$\boxed{1}$ Claim A normed vector space $\mathcal{X}$ with $\|\cdot\| < \infty$ is complete iff every absolutely convergent series converges.

pf) ($\Longrightarrow$) Let $\sum \|x_n\| < \infty$. Let $S_N = \sum_{n=1}^{N} x_n$. Then $\|S_N - S_M\| \le \sum_{n=M+1}^{N} \|x_n\| \to 0$ as $N > M \to \infty$.
Thus, $\{S_N\}$ is Cauchy so convergent.

($\Longleftarrow$) Let $\{x_n\}$ be Cauchy. Choose $n_1 < n_2 < \cdots$ s.t. $\|x_n - x_m\| < \frac{1}{2^j}$ for $n, m \ge n_j$.
Let $y_k = \begin{cases} x_{n_1} & \text{if } k=1 \\ x_{n_k} - x_{n_{k-1}} & \text{o.w.} \end{cases}$ Then $\sum_{k=1}^{\infty} \|y_k\| \le \|y_1\| + \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$.

Thus $\lim_{k\to\infty} x_{n_k} = \sum_{n=1}^{\infty} y_n$ convergent. Since $\{x_n\}$ Cauchy, $\lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k}$

Then we can show that L^p is complete.

① $1 \le p < \infty$

Let $\{f_k \in L^p(X)\}$ with $\sum \|f_k\|_p < \infty$. Let $\sum_{k=1}^{n} |f_k| = G_n$ & $\sum_{k=1}^{\infty} |f_k| = G$

Then $G_n \uparrow G$. And $\|G_n\|_p \le \sum \|f_k\|_p < \infty$.

By MCT, $\int G^p \cdot d\mu = \lim_{n\to\infty} \int G_n^p \cdot d\mu = \lim_{n\to\infty} \|G_n\|_p^p < \infty$

let $F = \sum_{n=1}^{\infty} f_n$. $|F - \sum_{n=1}^{k} f_n|^p \le (|F| + G_k)^p \le (2G)^p < \infty$.

$\|F - \sum_{k=1}^{n} f_n\|_p \longrightarrow 0$ as $n \to \infty$ by DCT$_{\,//}$

② $p = \infty$

Let $G_k = \sum_{n=1}^{k} |f_n|$ & $G = \sum_{n=1}^{\infty} |f_n|$. $\|G\|_\infty \le \sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$ a.e.

Let $F = \sum_{n=1}^{\infty} f_n$.

$\|F - \sum_{n=1}^{k} f_n\|_\infty \le \|\sum_{n=k+1}^{\infty} f_n\|_\infty \le \sum_{n=k+1}^{\infty} \|f_n\|_\infty \to 0$ as $k \to \infty$ ∎

$\boxed{2}$ ① $1 \le p < \infty$.

Let $f \in L^p(X)$ be non-negative. $\exists \{s_n : \text{simple, non-negative}\}$ s.t. $s_n \uparrow f$

Note that $s_n \in L^p(X)$ as $s_n \le f$ & $|s_n|^p \le |f|^p \in L^1$.

Also, $|f - s_n|^p \le |f|^p \in L^1$ $\therefore$ By DCT, $\lim_{n\to\infty} \|f - s_n\|_p = \left(\lim_{n\to\infty} \int |f - s_n|^p\right)^{1/p} = 0$.

Now let $f \in L^p(X)$, not necessarily non-negative.

For $f = f^+ - f^-$, $\exists \{s_n^+\}, \{s_n^-\}$ s.t. $s_n^+ \uparrow f^+$ & $s_n^- \uparrow f^-$ and $\|f^+ - s_n^+\|_p \to 0$, $\|f^- - s_n^-\|_p \to 0$ as $n \to \infty$

Then $\|f - (s_n^+ - s_n^-)\|_p \le \|f^+ - s^+\|_p + \|f^- - s^-\|_p \to 0$ as $n \to \infty$.

② $p = \infty$

Let $f \in L^\infty(X)$ be non-negative. $\exists \{s_n : \text{simple, non-negative}\}$ s.t. $s_n \uparrow f$ uniformly as f is bounded.

Formally, let $\forall \varepsilon > 0$ be given. Then, $\exists n_0 \in \mathbb{N}$ s.t. $|f - s_n| < \varepsilon$ $\forall n \ge n_0$ for $\forall x \in X$.

That is, $\|f - s_n\|_\infty = \inf \underbrace{\{M > 0 : |f - s_n| < M \text{ a.e.}\}}_{\varepsilon \text{ belongs to this set } \forall n \ge n_0} \le \varepsilon$ $\forall n \ge n_0$ $\therefore \lim_{n\to\infty} \|f - s_n\|_\infty = 0$.

With this result, we can show the result for general $f \in L^\infty(X)$ as $1 \le p < \infty$ case ∎

$\boxed{3}$ $\{(x,t) : |f(x)| \ge t\} = A$ and $A^t = \{x : |f(x)| \ge t\}$ for $t \in [0, \infty)$

$$\int_0^\infty p t^{p-1}\, m(|f| \ge t) \cdot dt$$

$$= \int_0^\infty \int_{-\infty}^{\infty} p t^{p-1} \chi_{A^t}(x) \cdot dx \cdot dt \underset{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_0^{|f(x)|} p t^{p-1} \cdot dt \cdot dx = \int |f(x)|^p \cdot dx$$ ∎

$\boxed{4}$ Recall that $\|f\|_p = \left(\int |f|^p\right)^{1/p}$

$\|f\|_\infty = \text{ess sup } |f| = \inf\{M > 0 : |f| \le M \text{ a.e.}\}$

Fix $\|f\|_\infty > \delta > 0$ & $E_\delta = \{x \in [0,1] : |f(x)| \ge \|f\|_\infty - \delta\}$.

