CHAPTER

Potential Flow

3

LEARNING OBJECTIVES

- Learn to calculate the air flow and pressure distribution around various body shapes.
- Learn more about the classical assumption of irrotational flow and its meaning that the vorticity is everywhere zero. Note that this also implies inviscid flow. Irrotational flows are potential fields.
- Learn about the potential function, known as the velocity potential. Learn that the velocity components can be determined from the velocity potential.
- Learn that the equations of motion for irrotational flow reduce to a single partial differential equation for velocity potential known as the Laplace equation.
- Examine the classical analytical techniques described for obtaining two-dimensional and axisymmetric solutions to the Laplace equation for aerodynamic applications.
- Learn how computational tools are applied for predicting the potential flows around arbitrary two-dimensional geometries.

3.1 TWO-DIMENSIONAL FLOWS

The problem of interest in this section is illustrated in Fig. 3.1, which illustrates the two-dimensional steady motion of a body moving at constant speed U in a quiescent fluid. The quiescent fluid is incompressible with uniform density ρ and uniform pressure p_∞. The figure shows this from the point of view of a coordinate system attached to the body. Hence, the body appears stationary in a uniform *onset* flow. In this coordinate system, we can assume steady-state conditions. Note that the x direction of the coordinate system illustrated in the figure was selected, for convenience, to be in the direction of the far-field velocity vector; the far-field velocity vector is $\mathbf{U}_\infty = U\mathbf{i}$.

The aerodynamic problem is to determine the velocity, $\mathbf{u} = (u, v)$, and pressure p for the flow illustrated in Fig. 3.1 as functions of the coordinates (x, y). Since the flow of interest is a two-dimensional incompressible flow, the continuity equation is

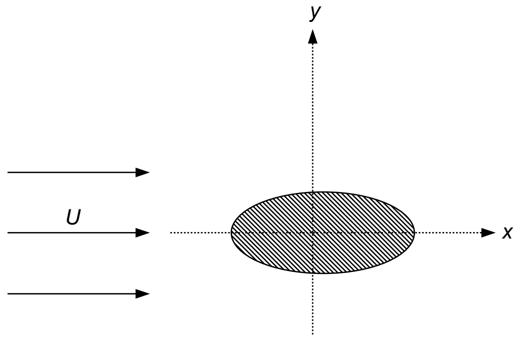

FIGURE 3.1

Arbitrary body in a uniform stream.

(as discussed in Section 2.4.2)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1}$$

We next assume that the fluid is inviscid (Euler's "perfect fluid"). We also assume steady flow and neglect body forces (e.g., gravity). In this case, the Euler equations (described in Section 2.6.1) reduce to the following form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.3}$$

Equations (3.1), (3.2), and (3.3) are in three unknowns. The unknowns are u, v, and p. Thus, they represent a complete system of equations that describes the flows of interest. We can simplify this system of equations as follows.

Equation (3.2) can be rearranged to read

$$\frac{\partial u^2/2}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p/\rho}{\partial x}$$

Adding and subtracting a term (the second and third term in the following), we get

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2} + \frac{v^2}{2}\right) - \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) + v\frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}\frac{p}{\rho}$$

Or, after rearranging terms,

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{p}{\rho}\right)=v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{3.4}$$

Similarly, we can operate on and rearrange Eq. (3.3) to get

$$\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{p}{\rho}\right)=-u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{3.5}$$

The equation for the z component of the vorticity (the only finite component of the vorticity vector in two-dimensional flow) is (as described in Section 2.7.4)

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\zeta$$

For irrotational flows, the condition for irrotationality is $\zeta=0$; hence

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \tag{3.6}$$

Applying Eq. (3.6) to Eqs. (3.4) and (3.5) and integrating the latter pair, we get

$$\frac{u^2}{2}+\frac{v^2}{2}+\frac{p}{\rho}=H \tag{3.7}$$

where H is a constant. This is Bernoulli's equation. If we evaluate H far upstream where the flow is uniform at speed U and at pressure p_∞, we get

$$\frac{u^2}{2}+\frac{v^2}{2}+\frac{p}{\rho}=\frac{U^2}{2}+\frac{p_\infty}{\rho}$$

Rearranging this equation,

$$C_p=\frac{p-p_\infty}{\frac{1}{2}\rho U^2}=1-\left(\frac{u^2+v^2}{U^2}\right) \tag{3.8}$$

where C_p is a dimensionless pressure coefficient.

Equation (3.6), the irrotationality condition, allows us to define a scalar function ϕ as follows:

$$u=\frac{\partial \phi}{\partial x} \quad \text{and} \quad v=\frac{\partial \phi}{\partial y} \tag{3.9}$$

Substituting this into Eq. (3.6), we get

$$\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial x\partial y}=0$$

which illustrates the fact that we can write the velocity vector $\mathbf{u}=(u,v)$ as the gradient of a scalar function ϕ. That is, if $\mathbf{u}=\nabla\phi$, then $\nabla\times\mathbf{u}=0$ (in two dimensions, these are Eqs. (3.9) and (3.6), respectively).

Substituting Eq. (3.9) into the continuity equation, Eq. (3.1), we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{3.10}$$

This is the Laplace equation for ϕ; hence ϕ is a *harmonic potential* function that we call the *velocity potential*. (Further discussion of this function and its relationship to the streamlines is given in Section 3.1.1.)

Equation (3.1), the continuity equation, also allows us to define the stream function ψ as follows:

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{3.11}$$

Substituting this into Eq. (3.6), the irrotationality condition, we get

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{3.12}$$

This indicates that the *stream function* ψ (already described in Section 2.5) is also a harmonic potential. In fact, since

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\quad\text{and}\quad\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

lines of the constant value of ϕ are perpendicular (or orthogonal) to lines of the constant value of ψ. The former are *equipotential lines*, and the latter are *streamlines*. These relationships are the *Cauchy-Riemann equations*, and they play an important role in the mathematical justification for the practical procedures we apply herein to solve potential-flow problems.

Equation (3.10) or (3.12) indicates that any solution to the Laplace equation is a possible potential flow. Since the Laplace equation is a linear partial differential equation, if, for example, two solutions are summed (*linearly superimposed*), a third solution is obtained. In other words, we can construct solutions by the superposition of elementary solutions to solve particular potential-flow problems. Before we examine elementary solutions to the Laplace equation, let us summarize the procedure to solve potential-flow problems in general.

Note that any streamline can be a boundary of a body on which we wish to predict the pressure distribution. Streamlines are lines tangent to the velocity vectors; therefore, in the direction normal (or perpendicular) to a streamline, the normal component of the velocity is zero:

$$\mathbf{n} \cdot \nabla\phi = \frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad S_B \tag{3.13}$$

where S_B is the streamline that represents the body surface or the wall of interest in an investigation, and **n** is the unit vector perpendicular to the streamline. An example of S_B is the boundary surrounding the cross-hatched region (or body) in Fig. 3.1.

To solve boundary-value problems associated with solving Eq. (3.10), the Laplace equation for ϕ, we need boundary conditions on all components of the boundary that completely surrounds the flow field of interest. In addition to a condition on S_B (i.e., Eq. 3.13), we need to specify the far-field condition, which for the problem illustrated in Fig. 3.1, is

$$\phi_\infty = Ux \tag{3.14}$$

which is the potential for a uniform stream in the x direction. This completes the specification of the boundary-value problem illustrated in Fig. 3.1. How we construct solutions to this class of problems is discussed in this chapter and in the two subsequent chapters.

The steps to solve aerodynamics problems can be summarized as follows. The first step is to construct the flow field by either solving for ϕ or for ψ. Once one of these fields (or functions) is known, then the velocity field is computed by solving Eq. (3.9) or (3.11) given ϕ or ψ, respectively. Knowing the velocity field, we compute the pressure field by applying Eq. (3.8). Finally, we integrate the pressure distribution on S_B appropriately to determine the force and moment acting on S_B due to pressure. Before we begin to deal with solving particular aerodynamic problems in Section 3.2, in the next three subsections we examine in more detail the velocity potential.

3.1.1 The Velocity Potential

The stream function (see Section 2.5) at a point has been defined as the quantity of fluid moving across some convenient imaginary line in the flow pattern, and lines of constant stream function (amount of flow or flux) may be plotted to give a picture of the flow pattern (see Section 2.5). Another mathematical definition, giving a different pattern of curves, can be obtained for the same flow system. In this case, an expression giving the amount of flow along the convenient imaginary line is found.

In a general two-dimensional fluid flow, consider any (imaginary) line OP joining the origin of a pair of axes to the point P(x, y). Again, the axes and this line do not impede the flow and are used only to form a reference datum. At a point Q on the line, let the local velocity q meet the line OP in β (Fig. 3.2(a)). Then the component of velocity parallel to δs is $q\cos\beta$. The amount of fluid flowing along δs is $q\cos\beta\delta s$. The total amount of fluid flowing along the line toward P is the sum of all such amounts

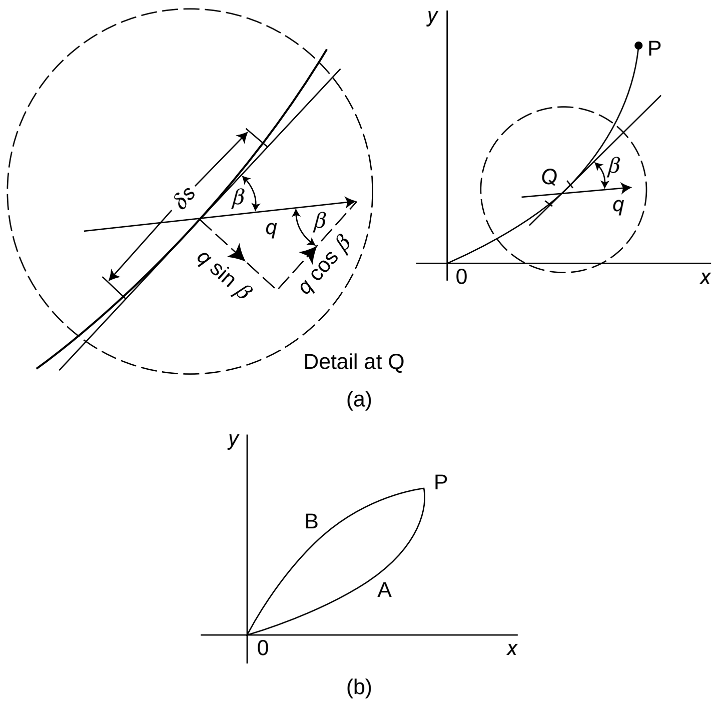

FIGURE 3.2

(a) Relationship between velocity vector **q** and line OP at its segment δs. (b) Two lines connecting points OP.

and is given mathematically as the integral $\int q \cos \beta \mathrm{d}s$. This function is called the velocity potential of P with respect to O and is denoted ϕ.

Now OQP can be any line between O and P and a necessary condition for $\int q \cos \beta \mathrm{d}s$ to be the velocity potential ϕ is that the value of ϕ is unique for the point P, irrespective of the path of integration. Then the velocity potential is

$$\phi = \int_{\mathrm{OP}} q \cos \beta \mathrm{d}s \tag{3.15}$$

If this were not the case, and integrating the tangential flow component from O to P via A (Fig. 3.2(b)) did not produce the same magnitude of ϕ as integrating from O to P via some other path such as B, there would be some flow components circulating in the circuit OAPBO. This in turn would imply that the fluid within the circuit possesses vorticity. The existence of a velocity potential must therefore imply zero vorticity in the flow or, in other words, a flow without circulation (see Section 2.7.7)—that is, an *irrotational* flow. Such flows are also called *potential* flows.

The sign convention for the velocity potential selected in this text is based on the following notion. The tangential flow along a curve is the product of the local velocity component and the elementary length of the curve. Now, if the velocity component is in the direction of integration, it is considered a *positive* increment of the velocity potential.

3.1.2 The Equipotential

Consider a point P having a velocity potential ϕ (ϕ is the integral of the flow component along OP), and let another point P_1 close to P have the same velocity potential ϕ. This means that the integral of flow along OP_1 equals the integral of flow along OP (Fig. 3.3). But, by definition, OPP_1 is another path of integration from O to P_1. Therefore,

$$\phi = \int_{OP} q\cos\beta\,\mathrm{d}s = \int_{OP_1} q\cos\beta\,\mathrm{d}s = \int_{OPP_1} q\cos\beta\,\mathrm{d}s$$

where q is the magnitude of $\mathbf{q}$; however, since the integral along OP equals that along OP_1, there can be no flow along the remaining portions of the path of the third integral, that is, along PP_1. Similarly for other points such as P_2 and P_3 having the same velocity potential, there can be no flow along the line joining P_1 to P_2.

The line joining P, P_1, P_2, and P_3 joins points having the same velocity potential and is called an *equipotential* or a line of constant velocity potential (i.e. a line of constant ϕ). The significant characteristic of an equipotential is that there is no flow along such a line.

The flow in the region of points P and P_1 should be investigated more closely. From the previous discussion, there can be no flow along the line PP_1, but there is fluid flowing in this region so it must be flowing in such a way that there is no component of velocity in the direction PP_1. Thus the flow can only be at right angles to PP_1, which means that the flow in the region PP_1 must be normal to PP_1. Now, the streamline in this region, the line to which the flow is tangential, must also be at right angles to PP_1, which is itself the local equipotential.

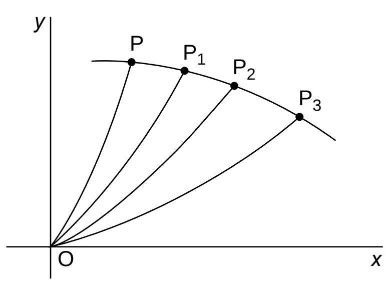

FIGURE 3.3

Equipotential perpendicular to the flow direction.

This relation applies at all points in a homogeneous continuous fluid and can be stated thus: Streamlines and equipotentials meet orthogonally (i.e., always at right angles. It follows from this statement that, for a given streamline pattern, there is a unique equipotential pattern for which the equipotentials are everywhere normal to the streamlines.

3.1.3 Velocity Components in Terms of ϕ

In Cartesian coordinates: Let a point $P(x,y)$ be on an equipotential ϕ and a neighboring point $Q(x+\delta x, y+\delta y)$ be on the equipotential $\phi+\delta\phi$ (Fig. 3.4). Then, by definition, the increase in velocity potential from P to Q is the line integral of the tangential velocity component along any path between P and Q. Taking PRQ as the most convenient path, where the local velocity components are u and v,

$$\delta\phi = u\delta x + v\delta y$$

but

$$\delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y$$

Thus, equating terms,

$$u = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v = \frac{\partial\phi}{\partial y} \tag{3.16}$$

In polar coordinates: Let a point $P(r,\theta)$ be on an equipotential ϕ and a neighboring point $Q(r+\delta r, \theta+\delta\theta)$ be on an equipotential $\phi+\delta\phi$ (Fig. 3.5). By definition, the increase $\delta\phi$ is the line integral of the *tangential* component of velocity along any path. For convenience, choose PRQ where point R is $(r+\delta r, \theta)$. Then, integrating

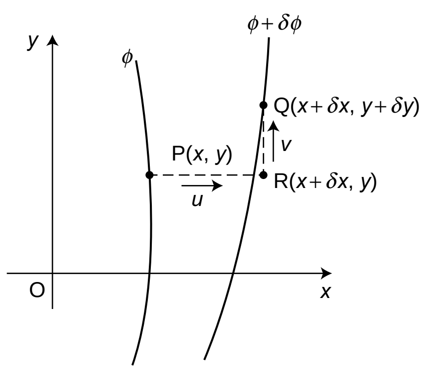

FIGURE 3.4

Relationship between equipotentials and velocity components in Cartesian coordinates.

