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CSE Exercises - Week 9

1. Let $X_1, \ldots, X_n \overset{IID}{\sim}$ Uniform (a, b), where $a < b$, i.e. the uniform distribution on the interval $[a, b]$.

(a) Find the ML estimators for a and b analytically.

(b) Let $\mu = E(X_1)$, i.e. the mean of the Uniform (a, b) distribution. Find the ML estimator $\hat{\mu}$ for μ.

(c) Suppose that $a = 5$, $b = 8$ and $n = 10$.

(i) Estimate the mean squared error of $\hat{\mu}$ by Monte Carlo simulation.

(ii) Let $\bar{X}$ be the sample mean of $X_1, \ldots, X_n$. Find the mean squared error of $\bar{X}$ analytically.

(iii) Compare and comment on your results in (i) and (ii).

(2) Let $X_1, \ldots, X_n \overset{IID}{\sim} \text{Normal}(\mu, \sigma^2)$. Let q be the 0.95 quantile of the $\text{Normal}(\mu, \sigma^2)$ distribution.

(a) Find the ML estimator, $\hat{q}$, analytically.

(b) Suppose that $n = 30$ and the data, $X_1, \ldots, X_n$, are given in the file, normal_data.mat.

(i) Find the ML estimate, $\hat{q}$.

(ii) Estimate the standard error of $\hat{q}$ using the parametric bootstrap.

(iii) Estimate the standard error of $\hat{q}$ using the delta method.

(iv) Compare and comment on your answers in (ii) and (iii).

(3) Recall the definition of the Beta(α, β) distribution in Exercise 5 of week 5.

Let $X_1, \ldots, X_n \overset{IID}{\sim} \text{Beta}(\alpha, \beta)$.

Suppose that $n = 25$ and the data, $X_1, \ldots, X_n$, are given in the file, beta_data.mat.

(a) Use fminsearch to find the ML estimates, $\hat{\alpha}$ and $\hat{\beta}$.

(b) When $\alpha > 1$ and $\beta > 1$, the mode of the Beta(α, β) density is at

$$m = \frac{\alpha - 1}{\alpha + \beta - 2}.$$

Find the ML estimate of m.

(c) Obtain approximate 95% confidence intervals for α, β and m using the parametric bootstrap.

Solutions

① $X_1, \dots, X_n \overset{\text{IID}}{\sim} \text{Uniform}(a, b)$, $a < b$.

(a) $$f(x; a, b) = \frac{1}{b-a} I_{[a,b]}(x).$$

$$f(x_1, \dots, x_n; a, b) = (b-a)^{-n} \prod_{i=1}^{n} I_{[a,b]}(x_i)$$

$$= \begin{cases} (b-a)^{-n} & \text{if } a \le x_1 \le b, \dots, a \le x_n \le b, \\ 0 & \text{otherwise}. \end{cases}$$

$$= \begin{cases} (b-a)^{-n} & \text{if } a \le x_1, \dots, a \le x_n,\ b \ge x_1, \dots, b \ge x_n \\ 0 & \text{otherwise}. \end{cases}$$

$$= \begin{cases} (b-a)^{-n} & \text{if } a \le \min\{x_1, \dots, x_n\},\ b \ge \max\{x_1, \dots, x_n\} \\ 0 & \text{otherwise}. \end{cases}$$

Hence, likelihood function will have maximum value when $a = \min\{x_1, \dots, x_n\}$ and $b = \max\{x_1, \dots, x_n\}$. Therefore, ML estimators for a and b are

$$\hat{a} = X_{(1)} \quad \text{and} \quad \hat{b} = X_{(n)}$$

i.e. the smallest and largest order statistics, respectively.

(b) Recall that $\mu = \frac{a+b}{2}$ and so by the equivariant property of ML estimators,

$$\hat{\mu} = \frac{\hat{a}+\hat{b}}{2} = \frac{X_{(1)} + X_{(n)}}{2} .$$

(c) Let $a = 5$, $b = 8$ and $n = 10$.
Then $\mu = (5+8)/2 = 6.5$.

(i) MSE of $\hat{\mu} = E[(\hat{\mu} - \mu)^2]$

$$\approx \frac{1}{N} \sum_{j=1}^{N} (\hat{\mu}_j - \mu)^2$$

where $\hat{\mu}_j$ is the ML estimate from realization j consisting of 10 sample values drawn from Uniform$(5, 8)$.

Using $N = 10{,}000$ realizations, we find

MSE of $\hat{\mu} \approx 0.0342$.

(ii) Since the sample mean $\bar{X}$ is an unbiased estimator for μ,

$$\text{MSE of } \bar{X} = V(\bar{X}) = \frac{V(X_1)}{n} = \frac{(b-a)^2}{12n}$$

$$= \frac{(8-5)^2}{12 \times 10} = \frac{3}{40} \approx 0.075$$

(iii) $\hat{\mu}$ has smaller MSE than $\bar{X}$.
In terms of MSE, $\hat{\mu}$ is a better estimator for μ than $\bar{X}$.

