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Normalizations for the multi-species pedestal code

The drift-kinetic equation for each species may be written

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{d0}\right)\cdot\left(\nabla f_{a1}\right)_{\mu,W_a} - C_a\left\{f_{a1}\right\} - S_a = -\mathbf{v}_m\cdot\nabla\psi\frac{\partial f_{a0}}{\partial\psi} \tag{1}$$

where $W_a = \upsilon^2/2 + Z_a e\Phi/m_a$,

$$f_{a0} = \frac{n_0}{\pi^{3/2}\upsilon_a}e^{-x_a^2}, \tag{2}$$

$\upsilon_a = \sqrt{2T_a/m_a}$, and $x_a = \upsilon/\upsilon_a$. Expanding the right-hand side of (1),

$$\begin{aligned}&\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{d0}\right)\cdot\left(\nabla f_{a1}\right)_{\mu,W_a} - C_a\left\{f_{a1}\right\} - S_a \\ &= \frac{m_a c I n_a}{2\pi^{3/2}\upsilon_a Z_a e B^3}\mathbf{B}\cdot\nabla\theta\left(\frac{\partial B}{\partial\theta}\right)\left(1+\xi^2\right)x^2 e^{-x^2}\left(\frac{1}{n_a}\frac{dn_a}{d\psi}+\frac{Z_a e}{T_a}\frac{d\Phi}{d\psi}+\left[x_a^2-\frac{3}{2}\right]\frac{1}{T_a}\frac{dT_a}{d\psi}\right).\end{aligned} \tag{3}$$

Just as in the local code, we normalize using a position-independent temperature $\bar{T}$, density $\bar{n}$, potential $\bar{\Phi}$, magnetic field $\bar{B}$, and length $\bar{R}$. We take $\bar{T}$ and $\bar{n}$ to be the same for all species. We thereby obtain the quantities

$$\hat{B} = B/\bar{B} \tag{4}$$

$$\bar{I} = I/\left(\bar{R}\bar{B}\right) \tag{5}$$

$$\hat{\Phi} = \Phi/\bar{\Phi} \tag{6}$$

$$\hat{J} = \frac{\mathbf{B}\cdot\nabla\theta}{\bar{B}/\bar{R}} \tag{7}$$

and

$$\psi = \psi_N\hat{\psi}_a\bar{R}^2\bar{B} \tag{8}$$

so

$$\frac{dX}{d\psi} = \frac{1}{\hat{\psi}_a\bar{R}^2\bar{B}}\frac{dX}{d\psi_N} \tag{9}$$

for any function X. Also, for each species,

$$\hat{n}_a = n_a/\bar{n} \tag{10}$$

and

$$\hat{T}_a = T_a/\bar{T}. \tag{11}$$

We define a species-independent reference mass $\bar{m}$ and reference thermal speed $\bar{\upsilon} = \sqrt{2\bar{T}/\bar{m}}$. Each species then has a normalized mass

$$\hat{m}_a = m_a/\bar{m} \tag{12}$$

and thermal speed

$$\upsilon_a = \bar{\upsilon}\sqrt{\hat{T}_a/\hat{m}_a}. \tag{13}$$

Two combinations of the normalization quantities will appear in the dimensionless equations:

$$\Delta = \frac{\bar{m} c \bar{\upsilon}}{e \bar{B} \bar{R}} \tag{14}$$

and

$$\omega = \frac{c\bar{\Phi}}{\bar{\upsilon}\bar{R}\bar{B}}. \tag{15}$$

The normalized distribution function of each species $\hat{f}_a$ is defined by

$$f_{a1} = \hat{f}_a \frac{\Delta \bar{n}}{\bar{\upsilon}^3}. \tag{16}$$

Notice the normalization is the same for all species.

To normalize the kinetic equation, we first multiply through by

$$\frac{\bar{\upsilon}^3}{\Delta \bar{n}} \frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \tag{17}$$

where the temperature dependence is included for historical reasons, for consistency with the single-species code. We then obtain

$$\frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \left(\upsilon_{\|} \mathbf{b} + \mathbf{v}_{d0} \right) \cdot \left(\nabla \hat{f}_a \right)_{\mu, W_a} - \frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} C_a \left\{ \hat{f}_a \right\} - \hat{S}_a = \hat{R} \tag{18}$$

where

$$\hat{S}_a = \frac{\bar{\upsilon}^3}{\Delta \bar{n}} \frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} S_a = \frac{\bar{\upsilon}^2 \bar{R}}{\Delta \bar{n}} \sqrt{\hat{m}_a} S_a \tag{19}$$

is the normalized source, and

$$\hat{R} = \frac{\hat{m}_a^2}{2\pi^{3/2} Z_a \hat{T}_a^{1/2}} \frac{\hat{n}_a \hat{I} \hat{J}}{\hat{B}^3 \hat{\psi}_a} \left(\frac{\partial \hat{B}}{\partial \theta} \right) \left(1 + \xi^2 \right) x^2 e^{-x^2} \left(\frac{1}{\hat{n}_a} \frac{d\hat{n}_a}{d\psi_N} + \frac{2Z_a}{\hat{T}_a} \frac{\omega}{\Delta} \frac{d\hat{\Phi}}{d\psi_N} + \left[x_a^2 - \frac{3}{2} \right] \frac{1}{\hat{T}_a} \frac{d\hat{T}_a}{d\psi_N} \right) \tag{20}$$

is the right-hand side.

