Time: 08.00-13.00. Total sum: 40p. Grades 3, 4 and 5 require 18p, 25p and 32p, respectively. Note:

- $\boxed{\text{Motivate solutions well but no more than necessary.}}$
- Just write on one side of the paper.
- Start new question on new page, and label the question number clearly next to your solution.
- Do not write with red-ink pen.
- Write the solutions in increasing order of question numbers.
- Hints and other information provided in a problem may also be useful for subsequent problems.

Permitted aids: Any books, notes, and pocket calculator.

Ex. 1 — *5p* – Eight earthquakes with the largest magnitude (in Richter scale) that hit Christchurch, NZ on 22nd February 2011 are:

$$5.8450, 4.6800, 5.9100, 5.0110, 4.7550, 4.8350, 4.6310, 4.6400$$

(a)— *1p* – Report the sample mean.

(b)— *1p* – Report the sample variance and sample standard deviation.

(c)— *1p* – Report the eight order statistics, as a point in $\mathbb{R}^8$, from minimum to maximum.

(d)— *2p* – Sketch the Empirical Distribution Function showing discontinuities in the function and clearly labeling the axes.

Ex. 2 — *5p* – Suppose you plan to obtain an independent and identically distributed (IID) sequence of n measurements from an instrument. This instrument has been calibrated so that the distribution of measurements made with it have population variance of $1/4$. Possibly useful fact: If $1-\alpha = 0.95$ and $Z \sim N(0,1)$ is the standard Normal RV, then from the Z Table, $z_{\alpha/2} = 1.96$, where, $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$. If $X \sim Poisson(\lambda)$ then $V(X) = E(X) = \lambda$]

(a)— *4p* – Your boss wants you to make a point estimate of the unknown population mean from an IID sample of size n. He also insists that the tolerance for error has to be $1/10$ and the probability of meeting this tolerance should be just above 95%. Use the Central Limit Theorem to

THIS IS ONLY A SAMPLE - FINAL EXAM WILL BE BASED ON ANY MATERIAL COVERED IN LECTURES AS PER SYLLABUS find how large should n be to meet the specifications of your boss. In summary,

$$X_1, X_2, \ldots, X_n \overset{IID}{\sim} X_1, \quad V(X_1) = 1/4$$

Find n such that $P(|\bar{X}_n - E(X_1)| < 1/10) = 0.95$.

(b)— *1p* – Making the further assumption that the IID samples are Poisson distributed random variables, find the Method of Moments Estimate for the mean parameter λ of the $Poisson(\lambda)$ RV X_i whose probability mass function is given by:

$$f(x;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k \in \{0,1,2,\ldots\}, \quad \lambda > 0$$

Ex. 3 — *5 p* – Assume that an independent and identically distributed sample, $X_1, X_2, \ldots, X_n$ is drawn from the distribution of X with PDF $f(x;\theta)$:

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

for a fixed and unknown parameter $\theta \in (0, \infty)$ and derive the maximum likelihood estimate of θ.

Hint: you only need to do Steps 1–5 to find the MLE: (Step 1:) find $\ell(\theta)$, the log-likelihood as a function of the parameter θ, (Step 2:) find $\frac{d}{d\theta}\ell(\theta)$, the first derivative of $\ell(\theta)$ with respect to θ, (Step 3:) solve the equation $\frac{d}{d\theta}\ell(\theta) = 0$ for θ and set this solution equal to $\widehat{\theta}_n$, (Step 4:) find $\frac{d^2}{d\theta^2}\ell(\theta)$, the second derivative of $\ell(\theta)$ with respect to θ and finally (Step 5:) $\widehat{\theta}_n$ is the MLE if $\frac{d^2}{d\theta^2}\ell(\theta) < 0$.

