UNIVERSITÄT BONN

Cellular Automata for Sandpiles

An Example of Self-organized criticality

Markus Frohne
markusfrohne@uni-bonn.de

Pascal Wolf
pwolf@uni-bonn.de

March 15th, 2018

Outline of talk

- Motivation
- Theory
  - Self-organized criticality (**SOC**) and Scale invariance
  - The `Bak-Tang-Wiesenfeld` (**BTW**) and Custom model
- Simulations
  - Visualization of simulated data
- Analysis
  - Methods
  - Results
  - Discussion & Comparison
- Summary

Motivation

Motivation

- Critical phenomena are studied widely

Motivation

- Critical phenomena are studied widely
- The Ising model
  - Main tunable model parameter: temperature T
  - Phase transition at critical temperature T_c
  - Cluster formation with characteristic properties (scale invariance!)

Motivation

- Critical phenomena are studied widely
- The Ising model
  - Main tunable model parameter: temperature T
  - Phase transition at critical temperature T_c
  - Cluster formation with characteristic properties (scale invariance!)

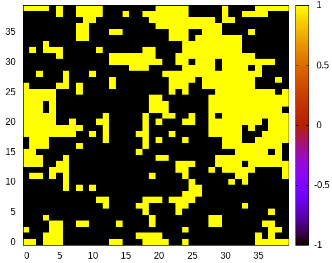


Fig. 1: Ising model cluster formation around the critical temperature.

Motivation

- Different types of critical phenomena
  - need tuning of parameter
  - don't need tuning of parameter ?

Motivation

- Different types of critical phenomena
  - need tuning of parameter
  - don't need tuning of parameter ?
- Critical phenomena without tuning exist:
  Examples from nature [1]
  - landslides
  - earthquakes
  - seacoasts
  - forest fires
  - **sandpiles**

Motivation

- Different types of critical phenomena
  - need tuning of parameter
  - don't need tuning of parameter ?
- Critical phenomena without tuning exist: Examples from nature [1]
  - landslides
  - earthquakes
  - seacoasts
  - forest fires
  - **sandpiles**

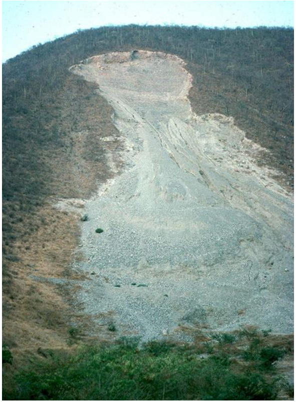


Fig. 2: A rock landslide in Guerrero, Mexico.
from: Wikimedia Commons, *File:Slide-guerrero1.JPG*

Motivation

- Different types of critical phenomena
  - need tuning of parameter
  - don't need tuning of parameter ?
- Critical phenomena without tuning exist: Examples from nature [1]
  - landslides
  - earthquakes
  - seacoasts
  - forest fires
  - **sandpiles**
- No tuning needed → "self-organized" criticality

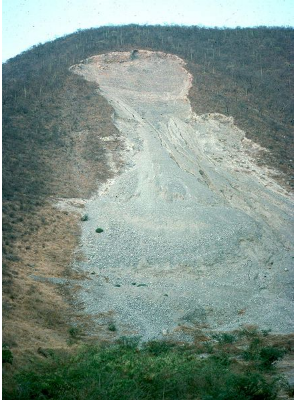


Fig. 2: A rock landslide in Guerrero, Mexico.
from: Wikimedia Commons, *File:Slide-guerrero1.JPG*

Theory

Theory – SOC and scale invariance

- Some definitions:
  - **SOC system**: System that naturally evolves into a critical equilibrium state by itself without parameter tuning
  - **Scale invariance**: Dynamics of the considered system do not change under rescaling

Theory – SOC and scale invariance

- Some definitions:
  - **SOC system**: System that naturally evolves into a critical equilibrium state by itself without parameter tuning
  - **Scale invariance**: Dynamics of the considered system do not change under rescaling

- Every SOC system is believed to show scale invariance
  - Observables $\hat{Y}$ follow power law distribution:

$$P^{(Y)}(y) \propto y^{-\rho}$$

  - Scaling factor λ retains form of power law:

$$P^{(Y)}(\lambda y) \propto (\lambda y)^{-\rho} \propto y^{-\rho} \propto P^{(Y)}(y)$$

