Department of Mathematics and Statistics

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

CSE Exercises – Week 8

(1) Revisiting Exercise 4(b) from last week:

(a) Find the ML estimate of c analytically.

(b) Find the ML estimate of c numerically using fminbnd.

(c) Explain why, in this case, the normal approximation should not be used to obtain a confidence interval for c? In spite of this, use the normal approximation to find a 95% confidence interval for c anyway. Comment on your result.

(d) We can use Monte Carlo simulation to get an approximate confidence interval for c:

(i) Use the ML estimate, $\hat{c}$, in the distribution, i.e. let $c = \hat{c}$ in $f(x; c)$.

(ii) Generate $X_1, X_2 \overset{IID}{\sim} f(x; \hat{c})$. Recall from exercise 3 in week 3 that we can do this by the inverse CDF method.

(iii) Find the ML estimate of c for the values of X_1 and X_2 from step (ii), i.e.

find $\hat{c}_1 = \underset{c>0}{\arg\max}\ f(X_1, X_2\,;\, c)$.

(iv) Repeat steps (ii) and (iii) N times to get $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_N$.

(v) Sort the estimates from (iv) in increasing order to get the order statistics, $\hat{c}_{(1)}, \hat{c}_{(2)}, \ldots, \hat{c}_{(N)}$.

(vi) An approximate 95% confidence interval is given by $(\hat{c}_{(0.025N)}, \hat{c}_{(0.975N)})$.

Use this procedure with $N = 1000$ to find an approximate 95% confidence interval for c.

Later on, you will learn about a method for constructing confidence intervals known as the bootstrap. The procedure that we have used here is sometimes called a "parametric bootstrap". It is worthwhile keeping in mind that the bootstrap is simulation-based.

(e) Next, we want to check that when the sample size n is "large enough", the normal approximation confidence interval and the parametric bootstrap confidence interval are about the same.

Suppose that $n = 30$, $X_1, \ldots, X_n \overset{\text{IID}}{\sim} f(x\,;\,c)$, and

$$\sum_{i=1}^{n} \log X_i = -7.7335 .$$

(i) Find the ML estimate of c.

(ii) Find a 95% confidence interval for c using the normal approximation for $\hat{c}$.

(iii) Find a 95% confidence interval for c using the "parametric bootstrap" procedure.

(2) Given that $X_1, \ldots, X_n \overset{\text{IID}}{\sim} f(x;\theta)$, where

$$f(x;\theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) \exp\left[-\exp\left(-\frac{x}{\theta}\right)\right],$$

for $x \in \mathbb{R}$ and $\theta > 0$. Now suppose that $n = 5$ and that $X_1 = -0.15$, $X_2 = 0.27$, $X_3 = 1.33$, $X_4 = -1.71$ and $X_5 = -0.89$.

(a) Find the ML estimate of θ using fminbnd.

(b) Use the "parametric bootstrap" procedure with $N = 1000$ to find an approximate 95% confidence interval for θ.

(c) Let σ^2 denote the variance of $X \sim f(x;\theta)$. Given that $\sigma^2 = \theta^2 \pi^2 / 6$, find the ML estimate of σ^2.

(d) By noting that σ^2 is a one-to-one monotone increasing function of θ for $\theta > 0$, find an approximate 95% confidence interval for σ^2.

(3) Given that $X_1, \ldots, X_n \overset{IID}{\sim} f(x;\theta)$, where

$$f(x;\theta) = \frac{1}{4\theta}\,\mathrm{sech}^2\left(\frac{x}{2\theta}\right),$$

for $x \in \mathbb{R}$ and $\theta > 0$. Now suppose that $n = 6$ and $X_1 = 0.32$, $X_2 = 1.56$, $X_3 = 3.11$, $X_4 = -0.01$, $X_5 = -2.27$ and $X_6 = -3.41$.

(a) Find the ML estimate of θ using fminbnd.

(b) Use the "parametric bootstrap" procedure to find an approximate 95% confidence interval for θ.

(c) Let $\lambda = 1/\sqrt{\theta}$. Find the ML estimate of λ.

(d) Find an approximate 95% confidence interval for λ.