Numerical Maths

CM1015

INTRO TO NUMBER BASES

In computing EVERYTHING is a number.

Binary is written in 1 and 0 and represents on and off switches, a computer has billions of these switches.

BASE10 – numbers we are used to working with

$$
\begin{aligned}
5192_{10} &= 5{,}000 + 100 + 90 + 2 \\
&= 5(1000) + 1(100) + 9(10) + 2(1) \\
&= 5(10^3) + (10^2) + 9(10^1) + 2(10^0)
\end{aligned}
$$

BASE 2 – this is binary

$$
\begin{aligned}
110101_2 &= 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0) \\
&= 1(32) + 1(16) + 0(8) + 1(4) + 0(2) + 1(1) \\
&= 32 + 16 + 4 + 1 \\
&= 53_{10}
\end{aligned}
$$

Using bases in computing we can represent anything in a cyclical manner. eg clock going from 60 to 0

We can convert between decimal and binary by looking at the values for each placeholder.

DECIMAL	64	32	16	8	4	2	1
BINARY	1	1	0	0	0	0	1

97 for example is made up of 64 + 32 + 1 so using the above translator in binary becomes 1100001

We can also use this to do the reverse by looking at the value of the binary placeholders where there is a 1.

To convert fractions the logic is basically the same in terms of placeholders, however the shortcut for decimals to binary is to multiply by 2. Take the integer as binary and then multiply the multiply the remainder.

$0.75_{10} = 0.11_2$

$0.75 \times 2 = 1.5$

$0.5 \times 2 = 1$

you continue this process until you end up with a whole number when x by 2.

ADDING BINARY

You stack the numbers as you would in school and you add each column as though it were a decimal number and then convert the decimal back to binary carrying the numbers as normal.

example 1.

$$\begin{array}{r} 1010 \\ +\quad 10 \\ \hline 1100 \end{array}$$

$0+0=0$
$1+1=2_{10}=10_2$
$0+1=1$
$1=1$

example 2

$$\begin{array}{r} 101110 \\ +\quad 11101 \\ \hline 1001011 \end{array}$$

$0+1=1$
$1+0=1$
$1+1=2_{10}=10_2$
$1+1+1=3_{10}=11_2$
$1+0+1=2_{10}=10_2$
$1+1=2_{10}=10_2$

SUBTRACTING BINARY

As with addition you lay this out as you would have done in school. If the number is too small you borrow from the number before. When you perform the subtraction you do it in binary and decimal. When you carry a number it goes from 0_2 to 10_2

Future log

example 1

$$\begin{array}{r} 1\cancel{1}\cancel{1}\cancel{0}\cancel{0}0 \\ -\ 101011 \\ \hline 001101 \end{array}$$

$10_2 - 1_2 = 1_2$
$1_2 - 1_2 = 0_2$

MULTIPLYING BINARY

When you multiply by 10_2 it is the same as multiplying by 10_{10} by adding a 0 to the end and shifting everything across.

example 1 $1101_2 \times 10_2 = 11010_2$

Otherwise when multiplying you would stagger it into different lines.

example 2

$$\begin{array}{r} 11011 \\ \times\quad 101 \\ \hline 11011 \\ 11011\cdot\cdot \\ \hline 10000111 \end{array}$$

we start with
11011 x 1
then we do
11011 x 100
and then add the two together

example 3

$$\begin{array}{r} 10111 \\ \times\quad 11 \\ \hline 10111 \\ 10111\cdot \\ \hline 1000101 \end{array}$$

OCTAL BASE - in powers of 8

512	64	8	1	1/8	1/64	1/512
8^3	8^2	8^1	8^0	8^{-1}	8^{-2}	8^{-3}

HEXADECIMAL

is the powers of 16, 10 above is represented as letters.

10	11	12	13	14	15
A	B	C	D	E	F

CONVERSION BINARY TO OCT TO HEX

To convert from binary to octal you group the binary into groups of three from right to left. You convert each number into decimal and the resulting answer is your octal number. The reverse applies

example 1

1 111 011 011

1731_8

example 2

4036_8

100 000 011 110

for conversion to hexadecimal, the same is true but in groups of 4.

example 3

11 1101 1001

3D9

Steganography

Is the art of coding secret messages into pictures. We can do this by very subtly adjusting the colours by hiding the colours code in the last few digits of the RGB code in binary

example 1 Red is

R 255 → 1111 1111
G 0 → 0000 0000
B 0 → 0000 0000

we replace the rightmost number with the binary of the number we want. Doing this changes the colour but almost imperceptible to the human eye.

Using this method we can hide a colour within another colour

R255
G150
B020

R162
G200
B140

R 255 → $\boxed{1111}$ 1111 $\boxed{1010}$ 0010 ← R162

$\boxed{1111}$ $\boxed{1010}$

G150 → $\boxed{1001}$ 0110 $\boxed{1100}$ 1000 ← G200

$\boxed{1001}$ $\boxed{1100}$

B020 → $\boxed{0001}$ 0100 $\boxed{1000}$ 1100 ← B140

$\boxed{0001}$ $\boxed{1000}$

You can do exactly the same in hex but with only two characters means you only take the rightmost digit for the secret code

MODULAR arithmetic

Modulo numbers only care about the remainder.

example 1 $62 \equiv 2 \pmod{60}$

example 2 $122 = 2 \pmod{60}$

ADDITIVE INVERSE

→ the additive identity is the number which when added to another does not change the result. For a normal number this will always be 0, for example $5+0=5$.

The additive inverse is any two numbers that add up to the additive inverse

In mod 5 this could be $1+4=0$ or $2+3=0$ for example.

You can also see $-1 \equiv 4 \pmod 5$ and $-2 \equiv 3 \pmod 5$ where $\equiv$ is congruent.

MULTIPLYING MOD NUMBERS

When multiplying modular numbers, you can either multiply then calculate, or you can simplify then multiply

example 1

$$11 \times 13 = ? \pmod 9$$
$$11 = 9 + 2 \equiv 2 \pmod 9$$
$$13 = 9 + 4 \equiv 4 \pmod 9$$

$$\therefore 11 \times 13 \equiv 2 \times 4 \equiv 8 \pmod 9$$

This works because when you are doing 11×13 you are doing

$$(9+2)(9+4) \quad \boxed{(9 \times 9)} + \boxed{(9 \times 4)} + \boxed{(2 \times 9)} + \boxed{(2 \times 4)}$$

all multiples of 9 so mod $9 \equiv 0$

not a multiple of 9

MULTIPLICATIVE IDENTITY

This is the same as the additive identity and is the number that does not change the number when multiplied by it. For normal numbers this will be 1. eg $5 \times 1 = 5$

The multiplicative inverse is then the two numbers which when multiplied equal the multiplicative identity

example 1 $\quad 3 \times 2 \equiv 6 \equiv 1 \pmod 5$

example 2 $\quad 4 \times 4 \equiv 16 \equiv 1 \pmod 5$

EXPONENTIAL MOD NUMBERS

When working with exponentials the numbers can become very large. In order to calculate the mod you can break it down and solve as follows:

Question $\rightarrow$ what is $163^{42} \bmod 49$?