Theory – Scale invariance of sandpiles

- Need observables

- Need observables:
    - Avalanche Area

Theory – Scale invariance of sandpiles

- Need observables:
  - Avalanche Area
  - Avalanche Duration

Theory – Scale invariance of sandpiles

- Need observables:
  - Avalanche Area
  - Avalanche Duration
  - Avalanche Size

Theory – Scale invariance of sandpiles

- Need observables:
  - Avalanche Area
  - Avalanche Duration
  - Avalanche Size
- Need distributions of observables:
  - Simulate sandpiles numerically
  - Use cellular automata:
    - Iterative algorithm acting on a discrete lattice
    - Lattice sites are repeatedly updated depending on current lattice state

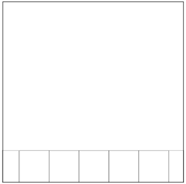
Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model

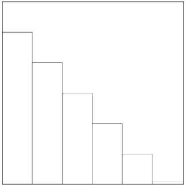
Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



(a) BTW lattice

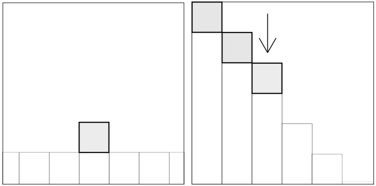


(b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The Bak-Tang-Wiesenfeld (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes

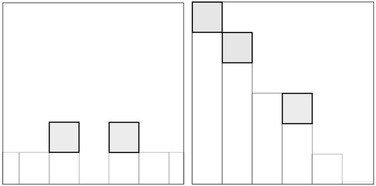


(a) BTW lattice (b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

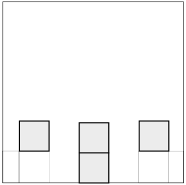
- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



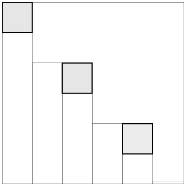
(a) BTW lattice (b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



(a) BTW lattice

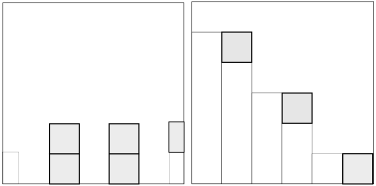


(b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



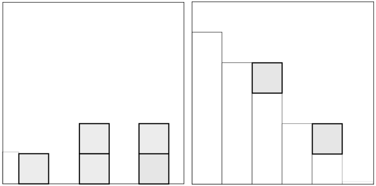
(a) BTW lattice (b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

UNIVERSITÄT BONN

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes

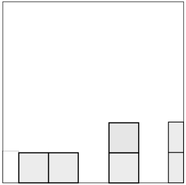


(a) BTW lattice (b) Sand grains

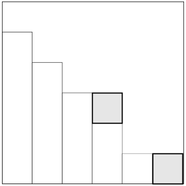
Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The Bak-Tang-Wiesenfeld (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



(a) BTW lattice

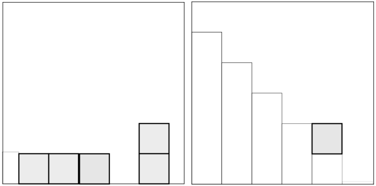


(b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes

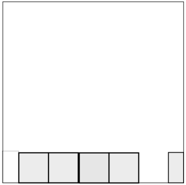


(a) BTW lattice (b) Sand grains

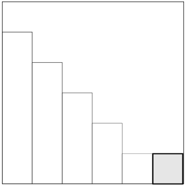
Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



(a) BTW lattice



(b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes

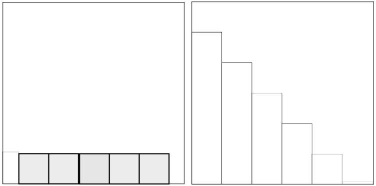
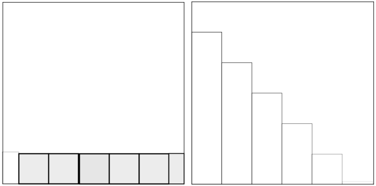

(a) BTW lattice (b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

- The `Bak-Tang-Wiesenfeld` (BTW) model
  - First model studying SOC in sandpile dynamics
  - Stores local slopes of the sandpile on a lattice
  - **Driving**: add sand to the pile ⇒ increase slope
  - **Relaxation**: Redistribute grains if slope is too large ⇒ change slopes



(a) BTW lattice (b) Sand grains

Fig. 3: Driving and relaxation of BTW lattice.

