

$$V = \{1, 3, 5\} = \{5, 1, 3\} = \{2n+1 \mid n \in \mathbb{Z} \text{ and } 0 \leq n \leq 3\}$$

N: set of natural numbers $\{1, 2, 3, \dots\}$

$$5 \in V, 6 \notin V$$

Z: set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$

Set unordered collection of unique objects

Cardinality: $|S|$ # of elements in a set

$$|V|=3 \quad \emptyset = \{\} \quad |\emptyset|=0$$

Q: set of rational numbers

Subset: $A \subseteq B$ all elements of A is in B

$$\{3, 5\} \subseteq \{1, 3, 5\}$$

R: set of real numbers

Powerset: $P(S)$ set containing all subsets of a given set

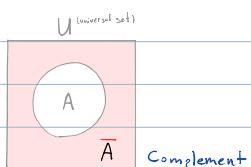
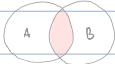
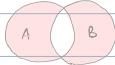
$$P(\emptyset) = \{\emptyset\}$$

$$S = \{1, 2, 3\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$|P(S)| = 2^{|S|}$$

$$|A|=n \rightarrow |P(P(A))| = 2^{2^n}$$

 $A \cup B$ union $A \cap B$ intersection $A - B$ difference $A \oplus B$ symmetric differenceA and B are disjointed if $A \cap B = \emptyset$

Partition: dividing a set into disjointed and comprehensive subsets

De Morgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Union

$$A \cup B = B \cup A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$A \cup \emptyset = A$$

$$A \cup U = U$$

$$A \cup \overline{A} = U$$

$$\overline{U} = \emptyset$$

$$\overline{\overline{A}} = A$$

$$A \cup (A \cap B) = A$$

$$A - B = A \cap \overline{B}$$

Name

$$\text{commutative}$$

$$\text{associative}$$

$$\text{distributive}$$

$$\text{De Morgan's Laws}$$

$$\text{identities}$$

$$\text{complement}$$

$$\text{double complement}$$

$$\text{absorption}$$

$$\text{set difference}$$

Intersection

$$A \cap B = B \cap A$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$A \cap \emptyset = \emptyset$$

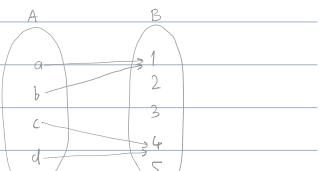
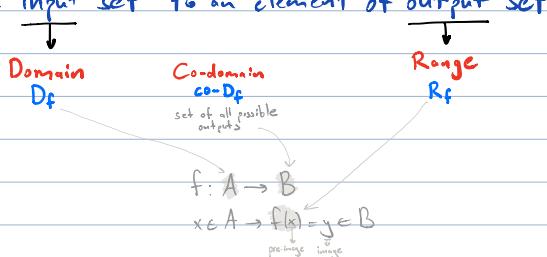
$$A \cap U = A$$

$$A \cap \overline{A} = \emptyset$$

$$\overline{\emptyset} = U$$

$$A \cap (A \cup B) = A$$

Function maps an element of an input set to an element of output set



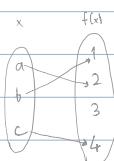
$$\text{Pre-images}(b) = \{c, d\}$$

$$\text{Image}(b) = 1$$

injective

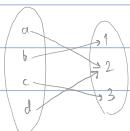
each x outputs a different y

"one-to-one"

> Prove: option #1 $\Rightarrow f(a) = f(b) \rightarrow a = b$ > option #2 $\Rightarrow a \neq b \rightarrow f(a) \neq f(b)$ > Disprove: counter example of $f(a) = f(b)$ and $a \neq b$

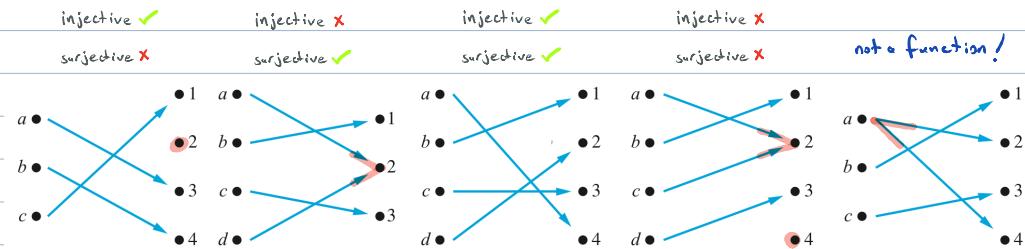
surjective each y in Range belongs to an x

"onto"
i.e. range = codomain
($R_f = \text{co-D}_f$)



