$\boxed{1}$ $X = \{0,1,2\}$

$M = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}\}$.

$\rightsquigarrow$ clearly not σ-algebra. $\{0\} \cap \{0,1\} \notin M$

But it is monotone class.

$X = \mathbb{Z}_{\geq 0}$

$M = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \cdots, X\}$

$\boxed{2}$ $X = \{0,1,2\}$.

$A_1 = \{\emptyset, \{0,1,2\}, \{0\}, \{1,2\}\}$

$A_2 = \{\emptyset, \{0,1,2\}, \{0,1\}, \{2\}\}$.

$> A_1 \cup A_2 = \{\emptyset, X, \{0\}, \{2\}, \{0,1\}, \{1,2\}\} \longrightarrow \{0\} \cup \{2\} \notin A_1 \cup A_2$

$\boxed{3}$ No.

Let $X = \mathbb{N}$

$A_n = \{A \mid A \subseteq \{1,2,\cdots,n\}$ or $A^c \subseteq \{1,\cdots,n\}\}$. $\rightsquigarrow$ clearly σ-algebra.

Then $A_n \uparrow A = \bigcup_{n=1}^{\infty} A_n$.

But A is not σ-algebra : $S_k = \{2k\} \rightsquigarrow S_k \in A$ $\forall k$ but $\bigcup_{k=1}^{\infty} S_k \notin A$ since $\forall n \in \mathbb{N}$, $\bigcup_{k=1}^{\infty} S_k \notin A_n$.

$\boxed{4}$ No.

$M_n = \mathcal{P}(\{1,2,\cdots,n\})$

$\Rightarrow M_n \uparrow M. = \bigcup_{n=1}^{\infty} M_n$.

Let $A_n = \{1,2,\cdots,n\}$ $\rightsquigarrow A_j \uparrow \mathbb{N}$.

But $\mathbb{N} \notin \bigcup_{n=1}^{\infty} M_n$ as $\mathbb{N} \notin M_n$ for $\forall n=1,2,\cdots$

$\boxed{5}$ $f: X \to Y$. A : σ-algebra. Reall that inverse map preserves countable union & intersection.

① $\emptyset \in \mathcal{B}$

: $\emptyset \in A \Rightarrow f^{-1}(\emptyset) = \emptyset \in \mathcal{B}$

② $B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$

For some $A \in A$, $B = f^{-1}(A) \Rightarrow B^c = f^{-1}(A^c)$.

③ $B_1, \cdots, B_n, \cdots \in \mathcal{B} \rightsquigarrow \bigcap_{i=1}^{\infty} B_i \in \mathcal{B}$

$\bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} f^{-1}(A_i) = f^{-1}(\bigcap_{i=1}^{\infty} A_i)$

1) $x \in \bigcap_{i=1}^{\infty} f^{-1}(A_i) \rightsquigarrow f(x) \in A_i$ $\forall i$.

$\rightsquigarrow f(x) \in \bigcap_{i=1}^{\infty} A_i$

2) $x \in f^{-1}(\cap_i A_i)$

$\rightsquigarrow f(x) \in A_i$ $\forall i$.

$\rightsquigarrow x \in f^{-1}(A_i)$ $\forall i$.

$\boxed{6}$ It is clear that A is not finite.

And any infinite σ-algebra is uncountable (exercise 2.8)

$\boxed{7}$ ① $\emptyset \in A$ as $\chi_\emptyset(x) = 0$. const func.

② $A \in A \rightsquigarrow \chi_{A^c}(x) = 1 - \chi_A(x) \in \mathcal{F}$. $\rightsquigarrow A^c \in A$

③ $A_i \in A$ $\forall i = 1,2,\cdots$.

$\chi_{\bigcap_{i=1}^{n} A_i}(x) \longrightarrow \chi_{\bigcap_{i=1}^{\infty} A_i}(x)$ point-wise $\therefore \chi_{\bigcap_{i=1}^{\infty} A_i} \in \mathcal{F}$ ∎

$\boxed{8}$ No.

Let $\mathcal{A}$ be a countable infinite σ-algebra containing a set X. Thus denote $\mathcal{A} = \{A_i \mid i \in \mathbb{N}\}$ ↗ must be ∞.

Define $f : X \to \mathcal{A} : f(x) = \bigcap_{x \in A_i} A_i$ i.e. $f(x)$ is the smallest element of $\mathcal{A}$ that contains $x \in X$. (*)

⇝ Then $f(x) \cap f(y) = \emptyset$ for $\forall x, y \in X$, $x \neq y$.

Spse not. Clearly, $f(x) \supseteq f(x) \cap f(y)$ But due to (*), $f(x) \subseteq f(x) \cap f(y)$.

Similarly, $f(y) = f(x) \cap f(y)$ $\therefore f(x) = f(y) \neq \emptyset$.

⇝ set arbitrary $x \in X$. Then $f(x) = X$ since $X \subseteq f(x)$ as $y \in X \to y \in f(y) = f(x)$. & $f(x) \subseteq X$ clearly.

Thus $\mathcal{A} = \{\emptyset, X\}$, which contradicts the assumption that $\mathcal{A}$ is infinite.

$\therefore$ There exists a partition of $\mathcal{A}$, denote $\mathcal{B} = \{f(x) \mid x \in X\} \cup \{\emptyset\}$.

i.e. $\bigcup_{B \in \mathcal{B}} B = \mathcal{A}$. • Note : hence $|\mathcal{A}| \leq 2^{|\mathcal{B}|}$

And for any $N \subset \mathbb{N}$, $\bigcup_{i \in N} \underbrace{B_i}_{\in \mathcal{A}} \in \mathcal{A}$. $\longrightarrow 2^{|\mathbb{N}|} \leq |\mathcal{A}|$

$\therefore$ σ-algebra must be finite or uncountably infinite.

$\boxed{9}$ (1) clear

(2) $A_i = \begin{cases} \{-1, 1\} & i = 2k \\ \{0\} & i = 2k-1 \end{cases}$

$\Rightarrow \limsup_i A_i = \{-1, 0, 1\}$. $\liminf A_i = \emptyset$.

(3) Let $\forall x$ be given.

If $x \in \liminf A_i \Rightarrow \chi_{\liminf A_i}(x) = 1$

$\Downarrow$

$\exists k \in \mathbb{N}$ s.t. $x \in \bigcap_{i=k}^{\infty} A_i$

Hence $\chi_{A_i}(x) = 1$ $\forall i \geq k$

⇝ $\liminf_i \chi_{A_i}(x) = \lim_{n \to \infty} (\inf_{m \geq n} \chi_{A_m}(x)) = 1$

If $x \notin \liminf A_i \Rightarrow \chi_{\liminf A_i}(x) = 0$

$\Downarrow$

For any $i \in \mathbb{N}$, $\exists j \geq i$ s.t. $x \notin A_j$ i.e. $\chi_{A_j}(x) = 0$.

Hence $\liminf_i \chi_{A_i}(x) = \lim_{n \to \infty} (\inf_{m \geq n} \chi_{A_m}(x)) = 0$.

Thus, for any x, $\chi_{\liminf A_i}(x) = \liminf_i \chi_{A_i}(x)$. limsup case can be proved in analogous way.