Set unordered collection of unique objects

$V = \{1,3,5\} = \{5,1,3\} = \{2n-1 \mid n \in \mathbb{Z} \text{ and } 0 < n \leq 3\}$

$5 \in V$, $6 \notin V$

$\mathbb{N}$: set of natural numbers $1, 2, 3, \dots$

$\mathbb{Z}$: set of integers $\dots -2, -1, 0, 1, 2 \dots$

$\mathbb{Q}$: set of rational numbers

$\mathbb{R}$: set of real numbers

Cardinality: $|S|$ # of elements in a set

$|V| = 3$ $\quad \emptyset = \{\}$ $\quad |\emptyset| = 0$

Subset: $A \subseteq B$ all elements of A is in B

$\{3,5\} \subseteq \{1,3,5\}$

Powerset: $P(S)$ set containing all subsets of a given set

$P(\emptyset) = \{\emptyset\}$

$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$

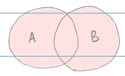
$S = \{1,2,3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

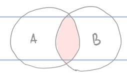
$$|P(S)| = 2^{|S|}$$

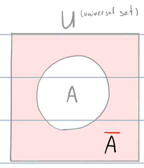
$|A| = n \rightarrow |P(P(A))| = 2^{2^n}$

$A \cup B$ union



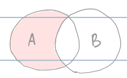
$A \cap B$ intersection



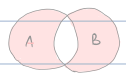


Complement

$A - B$ difference



$A \oplus B$ symmetric difference



A and B are disjointed if $A \cap B = \emptyset$

Partition: dividing a set into disjointed and comprehensive subsets

De Morgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Union	Name	Intersection
$A \cup B = B \cup A$	commutative	$A \cap B = B \cap A$
$(A \cup B) \cup C = A \cup (B \cup C)$	associative	$(A \cap B) \cap C = A \cap (B \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's Laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
$A \cup \emptyset = A$	identities	$A \cap \emptyset = \emptyset$
$A \cup U = U$		$A \cap U = A$
$A \cup \overline{A} = U$	complement	$A \cap \overline{A} = \emptyset$
$\overline{U} = \emptyset$		$\overline{\emptyset} = U$
$\overline{\overline{A}} = A$	double complement	
$A \cup (A \cap B) = A$	absorption	$A \cap (A \cup B) = A$
$A - B = A \cap \overline{B}$	set difference	

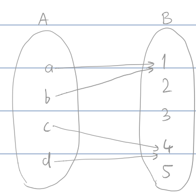
Function maps an element of an input set to an element of output set

Domain D_f $\qquad$ Co-domain co-D_f (set of all possible outputs) $\qquad$ Range R_f

$f: A \rightarrow B$

$x \in A \rightarrow f(x) = y \in B$

pre-image $\quad$ image



↳ $D_f = A = \{a, b, c, d\}$

Co-$D_f = B = \{1,2,3,4,5\}$

$R_f = \{1,4,5\}$

Pre-images$(4) = \{c, d\}$

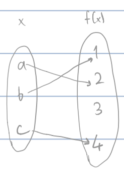
Image$(b) = 1$

requirements for function

↳ every input has to have an image (output)

↳ " " " " " one unique image

injective each x outputs a different y
"one-to-one" $\qquad$ f(x)



>Prove: option #1> $f(a) = f(b) \longrightarrow a = b$

option #2> $a \neq b \rightarrow f(a) \neq f(b)$

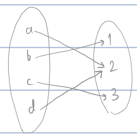
>Disprove: counter example of $f(a) = f(b)$ and $a \neq b$

surjective "onto"

each y in Range belongs to an x

i.e. range = codomain

$(R_f = co\text{-}D_f)$



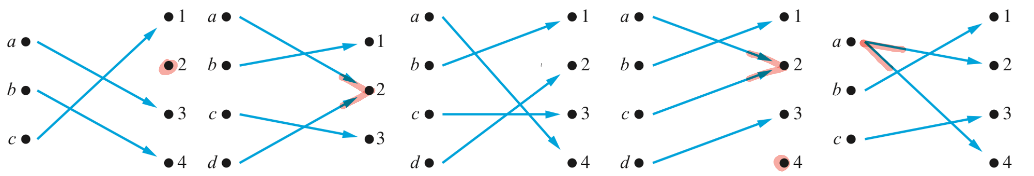
→ Prove: $f(x) = y \in \mathbb{R} \longrightarrow x = \cdots y \cdots \in \mathbb{R}$

bijective "invertible"

each y in co-Domain has exactly one pre-image x

both injective and surjective

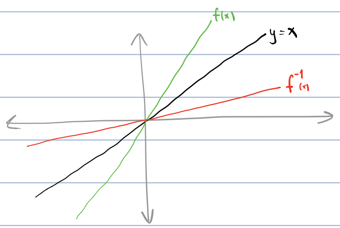
injective ✓ surjective ✗ | injective ✗ surjective ✓ | injective ✓ surjective ✓ | injective ✗ surjective ✗ | not a function!



