$\boxed{2}$ Let $\{s_n\}$ be non-negative simple functions increasing to f.

Write $s_n = \sum_{j=1}^{m_n} a_{nj} \chi_{A_{nj}}$

$\int s_n \cdot d\rho = \sum_{j=1}^{m_n} a_{nj} \mu(A_{nj} \cap E) = \int s_n \cdot \chi_E \, d\mu$ Note that $s_n \cdot \chi_E \uparrow f \cdot \chi_E$

$\therefore$ By monotone convergence thm, $\int f \cdot d\rho = \lim_{n\to\infty} \int s_n \cdot \chi_E \cdot d\mu = \int f \cdot \chi_E \, d\mu$ ■

$\boxed{4}$ (1) True.

$\subseteq$) $\{(r_1-\varepsilon_1, r_1+\varepsilon_1) \times (r_2-\varepsilon_2, r_2+\varepsilon_2) \mid r_i \in \mathbb{Q},\ \varepsilon_i \in \mathbb{Q}\}$ is an open basis of $\mathbb{R}^2$.

As its elements are measurable rectangles, $G_2 \subseteq \mathcal{B}_1 \times \mathcal{B}_1$ where G_2 is the collection of open sets in $\mathbb{R}^2$.

topology: can be written as union of elements of basis.

Since $\mathcal{B}_1 \times \mathcal{B}_1$ is σ-algebra, $\mathcal{B}_2 = \sigma(G_2) \subseteq \mathcal{B}_1 \times \mathcal{B}_1$

$\supseteq$) Define $\mathcal{C}(B) = \{A \subseteq \mathbb{R} : A\times B \in \mathcal{B}_2\}$ for $B \subseteq \mathbb{R}$.

Then it is a σ-algebra containing every open set for open set B in $\mathbb{R}$:
① product of open sets is open, i.e. $\in \mathcal{B}_2$
② $A \in \mathcal{C}(B) \to A^c \in \mathcal{C}(B)$ as $A^c \times B = (\mathbb{R}\times B) \cap (A\times B)^c$
③ Let $A_1, \cdots, A_n \in \mathcal{C}(B)$. $\left(\bigcap_{k=1}^{\infty} A_k\right) \times B = \bigcap_{k=1}^{\infty} (A_k \times B) \in \mathcal{B}_2$.

$\Rightarrow \mathcal{B}_1 \subseteq \mathcal{C}(B)$.

i.e. $\forall A \in \mathcal{B}_1$, $A\times B \in \mathcal{B}_2$.

Define $D = \{B \subseteq \mathbb{R} \mid \underline{\mathcal{B}_1 \subseteq \mathcal{C}(B)}\}$.

Then it is a σ-algebra containing every open set:
① $\emptyset \in D$ as $\emptyset$ is open in $\mathbb{R}$.
② $B \in D \to B^c \in D$ as $\forall A \in \mathcal{B}_1$, $A \times B^c = \underbrace{(A\times\mathbb{R})}_{\in \mathcal{B}_2} \cap \underbrace{(A\times B)^c}_{\in \mathcal{B}_2} \in \mathcal{B}_2$.
as $\mathbb{R}$ is open
③ Let $B_1, \cdots, B_n \in D$.
$\forall A \in \mathcal{B}_1$, $A \times \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} A \times B_k \in D$ clearly.

$\Rightarrow \mathcal{B}_1 \subseteq D$.

$\therefore \forall A, B \in \mathcal{B}_1$, $A\times B \in \mathcal{B}_2$. Note that $\mathcal{B}_2$ is σ-algebra.

$\leadsto \mathcal{B}_1 \times \mathcal{B}_1 = \sigma(\{A\times B\}) \subseteq \mathcal{B}_2$.

