

Normalizations for the multi-species pedestal code

The drift-kinetic equation for each species may be written

$$\left(\nu_{\parallel} \mathbf{b} + \mathbf{v}_{d0} \right) \cdot \left(\nabla f_{a1} \right)_{\mu, W_a} - C_a \{ f_{a1} \} - S_a = -\mathbf{v}_m \cdot \nabla \psi \frac{\partial f_{a0}}{\partial \psi} \quad (1)$$

where $W_a = \nu^2 / 2 + Z_a e \Phi / m_a$,

$$f_{a0} = \frac{n_0}{\pi^{3/2} \nu_a} e^{-x_a^2}, \quad (2)$$

$\nu_a = \sqrt{2T_a / m_a}$, and $x_a = \nu / \nu_a$. Expanding the right-hand side of (1),

$$\begin{aligned} & \left(\nu_{\parallel} \mathbf{b} + \mathbf{v}_{d0} \right) \cdot \left(\nabla f_{a1} \right)_{\mu, W_a} - C_a \{ f_{a1} \} - S_a \\ &= \frac{m_a c I n_a}{2 \pi^{3/2} \nu_a Z_a e B^3} \mathbf{B} \cdot \nabla \theta \left(\frac{\partial B}{\partial \theta} \right) (1 + \xi^2) x^2 e^{-x^2} \left(\frac{1}{n_a} \frac{dn_a}{d\psi} + \frac{Z_a e}{T_a} \frac{d\Phi}{d\psi} + \left[x_a^2 - \frac{3}{2} \right] \frac{1}{T_a} \frac{dT_a}{d\psi} \right). \end{aligned} \quad (3)$$

Just as in the local code, we normalize using a position-independent temperature \bar{T} , density \bar{n} , potential $\bar{\Phi}$, magnetic field \bar{B} , and length \bar{R} . We take \bar{T} and \bar{n} to be the same for all species. We thereby obtain the quantities

$$\hat{B} = B / \bar{B} \quad (4)$$

$$\hat{I} = I / (\bar{R} \bar{B}) \text{ (where } I(\psi) = R B_{\text{toroidal}} \text{)} \quad (5)$$

$$\hat{\Phi} = \Phi / \bar{\Phi} \quad (6)$$

$$\hat{J} = \frac{\mathbf{B} \cdot \nabla \theta}{\bar{B} / \bar{R}} \quad (7)$$

and

$$\psi = \psi_N \hat{\psi}_a \bar{R}^2 \bar{B} \quad (8)$$

so

$$\frac{dX}{d\psi} = \frac{1}{\hat{\psi}_a \bar{R}^2 \bar{B}} \frac{dX}{d\psi_N} \quad (9)$$

for any function X . Also, for each species,

$$\hat{n}_a = n_a / \bar{n} \quad (10)$$

and

$$\hat{T}_a = T_a / \bar{T}. \quad (11)$$

We define a species-independent reference mass \bar{m} and reference thermal speed

$$\bar{\nu} = \sqrt{2\bar{T} / \bar{m}}. \quad (12)$$

Each species then has a normalized mass

$$\hat{m}_a = m_a / \bar{m} \quad (13)$$

and thermal speed

$$\nu_a = \bar{\nu} \sqrt{\hat{T}_a / \hat{m}_a}. \quad (14)$$

Two combinations of the normalization quantities will appear in the dimensionless equations:

$$\Delta = \frac{\bar{m}c\bar{v}}{e\bar{B}\bar{R}} \quad (15)$$

(which resembles $\rho_* = \rho / R$) and

$$\alpha = \frac{e\bar{\Phi}}{\bar{T}} \quad (16)$$

$$\omega = \frac{c\bar{\Phi}}{\bar{v}\bar{R}\bar{B}} = \frac{\Delta\alpha}{2}. \quad (17)$$

The normalized distribution function of each species \hat{f}_a is defined by

$$f_{a1} = \hat{f}_a \frac{\Delta\bar{n}}{\bar{v}^3}. \quad (18)$$

Notice the normalization is the same for all species.

To normalize the kinetic equation, we first multiply through by

$$\frac{\bar{v}^3}{\Delta\bar{n}} \frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \quad (19)$$

where the temperature dependence is included for historical reasons, for consistency with the single-species code. We then obtain

$$\frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} (\nu_{\parallel} \mathbf{b} + \mathbf{v}_{d0}) \cdot (\nabla \hat{f}_a)_{\mu, W_a} - \frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} C_a \{ \hat{f}_a \} - \hat{S}_a = \hat{R} \quad (20)$$

where

$$\hat{S}_a = \frac{\bar{v}^3}{\Delta\bar{n}} \frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} S_a = \frac{\bar{v}^2 \bar{R}}{\Delta\bar{n}} \sqrt{\hat{m}_a} S_a \quad (21)$$

