Department of Mathematics and Statistics

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

CSE Exercises – Week 11

1. Let X be an exponential random variable with mean 2. Use Monte Carlo integration to estimate $E(\log X)$, with $n = 10^2$, 10^4 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

2. Let X be a Laplacian(0,1) random variable. Use Monte Carlo integration to estimate $P(X > 2)$, with $n = 10^3$, 10^4, 10^5 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

3. Consider the integral,

$$\int_0^1 [\cos(50x) + \sin(20x)]^2 \, dx .$$

(a) Find the exact answer analytically.

(b) Estimate the integral using Monte Carlo integration with $n = 10^2$, 10^4 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

4. Estimate the following integrals using Monte Carlo integration with $n = 10^2$, 10^4 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

(a) $\displaystyle\int_0^1 \frac{\ln x}{1-x} \, dx .$

(b) $\displaystyle\int_0^\infty \frac{1}{x^2 + 25} \, dx .$

(c) $\displaystyle\int_{-\infty}^\infty \frac{\cos 2x}{\cosh 3x} \, dx .$

5. Let X be a Cauchy random variable whose density is

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

We wish to estimate

$$P(X > 2) = \int_2^{\infty} f(x)dx, \tag{1}$$

which can be shown to be equivalent to

$$P(X > 2) = \int_0^{1/2} \frac{f(1/y)}{y^2} dy. \tag{2}$$

Compare the estimates of $P(X > 2)$ using

(a) direct Monte Carlo integration with equation (1),

(b) Monte Carlo integration with equation (2);

for = 10^2, 10^4 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

6. Let X be an exponential random variable with mean 1, i.e. the density of x is

$$f(x) = e^{-x},$$

for $x \geq 0$. We wish to estimate

$$P(X > 4) = \int_4^{\infty} f(x)dx. \tag{3}$$

(a) Show that

$$P(X > 4) = \int_0^{1/4} \frac{f(1/y)}{y^2} dy. \tag{4}$$

Hint: Use the substitution, $y = 1/x$.

(b) Compare the estimates of $P(X > 4)$ using

(i) direct Monte Carlo integration with equation (3),

(ii) Monte Carlo integration with equation (4),

(iii) importance sampling with a Cauchy distribution (defined in exercise 5) that is left-truncated at 4 as the importance distribution;

for = 10^2, 10^4 and 10^6. For each n, compute the estimate, the 0.95 confidence bound and approximate 0.95 confidence interval.

Solutions

1.

n	10^2	10^4	10^6
$E(\log X)$	0.0767	0.1148	0.1172
95% CB	0.2603	0.0248	0.0025
95% CI	(-0.1836, 0.3370)	(0.0900, 0.1397)	(0.1147, 0.1197)

2.

n	10^3	10^4	10^5	10^6
$P(X>2)$	0.0640	0.0665	0.0665	0.0679
95% CB	0.0152	0.0049	0.0015	0.0005
95% CI	(0.0488, 0.0792)	(0.0616, 0.0714)	(0.0650, 0.0681)	(0.0674, 0.0684)

3.

(a) $\int_0^1 [\cos 50x + \sin 20x]^2 \, dx$

$$= \int_0^1 \cos^2 50x + \sin^2 20x + 2 \sin 20x \cos 50x \, dx$$

$$= \left[\frac{x}{2} + \frac{\sin 100x}{200} + \frac{x}{2} - \frac{\sin 40x}{80} + \frac{\cos(-30x)}{30} - \frac{\cos 70x}{70} \right]_0^1$$

$$= \left[1 + \frac{\sin 100}{200} - \frac{\sin 40}{80} + \frac{\cos(-30)}{30} - \frac{\cos 70}{70} \right] - \left[\frac{1}{30} - \frac{1}{70} \right]$$

$$= 0.9652$$

(b)

n	10^2	10^4	10^6
Integral	1.0128	0.9554	0.9664
95% CB	0.1873	0.0203	0.0021
95% CI	(0.8255, 1.2001)	(0.9351, 0.9757)	(0.9643, 0.9685)

4.

(a)

n	10^2	10^4	10^6
Integral	-1.6916	-1.6420	-1.6455
95% CB	0.1732	0.0148	0.0015
95% CI	(-1.8648, -1.5185)	(-1.6569, -1.6272)	(-1.6470, -1.6440)

(b)

n	10^2	10^4	10^6
Integral	0.3256	0.3131	0.3147
95% CB	0.0661	0.0063	0.0006
95% CI	(0.2595, 0.3917)	(0.3067, 0.3194)	(0.3140, 0.3153)

(c)

n	10^2	10^4	10^6
Integral	0.7152	0.6613	0.6539
95% CB	0.2037	0.0194	0.0019
95% CI	(0.5115, 0.9189)	(0.6419, 0.6807)	(0.6520, 0.6559)

5.

(a)

n	10^2	10^4	10^6
Integral	0.1800	0.1478	0.1472
95% CB	0.0753	0.0070	0.0007
95% CI	(0.1047, 0.2553)	(0.1408, 0.1548)	(0.1465, 0.1479)

(b)

n	10^2	10^4	10^6
Integral	0.1477	0.1476	0.14759
95% CB	0.0018	0.0002	0.00002
95% CI	(0.1459, 0.1496)	(0.1474, 0.1478)	(0.14757, 0.14761)

6.

(a) $P(x > 4) = \int_4^\infty f(x)\, dx$

$$y = \frac{1}{x} \iff x = \frac{1}{y}$$

$$dx = -\frac{1}{y^2}\, dy$$

$$\therefore \quad P(x > 4) = -\int_{1/4}^{0} \frac{f(1/y)}{y^2}\, dy$$

$$= \int_0^{1/4} \frac{f(1/y)}{y^2}\, dy$$

(b) – (i)

n	10^2	10^4	10^6
Integral	0.0300	0.0176	0.0182
95% CB	0.0334	0.0026	0.0003
95% CI	(-0.0034, 0.0634)	(0.0150, 0.0202)	(0.0179, 0.0185)

(b) – (ii)

n	10^2	10^4	10^6
Integral	0.0177	0.0183	0.0183
95% CB	0.0214	0.0023	0.0002
95% CI	(-0.0037, 0.0391)	(0.0160, 0.0206)	(0.0181, 0.0185)

(b) – (iii) Importance density $\phi(x)$ is Cauchy $(0,1)$ left truncated at 4, so

$$\phi(x) = \frac{2}{\pi[1 + (x-4)^2]}, \quad x > 4$$

$$\therefore \quad \frac{f(x)}{\phi(x)} = \frac{1}{2} \pi e^{-x} [1 + (x-4)^2] .$$

n	10^2	10^4	10^6
Integral	0.0183	0.0182	0.01831
95% CB	0.0016	0.0002	0.00002
95% CI	(0.0167, 0.0199)	(0.0180, 0.0184)	(0.01829, 0.01833)