Ex. 4 — *15p* – Let $X_1, X_2, \ldots, X_n \overset{IID}{\sim} Normal(\mu, \sigma^2)$. Suppose that μ is known and σ is unknown. The parameter of interest is $\psi = \log(\sigma)$. Answer the following:

(1)— *2p* – Find the log-likelihood function $\ell(\sigma)$

(2)— *2p* – Find its derivative with respect to the unknown parameter σ

(3)— *2p* – Set the derivative equal to 0 and solve for σ

(4)— *2p* – Find the estimated standard error $\widehat{se}_n$ for the estimator of σ via Fisher Information

(5)— *2p* – Derive the estimated standard error of $\psi = \log(\sigma)$ via the Delta method

(6)— *2p* – Obtain the 95% confidence interval for ψ

(7)— *1p* – Suppose you observed $n = 110$ samples and a sample standard deviation of 12.4, Will you reject or fail to reject the null hypothesis $H_0 : \psi = 10.0$ versus $H_1 : \psi \neq 10.0$ using a size $\alpha = 0.05$ Wald test?

(8)— *1p* – Obtain the 95% confidence interval for σ based on $n = 110$ samples and a sample standard deviation of 12.4

(9)— *1p* – Will you reject or fail to reject the null hypothesis $H_0 : \sigma = 10.0$ versus $H_1 : \sigma \neq 10.0$ using a size $\alpha = 0.05$ Wald test?

Ex. 5 — *5p* – Let $X_1, X_2, \ldots, X_n \overset{IID}{\sim} Poisson(\lambda)$ where $\lambda = E(X_i) > 0$ is unknown. Show that the sample mean $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ is a sufficient statistic.

Ex. 6 — *5p* – Suppose $X_1, X_2, \ldots, X_m \overset{IID}{\sim} F$ where F is any distribution function of a real-valued random variable. Recall that a permutation can be defined as a bijection from a set onto itself, i.e., $\pi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$.

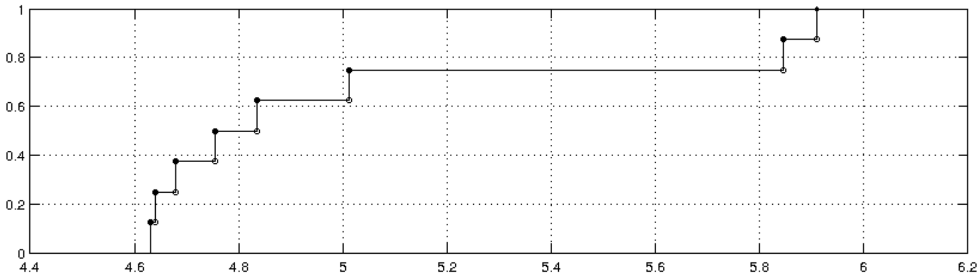
(a)– *2p* — Show that $\widehat{F}_n(x) := n^{-1} \sum_{i=1}^{n} \mathbf{1}(X_i \leq x)$ is an unbiased estimator of $F(x)$, for each $x \in \mathbb{R}$, where,

$$\mathbf{1}(X_i \leq x) = \begin{cases} 1 \text{ if } X_i \leq x \\ 0 \text{ if } X_i > x \end{cases}$$

(b)– *2p* — What is the Variance of the estimator $\widehat{F}_n(x)$ of $F(x)$, for each $x \in \mathbb{R}$.

(c)– *1p* — Why is $P(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m) = P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \ldots, X_m = x_{\pi(m)}) = 1/m!$? Explain your answer step by step.

Lycka till!



Figur 1: Empirical Distribution Function

Answer (Ex. 1) —

(a)sample mean $= 5.0384$

(b)sample variance and sample standard deviation are: $0.2837, 0.5326$, respectively.

(c)order statistics is:

$$(\ 4.6310, 4.6400, 4.6800, 4.7550, 4.8350, 5.0110, 5.8450, 5.9100\ )$$

(d)Empirical Distribution Function is given in Fig. 1.

Answer (Ex. 2) —

(a)Using the CLT and the given fact:

$$\begin{aligned} P(|\bar{X}_n - E(X_1)| < 0.10) &= 0.95 \\ P(-0.10 < \bar{X}_n - E(X_1) < 0.10) &= 0.95 \end{aligned}$$

(1p)

$$\begin{aligned} P\left(\frac{-0.10}{\sqrt{V(X_1)/n}} < \frac{\bar{X}_n - E(X_1)}{\sqrt{V(X_1)/n}} < \frac{0.10}{\sqrt{V(X_1)/n}}\right) &= 0.95 \\ P\left(\frac{-0.10}{\sqrt{1/4n}} < Z < \frac{0.10}{\sqrt{1/4n}}\right) &= 0.95 \end{aligned}$$