For $\forall p \in \mathbb{N}$, $\|f\|_p \ge \left(\int_{E_\delta} |\|f\|_\infty - \delta|^p\right)^{1/p} = (\|f\|_\infty - \delta) \cdot \underbrace{m(E_\delta)}_{\text{note that finite}}^{1/p}$

$\therefore$ As $\delta > 0$ arbitrary,

$\liminf_{p\to\infty} \|f\|_p = \sup_{p\to\infty} \inf_{k \ge p} \|f\|_k \ge \|f\|_\infty$.

For $\forall p \in \mathbb{N}$, $\|f\|_p \le \left(\int_0^1 \|f\|_\infty^p \cdot dm\right)^{1/p} = \|f\|_\infty$.

$\therefore \limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$ ∎

$\boxed{7}$ Spse $f \in L^r(X) \cap L^s(X)$.

$\rightsquigarrow \int |f|^r < \infty$, $\int |f|^s < \infty$.

$$\int |f|^p = \int_{\{|f|<1\}} |f|^p + \int_{\{|f|\ge 1\}} |f|^p < \int_{\{|f|<1\}} |f|^r + \int_{\{|f|\ge 1\}} |f|^s < \infty$$ ∎

$\boxed{9}$ (1) $\left|\int_0^1 f g_n\right| \le \int_0^1 |f g_n| \le \|f\|_2 \cdot \|g_n\|_2 = \frac{1}{\sqrt{n}}\|f\|_2 \longrightarrow 0$ as $n\to\infty$. ∎

$\boxed{15}$ Recall that $L^p(\mu) = \{\|f\|_p < \infty\}$

($\Leftarrow$) Assume $\sum_{n=1}^{\infty} (2^n)^p \mu\{|f|>2^n\} < \infty$. $\int_X |f|^p \cdot d\mu = \sum_{n=1}^{\infty} \int_{\{2^n<|f|<2^{n+1}\}} |f|^p \cdot d\mu + \int_{\{|f|<1\}} |f|^p \cdot d\mu$

$$< 2\sum_{n=1}^{\infty} (2^n)^p \cdot \mu(\{2^n<|f|<2^{n+1}\}) + \mu(\{|f|<2\})$$
$$\le 2\sum_{n=1}^{\infty} (2^n)^p \mu(\{2^n<|f|\}) + \mu(\{|f|<2\}) < \infty_{//}$$

($\Rightarrow$) Assume $\int |f|^p \cdot d\mu < \infty$. $\rightsquigarrow |f|^p < \infty$ a.e. Let $x\in X$ with $|f(x)|^p<\infty$.

$\rightsquigarrow \exists\, m \in \mathbb{N}$ s.t. $2^m < |f(x)|^p \le 2^{m+1}$

$$\sum_{n=1}^{\infty} (2^n)^p \chi_{\{|f|>2^n\}}(x) = \sum_{n=1}^{m} (2^n)^p \chi_{\{|f|>2^n\}}(x) = \sum_{n=1}^{m} 2^n \le 2^{m+1} < 2|f(x)|^p \text{ a.e.}$$

$\therefore$ By integrating both sides, done. ∎

$\boxed{19}$ ① $p=1$ Fubini's Theorem

② $p=\infty$

$\|f\|_{L^1(\nu)}(x) = \int_Y |f(x,y)| \cdot \nu(dy) \le \int_Y \|f(x,y)\|_{L^\infty(X)} \cdot \nu(dy)$. Done.

③ $1<p<\infty$

Let $H(x) = \int |f(x,y)| \cdot d\nu(y) = \|f\|_{L^1(\nu)}$

By Hölder's ineq. & Fubini's Thm,

$\|H\|_{L^p(\mu)}^{\cancel{p}} = \int \int |f(x,y)|\, H^{p-1}(x) \cdot \mu(dx) \cdot \nu(dy) \le \int_Y \left(\int_X |f(x,y)|^p \mu(dx)\right)^{\frac{1}{p}} \cdot \cancel{\|H\|_{L^p(\mu)}^{p-1}} \cdot \nu(dy)$

$\boxed{31}$ $\sum_{n\in\mathbb{Z}} |f(n+1)-f(n)|^p = \sum_{n\in\mathbb{Z}} \left|\int_n^{n+1} f'(x)\cdot dx\right|^p \overset{\text{Hölder}}{\le} \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f'(x)|^p \cdot dx$

$\forall x\in\mathbb{R}, \exists!\, n\in\mathbb{Z}$ s.t. $n<x\le n+1$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} |f'(x)|^p \cdot \sum_{n\in\mathbb{Z}} \chi_{(n,n+1]}(x)\cdot dx = \int_{\mathbb{R}} |f'(x)|^p \cdot dx < \infty.$$ ∎