1 Newton's Equations in a Rotating Coordinate System

In classical mechanics, Newton's laws hold in all systems moving uniformly relative to each other (i.e., inertial systems) if they hold in one system. However, this is no longer valid if a system undergoes accelerations. The new relations are obtained by establishing the equations of motion in a fixed system and transforming them into the accelerated system.

We first consider the *rotation* of a coordinate system (x', y', z') about the origin of the inertial system (x, y, z) where the two coordinate origins coincide. The inertial system is denoted by L ("laboratory system") and the rotating system by M ("moving system").

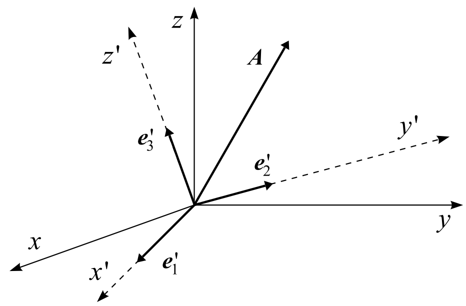

Fig. 1.1. Relative position of the coordinate systems x, y, z and x', y', z'

In the primed system the vector $\mathbf{A}(t) = A_1'\mathbf{e}_1' + A_2'\mathbf{e}_2' + A_3'\mathbf{e}_3'$ changes with time. For an observer resting in this system this can be represented as follows:

$$\left.\frac{d\mathbf{A}}{dt}\right|_M = \frac{dA_1'}{dt}\mathbf{e}_1' + \frac{dA_2'}{dt}\mathbf{e}_2' + \frac{dA_3'}{dt}\mathbf{e}_3'.$$

The index M means that the derivative is being calculated from the moving system. In the inertial system (x, y, z) $\mathbf{A}$ is also time dependent. Because of the rotation of the primed system the unit vectors $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ also vary with time; i.e., when differentiating the vector $\mathbf{A}$ from the inertial system, the unit vectors must be differentiated too:

$$\begin{aligned}\left.\frac{d\mathbf{A}}{dt}\right|_L &= \frac{dA_1'}{dt}\mathbf{e}_1' + \frac{dA_2'}{dt}\mathbf{e}_2' + \frac{dA_3'}{dt}\mathbf{e}_3' + A_1'\dot{\mathbf{e}}_1' + A_2'\dot{\mathbf{e}}_2' + A_3'\dot{\mathbf{e}}_3' \\ &= \left.\frac{d\mathbf{A}}{dt}\right|_M + A_1'\dot{\mathbf{e}}_1' + A_2'\dot{\mathbf{e}}_2' + A_3'\dot{\mathbf{e}}_3'.\end{aligned}$$

Generally the following holds: $(d/dt)(\mathbf{e}_\gamma' \cdot \mathbf{e}_\gamma') = \mathbf{e}_\gamma' \cdot \dot{\mathbf{e}}_\gamma' + \dot{\mathbf{e}}_\gamma' \cdot \mathbf{e}_\gamma' = (d/dt)(1) = 0$. Hence, $\mathbf{e}_\gamma' \cdot \dot{\mathbf{e}}_\gamma' = 0$. The derivative of a unit vector $\dot{\mathbf{e}}_\gamma$ is always orthogonal to the vector

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itself. Therefore the derivative of a unit vector can be written as a linear combination of the two other unit vectors:

$$\dot{\mathbf{e}}_1' = a_1\mathbf{e}_2' + a_2\mathbf{e}_3',$$
$$\dot{\mathbf{e}}_2' = a_3\mathbf{e}_1' + a_4\mathbf{e}_3',$$
$$\dot{\mathbf{e}}_3' = a_5\mathbf{e}_1' + a_6\mathbf{e}_2'.$$

Only 3 of these 6 coefficients are independent. To show this, we first differentiate $\mathbf{e}_1' \cdot \mathbf{e}_2' = 0$, and obtain

$$\dot{\mathbf{e}}_1' \cdot \mathbf{e}_2' = -\dot{\mathbf{e}}_2' \cdot \mathbf{e}_1'.$$

