CSE Exercises - Week 5

(1) Page 143 Exercise 107.

Change "Referring to Example 2.2.3 ..." to "Referring to Simulation 97 (page 133) ...".

(2) Page 143 Exercise 108.

(3) Page 143 Exercise 109.

(4) Page 137 Labwork 100.

(5) X has a Beta(α, β) distribution with parameters $\alpha > 0$ and $\beta > 0$ if its PDF is

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},$$

for $0 \leq x \leq 1$, and 0 elsewhere.

(a) Let (X_1, X_2) be a continuous random vector with joint PDF $f_X(x_1, x_2)$ and for which there exists an open subset $\mathcal{U} \subseteq \mathbb{R}^2$ such that $P((X_1, X_2) \in \mathcal{U}) = 1$. Let $(Y_1, Y_2) = g(X_1, X_2) = (g_1(X_1, X_2), g_2(X_1, X_2))$ be such that $g = (g_1, g_2)$ is a one-to-one function on $\mathcal{U}$ and

$$\det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} \neq 0,$$

where $\det A$ denotes the determinant of matrix A. Let $h = (h_1, h_2)$ be the pointwise inverse of g so that $(X_1, X_2) = h(Y_1, Y_2) = (h_1(Y_1, Y_2), h_2(Y_1, Y_2))$. Then (Y_1, Y_2) is a continuous random vector and its joint PDF is given by

$$f_Y(y_1, y_2) = f_X(h(y_1, y_2)) \left| \det \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} \right|, \quad (y_1, y_2) \in g(U).$$

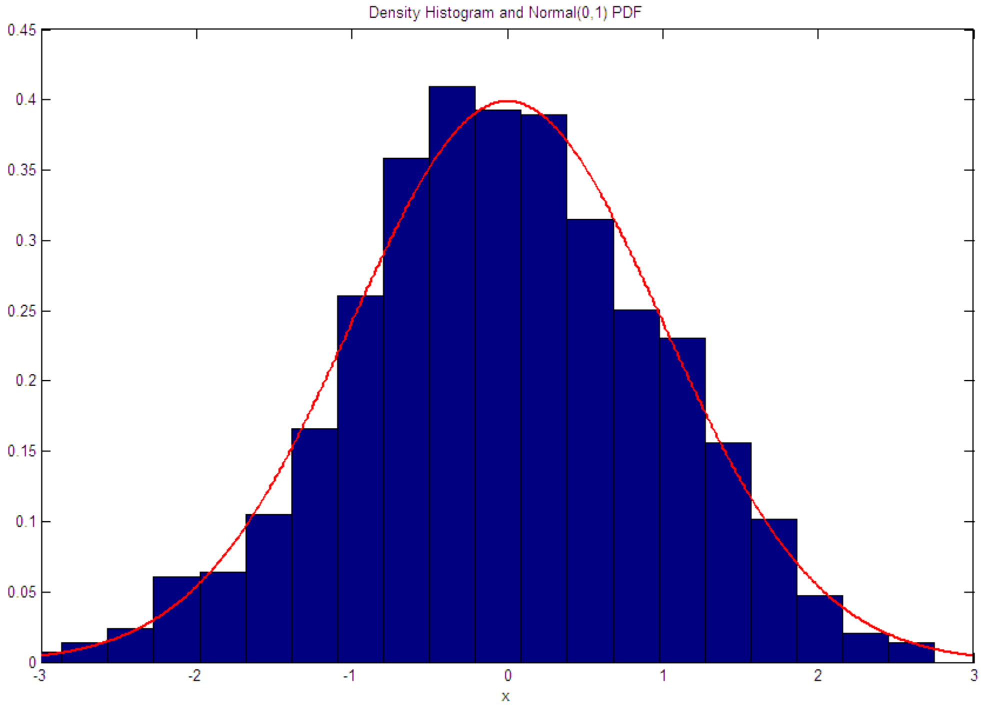
Now if X_1 and X_2 are independent random variables such that $X_1 \sim \text{Gamma}(1, a_1)$ and $X_2 \sim \text{Gamma}(1, a_2)$, use the above result, known as the transformation theorem, to show that $Y_1 = X_1/(X_1 + X_2)$ has a $\text{Beta}(a_1, a_2)$ distribution. Hint: Consider the transformation, $(Y_1, Y_2) = g(X_1, X_2) = (X_1/(X_1+X_2), X_2)$.