$f \circ g = f(g(x))$ $g: A \to B$ $f: B \to C \Rightarrow f \circ g: A \to C$ not comm.: $f \circ g \neq g \circ f$

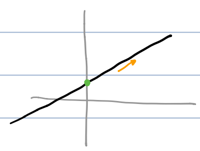
inverse function $f^{-1}: B \to A$ for a bijective $f: A \to B$

identity function: $f \circ f^{-1} = f^{-1} \circ f$

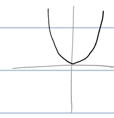


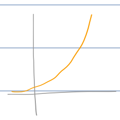
f and f^{-1} are symmetric over the straight line of $y = x$

Linear function: $f(x) = ax + b$



Quadratic function: $f(x) = ax^2 + bx + c$

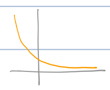




Exponential function: $f(x) = b^x$ (b: base function)

$b > 1$: exponential growth

$0 < b < 1$: exponential decay

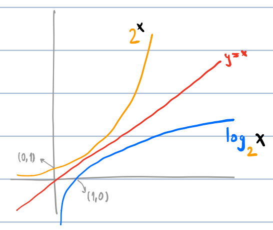


inverse: $\log_b x = y \longleftrightarrow x = b^y$

Logarithmic function: $f(x) = \log_b x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$



$\log(m \times n) = \log m + \log n$

$\log \frac{m}{n} = \log m - \log n$

$\log m^n = n \times \log m$

$\log 1 = 0$

$\log_b b = 1$

$\ln x = \log_e x$

$\log_{10} 100 = 2$

$\log_3 81 = 4$

$\log_3 \frac{1}{81} = -4$

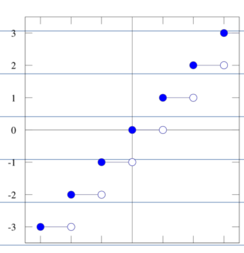
$\log_2 1 = 0$

Floor function: $n \leq x < n+1 \rightarrow \lfloor x \rfloor = n$

$\lfloor x \rfloor$

Ceiling function: $n < x \leq n+1 \rightarrow \lceil x \rceil = n+1$

$\lceil x \rceil$



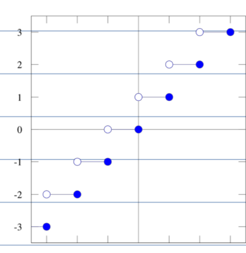
$\lfloor 10 \rfloor = 10$

$\lfloor 1.1 \rfloor = 1$

$\lfloor 1.99 \rfloor = 1$

$\lfloor -1.1 \rfloor = -2$

$\lfloor -1.99 \rfloor = -2$



$\lceil 10 \rceil = 10$

$\lceil 1.1 \rceil = 2$

$\lceil 1.99 \rceil = 2$

$\lceil -1.1 \rceil = -1$

$\lceil -1.99 \rceil = -1$

proposition must be true or false (ex: "x+2=3" is not a proposition!)

Truth Set set of elements in universe where proposition is true $S=\{1,2,3,\dots,10\}$, p: n is even, truth set: $P=\{2,4,6,8,10\}$

$\neg p$: NOT (negation)
$\wedge$: AND (conjunction)
$\vee$: OR (disjunction)

$\rightarrow$ IF THEN

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

(biconditional) $\leftrightarrow$ IF and only IF

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

(exclusive or) $\oplus$ XOR

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

operator precedence: $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$
First … Last

$$\boxed{p \rightarrow q \equiv \neg p \vee q \equiv \neg q \rightarrow \neg p}$$

equivalent
- $p \rightarrow q \wedge q \rightarrow p$
- same values in truth table

for $a \rightarrow b$
"if a, then b"
"a, only if b"

converse $b \rightarrow a$
contrapositive $\neg b \rightarrow \neg a \equiv a \rightarrow b$
inverse $\neg a \rightarrow \neg b$

