$\boxed{1}$ Since $\nu \ll \mu$, by exercise 13.10, $\nu^{\pm} \ll \mu$.
Hence we can apply Radon – Nikodym to both $\nu^{\pm}$.
$\nu = \nu^+ - \nu^- \rightsquigarrow$ Let $f^+ = \frac{d\nu^+}{d\mu}$, $f^- = \frac{d\nu^-}{d\mu}$.
Let $f = f^+ - f^-$. Then for any $A \in \mathcal{A}$, $\nu(A) = \nu^+(A) - \nu^-(A)$
$$= \int_A \underbrace{(f^+ - f^-)}_{\mu\text{-integrable.}} d\mu = \int_A f \cdot d\mu$$

(+) uniqueness
Let $\nu(A) = \int_A g \cdot d\mu$ for any $A \in \mathcal{A}$. Then $\int_A (f-g) \cdot d\mu = 0$ $\forall A \in \mathcal{A}$. By proposition 8.2. $f = g$ a.e.

$\boxed{2}$ State : Suppose μ is a σ-finite finite measure and ν is a finite signed measure. There exist unique signed measures λ, ρ such that $\nu = \lambda + \rho$ where $\rho \ll \mu$, $\lambda \perp \mu$.

Prove : Let $\nu = \nu^+ - \nu^-$ be Jordan decomposition.
$\exists !\, \lambda^+, \rho^+$ s.t. $\nu^+ = \lambda^+ + \rho^+$ $\quad \exists !\, \lambda^-, \rho^-$ s.t. $\nu^- = \lambda^- + \rho^-$

$\nu = (\lambda^+ - \lambda^-) + (\rho^+ - \rho^-)$
$\lambda^+ - \lambda^- \perp \mu$: $\lambda^+(E_1) = \mu(F_1) = \lambda^-(E_2) = \mu(F_2)$ $\quad \rho^+ - \rho^- \ll \mu$: trivial ■
Then $(\lambda^+ - \lambda^-)(E_1 \cap E_2) = 0$
$\mu(F_1 \cup F_2) = 0$
Uniqueness can be shown similarly with positive measure version.

$\boxed{5}$ (→) By Radon – Nikodym, $d\nu = f \cdot d\mu$ & $d\mu = g \cdot d\nu$ as μ, ν finite positive, $\mu \ll \nu$ and $\nu \ll \mu$. Note that $f \in L^1(\mu)$, $g \in L^1(\nu)$
By exercise 13.8, $\int_A 1 \cdot d\mu = \int_A g \cdot d\nu = \int_A fg \cdot d\mu$ for any $A \in \mathcal{A}$
$\therefore$ By proposition 8.2. $fg = 1$ μ-a.e. Since $f, g \geq 0$ from theorem, $f > 0$ μ-a.e.

(←) Let $g(x) = \begin{cases} 1/f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases} \Longrightarrow g = \frac{1}{f}$ μ-a.e.

By exercise 13.8, $\int_A g \cdot d\nu = \int_A g f \cdot d\mu = \int_A 1 \cdot d\mu$ $\quad \therefore d\mu = g \cdot d\nu$
$\nu(A) = 0 \longrightarrow \int_A g \cdot d\nu = 0 = \mu(A)$ as g is non-negative
$\mu(A) = 0 \longrightarrow \int_A f \cdot d\mu = 0 = \nu(A)$ ■

$\boxed{6}$ We have $\nu \ll \mu$, $\mu \leq \rho$, $\nu \leq \rho$
① $f > 0$ μ-a.e.
pf) Let $\underbrace{A = \{f = 0\}}_{\text{measurable}}$. Since $\mu(A) = \int_A f \cdot d\rho = 0$ and f is non-negative by theorem, done.

② $f + g = 1$ a.e.
pf) $\int_A (f+g) \cdot d\rho = \mu(A) + \nu(A) = \rho(A) = \int_A 1 \cdot d\rho$ for $\forall A \in \mathcal{A}$
By proposition 8.2., $f + g = 1$ a.e.

③ $d\nu = (g/f)\, d\mu$ $\quad$ possible as $f > 0$ μ-a.e. i.e, ρ-a.e. since $\nu \ll \mu$
pf) For any given $A \in \mathcal{A}$, $\nu(A) = \int_A g \cdot d\rho = \int_A \frac{g}{f} \cdot f \cdot d\rho = \int_A \frac{g}{f} \cdot d\mu$ by exercise 13.8 (7.27)
$\therefore$ By uniqueness, $g/f = d\nu / d\mu$ ■

$\boxed{7}$ 2 possible solutions.