Proof: $f(x) = y \in R \implies x = \dots \in \mathbb{R}$

bijective each y in co-Domain has exactly one pre-image x
"invertible"
both injective and surjective



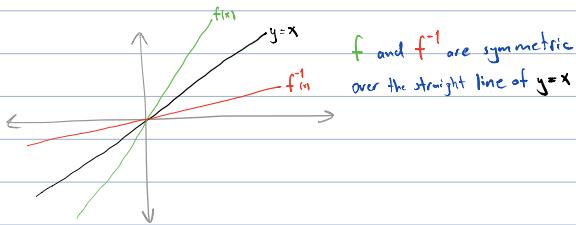
$$f \circ g = f(g(x))$$

$g: A \rightarrow B$ $f: B \rightarrow C \Rightarrow f \circ g: A \rightarrow C$

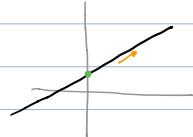
not comm.: $f \circ g \neq g \circ f$

inverse function $f^{-1}: B \rightarrow A$ for a bijective $f: A \rightarrow B$

identity function: $f \circ f^{-1} = f^{-1} \circ f$



Linear function: $f(x) = ax + b$

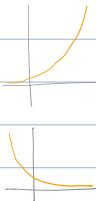


Quadratic function: $f(x) = ax^2 + bx + c$



Exponential function: $f(x) = b^x$

$b > 1$: exponential growth



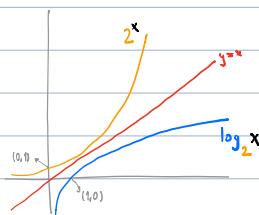
$0 < b < 1$: exponential decay



Logarithmic function: $f(x) = \log_b x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$



$$\log(m \cdot n) = \log m + \log n$$

$$\log \frac{m}{n} = \log m - \log n$$

$$\log m^n = n \cdot \log m$$

$$\log 1 = 0$$

$$\log_b b = 1$$

$$\ln x = \log_e x$$

$$\log_{10} 100 = 2$$

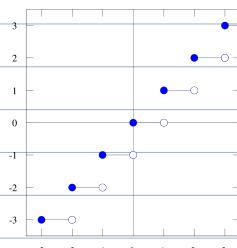
$$\log_3 81 = 4$$

$$\log_{10} \frac{1}{100} = -2$$

$$\log_{10} 1 = 0$$

Floor function: $n \leq x < n+1 \rightarrow \lfloor x \rfloor = n$

$\lfloor x \rfloor$



$$\lfloor 10 \rfloor = 10$$

$$\lfloor 1.1 \rfloor = 1$$

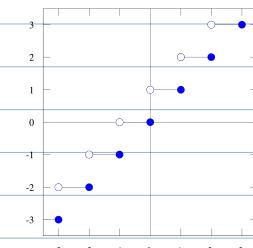
$$\lfloor 1.99 \rfloor = 1$$

$$\lfloor -1.1 \rfloor = -2$$

$$\lfloor -1.99 \rfloor = -2$$

Ceiling function: $n < x \leq n+1 \rightarrow \lceil x \rceil = n+1$

$\lceil x \rceil$



$$\lceil 10 \rceil = 10$$

$$\lceil 1.1 \rceil = 2$$

$$\lceil 1.99 \rceil = 2$$

$$\lceil -1.1 \rceil = -1$$

$$\lceil -1.99 \rceil = -1$$

proposition must be true or false (ex: ' $x+2=3$ ' is not a proposition!)

Truth Set set of elements in universe where proposition is true $S = \{1, 2, 3, \dots, 10\}$, $p: n$ is even, $\text{set}^{\text{true}}: P = \{2, 4, 6, 8, 10\}$

