1 (a) $\exists M \ge 0$ s.t. $|f|, |g| \le M$ as continuous function on the compact set is bounded.
Let $\forall \varepsilon > 0$ be given. Let $\delta_1, \delta_2 > 0$ be δ that satisfies the definition of absolute continuity corresponding to f & g with $\frac{\varepsilon}{2M}$, resp.

Set $\delta = \min(\delta_1, \delta_2)$ and let $\{(a_i, b_i)\}$ be a finite collection of disjoint intervals with $\sum_{i=1}^n |b_i - a_i| < \delta$.
For $\forall \varepsilon > 0$,

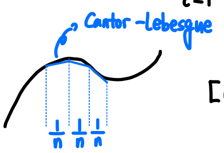
$$\sum_{i=1}^n |fg(b_i) - fg(a_i)| = \sum_{i=1}^n |fg(b_i) - f(a_i)g(b_i) + f(a_i)g(b_i) - fg(a_i)|$$
$$\le \sum_{i=1}^n |f(b_i) - f(a_i)|\cdot|g(b_i)| + |g(b_i) - g(a_i)|\cdot|f(a_i)| \le \varepsilon \blacksquare$$

(2) let $F(x) = f(x)\cdot g(x) \rightsquigarrow$ by (1), absolutely conti on $[a,b]$.
Done by $F(b) - F(a) = \int_a^b F'(x)\cdot dx$ & linearity of integral $\blacksquare$

2 By Lemma 14.4, it suffices to show that F is absolutely continuous.
Let $\forall \varepsilon > 0$ be given. As f is integrable, $|f| < \infty$ a.e. Thus we can find $M > 0$ s.t. $|f| < M$ a.e.
Set $\delta = \varepsilon / M > 0$ and let $\{(a_i, b_i)\}$ be a finite collection of disjoint intervals with $\sum_{i=1}^n |b_i - a_i| < \delta$.

$$\sum_{i=1}^n |F(b_i) - F(a_i)| = \sum_{i=1}^n \left|\int_{a_i}^{b_i} f(x)\cdot dx\right| \le \int_{\bigcup(a_i, b_i)} |f|\cdot dx \le M\cdot\sum_{i=1}^n |b_i - a_i| < \varepsilon \blacksquare$$

3 idea



$$[a,b] \Rightarrow \chi_{[a,b)}\cdot(f(b) - f(a))\cdot F\left(\frac{x-a}{b-a}\right) + f(b)\cdot\chi_{\{b\}}$$

Let F be Cantor-Lebesgue function
Since f is conti on $[0,1]$, f is uniform continuous: $\exists N \in \mathbb{N}$ s.t. $|x - y| \le \frac{1}{N} \longrightarrow |f(y) - f(x)| < \frac{\varepsilon}{2}$
Let $g(x) = \sum_{i=1}^N \left\{f\left(\frac{i-1}{N}\right) + \left[f\left(\frac{i}{N}\right) - f\left(\frac{i-1}{N}\right)\right] F(Nx - i + 1)\right\}\cdot\chi_{[\frac{i-1}{N}, \frac{i}{N})}(x) + f(1)\cdot\chi_{\{1\}}(x)$

Let $\forall x \in [0,1)$ be given. ($f(1) - g(1) = 0$ clearly)
$\rightsquigarrow \exists!\, j \in \{1, 2, \cdots, N\}$ s.t. $x \in \left[\frac{j-1}{N}, \frac{j}{N}\right)$

Then $|f(x) - g(x)| \le |f(x) - f(\frac{j}{N})|\cdot k + |f(x) - f(\frac{j-1}{N})|\cdot(1-k) < \frac{\varepsilon}{2}$

$\therefore \sup_{x\in[0,1]} |f(x) - g(x)| < \varepsilon \square$

4 ① No.

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & x \in (0,1] \\ 0 & x = 0 \end{cases} \rightsquigarrow f'(x) = \begin{cases} 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2} & x \in (0,1] \\ 0 & x = 0. \end{cases}$$

Thus for $\forall a \in (0,1)$, $f' \le M_a$ a.e. as f' is conti on $[a,1]$.
Hence f is absolutely conti on $(a,1]$: for disjoint $\{(a_j, b_j)\}$ with $\sum_{j=1}^n |b_j - a_j| < \delta = \frac{\varepsilon}{M_a}$.
then $\sum_{j=1}^n |f(b_j) - f(a_j)| = \sum_{j=1}^n \left|\int_{a_j}^{b_j} f'\right| \le \varepsilon$.

