Department of Mathematics and Statistics

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Te Whare Wānanga o Waitaha
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CSE Exercises – Week 3

① (Page 142 Exercise 104)

If $U \sim$ Uniform$(0,1)$, show that the distribution of $1-U$ is also Uniform$(0,1)$.

② (Updated version of Exercise 106 on page 142)

The Laplacian distribution is also called the double exponential distribution because it can be regarded as the extension of the exponential distribution for both positive and negative values. An easy way to generate a Laplacian(λ) random variable is to generate an exponential(λ) random variable and then change its sign to negative with probability 0.5.

(a) Write a Matlab function to generate n Laplacian(λ) random variables. Your function should call the ExpInvSam function that has already been created.

(b) Use your Matlab function to generate 10,000 Laplacian(1) random values.

(c) Plot a density histogram for the generated values from part (b). On the same plot, superimpose the Laplacian(1) PDF.

(3) Let X be a continuous random variable whose PDF is given by

$$f(x) = \begin{cases} c x^{c-1} & , \quad 0 \le x \le 1, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

where $c > 0$ is a parameter.

(a) Find the CDF and inverse CDF.

(b) Write a Matlab fuction to implement an inversion sampler for generating from this distribution for any value of $c > 0$. Use your function to generate 10,000 sample values from the distribution with $c = \frac{1}{2}$.

(c) Plot the density histogram for the values generated in part (b). On the same plot, superimpose the curve for the PDF.

(4) Revisiting week 1 exercise 8 and week 2 exercise 5,

(a) Implement an inversion sampler for generating from the triangle density defined in week 1 exercise 8. Use your sampler to generate 10,000 sample values.

(b) Plot a density histogram for the generated values from part (a). On the same plot, superimpose the curve for the PDF.

(5) The PDF and CDF of the Gumbel distribution are given by

$$f(x) = \frac{1}{b} \exp\left[-\frac{(x-a)}{b}\right] \exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\},$$

$$F(x) = 1 - \exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\},$$

for $x \in \mathbb{R}$, and where $a \in \mathbb{R}$ and $b > 0$ are parameters.

(a) Find the inverse CDF.

(b) Implement an inversion sampler for generating from this distribution for any values of a and $b>0$. Use your sampler to generate 10,000 sample values for the distribution with $a = 0$ and $b = 1$.

(c) Plot a density histogram for the generated values from part (b). On the same plot, superimpose the curve for the PDF.

(6) The logistic PDF is given by

$$f(x) = \frac{1}{4b} \operatorname{sech}^2\left(\frac{x-a}{2b}\right),$$

for $x \in \mathbb{R}$, and where $a \in \mathbb{R}$ and $b > 0$ are parameters.

(a) Find the CDF and inverse CDF.

(b) Implement an inversion sampler for generating from this distribution for any values of a and b. Use your sampler to generate 10,000 sample values from the distribution with $a = 0$ and $b = 1$.

(c) Plot a density histogram for the generated values from part (b). On the same plot, superimpose the curve for the PDF.

7. The CDF of the Weibull distribution is given by

$$F(x) = 1 - \exp\left[-\left(\frac{x}{b}\right)^a\right],$$

for $x \geq 0$, and where $a > 0$ and $b > 0$ are parameters.

(a) Find the PDF.

(b) Find the inverse CDF.

(c) Implement an inversion sampler to generate from this distribution for any values of a and b. Use your sampler to generate 10,000 sample values from the distribution with $a = 3$ and $b = 1$.

(d) Plot a histogram for your generated values from part (c). On the same plot, superimpose the curve for the PDF.

Solutions

① Let $U \sim$ Uniform $(0,1)$ and let F be the Uniform $(0,1)$ CDF so that $F(x) = x$ for $x \in [0,1]$. Now for $x \in [0,1]$,

$$
\begin{aligned}
P(1-U \leq x) &= P(U \geq 1-x) \\
&= 1 - P(U \leq 1-x) \\
&= 1 - F(1-x) \\
&= 1 - (1-x) \\
&= x = F(x).
\end{aligned}
$$

Hence, $1-U \sim$ Uniform $(0,1)$.

