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ictp *lecture notes*

TOPOLOGY OF HIGH-DIMENSIONAL MANIFOLDS

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Number 1

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ICTP *Lecture Notes*

TOPOLOGY OF HIGH-DIMENSIONAL MANIFOLDS

21 May - 8 June 2001

Editors

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TOPOLOGY OF HIGH-DIMENSIONAL MANIFOLDS – First edition

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PREFACE

One of the main missions of the Abdus Salam International Centre for Theoretical Physics in Trieste, Italy, founded in 1964 by Abdus Salam, is to foster the growth of advanced studies and research in developing countries. To this aim, the Centre organizes a large number of schools and workshops in a great variety of physical and mathematical disciplines.

Since unpublished material presented at the meetings might prove of great interest also to scientists who did not take part in the schools the Centre has decided to make it available through a new publication titled ICTP Lecture Note Series. It is hoped that this formally structured pedagogical material in advanced topics will be helpful to young students and researchers, in particular to those working under less favourable conditions.

The Centre is grateful to all lecturers and editors who kindly authorize the ICTP to publish their notes as a contribution to the series.

Since the initiative is new, comments and suggestions are most welcome and greatly appreciated. Information can be obtained from the Publications Section or by e-mail to "`pub_off@ictp.trieste.it`". The series is published in house and also made available on-line via the ICTP web site: "`http://www.ictp.trieste.it`".

M.A. Virasoro
Director

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Introduction

The School on High-Dimensional Manifold Topology took place at the Abdus Salam ICTP, Trieste from 21 May 2001 to 8 June 2001 under the direction of F.T. Farrell (State University of New York at Binghamton) and W. Lück (Westfälische Wilhelms-Universität Münster). The local organizer was Lothar Göttsche (ICTP).

The focus of the school was on the classification of manifolds and related aspects of K-theory, geometry, and operator theory. The topics covered included: surgery theory, algebraic K- and L-theory, controlled topology, homology manifolds, exotic aspherical manifolds, homeomorphism and diffeomorphism groups, and scalar curvature. Several conjectures were discussed including those due to: Borel, Novikov, Gromov-Lawson-Rosenberg, Baum-Connes, and Farrell-Jones. The school consisted of 2 weeks of lecture courses and one week of conference. This two-part lecture notes volume contains the notes of most of the lecture courses. The electronic version of these lecture notes is available at

http://www.ictp.trieste.it/~pub_off/lectures/

The proceedings of the conference will be published as a separate monograph by World Scientific Publishing Co.

The school was financially supported by the ICTP and by grants from the European commission, the National Science Foundation of the USA, and the Sonderforschungsbereich 478. We are very thankful for this support.

We, Tom Farrell and Wolfgang Lück, would like to take this opportunity to thank the Abdus Salam ICTP and Lothar Göttsche for the hospitality and excellent organization. We are grateful to all the lectures and speakers of the conference for their contribution to the success of the school.

F. Thomas Farrell
Wolfgang Lück
May, 2002

A Basic Introduction to Surgery Theory

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Lectures given at the
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Preface

This manuscript contains extended notes of the lectures presented by the author at the summer school "High-dimensional Manifold Theory" in Trieste in May/June 2001. It is written not for experts but for talented and well educated graduate students or Ph.D. students who have some background in algebraic and differential topology. Surgery theory has been and is a very successful and well established theory. It was initiated and developed by Browder, Kervaire, Milnor, Novikov, Sullivan, Wall and others and is still a very active research area. The idea of these notes is to give young mathematicians the possibility to get access to the field and to see at least a small part of the results which have grown out of surgery theory. Of course there are other good text books and survey articles about surgery theory, some of them are listed in the references.

The Chapters 1 and 2 contain interesting and beautiful results such as the s-Cobordism Theorem and the classification of lens spaces including their illuminating proofs. If one wants to start with the surgery machinery immediately, one may skip these chapters and pass directly to Chapters 3, 4 and 5. As an application we present the classification of homotopy spheres in Chapter 6. Chapters 7 and 8 contain material which is directly related to the main topic of the summer school.

I thank the members of the topology group at Münster and the participants of the summer school who made very useful comments and suggestions, in particular C.L. Douglas, J. Verrel, T. A. Kro, R. Sauer and M. Szymik. I thank the ICTP for its hospitality and financial support for the school and the conference.

Contents

Chapter 1

The s-Cobordism Theorem

Introduction

In this chapter we want to discuss and prove the following result

Theorem 1.1 (s-Cobordism Theorem) *Let M_0 be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M_0)$. Then*

1. *Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ vanishes;*

2. *For any $x \in \mathrm{Wh}(\pi)$ there is an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 with $\tau(W, M_0) = x \in \mathrm{Wh}(\pi)$;*

3. *The function assigning to an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of h-cobordisms over M_0 to the Whitehead group $\mathrm{Wh}(\pi)$.*

Here are some explanations. An *n-dimensional cobordism* (sometimes also called just bordism) $(W; M_0, f_0, M_1, f_1)$ consists of a compact oriented n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0 and M_1, a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ of the boundary ∂W of W and orientation preserving diffeomorphisms $f_0 \colon M_0 \to \partial W_0$ and $f_1 \colon M_1^- \to \partial W_1$. Here and in the sequel we denote by M_1^- the manifold M_1 with the reversed orientation and we use on ∂W the orientation with respect to the decomposition $T_x W = T_x \partial W \oplus \mathbb{R}$ coming from an inward normal field for the boundary. If we equip D^2 with the standard orientation coming from the standard orientation on $\mathbb{R}^2$, the induced orientation on $S^1 = \partial D^2$ corresponds to the anti-clockwise orientation on S^1. If we want to specify M_0, we say that W is a *cobordism over* M_0. If $\partial_0 W = M_0$, $\partial_1 W = M_1^-$ and f_0 and f_1 are given by the identity or if f_0 and f_1 are obvious from the context, we briefly write $(W; \partial_0 W, \partial_1 W)$. Two cobordisms (W, M_0, f_0, M_1, f_1) and $(W', M_0, f_0', M_1', f_1')$ over M_0 are *diffeomorphic relative* M_0 if there is an orientation preserving diffeomorphism $F \colon W \to W'$ with $F \circ f_0 = f_0'$. We call an h-cobordism over M_0 *trivial*, if it is diffeomorphic relative M_0 to the trivial h-cobordism $(M_0 \times [0,1]; M_0 \times \{0\}, (M_0 \times \{1\})^-)$. Notice that the choice of the diffeomorphisms f_i do play a role although they are often suppressed in the notation. We call a cobordism $(W; M_0, f_0, M_1, f_1)$ an *h-cobordism*, if the inclusions $\partial_i W \to W$ for $i = 0, 1$ are homotopy equivalences.

We will later see that the Whitehead group of the trivial group vanishes. Thus the s-Cobordism Theorem 1.1 implies

Theorem 1.2 (h-Cobordism Theorem) *Any h-cobordism $(W; M_0, f_0, M_1, f_1)$ over a simply connected closed n-dimensional manifold M_0 with* $\dim(W) \geq 6$ *is trivial.*

Theorem 1.3 (Poincaré Conjecture) *The Poincaré Conjecture is true for a closed n-dimensional manifold M with* $\dim(M) \geq 5$, *namely, if M is simply connected and its homology $H_p(M)$ is isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$, then M is homeomorphic to S^n.*

Proof : We only give the proof for $\dim(M) \geq 6$. Since M is simply connected and $H_*(M) \cong H_*(S^n)$, one can conclude from the Hurewicz Theorem and Whitehead Theorem [121, Theorem IV.7.13 on page 181 and Theorem IV.7.17 on page 182] that there is a homotopy equivalence $f \colon M \to S^n$. Let $D_i^n \subset M$ for $i = 0, 1$ be two embedded disjoint disks. Put $W = M - (\operatorname{int}(D_0^n) \coprod \operatorname{int}(D_1^n))$. Then W turns out to be a simply connected h-cobordism. Hence we can find a diffeomorphism $F \colon (\partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$ which is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$ and induces some (unknown) diffeomorphism $f_1 \colon \partial D_0^n \times \{1\} \to \partial D_1^n$. By the *Alexander trick* one can extend $f_1 \colon \partial D_0^n = \partial D_0^n \times \{1\} \to \partial D_1^n$ to a homeomorphism $\overline{f_1} \colon D_0^n \to D_1^n$. Namely, any homeomorphism $f \colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f} \colon D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h \colon D_0^n \times \{0\} \cup_{i_0}$

$\partial D_0^n \times [0,1] \cup_{i_1} D_0^n \times \{1\} \to M$ for the canonical inclusions $i_k \colon \partial D_0^n \times \{k\} \to \partial D_0^n \times [0,1]$ for $k = 0,1$ by $h|_{D_0^n \times \{0\}} = \mathrm{id}$, $h|_{\partial D_0^n \times [0,1]} = F$ and $h|_{D_0^n \times \{1\}} = \overline{f_1}$. Since the source of h is obviously homeomorphic to S^n, Theorem 1.3 follows.

In the case $\dim(M) = 5$ one uses the fact that M is the boundary of a contractible 6-dimensional manifold W and applies the s-cobordism theorem to W with an embedded disc removed. ∎

Remark 1.4 Notice that the proof of the Poincaré Conjecture in Theorem 1.3 works only in the topological category but not in the smooth category. In other words, we cannot conclude the existence of a diffeomorphism $h \colon S^n \to M$. The proof in the smooth case breaks down when we apply the Alexander trick. The construction of $\overline{f}$ given by coning f yields only a homeomorphism $\overline{f}$ and not a diffeomorphism even if we start with a diffeomorphism f. The map $\overline{f}$ is smooth outside the origin of D^n but not necessarily at the origin. We will see that not every diffeomorphism $f \colon S^{n-1} \to S^{n-1}$ can be extended to a diffeomorphism $D^n \to D^n$ and that there exist so called *exotic spheres*, i.e. closed manifolds which are homeomorphic to S^n but not diffeomorphic to S^n. The classification of these exotic spheres is one of the early very important achievements of surgery theory and one motivation for its further development (see Chapter 6).

Remark 1.5 In some sense the s-Cobordism Theorem 1.1 is one of the first theorems, where diffeomorphism classes of certain manifolds are determined by an algebraic invariant, namely the Whitehead torsion. Moreover, the Whitehead group $\mathrm{Wh}(\pi)$ depends only on the fundamental group $\pi = \pi_1(M_0)$, whereas the diffeomorphism classes of h-cobordisms over M_0 a priori depend on M_0 itself. The s-Cobordism Theorem 1.1 is one step in a program to decide whether two closed manifolds M and N are diffeomorphic what is in general a very hard question. The idea is to construct an h-cobordism $(W; M, f, N, g)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M what implies that M and N are diffeomorphic. So the *surgery program* would be:

1. Construct a homotopy equivalence $f \colon M \to N$;

2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id}) \colon (W; M, N) \to (N \times [0,1], N \times \{0\}, N \times \{1\})$;

3. Modify W and F relative boundary by so called surgery such that F becomes a homotopy equivalence and thus W becomes an h-cobordism. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial.

The advantage of this approach will be that it can be reduced to problems in homotopy theory and algebra which can sometimes be handled by well-known techniques. In particular one will get sometimes computable obstructions for two homotopy equivalent manifolds to be diffeomorphic. Often surgery theory has proved to be very useful when one wants to distinguish two closed manifolds which have very similar properties. The classification of homotopy spheres (see Chapter 6) is one example. Moreover, surgery techniques can be applied to problems which are of different nature than of diffeomorphism or homeomorphism classifications.

In this chapter we want to present the proof of the s-Cobordism Theorem and explain why the notion of Whitehead torsion comes in. We will encounter a typical situation in mathematics. We will consider an h-cobordism and try to prove that it is trivial. We will introduce modifications which we can apply to a handlebody decomposition without changing the diffeomorphism type and which are designed to reduce the number of handles. If we could get rid of all handles, the h-cobordism would be trivial. When attempting to cancel all handles, we run into an algebraic difficulty. A priori this difficulty could be a lack of a good idea or technique. But it will turn out to be the principal obstruction and lead us to the definition of the Whitehead torsion and Whitehead group.

The rest of this Chapter is devoted to the proof of the s-cobordism Theorem 1.1. Its proof is interesting and illuminating and it motivates the definition of Whitehead torsion. But we mention that it is not necessary to go through it in order to understand the following chapters.

1.1 Handlebody Decompositions

In this section we explain basic facts about handles and handlebody decompositions.

Definition 1.6 *The* n-dimensional handle of index q *or briefly* q*-handle is* $D^q \times D^{n-q}$. *Its* core *is* $D^q \times \{0\}$. *The* boundary of the core *is* $S^{q-1} \times \{0\}$. *Its* cocore *is* $\{0\} \times D^{n-q}$ *and its* transverse sphere *is* $\{0\} \times S^{n-q-1}$.

Let $(M, \partial M)$ be an n-dimensional manifold with boundary ∂M. If $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial M$ is an embedding, then we say that the manifold $M + (\phi^q)$ defined by $M \cup_{\phi^q} D^q \times D^{n-q}$ is obtained from M by attaching a handle of index q by ϕ^q.

Obviously $M + (\phi^q)$ carries the structure of a topological manifold. To get a smooth structure, one has to use the technique of straigthening the angle to get rid of the corners at the place, where the handle is glued to M. The boundary $\partial(M + (\phi^q))$ can be described as follows. Delete from ∂M the interior of the image of ϕ^q. We obtain a manifold with boundary together with a diffeomorphism from $S^{q-1} \times S^{n-q-1}$ to its boundary induced by $\phi^q|_{S^{q-1} \times S^{n-q-1}}$. If we use this diffeomorphism to glue $D^q \times S^{n-q-1}$ to it, we obtain a closed manifold, namely, $\partial(M + (\phi^q))$.

Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then we want to construct W from $\partial_0 W \times [0,1]$ by attaching handles as follows. Notice that the following construction will not change $\partial_0 W = \partial_0 W \times \{0\}$. If $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W$ is an embedding, we get by attaching a handle the compact manifold $W_1 = \partial_0 W \times [0,1] + (\phi^q)$ which is given by $W \cup_{\phi^q} D^q \times D^{n-q}$. Its boundary is a disjoint sum $\partial_0 W_1 \coprod \partial_1 W_1$, where $\partial_0 W_1$ is the same as $\partial_0 W$. Now we can iterate this process, where we attach a handle to $\partial_1 W_1$. Thus we obtain a compact manifold with boundary

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \ldots + (\phi_r^{q_r}),$$

whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$, where $\partial_0 W$ is just $\partial_0 W \times \{0\}$. We call such a description of W as above a *handlebody decomposition* of W relative $\partial_0 W$. We get from Morse theory [54, Chapter 6], [82, part I].

Lemma 1.7 *Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then W possesses a handlebody decomposition relative $\partial_0 W$, i.e. W is up to diffeomorphism relative $\partial_0 W = \partial_0 W \times \{0\}$ of the form*

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \ldots + (\phi_r^{q_r}).$$

If we want to show that W is diffeomorphic to $\partial_0 W \times [0,1]$ relative $\partial_0 W = \partial_0 W \times \{0\}$, we must get rid of the handles. For this purpose we have to find possible modifications of the handlebody decomposition which reduce the number of handles without changing the diffeomorphism type of W relative $\partial_0 W$.

Lemma 1.8 (Isotopy lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. If $\phi^q, \psi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ are isotopic embeddings, then there is a diffeomorphism $W + (\phi^p) \to W + (\psi^q)$ relative $\partial_0 W$.*

Proof : Let $i \colon S^{q-1} \times D^{n-q} \times [0,1] \to \partial_1 W$ be an isotopy from ϕ^q to ψ^q. Then one can find a diffeotopy $H \colon W \times [0,1] \to W$ with $H_0 = \mathrm{id}_W$ such that the composition of H with $\phi^q \times \mathrm{id}_{[0,1]}$ is i and H is stationary on $\partial_0 W$ [54, Theorem 1.3 in Chapter 8 on page 184]. Thus $H_1 \colon W \to W$ is a diffeomorphism relative $\partial_0 W$ and satisfies $H_1 \circ \phi^q = \psi^q$. It induces a diffeomorphism $W + (\phi^p) \to W + (\psi^q)$ relative $\partial_0 W$. ■

Lemma 1.9 (Diffeomorphism lemma) *Let W resp. W' be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$ resp. $\partial_0 W' \coprod \partial_1 W'$. Let $F \colon W \to W'$ be a diffeomorphism which induces a diffeomorphism $f_0 \colon \partial_0 W \to \partial_0 W'$. Let $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Then there is an embedding $\overline{\phi}^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W'$ and a diffeomorphism $F' \colon W + (\phi^q) \to W' + (\overline{\phi}^q)$ which induces f_0 on $\partial_0 W$.*

Proof : Put $\overline{\phi}^q = F \circ \phi^q$. ■

Lemma 1.10 *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Suppose that $V = W + (\psi^r) + (\phi^q)$ for $q \le r$. Then V is diffeomorphic relative $\partial_0 W$ to $V' = W + (\overline{\phi}^q) + (\psi^r)$ for an appropriate $\overline{\phi}^q$.*

Proof : By transversality and the assumption $(q-1) + (n-1-r) < n-1$ we can show that the embedding $\phi^q|_{S^{q-1} \times \{0\}} \colon S^{q-1} \times \{0\} \to \partial_1(W + (\psi^r))$ is isotopic to an embedding which does not meet the transverse sphere of the handle (ψ^r) attached by ψ^r [54, Theorem 2.3 in Chapter 3 on page 78]. This isotopy can be embedded in a diffeotopy on $\partial_1(W + (\psi^r))$. Thus the embedding $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1(W + (\psi^r))$ is isotopic to an embedding whose restriction to $S^{q-1} \times \{0\}$ does not meet the transverse sphere of the handle (ψ^r). Since we can isotope an embedding $S^{q-1} \times D^{n-q} \to W + (\psi^r)$ such that its image becomes arbitrary close to the image of $S^{q-1} \times \{0\}$, we can isotope $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1(W + (\psi^r))$ to an embedding which does not meet a closed neighborhood $U \subset \partial_1(W + (\psi^r))$ of the transverse sphere of the handle (ψ^r). There is an obvious diffeotopy on $\partial_1(W + (\psi^r))$ which is stationary on the transverse sphere of (ψ^r) and moves any point on $\partial_1(W + (\psi^r))$ which belongs to the handle (ψ^r) but not to U to a point outside the handle (ψ^r). Thus we can find an isotopy of ϕ^q to an embedding $\overline{\phi}^p$ which does not meet the handle (ψ^r) at all. Obviously $W+(\psi^r)+(\overline{\phi}^q)$ and $W + (\overline{\phi}^q) + (\psi^r)$ agree. By the Isotopy Lemma 1.8 there is a diffeomorphism relative $\partial_0 W$ from $W + (\psi^r) + (\overline{\phi}^q)$ to $W + (\psi^r) + (\phi^q)$. ■

Example 1.11 Here is a standard situation, where attaching first a q-handle and then a $(q+1)$-handle does not change the diffeomorphism type of an n-dimensional compact manifold W with the disjoint union $\partial_0 W \coprod \partial_1 W$ as boundary ∂W. Let $0 \le q \le n-1$. Consider an embedding

$$\mu\colon S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times S_+^{n-1-q}} D^q \times S_+^{n-1-q} \to \partial_1 W,$$

where S_+^{n-1-q} is the upper hemisphere in $S^{n-1-q} = \partial D^{n-q}$. Notice that the source of μ is diffeomorphic to D^{n-1}. Let $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be its restriction to $S^{q-1} \times D^{n-q}$. Let $\phi_+^{q+1}\colon S_+^q \times S_+^{n-q-1} \to \partial_1(W + (\phi^q))$ be the embedding which is given by

$$S_+^q \times S_+^{n-q-1} = D^q \times S_+^{n-q-1} \subset D^q \times S^{n-q-1} = \partial(\phi^q) \subset \partial_1(W + (\phi^q)).$$

It does not meet the interior of W. Let $\phi_-^{q+1}\colon S_-^q \times S_+^{n-1-q} \to \partial_1(W \cup (\phi^q))$ be the embedding obtained from μ by restriction to $S_-^q \times S_+^{n-1-q} = D^q \times S_+^{n-1-q}$. Then ϕ_-^{q+1} and ϕ_+^{q+1} fit together to yield an embedding $\psi^{q+1}\colon S^q \times D^{n-q-1} = S_-^q \times S_+^{n-q-1} \cup_{S^{q-1} \times S_+^{n-q-1}} S_+^q \times S_+^{n-q-1} \to \partial_1(W + (\phi^q))$. Then it is not difficult to check that $W + (\phi^q) + (\psi^{q+1})$ is diffeomorphic relative $\partial_0 W$ to W since up to diffeomorphism $W + (\phi^q) + (\psi^{q+1})$ is obtained from W by taking the boundary connected sum of W and D^n along the embedding μ of $D^{n-1} = S_+^{n-1} = S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times S_+^{n-1-q}} D^q \times S_+^{n-1-q}$ into $\partial_1 W$.

This cancellation of two handles of consecutive index can be generalized as follows.

Lemma 1.12 (Cancellation Lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Let $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Let $\psi^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ be an embedding. Suppose that $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of the handle (ϕ^q) and meets the transverse sphere in exactly one point. Then there is a diffeomorphism relative $\partial_0 W$ from W to $W + (\phi^q) + (\psi^{q+1})$.*

Proof : Given any neighborhood $U \subset \partial(\phi^q)$ of the tranverse sphere of (ϕ^q), there is an obvious diffeotopy on $\partial_1(W + (\phi^q))$ which is stationary on the transverse sphere of (ϕ^q) and moves any point on $\partial_1(W+(\phi^q))$ which belongs to the handle (ϕ^q) but not to U to a point outside the handle (ϕ^q). Thus we can achieve that ψ^{q+1} maps the lower hemisphere $S^q_- \times \{0\}$ to points outside (ϕ^q) and is on the upper hemisphere $S^q_+ \times \{0\}$ given by the obvious inclusion $D^q \times \{x\} \to D^p \times D^{n-q} = (\phi^q)$ for some $x \in S^{n-q-1}$ and the obvious identification of $S^q_+ \times \{0\}$ with $D^q \times \{x\}$. Now it is not hard to construct an diffeomorphism relative $\partial_0 W$ from $W + (\phi^q) + (\psi^{q+1})$ to W modelling the standard situation of Example 1.11. ■

The Cancellation Lemma 1.12 will be our only tool to reduce the number of handles. Notice that one can never get rid of one handle alone, there must always be involved at least two handles simultaneously. The reason is that the Euler characteristic $\chi(W, \partial_0 W)$ is independent of the handle decomposition and can be computed by $\sum_{q\geq 0}(-1)^q \cdot p_q$, where p_q is the number of q-handles (see Section 1.2).

We call an embedding $S^q \times D^{n-q} \to M$ for $q < n$ into an n-dimensional manifold *trivial* if it can be written as the composition of an embedding $D^n \to W$ and a fixed standard embedding $S^q \times D^{n-q} \to D^n$. We call an embedding $S^q \to M$ for $q < n$ *trivial* if it can be extended to a trivial embedding $S^q \times D^{n-q} \to M$. We conclude from the Cancellation Lemma 1.12

Lemma 1.13 *Let $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be a trivial embedding. Then there is an embedding $\phi^{q+1} \colon S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ such that W and $W + (\phi^q) + (\phi^{q+1})$ are diffeomorphic relative $\partial_0 W$.*

Consider a compact n-dimensional manifold W whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$. In view of Lemma 1.7 and Lemma 1.10 we can write it

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \ldots + \sum_{i=1}^{p_n} (\phi_i^n), \qquad (1.14)$$

where $\cong$ means diffeomorphic relative $\partial_0 W$.