(2) True. $\mathcal{L}_2 = \overline{\mathcal{B}_2} = \overline{\mathcal{B}_1 \times \mathcal{B}_1}$ by (1)

Remark. $\mathcal{L}_2 \neq \overline{\mathcal{B}_1} \times \overline{\mathcal{B}_1} = \mathcal{L} \times \mathcal{L}$

pf) Let V be a Vitali set on $\mathbb{R}$.
Set $N = V \times \{0\} \leadsto$ obviously $N \notin \mathcal{L}\times\mathcal{L}$
But since $\underset{\sim}{N \text{ is a } m_2\text{-null set}}$ and $\mathcal{L}_2$ complete, $N \in \mathcal{L}_2$
$N \subseteq \mathbb{R}\times\{0\} \subset \mathbb{R}^2$.

$\boxed{5}$ Let $A = \{(x,t) \mid 0 \le t \le |f(x)|\} \subseteq \mathbb{R}^2$. Define $\chi_A : \mathbb{R} \times \mathbb{R}_{\ge 0} \longrightarrow \{0,1\}$. non-negative

Note that $m(|f| \ge t) = \int_{-\infty}^{\infty} \chi_A \cdot dx$

$$\text{RHS} = \int_0^\infty \int_{-\infty}^\infty \chi_A \cdot dx \cdot dt \underset{\text{Fubini's thm}}{=} \int_{-\infty}^\infty \int_0^\infty \chi_A \cdot dt \cdot dx = \int_{-\infty}^\infty [\,t\,]_0^{|f(x)|} \cdot dx = \int_{-\infty}^\infty |f(x)| \cdot dx \ ■$$

$\boxed{6}$ $1 = m_2(A) = \int_{[0,1]^2} \chi_A \cdot dm_2 \underset{\text{Fubini}}{=} \int_{[0,1]} \int_{[0,1]} \chi_A(x,y) \, dy \cdot dx$

$= \int_{[0,1]} m(S_x(A)) \cdot dx$

$\therefore \int_0^1 [1 - m_2(S_x(A))] \cdot dx = 0$ & $m(S_x(A)) \le 1$. $\therefore m_2(S_x(A)) = 1$ a.e. by Thm 8.1.

$\boxed{9}$ $\int_0^1\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^{3/4}}\log(4+\sin x)\right|\cdot dy\cdot dx$

$\le \int_0^1\int_0^1 \frac{x^2+y^2}{(x^2+y^2)^{3/4}}\log 5\cdot dy\cdot dx = \int_0^1\int_0^1 (x^2+y^2)^{1/4}\log 5\cdot dy\cdot dx \le 2^{\frac{1}{4}}\log 5 < \infty$. Done by Fubini's theorem.

$\boxed{10}$ (1) a. Let $h(x,y) = x-y$. (continuous) $\rightsquigarrow$ Borel measurable.

$D = h^{-1}(\{0\})$

b. $D = \bigcap_{n=1}^{\infty}\bigcup_{i=1}^{n}\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{i-1}{n},\frac{i}{n}\right]$: $\subseteq$) Let $(x,x)\in D$. $\forall n\in\mathbb{N}$, $\exists i\in\{0,\cdots,n\}$ s.t. $\frac{i-1}{n}\le x\le\frac{i}{n}$ as $x\in[0,1]$.

$\supseteq$) Let $(x,y)\in$ RHS. Spse $x\ne y$. Then for large n with $\frac{1}{2^n}<|x-y|$,

$x\times y \notin \bigcup_{i=1}^{n}[(i-1)/2^n, i/2^n]\times[(i-1)/2^n, i/2^n]$ $(\rightarrow\leftarrow)$

(2) $\int_X\int_Y$ " $= 1$ as $D_x = \{(x,x)\}$.

$\int_Y\int_X$ " $= 0$ as $m(D^y) = 0$.

But it doesn't contradict thm as $(Y,\mathcal{B},\mu)$ isn't σ-finite. ∎

$\boxed{11}$ $\iint f(x,y)\,dy\,dx = 0$,

Note that $f(x,y) = \begin{cases} 1 & x\ge 0\ \&\ y-1<x\le y \\ -1 & x\ge 0\ \&\ y-2<x\le y-1 \\ 0 & \text{o.w.}\end{cases}$ $\iint f(x,y)\,dx\,dy = \int_{-\infty}^0 0\cdot dy + \int_0^1 y\cdot dy + \int_1^2 (1+1-y)\cdot dy + \int_2^\infty 0\cdot dy$