(b) Use the result in part (a) to generate 10,000 samples from the Beta (2, 4) distribution, starting from samples with gamma distributions. Use the Matlab function "gamrnd" to generate gamma distributed samples.

(c) Plot the density histogram for your generated Beta samples and superimpose on it the curve for the Beta(2,4) PDF.

Solutions

1 (b)



$$a = \sqrt{\frac{2}{\pi}} \exp\left(\tfrac{1}{2}\right) \simeq 1 \cdot 3155 .$$

Average number of iterations, $\bar{a} = 1 \cdot 295$.

Approximate 95% confidence interval for the expected number of iterations is $(1 \cdot 2585, 1 \cdot 3315)$, which contains $a = 1 \cdot 3155$.

(c) $$\tilde{f}(x)/\tilde{g}(x) = \exp\left(|x| - \frac{x^2}{2}\right)$$

$$\leq \exp\left(\tfrac{1}{2}\right) = \tilde{a} \approx 1 \cdot 6487 .$$

This is not within the confidence interval. Hence, $\tilde{a}$ is not the expected number of iterations.

(2)

$$f(x) = 20x(1-x)^3, \quad x \in [0,1].$$

$$g(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

(a) $f(x)/g(x) = 20x(1-x)^3$

Must find a such that $f(x)/g(x) \leq a$ $\forall$ $x \in [0,1]$, i.e. the maximum of $f(x)/g(x)$.

$$\frac{d}{dx}\left[f(x)/g(x)\right] = 20(1-x)^3 - 60x(1-x)^2$$

$$= 20(1-x)^2\left[(1-x) - 3x\right]$$

$$= 20(1-x)^2(1-4x)$$

Equating to 0,

$$20(1-x)^2(1-4x) = 0$$

$$\iff \quad x = 1 \text{ or } \frac{1}{4}$$

Differentiating again,

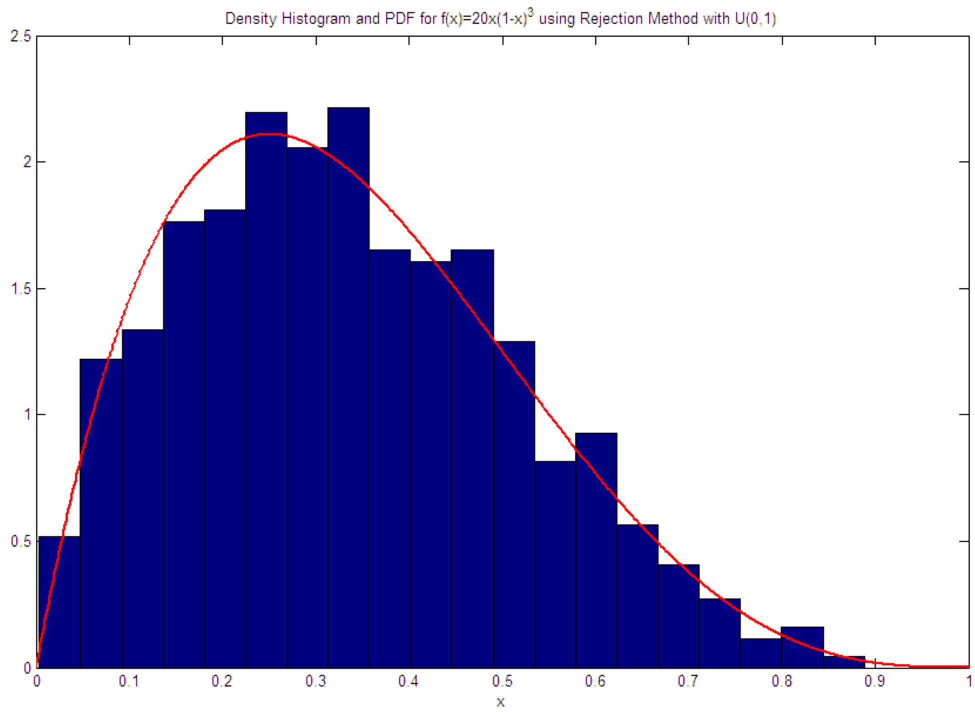
$$\frac{d^2}{dx^2}\left[f(x)/g(x)\right]$$