	Disjunction	Conjunction
Idempotent laws	$p \vee p \equiv p$	$p \wedge p \equiv p$
Commutative laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	$p \vee \mathbf{F} \equiv p$	$p \wedge \mathbf{T} \equiv p$
Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$	$p \wedge \mathbf{F} \equiv \mathbf{F}$
De Morgan's laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation laws	$p \vee \neg p \equiv \mathbf{T}^1$	$p \wedge \neg p \equiv \mathbf{F}^2$
Double negation law	$\neg\neg p \equiv p$	

predicate generalized propositions — predicate: $P(x) \equiv x > 4$ — proposition: $P(3)$

Quantifiers

Universal: $\forall x$ some formula : **for all** x, the formula is true
Existential: $\exists x$ some formula : **for some** (at least one) x, the formula is true
Uniqueness: $\exists! x$ some formula : **for a unique value** (single) x, the formula is true

"All P's are Q's" translates into $\forall x(P(x) \rightarrow Q(x))$
"No P's are Q's" translates into $\forall x(P(x) \rightarrow \neg Q(x))$
"Some P's are Q's" translates into $\exists x(P(x) \wedge Q(x))$
"Some P's are not Q's" translates into $\exists x(P(x) \wedge \neg Q(x))$

Nested Quantifier	Meaning
$\forall x \forall y P(x,y)$	$P(x,y)$ is true for every pair (x,y)
$\exists x \exists y P(x,y)$	There is a pair (x,y) for which $P(x,y) is true
$\forall x \exists y P(x,y)$	For all x, there is a y for which P(x,y) is true
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for all y

De Morgan's:

$\neg(\forall x\, P(x)) = \exists x\, \neg P(x)$
$\neg(\exists x\, P(x)) = \forall x\, \neg P(x)$

$$\neg\forall x \exists y \forall z P(x,y,z) \equiv \exists x \neg\exists y \forall z P(x,y,z)$$
$$\equiv \exists x \forall y \neg\forall z P(x,y,z)$$
$$\equiv \exists x \forall y \exists z \neg P(x,y,z)$$

Argument: premise, premise, ⋮, ∴ conclusion — is **valid** if: when all premises are true, conclusion must also be true.

Rules of inference

Modus Ponens	Modus Tollens	Conjuction	Simplification	Addition	Hypotetical Syllogism	Disjunctive Syllogism	Resolution
$p \rightarrow q$	$\neg q$	p			$p \rightarrow q$	$p \vee q$	$p \vee q$
p	$p \rightarrow q$	q	$p \wedge q$	p	$q \rightarrow r$	$\neg p$	$\neg p \vee r$
$\therefore q$	$\therefore \neg p$	$\therefore p \wedge q$	$\therefore p$	$\therefore p \vee q$	$\therefore p \rightarrow r$	$\therefore q$	$\therefore q \vee r$

w/ Quantifiers

(UI) Universal Instantiation	(UG) Universal Generalisation	(EI) Existential Instantiation	(EG) Existential Generalisation	Universal Modus Ponens	Universal Modus Tollens
				$\forall x P(x) \rightarrow Q(x)$	$\forall x P(x) \rightarrow Q(x)$
$\forall x P(x)$	$P(c)$	$\exists x P(x)$	$P(c)$	$P(a)$	$\neg Q(a)$
$\therefore P(c)$	$\therefore \forall x P(x)$	$\therefore P(c)$	$\therefore \exists x P(x)$	$\therefore Q(a)$	$\therefore \neg P(a)$

Boolean Algebra

$x \cdot y$ AND
$x + y$ OR
$\bar{x}$ NOT

precedence: NOT > AND > OR

De Morgan's: $\overline{x \cdot y} = \bar{x} + \bar{y}$, $\overline{x + y} = \bar{x} \cdot \bar{y}$

Duality
- Changing every + to ·
- Changing every · to +
- Changing every 0 to 1
- Changing every 1 to 0

For example $A + B \cdot C \equiv A \cdot (B + C)$.

