Department of Mathematics and Statistics

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CSE Exercises - Week 7

1. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $E(X_1) < \infty$ and $V(X_1) < \infty$. Let $\overline{X}_n$ denote the sample mean. We know, by the WLLN, that $\overline{X}_n$ is asymptotically consistent for $E(X_1)$. Verify this using Proposition 95 (page 199) by showing that

   (a) $\overline{X}_n$ is an unbiased estimator for $E(X_1)$; and

   (b) $V(\overline{X}_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

2. Let $X_1, \ldots, X_n \overset{IID}{\sim} \text{Bernoulli}(\theta^*)$ and let $S_n = X_1 + \cdots + X_n$. Consider the following estimator for θ^*:

$$\hat{\theta}_n := (S_n + 1)/(n+2).$$

   (a) Is $\hat{\theta}_n$ a biased or unbiased estimator for θ^*? If biased, find the bias.

   (b) Show clearly whether $\hat{\theta}_n$ is asymptotically consistent.

(3) Recall from Exercise 1 in week 2's exercises that for $X_1, \ldots, X_n$ IID with $E(X_1) < \infty$ and $V(X_1) < \infty$, the usual sample variance,

$$S_n^2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 ,$$

is an unbiased estimator for $V(X_1)$.

In this exercise, we will discover by simulation that the property of unbiasedness is generally not preserved under transformations. Specifically, we will see that S_n is <u>not</u> an unbiased estimator of the population standard deviation, $\sqrt{V(X_1)}$. Do the following:

(i) Generate $n = 10$ Uniform$(0,1)$ random values.

(ii) Compute S_n^2 and S_n and store the answers.

(iii) Repeat steps (i) and (ii) 10,000 times.

(iv) Use the 10,000 S_n^2 values to obtain a 95% confidence interval for $E(S_n^2)$.
Compare this with the Uniform$(0,1)$ population variance and comment on your result.

(v) Use the 10,000 S_n values to obtain a 95% confidence interval for $E(S_n)$.
Compare this with the Uniform$(0,1)$ population standard deviation and comment on your result.

4. Let X be a continuous random variable with PDF,

$$f(x;c) = \begin{cases} c\,x^{c-1}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $c > 0$ is a parameter.

(a) Suppose that $X = 0.25$ is observed. Write down the likelihood function and log-likelihood function and plot them.

(b) Suppose that $X_1, X_2 \overset{\text{IID}}{\sim} f(x;c)$, and $X_1 = 0.7$ and $X_2 = 0.9$ are observed. Write down the likelihood function and log-likelihood function and plot them.

(c) If $X_1, \ldots, X_n \overset{\text{IID}}{\sim} f(x;c)$, write down the likelihood function and log-likelihood function.