is the normalized source, and

$$\hat{R} = \frac{\hat{m}_a^2}{2\pi^{3/2} Z_a \hat{T}_a^{1/2}} \frac{\hat{n}_a \hat{J} \hat{J}}{\hat{B}^3 \hat{\psi}_a} \left(\frac{\partial \hat{B}}{\partial \theta} \right) (1 + \xi^2) x^2 e^{-x^2} \left(\frac{1}{\hat{n}_a} \frac{d\hat{n}_a}{d\psi_N} + \frac{2Z_a}{\hat{T}_a} \frac{\omega}{\Delta} \frac{d\hat{\Phi}}{d\psi_N} + \left[x_a^2 - \frac{3}{2} \right] \frac{1}{\hat{T}_a} \frac{d\hat{T}_a}{d\psi_N} \right) \quad (22)$$

is the right-hand side.

We next change the independent variables in the drift-kinetic operator from $\{\psi, \theta, \mu, W_a\}$ to $\{\psi, \theta, x_a, \xi\}$, to re-write (20) in the form

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta} \frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi} \frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} C_a \{ \hat{f}_a \} - \hat{S}_a = \hat{R}. \quad (23)$$

We begin with the streaming term:

$$\frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \nu_{\parallel} \mathbf{b} \cdot \nabla \psi = 0 \quad (24)$$

$$\frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \nu_{\parallel} \mathbf{b} \cdot \nabla \theta = \frac{\hat{J} \sqrt{\hat{T}_a}}{\hat{B}} x_a \xi \quad (25)$$

$$\frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \nu_{\parallel} \mathbf{b} \cdot (\nabla x)_{W_a} = 0 \quad (26)$$

$$\frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \nu_{\parallel} \mathbf{b} \cdot (\nabla \xi)_{\mu, W_a} = -\sqrt{\hat{T}_a} x_a \frac{(1-\xi^2)}{2\hat{B}^2} \hat{J} \frac{d\hat{B}}{d\theta}. \quad (27)$$

Moving on to the drift terms, we begin with

$$\dot{\psi}_N = \frac{\bar{R}}{\nu_a} \sqrt{\hat{T}_a} \mathbf{v}_m \cdot \nabla \psi_N. \quad (28)$$

which gives

$$\dot{\psi}_N = -\frac{\hat{m}_a^{1/2} \Delta \hat{T}_a \hat{I} \hat{J}}{2Z_a \hat{\psi}_a} x_a^2 (1+\xi^2) \frac{1}{\hat{B}^3} \frac{\partial \hat{B}}{\partial \theta}. \quad (29)$$

This result differs by a factor $\hat{m}_a^{1/2} / Z_a$ from the single-species code.

Next, we evaluate the contribution to $\dot{\theta}$:

$$\mathbf{v}_{d0} \cdot \nabla \theta = \frac{cI}{B^2} (\mathbf{B} \cdot \nabla \theta) \frac{d\Phi}{d\psi} + \frac{m_a c \nu_{\parallel}^2}{Z_a e B} \nabla \times \mathbf{b} \cdot \nabla \theta + \frac{m_a c \nu_{\perp}^2}{2Z_a e B^3} \mathbf{B} \times \nabla B \cdot \nabla \theta. \quad (30)$$

We can re-use two results from the single-species calculation:

$$\mathbf{B} \times \nabla B \cdot \nabla \theta = \frac{\bar{B}^2}{\bar{R}^2} \frac{\hat{I} \hat{J}}{\hat{\psi}_a} \frac{\partial \hat{B}}{\partial \psi_N} \quad (31)$$

and

$$(\nabla \times \mathbf{b}) \cdot \nabla \theta = \frac{1}{\bar{R}^2} \frac{\hat{J}}{\hat{\psi}_a} \left[\frac{\hat{I}}{\hat{B}^2} \frac{\partial \hat{B}}{\partial \psi_N} - \frac{1}{\hat{B}} \frac{d\hat{I}}{d\psi_N} \right]. \quad (32)$$

Then (30) becomes

$$\mathbf{v}_{d0} \cdot \nabla \theta = \frac{\bar{\nu}}{\bar{R}} \left[\frac{\omega \hat{I} \hat{J}}{\hat{\psi}_a \hat{B}^2} \frac{d\Phi}{d\psi_N} + \frac{\Delta \hat{T}_a}{Z_a \hat{B}} x_a^2 \xi^2 \frac{\hat{J}}{\hat{\psi}_a} \left[\frac{\hat{I}}{\hat{B}^2} \frac{\partial \hat{B}}{\partial \psi_N} - \frac{1}{\hat{B}} \frac{d\hat{I}}{d\psi_N} \right] + \frac{\Delta \hat{T}_a}{2Z_a \hat{B}^3} x_a^2 (1-\xi^2) \frac{\hat{I} \hat{J}}{\hat{\psi}_a} \frac{\partial \hat{B}}{\partial \psi_N} \right]. \quad (33)$$