Logical Sum	Theorem	Logical Product
$x + x = x$	**Idempotent Laws**	$x \cdot x = x$
$x + 1 = 1$	**Tautology and Contradiction**	$x \cdot 0 = 0$
$\bar{\bar{x}} = x$	**Involution**	
$(x+y)+z = x+(y+z)$	**Associative Laws**	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x + (x \cdot y) = x$	**Absorption Laws**	$x \cdot (x + y) = z$
if $y + x = 1$ and $y \cdot x = 0$, then $x = \bar{y}$	**Uniqueness of Complement**	
$\bar{0} = 1$ and $\bar{1} = 0$	**Inversion Law**	

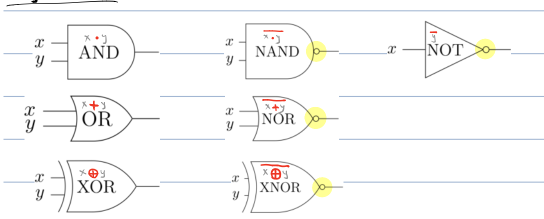
Function forms
- product-of-sums: $(x+y)\cdot(x+z)\cdot(y+z)$
- sum-of-products: $xy + xz + yz$ ← preferred!

x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	1

$\Rightarrow f(x,y) = \bar{x}y + x\bar{y} + xy$

- **Identity**
  The logical sum identity $x + 0 = x$ and the logical product identity $x \cdot 1 = x$.
- **Commutativity**
  Both logical sum and logical product are commutative, therefore $x + y = y + x$ and $x \cdot y = y \cdot x$.
- **Distributivity**
  Both logical sum and logical product and distributive between each other, therefore $x \cdot (y + z) = x \cdot y + x \cdot z$ and $x + (y \cdot z) = (x + y) \cdot (x + z)$.
- **Complement**
  Complements exist for all elements: $x + \bar{x} = 1$ and $x \cdot \bar{x} = 0$.

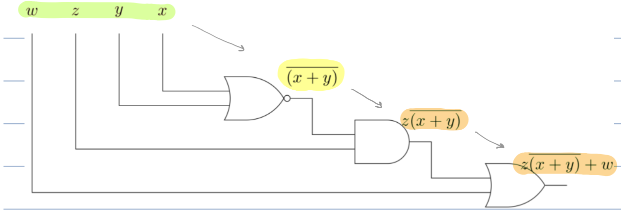
Logic Gates



"exclusive or" $x \oplus y$: $f(x,y) = \bar{x}y + x\bar{y}$
"implies" $x \rightarrow y$: $f(x,y) = \bar{x} + y$

Circuit → Boolean

1. **Label** all gate outputs that are a function of input variables
2. **Express** the Boolean functions for each gate in the first level
3. **Repeat** until all outputs of the circuit are written as Boolean expressions



Karnaugh maps

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

	$\bar{y}\bar{z}$	$\bar{y}z$	yz	$y\bar{z}$
$\bar{x}$	0	0	1	0
x	1	1	1	1

$\rightarrow x + yz$

Proofs

Theorem A formal statement that can be shown to be true

Axiom A statement assumed to be true to serve as a premise to further arguments

Lemma A proven statement used as a step to a larger result rather than as a statement of interest by itself

Corollary A theorem that can be established by a short proof from a theorem

Proof techniques

↳ **Direct Proof** Prove A→B: Assume A, prove B

↳ **Contrapositive** Prove A→B: (if A, then B) Prove ¬B→¬A (if not B, then not A)

↳ **Contradiction** Prove A: Assume A is false & arrive at a statement that contradicts this

↳ Math. **Induction**
1) Base case: Prove that P(1) is true
2) Induction hypothesis: Assume that P(k) is true
3) Inductive step: Prove that P(k+1) is true

Conclusion ∵ ∀n ∈ ℕ P(n) is true □

Theorem 10. $n < 3^n$, *for all* $n \in \mathbb{N}$

Proof. Let $P(n) = n < 3^n$, prove by induction that $P(n)$ is true for all n.

Basis: Prove $P(0)$ is true.

$$\begin{aligned} P(0) &= 0 < 3^0 \\ &= 0 < 1 \end{aligned}$$

Inductive Step: Prove $P(k) \to P(k+1)$.