But not BV([0,1]). Thus not absolutely conti on [0,1]
$x_0 = 0 < x_1 < x_2 < \cdots < x_{2N} = 1$. Let $x_{2n+1} = (2n\pi + \frac{\pi}{2})^{-2}$, $x_{2n} = (2n\pi)^{-2}$ $\rightsquigarrow \sum_{j=0}^{2N-1} |f(x_{j+1}) - f(x_j)| \ge \sum_{j=1}^N \frac{1}{2\pi j + \frac{\pi}{2}}$
$\longrightarrow \infty$ as $n \to \infty$.

② Yes

$f \in BV([0,1]) \rightsquigarrow f = f_1 - f_2$ for some increasing

$\rightsquigarrow \exists f'$ a.e. as $\exists f_1', f_2'$ a.e.

Also $f' \in L^1$ on $[0,1]$ as $f_1' \& f_2'$ locally integrable and $[0,1]$ is compact.

Thus $f' \in L^1$.

$$\Rightarrow \lim_{a\to0+} \int_a^x f' \cdot dt \overset{\because \text{ abs conti on } (a,1]}{=} \lim_{a\to0+} (f(x) - f(a)) \overset{\because \text{ conti on } [0,1]}{=} f(x) - f(0)$$

|| D.C.T.

$$\int \lim_{a\to0+} f' \cdot \chi_{[a,x]} = \int_0^x f'$$

$\therefore f(x)$ is abs conti on $[0,1]$ by 14.2 ■

$\boxed{5}$ $\rightarrow$) ① $|f'| \le M$

$$|f'(x)| = \left| \lim_{h\to0} \frac{f(x+h)-f(x)}{h} \right| \le M$$

② AC

Let any $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{M} > 0$.

$$\sum_{i=1}^n |f(b_i) - f(a_i)| \le \sum_{i=1}^n |b_i - a_i| \cdot M < \delta \cdot M = \varepsilon_{//}$$

($\leftarrow$) Let $y \ge x$.

$$|f(y) - f(x)| \underset{\because\, f \text{ is AC}}{=} \left| \int_x^y f' \right| \le \int_x^y |f'| \le M \cdot |y-x|$$

$\boxed{6}$ ① $F' \ge \Sigma F_n'$ a.e.

$F - \sum_{n=1}^N F_n = \sum_{n=N+1}^\infty F_n$ is increasing function.

$\therefore F' - \sum_{n=1}^N F_n' \ge 0$ a.e. By $N \to \infty$, $F' \ge \Sigma F_n'$ a.e.

② $F' \le \Sigma F_n'$ a.e.

Note that $\sum_{n=N+1}^\infty F_n$ is also increasing & right conti.

$\int_0^1 F' - \sum_{n=1}^N F_n' \le \sum_{n=N+1}^\infty F_n(1) - \sum_{n=N+1}^\infty F_n(0)$. As $F(0) \le F(1) < \infty$, RHS $\to 0$ as $N \to \infty$.

Hence for $\forall \varepsilon > 0$, for sufficiently large N, $\int_0^1 F' \le \int_0^1 \sum_{n=1}^N F_n' + \varepsilon$

As F_n increasing, $F_n' \ge 0$ a.e. thus $\int_0^1 F' \le \varepsilon + \int_0^1 \sum_{n=1}^\infty F_n'$ as $N \to \infty$ by M.C.T.

As ε arbitrary, $\int_0^1 F' \le \int_0^1 \sum_{n=1}^\infty F_n'$. As $F' \ge \sum_{n=1}^\infty F_n' \ge 0$ non-negative, $F' \le \sum_{n=1}^\infty F_n'$ a.e. ■

* Another solution

Let λ, λ_n be Lebesgue–Stieltjes measure w.r.t. $F \& F_n$ resp.

Then $\lambda((a,b]) = \sum_{n=1}^\infty F_n(b) - F_n(a) = \sum_{n=1}^\infty \lambda_n((a,b])$

Thus, by Exercise 4.1., $\lambda = \sum_{n=1}^\infty \lambda_n$

By Lebesgue decompostion w.r.t. m, $\lambda = \rho + \nu$, $\lambda_n = \rho_n + \nu_n$. From prop. 14.7 $\rho(A) = \int_A f \cdot dm$, $\rho_n(A) = \int_A f_n \cdot dm$ where $F' = f$, $F_n' = f_n$.

By Exercise 13.11, by uniqueness of Lebesgue decompostion, $F' = \sum_{n=1}^\infty F_n'$ a.e.

$\boxed{7}$ Let $^\forall \varepsilon>0$ be given.

Set $\delta>0$ that satisfies the definition of absolutely continuity of f.

As A is Lebesgue measurable, $^\exists$open set $G\supset A$ s.t. $m(G-A)<\delta$, i.e. $m(G)<\delta$.