① Let $X = E \cup F$ be Hahn decomposition w.r.t. μ.
Let $f = \chi_F - \chi_E$. For any $A \in \mathcal{A}$, $\mu(A) = \mu(A \cap F) + \mu(A \cap E)$
$= |\mu|(A \cap F) - |\mu|(A \cap E) = \int_A (\chi_F - \chi_E)\, d|\mu|$ ∎

② $|\mu| := \mu^+ + \mu^- = \sup\{\sum_{i=1}^n |\mu(A_i)| : A = \bigcup_{i=1}^n A_i, \text{ disjoint}\}$ by exercise 12.7.
It is obvious : $\mu \ll |\mu|$.
Then, by Radon–Nikodym $\exists f = \frac{d\mu}{d|\mu|} \in L^1(|\mu|)$.
Claim : $d\mu = f \cdot d\rho \rightsquigarrow d|\mu| = |f| \cdot d\rho$, ρ is positive

Then, $d|\mu| = |f|\, d|\mu|$ & $|f| = 1$ a.e. by proposition 8.2.

pf) WTS : $|\mu|(A) = \int_A |f| \cdot d\rho$ for $\forall A \in \mathcal{A}$
$\leq$) Let $A = \bigcup_{i=1}^n A_i$ disjoint. $\rightsquigarrow \sum_{i=1}^n |\mu(A_i)| = \sum_{i=1}^n |\int_{A_i} f \cdot d\rho| \leq \int_A |f| \cdot d\rho$ $\therefore |\mu|(A) \leq \int_A |f| \cdot d\rho$

$\geq$) For any $A \in \mathcal{A}$, define $A^+ = A \cap \{f \geq 0\}$, $A^- = A \cap \{f < 0\}$. $\exists$ non-negative simple $\{s_n\}$ s.t. $s_n \uparrow f$ on A^+

$\forall \varepsilon > 0$, $\exists$ non-negative simple s with $\int_{A^+} f \cdot d\rho - \int_{A^+} s \cdot d\rho < \frac{\varepsilon}{2}$. Write $s = \sum_{i=1}^n a_i \chi_{A_i}$, A_i disjoint and $\bigcup_{i=1}^n A_i = A^+$ (by construction), $a_i \geq 0$, see prop 5.14

Note that $\underbrace{\int_{A^+} |f| \cdot d\rho}_{= \int_{A^+} f \cdot d\rho} - \sum_{i=1}^n \underbrace{|\mu(A_i)|}_{|\int_{A_i} f \cdot d\rho|} \leq \int_{A^+} f \cdot d\rho - \sum_{i=1}^n a_i \rho(A_i) < \frac{\varepsilon}{2}$

$|\int_{A_i} f \cdot d\rho| > |\int_{A_i} s \cdot d\rho| \geq a_i \rho(A_i)$.

Similarly, $\int_{A^-} |f| \cdot d\rho - \sum_{j=1}^m |\mu(B_j)| < \frac{\varepsilon}{2}$ where $\bigcup B_j = A^-$, disjoint. Note that A_i & B_j pairwise disjoint and $\bigcup A_i \cup \bigcup B_j = A$.

$\therefore \int_A |f| \cdot d\rho \leq \varepsilon + |\mu|(A)$, $\varepsilon > 0$ arbitrary.

$\boxed{8}$ ① $f = \chi_E$. clear
② $f = \sum_{i=1}^n a_i \chi_{E_i}$ non-negative simple. Done by linearity of integral.
③ f non-negative. $\exists \{s_n : \text{non-negative simple}\}$ with $s_n \uparrow f$.
By MCT, $\int f \cdot d\mu = \lim_{n\to\infty} \int s_n \cdot g \cdot d\mu = \int f \cdot g \cdot d\mu$

※ Note
MCT doesn't require integrability of f.

$\boxed{9}$ $\rho \ll \mu$ trivial.
By exercise 13.8., for any $A \in \mathcal{A}$, $\rho(A) = \int_A \frac{d\rho}{d\nu} \cdot d\nu = \int_A \frac{d\rho}{d\nu} \cdot \frac{d\nu}{d\mu} \cdot d\mu$. Done by uniqueness.

$\boxed{10}$ ($\rightarrow$) Spse $\nu \ll \mu$.
Let $\nu = \nu^+ - \nu^-$ Jordan decomposition with $\nu^+(A) = \nu(A \cap F)$ $\nu^-(A) = \nu(A \cap E)$
Let $\mu(A) = 0$. Then $\nu^+(A) = \nu^-(A)$ and $\nu(A) = 0$.
Claim: $\nu^+(A) = 0$.
pf) Note that $\mu(A \cap F) = 0$. $\rightsquigarrow \nu(A \cap F) = 0$.

($\leftarrow$) Trivial.