$= \frac{1}{2} + 2 - 2 + \frac{1}{2} = 1$

But it doesn't contradict thm as $\iint |f(x,y)|\cdot dy\cdot dx = \infty$ ∎

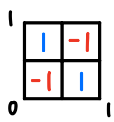
$\frac{1}{x} = \int_0^\infty e^{-xy}\cdot dy$

$\int_0^b\int_0^\infty e^{-xy}\sin x\cdot dy\cdot dx$

$\int_0^b\int_0^\infty e^{-xy}\sin x$

$\le$

$\boxed{12}$ Let $f(x,y) = \begin{cases} -1 & 0\le x<\frac{1}{2},\ 0\le y<\frac{1}{2} \\ 1 & \frac{1}{2}\le x\le 1,\ 0\le y<\frac{1}{2} \\ 1 & 0\le x<\frac{1}{2},\ \frac{1}{2}\le y\le 1 \\ -1 & \frac{1}{2}\le x\le 1,\ \frac{1}{2}\le y\le 1\end{cases}$



$\int_0^1 f(x,y)\cdot dy = \int_0^1 f(x,y)\cdot dx = 0$.

$\boxed{13}$ Let $A = \{(x,y)\mid x<y\le x+c\}$

$\{(x,y)\mid y\le x\}$ is measurable on $\mathcal{B}\times\mathcal{B}$.

$\int_{\mathbb{R}}[f(x+c)-f(x)]\cdot dx = \int_{\mathbb{R}}\int_{\mathbb{R}}\chi_A(x,y)\cdot d\mu(y)\cdot dx$

$= \int_{\mathbb{R}}\int_{\mathbb{R}}\chi_A(x,y)\cdot dx\cdot d\mu(y)$

$= \int_{\mathbb{R}}\underbrace{m([y-c,y))}_{c}\cdot d\mu(y) = c\mu(\mathbb{R})$ ∎

$\mu\ \&\ \nu$ finite.

$\mathcal{C} = \{E\in A\times\mathcal{B}\mid \mu\times\nu(E) = \lambda(E)\}$.

Claim: $\mathcal{C} = \mathfrak{M}(\mathcal{C}_0)$

pf) $E_n\uparrow E$

$\in \mathcal{C}$

$\mu\times\nu(E_n) =$

$\boxed{14}$ Note that $\int_0^\infty\int_0^\infty |e^{-xy}\cdot \sin x|\cdot dy \cdot dx \le \int_0^\pi \frac{|\sin x|}{x} < \infty$

① $\int_0^b\int_0^\infty e^{-xy}\sin x \cdot dy\cdot dx = \int_0^b \sin x \int_0^\infty e^{-xy}\cdot dy\cdot dx = \int_0^b \frac{\sin x}{x}\cdot dx.$

② $\int_0^b\int_0^\infty e^{-xy}\sin x \cdot dy\cdot dx = \int_0^\infty\int_0^b e^{-xy}\cdot \sin x\cdot dx\cdot dy$

$= \int_0^\infty \frac{1}{1+y^2}\cdot\left[e^{-by}\cdot(-y\sin b - \cos b)+1\right]\cdot dy$

$$\left|\frac{1}{1+y^2}\cdot\left[e^{-by}\cdot(-y\sin b-\cos b)+1\right]\right|$$
$$\le \left|\frac{e^{-by}}{1+y^2}\,y\right| + \frac{e^{-by}}{1+y^2} + \frac{1}{1+y^2}$$
$$\le \frac{e^{-y}}{1+y^2}\cdot y + \frac{2}{1+y^2} \quad \text{for } b=1,2,\cdots$$
$$\underset{1+y^2\ge 2y}{\le} \frac{1}{2}e^{-y} + \frac{2}{1+y^2} \longrightarrow \text{integrable on } [0,\infty)$$

$\therefore \lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\cdot dx = \int_0^\infty \frac{dy}{1+y^2} = [\arctan y]_0^\infty = \frac{\pi}{2}$ by DCT ■

$\boxed{15}$ $\iint f(x,y)\,\mu(dx)\,\mu(dy) = \int \sum_{k=1}^\infty f(k,y)\,\mu(dy) = 0$