This result is equal to (59) in the single-species technical manual, but with an extra factor of $1/Z_a$ in each of the magnetic drift terms. Combining with (25), we obtain

$$\dot{\theta} = \frac{\hat{J} \sqrt{\hat{T}_a}}{\hat{B}} x_a \xi + \frac{\sqrt{\hat{m}_a} \omega \hat{I} \hat{J}}{\hat{\psi}_a \hat{B}^2} \frac{d\Phi}{d\psi_N} + \frac{\sqrt{\hat{m}_a} \Delta \hat{T}_a}{2Z_a \hat{B}^3} x_a^2 (1+\xi^2) \frac{\hat{I} \hat{J}}{\hat{\psi}_a} \frac{\partial \hat{B}}{\partial \psi_N} - \frac{\sqrt{\hat{m}_a} \Delta \hat{T}_a}{Z_a \hat{B}^2} x_a^2 \xi^2 \frac{\hat{J}}{\hat{\psi}_a} \frac{d\hat{I}}{d\psi_N}. \quad (34)$$

This equation is the generalization of (60) in the single-species technical manual, with the new factors colored red.

To obtain \dot{x}_a , we first write

$$W_0 = \frac{\bar{\nu}^2}{2} \frac{\hat{T}_a}{\hat{m}_a} x_a^2 + \frac{Z_a}{m_a} e \Phi. \quad (35)$$

Differentiating,

$$\dot{x}_a = -\frac{1}{\hat{T}_a x_a} \left[\frac{1}{2} x_a^2 \frac{d\hat{T}_a}{d\psi_N} + \frac{Z_a}{\bar{m} \bar{\nu}^2} e \frac{d\Phi}{d\psi_N} \right] \dot{\psi}_N. \quad (36)$$

Applying (28),

$$\dot{x}_a = \left[\frac{\Delta}{Z_a} \frac{x_a^3}{2} \frac{d\hat{T}_a}{d\psi_N} + \omega x_a \frac{d\hat{\Phi}}{d\psi_N} \right] \frac{\hat{m}_a^{1/2} \hat{I} \hat{J} (1 + \xi^2)}{\hat{\psi}_a \hat{B}^3} \frac{\partial \hat{B}}{\partial \theta}. \quad (37)$$

To obtain $\mathbf{v}_{d0} \cdot (\nabla \xi)_{\mu, W_a}$, we first write

$$2\mu B = \bar{v}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 (1 - \xi^2) \quad (38)$$

Differentiating,

$$\begin{aligned} & \left[\bar{v}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 (1 - \xi^2) \right] \frac{1}{B} \left(\frac{\partial B}{\partial \theta} d\theta + \frac{\partial B}{\partial \psi_N} d\psi_N \right) \\ &= -2\bar{v}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 \xi d\xi + 2x_a \bar{v}^2 \frac{\hat{T}_a}{\hat{m}_a} (1 - \xi^2) dx_a + \bar{v}^2 \frac{1}{\hat{m}_a} x_a^2 (1 - \xi^2) \frac{d\hat{T}_a}{d\psi_N} d\psi_N \end{aligned} \quad (39)$$

Solving for $d\xi$,

$$d\xi = (1 - \xi^2) \frac{1}{\xi x_a} dx_a + \left[\frac{1}{\hat{T}_a} \frac{d\hat{T}_a}{d\psi_N} - \frac{1}{\hat{B}} \frac{\partial \hat{B}}{\partial \psi_N} \right] \frac{1}{2\xi} (1 - \xi^2) d\psi_N - (1 - \xi^2) \frac{1}{2\xi \hat{B}} \frac{\partial \hat{B}}{\partial \theta} d\theta \quad (40)$$

