$\boxed{1}$ Construct $\{g_j\} = \{f_{n_j}\}$ s.t. $\mu(A_j = \{|g_{j+1} - g_j| \ge \frac{1}{2^j}\}) < 2^{-j}$ for each j.

Let $F_k = \bigcup_{j=k}^{\infty} A_j$. Then $\mu(F_k) \le \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}$

And for $x \in F_k^c$, $|g_i(x) - g_j(x)| \le \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \le 1/2^{j-1}$ $\forall i \ge j \ge k$. $(*)$

Let $F = \limsup_j A_j$ and define $f(x) = \begin{cases} \lim_{j\to\infty} g_j(x) & x \in F^c \\ 0 & x \in F \end{cases}$ Then $\mu(F) = \lim_{k\to\infty} \mu(F_k) = 0$

Then $g_j \to f$ in measure as $\overset{(*)}{|f(x) - g_j(x)|} \le \frac{1}{2^{j-1}}$ $\forall j \ge k$ $\forall x \in F_k^c$ Thus $\mu\{|g_j - f| > \frac{1}{2^{j-1}}\} \le \mu(F_k) \to 0$ as $k \to \infty$

It follows that $f_n \to f$ in measure

$\because \{|f_n - f| > \varepsilon\} \subseteq \underbrace{\{|f_n - g_j| > \tfrac{1}{2}\varepsilon\} \cup \{|f - g_j| > \tfrac{1}{2}\varepsilon\}}_{\text{measure goes to } 0 \text{ as } n,j\to\infty}$ ■

$\boxed{2}$ ① d is metric

$d(f,g) \ge 0$, $d(f,g) = d(g,f)$ $\forall g, f \in L^1$, $\frac{1}{1+|f-h|} \ge \frac{1}{1+|f-g|+|g-h|+|f-g|\cdot|g-h|}$

$$1 - \frac{1}{1+|f-g|} \le \frac{|f-g|(1+|g-h|) + |g-h|}{1+|f-g|+|g-h|+|f-g|\cdot|g-h|} \le \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|}$$

② $f_n \to f$ in measure iff $d(f_n, f) \to 0$

$(\rightarrow)$ Let $A_n = \{\frac{|f_n - f|}{1+ \,''} > \varepsilon\} = \{|f_n - f| > \frac{\varepsilon}{1-\varepsilon}\}$ for $\forall \varepsilon \in (0,1)$ $\Rightarrow \mu(A_n) \to 0$

$$d(f_n, f) = \int_{A_n} '' + \int_{A_n^c} \le \int_{A_n} d\mu + \int_{A_n^c} \varepsilon \cdot d\mu = \mu(A_n) + \varepsilon \cdot \mu(X)$$

By letting ε to be zero & $n \to \infty$, $d(f_n, f) \to 0$.

$(\leftarrow)$ As $\mu(X) < \infty$, $d(f_n, f) = \mu(X) - \int \frac{d\mu}{1+|f_n - f|}$ $\therefore \int \frac{d\mu}{1+|f_n-f|} \to \mu(X)$

Let $A_n = \{|f_n - f| > \varepsilon\}$

$$\int \frac{d\mu}{1+|f_n - f|} \le \int_{A_n} \frac{d\mu}{1+\varepsilon} + \int_{A_n^c} d\mu = \frac{\mu(A_n)}{1+\varepsilon} + \mu(X) - \mu(A_n) \quad \therefore \mu(A_n) \to 0$$ ■

$\boxed{3}$ WLOG, assume that $\liminf_n \int f_n < \infty \rightsquigarrow f$ integrable.

$\exists\{n_j\}$ s.t. $\lim_{j\to\infty} \int f_{n_j} = \liminf_n \int f_n$ $(*)$ And from $f_n \to f$ in measure, $f_{n_j} \to f$ in measure too.

$\exists\{n_{j_k}\}$ s.t $f_{n_{j_k}} \to f$ a.e.

$$\int f = \int \lim_{k\to\infty} f_{n_{j_k}} \le \lim_{k\to\infty} \int f_{n_{j_k}} \overset{(*)}{=} \lim_{j\to\infty} \int f_{n_j} = \liminf_n \int f_n$$ ■

↳ Fatou's lemma.

$(*)$ proof

$\liminf_n a_n = \sup_N \inf_{n \ge N} a_n. = \lim_{N\to\infty} \,''$

$\rightsquigarrow$ Let $\alpha_N = \inf_{n\ge N} a_n$. $\rightsquigarrow \exists N_k$ s.t. $\liminf a_n - \frac{1}{2^k} < \alpha_{N_k} < \liminf a_n + \frac{1}{2^k}$

$\exists n_k \ge N_k$ s.t. $\alpha_{N_k} \le a_{n_k} < \liminf_n a_n + \frac{1}{2^k}$

$\therefore \forall k \in \mathbb{N}$, $\exists n_k$ s.t. $|a_{n_k} - \liminf_n a_n| < \frac{1}{2^k}$.

