$\boxed{1}$ ① $\alpha$ is increasing & right conti

$x<y \Rightarrow \alpha(y)-\alpha(x)=\mu((x,y])>0.$

$$\lim_{z\to x+}\alpha(z)=\begin{cases}\lim_{z\to x+}\mu((0,z]) = \mu((0,x]) \\ \lim_{z\to x+}\mu((-z,0]) \underset{\uparrow}{=} \mu((-x,0])\end{cases} = \alpha(x)$$

since $\mu$: measure

② Let $\mathcal{C}$ be a collection of $(a,b]$.
Let $\overline{\mathcal{C}}$ be a finite union of sets in $\mathcal{C}$. Define $\ell: \overline{\mathcal{C}} \to \mathbb{R}$ s.t. $\ell((a,b])=\alpha(b)-\alpha(a)$

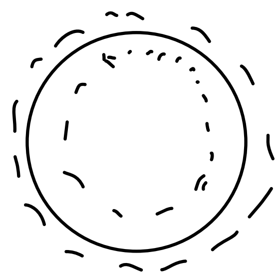
③ Note that $\overline{\mathcal{C}}$ is an algebra.
And, $\mu=\ell$ on $\overline{\mathcal{C}}$ as $\mu((a,b])=\alpha(b)-\alpha(a)=\ell((a,b])$

Let $K_n=(-n,n]$. Then $\bigcup K_n=\mathbb{R}$ & $\ell(K_n)=\mu(K_n)\le\mu(\underbrace{[-n,n]}_{\text{compact.}})<\infty$

$\therefore$ $\ell$ is $\sigma$-finite.

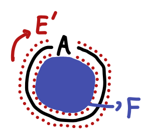
By Thm 4.17 (4), Lebesgue-Stieltjes measure $m$ corresponding to $\alpha$ is a unique extension on $\sigma(\mathcal{C})=\mathcal{B}$.
Since $\mu$ is measure on $\mathcal{B}$, $m=\mu$

$\boxed{2}$ For $A$, by definition of infimum, set $E=\bigcup_{i=1}^{\infty}(a_i,b_i]$ s.t. $A\subset E$ & $m(E)<m(A)+\varepsilon$.
Define $G=\bigcup(a_i, b_i+\frac{\varepsilon}{2^{i+1}})$
$\rightsquigarrow m(G)<m(A)+2\varepsilon.$



① A is bdd.
Note that $B=\underbrace{\overline{A}}_{\text{closed}}-A$ is lebesgue measurable. (both Lebesgue measurable.)

$\rightsquigarrow \exists$ open $E'\supset B$ s.t. $m(E')<m(B)+\varepsilon$.




Let $F=\overline{A}-E'$. Then $F$ is closed and bdd $\rightsquigarrow$ compact. Also, $F\subset A$

$$\begin{aligned} m(F) &= m(A)-m(A\cap E') \\ &= m(A)-[m(E')-m(E'-A)] \ge m(A)-m(E')+m(B) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = m(A)-\varepsilon. \end{aligned}$$

$\therefore m(G-F)\le m(G-A)+m(A-F)\le 2\varepsilon+\varepsilon=3\varepsilon.$

② A is unbdd
Consider $A_n=A\cap[n,n+1]$
By the same argument $\forall\varepsilon>0$ $F_i\subset A_i$ closed & bdd s.t. $m(F_i)>m(A_i)-\varepsilon\cdot\frac{1}{2^{|i|+1}}$

Set $H_n=\bigcup_{i=-n}^{n}F_i$ : closed & bdd $\subset A$. let $F=\bigcup_{i=-\infty}^{\infty}F_i$

$m(A-F)\le m(\bigcup_{i=-n}^{n}(A_i-F_i))$. By letting $n\to\infty$, we have $m(A-F)<\varepsilon$.

## 3 Outer measure

① $\mu^*(\emptyset)=0$ clearly

② monotone — must exist, e.g. $A=X$.

$K\subset L \subseteq X$. Let $\forall A\in\mathcal{A}$ s.t. $L\subset A$ be given. Then $K\subset A$ too.

As $\mu(A)\in\{\mu(B)\mid K\subset B, B\in\mathcal{A}\}$, $\mu^*(K)\le\mu(A)$ $\forall A\supset L \rightsquigarrow \mu^*(K)\le\mu^*(L)$

③ sub-additivity.

Let $A_1, A_2, \cdots \subseteq X$ be given. Fix $\forall\varepsilon>0$.

