

Time: 08.00-13.00. Total sum: 40p. Grades 3, 4 and 5 require 18p, 25p and 32p, respectively. Note:

- Motivate solutions well but no more than necessary.
- Just write on one side of the paper.
- Start new question on new page, and label the question number clearly next to your solution.
- Do not write with red-ink pen.
- Write the solutions in increasing order of question numbers.
- Hints and other information provided in a problem may also be useful for subsequent problems.

Permitted aids: Any books, notes, and pocket calculator.

Ex. 1 — 5p – Eight earthquakes with the largest magnitude (in Richter scale) that hit Christchurch, NZ on 22nd February 2011 are:

5.8450, 4.6800, 5.9100, 5.0110, 4.7550, 4.8350, 4.6310, 4.6400

- (a)— 1p – Report the sample mean.
- (b)— 1p – Report the sample variance and sample standard deviation.
- (c)— 1p – Report the five order statistics, as a point in \mathbb{R}^5 , from minimum to maximum.
- (d)— 2p – Sketch the Empirical Distribution Function showing discontinuities in the function and clearly labeling the axes.

Ex. 2 — 5p – Suppose you plan to obtain an independent and identically distributed (IID) sequence of n measurements from an instrument. This instrument has been calibrated so that the distribution of measurements made with it have population variance of $1/4$. Possibly useful fact: If $1 - \alpha = 0.95$ and $Z \sim N(0, 1)$ is the standard Normal RV, then from the Z Table, $z_{\alpha/2} = 1.96$, where, $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$. If $X \sim \text{Poisson}(\lambda)$ then $V(X) = E(X) = \lambda$

- (a)— 4p – Your boss wants you to make a point estimate of the unknown population mean from an IID sample of size n . He also insists that the tolerance for error has to be $1/10$ and the probability of meeting this tolerance should be just above 95%. Use the Central Limit Theorem to

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MATERIAL COVERED IN LECTURES AS PER SYLLABUS**

find how large should n be to meet the specifications of your boss. In summary,

$$X_1, X_2, \dots, X_n \stackrel{IID}{\sim} X_1, \quad V(X_1) = 1/4$$

Find n such that $P(|\bar{X}_n - E(X_1)| < 1/10) = 0.95$.

- (b)— $1p$ – Making the further assumption that the IID samples are Poisson distributed random variables, find the Method of Moments Estimate for the mean parameter λ of the *Poisson*(λ) RV X_i whose probability mass function is given by:

$$f(x; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \{0, 1, 2, \dots\}, \quad \lambda > 0$$

Ex. 3 — $5p$ – Assume that an independent and identically distributed sample, X_1, X_2, \dots, X_n is drawn from the distribution of X with PDF $f(x; \theta)$:

$$f(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

for a fixed and unknown parameter $\theta \in (0, \infty)$ and derive the maximum likelihood estimate of θ .

Hint: you only need to do Steps 1–5 to find the MLE: (Step 1:) find $\ell(\theta)$, the log-likelihood as a function of the parameter θ , (Step 2:) find $\frac{d}{d\theta} \ell(\theta)$, the first derivative of $\ell(\theta)$ with respect to θ , (Step 3:) solve the equation $\frac{d}{d\theta} \ell(\theta) = 0$ for θ and set this solution equal to $\hat{\theta}_n$, (Step 4:) find $\frac{d^2}{d\theta^2} \ell(\theta)$, the second derivative of $\ell(\theta)$ with respect to θ and finally (Step 5:) $\hat{\theta}_n$ is the MLE if $\frac{d^2}{d\theta^2} \ell(\theta) < 0$.