$\iint f(x,y)\,\mu(dy)\,\mu(dx) = \int \sum_{k=1}^\infty f(x,k)\,\mu(dx) = \sum_{x=1}^\infty \sum_{k=1}^\infty f(x,k) = 1$

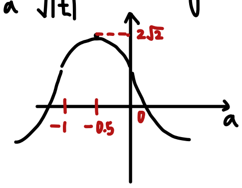
But it doesn't contradict thm as $\iint |f(x,y)|\,\mu(dx)\,\mu(dy) = \infty$ ■

$\boxed{16}$ Notice that it suffices to show $\sum_{n=1}^\infty \frac{|a_n|}{\sqrt{|x-r_n|}} < \infty$ $\forall x\in[k,k+1]$ a.e. for any $k\in\mathbb{N}$.

Then $m\left(\left\{x\in\mathbb{R} \,\middle|\, \sum_{n=1}^\infty \frac{|a_n|}{\sqrt{|x-r_n|}} = \infty\right\}\right) = \sum_{k=-\infty}^\infty m\left(\left\{x\in[k,k+1] \,\middle|\, {}'' \right\}\right) = 0.$

$$\int_k^{k+1}\left(\underbrace{\sum_{n=1}^\infty}_{\substack{\mu:\text{ counting measure on }\mathbb{N}.\\ (\sigma\text{-finite})}} \frac{|a_n|}{\sqrt{|x-r_n|}}\right)dx = \sum_{n=1}^\infty \int_k^{k+1} \frac{|a_n|}{\sqrt{|x-r_n|}}\,dx \le \sum_{n=1}^\infty 2\sqrt{2}\,|a_n| < \infty$$

$\int_a^{a+1}\frac{dt}{\sqrt{|t|}} = 2\left(\operatorname{sign}(a+1)\sqrt{|a+1|} - \operatorname{sign}(a)\sqrt{|a|}\right)$



$\therefore \int_a^{a+1}\frac{dt}{\sqrt{t}} \le 2\sqrt{2}.$

$\therefore$ By Exercise 6.1., $\sum_{n=1}^\infty \frac{|a_n|}{\sqrt{|x-r_n|}} < \infty$ a.e. on $[k,k+1]$ ■

$\boxed{17}$ Let $\mathcal{C} = \{E\in\mathcal{A}\times\mathcal{B} \mid \lambda(E) = \mu\times\nu(E)\}. \subseteq \mathcal{A}\times\mathcal{B}$

It's obvious that $\mathcal{C}_0 \subseteq \mathcal{C}$

Claim : $\mathcal{C}$ is monotone class. Then $\mathcal{C} = \mathcal{A}\times\mathcal{B}$ by monotone class theorem and $\mathcal{A}\times\mathcal{B} = \sigma(\mathcal{C}_0)$

pf) ① μ,ν finite

Let $E_n\in\mathcal{C}$ and $E_n\uparrow E$. $\lambda(E) = \lim_{n\to\infty}\lambda(E_n) = \lim_{n\to\infty}\mu\times\nu(E_n) = \mu\times\nu(E)$ by continuity from below of measures.

$\therefore E\in\mathcal{C}$

Let $E_n\in\mathcal{C}$ and $E_n\downarrow E$. $\lambda(E) = \lim_{n\to\infty}\lambda(E_n) = \lim_{n\to\infty}\mu\times\nu(E_n) = \mu\times\nu(E)$ by continuity from above of measures and μ,ν finite.

$\therefore E\in\mathcal{C}$

② μ,ν σ-finite

$\exists\{X_n\times Y_n\}$ with $X_n\uparrow X$, $Y_n\uparrow Y$ and $\mu\times\nu(X_n\times Y_n)<\infty$

For $E\in\mathcal{A}\times\mathcal{B}$, $\lambda(E) = \lim_{n,m\to\infty}\underbrace{\lambda(E\cap(X_n\times Y_m))}_{<\infty} = \lim_{n,m\to\infty}\mu\times\nu(E\cap(X_n\times Y_m)) = \mu\times\nu(E)$ ■