Plugging in our previous results,

$$\begin{aligned} \dot{\xi} &= -(1 - \xi^2) \frac{1}{2\xi \hat{B}} \frac{\partial \hat{B}}{\partial \theta} \frac{\hat{J} \sqrt{\hat{T}_a}}{\hat{B}} x_a \xi \\ &+ \xi (1 - \xi^2) \frac{\hat{m}_a^{1/2} \omega \hat{I} \hat{J}}{\hat{\psi}_a \hat{B}^3} \frac{\partial \hat{B}}{\partial \theta} \frac{d\hat{\Phi}}{d\psi_N} \\ &+ (1 - \xi^2) \frac{1}{\xi x_a} \frac{\Delta}{Z_a} \frac{x_a^3}{2} \frac{d\hat{T}_a}{d\psi_N} \frac{\hat{m}_a^{1/2} \hat{I} \hat{J} (1 + \xi^2)}{\hat{\psi}_a \hat{B}^3} \frac{\partial \hat{B}}{\partial \theta} \\ &- \left[\frac{1}{\hat{T}_a} \frac{d\hat{T}_a}{d\psi_N} - \frac{1}{\hat{B}} \frac{\partial \hat{B}}{\partial \psi_N} \right] \frac{1}{4\xi} (1 - \xi^4) \frac{\hat{m}_a^{1/2} \Delta \hat{T}_a \hat{I} \hat{J}}{Z_a \hat{\psi}_a} x_a^2 \frac{1}{\hat{B}^3} \frac{\partial \hat{B}}{\partial \theta} \\ &- (1 - \xi^2) \frac{1}{2\xi \hat{B}} \frac{\partial \hat{B}}{\partial \theta} \left[\frac{\sqrt{\hat{m}_a} \Delta \hat{T}_a}{2Z_a \hat{B}^3} x_a^2 (1 + \xi^2) \frac{\hat{I} \hat{J}}{\hat{\psi}_a} \frac{\partial \hat{B}}{\partial \psi_N} - \frac{\sqrt{\hat{m}_a} \Delta \hat{T}_a}{Z_a \hat{B}^2} x_a^2 \xi^2 \frac{\hat{J}}{\hat{\psi}_a} \frac{d\hat{I}}{d\psi_N} \right] \end{aligned} \quad (41)$$

Two pairs of terms cancel, leaving

$$\dot{\xi} = -x_a (1 - \xi^2) \frac{\hat{J} \sqrt{\hat{T}_a}}{2\hat{B}^2} \frac{\partial \hat{B}}{\partial \theta} + \xi (1 - \xi^2) \frac{\hat{m}_a^{1/2} \omega \hat{I} \hat{J}}{2\hat{\psi}_a \hat{B}^3} \frac{\partial \hat{B}}{\partial \theta} \frac{d\hat{\Phi}}{d\psi_N} + \xi (1 - \xi^2) x_a^2 \frac{\sqrt{\hat{m}_a} \Delta \hat{T}_a \hat{J}}{2\hat{\psi}_a Z_a \hat{B}^3} \frac{d\hat{I}}{d\psi_N} \frac{\partial \hat{B}}{\partial \theta}. \quad (42)$$

Collision operator

The linearized collision operator may be written

$$C_{ab} = C_{ab}^L + C_{ab}^E + C_{ab}^F, \quad (43)$$

where the Lorentz part of the collision term is

$$C_{ab}^L = \frac{\nu_{Dab}}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f_{a1}}{\partial \xi} \quad (44)$$

with

$$\nu_{Dab} = \frac{\Gamma_{ab} n_b}{\nu^3} [\text{erf}(x_b) - \Psi(x_b)] \quad (45)$$

$$\Gamma_{ab} = \frac{4\pi Z_a^2 Z_b^2 e^4 \ln \Lambda}{m_a^2} \quad (46)$$

$$\Psi(x_b) = \frac{\text{erf}(x_b) - x_b \text{erf}'(x_b)}{2x_b^2}. \quad (47)$$

The energy scattering contribution is

$$C_{ab}^E = \nu_{\parallel ab} \left[\frac{\nu^2}{2} \frac{\partial^2 f_{a1}}{\partial \nu^2} - x_b^2 \left(1 - \frac{m_a}{m_b} \right) \nu \frac{\partial f_{a1}}{\partial \nu} \right] + \nu_{Dab} \nu \frac{\partial f_{a1}}{\partial \nu} + 4\pi \Gamma_{ab} \frac{m_a}{m_b} f_{Mb} f_{a1} \quad (48)$$

where

$$\nu_{\parallel ab} = 2 \frac{\Gamma_{ab} n_b}{\nu^3} \Psi(x_b). \quad (49)$$

The field term is

$$C_{ab}^F = \Gamma_{ab} f_{Ma} \left[\frac{2\nu^2}{\nu_a^4} \frac{\partial^2 G_{b1}}{\partial \nu^2} - \frac{2\nu}{\nu_a^2} \left(1 - \frac{m_a}{m_b} \right) \frac{\partial H_{b1}}{\partial \nu} - \frac{2}{\nu_a^2} H_{b1} + 4\pi \frac{m_a}{m_b} f_{b1} \right] \quad (50)$$

where the potentials are defined by

$$\nabla_\nu^2 H_{b1} = -4\pi f_{b1} \quad (51)$$

and

$$\nabla_\nu^2 G_{b1} = 2H_{b1}. \quad (52)$$

We write the field term as

$$C_{ab}^F = C_{ab}^H + C_{ab}^G + C_{ab}^D \quad (53)$$

where

$$C_{ab}^G = \Gamma_{ab} f_{Ma} \frac{2\nu^2}{\nu_a^4} \frac{\partial^2 G_{b1}}{\partial \nu^2} \quad (54)$$

$$C_{ab}^H = \Gamma_{ab} f_{Ma} \left[-\frac{2\nu}{\nu_a^2} \left(1 - \frac{m_a}{m_b} \right) \frac{\partial H_{b1}}{\partial \nu} - \frac{2}{\nu_a^2} H_{b1} \right] \quad (55)$$

$$C_{ab}^D = \Gamma_{ab} f_{Ma} 4\pi \frac{m_a}{m_b} f_{b1} = \frac{\Gamma_{ab} n_a}{\nu_a^3} \exp(-x_a^2) \frac{4}{\pi^{1/2}} \frac{m_a}{m_b} f_{b1} \quad (56)$$