Theory – Sandpile cellular automata

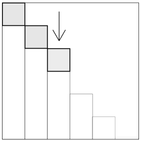
- The custom model

Theory – Sandpile cellular automata

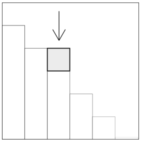
- The custom model
  - Different approach
  - Store local sandpile *heights* instead of *slopes* on the lattice
  - **Driving**: add one grain of sand to the random site instead of the whole upper hillside
  - **Relaxation**: Redistribute grains if slope is too large and consider direction of slope

Theory – Sandpile cellular automata

- The custom model
  - Different approach
  - Store local sandpile *heights* instead of *slopes* on the lattice
  - **Driving**: add one grain of sand to the random site instead of the whole upper hillside
  - **Relaxation**: Redistribute grains if slope is too large and consider direction of slope



(a) Non-conservative



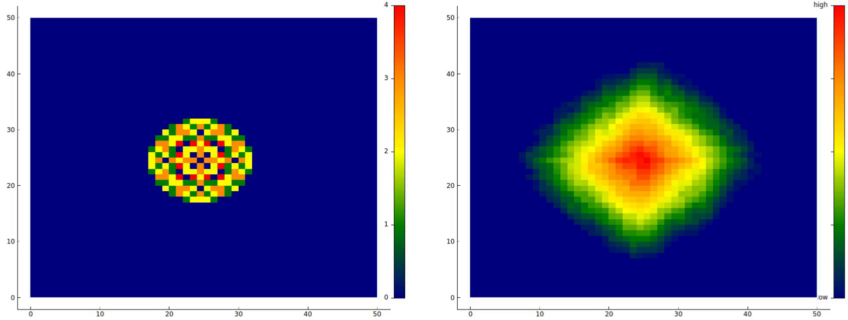
(b) Conservative

Fig. 4: Conservative and non-conservative driving.

Simulations

Visualization of data

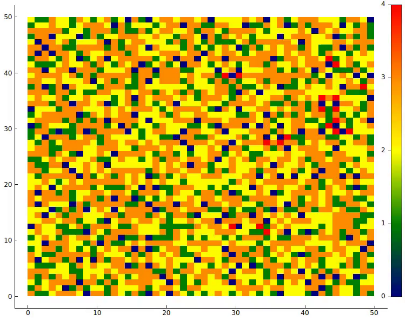
Sandpile dynamics – Avalanche evolution



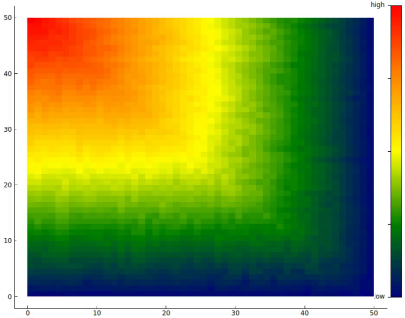
(a) BTW model (b) Custom model

Fig. 5: Avalanche evolution in a 2D sandbox of length 50 (**center** drives)

Sandpile dynamics – Avalanche evolution



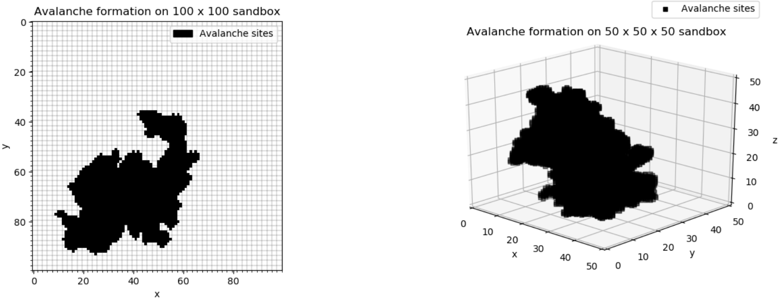
(a) BTW model



(b) Custom model

Fig. 5: Avalanche evolution in a 2D sandbox of length 50 (**random** drives)

Sandpile dynamics – Avalanche formations

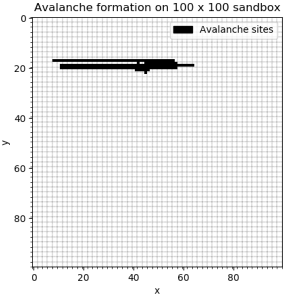


(a) 2D

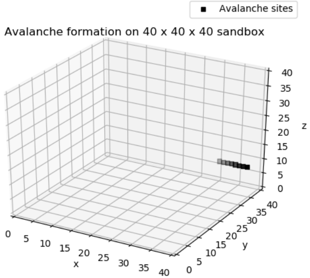
(b) 3D

Fig. 6: Avalanche formation in the BTW model.