Notation 1.15 *Put for* $-1 \leq q \leq n$

$$\begin{aligned} W_q &:= \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \ldots + \sum_{i=1}^{p_q}(\phi_i^q); \\ \partial_1 W_q &:= \partial W_q - \partial_0 W \times \{0\}; \\ \partial_1^\circ W_q &:= \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \phi_i^{q+1}(S^q \times \operatorname{int}(D^{n-1-q})). \end{aligned}$$

Notice for the sequel that $\partial_1^\circ W_q \subset \partial_1 W_{q+1}$.

Lemma 1.16 (Elimination Lemma) *Fix an integer* q *with* $1 \leq q \leq n-3$. *Suppose that* $p_j = 0$ *for* $j < q$, *i.e.* W *looks like*

$$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q}(\phi_i^q) + \sum_{i=1}^{p_{q+1}}(\phi_i^{q+1}) + \ldots + \sum_{i=1}^{p_n}(\phi_i^n).$$

Fix an integer i_0 *with* $1 \leq i_0 \leq p_q$. *Suppose that there is an embedding* $\psi^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ *with the following properties:*

1. $\psi^{q+1}|_{S^q \times \{0\}}$ *is isotopic in* $\partial_1 W_q$ *to an embedding* $\psi_1^{q+1}\colon S^q \times \{0\} \to \partial_1 W_q$ *which meets the transverse sphere of the handle* $(\phi_{i_0}^q)$ *transversally and in exactly one point and is disjoint from to the transverse sphere of* ϕ_i^q *for* $i \neq i_0$;

2. $\psi^{q+1}|_{S^q \times \{0\}}$ *is isotopic in* $\partial_1 W_{q+1}$ *to a trivial embedding* $\psi_2^{q+1}\colon S^q \times \{0\} \to \partial_1 W_{q+1}$.

Then W *is diffeomorphic relative* $\partial_0 W$ *to a manifold of the shape*

$$\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i\neq i_0}(\phi_i^q) + \sum_{i=1}^{p_{q+1}}(\overline{\phi}_i^{q+1}) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}}(\overline{\phi}_i^{q+2}) + \ldots + \sum_{i=1}^{p_n}(\overline{\phi}_i^n).$$

Proof : Since $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic to ψ_1^{q+1} and ψ_2^{q+1} is trivial, we can extend ψ_1^{q+1} and ψ_2^{q+1} to embeddings denoted in the same way $\psi_1^{q+1}\colon S^q \times D^{n-q-1} \to \partial_1 W_q$ and $\psi_2^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1^\circ W_{q+1}$ with the following properties [54, Theorem 1.5 in Chapter 8 on page 180]: ψ^{q+1} is isotopic to ψ_1^{q+1} in $\partial_1 W_q$, ψ_1^{q+1} does not meet the transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$, $\psi_1^{q+1}|_{S^q \times \{0\}}$ meets the transverse sphere of the handle $(\phi_{i_0}^q)$ transversally and in exactly one point, ψ^{q+1} is isotopic to ψ_2^{q+1} within $\partial_1 W_{q+1}$ and

ψ_2^{q+1} is trivial. Because of the Diffeomorphism Lemma 1.9 we can assume without loss of generality that there are no handles of index $\geq q+2$, i.e. $p_{q+2} = p_{q+3} = \ldots = p_n = 0$. It suffices to show for appropriate embeddings $\overline{\phi}_i^{q+1}$ and ψ^{q+2} that

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\
&\qquad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi^{q+2}),
\end{aligned}$$

where $\cong$ means diffeomorphic relative $\partial_0 W$. Because of Lemma 1.13 there is an embedding (ψ^{q+2}) satisfying

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\
&\qquad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}).
\end{aligned}$$

We conclude from the Isotopy Lemma 1.8 and the Diffeomorphism Lemma 1.9 for appropriate embeddings ψ_k^{q+1} for $k = 1, 2$

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}) \\
&\cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi^{q+1}) + (\psi_1^{q+2}) \\
&\cong \quad \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2}).
\end{aligned}$$

We get from the Diffeomorphism Lemma 1.9 and the Cancellation Lemma 1.12 for appropriate embeddings $\overline{\phi}_i^{q+1}$ and ψ_3^{q+2}

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2}) \\
&\qquad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi_3^{q+2}).
\end{aligned}$$

This finishes the proof of the Elimination Lemma 1.16. ∎

1.2 Handlebody Decompositions and CW-Structures

Next we explain how we can associate to a handlebody decomposition (1.14) a CW-pair $(X, \partial_0 W)$ such that there is a bijective correspondence between the q-handles of the handlebody decomposition and the q-cells of $(X, \partial_0 W)$. The key ingredient is the elementary fact that the projection $(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (D^q, S^{q-1})$ is a homotopy equivalence and actually – as we will explain later – a simple homotopy equivalence.

Recall that a (relative) CW-complex (X, A) consists of a pair of topological spaces (X, A) together with a filtration

$$X_{-1} = A \subset X_0 \subset X_1 \subset \ldots \subset X_q \subset X_{q+1} \subset \ldots \subset \cup_{q \geq 0} X_q = X$$

such that X carries the colimit topology with respect to this filtration and for any $q \geq 0$ there exists a pushout of spaces

$$\begin{array}{ccc} \coprod_{i \in I_q} S^{q-1} & \xrightarrow{\coprod_{i \in I_q} \phi_i^q} & X_{q-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_q} D^q & \xrightarrow[\coprod_{i \in I_q} \Phi_i^q]{} & X_q \end{array}$$

The map ϕ_i^q is called the *attaching map* and the map (Φ_i^q, ϕ_i^q) is called the *characteristic map* of the q-cell belonging to $i \in I_q$. The pushouts above are not part of the structure, only their existence is required. Only the filtration $\{X_q \mid q \geq -1\}$ is part of the structure. The path components of $X_q - X_{q-1}$ are called the *open cells*. The open cells coincide with the sets $\Phi_i^q(D^q - S^{q-1})$. The closure of an open cell $\Phi_i^q(D^q - S^{q-1})$ is called *closed cell* and turns out to be $\Phi_i^q(D^q)$.

Suppose that X is connected with fundamental group π. Let $p\colon \widetilde{X} \to X$ be the universal covering of X. Put $\widetilde{X_q} = p^{-1}(X_q)$ and $\widetilde{A} = p^{-1}(A)$. Then $(\widetilde{X}, \widetilde{A})$ inherits a CW-structure from (X, A) by the filtration $\{\widetilde{X_q} \mid q \geq -1\}$. The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X}, \widetilde{A})$ has as q-th $\mathbb{Z}\pi$-chain module the singular homology $H_q(\widetilde{X_q}, \widetilde{X_{q-1}})$ with $\mathbb{Z}$-coefficients and the π-action coming from the deck transformations. The q-th differential d_q is given by the composition

$$H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) \xrightarrow{\partial_q} H_{q-1}(\widetilde{X_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{X_{q-1}}, \widetilde{X_{q-2}}),$$

where ∂_q is the boundary operator of the long exact sequence of the pair $(\widetilde{X_q}, \widetilde{X_{q-1}})$ and i_q is induced by the inclusion. If we choose for each $i \in I_q$ a lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})\colon (D^q, S^{q-1}) \to (\widetilde{X_q}, \widetilde{X_{q-1}})$ of the characteristic map (Φ_i^q, ϕ_i^q), we obtain a $\mathbb{Z}\pi$-basis $\{b_i \mid i \in I_n\}$ for $C_n(\widetilde{X}, \widetilde{A})$, if we define b_i as the image of a generator in $H_q(D^q, S^{q-1}) \cong \mathbb{Z}$ under the map $H_q(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})\colon H_q(D^q, S^{q-1}) \to H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) = C_q(\widetilde{X}, \widetilde{A})$. We call $\{b_i \mid i \in I_n\}$ *the cellular basis.* Notice that we have made several choices in defining the cellular basis. We call two $\mathbb{Z}\pi$-bases $\{\alpha_j \mid j \in J\}$ and $\{\beta_k \mid k \in K\}$ for $C_q(\widetilde{X}, \widetilde{A})$ *equivalent* if there is a bijection $\phi\colon J \to K$ and elements $\epsilon_j \in \{\pm 1\}$ and $\gamma_j \in \pi$ for $j \in J$ such that $\epsilon_j \cdot \gamma_j \cdot \alpha_j = \beta_{\phi(j)}$. The equivalence class of the basis $\{b_i \mid i \in I_n\}$ constructed above does only depend on the CW-structure on (X, A) and is independent of all further choices such as (Φ_i^q, ϕ_i^q), its lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})$ and the generator of $H_n(D^n, S^{n-1})$.

Now suppose we are given a handlebody decomposition (1.14). We construct by induction over $q = -1, 0, 1, \ldots, n$ a sequence of spaces $X_{-1} = \partial_0 W \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n$ together with homotopy equivalences $f_q\colon W_q \to X_q$ such that $f_q|_{W_{q-1}} = f_{q-1}$ and $(X, \partial_0 W)$ is a CW-complex with respect to the filtration $\{X_q \mid q = -1, 0, 1, \ldots, n\}$. The induction beginning $f_1\colon W_{-1} = \partial_0 W \times [0,1] \to X_{-1} = \partial_0 W$ is given by the projection. The induction step from $(q-1)$ to q is done as follows. We attach for each handle (ϕ_i^q) for $i = 1, 2, \ldots, p_q$ a cell D^q to X_{q-1} by the attaching map $f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}$. In other words, we define X_q by the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} & \xrightarrow{\coprod_{i=1}^{p_q} f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}} & X_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q & \longrightarrow & X_q
\end{array}$$

Recall that W_q is the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} \times D^{n-q} & \xrightarrow{\coprod_{i=1}^{p_q} \phi_i^q} & W_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q \times D^{n-q} & \longrightarrow & W_q
\end{array}$$

Define a space Y_q by the pushout

$$\begin{array}{ccc} \coprod_{i=1}^{p_q} S^{q-1} & \xrightarrow{\coprod_{i=1}^{p_q} \phi_i^q|_{S^{q-1}\times\{0\}}} & W_{q-1} \\ \downarrow & & \downarrow \\ \coprod_{i=1}^{p_q} D^q & \longrightarrow & Y_q \end{array}$$

Define $(g_q, f_{q-1})\colon (Y_q, W_{q-1}) \to (X_q, X_{q-1})$ by the pushout property applied to homotopy equivalences given by $f_{q-1}\colon W_{q-1} \to X_{q-1}$ and the identity maps on S^{q-1} and D^q. Define $(h_q, \mathrm{id})\colon (Y_q, W_{q-1}) \to (W_q, W_{q-1})$ by the pushout property applied to homotopy equivalences given by the obvious inclusions $S^{q-1} \to S^{q-1} \times D^{n-q}$ and $D^q \to D^q \times D^{n-q}$ and the identity on W_{q-1}. The resulting maps are homotopy equivalences of pairs since the upper horizontal arrows in the three pushouts above are cofibrations (see [18, page 249]). Choose a homotopy inverse $(h_q^{-1}, \mathrm{id})\colon (W_q, W_{q-1}) \to (Y_q, W_{q-1})$. Define f_q by the composition $g_q \circ h_q^{-1}$.

In particular we see that the inclusions $W_q \to W$ are q-connected since the inclusion of the q-skeleton $X_q \to X$ is always q-connected for a CW-complex X.

Denote by $p\colon \widetilde{W} \to W$ the universal covering with $\pi = \pi_1(W)$ as group of deck transformations. Let $\widetilde{W_q}$ be the preimage of W_q under p. Notice that this is the universal covering for $q \geq 2$ since each inclusion $W_q \to W$ induces an isomorphism on the fundamental groups. Let $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ be the $\mathbb{Z}\pi$-chain complex whose q-th chain group is $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and whose q-th differential is given by the composition

$$H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \xrightarrow{\partial_p} H_q(\widetilde{W_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{W_{q-1}}, \widetilde{W_{q-2}}),$$

where ∂_q is the boundary operator of the long homology sequence associated to the pair $(\widetilde{W_p}, \widetilde{W_{p-1}})$ and i_q is induced by the inclusion. The map $f\colon W \to X$ induces an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$C_*(\widetilde{f})\colon C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} C_*(\widetilde{X}, \widetilde{\partial_0 W}). \tag{1.17}$$

Each handle (ϕ_i^q) determines an element

$$[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W}) \tag{1.18}$$

after choosing a lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})\colon (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (\widetilde{W_q}, \widetilde{W_{q-1}})$ of its characteristic map $(\Phi_i^q, \phi_i^q)\colon (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (W_q, W_{q-1})$, namely, the image of the preferred generator in $H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \cong H_0(\{*\}) = \mathbb{Z}$ under the map $H_q(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})$. This element is only well-defined up to multiplication with an element $\gamma \in \pi$. The elements $\{[\phi_i^q] \mid i = 1, 2, \dots, p_q\}$ form a $\mathbb{Z}\pi$-basis for $C_q(\widetilde{W}, \widetilde{\partial_0 W})$. Its image under the isomorphism (1.17) is a cellular $\mathbb{Z}\pi$-basis.

If W has no handles of index ≤ 1, i.e. $p_0 = p_1 = 0$, one can express $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ also in terms of homotopy groups as follows. Fix a base point $z \in \partial_0 W$ and a lift $\widetilde{z} \in \widetilde{\partial_0 W}$. All homotopy groups are taken with respect to these base points. Let $\pi_*(W_*, W_{*-1})$ be the $\mathbb{Z}\pi$-chain complex, whose q-th $\mathbb{Z}\pi$-module is $\pi_q(W_q, W_{q-1})$ for $q \geq 2$ and zero for $q \leq 1$ and whose q-th differential is given by the composition

$$\pi_q(W_q, W_{q-1}) \xrightarrow{\partial_q} \pi_{q-1}(W_{q-1}) \xrightarrow{\pi_{q-1}(i)} \pi_{q-1}(W_{q-1}, W_{q-2}).$$

The $\mathbb{Z}\pi$-action comes from the canonical $\pi_1(Y)$-action on the group $\pi_q(Y, A)$ [121, Theorem I.3.1 on page 164]. Notice that $\pi_q(Y, A)$ is abelian for any pair of spaces (Y, A) for $q \geq 3$ and is abelian also for $q = 2$ if A is simply connected or empty. For $q \geq 2$ the Hurewicz homomorphism is an isomorphism [121, Corollary IV.7.11 on page 181] $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and the projection $p\colon \widetilde{W} \to W$ induces isomorphisms $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to \pi_q(W_q, W_{q-1})$. Thus we obtain an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} \pi_*(W_*, W_{*-1}). \tag{1.19}$$

Fix a path w_i in W from a point in the transverse sphere of (ϕ_i^q) to the base point z. Then the handle (ϕ_i^q) determines an element

$$[\phi_i^q] \quad \in \quad \pi_q(W_q, W_{q-1}). \tag{1.20}$$

It is represented by the obvious map $(D^q \times \{0\}, S^{q-1} \times \{0\}) \to (W_q, W_{q-1})$ together with w_i. It agrees with the element $[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W})$ defined in (1.18) under the isomorphism (1.19) if we use the lift of the characteristic map determined by the path w_i.

1.3 Reducing the Handlebody Decomposition

In the next step we want to get rid of the handles of index zero and one in the handlebody decomposition (1.14).

Lemma 1.21 *Let W be an n-dimensional manifold for $n \geq 6$ whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$. Then the following statements are equivalent*

1. *The inclusion $\partial_0 W \to W$ is 1-connected;*
2. *We can find a diffeomorphism relative $\partial_0 W$*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \sum_{i=1}^{p_n} (\overline{\phi}_i^n).$$

Proof : (2) $\Rightarrow$ (1) has already been proved in Section 1.2. It remains to conclude (2) provided that (1) holds.

We first get rid of all 0-handles in the handlebody decomposition (1.14). It suffices to give a procedure to reduce the number of handles of index 0 by one. Since the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_0 W \to W_1$ induces a bijection on the set of path components. Given any index i_0, there must be an index i_1 such that the core of the handle $\phi_{i_1}^1$ is a path connecting a point in $\partial_0 W \times \{1\}$ with a point in $(\phi_{i_0}^0)$. We conclude from the Diffeomorphism Lemma 1.9 and the Cancellation Lemma 1.12 that (ϕ_{i_0}) and $(\phi_{i_1}^1)$ cancel one another, i.e. we have

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_0, i \neq i_0} (\phi_i^0) + \sum_{i=1,2,\ldots,p_1, i \neq i_1} (\phi_i^1) + \ldots + \sum_{i=1}^{p_n} (\phi_i^n).$$

Hence we can assume $p_0 = 0$ in (1.14).

Next we want to get rid of the 1-handles assuming that the inclusion $\partial_0 W \to W$ is 1-connected. It suffices to give a procedure to reduce the number of handles of index 1 by one. We want to do this by constructing an embedding $\psi^2 \colon S^1 \times D^{n-2} \to \partial_1^\circ W_1$ which satisfies the two conditions of the Elimination Lemma 1.16 and then applying the Elimination Lemma 1.16. Consider the embedding $\psi_+^2 \colon S_+^1 = D^1 = D^1 \times \{x\} \subset D^1 \times D^{n-1} = (\phi_1^1)$ for some fixed $x \in S^{n-2} = \partial D^{n-1}$. The inclusion $\partial_1^\circ W_0 \to \partial_1 W_0 = \partial_0 W \times \{1\}$ induces an isomorphism on the fundamental group since $\partial_1^\circ W_0$ is obtained from $\partial_1 W_0 = \partial_0 W \times \{1\}$ by removing the interior of a finite number of embedded $(n-1)$-dimensional disks. Since by assumption the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_1^\circ W_0 \to W$ induces an epimorphism on the fundamental groups. Therefore we can find an embedding $\psi_-^2 \colon S_-^1 \to \partial_1^\circ W_0$ with $\psi_-^2|_{S^0} = \psi_+^2|_{S^0}$ such that the map $\psi_0^2 \colon S^1 = S_+^1 \cup_{S^0} S_-^1 \to \partial_1 W_1$

given by $\psi_+^2 \cup \psi_-^2$ is nullhomotopic in W. One can isotope the attaching maps $\phi_i^2\colon S^1 \times D^{n-2} \to \partial_1 W_1$ of the 2-handles (ϕ_i^2) such that they do not meet the image of ψ_0^2 because the sum of the dimension of the source of ψ_0^2 and of $S^1 \times \{0\} \subset S^1 \times D^{n-2}$ is less than the dimension $(n-1)$ of $\partial_1 W_1$ and one can always shrink inside D^{n-2}. Thus we can assume without loss of generality by the Isotopy Lemma 1.8 and the Diffeomorphism Lemma 1.9 that the image of ψ_0^2 lies in $\partial_1^\circ W_1$. The inclusion $\partial_1 W_2 \to W$ is 2-connected. Hence ψ_0^2 is nullhomotopic in $\partial_1 W_2$. Let $h\colon D^2 \to \partial_1 W_2$ be a nullhomotopy for ψ_0^2. Since $2 \cdot \dim(D^2) < \dim(\partial_1 W_2)$, we can change h relative to S^1 into an embedding. (Here we need for the first time the assumption $n \geq 6$.) Since D^2 is contractible the normal bundle of h and thus of $\psi_0^2 = \psi_+^2 \cup \psi_-^2$ are trivial. Therefore we can extend ψ_0^2 to an embedding $\psi^2\colon S^1 \times D^{n-1} \to \partial_1^\circ W_1$ which is isotopic to a trivial embedding in $\partial_1 W_2$ and meets the transverse sphere of the handle (ϕ_1^1) transversally and in exactly one point and does not meet the transverse spheres of the handles (ϕ_i^1) for $2 \leq i \leq p_1$. Now Lemma 1.21 follows from the Elimination Lemma 1.16. ■

Now consider an h-cobordism $(W; \partial_0 W, \partial_1 W)$. Because of Lemma 1.21 we can write it as

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \dots .$$

Lemma 1.22 (Homology Lemma) *Suppose $n \geq 6$. Fix $2 \leq q \leq n-3$ and $i_0 \in \{1, 2, \dots, p_q\}$. Let $f\colon S^q \to \partial_1 W_q$ be an embedding. Then the following statements are equivalent*

1. *f is isotopic to an embedding $g\colon S^q \to \partial_1 W_q$ such that g meets the transverse sphere of $(\phi_{i_0}^q)$ transversally and in exactly one point and is disjoint from transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$;*

2. *Let $\widetilde{f}\colon S^q \to \widetilde{W_q}$ be a lift of f under $p|_{\widetilde{W_q}}\colon \widetilde{W_q} \to W_q$. Let $[\widetilde{f}]$ be the image of the class represented by $\widetilde{f}$ under the obvious composition*

$$\pi_q(\widetilde{W_q}) \to \pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) = C_q(\widetilde{W}).$$

Then there is $\gamma \in \pi$ with

$$[\widetilde{f}] = \pm \gamma \cdot [\phi_{i_0}^q].$$

Proof : (1) $\Rightarrow$ (2) We can isotop f such that $f|_{S^q_+} : S^q_+ \to \partial_1 W_q$ looks like the canonical embedding $S^q_+ = D^q \times \{x\} \subset D^q \times S^{n-1-q} = \partial(\phi^q_{i_0})$ for some $x \in S^{n-1-q}$ and $f(S^q_-)$ does not meet any of the handles (ϕ^q_i) for $i = 1, 2, \ldots, p_q$. One easily checks that then (2) is true.