Step 1 $\rightarrow$ simplify $163 = (3 \times 49) + 16$ so solve 16^{42}

Step 2 $\rightarrow$ $16^2 \equiv 256 - (49 \times 5) \equiv \boxed{11} \pmod{49}$

Step 3 $\rightarrow$ $16^4 \equiv (16^2)^2 \equiv 11^2 \equiv 121 - (2 \times 49) \equiv \boxed{23} \bmod 49$

Step 4 $\rightarrow$ $16^8 \equiv (16^4)^2 \equiv 23^2 \equiv 529 - (10 \times 49) \equiv 39 \bmod 49$ or $-10 \bmod 49$

Step 5 $\rightarrow$ $16^{10} \equiv 16^8 \times 16^2 \equiv -10 \times 11 \equiv \boxed{-12} \bmod 49$

Step 6 $\rightarrow$ $16^{40} = (16^{10})^4 = -12^4 \equiv -12^2 \times -12^2 \equiv 9 \bmod 49$

Step 7 $\rightarrow$ $16^{42} \equiv 16^{40} \times 16^2 = 9 \times 11 = 99 \equiv 1 \bmod 49$

$$\therefore 163^{42} \equiv 16^{42} \equiv 1 \bmod 49$$

A logical follow on from this point would be to calculate the multiplicative inverse. We know that it is the number which when multiplied by another equals 1

Since $163^{42} \equiv 1 \bmod 49$ then $163^{41} \times 163 \equiv 1$ which means 163^{41} is the inverse of 163 mod 49. This is also written as 163^{-1}

Using the above steps already done we can use this to calculate 163^{41}

$$16^{41} \equiv 16^{40} \times 16 = 9 \times 16 \equiv 46 \bmod 49$$

$$\boxed{\therefore 163^{-1} \text{ is } 46 \bmod 49}$$

Femats Theorum

$$a^p = a \pmod p$$

where p does not divide a and p is a prime number eg

$$3^7 = 3 \pmod 7$$

$a^{p-1} \equiv 1 \bmod p \;\therefore\; a^{p-2}$ is the multiplicative inverse

ENCRYPTION using modular arithmetic

$$C \text{ equiv } M^e \pmod P$$

Where e and p are public keys, M is the message you would like to send and C is the encrypted result.

To decrypt the message you can then use

$$M \text{ equiv } C^d \pmod P$$

Where d is the private key, p is the same as above. M is the decrypted result and C is the encrypted result.

In order to break the codes we need to know the bases for d and e. Someone discovered that in order to encrypt or decrypt messages then these numbers must be multiplicative inverses of the mod.

We can use a variation of fermats theorem to help with this. Since we know that for prime numbers apply we can use the below table to calculate this:

$P = 23$ and $e = 9$ we find the intersection where

$x + y = P - 1$ or $x + y = 22$

	0	1	2	3	4	5	6	7	8	9	10	11	12
0		1	1	2	2	4	2	6	4	6	4	10	
12	4	12	6	8	8	16	6	18	8	12	10	22	
24	8	20	12	18	12	28	8	30	16	20	16	24	

To find the inverse of 9 (mod 22) we do $9^{10-1} \pmod{22} = 5$ which gives us $e = 9$ and $d = 5$.

Algebraic Fractions

To simplify a fraction you can factorise it and then cancel out common denominators and numerators

$$\frac{5}{25 + 15x} = \frac{5}{5(5 + 3x)} = \frac{1}{5 + 3x}$$

To multiply fractions, you multiply each line then simplify

$$\frac{a}{c} \times \frac{b}{d} = \frac{a \times b}{c \times d}$$

Dividing is done by flipping the second fraction and then multiplying.

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{a \times d}{c \times b}$$

To add or subtract you must have a common denominator and then add or subtract the numerators.

$$\frac{3}{4} + \frac{1}{x} = \frac{3x}{4x} + \frac{4}{4x} = \frac{3x + 4}{4x}$$

SEQUENCES

A sequence is usually written as $a_n = n$ where anything after equals is the result and a_n where n is the position in the sequence.

Example 1 $a_n = 3n-2$ the first three numbers are

$$a_1 = 3\times1-2 = 1$$
$$a_2 = 3\times2-2 = 4$$
$$a_3 = 3\times3-2 = 7$$

ARITHETIC PROGRESSION is a sequence that increases by the same amount each time eg.

$$\underset{+2}{1}\quad\underset{+2}{3}\quad\underset{+2}{5}\quad\underset{+2}{7}\quad\underset{+2}{9}\quad\underset{+2}{11}\quad 13$$

An arithmetic sequence can be written as

$$a_n = a_1 + (d(n-1))$$ where d is the common difference

The above example would become $a_n = 1 + (2(n-1))$ or $a_n = 2n-1$

GEOMETRIC PROGRESSION is where we have an exponential as part of the expression. We can identify this when the ratio between the sequences is the same.

$3 \quad 6 \quad 12$ where $\frac{6}{3} = 2$ and $\frac{12}{6} = 2$

a geometric progression can be written as

$$a_n = a_1 \times r^{n-1}$$ where r is the common ratio

RANDOM NUMBERS can also be generated by an expression such as

$$r_{n+1} \equiv m \cdot r_n + i \pmod{c}$$

where $c > 0$ the modulus
m is the multiplier
i is the increment
$0 \leq r_0 < c$ the starting value

The maximum number of possible values is limited by the modulus, so a small mod will start repeating earlier than a larger number.

SERIES

Series use sigma notation: $\sum_{k=1}^{n} k$ This means we are summing the sequence k from 1 to n

For example with the sequence $a_n = n+1$ the first three terms are 2, 3, 4. As a series this would be

$$\sum_{n=1}^{3} n+1 = 2+3+4 = 9$$

You can split a sum into various components so long as splitting does not ~~resolve~~ result in a different answer

$$\sum_{n=1}^{4} (n+2^n) = \sum_{n=1}^{4} n + \sum_{n=1}^{4} 2^n$$

You can also write multiples as per below, this becomes important when looking at the proforma formulas

$$\sum_{n=2}^{4} 5(n+2^n) = 5\sum_{n=2}^{4} n+2^n$$

The solutions for various sequence types are as follows

$$\sum_{i=1}^{n} i = 1+2+3+4\ldots+n = \frac{n(n_1+n_n)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+4^2+n^2 = \frac{n(n+n_1)(2n+n_2)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3\ldots n^2 = \frac{n^2(n+n_1)^2}{4}$$

$$\sum_{i=1}^{n} 2^{i-1} = 2^0+2^1+2^2+2^3\ldots+2^n = \frac{8\times(r^n-1)}{r-1}$$ r - common ratio

You can imagine this by thinking of the sequence 1, 2, 3 ... 100 you can say you have 50 pairs of numbers that add up to 101 where you add the first and the last numbers together.

$$\frac{(1+100)\times 100}{2} = 5,050$$

For more complex series it can be easier to split the problem and solve each component part.

$$\sum_{i=1}^{10}(i^2+2i-3) = \sum_{i=1}^{10} i^2 + 2\sum_{i=1}^{10} i - \sum_{i=1}^{10} 3$$

$$= \frac{10(10+1)(2\times10+10)}{6} + 2\times\frac{(1\times10)\times10}{2} - 3\times10$$

$$= 385+110-30$$

$$= 465$$

For a sequence where the exponential is determined by the sequence position of 2^{n-1}, the first step is to multiply the sequence by the ratio.

$$i = 1+2+4+8+16\ldots 1024$$
$$2i = \cancel{1}\ 2+4+8+16\ldots 1024+2048$$

If we subtract these 2 numbers we end up with

$2i - i = 2048 - 1 = 2047$ as all the terms cancel each other out except the first and last.