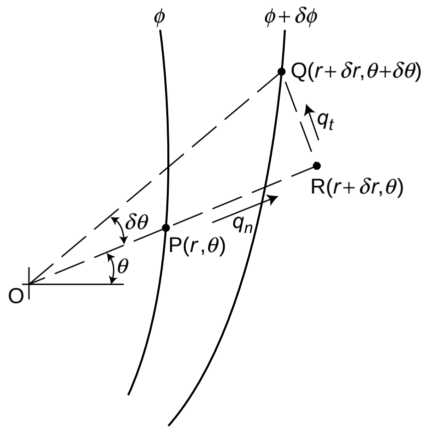

FIGURE 3.5

Relationship between equipotentials and velocity components in polar coordinates.

along PR and RQ where the velocities are q_n and q_t, respectively, and are both in the direction of integration,

$$\delta\phi = q_n \delta r + q_t (r + \delta r) \delta\theta \approx q_n \delta r + q_t r \delta\theta$$

where the right-hand side is a first-order approximation; it neglects terms that are products of small quantities. But, since ϕ is a function of two independent variables,

$$\delta\phi = \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta$$

and

$$q_n = \frac{\partial\phi}{\partial r} \quad \text{and} \quad q_t = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{3.17}$$

Again, in general, the velocity q in any direction s is found by differentiating the velocity potential ϕ partially with respect to the direction s of q:

$$q = \frac{\partial\phi}{\partial s}$$

3.2 STANDARD FLOWS IN TERMS OF ψ AND ϕ

There are four basic two-dimensional flow fields, from combinations of which all other steady-flow conditions may be modeled. The four flows are *uniform parallel* *stream*, *source (sink)*, *doublet*, and *point vortex*. Examples of how these are applied to construct useful flows are presented in this section.

3.2.1 Uniform Flow

In this subsection we examine a number of ways to represent a uniform flow. We first examine the velocity potential and stream function that represents a uniform stream in the x direction (this is the uniform stream upstream of the body illustrated in Fig. 3.1). Then we examine the same for a uniform stream in an arbitrary direction.

Flow of Constant Velocity Parallel to the x-Axis from Left to Right

The potentials for a uniform flow in the x direction are

$$\phi = Ux, \quad \psi = Uy \tag{3.18}$$

Thus, applying Eq. (3.9) or (3.11), we get

$$\mathbf{u} = (u, v) = (U, 0) \tag{3.19}$$

Note that, if ψ is constant, streamlines are parallel to the x-axis, as illustrated in Figure 3.6. Similarly, if ϕ is constant, equipotential lines are parallel to the y-axis. The pattern of the potentials for a uniform flow in the x direction is shown in the figure. The MATLAB script used to generate this figure is given in Table 3.1.

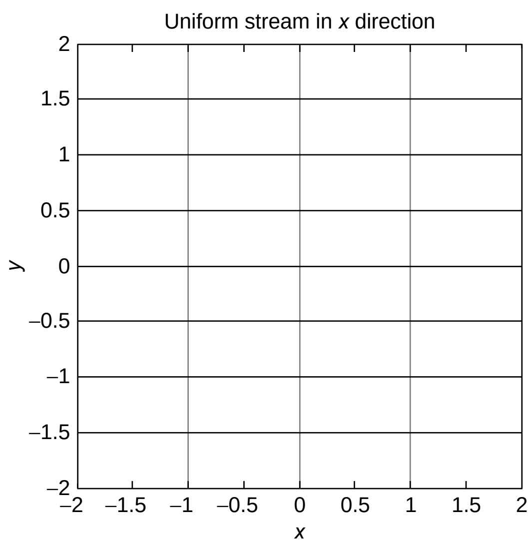

FIGURE 3.6

Vertical lines are equipotentials; horizontal lines are streamlines. The flow in this case is from left to right at uniform speed U.

Table 3.1 MATLAB Script Applied to Plot Figure 3.6

```
U = 1;     % Uniform stream
% GRID:
x = -2:.02:2;
y = -2:.02:2;
for m = 1:length(x)
    for n = 1:length(y)
        xx(m,n) = x(m); yy(m,n) = y(n);
    % Velocity potential function:
   phi_UniFlow(m,n) = U * x(m);
    % Stream function:
   psi_UniFlow(m,n) = U * y(n);
    end
end
% Plots
% Uniform stream in x direction:
contour(xx,yy,psi_UniFlow,[-2:.5:2],'k'),hold on
contour(xx,yy,phi_UniFlow,[-10:1:5],'r')
axis image,hold off
title('Uniform stream in x direction')
xlabel('x'),ylabel('y')
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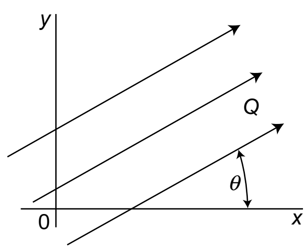


FIGURE 3.7

Uniform stream at an arbitrary angle θ from the x-axis.

Flow of Constant Velocity in Any Direction

Consider the flow streaming past the x- and y-axes at some velocity $\mathbf{Q}$, making an angle θ with the x-axis, as shown in Fig. 3.7. Velocity $\mathbf{Q}$ can be resolved into two components U and V parallel to the x- and y-axes, respectively, where $\mathbf{Q} = U\mathbf{i} + V\mathbf{j}$, $Q^2 = U^2 + V^2$, and $\tan\theta = V/U$. The potentials for this flow field are

$$\phi = Ux + Vy \quad \text{and} \quad \psi = -Vx + Uy \tag{3.20}$$

where U and V are constants. Thus the lines of constant ψ are streamlines defined by the straight lines

$$-Vx + Uy = \text{constant}$$

Assigning a different value of ψ for every streamline plotted leads to a set of parallel lines in the direction of $\mathbf{Q}$, as shown in Fig. 3.7.

Example 3.1

Interpret the flow given by the stream function (units: $\mathrm{m^2\,s^{-1}}$)

$$\psi = 6x + 12y$$

This represents a uniform stream with velocity vector $\mathbf{U} = (U, V)$. The constant component of the velocity in the x direction is $u = \partial\psi/\partial y = +12\ \mathrm{m\,s^{-1}}$. The constant component of the velocity in the y direction is $v = -\partial\psi/\partial x = -6\ \mathrm{m\,s^{-1}}$. Therefore, the flow equation represents uniform flow inclined to the x-axis by angle θ where $\tan\theta = -6/12$ (i.e., inclined downward).

The speed of flow is given by

$$Q = \sqrt{6^2 + 12^2} = \sqrt{180}\ \ \mathrm{m\,s^{-1}}$$

3.2.2 Two-Dimensional Flow from a Source (or towards a Sink)

Let us find cylindrically symmetric solutions to the Laplace equation, $\phi = \Phi(r)$. In this case, the Laplace equation in polar coordinates is useful:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

For cylindrically symmetric solutions this reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0$$

Integrating twice, we get

$$\Phi = C_1 \ln r + C_2$$

where C_1 and C_2 are arbitrary constants. For example, if we set $C_1 = 1$ and $C_2 = 0$, then

$$\Phi_o = \ln r$$

which is often called the *fundamental solution* to the two-dimensional Laplace equation. This formula satisfies the Laplace equation everywhere except at $r = 0$, where it is logarithmically singular. The source flow field it induces is examined next.

A source (sink) of strength $m(-m)$ is a point at which fluid is appearing (or disappearing) at a uniform rate of $m(-m)$ m^2 s^{-1}. Let us assume this source is located at $r = 0$; hence, the constant $C_1 = m/(2\pi)$ and the constant $C_2 = 0$. In this case, the velocity potential is

$$\phi = \frac{m}{2\pi} \ln r$$

We can interpret the flow field for this potential as follows. Consider the analogy of a small hole in a large flat plate through which fluid is welling (the source). If there is no obstruction and the plate is perfectly flat and level, the fluid puddle will get larger and larger, all the while remaining circular in shape. The path that any particle of fluid will trace out as it emerges from the hole and travels outward is a purely radial one; it cannot go sideways because its fellow particles are also moving outward.

Also, its velocity must lessen as it goes outwards. Fluid issues from the hole at a rate of m m^2 s^{-1}. The velocity of flow over a circular boundary of 1-m radius is $m/2\pi$ m s^{-1}. Over a circular boundary of 2-m radius it is $m/(2\pi \times 2)$ (i.e., half as much, and over a circle of diameter $2r$ the velocity is $m/2\pi r$ m s^{-1}. Therefore, the velocity of flow is inversely proportional to the distance of the particle from the source.

All the previous applies to a sink except that fluid is being drained away through the hole and is moving toward the sink radially, increasing in speed as the sink is approached. Thus the particles all move radially, and the streamlines must be radial lines with their origin at the source (or sink).

The stream function that describes the source at the origin of the coordinate system is the stream function

$$\psi = \frac{m}{2\pi} \tan^{-1} \frac{y}{x} \quad \text{or} \quad \psi = \frac{m}{2\pi}\theta \tag{3.21}$$

where $\theta = \tan^{-1}(y/x)$ (see Fig. 3.8). Again, we place the source (for convenience) at the origin of a system of axes, to which the point P is at (x, y) or (r, θ). The velocity potential for this flow is

$$\phi = \frac{m}{4\pi} \ln\left(x^2 + y^2\right) \quad \text{or} \quad \phi = \frac{m}{2\pi} \ln r \tag{3.22}$$

where $x = r\cos\theta$, $y = r\sin\theta$ and, hence, $r^2 = x^2 + y^2$ were used.

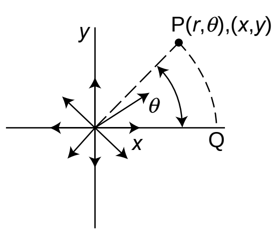


FIGURE 3.8

Source flow and polar coordinates.

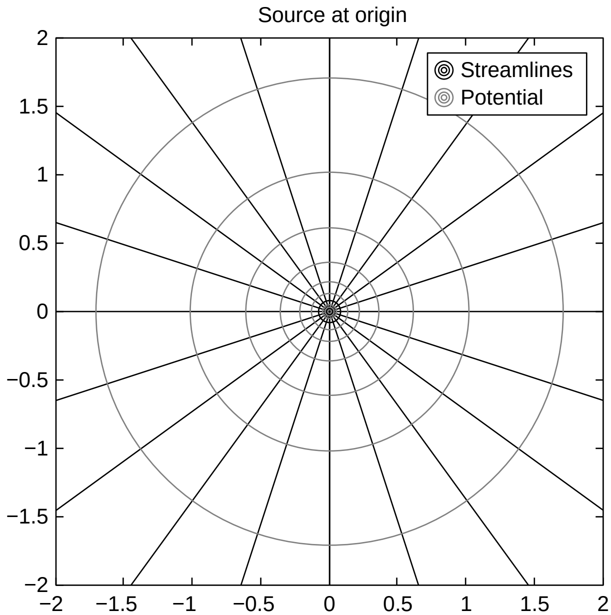


FIGURE 3.9

The potentials of a source: the radial lines emanating from the origin are the streamlines; the concentric circles surrounding the origin are lines of constant velocity potential.

Fig. 3.9 is an illustration of the pattern of the potentials for the source. The MATLAB file used to generate this figure is given in Table 3.2.

The components of the velocity vector due to the source at $(x, y) = (0, 0)$ are

$$u = \frac{\partial \phi}{\partial x} = \frac{m}{2\pi} \frac{x}{x^2 + y^2}, \quad v = \frac{\partial \phi}{\partial y} = \frac{m}{2\pi} \frac{y}{x^2 + y^2} \tag{3.23}$$

or

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r}, \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0 \tag{3.24}$$

where u_r and u_θ are the radial and circular components of the velocity in polar coordinates.

3.2.3 Doublet Located at $(x, y) = (0, 0)$

The doublet given in this section points in the $-x$ direction. The potentials that describe this flow field are

$$\phi = \mu \frac{x}{x^2 + y^2} \quad \text{and} \quad \psi = -\mu \frac{y}{x^2 + y^2} \tag{3.25}$$

Table 3.2 MATLAB Script to Plot Figure 3.9

```
sig = 1; % Source strength
% GRID:
x = -2:.02:2;
y = -2:.02:2;
for m = 1:length(x)
    for n = 1:length(y)
        xx(m,n) = x(m); yy(m,n) = y(n);
    % Velocity potential function:
   phi_Source(m,n) = (sig/4/pi) * log(x(m)^2+(y(n)+.01)^2);
    % Stream function:
   psi_Source(m,n) = (sig/2/pi) * atan2(y(n),x(m));
    end
end
% Plots
% Source at origin of coordinate systex:
contour(xx,yy,psi_Source,[-.5:.05:.5],'k'),hold on
contour(xx,yy,phi_Source,10,'r')
legend('streamlines' ,'potential')
title(' Source at origin')
axis image,hold off
```

The components of the corresponding velocity field are

$$u = \mu \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}, \quad v = -\mu \frac{2xy}{\left(x^2 + y^2\right)^2} \tag{3.26}$$

The potentials for this flow field are illustrated in Fig. 3.10. The MATLAB script used to generate this figure is given in Table 3.3.

3.2.4 Line (Point) Vortex

The flow induced by a line (point) vortex described next can be interpreted as the flow induced by a straight-line vortex that is infinitely long in the z direction. In the (x, y) plane, it is a point. The potentials for this solution of the Laplace equation, if the point vortex is located at $(x, y) = (0, 0)$, are

$$\phi = -\frac{\Gamma}{2\pi} \tan^{-1} \frac{y}{x}, \quad \psi = -\frac{\Gamma}{4\pi} \ln\left(x^2 + y^2\right) \tag{3.27}$$

The corresponding components of the velocity field are

$$u = \frac{\partial \phi}{\partial x} = -\frac{\Gamma}{2\pi} \frac{y}{x^2 + y^2}, \quad v = \frac{\partial \phi}{\partial y} = \frac{\Gamma}{2\pi} \frac{x}{x^2 + y^2} \tag{3.28}$$

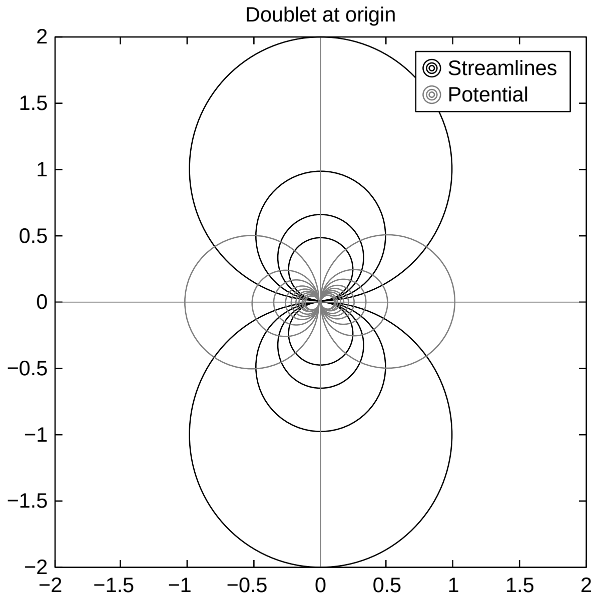


FIGURE 3.10

The potentials of an $-x$ directed doublet: streamlines and velocity potential lines are circles; flow along the x-axis is in the $-x$ direction.

Table 3.3 MATLAB Script Applied to Plot Figure 3.10