$\neg p$: NOT negation	\rightarrow IF THEN \leftrightarrow IF and only IF	\oplus XOR (exclusive or)	Operator precedence: \neg first \wedge \vee \rightarrow \leftrightarrow last																																				
\wedge : AND conjunction	$p \quad q \quad p \rightarrow q$ <table border="1"><tr><td>F</td><td>F</td><td>T</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>T</td><td>T</td><td>T</td></tr></table>	F	F	T	F	T	T	T	F	F	T	T	T	$p \quad q \quad p \leftrightarrow q$ <table border="1"><tr><td>F</td><td>F</td><td>T</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>T</td><td>T</td><td>T</td></tr></table>	F	F	T	F	T	F	T	F	F	T	T	T	$p \quad q \quad p \oplus q$ <table border="1"><tr><td>F</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>T</td><td>T</td><td>F</td></tr></table>	F	F	F	F	T	T	T	F	T	T	T	F
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\vee : OR disjunction	$p \rightarrow q \equiv \neg p \vee q$ $\equiv \neg p \rightarrow p$ for $a \rightarrow b$ "if a , then b " " a , only if b " converse $b \rightarrow a$ contrapositive $\neg b \rightarrow \neg a \equiv a \rightarrow b$ inverse $\neg a \rightarrow \neg b$	equivalent <ul style="list-style-type: none">$p \rightarrow q \wedge q \rightarrow p$some values in truth table	Disjunction Idempotent laws Commutative laws Associative laws Distributive laws Identity laws Domination laws De Morgan's laws Absorption laws Negation laws																																				
			Conjunction $p \vee p \equiv p$ $p \vee q \equiv q \vee p$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \vee \mathbf{F} \equiv p$ $p \vee \mathbf{T} \equiv \mathbf{T}$ $\neg(p \wedge q) \equiv \neg p \wedge \neg q$ $p \vee (p \wedge q) \equiv p$ $p \vee \neg p \equiv \mathbf{T}^1$																																				

$p \rightarrow q \equiv \neg p \vee q$	$\equiv \neg p \rightarrow p$	predicate	proposition
for $a \rightarrow b$ "if a , then b " " a , only if b "			
converse $b \rightarrow a$			
contrapositive $\neg b \rightarrow \neg a \equiv a \rightarrow b$			
inverse $\neg a \rightarrow \neg b$			
		$P(x) \equiv x > 4$	$P(3)$

Disjunction	Conjunction
$p \vee p \equiv p$	$p \wedge p \equiv p$
$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$p \vee \mathbf{F} \equiv p$	$p \wedge \mathbf{T} \equiv p$
$p \vee \mathbf{T} \equiv \mathbf{T}$	$p \wedge \mathbf{F} \equiv \mathbf{F}$
$\neg(p \wedge q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
$p \vee \neg p \equiv \mathbf{T}^1$	$p \wedge \neg p \equiv \mathbf{F}^2$
Double negation law	$\neg\neg p \equiv p$

predicate generalized propositions

Quantifiers

Universal: $\forall x$ <small>some formula</small> : for all x , the formula is true	"All P's are Q's" translates into $\forall x(P(x) \rightarrow Q(x))$	Nested Quantifier	Meaning
Existential: $\exists x$ <small>some formula</small> : for some x , the formula is true	"No P's are Q's" translates into $\forall x(P(x) \rightarrow \neg Q(x))$	$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair (x, y)
Uniqueness: $\exists ! x$ <small>some formula</small> : for a unique value x , the formula is true	"Some P's are Q's" translates into $\exists x(P(x) \wedge Q(x))$	$\exists x \exists y P(x, y)$	There is a pair (x, y) for which $P(x, y)$ is true
	"Some P's are not Q's" translates into $\exists x(P(x) \wedge \neg Q(x))$	$\forall x \exists y P(x, y)$	For all x , there is a y for which $P(x, y)$ is true
		$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for all y

De Morgan's:

$$\begin{aligned}\neg(\forall x P(x)) &\equiv \exists x \neg P(x) \\ \neg(\exists x P(x)) &\equiv \forall x \neg P(x)\end{aligned}$$

$$\begin{aligned}\neg \forall x \exists y \forall z P(x, y, z) &\equiv \exists x \neg \exists y \forall z P(x, y, z) \\ &\equiv \exists x \forall y \neg \forall z P(x, y, z) \\ &\equiv \exists x \forall y \exists z \neg P(x, y, z)\end{aligned}$$

Argument: premise premise conclusion is valid if: when all premises are true, conclusion must also be true.

Rules of inference

Modus Ponens	Modus Tollens	Conjunction	Simplification	Addition	Hypothetical Syllogism	Disjunctive Syllogism	Resolution
$p \rightarrow q$	$\neg q$	p	$p \wedge q$	p	$p \rightarrow q$	$p \vee q$	$p \vee q$
$\frac{p}{\therefore q}$	$\frac{p \rightarrow q}{\therefore \neg p}$	$\frac{p}{\therefore p \wedge q}$	$\frac{p \wedge q}{\therefore p}$	$\frac{p}{\therefore p \vee q}$	$\frac{q \rightarrow r}{\frac{p \rightarrow q}{\therefore p \rightarrow r}}$	$\frac{\neg p}{\therefore q}$	$\frac{\neg p}{\therefore q \vee r}$

v/Quantifiers

(UI) Universal Instantiation	(UG) Universal Generalisation	(EI) Existential Instantiation	(EG) Existential Generalisation	Universal Modus Ponens	Universal Modus Tollens
$\frac{}{\forall x P(x)}$	$\frac{}{P(c)}$	$\frac{}{\exists x P(x)}$	$\frac{P(c)}{\exists x P(x)}$	$\frac{\forall x P(x) \rightarrow Q(x)}{P(a)}$	$\frac{\forall x P(x) \rightarrow Q(x)}{\neg Q(a)}$
$\therefore P(c)$	$\therefore \forall x P(x)$	$\therefore P(c)$	$\therefore \exists x P(x)$	$\frac{P(a)}{\therefore Q(a)}$	$\frac{\neg Q(a)}{\therefore \neg P(a)}$