Ex. 4 — $15p$ – Let $X_1, X_2, \dots, X_n \stackrel{IID}{\sim} \text{Normal}(\mu, \sigma^2)$. Suppose that μ is known and σ is unknown. The parameter of interest is $\psi = \log(\sigma)$. Answer the following:

- (1)— $2p$ – Find the log-likelihood function $\ell(\sigma)$
- (2)— $2p$ – Find its derivative with respect to the unknown parameter σ
- (3)— $2p$ – Set the derivative equal to 0 and solve for σ
- (4)— $2p$ – Find the estimated standard error \widehat{se}_n for the estimator of σ via Fisher Information

- (5)— $2p$ — Derive the estimated standard error of $\psi = \log(\sigma)$ via the Delta method
- (6)— $2p$ — Obtain the 95% confidence interval for ψ
- (7)— $1p$ — Suppose you observed $n = 110$ samples and a sample standard deviation of 12.4, Will you reject or fail to reject the null hypothesis $H_0 : \psi = 10.0$ versus $H_1 : \psi \neq 10.0$ using a size $\alpha = 0.05$ Wald test?
- (8)— $1p$ — Obtain the 95% confidence interval for σ based on $n = 110$ samples and a sample standard deviation of 12.4
- (9)— $1p$ — Will you reject or fail to reject the null hypothesis $H_0 : \sigma = 10.0$ versus $H_1 : \sigma \neq 10.0$ using a size $\alpha = 0.05$ Wald test?

Ex. 5 — $5p$ — Let $X_1, X_2, \dots, X_n \stackrel{IID}{\sim} \text{Poisson}(\lambda)$ where $\lambda = E(X_i) > 0$ is unknown. Show that the sample mean $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ is a sufficient statistic.

Ex. 6 — $5p$ — Suppose $X_1, X_2, \dots, X_m \stackrel{IID}{\sim} F$ where F is any distribution function of a real-valued random variable. Recall that a permutation can be defined as a bijection from a set onto itself, i.e., $\pi : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}$.

- (a)— $2p$ — Show that $\hat{F}_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x)$ is an unbiased estimator of $F(x)$, for each $x \in \mathbb{R}$, where,

$$\mathbf{1}(X_i \leq x) = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$$

- (b)— $2p$ — What is the Variance of the estimator $\hat{F}_n(x)$ of $F(x)$, for each $x \in \mathbb{R}$.
- (c)— $1p$ — Why is $P(X_1 = x_1, X_2 = x_2, \dots, X_m = x_m) = P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \dots, X_m = x_{\pi(m)}) = 1/m!$? Explain your answer step by step.

Lycka till!

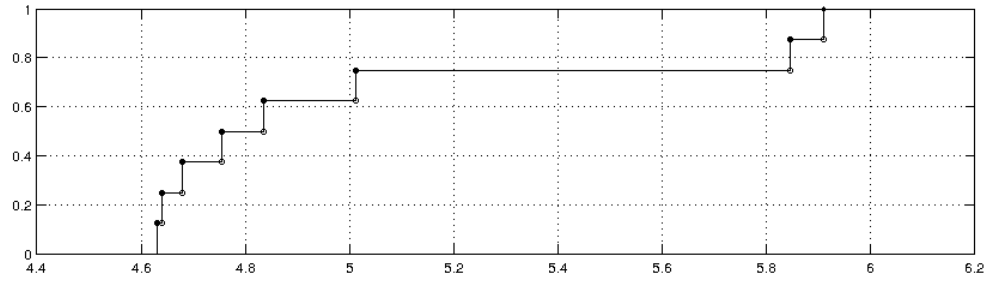


Figure 1: Empirical Distribution Function

Answer (Ex. 1) —

(a) sample mean = 5.0384

(b) sample variance and sample standard deviation are: 0.2837, 0.5326, respectively.

(c) order statistics is:

$$(4.6310, 4.6400, 4.6800, 4.7550, 4.8350, 5.0110, 5.8450, 5.9100)$$

(d) Empirical Distribution Function is given in Fig. 1.

Answer (Ex. 2) —

(a) Using the CLT and the given fact:

$$P(|\bar{X}_n - E(X_1)| < 0.10) = 0.95$$

$$P(-0.10 < \bar{X}_n - E(X_1) < 0.10) = 0.95$$

(1p)

$$P\left(\frac{-0.10}{\sqrt{V(X_1)/n}} < \frac{\bar{X}_n - E(X_1)}{\sqrt{V(X_1)/n}} < \frac{0.10}{\sqrt{V(X_1)/n}}\right) = 0.95$$

$$P\left(\frac{-0.10}{\sqrt{1/4n}} < Z < \frac{0.10}{\sqrt{1/4n}}\right) = 0.95$$