Convergence
and divergence

$1/n = 1/1, 1/2, 1/3, 1/4\ldots$ converges on 0 as n gets large
$0.1^n = 0.1, 0.01, 0.001, 0.0001\ldots \to 0$
$10^n = 10, 100, 1000\ldots \to +\infty$ or is divergent
$(-0.1)^n = -0.01, 0.001, -0.0001\ldots \to 0$
$(-10)^n = -10, 100, -1000 \to$ no limit and divergent
$\frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\ldots \to$ converges to 1.

It does not matter if the sequences never reaches the number it is converging on, but that it never exceeds it. Formally we write it as:

$\boxed{\frac{1}{n} \xrightarrow[n\to+\infty]{} 0}$ or $\boxed{\lim_{n\to+\infty} \frac{1}{n} = 0}$

$\boxed{\frac{n}{n+1} \xrightarrow[n\to+\infty]{} 1}$ or $\boxed{\lim_{n\to+\infty} \frac{n}{n+1} = 1}$

And a divergent sequence should be written as

$\boxed{\lim_{n\to+\infty} 10^n = +\infty}$ this will only ever be a positive number

$\boxed{\lim_{n\to+\infty} (-10)^n}$ = no limit because this has no limit positive or negative it has no bounds

Can the sum of infinite numbers ever be finite? While this may not seem possible under certain circumstances it can be done

$$S_\infty = \frac{a}{1-r}$$

so long as $-1 < r < 1$ and it is a geometric series

example 1

for a sequence where the first term is 2 and the ratio is $\frac{1}{3}$

$$\frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3$$

If you have two sequences that are convergent, then they are still convergent when you add them together, with the new limit being the sum of the 2 original limits.

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

If you multiply a sequence by a constant you will also multiply the limit by the constant to get it's new value

$$\lim_{n\to\infty} Ca_n = C\lim_{n\to\infty} a_n$$

Convergence is all about defining limits on a sequence. So we can sandwich it between other sequences for example

$$\lim 0 \leq \lim \frac{1}{n^2} \leq \frac{1}{n}$$

As $\frac{1}{n^2}$ will always be smaller than $\frac{1}{n}$ and both will converge on 0.

Functions

A function is something you put an input through and get an output out of. For it to be a function you must only get 1 output.

$x \longrightarrow$ add 2 $\longrightarrow x+2$

input $\longrightarrow$ function $\longrightarrow$ output

The input is the independent value (it can be anything), the output is the dependent value (as it is dependent on the input).

The syntax is usually: x = input
f = function
y = output

This means the above example becomes

$$f(x) = x + 2 \quad \text{or} \quad y = x + 2$$

A piecewise function is one where different rules apply for different values of x and can be written as

$$y(x) \begin{cases} 3x^2+1 & \text{when } -1 \leq x \leq 2 \\ 3x & \text{when } 2 < x \leq 6 \\ 2x+1 & \text{when } x > 6 \end{cases}$$

eg $y(4) = 4 \times 3 = 12$
$y(7) = 2\times 7 + 1 = 15$

COMPOSITE FUNCTIONS

is where we apply one function after the other

$x \longrightarrow +3 \longrightarrow x+3 \longrightarrow \times 2 \longrightarrow 2(x+3)$

input f(x) input/output g(x) output

A composite function is written as $g(f(x))$

INVERSE FUNCTION

is the revese of the orginal function

example 1 $f(x) = 2x$ then the inverse is $g(x) = \frac{x}{2}$

We can find the inverse by rearranging the formula aso that x becomes y and $f(x)$ becomes x,

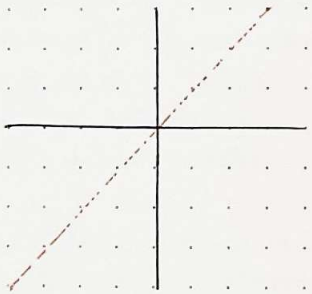
Some functions will not have an inverse. For example x^2 will always be positive but it could have a positive or negative input. This can't be reveated eg $3^2 = 9$ and $-3^2 = 9$

CO-ORDINATES

A closed interval is one that includes the limit $\leq$ and $\geq$ uses $[4, 1]$
An open interval does not include them for example $<$ and $>$ uses $(4, 1)$

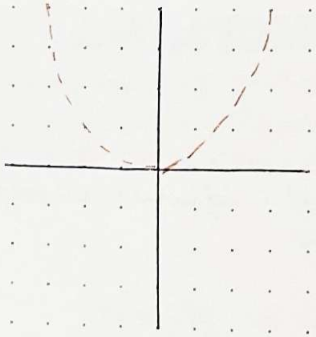
Set notation would write this as $x \in \mathbb{R},\ 4 \leq x < 1$
where $x \in \mathbb{R}$ means for real numbers not just integers

Straight line graph
$y = ax + c$



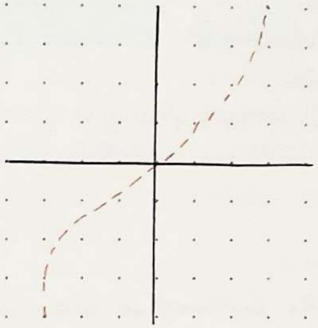
- include positive and negative integers
- include symmetrical values
- a is the gradient
- c is the y intercept

parabola
$y = x^2$



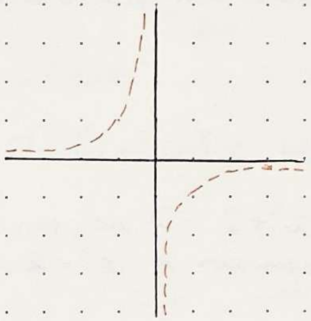
- include positive and negative integers
- include symetrical values
- Should be more values near x

Curved graph
$y = x^3$



- symetrical x values
- include negative and positive values
- close to 0 more values

asymtote
$y = \frac{2}{x}$



- also called a hyperbola
- include more values at 0
- include symetrical values
- will have asymtotes
- cannot calculate at the convergence
- domain $\mathbb{R} - \{0\}$

To find the y intercept for a quadratic graph you could use the formula

input formula = $ax^2 + bx + c$

quadratic equation = $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

example 1 $y = x^2 + 3x$ where $a = 1$, $b = 3$, $c = 0$

$= \frac{-3 + \sqrt{3^2 - 4 \times 1 \times 0}}{2 \times 1}$ $\quad = \frac{-3 - \sqrt{3^2 - 4 \times 1 \times 0}}{2 \times 1}$

$= 0$ $\quad = -3$

$\therefore$ the graph $y = x^2 + 3x$ has the following intercepts $(0, 0)$ $(-3, 0)$

To locate the vertex we know the above graph is symmetrical so the vertex is halfway between the x intercepts of 0 and -3. So we can insert -1.5 into the line formula

$$y = -1.5^2 + (3x - 1.5)$$
$$= -2.25$$

∴ vertex is at $(-1.5, -2.25)$

PLOTTING CUBICS the best way to plot cubics is to factorise and then each of the components can equal 0 to find the intercepts.

