$\boxed{1}$ Spse $\mu(\underbrace{\{x \mid f(x)=\infty\}}_{\text{Let } A}) > 0$.

Then, $s_n = n \cdot \chi_A$ is clearly $0 \le s_n \le f$ for all x for each n.

Thus $\int s_n \cdot d\mu = n \cdot \mu(A) \le \sup\{\int s \cdot d\mu : s \text{ simple, } 0 \le s \le f\} = \int f \cdot d\mu$ for $\forall n \in \mathbb{N}$.
Hence, $\int f \cdot d\mu = \infty$, which contradicts to the assumption ∎

Remark. Converse doesn't hold
$f(x) = \frac{1}{x}$

$\boxed{2}$ Skip (Check the author's solution)

$\boxed{3}$ Let any simple s s.t. $0 \le s \le f$ be given.

$$\int_{A_1 \cup A_2} s = \sum_{i=1}^{n} a_i \mu(E_i \cap (A_1 \cup A_2))$$

$$= \sum_{i=1}^{n} a_i \mu(E_i \cap A_1) + a_i \mu(E_i \cap A_2) = \int_{A_1} s + \int_{A_2} s \le \int_{A_1} f + \int_{A_2} f$$

$\therefore \int_{A_1 \cup A_2} f \le \int_{A_1} f + \int_{A_2} f$

Similarly, $\int_{A_1} s + \int_{A_2} s \le \int_{A_1 \cup A_2} s \le \int_{A_1 \cup A_2} f$. Therefore, if f is non-negative, $\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$

For integrable case, it's obvious from $f = f^+ - f^-$ and definition.

$\boxed{4}$ 1) f is non-negative.
Let any simple s with $0 \le s \le f$ be given.
Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$. Let $y \in E_{k_1}, \cdots, E_{k_m}$. $\Rightarrow s(y) = \sum_{j=1}^{m} a_{k_j} \le f(y)$

$$\int s \cdot d\delta_y = \sum_{i=1}^{n} a_i \, \delta_y(E_i) = \sum_{j=1}^{m} a_{k_j} = s(y) \le f(y)$$

Thus, $\int f \cdot d\delta_y \le f(y)$.
Let $s = f(y) \cdot \chi_{\{y\}}$. Then, s is a simple s.t. $0 \le s \le f$ & $\int s \cdot \delta_y = f(y) \le \int f \cdot \delta_y$.
$\therefore \int f \cdot d\delta_y = f(y)$.

2) f is integrable.
Obvious from $f = f^+ - f^-$ and definition

$\boxed{5}$ 1) $\sum f(k) < \infty$

Let any simple s with $0 \le s \le f$ be given.

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$.

$$\int_{\mathbb{N}} s \cdot d\mu = \sum_{i=1}^{n} a_i \mu(E_i \cap \mathbb{N}) = \sum_{k=1}^{\infty}\sum_{i=1}^{n} a_i \underbrace{\mu(E_i \cap \{k\})}_{\chi_{E_i}(k)}$$
$$= \sum_{k=1}^{\infty} s(k) \le \sum_{k=1}^{\infty} f(k).$$

Thus $\int_{\mathbb{N}} f \le \sum_{k=1}^{\infty} f(k)$

Let $g = \sum_{n=1}^{\infty} f(n) \cdot \chi_{\{n\}}$. $\rightsquigarrow$ simple function. that $0 \le g \le f$ on $\mathbb{N}$.

$\int_{\mathbb{N}} g \le \int_{\mathbb{N}} f$ and $\int_{\mathbb{N}} g = \sum_{n=1}^{\infty} g(n) = \sum_{n=1}^{\infty} f(n)$ $\quad \therefore \int_{\mathbb{N}} f = \sum_{k=1}^{\infty} f(k)$ ■

2) $\sum f(k) = \infty$

Let $s = \sum f(k) \cdot \chi_{\{k\}}$. $\rightsquigarrow \int s \cdot d\mu = \infty$

$\rightsquigarrow \int f \cdot d\mu \ge \sup\{\int s \cdot d\mu \mid 0 \le s \le f, s \text{ simple}\} = \infty$ ■

$\boxed{6}$ Let $X = \bigcup_{i=1}^{\infty} A_i$ where $A_1 \subset A_2 \subset \cdots$ and $\mu(A_n) < \infty$ for each n.