Let $\forall \varepsilon > 0$ given. $\exists k_0 \in \mathbb{N}$ s.t. $\frac{1}{2^k} < \varepsilon$ $\forall k \ge k_0$

$\therefore |a_{n_k} - \liminf_n a_n| < \varepsilon$ $\forall k \ge k_0$ ■

i.e., $|a_{n_{k_j}} - \liminf_n a_n| < \varepsilon$ $\forall j \ge j_0$ where $k_{j_0} \ge k_0$.

$\boxed{4}$ $\exists \{n_j\}$ s.t $\chi_{n_j} \to f$ a.e. i.e. on E where $\mu(E^c) = 0$.

And if $\chi_{n_j}(x) \to f(x)$, $f(x)$ must be 0 or 1.

Spse not. Let $f(x) = c \in (0,1)$.

Then, for $\varepsilon < \min\{c, 1-c\}$. Then, for any $n \in \mathbb{N}$, $|\chi_n(x) - f(x)| > \varepsilon$. (contradiction)

And we can prove when $c > 1$ in the analogous way. [1,

$\therefore f(x) \in \{0,1\}$ $\forall x \in E$. Let $A = f^{-1}(\{1\})$, measurable.

$\hookrightarrow$ f is measurable as χ_{n_j} is measurable.

Then $f = \chi_A$ a.e. as $\mu(E^c) = 0$ ■

$\boxed{5}$ 1) A. U. $\to$ A.E.

$\forall n \in \mathbb{N}$, $\exists E_n \in \mathcal{A}$ s.t. $\mu(E_n^c) < \frac{1}{n}$ & $f_n \longrightarrow f$ unif. on E_k.

Let $E = \bigcup_{n=1}^{\infty} E_n$. By continuity from above, $\mu(E^c) = \lim_{n\to\infty} \mu(E_n^c) = 0$ & $f_n \to f$ unif on E.

Thus $f_n \to f$ pointwisely on E ■

2) A. U. $\to$ M

$\exists N \in \mathbb{N}$ s.t. $|f_n - f| < \varepsilon$ $\forall n \geq N$. on E.

$\rightsquigarrow \mu(E^c) \geq \mu(\{|f_n - f| \geq \varepsilon\})$ $\forall n \geq N$ ■

$\boxed{6}$ WTS: $\mu(\{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$

$$x \in \{x : \lim_{n\to\infty} f_n(x) \neq f(x)\} \leftrightarrow x \in \{x : \lim_{n\to\infty} f_n(x) = f(x)\}^c$$

$$\leftrightarrow x \in \{x : \forall m \in \mathbb{N}, \exists n_0 \in \mathbb{N} \text{ s.t. } |f_n(x) - f(x)| \leq \tfrac{1}{m}\}^c$$

$$\leftrightarrow x \in \Big(\bigcap_{m=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{|f_n - f| \leq \tfrac{1}{m}\}\Big)^c \subseteq \Big(\bigcap_{m=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{|f_n - f_n| \leq \tfrac{1}{m}\}\Big)^c$$

$$= \bigcup_{m=1}^{\infty} \underbrace{\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} \{|f_n - f| > \tfrac{1}{m}\}}_{\text{increasing w.r.t. } m.}$$

Hence, $\mu(\{f_n \not\to f\}) = 0$ if $\forall m \in \mathbb{N}$, $\mu\Big(\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} \{|f_n - f| > \tfrac{1}{m}\}\Big) = 0$.

$\leftrightarrow \forall \varepsilon > 0$, $\mu\Big(\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} |f_n - f| > \varepsilon\Big) = 0$ ■

$\boxed{7}$ $\mu(g_n > \varepsilon) \to 0$. Thus $\mu\Big(\bigcup_{m=n}^{\infty} |f_m - f_n| > \varepsilon\Big) \to 0$ as $n \to \infty$.

A^c $\mu\Big(\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0} \bigcup_{m=n_0}$

$x \in A^c \rightsquigarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|f_m - f_N| < \varepsilon$ $\forall m \geq N$.

$\mu($

$\mu(|f_m(x) - f_n(x)| > \varepsilon.$ $\forall m \geq n)$

$\mu\Big(\bigcap_{m=n}^{\infty} |f_m - f_n| > \varepsilon\Big)$ $\forall n \geq \infty$

$\boxed{8}$ No.

$X = [0,1]$

$$f_n(x) = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd.} \end{cases} \quad \text{if } \frac{k-1}{2^n} \le x < \frac{k}{2^n} \quad \text{for } k = 1, 2, \cdots, 2^n$$

① $g = \chi_{(a,b]}$.