example 1

$y = -x^2 + 2x$ becomes $x(-x^2 + 2) = 0$

either $x = 0$ or $-x^2 + 2 = 0$ we can simplify the second expression to $x^2 = 2$ so $x = 0$ or $x = \sqrt{2}$

PLOTTING POLYNOMIALS a polynomial is a function made up multiple exponential functions, anything over 4 is classed as a higher order polynomial.

example 2

$y = x^5 + 9x^4 + 25x^3 - 15x^2 - 26x + 24$

we then factorise to get

$y = (x-4)(x-3)(x-2)(x-1)(x+1)$

$x = 0, +4, +3, +2, +1, -1$

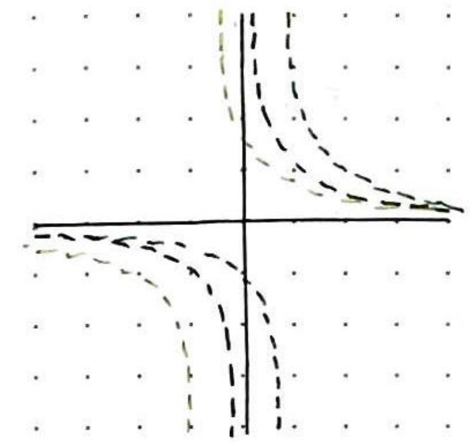
PLOTTING RECIPROCALS a reciprocal will converge on both the x and y axis.

$y = \frac{1}{x-5}$ will converge on $x = 5$ and $y = 0$ as no fraction will equal 0 and you cannot divide by 0.

TRANSFORMATION

is all about moving a graph, flipping it or squishing it. Basically transforming it into something slightly (but not fundamentally) different.

example 1



- $\frac{1}{x}$
- $\frac{1}{x+1}$
- $\frac{1}{x-1}$

★ we can move it right by adding 1 to x and left by subtracting 1 to x

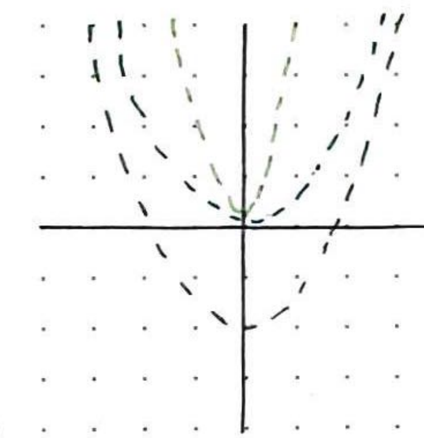
★ to make it a mirror image we add a minus to the x

★ to make the curve shallower we increase the numerator

★ a steep more angular curve is done by multiplying the denominator

for a squared graph we have

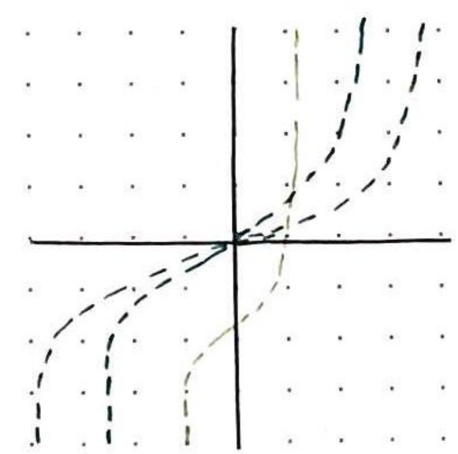
- x^2
- $5x^2$
- $x^2 - 5$



★ to make it a mirror image we reverse the x

★ to squish it we multiply the x component

★ to change the y intercept we add or subtract a constant



- x^3
- $0.1x^3$
- $x^3 - 2$

★ to make the line shallower we multiply x by < 0

★ to change the y intercept we add or subtract a constant

★ to make a mirror image we reverse the sign of x

With a graph that is a combination of 2 functions eg $x^3 - x^2$ the same logic will apply.

- ★ $(x^3-1) - x^2$ will move the y intercept down 1
- ★ $(x^3-1)-(x^2-1)$ will have no impact on the graph
- ★ $(x-1)^3 - (x-1)^2$ will translate the graph to the right

We can also scale the entire function by multiplying it by 5 to get $5(x^3-x^2)$ and we can shrink it by increasing that number on the y

To make it fatter we can divide x by the constant but it must be before applying the exponential eg $\left(\frac{x}{5}\right)^3 - \left(\frac{x}{5}\right)^2$

example 1

$y = \frac{1}{x^2+x}$ transformed $\boxed{3+}\frac{1}{(x\boxed{-2})^2+(x\boxed{-2})}$

moves the y

moves the x

example 2

$y = x^3 - 2x$ transformed $y = (2x)^3 - \boxed{4}x - \boxed{3}$

a stretch of 1/2 in the x

a shift of 3 in the y

example 3

$y = \sqrt{x}$ transformed $y = \boxed{3}\sqrt{x\boxed{+2}}$

a stretch of 3

shift of minus 2 in the x

When multiplying the whole expression will change it on the y plane with a positive numbers squashing it thinner and fraction making it fatter.
Multiplying just the x expression will change it on the x plane

Kinematics

Suvat equations of motions

$$v = u + at$$

$$s = ut + \tfrac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{u+v}{2}t$$

$$s = vt - \tfrac{1}{2}a^2t^2$$

s = distance (m)
u = initial velocity (ms^{-1})
v = final velocity (ms^{-1})
a = acceleration (ms^{-2})
t = time (seconds)

Triangles

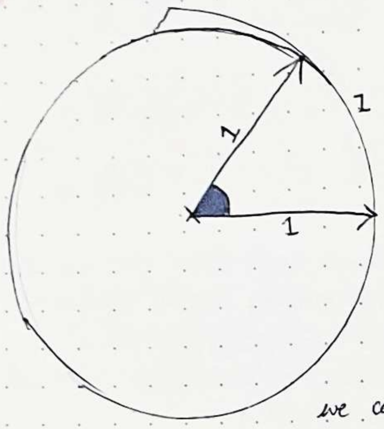
EQUILATERAL
- ★ All sides same length
- ★ All angles the same

ISOSELES
- ★ 2 sides the same
- ★ 2 angles the same

SCALENE
- ★ No sides the same
- ★ No angles the same

- A triangles 3 angles add up to 180° and the exterior angles all add up to 900°
- The area is ½ width × height
- The largest side is opposite the largest angle, the smallest side is opposite the smallest angle

RADIANS



A radian is the angle where the portion of the circumference is the same length as the radius

★ The circumference of the circle is

$$2\pi r$$

$\therefore$ 2π radians $= 360°$ and 1 radian $= \frac{360}{2\pi}$

we can also write this as :

$$\frac{360°}{\text{an angle in degrees}} = \frac{2\pi \text{ radians}}{\text{angle in radians}}$$

So we can use this to help convert between the two when we have limited information.