Multiplying $\dot{\mathbf{e}}_1' = a_1\mathbf{e}_2' + a_2\mathbf{e}_3'$ by $\mathbf{e}_2'$ and correspondingly $\dot{\mathbf{e}}_2' = a_3\mathbf{e}_1' + a_4\mathbf{e}_3'$ by $\mathbf{e}_1'$, one obtains

$$\mathbf{e}_2' \cdot \dot{\mathbf{e}}_1' = a_1 \quad \text{and} \quad \mathbf{e}_1' \cdot \dot{\mathbf{e}}_2' = a_3,$$

and hence $a_3 = -a_1$. Analogously one finds $a_6 = -a_4$ and $a_5 = -a_2$.

The derivative of the vector $\mathbf{A}$ in the inertial system can now be written as follows:

$$\begin{aligned}
\left.\frac{d\mathbf{A}}{dt}\right|_L &= \left.\frac{d\mathbf{A}}{dt}\right|_M + A_1'(a_1\mathbf{e}_2' + a_2\mathbf{e}_3') + A_2'(-a_1\mathbf{e}_1' + a_4\mathbf{e}_3') + A_3'(-a_2\mathbf{e}_1' - a_4\mathbf{e}_2') \\
&= \left.\frac{d\mathbf{A}}{dt}\right|_M + \mathbf{e}_1'(-a_1A_2' - a_2A_3') + \mathbf{e}_2'(a_1A_1' - a_4A_3') + \mathbf{e}_3'(a_2A_1' + a_4A_2').
\end{aligned}$$

From the evaluation rule for the vector product,

$$\begin{aligned}
\mathbf{C} \times \mathbf{A} &= \begin{vmatrix} \mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3' \\ C_1 & C_2 & C_3 \\ A_1' & A_2' & A_3' \end{vmatrix} \\
&= \mathbf{e}_1'(C_2A_3' - C_3A_2') - \mathbf{e}_2'(C_1A_3' - C_3A_1') + \mathbf{e}_3'(C_1A_2' - C_2A_1'),
\end{aligned}$$

it follows by setting $\mathbf{C} = (a_4, -a_2, a_1)$ that

$$\left.\frac{d\mathbf{A}}{dt}\right|_L = \left.\frac{d\mathbf{A}}{dt}\right|_M + \mathbf{C} \times \mathbf{A}.$$

We still have to show the physical meaning of the vector $\mathbf{C}$. For this purpose we consider the special case $d\mathbf{A}/dt|_M = 0$; i.e., the derivative of the vector $\mathbf{A}$ in the moving system vanishes. $\mathbf{A}$ moves (rotates) with the moving system; it is tightly "mounted" in the system. Let φ be the angle between the axis of rotation (in our special case the z-axis) and $\mathbf{A}$. The component parallel to the angular velocity $\boldsymbol{\omega}$ is not changed by the rotation.

The change of $\mathbf{A}$ in the laboratory system is then given by

$$dA = \omega\, dt\, A \sin\varphi \quad \text{or} \quad \left.\frac{dA}{dt}\right|_L = \omega A \sin\varphi.$$

This can also be written as

$$\left.\frac{d\mathbf{A}}{dt}\right|_L = \boldsymbol{\omega} \times \mathbf{A}.$$

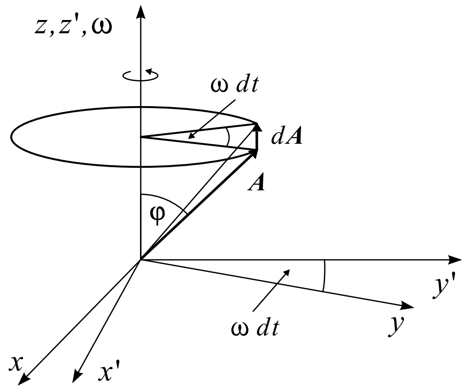

Fig. 1.2. Change of an arbitrary vector **A** tightly fixed to a rotating system

The orientation of $(\boldsymbol{\omega} \times \mathbf{A})\,dt$ also coincides with $d\mathbf{A}$ (see Fig. 1.2). Since the (fixed) vector **A** can be chosen arbitrarily, the vector **C** must be identical with the angular velocity $\boldsymbol{\omega}$ of the rotating system M. By insertion we obtain

$$\left.\frac{d\mathbf{A}}{dt}\right|_L = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega} \times \mathbf{A}. \tag{1.1}$$