Sandpile dynamics – Avalanche formations



(a) 2 D



(b) 3 D

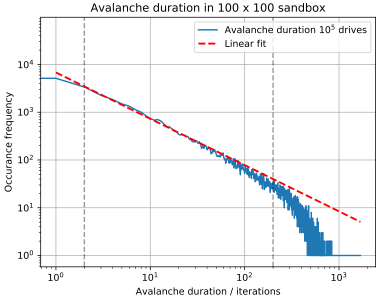
Fig. 6: Avalanche formation in the custom model.

Analysis

Analysis method – Naive fit

- Initial method: straight line fit of power-law scaling behavior:

$$P^{(Y)}(y) \propto y^{-\rho}$$



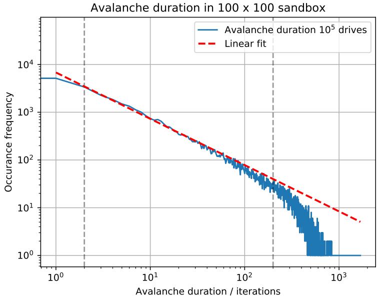
Analysis method – Naive fit

- Initial method: straight line fit of power-law scaling behavior:

$$P^{(Y)}(y) \propto y^{-\rho}$$

- $P^{(Y)}(y)$ deviates from linear trend due to e.g *finite size effects*
  - Straight line fit not sufficient to determine scaling exponents
  - No reasonable estimation of uncertainties on scaling exponents possible

$\Rightarrow$ More sophisticated approach: so-called *moment analysis*



Analysis method – Moment analysis

- Define the so-called n^{th} *moment* of the observable $\hat{Y}$ by

$$\left\langle y^n \right\rangle = \int_0^\infty dy\, y^n P^{(Y)}(y) \quad \text{where} \quad P^{(Y)} \equiv \text{PDF of } \hat{Y}$$

- For $L \to \infty$ (or approx. $L \gg 1$) the following relation holds:

$$\left\langle y^n \right\rangle \propto L^{K(1+n-\rho)} \qquad \text{with} \qquad K\,(1+n-\rho) \equiv \sigma_n$$

where K, ρ is the set of scaling exponents of the observable $\hat{Y}$.
Taking the logarithm, this relation exhibits a linear trend with slope σ_n.

- Determination of the set of scaling exponents K, ρ via linear fit of the 1st and 2nd moments, yielding σ_1 and σ_2, like:

$$K = \sigma_2 - \sigma_1 \quad , \quad \rho = \frac{2\sigma_2 - 3\sigma_1}{\sigma_2 - \sigma_1}$$

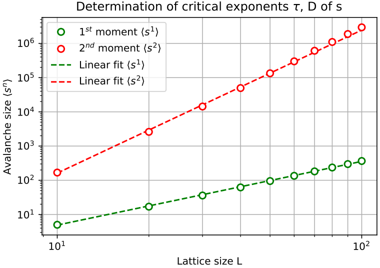
Analysis method – Moment analysis

- General approach of moment analysis:
  - Simulate R samples for N lattice sizes L_N, record PDF of observables and calculate $\left\langle y^1 \right\rangle$, $\left\langle y^2 \right\rangle$
  - Draw *random* bootstrap sample from $\left\langle y^1 \right\rangle_R$, $\left\langle y^2 \right\rangle_R$ for each L_N
  - Calculate covariance matrix for each bootstrap (**double bootstrap**)
  - Linear fit within each bootstrap sample to calculate and store $K,\ \rho$
  - $K,\ \rho = \langle K, \rho \rangle_{\mathrm{bootstraps}}$ and estimate uncertainty by $\sigma(K,\ \rho)_{\mathrm{bootstraps}}$

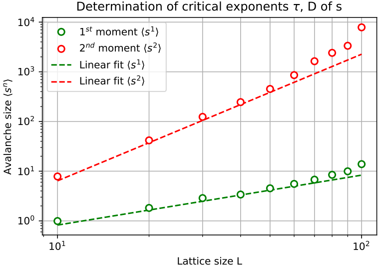
$\Rightarrow$ Actual numbers for analysis in 2 and 3 dimensions:

- $R = 10$
- $N = R$ with $L_N \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

Analysis method – Moment analysis



(a) BTW



(b) Custom

Fig. 7: 1$^{\mathrm{st}}$ and 2$^{\mathrm{nd}}$ moments of one *random* bootstrap sample for avalanche **size** and overall best linear fit in 2D.