(2) $\Rightarrow$ (1) We can isotop f such that it is transversal to the transverse spheres of the handles (ϕ^q_i) for $i = 1, 2, \ldots, p_q$. Since the sum of the dimension of the source of f and of the dimension of the transverse spheres is the dimension of $\partial_1 W_q$, the intersection of the image of f with the transverse sphere of the handle (ϕ^q_i) consists of finitely many points $x_{i,1}, x_{i,2}, \ldots, x_{i,r_i}$ for $i = 1, 2, \ldots, p_q$. Fix a base point $y \in S^q$. It yields a base point $z = f(y) \in W$. Fix for each handle (ϕ^q_i) a path w_i in W from a point in its transverse sphere to z. Let $u_{i,j}$ be a path in S^q with the property that $u_{i,j}(0) = y$ and $f(u_{i,j}(1)) = x_{i,j}$ for $1 \le j \le r_i$ and $1 \le i \le p_q$. Let $v_{i,j}$ be any path in the transverse sphere of (ϕ^q_i) from $x_{i,j}$ to $w_i(0)$. Then the composition $f(u_{i,j}) * v_{i,j} * w_i$ is a loop in W with base point z and thus represents an element denoted by $\gamma_{i,j}$ in $\pi = \pi_1(W, z)$. It is independent of the choice of $u_{i,j}$ and $v_{i,j}$ since S^q and the transverse sphere of each handle (ϕ^q_i) are simply connected. The tangent space $T_{x_{i,j}} \partial_1 W_q$ is the direct sum of $T_{f^{-1}(x_{i,j})} S^p$ and the tangent space of the transverse sphere $\{0\} \times S^{n-1-q}$ of the handle (ϕ^q_i) at $x_{i,j}$. All these three tangent spaces come with preferred orientations. We define elements $\epsilon_{i,j} \in \{\pm 1\}$ by requiring that it is 1 if these orientations fit together and -1 otherwise. Now one easily checks that

$$[\widetilde{f}] \;=\; \sum_{i=1}^{p_q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi^q_i],$$

where $[\phi^q_i]$ is the element associated to the handle (ϕ^q_i) after the choice of the path w_i (see (1.18) and (1.20)). We have by assumption $[\widetilde{f}] = \pm \cdot \gamma \cdot [\phi^q_{i_0}]$ for some $\gamma \in \pi$. We want to isotop f such that f does not meet the transverse spheres of the handles (ϕ^q_i) for $i \neq i_0$ and does meet the transverse sphere of $(\phi^q_{i_0})$ transversally and in exactly one point. Therefore it suffices to show in the case that the number $\sum_{i=1}^{p_q} r_i$ of all intersection points of f with the transverse spheres of the handles (ϕ^q_i) for $i = 1, 2, \ldots, p_i$ is bigger than one that we can change f by an isotopy such that this number becomes smaller. We have

$$\pm\gamma \cdot [\phi^q_{i_0}] \;=\; \sum_{i=1}^{p_q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi^q_i].$$

Recall that the elements $[\phi_i^q]$ for $i = 1, 2, \ldots, p_q$ form a $\mathbb{Z}\pi$-basis. Hence we can find an index $i \in \{1, 2, \ldots, p_q\}$ and two different indices $j_1, j_2 \in \{1, 2, \ldots, r_i\}$ such that the composition of the paths $f(u_{i,j_1}) * v_{i,j_1} * v_{i,j_2}^- * f(u_{i,j_2}^-)$ is nullhomotopic in W and hence in $\partial_1 W_q$ and the signs ϵ_{i,j_1} and ϵ_{i,j_2} are different. Now by the Whitney trick (see [83, Theorem 6.6 on page 71], [123]) we can change f by an isotopy such that the two intersection points x_{i,j_1} and x_{i,j_2} disappear, the other intersection points of f with transverse spheres of the handles (ϕ_i^q) for $i \in \{1, 2, \ldots, p_q\}$ remain and no further intersection points are introduced. For the application of the Whitney trick we need the assumption $n - 1 \geq 5$. This finishes the proof of the Homology Lemma 1.22. ■

Lemma 1.23 (Modification Lemma) *Let $f\colon S^q \to \partial_1^\circ W_q$ be an embedding and let $x_j \in \mathbb{Z}\pi$ be elements for $j = 1, 2 \ldots, p_{q+1}$. Then there is an embedding $g\colon S^q \to \partial_1^\circ W_q$ with the following properties:*

1. *f and g are isotopic in $\partial_1 W_{q+1}$;*
2. *For a given lifting $\widetilde{f}\colon S^q \to \widetilde{W_q}$ of f one can find a lifting $\widetilde{g}\colon S^q \to \widetilde{W_q}$ of g such that we get in $C_q(\widetilde{W})$*

$$[\widetilde{g}] \;=\; [\widetilde{f}] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi_j^{q+1}],$$

where d_{q+1} is the $(q+1)$-th differential in $C_*(\widetilde{W}, \widetilde{\partial_0 W})$.

Proof : Any element in $\mathbb{Z}\pi$ can be written as a sum of elements of the form $\pm\gamma$ for $\gamma \in \pi$. Hence it suffices to prove for a fixed number $j \in \{1, 2 \ldots, p_q\}$, a fixed element $\gamma \in \pi$ and a fixed sign $\epsilon \in \{\pm 1\}$ that one can find an embedding $g\colon S^q \to \partial_1^\circ W_q$ which is isotopic to f in $\partial_1 W_{q+1}$ and satisfies for an appropriate lifting $\widetilde{g}$

$$[\widetilde{g}] \;=\; [\widetilde{f}] + \epsilon \cdot \gamma \cdot d_{p+1}[\phi_j^{q+1}].$$

Consider the embedding $t_j\colon S^q = S^q \times \{z\} \subset S^q \times S^{n-2-q} \subset \partial(\phi_j^{q+1}) \subset \partial_1 W_q$ for some point $z \in S^{n-2-q} = \partial D^{n-1-q}$. It is in $\partial_1 W_{q+1}$ isotopic to a trivial embedding. Choose a path w in $\partial_1^\circ W_q$ connecting a point in the image of f with a point in the image of t_j. Without loss of generality we can arrange w to be an embedding. Moreover, we can thicken $w\colon [0,1] \to$

$\partial_1^\circ W_q$ to an embedding $\overline{w}[0,1] \times D^q \to \partial_1^\circ W_q$ such that $\overline{w}(\{0\} \times D^q)$ and $\overline{w}(\{1\} \times D^q)$ are embedded q-dimensional disks in the images of f and t_j and $\overline{w}((0,1) \times D^q)$ does not meet the images of f and t_j. Now one can form a new embedding, the connected sum $g := f\sharp_w t_j \colon S^q \to \partial_1^\circ W_q$. It is essentially given by restriction of f and t_j to the part of S^q, which is not mapped under f and t_j to the interior of the disks $\overline{w}(\{0\} \times D^q)$, $\overline{w}(\{1\} \times D^q)$, and $\overline{w}|_{[0,1]\times S^{q-1}}$. Since t_j is isotopic to a trivial embedding in $\partial_1 W_{q+1}$, the embedding g is isotopic in $\partial_1 W_{q+1}$ to f. Recall that we have fixed a lifting $\widetilde{f}$ of f. This determines a unique lifting of $\widetilde{g}$, namely, we require that $\widetilde{f}$ and $\widetilde{g}$ coincide on those points, where f and g already coincide. For an appropriate element $\gamma' \in \pi$ one gets $[\widetilde{g}] = [\widetilde{f}] + \gamma' \cdot d_{q+1}([\phi_j^{q+1}])$, since $t_j \colon S^q \to \partial_1 W_q \subset W_q$ is homotopic to $\phi_j^{q+1}|_{S^q \times \{0\}} \colon S^q \times \{0\} = S^q \to W_q$ in W_q. We can change the path w by composing it with a loop representing $\gamma \cdot (\gamma')^{-1} \in \pi$. Then we get for the new embedding g that

$$[\widetilde{g}] = [\widetilde{f}] + \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

If we compose t_j with a diffeomorphism $S^q \to S^q$ of degree -1, we still get an embedding g which is isotopic to f in $\partial_1 W_{q+1}$ and satisfies

$$[\widetilde{g}] = [\widetilde{f}] - \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

This finishes the proof of the Modification Lemma 1.23. ■

Lemma 1.24 (Normal Form Lemma) *Let $(W; \partial_0 W, \partial_1 W)$ be an oriented compact h-cobordism of dimension $n \geq 6$. Let q be an integer with $2 \leq q \leq n-3$. Then there is a handlebody decomposition which has only handles of index q and $(q+1)$, i.e. there is a diffeomorphism relative $\partial_0 W$*

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Proof : In the first step we show that we can arrange $W_{-1} = W_{q-1}$, i.e. $p_r = 0$ for $r \leq q-1$. We do this by induction over q. The induction beginning $q = 2$ has already been carried out in Lemma 1.21. In the induction step from q to $(q+1)$ we must explain how we can decrease the number of q-handles provided that there are no handles of index $< q$. In order to get rid of the handle (ϕ_1^q) we want to attach a new $(q+1)$-handle and a new $(q+2)$-handle such that (ϕ_1^q) and the new $(q+1)$-handle cancel and the new

$(q+1)$-handle and the new $(q+2)$-handle cancel each other. The effect will be that the number of q-handles is decreased by one at the cost of increasing the number of $(q+2)$-handles by one.

Fix a trivial embedding $\overline{\psi}^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1^\circ W_q$. Since the inclusion $\partial_0 W \to W$ is a homotopy equivalence, $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$ for all $p \geq 0$. Since the p-th homology of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$, the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Since $C_{q-1}(\widetilde{W}, \widetilde{\partial_0 W})$ is trivial, the q-th differential of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is zero and hence the $(q+1)$-th differential d_{q+1} is surjective. We can choose elements $x_j \in \mathbb{Z}\pi$ such that

$$[\phi_1^q] \;=\; \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]).$$

Since $\alpha := \overline{\psi}^{q+1}|_{S^q \times \{0\}} \to \partial_1^\circ W_q$ is nullhomotopic, $[\widetilde{\alpha}] = 0$ in $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. Because of the Modification Lemma 1.23 we can find an embedding $\psi^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ such that $\beta := \psi^q|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to α and we get

$$[\widetilde{\beta}] \;=\; [\widetilde{\alpha}] + \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]) \;=\; [\phi_1^q].$$

Because of the Homology Lemma 1.22 the embedding $\beta = \psi^q|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\gamma\colon S^q \to \partial_1 W_q$ which meets the transverse sphere of (ϕ_1^q) transversally and in exactly one point and is disjoint from the transverse spheres of all other handles of index q. By construction ψ^{q+1} is isotopic in $\partial_1 W_{q+1}$ to the trivial embedding $\overline{\psi}^{q+1}$. Now we can apply the Elimination Lemma 1.16. This finishes the proof that we can arrange $W_{-1} = W_{q-1}$.

Next we explain the *dual handlebody decomposition*. Suppose that W is obtained from $\partial_0 W \times [0,1]$ by attaching one q-handle (ϕ^q), i.e. $W = \partial_0 W \times [0,1] + (\phi^q)$. Then we can interchange the role of $\partial_0 W$ and $\partial_1 W$ and try to built W from $\partial_1 W$ by handles. It turns out that W can be written as

$$W \;=\; \partial_1 W \times [0,1] + (\psi^{n-q}) \tag{1.25}$$

by the following argument.

Let M be the manifold with boundary $S^{q-1} \times S^{n-1-q}$ obtained from $\partial_0 W$ by removing the interior of $\phi^q(S^{q-1} \times D^{n-q})$. We get

$$\begin{aligned} W &\cong \partial_0 W \times [0,1] \cup_{S^{q-1} \times D^{n-q}} D^q \times D^{n-q} \\ &= M \times [0,1] \cup_{S^{q-1} \times S^{n-2-q} \times [0,1]} \\ &\qquad \left(S^{q-1} \times D^{n-1-q} \times [0,1] \cup_{S^{q-1} \times D^{n-q} \times \{1\}} D^q \times D^{n-q}\right). \end{aligned}$$

Inside $S^{q-1} \times D^{n-1-q} \times [0,1] \cup_{S^{q-1} \times D^{n-q} \times \{1\}} D^q \times D^{n-q}$ we have the following submanifolds

$$\begin{aligned} X &:= S^{q-1} \times 1/2 \cdot D^{n-1-q} \times [0,1] \cup_{S^{q-1} \times 1/2 \cdot D^{n-q} \times \{1\}} D^q \times 1/2 \cdot D^{n-q}; \\ Y &:= S^{q-1} \times 1/2 \cdot S^{n-1-q} \times [0,1] \cup_{S^{q-1} \times 1/2 \cdot S^{n-q} \times \{1\}} D^q \times 1/2 \cdot S^{n-q}. \end{aligned}$$

The pair (X, Y) is diffeomorphic to $(D^q \times D^{n-q}, D^q \times S^{n-1-q})$, i.e. it is a handle of index $(n-q)$. Let N be obtained from W by removing the interior of X. Then W is obtained from N by adding a $(n-q)$-handle, the so called *dual handle*. One easily checks that N is diffeomorphic to $\partial_1 W \times [0,1]$ relative $\partial_1 W \times \{1\}$. Thus (1.25) follows.

Suppose that W is relatively $\partial_0 W$ of the shape

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \ldots + \sum_{i=1}^{p_n} (\phi_i^n),$$

Then we can conclude inductively using the Diffeomorphism Lemma 1.9 and (1.25) that W is diffeomorphic relative to $\partial_1 W$ to

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n} (\overline{\phi}_i^0) + \sum_{i=1}^{p_{n-1}} (\overline{\phi}_i^1) + \ldots + \sum_{i=1}^{p_0} (\overline{\phi}_i^n). \tag{1.26}$$

This corresponds to replacing a Morse function f by $-f$. The effect is that the number of q-handles becomes now the number of $(n-q)$-handles.

Now applying the first step to the dual handlebody decomposition for q replaced by $(n-q-1)$ and then considering the dual handlebody decomposition of the result finishes the proof of the Normal Form Lemma 1.24.

■

1.4 Handlebody Decompositions and Whitehead Groups

Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact oriented h-cobordism for $n \geq 6$. By the Normal Form Lemma 1.24 we can fix a handlebody decomposition for some fixed number $2 \leq q \leq n-3$

$$W \;\cong\; \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Recall that the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Hence the only non-trivial differential $d_{q+1}\colon H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ is bijective. Recall that $\{[\phi_i^{q+1}] \mid i = 1, 2 \ldots, p_{q+1}\}$ is a $\mathbb{Z}\pi$-basis for $H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q})$ and $\{[\phi_i^q] \mid i = 1, 2 \ldots, p_q\}$ is a $\mathbb{Z}\pi$-basis for $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. In particular $p_q = p_{q+1}$. The matrix A, which describes the differential d_{q+1} with respect to these basis, is an invertible (p_q, p_q)-matrix over $\mathbb{Z}\pi$. Since we are working with left modules, d_{q+1} sends an element $x \in (\mathbb{Z}G)^n$ to $x \cdot A \in \mathbb{Z}G^n$, or equivalently, $d_{q+1}([\phi_i^{q+1}]) = \sum_{j=1}^n a_{i,j}[\phi_j^q]$.

Next we define an abelian group $\mathrm{Wh}(\pi)$ as follows. It is the set of equivalence classes of invertible matrices of arbitrary size with entries in $\mathbb{Z}\pi$, where we call an invertible (m,m)-matrix A and an invertible (n,n)-matrix B over $\mathbb{Z}\pi$ equivalent, if we can pass from A to B by a sequence of the following operations:

1. B is obtained from A by adding the k-th row multiplied with x from the left to the l-th row for $x \in \mathbb{Z}\pi$ and $k \neq l$;
2. B is obtained by taking the direct sum of A and the $(1,1)$-matrix $I_1 = (1)$, i.e. B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
3. A is the direct sum of B and I_1. This is the inverse operation to (2);
4. B is obtained from A by multiplying the i-th row from the left with a trivial unit , i.e. with an element of the shape $\pm\gamma$ for $\gamma \in \pi$;
5. B is obtained from A by interchanging two rows or two columns.

The group structure is given on representatives A and B as follows. By taking the direct sum $A \oplus I_m$ and $B \oplus I_n$ with the identity matrices I_m and

I_n of size m and n for appropriate m and n one can arrange that $A \oplus I_m$ and $B \oplus I_n$ are invertible matrices of the same size and can be multiplied. Define $[A] \cdot [B]$ by $[(A \oplus I_m) \cdot (B \oplus I_n)]$. The zero element $0 \in \mathrm{Wh}(\pi)$ is represented by I_n for any positive integer n. The inverse of $[A]$ is given by $[A^{-1}]$. We will show later in Lemma 2.4 that the multiplication is well-defined and yields an abelian group $\mathrm{Wh}(\pi)$.

Lemma 1.27 *1. Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact oriented h-cobordism for $n \geq 6$ and A be the matrix defined above. If $[A] = 0$ in $\mathrm{Wh}(\pi)$, then the h-cobordism W is trivial relative $\partial_0 W$;*

2. Consider an element $u \in \mathrm{Wh}(\pi)$, a closed oriented manifold M of dimension $n - 1 \geq 5$ with fundamental group π and an integer q with $2 \leq q \leq n - 3$. Then we can find an h-cobordism of the shape

$$W = M \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

such that $[A] = u$.

Proof : (1) Let B be a matrix which is obtained from A by applying one of the operations (1), (2), (3), (4) and (5). It suffices to show that we can modify the given handlebody decomposition in normal form of W with associated matrix A such that we get a new handlebody decomposition in normal form whose associated matrix is B.

We begin with (1). Consider $W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{j=1, j \neq l}^{p_{q+1}} (\phi_j^{q+1})$. Notice that we get from W' our h-cobordism W if we attach the handle (ϕ_l^{q+1}). By the Modification Lemma 1.23 we can find an embedding $\overline{\phi}_l^{q+1} \colon S^q \times D^{n-1-q} \to \partial_1 W'$ such that $\overline{\phi}_l^{q+1}$ is isotopic to ϕ_l^{q+1} and we get

$$\left[\widetilde{\overline{\phi}_l^{q+1}|_{S^q \times \{0\}}} \right] = \left[\widetilde{\phi_l^{q+1}|_{S^q \times \{0\}}} \right] + x \cdot d_{q+1}([\phi_k^{q+1}]).$$

If we attach to W' the handle $(\overline{\phi}_l^{q+1})$, the result is diffeomorphic to W relative $\partial_0 W$ by the Isotopy Lemma 1.8. One easily checks that the associated invertible matrix B is obtained from A by adding the k-th row multiplied with x from the left to the the l-th row.

The claim for the operations (2) and (3) follow from the Cancellation Lemma 1.12 and the Homology Lemma 1.22. The claim for the operation

(4) follows from the observation that we can replace the attaching map of a handle $\phi^q\colon S^q \times D^{n-1-q} \to \partial_1 W_q$ by its composition with $f \times \mathrm{id}$ for some diffeomorphism $f\colon S^q \to S^q$ of degree -1 and that the base element $[\phi_i^q]$ can also be changed to $\gamma \cdot [\phi_i^q]$ by choosing a different lift along $\widetilde{W_q} \to W_q$. Operation (5) can be realized by interchanging the numeration of the q-handles and $(q+1)$-handles.

(2) Fix an invertible matrix $A = (a_{i,j}) \in GL(n, \mathbb{Z}\pi)$. Choose trivial pairwise disjoint embeddings $\phi_i^2\colon S^1 \times D^{n-2} \to M_0 \times \{1\}$. Consider

$$W_2 = M_0 \times [0,1] + (\phi_1^2) + (\phi_2^2) + \ldots + (\phi_n^2).$$

Since the embeddings ϕ_i^2 are trivial, we can construct embeddings $\phi_i^3\colon S^2 \times D^{n-3} \to \partial_1 W_2$ and lifts $\widetilde{\phi_i^3}\colon S^2 \times D^{n-3} \to \widetilde{\partial_1 W_2}$ such that in $\pi_2(\widetilde{W_2}, \widetilde{\partial_0 W})$

$$[\widetilde{\phi_i^3}|_{S^2\times\{0\}}] = \sum_{j=1}^{n} a_{i,j} \cdot [\phi_j^2].$$

Put $W = W_2 + (\phi_1^3) + (\phi_2^3) + \ldots + (\phi_n^3)$. One easily checks that W is an h-cobordism over M_0 with a handlebody decomposition which realizes the matrix A. This finishes the proof Lemma 1.27. ■

Remark 1.28 If π is trivial, then $\mathrm{Wh}(\pi)$ is trivial. This follows from the fact that any invertible matrix over the integers can be reduced by elementary column operations, permutations of columns and rows and multiplication of a row with ± 1 to the identity matrix. This is essentially a consequence of the existence of an Euclidean algorithm for $\mathbb{Z}$. Hence Lemma 1.27 (2) implies already the h-Cobordism Theorem 1.2. As soon as we have shown that $[A] \in \mathrm{Wh}(\pi)$ agrees with the Whitehead torsion $\tau(W, M_0)$ of the h-cobordism W over M_0 and that this invariant depends only on the diffeomorphism type of W relative M_0, the s-Cobordism Theorem 1.1 (1) will follow.

Obviously Lemma 1.27 (2) implies the s-Cobordism Theorem 1.1 (2). We will later see that assertion (3) of the s-Cobordism Theorem 1.1 follows from assertions (1) and (2) if we have more information about the Whitehead torsion, namely the sum and the composition formulas.

1.5 Miscellaneous

The s-Cobordism Theorem 1.1 is due to Barden, Mazur, Stallings. Its topological version was proved by Kirby and Siebenmann [59, Essay II]. More information about the s-cobordism theorem can be found for instance in [57], [83] [100, page 87-90]. The s-cobordism theorem is known to be false for $n = \dim(M_0) = 4$ in general, by the work of Donaldson [35], but it is true for $n = \dim(M_0) = 4$ for so called "good" fundamental groups in the topological category by results of Freedman [46], [47]. The trivial group is an example of a "good" fundamental groups. Counterexamples in the case $n = \dim(M_0) = 3$ are constructed by Cappell and Shaneson [22]. The Poincaré Conjecture (see Theorem 1.3) is at the time of writing known in all dimensions except dimension 3.

Chapter 2

Whitehead Torsion

Introduction

In this section we will assign to a homotopy equivalence $f\colon X \to Y$ of finite CW-complexes its Whitehead torsion $\tau(f)$ in the Whitehead group $\mathrm{Wh}(\pi(Y))$ associated to Y. The main properties of this invariant are summarized in the following

Theorem 2.1 *1. Sum formula*

Let the following two diagrams be cellular pushouts of finite CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array} \qquad\qquad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ Y_2 & \xrightarrow{l_2} & Y \end{array}$$

Put $l_0 = l_1 \circ k_1 = l_2 \circ k_2\colon Y_0 \to Y$. Let $f_i\colon X_i \to Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f\colon X \to Y$ the map induced by f_0, f_1 and f_2 and the pushout property. Then f is a homotopy equivalence and

$$\tau(f) \;=\; (l_1)_*\tau(f_1) + (l_2)_*\tau(f_2) - (l_0)_*\tau(f_0);$$

2. Homotopy invariance

Let $f \simeq g\colon X \to Y$ be homotopic maps of finite CW-complexes. Then *the homomorphisms* $f_*, g_* \colon \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y))$ *agree. If additionally* f *and* g *are homotopy equivalences, then*

$$\tau(g) \ = \ \tau(f);$$

3. *Composition formula*

   Let $f \colon X \to Y$ *and* $g \colon Y \to Z$ *be homotopy equivalences of finite CW-complexes. Then*

$$\tau(g \circ f) \ = \ g_* \tau(f) + \tau(g);$$

4. *Product formula*

   Let $f \colon X' \to X$ *and* $g \colon Y' \to Y$ *be homotopy equivalences of connected finite CW-complexes. Then*

$$\tau(f \times g) \ = \ \chi(X) \cdot j_* \tau(g) + \chi(Y) \cdot i_* \tau(f),$$

   where $\chi(X), \chi(Y) \in \mathbb{Z}$ *denote the Euler characteristics,* $j_* \colon \mathrm{Wh}(\pi(Y)) \to \mathrm{Wh}(\pi(X \times Y))$ *is the homomorphism induced by* $j \colon Y \to X \times Y, y \mapsto (y, x_0)$ *for some base point* $x_0 \in X$ *and* i_* *is defined analogously;*

5. *Topological invariance*

   Let $f \colon X \to Y$ *be a homeomorphism of finite CW-complexes. Then*

$$\tau(f) \ = \ 0.$$

Given an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0, we define its Whitehead torsion $\tau(W, M_0)$ by the Whitehead torsion of the inclusion $\partial_0 W \to W$ (see (2.14)). This is the invariant appearing in the s-Cobordism Theorem 1.1. We will give some information about the Whitehead group $\mathrm{Wh}(\pi(Y))$ in Section 2.1. We will present the algebraic definition of Whitehead torsion and the proof of Theorem 2.1 in Section 2.2. A geometric approach to the Whitehead torsion is summarized in Section 2.3. A similar invariant, the Reidemeister torsion, will be treated in Section 2.4. It will be used to classify lens spaces. In order to understand the following chapters, it suffices to comprehend the statements in the s-Cobordism Theorem 1.1 and Theorem 2.1

2.1 Whitehead Groups

In this section we define $K_1(R)$ for an associative ring R with unit and the Whitehead group $\mathrm{Wh}(G)$ of a group G and relate the definitions of this section with the one of Section 1.4. Furthermore we give some basic information about its computation.