We next change the independent variables in the drift-kinetic operator from $\{\psi, \theta, \mu, W_a\}$ to $\{\psi, \theta, x_a, \xi\}$, to re-write (18) in the form

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta} \frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi} \frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} C_a \left\{ \hat{f}_a \right\} - \hat{S}_a = \hat{R}. \tag{21}$$

We begin with the streaming term:

$$\frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \upsilon_{\|} \mathbf{b} \cdot \nabla \psi = 0 \tag{22}$$

$$\frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \upsilon_{\|} \mathbf{b} \cdot \nabla \theta = \frac{\hat{J} \sqrt{\hat{T}_a}}{\hat{B}} x_a \xi \tag{23}$$

$$\frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \upsilon_{\|} \mathbf{b} \cdot \left(\nabla x \right)_{W_a} = 0 \tag{24}$$

$$\frac{\bar{R}}{\upsilon_a} \sqrt{\hat{T}_a} \upsilon_{\|} \mathbf{b} \cdot \left(\nabla \xi \right)_{\mu, W_a} = -\sqrt{\hat{T}_a} x_a \frac{\left(1 - \xi^2 \right)}{2\hat{B}^2} \hat{J} \frac{d\hat{B}}{d\theta}. \tag{25}$$

Moving on to the drift terms, we begin with

$$\dot{\psi}_N = \frac{\bar{R}}{\upsilon_a}\sqrt{\hat{T}_a}\mathbf{v}_m\cdot\nabla\psi_N . \tag{26}$$

which gives

$$\boxed{\dot{\psi}_N = -\frac{\hat{m}_a^{1/2}\Delta\hat{T}_a\hat{I}\hat{J}}{2Z_a\hat{\psi}_a}x_a^2\left(1+\xi^2\right)\frac{1}{\hat{B}^3}\frac{\partial\hat{B}}{\partial\theta}}. \tag{27}$$

This result differs by a factor $\hat{m}_a^{1/2}/Z_a$ from the single-species code.

Next, we evaluate the contribution to $\dot{\theta}$:

$$\mathbf{v}_{d0}\cdot\nabla\theta = \frac{cI}{B^2}\left(\mathbf{B}\cdot\nabla\theta\right)\frac{d\Phi}{d\psi}+\frac{m_a c\upsilon_\parallel^2}{Z_a eB}\nabla\times\mathbf{b}\cdot\nabla\theta+\frac{m_a c\upsilon_\perp^2}{2Z_a eB^3}\mathbf{B}\times\nabla B\cdot\nabla\theta . \tag{28}$$

We can re-use two results from the single-species calculation:

$$\mathbf{B}\times\nabla B\cdot\nabla\theta = \frac{\bar{B}^2}{\bar{R}^2}\frac{\hat{I}\hat{J}}{\hat{\psi}_a}\frac{\partial\hat{B}}{\partial\psi_N} \tag{29}$$

and

$$\left(\nabla\times\mathbf{b}\right)\cdot\nabla\theta = \frac{1}{\bar{R}^2}\frac{\hat{J}}{\hat{\psi}_a}\left[\frac{\hat{I}}{\hat{B}^2}\frac{\partial\hat{B}}{\partial\psi_N}-\frac{1}{\hat{B}}\frac{d\hat{I}}{d\psi_N}\right]. \tag{30}$$

Then (28) becomes

$$\mathbf{v}_{d0}\cdot\nabla\theta = \frac{\bar{\upsilon}}{\bar{R}}\left[\frac{\omega\hat{I}\hat{J}}{\hat{\psi}_a\hat{B}^2}\frac{d\hat{\Phi}}{d\psi_N}+\frac{\Delta\hat{T}_a}{Z_a\hat{B}}x_a^2\xi^2\frac{\hat{J}}{\hat{\psi}_a}\left[\frac{\hat{I}}{\hat{B}^2}\frac{\partial\hat{B}}{\partial\psi_N}-\frac{1}{\hat{B}}\frac{d\hat{I}}{d\psi_N}\right]+\frac{\Delta\hat{T}_a}{2Z_a\hat{B}^3}x_a^2\left(1-\xi^2\right)\frac{\hat{I}\hat{J}}{\hat{\psi}_a}\frac{\partial\hat{B}}{\partial\psi_N}\right]. \tag{31}$$

This result is equal to (59) in the single-species technical manual, but with an extra factor of $1/Z_a$ in each of the magnetic drift terms. Combining with (23), we obtain

$$\boxed{\dot{\theta} = \frac{\hat{J}\sqrt{\hat{T}_a}}{\hat{B}}x_a\xi+\frac{\sqrt{\hat{m}_a}\omega\hat{I}\hat{J}}{\hat{\psi}_a\hat{B}^2}\frac{d\hat{\Phi}}{d\psi_N}+\frac{\sqrt{\hat{m}_a}\Delta\hat{T}_a}{2Z_a\hat{B}^3}x_a^2\left(1+\xi^2\right)\frac{\hat{I}\hat{J}}{\hat{\psi}_a}\frac{\partial\hat{B}}{\partial\psi_N}-\frac{\sqrt{\hat{m}_a}\Delta\hat{T}_a}{Z_a\hat{B}^2}x_a^2\xi^2\frac{\hat{J}}{\hat{\psi}_a}\frac{d\hat{I}}{d\psi_N}}. \tag{32}$$

This equation is the generalization of (60) in the single-species technical manual, with the new factors colored red.