Assuming $P(k) = k < 3^k$ to be true, prove that $P(k+1) = (k+1) < 3^{k+1}$ is true.

$$\begin{aligned} k &< 3^k \\ k+1 &< 3^k+1 \\ k+1 &< 3^k+1 < 3^k+3^k+3^k \\ k+1 &< 3^k+1 < 3\cdot 3^k \\ k+1 &< 3^k+1 < 3^{k+1} \\ k+1 &< 3^{k+1} \end{aligned}$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} n < 3^n$ is true. □

↳ **Strong Induction**
1) Base case: Prove that P(1) is true
2) Induction hypothesis: Assume that P(1), P(2), ..., P(k) are true
3) Inductive step: Prove that P(k+1) is true

Conclusion ∵ ∀n ∈ ℕ P(n) is true □

EXAMPLE

[Fundamental Theorem of Arithmetic]

Every integer $n \geq 2$ can be written uniquely as the product of prime numbers.

Base case: This is clearly true for $n = 2$.

Inductive step: Suppose the statement is true for $n = 2, 3, 4, \ldots, k$.
If $(k+1)$ is prime, then we are done. Otherwise, $(k+1)$ has a smallest prime factor, which we denote by p. Let $k+1 = p \times N$. Since $N < k$, by our inductive hypothesis, N can be written uniquely as the product of prime numbers. That means $k+1 = p \times N$ can also be written uniquely as a product of primes. We're done! □

Well-ordering property

It is an axiom about ℕ that we assume to be true. The axioms about ℕ are as follows:

1. The number 1 is a positive integer
2. If $n \in \mathbb{N}$, then $n+1$ is also a positive integer
3. Every positive integer other than 1, is the successor of a positive integer
4. The Well-ordering property: every non-empty subset of the set of positive integers has a least element

Math. induction ≡ Strong induction ≡ Well-ordering property

Recursive

	function	set
Basis step	initial value	initial elements
Recursive step	find further values from previous values	generate further elements from previous elements

Algorithm 1 Computing $n!$

```
1: function FACTORIAL(n)
2:     if n = 0 then
3:         return 1
4:     return n× FACTORIAL(n − 1)
```

Recurrence relations sequence generating next term as a function of previous term

Infinite sequence function from positive integers to real numbers

Linear recurrence relation where each term of the sequence is a linear function of earlier terms

↳ Linear Homogenous Recurrences $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$. $c_1, \ldots, c_k \in \mathbb{R}$. k is the degree of the relation.

↳ Linear Non-homogenous Recurrences $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$. $c_1, \ldots, c_k \in \mathbb{R}$. k is the degree of the relation.

Arithmetic sequence difference between consecutive terms is constant c — Ex: $a_{n+1} = a_n + c$

Geometric sequence ratio " " " " " r — $a_{n+1} = r \cdot a_n$

Divide & Conquer Algorithm

divide problem into subproblems

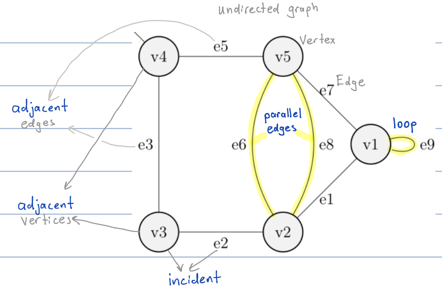
solve subproblems recursively

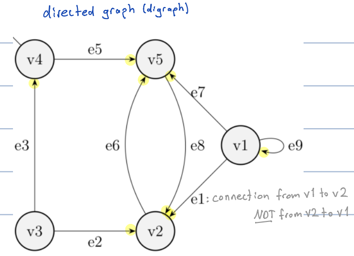
combine solutions

Graph

$G = \{V, E\}$

Vertices Edges





	Vertices	Edges
	repeatable?	
Walk	✓	✓
Trail	✓	✗
Circuit "closed trail" $v_{start} = v_{end}$	✓	✗
Path	✗	✗
Cycle "closed path" $v_{start} = v_{end}$	✗	✗

Connectivity

undirected Graph → connected

directed Graph → strongly connected

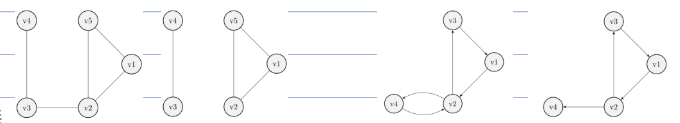


Figure 23: A Connected Graph Figure 24: Not A Connected Graph Figure 25: A Strongly Connected Graph Figure 26: Not A Strongly Connected Graph

Euler Path visits each **edge** precisely once → traversable

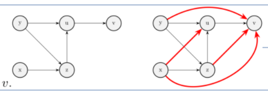
Hammiltonian Path visits each **vertex** precisely once → traceable

Hammiltonian Cycle visits each **vertex** precisely once & returns to the start $v_{start} = v_{end}$ → Hamm. Graph
↳ visited twice

Transitive Closure

Given a digraph G, the Transitive Closure of G is the digraph G^* such that:

- G^* has the same vertices as G
- If G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v.