Then $\exists B_i\in\mathcal{A}$ & $A_i\subset B_i$ s.t. $\mu^*(A_i)+\frac{\varepsilon}{2^i}>\mu(B_i)$ for each $i$.

by countable monotonicity of measure

$\rightsquigarrow \bigcup_i B_i\in\mathcal{A}$ and $\bigcup_i A_i\subset\bigcup_i B_i \rightsquigarrow \mu^*(\bigcup_i A_i)\le\mu(\bigcup_i B_i)\le\sum_{i=1}^{\infty}\mu(B_i)\le\sum_{i=1}^{\infty}\mu^*(A_i)+\varepsilon.$

Since $\varepsilon$ is arbitrary, done. ∎

### Each set in $\mathcal{A}$ is $\mu^*$-measurable.

Let $\forall B\in\mathcal{A}$ be given. For any subset $E$ of $X$, want to show $\mu^*(E)\ge\mu^*(E\cap B)+\mu^*(E\cap B^c)$

Let $\forall\varepsilon>0$ be given. $\exists K\in\mathcal{A}$ s.t. $K\supset E$ & $\mu(K)<\mu^*(E)+\varepsilon$.

$$\left.\begin{array}{l}\text{And } E\cap B\subset K\cap B\in\mathcal{A}\\ \quad\;\; E\cap B^c\subset K\cap B^c\in\mathcal{A}\end{array}\right)\ \mu^*(E\cap B)+\mu^*(E\cap B^c)\le\mu(K\cap B)+\mu(K\cap B^c)=\mu(K)\le\mu^*(E)+\varepsilon$$

Since $\varepsilon$ arbitrary, done. ∎

### $\mu^*(B)=\mu(B)$ $\forall B\in\mathcal{A}$

Spse not, i.e. $\mu(B)-\mu^*(B)=\varepsilon>0$.

Then, $\exists K\in\mathcal{A}$ s.t. $B\subset K$ & $\mu(K)<\mu^*(B)+\varepsilon=\mu(B)$

contradiction ∎

## 4

$$m(\{x\}) = m\left(\bigcap_{n=1}^{\infty}(x-\tfrac{1}{n},x]\right)$$

$$= \lim_{n\to\infty} m\left((x-\tfrac{1}{n},x]\right) \quad \text{by } m((x-1,x])=1 \text{ \& continuity from above.}$$

$$= \lim_{n\to\infty} \alpha(x)-\alpha(x-\tfrac{1}{n}) = \alpha(x)-\alpha(x-)$$

## 5

Note that collection of open intervals is invariant under translation & dilation $\rightsquigarrow$ so does $\mathcal{B}$.

For $\mathcal{B}$, let $m_x(A)=m(x+A)$

$m^c(A)=m(cA)$.

Let $\mathcal{C}=\{(a,b]\mid a,b\in\mathbb{R}\}$, $\overline{\mathcal{C}}=$ a finite union of sets in $\mathcal{C}$. $\rightsquigarrow m_s=m$ & $m^c=|c|\cdot m$ on $\overline{\mathcal{C}}$.

↳ algebra.

Thus by Carathéodory extension & $\ell$ is $\sigma$-finite on $\mathbb{R}$, $m_s=m$ & $m^c=|c|\cdot m$ on $\mathcal{B}$.

Since $\forall E\subset\mathbb{R}$ has a $G_\delta$ set s.t. $m(G_\delta-E)=0$ & $E\subset G_\delta$, $G_\delta\in\mathcal{B}$.

Note that for any Lebesgue measure zero set $E$, $m(E)=m(G_\delta)=0$. So, $m_s=m$ & $m^c=|c|\cdot m$ on Lebesgue null sets.

$$\therefore\ m(E)=m(G_\delta)-m(G_\delta-E)$$
$$= m_s(G_\delta)-m_s(G_\delta-E)=m_s(E)\ ∎$$

$\boxed{6}$ Note that $B = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$

(1) Since the collection of Lebesgue measurable sets are $\sigma$-algebra, $B$ is also Lebesgue measurable.

(2) Let $B_m = \bigcap_{n=1}^{m} \bigcup_{i=n}^{\infty} A_i$. $\rightsquigarrow B_m \downarrow B$ and $m(B_1) = m(\bigcup_{i=1}^{\infty} A_i \subseteq [0,1]) \le 1 < \infty$.

And for $\forall m \in \mathbb{N}, B_m \supseteq A_m$ so that $m(B_m) > \delta$.

