Report on the Use of Hawkes processes in a Cramér-Lundberg type model

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Introduction

In many domains, it is important to have a model that reflects the occurrence of some events occurring over time, for example to prevent a potential risk. This risk can take several forms: in the field of geology we can note the earthquakes, in the financial field, the fall of the stock market, and in the insurance field, a claim for the insured.

To model the risk in insurance, we often use the Cramér-Lundberg model, based on Poisson processes. A Poisson process is a counting process, it models events occurring over time. This model is well suited when there is no link between events, but when they tend to appear in cluster (snowball effect), this model does not fit anymore. Hawkes processes try to solve this issue, by modeling this chain effect we can observe when dealing with events such as drop in financial market, earthquakes, visibility of a post in a social media...

This report will start with a theoretical presentation of Hawkes processes, before moving to numerical simulations and the study of the parameters. We will then compare them with Poisson processes in the context of the Cramer-Lundberg model. Finally, we will try to fit a Hawkes process on real data from the evolution of the CAC 40 since the 90s.

1 Presentation of Hawkes processes

Before defining the Hawkes process, we will first introduce some concepts and notations that are essential to its understanding. Among these concepts, we can note the notions of point process, counting process, filtration function and intensity function. The following definitions are adapted from the Mélisande Albert's lecture notes[1] and a paper from Yuanda Chen[2].

Definition (Counting process)**.** *A counting process noted $(N_t)_{t\geq 0}$, is a stochastic process modelling the number of events that occurred from time 0 to time t. It is a piecewise constant and non-decreasing function with values in $\mathbb{N}$. That process is also right-continuous and left-limited.*

Example. *N_t may represent the number of accidents an insurance company has to cover by time t. Here an event is a claim.*

Definition (Point process)**.** *A point process is a subset of $\mathbb{R}_+^*$ where each elements represents the time of the occurrence of an event. It is noted as the following :* $0 < T1 < T2 < T3 < \dots$

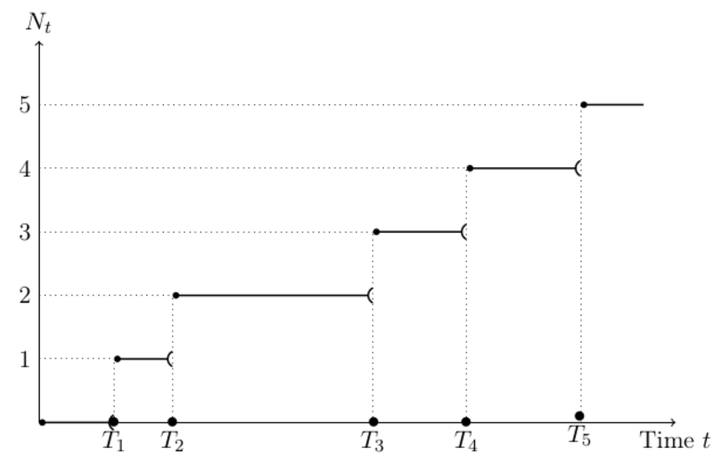

Figure 1: Counting and Point process (figure extracted from Mélisande Albert's lecture notes[1])

Definition (Filtration function $\mathcal{F}_t^N$)**.** *A filter function is a σ-algebra constructed from a counting process N_t by time t. This filter function corresponds to the process history.*

$$\mathcal{F}_t^N = \sigma\{N_s;\ s \leq t\}$$

Definition (Stochastic intensity function of a point process)**.** *Let N_t be a Poisson process and $\mathcal{F}_t^N$, the corresponding filter function. The left-continuous intensity process is defined as :*

$$\lambda(t|\mathcal{F}_{t^-}^N) = \lim_{h\to\ 0^+} \frac{P\{N_{t+h} - N_t > 0|\mathcal{F}_{t^-}^N\}}{h}$$

[1]Mélisande Albert. *Lecture notes on Poisson processes - INSA Toulouse.* 2021.
[2]Yuanda Chen. *Thinning Algorithms for Simulating Point Processes.* Link. 2016.

Definition (Hawkes process)**.** *A Hawkes process is a counting process, which is self-exciting and takes into account the history of the events. It satisfies the following properties:*

1. $N_0 = 0$.
2. $\lambda(t)$ *is a left-continuous stochastic process given by the Stieltjes integral*
$$\lambda(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} \, \mathrm{d}N_s$$
where $\mu > 0$ *and* $0 < \alpha < \beta$
3. $\lambda(t)$ *is the stochastic intensity of the point process*
$$P\{N_{t+h} - N_t = 1 | \mathcal{F}_{t^-}^N\} = \lambda(t)h + o(h)$$
4. *The point process is orderly*
$$P\{N_{t+h} - N_t) \geq 2 | \mathcal{F}_{t^-}^N\} = o(h)$$
is called a univariate Hawkes process with exponential decay on $[0, \infty)$.

You can find in figure 2 an example of a Hawkes process. The first graph represents the counting process, while the second one represent the corresponding intensity.