Since G is open, it can be written as disjoint union of open intervals : $G=\bigcup_{i=1}^{\infty}(a_i,b_i)$

Then, as $m(G)=\sum_{i=1}^{\infty}(b_i-a_i)<\delta$, $\sum_{i=1}^{\infty}|f(b_i)-f(a_i)|<\varepsilon$ by abs conti.

As f is conti on $[a_i,b_i]$, it has the maximum, minimum at q_i, p_i resp. where $[p_i,q_i]\subset[a_i,b_i]$ or $[q_i,p_i]$

Note that $f(A\cap[a_i,b_i])\subseteq f([a_i,b_i])\subseteq[f(p_i),f(q_i)]$

$$\therefore\ m(f(A))=m\Big(f\Big(\bigcup_{i=1}^{\infty}A\cap[a_i,b_i]\Big)\Big)=m\Big(\bigcup_{i=1}^{\infty}f(A\cap[a_i,b_i])\Big)\le\sum_{i=1}^{\infty}m(f(A\cap[a_i,b_i]))$$

$$\le\sum_{i=1}^{n}|f(q_i)-f(p_i)|<\varepsilon \text{ by abs conti. } (\because \textstyle\sum|p_i-q_i|\le\sum|a_i-b_i|)$$

$\boxed{8}$ $f(x)=\begin{cases}x^2\sin(1/x^2) & x\neq 0\\ 0 & x=0\end{cases}$ $\rightsquigarrow$ $f'(x)=\begin{cases}2x\sin(1/x^2)-\frac{1}{x}\cos(1/x^2) & x\neq 0.\\ 0 \quad x=0 \rightsquigarrow \lim_{h\to 0}\frac{f(h)}{h}=\lim_{h\to 0}h\sin\left(\frac{1}{h^2}\right)=0\end{cases}$

But it is not bounded variation on $[0,1]$.

Consider the following partition $0<x_0<x_1<\cdots<x_{2N}<1$ where $x_{2n+1}=\left(2n\pi+\frac{\pi}{2}\right)^{-1/2}$, $x_{2n}=(2n\pi)^{-1/2}$

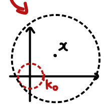
Then $\sum|f(x_{2n+1})-f(x_{2n})|\ge\sum_{n=1}^{N}\frac{1}{2n\pi+\pi/2}$

Since RHS diverges as $N\to\infty$, f can't be of bounded variation ∎

$\boxed{9}$ Note that $\{f\neq 0\}=\bigcup_{k,l\in\mathbb{N}}\{x\in B(0,k):|f(x)|>\frac{1}{l}\}$

As $m(\{f\neq0\})>0$, $^\exists k_0,l_0\in\mathbb{N}$ and $^\exists\varepsilon>0$ s.t. $m(\{x\in B(0,k_0):|f(x)|>\frac{1}{l_0}\})>\varepsilon$.

Then $Mf(x)\ge\frac{1}{m(B(x,|x|+k_0))}\int_{B(x,|x|+k_0)}|f(y)|\cdot dy\ \ge\ \frac{1}{m(B(x,|x|+k_0))}\int_{B(0,k_0)}|f(y)|\cdot dy>\frac{\varepsilon/l_0}{m(B(x,|x|+k_0))}$



Hence, $\int_{\mathbb{R}^n}Mf(x)\cdot dx>\int_{\mathbb{R}^n}\frac{\varepsilon/l_0}{m(B(x,|x|+k_0))}\propto\int\frac{\overset{\rightsquigarrow\text{const.}}{c_n\cdot(\varepsilon/l_0)}}{(|x|+k_0)^n}\cdot dx\ \underset{\text{polar}}{=}\ \int_0^\infty\frac{c_n}{(k_0+r)^n}\cdot\frac{\varepsilon}{l_0}\cdot r^{n-1}\cdot s_n\cdot dr$

$$\ge\int_{k_0}^{\infty}\frac{\varepsilon\cdot c_n\cdot s_n}{l_0}\cdot\frac{r^{n-1}}{2^n\cdot r^n}\cdot dr=\infty$$ ∎

$\boxed{10}$ $f(x) = \sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{2^n}$ on $[0,1]$ ⇝ not constant on any open interval.

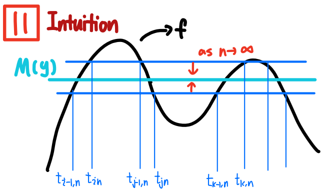
$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\lim_{N\to\infty}\frac{1}{h}\sum_{n=1}^{N}\left(\lfloor 2^n(x+h)\rfloor - \lfloor 2^n x\rfloor\right)$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N} \lim_{h\to0}\frac{\lfloor 2^n(x+h) - 2^n x\rfloor}{h}$$

Note that floor function $\lfloor x \rfloor$ is conti except on integer values of x.