(1p)

We want $1 - \alpha = 0.95$, and from the standard Normal Table we know that the corresponding $z_{\alpha/2} = 1.96$. So, $\frac{0.10}{\sqrt{1/4n}} = z_{\alpha/2} = 1.96$

(1p)

and therefore we can get the right sample size n as follows:

$$\begin{aligned} n &= \left((\sqrt{1/4} \times 1.96) / (1/10) \right)^2 \\ &= (((1/2) \times 1.96) / (1/10))^2 = (0.98 \times 10)^2 = 9.8^2 = 96.04 \end{aligned}$$

Finally, by rounding 96.04 up to the next largest integer we need $n = 97$ measurements to meet the specifications of your boss (at least up to the approximation provided by the CLT).

(1p)

(b)

$$E(X_i; \lambda) = \lambda = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

The Method of Moment Estimator of λ is \bar{X}_n .

(1p)

Answer (Ex. 3) — Step 1: If $x_i \in (0, 1)$ for each $i \in \{1, 2, \dots, n\}$, i.e. when each data point lies inside the open interval $(0, 1)$, the log-likelihood is

$$\begin{aligned} \ell(\theta) &= \sum_{i=1}^n \log(f_X(x_i; \theta)) = \sum_{i=1}^n \left(\log(\theta x_i^{\theta-1}) \right) = \sum_{i=1}^n \left(\log(\theta) + \log(x_i^{\theta-1}) \right) \\ &= \sum_{i=1}^n (\log(\theta) + (\theta - 1)(\log(x_i))) = \sum_{i=1}^n (\log(\theta) + \theta \log(x_i) - \log(x_i)) \\ &= \sum_{i=1}^n \log(\theta) + \sum_{i=1}^n \theta \log(x_i) - \sum_{i=1}^n \log(x_i) = n \log(\theta) + \theta \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \log(x_i) \end{aligned}$$

Step 2:

$$\frac{d}{d\theta}(\ell(\theta)) = \frac{d}{d\theta} \left(n \log(\theta) + \theta \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \log(x_i) \right) = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i) - 0 = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i)$$

Step 3:

$$\frac{d}{d\theta}(\ell(\theta)) = 0 \iff \frac{n}{\theta} + \sum_{i=1}^n \log(x_i) = 0 \iff \frac{n}{\theta} = - \sum_{i=1}^n \log(x_i) \iff \theta = - \frac{n}{\sum_{i=1}^n \log(x_i)}$$

Let

$$\hat{\theta}_n = -\frac{n}{\sum_{i=1}^n \log(x_i)} .$$

Step 4:

$$\begin{aligned} \frac{d^2}{d\theta^2} \ell(\theta) &= \frac{d}{d\theta} \left(\frac{d}{d\theta} (\ell(\theta)) \right) = \frac{d}{d\theta} \left(\frac{n}{\theta} + \sum_{i=1}^n \log(x_i) \right) = \frac{d}{d\theta} \left(n\theta^{-1} + \sum_{i=1}^n \log(x_i) \right) \\ &= -n\theta^{-2} + 0 = -\frac{n}{\theta^2} \end{aligned}$$

Step 5: The problem states that $\theta > 0$. Since $\theta^2 > 0$ and $n \geq 1$, we have indeed checked that

