$\boxed{1}$ Let $^{\forall} a \in \mathbb{R}$ be given.
Let $R = \{ q \mid q \in \mathbb{Q},\ q < a \} \rightsquigarrow$ countable

Claim: $\{x \mid f(x) \geq a\} = \bigcap_{r \in R} \{x \mid f(x) > r\}$

pf) $\subseteq$) clear

$\supseteq$) Let $y \in$ RHS. Spse $y \notin$ LHS, i.e. $f(y) < a$.
Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $^{\exists} q \in \mathbb{Q}$ s.t. $f(y) < q < a$.
$\Longrightarrow q \in R$, $y \notin \{x \mid f(x) > q\}$ (contradiction) //

Since $\{x \mid f(x) \geq a\}$ is a countable intersection of elements in $\mathcal{A}$, $\{x \mid f(x) \geq a\} \in \mathcal{A}$ ■

$\boxed{2}$ For each $x \in (0,1)$, we can choose $r_x > 0$ s.t. $(x - r_x, x + r_x) \subseteq (0,1)$ & $f = g$ on $(x - r_x, x + r_x)$.
Then $\{(x - r_x, x + r_x) \mid x \in (0,1)\}$ covers $(0,1)$
By Lindelöf lemma, there exists a countable union that covers $(0,1)$.
Denote them by $I_1, I_2, \cdots$ $\rightsquigarrow$ Borel measurable as they are open sets.

Construct $J_n = I_n - \bigcup_{i=1}^{n-1} I_i \Rightarrow J_n$ pairwise disjoint & $J_n \uparrow (0,1)$
Then, $f = g$ on each J_n.

Define $f\chi_{J_n}(x) = \begin{cases} f(x) & x \in J_n \\ 0 & \text{o.w.} \end{cases}$ $\rightsquigarrow f\chi_{J_n} = g\chi_{J_n}$ & $g\chi_{J_n}$ is Borel measurable.
$\rightsquigarrow f\chi_{J_n}$

$\therefore f = \sum_{n=1}^{\infty} f\chi_{J_n}$ is also Borel measurable.

Or, it suffices to show that $f^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} (x_n - r_{x_n}, x_n + r_{x_n}) \cap (0,1) \cap g_{x_n}^{-1}((a,\infty))$ ■

$\boxed{3}$ $g^{-1}((a,\infty)) = \begin{cases} \emptyset & \in \mathcal{A} \quad (a<0) \\ X & \in \mathcal{A} \quad (a=0) \\ f^{-1}((0,\frac{1}{a})) & \in \mathcal{A} \quad (a>0) \end{cases}$

as f is measurable ■

$\boxed{4}$ sol 1) lim sup = lim inf
$A = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists}\} = \{x \mid \limsup_n f_n(x) = \liminf_n f_n(x)\}$

Let $g(x) = \begin{cases} 0 & \text{if } \limsup_n f_n(x) = \liminf_n f_n(x) = \pm\infty \\ \limsup_n f_n(x) - \liminf_n f_n(x) & \text{otherwise.} \end{cases}$

Hence, g is measurable.
Then $A = g^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} g^{-1}([0,\frac{1}{n})) \in \mathcal{A}$ measurable ■

sol 2) Cauchy complete
$A = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$
$= \{x \mid {}^{\forall} n \in \mathbb{N}, {}^{\exists} k_0 \in \mathbb{N} \text{ s.t. } |f_k(x) - f_m(x)| < \frac{1}{n} \ \ {}^{\forall} k, m \geq k_0\}$

$\therefore A = \bigcap_{n=1}^{\infty} \bigcup_{k_0=1}^{\infty} \bigcap_{k=k_0}^{\infty} \bigcap_{m=k_0}^{\infty} \underbrace{\{x \mid g_{k,m}^{-1}((-\frac{1}{n}, \frac{1}{n}))\}}_{\in \mathcal{A}}$ where $g_{k,m} = f_k - f_m \rightsquigarrow$ measurable

$\because (-1/n, 1/n) = (-\frac{1}{n}, \infty) \cap \left(\bigcup_{k=1}^{\infty} (\frac{1}{n} + \frac{1}{k}, \infty) \right)^c$
and inverse image preserves countable union & intersection & complement.