```
mu = 1;  % Doublet strength
% GRID:
x = -2:.02:2;
y = -2:.02:2;
for m = 1:length(x)
    for n = 1:length(y)
        xx(m,n) = x(m); yy(m,n) = y(n);
    % Velocity potential function:
   phi_Doublet(m,n) = mu * x(m)/(x(m)^2+(y(n)+.01)^2);
     % Stream function:
   psi_Doublet(m,n) = - mu * y(n)/(x(m)^2+(y(n)+.01)^2);
     end
end
% Plots
% Doublet at origin of coordinate system:\
figure(4)
contour(xx,yy,psi_Doublet,[-2:.5:2],'k'),hold on
contour(xx,yy,phi_Doublet,[-10:1:10],'r')
legend('streamlines' ,'potential')
title(' Doublet at origin')
axis image,hold off
```

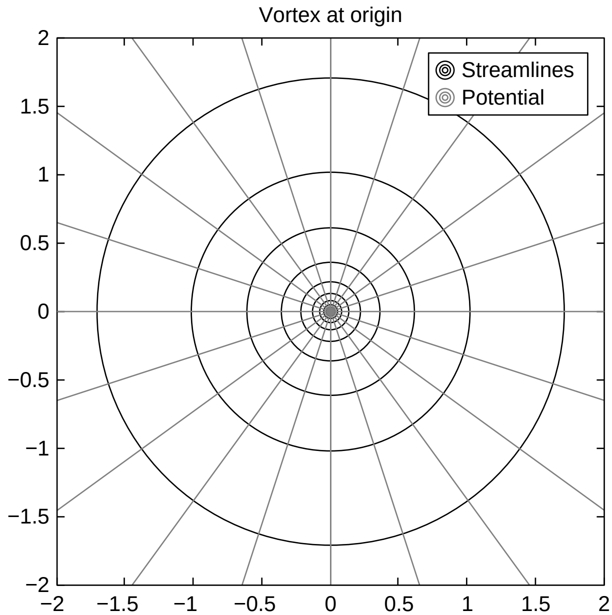


FIGURE 3.11

The potentials of a point vortex: streamlines are concentric circles; velocity potential lines are radial lines.

The potentials for this flow field are illustrated in Fig. 3.11. The MATLAB script used to generate this figure is given in Table 3.4.

Since the flow due to a line vortex gives streamlines that are concentric circles, $u_r = 0$ and u_θ is finite. In polar coordinates, the potentials are as follows:

$$\phi = \frac{\Gamma}{2\pi}\theta, \quad \psi = -\frac{\Gamma}{2\pi}\ln r \tag{3.29}$$

The corresponding components of the velocity field are

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \tag{3.30}$$

3.2.5 Solid Boundaries and Image Systems

The fact that the flow is always along a streamline and not through it has an important fundamental consequence. A streamline of an *inviscid* flow can be replaced by a solid boundary of the same shape without affecting the remainder of the flow pattern. If, as is often the case, a streamline forms a closed curve that separates the flow pattern into two separate streams, one inside and one outside, then a solid body can replace

Table 3.4 MATLAB Script Applied to Plot Figure 3.11

```
gam = 1; % Vortex strength
% GRID:
x = -2:.02:2;
y = -2:.02:2;
for m = 1:length(x)
    for n = 1:length(y)
        xx(m,n) = x(m); yy(m,n) = y(n);
     % Velocity potential function:
    phi_Vortex(m,n) = (gam/2/pi)*atan2(y(n)+.01,x(m));
     % Stream function:
    psi_Vortex(m,n) = -(gam/4/pi)*log(x(m)^2+(y(n)+.01)^2);
     end
end
% Plots
% Vortex at origin of coordinate system:
contour(xx,yy,psi_Vortex,10,'k'),hold on
contour(xx,yy,phi_Vortex,[-.5:.05:.5],'r')
legend('streamlines' ,'potential')
title(' Vortex at origin')
axis image
```

the closed curve and the flow made outside without altering its shape. An example of this is illustrated in Fig. 3.12(a); this is the Rankine oval described in detail in Section 3.2.7. To represent the flow in the region of a contour or body, it is only necessary to replace the contour by a similarly shaped streamline; in this case, the closed streamline is oval shaped because it divides the flow between a source, S_o, of strength m followed by a sink, S_i, of strength $-m$ along the x-axis in a uniform stream flowing in the direction of the x-axis. All of the source flow is consumed by the sink; hence, a closed dividing streamline is formed that can be used to model a body of the same shape in a uniform inviscid flow. The following sections contain examples of simple flows that provide continuous streamlines in the shapes of circles and airfoils, and these emerge as consequences of the combinations of elementary solutions selected.

When arbitrary contours and their adjacent flows have to be replaced by identical flows containing similarly shaped streamlines, image systems must be placed within the contour that reflect the external flow system in the solid streamline.

Figure 3.12(b) shows the simple case of a source m placed a short distance A from an infinite plane wall. The effect of the solid boundary on the flow from the source is exactly represented by considering the effect of the image source m placed a distance A′ reflected in the wall; the magnitude of the distance is the same as A. The source pair has a long straight streamline (i.e., the vertical axis of symmetry) that separates the flows from the two sources and that may be replaced by a solid boundary without

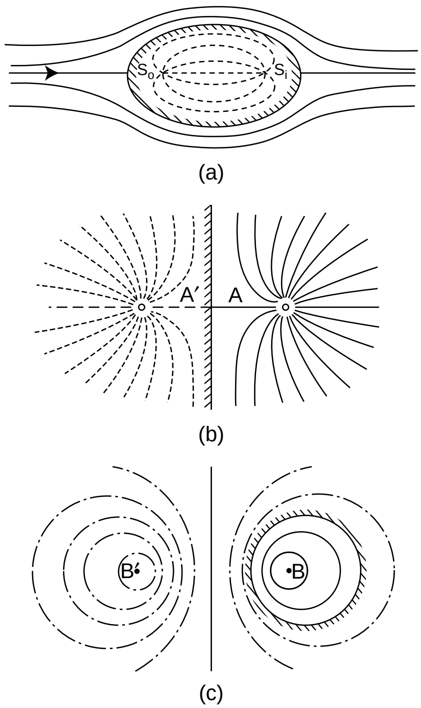


FIGURE 3.12

Solid boundaries constructed by image systems of singular solutions.

affecting the flow. Assume that the two sources are located on the x-axis and that the y-axis ($x = 0$) is the wall. The potential for this flow field is thus

$$\phi = \frac{m}{4\pi} \ln\left((x-A)^2 + y^2\right) + \frac{m}{4\pi} \ln\left((x+A)^2 + y^2\right)$$

where the second term on the right-hand side is the image source required to model the flow field of a source adjacent to the wall on the right side of the wall. The source is located at $(x, y) = (A, 0)$. Its image is located at $(x, y) = (-A, 0)$. Thus by symmetry the wall is located at $x = 0$. We can demonstrate that a wall is located at this position by solving for the velocity vector normal to the y-axis for all y as follows:

$$u = \frac{\partial \phi}{\partial x}|_{x=0} = \left[-\frac{m}{2\pi} \frac{x-A}{(x-A)^2 + y^2} - \frac{m}{2\pi} \frac{x+A}{(x+A)^2 + y^2} \right]_{x=0} = 0$$

Thus, since the normal component of the velocity is zero at $y = 0$, the y-axis is a streamline and hence a model of a wall in inviscid flow. Examination of further details of this flow is left as an exercise. It is useful for the student to compute the velocity along the wall $v = \partial\phi/\partial y$, the pressure distribution by applying the Bernoulli equations and integrating the pressure coefficient along the wall to show that the force of the source acting on the wall is a suction towards the source.

Figure 3.12(c) shows the flow in the cross-section of a vortex lying parallel to the axis of a circular duct. The circular duct wall can be replaced by the corresponding streamline in the vortex-pair system given by the original vortex B and its image B′. It can easily be shown that B′ is a distance r^2/s from the center of the duct on the diameter produced passing through B, where r is the radius of the duct and s is the distance of the vortex axis from the center of the duct. This flow field is left as an exercise that follows along the same lines as the problem of a source near a wall described previously.

To help the student with the exercises suggested as homework in this section, we will examine similar problems in detail in the rest of the sections in this chapter. For example, in Section 3.2.7 we examine the details of the flow illustrated in Fig. 3.12(a). In the next section we examine a simpler flow, viz., the Rankine leading edge.

3.2.6 Rankine Leading Edge

The Rankine nose (or Rankine leading edge) is an example of a flow around the leading edge of a symmetric aerodynamic body (symmetric about the x-axis). This flow is constructed by adding to a uniform stream a source at the origin of the coordinates. The potentials for this flow are

$$\phi = U_\infty x + \frac{m}{4\pi}\ln\left(x^2+y^2\right), \quad \psi = U_\infty y + \frac{m}{2\pi}\tan^{-1}\frac{y}{x} \tag{3.31}$$

Thus the velocity field for this flow is

$$u = U_\infty + \frac{m}{2\pi}\frac{x}{x^2+y^2}, \quad v = \frac{m}{2\pi}\frac{y}{x^2+y^2} \tag{3.32}$$

Applying the Bernoulli theorem, we get the pressure field

$$p = p_\infty + \rho\left(\frac{U_\infty^2}{2} - \frac{u^2+v^2}{2}\right) \tag{3.33}$$

There are a number of ways to solve the problem of determining the geometry of the Rankine nose. We will examine the problem in polar coordinates. The potential, the stream function, and the velocity components—Eqs. (3.28) and (3.29)—in polar coordinates are

$$\phi = U_\infty r\cos\theta + \frac{m}{2\pi}\ln r, \quad \psi = U_\infty r\sin\theta + \frac{m}{2\pi}\theta \tag{3.34}$$

$$u_r = \frac{\partial\phi}{\partial r} = U_\infty\cos\theta + \frac{m}{2\pi r}, \quad u_\theta = -U_\infty\sin\theta \tag{3.35}$$

There is one stagnation point in this flow field. It is located on the x-axis at $\theta = \pi$. Substituting this into Eq. (3.32) and setting $u_r = 0$, we get

$$r_{stg} = \frac{m}{2\pi U_\infty}$$

This is the distance upstream of the origin of the coordinate system where the leading edge begins. The thickness of the Rankine nose at $x = 0$ or at $\theta = \pi/2$ is determined as follows:

$$\frac{m}{4} = -\int_0^{r_t} u_\theta\, dr = U_\infty r_t$$

Thus

$$r_t = \frac{m}{4U_\infty}$$

With two points along the dividing streamline that divides the upstream flow from the source flow, we can evaluate and check the constant ψ_d associated with the Rankine nose and so find the formula for its geometry. Substituting either $(r,\theta) = (r_{stg},\pi)$ or $(r,\theta) = (r_t,\pi/2)$, we find that $\psi_d = m/2$. Thus the equation describing the shape of the Rankine nose is

$$\psi_d = \frac{m}{2} = U_\infty r\sin\theta + \frac{m}{2\pi}\theta$$

Rearranging this equation, we get

$$r = \frac{m}{2\pi U_\infty}\left(\frac{\pi-\theta}{\sin\theta}\right) \tag{3.36}$$

This is an exact formula for the shape of the Rankine nose. If you want to plot it in (x,y) coordinates, compute r for a range of θ. Next compute the corresponding Cartesian coordinates $(x_d,y_d) = (r\cos\theta, r\sin\theta)$. This was done and the result is plotted in Fig. 3.13.

We can write the equation for the pressure in polar coordinates as follows:

$$C_p = \frac{p-p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{1}{U_\infty^2}\left(u_r^2 + u_\theta^2\right)$$

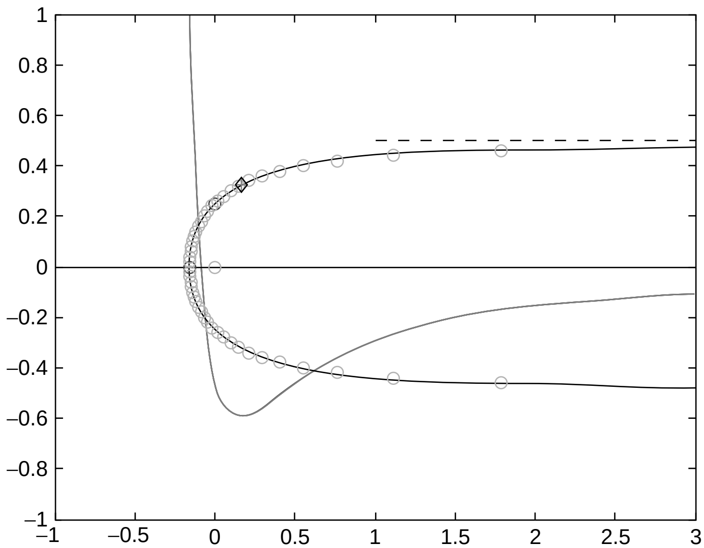


FIGURE 3.13

Rankine nose complete solution graphically displayed.

Substituting the velocity components, we get

$$C_p = -\left(\frac{m}{\pi U_\infty}\frac{\cos\theta}{r} + \frac{m^2}{4\pi^2 U_\infty^2}\frac{1}{r^2}\right)$$

From Eq. (3.33) on the Rankine nose, we find that

$$\frac{m}{U_\infty} = \frac{2\pi r \sin\theta}{\pi - \theta}$$

Substituting this into the equation for C_p, we get the pressure distribution on the Rankine nose:

$$-C_{pd} = \frac{2\sin\theta\cos\theta}{\pi - \theta} + \frac{\sin^2\theta}{(\pi - \theta)^2}$$

With this equation, we can determine the location of the minimum pressure and hence the minimum pressure exactly. To do this we take the derivative of this formula with respect to θ and set it equal to zero. The result of this operation (which is left as an exercise) leads to

$$\theta_m = \tan^{-1}\left(\frac{\pi - \theta_m}{\pi - \theta_m - 1}\right) \tag{3.37}$$

which may be solved by successive approximation with the initial guess of $\theta_m = 0$. The result is 62.9569 degrees and the minimum pressure is $C_{pmin} = -0.5866$. The numerical results compare with these results to within the significant figures shown. Table 3.5 presents the MATLAB solution to the entire problem.

The Rankine nose is a geometrically similar object because increasing or decreasing m and/or U_∞ does not change the location and the value of the minimum pressure coefficient. This is because changing the ratio m/U_∞ does not change the variation in curvature of the surface at any location on the body. The length of the nose and the thickness change proportionally when m and/or U_∞ are changed. Thus, we can say that all Rankine noses are geometrically similar.

3.2.7 Rankine Oval

A source of strength m upstream of a sink of strength $-m$ in a uniform stream U is used to construct the Rankine oval. The stream function due to this combination is

$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2+y^2-c^2} - Uy \tag{3.38}$$

The uniform stream in this case is coming from the $-x$ direction. Also, the first term represents a source and sink combination set with the source to the right of the sink. For the source to be upstream of the sink, the uniform stream must be from right to left (i.e., flowing in the negative x direction). If the source is placed downstream of the sink, an entirely different stream pattern is obtained.

The velocity potential at any point in the flow due to this combination is given by

$$\phi = \frac{m}{2\pi}\ln\frac{r_1}{r_2} - Ur\sin\theta \tag{3.39}$$

or

$$\phi = \frac{m}{4\pi}\ln\frac{x^2+y^2+c^2-2xc}{x^2+y^2+c^2+2xc} - Ux \tag{3.40}$$

The streamline $\psi = 0$ gives a closed oval curve (not an ellipse) that is symmetrical about the x- and y-axes. Flow of stream function ψ greater than $\psi = 0$ shows the flow around such an oval set at zero incidence in a uniform stream. Streamlines can be obtained by plotting or by superposition of the separate standard flows (Fig. 3.14). The streamline $\psi = 0$ again separates the flow into two distinct regions. The first is wholly contained within the closed oval and consists of the flow out of the source and into the sink. The second is the approaching uniform stream that flows around the oval curve and returns to its uniformity. Again replacing $\psi = 0$ by a solid boundary, or indeed a solid body whose shape is given by $\psi = 0$, does not influence the flow pattern in any way.

Thus the stream function ψ of Eq. (3.35) can be used to represent the flow around a long cylinder of oval section set with its major axis parallel to a steady stream. To find the stream function representing a flow around such an oval cylinder, it must be

Table 3.5 MATLAB Script of Complete Solution of Rankine Nose Problem