(2) $X_1, \ldots, X_n \overset{IID}{\sim} \text{Normal}(\mu, \sigma^2)$.

(a) From the lectures, the ML estimators for μ and σ^2 are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \hat{\mu})^2 .$$

For the Normal (μ, σ^2) distribution, the 0.95 quantile is

$$q = \mu + 1.6449\,\sigma .$$

Therefore, by the equivariant property of ML estimators,

$$\hat{q} = \hat{\mu} + 1.6449\,\hat{\sigma} .$$

(b) (i) Using the given data, $\hat{\mu} = 0.7298$ and $\hat{\sigma} = 1.7143$, and so

$$\hat{q} = 0.7298 + 1.6449 \times 1.7143$$
$$= 3.5497 .$$

(ii) Standard error of $\hat{q}$

= standard deviation of $\hat{q}$

= sample standard deviation of $\hat{q}_1, \ldots, \hat{q}_N$

where $\hat{q}_j$ is the ML estimate from realization j consisting of $n = 30$ sample values drawn from Normal$(\hat{\mu}, \hat{\sigma}^2)$.

Using $N = 10{,}000$ realizations, I got standard error of $\hat{q} \approx 0.4827$.

(iii) Recall from page 231 in the notes that

$$I^{-1}(\mu, \sigma) = \frac{1}{n}\begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{\sigma^2}{2} \end{bmatrix}.$$

Now $q = g(\mu, \sigma) = \mu + 1.6449\,\sigma$

and so $\nabla g(\mu, \sigma) = \begin{pmatrix} 1 \\ 1.6449 \end{pmatrix}$

Estimate of standard error of $\hat{q}$ is

$$\sqrt{(\nabla g)^T I^{-1}(\hat{\mu}, \hat{\sigma})\, \nabla g}$$

$$= \sqrt{\begin{pmatrix} 1 & 1.6449 \end{pmatrix} \begin{pmatrix} \hat{\sigma}^2/30 & 0 \\ 0 & \hat{\sigma}^2/60 \end{pmatrix} \begin{pmatrix} 1 \\ 1.6449 \end{pmatrix}}$$

≈ 0.4801.

(iv) The estimates of standard error of $\hat{q}$ are close. Thus, the sample size of $n = 30$ is large enough here for using the asymptotic normality of $(\hat{\mu}, \hat{\sigma})$.

(3) $X_1, \ldots, X_n \overset{IID}{\sim} \text{Beta}(\alpha, \beta)$.

(a) Recall from week 5 Exercise 5 that

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

for $0 \le x \le 1$, $\alpha > 0$ and $\beta > 0$.

$$L_n(\alpha,\beta) = \prod_{i=1}^{n} f(x_i;\alpha,\beta)$$

$$\ell_n(\alpha,\beta) = \sum_{i=1}^{n} \log f(x_i;\alpha,\beta)$$

$$= n\left[\log\Gamma(\alpha+\beta) - \log\Gamma(\alpha) - \log\Gamma(\beta)\right] + (\alpha-1)\sum_{i=1}^{n}\log x_i + (\beta-1)\sum_{i=1}^{n}\log(1-x_i)$$

Using the given data with fminsearch, the ML estimates are

$$\hat{\alpha} = 1.6588, \quad \hat{\beta} = 5.6497.$$

(b) Since mode $m = \dfrac{\alpha-1}{\alpha+\beta-2}$, by the equivariant property of ML estimators, the ML estimate of the mode is

$$\hat{m} = \frac{\hat{\alpha}-1}{\hat{\alpha}+\hat{\beta}-2} = \frac{1.6588-1}{1.6588+5.6497-2}$$

$$= 0.1241.$$

(c) Generating $N = 1000$ realizations, each with $n = 25$ sample values from the Beta$(\hat{\alpha}, \hat{\beta})$ distribution, the resulting ML estimates are

$\hat{\alpha}_1, \ldots, \hat{\alpha}_N$, $\hat{\beta}_1, \ldots, \hat{\beta}_N$, and $\hat{m}_1, \ldots, \hat{m}_N$, where

$$\hat{m}_j = \frac{\hat{\alpha}_j - 1}{\hat{\alpha}_j + \hat{\beta}_j - 2}.$$

The required approximate 95% confidence intervals are

$$(\hat{\alpha}_{(25)}, \hat{\alpha}_{(975)}) = (1.0743, 3.1035),$$

$$(\hat{\beta}_{(25)}, \hat{\beta}_{(975)}) = (3.4486, 11.4244),$$

$$(\hat{m}_{(25)}, \hat{m}_{(975)}) = (0.0245, 0.2109).$$