Let R be an associative ring with unit. Denote by $GL(n,R)$ the group of invertible (n,n)-matrices with entries in R. Define the group $GL(R)$ by the colimit of the system indexed by the natural numbers $\ldots \subset GL(n,R) \subset GL(n+1,R) \subset \ldots$, where the inclusion $GL(n,R)$ to $GL(n+1,R)$ is given by stabilization

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Define $K_1(R)$ by the abelianization $GL(R)/[GL(R),GL(R)]$ of $GL(R)$. Let $\widetilde{K}_1(R)$ be the cokernel of the map $K_1(\mathbb{Z}) \to K_1(R)$ induced by the canonical ring homomorphism $\mathbb{Z} \to R$. The homomorphism $\det\colon K_1(\mathbb{Z}) \to \{\pm 1\}$, $[A] \mapsto \det(A)$ is a bijection, because $\mathbb{Z}$ is a ring with Euclidian algorithm. Hence $\widetilde{K}_1(R)$ is the same as the quotient of $K_1(R)$ by the cyclic subgroup of at most order two generated by the class of the $(1,1)$-matrix (-1). Define the *Whitehead group* $\mathrm{Wh}(G)$ of a group G to be the cokernel of the map $G \times \{\pm 1\} \to K_1(\mathbb{Z}G)$ which sends $(g,\pm 1)$ to the class of the invertible $(1,1)$-matrix $(\pm g)$. This will be the group, where Whitehead torsion will take its values in.

The Whitehead group $\mathrm{Wh}(G)$ is known to be trivial if G is the free abelian group $\mathbb{Z}^n$ of rank n [8] or the free group $*_{i=1}^n \mathbb{Z}$ of rank n [109]. There is the conjecture that it vanishes for any torsionfree group. This has been proved by Farrell and Jones [37], [38], [39] for a large class of groups. This class contains any subgroup $G \subset G'$, where G' is a discrete cocompact subgroup of a Lie group with finitely many path components, and any group G which is the fundamental group of a non-positively curved closed Riemannian manifold or of a complete pinched negatively curved Riemannian manifold. Recall that a G-space X is called *cocompact* if its orbit space $G\backslash X$ is compact. The Whitehead group satisfies $\mathrm{Wh}(G*H) = \mathrm{Wh}(G) \oplus \mathrm{Wh}(H)$ [109].

If G is finite, then $\mathrm{Wh}(G)$ is very well understood (see [91]). Namely, $\mathrm{Wh}(G)$ is finitely generated, its rank as abelian group is the number of conjugacy classes of unordered pairs $\{g,g^{-1}\}$ in G minus the number of conjugacy classes of cyclic subgroups, and its torsion subgroup is isomorphic to the kernel $SK_1(G)$ of the change of coefficient homomorphism $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$. For a finite cyclic group G the Whitehead group $\mathrm{Wh}(G)$ is torsionfree. For instance the Whitehead group $\mathrm{Wh}(\mathbb{Z}/p)$ of a cyclic group of order p for an odd prime p is the free abelian group of rank $(p-3)/2$ and $\mathrm{Wh}(\mathbb{Z}/2) = 0$. The

Whitehead group of the symmetric group S_n is trivial. The Whitehead group of $\mathbb{Z}^2 \times \mathbb{Z}/4$ is not finitely generated as abelian group. Next we want to relate the definitions above to the one of Section 1.4. Denote by $E_n(i,j)$ for $n \geq 1$ and $1 \leq i,j \leq n$ the (n,n)-matrix whose entry at (i,j) is one and is zero elsewhere. Denote by I_n the identity matrix of size n. An elementary (n,n)-matrix is a matrix of the form $I_n + r \cdot E_n(i,j)$ for $n \geq 1$, $1 \leq i,j \leq n$, $i \neq j$ and $r \in R$. Let A be a (n,n)-matrix. The matrix $B = A \cdot (I_n + r \cdot E_n(i,j))$ is obtained from A by adding the i-th column multiplied with r from the right to the j-th column. The matrix $C = (I_n + r \cdot E_n(i,j)) \cdot A$ is obtained from A by adding the j-th row multiplied with r from the left to the i-th row. Let $E(R) \subset GL(R)$ be the subgroup generated by all elements in $GL(R)$ which are represented by elementary matrices.

Lemma 2.2 *We have $E(R) = [GL(R), GL(R)]$. In particular $E(R) \subset GL(R)$ is a normal subgroup and $K_1(R) = GL(R)/E(R)$.*

Proof : For $n \geq 3$, pairwise distinct numbers $1 \leq i,j,k \leq n$ and $r \in R$ we can write $I_n + r \cdot E_n(i,k)$ as a commutator in $GL(n,R)$, namely

$$\begin{aligned} I_n + r \cdot E_n(i,k) &= (I_n + r \cdot E_n(i,j)) \cdot (I_n + E_n(j,k)) \cdot \\ &\qquad (I_n + r \cdot E_n(i,j))^{-1} \cdot (I_n + E_n(j,k))^{-1}. \end{aligned}$$

This implies $E(R) \subset [GL(R), GL(R)]$.

Let A and B be two elements in $GL(n,R)$. Let $[A]$ and $[B]$ be the elements in $GL(R)$ represented by A and B. Given two elements x and y in $GL(R)$, we write $x \sim y$ if there are elements e_1 and e_2 in $E(R)$ with $x = e_1 y e_2$, in other words, if the classes of x and y in $E(R)\backslash GL(R)/E(R)$ agree. One easily checks

$$\begin{aligned} [AB] \sim \left[\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix}\right] &\sim \left[\begin{pmatrix} AB & A \\ 0 & I_n \end{pmatrix}\right] \\ &\sim \left[\begin{pmatrix} 0 & A \\ -B & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right], \end{aligned}$$

since each step is given by multiplication from the right or left with a block matrix of the form $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$ or $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$ and such a block matrix is obviously obtained from I_{2n} by a sequence of column and row operations and hence its class in $GL(R)$ belongs to $E(R)$. Analogously we get

$$[BA] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

Since the element in $GL(R)$ represented by $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ belongs to $E(R)$, we conclude

$$\left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right] \sim \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

This shows

$$[AB] \quad \sim [BA]. \tag{2.3}$$

This implies for any element $x \in GL(R)$ and $e \in E(R)$ that $xex^{-1} \sim ex^{-1}x = e$ and hence $xex^{-1} \in E(R)$. Therefore $E(R)$ is normal. Given a commutator $xyx^{-1}y^{-1}$ for $x, y \in GL(R)$, we conclude for appropriate elements e_1, e_2, e_3 in $E(R)$

$$xyx^{-1}y^{-1} = e_1yxe_2x^{-1}y^{-1} = e_1yxx^{-1}y^{-1}(yx)e_2(yx)^{-1} = e_1e_3 \in E(R).$$

This finishes the proof of Lemma 2.2. ■

Lemma 2.4 *The definition of* $\mathrm{Wh}(G)$ *of Section 1.4 makes sense and yields an abelian group which can be identified with the definition of* $\mathrm{Wh}(G)$ *given in this section above.*

Proof : Notice that the operation (4) appearing in the definition of $\mathrm{Wh}(G)$ in Section 1.4 corresponds to multiplication with an elementary matrix from the left. Since $E(R)$ is normal by Lemma 2.2, two invertible matrices A and B over $\mathbb{Z}G$ are equivalent under the equivalence relation appearing in the definition of $\mathrm{Wh}(G)$ as explained in Section 1.4 if and only their classes $[A]$ and $[B]$ in $\mathrm{Wh}(G)$ as defined in this section agree. Now the claim follows from Lemma 2.2. ■

Remark 2.5 Often $K_1(R)$ is defined in a more conceptual way in terms of automorphisms as follows. Namely, $K_1(R)$ is defined as the abelian group whose generators $[f]$ are conjugacy classes of automorphisms $f\colon P \to P$ of finitely generated projective R-modules P and which satisfies the following relations. For any commutative diagram of finitely generated projective R-modules with exact rows and automorphisms as vertical arrows

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "i"] \arrow[d, "f_0"', "\cong"] & P_1 \arrow[r, "p"] \arrow[d, "f_1"', "\cong"] & P_2 \arrow[r] \arrow[d, "f_2"', "\cong"] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "i"] & P_1 \arrow[r, "p"] & P_2 \arrow[r] & 0
\end{tikzcd}$$

we get the relation $[f_0]-[f_1]+[f_2]=0$. If $f,g\colon P\to P$ are automorphisms of a finitely generated projective R-module P, then $[g\circ f]=[g]+[f]$. Using Lemma 2.2 one easily checks that sending the class $[A]$ of an invertible (n,n)-matrix A to the class of the automorphism $R_A\colon R^n\to R^n,\ x\mapsto xA$ defines an isomorphism from $GL(R)/[GL(R),GL(R)]$ to the abelian group defined above.

2.2 Algebraic Approach to Whitehead Torsion

In this section we give the definition and prove the basic properties of the Whitehead torsion using an algebraic approach via chain complexes. This will enable us to finish the proof of the s-Cobordism Theorem 1.1. The idea which underlies the notion of Whitehead torsion will become more transparent in Section 2.3, where we will develop a geometric approach to Whitehead torsion and link it to the strategy of proof of the s-Cobordism Theorem 1.1.

We begin with some input about chain complexes. Let $f_*\colon C_*\to D_*$ be a chain map of R-chain complexes for some ring R. Define $\operatorname{cyl}_*(f_*)$ to be the chain complex with p-th differential

$$C_{p-1}\oplus C_p\oplus D_p \xrightarrow{\begin{pmatrix} -c_{p-1} & 0 & 0\\ -\mathrm{id} & c_p & 0\\ f_{p-1} & 0 & d_p\end{pmatrix}} C_{p-2}\oplus C_{p-1}\oplus D_{p-1}.$$

Define $\operatorname{cone}_*(f_*)$ to be the quotient of $\operatorname{cyl}_*(f_*)$ by the obvious copy of C_*. Hence the p-th differential of $\operatorname{cone}_*(f_*)$ is

$$C_{p-1}\oplus D_p \xrightarrow{\begin{pmatrix} -c_{p-1} & 0\\ f_{p-1} & d_p\end{pmatrix}} C_{p-2}\oplus D_{p-1}.$$

Given a chain complex C_*, define ΣC_* to be the quotient of $\operatorname{cone}_*(\mathrm{id}_{C_*})$ by the obvious copy of C_*, i.e. the chain complex with p-th differential

$$C_{p-1}\xrightarrow{-c_{p-1}} C_{p-2}.$$

Definition 2.6 *We call* $\operatorname{cyl}_*(f_*)$ *the* mapping cylinder, $\operatorname{cone}_*(f_*)$ *the* mapping cone *of the chain map* f_* *and* ΣC_* *the* suspension *of the chain complex* C_*.

These algebraic notions of mapping cylinder, mapping cone and suspension are modelled on their geometric counterparts. Namely, the cellular chain complex of a mapping cylinder of a cellular map of CW-complexes is the mapping cylinder of the chain map induced by f. From the geometry it is also clear why one obtains obvious exact sequences such as $0 \to C_* \to \operatorname{cyl}(f_*) \to \operatorname{cone}(f_*) \to 0$ and $0 \to D_* \to \operatorname{cone}_*(f_*) \to \Sigma C_* \to 0$.

A *chain contraction* γ_* for an R-chain complex C_* is a collection of R-homomorphisms $\gamma_p \colon C_p \to C_{p+1}$ for $p \in \mathbb{Z}$ such that $c_{p+1} \circ \gamma_p + \gamma_{p-1} \circ c_p = \mathrm{id}_{C_p}$ holds for all $p \in \mathbb{Z}$. We call an R-chain complex C_* *finite* if there is a number N with $C_p = 0$ for $|p| > N$ and each R-chain module C_p is a finitely generated R-module. We call an R-chain complex C_* *projective* resp. *free* resp. *based free* if each R-chain module C_p is projective resp. free resp. free with a preferred basis. Suppose that C_* is a finite based free R-chain complex which is *contractible*, i.e. which possesses a chain contraction. Put $C_{\mathrm{odd}} = \oplus_{p\in\mathbb{Z}} C_{2p+1}$ and $C_{\mathrm{ev}} = \oplus_{p\in\mathbb{Z}} C_{2p}$. Let γ_* and δ_* be two chain contractions. Define R-homomorphisms

$$\begin{array}{rcl} (c_* + \gamma_*)_{\mathrm{odd}} : C_{\mathrm{odd}} & \to & C_{\mathrm{ev}}; \\ (c_* + \delta_*)_{\mathrm{ev}} : C_{\mathrm{ev}} & \to & C_{\mathrm{odd}}. \end{array}$$

Let A be the matrix of $(c_* + \gamma_*)_{\mathrm{odd}}$ with respect to the given bases. Let B be the matrix of $(c_* + \delta_*)_{\mathrm{ev}}$ with respect to the given bases. Put $\mu_n := (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n$ and $\nu_n := (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n$. One easily checks that $(\mathrm{id} + \mu_*)_{\mathrm{odd}}$, $(\mathrm{id} + \nu_*)_{\mathrm{ev}}$ and both compositions $(c_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (c_* + \delta_*)_{\mathrm{ev}}$ and $(c_* + \delta_*)_{\mathrm{ev}} \circ (\mathrm{id} + \nu_*)_{\mathrm{ev}} \circ (c_* + \gamma_*)_{\mathrm{odd}}$ are given by upper triangular matrices whose diagonal entries are identity maps. Hence A and B are invertible and their classes $[A], [B] \in \widetilde{K}_1(R)$ satisfy $[A] = -[B]$. Since $[B]$ is independent of the choice of γ_*, the same is true for $[A]$. Thus we can associate to a finite based free contractible R-chain complex C_* an element

$$\tau(C_*) \quad = \quad [A] \quad \in \widetilde{K}_1(R). \tag{2.7}$$

Let $f_* \colon C_* \to D_*$ be a homotopy equivalence of finite based free R-chain complexes. Its mapping cone $\operatorname{cone}(f_*)$ is a contractible finite based free R-chain complex. Define the *Whitehead torsion* of f_* by

$$\tau(f_*) \quad := \quad \tau(\operatorname{cone}_*(f_*)) \quad \in \widetilde{K}_1(R). \tag{2.8}$$

We call a sequence of finite based free R-chain complexes $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ *based exact* if for any $p \in \mathbb{Z}$ the basis B for D_p can be written as a disjoint union $B' \coprod B''$ such that the image of the basis of C_p under i_p is B' and the image of B'' under q_p is the basis for E_p.

Lemma 2.9 *1. Consider a commutative diagram of finite based free R-chain complexes whose rows are based exact.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \\ & & {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle h_*}\downarrow & & \\ 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \end{array}$$

Suppose that two of the chain maps f_*, g_* and h_* are R-chain homotopy equivalences. Then all three are R-chain homotopy equivalences and

$$\tau(f_*) - \tau(g_*) + \tau(h_*) \;=\; 0;$$

2. Let $f_* \simeq g_* \colon C_* \to D_*$ be homotopic R-chain homotopy equivalences of finite based free R-chain complexes. Then

$$\tau(f_*) \;=\; \tau(g_*);$$

3. Let $f_* \colon C_* \to D_*$ and $g_* \colon D_* \to E_*$ be R-chain homotopy equivalences of based free R-chain complexes. Then

$$\tau(g_* \circ f_*) = \tau(g_*) + \tau(f_*).$$

Proof : (1) A chain map of projective chain complexes is a homotopy equivalence if and only if it induces an isomorphism on homology. The five-lemma and the long homology sequence of a short exact sequence of chain complexes imply that all three chain maps f_*, h_* and g_* are chain homotopy equivalences if two of them are.

To prove the sum formula, it suffices to show for a based free exact sequence $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ of contractible finite based free R-chain complexes that

$$\tau(C_*) - \tau(D_*) + \tau(E_*) \;=\; 0. \tag{2.10}$$

Let $u_* \colon F_* \to G_*$ be an isomorphism of contractible finite based free R-chain complexes. Since the choice of chain contraction does not affect the values of the Whitehead torsion, we can compute $\tau(F_*)$ and $\tau(G_*)$ with respect to chain contractions which are compatible with u_*. Then one easily checks in $\widetilde{K}_1(R)$

$$\tau(G_*) - \tau(F_*) \quad = \quad \sum_{p \in \mathbb{Z}} (-1)^p \cdot [u_p], \tag{2.11}$$

where $[u_p]$ is the element represented by the matrix of u_p with respect to the given bases.

Let ϵ_* be a chain contraction for E_*. Choose for any $p \in \mathbb{Z}$ an R-homomorphism $\sigma_p \colon E_p \to D_p$ satisfying $p_q \circ \sigma_q = \mathrm{id}$. Define $s_p \colon E_p \to D_p$ by $d_{p+1} \circ \sigma_{p+1} \circ \epsilon_p + \sigma_p \circ \epsilon_{p-1} \circ e_p$. One easily checks that the collection of the s_p-s defines a chain map $s_* \colon E_* \to D_*$ with $q_* \circ s_* = \mathrm{id}$. Thus we obtain an isomorphism of contractible based free R-chain complexes

$$i_* \oplus q_* \colon C_* \oplus E_* \to D_*.$$

Since the matrix of $i_p \oplus s_p$ with respect to the given basis is a block matrix of the shape $\begin{pmatrix} I_m & * \\ 0 & I_n \end{pmatrix}$ we get $[i_p \oplus s_p] = 0$ in $\widetilde{K}_1(R)$. Now (2.11) implies $\tau(C_* \oplus D_*) = \tau(E_*)$. Since obviously $\tau(C_* \oplus D_*) = \tau(C_*) + \tau(D_*)$, (2.10) and thus assertion (1) follows.

(2) If $h_* \colon f_* \simeq g_*$ is a chain homotopy, we obtain an isomorphism of based free R-chain complexes

$$\begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix} \colon \operatorname{cone}_*(f_*) = C_{*-1} \oplus D_* \to \operatorname{cone}_*(g_*) = C_{*-1} \oplus D_*.$$

We conclude from (2.11)

$$\tau(g_*) - \tau(f_*) \quad = \quad \sum_{p \in \mathbb{Z}} (-1)^p \cdot \left[\begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix} \right] \quad = \quad 0.$$

(3) Define a chain map $h_* \colon \Sigma^{-1} \operatorname{cone}_*(g_*) \to \operatorname{cone}_*(f_*)$ by

$$\begin{pmatrix} 0 & 0 \\ -\mathrm{id} & 0 \end{pmatrix} \;:\; D_p \oplus E_{p+1} \to C_{p-1} \oplus D_p.$$

There is an obvious based exact sequence of contractible finite based free R-chain complexes $0 \to \operatorname{cone}_*(f_*) \to \operatorname{cone}(h_*) \to \operatorname{cone}(g_*) \to 0$. There is also a based exact sequence of contractible finite based free R-chain complexes

$0 \to \text{cone}_*(g_* \circ f_*) \xrightarrow{i_*} \text{cone}_*(h_*) \to \text{cone}_*(\text{id}\colon D_* \to D_*) \to 0$, where i_p is given by

$$\begin{pmatrix} f_{p-1} & 0 \\ 0 & \text{id} \\ \text{id} & 0 \\ 0 & 0 \end{pmatrix} \; : \; C_{p-1} \oplus E_p \to D_{p-1} \oplus E_p \oplus C_{p-1} \oplus D_p.$$

We conclude from assertion (1)

$$\begin{aligned} \tau(h_*) &= \tau(f_*) + \tau(g_*); \\ \tau(h_*) &= \tau(g_* \circ f_*) + \tau(\text{id}_*\colon D_* \to D_*); \\ \tau(\text{id}_*\colon D_* \to D_*) &= 0. \end{aligned}$$

This finishes the proof of Lemma 2.9. ■

Now we can pass to CW-complexes. Let $f\colon X \to Y$ be a homotopy equivalence of connected finite CW-complexes. Let $p_X\colon \widetilde{X} \to X$ and $p_Y\colon \widetilde{Y} \to Y$ be the universal coverings. Fix base points $\widetilde{x} \in \widetilde{X}$ and $\widetilde{y} \in \widetilde{Y}$ such that f maps $x = p_X(\widetilde{x})$ to $y = p_Y(\widetilde{y})$. Let $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ be the unique lift of f satisfying $\widetilde{f}(\widetilde{x}) = \widetilde{y}$. We abbreviate $\pi = \pi_1(Y, y)$ and identify $\pi_1(X, x)$ in the sequel with π by $\pi_1(f, x)$. After the choice of the base points $\widetilde{x}$ and $\widetilde{y}$ we get unique operations of π on $\widetilde{X}$ and $\widetilde{Y}$. The lift $\widetilde{f}$ is π-equivariant. It induces a $\mathbb{Z}\pi$-chain homotopy equivalence $C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$. We can apply (2.8) to it and thus obtain an element

$$\tau(f) \in \text{Wh}(\pi_1(Y, y)). \qquad (2.12)$$

So far this definition depends on the various choices of base points. We can get rid of these choices as follows. If y' is a second base point, we can choose path w from y to y' in Y. Conjugation with w yields a homomorphism $c_w\colon \pi_1(Y, y) \to \pi_1(Y, y')$ which induces $(c_w)_*\colon \text{Wh}(\pi_1(Y, y)) \to \text{Wh}(\pi_1(Y, y'))$. If v is a different path from y to y', then c_w and c_v differ by an inner automorphism of $\pi_1(Y, y)$. Since an inner automorphism of $\pi_1(Y, y)$ induces the identity on $\text{Wh}(\pi_1(Y, y))$, we conclude that $(c_w)_*$ and $(c_v)_*$ agree. Hence we get a unique isomorphism $t(y, y')\colon \text{Wh}(\pi_1(Y, y)) \to \text{Wh}(\pi_1(Y, y'))$ depending only on y and y'. Moreover $t(y, y) = \text{id}$ and $t(y, y'') = t(y', y'') \circ t(y, y')$. Therefore we can define $\text{Wh}(\pi(Y))$ independently of a choice of a base point by $\coprod_{y \in Y} \text{Wh}(\pi_1(Y, y))/\sim$, where $\sim$ is the obvious equivalence relation generated by $a \sim b \Leftrightarrow t(y, y')(a) = b$ for $a \in \text{Wh}(\pi_1(Y, y))$ and $b \in \text{Wh}(\pi_1(Y, y'))$. Define $\tau(f) \in \text{Wh}(\pi_1(Y))$ by the element represented by the element introduced in (2.12). Notice that $\mathrm{Wh}(\pi(Y))$ is isomorphic to $\mathrm{Wh}(\pi_1(Y,y))$ for any base point $y \in Y$. It is not hard to check using Lemma 2.9 that $\tau(f)$ depends only on $f\colon X \to Y$ and not on the choice of the universal coverings and base points. Finally we want to drop the assumption that Y is connected. Notice that f induces a bijection $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$.

Definition 2.13 *Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Define the Whitehead group* $\mathrm{Wh}(\pi(Y))$ *of Y and the Whitehead torsion* $\tau(f) \in \mathrm{Wh}(\pi(Y))$ *by*

$$\begin{aligned} \mathrm{Wh}(\pi(Y)) &= \oplus_{C\in\pi_0(Y)} \mathrm{Wh}(\pi_1(C)); \\ \tau(f) &= \oplus_{C\in\pi_0(Y)} \tau\left(f|_{\pi_0(f)^{-1}(C)}\colon \pi_0(f)^{-1}(C) \to C\right). \end{aligned}$$

In the notation $\mathrm{Wh}(\pi(Y))$ one should think of $\pi(Y)$ as the fundamental groupoid of Y. Notice that a map $f\colon X \to Y$ induces a homomorphism $f_*\colon \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y))$ such that $\mathrm{id}_* = \mathrm{id}$, $(g \circ f)_* = g_* \circ f_*$ and $f \simeq g \Rightarrow f_* = g_*$.