This can also be seen as follows (see Fig. 1.3): If the rotational axis of the primed system coincides during a time interval dt with one of the coordinate axes of the nonprimed system, e.g., $\boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3$, then

$$\dot{\mathbf{e}}'_1 = \dot{\varphi}\mathbf{e}'_2 \quad \text{and} \quad \dot{\mathbf{e}}'_2 = -\dot{\varphi}\mathbf{e}'_1,$$

i.e.,

$$a_1 = \dot{\varphi}, \quad a_2 = a_4 = 0, \quad \text{and hence} \quad \mathbf{C} = \dot{\varphi}\mathbf{e}'_3 = \boldsymbol{\omega}.$$

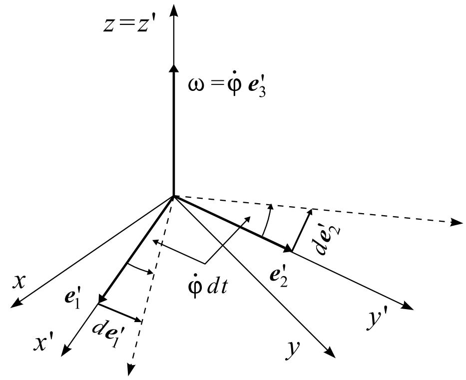

Fig. 1.3. $|d\mathbf{e}'_1| = |d\mathbf{e}'_2| = \dot{\varphi} \cdot dt$

In the general case $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$, one decomposes $\boldsymbol{\omega} = \sum \boldsymbol{\omega}_i$ with $\boldsymbol{\omega}_i = \omega_i\mathbf{e}_i$, and by the preceding consideration one finds

$$\mathbf{C}_i = \boldsymbol{\omega}_i; \quad \text{i.e.,} \quad \mathbf{C} = \sum_i \mathbf{C}_i = \sum_i \boldsymbol{\omega}_i = \boldsymbol{\omega}.$$

1.1 Introduction of the Operator $\widehat{D}$

To shorten the expression $\partial F(x,\ldots,t)/\partial t = \partial F/\partial t$, we introduce the operator $\widehat{D} = \partial/\partial t$. The inertial system and the accelerated system will be distinguished by the indices L and M, so that

$$\widehat{D}_L = \left.\frac{\partial}{\partial t}\right|_L \quad \text{and} \quad \widehat{D}_M = \left.\frac{\partial}{\partial t}\right|_M .$$

The equation

$$\left.\frac{d\mathbf{A}}{dt}\right|_L = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A}$$

then simplifies to

$$\widehat{D}_L\mathbf{A} = \widehat{D}_M\mathbf{A} + \boldsymbol{\omega}\times\mathbf{A}.$$

If the vector $\mathbf{A}$ is omitted, the equation is called an operator equation

$$\widehat{D}_L = \widehat{D}_M + \boldsymbol{\omega}\times,$$

which can operate on arbitrary vectors.

EXAMPLE

1.1 Angular Velocity Vector $\boldsymbol{\omega}$

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_L = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M + \boldsymbol{\omega}\times\boldsymbol{\omega}.$$

Since $\boldsymbol{\omega}\times\boldsymbol{\omega} = 0$, it follows that

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_L = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M .$$

These two derivatives are evidently identical for all vectors that are parallel to the rotational plane, since then the vector product vanishes.

EXAMPLE

1.2 Position Vector r

$$\left.\frac{d\mathbf{r}}{dt}\right|_L = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{r},$$

in operator notation this becomes

$$\widehat{D}_L\mathbf{r} = \widehat{D}_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r},$$

where $(d\mathbf{r}/dt)|_M$ is called the *virtual* velocity and $(d\mathbf{r}/dt)|_M + \boldsymbol{\omega} \times \mathbf{r}$ the *true* velocity. The term $\boldsymbol{\omega} \times \mathbf{r}$ is called the *rotational velocity.* *Example 1.2*

1.2 Formulation of Newton's Equation in the Rotating Coordinate System

Newton's law $m\ddot{\mathbf{r}} = \mathbf{F}$ holds only in the inertial system. In accelerated systems, there appear additional terms. First we consider again a pure rotation.