② (a) Here are the steps that should be in your function:

(i) Generate $U_1, \ldots, U_n \overset{iid}{\sim}$ Uniform $(0,1)$.

(ii) Use ExpInvSam to generate

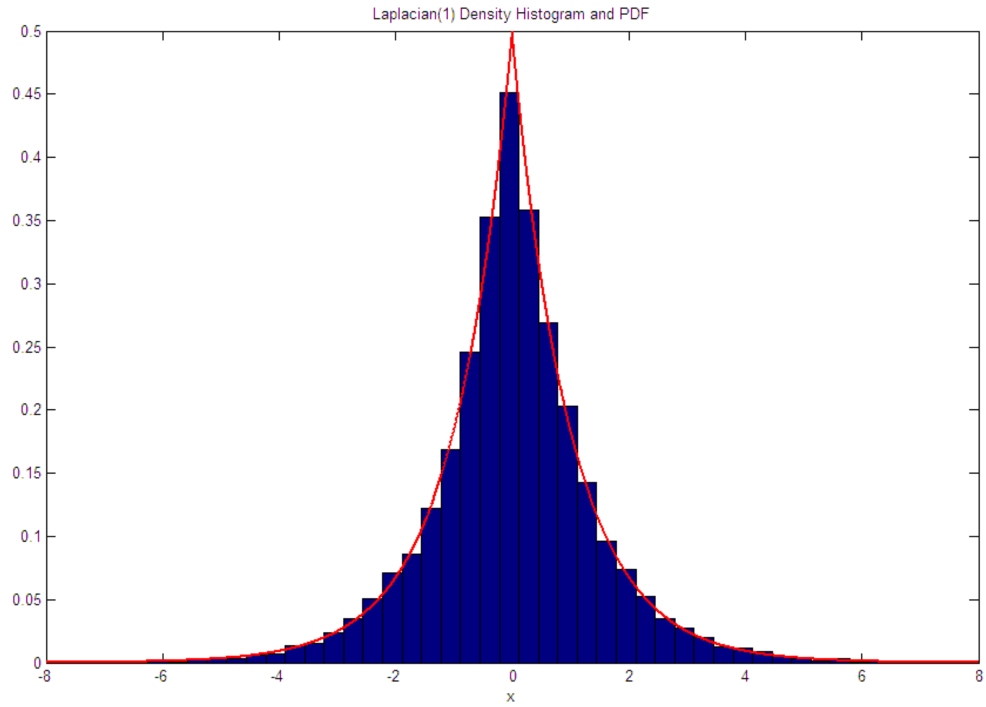
$$X_1, \ldots, X_n \overset{iid}{\sim} \text{exponential}(\lambda).$$

(iii) Generate $V_1, \ldots, V_n \overset{iid}{\sim}$ Uniform $(0,1)$.

(iv) For $i = 1, \ldots, n$: If $V_i < 0.5$, then set $X_i = -X_i$.

(v) Return $X_1, \ldots, X_n$.

(c)



Note: You may have to experiment with the number of bins in your density histogram to get a nice plot.

③ Given $f(x) = c x^{c-1}$, for $0 \le x \le 1$.