To obtain $\dot{x}_a$, we first write

$$W_0 = \frac{\bar{\upsilon}^2}{2}\frac{\hat{T}_a}{\hat{m}_a}x_a^2+\frac{Z_a}{m_a}e\Phi . \tag{33}$$

Differentiating,

$$\dot{x}_a = -\frac{1}{\hat{T}_a x_a}\left[\frac{1}{2}x_a^2\frac{d\hat{T}_a}{d\psi_N}+\frac{Z_a}{\bar{m}\bar{\upsilon}^2}e\frac{d\Phi}{d\psi_N}\right]\dot{\psi}_N . \tag{34}$$

Applying (26),

$$\boxed{\dot{x}_a = \left[\frac{\Delta}{Z_a}\frac{x_a^3}{2}\frac{d\hat{T}_a}{d\psi_N}+\omega x_a\frac{d\hat{\Phi}}{d\psi_N}\right]\frac{\hat{m}_a^{1/2}\hat{I}\hat{J}}{\hat{\psi}_a\hat{B}^3}\frac{\left(1+\xi^2\right)}{2}\frac{\partial\hat{B}}{\partial\theta}}. \tag{35}$$

To obtain $\mathbf{v}_{d0}\cdot\left(\nabla\xi\right)_{\mu,W_a}$, we first write

$$2\mu B = \bar{\upsilon}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 \left(1-\xi^2\right) \tag{36}$$

Differentiating,

$$\begin{aligned} &\left[\bar{\upsilon}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 \left(1-\xi^2\right)\right] \frac{1}{B}\left(\frac{\partial B}{\partial \theta} d\theta + \frac{\partial B}{\partial \psi_N} d\psi_N\right) \\ &= -2\bar{\upsilon}^2 \frac{\hat{T}_a}{\hat{m}_a} x_a^2 \xi d\xi + 2 x_a \bar{\upsilon}^2 \frac{\hat{T}_a}{\hat{m}_a}\left(1-\xi^2\right) dx_a + \bar{\upsilon}^2 \frac{1}{\hat{m}_a} x_a^2 \left(1-\xi^2\right) \frac{d\hat{T}_a}{d\psi_N} d\psi_N \end{aligned} \tag{37}$$

Solving for $d\xi$,

$$d\xi = \left(1-\xi^2\right)\frac{1}{\xi x_a} dx_a + \left[\frac{1}{\hat{T}_a}\frac{d\hat{T}_a}{d\psi_N} - \frac{1}{\hat{B}}\frac{\partial \hat{B}}{\partial \psi_N}\right]\frac{1}{2\xi}\left(1-\xi^2\right) d\psi_N - \left(1-\xi^2\right)\frac{1}{2\xi \hat{B}}\frac{\partial \hat{B}}{\partial \theta} d\theta \tag{38}$$

Plugging in our previous results,

$$\begin{aligned} \dot{\xi} = &-\left(1-\xi^2\right)\frac{1}{2\xi\hat{B}}\frac{\partial \hat{B}}{\partial \theta}\frac{\hat{J}\sqrt{\hat{T}_a}}{\hat{B}} x_a \xi \\ &+\xi\left(1-\xi^2\right)\frac{\hat{m}_a^{1/2}\omega \hat{I}\hat{J}}{\hat{\psi}_a \hat{B}^3}\frac{\partial \hat{B}}{\partial \theta}\frac{d\hat{\Phi}}{d\psi_N} \\ &+\left(1-\xi^2\right)\frac{1}{\xi x_a}\frac{\Delta}{Z_a}\frac{x_a^3}{2}\frac{d\hat{T}_a}{d\psi_N}\frac{\hat{m}_a^{1/2}\hat{I}\hat{J}}{\hat{\psi}_a \hat{B}^3}\frac{\left(1+\xi^2\right)}{2}\frac{\partial \hat{B}}{\partial \theta} \\ &-\left[\frac{1}{\hat{T}_a}\frac{d\hat{T}_a}{d\psi_N} - \frac{1}{\hat{B}}\frac{\partial \hat{B}}{\partial \psi_N}\right]\frac{1}{4\xi}\left(1-\xi^4\right)\frac{\hat{m}_a^{1/2}\Delta \hat{T}_a \hat{I}\hat{J}}{Z_a \hat{\psi}_a} x_a^2 \frac{1}{\hat{B}^3}\frac{\partial \hat{B}}{\partial \theta} \\ &-\left(1-\xi^2\right)\frac{1}{2\xi\hat{B}}\frac{\partial \hat{B}}{\partial \theta}\left[\frac{\sqrt{\hat{m}_a}\Delta \hat{T}_a}{2 Z_a \hat{B}^3} x_a^2 \left(1+\xi^2\right)\frac{\hat{I}\hat{J}}{\hat{\psi}_a}\frac{\partial \hat{B}}{\partial \psi_N} - \frac{\sqrt{\hat{m}_a}\Delta \hat{T}_a}{Z_a \hat{B}^2} x_a^2 \xi^2 \frac{\hat{J}}{\hat{\psi}_a}\frac{d\hat{I}}{d\psi_N}\right] \end{aligned} \tag{39}$$

