$\boxed{1}$ State : f_n is measurable, complexed-value function.
$f_n \to f$ for all $x \in X$ and $\exists g$ s.t. non-negative integrable with $|f_n| \le g$.
Then $\int f_n \to \int f$.

Proof : Let $f_n = u_n + i \cdot v_n$, $f = u + iv$. where $u_n, v_n, u, v : X \to \mathbb{R}$.
Then $|u_n| \le g$, $|v_n| \le g$. $\rightsquigarrow$ $\int u_n \to \int u$, $\int v_n \to \int v$.
$\therefore \int f_n = \int u_n + i \cdot \int v_n \longrightarrow \int u + i \int v$ ■

$\boxed{2}$ $2g - |f_n - f| \ge 0 \to$ By Fatou's lemma $\int 2g - \int \cancel{\liminf |f_n - f|} \overset{0}{\le} 2\int g - \liminf_n \int |f_n - f| \rightsquigarrow \limsup_n \int |f_n - f| \le 0$

$2g + |f_n - f| \ge 0 \to$ By Fatou's lemma $\int 2g + \int \liminf |f_n - f| \le 2\int g + \liminf_n \int |f_n - f| \rightsquigarrow \liminf_n \int |f_n - f| \ge 0$ as $\int 2g < \infty$

Claim : $\lim_{n\to\infty} \int |f_n - f| = 0 \Longrightarrow \int f_n \to \int f$ (converse doesn't hold : $f_n = n\chi_{(0,\frac{1}{n})}$, $f = 1$)
pf) $\left| \int f_n - f \right| \le \int |f_n - f| \to 0$.

$\boxed{3}$ $f \cdot \chi_{A_n} \uparrow f \cdot \chi_A$ clearly.
And $|f \cdot \chi_{A_n}| \le g = |f|$
By dominated convergence theorem, done ■

In case of $A_n \downarrow A$, let $B_n = A_n^c \uparrow B = A^c$.
Since f integrable
$\int_{A_n} f = \int f - \int_{B_n} f$
$\to \int f - \int_B f = \int_A f$ ■

$\boxed{4}$ ① g is integrable.
From proposition 7.6., $\int \underbrace{\sum_{n=1}^{\infty} |f_n|}_{\text{Let } h} = \sum_{n=1}^{\infty} \int |f_n| < \infty$ Thus $h \in L^1$
$\int \left| \sum_{n=1}^{\infty} f_n \right| \le \int \sum_{n=1}^{\infty} |f_n| < \infty$

Thus $g = \sum_{n=1}^{\infty} f_n \in L^1$
② Converges absolutely : should show $\sum_{n=1}^{\infty} |f_n(x)| < \infty$. a.e.
By exercise 6.1. $\mu\left(\sum_{n=1}^{\infty} |f_n| = \infty\right) = 0$. $\therefore \sum_{n=1}^{\infty} |f_n| < \infty$ a.e.
③ $\int \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \int f_n(x)$.
Let $h_N = \sum_{k=1}^{N} f_k \rightsquigarrow |h_N| \le h \in L^1$
By DCT $\lim_{N\to\infty} \sum_{k=1}^{N} \int f_k = \int \lim_{N\to\infty} \sum_{k=1}^{N} f_k$ ■

$\boxed{5}$ $g_n - f_n \ge 0 \rightsquigarrow \int \liminf_n (g_n - f_n) = \int g - f \le \liminf_n \left(\int g_n - \int f_n \right) \le \liminf_n \left(\int g - \int f_n \right)$
$\le \int g + \liminf_n \int -f_n$
As $g \in L^1$, $\int f \ge \limsup_n \int f_n$

$g_n + f_n \ge 0 \rightsquigarrow \int \liminf_n (g_n + f_n) = \int g + f \le \liminf_n \left(\int g_n + \int f_n \right) \le \liminf_n \left(\int g + \int f_n \right)$
$\le \int g + \liminf_n \int f_n$
As $g \in L^1$, $\int f \le \liminf_n \int f_n$ ■

6 $f_n = n\cdot\chi_{(0,\frac{1}{n^2}]}$

$\rightsquigarrow \int f_n = \frac{1}{n} \to 0$, $f_n \to 0$ $\forall x\in\mathbb{R}$

But no g s.t. $|f_n|\leq g$.