(a) For $0 \le x \le 1$,

$$F(x) = \int_0^x c y^{c-1} \, dy$$

$$= y^c \Big|_0^x = x^c .$$

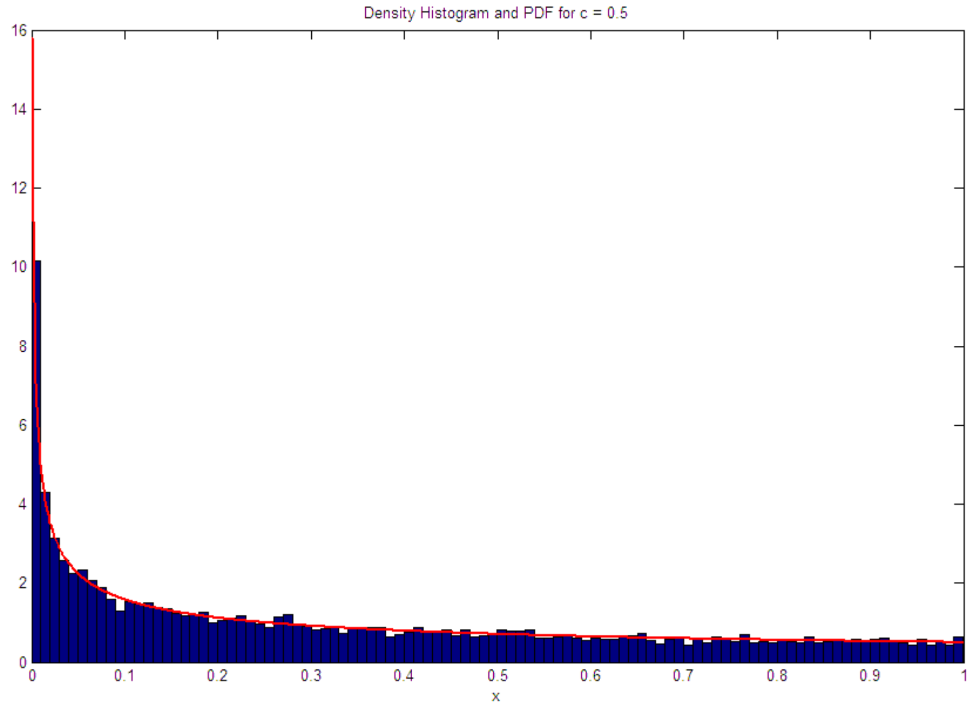
Hence,

$$F(x) \begin{cases} 0 & , \quad x < 0 , \\ x^c & , \quad 0 \leq x \leq 1 , \\ 1 & , \quad x > 1 . \end{cases}$$

Since F is continuous, the inverse CDF is given by the pointwise inverse, i.e.

$$F^{[-1]}(q) = q^{1/c} , \qquad 0 < q \leq 1 .$$

(c)



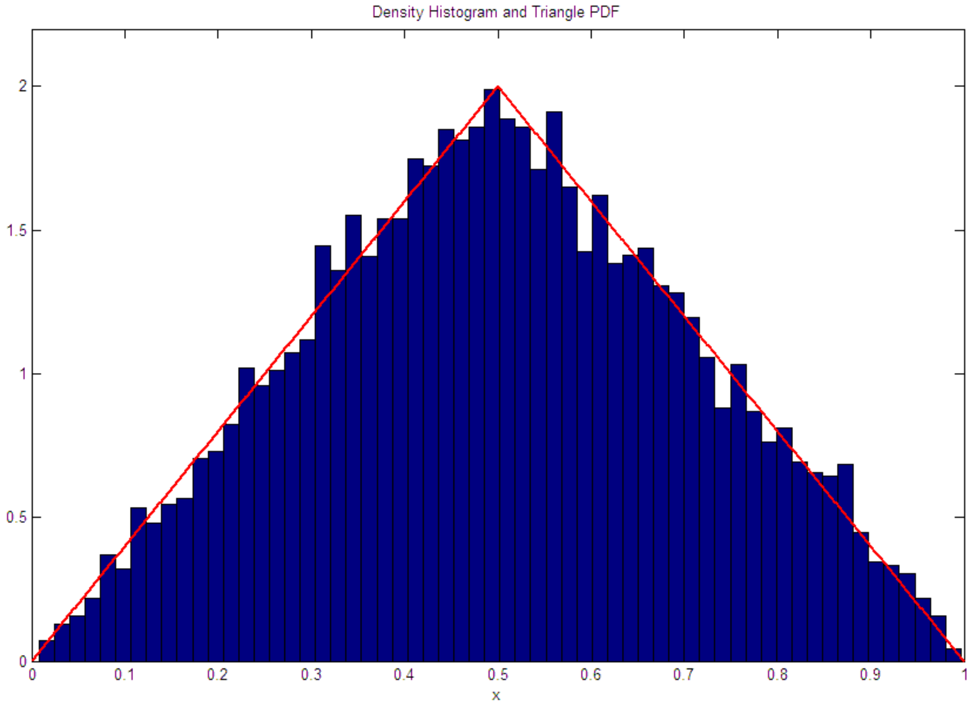
(4) Recall, from week 1 exercise 8, that the triangle PDF is