Thus, $A \in \mathcal{A}$ ■

$\boxed{5}$ Without loss of generality, we may assume $f \geq 0$.

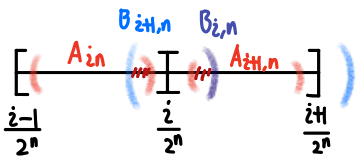
(Any function $f = f^+ - f^-$. Hence, if we find $g^+ = f^+$ a.e. & $g^- = f^-$ a.e.
$f = g = g^+ - g^-$ a.e.

The high-level idea is this : 1) $f : \mathbb{R} \to \mathbb{R}^+$. We will divide co-domain $\mathbb{R}$ into intervals.
Then, inverse image of these intervals will be Lebesgue measurable.
2) Environ these Lebesgue measurable sets with Borel measurable sets within measure zero.

By regularity of Lebesgue measure, for each $i \in \mathbb{N}$

$\exists \underbrace{A_{in}}_{G_\delta}$ & $\underbrace{B_{in}}_{F_\sigma}$ s.t. $A_{in} \subset f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})) \subset B_{in}$ & $m(B_{in} - A_{in}) = 0$.

Since A_{in} & B_{jn} may be overlapped, we'll discard these regions.



Let $C_n = \bigcup_{i=1}^{\infty} (B_{in} - A_{in}) \rightsquigarrow \mu(C_n) = 0$.

Let $A_{in}' = A_{in} - C_n$. Then we define a simple func. that approximates f on $\bigcup_i A_{in}'$

Define $S_n = \sum_{i=1}^{\infty} \frac{i-1}{2^n} \chi_{A_{in}'}$: Since A_{in}' are pairwise disjoint, it is well defined.
Also, $\chi_{A_{in}'}$ is Borel measurable for each $i \in \mathbb{N}$, so does $S_n(x) = \frac{i_x - 1}{2^n} \chi_{A_{i_x n}'}(x)$ for $x \in A_{i_x n}'$

$\rightsquigarrow |S_n(x) - f_n(x)| \leq \frac{1}{2^n}$ for $\forall x \in \bigcup_{i=1}^{\infty} A_{in}' = C_n^c$

Let $g(x) = \begin{cases} \lim_{n\to\infty} S_n(x) & \text{if } x \in (\bigcup_{n=1}^{\infty} C_n)^c \\ 0 & \text{o.w.} \end{cases}$

Then, $g(x) = \lim_{n\to\infty} S_n(x) = f(x)$ on $x \in (\bigcup_{n=1}^{\infty} C_n)^c$, and $f \neq g$ on $\underbrace{\bigcup_{n=1}^{\infty} C_n}_{\text{measure } 0}$

Also, g is clearly Borel measurable since C_n, S_n is Borel measurable

$\boxed{6}$ (1) $|f| = \sqrt{g^2 + h^2}$ Borel measurable by proposition 5.7.

(2) Note that $\arg z = \tan^{-1}\left(\frac{\text{Im } z}{\text{Re } z}\right)$

$\{z \mid \arg z > \theta\}$ $(0 \leq \theta < 2\pi)$

$= \{z \mid \frac{\text{Im } z}{\text{Re } z} > \tan\theta\}$

$\boxed{7}$ f is continuous. Hence, it's obvious that f is Borel measurable by proposition 5.6.

Let $f_n(x) = f(x+\frac{1}{n})$ $\rightsquigarrow$ also measurable.

Then $g_n = \dfrac{f_n - f}{1/n} \longrightarrow f'$ pointwisely $\forall x \in \mathbb{R}$. And g_n is measurable for each n.

Since $f'(x) = \lim_{n\to\infty} g_n(x) = \limsup_n g_n(x)$ and not diverge, it is Borel measurable.