Boolean Algebra

$x \cdot y$ AND precedence: NOT > AND > OR
 $x + y$ OR De Morgan's: $\overline{x \cdot y} = \overline{x} + \overline{y}$
 \overline{x} NOT $\overline{x+y} = \overline{x} \cdot \overline{y}$

Duality

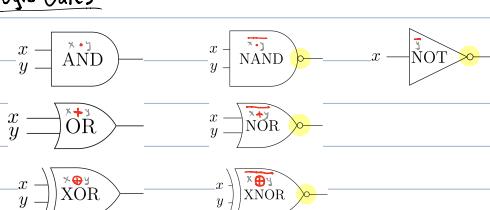
- Changing every '+' to '-'
 - Changing every '-' to '+'
 - Changing every 0 to 1
 - Changing every 1 to 0
- For example $A + B \cdot C \equiv A \cdot (B + C)$.

Logical Sum	Theorem	Logical Product
$x + x = x$	Idempotent Laws	$x \cdot x = x$
$x + 1 = 1$	Tautology and Contradiction	$x \cdot 0 = 0$
$\bar{x} = x$	Involution	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$(x + y) + z = x + (y + z)$	Associative Laws	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$x + (x \cdot y) = x$	Absorption Laws	$x \cdot (x + y) = x$
$\text{if } y + x = 1 \text{ and } y \cdot x = 0,$ then $x = \bar{y}$	Uniqueness of Complement	
$\bar{0} = 1$ and $\bar{1} = 0$	Inversion Law	

product-of-sums: $(x+y)(x+z)(y+z)$
sum-of-products: $xy + xz + yz \leftarrow \text{preferred!}$

- Identity: The logical sum identity $x + 0 = x$ and the logical product identity $x \cdot 1 = x$.
- Commutativity: Both logical sum and logical product are commutative, therefore $x + y = y + x$ and $x \cdot y = y \cdot x$.

Logic Gates



$$\begin{array}{c|c|c|c}
x & y & f(x, y) \\
\hline
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{array} \Rightarrow f(x, y) = \overline{xy} + xy + xy$$

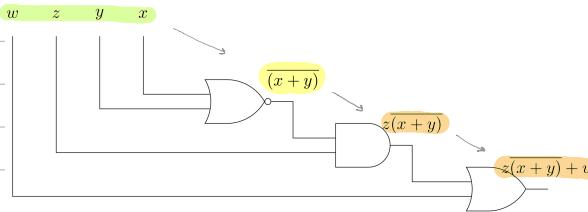
- Distributivity: Both logical sum and logical product are distributive between each other, therefore $x \cdot (y + z) = x \cdot y + x \cdot z$ and $x + (y \cdot z) = (x + y) \cdot (x + z)$.
- Complement: Complements exist for all elements: $x + \bar{x} = 1$ and $x \cdot \bar{x} = 0$.

"exclusive or" $x \oplus y : f(x, y) = \overline{xy} + xy$

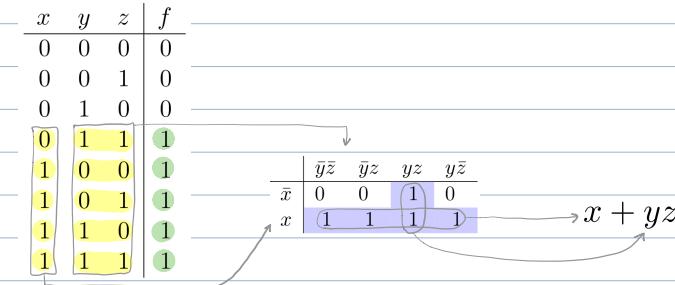
"implies" $x \rightarrow y : f(x, y) = \overline{x} + y$

Circuit → Boolean

- Label all gate outputs that are a function of input variables
- Express the Boolean functions for each gate in the first level
- Repeat until all outputs of the circuit are written as Boolean expressions



Karnaugh maps



Proofs

Theorem A formal statement that can be shown to be true.

