$\boxed{1}$ Let $A_1, \cdots, A_n$ be pairwise disjoint.

$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

Since $\bigcup_{i=1}^{n} A_i$ is increasing, $\mu(\underbrace{\bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{n} A_i)}_{\bigcup_{i=1}^{\infty} A_i}) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ ∎

$\boxed{2}$ Let $A = \bigcup_{n=1}^{\infty} A_n$

Let $B_n = A - \bigcup_{k=1}^{n} A_k \downarrow \emptyset$

$\mu(B_n) \underset{\text{finite additivity}}{=} \mu(A) - \sum_{k=1}^{n} \mu(A_k)$ Done by $\lim_{n\to\infty} \mu(B_n) = 0$ ∎

$\boxed{3}$ $\mu(\emptyset) = 0$.

Let $A_1, \cdots, A_n, \cdots$ be pairwise disjoint. Note that only 2 cases are possible: $\begin{cases} \text{all } A_n \text{ countable.} \\ \text{only one } A_i \text{ is uncountable.} \end{cases}$

: if not, spse $A_i, A_j \in \mathcal{A}$ be uncountable. & disjoint. $\rightsquigarrow A_i \cap A_j = \emptyset$.

Hence $A_i \subseteq \overset{\text{countable}}{A_j^c}$ (contradiction).

$\therefore \mu$ is σ-additive.

$\boxed{4}$ $\mu(A \cup B) = \mu(A \cup (B \cap A^c))$

$= \mu(A) + \mu(A^c \cap B) = \mu(A) + \mu(B) - \mu(A \cap B)$

$\mu(B) = \mu(A \cap B) + \mu(A^c \cap B)$

$\boxed{5}$ $\sum_{n=1}^{\infty} a_n \mu_n(\emptyset) = 0$.

$\sum_{n=1}^{\infty} a_n \mu_n(\bigcup_{i=1}^{\infty} A_i) = \sum_{n=1}^{\infty} a_n \sum_{i=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} (\sum_{n=1}^{\infty} a_n \mu_n(A_i))$

$\boxed{6}$ $\nu(\emptyset) = 0$. clearly.

$\nu(\bigcup_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty} A_i \cap B) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \nu(A_i)$ for pairwise disjoint A_i ∎

$\boxed{7}$ $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu_n(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ as $\mu_n(A_i) \uparrow \mu(A_i)$

Now spse $\mu_n(A) \downarrow$ & $\mu_1(X) < \infty$

$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu_n(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i)$

As $\forall n \in \mathbb{N}$, $\mu_n(A_i) < \infty$ & $\sup_n \mu_n(A_i) = \mu_1(A_i) < \infty$ $\forall i$, $\sum_{i=1}^{\infty} \mu_1(A_i) = \mu_1(\bigcup_{i=1}^{\infty} A_i) < \infty$.

Hence, by (2), $\lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

$\boxed{8}$ Let $\mathcal{C} = \{B \mid B = A \cup N \text{ for some } A \in \mathcal{A}, N \in \mathcal{N}\}$. We'll show that $\mathcal{C}$ is the completion of $\mathcal{A}$.

Claim: $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N}) = \mathcal{C}$

pf) ($\subseteq$) To prove this, we show $\mathcal{C}$ is σ-algebra containing $\mathcal{A} \cup \mathcal{N}$.

① $\emptyset \in \mathcal{C}$ clearly.

② $B \in \mathcal{C} \to B = A \cup N \to B^c = A^c \cap N^c$

Note that $\exists M \in \mathcal{A}$ s.t. $N \subseteq M$ where $\mu(M) = 0$.

Then $B^c = A^c \cap N^c = A^c \cap ((M^c \cap N^c) \cup (M \cap N^c))$

$= (A^c \cap M^c \cap N^c) \cup (A^c \cap M \cap N^c)$

$= \underbrace{(A^c \cap M^c)}_{\in \mathcal{A}} \cup \underbrace{(A^c \cap N^c \cap M)}_{\subseteq M \ \therefore \in \mathcal{N}}$

③ Let $B_i \in \mathcal{C}$, pairwise disjoint

Let $B_i = A_i \cup N_i$ & $N_i \subseteq M_i$.

where $M_i \in \mathcal{A}$ & $\mu(M_i) = 0$ for $\forall i$.