```
%
% The Rankine nose (or leading edge) Feb. 2, 2010
% Source in a uniform stream: A 2D potential flow
%
   clear;clc
    disp(' Example: Rankine nose')
    m = 1; % Source strength for source at (x,y) = (0,0).
    V = 1; % Free stream velocity in the x-direction
    disp('    V     m ')
    disp([V m])
    disp(' Velocity potential:')
    disp('    phi = V*x + (m/4/pi)*log(x^2+y^2)')
    disp('Stream function:')
    disp(' psi = V*y + (m/2/pi)*atan2(y,x)')
    disp('The (x,y) components of velocity (u,v):')
    disp('     u = V + m/2/pi * xc/(x^2+y^2)')
    disp('     v = m/2/pi * y/(x^2+y^2)')
    %
    xstg = - m/2/pi/V; ystg = 0; % Location of stagnation point.
    %
    N = 1000;
    xinf = 3;
    xd = xstg:xinf/N:xinf;
    for n = 1:length(xd)
        if n==1
           yd(1) = 0;
        else
        yd(n) = m/2/V;
        for it = 1:2000
           yd(n) = (m/2/V)*( 1 - 1/pi*atan2(yd(n),xd(n)) );
        end
        end
    end
    xL(1) = xd(end); yL = -yd(end);
    for nn = 2:length(xd)-1
    xL(nn) = xd(end-nn); yL(nn) = -yd(end-nn);end
    plot([xd xL],[yd yL],'k',[-1 3],[0 0],'k'),axis([-1 3 -1 1])
     u = V + m/2/pi * xd./(xd.^2+yd.^2);
     v = m/2/pi * yd./(xd.^2+yd.^2);
     Cp = 1 - (u.^2+v.^2)/V^2;
     hold on
     plot(xd,Cp),axis([-1 3 -1 1])
    plot(0,m/V/4,'o')
    plot(xstg,ystg,'o')
    plot([1 3],[m/2/V m/2/V],'--k')
```

Table 3.5 (*Continued*)

```
[Cpmin ixd] = min(Cp);
xmin = xd(ixd);
ymin = yd(ixd);
plot(xmin,ymin,'+r')
Cpmin
% Computation of normal and tangential velocity on (xd,yd):
phi = V*xd + (m/4/pi).*log(xd.^2+yd.^2);
dx = diff(xd); dy = diff(yd); ds = sqrt(dx.^2 + dy.^2);
dph = diff(phi); ut = dph./ds; xm = xd(1:end-1) + dx/2;
psi = V*yd + (m/2/pi).*atan2(yd,xd);
plot(xm,1-ut.^2/V^2,'r')
%
% Check on shape equation
%
th = 0:pi/25:2*pi;
r = (m/2/pi/V)*(pi - th)./sin(th);
xb = r.*cos(th);
yb = r.*sin(th);
plot(xb,yb,'om')
%
% Exact location of minimum pressure
thm = 0;
for nit = 1:1000
    thm = atan2(pi-thm,pi-thm-1);
end
thdegrees = thm*180/pi
   rm = (m/2/pi/V)*(pi - thm)/sin(thm)
      xm = rm*cos(thm);
         ym = rm*sin(thm);
plot(xm,ym,'dk')
um = V + m/2/pi * xmin/(xmin^2+ymin^2);
    vm = m/2/pi * ymin/(xmin^2+ymin^2);
       Cpm = 1 - (um^2+vm^2)/V^2
```

possible to obtain m and c (the strengths of the source and sink and distance apart) in terms of the size of the body and the speed of the incident stream.

Suppose there is an oval of breadth $2b_0$ and thickness $2t_0$ set in a uniform flow of U. The problem is to find m and c in the stream function, Eq. (3.35), which will then represent the flow around the oval.

1. The oval must conform to Eq. (3.35):

$$\psi = 0 = \frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2+y^2-c^2} - Uy$$

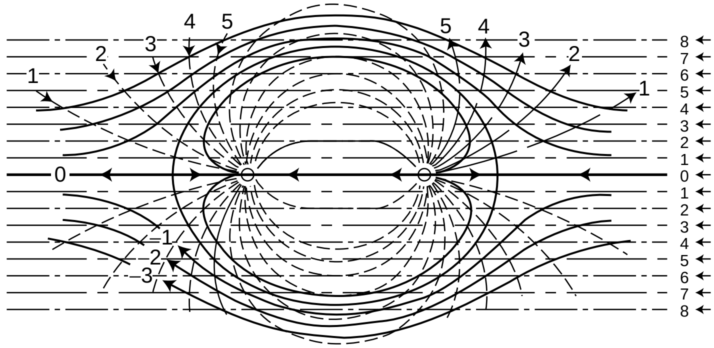


FIGURE 3.14

Rankine oval.

2. On streamline $\psi = 0$, maximum thickness t_0 occurs at $x = 0$, $y = t_0$. Therefore, substituting in the above equation, the oval must conform to Eq. (3.35):

$$0 = \frac{m}{2\pi} \tan^{-1} \frac{2ct_0}{t_0^2 - c^2} - Ut_0$$

and rearranging

$$\tan \frac{2\pi\, Ut + 0}{m} = \frac{2ct_0}{t_0^2 - c^2} \tag{3.41}$$

3. A stagnation point (a point where the local velocity is zero) is situated at the “nose” of the oval (i.e., at the point $y = 0$, $x = b_0$):

$$u = 0 = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{m}{2\pi} \tan^{-1} \frac{2cy}{x^2 + y^2 - c^2} - Uy\right)$$

$$\frac{\partial \psi}{\partial y} = \frac{m}{2\pi} \frac{1}{1 + \left(\frac{2cy}{x^2+y^2-c^2}\right)^2} \frac{\left(x^2 + y^2 - c^2\right) 2c - 2y\, 2cy}{\left(x^2 + y^2 - c^2\right)^2} - U$$

and putting $y = 0$ and $x = b_0$ with $\partial\psi/\partial y = 0$:

$$0 = \frac{m}{2\pi} \frac{\left(b_0^2 - c^2\right) 2c}{\left(b_0^2 - c^2\right)^2} - U = \frac{m}{2\pi} \frac{2c}{b_0^2 - c^2} - U$$

Therefore,

$$m = \pi U \frac{b_0^2 - c^2}{c} \tag{3.42}$$

The simultaneous solution of Eqs. (3.38) and (3.39) will furnish values of m and c to satisfy any given set of conditions. Alternatively (a), (b), and (c) can be used to find the thickness and length of the oval formed by the streamline $\psi = 0$.

3.2.8 Circular Cylinder with Circulation in a Cross Flow

This section is on the detailed analysis of the potential flow around a circular cylinder in a cross flow. We construct the potential, solve for the velocity and pressure distributions around the cylinder, and integrate the latter to determine the lift coefficient.

The potential for the flow around a cylinder in a cross flow with circulation is the superposition of a uniform stream in the x direction, a doublet located at the origin pointing in the $-x$ direction, and a vortex located at the origin. Thus, the potentials for this system are

$$\phi' = Vx + \mu \frac{x}{x^2+y^2} + \frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x}$$

and

$$\psi' = Vy - \mu \frac{y}{x^2+y^2} - \frac{\Gamma}{4\pi}\ln(x^2+y^2)$$

where $U_\infty = V$ in this example. For convenience, we scale the problem such that all velocities are divided by the free-stream speed V; hence, the scaled free-stream speed is $V = 1$. We scale all distances by the radius of the cylinder, $R = \sqrt{\mu/V}$; hence, $R = 1$ implies $\mu = 1$. The potential may then be rewritten as

$$\phi = x + \frac{x}{x^2+y^2} + \frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x} \tag{3.43}$$

and the stream function as

$$\psi = y - \frac{y}{x^2+y^2} - \frac{\Gamma}{4\pi}\ln\left(x^2+y^2\right) \tag{3.44}$$

Taking the gradient of the velocity potential, ϕ, we get the components of the velocity:

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = 1 + \frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2} \tag{3.45}$$

and

$$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = -\frac{2yx}{\left(x^2+y^2\right)^2} + \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{3.46}$$

Finally, from the Bernoulli theorem, we write the pressure coefficient as follows:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = 1 - \left(u^2 + v^2\right) \tag{3.47}$$

A program to solve these equations around the unit circle is given in Table 3.6. Fig. 3.15 shows the pressure distribution around the cylinder with zero circulation. The minimum pressure coefficient is $C_p = -3$ located dead center at the top and bottom. The maximum pressure coefficient is $C_p = 1$ located at the leading- and trailing-edge stagnation points. The pressure distribution along the top of the cylinder is the same as the pressure distribution along the bottom, so only one curve is shown in the figure.

The following are some important points to note about the pressure distribution for the potential flow around a cylinder illustrated in Fig. 3.15.

Table 3.6 MATLAB File Used to Produce Figures 3.15 and 3.17

```
V = 1; mu = 1; Gamma = -2; R = sqrt(mu/V);
N = 360;
   dthe = 2*pi/N;
       the = 0:dthe:2*pi;
for n = 1:length(the)-1
  them(n) = the(n) + dthe/2;
  x(n) = R*cos(them(n));
  y(n) = R*sin(them(n));
  u(n) = V + V*R^2*(y(n)^2 - x(n)^2)/(x(n)^2 + y(n)^2)^2 ...
           - Gamma/2/pi*y(n)/(x(n)^2+y(n)^2);
  v(n) = -2*V*R^2*y(n)*x(n)/(x(n)^2 + y(n)^2)^2 ...
         + Gamma/2/pi*x(n)/(x(n)^2+y(n)^2);
    ut(n) = -u(n)*sin(them(n)) + v(n)*cos(them(n));
    ur(n) = u(n)*cos(them(n)) + v(n)*sin(them(n));
    Cp(n) = 1 - (u(n)^2+v(n)^2)/V^2;
end
plot(x,y,'k',[-1 1],[0 0],'k'),axis image
      hold on
plot(x,Cp),axis([-2 2 -5 1])
CL = -sum(Cp.*sin(them))*dthe
CLexact = -2*Gamma/R/V
for mm=1:length(the)-1
    them(mm) = the(mm) + dthe/2;
    uthe(mm) = -2*sin(them(mm)) + Gamma/2/pi/R/V;
    Cp2(mm) = 1 - uthe(mm)^2;
end
CL2 = -sum(Cp2.*sin(them))*dthe
```

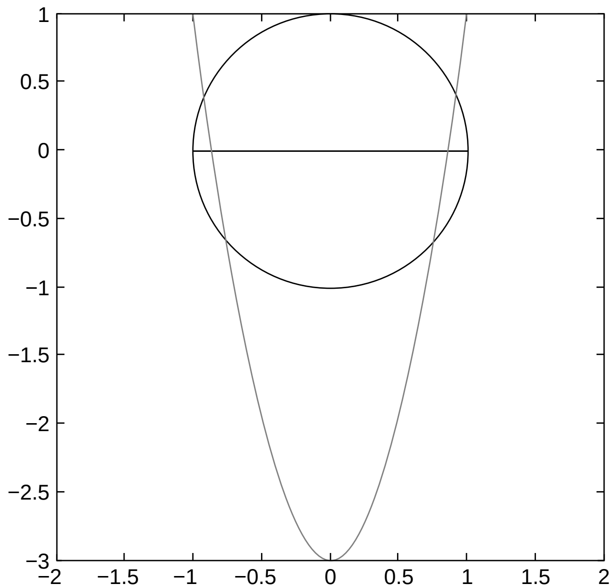


FIGURE 3.15

Pressure distribution around a circle; this is a plot of C_p versus x; the definition of C_p is given in Figure 3.16.

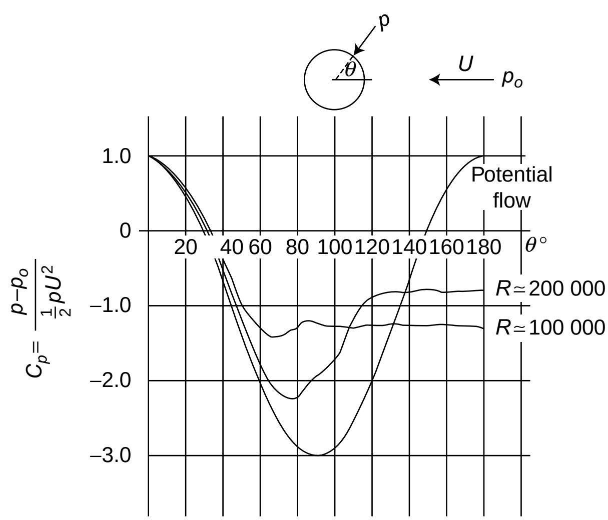


FIGURE 3.16

Pressure distribution around a circle; $C_L = 0$.

1. At the stagnation points (0° and 180°), the pressure difference $(p - p_0)$ is positive and equal to $\rho U^2/2$.
2. At 30 and 150 degrees, where $\sin\theta = 1/2$, $(p - p_0)$ is zero, and at these points the local velocity is the same as that of the free stream.
3. Between 30 and 150 degrees, C_p is negative, showing that p is less than p_0.
4. The pressure distribution is symmetrical about the vertical axis and there is therefore no drag force.

Fig. 3.16 compares this ideal pressure distribution with that obtained by experiment; it shows that the actual pressure distribution is similar to the theoretical value up to about 70 degrees, but departs radically from it thereafter. Furthermore, it can be seen that the pressure coefficient over the rear portion of the cylinder remains negative. This destroys the symmetry about the vertical axis and produces a force in the direction of the flow. This does *not* mean that the potential flow around a circular cylinder is useless. It is useful because this flow field can be mapped to the flow field around an airfoil as illustrated in Section 3.2.9.

Now we examine the same problem in polar coordinates. In addition, we explicitly keep V and R in the problem. This will help us interpret the meaning of the scaling applied previously. In polar coordinates, the velocity potential for this problem can be written as follows:

$$\phi = V\left(r + \frac{\mu}{rV}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

If we define the parameter $R = \sqrt{\mu/V}$, then

$$\phi = V\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \tag{3.48}$$

The radial and tangential components of the velocity field are, respectively,

$$u_r = \frac{\partial\phi}{\partial r} = \left(1 - \frac{R^2}{r^2}\right)V\cos\theta \tag{3.49}$$

and

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\left(1 + \frac{R^2}{r^2}\right)V\sin\theta + \frac{\Gamma}{2\pi r} \tag{3.50}$$

At $r = R$ the radial component of the velocity is zero. Hence, $r = R$ is the radius of a closed circular streamline and thus the surface of the cylinder in a cross flow V. On this surface $u_\theta = -2V\sin\theta + \Gamma/2\pi R$. Substituting this into the Bernoulli theorem, we get the exact solution of the pressure distribution on the surface of the cylinder, which is

$$p = p_\infty + \frac{1}{2}\rho\left[V^2 - \left(-2V\sin\theta + \frac{\Gamma}{2\pi R}\right)^2\right] \tag{3.51}$$

The lift and drag acting on the cylinder are defined as follows:

$$L = -\int_0^{2\pi} p \sin\theta \, R d\theta$$

$$D = -\int_0^{2\pi} p \cos\theta \, R d\theta$$

Substituting Eq. (3.48) into the integrals and doing the integrations, we get

$$L = -\rho V \Gamma \tag{3.52}$$

$$D = 0 \tag{3.53}$$

Equation (3.52) is the Kutta-Joukowski theorem. Equation (3.53) is what Euler called D'Alembert's paradox.

Let us rearrange Eq. (3.51) to get a formula for the dimensionless pressure coefficient:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = 1 - \left(-2\sin\theta + \frac{\Gamma}{2\pi R V}\right)^2 \tag{3.54}$$

If $\Gamma = 0$, then

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = 1 - 4\sin^2\theta$$

At $\theta = 0$ and π (the trailing and leading edges, respectively), this is unity. At $\theta = \pi/2$ and $3\pi/2$ (the top and bottom dead centers), this is -3. These results check with the results already given in Fig. 3.15.

Table 3.6 lists the MATLAB script for the case where R and V are explicitly allowed to be changed by the users. Figure 3.17 is the pressure distribution on the cylinder with $\Gamma = -1$. Note that the lift coefficient is equal to twice the magnitude of Γ because $V = 1$ and $R = 1$ were used. Further discussion on the lift coefficient and, in particular, the derivation of the exact result is described soon.

Circulation is not a geometric parameter. However, it can be related to a geometric parameter, α, illustrated in Fig. 3.18, the streamline plot of the flow around a cylinder. The script used to plot the streamlines is given in Table 3.7. The angle α is the angular location of the rear stagnation point as measured from the horizontal axis. It is related to the circulation in the following way. On the cylindrical surface the velocity is

$$\mathbf{u} = (u_r, u_\theta) = (0, -2V\sin\theta + \Gamma/2\pi R)$$

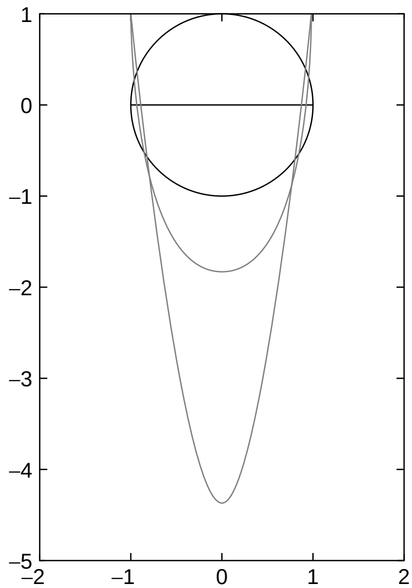


FIGURE 3.17

Pressure distribution around a circle with circulation; $C_L = 4$.

Table 3.7 MATLAB File Used to Produce Figure 3.18