Axiom A statement assumed to be true to serve as a premise to further arguments

Lemma A proven statement used as a step to a larger result rather than as a statement of interest by itself

Corollary A theorem that can be established by a short proof from a theorem

Proof techniques

↳ **Direct Proof**: Prove $A \rightarrow B$: Assume A , prove B

↳ **Contrapositive**: Prove $A \rightarrow B$: Prove $\neg B \rightarrow \neg A$
 $(\text{if } A \text{ then } B) \equiv (\text{if not } B \text{ then not } A)$

↳ **Contradiction**: Prove A : Assume A is false & prove at a statement that contradicts this

↳ **Induction** math.

- 1) **Base case:** Prove that $P(1)$ is true
- 2) **Induction hypothesis:** Assume that $P(k)$ is true
- 3) **Inductive step:** Prove that $P(k+1)$ is true

Conclusion: $\therefore \forall n \in \mathbb{N} P(n)$ is true \square

Theorem 10. $n < 3^n$, for all $n \in \mathbb{N}$

Proof. Let $P(n) = n < 3^n$, prove by induction that $P(n)$ is true for all n .

Basis: Prove $P(0)$ is true.

$$\begin{aligned} P(0) &= 0 < 3^0 \\ &= 0 < 1 \end{aligned}$$

Inductive Step: Prove $P(k) \rightarrow P(k+1)$:

Assuming $P(k) = k < 3^k$ to be true, prove that $P(k+1) = (k+1) < 3^{k+1}$ is true.

$$\begin{aligned} k &< 3^k \\ k+1 &< 3^k + 1 \\ k+1 &< 3^k + 1 < 3^k + 3^k + 3^k \\ k+1 &< 3^k + 1 < 3 \cdot 3^k \\ k+1 &< 3^k + 1 < 3^{k+1} \\ k+1 &< 3^{k+1} \end{aligned}$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} n < 3^n$ is true. \square

↳ **Strong Induction**

- 1) **Base case:** Prove that $P(1)$ is true

2) **Induction hypothesis:** Assume that $P(1), P(2), \dots, P(k)$ are true

3) **Inductive step:** Prove that $P(k+1)$ is true

Conclusion: $\therefore \forall n \in \mathbb{N} P(n)$ is true \square

EXAMPLE
[Fundamental Theorem of Arithmetic]

Every integer $n \geq 2$ can be written uniquely as the product of prime numbers.

Base case: This is clearly true for $n = 2$.

Inductive step: Suppose the statement is true for $n = 2, 3, 4, \dots, k$. If $(k+1)$ is prime, then we are done. Otherwise, $(k+1)$ has a smallest prime factor, which we denote by p . Let $k+1 = p \times N$. Since $N < k$, by our inductive hypothesis, N can be written uniquely as the product of prime numbers. That means $k+1 = p \times N$ can also be written uniquely as a product of primes. We're done! \square

Well-ordering property

It is an axiom about \mathbb{N} that we assume to be true. The axioms about \mathbb{N} are as follows:

1. The number 1 is a positive integer
2. If $n \in \mathbb{N}$, then $n+1$ is also a positive integer
3. Every positive integer other than 1, is the successor of a positive integer
4. The Well-ordering property: every non-empty subset of the set of positive integers has a least element

Math. induction ≡ Strong induction ≡ Well-ordering property

Recursive

	function	set	
Basis step	initial value	initial elements	Algorithm 1 Computing $n!$ 1: function FACTORIAL(n) 2: if $n = 0$ then 3: return 1 4: else return $n \times \text{FACTORIAL}(n - 1)$
Recursive step	find further values from previous values	generate further elements from previous elements	

Recurrence relations sequence generating next term as a function of previous term

Infinite sequence function from positive integers to real numbers

Linear recurrence relation where each term of the sequence is a linear function of earlier terms

↳ **Linear Homogenous Recurrences** $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$, $c_1, \dots, c_k \in \mathbb{R}$.
 k is the degree of the relation.

↳ **Linear Non-homogenous Recurrences** $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$,
 $c_1, \dots, c_k \in \mathbb{R}$. k is the degree of the relation.