$$f(x) = \begin{cases} 4x, & 0 \leq x < 0.5, \\ 4(1-x), & 0.5 \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Also, from week 2 exercise 5, the inverse CDF is

$$F^{[-1]}(q) = \begin{cases} \sqrt{q/2}, & 0 < q \leq 0.5, \\ 1 - \sqrt{1 - \frac{(q+1)}{2}}, & 0.5 < q \leq 1. \end{cases}$$

(b)



(5) Given :

$$f(x) = \frac{1}{b}\exp\left[-\frac{(x-a)}{b}\right]\exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\},$$

$$F(x) = 1 - \exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\},$$

for $x \in \mathbb{R}$, and where $a \in \mathbb{R}$ and $b > 0$.

(a) Since this is continuous, the inverse CDF is given by the pointwise inverse. Let

$$q = 1 - \exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\}$$

$$\iff \exp\left\{-\exp\left[-\frac{(x-a)}{b}\right]\right\} = 1 - q$$

$$\iff \exp\left[-\frac{(x-a)}{b}\right] = -\log_e(1-q)$$

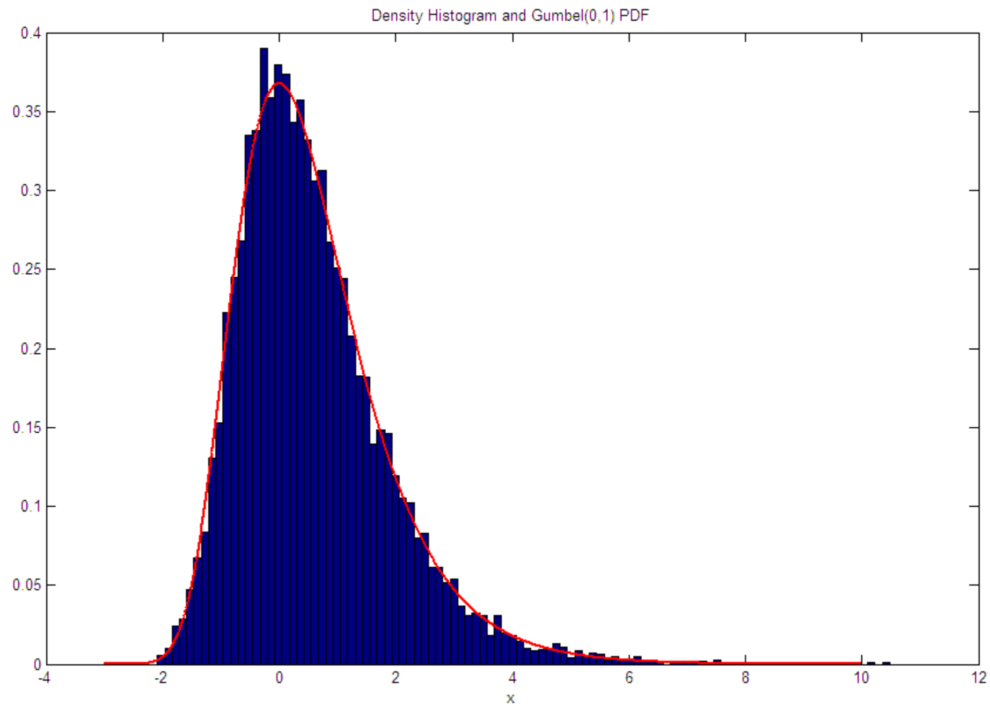
$$\iff \frac{x-a}{b} = -\log_e\left[-\log_e(1-q)\right]$$

$$\iff x = a - b\log_e\left[-\log_e(1-q)\right].$$

Therefore,

$$F^{[-1]}(q) = a - b\log_e\left[-\log_e(1-q)\right].$$

(c)