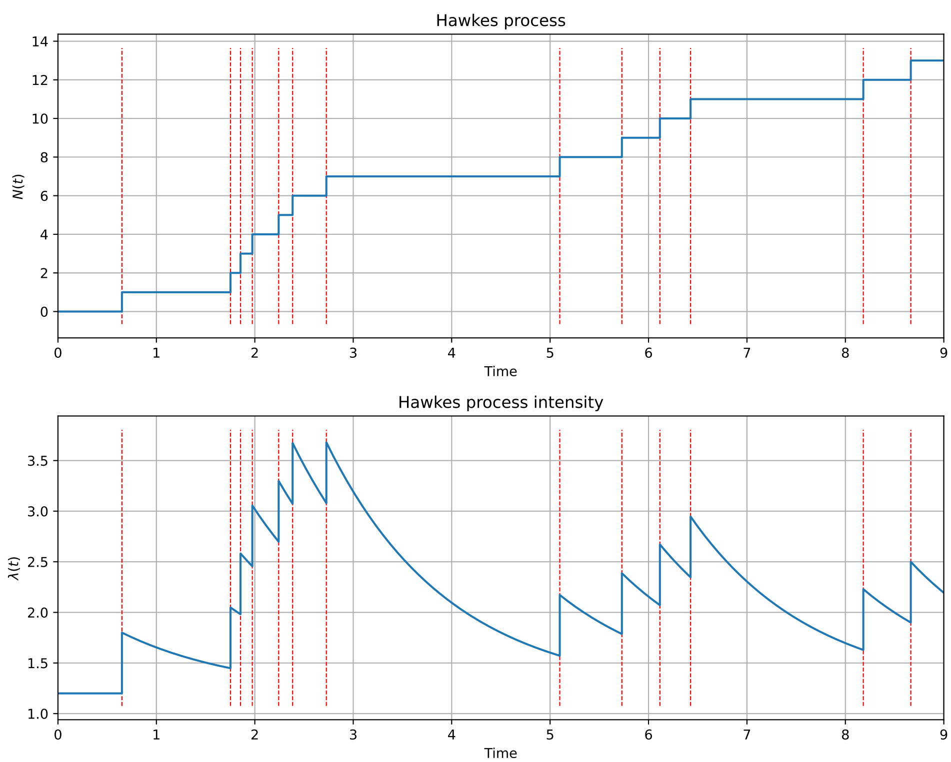

Figure 2: Example of Hawkes process with $\mu = 1.2$, $\alpha = 0.6$, and $\beta = 0.8$

As you can see, each event causes a jump in the intensity, making another event occurring shortly after more likely. This is why this process is called self-exciting, and this is why it is well suited to model events that usually happen in clusters. Note that after some time without events, the intensity progressively decrease, as the model "forgets" previous events.

2 Simulation

2.1 Thinning algorithm

To simulate numerically Hawkes processes, we used a thinning method. This kind of algorithm is usually aimed at simulating inhomogeneous Poisson processes, but it can be easily adapted to Hawkes processes. We based our algorithm on the one introduced in a Yuanda Chen's paper[3].

The first event is simulated like an homogeneous Poisson process, as the intensity is constant. After this first event, the idea is to consider potential candidate events, and to choose to keep or drop them depending on the current intensity at that point. This choice is exactly the same as the one we would do to simulate a non-homogeneous Poisson process with an decreasing exponential intensity.

For each candidate event, we generate a random variable uniformly distributed between 0 and the intensity at the previous candidate event[4]. Once the variable generated, we compare its value to the value of the intensity at that point. If it is lower, we keep this event (and the intensity jumps again). If it is higher, we skip this event and reiterate the process from that point: generation of a exponential time to get the next candidate event, simulation of a uniform random variable to keep of drop the event.

An example of this algorithm is presented in figure 3.

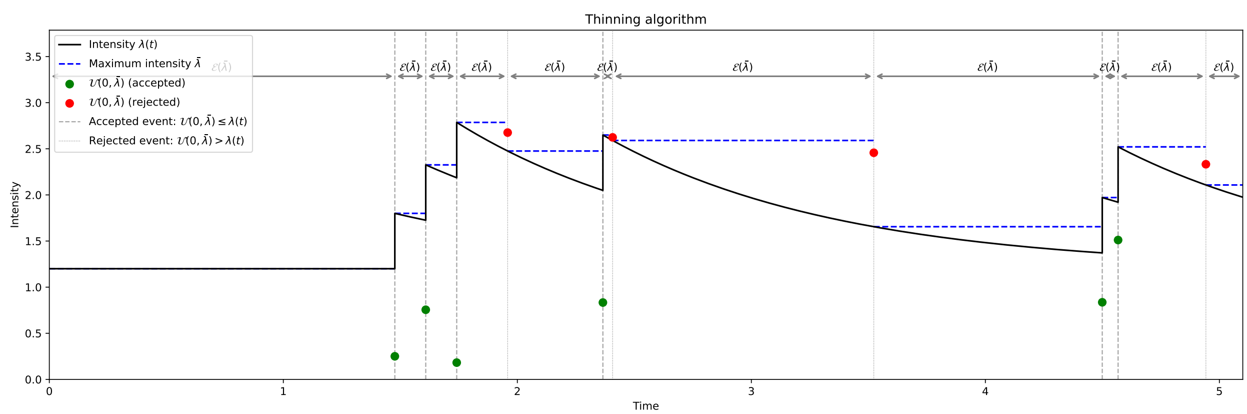

Figure 3: Thinning algorithm for a Hawkes process

2.2 Influence of model's parameters

We recall the formula of the intensity of a Hawkes process:

$$\lambda(t) = \mu + \int_0^t \varphi(t-u) dN_u \text{ with } \varphi(s) = \alpha e^{-\beta s}$$

We have three different parameters for the model: μ, α and β.