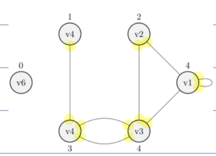


Degree

Undirected Graphs

Degree ($deg(v)$) the number of edges **incident** on v

- A loop contributes **twice** to the degree
- An isolated vertex has degree 0

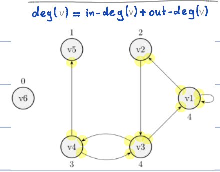


Directed Graphs

In-degree: #edges going in

Out-degree: #edges going out +

deg(v) = in-deg(v) + out-deg(v)

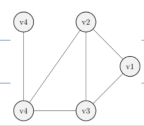


Degree sequence sequence of a graph's degrees from max to min Ex: 4, 4, 3, 2, 1, 0

↳ sum of sequence is always → even!
→ = 2 × number of edges

Simple Graph

- no loops
- no parallel edges



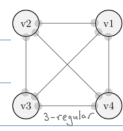
degree of any vertex ≤ #vertices − 1

Q: is ●, ●, ●, ... (degree seq) simple graph?

A: - check: is sum of sequence even?
- check: is max degree < #vertices?

Regular Graph

- all degrees are same = r
- called r-regular graph

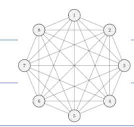


Given an r-regular graph G with n vertices, the following is true:

- Degree sequence of G is $r, r, r, \ldots$ (repeated n times)
- Sum of degree sequence is $r \cdot n$
- Number of edges in G is $\frac{r \cdot n}{2}$

Complete Graph

- simple graph where all vertices are adjacent
- n vertices → K_n

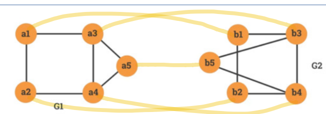


A complete graph with n vertices, K_n has the following properties:

- Every vertex has degree $n - 1$
- Sum of degree sequences is $n(n-1)$
- Number of edges is $\frac{n(n-1)}{2}$

Isomorphism

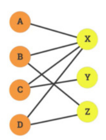
Graphs G_1 and G_2 are isomorphic if there is a bijection $f: G_1 \to G_2$
where if u & v are adjacent in G_1
f(u) & f(v) are also adjacent in G_2



$f: G_1 \to G_2$

G_1	a_1	a_2	a_3	a_4	a_5
$f(G_1)=G_2$	b_1	b_2	b_3	b_4	b_5

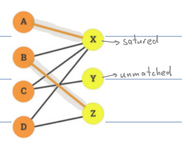
Bipartite graph vertices divided into 2 groups
where all edges are between groups



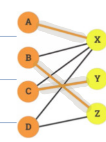
↳ 2-colorable
↳ no odd-length cycles

$V_1 = \{A,B,C,D\}$ $V_2 = \{X,Y,Z\}$

Matching edges without shared endpoints



Maximum Matching no further edge can be added



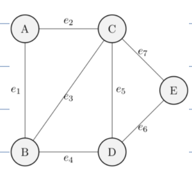
⇒ to find: Hopcroft-Karp Algorithm

1. $M \leftarrow \emptyset$
2. While there is an augmenting path P
   a) Use BFS to build layers that terminate at free vertices
   b) Start at free vertices in C, use DFS
3. Return M

Adjacency List

Vertex	Adjacent Vertices
A	B, C
B	A, C, D
C	A, B, D, E
D	B, C, E
E	C, D

⇒



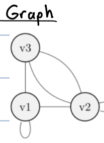
Adjacency Matrix

$$M(G) = \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix}$$ → loops

↳ **undirected Graph**

$$M(G) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

symmetric



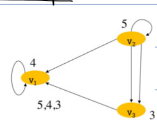
all edges (except loops) represented twice

sum of matrix elements = 2× #edges
= sum of degree sequence

↳ **directed Graph**

$$M(G) = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 1 & 0 & 0 \\ v_2 & 1 & 1 & 2 \\ v_3 & 1 & 0 & 0 \end{array}$$