$f_n = \begin{cases} 1 & n=1 \\ 0 & n\geq 2 \end{cases}$ Let $X=\mathbb{R}$.

$\rightsquigarrow g\geq 1 \Rightarrow \int g \geq \mu(X)$

7 First, note that $\int f \leq \int f_n < \infty$.

Set $g_n = f + f_1 - f_n \uparrow f_1$ $\rightsquigarrow$ MCT $\int f_n \to \int f$.

$|f_n| \leq f_1 \in L^1$

$\therefore$ By DCT $\int f_n \to \int f$ ■

8 Let $|f_n(x)| \leq M$ $\forall x\in X, n\in\mathbb{N}$.

Then let $g = M\cdot\chi_X$ $\rightsquigarrow \int g = M\cdot\mu(X)<\infty$: non-negative & integrable.

Done by dominated convergence theorem.

$\int |f_n| < \int_{cm} \cdot |f_m|$

$= m\cdot\mu(x)$

9 $f_n\cdot\chi_A \to f\cdot\chi_A$

And $|f_n\cdot\chi_A(x)| \leq f_n(x)$ $\forall x\in X$.

$\therefore$ Done by generalized dominated convergence theorem.

Another solution

Claim: $\int f_n \to \int f$ and $f_n\to f$ a.e. $(f_n, f\in L^1) \Rightarrow \int |f_n - f| = 0$ $\longrightarrow$ Then, $\left|\int_A f_n - f\right| \leq \int_A |f_n - f| \leq \int |f_n - f| \longrightarrow 0$ //

pf) $|f_n - f| \leq |f_n| + |f|$

By generalized DCT $\int |f_n - f| = 0$ ■

10 $|f_n - f| \leq \underbrace{|f_n|+|f|}_{\text{non-negative, integrable.}}$ & $\int |f_n| + |f| \longrightarrow \int 2|f|$

$\therefore$ Done by generalized dominated convergence theorem.

11 Let $\forall\varepsilon>0$ be given.

By exercise 6.8., $\exists\delta>0$ s.t. $\forall m([x,y]) = |y-x|<\delta$, $\int_x^y |f|\cdot dx < \varepsilon$.

$\therefore \left|\int_x^y f(x)\cdot dx\right| < \int_x^y |f(x)|\,dx < \varepsilon$ whenever $|x-y|<\delta$ ■

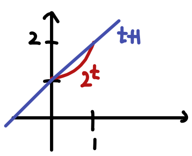
$\left|\int_x^y f(x)\cdot dx\right| = |F(y)-F(x)|$

12 No.

$f_n = n\,\chi_{[0,\frac{1}{\sqrt{n}})}$ $\limsup_{n\to\infty} f_n = 0$ $\forall x\in X$.

$\int f_n = \sqrt{n}$ $\limsup_{n\to\infty} \int f_n = \infty$.

13 Note that $2^t \leq 1+t$ for $\forall t\in[0,1]$



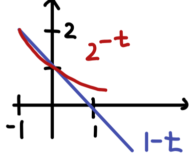
$\rightsquigarrow 2^{-nt} \geq (1+t)^{-n}$, $t=\frac{x}{n}\in[0,1]$.

$\therefore (1+\frac{x}{n})^{-n} \leq 2^{-x}$ & 2^{-x} integrable.

As $\log(2+\cos(\frac{x}{n})) \leq \log 3$

$\therefore \lim_{n\to\infty} \int (1+\frac{x}{n})^{-n}\cdot\log(2+\cos\frac{x}{n})\cdot\chi_{[0,n]} = \int e^{-x}\cdot\log 3\cdot\chi_{[0,\infty)} = \log 3$ by D.C.T. ■

14 Note that $2^{-t} \le 1-t$ for $\forall t \in [0,1]$



$\rightsquigarrow 2^{nt} \ge (1-t)^n$, $t = \frac{x}{n} \in [0,1]$.