$$= -80(1-x)^2 - 40(1-x)(1-4x)$$

which is 0 when $x = 1$ and -45 when $x = 1/4$. Therefore, $f(x)/g(x)$ is max when $x = 1/4$ and the max value is

$$20 \times 1/4 \times (3/4)^3 = \frac{5\times 27}{64} = 2.1094 = a.$$

(c)



Average number of iterations, $\bar{a} = 2.067$.

Approximate 95% confidence interval for expected number of iterations is $(1.9715, 2.1625)$, which contains $a = 2.1094$.

(3) $$f(x) = \frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}, \quad x \geq 0.$$

$$g(x) = \frac{1}{m} e^{-x/m}, \quad x \geq 0, \quad m > 0.$$

(a) To find $a \geq f(x)/g(x)$, $x \geq 0$.

$$f(x)/g(x) = \frac{2m}{\sqrt{\pi}} x^{1/2} e^{-x\left(\frac{m-1}{m}\right)}, \quad x \geq 0$$

$$\frac{d}{dx}\left[f(x)/g(x)\right] = \frac{2m}{\sqrt{\pi}}\left[-x^{1/2} e^{-x\left(\frac{m-1}{m}\right)}\left(\frac{m-1}{m}\right) + e^{-x\left(\frac{m-1}{m}\right)} \frac{1}{2} x^{-1/2}\right]$$

Equating to 0 : $\frac{1}{2} x^{-1/2} = \left(\frac{m-1}{m}\right) x^{1/2} \iff x = \frac{m}{2(m-1)}$

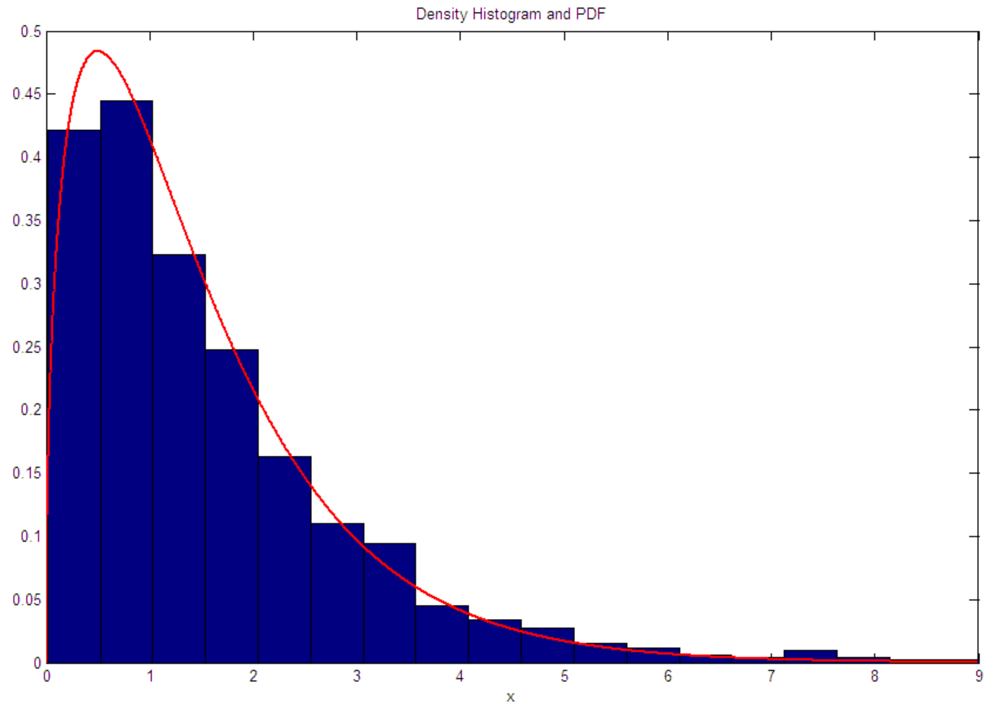
$$\therefore a = \frac{2m}{\sqrt{\pi}} \sqrt{\frac{m}{2(m-1)}}\, e^{-1/2} = \sqrt{\frac{2}{\pi e}} \frac{m^{3/2}}{(m-1)^{1/2}}.$$