(1p)

We want $1-\alpha = 0.95$, and from the standard Normal Table we know that the corresponding $z_{\alpha/2} = 1.96$. So, $\frac{0.10}{\sqrt{1/4n}} = z_{\alpha/2} = 1.96$ (1p)

and therefore we can get the right sample size n as follows:

$$\begin{aligned} n &= \left((\sqrt{1/4} \times 1.96)/(1/10)\right)^2 \\ &= (((1/2) \times 1.96)/(1/10))^2 = (0.98 \times 10)^2 = 9.8^2 = 96.04 \end{aligned}$$

Finally, by rounding 96.04 up to the next largest integer we need $n = 97$ measurements to meet the specifications of your boss (at least up to the approximation provided by the CLT). (1p)

(b)

$$E(X_i; \lambda) = \lambda = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The Method of Moment Estimator of λ is $\bar{X}_n$. (1p)

Answer (Ex. 3) — Step 1: If $x_i \in (0,1)$ for each $i \in \{1, 2, \ldots, n\}$, i.e. when each data point lies inside the open interval $(0,1)$, the log-likelihood is

$$\begin{aligned} \ell(\theta) &= \sum_{i=1}^{n} \log(f_X(x_i;\theta)) = \sum_{i=1}^{n}\left(\log\left(\theta x_i^{\theta-1}\right)\right) = \sum_{i=1}^{n}\left(\log(\theta) + \log\left(x_i^{\theta-1}\right)\right) \\ &= \sum_{i=1}^{n}(\log(\theta) + (\theta-1)(\log(x_i))) = \sum_{i=1}^{n}(\log(\theta) + \theta\log(x_i) - \log(x_i)) \end{aligned}$$

$$= \sum_{i=1}^{n}\log(\theta) + \sum_{i=1}^{n}\theta\log(x_i) - \sum_{i=1}^{n}\log(x_i) = n\log(\theta) + \theta\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log(x_i)$$

Step 2:

$$\frac{d}{d\theta}(\ell(\theta)) = \frac{d}{d\theta}\left(n\log(\theta) + \theta\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log(x_i)\right) = \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i) - 0 = \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i)$$

Step 3:

$$\frac{d}{d\theta}(\ell(\theta)) = 0 \iff \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i) = 0 \iff \frac{n}{\theta} = -\sum_{i=1}^{n}\log(x_i) \iff \theta = -\frac{n}{\sum_{i=1}^{n}\log(x_i)}$$

Let

$$\widehat{\theta}_n = -\frac{n}{\sum_{i=1}^n \log(x_i)} \quad .$$

Step 4:

$$\frac{d^2}{d\theta^2}\ell(\theta) = \frac{d}{d\theta}\left(\frac{d}{d\theta}(\ell(\theta))\right) = \frac{d}{d\theta}\left(\frac{n}{\theta} + \sum_{i=1}^n \log(x_i)\right) = \frac{d}{d\theta}\left(n\theta^{-1} + \sum_{i=1}^n \log(x_i)\right)$$
$$= -n\theta^{-2} + 0 = -\frac{n}{\theta^2}$$

Step 5: The problem states that $\theta > 0$. Since $\theta^2 > 0$ and $n \geq 1$, we have indeed checked that

$$\frac{d^2}{d\theta^2}\ell(\theta) = -\frac{n}{\theta^2} < 0$$

and therefore the MLE is indeed

$$\widehat{\theta}_n = \frac{-n}{\sum_{i=1}^n \log(x_i)} \quad .$$

Answer (Ex. 4) — (a)— 2p –

$$\ell(\sigma) := \log(L(\sigma)) := \log(L(x_1, x_2, \ldots, x_n; \sigma)) = \log\left(\prod_{i=1}^n f(x_i; \sigma)\right) = \sum_{i=1}^n \log(f(x_i; \sigma))$$
$$= \sum_{i=1}^n \log\left(\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right)\right)$$
$$= \sum_{i=1}^n \left(\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \log\left(\exp\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right)\right)\right)$$
$$= \sum_{i=1}^n \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \sum_{i=1}^n\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right) = n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \left(-\frac{1}{2\sigma^2}\right)\sum_{i=1}^n (x_i - \mu)^2$$
$$= n\left(\log\left(\frac{1}{\sqrt{2\pi}}\right) + \log\left(\frac{1}{\sigma}\right)\right) - \left(\frac{1}{2\sigma^2}\right)\sum_{i=1}^n (x_i - \mu)^2$$
$$= n\log\left(\sqrt{2\pi}^{-1}\right) + n\log\left(\sigma^{-1}\right) - \left(\frac{1}{2\sigma^2}\right)\sum_{i=1}^n (x_i - \mu)^2$$
$$= -n\log\left(\sqrt{2\pi}\right) - n\log(\sigma) - \left(\frac{1}{2\sigma^2}\right)\sum_{i=1}^n (x_i - \mu)^2$$

