FCS Week 5 Lecture Note

Notebook:	Fundamentals of Computer Science		
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Cornell Notes	**Topic:** Basic Combinatorial Principles: Part 1	Course: BSc Computer Science Class: CM1025 Fundamentals of Computer Science[Lecture] Date: April 21, 2021

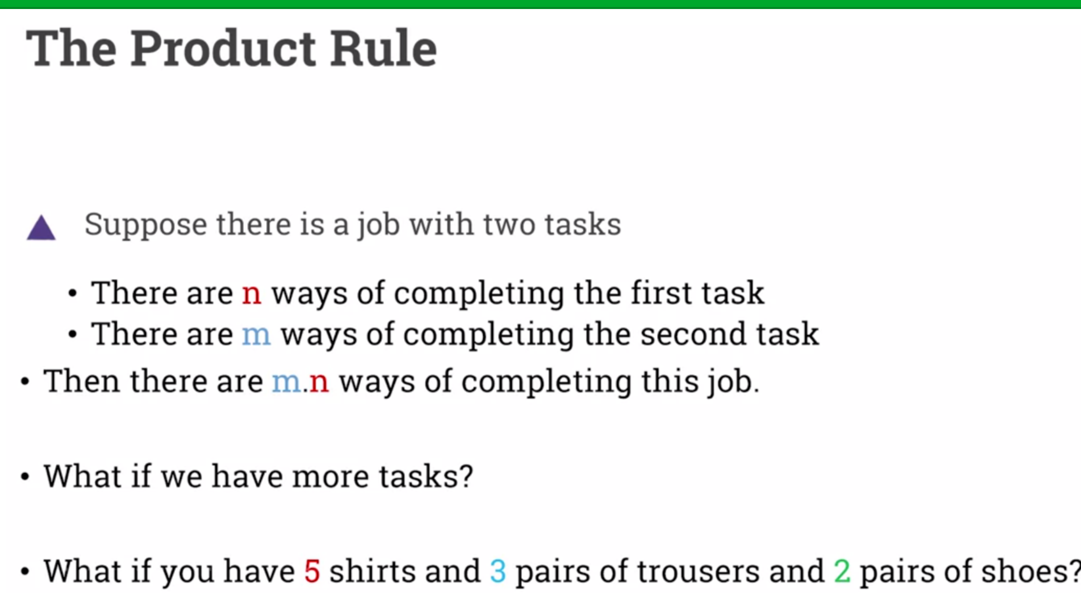
Essential Question:

What is counting?

Questions/Cues:

- What is the product rule?
- What is the sum rule?
- What is the subtraction rule?
- What is the pigeonhole principle?

Notes



What if you have five shirts and three pairs of trousers and two pairs of shoes?

Product rule for more tasks

- Suppose there is a job with k tasks.
- If there are n_i ways of completing task i
- Then there are $n_1 . n_2 \dots n_k$ ways of completing this job.

- What if you have 5 shirts, 3 pairs of trousers and 2 pairs of shoes?
- There are 5 x 3 x 2 = 30 ways of choosing an outfit.

Example

▲ How many sequences of length 5 can you make using uppercase English letters?

26 x 26 x 26 x 26 x 26

26 uppercase letters

$$26^5 = 11881376$$

Example

▲ How many sequence of length 5 can you make, where the first 3 letters are from uppercase English letters followed by 2 digits?

26 x 26 x 26 x 10 x 10

26^3x100 = 1757600

26 uppercase letters

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Either / or

- We need to choose an item of clothing to donate to a charity. Suppose we have 5 pairs of trousers and 7 shirts. The item could be either a pair of trousers or a shirt. How many choices are there?

The Sum Rule

- If a job can be done either in n ways **OR** in m ways, then the job can be completed in $m + n$ ways.

So, if we have 5 pairs of trousers and 7 shirts, we have (5+7) = 12 ways of choosing an item from them.