$\boxed{11}$ Let any $A \in \mathcal{A}$ be given.

$$\lambda(A) = \sum_{n=1}^{\infty} \lambda_n(A) = \sum_{n=1}^{\infty} \int_A \underbrace{f_n}_{\text{non-negative}} \cdot d\mu + \sum_{n=1}^{\infty} \nu_n(A)$$

$$\underset{\text{Fubini}}{=} \int_A \left(\sum_{n=1}^{\infty} f_n \right) \cdot d\mu + \sum_{n=1}^{\infty} \nu_n(A)$$

Note that $\sum \nu_n \perp \mu$: $\nu_n \perp \mu \rightsquigarrow \exists X = E_n \cup F_n$ $\nu_n(E_n) = 0$ $\mu(F_n) = 0$.

Let $E = \bigcap_{n=1}^{\infty} E_n$ $F = \bigcup_{n=1}^{\infty} F_n$

$\sum \nu_n(E) = 0$, $\mu(F) = 0$ & $X = E \cup F$ & $E \cap F = \emptyset$ ■

Also $\int_A \left(\sum_{n=1}^{\infty} f_n \right) \cdot d\mu \ll \mu$: $\mu(A) = 0 \longrightarrow \int_A f_n \cdot d\mu = 0$ $\forall n \in \mathbb{N}$ as $\int f_n \cdot d\mu \ll \mu$.

$\longrightarrow \sum_{n=1}^{\infty} \int_A f_n \cdot d\mu = 0$ ■

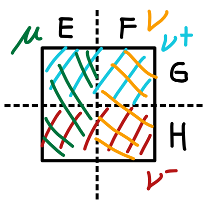
$\boxed{12}$ $\nu \perp \mu \rightsquigarrow \exists E, F$ s.t. $X = E \cup F$, $\mu(F) = \nu(E) = 0$.

Let $\forall A \in \mathcal{A}$ be given. $\nu(A) = \underbrace{\cancel{\nu(A \cap E)}}_{\le \nu(E)}^{0} + \underbrace{\cancel{\nu(A \cap F)}}_{\substack{\mu(A \cap F) = 0 \\ \& \ \nu \ll \mu}}^{0} = 0$ ■

* Still holds when ν : signed-measure.

Let $G \cup H = X$ be Hahn decomposition w.r.t. ν. $\nu^{\pm} \ll \mu$ by exercise 13.10.

$\nu^+ = \nu|_G$, $\nu^- = -\nu|_H$



ν^+: finite positive & $\nu^+ \perp \mu$ as $\mu(F \cap G) \le \mu(F) = 0$

$\nu^+(E \cup H) = \nu^+(E) = \nu(E \cap G)$

$\hookrightarrow \nu^+(F \cap G) = \nu(F \cap G) = 0.$

$\underbrace{\nu(G)}_{\nu^+(\emptyset) = 0} = \nu(E \cap G) = \nu^+(E)$

ν^-: finite positive & $\nu^- \perp \mu$ as $\mu(F \cap H) \le \mu(F) = 0$

$\nu^-(E \cup G) = \nu^-(E) = -\nu(E \cap H)$

$\hookrightarrow \nu^-(F \cap H) = -\nu(F \cap H) = 0.$

$-\underbrace{\nu(G)}_{\nu^-(\emptyset) = 0} = -\nu(E \cap H) = \nu^-(E).$

$\therefore \nu^{\pm} = 0$. Thus $\nu = 0$ ■

$\boxed{13}$ ① Using * in the previous exercise

$\because \nu = \lambda_1 + \rho_1 = \lambda_2 + \rho_2$

$\rightsquigarrow \underbrace{\lambda_1 - \lambda_2}_{\perp \mu} = \underbrace{\rho_2 - \rho_1}_{\ll \mu}$ $\quad \therefore \lambda_1 - \lambda_2 = \rho_2 - \rho_1 = 0$ ■

② $\nu = \lambda_1 + \rho_1 = \lambda_2 + \rho_2$

$\rightsquigarrow \underbrace{\lambda_1 - \lambda_2}_{\perp \mu} = \underbrace{\rho_2 - \rho_1}_{\ll \mu}$

Let $\lambda_1(E_1) = \mu(F_1) = \lambda_2(E_2) = \mu(F_2) = 0$.

Then $E_1 \cap E_2$ is a null set for $\lambda_1 - \lambda_2$

Since $\mu(F_1 \cup F_2) = 0$, $(E_1 \cap E_2)^c$ is a null set for $\rho_2 - \rho_1 = \lambda_1 - \lambda_2$

$\therefore \lambda_1 = \lambda_2$ & $\rho_2 = \rho_1$ ■