$$\frac{d^2}{d\theta^2} \ell(\theta) = -\frac{n}{\theta^2} < 0$$

and therefore the MLE is indeed

$$\hat{\theta}_n = \frac{-n}{\sum_{i=1}^n \log(x_i)} .$$

Answer (Ex. 4) — (a)— 2p —

$$\begin{aligned} \ell(\sigma) &:= \log(L(\sigma)) := \log(L(x_1, x_2, \dots, x_n; \sigma)) = \log \left(\prod_{i=1}^n f(x_i; \sigma) \right) = \sum_{i=1}^n \log(f(x_i; \sigma)) \\ &= \sum_{i=1}^n \log \left(\frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{1}{2\sigma^2} (x_i - \mu)^2 \right) \right) \\ &= \sum_{i=1}^n \left(\log \left(\frac{1}{\sigma\sqrt{2\pi}} \right) + \log \left(\exp \left(-\frac{1}{2\sigma^2} (x_i - \mu)^2 \right) \right) \right) \\ &= \sum_{i=1}^n \log \left(\frac{1}{\sigma\sqrt{2\pi}} \right) + \sum_{i=1}^n \left(-\frac{1}{2\sigma^2} (x_i - \mu)^2 \right) = n \log \left(\frac{1}{\sigma\sqrt{2\pi}} \right) + \left(-\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \\ &= n \left(\log \left(\frac{1}{\sqrt{2\pi}} \right) + \log \left(\frac{1}{\sigma} \right) \right) - \left(\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \\ &= n \log \left(\sqrt{2\pi}^{-1} \right) + n \log \left(\sigma^{-1} \right) - \left(\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \\ &= -n \log \left(\sqrt{2\pi} \right) - n \log(\sigma) - \left(\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \end{aligned}$$

(b)— $2p -$

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \ell(\sigma) &:= \frac{\partial}{\partial \sigma} \left(-n \log(\sqrt{2\pi}) - n \log(\sigma) - \left(\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \right) \\
&= \frac{\partial}{\partial \sigma} \left(-n \log(\sqrt{2\pi}) \right) - \frac{\partial}{\partial \sigma} (n \log(\sigma)) - \frac{\partial}{\partial \sigma} \left(\left(\frac{1}{2\sigma^2} \right) \sum_{i=1}^n (x_i - \mu)^2 \right) \\
&= 0 - n \frac{\partial}{\partial \sigma} (\log(\sigma)) - \left(\frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2 \right) \frac{\partial}{\partial \sigma} (\sigma^{-2}) \\
&= -n\sigma^{-1} - \left(\frac{1}{2} \sum_{i=1}^n (x_i - \mu)^2 \right) (-2\sigma^{-3}) = -n\sigma^{-1} + \sigma^{-3} \sum_{i=1}^n (x_i - \mu)^2
\end{aligned}$$

(c)— $2p -$

$$\begin{aligned}
0 = -n\sigma^{-1} + \sigma^{-3} \sum_{i=1}^n (x_i - \mu)^2 &\iff n\sigma^{-1} = \sigma^{-3} \sum_{i=1}^n (x_i - \mu)^2 \iff n\sigma^{-1}\sigma^{+3} = \sum_{i=1}^n (x_i - \mu)^2 \\
&\iff n\sigma^{-1+3} = \sum_{i=1}^n (x_i - \mu)^2 \iff n\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 \\
&\iff \sigma^2 = \left(\sum_{i=1}^n (x_i - \mu)^2 \right) / n \iff \sigma = \sqrt{\sum_{i=1}^n (x_i - \mu)^2 / n}
\end{aligned}$$

So MLE $\widehat{\sigma}_n = \sqrt{\sum_{i=1}^n (x_i - \mu)^2 / n}$.

(d)— $2p -$ The Log-likelihood function of σ , based on one sample from the $Normal(\mu, \sigma^2)$ RV with known μ is,

$$\log f(x; \sigma) = \log \left(\frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{1}{2\sigma^2} (x - \mu)^2 \right) \right) = -\log(\sqrt{2\pi}) - \log(\sigma) - \left(\frac{1}{2\sigma^2} \right) (x - \mu)^2$$

Therefore, in much the same way as in part (2) earlier,

$$\begin{aligned}
\frac{\partial^2 \log f(x; \sigma)}{\partial^2 \sigma} &:= \frac{\partial}{\partial \sigma} \left(\frac{\partial}{\partial \sigma} \left(-\log(\sqrt{2\pi}) - \log(\sigma) - \left(\frac{1}{2\sigma^2} \right) (x - \mu)^2 \right) \right) \\
&= \frac{\partial}{\partial \sigma} \left(-\sigma^{-1} + \sigma^{-3} (x - \mu)^2 \right) = \sigma^{-2} - 3\sigma^{-4} (x - \mu)^2
\end{aligned}$$