Let $\frac{k_a - 1}{2^n} \le a < \frac{k_a}{2^n}$ Then $\int f_n \cdot g \le \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n}$

$\frac{k_b - 1}{2^n} \le b < \frac{k_b}{2^n}$.

② $g = \sum a_n \chi_{B_n}$

$\int f_n \cdot g = \sum a_n \int f_n \quad \le$

③ g integrable.

$$f_{n_j} \to \liminf f_n = f.$$

$$\int f \le \liminf_{j \to \infty} \int f_{n_j}$$

$|f_n - f| < \varepsilon \quad \forall n \ge N.$

$A_{nk} = \bigcup_{m=n}^{\infty} |f_m - f| > \frac{1}{k}.$

$A_{nk} \downarrow \underbrace{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} |f_m - f| > \frac{1}{k}}_{\to 0.}$

$F_j^c < \frac{\varepsilon}{2^j} \quad |f_n - f| < \varepsilon \quad \forall n \ge N_j$

$\therefore \mu(A_{nk}) < \frac{\varepsilon}{2^k}.$

$F^c = \bigcup_{j=1}^{\infty} F_j^c < \varepsilon.$

$A = \bigcup_{k=1}^{\infty}$

$F = \bigcap_{j=1}^{\infty} F_j^c$

9 With countable X and σ-algebra $\mathcal{A}$, define an equivalence class $\sim$ on X

: $x_1 \sim x_2$ if $\forall A \in \mathcal{A}$ $x_1 \in A$ iff $x_2 \in A$

For each $x \in X$, denote A_x the equiv class of x. $\rightarrow$ $A_x \cap A_y = \emptyset$ if $x \neq y$.

① $A_x \in \mathcal{A}$ and no proper subset (except $\emptyset$) of A_x is in $\mathcal{A}$.

pf) For $\forall y \notin A_x$ there exists $B_y \in \mathcal{A}$ s.t. $y \in B_y$ & $x \notin B_y$

$\rightsquigarrow A_x = \Big(\underbrace{\bigcup_{y \in X - A_x}}_{\text{countable}} B_y\Big)^c \in \mathcal{A}$: $\subseteq$) Let $z \in$ LHS. Spse $\exists y \in A_x^c$ s.t. $z \in B_y$.

As $B_y \in \mathcal{A}$, $x \in B_y$ by definition of $\sim$ (contradiction)

$\supseteq$) Let $z \in$ RHS. Spse $z \in A_x^c$.

Then, $\exists B_z$ s.t $z \in B_z$ & $x \notin B_z$ but $B_z \cap \text{RHS} = \emptyset$

Also, from definition of $\sim$, no proper subset (except $\emptyset$) of A_x is in $\mathcal{A}$.

If $A \subset A_x$ & $A \in \mathcal{A}$, $A \neq \emptyset$, $\exists z \in A \subseteq A_x$. $\forall y \in A_x$, as $z \in A$, $y \in A_x \rightsquigarrow A$ must be A_x.

② Every $A \in \mathcal{A}$ is a finite or countable disjoint union of equivalence classes.

: $\bigcup_{x \in A} A_x \subseteq A$

$\therefore \mathcal{A} = \sigma(\{A_x\})$ Note that as X is countable, $\{A_x\}$ is countable.

Claim: f_n is constant on each A_x.

pf) $\forall c \in \mathbb{R}$, as f_n is measurable, $f_n^{-1}(\{c\}) \in \mathcal{A}$ thus $f_n^{-1}(\{c\}) = \bigcup_{f_n(x)=c} A_x$.

So, for any given A_y, find $f_n(y)$. Then $A_y \subseteq f_n^{-1}(\{f_n(y)\})$

Hence, w.l.o.g. we assume $\mathcal{A} = \mathcal{P}(X)$

Denote $X = \{x_n \mid n \in \mathbb{N}\}$. Collect a subsequence $\{y_n\}$ by collecting any $x_n \in X$ with $\mu(\{x_n\}) > 0$.

Then, from $\mathcal{A} = \mathcal{P}(X)$, It is clear that $\mu(X) = \mu(Y)$ where $Y = \{y_n \mid n \in \mathbb{N}\}$.

Thus it suffices to show $f_n(y_k) \longrightarrow f(y_k)$ $\forall k \in \mathbb{N}$.

As $f_n \longrightarrow f$ in measure, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\mu(\{|f_n - f| > \varepsilon\}) < \mu(y_k)$ $\forall n \geq N$.

This implies $y_k \notin \{|f_n - f| > \varepsilon\}$ $\forall n \geq N$

$\longleftrightarrow |f_n(y_k) - f(y_k)| \leq \varepsilon$ $\forall n \geq N$ ∎