FEASIBLE TRIANGLES

You can identify if a triangle is feasible without drawing it as

- The sum of all its angles must be 180°
- The sum of two lengths must always be longer than the third

COMPUTING WITH RADICALS AND SURDS

A surd is a root of a number that gives a decimal that repeats forever eg $\sqrt{2}$
When computing with them we can simplify as follows

examples 1 $(\sqrt{2} - \sqrt{5})(2\sqrt{2} + 1) = \boxed{2\sqrt{2}\sqrt{2}} + \boxed{\sqrt{2}} - \boxed{\sqrt{2}\sqrt{5}} - \boxed{\sqrt{5}}$

$$\boxed{2\sqrt{4}} + \boxed{\sqrt{2}} - \boxed{2\sqrt{10}} - \boxed{\sqrt{5}}$$

$$\boxed{4} \quad \boxed{\sqrt{2}} - \boxed{2\sqrt{10}} - \boxed{\sqrt{5}}$$

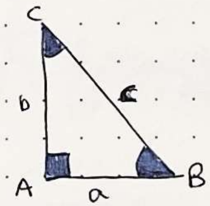
When converting into its simplest form we know $\boxed{\sqrt{ab} = \sqrt{a} \times \sqrt{b}}$ we can then find the smallest common denominator that will give us a non-surd root.

example 2

$$\sqrt{48} = \sqrt{16} \times \sqrt{3}$$
$$= 4\sqrt{3}$$

we can also say $\boxed{\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}}$

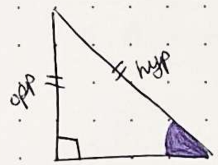
PYTHAGORAS THEORUM



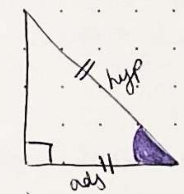
$$\boxed{a^2 + b^2 = c^2}$$

Hypotenuse is the longest side, c.

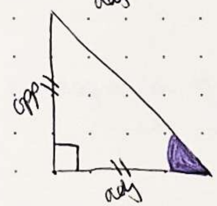
Trigonometric Ratios



Sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ when you want to find the blue angle and you have the length of the opposite side and hypotenuse



Cosine = $\frac{\text{adjacent}}{\text{hypotenuse}}$ when you have the length of the adjacent side and the hypotenuse

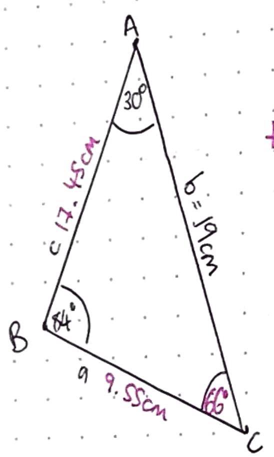


Tan = $\frac{\text{opposite}}{\text{adjacent}}$ For when you have opposite and adjacent lengths marked

Sine Rule

Many triangles do not have a right angle and so other rules must be used. The sine rule states that

$$\frac{a}{\sin a} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



For example, with this triangle we are given
$A = 30°$, $B = 84°$, $b = 19cm$.

★ straight away the remaining angle is 66° as
$180° - 30° - 84° = 66°$

★ substituting values into the sine rule we get

$$\frac{a}{\sin 30°} = \frac{19°}{\sin 84°} = \frac{c}{\sin 66}$$

we rearrange the equation to solve for a:

$$a = \frac{19 \times \sin 30°}{\sin 84°} = 9.55cm \quad \text{and} \quad c = \frac{19 \times \sin 66°}{\text{sine } 84°} = 17.45\,cm$$

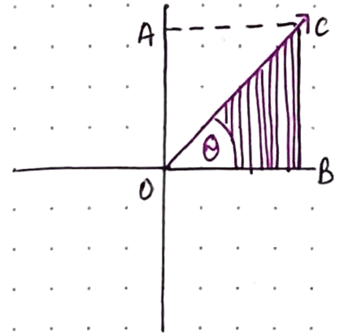
Cosine Rule

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}}$$

For use when we have three sides or 2 sides and the included angle (the angle between the two sides given).

When using the quadratic equation with the cosine rule, it is important to note that if the number in the square is negative then the TRIANGES IS NOT FEASIBLE.

When trying to work out the angles of certain lines on an x/y graph you can use the properties of a right angled triangle



$$\sin \theta = \frac{AO}{OC} = \frac{CB}{OC}$$

$$\cos \theta = \frac{OB}{OC} = \frac{AC}{OB}$$

$$\tan \theta = \frac{OA}{OB} = \frac{BC}{AC}$$

noting that each quadrant of the graph the line passes through you need to add 90°

Trigonometric Identities

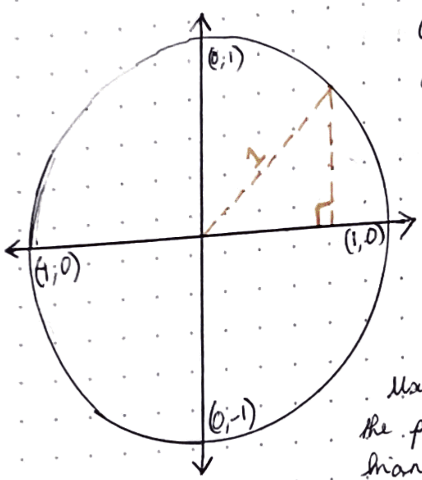
An identity is something that is true for all known values. An equation is something to solve

$$\boxed{\frac{\sin \theta}{\cos \theta} = \tan \theta}$$

$$\boxed{\cos^2 A + \sin^2 A = 1 \quad \text{where } \cos^2 A = (\cos A)^2}$$

$$\boxed{\begin{aligned} \sin \theta &= -\sin(\theta - 180°) \\ \cos \theta &= -\cos(\theta - 180°) \\ \tan \theta &= \tan(\theta - 180°) \end{aligned}}$$

Unit Circle

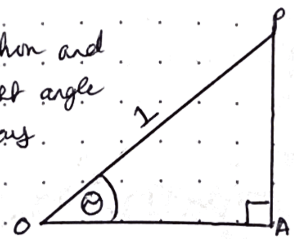


Consider the unit circle given by the formula

$$x^2 + y^2 = 1$$

Using this circle we can create triangles. The hypotenuse of the triangle will always be 1 as the radius of the circle is one

Using this information and the properties of a right angle triangle we can say



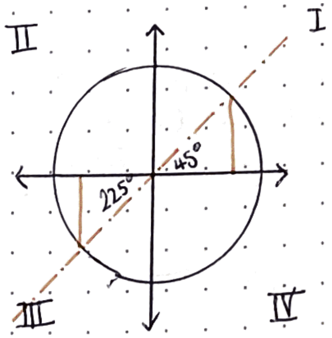
$\sin \theta = \frac{\text{length } PA}{1}$ or length PA

$\cos \theta = \frac{\text{length } OA}{1}$ or length OA

$\tan \theta = \frac{\text{length } PA}{\text{length } OA}$

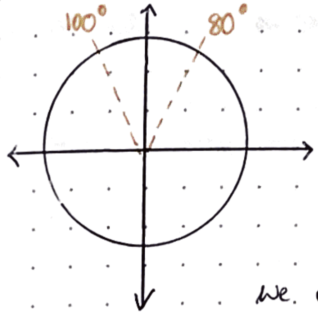
We can extrapolate this information for wider angles and relate it back to quadrant I.