Analysis results – Moment analysis

	Model	Size		Duration		Area	
		τ	D	α	Z	κ	T
2D	BTW	1.19(9)	2.3(2)	1.21(2)	1.34(3)	1.23(7)	1.9(2)
	CST_{C5}	1.3(2)	1.6(2)	1.2(2)	0.7(2)	1.3(2)	0.8(2)
	CST_{C7}	1.4(2)	1.6(4)	1.3(1)	0.81(8)	1.4(1)	1.0(1)
3D	BTW	1.3(1)	2.5(3)	1.45(3)	1.41(6)	1.4(3)	2.6(7)
	$\mathrm{CST}_{C5}{}^{*}$	1.23(1)	1.99(2)	1.29(1)	1.17(1)	1.29(1)	1.43(1)

Table 1: Scaling exponents for avalanche size, duration and area.

*No bootstrap fitting due to too few simulated samples

Analysis discussion – Scaling exponents

- Comparison of results to literature possible, but:

Analysis discussion – Scaling exponents

- Comparison of results to literature possible, but:
  - Many different literature values of scaling exponents from various papers from the early 90's until today
  - Many different models used (**Abelian BTW**, Non-abelian BTW, etc.)
  - Usually only the sets τ, D and α, Z are stated

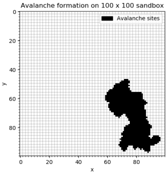
Analysis discussion – Scaling exponents

- Comparison of results to literature possible, but:
  - Many different literature values of scaling exponents from various papers from the early 90's until today
  - Many different models used (**Abelian BTW**, Non-abelian BTW, etc.)
  - Usually only the sets τ, D and α, Z are stated
- Generally, scaling exponents should satisfy $\tau, \alpha \geq 1$ for the BTW model
- Example in 2D: $\tau = 1.2(1)$, $\alpha = 1.16(3)$ from [2] coincide with our simulations $\tau = 1.19(9)$, $\alpha = 1.21(2)$.
- Scaling exponents of the BTW model determined in this project generally coincide with most of the BTW literature values within their σ

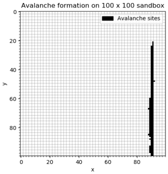
Analysis discussion – Scaling exponents

- Custom model partially deviates from BTW model: This is most prominent in the dimension exponents e.g. of the avalanche area:
  - 2D area dimension exponent $T_{\mathrm{BTW}} = 1.9(2)$ vs. $T_{\mathrm{CST}} = 1.0(1)$

  $\Rightarrow$ Custom avalanches are mostly 1 dimensional; That's also what we see:



(a) BTW 2D



(b) CST 2D

- Scaling exponents τ, α of the custom model coincide with BTW

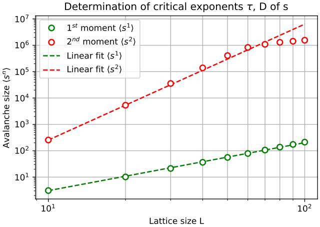
Analysis discussion – Issues

$\Rightarrow$ The main issue is the lack of computational resources: Inherits several consequences:

- Deviation from linear trend of 2nd moment for large lattices $L \geq 80$ in 3D:

$\Rightarrow$ # of drives of the simulation < # lattice sites: Not sufficient

$\Rightarrow$ 1 simulation of 100^3 lattice, 10^5 drives $\approx$ **28 hours**



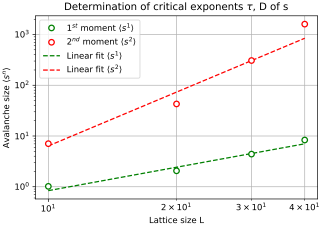
Analysis discussion – Issues

⇒ The main issue is the lack of computational resources: Inherits several consequences:

- Only samples for $L \leq 40$ for custom model in 3D:

⇒ Bootstrap fitting failed, external fitting library used

⇒ No reliable estimation of uncertainty possible



Analysis discussion – Issues

⇒ The main issue is the lack of computational resources: Inherits several consequences:

- 10 samples for each lattice size L_N already very few:

⇒ Double bootstrap needed

⇒ Many bootstrap samples need to be drawn

Summary and Conclusion

Within this project:

- Two models of cellular automata for sandpile dynamics have been implemented
  - The BTW model
  - A custom model
- The characteristics, namely their sets of critical scaling exponents, have been determined
- The statistics were gathered for 2 and 3 dimensional lattices
- The obtained results coincide with most of the values given in the literature

Summary and Conclusion

Conclusions:

- More computational recourses would improve results:
  - Simulate more samples and increase number of total drives
  - Go to higher dimensions and lattice sites
- More detailed look / more configurations of the custom model would be interesting

Thank you!

