$\boxed{1}$ ① $\int f(x+a) = \int f(x)$

$\leq$) Let s be any simple s.t. $0 \leq s(x) \leq f(x+a)$.

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i} \rightsquigarrow t = \sum_{i=1}^{n} a_i \chi_{E_i - a}$ is a simple s.t. $0 \leq t \leq f(x)$

$\therefore \int s \cdot dx = \int t \cdot dx \leq \int f(x) \cdot dx$ $\quad \therefore \int f(x+a) \cdot dx \leq \int f(x) \cdot dx$

$\geq$) It can be shown in the analogous way. ∎

② $\int f(ax) \cdot dx = \int f(x) \cdot dx \; \frac{1}{|a|}$

$\leq$) Let s be any simple s.t. $0 \leq s(x) \leq f(ax)$

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i} \rightsquigarrow t = \sum_{i=1}^{n} a_i \chi_{\frac{1}{a} E_i}$ is a simple s.t. $0 \leq t \leq f(x)$

$\therefore \int s \cdot dx = \frac{1}{|a|} \int t \cdot dx \leq \frac{1}{|a|} \int f(x) \cdot dx$ $\quad \therefore \int f(ax) \cdot dx \leq \int f(x) \cdot dx \cdot \frac{1}{|a|}$

$\geq$) It can be shown in the analogous way. ∎

$\boxed{2}$ σ - finite $\rightsquigarrow$ exists $\{A_i\}$ s.t. $A_i \uparrow X$ and $\mu(A_i) < \infty$.

Let $f_n = f \cdot \chi_{A_n}$

Note that $|f_n| \leq f$. and f_n increasing to f

$\therefore$ By DCT, $\lim_{n \to \infty} \int f_n = \lim_{n \to \infty} \int_{A_n} f = \int f$

$\therefore \exists n \in \mathbb{N}$ s.t. $\int_{A_n} f + \varepsilon > \int f$.

$\boxed{3}$ $\exists$ open set G s.t. $m(G - A) < \varepsilon'$. $G \supseteq A$

$G = \bigcup_{i=1}^{\infty} (a_i, b_i)$ for pairwise disjoint (a_i, b_i)

Construct $f_n(x) = \begin{cases} 1 & x \in [a_n + \delta_n, b_n - \delta_n] \\ 0 & x \in (-\infty, a_n] \cup [b_n, \infty) \\ \text{linear} & \text{o.w.} \end{cases}$ where $\delta_n = \frac{\varepsilon}{2^{n+1}} \wedge \frac{(b_n - a_n)}{3}$

Let $f(x) = \begin{cases} f_n(x) & \text{if } x \in (a_n, b_n) \text{ for some } n \in \mathbb{N}. \text{ i.e. } x \in G. \\ 0 & \text{o.w.} \end{cases}$

$\rightsquigarrow m\{x \mid f(x) \neq \chi_A(x)\} \leq m\{x \mid f(x) \neq \chi_G(x)\} \leq \sum_{n=1}^{\infty} 2\delta_n < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

$\boxed{4}$ $0 \leq t\mu(f^{-1}[t, \infty)) = \int_{f(x) \geq t} t \cdot d\mu$

$\leq \int_{f(x) \geq t} f \cdot d\mu = \int f \cdot d\mu - \int_{f<t} f \cdot d\mu$.

Claim: $\lim_{t \to \infty} \int_{f<t} f \cdot d\mu = \int f \cdot d\mu$

pf) Let $f_n = f \cdot \chi_{f<n}$.

$\rightsquigarrow f_n \leq f_{n+1}$, non-negative, and $f_n \to f$ a.e. (Since f is non-negative integrable, by exercise 6.1. $f < \infty$ a.e.)

$\therefore$ By MCT, done. ∎

$\boxed{5}$ Let $a_n = \frac{1}{n \cdot 2^n}$, $I_n = (a_{n+1}, a_n)$

$f(x) = \sum_{n=1}^{\infty} 2^n \cdot \chi_{I_n}(x) \geq 0$.