$\therefore (1-\frac{x}{n})^n \le 2^x$ & 2^x integrable.

As $\log(2+\cos(\frac{x}{n})) \le \log 3$

$\therefore \lim_{n\to\infty} \int (1-\frac{x}{n})^n \cdot \log(2+\cos\frac{x}{n}) \cdot \chi_{[0,n]} = \int e^{-x} \cdot \log 3 \cdot \chi_{[0,\infty)} = \log 3$ by D.C.T. ■

15 We see that $1+nx^2 \le (1+x^2)^n$ for $n \ge 1$ & $\forall 0 \le x \le 1$

pf) It suffices to show $\log(1+nx^2) \le n\log(1+x^2)$

Let $h(x) = n\log(1+x^2) - \log(1+nx^2)$

$h'(x) = \frac{2nx}{1+x^2} - \frac{2nx}{1+nx^2} \ge 0$ for $\forall n \ge 1, 0 \le x \le 1$. $h(0) = 0$ $\therefore h(x) \ge 0$ //

$\log(2+\cos\frac{x}{n}) \le \log 3$ $\forall x \in [0,1]$

$\therefore \frac{1+nx^2}{(1+x^2)^n} \log(2+\cos\frac{x}{n}) \le 1$ $\forall n \in \mathbb{N}, x \in [0,1]$. $\therefore$ It converges to $\int \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} \log(2+\cos\frac{x}{n}) \cdot dx$

$= \int 0 = 0$ ■ by bdd convergence theorem.

16 $\int_0^\infty ne^{-nx} \cdot dx = 1$ (integrable)

$0 \le f_n(x) \le ne^{-nx}$ $\forall n \in \mathbb{N}$.

$\left| \int_0^\infty ne^{-nx} - \int_0^\infty ne^{-nx} \cdot \frac{x+1}{x^2+x+1} \right| = \int_0^\infty \frac{nxe^{-nx}}{x^2+x+1} \cdot dx$ te^{-t} $(t>0)$ has the maximum e^{-1}

Thus $0 \le \frac{nxe^{-nx}}{x^2+x+1} \le \frac{e^{-1}}{x^2+1}$ $\forall n \in \mathbb{N}$ & $\forall x > 0$. $\frac{1}{x^2+1}$ integrable.

$\therefore$ By D.C.T, $\int_0^\infty \frac{nxe^{-nx}}{x^2+x+1} \cdot dx \longrightarrow 0$ $\therefore 1$

17 $\lim_{n\to\infty} f(1+\frac{x}{n^2}) = f(1)$ as f is continuous at 1

Spse $|f| \le M$. Then $|f(1+\frac{x}{n^2})g(x)| \le M \cdot |g|$, $|g|$ is non-negative integrable.

$\therefore$ By dominated convergence theorem, it converges to $\int_{-n}^{n} f(1) \cdot g(x) \cdot dx$ ■

$\boxed{18}$ $\forall \varepsilon>0,\ \exists N\in\mathbb{N}$ s.t. $|f_n-f|<\varepsilon \quad \forall n\geq N.$

$\rightarrow |f| \leq |f-f_n| + |f_n| \quad \therefore \int|f| \leq \int \varepsilon + \int|f_n| = \varepsilon\cdot\mu(X) + \int|f_n| < \infty.$ Thus f is integrable

Since $f_n\rightarrow f$ unif. $\exists N_0\in\mathbb{N}$ st. $|f_n-f| < \varepsilon\cdot\mu(X)^{-1}<\infty \quad \forall n\geq N_0 \quad \forall x\in X.$

Let $n_0=N_0$

Then, $\left|\int f_n - \int f\right| = \left|\int f_n-f\right| \leq \int|f_n-f| < \varepsilon \quad \forall n\geq n_0.$