(b)— *2p* –

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\ell(\sigma) &:= \frac{\partial}{\partial\sigma}\left(-n\log\left(\sqrt{2\pi}\right) - n\log(\sigma) - \left(\frac{1}{2\sigma^2}\right)\sum_{i=1}^{n}(x_i-\mu)^2\right) \\
&= \frac{\partial}{\partial\sigma}\left(-n\log\left(\sqrt{2\pi}\right)\right) - \frac{\partial}{\partial\sigma}(n\log(\sigma)) - \frac{\partial}{\partial\sigma}\left(\left(\frac{1}{2\sigma^2}\right)\sum_{i=1}^{n}(x_i-\mu)^2\right) \\
&= 0 - n\frac{\partial}{\partial\sigma}(\log(\sigma)) - \left(\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)\frac{\partial}{\partial\sigma}\left(\sigma^{-2}\right) \\
&= -n\sigma^{-1} - \left(\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)\left(-2\sigma^{-3}\right) = -n\sigma^{-1} + \sigma^{-3}\sum_{i=1}^{n}(x_i-\mu)^2
\end{aligned}$$

(c)— *2p* –

$$\begin{aligned}
0 = -n\sigma^{-1} + \sigma^{-3}\sum_{i=1}^{n}(x_i-\mu)^2 \iff n\sigma^{-1} = \sigma^{-3}\sum_{i=1}^{n}(x_i-\mu)^2 &\iff n\sigma^{-1}\sigma^{+3} = \sum_{i=1}^{n}(x_i-\mu)^2 \\
&\iff n\sigma^{-1+3} = \sum_{i=1}^{n}(x_i-\mu)^2 \iff n\sigma^2 = \sum_{i=1}^{n}(x_i-\mu)^2 \\
&\iff \sigma^2 = \left(\sum_{i=1}^{n}(x_i-\mu)^2\right)/n \iff \sigma = \sqrt{\sum_{i=1}^{n}(x_i-\mu)^2/n}
\end{aligned}$$

So MLE $\widehat{\sigma_n} = \sqrt{\sum_{i=1}^{n}(x_i-\mu)^2/n}$.

(d)— *2p* – The Log-likelihood function of σ, based on one sample from the $Normal(\mu,\sigma^2)$ RV with known μ is,

$$\log f(x;\sigma) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)\right) = -\log\left(\sqrt{2\pi}\right) - \log(\sigma) - \left(\frac{1}{2\sigma^2}\right)(x-\mu)^2$$

Therefore, in much the same way as in part (2) earlier,

$$\begin{aligned}
\frac{\partial^2 \log f(x;\sigma)}{\partial^2\sigma} &:= \frac{\partial}{\partial\sigma}\left(\frac{\partial}{\partial\sigma}\left(-\log\left(\sqrt{2\pi}\right) - \log(\sigma) - \left(\frac{1}{2\sigma^2}\right)(x-\mu)^2\right)\right) \\
&= \frac{\partial}{\partial\sigma}\left(-\sigma^{-1} + \sigma^{-3}(x-\mu)^2\right) = \sigma^{-2} - 3\sigma^{-4}(x-\mu)^2
\end{aligned}$$