```
mu = 1; gam = -2; V=1;
x = -3:.02:3;
y = -2:.02:2;
for m = 1:length(x)
 for n = 1:length(y)
  xx(m,n) = x(m); yy(m,n) = y(n);
  psis(m,n) = V * y(n) - mu * y(n)/(x(m)^2+(y(n)+.01)^2) ...
        - (gam/4/pi)*log(x(m)^2+(y(n)+.01)^2);
 end
end
contour(xx,yy,psis,[-3:.3:3],'k'), axis image
```

The stagnation points, if they occur on the cylinder, are located where $u_\theta = 0$. Thus, they are located at

$$\sin\theta \equiv \sin\alpha = \frac{\Gamma}{4\pi RV}$$

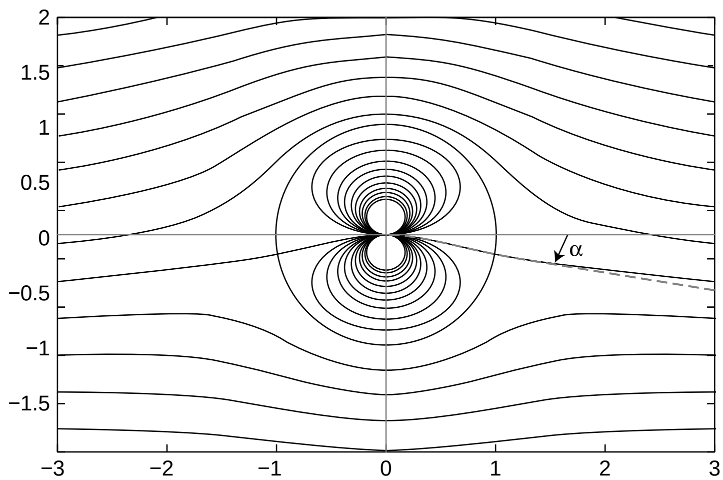


FIGURE 3.18

Streamlines and the definition of α.

Therefore,

$$\alpha = \sin^{-1}\left(\frac{\Gamma}{4\pi RV}\right) \tag{3.55}$$

or

$$\Gamma = 4\pi RV \sin\alpha \tag{3.56}$$

Substituting Eq. (3.56) into the Kutta-Joukowski theorem, we get

$$L = \rho V\Gamma = 8\pi \frac{1}{2}\rho V^2 R \sin\alpha$$

Thus, the lift coefficient in terms of α is

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 R} = 8\pi \sin\alpha \tag{3.57}$$

Substituting for $\sin\alpha$ from Eq. (3.55) or (3.56), we can also write the lift coefficient as

$$C_L = 2\frac{\Gamma}{RV} \tag{3.58}$$

This coefficient of lift is the conventional definition for a cylinder with R used as the length scale. This illustrates the fact that there is zero lift if there is zero circulation.

3.2.9 Joukowski Airfoil and the Circular Cylinder

The solution of the flow around a circular cylinder with circulation in a cross flow can be used to predict the flow around thin airfoils. We can transform the local geometry of the cylinder into an ellipse, an airfoil, or a flat plate without influencing the geometry of the far-field flow. This procedure is known as conformal mapping. If we interpret the Cartesian coordinates as the coordinates of the plane of complex numbers $z = x + iy$, where x and y are real numbers, $i = \sqrt{-1}$, x is the real part of z, and y is the imaginary part of z, then doing this allows us to use complex variable theory to solve two-dimensional potential-flow problems. We are *not* going to examine complex variable theory here. Instead, we will give an example of applying one of the ideas to illustrate that the circle can be transformed into an airfoil. Table 3.8 is a MATLAB script that does this. MATLAB is very useful here because of its complex arithmetic capabilities. The steps are outlined in the table. The result of executing this code is illustrated in Fig. 3.19. The figure shows the circle and the airfoil onto which it is mapped. Each point on the circle corresponds to a unique point on the airfoil. The potential at each point on the cylinder is the same as the corresponding point on the airfoil. What is different is the distance between points, and thus the velocity and pressure distributions on the airfoil must be determined from the mapped distribution of the potential. Since the far field is not affected by the transformation, the lift on the airfoil is $L = -\rho V \Gamma$ (i.e., the same as the lift on the cylinder). However, the lift coefficient is not the same because the length of the airfoil is nearly four times the cylinder's radius. Using this fact, we multiply Eq. (3.54) by $R/c = 1/4$, where $R = 1$ is the radius of the unit circle examined in the previous section and $c = 4R$ is the length of the airfoil chord. Thus, for an airfoil with zero camber, we predict

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 c} = 2\pi\alpha \tag{3.59}$$

where $\alpha \approx \sin\alpha$ for small angles was used. This formula illustrates the influence of angle of attack on the performance of an airfoil. The circulation was chosen such that the flow left the trailing edge smoothly (the Kutta condition).

This brief discussion is, of course, an overview of a connection between the flow around a circle and the flow around an airfoil. In the next chapter, we will deal with the details of thin-airfoil theory and show that this formula for lift is theoretically well founded. However, it illustrates only part of the story. We will also investigate the other geometric feature that contributes to lift—camber. It can be shown that, for cambered thin airfoils at small angles of attack,

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 c} = 2\pi\left(\alpha + 2\frac{f}{c}\right)$$

where f is the maximum camber at midchord of a camber line that is symmetric fore and aft of the midchord. Aeronautical engineers find this equation and Eq. (3.59) quite useful, particularly in preliminary design and performance estimation analyses.

Table 3.8 MATLAB File Used to Produce Figure 3.19

