} = \bar{x} \cdot \bar{y}$$

2.3. Principles of Duality

Starting with a boolean relation, we can build another equivalent boolean relation by:

- changing each OR(+) to an AND(.)
- changing each AND(.) to an OR(+)
- changing each 0 to 1 and each 1 to 0.

For example:

$$(a \cdot 1) \cdot (0 + \bar{a}) = 0$$

$$\equiv$$

$$(a + 0) + (1 \cdot \bar{a}) = 1$$

3. Boolean Functions

3.1. Standardized Forms of a Functions

The two most common standardized forms are **sum-of-products** and **product-of-sums**.

The **sum-of-products form**: * $f(x, y, z) = xy + xz + yz$

The **product-of-sums form**: * $f(x, y, z) = (x + y)(x + z)(y + z)$

For example:

$$f(x, y) = x'y + xy' + xy$$

3.2. Useful Functions

The **exclusive-or** function:

$$x \oplus y = x'y + xy'$$

The **implies** function:

$$x \rightarrow y = x' + y$$