As $\lim_{k\to\infty} m(B_k) = m(B)$ by continuity from above, $m(B) \ge \delta$ by Sandwich Theorem $\blacksquare$

(3) From $\sum_{n=1}^{\infty} m(A_n) < \infty$, $m(\bigcup_{n=1}^{\infty} A_n) < \infty$ by countable additivity. (hence $m(B_1) < \infty$)

And $m(B) \le m(\bigcup_{k=n}^{\infty} A_k) \le \sum_{k=n}^{\infty} m(A_k)$

$$= \sum_{k=1}^{\infty} m(A_k) - \sum_{k=1}^{n-1} m(A_k) \longrightarrow 0 \text{ as } n\to\infty.$$

(4) $A_n = [0, \frac{1}{n}]$. $\rightsquigarrow B = \{0\}$. therefore $m(B) = 0$

$m(A_n) = \frac{1}{n} \rightsquigarrow \sum_{i=1}^{\infty} m(A_n) = \infty$

$\boxed{7}$ $\mathbb{Q} \cap [0,1] \cup (0, \epsilon]$

Since $\overline{\mathbb{Q}} = \mathbb{R}$ ($\because \mathbb{Q}$ is dense in $\mathbb{R}$), obvious.

$\boxed{8}$ By proposition 1.5., an open set $(-1,2) \cap F^c = \bigcup_{i=1}^{\infty} (a_i, b_i)$ & $I_i = (a_i, b_i)$ disjoint.

Define $\alpha(x) = \begin{cases} a_i & (x \in I_i) \\ x & (x \in F) \end{cases}$ $\rightsquigarrow$ right conti on $[0,1]$ $\begin{cases} \text{If } x \in F^c, \text{ i.e. } x \in I_j \text{ for some } j, \ \exists \delta > 0 \text{ s.t. } \alpha(y) - \alpha(x) < \varepsilon \text{ whenever } y - x < \delta \\ \quad \text{for } \forall \varepsilon > 0 \text{ by defintion of open set.} \\ \text{If } x \in F, \text{ if } x \text{ is interior pt, clear.} \\ \quad \text{if } x \text{ is bdry pt, } \exists I_j = (x, b_j) \ \& \ \alpha(y) - \alpha(x) = 0 \text{ whenever} \\ \quad y - x < \delta = \frac{1}{2}(b_j - x). \end{cases}$

& increasing (clearly) $0 \le x < y \le 1 \rightsquigarrow \alpha(x) \le x < \begin{cases} y & (y \in F) \\ a_i & (y \in F^c) \end{cases}$

$\hookrightarrow x \le a_i < y$.

Let $m$ be a Lebesgue-Stieltjes measure corresponding to $\alpha$.

We show that $m(F^c) = 0$ & $F$ is support.

1) $m(F^c) = 0$.

$m((a_i, b_i)) = m(\bigcup_{k=1}^{\infty} (a_i + \frac{1}{k}, b_i - \frac{1}{k}]) \le \sum_{k=1}^{\infty} \underbrace{m((a_i + \frac{1}{k}, b_i - \frac{1}{k}])}_{\alpha(b_i - \frac{1}{k}) - \alpha(a_i + \frac{1}{k}) = 0} = 0.$ As $\underbrace{F^c}_{[0,1]-F}$ is in the disjoint union of $(a_i, b_i)$, done.

2) $F$ is the smallest.

Spse not, i.e. $K \subset F$ & $m(K^c) = 0$. Then $x \in F - K$. As $K^c$ is open, $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subset K^c$. (Hence $\alpha(x) = x$)

$\rightsquigarrow (x - \varepsilon, x] \subset K^c \ \therefore m(K^c) \ge m((x - \varepsilon, x]) = \alpha(x) - \alpha(x - \varepsilon) = x - \alpha(x - \varepsilon)$

$$\ge x - x + \varepsilon = \varepsilon > 0 \text{ (contradiction} \blacksquare)$$

$\boxed{10}$ By proposition 4.14., set $G : m(G) - m(A) < C$ & $m(G \cap A) = m(A)$, $G$ is open.

By proposition 1.5., $G = \bigcup_{i=1}^{\infty} (a_i, b_i) = I_i$ for disjoint open intervals $I_i$

$$m(A) = m\left(\bigcup_{i=1}^{\infty} A \cap I_i\right)$$
$$= \sum_{i=1}^{\infty} m(A \cap I_i) \le (1-\varepsilon) m(G) = (1-\varepsilon) m(A) + C(1-\varepsilon)$$

$\therefore m(A) < \frac{1-\varepsilon}{\varepsilon} \cdot C$ as $0 < \varepsilon < 1$

Since $C > 0$ is arbitrary, $m(A) = 0$ ∎

$\boxed{11}$ We'll use 4.10.