Suppose that the following diagram is a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y \end{array}$$

the map i is an inclusion of CW-complexes and f is a *cellular* map of CW-complexes, i.e. respects the filtration given by the CW-structures. Then Y inherits a CW-structure by defining Y_n as the union of $j(B_n)$ and $g(X_n)$. If we equip Y with this CW-structure, we call the pushout above a *cellular pushout*.

Next we give the proof of Theorem 2.1.

(1), (2) and (3) follow from Lemma 2.9.

(4) Because of assertion (3) we have

$$\tau(f \times g) = \tau(f \times \mathrm{id}_Y) + (f \times \mathrm{id}_Y)_* \tau(\mathrm{id}_X \times g).$$

Hence it suffices to treat the case $g = \mathrm{id}_Y$. Now one proceeds by induction over the cells of Y using assertions (1), (2) and (3).

(5) This (in comparision with the other assertions much deeper result) is due to Chapman [25], [26]. This finishes the proof of Theorem 2.1. ■

We define the Whitehead torsion of an h-cobordism $(W; M_0, M_1, f_0, f_1)$

$$\tau(W, M_0) \quad \in \quad \mathrm{Wh}(\pi(M_0)). \tag{2.14}$$

by $\tau(W, M_0) := (i_0 \circ f_0)_*^{-1} (\tau(i_0 \circ f_0 \colon M_0 \to W))$, where we equip W and M_0 with some CW-structure, for instance one coming from a smooth triangulation. This is independent of the choice of CW-structure by Theorem 2.1 (5). Because of Theorem 2.1 (5) two h-cobordisms over M_0 which are diffeomorphic relative M_0 have the same Whitehead torsion.

Let R be a *ring with involution* $i \colon R \to R,\ r \to \overline{r}$, i.e. a map satisfying $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$ and $\overline{1} = 1$. Given a (m,n)-matrix $A = (a_{i,j})$ define the (n,m)-matrix A^* by $(\overline{a_{j,i}})$. We obtain an involution

$$* \colon K_1(R) \to K_1(R), \qquad [A] \mapsto [A^*]. \tag{2.15}$$

Let P be a left R-module. Define the *dual R-module* P^* to be the left R-module whose underlying abelian group is $P^* = \hom(P, R)$ and whose left R-module structure is given by $(rf)(x) := f(x)\overline{r}$ for $f \in P^*$ and $x \in P$. Then the induced involution on $K_1(R)$ corresponds to $[f \colon P \to P] \mapsto [f^* \colon P^* \to P^*]$ if one defines $K_1(R)$ as in Remark 2.5. We equip $\mathbb{Z}G$ with the involution $\overline{\sum_{g \in G} \lambda_g \cdot g} = \sum_{g \in G} \lambda_g \cdot g^{-1}$. Thus we get an involution on $K_1(\mathbb{Z}G)$ which induces an involution $* \colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$.

Lemma 2.16 *1. Let $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0', f_0', M_1', f_1')$ be h-cobordisms over M_0 and M_0' and let $g \colon M_1 \to M_0'$ be a diffeomorphism. Let $W \cup W'$ be the h-cobordism over M_0 obtained from W and W' by glueing with the diffeomorphism $f_0' \circ g \circ f_1^{-1} \colon \partial_1 W \to \partial_0 W'$. Let $u \colon \mathrm{Wh}(M_0') \to \mathrm{Wh}(M_0)$ be the isomorphism given by the composition $(f_0)_*^{-1} \circ (i_0)_*^{-1} \circ (i_1)_* \circ (f_1)_* \circ (g_*)^{-1}$, where $i_k \colon \partial_k W \to W$ is the inclusion for $k = 0, 1$. Then*

$$\tau(W \cup W', M_0) = \tau(W, M_0) + u\left(\tau(W', M_0')\right).$$

2. Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Let $v \colon \mathrm{Wh}(M_1) \to \mathrm{Wh}(M_0)$ be the isomorphism given by the composition $(f_0)_*^{-1} \circ (i_0)_*^{-1} \circ (i_1)_* \circ (f_1)_*$. Then

$$*(\tau(W, M_0)) = (-1)^{\dim(M_0)} \cdot v(\tau(W, M_1)).$$

Proof : (1) follows from Theorem 2.1.

(2) Let $C_*(\widetilde{W}, \widetilde{\partial_k W})$ be the cellular $\mathbb{Z}\pi$-chain complex with respect to a triangulation of W of the universal covering $\widetilde{W} \to W$ for $\pi = \pi_1(W) = \pi_1(\partial_0 W) = \pi_1(\partial_1 W)$. Put $n = \dim(W)$. The dual $\mathbb{Z}\pi$-chain complex $C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W})$ has as p-th chain module the dual module of $C_{n-p}(\widetilde{W}, \widetilde{\partial_1 W})$ and its p-th differential is the dual of $c_{n-p+1}\colon C_{n-p+1}(\widetilde{W}, \widetilde{\partial_1 W}) \to C_{n-p}(\widetilde{W}, \widetilde{\partial_1 W})$. Poincaŕe duality yields a $\mathbb{Z}\pi$-chain homotopy equivalence for $n = \dim(W)$

$$\cap[W, \partial W]\colon C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W}) \to C_*(\widetilde{W}, \widetilde{\partial_0 W}).$$

Inspecting the proof of Poincaré duality using dual cells [67] shows that this is a base preserving chain map if one passes to subdivisions. This implies that this $\mathbb{Z}\pi$-chain homotopy equivalence has trivial Whitehead torsion. We conclude from Lemma 2.9

$$\tau(C_*(\widetilde{W}, \widetilde{\partial_0 W})) \;=\; \tau(C^{n-*}(\widetilde{W}, \widetilde{\partial_1 W})) \;=\; (-1)^{n-1} \cdot * \left(\tau(C_*(\widetilde{W}, \widetilde{\partial_1 W}))\right).$$

This finishes the proof of Lemma 2.16 ∎

Next we can finish the proof of the s-Cobordism Theorem 1.1. As mentioned already in Remark 1.28, Theorem 1.1 (1) follows if we can show that the class of the matrix $[A] \in \mathrm{Wh}(\pi)$ agrees with $\tau(W, M_0)$ and that $\tau(W, M_0)$ depends only on the diffeomorphism type of W relative M_0 (see Lemma 1.27 (1)). Using Theorem 2.1 one can show that the homotopy equivalence $f\colon W \to X$ of Section 1.2 satisfies $\tau(f) = 0$ and hence $\tau(W, M_0) = \tau(C_*(\widetilde{X}, \widetilde{\partial_0 W}))$. We conclude $[A] = \tau(C_*(\widetilde{X}, \widetilde{\partial_0 W}))$ from the existence of the base preserving isomorphism (1.17). In view of Remark 1.28 and Theorem 1.1 assertions (1) and (2) of Theorem 1.1 follow. In order to prove Theorem 1.1 (3), we must show for two h-cobordisms $(W; M_0, M_1, f_0, f_1)$ and $(W'; M_0, M_1', f_0', f_1')$ over M_0 with $\tau(W, M_0) = \tau(W', M_0)$ that they are diffeomorphic relative M_0. Choose an h-cobordism $(W''; M_1, M_2', f_1', f_2')$ over M_1 such that $\tau(W''; M_1, M_2, f_1', f_2) \in \mathrm{Wh}(\pi(M_1))$ is the image of $\tau(W, M_0)$ under the isomorphism $(f_1 \circ i_1)_*^{-1} \circ (f_0 \circ i_0)_*\colon \mathrm{Wh}(\pi(M_0)) \to \mathrm{Wh}(\pi(M_1))$, where $i_k\colon \partial_k W \to W$ for $k = 0, 1$ is the inclusion. We can glue W and W'' along M_1 to get an h-cobordism $W \cup_{M_1} W''$ over M_0. From Lemma 2.16 (1) we get $\tau(W \cup_{M_1} W'', M_0) = 0$. Hence there is a diffeomorphism $G\colon W \cup_{M_1} W'' \to M \times [0,1]$ which induces the identity on $M_0 = M_0 \times \{0\}$ and a diffeomorphism $g_1\colon M_2 \to M_0 \times \{1\} = M_0$. Now we can form the

h-cobordism $W \cup_{M_1} W'' \cup_{g_1} W'$. Using G we can construct a diffeomorphism relative M_0 from $W \cup_{M_1} W'' \cup_{g_1} W'$ to W'. Similarly one can show that $W'' \cup_{g_1} W'$ is diffeomorphic relative M_1 to the trivial h-cobordism over M_1. Hence there is also a diffeomorphism relative M_0 from $W \cup_{M_1} W'' \cup_{g_1} W'$ to W. Hence W and W' are diffeomorphic relative M_0. This finishes the proof of the s-Cobordism Theorem 1.1. ∎

2.3 The Geometric Approach to Whitehead Torsion

In this section we introduce the concept of a simple homotopy equivalence $f\colon X \to Y$ of finite CW-complexes geometrically. We will show that the obstruction for a homotopy equivalence $f\colon X \to Y$ of finite CW-complexes to be simple is the Whitehead torsion. This proof is a CW-version or homotopy version of the proof of the s-Cobordism Theorem 1.1.

We have the inclusion of spaces $S^{n-2} \subset S^{n-1}_+ \subset S^{n-1} \subset D^n$, where $S^{n-1}_+ \subset S^{n-1}$ is the upper hemisphere. The pair (D^n, S^{n-1}_+) carries an obvious relative CW-structure. Namely, attach an $(n-1)$-cell to S^{n-1}_+ by the attaching map $\mathrm{id}\colon S^{n-2} \to S^{n-2}$ to obtain S^{n-1}. Then we attach to S^{n-1} an n-cell by the attaching map $\mathrm{id}\colon S^{n-1} \to S^{n-1}$ to obtain D^n. Let X be a CW-complex. Let $q\colon S^{n-1}_+ \to X$ be a map satisfying $q(S^{n-2}) \subset X_{n-2}$ and $q(S^{n-1}_+) \subset X_{n-1}$. Let Y be the space $D^n \cup_q X$, i.e. the pushout

$$\begin{array}{ccc} S^{n-1}_+ & \xrightarrow{q} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ D^n & \xrightarrow[g]{} & Y \end{array}$$

where i is the inclusion. Then Y inherits a CW-structure by putting $Y_k = j(X_k)$ for $k \leq n-2$, $Y_{n-1} = j(X_{n-1}) \cup g(S^{n-1})$ and $Y_k = j(X_k) \cup g(D^n)$ for $k \geq n$. Notice that Y is obtained from X by attaching one $(n-1)$-cell and one n-cell. Since the map $i\colon S^{n-1}_+ \to D^n$ is a homotopy equivalence and cofibration, the map $j\colon X \to Y$ is a homotopy equivalence and cofibration. We call j an *elementary expansion* and say that Y is obtained from X by an elementary expansion. There is a map $r\colon Y \to X$ with $r \circ j = \mathrm{id}_X$. This map is unique up to homotopy relative $j(X)$. We call any such map an *elementary collaps* and say that X is obtained from Y by an elementary collaps.

An elementary expansion is the CW-version or homotopy version of the construction in Example 1.11, where we have added a q-handle and a $(q+1)$-handle to W without changing the diffeomorphism type of W. This corresponds to an elementary expansion for the CW-complex X which we have assigned to W in Section 1.2.

Definition 2.17 *Let $f\colon X \to Y$ be a map of finite CW-complexes. We call it a simple* homotopy equivalence *if there is a sequence of maps*

$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \ldots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each f_i is an elementary expansion or elementary collaps and f is homotopic to the composition of the maps f_i.

The idea of the definition of a simple homotopy equivalence is that such a map can be written as a composition of elementary maps which are obviously homotopy equivalences. This is similar to the idea in knot theory that two knots are equivalent if one can pass from one knot to the other by a sequence of elementary moves, the so called Reidemeister moves. A Reidemeister move obviously does not change the equivalence class of a knot and, indeed, it turns out that one can pass from one knot to a second knot by a sequence of Reidemeister moves if the two knots are equivalent. The analogous statement is not true for homotopy equivalences $f\colon X \to Y$ of finite CW-complexes because there is an obstruction for f to be simple, namely its Whitehead torsion.

Lemma 2.18

1. *Let $f\colon X \to Y$ be a simple homotopy equivalence. Then its Whitehead torsion $\tau(f) \in \mathrm{Wh}(Y)$ vanishes;*
2. *Let X be a finite CW-complex. Then for any element $x \in \mathrm{Wh}(\pi(X))$ there is an inclusion $i\colon X \to Y$ of finite CW-complexes such that i is a homotopy equivalence and $i_*^{-1}(\tau(i)) = x$.*

Proof : (1) Because of Theorem 2.1 it suffices to prove for an elementary expansion $j\colon X \to Y$ that its Whitehead torsion $\tau(j) \in \mathrm{Wh}(Y)$ vanishes. We can assume without loss of generality that Y is connected. In the sequel we write $\pi = \pi_1(Y)$ and identify $\pi = \pi_1(X)$ by $\pi_1(f)$. The following diagram of based free finite $\mathbb{Z}\pi$-chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(\widetilde{X}) & \xrightarrow{C_*(\widetilde{j})} & C_*(\widetilde{Y}) & \xrightarrow{\mathrm{pr}_*} & C_*(\widetilde{Y},\widetilde{X}) & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}_*}\uparrow & & {\scriptstyle C_*(\widetilde{j})}\uparrow & & {\scriptstyle 0_*}\uparrow & & \\
0 & \longrightarrow & C_*(\widetilde{X}) & \xrightarrow{\mathrm{id}_*} & C_*(\widetilde{X}) & \xrightarrow{\mathrm{pr}_*} & 0 & \longrightarrow & 0
\end{array}$$

has based exact rows and $\mathbb{Z}\pi$-chain homotopy equivalences as vertical arrows. We conclude from Lemma 2.9 (1)

$$\begin{aligned}
\tau\left(C_*(\widetilde{j})\right) &= \tau\left(\mathrm{id}_*\colon C_*(\widetilde{X}) \to C_*(\widetilde{X})\right) + \tau\left(0_*\colon 0 \to C_*(\widetilde{Y},\widetilde{X})\right) \\
&= \tau\left(C_*(\widetilde{Y},\widetilde{X})\right).
\end{aligned}$$

The $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ is concentrated in two consecutive dimensions and its only non-trivial differential is $\mathrm{id}\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ if we identify the two non-trivial $\mathbb{Z}\pi$-chain modules with $\mathbb{Z}\pi$ using the cellular basis. This implies $\tau(C_*(\widetilde{Y},\widetilde{X})) = 0$ and hence $\tau(j) := \tau(C_*(\widetilde{j})) = 0$.

(2) We can assume without loss of generality that X is connected. Put $\pi = \pi_1(X)$. Choose an element $A \in GL(n,\mathbb{Z}\pi)$ representing $x \in \mathrm{Wh}(\pi)$. Choose $n \geq 2$. In the sequel we fix a zero-cell in X as base point. Put $X' = X \vee \vee_{j=1}^n S^n$. Let $b_j \in \pi_n(X')$ be the element represented by the inclusion of the j-th copy of S^n into X for $j = 1, 2 \ldots, n$. Recall that $\pi_n(X')$ is a $\mathbb{Z}\pi$-module. Choose for $i = 1, 2 \ldots, n$ a map $f_i\colon S^n \to X'_n$ such that $[f_i] = \sum_{j=1}^n a_{i,j} \cdot b_j$ holds in $\pi_n(X')$. Attach to X' for each $i \in \{1, 2 \ldots, n\}$ an $(n+1)$-cell by $f_i\colon S^n \to X'_n$. Let Y be the resulting CW-complex and $i\colon X \to Y$ be the inclusion. Then i is an inclusion of finite CW-complexes and induces an isomorphism on the fundamental groups. In the sequel we identify π and $\pi_1(Y)$ by $\pi_1(i)$. The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ is concentrated in dimensions n and $(n+1)$ and its $(n+1)$-differential is given by the matrix A with respect to the cellular basis. Hence $C_*(\widetilde{Y},\widetilde{X})$ is a contractible finite based free $\mathbb{Z}\pi$-chain complex with $\tau(C_*(\widetilde{Y},\widetilde{X})) = [A]$ in $\mathrm{Wh}(\pi)$. This implies that $i\colon X \to Y$ is a homotopy equivalence with $i_*^{-1}(\tau(i)) = x$. This finishes the proof of Lemma 2.18. ■

Notice that Lemma 2.18 (2) is the CW-analog of Theorem 1.1 (2).

Recall that the *mapping cylinder* $\mathrm{cyl}(f)$ of a map $f\colon X \to Y$ is defined by the pushout

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & \mathrm{cyl}(f) \end{array}$$

There are natural inclusions $i_X\colon X = X \times \{1\} \to \mathrm{cyl}(f)$ and $i_Y\colon Y \to \mathrm{cyl}(f)$ and a natural projection $p\colon \mathrm{cyl}(f) \to Y$. Notice that i_X is a cofibration and $p \circ i_X = f$ and $p_Y \circ Y = \mathrm{id}_Y$. Define *the mapping cone* $\mathrm{cone}(f)$ by the quotient $\mathrm{cyl}(f)/i_X(X)$.

Lemma 2.19 *Let $f\colon X \to Y$ be a cellular map of finite CW-complexes and $A \subset X$ be a CW-subcomplex. Then the inclusion $\mathrm{cyl}(f|_A) \to \mathrm{cyl}(f)$ and (in the case $A = \emptyset$) $i_Y\colon Y \to \mathrm{cyl}(f)$ is a composition of elementary expansions and hence a simple homotopy equivalence.*

It suffices to treat the case, where X is obtained from A by attaching an n-cell by an attaching map $q\colon S^{n-1} \to X$. Then there is an obvious pushout

$$\begin{array}{ccc} S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} & \longrightarrow & \mathrm{cyl}(f|_A) \\ \downarrow & & \downarrow \\ D^n \times [0,1] & \longrightarrow & \mathrm{cyl}(f) \end{array}$$

and an obvious homeomorphism $(D^n \times [0,1], S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\}) \to (D^{n+1}, S^n_+)$. ■

Lemma 2.20 *A map $f\colon X \to Y$ of finite CW-complexes is a simple homotopy equivalence if and only if $i_X\colon X \to \mathrm{cyl}(f)$ is a simple homotopy equivalence.*

Proof : follows from Lemma 2.19 since a composition of simple homotopy equivalence and a homotopy inverse of a simple homotopy equivalence is again a simple homotopy equivalence. ■

Fix a finite CW-complex X. Consider to pairs of finite CW-complexes (Y, X) and (Z, X) such that the inclusions of X into Y and Z are homotopy equivalent. We call them equivalent, if there is a chain of pairs of finite CW-complexes

$$(Y, X) = (Y[0], X), (Y[1], X), (Y[2], X), \dots , (Y[n], X) = (Z, X),$$

such that for each $k \in \{1, 2, \ldots, n\}$ either $Y[k]$ is obtained from $Y[k-1]$ by an elementary expansion or $Y[k-1]$ is obtained from $Y[k]$ by an elementary expansion. Denote by $\mathrm{Wh}^{\mathrm{geo}}(X)$ the equivalence classes $[Y, X]$ of such pairs (Y, X). This becomes an abelian group under the addition $[Y, X] + [Z, X] := [Y \cup_X Z, X]$. The zero element is given by $[X, X]$. The inverse of $[Y, X]$ is constructed as follows. Choose a map $r \colon Y \to X$ with $r_X = \mathrm{id}$. Let $p \colon X \times [0,1] \to X$ be the projection. Then $[(\mathrm{cyl}(r) \cup_p X) \cup_r X, X] + [Y, X] = 0$. A map $g \colon X \to X'$ induces a homomorphism $g_* \colon \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}^{\mathrm{geo}}(X')$ by sending $[Y, X]$ to $[Y \cup_g X', X']$. We obviously have $\mathrm{id}_* = \mathrm{id}$ and $(g \circ h)_* = g_* \circ h_*$. In other words, we obtain a covariant functor on the category of finite CW-complexes with values in abelian groups.

The next result may be viewed as the homotopy theoretic analog of the s-Cobordism Theorem 1.1 (3), where $\mathrm{Wh}^{\mathrm{geo}}(X)$ plays the role of the set of the diffeomorphism classes relative M_0 of h-cobordism over M_0

Theorem 2.21 *1. Let X be a finite CW-complex. The map*

$$\tau \colon \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}(X)$$

sending $[Y, X]$ to $i_*^{-1}\tau(i)$ for the inclusion $i \colon X \to Y$ is a natural isomorphism;

2. A homotopy equivalence $f \colon X \to Y$ is a simple homotopy equivalence if and only if $\tau(f) \in \mathrm{Wh}(Y)$ vanishes.

Proof : (1) The map τ is a well-defined homomorphism by Theorem 2.1 and Lemma 2.18 (1). It is surjective by Lemma 2.18 (2).

We give only a sketch of the proof of injectivity which is similar but much easier than the proof of s-Cobordism Theorem 1.1 (1). Consider an element $[Y, X]$ in $\mathrm{Wh}^{\mathrm{geo}}(X)$ with $i_*^{-1}\tau(i) = 0$ for the inclusion $i \colon X \to Y$. We want to show that $[Y, X] = [X, X]$ by reducing the number of cells, which must be attached to X to obtain Y, to zero without changing the class $[Y, X] \in \mathrm{Wh}^{\mathrm{geo}}(X)$. This corresponds in the proof of the s-Cobordism Theorem 1.1 (1) to reducing the number of handles in the handlebody decomposition to zero without changing the diffeomorphism type of the s-cobordism.

In the first step one arranges that Y is obtained from X by attaching only cells in two dimensions r and $(r+1)$ for some integer r. This is analogous to, but much easier to achieve than in the case of the s-Cobordism Theorem 1.1 (1) (see Normal Form Lemma 1.24). Details of this construction for CW-complexes can be found in [28, page 25-26].

Let $A \in GL(n, \mathbb{Z}\pi)$ be the matrix describing the $(r+1)$-differential in the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y}, \widetilde{X})$. As in the proof of the s-Cobordism Theorem 1.1 (1) (see also [28, chapter II, Section §8]) one shows that we can modify (Y, X) without changing its class $[Y, X] \in \mathrm{Wh}^{\mathrm{geo}}(X)$ such that the new matrix B is obtained from A by applying one of the operations (1), (2), (3), (4) and (5) introduced in Section 1.4. Since one can reduce A by a sequence of these operation to the trivial matrix if and only if its class $[A] \in \mathrm{Wh}(X)$ vanishes and this class $[A]$ is $i_*^{-1}\tau(i)$, the map τ is injective. Hence τ is a natural isomorphism of abelian groups.

(2) follows from Lemma 2.18 (1), Lemma 2.20 and the obvious fact that $i\colon X \to Y$ is a simple homotopy equivalence if $[Y, X] = 0$ in $\mathrm{Wh}^{\mathrm{geo}}(X)$. This finishes the proof of Theorem 2.21.

2.4 Reidemeister Torsion and Lens Spaces

In this section we deal with Reidemeister torsion which was defined earlier than Whitehead torsion and motivated the definition of Whitehead torsion. Reidemeister torsion was the first invariant in algebraic topology which could distinguish between spaces which are homotopy equivalent but not homeomorphic. Namely, it can be used to classify lens spaces up to homeomorphism.