Now, we compute the Fisher Information of one sample as an expectation

of the continuous RV X over $(-\infty, \infty)$ with density $f(x; \sigma)$,

$$\begin{aligned}
I_1(\sigma) &= - \int_{x \in (-\infty, \infty)} \left(\frac{\partial^2 \log f(x; \sigma)}{\partial^2 \lambda} \right) f(x; \sigma) dx = - \int_{-\infty}^{\infty} (\sigma^{-2} - 3\sigma^{-4}(x - \mu)^2) f(x; \sigma) dx \\
&= \int_{-\infty}^{\infty} -\sigma^{-2} f(x; \sigma) dx + \int_{-\infty}^{\infty} 3\sigma^{-4}(x - \mu)^2 f(x; \sigma) dx \\
&= -\sigma^{-2} \int_{-\infty}^{\infty} f(x; \sigma) dx + 3\sigma^{-4} \int_{-\infty}^{\infty} (x - \mu)^2 f(x; \sigma) dx \\
&= -\sigma^{-2} + 3\sigma^{-4}\sigma^2 \quad \because \sigma^2 = V(X) = E(X - E(X))^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x; \sigma) dx \\
&= -\sigma^{-2} + 3\sigma^{-4+2} = -\sigma^{-2} + 3\sigma^{-2} = 2\sigma^{-2}
\end{aligned}$$

Therefore, the estimated standard error of the estimator of the unknown σ is

$$\frac{1}{\sqrt{I_n(\hat{\sigma}_n)}} = \frac{1}{\sqrt{nI_1(\hat{\sigma}_n)}} = \frac{1}{\sqrt{n2(\hat{\sigma}_n)^{-2}}} = \frac{\hat{\sigma}_n}{\sqrt{2n}}.$$

(e)— $2p$ –

$$\widehat{\text{se}}_n(\hat{\Psi}_n) = |g'(\sigma)|\widehat{\text{se}}_n(\hat{\sigma}_n) = \left| \frac{\partial}{\partial \sigma} \log(\sigma) \right| \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sigma} \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sqrt{2n}}.$$

(f)— $2p$ – Finally, the 95% confidence interval for ψ is

$$\hat{\psi}_n \pm 1.96\widehat{\text{se}}_n(\hat{\Psi}_n) = \log(\hat{\sigma}_n) \pm 1.96 \frac{1}{\sqrt{2n}}.$$

(g)— $1p$ – Since $n = 110$ and $\hat{\sigma}_n = 12.4$, the $(1 - \alpha) = 95\%$ confidence interval for ψ is

$$\log(\hat{\sigma}_n) \pm 1.96 \frac{1}{\sqrt{2n}} = \log(12.4) \pm 1.96 \frac{1}{\sqrt{2 \times 110}} = 0.8754 \pm 0.132 = [0.7433, 1.0076]$$

Since $(1 - \alpha) = 95\%$ confidence interval for ψ does not contain 10.0, we reject $H_0 : \psi = 10.0$ in favour of $H_1 : \psi \neq 10.0$ using a size $\alpha = 0.05$ Wald test.

(h)— $1p$ – The 95% confidence interval for σ based on $n = 110$ samples and a sample standard deviation of 12.4 is:

$$\hat{\sigma}_n \pm 1.96 \frac{\hat{\sigma}_n}{\sqrt{2n}} = 12.4 \pm 1.96 \frac{12.4}{\sqrt{2 \times 110}} = [10.76, 14.04]$$

- (i) — $1p$ — Since the 95% confidence interval for σ does contain 10.0, we fail to reject the null hypothesis $H_0 : \sigma = 10.0$ versus $H_1 : \sigma \neq 10.0$ using a size $\alpha = 0.05$ Wald test.

Answer (Ex. 5) — — $1p$ — First set up what you need to show:

We need to show that: $P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n | \bar{X}_n)$ is independent of λ for any $(x_1, x_2, \dots, x_n) \in \{0, 1, 2, \dots\}^n$ and any $\bar{x}_n \geq 0$.