```
      % Joukowski transformation MATLAB code
      %
      % Example of conformal mapping of a circle to an airfoil
      %                  AE425-ME425 Aerodynamics
      % Daniel T. Valentine ................... January 2009
      % Circle in (xp,yp) plane: R = sqrt(xp^2 + yp^2), R > 1
      % Complex variables of three complex planes of interest:
      %   zp = xp + i*yp ==> Circle plane
      %   z = x + i*y    ==> Intermediate plane
      %   w = u + i*v    ==> Airfoil (or physical) plane
      clear;clc
% Step 1: Select the parameters that define the airfoil of interest.
% (1) Select the a == angle of attack alpha
           a = -2;       % in degrees
           a = a*pi/180; % Conversion to radians
% (2) Select the parameter related to thichkness of the airfoil:
           e = .1;
% (3) Select the shift of y-axis related to camber of the airfoil:
           f = .1;
% (4) Select the trailing edge angle parameter:
        te = .05;     % 0 < te < 1 (0 ==> cusped trailing edge)
        n = 2 - te;      % Number related to trailing edge angle.
        tea = (n^2-1)/3;    % This is a Karman-Trefftz extension.
% Step 2: Compute the coordinates of points on circle in zp-plane:
           R = 1 + e;
           theta = 0:pi/200:2*pi;
           yp = R * sin(theta);
           xp = R * cos(theta);
% Step 3: Transform coordinates of circle from zp-plane to z-plane:
           z = (xp - e) + i.*(yp + f);
% Step 4: Transform circle from z-plane to airfoil in w-plane
% (the w-plane is the "physical" plane of the airfoil):
   rot = exp(i*a); % Application of angle of attack.
   w = rot .* (z + tea*1./z); % Joukowski transformation.
% Step 5: Plot of circle in z-plane on top of airfoil in w-plane
           plot(xp,yp), hold on
           plot(real(w),imag(w),'r'),axis image, hold off
```

3.3 AXISYMMETRIC FLOWS (INVISCID AND INCOMPRESSIBLE FLOWS)

Consider now axisymmetric potential flows—that is, the flows around bodies such as cones aligned to the flow and spheres. To analyze and, for that matter, to define,

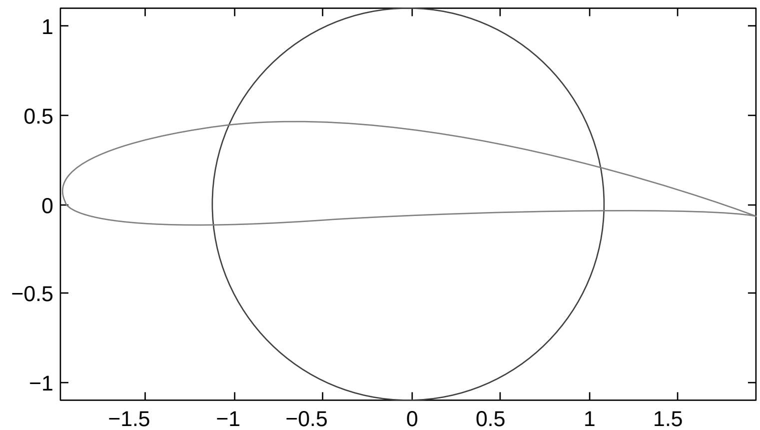


FIGURE 3.19

Streamlines and the definition of α.

axisymmetric flows, it is necessary to introduce cylindrical and spherical coordinate systems. Unlike the Cartesian system, these systems can exploit the underlying symmetry of the flows.

3.3.1 Cylindrical Coordinate System

The cylindrical coordinate system is illustrated in Fig. 3.20. The three coordinate surfaces are the planes $z =$ constant and $\theta =$ constant, with the surface of the cylinder having radius r. For the Cartesian system, in contrast, all three coordinate surfaces are planes. As a consequence for the Cartesian system, the directions (x, y, z) of the velocity components are fixed throughout the flow field. For the cylindrical coordinate system, though, only one of the directions (z) is fixed throughout the flow field; the other two (r and θ) vary depending on the value of the angular coordinate θ. In this respect there is a certain similarity to the polar coordinates introduced earlier in the chapter. The velocity component q_r is always locally perpendicular to the cylindrical coordinate surface, and q_θ is always tangential to that surface. Once this elementary fact is properly understood, cylindrical and Cartesian coordinates are equally easy to use.

Like the relationships between velocity potential and velocity components derived for polar coordinates (see Section 3.1.3), the following relationships are obtained for cylindrical coordinates:

$$q_r = \frac{\partial \phi}{\partial r},\ q_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta},\ q_z = \frac{\partial \phi}{\partial z} \tag{3.60}$$

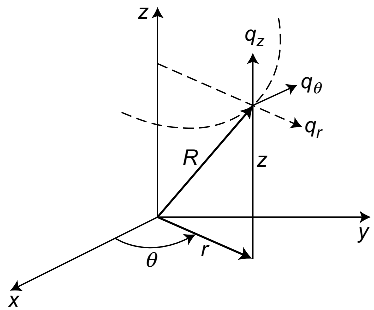


FIGURE 3.20

Cylindrical coordinates.

An axisymmetric flow is defined as one for which the flow variables (i.e., velocity and pressure) do not vary with the angular coordinate θ. This would be so, for example, for a body of revolution about the z-axis, along which the oncoming flow is directed. For such an axisymmetric flow a stream function can be defined. The continuity equation for axisymmetric flow in cylindrical coordinates can be derived in a similar manner as for two-dimensional flow in polar coordinates (see Section 2.4.3); it takes the form

$$\frac{1}{r}\frac{\partial rq_r}{\partial r} + \frac{\partial q_z}{\partial z} = 0 \tag{3.61}$$

The relationship between stream function and velocity component must be such as to satisfy Eq. (3.61); hence it can be seen that

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \; q_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{3.62}$$

3.3.2 Spherical Coordinates

For analyzing certain two-dimensional flows—for example, the flow over a circular cylinder with and without circulation—it is convenient to work with polar coordinates. The axisymmetric equivalents of polar coordinates are spherical coordinates, such as those used for analyzing the flow around spheres. Spherical coordinates are illustrated in Fig. 3.21. In this case none of the coordinate surfaces are plane and the directions of all three velocity components vary over the flow field, depending on the values of the angular coordinates θ and φ. The relationships between the velocity components and potential here are given by

$$q_R = \frac{\partial \phi}{\partial R}, \quad q_\theta = \frac{1}{R\sin\varphi}\frac{\partial \phi}{\partial \theta}, \quad q_\varphi = \frac{1}{R}\frac{\partial \phi}{\partial \varphi} \tag{3.63}$$

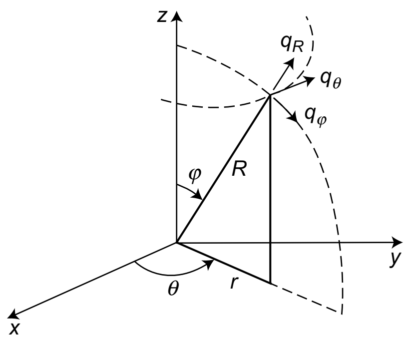


FIGURE 3.21

Spherical coordinates.

For axisymmetric flows, the variables are independent of θ, and in this case the continuity equation takes the form

$$\frac{1}{R^2}\frac{\partial\left(R^2 q_R\right)}{\partial R}+\frac{1}{R\sin\varphi}\frac{\left(\sin\varphi\, q_\varphi\right)}{\partial\varphi}=0 \tag{3.64}$$

Again, the relationship between the stream function and the velocity components must be such as to satisfy the continuity equation (Eq. 3.64), so

$$q_R=\frac{1}{R^2\sin\varphi}\frac{\partial\psi}{\partial\varphi},\quad q_\varphi=-\frac{1}{R\sin\varphi}\frac{\partial\psi}{\partial R} \tag{3.65}$$

3.3.3 Axisymmetric Flow from a Point Source (or towards a Point Sink)

The point source and sink are similar to the line source and sink discussed in Section 3.2. A close physical analogy can be found if one imagines the flow into or out of a very (strictly infinitely) thin round pipe, as depicted in Fig. 3.22. As suggested in this figure the streamlines would be purely radial in direction.

We will suppose that the flow rate out of the point source is given by Q. Q is usually referred to as the strength of the point source. Now, since the flow is purely radial away from the source, the total flow rate across the surface of any sphere having its center at the source is also Q. (Note that this sphere is purely notional and does not represent a solid body or in any way hinder the flow.) Thus the radial velocity component at any radius R is related to Q as follows:

$$4\pi R^2 q_R = Q$$

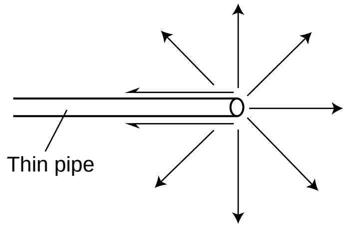


FIGURE 3.22
Illustration of a spherical source.

It therefore follows from Eq. (3.63) that

$$q_R = \frac{\partial \phi}{\partial R} = \frac{Q}{4\pi R^2}$$

Integration then gives the expression for the velocity potential of a point source as

$$\phi = -\frac{Q}{4\pi R} \tag{3.66}$$

In a similar fashion, an expression for the stream function can be derived using Eq. (3.65), giving

$$\psi = -\frac{Q}{4\pi}\cos\varphi \tag{3.67}$$

3.3.4 Point Source and Sink in a Uniform Axisymmetric Flow

Placing a point source and/or sink in a uniform horizontal stream of $-U$ leads to results very similar to those found in Section 3.2.5 for the two-dimensional case with line sources and sinks.

First the velocity potential and stream function for uniform flow $-U$ in the z direction must be expressed in spherical coordinates. The velocity components q_R and q_φ are related to $-U$ as follows:

$$q_R = -U\cos\varphi \quad \text{and} \quad q_\varphi = U\sin\varphi$$

Using Eq. (3.63) followed by integration then gives

$$\frac{\partial \phi}{\partial R} = -U\cos\varphi \rightarrow \phi = -UR\cos\varphi + f(\varphi)$$

$$\frac{\partial \phi}{\partial \varphi} = -UR\sin\varphi \rightarrow \phi = -UR\cos\varphi + g(R)$$

$f(\varphi)$ and $g(R)$ are arbitrary functions that take the place of constants of integration when partial integration is carried out. Plainly, in order for the two expressions for ϕ derived previously to be in agreement, $f(\varphi) = g(R) = 0$. The required expression for the velocity potential is thereby given as

$$\phi = -UR\cos\varphi \tag{3.68}$$

Similarly, using Eq. (3.65) followed by integration gives

$$\frac{\partial\psi}{\partial\varphi} = -UR^2\cos\varphi\sin\varphi \rightarrow \psi = -\frac{UR^2}{4}\cos 2\varphi + f(R)$$

$$\frac{\partial\psi}{\partial R} = -UR\sin^2\varphi \rightarrow \psi = -\frac{UR^2}{2}\sin^2\varphi + g(\varphi)$$

Recognizing that $\cos 2\varphi = 1 - 2\sin^2\varphi$, it can be seen that the two expressions given for ψ agree if the arbitrary functions of integration take the values $f(R) = -UR^2/4$ and $g(\varphi) = 0$. The required expression for the stream function is thus given as

$$\psi = -\frac{UR^2}{2}\sin^2\varphi \tag{3.69}$$

Using Eqs. (3.66) and (3.68) and Eqs. (3.67) and (3.69), it can be seen that for a point source at the origin placed in a uniform flow $-U$ along the z-axis,

$$\phi = -UR\cos\varphi - \frac{Q}{4\pi R} \quad \psi = -\frac{1}{2}UR^2\sin^2\varphi - \frac{Q}{4\pi}\cos\varphi \tag{3.70}$$

The flow field represented by Eqs. (3.70) corresponds to the potential flow around a semi-finite body of revolution—very much like its two-dimensional counterpart described in Section 3.2.6. Similarly to the procedure described in Section 3.2.6, it can be shown that the stagnation point occurs at the point $(-a, 0)$, where

$$a = \sqrt{\frac{Q}{4\pi U}} \tag{3.71}$$

and that the streamlines passing through this stagnation point define a body of revolution given by

$$R^2 = 2a^2(1+\cos\varphi)/\sin^2\varphi \tag{3.72}$$

The derivation of Eqs. (3.71) and (3.72) are left as an exercise (see Exercise 3.19 at the end of the chapter).

As in the two-dimensional case described in Section 3.2.7, a point source placed on the z-axis at $z = -a$, combined with an equal-strength point sink also placed on

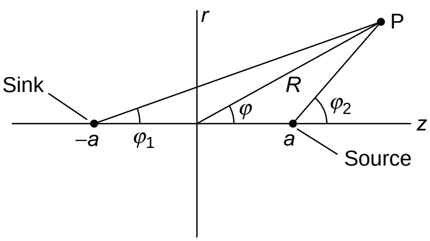


FIGURE 3.23

Source-sink pair.

the z-axis at $z = a$ (see Fig. 3.23) gives the following velocity potential and stream function at point P:

$$\phi = \frac{Q}{4\pi\left[(R\cos\varphi + a)^2 + R^2\sin^2\varphi\right]^{1/2}} - \frac{Q}{4\pi\left[(R\cos\varphi - a)^2 + R^2\sin^2\varphi\right]^{1/2}} \tag{3.73}$$

$$\psi = \frac{Q}{4\pi}(\cos\varphi_1 - \cos\varphi_2) \tag{3.74}$$

where

$$\cos\varphi_1 = \frac{R\cos\varphi + a}{\left[(R\cos\varphi + a)^2 + R^2\sin^2\varphi\right]^{1/2}}$$

$$\cos\varphi_2 = \frac{R\cos\varphi - a}{\left[(R\cos\varphi - a)^2 + R^2\sin^2\varphi\right]^{1/2}}$$

If this source-sink pair is placed in a uniform stream $-U$ in the z direction, it generates the flow around a body of revolution known as a Rankine body. The shape is very similar to the two-dimensional Rankine oval shown in Fig. 3.14 and described in Section 3.2.7.

3.3.5 The Point Doublet and the Potential Flow around a Sphere

A point doublet is produced when the source-sink pair in Fig. 3.23 become infinitely close together. This is closely analogous to a line doublet, which was described in Section 3.2.8. Mathematically, the expressions for velocity potential and stream function for a point doublet can be derived, respectively, from Eqs. (3.73) and (3.74) by allowing $a \to 0$, keeping $\mu = 2Qa$ fixed. The latter quantity is known as the strength of the doublet.

If a is very small, a^2 may be neglected compared to $2Ra\cos\varphi$ in Eq. (3.73) and can then can be written as

$$\phi = \frac{Q}{4\pi R}\left[\frac{1}{[1+2(a/R)\cos\varphi]^{1/2}} - \frac{1}{[1-2(a/R)\cos\varphi]^{1/2}}\right] \tag{3.75}$$

On expanding,

$$\frac{1}{\sqrt{1\pm x}} = 1 \mp \frac{1}{2}x + \ldots$$

Therefore, as $a \to 0$, Eq. (3.75) reduces to

$$\phi = -\frac{\mu}{4\pi R^2}\cos\varphi \tag{3.76}$$

Similarly, we can show that

$$\psi = \frac{\mu}{4\pi R}\sin^2\varphi \tag{3.77}$$

The streamline patterns corresponding to the point doublet are similar to those depicted in Fig. 3.10. It is apparent from this streamline pattern and from the form of Eq. (3.77) that, unlike the point source, the flow field for the doublet is not omnidirectional. On the contrary, it is strongly directional. Moreover, the case analyzed previously is a somewhat special case in that the source-sink pair lie on the z-axis. In fact, the *axis* of the doublet can be in any direction in three-dimensional space.

For two-dimensional flow, it was shown in Section 3.2.8 that the line doublet placed in a uniform stream produces potential flow around a circular cylinder. It is similarly shown here that a point doublet placed in a uniform stream corresponds to the potential flow around a sphere.

From Eqs. (3.68) and (3.76), the velocity potential for a point doublet in a uniform stream, with both the stream and doublet axis aligned in the negative z direction, is given by

$$\phi = -UR\cos\varphi - \frac{\mu}{4\pi R^2}\cos\varphi \tag{3.78}$$

From Eq. (3.63), the velocity components are given by

$$q_R = \frac{\partial\phi}{\partial R} = -\left(U - \frac{\mu}{2\pi R^3}\right)\cos\varphi \tag{3.79}$$

$$q_\varphi = \frac{1}{R}\frac{\partial\phi}{\partial\varphi} = \left(U + \frac{\mu}{4\pi R^3}\right)\sin\varphi \tag{3.80}$$

The stagnation points are defined by $q_R = q_\varphi = 0$. Let the coordinates of the stagnation points be denoted (R_s, φ_s). Then, from Eq. (3.80), it can be seen that

$$R_s^3 = -\frac{\mu}{4\pi U} \quad \text{or} \quad \sin\varphi_s = 0$$

The first of these two equations cannot be satisfied because it implies that R_s is not a positive number. Accordingly, the second of the two equations must hold, implying that

$$\phi_s = 0 \text{ and } \pi$$

It now follows from Eq. (3.76) that

$$R_s = \left(\frac{\mu}{2\pi U}\right)^{1/3} \tag{3.81}$$

Thus there are two stagnation points on the z-axis at equal distances from the origin.

From Eqs. (3.69) and (3.77), the stream function for a point doublet in a uniform flow is given by

$$\psi = -\frac{UR^2}{2}\sin^2\varphi + \frac{\mu}{4\pi R}\sin^2\varphi \tag{3.82}$$

It follows from substituting Eq. (3.81) into Eq. (3.82) that at the stagnation points $\psi = 0$. So the streamlines passing through the stagnation points are described by

$$\psi = -\left(\frac{UR^2}{2} - \frac{\mu}{4\pi R}\right)\sin^2\varphi = 0 \tag{3.83}$$

Equation (3.82) shows that, when $\varphi \neq 0$ or π, the radius R of the stream surface, containing the streamlines that pass through the stagnation points, remains fixed equal to R_s. R can take any value when $\phi = 0$ or π. Thus these streamlines define the surface of a sphere of radius R_s. This is very similar to the two-dimensional case of flow over a circular cylinder described in Section 3.2.8.

From Eqs. (3.80) and (3.81), it follows that the velocity on the surface of the sphere is given by

$$q = \frac{3}{2}U\sin\varphi$$

so that using the Bernoulli equation gives

$$p_0 + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho q^2 = p + \frac{1}{2}\rho\left(\frac{3}{2}U\sin\varphi\right)^2$$

Therefore, the pressure variation over the sphere's surface is given by

$$p - p_0 = \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\varphi\right) \tag{3.84}$$

Again, this result is quite similar to that for the circular cylinder described in Section 3.2.8.

3.3.6 Flow around Slender Bodies

It was shown that the flow around a class of bodies of revolution can be modeled by the use of a source and sink of equal strength. Accordingly, it is natural to speculate whether the flow around more general body shapes can be obtained using several sources and sinks or a distribution of them along the z-axis. It is indeed possible to do this as Fuhrmann first demonstrated [5]. Two examples similar to those Fuhrmann presented are shown in Fig. 3.24. Although Fuhrmann's method could model the flow around realistic-looking bodies, it suffered an important defect from a design perspective. One could calculate the body of revolution corresponding to a specified distribution of sources and sinks, but a designer would want to solve the inverse problem of selecting the variation of source strength in order to obtain the flow around a given shape. This more practical approach became possible after Munk [6] introduced his slender-body theory for calculating the forces on airship hulls. We briefly describe this approach next.

For Munk's slender-body theory, it is assumed that the radius of the body is very much smaller than its total length. The flow is modeled by a distribution of sources and sinks placed on the z-axis as depicted in Fig. 3.25. In many respects, this theory is analogous to the theory for calculating the two-dimensional flow around symmetric wing sections, known as the thickness problem.

For an element of source distribution located at $z = z_1$, the velocity induced at point P(r, z) is

$$q_R = \frac{\sigma(z_1)}{4\pi R^2}\, dz_1 \tag{3.85}$$

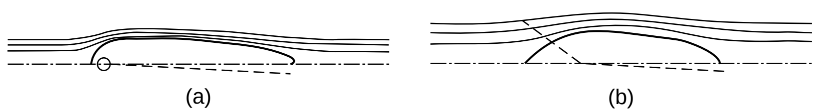


FIGURE 3.24

Two examples of flows around bodies of revolution generated by (a) point source plus linear distribution of sink strength, and (b) two linear distributions of source/sink strength. The distributions are denoted by broken lines.

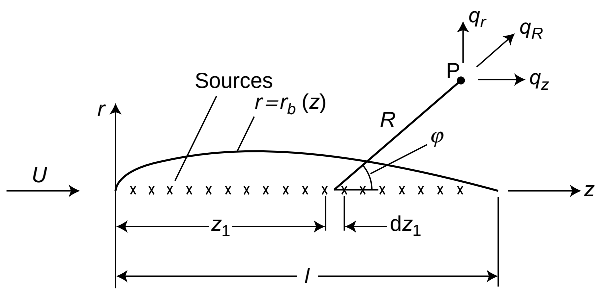


FIGURE 3.25

Flow over a slender body of revolution modeled by source distribution.

where $\sigma(z_1)$ is the source strength per unit length and $\sigma(z_1)dz_1$ takes the place of Q in Eq. (3.63). Thus, to obtain the velocity components in the r and z directions at P due to all sources, we resolve the velocity given by Eq. (3.82) in the two coordinate directions and integrate along the length of the body. Thus

$$q_r = \int_0^l q_R \sin\varphi\, dz_1 = \frac{1}{4\pi}\int_0^l \sigma(z_1)\frac{r}{\left[(z-z_1)^2+r^2\right]^{3/2}}\,dz_1 \tag{3.86}$$

$$q_z = \int_0^l q_R \cos\varphi\, dz_1 = \frac{1}{4\pi}\int_0^l \sigma(z_1)\frac{z-z_1}{\left[(z-z_1)^2+r^2\right]^{3/2}}\,dz_1 \tag{3.87}$$

Source strength can be related to body geometry by the following physical argument. Consider the elemental length of the body as shown in Fig. 3.26. If the body radius r_b is very small compared to the length, l, then the limit $r \to 0$ can be considered. For this limit, the flow from the sources may be considered purely radial so that the flow across the body surface of the element is entirely due to the sources within the element itself. Accordingly,

$$2\pi r q_r dz_1 = \sigma(z_1)dz_1 \text{ at } r = r_b \quad \text{provided} \quad r_b \to 0$$

But the effects of the oncoming flow as well as the sources must be considered. The net perpendicular velocity on the body surface due to both the oncoming flow and the sources must be zero. Provided that the slope of the body contour is very small (i.e., $dr_b/dz \ll 1$), the perpendicular and radial velocity components may be considered the same. Thus the requirement that the net normal velocity be zero becomes (see Fig. 3.26)

$$q_r = U\sin\beta = U\frac{dr_b}{dz_1}$$

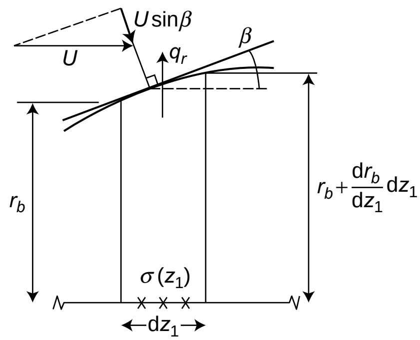


FIGURE 3.26

Element of slender body of revolution.

where q_r is associated with the sources and the two parts on the right-hand side are associated with the oncoming flow. The above relationship is such that the source strength per unit length and body shape are related as follows:

$$\sigma(z) = U\frac{dS}{dz} \tag{3.88}$$

where S is the frontal area of a cross-section and is given by $S = \pi r_b^2$.

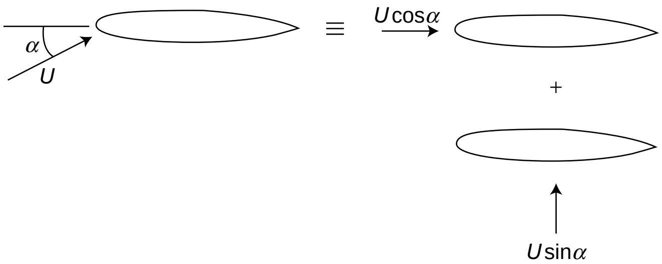
In the limit, as $r \to 0$, Eq. (3.87) simplifies to

$$q_z = \frac{1}{4\pi}\int_0^l \frac{\sigma(z+1)}{(z-z_1)^2}dz_1 \tag{3.89}$$

Thus, once the variation of source strength per unit length has been determined according to Eq. (3.88), the axial velocity can be obtained by evaluating Eq. (3.89), and hence the pressure can be evaluated from the Bernoulli equation.

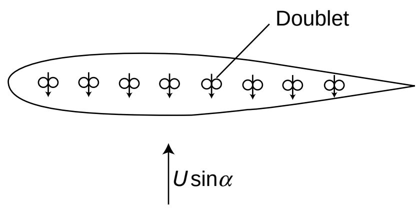
We see from the derivation of Eq. (3.89) that both r_b and dr_b/dz must be very small. Plainly, the latter requirement would be violated in the vicinity of $z = 0$ if the body had a rounded nose. This is a major drawback of the method.

The slender-body theory was extended by Munk [9] to the case of a body at an angle of incidence or yaw. This case is treated as a superposition of two distinct flows, as shown in Fig. 3.27(a). One of these is the slender body at zero angle of incidence as discussed previously. The other is the slender body in a cross flow. For such a slender body, the flow around a particular cross-section is closely analogous to that around a circular cylinder (see Section 3.2.8). Accordingly, this flow can be modeled by a distribution of point doublets with axes aligned in the direction of the cross flow,



Flow at angle of yaw around a body of revolution as the superposition of two flows

(a)



Cross flow over slender body of revolution modeled as distribution of doublets

(b)

FIGURE 3.27

Slender body in an arbitrary onset flow: at top (a) an angle of attack; at bottom (b) in a cross-flow.

as depicted in Fig. 3.27(b). Slender-body theory will not be taken further here. The student is referred to Thwaites and Karamcheti for further details [8, 9].

3.4 COMPUTATIONAL (PANEL) METHODS

In Section 3.2.7, it was shown how the two-dimensional potential flow around an oval-shaped contour, the Rankine oval, can be generated by the superposition of a source and sink on the x-axis and a uniform flow. An analogous three-dimensional flow can also be generated around a Rankine body (see Section 3.3.4) by using a point source and sink. Thus it can be demonstrated that the potential flow around certain bodies can be modeled by placing sources and sinks in the body's interior. However, it is only possible to deal with particular cases in this way. We can model the potential flow around slender bodies of any shape by distributing sources lying along the x-axis in the interior of the body. This slender-body theory is discussed in

Section 3.3.6. However, calculations based on this theory are only approximate unless the body is infinitely thin and the slope of the body contour is very small. Even in this case, the theory breaks down if the nose or leading edge is rounded because there the slope of the contour is infinite. The panel methods described here model the potential flow around a body by distributing sources over the body surface. In this way, the potential flow around a body of any shape can be calculated to a very high degree of precision. The method was developed by Hess and Smith [10] at Douglas Aircraft.

If a body is placed in a uniform flow of speed U, in exactly the same way as for the Rankine oval of Section 3.2.7, the velocity potential for the flow may be superimposed on that for the disturbed flow around the body to obtain a total velocity potential of the form

$$\Phi = Ux + \phi \tag{3.90}$$

where ϕ denotes the so-called disturbance potential (i.e., the departure from free-stream conditions). It can be shown that the disturbance potential flow around a body of any given shape can be modeled by a distribution of sources over its surface (Fig. 3.28). Let the source strength per unit arc of contour be σ_Q. In the two-dimensional case, $\sigma_Q \mathrm{d}s_Q$ would replace $m/2\pi$ in Eq. (3.22). Thus the velocity potential at P due to sources on an element $\mathrm{d}s_Q$ of arc of contour centered at point Q is given by

$$\phi_{PQ} = \sigma_Q \ln R_{PQ} \, \mathrm{d}s_Q \tag{3.91}$$

where R_{PQ} is the distance from P to Q. The velocity potential due to all of the sources on the surface is obtained by integrating Eq. (3.91), over the surface. Thus, following Eq. (3.90), the total velocity potential at P can be written (for the two-dimensional case) as

$$\phi_P = Ux + \oint \sigma_Q \ln R_{PQ} \mathrm{d}s_Q \tag{3.92}$$

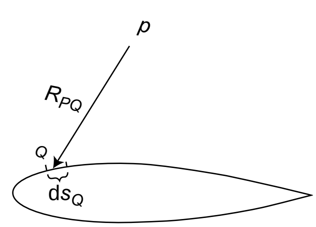


FIGURE 3.28

Surface distribution of singularities on a two-dimensional body.

where the integral is understood as being carried out over the contour of the body. With the power of desktop computers today, this is the basis of a class of computational techniques that is quite useful in aerodynamic design.

In order to use Eq. (3.92) for numerical modeling, it is first necessary to "discretize" the surface—that is, break it down into a finite but quite possibly large number of separate parts. This is achieved by representing the two-dimensional shape of the surface by a series of straight line segments (see Fig. 3.29).

The use of panel methods to calculate the potential flow around a body may be best illustrated with a concrete example. For this we select the two-dimensional flow around a symmetric airfoil (see Fig. 3.29).

The first step is to number all end points or *nodes* of the panels from 1 to N, as indicated in Fig. 3.29. Individual panels are assigned the same number as the node located to the left when facing in the outward direction from the panel. The midpoints of each panel are chosen as *collocation* points. We will see that the boundary condition of zero flow perpendicular to the surface is applied at these points. Then we define for each panel the unit normal and tangential vectors, $\hat{n}_i$ and $\hat{t}_i$, respectively. Consider panels i and j in the Fig. 3.29. The sources distributed over panel j induce a velocity, denoted by the vector $\mathbf{v}_{ij}$ at the collocation point of panel i. The components of $\mathbf{v}_{ij}$ perpendicular and tangential to the surface at the collocation point i are given by the scalar (or dot) products $\mathbf{v}_{ij} \cdot \hat{n}_i$ and $\mathbf{v}_{ij} \cdot \hat{t}_i$, respectively. Both of these quantities are proportional to the strength of the sources on panel j and therefore can be written in the forms

$$\mathbf{v}_{ij} \cdot \hat{n}_i = \sigma_j N_{ij} \text{ and } \mathbf{v}_{ij} \cdot \hat{t}_i = \sigma_j T_{ij} \tag{3.93}$$

N_{ij} and T_{ij} are the perpendicular and tangential velocities induced at the collocation point of panel i by sources of unit strength distributed over panel j; they are known as the *normal* and *tangential influence coefficients*.

The actual velocity perpendicular to the surface at collocation point i is the sum of the perpendicular velocities induced by each of the N panels plus the contribution

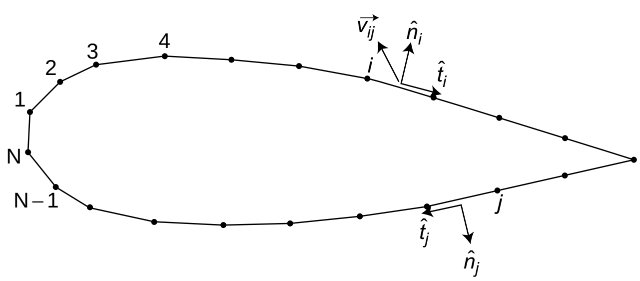


FIGURE 3.29

Discretization of a surface distribution of singularities.

due to the free stream. It is given by

$$v_{n_i} = \sum_{j=1}^{N} \sigma_j N_{ij} + \mathbf{U} \cdot \hat{n}_i \tag{3.94}$$

In a similar fashion, the tangential velocity at collocation point i is given by

$$v_{S_i} = \sum_{j=1}^{N} \sigma_j T_{ij} + \mathbf{U} \cdot \hat{t}_i \tag{3.95}$$

If the surface represented by the panels is to correspond to a solid surface, the actual perpendicular velocity at each collocation point must be zero. This condition may be expressed mathematically as $v_{n_i} = 0$. Equation (3.94) then becomes

$$\sum_{j=1}^{N} \sigma_j N_{ij} = -\mathbf{U} \cdot \hat{n}_i \;\; (i = 1, 2, \ldots, N) \tag{3.96}$$

Equation (3.96) is a system of linear algebraic equations for the N unknown source strengths, $\sigma_i \, (i = 1, 2, \ldots, N)$. It takes the form of a matrix equation

$$\mathbf{N}\sigma = \mathbf{b} \tag{3.97}$$

where $\mathbf{N}$ is an $N \times N$ matrix composed of the elements N_{ij}, σ is a column matrix composed of the N elements σ_i, and $\mathbf{b}$ is a column matrix composed of the N elements $-\mathbf{U} \cdot \hat{n}_i$. Assuming for the moment that the perpendicular influence coefficients N_{ij} have been calculated and that the elements of the right-hand column matrix $\mathbf{b}$ have also been calculated, Eq. (3.97) may, in principle at least, be solved for the source strengths comprising the elements of the column matrix σ. Systems of linear equations such as (3.97) can be readily solved numerically using standard methods. For the results presented here, one of the matrix inversion procedures available in MATLAB was used to solve for the source strengths. This method is described by Press et al. [9], who also give listings for the necessary computational routines.

Once the influence coefficients N_{ij} have been calculated, the source strengths can be determined by solving the system of Eq. (3.96) by some standard numerical technique. If the tangential influence coefficients T_{ij} have also been calculated, then, once the source strengths have been determined, the tangential velocities may be obtained from Eq. (3.95). The Bernoulli equation can now be used to calculate the pressure acting at collocation point i; in particular, the coefficient of pressure is given by Eq. (2.24) as

$$C_{p_i} = 1 - \left(\frac{v_{S_i}}{U}\right)^2 \tag{3.98}$$

The calculation of the influence coefficient is a central and essential part of the panel method. This is the question now addressed. As a first step, consider the calculation of the velocity induced at a point P by sources of unit strength distributed over a panel centered at point Q.

In terms of a coordinate system (x_Q, y_Q) measured relative to the panel (Fig. 3.30), the disturbance potential is given by integrating Eq. (3.91) over the panel. Mathematically this is expressed as

$$\phi_{PQ} = \int_{-\Delta s/2}^{\Delta s/2} \ln\sqrt{(x_Q - \xi)^2 + y_Q^2}\, d\xi \tag{3.99}$$

The corresponding velocity components at P in the x_Q and y_Q directions can be readily obtained from Eq. (3.99) as

$$\begin{aligned} v_{x_Q} = \frac{\partial \phi_{PQ}}{\partial x_Q} &= \int_{-\Delta s/2}^{\Delta s/2} \frac{x_Q - \xi}{(x_Q - \xi)^2 + y_Q^2} d\xi \\ &= -\frac{1}{2}\ln\left[\frac{(x_Q + \Delta s/2)^2 + y_Q^2}{(x_Q - \Delta s/2)^2 + y_Q^2}\right] \end{aligned} \tag{3.100}$$

$$\begin{aligned} v_{y_Q} = \frac{\partial \phi_{PQ}}{\partial y_Q} &= \int_{-\Delta s/2}^{\Delta s/2} \frac{y_Q}{(x_Q - \xi)^2 + y_Q^2} d\xi \\ &= -\left[\tan^{-1}\left(\frac{x_Q + \Delta s/2}{y_Q}\right) = \tan^{-1}\left(\frac{x_Q - \Delta s/2}{y_Q}\right)\right] \end{aligned} \tag{3.101}$$

Armed with these results for the velocity components induced at point P due to the sources on a panel centered at point Q, we now return to the problem of calculating the influence coefficients. Suppose that points P and Q are chosen to be the collocation points i and j, respectively. Equations (3.100) and (3.101) give the velocity components in a coordinate system relative to panel j; whereas the velocity components perpendicular and tangential to panel i are required. In vector form, the velocity at collocation point i is given by

$$\mathbf{v}_{PQ} = v_{x_Q}\hat{t}_j + v_{y_Q}\hat{n}_j$$

Therefore, to obtain the components of this velocity vector perpendicular and tangential to panel i, we take the scalar product of the velocity vector with $\hat{n}_i$ and $\hat{t}_i$, respectively, to obtain

$$N_{ij} = \mathbf{v}_{PQ} \cdot \hat{n}_i = v_{x_Q}\hat{n}_i \cdot \hat{t}_j + v_{y_Q}\hat{n}_i \cdot \hat{n}_j \tag{3.102a}$$

$$T_{ij} = \mathbf{v}_{PQ} \cdot \hat{t}_i = v_{x_Q}\hat{t}_i \cdot \hat{t}_j + v_{y_Q}\hat{t}_i \cdot \hat{n}_j \tag{3.102b}$$

A COMPUTATIONAL ROUTINE IN MATLAB

To see how calculation of the influence coefficients works in practice, a function routine written in MATLAB is given in Table 3.9. This script file solves for the flow around an ellipse in a uniform stream. With the parameters selected, as given in the table, the ellipse is actually a circular cylinder in order to check the surface distribution of sources model of a cylinder without circulation. The results in Fig. 3.31 illustrate that the identical pressure distribution is obtained as compared with the model, previously examined, based on a doublet at the origin in a uniform stream.

The routine in this function, step by step, performs the following:

1. The surface is discretized by assigning numbers from 1 to N to points on the surface of the airfoil, as suggested in Fig. 3.29. The x and y coordinates of these points are function inputs XP and YP.
2. For each of the N panels, the collocation points are calculated by taking an average of the coordinates at either end of the panel in question.
3. For each of the panels, the length $S(J) = \Delta s_j$ is calculated.
4. The x and y components of the unit tangent vectors for each panel are calculated as follows:

$$t_{jx} = \frac{x'_j - x'_{j-1}}{\Delta s_j}, \quad t_{jy} = \frac{y'_j - y'_{j-1}}{\Delta s_j}$$

5. The unit normal vectors are then calculated from $n_{j_x} = -t_{j_y}$ and $n_{j_y} = t_{j_x}$. The main task of the function, that of calculating the influence coefficients, now begins.
6. For each possible combination of panels, I and $J = 1$ to N, the first step is dealing with the special case when $I = J$—that is, the velocity induced by the sources on the panel itself at its collocation point. From Eqs. (3.100), (3.101), (3.102a), and (3.102b) we see that, when $x_Q = y_Q = 0$,

$$v_{PQ_x} = \ln(1) = 0, v_{PQ_y} = \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \pi$$

   When $i \neq j$, the influence coefficients have to be calculated from Eqs. (3.100), (3.101), (3.102a), and (3.102b).
7. The components DX and DY of R_{PQ} are calculated in terms of the x and y coordinates.
8. The components of R_{PQ} in terms of the coordinate system based on panel j are then calculated as

$$X_Q = \vec{R}_{PQ} \cdot \hat{t}_j, \; Y_Q = \vec{R}_{PQ} \cdot \hat{n}_j$$

9. VX and VY (i.e., v_{x_Q} and v_{y_Q}) are evaluated using Eqs. (3.100) and (3.101).
10. $\hat{n}_i \cdot \hat{t}_j$, $\hat{n}_i \cdot \hat{n}_j$, $\hat{t}_i \cdot \hat{t}_j$, and $\hat{t}_i \cdot \hat{n}_j$ are evaluated.
11. Finally, the influence coefficients are evaluated from Eq. (3.102b).

The function just presented is primarily intended for educational purposes and has not been optimized to economize on computing time. Nevertheless, the MATLAB program in Table 3.10 produced the calculation for an NACA 0024 airfoil presented in Fig. 3.31. In this case, 160 panels were used for the complete airfoil consisting of upper and lower surfaces. The results compare exactly with results obtained from other potential-flow codes. For more details on other two-dimensional methods, on the lifting body problem (discussed at the end of the next chapter), and three-dimensional panel and other *boundary-integral* methods the book by Katz and Plotkin [12] is quite useful.

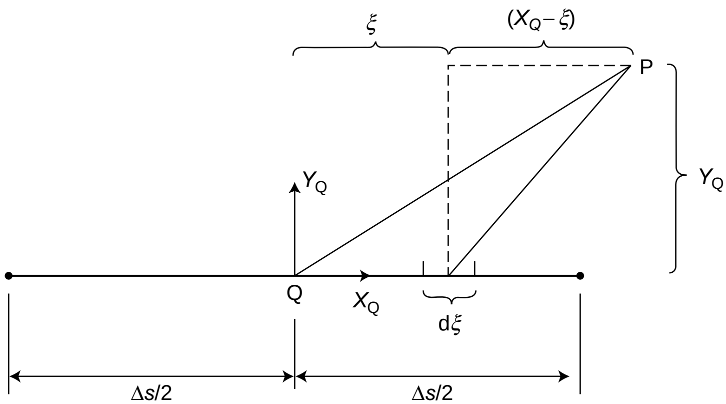


FIGURE 3.30

Local coordinate system of a surface panel.

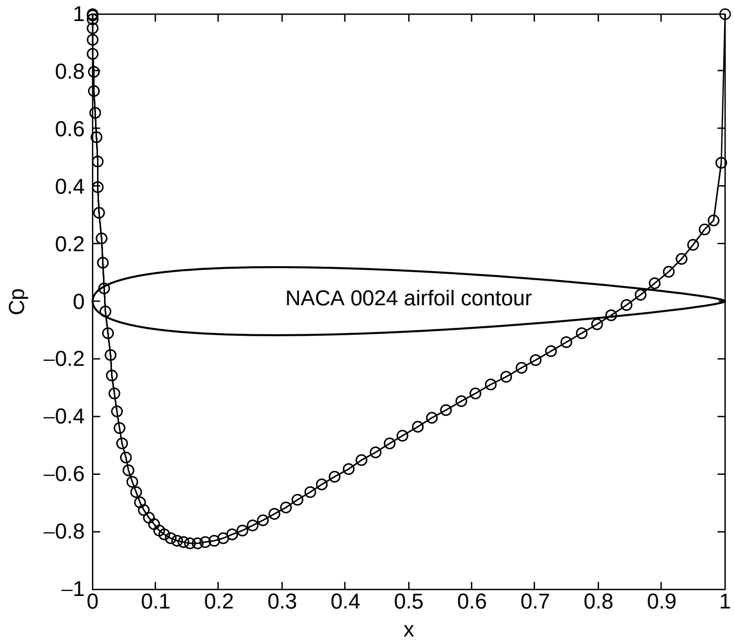


FIGURE 3.31

Comparison of pressure distributions.

Table 3.9 MATLAB Function to Compute Influence Coefficients of a Source Distribution