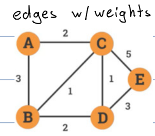
not symmetric ($v_1v_2 \neq v_2v_1$)



sum of matrix elements = #edges

Weighted Graph

edges w/ weights



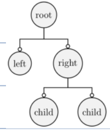
find shortest path between nodes
Dijkstra's Algorithm

Tree
connected, acyclic, undirected graph

no loops & no parallel edges

↳ has unique simple path between two vertices

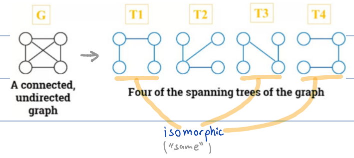
↳ has n vertices → n-1 edges



Forest:
disconnected, acyclic, undirected graph

(contains multiple trees)

Spanning tree
connected subgraph that contains all vertices of a graph without cycles



To get a spanning tree of graph G

1. Keep all vertices of G
2. Break all the cycles but keep the tree connected

Minimum-cost spanning tree
spanning tree w/ lowest cost

of a weighted graph

↳ greedy algorithms to find MST:

Kruskal's Algo
- **Start** w/ cheapest edge in spanning tree
- **Add** cheapest edge that doesn't create cycle & **Repeat**

Prim's Algo
- **Start** w/ any one node
- **Add** cheapest edge and the node it leads to & **Repeat**

Rooted tree:
every edge is directed away from the root vertex

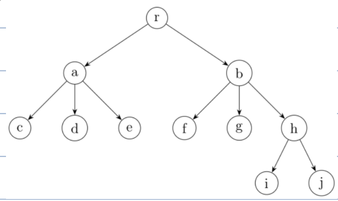
root only!

in-degree = 0

in-degree = 1

in-degree = 2

in-degree = 3



- r is the root of the tree
- a is the parent of vertices c, d, e
- f, g, h are the children of b
- r and b are ancestors of i and j
- i and j are sibblings
- h and b are internal nodes
- c, d, e, f, g, i, j are external nodes

- depth of r is 0
- depth of a and b is 1
- depth of c, d, e, f, g, h is 2
- depth of i and j is 3
- height of r is 3
- height of b is 2
- height of a and h is 1
- height of c, d, e, f, g, i, j is 0

Binary Tree A tree in which every vertex has 2 or fewer children

Ternary Tree A tree in which every vertex has 3 or fewer children

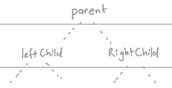
m-ary Tree A tree in which every vertex has m or fewer children

m-ary tree: - is regular if every internal node has exactly m children

- has at most m^h vertices at level h

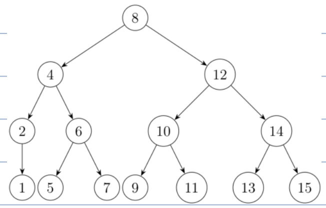
isomorphic as rooted trees if bijection maps root-to-root

Binary Search Tree
vertices w/ labels



parent's value ≥ all keys in left subtree

parent's value ≤ all keys in right subtree



for n elements in BST: $\underbrace{\text{height}}_{\text{of tree}} = \lceil \log_2(n+1) \rceil$ → maximum numbers of comparisons in search

- Start at root
- Proceed to appropriate subtree & Repeat until found

Relation R
$\underset{x \in A \;\; y \in B}{A \text{ to } B} \to xRy$

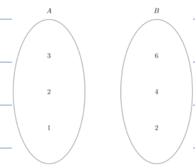
Cartesian Product
$A \times B = \{(a, b) \mid a \in A \land b \in B\}$

↳ R: A to B → $R \subseteq A \times B$

↳ R on A → $R \subseteq A \times A$ → xRy only if $x < y$

Ex: $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$



$A \times B = \{(1,2), (1,4), (1,6), (2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$

Matrix representation

A \ B	CS100	CS101	CS102	CS103
Sofia	X	X		
Samir		X	X	
Sarah	X		X	

$$\longrightarrow A_{m,n} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\!\!\begin{array}{c} \begin{matrix} B_1 & B_2 & B_3 & B_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Union (or)

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

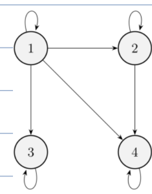
Intersection (and)