The Poisson equations that define the potentials are

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial H_{b1}}{\partial x_b} - \ell(\ell+1) H_{b1} = -4\pi \nu^2 f_{b1} \quad (57)$$

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial G_{b1}}{\partial x_b} - \ell(\ell+1) G_{b1} = 2\nu^2 H_{b1}. \quad (58)$$

Let us define

$$\hat{H}_{b1} = H_{b1} / \nu_b^2 \quad (59)$$

$$\hat{G}_{b1} = G_{b1} / \nu_b^4 \quad (60)$$

so the defining equations for the potentials become

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial \hat{H}_{b1}}{\partial x_b} - \ell(\ell+1) \hat{H}_{b1} = -4\pi x_b^2 f_{b1} \quad (61)$$

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial \hat{G}_{b1}}{\partial x_b} - \ell(\ell+1) \hat{G}_{b1} = 2x_b^2 \hat{H}_{b1}. \quad (62)$$

It is convenient to specify the collisionality at the reference parameters:

$$\nu_r = \frac{\bar{\nu} \bar{R}}{\bar{\nu}} \quad (63)$$

where

$$\bar{\nu} = \frac{4\sqrt{2\pi} e^4 \bar{n} \ln \Lambda}{3\sqrt{\bar{m}} \bar{T}^{3/2}} \quad (64)$$

The kinetic equation (20) may then be written

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta} \frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi} \frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \nu_r \sqrt{\hat{m}_a} \frac{1}{\bar{\nu}} \sum_b C_{ab} \{ \hat{f}_a \} - \hat{S}_a = \hat{R}. \quad (65)$$

Next, it is convenient to note

$$\sqrt{\hat{m}_a} \frac{\Gamma_{ab}}{\bar{\nu}} = \frac{3\sqrt{\pi}}{4} \frac{1}{\bar{n}} \frac{Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \nu_a^3 \quad (66)$$

Then we may write the kinetic equation (65) as

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta} \frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi} \frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \nu_r \sum_b \hat{C}_{ab} \{ \hat{f}_a \} - \hat{S}_a = \hat{R} \quad (67)$$

where

$$\hat{C}_{ab} = \sqrt{\hat{m}_a} \frac{1}{\bar{\nu}} C_{ab}. \quad (68)$$

Expanding $\hat{C}_{ab} = \hat{C}_{ab}^L + \hat{C}_{ab}^E + \hat{C}_{ab}^F$ as before,

$$\hat{C}_{ab}^L = \frac{\hat{\nu}_{Dab}}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial \hat{f}_{a1}}{\partial \xi} \quad (69)$$

where

$$\hat{\nu}_{Dab} = \frac{3\sqrt{\pi}}{4} \frac{\hat{n}_b Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \frac{1}{x_a^3} [\text{erf}(x_b) - \Psi(x_b)] \quad (70)$$

The energy scattering component is

$$\hat{C}_{ab}^E \{f_{a1}\} = \frac{3\sqrt{\pi}}{4} \frac{\hat{n}_b Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left[\frac{1}{x_a} \Psi(x_b) \frac{\partial^2 f_{a1}}{\partial x_a^2} + \left\{ -2 \frac{\hat{T}_a}{\hat{T}_b} \frac{\hat{m}_b}{\hat{m}_a} \Psi(x_b) \left(1 - \frac{\hat{m}_a}{\hat{m}_b} \right) + \frac{1}{x_a^2} [\text{erf}(x_b) - \Psi(x_b)] \right\} \frac{\partial f_{a1}}{\partial x_a} + \frac{4}{\sqrt{\pi}} \left(\frac{\hat{T}_a}{\hat{T}_b} \right)^{3/2} \left(\frac{\hat{m}_b}{\hat{m}_a} \right)^{1/2} e^{-x_b^2} f_{a1} \right] \quad (71)$$

The diagonal term is

$$\hat{C}_{ab}^D = 3 \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \frac{\hat{m}_a}{\hat{m}_b} \exp(-x_a^2) f_{b1} \quad (72)$$

In the cross-species case, this term is no longer identical to the f_{a1} term in energy scattering (as it is in the same-species case).