As $m(A) > 0$, $\exists I$ s.t. $m(A \cap I) \ge \frac{9}{10} m(I)$ $(*)$

Let $x \in \left(-\frac{1}{2} m(I), \frac{1}{2} m(I)\right)$

Define $K = A \cap I$, $L = (A \cap I) + x$. By $(*)$, $K \cap L \ne \emptyset$.

If not, $m(K \cup L) = m(K) + m(L) = 2m(A \cap I) \ge \frac{9}{5} m(I)$

But $m(K \cup L) \le m(I \cup (I + x)) \le \frac{3}{2} m(I)$

Thus let $y \in K \cap L \rightsquigarrow y \in A \cap (A + x)$ too.

Then, $\exists z \in A$ s.t. $y = z + x \in A$

$\therefore x = z - y$ & $z, y \in A$.

As $x$ is an arbitrary point in $\left(-\frac{1}{2} m(I), \frac{1}{2} m(I)\right)$, $\left(-\frac{1}{2} m(I), \frac{1}{2} m(I)\right) \in A - A = B$ ∎

$\boxed{12}$ $C$ : generalized Cantor set with Lebesgue measure $\frac{1}{2}$.

Write $C^c = \bigcup_{i=1}^{\infty} (a_i, b_i)$.

Let $D_i = a_i + (b_i - a_i) C \rightsquigarrow$ closed

and $C_1 = C \cup \left(\bigcup_{i=1}^{\infty} D_i\right) \rightsquigarrow$ closed.

Write $C_1^c = \bigcup_{i=1}^{\infty} (a_i', b_i')$, and put a scaled version into each interval & construct $C_2$.

Continue & let $A = \bigcup_{j=1}^{\infty} C_j$. (Note that $C_j \uparrow A$)

Then any interval $I$ will contain some interval $J$ in some $C_j^c$ as the length of intervals in $C_j^c$ goes to $0$ as $j \to \infty$

Hence $m(I \cap J) < m(I)$

$$m(I) = m(A \cap I) + m(A^c \cap I)$$
$$\le m(A \cap I) + m(I \cap J)$$
$$< m(A \cap I) + m(I)$$

$\therefore 0 < m(A \cap I)$

$m(C_k^c \cap A) > 0 \quad \forall k \ge j$

$\rightsquigarrow m(A^c \cap I) > 0$

$\therefore m(A \cap I) < m(I)$

$\boxed{13}$ $N$ is a Vitali set on $[0,1]$

Since $A \subset N$, $\bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A+q) \subset \underbrace{[-1,2]}_{m(\,''\,)=3}$ & $A+q$ are disjoint for different $q$. (since $A+q \subset N+q$)

$\therefore$ $m(A+q) = m(A)$ must be $0$. ■

$\boxed{14}$ We may assume $A \subset [0,1]$ : $\bigcup_{n=-\infty}^{\infty} (A \cap [n,n+1]) = A$ & $m(A) > 0$. $\rightsquigarrow$ $\exists n$ s.t. $m(A \cap [n,n+1]) > 0$.

Hence if we show $A \cap [n,n+1]$ contains non-measurable set, done.

Let $A \subset [0,1]$.

Define equivalence relation $\sim$ on $A$ where $x \sim y$ if $x-y \in \mathbb{Q}$.

Construct $N$ by collecting elements of $A$ out of $\sim$.

Then, $\underbrace{m(A)}_{>0.} \le m\left(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} N+q\right) = \underset{''}{\sum} m(N+q) \le m([-1,2]) = 3.$

$\therefore$ $N$ is non-measurable & $N \subset A$.

$\boxed{15}$ Let $\forall E \subseteq X$ be given. Set arbitrary $\varepsilon > 0$. $\mu^*(E \cap A^c) \le \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)$

$\le \mu^*(B \cap A^c) + \mu^*(E \cap B^c)$

$\mu^*(E \cap A) + \mu^*(E \cap A^c)$

$\le \mu^*(E \cap B) + \mu^*(E \cap B^c) + \varepsilon = \mu^*(E) + \varepsilon$. As $\varepsilon > 0$ arbitrary, done. ■

$\boxed{16}$ $(\rightarrow)$ $\underbrace{\mu^*(X)}_{X \in \mathcal{A} \,\rightsquigarrow\, \mu^*(X) = \ell(X)} = \mu^*(A) + \mu^*(A^c)$ $\therefore$ $\mu^*(A) = \ell(X) - \mu^*(A^c)$

$(\leftarrow)$ Let $\forall E \subseteq X$ be given

Claim: $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$

pf) First, $\exists$ set $C$ & $D \in \mathcal{A}$ s.t. $C \supset A$ & $D \supset A^c$ where $\ell(C) - \mu^*(A) < \varepsilon$. for $\forall \varepsilon > 0$ by property of infimum

$\ell(D) - \mu^*(A^c) < \varepsilon$.