Two pairs of terms cancel, leaving

$$\boxed{\dot{\xi} = -x_a\left(1-\xi^2\right)\frac{\hat{J}\sqrt{\hat{T}_a}}{2\hat{B}^2}\frac{\partial \hat{B}}{\partial \theta} + \xi\left(1-\xi^2\right)\frac{\hat{m}_a^{1/2}\omega \hat{I}\hat{J}}{2\hat{\psi}_a \hat{B}^3}\frac{\partial \hat{B}}{\partial \theta}\frac{d\hat{\Phi}}{d\psi_N} + \xi\left(1-\xi^2\right)x_a^2\frac{\sqrt{\hat{m}_a}\Delta \hat{T}_a \hat{J}}{2\hat{\psi}_a Z_a \hat{B}^3}\frac{d\hat{I}}{d\psi_N}\frac{\partial \hat{B}}{\partial \theta}}. \tag{40}$$

Collision operator

The linearized collision operator may be written

$$C_{ab} = C_{ab}^L + C_{ab}^E + C_{ab}^F, \tag{41}$$

where the Lorentz part of the collision term is

$$C_{ab}^L = \frac{\nu_{Dab}}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_{a1}}{\partial \xi} \tag{42}$$

with

$$\nu_{Dab} = \frac{\Gamma_{ab} n_b}{\upsilon^3}\left[\mathrm{erf}\left(x_b\right) - \Psi\left(x_b\right)\right] \tag{43}$$

$$\Gamma_{ab} = \frac{4\pi Z_a^2 Z_b^2 e^4 \ln\Lambda}{m_a^2} \tag{44}$$

$$\Psi\left(x_b\right) = \frac{\operatorname{erf}\left(x_b\right) - x_b \operatorname{erf}'\left(x_b\right)}{2x_b^2}. \tag{45}$$

The energy scattering contribution is

$$C_{ab}^E = \nu_{\|ab}\left[\frac{\upsilon^2}{2}\frac{\partial^2 f_{a1}}{\partial\upsilon^2} - x_b^2\left(1-\frac{m_a}{m_b}\right)\upsilon\frac{\partial f_{a1}}{\partial\upsilon}\right] + \nu_{Dab}\upsilon\frac{\partial f_{a1}}{\partial\upsilon} + 4\pi\Gamma_{ab}\frac{m_a}{m_b} f_{Mb} f_{a1} \tag{46}$$

where

$$\nu_{\|ab} = 2\frac{\Gamma_{ab} n_b}{\upsilon^3}\Psi\left(x_b\right). \tag{47}$$

The field term is

$$C_{ab}^F = \Gamma_{ab} f_{Ma}\left[\frac{2\upsilon^2}{\upsilon_a^4}\frac{\partial^2 G_{b1}}{\partial\upsilon^2} - \frac{2\upsilon}{\upsilon_a^2}\left(1-\frac{m_a}{m_b}\right)\frac{\partial H_{b1}}{\partial\upsilon} - \frac{2}{\upsilon_a^2}H_{b1} + 4\pi\frac{m_a}{m_b} f_{b1}\right] \tag{48}$$

where the potentials are defined by

$$\nabla_\upsilon^2 H_{b1} = -4\pi f_{b1} \tag{49}$$

and

$$\nabla_\upsilon^2 G_{b1} = 2H_{b1}. \tag{50}$$

We write the field term as

$$C_{ab}^F = C_{ab}^H + C_{ab}^G + C_{ab}^D \tag{51}$$

where

$$C_{ab}^G = \Gamma_{ab} f_{Ma}\frac{2\upsilon^2}{\upsilon_a^4}\frac{\partial^2 G_{b1}}{\partial\upsilon^2} \tag{52}$$

$$C_{ab}^H = \Gamma_{ab} f_{Ma}\left[-\frac{2\upsilon}{\upsilon_a^2}\left(1-\frac{m_a}{m_b}\right)\frac{\partial H_{b1}}{\partial\upsilon} - \frac{2}{\upsilon_a^2}H_{b1}\right] \tag{53}$$

$$C_{ab}^D = \Gamma_{ab} f_{Ma} 4\pi\frac{m_a}{m_b} f_{b1} = \frac{\Gamma_{ab} n_a}{\upsilon_a^3}\exp\left(-x_a^2\right)\frac{4}{\pi^{1/2}}\frac{m_a}{m_b} f_{b1} \tag{54}$$