μ is called the baseline intensity. It represent the starting value of the process intensity. Note that the intensity will never be lower than this value, it is a lower bound for the function. After the first event, the lower bound is strict, and will never be reached again (even if we can get very close if no event occur in a long time).

[3]Chen, *Thinning Algorithms for Simulating Point Processes*.

[4]Note that for an arbitrary inhomogenous Poisson process, we would have to compute the maximum of the intensity over the interval in order to use it for the upper bound of the uniform law, but as in our case we use an exponential decay, the maximum is necessarily reached for the previous candidate event.

α represents the size of the jump in intensity at each event. The higher the α, the higher the jump, and the more likely there will be another event shortly after.

β represents the memory of the model. For low β, the memory of the system is long: each event will have a impact on the model for a long period of time. It will result in the intensity getting higher and higher as events continue to occur. When β is high, the memory is very short, and the intensity will decrease rather quickly after each event, meaning that the clustering effect will be less important.

You can find in figure 4 two Hawkes processes with different parameters. On the first one, we can see that the baseline intensity (μ) and the jump size (α) are lower than in the second one. In the second process, the decay parameter (β) is higher, meaning that the system forgets faster the events (you can note that the intensity falls quickly to the baseline when there is no events in a long period of time.

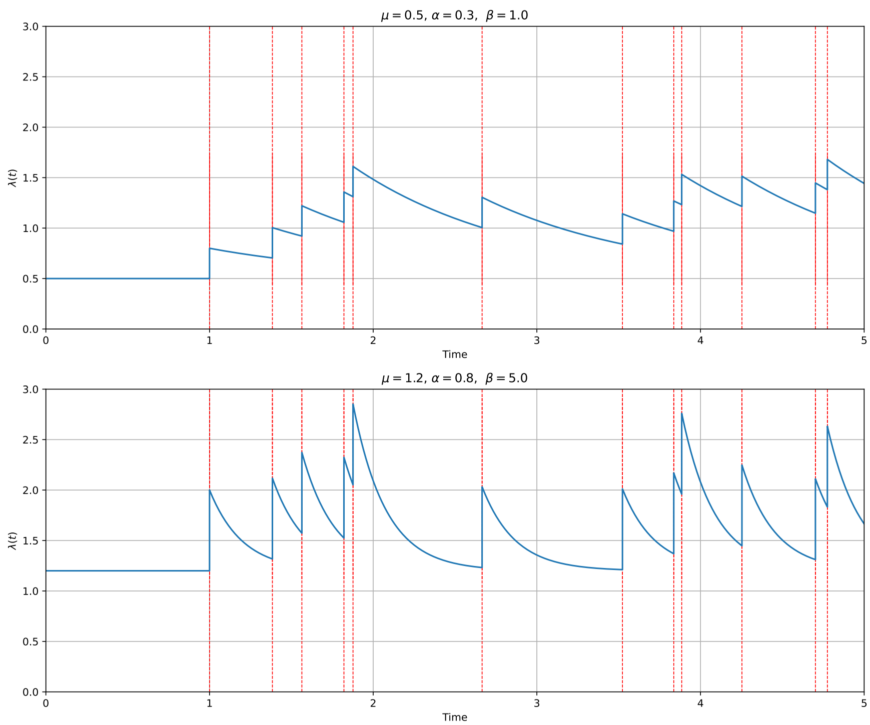

Figure 4: Two Hawkes processes with different parameters

2.3 Intensity expectation

As the intensity is random (because depending on previous events), we wondered if we could compute a expectation function to see the average behaviour. We based our calculations on a paper from 2020[5].

[5] Laurent Lesage et al. *Hawkes processes framework with a Gamma density as excitation function: application to natural disasters for insurance*. Link. 2020.

$$
\begin{aligned}
g(t) = \mathbb{E}[\lambda(t)] = \mathbb{E}\left[\mu + \int_0^t \varphi(t-u)dN_u\right] \\
= \mu + \int_0^t \varphi(t-u)\mathbb{E}[dN_u] \\
= \mu + \int_0^t \varphi(t-u)\mathbb{E}[\lambda(u)]du \\
= \mu + \int_0^t \varphi(t-u)g(u)du
\end{aligned}
$$

We recognize here a convolutional product, that is why we are going to use the Lagrange transform in order to facilitate working with this expression.

$$
\mathcal{L}\{g\}(s) = \mathcal{L}\{\mu\}(s) + \mathcal{L}\{\varphi\}(s) \times \mathcal{L}\{g\}(s) \implies \mathcal{L}\{g\}(s) = \frac{\mathcal{L}\{\mu\}(s)}{1 - \mathcal{L}\{\varphi\}(s)}
$$

As we have $\mathcal{L}\{\mu\}(s) = \dfrac{\mu}{s}$ and $\mathcal{L}\{\varphi\}(s) = \dfrac{\alpha}{s+\beta}$, we deduce

$$
\mathcal{L}\{g\}(s) = \frac{\mu}{s\left(1 - \frac{\alpha}{s+\beta}\right)} = \frac{\mu(s+\beta)}{s(s+\beta-\alpha)}
$$

Using a simple partial fraction decomposition, we obtain finally:

$$
\mathcal{L}\{g\}(s) = \frac{\frac{\mu\beta}{\beta-\alpha}}{s} - \frac{\frac{\mu\alpha}{\beta-\alpha}}{s+\beta-\alpha}
$$

We can finally compute our inverse transform:

$$
\mathbb{E}[\lambda(t)] = g(t) = \frac{\mu\beta}{\beta-\alpha} - \frac{\mu\alpha}{\beta-\alpha}e^{(\alpha-\beta)t}
$$

We can see that when t goes to infinity, the average intensity becomes constant (as $\alpha < \beta$).