Note that $C, D, B = C \cap D$ $\mu^*$-measurable & we assume $\ell(X) = \mu^*(A) + \mu^*(A^c)$ & $\ell(X) < \infty$.

Then $\underbrace{\mu^*(A^c)}_{\ell(X)-\mu^*(A)} = \mu^*(A^c \cap C) + \mu^*(\underbrace{A^c \cap C^c}_{C^c}) = \mu^*(C - A) + \underbrace{\ell(C^c)}_{\ell(X)-\mu^*(C)}$ $\therefore$ $\ell(C) - \mu^*(A) = \mu^*(C-A)$

$\underbrace{\mu^*(A)}_{\ell(X)-\mu^*(A^c)} = \mu^*(A \cap D) + \mu^*(\underbrace{A \cap D^c}_{D^c}) = \mu^*(D - A^c) + \underbrace{\ell(D^c)}_{\ell(X)-\mu^*(D)}$ $\therefore$ $\ell(D) - \mu^*(A^c) = \mu^*(D - A^c)$

Note that $B = C \cap D$ is $\ell$-measurable. $\underset{\mu^*(B)}{\underset{''}{\ell(B)}} \le \mu^*(D - A^c) + \mu^*(C - A) < 2\varepsilon$.

For $\forall F = \bigcup_{i=1}^{\infty} F_i \supseteq E$ where $F_i \in \mathcal{A}$ for each $i$,

$\ell(F) \ge \ell(F \cap C) + \ell(F \cap D) - \ell(F \cap B) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c) - 2\varepsilon$.

$\hookrightarrow \ell(F) \ge \ell((F \cap C) \cup (F \cap D))$

By taking infimum, $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c) - 2\varepsilon$.

$\varepsilon$ arbitrary, done. ■

$\boxed{17}$ (1) $\mu^*(A) = \begin{cases} 0 & A=\emptyset \\ 1 & A \text{ finite} \\ 2 & A \text{ infinite} \end{cases}$ $\quad A_n = \{1,2,\cdots,n\} \uparrow \mathbb{N}$

$B_n = \{n, n+1, n+2, \cdots\} \downarrow \emptyset.$

(2) $\mu^*(A) = \inf\{\mu(B) \mid A \subset B, B \in \mathcal{A}\}$

Spse $A_n \uparrow A$. Choose $B_n \in \mathcal{A}$, $A_n \subset B_n$ s.t. $\mu^*(B_n) \le \mu^*(A_n) + \frac{1}{n}$

Let $C_n = \bigcap_{i=n}^{\infty} B_i \uparrow C$. Note that $\mu^*$ is measure on $\mathcal{A}$ and $C_n, C \in \mathcal{A}$ $\forall n \in \mathbb{N}$.

$x \in A \to x \in A_n$ $\forall n \ge N >> 1 \longrightarrow x \in B_n$ $\forall n \ge N \longrightarrow x \in C_n$ $\forall n \ge N$.

$\therefore A \subset C$. $\lim_{n\to\infty} \mu^*(A_n) \ge \liminf_n \left(\mu^*(B_n) - \frac{1}{n}\right) \ge \lim_{n\to\infty} \left(\mu^*(C_n) - \frac{1}{n}\right) = \mu^*(C) \ge \mu^*(A)$

$\therefore \mu(A) = \lim \mu^*(A_n)$.

$\boxed{18}$ $\bigcup_{x\in A} (x-1, x+1) \rightsquigarrow$ open set ($\because$ arbitrary union of open sets is open set) $\rightsquigarrow$ Lebesgue measurable

$-1 + A$ & $1+A$ measurable. as $A$ is measurable.

$\therefore B = (-1+A) \cup (1+A) \cup \{\bigcup_{x\in A}(x-1, x+1)\}$ Lebesgue measurable ■

$\boxed{19}$ Spse $\forall c \in \mathbb{R}$, $A \cap c + \mathbb{Q} \neq \emptyset$ Then, $\forall x \in \mathbb{R}$, $\exists q \in \mathbb{Q}$ s.t. $x + q \in A$

$x \in c + \mathbb{Q}$ $\qquad \rightsquigarrow x \in A - q \subset \bigcup_{q\in\mathbb{Q}} (A+q)$

$\mathbb{R} \subseteq \bigcup_{q \in \mathbb{Q}} (A+q)$

$m(\mathbb{R}) \le \sum_{q\in\mathbb{Q}} \underbrace{m(A+q)}_{0}$ contradiction ■