FCS Week 3 Lecture Note

Notebook:	Fundamentals of Computer Science
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Cornell Notes	**Topic:** Proof Techniques: Part 1	Course: BSc Computer Science Class: CM1025 Fundamentals of Computer Science[Lecture] Date: April 19, 2021

Essential Question:

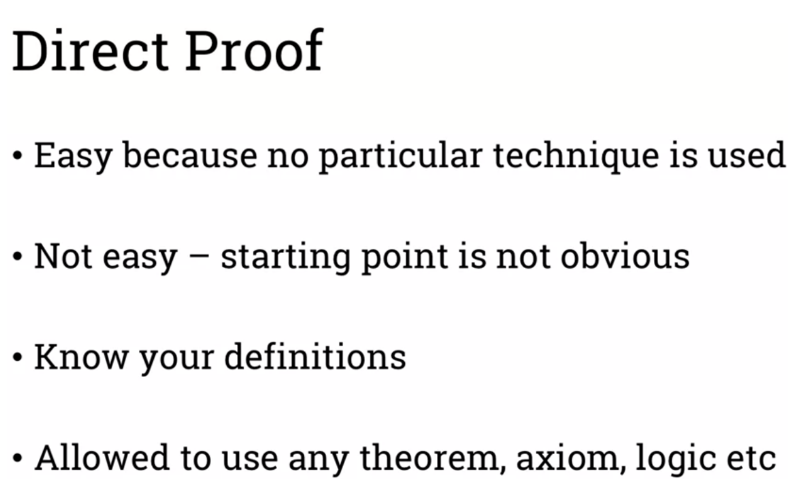
What is a proof?

Questions/Cues:

- What is a proof?
- What is a direct proof?
- What is proof by contradiction or an indirect proof?
- What is proof by contrapositive?
- What are two ways in which we can prove A implies B?

Notes

- Proof = sequence of connected logical statements that explains why a statement is true
- Direct proof = proof that exploits definitions and mathematical theorems, then using logical steps it arrives at the desired statement



Theorem: If **n** and **m** are both even, then **n+m** is even

- Proof: what does even mean?
- If an integer is even, it is twice another integer
- n = 2k, m = 2l, l and k are integers
- m+n = 2k+2l
- Factorise m+n = 2(k+l)
- Integers are closed under addition
- k+l is an integer, call it t
- m+n = 2t. So it is even.

Theorem: For $n \in \mathbb{N}, n^2 + n$ is even

- Proof:
- If n is even, $n = 2k$
- $n^2 + n = (2k)^2 + 2k$ Even

- If n is odd, $n = 2k + 1$
- $n^2 + n = (2k+1)^2 + 2k + 1$

  $= 4k^2 + 4k + 1 + 2k + 1$

  $= 4k^2 + 6k + 2$ Even

  - Adding two even numbers results in an even number. $4k^2$ is even, $6k$ is even and also 2 is even. Therefore, $4k^2 + 6k + 2$ is even.

Theorem: If $a < b < 0$ then $a^2 > b^2$

- Proof
- $a < b$ and $a < 0$, so multiply the inequality by a
- $a.a > b.a$ so $a^2 > b.a$
- $a < b$ and $b < 0$, so multiply the inequality by b
- $a.b > b.b$ so $a.b > b^2$
- *Commutative property* of multiplication: $a.b = b.a$
- $a^2 > a.b > b^2$
- $a^2 > b^2$

Theorem: For all $x \in \mathbb{N}, 2x^3 + x$ is a multiple of 3

- Proof
- $2x^3 + x = x(2x^2 + 1)$
- If x is multiple of 3, we are complete!
- If $x = 3k + 1$
  - $x(2x^2 + 1) = (3\mathrm{k} + 1)[2(3\mathrm{k} + 1)^2 + 1] = (3\mathrm{k} + 1)(18k^2 + 12k + 3)$
    $= 3(3\mathrm{k} + 1)(6k^2 + 4k + 1)$
- If $x = 3k + 2$
  - $x(2x^2 + 1) = (3\mathrm{k} + 2)[2(3\mathrm{k} + 2)^2 + 1] = (3\mathrm{k} + 2)(18k^2 + 24k + 9)$
    $= 3(3\mathrm{k} + 2)(6k^2 + 8k + 3)$

Proof by Contradiction – Indirect proof

- Simple concept: prove statement A is true
- Start with assuming A is false
- Use mathematical steps and logic like direct proof
- Arrive at a statement that contradicts our assumption
- Our assumption must be WRONG, therefore, A is True.