The Poisson equations that define the potentials are

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial H_{b1}}{\partial x_b} - \ell\left(\ell+1\right) H_{b1} = -4\pi\upsilon^2 f_{b1} \tag{55}$$

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial G_{b1}}{\partial x_b} - \ell\left(\ell+1\right) G_{b1} = 2\upsilon^2 H_{b1}. \tag{56}$$

Let us define

$$\hat{H}_{b1} = H_{b1} / \upsilon_b^2 \tag{57}$$

$$\hat{G}_{b1} = G_{b1} / \upsilon_b^4 \tag{58}$$

so the defining equations for the potentials become

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial \hat{H}_{b1}}{\partial x_b} - \ell\left(\ell+1\right)\hat{H}_{b1} = -4\pi x_b^2 f_{b1} \tag{59}$$

$$\frac{\partial}{\partial x_b} x_b^2 \frac{\partial \hat{G}_{b1}}{\partial x_b} - \ell\left(\ell+1\right)\hat{G}_{b1} = 2x_b^2 \hat{H}_{b1}. \tag{60}$$

It is convenient to specify the collisionality at the reference parameters:

$$\nu_r = \frac{\bar{\nu}\bar{R}}{\bar{\upsilon}} \tag{61}$$

where

$$\bar{\nu} = \frac{4\sqrt{2\pi} e^4 \bar{n} \ln\Lambda}{3\sqrt{\bar{m}}\bar{T}^{3/2}} \tag{62}$$

The kinetic equation (18) may then be written

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta}\frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi}\frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \nu_r \sqrt{\hat{m}_a}\frac{1}{\bar{\nu}}\sum_b C_{ab}\left\{\hat{f}_a\right\} - \hat{S}_a = \hat{R}. \tag{63}$$

Next, it is convenient to note

$$\sqrt{\hat{m}_a}\frac{\Gamma_{ab}}{\bar{\nu}} = \frac{3\sqrt{\pi}}{4}\frac{1}{\bar{n}}\frac{Z_a^2 Z_b^2}{\hat{T}_a^{3/2}}\upsilon_a^3 \tag{64}$$

Then we may write the kinetic equation (63) as

$$\dot{\psi}_N \frac{\partial \hat{f}_a}{\partial \psi_N} + \dot{\theta}\frac{\partial \hat{f}_a}{\partial \theta} + \dot{\xi}\frac{\partial \hat{f}_a}{\partial \xi} + \dot{x}_a \frac{\partial \hat{f}_a}{\partial x_a} - \nu_r \sum_b \hat{C}_{ab}\left\{\hat{f}_a\right\} - \hat{S}_a = \hat{R} \tag{65}$$

where

$$\hat{C}_{ab} = \sqrt{\hat{m}_a}\frac{1}{\bar{\nu}}C_{ab}. \tag{66}$$

Expanding $\hat{C}_{ab} = \hat{C}_{ab}^L + \hat{C}_{ab}^E + \hat{C}_{ab}^F$ as before,

$$\hat{C}_{ab}^L = \frac{\hat{\nu}_{Dab}}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_{a1}}{\partial \xi} \tag{67}$$

where

$$\hat{\nu}_{Dab} = \frac{3\sqrt{\pi}}{4}\frac{\hat{n}_b Z_a^2 Z_b^2}{\hat{T}_a^{3/2}}\frac{1}{x_a^3}\left[\operatorname{erf}\left(x_b\right) - \Psi\left(x_b\right)\right] \tag{68}$$

The energy scattering component is

$$\hat{C}_{ab}^E\left\{f_{a1}\right\} = \frac{3\sqrt{\pi}}{4}\frac{\hat{n}_b Z_a^2 Z_b^2}{\hat{T}_a^{3/2}}\left[\begin{array}{l} \dfrac{1}{x_a}\Psi\left(x_b\right)\dfrac{\partial^2 f_{a1}}{\partial x_a^2} \\ +\left\{-2\dfrac{\hat{T}_a}{\hat{T}_b}\dfrac{\hat{m}_b}{\hat{m}_a}\Psi\left(x_b\right)\left(1-\dfrac{\hat{m}_a}{\hat{m}_b}\right) + \dfrac{1}{x_a^2}\left[\operatorname{erf}\left(x_b\right)-\Psi\left(x_b\right)\right]\right\}\dfrac{\partial f_{a1}}{\partial x_a} \\ +\dfrac{4}{\sqrt{\pi}}\left(\dfrac{\hat{T}_a}{\hat{T}_b}\right)^{3/2}\left(\dfrac{\hat{m}_b}{\hat{m}_a}\right)^{1/2} e^{-x_b^2} f_{a1} \end{array}\right] \tag{69}$$

The diagonal term is

$$\hat{C}_{ab}^{D} = 3\frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \frac{\hat{m}_a}{\hat{m}_b} \exp\left(-x_a^2\right) f_{b1} \tag{70}$$

In the cross-species case, this term is no longer identical to the f_{a1} term in energy scattering (as it is in the same-species case).