$\bigcup_i B_i = \underbrace{(\bigcup_i A_i)}_{\in \mathcal{A} (\because \sigma\text{-algebra})} \cup (\bigcup_i N_i)$

And $\bigcup_i N_i \subseteq \bigcup_i M_i$

but $0 \leq \mu(\bigcup_i M_i) \leq \sum_i \mu(M_i) = 0$.

$\therefore \bigcup_i N_i \in \mathcal{N}$.

$\therefore \bigcup_i B_i \in \mathcal{C}$.

($\supseteq$) Clear: $B \in \mathcal{C} \rightsquigarrow \exists A \in \mathcal{A}, N \in \mathcal{N}$ s.t. $A \cup N = B \in \mathcal{B}$ ∎

Well-definedness of $\overline{\mu}$.

Spse $\overline{\mu}(B) = \mu(A_1) \neq \mu(A_2)$ where $B = A_1 \cup N_1 = A_2 \cup N_2$ & $N_i \subseteq M_i \in \mathcal{A}$ and $\mu(M_i) = 0$. for $i = 1, 2$.

Then $A_1 \subseteq A_1 \cup N_1 = B = A_2 \cup N_2 \subseteq A_2 \cup M_2$. $\rightsquigarrow \mu(A_1) \leq \mu(A_2) + \overbrace{\cancel{\mu(M_2)}}^{0}$

Similarly, $\mu(A_2) \leq \mu(A_1)$, thus $\mu(A_1) = \mu(A_2)$ (contradiction) $\therefore \overline{\mu}$ well-defined.

$\overline{\mu}$ is a measure on $\mathcal{B}$

$\overline{\mu}(\emptyset) = 0$ clearly.

Let $B_i \in \mathcal{B}$, pairwise disjoint. $\overline{\mu}(\cup_i B_i) = \overline{\mu}(\cup_i (A_i \cup N_i)) = \overline{\mu}(\cup_i A_i \cup \cup_i N_i) = \mu(\cup_i A_i) = \sum_i \overline{\mu}(B_i)$

$(X, \mathcal{B}, \overline{\mu})$ is complete.

Let $S \subset X$ and $B = A \cup N \in \mathcal{B}$. and $\overline{\mu}(B) = \mu(A) = 0$ where $S \subseteq B$. (i.e. let S be any $\overline{\mu}$-null set.

And since $N \in \mathcal{N}$, $\exists E \in \mathcal{A}$ s.t. $\mu(E) = 0$. and $N \subseteq E$ Then $A \cup E \in \mathcal{A}$ satisfies $0 \leq \mu(A \cup E) \leq \mu(A) + \mu(E) = 0$.

Thus any subset S of B, that is $\subset A \cup E$, is a μ-null set in $\mathcal{N}$. But for any S, $S = \underbrace{\emptyset}_{\in \mathcal{A}} \cup \underbrace{S}_{\in \mathcal{N}}$ thus $S \in \mathcal{B}$ ∎

uniqueness

Spse ν be another extension of μ to $\mathcal{B}$.

Let $B = A \cup N \in \mathcal{B}$ and $N \subseteq M$ where $\mu(M) = 0$ and $M \in \mathcal{A}$

Since ν & $\overline{\mu}$ are extension, $\nu(A) = \overline{\mu}(A) = \mu(A)$ $\forall A \in \mathcal{A}$

From this, $\nu(B) \leq \nu(\underbrace{A \cup M}_{\in \mathcal{A}}) = \overline{\mu}(A \cup M) \leq \mu(A)$

Since ν is measure on $\mathcal{B}$ and $A \in \mathcal{B}$, $\nu(A) \leq \nu(A \cup N) = \nu(B)$

$\therefore \nu(B) = \mu(A)$

Similarly, we have $\overline{\mu}(B) = \mu(A)$ $\therefore \nu = \overline{\mu}$ ∎

9 * Note $m((a,b)) = n((a,b)) \rightsquigarrow m([a,b]) = n([a,b])$ as $[a,b] = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$
$m((a,b]) = n((a,b])$ $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$

Fix $k \in \mathbb{N}$.