The G term in the collision operator is

$$\begin{aligned} \hat{C}_{ab}^G &= \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b} \right)^2 e^{-x_a^2} x_b^2 \frac{\partial^2 \hat{G}_{b1}}{\partial x_b^2} \\ &= \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b} \right)^2 e^{-x_a^2} x_a^2 \frac{\partial^2 \hat{G}_{b1}}{\partial x_b^2}. \end{aligned} \quad (73)$$

Although in principle we would also be free to write

$$\hat{C}_{ab}^G = \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b} \right)^2 e^{-x_a^2} \mathbf{x}_a^2 \frac{\partial^2 \hat{G}_{b1}}{\partial \mathbf{x}_a^2} \quad (74)$$

(i.e. replacing $x_b \rightarrow x_a$ in two places), the resulting expression is less convenient because we compute \hat{G}_{b1} on the x_b grid, and so it is easier to differentiate with respect to x_b .

The H collision term is

$$\hat{C}_{ab}^H = -\frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b} e^{-x_a^2} \left[\left(1 - \frac{\hat{m}_a}{\hat{m}_b} \right) x_b \frac{\partial \hat{H}_{b1}}{\partial x_b} + \hat{H}_{b1} \right] \quad (75)$$

Right-hand side

Applying

$$\frac{2L+1}{2} \int_{-1}^1 d\xi P_L(\xi) (\quad) \quad (76)$$

to (23) and using the identity

$$\frac{2L+1}{2} \int_{-1}^1 d\xi P_L(\xi) (1 + \xi^2) = \frac{4}{3} \delta_{L,0} + \frac{2}{3} \delta_{L,2}, \quad (77)$$

we find the right-hand side of the linear system is

$$\boxed{\frac{\hat{m}_a^2}{2\pi^{3/2} Z_a \hat{T}_a^{1/2}} \frac{\hat{n}_a \hat{I} \hat{J}}{\hat{B}^3 \hat{\psi}_a} \left(\frac{\partial \hat{B}}{\partial \theta} \right) x_a^2 e^{-x_a^2} \left(\frac{1}{n_a} \frac{dn_a}{d\psi_N} + \frac{2Z_a}{\hat{T}_a} \frac{\omega}{\Delta} \frac{d\hat{\Phi}}{d\psi_N} + \left[x_a^2 - \frac{3}{2} \right] \frac{1}{T_a} \frac{dT_a}{d\psi_N} \right) \left(\frac{4}{3} \delta_{L,0} + \frac{2}{3} \delta_{L,2} \right)}. \quad (78)$$

Outputs

Normalized scale lengths

$$\delta_{T_a} = -\frac{I\nu_a}{\Omega_a T_a} \frac{dT_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a \hat{m}_a} \hat{I}}{\hat{\psi}_a \hat{Z}_a \hat{B}} \frac{1}{\hat{T}_a} \frac{d\hat{T}_a}{d\psi_N} \quad (79)$$

$$\delta_{n_a} = -\frac{I\nu_a}{\Omega_a n_a} \frac{dn_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a \hat{m}_a} \hat{I}}{\hat{\psi}_a \hat{Z}_a \hat{B}} \frac{1}{\hat{n}_a} \frac{d\hat{n}_a}{d\psi_N} \quad (80)$$

$$\delta_{\eta_a} = -\frac{I\nu_a}{\Omega_a \eta_a} \frac{d\eta_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a \hat{m}_a} \hat{I}}{\hat{\psi}_a \hat{Z}_a \hat{B}} \frac{1}{\hat{\eta}_a} \frac{d\hat{\eta}_a}{d\psi_N} \quad (81)$$

Normalized electric field

$$U_a = \frac{cI}{B\nu_a} \frac{d\Phi}{d\psi} = \frac{\omega \hat{I}}{\hat{\psi}_a \hat{B}} \sqrt{\frac{\hat{m}_a}{\hat{T}_a}} \frac{d\hat{\Phi}}{d\psi_N} \quad (82)$$

Density perturbation

$$\text{densityPerturbation} = \frac{1}{n_a} \int d^3\nu f_{a1} = \frac{4\pi\Delta\hat{T}_a^{3/2}}{\hat{n}_a \hat{m}_a^{3/2}} \int_0^\infty dx_a x_a^2 \hat{f}_{a,L=0} \quad (83)$$

Parallel flow

$$\begin{aligned} V_{a\parallel} &= \frac{1}{n_a} \int d^3\nu \nu_{\parallel} f_{a1} = 2\pi\Delta\bar{\nu} \frac{\hat{T}_a^2}{\hat{n}_a \hat{m}_a^2} \int_{-1}^1 d\xi \int_0^\infty dx_a x_a^3 \xi \hat{f}_a \\ &= \frac{4}{3} \pi\Delta\bar{\nu} \frac{\hat{T}_a^2}{\hat{n}_a \hat{m}_a^2} \int_0^\infty dx_a x_a^3 \hat{f}_{a,L=1} \\ &= \Delta\bar{\nu} (\text{flow}) \end{aligned} \quad (84)$$

where

$$\text{flow} = \frac{4}{3} \pi \frac{\hat{T}_a^2}{\hat{n}_a \hat{m}_a^2} \int_0^\infty dx_a x_a^3 \hat{f}_{a,L=1}. \quad (85)$$

