$\boxed{1}$ Let $\mu=\mu^+-\mu^-$, $X=E\cup F$ & $E\cap F=\phi$, $\mu^-(F)=\mu^+(E)=0$. from Jordan decomposition

$\rightarrow)$ Spse A is null set for μ.

Then trivially $A\cap E$ and $A\cap F$ are null sets

$\rightsquigarrow |\mu|(A)=\mu(A\cap E)+\mu(A\cap F)=0.$

$\leftarrow)$ $|\mu|(A)=0.$

Since μ^+ & μ^- are positive measures, $\mu^+(A)=\mu^-(A)=0.$

For any $B\subseteq A$ and $B\in\mathcal{A}$, we have $\mu^+(B)\le\mu^+(A)=0$ and $\mu^-(B)\le\mu^-(A)=0$

$\mu(B)=\mu^+(B)-\mu^-(B)=0$ ■

$\boxed{2}$ ① f is non-negative.

Let $f=\chi_A$. Then $\left|\int f\cdot d\mu\right| \underset{\text{by definition}}{=} \left|\mu^+(A)-\mu^-(A)\right| \underset{\mu^+,\mu^-\text{ positive}}{\le} \mu^+(A)+\mu^-(A)=\int|f|\cdot d|\mu|$

Let $f=\sum_{i=1}^n a_i\chi_{E_i}$ be a non-negative simple. Then $\left|\int f\cdot d\mu^+-\int f\cdot d\mu^-\right|=\left|\sum_{j=1}^n a_j\left(\mu^+(E_j)-\mu^-(E_j)\right)\right|$

$\le \sum_{j=1}^n\left|a_j\cdot|\mu|(E_j)\right|=\int|f|\cdot d|\mu|$

There exists a sequence of non-negative simple $\{s_n\}$ s.t. $s_n\uparrow f$.

Then $\left|\int s_n\cdot d\mu\right|\le\int|s_n|\cdot d|\mu| \Rightarrow \underbrace{\left|\int s_n\cdot d\mu\right|}_{\left|\int s_n\cdot d\mu^+-\int s_n\cdot d\mu^-\right|}\le\int|f|\cdot d|\mu|$ by taking a limit to RHS and M.C.T.

$\Rightarrow \left|\int f\cdot d\mu\right|\le\int|f|\cdot d|\mu|$ by taking a limit to LHS and M.C.T.

② $f=f^+-f^-$ integrable.

$\int f\cdot d\mu=\int f\cdot d\mu^+-\int f\cdot d\mu^- = \int f^+d\mu^++\int f^-d\mu^--\int f^-d\mu^+-\int f^+d\mu^-$

$\le \int f^+d\mu^++\int f^-d\mu^-$

$\le \int|f|\cdot d\mu^++\int|f|d\mu^- = \int|f|\cdot d|\mu|$

Claim. $\int f\cdot d|\mu|=\int f\cdot d\mu^++\int f\cdot d\mu^-$

pf) ① $f\ge0$

Let $f=\chi_A$. LHS $=|\mu|(A)=\mu^+(A)+\mu^-(A)=$ RHS clearly.

It holds for non-negative simple

Since $\exists\{s_n\}$ s.t. $s_n\ge0$ and $s_n\uparrow f$, done by M.C.T.

② f integrable.

Trivial from $f=f^+-f^-$.

Similarly, $-\int f\cdot d\mu \le \int f^-d\mu^++\int f^+d\mu^-$

$\le \int|f|\cdot d\mu^++\int|f|d\mu^-$

$=\int|f|\cdot d|\mu|.$

③ f is complex-valued. $\int f\cdot d\mu\in\mathbb{C}$

Let $\int f\cdot d\mu=re^{-i\theta}$ where $\theta=\arg\left(\int f\cdot d\mu\right)$. Let $c=e^{i\theta}$

Then $\left|\int f\cdot d\mu\right|=r=c\int f\cdot d\mu=c\int f\cdot d\mu^+-c\int f\cdot d\mu^-$

$=\int cf\cdot d\mu$

As $r\in\mathbb{R}$, $\int cf\cdot d\mu=\int\mathrm{Re}(cf)\cdot d\mu\le\int|\mathrm{Re}(cf)|\cdot d|\mu| \underset{|c|=1}{\le} \int|f|\cdot d|\mu|$ ■