The G term in the collision operator is

$$\begin{aligned}\hat{C}_{ab}^{G} &= \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b}\right)^2 e^{-x_a^2} x_b^2 \frac{\partial^2 \hat{G}_{b1}}{\partial x_b^2} \\ &= \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b}\right) e^{-x_a^2} x_a^2 \frac{\partial^2 \hat{G}_{b1}}{\partial x_b^2}.\end{aligned} \tag{71}$$

Although in principle we would also be free to write

$$\hat{C}_{ab}^{G} = \frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \left(\frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b}\right)^2 e^{-x_a^2} x_a^2 \frac{\partial^2 \hat{G}_{b1}}{\partial x_a^2} \tag{72}$$

(i.e. replacing $x_b \to x_a$ in two places), the resulting expression is less convenient because we compute $\hat{G}_{b1}$ on the x_b grid, and so it is easier to differentiate with respect to x_b.

The H collision term is

$$\hat{C}_{ab}^{H} = -\frac{3}{2\pi} \frac{\hat{n}_a Z_a^2 Z_b^2}{\hat{T}_a^{3/2}} \frac{\hat{T}_b}{\hat{T}_a} \frac{\hat{m}_a}{\hat{m}_b} e^{-x_a^2} \left[\left(1 - \frac{\hat{m}_a}{\hat{m}_b}\right) x_b \frac{\partial \hat{H}_{b1}}{\partial x_b} + \hat{H}_{b1}\right] \tag{73}$$

Right-hand side

Applying

$$\frac{2L+1}{2} \int_{-1}^{1} d\xi P_L(\xi) (\quad) \tag{74}$$

to (21) and using the identity

$$\frac{2L+1}{2} \int_{-1}^{1} d\xi \ P_L(\xi) \left(1+\xi^2\right) = \frac{4}{3}\delta_{L,0} + \frac{2}{3}\delta_{L,2}, \tag{75}$$

we find the right-hand side of the linear system is

$$\boxed{\frac{\hat{m}_a^2}{2\pi^{3/2} Z_a \hat{T}_a^{1/2}} \frac{\hat{n}_a \hat{I}\hat{J}}{\hat{B}^3 \hat{\psi}_a} \left(\frac{\partial \hat{B}}{\partial \theta}\right) x_a^2 e^{-x_a^2} \left(\frac{1}{n_a}\frac{dn_a}{d\psi_N} + \frac{2Z_a}{\hat{T}_a}\frac{\omega}{\Delta}\frac{d\hat{\Phi}}{d\psi_N} + \left[x_a^2 - \frac{3}{2}\right]\frac{1}{T_a}\frac{dT_a}{d\psi_N}\right)\left(\frac{4}{3}\delta_{L,0} + \frac{2}{3}\delta_{L,2}\right)}. \tag{76}$$

Outputs

Normalized scale lengths

$$\delta_{T_a} = -\frac{I\upsilon_a}{\Omega_a T_a}\frac{dT_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a \hat{m}_a}\,\hat{I}}{\hat{\psi}_a Z_a \hat{B}} \frac{1}{\hat{T}_a}\frac{d\hat{T}_a}{d\psi_N} \tag{77}$$

$$\delta_{n_a} = -\frac{I\upsilon_a}{\Omega_a n_a}\frac{dn_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a\hat{m}_a}\,\hat{I}}{\hat{\psi}_a Z_a \hat{B}}\frac{1}{\hat{n}_a}\frac{d\hat{n}_a}{d\psi_N} \tag{78}$$

$$\delta_{\eta_a} = -\frac{I\upsilon_a}{\Omega_a \eta_a}\frac{d\eta_a}{d\psi} = -\frac{\Delta\sqrt{\hat{T}_a\hat{m}_a}\,\hat{I}}{\hat{\psi}_a Z_a \hat{B}}\frac{1}{\hat{\eta}_a}\frac{d\hat{\eta}_a}{d\psi_N} \tag{79}$$

Normalized electric field

$$U_a = \frac{cI}{B\upsilon_a}\frac{d\Phi}{d\psi} = \frac{\omega\hat{I}}{\hat{\psi}_a\hat{B}}\sqrt{\frac{\hat{m}_a}{\hat{T}_a}}\frac{d\hat{\Phi}}{d\psi_N} \tag{80}$$

Density perturbation

$$\texttt{densityPerturbation} = \frac{1}{n_a}\int d^3\upsilon\ f_{a1} = \frac{4\pi\Delta\hat{T}_a^{3/2}}{\hat{n}_a\hat{m}_a^{3/2}}\int_0^\infty dx_a x_a^2 \hat{f}_{a,L=0} \tag{81}$$

Parallel flow

$$\begin{aligned} V_{a\|} &= \frac{1}{n_a}\int d^3\upsilon\upsilon_\| f_{a1} = 2\pi\Delta\bar{\upsilon}\frac{\hat{T}_a^2}{\hat{n}_a\hat{m}_a^2}\int_{-1}^{1} d\xi\int_0^\infty dx_a x_a^3 \xi \hat{f}_a \\ &= \frac{4}{3}\pi\Delta\bar{\upsilon}\frac{\hat{T}_a^2}{\hat{n}_a\hat{m}_a^2}\int_0^\infty dx_a x_a^3 \hat{f}_{a,L=1} \\ &= \Delta\bar{\upsilon}\left(\texttt{flow}\right) \end{aligned} \tag{82}$$

where

$$\texttt{flow} = \frac{4}{3}\pi\frac{\hat{T}_a^2}{\hat{n}_a\hat{m}_a^2}\int_0^\infty dx_a x_a^3 \hat{f}_{a,L=1}\,. \tag{83}$$