Example – Contradiction

- Theorem: $\sqrt{2}$ is irrational
- Assume $\sqrt{2}$ is rational
  - A rational number is a number that can be made by dividing two integers
  - It means $\sqrt{2} = \frac{p}{q}$ for a $p, q \in \mathbb{N}, q \neq 0, (p, q) = 1$
  - $\sqrt{2} = \frac{p}{q} \rightarrow 2 = \frac{p^2}{q^2} \rightarrow 2q^2 = p^2 \rightarrow p\ is\ even \rightarrow p = 2k$
  - $2q^2 = p^2 \rightarrow 2q^2 = (2k)^2 = 4k^2 \rightarrow q^2 = 2\,k^2 \rightarrow q\ is\ even$
  - So $(p, q) = 2$, this contradicts our initial assumption
  - Therefore $\sqrt{2}$ is irrational.

Proof by Contradiction – Example

- Theorem: There is an infinite number of prime numbers
- Assume there are finite number of prime numbers
  - Let's call these prime numbers: $p_1, p_2, p_3, p_{4,\ldots,} p_n$
  - Now, let number N be $P_1 . P_2 . P_3 \ldots P_n + 1$
  - Do you think N is divisible by any P_i, $0 < i \leq n$?
  - The answer is negative, remainder of N divided by any P_i is 1
  - Is N prime?
  - Yes and $N \neq P_i$, for all i, $0 < i \leq n$
  - Now we have a new prime number that is not in the list. Contradiction!

Proof by Contrapositive – Indirect proof

- Simple concept:
- Prove $A \rightarrow B$ is true

- $A \rightarrow B$ is equivalent to $\neg B \rightarrow \neg A$ (Contrapositive)

- Prove $\neg B \rightarrow \neg A$ is true

Example – Contrapositive

- Theorem: For all $n \in \mathbb{N}$, $n^3 + 1$ is *odd* $\rightarrow n$ *even*
- Prove for all $n \in \mathbb{N}$, n *odd* $\rightarrow n^3 + 1$ is *even*
- Assume n is *odd*, n = 2k+1 for a k ∈ ℕ
- Prove $n^3 + 1$ is *even*
  - $n^3 + 1 = (2k+1)^3 + 1$

$$= \underbrace{8k^3 + 12k^2 + 6k}_{\text{Even}} \underbrace{+ 1 + 1}_{\text{Even}}$$

  - So $n^3 + 1$ is *even*

Proving $A \rightarrow B$

- Either direct proof
  Assume A is true, show B is true

- Or indirect proof
  Assume not B is true, show not A is true

Example – Contrapositive

- Theorem: Suppose $x, y \in \mathbb{R}$, $y^3 + yx^2 \leq x^3 + xy^2 \rightarrow y \leq x$
- Contrapositive: $y > x \rightarrow y^3 + yx^2 > x^3 + xy^2$
  - Assume $y > x$ so $y - x > 0$, multiply both sides by the positive value $y^2 + x^2$

$$(y - x)(y^2 + x^2) > 0(y^2 + x^2)$$

$$y^3 + yx^2 - xy^2 - x^3 > 0$$

$$y^3 + yx^2 > x^3 + xy^2$$

Example of proof by contradiction

- Claim: If 5n+2 is even then n is even, for n
- Proof: Assume the claim is false

p then q is false means p is true and q is false

- 5n+2 is even and n is odd
- n is odd, n = 2k+1, for some k
- 5n+2 = 5(2k+1)+2 = 10k+5+2

This is odd!

- Contradiction! So the claim is true

Example of proof by contrapositive

- Claim: If 5n+2 is even then n is even, for n

$p \to q$ is equivalent to $\neg q \to \neg p$

- Prove: if n is odd then 5n+2 is odd
- Assume n is odd, n = 2k+1 for some k
- 5n+2 = 5(2k+1)+2 = 10k+5+2
- We proved the contrapositive of the claim!

Summary

In this week, we learned about what is a proof is and the methods of proving like the contrapositive.