$2^k \, m(f \geq 2^k) = \sum_{n=k}^{\infty} 2^k \cdot m(I_n) = 2^k \sum_{n=k}^{\infty} \left(\frac{1}{n \cdot 2^n} - \frac{1}{(n+1) \cdot 2^{n+1}} \right)$

$= 2^k \cdot \frac{1}{k \cdot 2^k} = \frac{1}{k}$.

And f is not integrable : $\int f\cdot dx = \sum_{n=1}^{\infty} \underbrace{2^n \cdot m(I_n)}_{\frac{1}{n}-\frac{1}{2(n+1)}=\frac{n+2}{2n(n+1)}\geq\frac{1}{2n}} > \infty$. ∎

$\boxed{6}$ $\rightarrow)$ Spse f is integrable.

Let $A_n = \{x \mid n \leq f(x) < n+1\}$. $\mu(X) = \sum_{n=1}^{\infty} \mu(A_n)$ $\{x \mid f(x) \geq n\} = \bigcup_{k=n}^{\infty} A_k$

$\infty > \int_X f \cdot d\mu \geq \sum_{n=1}^{\infty} n\mu(A_n). = \sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\})$

$\leftarrow)$

Let $A_n = \{x \mid n \leq f(x) < n+1\}$.

Then $\mu(X) = \sum_{n=1}^{\infty} \mu(A_n)$, $\underbrace{\{x \mid f(x) \geq n\}}_{\text{Let } B_n} = \bigcup_{k=n}^{\infty} A_k \rightsquigarrow \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} n\mu(A_n) < \infty$.

$$\int f = \int_{\cup_n A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f = \sum_{n=1}^{\infty} (n+1)\mu(A_n) = \lim_{N\to\infty} \sum_{n=1}^{N} (n+1)\mu(A_n)$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} n\mu(A_n) + \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n)$$
$$= \mu(X) + \sum_{n=1}^{\infty} \mu(A_n) < \infty$$ ∎

$\boxed{7}$ Let $F_n = \{x : |f(x)| \geq \frac{1}{n}\}$. $\rightsquigarrow$ $F_n \uparrow A$ clearly.

$\frac{1}{n}\mu(F_n) \leq \int_{F_n} |f| < \int |f|$

Thus, for $\forall n \in \mathbb{N}$, $\mu(F_n) < n \cdot \int |f| < \infty$ ∎

$\boxed{8, 9}$ Check the solution

$\boxed{10}$ Denote $\overline{h} = \int_0^1 h$ for any functions h

For any continuous F, right $f = F - \overline{F}$

$\longrightarrow 0 = \int_0^1 fg = \int_0^1 Fg - \overline{F}\int_0^1 g = \int_0^1 Fg - \overline{F}\cdot\overline{g}$.

Then $\int_0^1 F(g-\overline{g}) = \int_0^1 Fg - \overline{F}\,\overline{g} = 0$. For any measurable A, χ_A is continuous a.e., hence we can plug χ_A in F.

$\rightsquigarrow \int_A g - \overline{g} = 0$ $\forall$measurable $A \subseteq [0,1]$, and $\int_0^1 g - \overline{g} < \infty$. integrable. $\therefore$ $g = \int_0^1 g$ a.e.

$\boxed{12}$ There exists a continuous g s.t. $\int |f(x) - g(x)| \cdot dx < \varepsilon/2$ with compact support K.

Then $\int |f(x+h) - f(x)| \leq \int |f(x+h) - g(x+h)| \cdot dx + \int |f(x) - g(x)| \cdot dx + \int |g(x+h) - g(x)|$
$$= 2 \cdot \int |f(x) - g(x)| \cdot dx + \int |g(x+h) - g(x)|$$
$$< \varepsilon + \int |g(x+h) - g(x)|$$

$\int |g(x+h) - g(x)| \leq \sup_K |g(x+h) - g(x)| \cdot m(K) \longrightarrow 0$ as $h \to 0$.

$\therefore$ $\int |f(x+h) - f(x)| < \varepsilon$ for $\forall \varepsilon > 0$. Done ∎