Define $E_{in} = \{x : (i-1)/2^n \le f(x) < i/2^n\}$

$F_n = \{x : f(x) \ge n\}$.

Let $t_n = \left(\sum_{i=1}^{n2^n} \frac{i-1}{2^n} \cdot \chi_{E_{in}}\right) + n \cdot \chi_{F_n}$ $\quad$ Define $s_n = t_n \cdot \chi_{A_n} = \left(\sum_{i=1}^{n2^n} \frac{i-1}{2^n} \cdot \chi_{E_{in} \cap A_n}\right) + n \cdot \chi_{F_n \cap A_n}$

Then $\forall x \in X$, $s_n(x) \le s_{n+1}(x)$ from the construction of t_n & $\chi_{A_n} \le \chi_{A_{n+1}}$

And from $t_n \to f$ & $\chi_{A_n} \to \chi_X$, $s_n \to f$.

Also $\{x \mid s_n(x) \ne 0\} = \bigcup_{i=2}^{n2^n} (E_{in} \cap A_n) \cup (F_n \cap A_n) \subseteq A_n$ thus $\mu(\,''\,) < \infty$ ■

$\boxed{7}$ Note that $f \wedge n$ stands for $\min(f, n)$.

For all $n \in \mathbb{N}$, $\int f \wedge n \le \int f$ Thus $\lim_{n\to\infty} \int f \wedge n \le \int f$.

For $\ge$, let s be simple s.t. $0 \le s \le f$.

Let $s = \sum_{i=1}^{l} a_i \chi_{E_i}$ $\rightsquigarrow \exists M > 0$ s.t. $s(x) < M$ $\forall x \in X$.

And $\exists n_0 \ge M$. $\underset{\displaystyle \int s}{\underset{\shortparallel}{\int s \wedge n_0}} \le \int f \wedge n_0$ $\quad \therefore \int f \le \int f \wedge n$ for $\forall n \ge n_0$. $\Rightarrow \int f \le \lim_{n\to\infty} \int f \wedge n$ ■

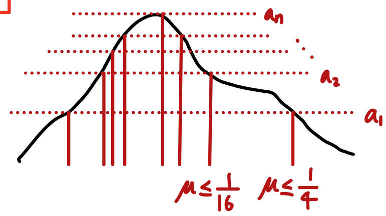
$\boxed{8}$ Denote $g = |f|$ and let $\forall \varepsilon > 0$ be given.

Then by exercise 6.7. $\exists n \in \mathbb{N}$ s.t. $\int g \le \int g \wedge n + \frac{\varepsilon}{2}$

Set $\delta = \frac{\varepsilon}{2n}$ & $\mu(A) < \delta$.

$$\int_A g = \int g - \int_{A^c} g \le \int g \wedge n - \int_{A^c} g \wedge n + \frac{\varepsilon}{2}$$
$$= \int_A g \wedge n + \frac{\varepsilon}{2} \le n\mu(A) + \frac{\varepsilon}{2} < \varepsilon$$ ■

$\boxed{9}$



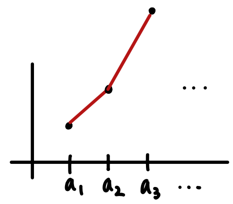
Let $a_0 = 0$ and set a_n to be $\mu(\{f(x) \ge a_n\}) \le \frac{1}{4^n}$

- this is possible as f is finite
- $\{a_n\}$ increasing

Set $A_0 = \{x \mid f(x) = 0\}$

$A_n = \{x \mid a_{n-1} \le f(x) \le a_n\}$

Define $g(a_n) = 2^n$ & connect points linearly :



For each $M \in \mathbb{N}$,

$$\int (g \circ f) \wedge 2^M = \sum_{n=0}^{M} \int_{A_n} (g \circ f) \le \sum_{n=1}^{\infty} \int_{A_n} g \circ f \le \sum_{n=1}^{\infty} 2^n \cdot \mu(A_n) < \sum_{n=1}^{\infty} \frac{2^n}{4^{n-1}} < \infty \blacksquare$$