Example

- A teacher can choose a student to be her assistant from 5 classes.
- The classes contain 28, 21, 24, 25 and 27 students.
- Each student belongs to only one class.
- How many possible assistants are there to choose from?

- The Sum Rule:
  - 28 + 21 + 24 + 25 + 27 = 125

How many valid passwords are there?

- It has to be five to seven characters drawn from uppercase letters or digits. Every password must contain at least one uppercase letter.

5 or 6 or 7

The opposite/complement of a password with no letters

Passwords of length 5 **or** 6 **or** 7

Passwords

- Length five
  - All Passwords: 36^5
  - No letters: 10^5
  - Valid passwords: $36^5 - 10^5$

    60,366,176
- Length six
  - All Passwords: 36^6
  - No letters: 10^6
  - Valid passwords: $36^6 - 10^6$

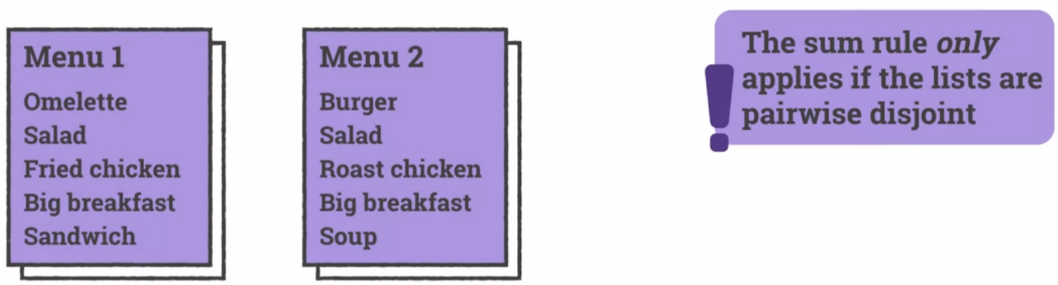
    2,175,782,336
- Length seven
  - All Passwords: 36^7
  - No letters: 10^7
  - Valid passwords: $36^7 - 10^7$

    78,354,164,096

Total number of valid passwords :

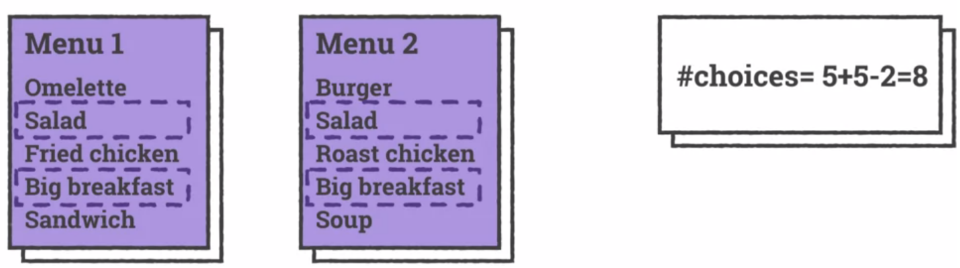
60,366,176 + 2,175,782,336 + 78,354,164,096 = 80,590,312,608

The sum rule with a minor defect



- Subtraction rule = applies when lists or sets have items in common
  - also known inclusion-exclusion principle
  - Formally, it states that if a choice can be made from two lists containing n and m items, then the number of ways to make a choice from these two lists is n + m - $n \cap m$(number of items in common)

Number of choices for a meal



Example

- How many integers less than 100 are divisible by either 2 or 3?
- Divisible by 2: $\lfloor\frac{99}{2}\rfloor = \lfloor 49.5 \rfloor = 49$
- Divisible by 3: $\lfloor\frac{99}{3}\rfloor = \lfloor 33 \rfloor = 33$
- Divisible by 6: $\lfloor\frac{99}{6}\rfloor = \lfloor 16.5 \rfloor = 16$

! $\lfloor x \rfloor$ is the greatest integer that is smaller or equal to x

! This is what is in common in two lists

- Divisible by either 2 or 3: 49 + 33 − 16 = 66

The pigeonhole principle

> If there are k+1 or more objects to be placed in k boxes, there is at least one box containing more than one object.