Now, we compute the Fisher Information of one sample as an expectation of the continuous RV X over $(-\infty, \infty)$ with density $f(x;\sigma)$,

$$
\begin{aligned}
I_1(\sigma) &= -\int_{x \in (-\infty,\infty)} \left(\frac{\partial^2 \log f(x;\sigma)}{\partial^2 \lambda} \right) f(x;\sigma)\ dx = -\int_{-\infty}^{\infty} \left(\sigma^{-2} - 3\sigma^{-4}(x-\mu)^2 \right) f(x;\sigma)\ dx \\
&= \int_{-\infty}^{\infty} -\sigma^{-2} f(x;\sigma)\ dx + \int_{-\infty}^{\infty} 3\sigma^{-4}(x-\mu)^2 f(x;\sigma)\ dx \\
&= -\sigma^{-2} \int_{-\infty}^{\infty} f(x;\sigma)\ dx + 3\sigma^{-4} \int_{-\infty}^{\infty} (x-\mu)^2 f(x;\sigma)\ dx \\
&= -\sigma^{-2} + 3\sigma^{-4}\sigma^2 \qquad \because \sigma^2 = V(X) = E(X - E(X))^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x;\sigma)\ dx \\
&= -\sigma^{-2} + 3\sigma^{-4+2} = -\sigma^{-2} + 3\sigma^{-2} = 2\sigma^{-2}
\end{aligned}
$$

Therefore, the estimated standard error of the estimator of the unknown σ is

$$\frac{1}{\sqrt{I_n(\widehat{\sigma}_n)}} = \frac{1}{\sqrt{n I_1(\widehat{\sigma}_n)}} = \frac{1}{\sqrt{n 2 (\widehat{\sigma}_n)^{-2}}} = \frac{\widehat{\sigma}_n}{\sqrt{2n}} \ .$$

(e)— *2p* –

$$\widehat{\mathsf{se}}_n(\widehat{\Psi}_n) = |g'(\sigma)| \widehat{\mathsf{se}}_n(\widehat{\sigma}_n) = \left| \frac{\partial}{\partial \sigma} \log(\sigma) \right| \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sigma} \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sqrt{2n}} \ .$$

(f)— *2p* – Finally, the 95% confidence interval for ψ is

$$\widehat{\psi}_n \pm 1.96 \widehat{\mathsf{se}}_n(\widehat{\Psi}_n) = \log(\widehat{\sigma}_n) \pm 1.96 \frac{1}{\sqrt{2n}}.$$

(g)— *1p* – Since $n = 110$ and $\widehat{\sigma}_n = 12.4$, the $(1-\alpha) = 95\%$ confidence interval for ψ is

$$\log(\widehat{\sigma}_n) \pm 1.96 \frac{1}{\sqrt{2n}} = \log(12.4) \pm 1.96 \frac{1}{\sqrt{2 \times 110}} = 0.8754 \pm 0.132 = [0.7433, 1.0076]$$

Since $(1-\alpha) = 95\%$ confidence interval for ψ does not contain 10.0, we reject $H_0 : \psi = 10.0$ in favour of $H_1 : \psi \neq 10.0$ using a size $\alpha = 0.05$ Wald test.

(h)— *1p* – The 95% confidence interval for σ based on $n = 110$ samples and a sample standard deviation of 12.4 is:

$$\widehat{\sigma}_n \pm 1.96 \frac{\widehat{\sigma}_n}{\sqrt{2n}} = 12.4 \pm 1.96 \frac{12.4}{\sqrt{2 \times 110}} = [10.76, 14.04]$$

(i)— *1p* – Since the 95% confidence interval for σ does not contain 10.0, we reject the null hypothesis $H_0 : \sigma = 10.0$ versus $H_1 : \sigma \neq 10.0$ using a size $\alpha = 0.05$ Wald test.

Answer (Ex. 5) — — *1p* – First set up what you need to show:
We need to show that: $P_\lambda(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|\bar{X}_n)$ is independent of λ for any $(x_1, x_2, \ldots, x_n) \in \{0, 1, 2, \ldots\}^n$ and any $\bar{x}_n \geq 0$.