(b) The best $g(x)$ to use is the one that gives the smallest a, so that the resulting rejection algorithm will have the smallest expected number of iterations.

$$\frac{da}{dm} = \sqrt{\frac{2}{\pi e}}\left[m^{3/2}\left(-\frac{1}{2}\right)(m-1)^{-3/2} + (m-1)^{-1/2}\left(\frac{3}{2}\right) m^{1/2}\right]$$

Equating to 0 : $m(m-1)^{-1} = 3 \iff m = \frac{3}{2}$.

(d) Using $m = \frac{3}{2}$, $a = 1.2573$.



Average number of iterations, $\bar{a} = 1.29$.

Approximate 95% confidence interval for the expected number of iterations is $(1.2509, 1.3291)$, which contains $a = 1.2573$.

4

$$f(x|\mu,\tau,\sigma^2) = \frac{exp(-(x-\mu)^2/2\sigma^2)}{\sqrt{2\pi}\sigma[1-\Phi((\tau-\mu)/\sigma)]}\mathbb{1}_{x\geq\tau}.$$

$$g(y|\lambda,\tau) = \lambda exp(-\lambda(y-\tau))\mathbf{1}_{y\geq\tau}.$$

(1) Let $\mu = 0$, $\sigma^2 = 1$ and $\tau \geq 0$. Then for $x \geq \tau$,

$$f(x|0,\tau,1)/g(x|\lambda,\tau)$$

$$= \frac{1}{\lambda\sqrt{2\pi}[1-\Phi(\tau)]} \exp\left[\lambda(x-\tau) - \frac{x^2}{2}\right]$$

$$\frac{d}{dx}\left[f(x|0,\tau,1)/g(x|\lambda,\tau)\right]$$

$$= \frac{1}{\lambda\sqrt{2\pi}[1-\Phi(\tau)]} \exp\left[\lambda(x-\tau) - \frac{x^2}{2}\right](\lambda - x).$$

Equating to 0 gives $x = \lambda$, and so

$$a = f(\lambda|0,\tau,1)/g(\lambda|\lambda,\tau)$$

$$= \frac{1}{\lambda\sqrt{2\pi}[1-\Phi(\tau)]} \exp\left(\frac{\lambda^2}{2} - \lambda\tau\right).$$

The best choice of λ is the one that gives the smallest a:

$$\frac{da}{d\lambda} = \frac{1}{\sqrt{2\pi}[1-\Phi(\tau)]} \exp\left(\frac{\lambda^2}{2} - \lambda\tau\right)\left[-\lambda^{-2} + \lambda^{-1}(\lambda-\tau)\right]$$

Equating to 0 : $\lambda^{-1} = \lambda - \tau$

$$\iff \lambda^2 - \tau\lambda - 1 = 0$$

$$\iff \lambda = \tfrac{1}{2}\left(\tau + \sqrt{\tau^2+4}\right) \text{ or } \tfrac{1}{2}\left(\tau - \sqrt{\tau^2+4}\right)$$

Since we must have $\lambda > 0$, the best choice is

$$\lambda = \tfrac{1}{2}\left(\tau + \sqrt{\tau^2+4}\right).$$

(2) For each value of τ, the maximum expected acceptance probability is given by $1/a$, where

$$a = \frac{1}{\lambda\sqrt{2\pi}\,[1 - \Phi(\tau)]} \exp\left(\frac{\lambda^2}{2} - \lambda\tau\right)$$

Normal (0,1) CDF

and $\lambda = \tfrac{1}{2}\left(\tau + \sqrt{\tau^2+4}\right)$.

Therefore,

τ	λ	a	$1/a$
0	1.0000	1.3155	0.7602
0.5000	1.2808	1.2084	0.8275
1.0000	1.6180	1.1409	0.8765
1.5000	2.0000	1.0984	0.9104
2.0000	2.4142	1.0711	0.9336
2.5000	2.8508	1.0530	0.9497
3.0000	3.3028	1.0407	0.9609

The acceptance probability increases as τ increases and so the algorithm becomes more efficient as τ gets further out into the right tail.