$\boxed{8}$ Let A be a Vitali set in $[0,1]$.

$f_\alpha(x) = \chi_{\{\alpha\}}(x)$

$\rightsquigarrow g(x) = \chi_A(x)$: finite, but non-measurable ∎

$\boxed{9}$ 1) g: conti, f: Lebesgue measurable.

Let $\forall B \in \mathcal{B}$ be given

$(g\circ f)^{-1}(B) = f^{-1}\circ g^{-1}(B)$ Since g is conti, g is Borel measurable. $\rightsquigarrow$ $g^{-1}(B) \in \mathcal{B}$. also Borel set.

Since f is Lebesgue measurable, $f^{-1}(g^{-1}(B))$ is Lebesgue measurable. Thus, $g\circ f$ is Lebesgue measurable.

2) g: Borel measurable. f: Lebesgue measurable.

True (by same argument in 1))

3) g, f: Lebesgue measurable

No.

Let F be a Cantor function. Define $G(x) = F(x) + x$ on $[0,1]$.

Then G is 1) strictly increasing ($\therefore$ bijective) And G^{-1} is conti from $[0,2]$ to $[0,1]$ as
2) continuous from $[0,1]$ to $[0,2]$.

Note that $m(G(C)) = 1$ since $m([0,2]) = 2 = m(G(C)) + \underbrace{m([0,2] - G(C))}_{= m(G([0,1]-C))}$

From the construction of Cantor set, for disjoint closed sets I_n, $[0,1] - C = \bigcup_{n=1}^{\infty} I_n$.

Note that $G(I_n)$ is just shift of I_n by definition of Cantor set (constant on removed intervals.)

$\rightsquigarrow m(G(I_n)) = m(I_n) \rightsquigarrow m(G([0,1]-C)) = 1$

$\therefore$ $\exists$ non-measurable $A \subset G(C)$

Since $G^{-1}(A) \subset C$ & $m(C) = 0$, $m(G^{-1}(A)) = 0$. Since Lebesgue measure m is complete,

$\underbrace{G^{-1}(A)}_{\text{Let } B}$ is Lebesgue measurable.

Now, let $f = G^{-1}$ and $g = \chi_B$. Then g is clearly Lebesgue measurable, and since f is continuous & Lebesgue σ-algebra contains any open set, f is also Lebesgue measurable.

But $(g\circ f)^{-1}((\frac{1}{2}, \infty))$

$= f^{-1}\circ g^{-1}(\ \text{''}\ )$

$= f^{-1}(B) = A$: non-measurable ∎

11 Let $\forall \varepsilon > 0$ be given.

Claim : $f(nx) = nf(x)$ $\forall n \in \mathbb{Z}$ & $\forall x \in \mathbb{R}$.

pf) By induction, $f((n+1)x) = f(nx) + f(x) = (n+1)f(x)$

$f(-(n+1)x) = f(-nx) + f(-x) = -(n+1)f(x)$

$\rightsquigarrow f(-x) = -f(x)$

Borel measurable.

By Lusin's Theorem, $\exists$ closed $F \subset [0,1]$ (hence F is $c^{q}pt$) s.t. $m(F) > 0$ & $f|_F$ is continuous.

Since F is compact, $f|_F$ is uniform continuous on F, i.e. $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$. and $x, y \in F$

By Exercise 4.11., $\exists \delta' > 0$ s.t. $(-\delta', \delta') \subseteq F - F = \{x-y \mid x, y \in F\}$. Then, set $\delta = \min(\delta, \delta')$

Let $\forall h \in (-\delta, \delta)$ be given. Then, $\exists x, y \in F$ s.t. $x - y = h$

And $\forall a \in \mathbb{R}$, $f(a+h) - f(a) = f(h) = f(x-y) = f(x) - f(y)$

In other words, $\exists \delta > 0$ s.t. for $\forall a \in \mathbb{R}$ $|f(a+h) - f(a)| < \varepsilon$ for any $|h| < \delta$.

$\therefore f$ is continuous.