$\therefore \int f_n \rightarrow \int f$

And from the proof we can see that $\mu(X)<\infty$ necessary.

counterex)

$f_n=\frac{1}{n}\chi_{[0,n]},\ f=0. \quad \rightsquigarrow \forall\varepsilon>0,$ for $n>>1$ $\frac{1}{n}<\varepsilon$

$\rightsquigarrow |f_n(x)-f(x)|<\varepsilon \quad \forall x\in\mathbb{R}^+$

But $\int f_n=1,\ \int f=0$

$\boxed{19}$ $\frac{1}{1-x} = 1+x+x^2+\cdots = \Sigma_{k=1}^{\infty} x^{k-1}$ for $x<1.$

$$\frac{-x^p}{1-x}\log x = \sum_{k=1}^{\infty} -x^{k+p-1}\cdot\log x$$

$$\therefore \int_0^{1-\frac{1}{n}} \frac{-x^p}{1-x}\log x\cdot dx = \sum_{k=1}^{\infty}\int_0^{1-\frac{1}{n}} \underbrace{-x^{k+p-1}\cdot\log x}_{>0}\cdot dx$$

By MCT for n, $-\int_0^1 \frac{x^p}{1-x}\log x = \sum_{k=1}^{\infty}\int_0^1 -x^{k+p-1}\cdot\log x\cdot dx$

$$\cancel{\left[-x^{k+p}\cdot\log x\cdot\frac{1}{p+k}\right]_0^1}^{\,0} + \int_0^1 x^{k+p-1}\cdot\frac{1}{p+k}\cdot dx$$

$$= \frac{1}{(p+k)^2}\ \blacksquare$$

$\boxed{20}$

$$\int_{|f>M|} |f| < \varepsilon.$$

$\longleftrightarrow \sup\int|f_n|\cdot d\mu<\infty$

&

$L=\sup_n\int|f_n|\cdot d\mu.$ $\qquad M>>1,$

$$\mu\{|\,|f|>M\,|\} \leq \int\frac{|f|}{M}\cdot d\mu.$$

$$= \frac{L}{M} \rightarrow 0 \text{ as } M\rightarrow\infty$$

$\boxed{21}$ $\longrightarrow$) Let $\forall \varepsilon>0$ be given. $\exists M$ s.t. $\int_{\{x:|f_n(x)|>M\}} |f_n|\cdot d\mu < \varepsilon$.

$$\Rightarrow \int |f_n|\, d\mu = \int_{\{x:|f_n(x)|>M\}} |f_n|\cdot d\mu + \int_{\{x:|f_n(x)|\le M\}} |f_n|\cdot d\mu$$
$$< \varepsilon + M\cdot\mu(X) < \infty \quad \forall n\in\mathbb{N}$$

$$\therefore \sup_n \int |f_n|\, d\mu < \varepsilon + M\cdot\mu(X) < \infty$$

Also, for $\frac{\varepsilon}{2}>0$, $\exists M$ s.t. $\int_{\{x:|f_n(x)|>M\}} |f_n|\cdot d\mu < \varepsilon/2$. Let $\delta = \frac{\varepsilon}{2M} > 0$.

Then $\int_A |f_n|\cdot d\mu = \int_{A\cap\{x:|f_n(x)|>M\}} |f_n|\cdot d\mu + \int_{A\cap\{x:|f_n(x)|\le M\}} |f_n|\cdot d\mu$
$$< \frac{1}{2}\varepsilon + M\cdot\mu(A) < \varepsilon.$$

$$\therefore \left|\int_A f_n\cdot d\mu\right| \le \int_A |f_n|\cdot d\mu < \varepsilon \blacksquare$$

$\longleftarrow$) Let $\forall\varepsilon>0$ be given. $\rightsquigarrow$ $\exists\delta>0$ s.t. $\left|\int_A f_n\right| < \frac{\varepsilon}{2}$ whenever $\mu(A)<\delta$.

Let $L = \sup_n \int |f_n|\cdot d\mu < \infty$.