(6) Given the logistic PDF :

$$f(x) = \frac{1}{4b} \operatorname{sech}^2\left(\frac{x-a}{2b}\right),$$

for $x \in \mathbb{R}$, and where $a \in \mathbb{R}$ and $b > 0$.

(a) $$F(x) = \frac{1}{4b} \int_{-\infty}^{x} \operatorname{sech}^2\left(\frac{y-a}{2b}\right) dy.$$

Let $z = \frac{y-a}{2b}$. Then $\frac{dz}{dy} = \frac{1}{2b}$ and so

$$F(x) = \frac{1}{2} \int_{-\infty}^{(x-a)/2b} \operatorname{sech}^2(z)\, dz$$

$$= \frac{1}{2} \tanh(z) \Big|_{-\infty}^{(x-a)/2b}$$

$$= \frac{1}{2}\left[\tanh\left(\frac{x-a}{2b}\right) + 1\right].$$

Now to find $F^{[-1]}$, let

$$q = \frac{1}{2}\left[\tanh\left(\frac{x-a}{2b}\right) + 1\right]$$

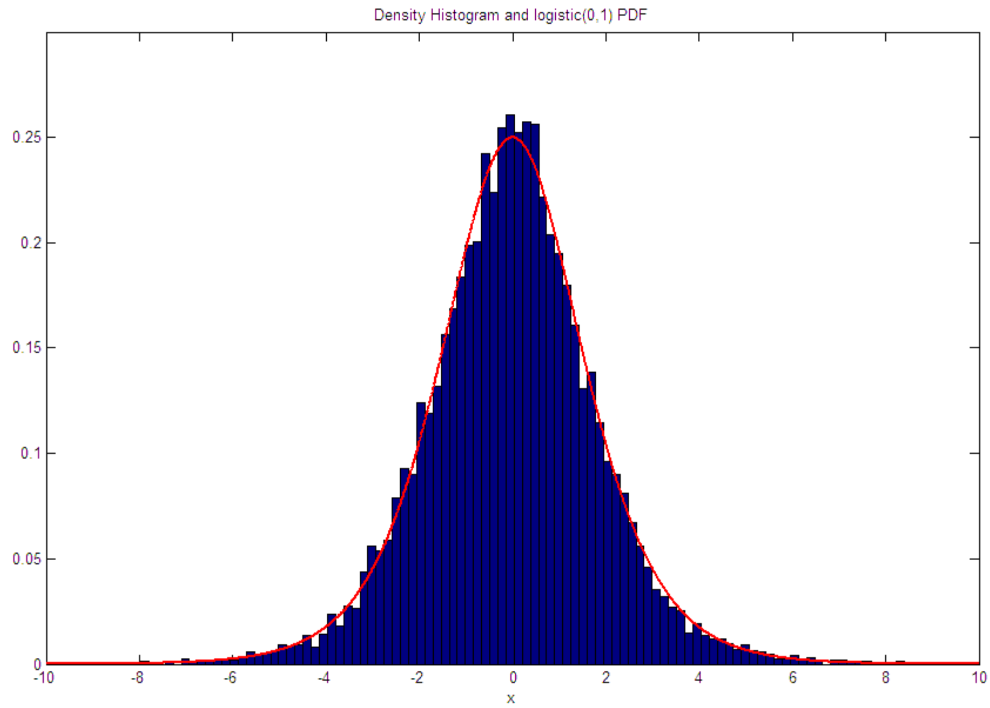
$$\Leftrightarrow \quad \tanh\left(\frac{x-a}{2b}\right) = 2q - 1$$

$$\Leftrightarrow \quad x = a + 2b\tanh^{-1}(2q-1).$$

Hence,

$$F^{[-1]}(q) = a + 2b\tanh^{-1}(2q-1).$$

(c)



Note that the Matlab built-in function for $\tanh^{-1}$ is atanh.