(3) To get $X \sim N_+(\mu, \tau, \sigma^2)$:

Generate $Y \sim N_+(0, \frac{\tau - \mu}{\sigma}, 1)$.

Compute $X = \mu + \sigma Y$.

Return X .

(4) To get $X \sim N_-(\mu, \tau, \sigma^2)$:

Generate $Y \sim N_+(-\mu, -\tau, \sigma^2)$.

Compute $X = -Y$.

Return X .

(5) $X \sim \text{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1.$$

gamma function
(see page 139)

(a) Let $X_1 \sim \text{Gamma}(1, a_1)$, $X_2 \sim \text{Gamma}(\underset{\text{scale}}{1}, \underset{\text{shape}}{a_2})$, X_1 and X_2 independent. Gamma $(1, a)$ PDF is

$$f(x;1,a) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x}, \quad x \geq 0.$$

Let $(Y_1, Y_2) = g(X_1, X_2) = \left(\frac{X_1}{X_1 + X_2}, X_2\right)$, and so

$$(X_1, X_2) = h(Y_1, Y_2) = \left(\frac{Y_1 Y_2}{1 - Y_1}, Y_2\right).$$

By the transformation theorem, the joint density of $Y = (Y_1, Y_2)$ is

$$f_Y(y_1, y_2) = f_X(h(y_1, y_2)) \left| \det \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} \right|.$$

Since X_1 and X_2 are independent, the joint density of $X = (X_1, X_2)$ is

$$f_X(x_1, x_2) = f_{X_1}(x_1; 1, a_1)\, f_{X_2}(x_2; 1, a_2)$$
$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)} x_1^{a_1 - 1} x_2^{a_2 - 1} e^{-(x_1 + x_2)}.$$

Hence, $f_X(h(y_1, y_2))$

$$= f_X\left(\frac{y_1 y_2}{1-y_1}, y_2\right)$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \left(\frac{y_1 y_2}{1-y_1}\right)^{a_1-1} y_2^{a_2-1} \exp\left[-\left(\frac{y_2}{1-y_1}\right)\right].$$

Now

$$\frac{\partial h_1}{\partial y_1} = y_2\left[(1-y_1)^{-1} + y_1(1-y_1)^{-2}\right]$$

$$= y_2(1-y_1)^{-1}\left[1 + y_1(1-y_1)^{-1}\right]$$

$$\frac{\partial h_1}{\partial y_2} = y_1(1-y_1)^{-1}$$

$$\frac{\partial h_2}{\partial y_1} = 0$$

$$\frac{\partial h_2}{\partial y_2} = 1$$

$$\det\begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} = \frac{y_2}{1-y_1}\left(1 + \frac{y_1}{1-y_1}\right)$$

$$= \frac{y_2}{(1-y_1)^2} \geq 0 .$$

Therefore,

$$f_Y(y_1, y_2) = \frac{1}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1-1} (1-y_1)^{a_2-1} \cdot \underbrace{\left(\frac{1}{1-y_1}\right)^{a_1+a_2} y_2^{a_1+a_2-1} \exp\left[-\left(\frac{y_2}{1-y_1}\right)\right]}_{\text{Notice that this is equal to } \Gamma(a_1+a_2)\, \underbrace{f\left(y_2; \frac{1}{1-y_1}, a_1+a_2\right)}_{\text{Gamma}\left(\frac{1}{1-y_1},\, a_1+a_2\right) \text{ PDF}}}$$

and so

$$f_{Y_1}(y_1) = \int_0^\infty f_Y(y_1, y_2)\, dy_2$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1-1}(1-y_1)^{a_2-1} \cdot \int_0^\infty \left(\frac{1}{1-y_1}\right)^{a_1+a_2} y_2^{a_1+a_2-1} \exp\left[-\left(\frac{y_2}{1-y_1}\right)\right] dy_2$$

$$= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1-1}(1-y_1)^{a_2-1},$$

which is the Beta(a_1, a_2) PDF.

(c)