Then, we have

$\mu(\{x: |f_n(x)|>M\}) \le \int_{\{x:|f_n(x)|>M\}} \frac{|f_n(x)|}{M}\cdot d\mu \le \frac{L}{M} < \delta$ for sufficiently large $M>0$

$$\therefore \int_{|f_n|>M} |f_n| = \int_{f_n>M} f_n + \int_{f_n<-M} -f_n$$
$$\le \left|\int_{f_n>M} f_n\right| + \left|\int_{f_n<-M} -f_n\right| < \varepsilon \blacksquare$$

$\boxed{22}$ Let $\forall\varepsilon>0$ be given.

① f_n, f is integrable.

$\int|f| = \int \liminf_n |f_n| \le \liminf_n \int |f_n|$ & $\int|f_n| = \int_{|f_n|>M}|f_n| + \int_{|f_n|<M} |f_n| < \varepsilon + M\cdot\mu(X) < \infty$.

Thus $|f|<\infty$ a.e. by exercise 6.1.

$\rightsquigarrow$ By DCT $\int_{\{x:|f|>M\}} |f|\, d\mu \longrightarrow 0$ as $M\to\infty$.

② $f_n \to f$ in L^1 metric

Let $A_n = \{x: |f_n(x) - f(x)| \le \frac{\varepsilon}{5\mu(X)}\}$ As $f_n\to f$ a.e. & μ finite $\mu(A_n^c)\longrightarrow 0$ as $n\to\infty$

Choose M s.t. $\int_{\{x:|f_n|>M\}} |f_n|\cdot d\mu < \frac{\varepsilon}{5}$ $\Rightarrow$ $\exists N$ s.t. $\mu(A_n^c) < \frac{\varepsilon}{5M}$ $\forall n\ge N$.

$\int_{\{x:|f|>M\}} |f|\cdot d\mu < \frac{\varepsilon}{5}$

$$\int |f_n - f|\cdot d\mu = \int_{A_n} |f_n - f| + \int_{A_n^c} |f_n - f|$$
$$\le \int_{A_n} |f_n - f| + \int_{A_n^c} |f_n| + \int_{A_n^c} |f|$$
$$\le \int_{A_n} |f_n - f| + \int_{A_n^c\cap\{|f_n|>M\}} |f_n| + \int_{A_n^c\cap\{|f_n|\le M\}} |f_n| + \int_{A_n^c\cap\{|f|>M\}} |f| + \int_{A_n^c\cap\{|f|\le M\}} |f|$$
$$\le \varepsilon \quad \forall n\ge N.$$

23 Let $\forall \varepsilon > 0$ be given.

As $|f| \wedge M \uparrow |f|$, by MCT $\exists$ large M s.t. $\int |f| - \int |f| \wedge M = \int_{|f|>M} |f| < \frac{\varepsilon}{3}$

Also, as $\int |f_n - f| \longrightarrow 0$ $\exists$ large N s.t. $\int |f_n - f| < \frac{\varepsilon}{3}$ $\forall n \geq N$.

For $\forall n \geq M$, let $A_n = \{ |f_n| > 2M \;\&\; |f| \leq M \}$

$\rightsquigarrow$ in A_n, $|f_n - f| > M$

Thus $M \cdot \mu(A_n) \leq \int_{A_n} |f_n - f| \leq \int |f_n - f| < \frac{\varepsilon}{3}$

$$\therefore \int_{|f_n|>2M} |f_n| \leq \int_{|f_n|>2M} |f_n - f| + |f| = \int_{|f_n|>2M} |f_n - f| + \int_{A_n} |f| + \int_{|f_n|>2M,\, |f|>M} |f|$$

$$\leq \varepsilon \quad \forall n \geq N.$$

In case of $n = 1, 2, \cdots, N-1$,

by MCT for each $n = 1, 2, \cdots, N-1$, $\exists M_n$ s.t. $\int_{|f_n|>M_n} |f_n| < \varepsilon$.