Mach number

$$\texttt{Mach} = \frac{V_{a\|}}{\upsilon_a} = \Delta\sqrt{\frac{\hat{m}_a}{\hat{T}_a}}\left(\texttt{flow}\right). \tag{84}$$

Pressure perturbation

$$\texttt{pressurePerturbation} = \frac{1}{n_a T_a}\frac{m_a}{3}\int d^3\upsilon\ \upsilon^2 f_{a1} = \frac{8\pi\Delta\hat{T}^{3/2}}{3\hat{n}_a\hat{m}_a^{3/2}}\int_0^\infty dx_a x_a^4 \hat{f}_{a,L=0} \tag{85}$$

Average density perturbation

$$\texttt{FSADensityPerturbation} = \frac{1}{n_a}\left\langle\int d^3\upsilon f_{a1}\right\rangle = \left\langle\texttt{densityPerturbation}\right\rangle \tag{86}$$

Average parallel flow

$$\begin{aligned} \left\langle V_{a\|}B\right\rangle &= \left\langle B\int d^3\upsilon\upsilon_\| f_{a1}\right\rangle = \Delta\bar{\upsilon}\bar{B}\left\langle\hat{B}\left(\texttt{flow}\right)\right\rangle \\ &= \Delta\bar{\upsilon}\bar{B}\left(\texttt{FSABFlow}\right) \end{aligned} \tag{87}$$

where

$$\texttt{FSABFlow} = \left\langle \hat{B}(\texttt{flow}) \right\rangle. \tag{88}$$

Parallel flow coefficient

$$V_{a\|} = -\frac{cI}{Z_a eB}\left[\frac{1}{n_a}\frac{dp_a}{d\psi} + Z_a e\frac{d\Phi}{d\psi} - k_\| \frac{B^2}{\left\langle B^2\right\rangle}\frac{dT_a}{d\psi}\right] \tag{89}$$

$$k_\| = \left(\frac{dT_a}{d\psi}\right)^{-1}\frac{\left\langle B^2\right\rangle}{B^2}\left(\frac{Z_a eB}{cI}V_{a\|} + \frac{1}{n_a}\frac{dp_a}{d\psi} + Z_a e\frac{d\Phi}{d\psi}\right) \tag{90}$$

$$\texttt{kPar} = k_\| = \left(\frac{d\hat{T}_a}{d\psi_N}\right)^{-1}\frac{\left\langle \hat{B}^2\right\rangle}{\hat{B}^2}\left(2\frac{Z_a\hat{\psi}_a\hat{B}}{\hat{I}}(\texttt{flow}) + \frac{d\hat{T}_a}{d\psi_N} + \frac{\hat{T}_a}{\hat{n}_a}\frac{d\hat{n}_a}{d\psi_N} + 2Z_a\frac{\omega}{\Delta}\frac{d\hat{\Phi}}{d\psi_N}\right) \tag{91}$$

Average pressure perturbation

$$\begin{aligned}\texttt{FSAPressurePerturbation} &= \frac{1}{n_a T_a}\frac{m_a}{3}\left\langle \int d^3\upsilon\, \upsilon^2 f_{a1}\right\rangle \\ &= \left\langle \texttt{pressurePerturbation}\right\rangle\end{aligned} \tag{92}$$

Particle flux

Define

$$\Gamma_a = V'\left\langle \int d^3\upsilon\, f_{a1}\mathbf{v}_{ma}\cdot\nabla\psi_N\right\rangle. \tag{93}$$

Recall

$$\mathbf{v}_{ma}\cdot\nabla\psi_N = -\frac{1}{\hat{\psi}_a\bar{R}^2\bar{B}}\frac{m_a cI\upsilon_a^2}{2Z_a eB^3}\mathbf{B}\cdot\nabla\theta\left(\frac{\partial B}{\partial\theta}\right)\left(1+\xi^2\right)x_a^2 \tag{94}$$

Manipulation gives

$$\Gamma_a = -\frac{\pi\Delta^2}{\hat{\psi}_a}\frac{\bar{n}\bar{\upsilon}}{\bar{B}}\frac{\hat{m}_a\hat{I}}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^{5/2}\int_0^{2\pi}d\theta\frac{1}{\hat{B}^3}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_{-1}^{1}d\xi\int_0^\infty dx_a\,\hat{f}_a\left(1+\xi^2\right)x_a^4. \tag{95}$$