∴ For $x\notin\{\frac{1}{2^n} \mid n\in\mathbb{N}\}$, $f'=0$ ∎

$\boxed{11}$ Intuition



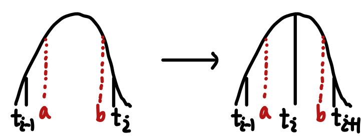
Let $g_n(y) = \sum_{i=1}^{2^n} \chi_{\underbrace{f([t_{i-1,n},\, t_n))}_{\text{Borel}}}(y) + \chi_{\underbrace{f(\{b\})}_{\text{Borel}}}(y)$ ⇝ Borel measurable.

$t_{in} = a + \frac{b-a}{2^n} i$

Note that $g_n \le g_{n+1}$ from the fact that $\chi_{A\cup B}\le \chi_A+\chi_B$

Also, $\lim_{n\to\infty} g_n = M$ as we expect : for $f^{-1}\{y\}=\{a_1,a_2,\cdots\}$, we can set partition by setting sufficiently large n. $\{[t_{i-1,n}, t_{in}]\}$ s.t. $a_j\in[t_{i_j-1,n}, t_{i_j n}]$ for each $j\in\mathbb{N}$ and i_js are different for different j.

e.g.



Claim : $V_f([a,b]) = \int M\cdot dy \overset{MCT}{=} \sup\int g_n\cdot dy$

pf) ≤) Take any partition $x_0=a<x_1<\cdots<x_n=b$.

Then variation $\sum_{i=0}^{n}|f(x_i)-f(x_{i-1})| \le \sum_{i=0}^{n-1} m(f([x_{i-1},x_i))) = \sum_{i=0}^{n-1}\int \chi_{f[x_{i-1},x_i)}\,dm$

$$\underset{\text{Fubini}}{=} \int\sum_{i=0}^{n-1}\chi_{f([x_{i-1},x_i))}(y)\cdot dy \le \int M(y)\cdot dy$$

since $\sum_{i=0}^{n-1}\chi_{f([x_i,x_{i+1}))}(y)\le M(y)=|f^{-1}\{y\}|$

∵ if $a_j\in f^{-1}(\{y\})$, then each a_j falls into at most one subinterval partitioned by x_i.

Thus it follows that the total # of such subintervals would be no more than $|f^{-1}\{y\}|$. It could be lesser since some a_j may share a subinterval.

≥) For $\forall n\in\mathbb{N}$, f conti on $[t_{i-1,n}, t_{in}]$.

⇒ For each i, let x_{2i-1} be minimum pt (exist)
x_{2i} be maximum pt.

And $x_0=a$, $x_{2^n+1}=b$. WLOG, let $x_{2i-1}<x_{2i}$

$$\int g_n(y)\cdot dy \le \sum_{i=1}^{2^n-1}|f(x_{2i})-f(x_{2i-1})|$$

$$\le\sum_{i=1}^{2^n+1}|f(x_i)-f(x_{i-1})|\le V_f[a,b]$$ ∎

$\boxed{12}$ Let $E_n = \left(-\frac{1}{n+1}-\frac{\alpha}{n(n+1)}, -\frac{1}{n+1}\right)\cup\left(\frac{1}{n+1},\frac{1}{n+1}+\frac{\alpha}{n(n+1)}\right)$ ⇝ $\{E_n \mid n\in\mathbb{N}\}$ pairwise disjoint.

$E=\bigcup_{n=1}^\infty E_n$. From $m(E_k\cap[-\frac1n,\frac1n]) = \frac{2\alpha}{k(k+1)}$ $\forall k\ge n$ (otherwise 0)

① $r=\frac1n$ for some $n\in\mathbb{N}$

$$\frac{m(E\cap[-\frac1n,\frac1n])}{2/n} = \frac n2\sum_{k=n}^\infty\frac{2\alpha}{k(k+1)}=\alpha.$$

② $r\in\left(\frac1{n+1},\frac1n\right)$ for some $n\in\mathbb{N}$

$m(E\cap[-r,r])\ge\frac{2\alpha}{n+1}$
$m(E\cap[-r,r])\le\frac{2\alpha}{n}$

As $n\in\mathbb{N}$, $\frac1n\le\frac2{n+1}<2r$ ⇝ $\frac1n=\frac1{n+1}\left(\frac1n+1\right)\le r(1+2r)$

$\frac1{n+1}=\frac1n\left(1-\frac1{n+1}\right)\ge r(1-2r)$

$$∴ \alpha(1-2r)\le\frac{m(E\cap[-r,r])}{2r}\le\alpha(1+2r)$$

$$∴ \lim_{r\to0+}\frac{m(E\cap[-r,r])}{2r}=\alpha$$