Other formulas can be derived for the variance of the intensity $\mathrm{Var}(\lambda_t)$, as well as the expectation and the variance of the process ($\mathbb{E}[N_t]$ and $\mathrm{Var}(N_t)$). All these formulas can be found in a paper from Andrew Daw and Jamol Pender[6].

We can then plot the mean function and the corresponding 95% confidence interval.

[6] Andrew Daw and Jamol Pender. *Queues Driven by Hawkes Processes.* Link. 2018.

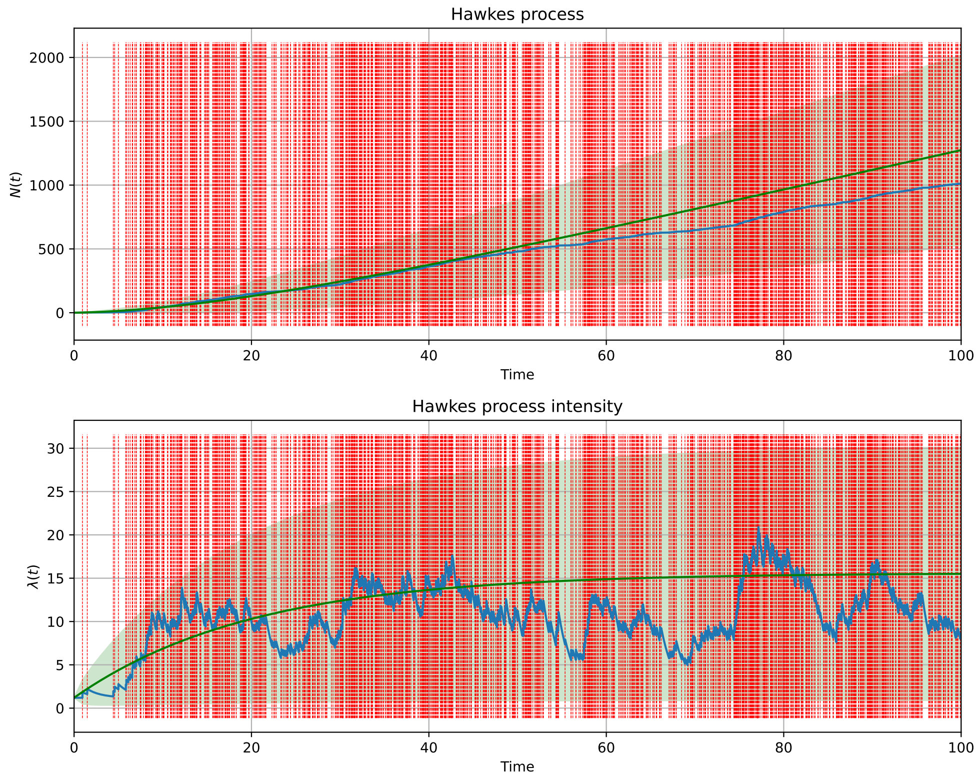

Figure 5: Expectation and 95% confidence interval

As we saw in the formula, the average intensity increase at the beginning, before reaching a stable value.

3 An overview of the ruin/risk theory

3.1 The Cramér-Lundberg model

In actuarial science, it is important to quantify the risk for insurances. In effect, to be able to take into account the claims of an insured person, the insurer must have enough reserves so that they are not ruined.

To measure the risk, Lundberg in 1903 and then Cramér in the 30s have developed a model named the *Cramer-Lundberg* model which a probability evaluation that adverse events occur for an insurance company. The model is defined as follow :

$$R(t) = u + pt - \sum_{i=1}^{N_t} X_i \text{ for } t \geq 0$$

3.2 The parameters of the Cramér-Lundberg model

The model is defined by four different parameters which are :

- $u \geq 0$: the initial amount of the insurer's reserves ;
- $p > 0$: premium that receive the insurance for the customers ;
- X_i: the amount of the i-th claims (real random variable) ;
- N_t: A counting process (originally a Poisson process).

The tuning of these parameters is very important depending on the problem we are trying to solve.

Firstly, the initial amount u has not to be to low so that the insurer is not ruined to soon.

Secondly, the premium p has to be big enough because the arrival of a claim can cause important damage and lead to large costs. On the contrary, a premium to high is bad for an insurer because nobody will be able to pay the cost of the premium and then the insurer will not have customers anymore.

The third parameter X_i is a random variable following a given distribution (uniform, normal...)

The counting process is very important as it models the claims to cover. The Poisson process is suited for a lot of cases, but it fails to catch potential chain effects in events. This is why we wanted to study a Cramér-Lundberg-type model, but that would use a Hawkes process to replace the Poisson process.