```
function [AN,AT,XC,YC,NHAT,THAT] = InfluSour(XP,YP,N)
% Influence coefficients for source distribution over a
% symmetric body.
% AN is the Nij matrix and AT is the Tij matix in Eqn (3.99).
%
for J = 1:N
if J==1
  XPL = XP(N);
  YPL = YP(N);
else
  XPL = XP(J-1);
  YPL = YP(J-1);
end
XC(J) = 0.5*(XP(J) + XPL);
YC(J) = 0.5*(YP(J) + YPL);
S(J) = sqrt( (XP(J) - XPL)^2 + (YP(J) - YPL)^2 );
THAT(J,1) = (XP(J) - XPL)/S(J);
THAT(J,2) = (YP(J) - YPL)/S(J);
NHAT(J,1) = - THAT(J,2);
NHAT(J,2) = THAT(J,1);
end
%Calculation of the influence coefficients.
for I = 1:N
for J = 1:N
if I==J
AN(I,J) = pi;
AT(I,J) = 0;
else
DX = XC(I) - XC(J);
DY = YC(I) - YC(J);
XQ = DX*THAT(J,1) + DY*THAT(J,2);
YQ = DX*NHAT(J,1) + DY*NHAT(J,2);
VX = -0.5*( log( (XQ + S(J)/2 )^2 + YQ*YQ )...
   -log( (XQ - S(J)/2 )^2 + YQ*YQ ) );
VY = -( atan2( (XQ + S(J)/2),YQ) - atan2((XQ - S(J)/2),YQ));
NTIJ = 0;
NNIJ = 0;
TTIJ = 0;
TNIJ = 0;
for K = 1:2
NTIJ = NHAT(I,K)*THAT(J,K) + NTIJ;
NNIJ = NHAT(I,K)*NHAT(J,K) + NNIJ;
TTIJ = THAT(I,K)*THAT(J,K) + TTIJ;
TNIJ = THAT(I,K)*NHAT(J,K) + TNIJ;
end
```

Table 3.9 (*Continued*)