With the angle to the right of 225° we can establish its equivalent in quad I is $180° + 45° = 225°$



$$\boxed{\begin{array}{l} \text{Sine } 225° = -\text{Sine } 45° \\ \text{Cosine } 225° = -\text{Cosine } 45° \\ \text{Tan } 225° = \text{Tan } 45° \end{array}}$$

Trigonometric equations



So we are told the equation:
$\sin(x - 30°) = \sin(80°)$

One of the answers will be 110° as $110° - 30° = 80°$, however we will need to find other solutions.

We can do this by locating where the length will be the same eg 100° or $x - 30° = 100°$.

Therefore $x = 110°$ or $x = 130°$. This is true where $x \in [0, 360°]$ but if it where for $x \in [-360°, 360°]$

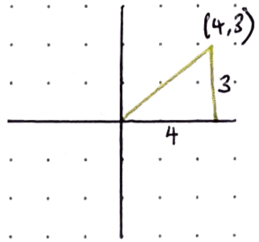
In this instance $x = 110° + (360° \times \text{turns})$ or
$x = 130° + (360° \times \text{turns})$

where the number of turns must not exceed the total number of degrees in the limit eg $110° + 360° \times 1 = 470°$ which is outside the limit

However $110° + 360° \times -1 = -250°$ which is within the limit

POLAR CO-ORD

Polar co-ordinates work by giving a radius and an angle in radians to convert cartesian co-ordinates to polar co-ordinates you just make a triangle and then solve from there



The rule for the other way around is

(radius × cos θ, radius × sin θ)

EXPONENTIALS

One of the most commonly used bases for exponentials is 'e' where 'e' is the exponential constant 2.71828

The expression e is found in many natural phenomena such as bacteria growth and radioactive decay.

Exponential functions have the following rules:

$\boxed{e^a \times e^b = e^{a+b}}$

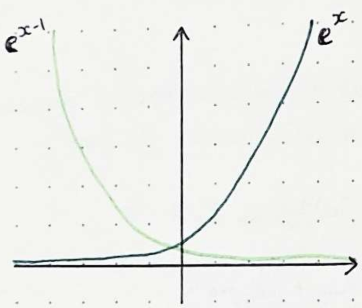
$\boxed{(e^a)^b = e^{ab}}$

$\boxed{e^0 = 1}$

$\boxed{e^{\frac{a}{b}} = \sqrt[b]{e^a}}$

$\boxed{\dfrac{e^a}{e^b} = e^{a-b}}$

* note the e function is never a negative



e^x increases without bounds as x increases. Aka it increases exponentially

as x becomes large and negative e^x becomes closer and closer to 0

1 TO 1 PROPERTY

you can solve exponentials by using the one to one property

Original equation	$9 = 3^{x+1}$
Make integers the same	$3^2 = 3^{x+1}$
Cancel them out	$2 = x + 1$
Solve for x	$x = 1$

Original equation	$9^x = 27$
Make integers same	$(3^2)^x = 3^3$
Cancel them out	$2x = 3$
Solve for x	$x = 1.5$

transformations of exponents

3^{x+1} → a horizontal shift to the left on the x axis

$3^x - 2$ → a vertical shift down 2 on the y axis

-3^x → a mirror image reflected on the x-axis

3^{-x} → a mirror image on the y axis

$\frac{1}{3}^x$ → a mirror image on the y axis

When trying to solve equations such as $e^x = 1.5$ you can do an approximation by looking at graph and finding the x value where $y = 1.5$

Logarithm

A logarithm can be used to write an exponential in another form

example 1 $\boxed{125 = 5^3}$ is the same $\boxed{\log_5 125 = 3}$

The base is the subscript number, in the above example $\log_5$, 5 is the base.

The base can be any positive number that is not 1. Most common used bases are 10 and e

$\boxed{\log_e = \ln}$ $\boxed{\log = \log_{10}}$

Solving between logs can be done using the following formula

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

example 1 $\log_6 19 = \dfrac{\log 19}{\log 6} = 1.6433$

It is worth noting that if $x = a$ then we get

$$\log_a a = \frac{\log a}{\log a} = 1$$

therefore $\log_a a = 1$ and $\log_a 1 = 0$

Laws of logarithms

$$\log A + \log B = \log AB$$
$$\log A - \log B = \log (A/B)$$
$$n \log a = \log a^n$$

establishing how many digits a number has can be done by using these laws

$2^{2001} = 10^{?}$ (seeing as each multiplication of 10 adds an extra digit)

$$\log_{10}(2^{2001}) = 2001 \log_{10} 2$$
$$= 602.36 \text{ or } 603 \text{ digits}$$

Other conversions that are good to be aware of are

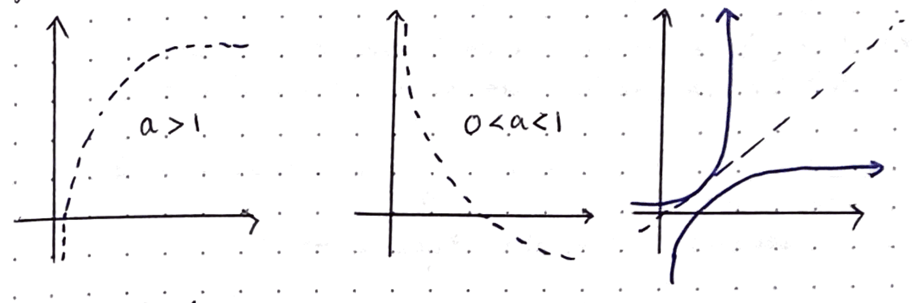
$$e^{x-3} = \frac{e^x}{e^3} = \frac{1}{e^3} \times e^x$$

$$\log_a y = -\log_{\frac{1}{a}} y$$

examples include

$$\ln 72 = \ln(9 \times 8) = \ln 9 + \ln 8$$
$$\ln 3 = \ln 9^{\frac{1}{2}} = \frac{1}{2} \ln 9$$
$$\ln(9^8) = 8 \times \ln 9$$
$$\ln 1 = \ln 9 - \ln 9 = \ln 9/9 = 0$$
$$\ln \tfrac{9}{8} = \ln 9 - \ln 8$$

logarithmic graph examples are as follows



★ has a vertical asymptote at 0 as $\log_a 0$ and $\log_a -x$ cannot be done

★ domain = x axis
range = y axis

★ domain = $(0, +\infty)$
range = $(-\infty, +\infty)$

★ the inverse of e^x is $\ln x$

★ inverse of 10^x is $\log x$

Inverting - expos and logs

example 1 $\quad f(x) = \log_4\left(\frac{x-1}{3}\right)$

$\boxed{\text{Step 1}}$ - write the expression switching y and x

$$x = \log_4\left(\frac{y-1}{3}\right)$$

$\boxed{\text{Step 2}}$ - convert into exponential form

$$\frac{y-1}{3} = 4^x$$ then isolate y

$\boxed{\text{Step 3}}$ - Isolate y to get final function

$$y = 4^x \times 3 - 1 \quad \text{or} \quad y = 3(4^x) - 1$$

Limits of Functions

$\lim_{x \to 3} \left(\frac{1}{x-2}\right)$ substitute in the 3 to get $\frac{1}{3-2} = 1$ therefore $\boxed{\lim_{x \to 2} \left(\frac{1}{x-2}\right) = 1}$