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

DiGraph representation

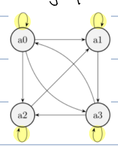
$A = \{1, 2, 3, 4\}$

$R = \{(x, y) \mid \underbrace{x|y}_{\text{"x divides y"}}\}$ →



reflexiv
xRx for all x

in graph



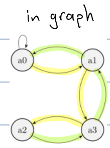
in matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

symmetric
(all edges have parallels)

xRy → yRx

in graph



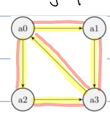
in matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

anti-symmetric
(no parallel edges) loops allowed

$\underset{(x \neq y)}{xRy} \to \cancel{yRx}$

in graph



in matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

transitiv
aRb and bRc → aRc

a → b → c ⇒ a → b → c

transitiv closure:

after adding necessary edges to make a relation transitiv

Equivalent
reflexive, symmetric and transitive

Partial Order
reflexive, anti-symmetric and transitive

Equivalent Class
subset where all elements are related to element **a** through an equivalent relation

Example

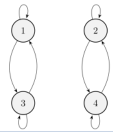
Let $S = \{1, 2, 3, 4\}$ and R be a relation on elements in S:

$R = \{(a, b) \in S^2 \mid a \mod 2 = b \mod 2\}$

R is an equivalence relation with 2 equivalence classes:

- $[1] = [3] = \{1, 3\}$
- $[2] = [4] = \{2, 4\}$

Graphically:



Total Order
is partial order

+

$\forall a, b \in \mathbb{Z}(a \leq b \vee b \leq a)$

Counting

↳ Product rule: (combination)

m ways for task 1
n " " " 2
⋮
→ job can be done in $m \times n \times \ldots$ ways

↳ Sum rule: (addition)

job can be done in m or n or ... ways → it can be done in $m+n+\ldots$ ways
(independent tasks)

Let's assume that a label in a programming language can be either a single letter or a letter followed by 2 digits. How many possible labels are there?

$$\begin{aligned} single\_letter + letter\_2\_digits &= 26 + (letters \cdot digits^2) \\ &= 26 + (26 \cdot 10^2) \\ &= 26 + (26 \cdot 100) \\ &= 26 + 2600 \\ &= 2626 \end{aligned}$$

↳ Subtraction rule: (Inclusion-Exclusion principle)

if m and n have items in common → $m + n -$ common items

↳ Division rule:

task can be done in n ways. For every way, d of n ways correspond → $\frac{n}{d}$ ways

In how many ways can we **seat 4 people** around a table, where two seating arrangements are considered the same when each person has the same left and right neighbour?

Solution:

Let's first number the seats around the table from 1 to 4 proceeding clockwise:

- There are **four ways** to select the person for seat 1, three for seat 2, two for seat 3, and one for seat 4
- Thus there are 4.3.2.1 = 24 ways to order the four people
- Since two seating arrangements are the same when each person has the same left and right neighbour, for every choice for seat 1, we get the same seating
- Therefore, **by the division rule**, there are 24/4 = 6 different seating arrangements.

Pigeonhole priciple

"if you have 4 pigeons and 3 holes, you'll have at least one hole with min. 2 pigeons"

ceiling function

$$\left\lceil \frac{\#\text{ items}}{\#\text{ boxes}} \right\rceil = X$$

at least one box with at least X items in it

A bag contains 6 blue balls, 12 red balls and 10 green balls. How many balls (minimum) must be selected to guarantee that at least 5 balls are the same colour?

↳ #boxes = 3 (r, g, b)
↳ one box with at least = 5

$$\Rightarrow \left\lceil \frac{\bigcirc}{3} \right\rceil = 5, \quad \bigcirc = 13$$

Select 5 integers from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$; show that at least two integers add up to 9.

1+8 = 9
2+7 = 9
⋮
a, b, c, d, d, c, b, a

↳ #boxes = 4 (a, b, c, d)
↳ #items = 5

$$\Rightarrow \left\lceil \frac{5}{4} \right\rceil = \bigcirc, \quad \bigcirc = 2$$

Permutation (unique, w/order)

→ of n distinct items $= n!$

→ of r elements from a set of size n: $P(n,r) = \frac{n!}{(n-r)!}$ "r-Permutation"

r-Combination (w/o order)

$$C(n,r) = \frac{n!}{r! \times (n-r)!}$$

	Ordered (permutations)	Unordered (combinations)
Repetition is not permitted	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
Repetition is permitted	n^k	$\frac{(n+k-1)!}{k!(n-1)!}$