$\therefore$ Done by setting $M = \max\{M_1, \cdots, M_{n-1}, 2M\}$ ∎

24 Let $K = \sup_n \int |f_n|^{1+\gamma} < \infty$. and any $\varepsilon > 0$ be given.

For any $M > 0$,

$\int_{|f_n| \geq M} |f_n| \leq \int_{|f_n| \geq M} |f_n| \cdot \left(\frac{|f_n|}{M}\right)^{\gamma} \leq \frac{K}{M^{\gamma}}$ Since $\gamma > 0$, $\exists$ large $M > 0$ s.t. $\frac{K}{M^{\gamma}} < \varepsilon$ ∎

$\boxed{27}$ (1) $\nu(\emptyset) = \int_{\emptyset} f \cdot d\mu = 0.$

Let $\{A_j\}$ be pairwise disjoint.

$\nu(\bigcup_{j=1}^{\infty} A_j) = \int f \cdot \chi_{\bigcup_j A_j} \cdot d\mu = \sum_{j=1}^{\infty} \int f \cdot \chi_{A_j} \cdot d\mu$ by proposition 7.6.

(2) Without loss of generality, let g be non-negative.

Let s be any simple s.t. $0 \leq s \leq g$. $\left(s = \sum_{j=1}^{n} a_j \chi_{E_j} \right)$

$\int s \cdot d\nu = \sum_{j=1}^{n} a_j \nu(E_j) = \int f \cdot s \cdot d\mu \leq \int f \cdot g \, d\mu$

Since g is integrable, i.e. measurable, $\exists$ sequence of non-negative simple functions $\{s_n\}$ increasing to g.

$\therefore$ By MCT $\int g \cdot d\nu = \lim_{n\to\infty} \int s_n \cdot d\nu$

$= \lim_{n\to\infty} \int f \cdot s_n \cdot d\mu = \int f \cdot g \cdot d\mu$ ∎

$\boxed{28}$ $E \in \mathcal{B}$.

$\exists$ open O, closed F s.t. $F \subseteq E \subseteq O$ $\mu(O - F) < \varepsilon$

$\exists$ open O', closed F' s.t. $F' \subseteq E \subseteq O'$ $\nu(O' - F') < \varepsilon$

Let $f = \begin{cases} 1 & \text{on } F \cup F' \\ 0 & \text{on } O^c \cup O'^c \\ \text{conti} & \text{o.w.} \end{cases}$ by Urysohn's lemma. $\rightsquigarrow$ continuous.

$\left| \int f \cdot d\mu - \mu(E) \right| < \varepsilon/2$

$\because F \cup F' \subset E$ $\mu(E - F' \cup F) \leq \mu(O - F) < \varepsilon/2$

Similarly, $\left| \int f \cdot d\nu - \nu(E) \right| < \varepsilon/2$

$\therefore \left| \mu(E) - \nu(E) \right| \leq \left| \int f \cdot d\mu - \mu(E) \right| + \left| \int f \cdot d\nu - \nu(E) \right| < \varepsilon.$

By letting $\varepsilon \to 0$, $\mu = \nu$ ∎

$\boxed{32}$ Note that for $g(x) = e^{\sin x}$, $|g'(x)| = |e^{\sin x} \cdot \cos x| \le e$

Let $x, y, h \in \mathbb{R}$.

$$\left| \frac{e^{\sin(x+h+y)} - e^{\sin(x+y)}}{h} \right| \overset{\text{Fundament Thm of Calculus}}{=} \left| \frac{1}{h} \int_0^h g'(x+y+t) \cdot dt \right| \le e$$

$$\frac{F(x+h) - F(x)}{h} = \int_{\mathbb{R}} e^{-y^2} \frac{e^{\sin(x+h+y)} - e^{\sin(x+y)}}{h} \cdot dy \le \int e \cdot e^{-y^2} \cdot dy < \infty. \rightsquigarrow F'(x) \text{ is finite}$$

$\therefore$ By D.C.T., $F'(x) = \int_{\mathbb{R}} e^{-y^2 + \sin(x+y)} \cdot \cos(x+y) \cdot dy$.