We may apply the identity

$$\int_{-1}^{1}d\xi\left(1+\xi^2\right)P_L(\xi) = \frac{8}{3}\delta_{L,0} + \frac{4}{15}\delta_{L,2} \tag{96}$$

to obtain

$$\Gamma_a = -\frac{\pi\Delta^2}{\hat{\psi}_a}\frac{\bar{n}\bar{\upsilon}}{\bar{B}}\frac{\hat{m}_a\hat{I}}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^{5/2}\int_0^{2\pi}d\theta\frac{1}{\hat{B}^3}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_0^\infty dx_a\left(\frac{8}{3}\hat{f}_{a,L=0} + \frac{4}{15}\hat{f}_{a,L=2}\right)x_a^4. \tag{97}$$

The dimensionless “particleFlux” quantity computed in the code is

$$\texttt{particleFlux} = -\frac{\hat{m}_a\hat{I}}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^{5/2}\int_0^{2\pi}d\theta\frac{1}{\hat{B}^3}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_0^\infty dx_a\left(\frac{8}{3}\hat{f}_{a,L=0} + \frac{4}{15}\hat{f}_{a,L=2}\right)x_a^4 \tag{98}$$

so

$$\Gamma_a = \frac{\pi\Delta^2}{\hat{\psi}_a}\frac{\bar{n}\bar{\upsilon}}{\bar{B}}(\texttt{particleFlux}). \tag{99}$$

Momentum flux

Define a momentum flux Π_a by

$$\Pi_a = V'\left\langle \int d^3\upsilon f_{a1}\frac{I\upsilon_\parallel}{B}\mathbf{v}_{ma}\cdot\nabla\psi_N\right\rangle. \tag{100}$$

Manipulation gives

$$\Pi_a = -\frac{\pi\Delta^2}{\hat{\psi}_a}\frac{\bar{n}\bar{\upsilon}^2\bar{R}}{\bar{B}}\frac{\hat{m}_a\hat{I}^2}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^3\int_0^{2\pi}d\theta\frac{1}{\hat{B}^4}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_{-1}^{1}d\xi^2\int_0^\infty dx_a\hat{f}_a\left(1+\xi^2\right)x_a^5. \tag{101}$$

We may apply the identity

$$\int_{-1}^{1}d\xi\xi\left(1+\xi^2\right)P_L(\xi) = \frac{16}{15}\delta_{L,1}+\frac{4}{35}\delta_{L,3}. \tag{102}$$

Thus, the dimensionless quantity computed in the code is

$$\texttt{momentumFlux} = -\frac{\hat{m}_a\hat{I}^2}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^3\int_0^{2\pi}d\theta\frac{1}{\hat{B}^4}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_0^\infty dx_a\left(\frac{16}{15}\hat{f}_{a,L=1}+\frac{4}{35}\hat{f}_{a,L=3}\right)x_a^5. \tag{103}$$

where

$$\Pi_a = \frac{\pi\Delta^2}{\hat{\psi}_a}\frac{\bar{n}\bar{\upsilon}^2\bar{R}}{\bar{B}}(\texttt{momentumFlux}). \tag{104}$$

Heat flux

Define a heat flux Q_a by

$$Q_a = V'\left\langle \int d^3\upsilon f_{a1}\frac{m_a\upsilon^2}{2}\mathbf{v}_{ma}\cdot\nabla\psi_N\right\rangle. \tag{105}$$

Manipulation gives

$$Q_a = -\frac{\pi\Delta^2}{2\hat{\psi}_a}\frac{\bar{n}\bar{m}\bar{\upsilon}^3}{\bar{B}}\frac{\hat{m}_a\hat{T}_a\hat{I}}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^{5/2}\int_0^{2\pi}d\theta\frac{1}{\hat{B}^3}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_{-1}^{1}d\xi\int_0^\infty dx_a\hat{f}_a\left(1+\xi^2\right)x_a^6. \tag{106}$$

We may again apply (96). The dimensionless quantity computed in the code is

$$\texttt{heatFlux} = -\frac{\hat{m}_a\hat{T}_a\hat{I}}{Z_a}\left(\frac{\hat{T}_a}{\hat{m}_a}\right)^{5/2}\int_0^{2\pi}d\theta\frac{1}{\hat{B}^3}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\int_0^\infty dx_a\left(\frac{8}{3}\hat{f}_{a,L=0}+\frac{4}{15}\hat{f}_{a,L=2}\right)x_a^6 \tag{107}$$

so

$$Q_a = \frac{\pi\Delta^2}{2\hat{\psi}_a}\frac{\bar{n}\bar{m}\bar{\upsilon}^3}{\bar{B}}(\texttt{heatFlux}). \tag{108}$$

Perturbed potential

Assuming a pure plasma in which only the ions are simulated in PERFECT, the quasineutrality equation is

$$n\frac{e\tilde{\Phi}}{T_e} = -n\frac{e\tilde{\Phi}}{T_i} + \int d^3\upsilon f_i \tag{109}$$

where $n = n_i = n_e$ is the unperturbed density. Further assuming $T_e = T_i$, then

$$\tilde{\Phi} = \frac{T}{2en}\int d^3\upsilon f_i \tag{110}$$

so

$$(\texttt{potentialPerturbation}) = \frac{\tilde{\Phi}}{\bar{\Phi}} = \frac{\Delta\hat{T}}{4\omega}(\texttt{densityPerturbation}). \tag{111}$$