This is the most direct method and is known as direct substitution. But it cannot be done in all instances example 1

$\lim_{x\to 2}\left(\frac{1}{x-2}\right)$ cannot be done as $\frac{1}{0}$ cannot be defined so we need to check the limits just either side

$\lim_{x\to 2^-}\left(\frac{1}{x-2}\right)$ we look at the behaviour just before 2 or 2^- so $x = 1.9, 1.99, 1.999$ and so on so becomes $= -10, -100, -1000 \ldots \to -\infty$

$\lim_{x\to 2^+}\left(\frac{1}{x-2}\right)$ we then look at the behaviour just after 2 or 2^+ so $x = 2.1, 2.01, 2.001$ and so on $= 10, 100, 1000 \ldots \to +\infty$

Some limits can be done using direct substitution but only once they've been factorised.

$$\lim_{x\to -2}\left(\frac{x+2}{x^2-4}\right) = \frac{-2+2}{4-4} = \frac{0}{0}$$

However if we factorise first you get:

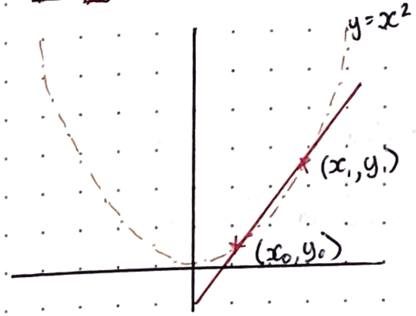
$$\frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2}$$ where $x \neq -2$ therefore

$$\boxed{\lim_{x\to -2}\left(\frac{x+2}{x^2-4}\right) = \lim_{x\to -2}\left(\frac{1}{x-2}\right) = \frac{1}{-4}}$$

$\lim_{x\to a} f(x)$ } can be: finite, $\pm$ infinity with an asymptote, lateral limits may differ

$\lim_{x\to \infty} f(x)$ } can be: finite with horizontal asymptote $\infty^{\pm}$

TANGENT LINES



$$\text{Gradient} = \frac{y_1 - y_0}{x_1 - x_0}$$

To find the tangent we move the co-ordinates closer and closer together or

The gradient at point $B = \lim_{A\to B}$ gradient of AB

So we have the following points on the line $y = x^2$

$A = (A, A^2)$

$B = (1, 1)$

Substituting these into the gradient formula you get

$$\lim_{a\to 1}\left(\frac{1-a^2}{1-a}\right) = \frac{(1+a)\cancel{(1-a)}}{\cancel{1-a}} = 1+a = 1+1 = \boxed{2}$$

Therefore we can show the gradient of $x^2 = y$ at point 1, 1 is 2.

This can further be simplified to say $2x_0$ = tangent gradient

$$\lim_{a\to x_0}(x_0 + a) = x_0 + x_0 = 2x_0$$

differentiation

Differential calculus is all about what is happening right now, aka at this exact point.

$$\text{POWER RULE} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

This can also be written as $f(x) = x^n$, $f'(x) = nx^{n-1}$ where $x \neq 0$

And also as $f(x) = ax^n$
$f'(x) = a\frac{dy}{dx}x^n$

example 1 $2x^5 = 2\boxed{\frac{dy}{dx}x^5} = 2\times 5x^4 = 10x^4$

$$\frac{dy}{dx}\left[f(x)+g(x)\right] = \frac{dy}{dx}\left[f(x)\right] + \frac{dy}{dx}\left[g(x)\right] \text{ or } f'(x)+g'(x)$$

MULTIPLYING DERIVATIVES

$$y = u \times v \qquad y' = u'v + uv'$$

example 1 $y = \underbrace{(x^2+3x)}_{u}\underbrace{\sqrt{x}}_{v}$

$\boxed{u' = 2x+3}$ $\boxed{v' = \frac{1}{2}x^{-\frac{1}{2}} \text{ or } \frac{1}{2\sqrt{x}}}$

$$y' = \underbrace{(2x+3)}_{u'}\underbrace{\sqrt{x}}_{v} + \underbrace{(x^2+3x)}_{u}\underbrace{\frac{1}{2\sqrt{x}}}_{v'}$$

DIVIDING DERIVATIVES

$$y = \frac{u}{v} \qquad y' = \frac{u'v - uv'}{v^2}$$

example 1 $y = \frac{2x+3}{x^2-1} \begin{matrix}\rightarrow u \\ \rightarrow v\end{matrix}$

$\boxed{u' = 2}$ $\boxed{y' = 2x}$

$$y' = \frac{\overset{u'}{2}\overset{v}{(x^2-1)} - \overset{u}{(2x+3)}\overset{v'}{2x}}{\underbrace{(x^2-1)^2}_{v}}$$

Chain Rule

When you are looking at differentiating multiple functions of a formula you can consider them seperatley

example 1 $\frac{1}{3x^2-5x+2}$ can be split into $f(x) = \frac{1}{x}$ $g(x) = 3x^2-5x+2$

$f'(x) = -\frac{1}{x^2}$ or $-x^{-2}$

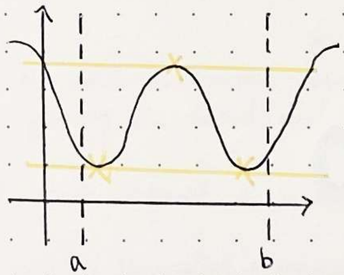
$g'(x) = 6x-5$

We can then combine the two together to get

$$-\frac{1}{g(x)} \times g'(x) = -\frac{1}{(3x^2-5x+2)^2} \times 6x-5$$

EVALUATING FUNCTIONS

You can use differentiation to find different parts of a function for example, locating the points at which the tangent has a gradient of 0 will show you where the maxima and minima points are.



within the range (a, b) if you find where $f'(x) = 0$ you will locate the maxima and minima or $f(x)$ is undefined.

You can find decreasing intervals using the same logic if you have a function

$$f(x) = x^6 - 3x^5$$
$$f'(x) = 6x^5 - 15x^4$$

When the graph is decreasing $f'(x) < 0$. You can then factorise the function which will assist you when finding where this occurs.

$$f'(x) = (3x^4)(2x-5)$$

And so either $3x^4$ must be < 0 or $2x - 5$ is less than 0 but both cannot be < 0.

