Model ## (Beta(α,β)) X has a Beta(α,β) distribution with parameters $\alpha,\beta>0$ if its PDF is

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0\leq x\leq 1 \tag{6.##}$$

Exercise ## (A challenging exercise) Let (X_1, X_2) be a continuous random vector with joint PDF $f_X(x_1,x_2)$ and for which there exists an open subset $U \subseteq \mathbb{R}^2$ such that $P((X_1,X_2)\in U)=1$. Let $(Y_1,Y_2) = g(X_1,X_2) = (g_1(X_1,X_2), g_2(X_1,X_2))$ be such that $g=(g_1,g_2)$ is a one-to-one function on U and

$$\det\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} \neq 0,$$

where $\det A$ denotes the determinant of matrix A. Let $h=(h_1,h_2)$ be the pointwise inverse of g so that $(X_1,X_2) = h(Y_1,Y_2) = (h_1(Y_1,Y_2), h_2(Y_1,Y_2))$. Then (Y_1,Y_2) is a continuous random vector and its joint PDF is given by

$$f_Y(y_1,y_2) = f_X(h(y_1,y_2))\left|\det\begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix}\right|, \quad (y_1,y_2)\in g(U).$$

Now if X_1 and X_2 are independent random variables such that $X_1 \sim$ Gamma$(a_1,1)$ and $X_2\sim$ Gamma$(a_2,1)$, use the above result, known as the transformation theorem, to show that $Y_1 = X_1/(X_1+X_2)$ has a Beta(a_1,a_2) distribution. Hint: Consider the transformation, $(Y_1,Y_2) = g(X_1,X_2) = (X_1/(X_1+X_2), X_2)$.

Labwork ## (Sampling from the Beta distribution)

1. Use the result in the previous Exercise to generate 1000 samples from the Beta (2, 4) distribution by using the MATLAB gamrnd function for generating samples from the Gamma distribution.

2. Obtain the histogram for the generated Beta samples and superimpose on it the curve for the Beta(2,4) PDF (which can be evaluated using the MATLAB betapdf function).