Arithmetic sequence difference between consecutive terms is constant c

$$\frac{\text{Ex:}}{a_{n+1} = a_n + c}$$

Geometric sequence ratio

$$\frac{\text{Ex:}}{a_n = r \cdot a_{n-1}}$$

Divide & Conquer Algorithm

divide problem into subproblems

solve subproblems recursively

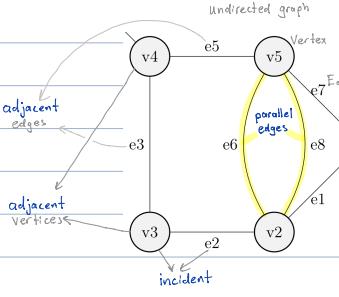
combine solutions

Graph

$$G = \{V, E\}$$

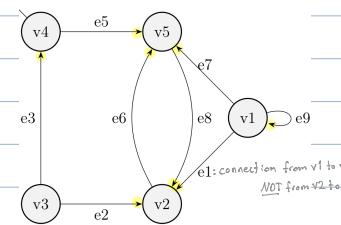
Vertices

Edges



undirected graph

directed graph (digraph)



Connectivity

undirected Graph

connected

directed Graph

strongly connected

	Vertices repeatable?	Edges
Walk	✓	✓
Trail	✓	✗
Circuit	✓	✗
Path	✗	✗
Cycle	✗	✗

Euler Path visits each edge precisely once → traversable

Hammiltonian Path visits each vertex precisely once → traceable

Hammiltonian Cycle visits each vertex precisely once & returns to the start $v_{start} = v_{end}$ → Hamm. Graph

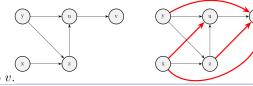


Figure 23: A Connected Graph



Figure 24: Not a Connected Graph



Figure 25: A Strongly Connected Graph

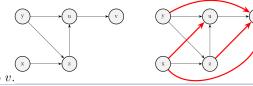


Figure 26: Not a Strongly Connected Graph

Transitive Closure

Given a digraph G , the Transitive Closure of G is the digraph G^* such that:

- G^* has the same vertices as G
- If G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v .

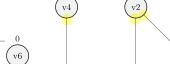


Degree

Undirected Graphs

Degree ($\deg(v)$) the number of edges incident on v

- A loop contributes twice to the degree
- An isolated vertex has degree 0



Directed Graphs

In-degree: edges going in
Out-degree: edges going out
 $\deg(v) = \text{in-deg}(v) + \text{out-deg}(v)$



Simple Graph

- no loops

- no parallel edges

(v_1 , v_2 , v_3 , v_4)

degree of my vertex $\leq \# \text{vertices} - 1$

Q: Is $\bullet\bullet\bullet\bullet\bullet$ (degree seq) simple graph?
A: - check: is sum of sequence even?
- check: is max degree $< \# \text{vertices}$?

Regular Graph

- all degrees are same = r

- called r -regular graph

(v_1 , v_2 , v_3 , v_4)

Given an r -regular graph G with n vertices, the following is true:

- Degree sequence of G is r, r, r, \dots (repeated n times)
- Sum of degree sequence is $r \cdot n$
- Number of edges in G is $\frac{rn}{2}$

Complete Graph

- simple graph where all vertices are adjacent

- n vertices $\rightarrow K_n$



A complete graph with n vertices, K_n has the following properties:

- Every vertex has degree $n - 1$
- Sum of degree sequences is $n(n - 1)$
- Number of edges is $\frac{n(n-1)}{2}$

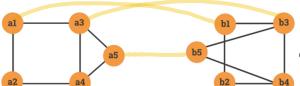
Degree sequence sequence of a graph's degrees from max to min Ex: 4, 4, 3, 2, 1, 0

↳ sum of sequence is always even!
 $\Leftrightarrow 2 \times \text{number of edges}$

Isomorphism

Graphs G_1 and G_2 are isomorphic if there is a bijection $f: G_1 \rightarrow G_2$

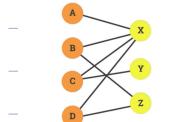
where if $u \& v$ are adjacent in G_1 , $f(u) \& f(v)$ are also adjacent in G_2



$$f: G_1 \rightarrow G_2$$

G_1	a_1	a_2	a_3	a_4	a_5
G_2	b_1	b_2	b_3	b_4	b_5

Bipartite graph vertices divided into 2 groups where all edges are between groups

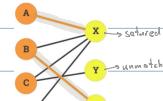


$V_1 = \{A, B, C, D\}$

$V_2 = \{X, Y, Z\}$

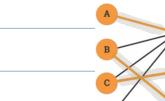
Matching

edges without shared endpoints



Maximum Matching

no further edge can be added

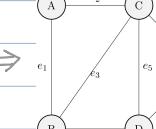


→ to find: Hopcroft-Karp Algorithm

- $M \leftarrow \emptyset$
- While there is an augmenting path P
 - Use BFS to build layers that terminate at free vertices
 - Start at free vertices in C , use DFS
- Return M