— $1p$ — Realising the following equality

$$P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n | \bar{X}_n) = \frac{P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, \bar{X}_n = \bar{x}_n)}{P_\lambda(\bar{X}_n = \bar{x}_n)}$$

— $1p$ — Realising $n\bar{X}_n = \sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$ and noting that

$$P_\lambda(n\bar{X}_n = n\bar{x}_n) = e^{-n\lambda} \frac{(n\lambda)^{\sum_{i=1}^n x_i}}{(\sum_{i=1}^n x_i)!} = e^{-n\lambda} \frac{(n\lambda)^k}{k!}, \quad k := n\bar{x}_n$$

— $1p$ — Realising the last equality below, with say $k = n\bar{x}_n \geq 0$

$$\frac{P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, \bar{X}_n = \bar{x}_n)}{P_\lambda(\bar{X}_n = \bar{x}_n)} = \frac{P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, n\bar{X}_n = k)}{P_\lambda(n\bar{X}_n = k)} = \frac{P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)}{P_\lambda(n\bar{X}_n = k)}$$

— $1p$ — Realising the following equality

$$\frac{P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)}{P_\lambda(n\bar{X}_n = k)} = \frac{\prod_{i=1}^n \left(e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right)}{e^{-n\lambda} \frac{(n\lambda)^k}{k!}} = \frac{e^{-n\lambda} \prod_{i=1}^n \frac{\lambda^{x_i}}{x_i!}}{e^{-n\lambda} \frac{(n\lambda)^k}{k!}}$$

— $1p$ — Finally, showing that the conditional probability is independent of parameter explicitly:

$$P_\lambda(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n | \bar{X}_n) = \frac{k!}{n^k \prod_{i=1}^n x_i!} = \left(\sum_{i=1}^n x_i \right)! \left(n^{\sum_{i=1}^n x_i} \prod_{i=1}^n x_i! \right)^{-1}$$

Answer (Ex. 6) — (a) — $1p$ — For writing what needs to be shown:

Fix any x . Bias is the expected value of the estimator, so we need to show that $E(\hat{F}_n(x)) - F(x) \rightarrow 0$ as $n \rightarrow \infty$.

And $1p$ — For showing:

$$\begin{aligned} E(\hat{F}_n(x)) &= E(n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x)) = n^{-1} \sum_{i=1}^n E(\mathbf{1}(X_i \leq x)) \\ &= n^{-1} \sum_{i=1}^n P(X_i \leq x) = n^{-1} \sum_{i=1}^n F(x) = F(x). \end{aligned}$$

Thus $\hat{F}_n(x)$ is an unbiased estimator of $F(x)$ for any x .

- (b)– $2p$ — Realise that $Y_i := \mathbf{1}(X_i \leq x) \sim \text{Bernoulli}(\theta = F(x))$ RV and use IID sum of Bernoulli RVs

$$V(\hat{F}_n(x)) = V\left(n^{-1} \sum_{i=1}^n Y_i\right) = V\left(\sum_{i=1}^n n^{-1} Y_i\right) = n^{-2} \sum_{i=1}^n V(X)$$

Since variance of a *Bernoulli*(θ) RV is $\theta(1-\theta)$, with $\theta = F(x)$ for us now, we get the following continuation of equalities from the previous line:

$$n^{-2} \sum_{i=1}^n V(X) = n^{-2} n V(X) = n^{-1} V(X) = \frac{V(X)}{n} = \frac{F(x)(1-F(x))}{n}$$

- (c)– $1p$ — Realizing IID likelihood is constant over the permutations of the m data points:

$X_1, \dots, X_m \stackrel{IID}{\sim} F \implies$, i.e., implies, the following equality:

$$P(X_1 = x_1, \dots, X_m = x_m) = \prod_{i=1}^m P(X_i = x_i) = \prod_{i=1}^m P(X_i = x_{\pi(i)})$$

And since there are $m!$ possible permutations and each of them is equally likely gives $P(X_1 = x_1, X_2 = x_2, \dots, X_m = x_m) = P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \dots, X_m = x_{\pi(m)}) = 1/m!$. [To properly understand what is going on here, you may want to recall the scribed example from last couple weeks' lectures on (nonparametric and exact) permutation test with 2 samples from one and 1 sample from a possibly different population with the probabilities being equally $1/6$ for all $6 = (2+1)!$ permutations.]