Let X be a finite CW-complex with fundamental group π. Let U be an orthogonal finite-dimensional π-representation. Denote by $H_p(X; U)$ the homology of X with coefficients in U, i.e. the homology of the $\mathbb{R}$-chain complex $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$. Suppose that X is *U-acyclic*, i.e. $H_p(X; U) = 0$ for all $p \geq 0$. If we fix a cellular basis for $C_*(\widetilde{X})$ and some orthogonal $\mathbb{R}$-basis for U, then $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ is a contractible based free finite $\mathbb{R}$-chain complex and defines an element $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ (see (2.7)). Define the Reidemeister torsion

$$\rho(X; U) \in \mathbb{R}^{>0} \tag{2.22}$$

to be the image of $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ under the homomorphism $\widetilde{K}_1(\mathbb{R}) \to \mathbb{R}^{>0}$ sending the class $[A]$ of $A \in GL(n, \mathbb{R})$ to $\det(A)^2$. Notice that for any trivial unit $\pm\gamma$ the automorphism of U given by multiplication with $\pm\gamma$ is orthogonal and that the square of the determinant of any orthogonal automorphism of U is 1. Therefore $\rho(X; U) \in \mathbb{R}^{>0}$ is independent of the choice of cellular basis for $C_*(\widetilde{X})$ and the orthogonal basis for U and hence is an invariant of the CW-complex X and U.

Lemma 2.23 *Let $f\colon X \to Y$ be a homotopy equivalence of connected finite CW-complexes and let U be an orthogonal finite-dimensional $\pi = \pi_1(Y)$-representation. Suppose that Y is U-acyclic. Let f^*U be the orthogonal $\pi_1(X)$-representation obtained from U by restriction with the isomorphism $\pi_1(f)$. Let $\det_U\colon \mathrm{Wh}(\pi(Y)) \to \mathbb{R}^{>0}$ be the map sending the class $[A]$ of $A \in GL(n, \mathbb{Z}\pi_1(Y))$ to $\det(\mathrm{id}_U \otimes_{\mathbb{Z}\pi} R_A\colon U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n \to U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n)^2$, where $R_A\colon \mathbb{Z}\pi^n \to \mathbb{Z}\pi^n$ is the $\mathbb{Z}\pi$-automorphism induced by A. Then*

$$\frac{\rho(Y,U)}{\rho(X, f^*U)} = \det_U(\tau(f)).$$

Proof: This follows from Lemma 2.9 (1) applied to the based exact sequence of contractible based free finite $\mathbb{R}$-chain complexes $0 \to U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{Y}) \to U \otimes_{\mathbb{Z}\pi} \mathrm{cone}_*(C_*(\widetilde{f})) \to \Sigma\left(U \otimes_{\mathbb{Z}\pi_1(f)} C_*(\widetilde{X})\right) \to 0$. ■

Next we introduce the family of spaces which we want to classify completely using Reidemeister torsion. Let G be a cyclic group of finite order $|G|$. Let V be a unitary finite-dimensional G-representation. Define its *unit sphere* SV and its *unit disk* DV to be the G-subspaces $SV = \{v \in V \mid ||v|| = 1\}$ and $DV = \{v \in V \mid ||u|| \leq 1\}$ of V. Notice that a complex finite-dimensional vector space has a preferred orientation as real vector space, namely the one given by the $\mathbb{R}$-basis $\{b_1, ib_1, b_2, ib_2, \ldots, b_n, ib_n\}$ for any $\mathbb{C}$-basis $\{b_1, b_2, \ldots, b_n\}$. Any $\mathbb{C}$-linear automorphism of a complex finite-dimensional vector space preserves this orientation. Thus SV and DV are compact oriented Riemannian manifolds with isometric orientation preserving G-action. We call a unitary G-representation V *free* if the induced G-action on its unit sphere $SV = \{v \in V \mid ||u|| = 1\}$ is free. Then $SV \to G\backslash SV$ is a covering and the quotient space $L(V) := G\backslash SV$ inherits from SV the structure of an oriented closed Riemannian manifold.

Definition 2.24 *We call the closed oriented Riemannian manifold $L(V)$ the* lens space *associated to the finite-dimensional unitary representation V of the finite cyclic group G.*

One can specify these lens spaces also by numbers as follows.

Notation 2.25 *Let $\mathbb{Z}/t$ be the cyclic group of order $t \geq 2$. The 1-dimensional unitary representation V_k for $k \in \mathbb{Z}/t$ has as underlying vector space $\mathbb{C}$ and $l \in \mathbb{Z}/t$ acts on it by multiplication with $\exp(2\pi ikl/t)$. Notice that V_k is free if and only if $k \in \mathbb{Z}/t^*$, and is trivial if and only if $k = 0$ in $\mathbb{Z}/t$. Define the* *lens space* $L(t;k_1,\dots,k_c)$ *for an integer* $c \geq 1$ *and elements* $k_1, \dots, k_c$ *in* $\mathbb{Z}/t^*$ *by* $L(\oplus_{i=1}^c V_{k_i})$.

These lens spaces form a very interesting family of manifolds which can be completely classified as we will see. Two lens spaces $L(V)$ and $L(W)$ of the same dimension $n \geq 3$ have the same homotopy groups, namely their fundamental group is G and their p-th homotopy group is isomorphic to $\pi_p(S^n)$. They also have the same homology with integral coefficients, namely $H_p(L(V)) \cong \mathbb{Z}$ for $p = 0, 2n-1$, $H_p(L(V)) \cong G$ for p odd and $1 \leq p < n$ and $H_p(L(V)) = 0$ for all other values of p. Also their cohomology groups agree. Nevertheless not all of them are homotopic. Moreover, there are homotopic lens spaces which are not diffeomorphic (see Example 2.41).

Suppose that $\dim_{\mathbb{C}}(V) \geq 2$. We want to give an explicit identification

$$\pi_1(L(V),x) \quad = \quad G. \tag{2.26}$$

Given a point $x \in L(V)$, we obtain an isomorphism $s(x)\colon \pi_1(L(V),x) \xrightarrow{\cong} G$ by sending the class of a loop w in $L(V)$ with base point x to the element $g \in G$ for which there is a lift $\widetilde{w}$ in SV of w with $\widetilde{w}(1) = g\cdot\widetilde{w}(0)$. One easily checks using elementary covering theory that this is a well-defined isomorphism. If y is another base point, we obtain a homomorphism $t(x,y)\colon \pi_1(L(V),x) \to \pi_1(L(V),y)$ by conjugation with any path v in $L(V)$ from x to y. Since $\pi_1(L(V),x)$ is abelian, $t(x,y)$ is independent of the choice of v. One easily checks $t(x,x) = \mathrm{id}$, $t(y,z) \circ t(x,y) = t(x,z)$ and $s(y) \circ t(x,y) = s(x)$. Hence we can in the sequel identify $\pi_1(L(V),x)$ with G and ignore the choice of the base point $x \in L(V)$.

Let $p\colon EG \to BG$ be a model for the *universal principal G-bundle.* It has the property that for any principal G-bundle $q\colon E \to B$ there is a map $f\colon B \to BG$ called *classifying map of q* which is up to homotopy uniquely determined by the property that the pull back of p with f is isomorphic over B to q. Equivalently p can be characterized by the property that BG is a CW-complex and EG is contractible. The space BG is called the *classifying space for G.* Let $f(V)\colon L(V) \to BG$ be the classifying map of the principal G-bundle $SV \to L(V)$. Put $n = \dim(L(V)) = 2 \cdot \dim_{\mathbb{C}}(V) - 1$. Define the element

$$l(V) \quad \in \quad H_n(BG) \tag{2.27}$$

by the image of the fundamental class $[L(V)] \in H_n(L(V))$ associated to the preferred orientation of $L(V)$ under the map $H_n(f(V))\colon H_n(L(V)) \to$

$H_n(BG)$ induced by $f(V)$ on homology with integer coefficients. The map $f(V)\colon L(V) \to BG$ is n-connected since its lift $SV \to EG$ is n-connected. Hence $H_n(f(V))$ is surjective. As $H_n(L(V))$ is infinite cyclic with $[L(V)]$ as generator, $l(V)$ generates $H_n(BG)$. Notice that n is odd and that $H_n(BG)$ is isomorphic to $\mathbb{Z}/|G|$ for a cyclic group G of finite order $|G|$. A map $f\colon L(V) \to L(W)$ of lens spaces of the same dimension $n = \dim(L(V)) = \dim(L(W))$ for two free unitary G-representations V and W induces a homomorphism $\pi_1(f,x)\colon \pi_1(L(V)) \to \pi_1(L(W))$. Under the identification (2.26) this is an endomorphism $\pi_1(f)$ of G. Define

$$e(f) \quad \in \quad \mathbb{Z}/|G| \tag{2.28}$$

to be the element for which $\pi_1(f)$ sends $g \in G$ to $g^{e(f)}$. Notice that $e(f)$ depends only on the homotopy class of f and satisfies $e(g \circ f) = e(g) \cdot e(f)$ and $e(\mathrm{id}) = 1$. In particular $e(f) \in \mathbb{Z}/|G|^*$ for a homotopy equivalence $f\colon L(V) \to L(W)$. Define the *degree*

$$\deg(f) \quad \in \quad \mathbb{Z} \tag{2.29}$$

of f to be the integer for which $H_n(f)$ sends $[L(V)] \in H_n(L(V))$ to $\deg(f) \cdot [L(W)] \in H_n(L(W))$.

Given two spaces X and Y, define their *join* $X * Y$ by the pushout

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times \mathrm{cone}(Y) \\ \downarrow & & \downarrow \\ \mathrm{cone}(X) \times Y & \longrightarrow & X * Y \end{array}$$

If X and Y are G-spaces, $X * Y$ inherits a G-operation by the diagonal operation. Given two free finite-dimensional unitary G-representations, there is a G-homeomorphism

$$S(V \oplus W) \quad \cong \quad SV * SW. \tag{2.30}$$

Theorem 2.31 (Homotopy Classification of Lens Spaces) *Let $L(V)$ and $L(W)$ be two lens spaces of the same dimension $n \geq 3$. Then*

1. *The map*

$$e \times \deg\colon [L(V), L(W)] \to \mathbb{Z}/|G| \times \mathbb{Z}, \qquad [f] \;\mapsto\; (e(f), \deg(f))$$

is injective, where $[L(V), L(W)]$ is the set of homotopy classes of maps from $L(V)$ to $L(W)$;

2. *An element* $(u, d) \in \mathbb{Z}/|G| \times \mathbb{Z}$ *is in the image of* $e \times \deg$ *if and only if we get in* $H_n(BG)$

$$d \cdot l(W) = u^{\dim_{\mathbb{C}}(V)} \cdot l(V);$$

3. $L(V)$ *and* $L(W)$ *are homotopy equivalent if and only if there is an element* $e \in \mathbb{Z}/|G|^*$ *satisfying in* $H_n(BG)$

$$\pm l(W) = e^{\dim_{\mathbb{C}}(V)} \cdot l(V).$$

   $L(V)$ *and* $L(W)$ *are oriented homotopy equivalent if and only if there is an element* $e \in \mathbb{Z}/|G|^*$ *satisfying in* $H_n(BG)$

$$l(W) = e^{\dim_{\mathbb{C}}(V)} \cdot l(V);$$

4. *The lens spaces* $L(t; k_1, \ldots, k_c)$ *and* $L(t; l_1, \ldots, l_c)$ *are homotopy equivalent if and only if there is* $e \in \mathbb{Z}/t^*$ *such that we get* $\prod_{i=1}^c k_i = \pm e^c \cdot \prod_{i=1}^c l_i$ *in* $\mathbb{Z}/t^*$.

   The lens spaces $L(t; k_1, \ldots, k_c)$ *and* $L(t; l_1, \ldots, l_c)$ *are oriented homotopy equivalent if and only if there is* $e \in \mathbb{Z}/t^*$ *such that we get* $\prod_{i=1}^c k_i = e^c \cdot \prod_{i=1}^c l_i$ *in* $\mathbb{Z}/t^*$.

Proof : (1) Obviously $e(f)$ and $\deg(f)$ depend only on the homotopy type of f. Consider two maps $f_0, f_1 \colon : L(V) \to L(W)$ with $e(f_0) = e(f_1)$ and $\deg(f_0) = \deg(f_1)$. Choose lifts $\widetilde{f_0}, \widetilde{f_1} \colon SV \to SW$. Let $\alpha \colon G \to G$ be the automorphism sending g to $g^{e(f_0)} = g^{e(f_1)}$. Then both lifts $\widetilde{f_k}$ are α-equivariant. Since G acts orientation preserving and freely on SV and SW, the projection induces a map $H_n(SV) \to H_n(L(V))$ resp. $H_n(SW) \to H_n(L(W))$ which send the fundamental class $[SV]$ resp. $[SW]$ to $|G| \cdot [L(V)]$ resp. $|G| \cdot [L(W)]$. Hence $\deg(f_k) = \deg(\widetilde{f_k})$ for $k = 0, 1$. Thus $\deg(f_0) = \deg(f_1)$ implies $\deg(\widetilde{f_0}) = \deg(\widetilde{f_1})$. Let $\alpha^* SW$ be the G-space obtained from SW by restricting the group action with α. Then $\widetilde{f_0}$ and $\widetilde{f_1} \colon SV \to \alpha^* SW$ are G-maps with $\deg(\widetilde{f_0}) = \deg(\widetilde{f_1})$. It suffices to show that they are G-homotopic.

We outline an elementary proof of this fact. There is a G-CW-structure on SV such that there is exactly one equivariant cell $G \times D^k$ in each dimension. It induces a G-CW-structure on $SV \times [0, 1]$ using the standard CW-structure on $[0, 1]$. We want to define inductively for $k = -1, 0, \ldots, n$ G-maps $h_k \colon SV \times \{0, 1\} \cup SV_k \times [0, 1] \to \alpha^* SW$ such that h_k extends h_{k-1}

and $h_{-1} = \widetilde{f}_0 \coprod \widetilde{f}_1 \colon SV \times \{0,1\} \to \alpha^* SW$. Then h_n will be the desired G-homotopy between f_0 and f_1. Notice that $SV \times \{0,1\} \cup SV_k \times [0,1]$ is obtained from $SV \times \{0,1\} \cup SV_{k-1} \times [0,1]$ by attaching one equivariant cell $G \times D^{k+1}$ with some attaching G-map $q_k \colon G \times S^k \to SV$. In the induction step we must show that the composition $h_{k-1} \circ q_k \colon G \times S^k \to SW$ can be extended to a G-map $G \times D^{k+1} \to SW$. This is possible if and only if its restriction to $S^k = \{1\} \times S^k$ can be extended to a (non-equivariant) map $D^{k+1} \to SW$. Since SW is n-connected, this can be done for any map $S^k \to SW$ for $k < n$. In the final step we run into the obstruction that a map $S^n \to SW$ can be extended to a map $D^{n+1} \to SW$ if and only if its degree is zero. Now one has to check that the degree of the map $(h_{n-1} \circ q_n)|_{\{1\} \times S^n} \colon \{1\} \times S^n = S^n \to SW$ is exactly the difference $\deg(f_0) - \deg(f_1)$ and hence zero.

(2) Given a map $f \colon LV \to LW$, let $\pi_1(f) \colon G \to G$ be the map induced on the fundamental groups under the identification (2.26). Then the following diagram commutes

$$\begin{array}{ccc}
H_n(LV) & \xrightarrow{H_n(f)} & H_n(LW) \\
{\scriptstyle H_n(f(V))}\downarrow & & \downarrow{\scriptstyle H_n(f(W))} \\
H_n(BG) & \xrightarrow[H_n(B\pi_1(f))]{} & H_n(BG)
\end{array}$$

This implies

$$\deg(f) \cdot l(W) \quad = \quad H_n(B\pi_1(f))(l(V)). \tag{2.32}$$

Given $e \in \mathbb{Z}$, let $m_e \colon \mathbb{Z}/t \to \mathbb{Z}/t$ be multiplication with e. Let e, $k_1, \ldots,$ k_c, $l_1, \ldots, l_c$ be integers which are prime to t. Fix integers $k_1', \ldots, k_c'$ such that we get in $\mathbb{Z}/t$ the equation $k_i \cdot k_i' = 1$. Define a m_e-equivariant map $d_i \colon V_{k_i} \to V_{l_i}$ by $z \mapsto z^{k_i' l_i e}$. It has degree $k_i' l_i e$. The m_e-equivariant map $*_{i=1}^c d_i \colon *_{i=1}^c SV_{k_i} \to *_{i=1}^c SV_{l_i}$ yields under the identification (2.30) a m_e-equivariant map $\widetilde{f} \colon S(\oplus_{i=1}^c V_{k_i}) \to S(\oplus_{i=1}^c V_{l_i})$ of degree $e^c \cdot \prod_{i=1}^c k_i' l_i$. By taking the quotient under the G-action yields a map $f \colon L(t; k_1, \ldots, k_c) \to L(t; l_1, \ldots, l_c)$ of $n = (2c-1)$-dimensional lens spaces with $\deg(f) = e^c \cdot \prod_{i=1}^c k_i' l_i$ and $e(f) = e$. We conclude from (2.32) in the special case $k_i = l_i$ for $i = 1, \ldots, c$ that $H_n(Bm_e) \colon H_n(B\mathbb{Z}/t) \to H_n(B\mathbb{Z}/t)$ is multiplication with e^c since $l(t; k_1 \ldots, k_c) := l(\oplus_{i=1}^c V_{k_i})$ is always a generator of $H_n(BG)$. Thus (2.32) becomes

$$\deg(f) \cdot l(W) \quad = \quad e(f)^{\dim_{\mathbb{C}}(V)} \cdot l(V). \tag{2.33}$$

We conclude from (2.33) in the special case $e = 1$ that we get in $H_n(B(\mathbb{Z}/t))$ the equation

$$\prod_{i=1}^{c} l_i \cdot l(t; l_1, \ldots, l_c) \quad = \quad \prod_{i=1}^{c} k_i \cdot l(t; k_1, \ldots, k_c). \tag{2.34}$$

Next we show for a map $f\colon L(V) \to L(W)$ and $m \in \mathbb{Z}$ that we can find another map $f'\colon LV \to LW$ with $e(f) = e(f')$ and $\deg(f') = \deg(f) + m \cdot |G|$. Fix some embedded disk $D^n \subset SV$ such that $g \cdot D^n \cap D^n \neq \emptyset$ implies $g = 1$. Let $\widetilde{f}\colon SV \to SW$ be a lift of f. Recall that f is m_e-equivariant for $m_e\colon G \to G,\ g \mapsto g^e$. Define a new m_e-equivariant map $\widetilde{f'}\colon SV \to SW$ as follows. Outside $G \cdot D^n$ the maps $\widetilde{f}$ and $\widetilde{f'}$ agree. Let $\frac{1}{2}D^n$ be $\{x \in D^n \mid ||x|| \leq 1/2\}$. Define $\widetilde{f'}$ on $G \cdot \left(D^n - \frac{1}{2}D^n\right)$ by sending $(g, t \cdot x)$ for $g \in G$, $t \in [1/2, 1]$ and $x \in S^{n-1} = \partial D^n$ to $g^e \cdot \widetilde{f}((2t-1)x)$. Let $c\colon (\frac{1}{2}D^n, \partial\frac{1}{2}D^n) \to (SW, \widetilde{f}(0))$ for $0 \in D^n$ the origin be a map such that the induced map $(\frac{1}{2}D^n)/\partial(\frac{1}{2}D^n) \to SW$ has degree m. Define $f'|_{G \cdot \frac{1}{2}D^n}\colon G \cdot \frac{1}{2}D^n \to SW$ by sending (g, x) to $g^e \cdot c(x)$. One easily checks that $\widetilde{f'}$ has degree $\deg(\widetilde{f}) + m \cdot |G|$. Then $G\backslash\widetilde{f'}$ is the desired map f'.

Notice that there is at least one map $f\colon L(V) \to L(W)$ with $e(f) = e$ for any given $e \in \mathbb{Z}/|G|$. This follows by an argument similar to the one above, since G acts freely on SV and SW is $(\dim(SV) - 1)$-connected. Now assertion (2) follows.

(3) Let $f\colon SV \to SW$ be a map. Then f is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups. This is the case if and only if $\pi_1(f)$ is an automorphism and $\widetilde{f}\colon SV \to SW$ induces an isomorphism on all homology groups. This follows from the Hurewicz Theorem [121, Theorem IV.7.13 on page 181] and the fact that a covering induces an isomorphism on π_n for $n \geq 2$. Hence f is a homotopy equivalence if and only if $e(f) \in \mathbb{Z}/|G|^*$ and $\deg(f) = \deg(\widetilde{f}) = \pm 1$. Recall that f is an oriented homotopy equivalence if and only if f is a homotopy equivalence and $\deg(f) = 1$. Now the claim follows from assertion (2).

(4) follows from (2.34) and assertion (3). This finishes the proof of Theorem 2.31. ■

Lemma 2.35 *Let G be a cyclic group of finite order $|G|$.*

1. *Let V be a free unitary finite-dimensional G-representation. Let U be an orthogonal finite-dimensional G-representation with $U^G = 0$. Then*

$L(V)$ is U-acyclic and the Reidemeister torsion $\rho(L(V);U) \in \mathbb{R}^{>0}$ is defined;

2. *Let V,V_1, V_2 be free unitary finite-dimensional G-representations. Let U, U_1 and U_2 be orthogonal finite-dimensional G-representations with $U^G = U_1^G = U_2^G = 0$. Then*

$$\begin{aligned} \rho(L(V_1 \oplus V_2), U) &= \rho(L(V_1);U) \cdot \rho(L(V_2);U); \\ \rho(L(V), U_1 \oplus U_2) &= \rho(L(V);U_1) \cdot \rho(L(V);U_2). \end{aligned}$$

Proof : (1) Let X be a finite CW-complex with fundamental group G. We show that X is U-acyclic if G acts trivial on $H_p(\widetilde{X})$ and U is an orthogonal finite-dimensional G-representation with $U^G = 0$. We have to show that $H_p(C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} U) \cong H_p(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R} \otimes_{\mathbb{R}G} U$ vanishes. By assumption $H_p(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a direct sum of copies of the trivial G-representation $\mathbb{R}$ and U is a direct sum of non-trivial irreducible representations. Since for any non-trivial irreducible G-representation W the tensor product $\mathbb{R} \otimes_{\mathbb{R}G} W$ is trivial, the claim follows.

(2) The claim for the second equation is obvious, it remains to prove the first. Since G-acts trivially on the homology of SV and SW and hence on the homology of $SV \times SW$, $SV \times DV$ and $DV \times SW$, we conclude from Theorem 2.1 (1), Lemma 2.9 (1), Lemma 2.23 and (2.30)

$$\begin{aligned} &\rho(L(V \oplus W);U) \\ &\quad = \rho(G\backslash(DV \times SW);U) \cdot \rho(G\backslash(SV \times DW);U) \cdot (\rho(G\backslash(SV \times SW);U))^{-1}. \end{aligned}$$

Hence it remains to show

$$\begin{aligned} \rho(G\backslash(DV \times SW);U) &= \rho(L(W);U); \\ \rho(G\backslash(SV \times DW);U) &= \rho(L(V);U); \\ \rho(G\backslash(SV \times SW);U) &= 1. \end{aligned}$$

These equations will follow from the following slightly more general formula (2.36) below. Let D_* be a finite $\mathbb{R}G$-chain complexes such that G-acts trivially on its homology. Assume that D_* comes with a $\mathbb{R}$-basis. Then $C_*(SV) \otimes_{\mathbb{Z}} D_*$ with the diagonal G-action is a finite $\mathbb{R}G$-chain complex such that G acts trivially on its homology and there is a preferred $\mathbb{R}G$-basis. Let $\chi_{\mathbb{R}}(D_*)$ be the Euler characteristic of D_*, i.e.

$$\chi_{\mathbb{R}}(D_*) := \sum_{p \in \mathbb{Z}} (-1)^p \cdot \dim_{\mathbb{R}}(D_p) = \sum_{p \in \mathbb{Z}} (-1)^p \cdot \dim_{\mathbb{R}}(H_p(D_*)).$$

Then

$$\rho(C_*(SV)\otimes_{\mathbb{Z}} D_*\otimes_{\mathbb{R}G} U) \quad = \quad \rho(L(V);U)^{\chi_{\mathbb{R}}(D_*)}, \tag{2.36}$$

where $\rho(C_*(SV)\otimes_{\mathbb{Z}} D_*\otimes_{\mathbb{R}G} U)\in\mathbb{R}^{>0}$ is the Reidemeister torsion of the acyclic based free finite $\mathbb{R}$-chain complex $C_*(SV)\otimes_{\mathbb{Z}} D_*\otimes_{\mathbb{R}G} U$ which is defined to be the image of the Whitehead torsion $\tau(C_*(SV)\otimes_{\mathbb{Z}} D_*\otimes_{\mathbb{R}G} U)\in\widetilde{K}_1(\mathbb{R})$ (see (2.7)) under the homomorphism $\widetilde{K}_1(\mathbb{R})\to\mathbb{R}^{>0}$, $[A]\mapsto\det(A)^2$.