For the acceleration we have

$$\begin{aligned}\ddot{\mathbf{r}}_L = \frac{d}{dt}(\dot{\mathbf{r}})_L &= \widehat{D}_L(\widehat{D}_L\mathbf{r}) = (\widehat{D}_M + \boldsymbol{\omega}\times)(\widehat{D}_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \widehat{D}_M^2\mathbf{r} + \widehat{D}_M(\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \widehat{D}_M\mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \widehat{D}_M^2\mathbf{r} + (\widehat{D}_M\boldsymbol{\omega}) \times \mathbf{r} + 2\boldsymbol{\omega} \times \widehat{D}_M\mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).\end{aligned}$$

We replace the operator by the differential quotient:

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_L = \left.\frac{d^2\mathbf{r}}{dt^2}\right|_M + \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M \times \mathbf{r} + 2\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{1.2}$$

The expression $(d\boldsymbol{\omega}/dt)|_M \times \mathbf{r}$ is called the *linear acceleration,* $2\boldsymbol{\omega} \times (d\mathbf{r}/dt)|_M$ the *Coriolis acceleration*, and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ the *centripetal acceleration.*

Multiplication by the mass m yields the force $\mathbf{F}$:

$$m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_M + m\left.\frac{d\boldsymbol{\omega}}{dt}\right|_M \times \mathbf{r} + 2m\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_M + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{F}.$$

The basic equation of mechanics in the rotating coordinate system therefore reads (with the index M being omitted):

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{1.3}$$

The additional terms on the right-hand side of (1.3) are *virtual forces* of a dynamical nature, but actually they are due to the acceleration term. For experiments on the earth the additional terms can often be neglected, since the angular velocity of the earth $\omega = 2\pi/T$ $(T = 24\ \text{h})$ is only $7.27 \cdot 10^{-5}\ \text{s}^{-1}$.

1.3 Newton's Equations in Systems with Arbitrary Relative Motion

We now drop the condition that the origins of the two coordinate systems coincide. The general motion of a coordinate system is composed of a rotation of the system and a translation of the origin. If $\mathbf{R}$ points to the origin of the primed system, then the position vector in the nonprimed system is $\mathbf{r} = \mathbf{R} + \mathbf{r}'$.

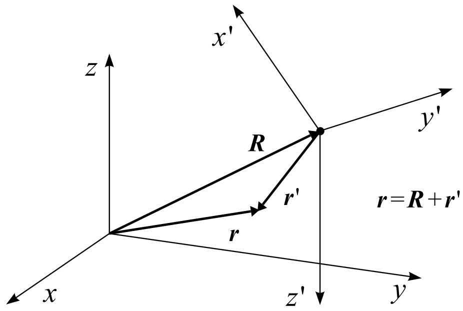

Fig. 1.4. Relative position of the coordinate systems x, y, z and x', y', z'

For the velocity we have $\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\mathbf{r}}'$, and in the inertial system we have as before

$$m \frac{d^2\mathbf{r}}{dt^2}\bigg|_L = \mathbf{F}\bigg|_L = \mathbf{F}.$$

By inserting $\mathbf{r}$ and differentiating, we obtain

$$m \frac{d^2\mathbf{r}'}{dt^2}\bigg|_L + m \frac{d^2\mathbf{R}}{dt^2}\bigg|_L = \mathbf{F}.$$

The transition to the accelerated system is performed as above (see (1.3)), but here we still have the additional term $m\ddot{\mathbf{R}}$:

$$m \frac{d^2\mathbf{r}'}{dt^2}\bigg|_M = \mathbf{F} - m \frac{d^2\mathbf{R}}{dt^2}\bigg|_L - m \frac{d\boldsymbol{\omega}}{dt}\bigg|_M \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}_M - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'). \quad (1.4)$$