3.3 Arrival time of the ruin

Now that we have defined the risk, it is important to know if the insurer will be ruined, and if so when. The ruin probability is defined as follow :

$$\Phi(u) = \mathbb{P}\left(\inf_{t \geq 0} R_t < 0 | R_0 = u\right)$$

The ruin arrival T is defined as the moment where the resources of the insurer fall below 0, we can write it as follow :

$$T = \inf \{t > 0 | R(t) < 0\}$$

3.4 Graphical visualization of the risk for a Poisson and a Hawkes process

We can have graphical visualization of the evolution of the risk over time, depending on the initial amount of resource u and the premium p.

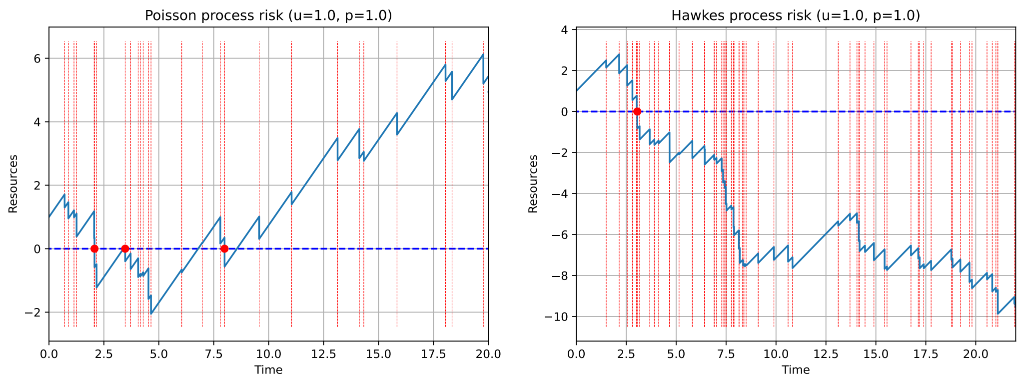

Figure 6: Risk evolution over time for a Poisson and a Hawkes process for $u = 1$ and $p = 1$

Here we can see with the red lines the occurrence of the claims. Then, when the resources $R(t)$ cross the 0 abscissa the insurer is ruined.

3.5 The influence of the parameters u and p on the ruin arrival time

To evaluate the influence of the parameters u and p on the ruin arrival time, we can can define a list of u and a list of p. Then, for each pair of (u, p), we run 100 simulations of Hawkes processes and then take mean arrival time on these 100 simulations. Finally, we can have look on the results:

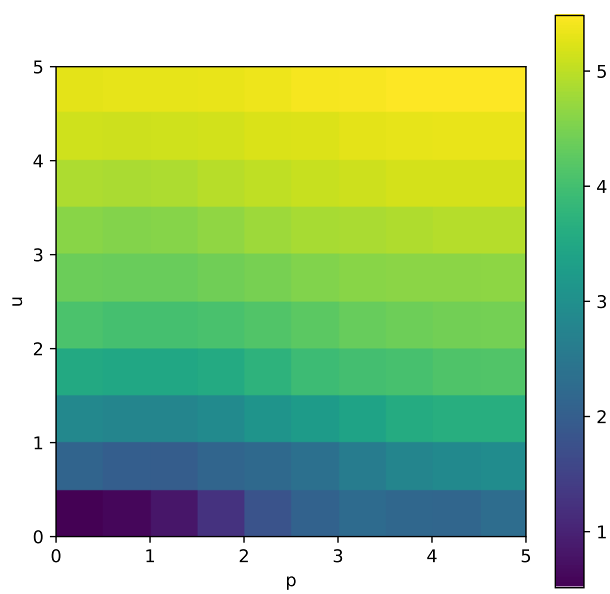

Figure 7: Time of arrival of the ruin as a function of u and p

As we could expect small values of u and p lead us quickly to the ruin. On the other hand, high values of u and p result in ruin reached later.

It seems more relevant to have a significant amount of resources at the beginning (so an high value of u), and over time adjust the value of p so that we do not reach to the ruin, and to not demand an important premium p to customers when they have to pay.

4 Application to real data from the CAC 40

To illustrate Hawkes process in a concrete case, we have downloaded the stock market evolution of the CAC40 (continuous assisted contribution) from the year 1990 to the year 2021.

The CAC 40 is the main stock market index of the Paris stock exchange. It is a bucket of 40 French companies selected from the 100 French companies with the highest trading volumes. The stock market value of this index undergoes high variations during certain periods of time.

You can see below an example of its evolution:

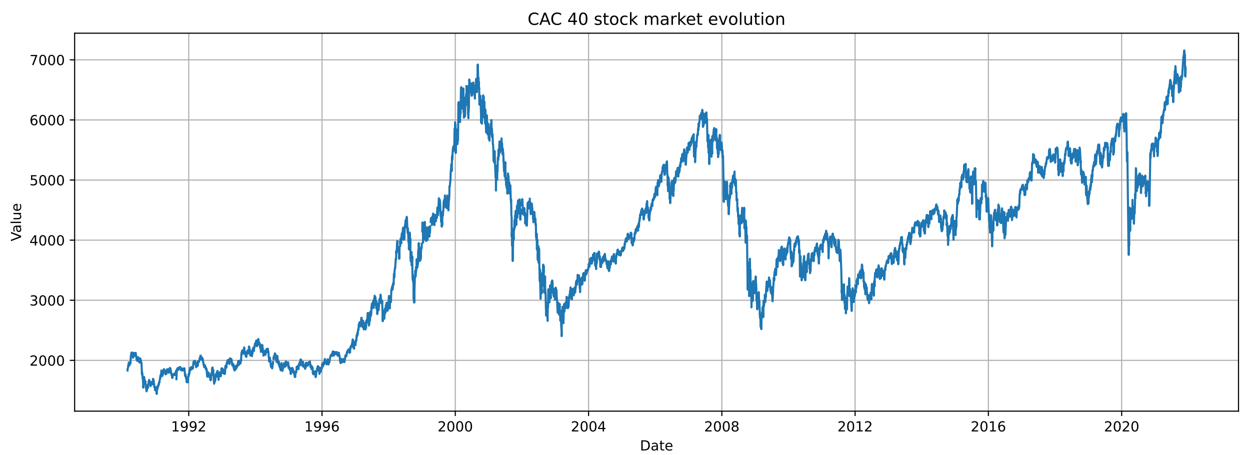

Figure 8: CAC 40 evolution

We can also note that the different values for each day are not independent and related to the history of the market. Indeed, the curve is globally increasing on certain periods (for example from 2000 to 2008, before the subprime crisis) and decreasing on others (for example from January 2020 to March 2020 due to the covid19 virus).

All these elements suggest that the use of the Hawkes process would be ideal to model the evolution of the stock market.

4.1 Methods

To realize this study, it was necessary to define what corresponds to an event in our process. We considered an event each time the stock exchange share of the CAC 40 dropped by at least 150 from the closure of a day to the closure of the following day. We can visualize these events in the figure 9.

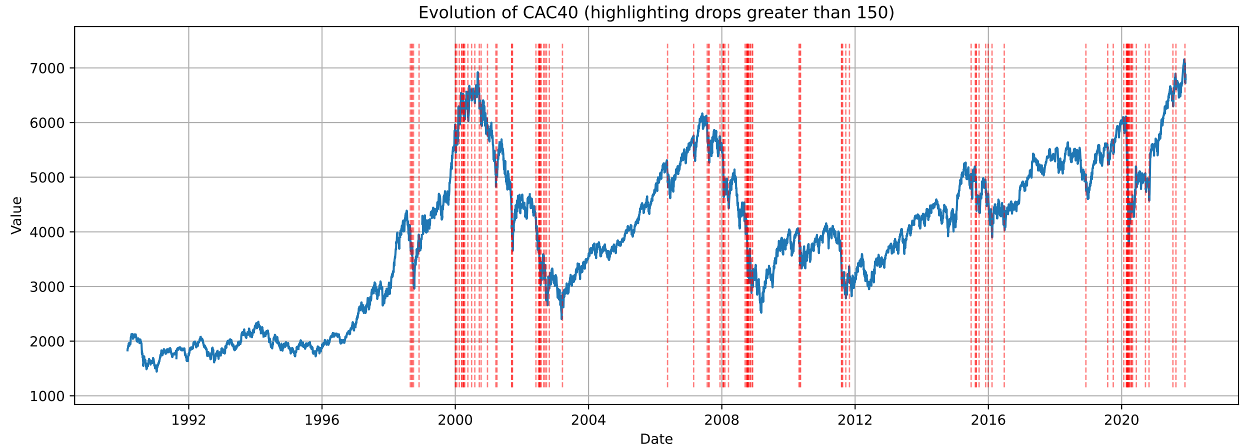

Figure 9: Drop by at least 150 in the CAC 40 evolution

Then, to be able to fit this model by a Hawkes process we used the python library `tick`. The module `Hawkes` of `tick` is a "module that proposes a comprehensive set of tools for the inference and the simulation of Hawkes processes, with both parametric and non-parametric estimation techniques and flexible tools for simulation."[7].

This library offers the possibility to choose between multiple kernel. In this study we used the exponential kernel (`tick.hawkes.HawkesExpKern`). However, while it proposes to estimate the parameters μ and α, the parameters β has to be provided. Therefore, we attempted to find by ourselves the best β by fitting several models and computing their log-likelihood. We used 100 different values for β in the interval $10^{-}4$ and 10.

You can find in figure 10 the log-likelihood as a function of β. The best value was obtained for $\beta = 0.060$, associated with the parameters $\alpha = 0.048$ and $\mu = 0.0024$.

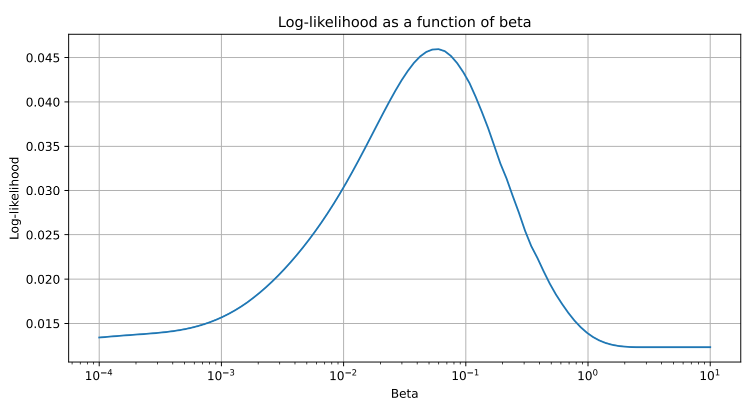

Figure 10: Log-likelihood of the Hawkes Process as function of betas

4.2 Results and validation

Now that we have the parameters, we are able to plot the intensity of the process :