- This is also known as Dirichlet Drawer Principle

The generalised pigeonhole principle

> If there are N objects to be placed in k boxes, there is at least one box containing at least $\lceil N/k \rceil$ objects.

! Proof:
Use contradiction!

- Assume no box contains more than $\lceil N/k \rceil$ -1 objects.
- Number of objects $\leq k(\lceil N/k \rceil - 1) < k(N/k + 1 - 1) = N$
- Number of objects < N

Example

- How many cards from a standard deck of 52 cards must be selected to guarantee that 5 cards are from the same suit?
- How many suits are there? 4 ♣ ♥ ♠ ♦
- If we pick 16 cards: ♣ ♣ ♣ ♣ ♥ ♥ ♥ ♥
  ♠ ♠ ♠ ♠ ♦ ♦ ♦ ♦
- When we pick the 17th card, whatever it is, it makes 5 cards from the same suit.
- $\lceil N/4 \rceil = 5$ So $N = 17$

1. In a group of 4 integers, show that there are at least two with the same remainder when divided by 3.

- Remainders when divided by 3: 0, 1, 2
- It means we have three boxes and four objects.

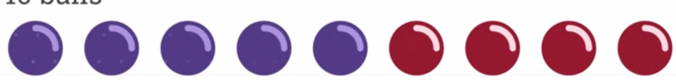
The Pigeonhole Principle: at least two have the same remainder.

- Can be generalised to k+1 integers when divided by k.

2. A bag contains 7 blue balls and 4 red balls. How many balls must be selected to guarantee that 3 balls are the same colour?

- Let us see how we can avoid three balls of same colour.

- As soon as we select the 5th ball, it is either red or blue. Then we will have three balls of same colour. $\lceil x/2 \rceil=3$, $x=5$
- What if we need to guarantee six balls of the same colour?
  - They cannot be red
  - 10 balls



3. Select 5 integers from {1, 2, 3, 4, 5, 6, 7, 8}; show that at least two integers add up to 9.

- Let's see the pairs of integers making up 9.

(1, 8)	(2, 7)	(3, 6)	(4, 5)
A	B	C	D

- Any integer from the list belongs to one label.
- There are 5 objects and 4 boxes.

The Pigeonhole Principle: at least two objects will be in the same box.

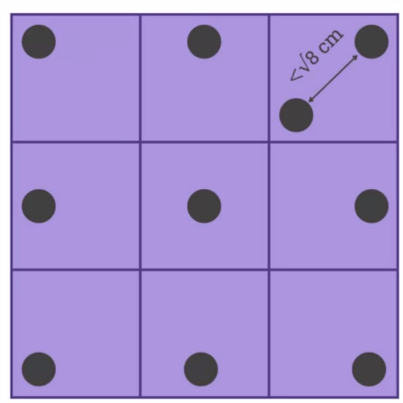
4. There are n people in the room; every pair is either friends or not. Show that there are at least two people with the same number of friends.

- The possibilities of number of friends
  - Either 1, 2, ..., n-1
  - Or 0, 1,.., n-2
- Because, if one person has no friends, then the maximum numbers of friends one can have in this group is n-2
- In both cases, we have n-1 boxes and n objects

The Pigeonhole Principle: at least two people have same number of friends.

5. Show that if there are 10 dots on a square of 6cmx6cm, there are at least two dots within $\sqrt{8}$ cm.

- Split the square into a grid of 3x3
- Each little square is 2cmx2cm
- The longest distance in the little square is $\sqrt{8}$ cm
- Ten dots, nine squares, one must contain at least two
- These two dots cannot be too far apart



Summary

In this week, we learned about the product, sum and subtraction rules for counting and the pigeonhole principle.