7 Given the Weibull CDF:

$$F(x) = 1 - \exp\left[-\left(\frac{x}{b}\right)^a\right],$$

for $x \geq 0$, and where $a > 0$ and $b > 0$.

(a) For $x \geq 0$,

$$f(x) = \frac{d\,F(x)}{dx}$$

$$= \frac{a x^{a-1}}{b^a} \exp\left[-\left(\frac{x}{b}\right)^a\right].$$

(b) To find $F^{[-1]}$, let

$$q = 1 - \exp\left[-\left(\frac{x}{b}\right)^a\right]$$

$$\Longleftrightarrow \quad \exp\left[-\left(\frac{x}{b}\right)^a\right] = 1 - q$$

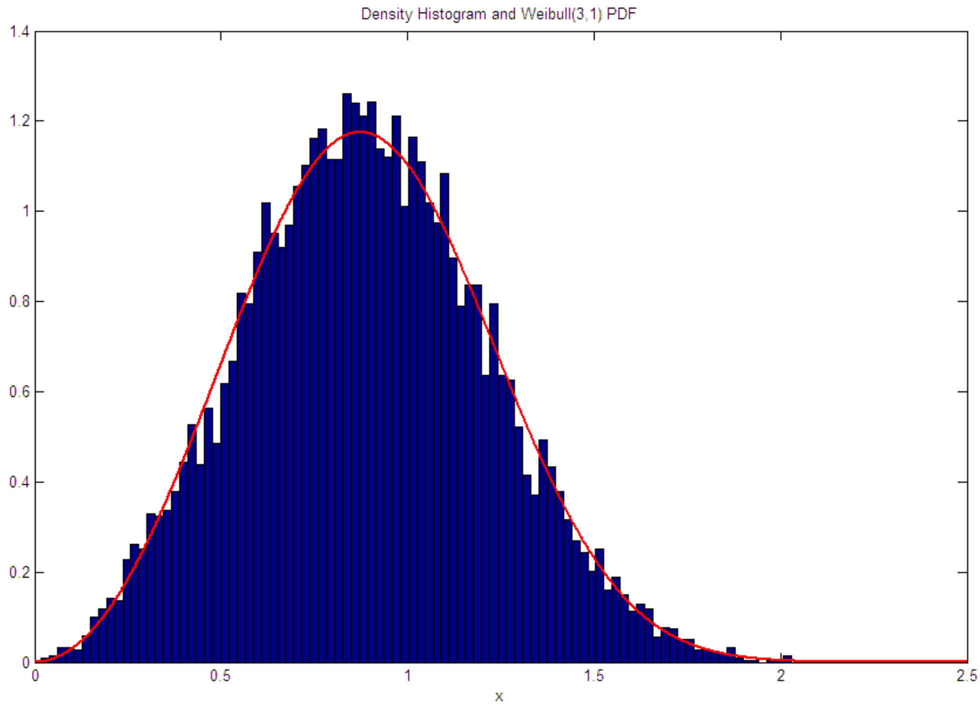
$$\Longleftrightarrow \quad \left(\frac{x}{b}\right)^a = -\log_e(1-q)$$

$$\Longleftrightarrow \quad x = b\left[-\log_e(1-q)\right]^{1/a}.$$

Hence,

$$F^{[-1]}(q) = b\left[-\log_e(1-q)\right]^{1/a}.$$

(d)


Density Histogram and Weibull(3,1) PDF
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
x