Mach number

$$\text{Mach} = \frac{V_{a\parallel}}{\nu_a} = \Delta \sqrt{\frac{\hat{m}_a}{\hat{T}_a}} (\text{flow}). \quad (86)$$

Pressure perturbation

$$\text{pressurePerturbation} = \frac{1}{n_a T_a} \frac{m_a}{3} \int d^3\nu \nu^2 f_{a1} = \frac{8\pi\Delta\hat{T}_a^{3/2}}{3\hat{n}_a \hat{m}_a^{3/2}} \int_0^\infty dx_a x_a^4 \hat{f}_{a,L=0} \quad (87)$$

Average density perturbation

$$\text{FSADensityPerturbation} = \frac{1}{n_a} \left\langle \int d^3v f_{a1} \right\rangle = \langle \text{densityPerturbation} \rangle \quad (88)$$

Average parallel flow

$$\begin{aligned} \langle V_{a\parallel} B \rangle &= \left\langle B \int d^3v v_{\parallel} f_{a1} \right\rangle = \Delta \bar{v} \bar{B} \langle \hat{B}(\text{flow}) \rangle \\ &= \Delta \bar{v} \bar{B} (\text{FSABFlow}) \end{aligned} \quad (89)$$

where

$$\text{FSABFlow} = \langle \hat{B}(\text{flow}) \rangle. \quad (90)$$

Parallel flow coefficient

$$V_{a\parallel} = -\frac{cI}{Z_a e B} \left[\frac{1}{n_a} \frac{dp_a}{d\psi} + Z_a e \frac{d\Phi}{d\psi} - k_{\parallel} \frac{B^2}{\langle B^2 \rangle} \frac{dT_a}{d\psi} \right] \quad (91)$$

$$k_{\parallel} = \left(\frac{dT_a}{d\psi} \right)^{-1} \frac{\langle B^2 \rangle}{B^2} \left(\frac{Z_a e B}{cI} V_{a\parallel} + \frac{1}{n_a} \frac{dp_a}{d\psi} + Z_a e \frac{d\Phi}{d\psi} \right) \quad (92)$$

$$\text{kPar} = k_{\parallel} = \left(\frac{d\hat{T}_a}{d\psi_N} \right)^{-1} \frac{\langle \hat{B}^2 \rangle}{\hat{B}^2} \left(2 \frac{Z_a \hat{\psi}_a \hat{B}}{\hat{I}} (\text{flow}) + \frac{d\hat{T}_a}{d\psi_N} + \frac{\hat{T}_a}{\hat{n}_a} \frac{d\hat{n}_a}{d\psi_N} + 2Z_a \frac{\omega}{\Delta} \frac{d\hat{\Phi}}{d\psi_N} \right) \quad (93)$$

Average pressure perturbation

$$\begin{aligned} \text{FSAPressurePerturbation} &= \frac{1}{n_a T_a} \frac{m_a}{3} \left\langle \int d^3v v^2 f_{a1} \right\rangle \\ &= \langle \text{pressurePerturbation} \rangle \end{aligned} \quad (94)$$

Particle flux

Define

$$\Gamma_a = V' \left\langle \int d^3v f_{a1} \mathbf{v}_{ma} \cdot \nabla \psi_N \right\rangle. \quad (95)$$

Recall

$$\mathbf{v}_{ma} \cdot \nabla \psi_N = -\frac{1}{\hat{\psi}_a \bar{R}^2 \bar{B}} \frac{m_a c I v_a^2}{2 Z_a e B^3} \mathbf{B} \cdot \nabla \theta \left(\frac{\partial B}{\partial \theta} \right) (1 + \xi^2) x_a^2 \quad (96)$$

Manipulation gives

$$\Gamma_a = -\frac{\pi \Delta^2}{\hat{\psi}_a} \frac{\bar{n} \bar{v}}{\bar{B}} \frac{\hat{m}_a}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^{5/2} \int_0^{2\pi} d\theta \frac{1}{\hat{B}^3} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_{-1}^1 d\xi \int_0^\infty dx_a \hat{f}_a (1 + \xi^2) x_a^4. \quad (97)$$

We may apply the identity

$$\int_{-1}^1 d\xi (1 + \xi^2) P_L(\xi) = \frac{8}{3} \delta_{L,0} + \frac{4}{15} \delta_{L,2} \quad (98)$$

to obtain

$$\Gamma_a = -\frac{\pi\Delta^2}{\hat{\psi}_a} \frac{\bar{n}\bar{v}}{\bar{B}} \frac{\hat{m}_a \hat{I}}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^{5/2} \int_0^{2\pi} d\theta \frac{1}{\hat{B}^3} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_0^\infty dx_a \left(\frac{8}{3} \hat{f}_{a,L=0} + \frac{4}{15} \hat{f}_{a,L=2} \right) x_a^4. \quad (99)$$