$\boxed{3}$ $(\ge)$ Claim : for any measurable f with $|f|\le1$, $\left|\int_A f\cdot d\mu\right|\le|\mu|(A)$ for $\forall A\in\mathcal{A}$.

pf) $\left|\int_A f\cdot d\mu\right|\le\int_A|f|\cdot d|\mu|\le\int\chi_A\cdot d|\mu|=|\mu|(A)$

$(\le)$ Let $\mu=\mu^+-\mu^-$, $X=E\cup F$ & $E\cap F=\phi$, $\mu^-(F)=\mu^+(E)=0$.

Recall that $\mu^+=\mu|_F$ & $\mu^-=-\mu|_E$

Define $f(x)=\begin{cases}1 & x\in F\\ -1 & x\in E\end{cases}$ Then $|f|\le1$ and $\int_A f\cdot d\mu=\int_A f\cdot d\mu^+-\int_A f\cdot d\mu^-$

$=|\mu|(A)$ ■

$\boxed{4}$ ① λ is signed measure

- $\lambda(\emptyset) = 0$
- disjoint $\{E_n\}$

finite

$$\lambda(\cup_n E_n) = \mu(\cup_n E_n) - \nu(\cup_n E_n) \quad (\because \text{Def of } \lambda)$$

$$= \sum_{n=1}^{\infty} \mu(E_n) - \sum_{n=1}^{\infty} \nu(E_n)$$

$$= \sum_{n=1}^{\infty} \mu(E_n) - \nu(E_n) \quad (\because \text{absolutely converges})$$

$$= \sum_{n=1}^{\infty} \lambda(E_n).$$ Note that $\mu = \lambda - \nu$ finite. And $\sum |\lambda(E_n)| < \sum |\mu(E_n)| + \sum |\nu(E_n)| < \infty$.

② Let $\lambda = \lambda^+ - \lambda^-$ be Jordan decomposition, and $X = E \cup F$ & $\emptyset = E \cap F$

$$\mu(A) \geq \mu(A \cap F) \geq \mu(A \cap F) - \nu(A \cap F) = \lambda(A \cap F) = \lambda^+(A)$$

$$\nu(A) \geq \nu(A \cap E) \geq \nu(A \cap E) - \mu(A \cap E) = -\lambda(A \cap E) = \lambda^-(A)$$

$\boxed{5}$ ① $\mu + \nu$ is signed measure

trivial

② Let $\mu + \nu = \lambda$. Note that $\lambda = (\mu^+ + \nu^+) - (\mu^- + \nu^-)$.

finite positive measures.

$|\lambda|(A) = \lambda^+(A) + \lambda^-(A)$.

By 12.4, $\lambda^+(A) \leq (\mu^+ + \nu^+)(A)$ Done.

$\lambda^-(A) \leq (\mu^- + \nu^-)(A)$

$\boxed{6}$ Let $\mu = \mu^+ - \mu^-$, $X = E \cup F$ & $E \cap F = \emptyset$, $\mu^+(F) = \mu^-(E) = 0$.

$\leq$) Let $B = A \cap F \in \mathcal{A}$.

$\mu(B) = \mu^+(A) \leq$ RHS.

$\geq$) Let any $B \in \mathcal{A}$ with $B \subset A$ be given.

$\mu(B) = \mu^+(B) - \mu^-(B) \leq \mu^+(B) \leq \mu^+(A)$

$\boxed{7}$ Let $\mu = \mu^+ - \mu^-$, $X = E \cup F$ & $E \cap F = \emptyset$, $\mu^+(F) = \mu^-(E) = 0$.

$\leq$) Let $B_1 = A \cap F$ and $B_2 = A \cap E$

$|\mu(B_1)| + |\mu(B_2)| = \mu^+(A) + \mu^-(A) = |\mu|(A) \leq$ RHS

$\geq$) Let $A = \bigcup_{k=1}^{n} B_k$ and $\{B_k\}$ are disjoint

$$\sum_{k=1}^{n} |\mu(B_k)| \leq \sum_{k=1}^{n} |\mu|(B_k) = |\mu| \left(\bigcup_{k=1}^{n} B_k = A\right)$$