It remains to prove (2.36). Since $\mathbb{R}G$ is semi-simple, there is a $\mathbb{R}G$-chain homotopy equivalence $p_*\colon D_*\to H_*(D_*)$, where we consider $H_*(D_*)$ as $\mathbb{R}G$-chain complex with the trivial differential.

Equip $H_*(D_*)$ with an $\mathbb{R}$-basis. Then we get in $\widetilde{K_1}(\mathbb{R}G)$ from Lemma 2.9 (1) and $\chi(L(V))=0$

$$\begin{aligned}
&\tau\left(\mathrm{id}_{C_*(SV)}\otimes p_*\colon C_*(SV)\otimes_{\mathbb{Z}} D_*\to C_*(SV)\otimes_{\mathbb{Z}} H_*(D_*)\right)\\
&= \sum_{q\in\mathbb{Z}}(-1)^q\cdot\tau\left(\mathrm{id}_{C_q(SV)}\otimes p_*\colon C_q(SV)\otimes_{\mathbb{Z}} D_*\to C_q(SV)\otimes_{\mathbb{Z}} H_*(D_*)\right)\\
&= \sum_{q\in\mathbb{Z}}(-1)^q\cdot\dim_{\mathbb{Z}G}(C_p(SV))\cdot\tau\left(\mathrm{id}_{\mathbb{Z}G}\otimes p_*\colon \mathbb{Z}G\otimes_{\mathbb{Z}} D_*\to\mathbb{Z}G\otimes_{\mathbb{Z}} H_*(D_*)\right)\\
&= \chi(L(V))\cdot\tau\left(\mathrm{id}_{\mathbb{Z}G}\otimes p_*\colon \mathbb{Z}G\otimes_{\mathbb{Z}} D_*\to\mathbb{Z}G\otimes_{\mathbb{Z}} H_*(D_*)\right)\\
&= 0.
\end{aligned}$$

The chain complex version of Lemma 2.23 shows

$$\rho(C_*(SV)\otimes_{\mathbb{Z}} D_*\otimes_{\mathbb{R}G} U) \quad = \quad \rho(C_*(SV)\otimes_{\mathbb{Z}} H_*(D_*)\otimes_{\mathbb{R}G} U).$$

Obviously

$$\rho(C_*(SV)\otimes_{\mathbb{Z}}H_*(D_*)\otimes_{\mathbb{R}G}U) \;=\; \rho(C_*(SV)\otimes_{\mathbb{Z}G}U)^{\chi_{\mathbb{R}}(D_*)} \;=\; \rho(L(V);U)^{\chi_{\mathbb{R}}(D_*)}.$$

This finishes the proof of Lemma 2.35. ■

Theorem 2.37 (Diffeomorphism Classification of Lens Spaces) *Let $L(V)$ and $L(W)$ be two lens spaces of the same dimension $n\geq 3$. Then the following statements are equivalent.*

1. *There is an automorphism $\alpha\colon G\to G$ such that V and α^*W are isomorphic as orthogonal G-representations;*

2. *There is an isometric diffeomorphism $L(V)\to L(W)$;*

3. *There is a diffeomorphism* $L(V) \to L(W)$*;*

4. *There is homeomorphism* $L(V) \to L(W)$*;*

5. *There is simple homotopy equivalence* $L(V) \to L(W)$*;*

6. *There is an automorphism* $\alpha\colon G \to G$ *such that for any orthogonal finite-dimensional representation* U *with* $U^G = 0$

$$\rho(L(W), U) = \rho(L(V), \alpha^* U)$$

holds.

7. *There is an automorphism* $\alpha\colon G \to G$ *such that for any non-trivial* 1*-dimensional unitary* G*-representation* U

$$\rho(L(W), \operatorname{res} U) = \rho(L(V), \alpha^* \operatorname{res} U)$$

holds, where the orthogonal representation $\operatorname{res} U$ *is obtained from* U *by restricting the scalar multiplication from* $\mathbb{C}$ *to* $\mathbb{R}$.

Proof : The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious or follow directly from Theorem 2.1 (5) and Lemma 2.23. Hence it remains to prove the implication (7) $\Rightarrow$ (1).

Fix a generator $g \in G$, or equivalently, an identification $G = \mathbb{Z}/|G|$. Choose $e \in \mathbb{Z}/|G|^*$ such that $\alpha\colon G \to G$ sends g to g^e. Recall that V_k denotes the 1-dimensional unitary G-representation for which g acts by multiplication with $\exp(2\pi i k/|G|)$. The based $\mathbb{Z}G$-chain complex of SV_k is concentrated in dimension 0 and 1 and its first differential is $g^k - 1\colon \mathbb{Z}G \to \mathbb{Z}G$. If g acts on U by multiplication with the $|G|$-th root of unity ζ, then we conclude

$$\begin{aligned} \rho(LV_k; U) &= ||\zeta^k - 1||^2; \\ \rho(LV_k; \alpha^* U) &= ||\zeta^{ek} - 1||^2. \end{aligned}$$

We can write for appropriate numbers $c \in \mathbb{Z}$, $c \geq 1$, $k_i, l_j \in \mathbb{Z}/G^*$

$$\begin{aligned} V &= \oplus_{i=1}^c V_{k_i}; \\ W &= \oplus_{j=1}^c V_{l_j}. \end{aligned}$$

We conclude from Lemma 2.35

$$\begin{aligned}\rho(LV;U) &= \prod_{i=1}^{c}(\zeta^{k_i}-1)\cdot(\zeta^{-k_i}-1);\\ \rho(LW;\alpha^*U) &= \prod_{i=1}^{c}(\zeta^{el_j}-1)\cdot(\zeta^{-el_j}-1).\end{aligned}$$

This implies that for any $|G|$-th root ζ of unity with $\zeta \neq 1$ the following equation holds

$$\prod_{i=1}^{c}(\zeta^{k_i}-1)\cdot(\zeta^{-k_i}-1) = \prod_{i=1}^{c}(\zeta^{el_j}-1)\cdot(\zeta^{-el_j}-1). \tag{2.38}$$

We will need the following number theoretic result due to Franz whose proof can be found for instance in [33].

Lemma 2.39 (Franz' Independence Lemma) *Let $t \geq 2$ be an integer and $S = \{j \in \mathbb{Z} \mid 0 < j < t, (j,t) = 1\}$. Let $(a_j)_{j\in S}$ be a sequence of integers indexed by S such that $\sum_{j\in S} a_j = 0$, $a_j = a_{t-j}$ for $j \in S$ and $\prod_{j\in S}(\zeta^j - 1)^{a_j} = 1$ holds for every t-th root of unity $\zeta \neq 1$. Then $a_j = 0$ for $j \in S$.*

Put $t = |G|$. For $j \in S = \{j \in \mathbb{Z} \mid 0 < j < |G|, (j, |G|) = 1\}$ let x_j be the number of elements in the sequence k_1, $-k_1$, k_2, $-k_2$, $\ldots$, k_c, $-k_c$, which are congruent j modulo $|G|$. Each of the elements k_i is prime to $|G|$ and hence $\pm x_j$ is congruent mod $|G|$ to some $j \in S$. This implies $\sum_{j\in S} x_j = 2c$. Obviously $x_j = x_{|G|-j}$ for $j \in S$. Define analogously a sequence $(y_j)_{j\in S}$ for the sequence el_1, $-el_1$, el_2, $-el_2$, $\ldots$, el_c, $-el_c$. Put $a_j = x_j - y_j$ for $j \in S$. Then $\sum_{\in S} a_j = 0$, $a_j = a_{|G|-j}$ for $j \in S$ and we conclude from (2.38) that $\prod_{j\in S}(\zeta^j - 1)^{a_j} = 1$ for any $|G|$-th root of unity $\zeta \neq 1$. We conclude from Franz' Independence Lemma 2.39 that $a_j = 0$ and hence $x_j = y_j$ holds for $j \in S$. This implies that there is a permutation $\sigma \in \Sigma_c$ together with signs $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2, \ldots , c$ such that k_i and $\epsilon_i \cdot l_{\sigma(i)}$ are congruent modulo $|G|$. But this implies that V_i and $\alpha^* W_{\sigma(i)}$ are isomorphic as orthogonal G-representations. Hence V and $\alpha^* W$ are isomorphic as orthogonal representations. This finishes the proof of Theorem 2.37. ■

Corollary 2.40 *Two lens spaces $L(t; k_1, \ldots , k_c)$ and $L(t; l_1, \ldots , l_c)$ are homeomorphic if and only there are $e \in \mathbb{Z}/t^*$, signs $\epsilon_i \in \{\pm 1\}$ and a permutation $\sigma \in \Sigma_c$ such that $k_i = \epsilon_i \cdot e \cdot l_{\sigma(i)}$ holds in $\mathbb{Z}/t^*$ for $i = 1, 2, \ldots , c$.*

Notice that $\oplus_{i=1}^c V_{k_i}$ and $\oplus_{i=1}^c V_{l_i}$ are isomorphic as orthogonal representations if and only if there are signs $\epsilon_i \in \{\pm 1\}$ and a permutation $\sigma \in \Sigma_c$ such that $k_i = \epsilon_i \cdot l_{\sigma(i)}$ holds in $\mathbb{Z}/t^*$ for $i = 1, 2, \ldots, c$. If $m_e \colon \mathbb{Z}/t \to \mathbb{Z}/t$ is multiplication with $e \in \mathbb{Z}/t^*$, then the restriction $m_e^*(\oplus_{i=1}^c V_{l_i})$ is $\oplus_{i=1}^c V_{el_i}$. Now apply Theorem 2.37. ■

Example 2.41 We conclude from Theorem 2.31 and Corollary 2.40 the following facts:

1. Any homotopy equivalence $L(7; k_1, k_2) \to L(7; k_1, k_2)$ has degree 1. Thus $L(7; k_1, k_2)$ possesses no orientation reversing selfdiffeomorphism;

2. $L(5; 1, 1)$ and $L(5; 2, 1)$ have the same homotopy groups, homology groups and cohomology groups, but they are not homotopy equivalent;

3. $L(7; 1, 1)$ and $L(7; 2, 1)$ are homotopy equivalent, but not homeomorphic.

2.5 Miscellaneous

We mention that lens spaces are the only closed manifolds M which carry a Riemannian metric with sectional curvature which is constant 1, provided $\pi_1(M)$ is cyclic. Reidemeister torsion can be used to classify all such manifolds without any assumption on $\pi_1(M)$ and to show that two finite-dimensional (not necessarily free) orthogonal representations V and W have G-diffeomorphic unit spheres SV and SW if and only if they are isomorphic as orthogonal representations (see [32]). The corresponding statement is false if one replaces G-diffeomorphic by G-homeomorphic (see [21],[23], [49]).

Let (W, L, L') be an h-cobordism of lens spaces which is compatible with the orientations and the identifications of $\pi_1(L)$ and $\pi_1(L')$ with G. Then W is diffeomorphic relative L to $L \times [0, 1]$ and L and L' are diffeomorphic [84, Corollary 12.13 on page 410].

We refer to [28] and [84] for more information about Whitehead torsion and lens spaces.

Chapter 3

Normal Maps and the Surgery Problem

Introduction

In this chapter we want to take the first step for the solution of the following problem

Problem 3.1 *Let X be a topological space. When is X homotopy equivalent to a closed manifold?*

We will begin with discussing Poincaré duality, which is the first obstruction from algebraic topology for a positive answer to Problem 3.1 above, in Section 3.1. In Section 3.2 we will deal with the Spivak normal fibration of a finite Poincaré complex X which is a reminiscence of the normal bundle of the embedding of a closed manifold into some high-dimensional Euclidean space. We will explain and motivate the notion of a normal map of degree one in Section 3.3. This is a map of degree one $f\colon M \to X$ from a closed manifold M to a finite Poincaré complex X covered by some bundle data. The surgery problem is to change it to a homotopy equivalence leaving the target fixed and changing the source without loosing the structure of a closed manifold. In Section 3.4 we will introduce the surgery step. This is the manifold analog of the process in the world of CW-complexes which is given by adding a cell in order to kill an element in the homotopy group. We will show that we can make a normal map $f\colon M \to X$ highly connected by carrying out a finite number of surgery steps. The surgery obstructions will later arise as obstructions to make f connected in the middle dimension. If

f is also connected in the middle dimension, then Poincaré duality implies that f is a homotopy equivalence from a closed manifold to X.

We mention that Problem 3.1 is an important problem but the main success of surgery comes from its contribution to the question whether two closed manifolds are diffeomorphic (see the surgery program in Remark 1.5) and the construction of exotic structures, i.e. different smooth structures on the same topological manifold. We will restrict our attention for some time to Problem 3.1 because it is a good motivation for certain techniques such as the surgery step, for certain notions such as normal maps of degree one and for surgery obstructions and surgery obstruction groups such as the signature and the L-groups. After we have developed the machinery which allows us to solve Problem 3.1 in principle, we will develop it further in order to carry out the surgery program.

3.1 Poincaré Duality

In order to state Poincaré duality on the level we will need we have to introduce some algebra. Recall that a *ring with involution* R is an associative ring R with unit together with an *involution of rings* $\overline{}\colon R \to R,\ \ r \mapsto \overline{r}$, i.e. a map satisfying $\overline{\overline{r}} = r$, $\overline{r+s} = \overline{r}+\overline{s}$, $\overline{rs} = \overline{s}\,\overline{r}$ and $\overline{1} = 1$ for $r, s \in R$. Our main example will be the group ring AG for some commutative associative ring A with unit and a group G together with a homomorphism $w\colon G \to \{\pm 1\}$. The so called *w-twisted involution* sends $\sum_{g\in G} a_g \cdot g$ to $\sum_{g\in G} w(g)\cdot a_g \cdot g^{-1}$. Let M be a left R-module. Then $M^* := \hom_R(M, R)$ carries a canonical right R-module structure given by $(fr)(m) = f(m)\cdot r$ for a homomorphism of left R-modules $f\colon M \to R$ and $m \in M$. The involution allows us to view $M^* = \hom_R(M;R)$ as a left R-module, namely define rf for $r \in R$ and $f \in M^*$ by $(rf)(m) := f(m)\cdot\overline{r}$ for $m \in M$. Given an R-chain complex of left R-modules C_* and $n \in \mathbb{Z}$, we define its dual chain complex C^{n-*} to be the chain complex of left R-modules whose p-th chain module is $\hom_R(C_{n-p}, R)$ and whose p-th differential is given by

$$\begin{aligned}\hom_R(c_{n-p+1}, \mathrm{id})\colon (C^{n-*})_p = \hom_R(C_{n-p}, R)&\\ \to (C^{n-*})_{p-1} = \hom_R(C_{n-p+1}, R).&\end{aligned}$$

Consider a connected finite CW-complex X with fundamental group π and a group homomorphism $w\colon \pi \to \{\pm 1\}$. In the sequel we use the w-twisted involution. Denote by $C_*(\widetilde{X})$ the cellular $\mathbb{Z}\pi$-chain complex of the universal covering. Recall that this is a free $\mathbb{Z}\pi$-chain complex and the cellular structure on X determines a cellular $\mathbb{Z}\pi$-basis on it such that each basis element corresponds to a cell in X. This basis is not quite unique but its equivalence class depends only on the CW-structure of X (see Section 1.2). The product $\widetilde{X} \times \widetilde{X}$ equipped with the diagonal π-action is again a π-CW-complex. The diagonal map $D\colon \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ sending $\widetilde{x}$ to $(\widetilde{x}, \widetilde{x})$ is π-equivariant but not cellular. By the equivariant cellular Approximation Theorem (see for instance [69, Theorem 2.1 on page 32]) there is up to cellular π-homotopy precisely one cellular π-map $\overline{D}\colon \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ which is π-homotopic to D. It induces a $\mathbb{Z}\pi$-chain map unique up to $\mathbb{Z}\pi$-chain homotopy

$$C_*(\overline{D})\colon C_*(\widetilde{X}) \to C_*(\widetilde{X} \times \widetilde{X}). \tag{3.2}$$

There is a natural isomorphism of based free $\mathbb{Z}\pi$-chain complexes

$$i_*\colon C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(\widetilde{X}) \xrightarrow{\cong} C_*(\widetilde{X} \times \widetilde{X}). \tag{3.3}$$

Denote by $\mathbb{Z}^w$ the $\mathbb{Z}\pi$-module whose underlying abelian group is $\mathbb{Z}$ and on which $g \in G$ acts by $w(g) \cdot \mathrm{id}$. Given two projective $\mathbb{Z}\pi$-chain complexes C_* and D_* we obtain a natural $\mathbb{Z}$-chain map unique up to $\mathbb{Z}$-chain homotopy

$$s\colon \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} (C_* \otimes_{\mathbb{Z}} D_*) \to \hom_{\mathbb{Z}\pi}(C^{-*}, D_*) \tag{3.4}$$

by sending $1 \otimes x \otimes y \in \mathbb{Z} \otimes C_p \otimes D_q$ to

$$s(1 \otimes x \otimes y)\colon \hom_{\mathbb{Z}\pi}(C_p, \mathbb{Z}\pi) \to D_q, \qquad (\phi\colon C_p \to \mathbb{Z}\pi) \;\mapsto\; \overline{\phi(x)} \cdot y.$$

The composite of the chain map (3.4) for $C_* = D_* = C_*(\widetilde{X})$, the inverse of the chain map (3.3) and the chain map (3.2) yields a $\mathbb{Z}$-chain map

$$\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) \to \hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X})).$$

Notice that the n-homology of $\hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X}))$ is the set of $\mathbb{Z}\pi$-chain homotopy classes $[C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi}$ of $\mathbb{Z}\pi$-chain maps from $C^{n-*}(\widetilde{X})$ to $C_*(\widetilde{X})$. Define $H_n(X; \mathbb{Z}^w) := H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$. Taking the n-th homology group yields a well-defined $\mathbb{Z}$-homomorphism

$$\cap\colon H_n(X; \mathbb{Z}^w) \to [C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi} \tag{3.5}$$

which sends a class $x \in H_n(X; \mathbb{Z}^w) = H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$ to the $\mathbb{Z}\pi$-chain homotopy class of a $\mathbb{Z}\pi$-chain map denoted by $? \cap x\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$.

Definition 3.6 *A* connected finite n-dimensional Poincaré complex *is a connected finite CW-complex of dimension n together with a group homomorphism $w = w_1(X)\colon \pi_1(X) \to \{\pm 1\}$ called* orientation homomorphism *and an element $[X] \in H_n(X;\mathbb{Z}^w)$ called* fundamental class *such that the $\mathbb{Z}\pi$-chain map $? \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence. We will call it the* Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence.

The orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ is uniquely determined by the homotopy type of X by the following argument. Denote by $C^{n-*}(\widetilde{X})_{\mathrm{untw}}$ the $\mathbb{Z}\pi$-chain complex which is analogously defined as $C^{n-*}(\widetilde{X})$, but now with respect to the untwisted involution. Its n-th homology $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$ depends only on the homotopy type of X. If X carries the structure of a Poincaré complex with respect to $w\colon \pi_1(X) \to \{\pm 1\}$, then the Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence induces a $\mathbb{Z}\pi$-isomorphism $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}}) \cong \mathbb{Z}^w$. Thus we rediscover w from $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$. Obviously there are two possible choices for $[X]$, since it has to be a generator of the infinite cyclic group $H_n(X,\mathbb{Z}^w) \cong H^0(X;\mathbb{Z}) \cong \mathbb{Z}$. A choice of $[X]$ will be part of the Poincaré structure.

The connected finite n-dimensional Poincaré complex X is called *simple* if the Whitehead torsion (see (2.8)) of the $\mathbb{Z}\pi$-chain homotopy equivalence of finite based free $\mathbb{Z}\pi$-chain complexes $? \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ vanishes.

Theorem 3.7 *Let M be a connected closed manifold of dimension n. Then M carries the structure of a simple connected finite n-dimensional Poincaré complex.*

For a proof we refer for instance to [119, Theorem 2.1 on page 23]. We explain at least the idea and give some evidence for it.

Any closed manifold admits a smooth triangulation $h\colon K \to M$, i.e. a finite simplicial complex together with a homeomorphism $h\colon K \to M$ such that h restricted to a simplex is a smooth C^∞-embedding. Two such smooth triangulations have common subdivisions. In particular M is homeomorphic to a finite CW-complex. Fix such a triangulation K. Denote by K' its barycentric subdivision. The vertices of K' are the barycenters $\widehat{\sigma^r}$ of simplices σ^r in K. A p-simplex in K' is given by a sequence $\widehat{\sigma^{i_0}}\widehat{\sigma^{i_1}}\ldots\widehat{\sigma^{i_p}}$, where σ^{i_j} is a face of $\sigma^{i_{j+1}}$. Next we define the dual CW-complex K^* as follows. It is not a simplicial complex but shares the property of a simplicial complex that all attaching maps are embeddings. Each p-simplex σ in K determines a $(n-p)$-dimensional cell σ^* of K^* which is the union of all simplices in K'

which begin with $\widehat{\sigma^p}$. So K has as many p-simplices as K^* has $(n-p)$-cells. The cap product with the fundamental cycle, which is given by the sum of the n-dimensional simplices, yields an isomorphism of $\mathbb{Z}\pi$-chain complexes $C^{n-*}(\widetilde{K^*}) \to C_*(\widetilde{K})$. It preserves the cellular $\mathbb{Z}\pi$-bases and in particular its Whitehead torsion is trivial. Since K' is a common subdivision of K and K^*, there are canonical $\mathbb{Z}\pi$-chain homotopy equivalences $C_*(\widetilde{K'}) \to C_*(\widetilde{K})$ and $C_*(\widetilde{K'}) \to C_*(\widetilde{K^*})$ which have trivial Whitehead torsion. Thus we can write the $\mathbb{Z}\pi$-chain map $? \cap [M]\colon C^{n-*}(\widetilde{K'}) \to C_*(\widetilde{K'})$ as a composite of three simple $\mathbb{Z}\pi$-chain homotopy equivalences. Hence it is a simple $\mathbb{Z}\pi$-chain homotopy equivalence.

Remark 3.8 Theorem 3.7 gives us the first obstruction for a topological space X to be homotopy equivalent to a connected closed n-dimensional manifold (see Problem 3.1). Namely, X must be homotopy equivalent to a connected finite simple n-dimensional Poincaré complex.