— *1p* – Realising the following equality
$P_\lambda(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|\bar{X}_n) = \frac{P_\lambda(X_1=x_1,X_2=x_2,\ldots,X_n=x_n,\bar{X}_n=\bar{x}_n)}{P_\lambda(\bar{X}_n=\bar{x}_n)}$

— *1p* – Realising $n\bar{X}_n = \sum_{i=1}^n X_i \sim Poisson(n\lambda)$ and noting that
$P_\lambda(n\bar{X}_n = n\bar{x}_n) = e^{-n\lambda}\frac{(n\lambda)^{\sum_{i=1}^n x_i}}{(\sum_{i=1}^n x_i)!} = e^{-n\lambda}\frac{(n\lambda)^k}{k!}, \quad k := n\bar{x}_n$

— *1p* – Realising the last equality below, with say $k = n\bar{x}_n \geq 0$

$$\frac{P_\lambda(X_1=x_1,X_2=x_2,\ldots,X_n=x_n,\bar{X}_n=\bar{x}_n)}{P_\lambda(\bar{X}_n=\bar{x}_n)} = \frac{P_\lambda(X_1=x_1,X_2=x_2,\ldots,X_n=x_n,n\bar{X}_n=k)}{P_\lambda(n\bar{X}_n=k)} = \frac{P_\lambda(X_1=x_1,X_2=x_2,\ldots,X_n=x_n)}{P_\lambda(n\bar{X}_n=k)}$$

— *1p* – Realising the following equality

$$\frac{P_\lambda(X_1=x_1,X_2=x_2,\ldots,X_n=x_n)}{P_\lambda(n\bar{X}_n=k)} = \frac{\prod_{i=1}^n\left(e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}\right)}{e^{-n\lambda}\frac{(n\lambda)^k}{k!}} = \frac{e^{-n\lambda}\frac{\lambda^k}{\prod_{i=1}^n x_i!}}{e^{-n\lambda}\frac{(n\lambda)^k}{k!}}$$

— *1p* – Finally, showing that the conditional probability is independent of parameter explicitly:

$$P_\lambda(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|\bar{X}_n) = \frac{k!}{n^k\prod_{i=1}^n x_i!} = \left(\sum_{i=1}^n x_i\right)!\left(n^{(\sum_{i=1}^n x_i)}\prod_{i=1}^n x_i!\right)^{-1}$$

Answer (Ex. 6) — (a)– *1p* — For writing what needs to be shown:
Fix any x. Bias is the expected value of the estimator, so we need to show that $E(\widehat{F}_n(x)) - F(x) \to 0$ as $n \to \infty$.

And *1p* — For showing:
$E(\widehat{F}_n(x)) = E(n^{-1}\sum_{i=1}^n \mathbf{1}(X_i \leq x)) = n^{-1}\sum_{i=1}^n E(\mathbf{1}(X_i \leq x))$
$= n^{-1}\sum_{i=1}^n P(X_i \leq x) = n^{-1}\sum_{i=1}^n F(x) = F(x)$.

Thus $\widehat{F}_n(x)$ is an unbiased estimator of $F(x)$ for any x.

(b)– *2p* — Realise that $Y_i := \mathbf{1}(X_i \leq x) \sim Bernoulli(\theta = F(x))$ RV and use IID sum of Bernoulli RVs

$$V(\widehat{F}_n(x)) = V\left(n^{-1}\sum_{i=1}^{n} Y_i\right) = V\left(\sum_{i=1}^{n} n^{-1}Y_i\right) = n^{-2}\sum_{i=1}^{n} V(X)$$

Since variance of a $Bernoulli(\theta)$ RV is $\theta(1-\theta)$, with $\theta = F(x)$ for us now, we get the following continuation of equalities from the previous line:

$$n^{-2}\sum_{i=1}^{n} V(X) = n^{-2}nV(X) = n^{-1}V(X) = \frac{V(X)}{n} = \frac{F(x)(1-F(x))}{n}$$

(c)– *1p* — Realizing IID likelihood is constant over the permutations of the m data points:

$$X_1, \ldots, X_m \overset{IID}{\sim} F \implies \text{,i.e., implies, the following equality:}$$

$$P(X_1 = x_1, \ldots, X_m = x_m) = \prod_{i=1}^{m} P(X_i = x_i) = \prod_{i=1}^{m} P(X_i = x_{\pi(i)})$$

And since there are $m!$ possible permutations and each of them is equally likely gives $P(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m) = P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \ldots, X_m = x_{\pi(m)}) = 1/m!$. [To properly understand what is going on here, you may want to recall the scribed example from last couple weeks' lectures on (nonparametric and exact) permutation test with 2 samples from one and 1 sample from a possibly different population with the probabilities being equally 1/6 for all $6 = (2+1)!$ permutations.]