The dimensionless “particleFlux” quantity computed in the code is

$$\text{particleFlux} = -\frac{\hat{m}_a \hat{I}}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^{5/2} \int_0^{2\pi} d\theta \frac{1}{\hat{B}^3} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_0^\infty dx_a \left(\frac{8}{3} \hat{f}_{a,L=0} + \frac{4}{15} \hat{f}_{a,L=2} \right) x_a^4 \quad (100)$$

so

$$\Gamma_a = \frac{\pi\Delta^2}{\hat{\psi}_a} \frac{\bar{n}\bar{v}}{\bar{B}} (\text{particleFlux}). \quad (101)$$

Momentum flux

Define a momentum flux Π_a by

$$\Pi_a = V' \left\langle \int d^3v f_{a1} \frac{I v_{\parallel}}{B} \mathbf{v}_{ma} \cdot \nabla \psi_N \right\rangle. \quad (102)$$

Manipulation gives

$$\Pi_a = -\frac{\pi\Delta^2}{\hat{\psi}_a} \frac{\bar{n}\bar{v}^2 \bar{R}}{\bar{B}} \frac{\hat{m}_a \hat{I}^2}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^3 \int_0^{2\pi} d\theta \frac{1}{\hat{B}^4} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_{-1}^1 d\xi^2 \int_0^\infty dx_a \hat{f}_a (1 + \xi^2) x_a^5. \quad (103)$$

We may apply the identity

$$\int_{-1}^1 d\xi \xi (1 + \xi^2) P_L(\xi) = \frac{16}{15} \delta_{L,1} + \frac{4}{35} \delta_{L,3}. \quad (104)$$

Thus, the dimensionless quantity computed in the code is

$$\text{momentumFlux} = -\frac{\hat{m}_a \hat{I}^2}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^3 \int_0^{2\pi} d\theta \frac{1}{\hat{B}^4} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_0^\infty dx_a \left(\frac{16}{15} \hat{f}_{a,L=1} + \frac{4}{35} \hat{f}_{a,L=3} \right) x_a^5. \quad (105)$$

where

$$\Pi_a = \frac{\pi\Delta^2}{\hat{\psi}_a} \frac{\bar{n}\bar{v}^2 \bar{R}}{\bar{B}} (\text{momentumFlux}). \quad (106)$$

Heat flux

Define a heat flux Q_a by

$$Q_a = V' \left\langle \int d^3v f_{a1} \frac{m_a v^2}{2} \mathbf{v}_{ma} \cdot \nabla \psi_N \right\rangle. \quad (107)$$

Manipulation gives

$$Q_a = -\frac{\pi\Delta^2}{2\hat{\psi}_a} \frac{\bar{n}\bar{m}\bar{v}^3}{\bar{B}} \frac{\hat{m}_a \hat{I}}{Z_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^{5/2} \int_0^{2\pi} d\theta \frac{1}{\hat{B}^3} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_{-1}^1 d\xi \int_0^\infty dx_a \hat{f}_a (1 + \xi^2) x_a^6. \quad (108)$$

We may again apply (98). The dimensionless quantity computed in the code is

$$\text{heatFlux} = -\frac{\hat{m}_a \hat{T}_a \hat{I}}{\hat{Z}_a} \left(\frac{\hat{T}_a}{\hat{m}_a} \right)^{5/2} \int_0^{2\pi} d\theta \frac{1}{\hat{B}^3} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \int_0^\infty dx_a \left(\frac{8}{3} \hat{f}_{a,L=0} + \frac{4}{15} \hat{f}_{a,L=2} \right) x_a^6 \quad (109)$$

so

$$Q_a = \frac{\pi \Delta^2}{2 \hat{\psi}_a} \frac{\bar{n} \bar{m} \bar{\omega}^3}{\bar{B}} (\text{heatFlux}). \quad (110)$$

Perturbed potential

Assuming a pure plasma in which only the ions are simulated in PERFECT, the quasineutrality equation is

$$n \frac{e \tilde{\Phi}}{T_e} = -n \frac{e \tilde{\Phi}}{T_i} + \int d^3 v f_i \quad (111)$$

where $n = n_i = n_e$ is the unperturbed density. Further assuming $T_e = T_i$, then

$$\tilde{\Phi} = \frac{T}{2en} \int d^3 v f_i \quad (112)$$

so

$$(\text{potentialPerturbation}) = \frac{\tilde{\Phi}}{\bar{\Phi}} = \frac{\Delta \hat{T}}{4\omega} (\text{densityPerturbation}). \quad (113)$$