Step 1. Prove that $m(A) = n(A)$ whenever $A \in \mathcal{B}$ & $A \subset (-k,k]$, i.e. m & n agrees on $(-k,k]$

Let $\mathcal{B}_k = \{A \in \mathcal{B} \mid A \subset (-k,k]$ & $m(A) = n(A)\}$. on $(-k,k]$

$\mathcal{A}_k = \{\bigcup_{i=1}^{n} (a_i, b_i] \mid a_i, b_i \in \mathbb{R}, (a_i, b_i] \subset (-k,k], n \in \mathbb{N}\}$: clearly algebra on $(-k,k]$.

Note that m & n agrees on $\mathcal{A}_k$. by Note

$\therefore \mathcal{A}_k \subseteq \mathcal{B}_k$.

Claim: $\mathcal{B}_k$ monotone class (containing $\mathcal{A}_k$)

pf) Spse $A_i \uparrow A$, $A_i \in \mathcal{B}_k$. $\rightsquigarrow A \subset (-k,k]$ clearly & $m(A) = m(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} n(A_n) = n(A)$

Thus $A \in \mathcal{B}_k$.

Here, it indicates $(-k,k] - S$.

Note that if $S \subset (-k,k]$ & $S \in \mathcal{B}_k$, $S^c \in \mathcal{B}_k$ also, since $m(S) = n(S) \to -m(S^c) = -n(S^c)$

Spse $A_i \downarrow A$, $A_i \in \mathcal{B}_k$. Set $B_i = A_i^c$ Then $B = \bigcup_i B_i = \bigcup_i A_i^c = A^c$

$\rightsquigarrow B_i \uparrow A^c$, thus $A^c \in \mathcal{B}_k \to A \in \mathcal{B}_k$.

$\therefore$ By monotone class thm, $\mathcal{B}_k$ is a $\underbrace{\sigma\text{-algebra containing } \mathcal{A}_k}_{\text{Borel } \sigma\text{-algebra on } (-k,k]}$.

Hence m & n agrees on Borel sets in $(-k,k]$.

Step 2. Use continuity from below & show $m(A) = n(A)$ $\forall A \in \mathcal{B}$.

$m(A) = \lim_{k\to\infty} m(\underbrace{A \cap (-k,k]}_{\text{Borel sets in } (-k,k]}) = \lim_{k\to\infty} n(A \cap (-k,k]) = n(A)$

$A \cap (-k,k] \uparrow A$.

10 No (Both)

σ-finite

$X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, μ: counting-measure. ν: $2\times$ counting measure. $\Rightarrow$ σ-finite as $X = \bigcup_{n=1}^{\infty} \{n\}$.

$\mathcal{C} = \{\{k, k+1, \cdots\} \mid k \in \mathbb{N}\}$.

$\rightsquigarrow \forall S \in \mathcal{C}$, $\mu(S) = \nu(S) = \infty$.

But $\{1\} \in \sigma(\mathcal{C})$, which $\mu(\{1\}) = 1 \neq \nu(\{1\}) = 2$.

Finite

$X = \{1, 2, 3\}$, $\mathcal{A} = \mathcal{P}(X)$

$\mathcal{C} = \{\{1,2\}, \{2,3\}\}$

Let $m(\{1\}) = m(\{3\}) = 0$ $m(\{2\}) = 1$ $\rightsquigarrow$ $m(\{1,2\}) = n(\{1,2\}) = 1$
$n(\{1\}) = n(\{3\}) = 1$ $n(\{2\}) = 0$ $m(\{2,3\}) = n(\{2,3\}) = 1$

But $X \in \sigma(\mathcal{C})$ but $m(X) = 1$
$n(X) = 2$.