FCS Week 6 Lecture Note

Notebook:	Fundamentals of Computer Science		
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Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Basic Combinatorial Principles: Part 2	Course: BSc Computer Science Class: CM1025 Fundamentals of Computer Science[Lecture] Date: April 22, 2021

Essential Question:

What is counting?

Questions/Cues:

- What is permutation?
- What is combination?

Notes

Definition

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- There are $n!$ unique permutations for n distinct objects.

$$n! = n.(n-1)....3.2.1$$

- Arranging only r elements of the set; this is called r-permutation.
- r-permutation of set of n elements is denoted by $P(n, r)$

We wish to arrange 5 people to queue to borrow a book. How many unique ways are there to make up this queue?

- Let us call these 5 people a, b, c, d, e
- We have 5 choices for the first position
- We are left with 4 choices for the second
- We now have 3 choices for the third
- We only have 2 choices for the fourth
- For the fifth position, we only have 1 choice
- So we have 5x4x3x2x1=5! = 120 unique ways

5				
5	4			
5	4	3		
5	4	3	2	
5	4	3	2	1

A simple example

- Assume we have a set {cat, mouse, dog, rabbit}
- Wish to take a side-by-side photo of two animals
- How many unique ways of photo arrangement are there?
  - cat-mouse and mouse-cat
  - cat-dog and dog-cat
  - cat-rabbit and rabbit-cat
  - mouse-dog and dog-mouse
  - mouse-rabbit and rabbit-mouse
  - dog-rabbit and rabbit-dog

! 12 unique arrangements

A useful formula

Theorem: For two integers n, r, $0 \leq r \leq n$. There are

$$P(n,r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set of n distinct elements.

- Photo of our animals:
- {cat, mouse, dog, rabbit}
- $n = 4, r = 2$
- $P(4,2)=4(4-1)\ldots(4-2+1)=4.3=12$

Example 2

- From a group of 20 actors, how many ways are there to select three actors to play a thief, a police officer and a shop owner?
- Answer: there are 20 actors but only three roles

20	19	18
Police Officer	Thief	Shop Owner

- There are P(20,3) = $\frac{20!}{17!}$ = 20.19.18 = 6,840 unique ways

Example 3

- How many permutations of the letters {a, b, c, d, e, f, g} contain "bad"?

Answer:

- "bad" must occur without a gap
- there can be letters before it or after it
- reduce the set to {c, e, f, g, "bad"}
- how many permutations are there for this new set?
- 5! = 120

How many words on {a, b, c, d, e, f, g} are there containing "bad"?

- Reduce the set {c, e, f, g} then the word bad can fit in the gaps
  - Length 3:1
  - Length 4: $P(4,1) = \frac{4!}{3!} = 4$
    - there are 2 positions to put "bad" in: bad · · bad
    - 4.2 = 8 words of length 4 containing "bad"
  - Length 5: $P(4,2) = \frac{4!}{2!} = 4.3 = 12$
    - there are 3 positions to put "bad" in:

      bad · · · bad · · bad
    - 12.3 = 36 words of length 5 containing "bad"

How many words on {a, b, c, d, e, f, g} are there containing "bad"?

- Reduce the set {c, e, f, g} then the word bad can fit in the gaps
  - Length 6: $P(4\,,3) = \frac{4!}{1!} = 4.3.2 = 24$
    - there are 4 positions to put "bad" in:

      bad ✸✸✸ ✸ bad ✸✸ ✸✸ bad ✸ ✸✸✸ bad

    - 24.4 = 96 words of length 6 containing "bad"
  - Length 7: Example 3, 120
  - 1 + 8 + 36 + 96 + 120 = 261 words containing "bad"

Definition

- A combination of a set of distinct objects is an unordered arrangement of these objects
- There is only one combination of n elements
- When you shuffle, only the order changes and with combination, **the order does not matter!**

- r-combination of elements of a set is an **unordered** selection of r elements from this set

Another useful formula

Theorem: For two integers $n, r\,, 0 \leq r \leq n$. There are

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

r combinations of a set of n distinct elements.

- Photo of our animals: {cat, mouse, dog, rabbit}
- n=4, r=2
- $C\,(4, 2) = \frac{4!}{2!(4-2)!} = \frac{4.3.2.1}{2.2} = 6$

Tricks with factorial

- $\frac{7!}{3!} = \frac{7.6.5.4.3.2.1}{3.2.1} = \frac{7.6.5.4.\cancel{3.2.1}}{\cancel{3.2.1}} = 7.6.5.4$
- $\frac{n!}{r!} = \frac{n...2.1}{r...2.1} = \frac{n...(r+1)\cancel{r....1}}{\cancel{r...2.1}} = n\ ...\ (r+1)$
- $\frac{n!}{r!(n-r)!} = \frac{n...(n-r+1)\cancel{(n-r)....r...1}}{(r...1)[\cancel{(n-r)(n-r-1)..r..1}]} = \frac{n...(n-r+1)}{r...1}$

Pick the one that is bigger

Example 1

- How many hands of 7 cards can be dealt from a standard deck of 52 cards?
- First to establish: No order! So this is a combination
- n = 52 r = 7
- $C(52,7) = \frac{52!}{7!(45)!} = \frac{52.51.50.49.48.47.46}{7.6.5.4.3.2}$

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Example 1

- What about hands of 45 cards?
- $C(52,45) = \frac{52!}{45!(7)!} = \frac{52.51.50.49.48.47.46}{7.6.5.4.3.2}$

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

$$C(n,r) = C(n,n-r) = \frac{n!}{r!\,(n-r)!}$$

Example 2

- How many ways are there to choose 11 football players from 16 to form a team?
- First: does order matter? NO
- n=16, r=11
- $C(16{,}11) = \frac{16!}{11!(5)!} = \frac{16.15.14.13.12}{5.4.3.2} = 4368$
- Equivalent to choosing 5 football players from 16 to stay behind!

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Example 3

- How many binary strings of length 8 contain equal number of 0s and 1s?
- Does order matter? No
- There are four "1"s
- There are 8 positions
- $C(8{,}4) = \frac{8!}{4!(4)!} = \frac{8.7.6.5}{4.3.2.1} = 70$

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Example 4

- How many binary strings of length 8 contain at most three 1s?
- At most three means: 0, 1, 2, or 3
- There is zero 1s: 1
- There is one 1: $C(8{,}1)$ = 8
- There are two 1s: $C(8{,}2) = \frac{8!}{2!(6)!} = \frac{8.7}{2.1} = 28$
- There are three 1s: $C(8{,}3) = \frac{8!}{3!(5)!} = \frac{8.7.6}{3.2.1} = 56$
- At most three 1s =1 + 8 + 28 + 56 = 93

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Example 5

- How many binary strings of length 8 contain at least five 1s?
- At least 5 means: 5 or 6 or 7 or 8
- There are five 1s: $C(8,5) = \frac{8!}{5!(3)!} = \frac{8.7.6}{3.2.1} = 56$
- There are six 1s: $C(8,6) = \frac{8!}{6!(2)!} = \frac{8.7}{2.1} = 28$
- There are seven 1s: $C(8,7) = \frac{8!}{7!(1)!} = 8$
- There are eight 1s: 1
- At least five 1s = 56 + 28 + 8 + 1 = 93

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Summary

In this week, we learned about permutations and combinations.