Since anything raised to the power of 4 is going to be a positive number we can focus on the second section. Solving this we get

$2x - 5$ becomes $x < \frac{5}{2}$ is where the line decreases

DIFF TRIG FUNCTIONS

$y = \sin x$ $\quad$ $y = \cos x$ $\quad$ $y = \tan x$ or $\sin x / \cos x$

$y' = \cos x$ $\quad$ $y' = -\sin x$ $\quad$ $y' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$ or $\frac{1}{\cos^2 x}$

example 1

$y = \sin(3x)$ can be written as $\sin(u)$ where $u = 3x$ and then the chain rule is used to differentiate

$y' = \cos(3x) \times 3$ eg $\sin'(3x) \times (3x')$

example 2

$y = \cos^2(x)$ or u^2 where $u = \cos x$

$y' = 2\cos x \times -\sin x$

DIFF EXPO

$y = e^x$ and $y' = e^x$ the only exponent that behaves this way

$y = 2^x$ is the same as $e^{\ln 2x}$. Using the chain rule e^u $u = \ln 2x$

$y' = e^{\ln 2x} \times \ln 2$ or $y = a^x$ then $y' = \ln a \times a^x$ $a > 0$

$= 2^x \times \ln 2$

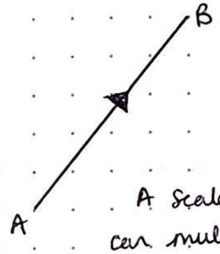
DIFF LOG

$y = \ln x$ $\quad$ $y = \log_a x$

$y' = \frac{1}{x}$ $\quad$ $y' = \frac{1}{\ln a x}$

VECTORS

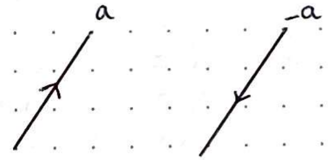
Vector has both a magnitude and a direction eg 70mph due North



This is written as $\overrightarrow{AB}$ where A is the tail and B is the head. The magnitude is written as $|\overrightarrow{AB}|$ or $|a|$

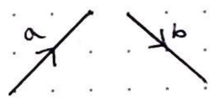
A Scalar is a singular number eg temperature, speed. You can multiply a vector by a scalar number and this will increase the magnitude but the direction will remain the same.

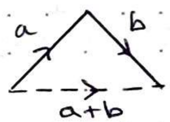
However multiplying it by a negative number will change the direction to the opposite



the magnitude is the same the direction is different

When we add two vectors together we do it by joining the head and tail together.





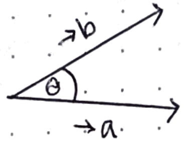
In the above example a is written as $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ which is 2 across 2 up

You can tell if vectors are parallel because they will have factors

$\vec{v}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{w}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\frac{4}{2} = 2$ and $\frac{3}{1} = 3$ as $2 \neq 3$ these are not parallel

DOT PRODUCT

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$



example 1. $\vec{a}\cdot\vec{b}$ when $\vec{a}\begin{pmatrix}2\\1\end{pmatrix}$ $\vec{b}\begin{pmatrix}3\\5\end{pmatrix}$

$2\times3+1\times5=11$

MATRICES

matrix $\begin{pmatrix}2&3&5&6\\-1&0&\boxed{-2}&1\\1&4&-1&1\end{pmatrix}$ The entry (2,3) is -2. Aka 2 down, 3 across the opposite to a standard grid reference

$\boxed{\text{MULTIPLYING}}$ - if you multiply the whole matrix by 3 then you multiply each number by 3.

$\boxed{\text{TRANSPOSE}}$ - To transpose the above grid you convert the columns into rows

$$\begin{pmatrix}2&-1&1\\3&0&4\\5&-2&-1\\6&1&1\end{pmatrix}$$

$\boxed{\text{IDENTITY MATRIX}}$ is an $x\times x$ grid with 1 dividing it

$id_2\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $id_3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $id_4\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$\boxed{\text{ADDING}}$ the matrices together means you must add the equivalent numbers THE DIMENSIONS MUST BE THE SAME

example 1 $A=\begin{pmatrix}1&0&3\\-1&4&0.5\end{pmatrix}$ $B=\begin{pmatrix}-1&1&2\\3&4&5\end{pmatrix}$ $A+B=\begin{pmatrix}0&1&5\\2&8&5.5\end{pmatrix}$

MULTIPLYING MATRICES TOGETHER

$A=\begin{pmatrix}\boxed{1}&\boxed{2}&\boxed{3}\\4&5&6\end{pmatrix}$ $B=\begin{pmatrix}\boxed{-2}&1&-7&1\\\boxed{1}&0&0.5&8\\\boxed{4}&0&-2&9\end{pmatrix}$ can only be multiplied if columns in A is equal to the rows in B. A×B = okay but B×A = not okay.

$A\times B=\begin{pmatrix}\boxed{12}&\boxed{1}&\boxed{2}&44\\21&4&37.5&78\end{pmatrix}$

$\boxed{\text{Row1, Col1}\quad(1\times-2)+(2\times1)+(3\times4)}$

$\boxed{\text{Row1, col2}\quad(1\times1)+(2\times0)+(3\times0)}$

$\boxed{\text{Row1, col 3}\quad(-1\times-7)+(2\times\tfrac{1}{2})+(3\times-2)}$

The inverse of a matrix is the matrix that when multiplied by another gives the identity matrix

The determinant is the product of the diagonals, if it does not equal 0 then it can be inverted.

$\begin{pmatrix}1&0\\2&1\end{pmatrix}$ $(2\times0)+(1\times1)$
$=0+1$
$=1$

To do a determinant on a 3×3 matrix you look at it like this

$\begin{pmatrix}1&2&3\\4&\boxed{5}&\boxed{6}\\-1&\boxed{-2}&\boxed{-3}\end{pmatrix}$ $\begin{pmatrix}1&2&3\\\boxed{4}&5&\boxed{6}\\\boxed{-1}&-2&\boxed{-3}\end{pmatrix}$ $\begin{pmatrix}1&2&3\\\boxed{4}&\boxed{5}&6\\\boxed{-1}&\boxed{-2}&-3\end{pmatrix}$

$+\boxed{1}\begin{pmatrix}5&6\\-2&-3\end{pmatrix}-\boxed{2}\begin{pmatrix}4&6\\-1&-3\end{pmatrix}+\boxed{3}\begin{pmatrix}4&5\\-1&-2\end{pmatrix}$ the signs alternate so col 4 would be a negative

delete row 1 column $\boxed{1}$ — delete row 1 column $\boxed{2}$ — delete row 1 column $\boxed{3}$

$=1\times((5\times-3)-(6\times-2))-2((4\times-3)-(6\times-1))+3((4\times-2)-(5\times-1))$
$=0$

DETERMINENTS ARE ONLY DONE ON SQUARE MATRICIES

In other words $\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}=a\begin{vmatrix}e&f\\h&i\end{vmatrix}-b\begin{vmatrix}d&f\\g&i\end{vmatrix}+c\begin{vmatrix}d&e\\g&h\end{vmatrix}$

Probability

$\boxed{\text{COMPLIMENTARY EVENTS}}$ - when events are complimentary it means if one event happens the other will not eg a coin toss, if it is a heads then it cannot be a tails
The sum of the probility for each outcome must be 1

When all events are likely probability = $\frac{\text{number of ways event can occur}}{\text{total number of possibilities}}$

eg probability of rolling 5 on a dice is $\frac{1}{6}$ the same as rolling a 1

$\boxed{\text{INDEPENDENT EVENTS}}$ not determined by the event before eg rolling a dence, you multiply the probability of each individual event eg rolling a dice and getting a 6 twice $1/6 \times 1/6 = 1/36$

$\boxed{\text{EXPERIMENTAL PROBABILITY}}$ is where you perform the event to establish the likelyhood of the outcome. The more times = more accurate.

INVERSE MATRIX

STEP 1 → Check the matrix is square
STEP 2 → Check determinent ≠ 0
STEP 3 → Compute co-factor matrix
STEP 4 → Transpose co-factor matrix

COFACTOR with the matrix below

$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ take the determinent of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and then put it into the new matrix $\begin{pmatrix} \square & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$