Adjacency List

Vertex	Adjacent Vertices
A	B, C
B	A, C, D
C	A, B, D, E
D	B, C, E
E	C, D



Adjacency Matrix

$$M(G) = \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} \xrightarrow{\text{loops}} \text{loops}$$

↳ undirected Graph

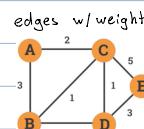
$$M(G) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{Symmetric}}$$

all edges (except loops) represented twice

sum of matrix elements = $2 \times \# \text{edges}$

= sum of degree sequence

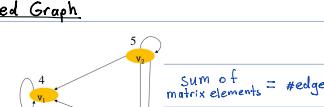
Weighted Graph



edges w/ weights
find shortest path between nodes
Dijkstra's Algorithm

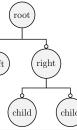
$$M(G) = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_2 & 1 & 0 & 2 \\ v_3 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{not symmetric}} \text{not symmetric}$$

sum of matrix elements = #edges

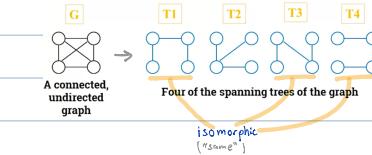


Tree

connected, acyclic, undirected graph
no loops & no parallel edges
↳ has unique simple path between two vertices
↳ has n vertices $\rightarrow n-1$ edges



Spanning tree



To get a spanning tree of graph G

1. Keep all vertices of G
2. Break all the cycles but keep the tree connected

Minimum-cost spanning tree

spanning tree w/ lowest cost
of a weighted graph

↳ greedy algorithms:
to find MST

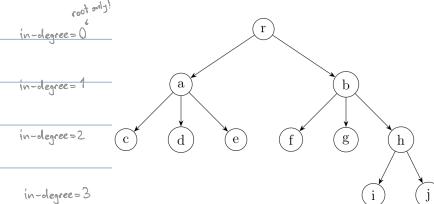
Kruskal's Algo

- Start w/ cheapest edge in spanning tree
- Add cheapest edge that doesn't create cycle & Repeat

Prim's Algo

- Start w/ any one node
- Add cheapest edge and the node it leads to & Repeat

Rooted tree: every edge is directed away from the root vertex



- r is the root of the tree
- a is a parent of vertices c, d, e
- f, g, h are the children of b
- r and b are ancestors of i and j
- i and j are siblings
- h and b are internal nodes
- c, d, e, f, g, i, j are external nodes
- depth of r is 0
- depth of a, d, e, f, g, h is 1
- depth of c and j is 2
- height of r is 3
- height of b is 2
- height of a and h is 1
- height of c, d, e, f, g, i, j is 0

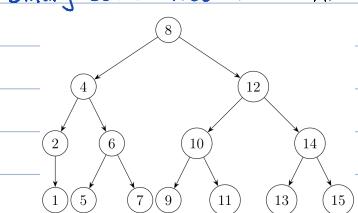
Binary Tree A tree in which every vertex has 2 or fewer children

Ternary Tree A tree in which every vertex has 3 or fewer children

m-ary Tree A tree in which every vertex has m or fewer children

isomorphic as rooted trees if bijection maps root-to-root

Binary Search Tree



vertices w/ labels
parent
parent's value \geq all keys in left subtree
parent's value \leq all keys in right subtree
left Child Right Child

for n elements: $\text{height} = \lceil \log_2(n+1) \rceil$ \rightarrow maximum numbers of comparisons in search
- Start at root
- Proceed to appropriate subtree & Repeat until found

Relation R

$$A \times B \rightarrow xRy$$

$x \in A, y \in B$

Cartesian Product $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

↳ $R: A \rightarrow B \rightarrow R \subseteq A \times B$

↳ R on $A \rightarrow R \subseteq A \times A \rightarrow xRy \text{ only if } x \leq y$

$$\exists x: A = \{1, 2, 3, 4\}$$

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

$$A \times B = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)\}$$

Matrix representation

		B			
		CS100	CS101	CS102	CS103
A	Sofia	X	X	X	
	Samir	X	X	X	
	Sarah	X			

$$A_{m,n} \Rightarrow A_m^1 \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Union (or)

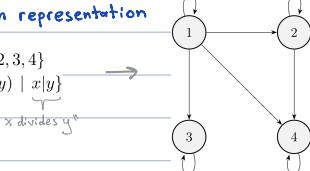
Intersection (and)