Bibliography I

Stefan Hergarten. “Landslides, sandpiles, and self-organized criticality”. In: 3 (Nov. 2003).

G. Pruessner. *Self-Organized Criticality; Theory, Models and Characterisation*. 2012.

Backup slides

Literature values

Fig. 8: Literature values from various papers gathered in [2]

Table 4.1 Exponents of the Abelian BTW Model. $\mathcal{P}^{(s)}(s) = a^{(s)} s^{-\tau} \mathcal{G}^{(s)}(s/(b^{(s)}L^{D}))$ is the avalanche size distribution and $\mathcal{P}^{(T)}(T) = a^{(T)} T^{-\alpha} \mathcal{G}^{(T)}(T/(b^{(T)}L^{z}))$ is the avalanche duration distribution, Sec. 1.3. Scaling relations: $D(2-\tau) = \sigma_1^{(s)} = 2$ (bulk drive) and $(1-\tau)D = (1-\alpha)z$ (sharply peaked joint distribution, widely assumed) in all dimensions (Sec. 2.2.2).

d	τ		D		α		z	
1	1	(*a*)	2	(*a*)	1	(*a*)	1	(*a*)
2	0.98 ∗	(*b*)	2.56	(*o*)	0.97 ∗	(*b*)	1.234	(*f*)
	1.22 ▷	(*c*)	2.50(5)	(*p*)	1.38	(*c*)	1.168	(*g*)
	1.21 ▷∗	(*d*)	2.73(2)	(*k*)	1.316(30)	(*e*)	1.52(2)	(*k*)
	1 ▷	(*d*)			1.480(11)	(*i*)	1.02(5)	(*q*)
	1.2007(50)	(*e*)			1.16(3)	(*n*)	1.082 −1.284	(*m*)
	1.253	(*f*)						
	1.253	(*g*)						
	6/5	(*h*)						
	1.293(9)	(*i*)						
	1.27(1)	(*k*)						
	1.122 −1.367	(*m*)						
	1.13(3)	(*n*)						
3	1.35 ∗	(*b*)	2.96	(*r*)	1.59 ∗	(*b*)	1.6	(*r*)
	1.33 ▷	(*o*)	3.004	(*t*)	1.625	(*d*)	1.618	(*t*)
	1.47 ▷∗	(*d*)			1.597(12)	(*s*)		
	1.37 ▷	(*d*)						
	1.35 ▷	(*r*)						
	1.333(7)	(*s*)						
4	1.61 ▷∗	(*d*)	4	(*s*)	2	(*s*)	2	(*s*)
	1.5 ▷	(*d*)						
	3/2	(*s*)						
MFT	3/2 ▷	(*u*)	4	(*w*)	1	(*u*)	2	(*w*)
	3/2 ▷	(*v*)			2	(*v*)		

∗ Based on the (original) non-Abelian BTW Model.
▷ Reported as $\tau + 1$.
a Ruelle and Sen (1992), *b* Bak *et al.* (1987), *c* Manna (1990), *d* Christensen *et al.* (1991), *e* Manna (1991a), *f* Pietronero, Vespignani, and Zapperi (1994), *g* Vespignani and Zapperi (1995), *h* Priezzhev, Ktitarev, and Ivashkevich (1996b), *i* Lübeck and Usadel (1997b), *k* Chessa *et al.* (1999a), *m* Lin and Hu (2002), *n* Bonachela (2008), *o* Grassberger and Manna (1990), *p* De Menech *et al.* (1998), *q* De Menech and Stella (2000), *r* Ben-Hur and Biham (1996), *s* Lübeck and Usadel (1997a), *t* Lübeck (2000), *u* Tang and Bak (1988a), *v* Vergeles, Maritan, and Banavar (1997), *w* Tang and Bak (1988b).