[7]Extracted from `https://x-datainitiative.github.io/tick/modules/hawkes.html`

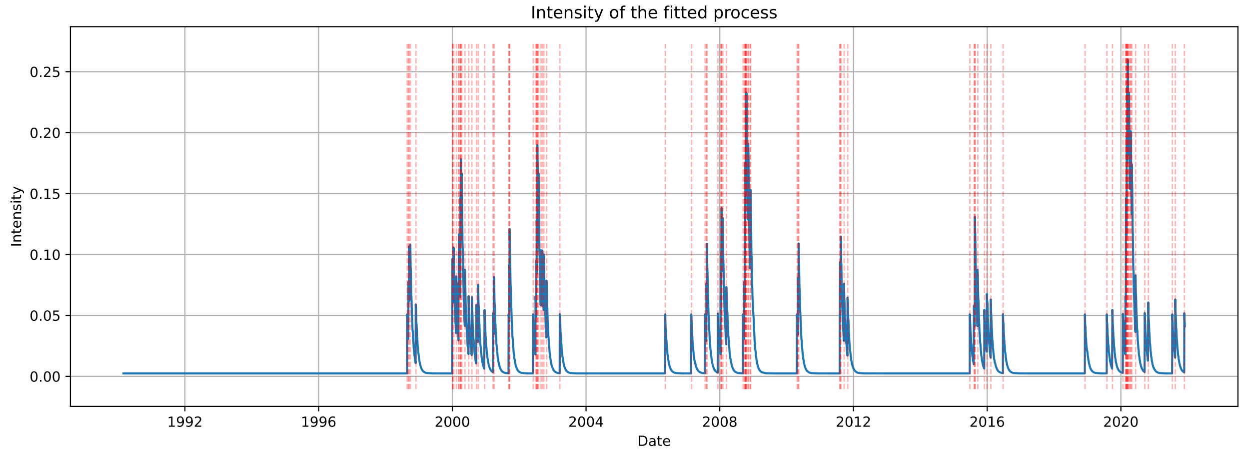

Figure 11: Hawkes process intensity

The analysis of this curve shows that the two major peak in the 2000's are located in 2008 and in 2020, that corresponds respectively to the subprime and Covid-19 crisis. We can observe that the process intensity is indeed self-exiting on those peaks.

We can also see it on the cumulative intensity function below :

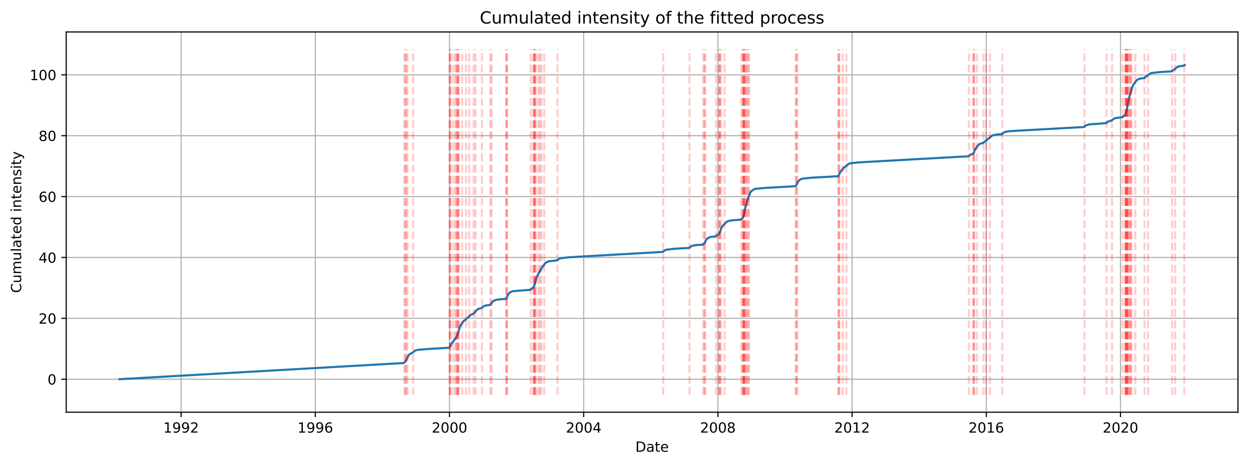

Figure 12: Cumulative intensity function

The curve undergoes large jumps when several events occurred during a short time interval, the chain effect is clearly visible.

Now, we would like to validate our assertions using a graphical method to visualize the goodness of fit. We rely on the Random time change theorem[8].

Definition (Theorem: Random time change theorem)**.** *Consider a point process with conditional intensity $\lambda^*(.)$. Let us note $\{t_1, t_2, ..., t_k\}$ these realizations in the interval $[0, T]$. If $\lambda^*(.)$ is positive in the interval $[0, T]$ and the cumulative intensity function is bounded. Then the points $\{\Lambda(t_1), \Lambda(t_2), ..., \Lambda(t_k)\}$ follow a Poisson process with intensity 1.*

[8]Patrick Laub. *Hawkes Processes: Simulation, Estimation, and Validation.* Link. 2014.