DiGraph representation

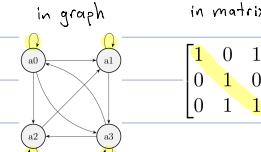
$$A = \{1, 2, 3, 4\}$$

$$R = \{(x, y) \mid x|y\}$$

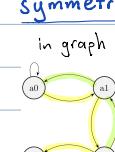
$\therefore x$ divides y



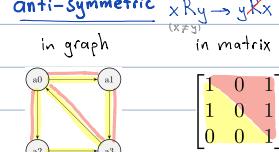
reflexiv xRx for all x



(all edges have parallel)
 $xRy \rightarrow yRx$



(no parallel edges)
 $xRy \rightarrow yRx$



transitiv

$$aRb \text{ and } bRc \rightarrow aRc$$

$$a \rightarrow b \rightarrow c \Rightarrow a \rightarrow b \rightarrow c$$

transitiv closure:
after adding necessary edges to make a relation transitiv

Equivalent

reflexive, symmetric and transitive

Equivalent Class

subset where all elements are related to element a through an equivalent relation

Example

Let $S = \{1, 2, 3, 4\}$ and R be a relation on elements in S :

$$R = \{(a, b) \in S^2 \mid a \bmod 2 = b \bmod 2\}$$

R is an equivalence relation with 2 equivalence classes:

- $[1] = [3] = \{1, 3\}$
- $[2] = [4] = \{2, 4\}$

Graphically:



Partial Order

reflexive, anti-symmetric and transitive

Total Order

is partial order

+

$$\forall a, b \in \mathbb{Z} (a \leq b \vee b \leq a)$$

Counting

↳ **Product rule:** m ways for task 1
 (Combination) $n \times n \times \dots \times 2$ job can be done in $m \times n \times \dots$ ways

↳ **Sum rule:** job can be done in m or n or... ways \rightarrow it can be done in $m+n+\dots$ ways
 (addition) (independent tasks)

Let's assume that a label in a programming language can be either a single letter or a letter followed by 2 digits. How many possible labels are there?

$$\begin{aligned} \text{single_letter + letter_2_digits} &= 26 + (\text{letters} \cdot \text{digits}^2) \\ &= 26 + (26 \cdot 10^2) \\ &= 26 + 2600 \\ &= 2626 \end{aligned}$$

In how many ways can we seat 4 people around a table where two seating arrangements are considered the same when each person has the same left and right neighbour?

Solution

Let's first number the seats around the table from 1 to 4 proceeding clockwise:

- There are four ways to select the person for seat 1, three for seat 2, two for seat 3, and one for seat 4
- Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to order the four people
- Since two seating arrangements are the same when each person has the same left and right neighbour, for every choice for seat 1, we get the same seating
- Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Pigeonhole principle "if you have 6 pigeons and 3 holes, you'll have at least one hole with min. 2 pigeons"

$$\left[\frac{\# \text{ items}}{\# \text{ boxes}} \right] = X \quad \text{at least one box with at least } \text{items in it}$$

A bag contains 6 blue balls, 12 red balls and 10 green balls. How many balls (minimum) must be selected to guarantee that at least 5 balls are the same colour?

$$\begin{aligned} \text{L}_1 \# \text{boxes} &= 3 \\ (r_1, g_1, b_1) &\Rightarrow \boxed{O} \quad \boxed{T} \quad \boxed{B} \\ \text{L}_2 \text{ one box with} &= 5 \\ \text{at least} &\Rightarrow \boxed{3} \\ \text{L}_3 \# \text{items} &= 5 \end{aligned}$$

Select 5 integers from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$; show that at least two integers add up to 9.

$$\begin{aligned} \text{L}_1 \# \text{sets} &= 3 \\ (1, 2, 3) &= 6 \\ (2, 3, 4) &= 9 \\ \vdots & \\ \text{L}_2 \# \text{sets} &= 4 \\ (1, 2, 3, 4) &= 10 \\ \text{L}_3 \# \text{items} &= 5 \end{aligned}$$

$$\begin{aligned} \text{L}_1 \# \text{sets} &= 3 \\ (1, 2, 3, 4) &= 10 \\ \text{L}_2 \# \text{sets} &= 4 \\ (1, 2, 3, 4) &= 10 \\ \text{L}_3 \# \text{items} &= 5 \end{aligned}$$

Permutation \rightarrow of n distinct items = $n!$

w/ order \rightarrow of r elements from a set of size n : $P(n, r) = \frac{n!}{(n-r)!}$

r-Combination $C(n, r) = \frac{n!}{r! \times (n-r)!}$

	Ordered (permutations)	Unordered (combinations)
Repetition is not permitted	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
Repetition is permitted	n^k	$\frac{(n+k-1)!}{k!(n-1)!}$