```
AN(I,J) = VX*NTIJ + VY*NNIJ;
AT(I,J) = VX*TTIJ + VY*TNIJ;
end
end
end
```

Table 3.10 MATLAB Script to Compute Pressure Distribution on a NACA 0024 Airfoil as illustrated in Figure 3.31

```
% This is the NACA 0024 offsets:
XY =[1.000000 0.000252
     0.990966 0.005041
     0.976102 0.009129
       ...      ...
     0.000160 0.004489
     0.000017 0.001486
     0.000017 -0.001486
     0.000160 -0.004489
       ...      ...
     0.941520 -0.018360
     0.959610 -0.013579
     0.976102 -0.009129
     0.990966 -0.005041
     1.000000 -0.000252];
 XP = XY(:,1); YP = XY(:,2); N = length(XP);
[AN,AT,XC,YC,NHAT,THAT] = InfluSour(XP,YP,N);
  uinfty = 1;
  for j=1:N
  b(j,1) = uinfty*NHAT(j,1) +0*NHAT(j,2);
  end
Sources = AN\b; ut = AT*Sources - THAT(:,1);
cp = 1 - ut.^2; plot(XC,cp,'-ok',XP,YP,'k')
xlabel(' x '),ylabel(' Cp ')
```

3.5 EXERCISES

3.1 Define vorticity in a fluid and obtain an expression for it at a point with polar coordinates (r,θ), the motion being assumed two-dimensional. From the definition of a line vortex as irrotational flow in concentric circles determine the variation of velocity with radius and thus obtain the stream function (ψ), and the velocity potential (ϕ), for a line vortex. (U of L)

3.2 A sink of strength 120 $m^2\,s^{-1}$ is situated 2 m downstream from a source of equal strength in an irrotational uniform stream of 30 $m\,s^{-1}$. Find the fineness ratio of the oval formed by the streamline $\psi = 0$.
(*Answer:* 1.51) (CU)

3.3 A sink of strength 20 $m^2\,s^{-1}$ is situated 3 m upstream of a source of 40 $m^2\,s^{-1}$ in a uniform irrotational stream. It is found that, at a point 2.5 m equidistant from both source and sink, the local velocity is normal to the line joining the source and sink. Find the velocity at this point and the velocity of the undisturbed stream.
(*Answer:* 1.02 $m\,s^{-1}$, 2.29 $m\,s^{-1}$) (CU)

3.4 A line source of strength m and a sink of strength $2m$ are separated a distance c. Show that the field of flow consists in part of closed curves. Locate any stagnation points and sketch the field of flow. (U of L)

3.5 Derive the expression giving the stream function for irrotational flow of an incompressible fluid past a circular cylinder of infinite span. In this way, determine the position of generators on the cylinder at which the pressure is equal to that of the undisturbed stream.
(*Answer:* $\pm 30°$, $\pm 150°$) (U of L)

3.6 Determine the stream function for a two-dimensional source of strength m. Sketch the resultant field of flow due to three such sources, each of strength m, located at the vertices of an equilateral triangle. (U of L)

3.7 Derive the irrotational flow formula

$$p = p_0 = \frac{1}{2}\rho U^2(1 - 4\sin^2\theta)$$

giving the intensity of normal pressure p on the surface of a long circular cylinder, set at right angles to a stream of velocity U. The undisturbed static pressure in the fluid is p_0 and θ is the angular distance from the stagnation point. Describe briefly an experiment to test the accuracy of the formula and comment on the results obtained. (U of L)

3.8 A long right circular cylinder of diameter α m is set horizontally in a steady stream of velocity U $m\,s^{-1}$ and caused to rotate at ω rad s^{-1}. Obtain an expression in terms of ω and U for the ratio of the pressure difference between the top and bottom of the cylinder to the dynamic pressure of the stream. Describe briefly the behavior of the stagnation lines of such a system as ω increases from zero, keeping U constant.
(*Answer:* $8a\omega/U$) (CU)

3.9 A line source is immersed in a uniform stream. Show that the resultant flow, if irrotational, may represent the flow past a two-dimensional fairing. If a maximum thickness of the fairing is 0.15 m and the undisturbed velocity of the

stream is 6.0 m/s, determine the strength and location of the source. Also obtain an expression for the pressure at any point on the surface of the fairing, taking the pressure at infinity as a datum.
(*Answer:* 0.9 $m^2\ s^{-1}$, 0.0237 m) (U of L)

3.10 A long right circular cylinder of radius a m is held with its axis normal to an irrotational inviscid stream of U. Obtain an expression for the drag force acting on a unit length of the cylinder due to pressures exerted on the front half only.
(*Answer:* $-\rho U^2 a/3$) (CU)

3.11 Show that a velocity potential exists in a two-dimensional steady irrotational incompressible fluid motion. The stream function of a two-dimensional motion of an incompressible fluid is given by

$$\psi = \frac{a}{2}x^2 + bxy - \frac{c}{2}y^3$$

where a, b, and c are arbitrary constants. Show that, if the flow is irrotational, the lines of constant pressure never coincide with either the streamlines or the equipotential lines. Is this possible for rotational motion? (U of L)

3.12 State the stream function and velocity potential for each of the motions induced by a source, vortex, and doublet in a two-dimensional incompressible fluid. Show that a doublet may be regarded as either of the following:
(a) The limiting case of a source and sink.
(b) The limiting case of equal and opposite vortices, clearly indicating the direction of the resultant doublet. (U of L)

3.13 Define the stream function, the irrotational flow, and the velocity potential for two-dimensional motion of an incompressible fluid, indicating the conditions under which they exist. Determine the stream function for a point source of strength σ at the origin. Hence, or otherwise, show that, for the flow due to any number of sources at points on a circle, the circle is a streamline provided that the algebraic sum of the strengths of the sources is zero. (U of L)

3.14 A line vortex of strength Γ is mechanically fixed at the point $(l, 0)$ in space described by a Cartesian coordinate system. It is in an inviscid incompressible fluid at rest at infinity bounded by a plane wall coincident with the y-axis. Find the velocity in the fluid at the point $(0, y)$, and determine the force that acts on the wall (per unit depth) if the pressure on the other side of the wall is the same as at infinity. Bearing in mind that this must be equal and opposite to the force acting on a unit length of the vortex, show that your result is consistent with the Kutta-Joukowski theorem. (U of L)

3.15 Write the velocity potential for the two-dimensional flow about a circular cylinder with a circulation Γ in an otherwise uniform stream of velocity U. Hence show that the lift on a unit span of the cylinder is $\rho U \Gamma$. Produce a brief but plausible argument that the same result should hold for the lift on a cylinder of arbitrary shape, basing your argument on consideration of the flow at large distances from the cylinder. (U of L)

3.16 Define the terms *velocity potential*, *circulation*, and *vorticity* as used in two-dimensional fluid mechanics, and show how they are related. The velocity distribution in the laminar boundary layer of a wide flat plate is given by

$$u = u_0\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

where u_0 is the velocity at the edge of the boundary layer where y equals δ. Find the vorticity on the surface of the plate.
[*Answer:* $-(3/2)(u_0/\delta)$] (U of L)

3.17 A two-dimensional fluid motion is represented by a point vortex of strength Γ set at unit distance from an infinite straight boundary. Draw the streamlines and plot the velocity distribution on the boundary when $\Gamma = \pi$. (U of L)

3.18 The velocity components of a two-dimensional inviscid incompressible flow are given by

$$u = 2y - \frac{y}{\left(x^2+y^2\right)^{1/2}}, \quad v = -2x - \frac{x}{\left(x^2+y^2\right)^{1/2}}$$

Find the stream function and the vorticity, and sketch the streamlines.
[*Answer:* $\psi = x^2 + y^2 + (x^2+y^2)^{1/2}$, $\zeta = -(4 + 1/\sqrt{(x^2+y^2)})$] (U of L)

3.19 **(a)** Given that the velocity potential for a point source takes the form

$$\phi = -\frac{Q}{4\pi R}$$

where, in axisymmetric cylindrical coordinates (r,z), $R = \sqrt{z^2+r^2}$, show that, when a uniform stream U is superimposed on a point source located at the origin, there is a stagnation point located on the z-axis upstream of the origin at distance

$$a = \sqrt{\frac{Q}{4\pi U}}$$

(b) Given that, in axisymmetric spherical coordinates (R,φ), the stream function for the point source takes the form

$$\psi = -\frac{Q}{4\pi R}$$

show that the streamlines passing through the stagnation point found in (a) define a body of revolution given by

$$R^2 = \frac{2a^2\,(1+\cos\varphi)}{\sin^2\varphi}$$

Make a rough sketch of this body.