Thus, we applied our empirical cumulative intensity function to our events to see if they follow a Poisson process.

In order to perform the test, we considered the interarrival times, noted : $\{\tau_1, \tau_2 ...\} = \{t_1^*, t_2^* - t_1^*, \dots\}$ (with $\tau_i \sim \mathcal{E}(1)$ independent).

From these interarrival times, we plotted a Q-Q plot for τ_i using an exponential distribution.

To check the independence of the events we plotted the points (Z_{i+1}, Z_i), with $Z_i = F_{\mathcal{E}(1)}(t_k^* - t_{k-1}^*)$ where $F_{\mathcal{E}(1)}$ is the cdf. As the transformed process is supposed memoryless, if the distribution of these points should be random and not follow any pattern.

Here are the graphical results obtained :

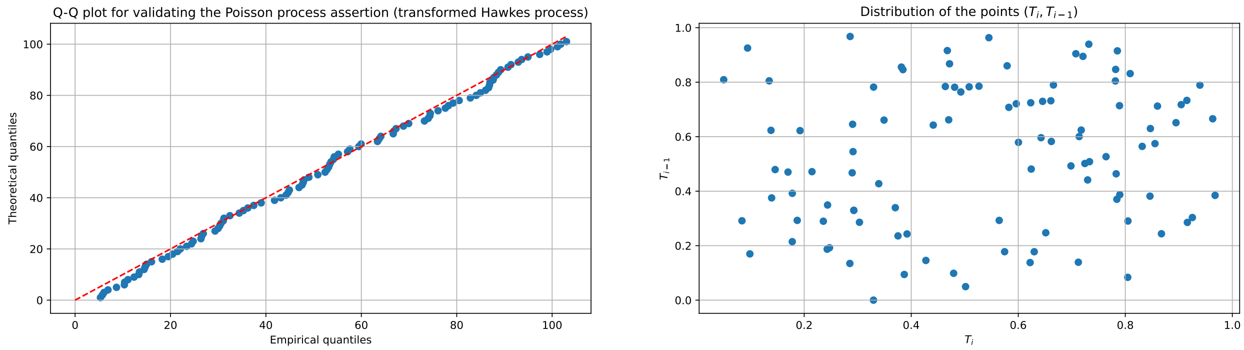

Figure 13: Test to validate the Poisson process fit of the transformed events

The Q-Q plot looks very good, as well as the independence plot. Visually, it looks like the transformed times follow a Poisson process, and therefore that the original event follow a Hawkes process.

To verify that the original events do not follow a Poisson process, we plotted the same graphical tests:

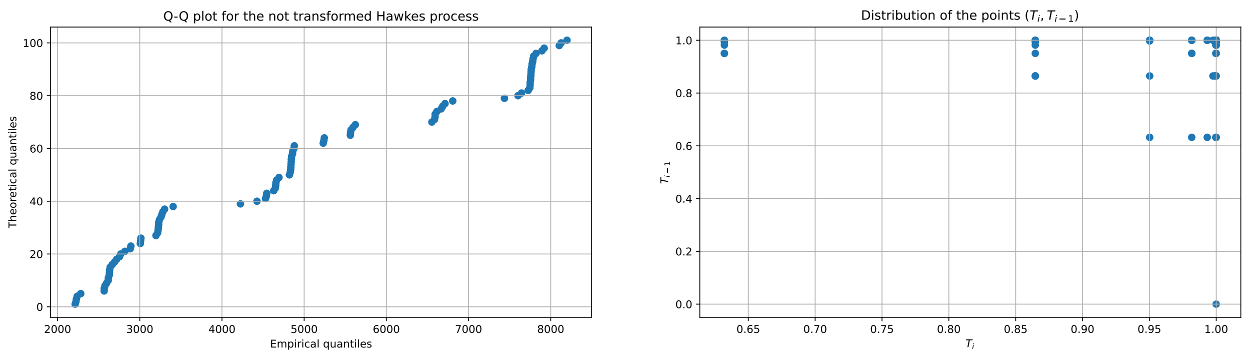

Figure 14: Test to validate the Poisson process fit of the original events

As we can see, neither the Q-Q plot nor the independence plot looks good. A Poisson process is not suited to describe the drops in the CAC 40 market shares.

Conclusion

The study of Hawkes processes allowed us to discover a new counting process. This process is more suited to some situations where events tends to occur in clusters, like when one events lead to another.

As we have seen, Hawkes processes are usefull in finance and insurance are domains? They are also used in geological domain to model earthquakes for example.

When used in the context of the Cramér-Lundberg model, Hawkes processes tend to decrease the time of ruin compared to Poisson processes. Clusters of claims are indeed quite difficult to deal with for an incurance company.

Finally, we saw that the evolution of a stock market can be accurately modeled with Hawkes processes, while Poisson processes fail to catch the cluster of events that can occur.

References

Albert, Mélisande. *Lecture notes on Poisson processes - INSA Toulouse*. 2021.

Chen, Yuanda. *Thinning Algorithms for Simulating Point Processes*. Link. 2016.

Daw, Andrew and Jamol Pender. *Queues Driven by Hawkes Processes*. Link. 2018.

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