Remark 3.9 Suppose that X is a Poincaré complex with respect to the trivial orientation homomorphism. Definition 3.6 of Poincaré duality implies that Poincaré duality holds for any G-covering $\overline{X} \to X$ and yields Poincaré duality for all possible coefficient systems. In particular we get a $\mathbb{Z}$-chain homotopy equivalence

$$\mathbb{Z} \otimes_{\mathbb{Z}\pi} (? \cap [X])\colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} C^{n-*}(\widetilde{X}) = C^{n-*}(X) \;\to\; \mathbb{Z} \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = C_*(X),$$

which induces for any commutative ring R an R-isomorphism on (co-)homology with R-coefficients

$$? \cap [X]\colon H^{n-*}(X;R) \xrightarrow{\cong} H_*(X;R).$$

Remark 3.10 From a Morse theoretic point of view Poincaré duality corresponds to the dual handlebody decomposition of a manifold which we have already described in (1.26).

Remark 3.11 From the analytic point of view Poincaré duality can be explained as follows. Let M be a connected closed oriented Riemannian manifold. Let $(\Omega^*(M), d^*)$ be the de Rham complex of smooth p-forms on M. The p-th Laplacian is defined by $\Delta_p = (d^p)^* d^p + d^{p-1}(d^{p-1})^*\colon \Omega^p(M) \to \Omega^p(M)$, where $(d^p)^*$ is the adjoint of the p-th differential d^p. The kernel of the p-th Laplacian is the space $\mathcal{H}^p(M)$ of harmonic p-forms on M. The Hodge-de Rham Theorem yields an isomorphism

$$A^p\colon \mathcal{H}^p(M) \xrightarrow{\cong} H^p(M;\mathbb{R}) \tag{3.12}$$

from the space of harmonic p-forms to the singular cohomology of M with coefficients in $\mathbb{R}$. Let $[M]_{\mathbb{R}} \in H^n(M;\mathbb{R})$ be the fundamental cohomology class with $\mathbb{R}$-coefficients which is characterized by the property $\langle [M]_{\mathbb{R}}, i_*([M])\rangle = 1$ for $\langle\ ,\ \rangle$ the Kronecker product and $i_*\colon H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$ the change of rings homomorphism. Then A^n sends the volume form $d\mathrm{vol}$ to the class $\frac{1}{\mathrm{vol}(M)} \cdot [M]_{\mathbb{R}}$. The Hodge-star operator $*\colon \Omega^{n-p}(M) \to \Omega^p(M)$ induces isomorphisms

$$*\colon \mathcal{H}^{n-p}(M) \xrightarrow{\cong} \mathcal{H}^p(M). \tag{3.13}$$

We obtain from (3.12) and (3.13) isomorphisms

$$H^{n-p}(M;\mathbb{R}) \xrightarrow{\cong} H^p(M;\mathbb{R}).$$

This is the analytic version of Poincaré duality. It is equivalent to the claim that the bilinear pairing

$$P^p\colon \mathcal{H}^p(M) \otimes_{\mathbb{R}} \mathcal{H}^{n-p}(M) \to \mathbb{R}, \qquad (\omega,\eta) \mapsto \int_M \omega \wedge \eta \tag{3.14}$$

is non-degenerate. Recall that for any commutative ring R with unit we have the intersection pairing

$$I^p\colon H^p(M;R) \otimes_R H^{n-p}(M;R) \to R, \quad (x,y) \mapsto \langle x \cup y, i_*[M]\rangle, \tag{3.15}$$

where i_* is the change of coefficients map associated to $\mathbb{Z} \to R$. The fact that the intersection pairing is non-degenerate is for R a field equivalent to the bijectivity of the homomorphism $? \cap [X]\colon H^{n-*}(X;R) \to H_*(X;R)$ appearing in Remark 3.9. If we take $R = \mathbb{R}$, then the pairings (3.14) and (3.15) agree under the Hodge-de Rham isomorphism (3.12).

One basic invariant of a finite CW-complex X is its *Euler characteristic* $\chi(X)$. It is defined by $\chi(X) := \sum_p (-1)^p \cdot n_p$, where n_p is the number of p-cells. Equivalently it can be defined in terms of its homology by $\chi(X) = \sum_p (-1)^p \cdot \dim_{\mathbb{Q}} H_p(X;\mathbb{Q})$. A basic invariant for Poincaré complexes is defined next.

Consider a symmetric bilinear non-degenerate pairing $s\colon V \otimes_{\mathbb{R}} V \to \mathbb{R}$ for a finite-dimensional real vector space V. Choose a basis for V and let A be the square matrix describing s with respect to this basis. Since s is symmetric and non-degenerate, A is symmetric and invertible. Hence A can be diagonalized by an orthogonal matrix U to a diagonal matrix whose entries on the diagonal are non-zero real numbers. Let n_+ be the number of positive entries and n_- be the number of negative entries on the diagonal. These two numbers are independent of the choice of the basis and the orthogonal matrix U. Namely n_+ is the maximum of the dimensions of subvector spaces $W \subset V$ on which s is positive-definite, and analogous for n_-. Obviously $n_+ + n_- = \dim_{\mathbb{R}}(V)$. Define the *signature* of s to be the integer $n_+ - n_-$.

Definition 3.16 *Let X be a finite connected Poincaré complex. Suppose that X is orientable, i.e. $w_1(X)\colon \pi_1(X) \to \{\pm 1\}$ is trivial and that its dimension $n = 4k$ is divisible by four. Define its* intersection pairing *to be the symmetric bilinear non-degenerate pairing*

$$I\colon H^{2k}(X;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(X;\mathbb{R}) \xrightarrow{\cup} H^n(X;\mathbb{R}) \xrightarrow{\langle -,[X]_{\mathbb{R}}\rangle} \mathbb{R}.$$

Define the signature $\operatorname{sign}(X)$ *to be the signature of the intersection pairing.*

Remark 3.17 The notion of a Poincaré complex can be extended to pairs as follows. Let X be a connected finite n-dimensional CW-complex with fundamental group π together with a subcomplex $A \subset X$ of dimension $(n-1)$. Denote by $\widetilde{A} \subset \widetilde{X}$ the preimage of A under the universal covering $\widetilde{X} \to X$. We call (X, A) a finite n-dimensional Poincaré pair with respect to the orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ if there is a fundamental class $[X, A] \in H_n(X, A; \mathbb{Z}^w)$ such that the $\mathbb{Z}\pi$-chain maps $? \cap [X, A]\colon C^{n-*}(\widetilde{X}, \widetilde{A}) \to C_*(\widetilde{X})$ and $? \cap [X, A]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}\pi$-chain equivalences. We call (X, A) *simple* if the Whitehead torsion of these $\mathbb{Z}\pi$-chain maps vanish.

Each component C of the space A inherits from (X, A) the structure of a finite $(n-1)$-dimensional Poincaré complex. Its orientation homomorphisms $w_1(C)$ is obtained from $w_1(X)$ by restriction with the homomorphism $\pi_1(C) \to \pi_1(X)$ induced by the inclusion. The fundamental classes $[C]$ of the components $C \in \pi_0(A)$ are given by the image of the fundamental class $[X, A]$ under the boundary map $H_n(X, A, \mathbb{Z}^w) \to H_{n-1}(A, \mathbb{Z}^{w_1(A)}) \cong$

$\oplus_{C\in\pi_0(A)} H_{n-1}(C, \mathbb{Z}^{w_1(C)})$. If M is a compact connected manifold of dimension n with boundary ∂M, then $(M, \partial M)$ is a simple finite n-dimensional Poincaré pair.

The signature will be the first and the most elementary surgery obstruction which we will encounter. This is due to the following lemma.

Lemma 3.18 *Let (X, A) be a $(4k+1)$-dimensional oriented finite Poincaré pair. Then*

$$\sum_{C\in\pi_0(A)} \operatorname{sign}(C) = 0.$$

For its proof and later purposes we need the following lemma.

Lemma 3.19 *Let $s\colon V \otimes_{\mathbb{R}} V \to \mathbb{R}$ be a symmetric bilinear non-degenerate pairing for a finite-dimensional real vector space V. Then $\operatorname{sign}(s) = 0$ if and only if there exists a subvector space $L \subset V$ such that $\dim_{\mathbb{R}}(V) = 2\cdot\dim_{\mathbb{R}}(L)$ and $s(a, b) = 0$ for $a, b \in L$.*

Proof : Suppose that $\operatorname{sign}(s) = 0$. Then one can find an orthonormal (with respect to s) basis $\{b_1, b_2, \dots, b_{n_+}, c_1, c_2, \dots, c_{n_-}\}$ such that $s(b_i, b_i) = 1$ and $s(c_j, c_j) = -1$ holds. Since $0 = \operatorname{sign}(s) = n_+ - n_-$, we can define L to be the subvector space generated by $\{b_i + c_i \mid i = 1, 2, \dots, n_+\}$. One easily checks that L has the desired properties.

Suppose such an $L \subset V$ exists. Choose subvector spaces V_+ and V_- of V such that s is positive-definite on V_+ and negative-definite on V_- and V_+ and V_- are maximal with respect to this property. Then $V_+ \cap V_- = \{0\}$ and $V = V_+ \oplus V_-$. Obviously $V_+ \cap L = V_- \cap L = \{0\}$. From

$$\dim_{\mathbb{R}}(V_\pm) + \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(V_\pm \cap L) \le \dim_{\mathbb{R}}(V)$$

we conclude $\dim_{\mathbb{R}}(V_\pm) \le \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(L)$. Since $2\cdot\dim_{\mathbb{R}}(L) = \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V_+) + \dim_{\mathbb{R}}(V_-)$ holds, we get $\dim_{\mathbb{R}}(V_\pm) = \dim_{\mathbb{R}}(L)$. This implies

$$\operatorname{sign}(s) \;=\; \dim_{\mathbb{R}}(V_+) - \dim_{\mathbb{R}}(V_-) \;=\; \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(L) = 0. \qquad \blacksquare$$

Now we can give the proof of Lemma 3.18.

Proof : Let $i\colon A \to M$ be the inclusion. Then the following diagram commutes for $n = 4k$.

$$\begin{array}{ccccc}
H^{2k}(X;\mathbb{R}) & \xrightarrow{H^{2k}(i)} & H^{2k}(A;\mathbb{R}) & \xrightarrow{\delta^{2k}} & H^{2k+1}(X,A;\mathbb{R}) \\
{\scriptstyle ?\cap[X,A]}\downarrow\cong & & {\scriptstyle ?\cap\partial_{4k+1}([X,A])}\downarrow\cong & & {\scriptstyle ?\cap[X,A]}\downarrow\cong \\
H_{2k+1}(X,A;\mathbb{R}) & \xrightarrow{\partial_{2k+1}} & H_{2k}(A;\mathbb{R}) & \xrightarrow{H_{2k}(i)} & H_{2k}(X;\mathbb{R})
\end{array}$$

This implies $\dim_{\mathbb{R}}(\ker(H_{2k}(i))) = \dim_{\mathbb{R}}(\mathrm{im}(H^{2k}(i)))$. Since $\mathbb{R}$ is a field, we get from the Kronecker pairing an isomorphism $H^{2k}(X;\mathbb{R}) \cong (H_{2k}(X;\mathbb{R}))^*$ and analogously for A. Under these identifications $H^{2k}(i)$ becomes $(H_{2k}(i))^*$. Hence $\dim_{\mathbb{R}}(\mathrm{im}(H_{2k}(i))) = \dim_{\mathbb{R}}(\mathrm{im}(H^{2k}(i)))$. From

$$\dim_{\mathbb{R}}(H_{2k}(A;\mathbb{R})) = \dim_{\mathbb{R}}(\ker(H_{2k}(i))) + \dim_{\mathbb{R}}(\mathrm{im}(H_{2k}(i)))$$

we conclude

$$\dim_{\mathbb{R}}(H^{2k}(A;\mathbb{R})) = 2 \cdot \dim_{\mathbb{R}}(\mathrm{im}(H^{2k}(i))).$$

We have for $x, y \in H^{2k}(M;\mathbb{R})$

$$\begin{aligned}
\langle H^{2k}(i)(x) \cup H^{2k}(i)(y), \partial_{4k+1}([X,A])\rangle &= \langle H^{2k}(i)(x \cup y), \partial_{4k+1}([X,A])\rangle \\
&= \langle x \cup y, H_{2k}(i) \circ \partial_{4k+1}([X,A])\rangle \\
&= \langle x \cup y, 0\rangle \;=\; 0.
\end{aligned}$$

If we apply Lemma 3.19 to the non-degenerate symmetric bilinear pairing

$$H^{2k}(A;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(A;\mathbb{R}) \xrightarrow{\cup} H^{4k}(A;\mathbb{R}) \xrightarrow{\langle ?, \partial_{4k+1}([X,A])\rangle} H^0(A;\mathbb{R}) \cong \oplus_{\pi_0(A)}\mathbb{R} \xrightarrow{\Sigma} \mathbb{R}$$

with L the image of $H^{2k}(i)\colon H^{2k}(X;\mathbb{R}) \to H^{2k}(A;\mathbb{R})$, we see that the signature of this pairing is zero. One easily checks that its signature is the sum of the signatures of the components of A. ■

The signature has the following further properties.

Lemma 3.20 *1. Let M and N be compact oriented manifolds and $f\colon \partial M \to \partial N$ be an orientation reversing diffeomorphism. Then $M \cup_f N$ inherits an orientation from M and N and*

$$\mathrm{sign}(M \cup_f N) = \mathrm{sign}(M) + \mathrm{sign}(N);$$

2. *Let* $p\colon \overline{M} \to M$ *be a finite covering with* d *sheets of closed oriented manifolds. Then*
$$\operatorname{sign}(\overline{M}) \;=\; d \cdot \operatorname{sign}(M).$$

Proof : (1) is due to Novikov. For a proof see for instance [5, Proposition 7.1 on page 588].

(2) For a smooth manifold M this follows from Atiyah's L^2-index theorem [4, (1.1)]. Topological closed manifolds are treated in [102, Theorem 8].

■

Example 3.21 Wall has constructed a finite connected Poincaré space X together with a finite covering with d sheets $\overline{X} \to X$ such that the signature does not satisfy $\operatorname{sign}(\overline{X}) = d \cdot \operatorname{sign}(X)$ (see [98, Example 22.28], [118, Corollary 5.4.1])). Hence X cannot be homotopy equivalent to a closed manifold by Lemma 3.20.

3.2 The Spivak Normal Fibration

In this section we introduce the Spivak normal fibration which is the homotopy theoretic analog of the normal fiber bundle of an embedding of a manifold into some Euclidean space. In order to motivate the construction we briefly recall the Pontrjagin-Thom construction which we will need later anyway.

3.2.1 Pontrjagin-Thom Construction

Let (M, i) be an embedding $i\colon M^n \to \mathbb{R}^{n+k}$ of a closed n-dimensional manifold M into $\mathbb{R}^{n+k}$. Notice that $T\mathbb{R}^{n+k}$ comes with an explicit trivialization $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \xrightarrow{\cong} T\mathbb{R}^{n+k}$ and the standard Euclidean inner product induces a Riemannian metric on $T\mathbb{R}^{n+k}$. Denote by $\nu(M) = \nu(M,i)$ the *normal bundle* of i which is the orthogonal complement of TM in $i^*T\mathbb{R}^{n+k}$. For a vector bundle $\xi\colon E \to X$ with Riemannian metric define its *disk bundle* $p_{DE}\colon DE \to X$ by $DE = \{v \in E \mid ||v|| \leq 1\}$ and its *sphere bundle* $p_{SE}\colon SE \to X$ by $SE = \{v \in E \mid ||v|| = 1\}$, where p_{DE} and p_{SE} are the restrictions of p. Its *Thom space* $\operatorname{Th}(\xi)$ is defined by DE/SE. It has a preferred base point $\infty = SE/SE$. The Thom space can be defined without choice of a Riemannian metric as follows. Put $\operatorname{Th}(\xi) = E \cup \{\infty\}$ for some extra point ∞. Equip $\operatorname{Th}(\xi)$ with the topology for which $E \subset \operatorname{Th}(E)$ is an open subset and a basis of open neighborhoods for ∞ is given by the complements of closed subsets $A \subset E$ for which $A \cap E_x$ is compact for each fiber E_x. If X is compact, E is locally compact and $\mathrm{Th}(\xi)$ is the one-point-compactification of E. The advantage of this definition is that any bundle map $(\overline{f}, f)\colon \xi_0 \to \xi_1$ of vector bundles $\xi_0\colon E_0 \to X_0$ and $\xi_1\colon E_1 \to X_1$ induces canonically a map $\mathrm{Th}(\overline{f})\colon \mathrm{Th}(\xi_0) \to \mathrm{Th}(\xi_1)$. Notice that we require that $\overline{f}$ induces a bijective map on each fiber. Denote by $\underline{\mathbb{R}^k}$ the trivial vector bundle with fiber $\mathbb{R}^k$. We mention that there are homeomorphisms

$$\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta); \tag{3.22}$$

$$\mathrm{Th}(\xi \oplus \underline{\mathbb{R}^k}) \cong \Sigma^k\,\mathrm{Th}(\xi), \tag{3.23}$$

where $\wedge$ stands for the *smash product* of pointed spaces

$$X \wedge Y := (X \times Y) \,/\, (X \times \{y\} \cup \{x\} \times Y) \tag{3.24}$$

and $\Sigma^k Y = S^k \wedge Y$ is the *(reduced) suspension.* Let $(N(M), \partial N(M))$ be a tubular neighborhood of M. Recall that there is a diffeomorphism

$$u\colon (D\nu(M), S\nu(M)) \to (N(M), \partial N(M))$$

which is up to isotopy relative M uniquely determined by the property that its restriction to M is i and its differential at M is $\epsilon \cdot \mathrm{id}$ for small $\epsilon > 0$ under the canonical identification $T(D\nu(M))|_M = TM \oplus \nu(M) = i^*T\mathbb{R}^{n+k}$. The *collapse map*

$$c\colon S^{n+k} = \mathbb{R}^{n+k} \coprod \{\infty\} \to \mathrm{Th}(\nu(M)) \tag{3.25}$$

is the pointed map which is given by the diffeomorphism u^{-1} on the interior of $N(M)$ and sends the complement of the interior of $N(M)$ to the preferred base point ∞. The homology group $H_{n+k}(\mathrm{Th}(TM)) \cong H_{n+k}(N(M), \partial N(M))$ is infinite cyclic, since $N(M)$ is a compact orientable (n+k)-dimensional manifold with boundary $\partial N(M)$. The Hurewicz homomorphism
$h\colon \pi_{n+k}(Th(TM)) \to H_{n+k}(\mathrm{Th}(TM))$ sends the class $[c]$ of c to a generator. This follows from the fact that any point in the interior of $N(M)$ is a regular value of c and has precisely one point in his preimage.

Before we deal with the Spivak normal fibration, we apply this construction to bordism. Fix a space X together with a k-dimensional vector bundle $\xi\colon E \to X$. Let us recall the definition of the bordism set $\Omega_n(\xi)$. An element in it is represented by a quadruple $(M, i, f, \overline{f})$ which consists of a closed n-dimensional manifold M, an embedding $i\colon M \to \mathbb{R}^{n+k}$, a map $f\colon M \to X$ and a bundle map $\overline{f}\colon \nu(M) \to \xi$ covering f. We briefly explain what a bordism $(W, I, F, \overline{F})$ from one such quadruple $(M_0, i_0, f_0, \overline{f_0})$ to another quadruple $(M_1, i_1, f_1, \overline{f_1})$ is. We need a compact $(n+1)$-dimensional manifold W together with a map $F\colon W \to X \times [0,1]$. Its boundary ∂W is written as a disjoint sum $\partial_0 W \coprod \partial_1 W$ such that F maps $\partial_0 W$ to $X \times \{0\}$ and $\partial_1 W$ to $X \times \{1\}$. There is an embedding $I\colon W \to \mathbb{R}^{n+k} \times [0,1]$ such that $I^{-1}(\mathbb{R}^{n+k} \times \{j\}) = \partial_j W$ holds for $j = 0, 1$ and W meets $\mathbb{R}^{n+k} \times \{j\}$ for $j = 0, 1$ transversally. We require a bundle map $(\overline{F}, F)\colon \nu(W) \to \xi \times [0,1]$. Moreover for $j = 0, 1$ there is a diffeomorphism $U_j\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k} \times \{j\}$ which maps M_j to $\partial_j W$. It satisfies $F \circ U_j|_{M_j \times \{j\}} = f_j$. Notice that U_j induces a bundle map $\nu(U_j)\colon \nu(M_j) \to \nu(W)$ covering $U_j|_{M_j}$. The composition of $\overline{F}$ with $\nu(U_j)$ is required to be $\overline{f_j}$.

Theorem 3.26 (Pontrjagin-Thom Construction) *Let $\xi\colon E \to X$ be a k-dimensional vector bundle over a CW-complex X. Then the map*

$$P_n(\xi)\colon \Omega_n(\xi) \xrightarrow{\cong} \pi_{n+k}(\mathrm{Th}(\xi)),$$

which sends the class of $(M, i, f, \overline{f})$ to the class of the composite $S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M)) \xrightarrow{\mathrm{Th}(\overline{f})} \mathrm{Th}(\xi)$ is a well-defined bijection and natural in ξ.

Proof : The details can be found in [15, Satz 3.1 on page 28, Satz 4.9 on page 35,]. The basic idea becomes clear after we have explained the construction of the inverse for a finite CW-complex X. Consider a pointed map $(S^{n+k}, \infty) \to (\mathrm{Th}(\xi), \infty)$. We can change f up to homotopy relative $\{\infty\}$ such that f becomes transverse to X. Notice that transversality makes sense although X is not a manifold, one needs only the fact that X is the zero-section in a vector bundle. Put $M = f^{-1}(X)$. The transversality construction yields a bundle map $\overline{f}\colon \nu(M) \to \xi$ covering $f|_M$. Let $i\colon M \to \mathbb{R}^{n+k} = S^{n+k} - \{\infty\}$ be the inclusion. Then the inverse of $P_n(\xi)$ sends the class of f to the class of $(M, i, f|_M, \overline{f})$. ■

Let $\Omega_n(X)$ be the bordism group of pairs (M, f) of oriented closed n-dimensional manifolds M together with reference maps $g\colon M \to X$. Let $\xi_k\colon E_k \to BSO(k)$ be the universal oriented k-dimensional vector bundle. In the sequel we will denote for a finite-dimensional vector space V by $\underline{V}$ the trivial bundle with fiber V. Let $\overline{j_k}\colon \xi_k \oplus \underline{\mathbb{R}} \to \xi_{k+1}$ be a bundle map covering a map $j_k\colon BSO(k) \to BSO(k+1)$. Up to homotopy of bundle maps this map is unique. Denote by γ_k the bundle $X \times E_k \to X \times BSO(k)$

and by $(\overline{i_k}, i_k)\colon \gamma_k \oplus \underline{\mathbb{R}} \to \gamma_{k+1}$ the bundle map $\mathrm{id}_X \times (\overline{j_k}, j_k)$. The bundle map $(\overline{i_k}, i_k)$ is unique up to homotopy of bundle maps and hence induces a well-defined map $\Omega_n(\overline{i_k})\colon \Omega_n(\gamma_k) \to \Omega_n(\gamma_{k+1})$, which sends the class of $(M, i, f, \overline{f})$ to the class of the quadruple which comes from the embedding $j\colon M \xrightarrow{i} \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$ and the canonical isomorphism $\nu(i) \oplus \underline{\mathbb{R}} = \nu(j)$. Consider the homomorphism

$$V_k\colon \Omega_n(\gamma_k) \to \Omega_n(X)$$

which sends the class of $(M, i, f, \overline{f})$ to $(M, \mathrm{pr}_X \circ f)$, where we equip M with the orientation determined by $\overline{f}$. Let $\mathrm{colim}_{k\to\infty} \Omega_n(\gamma_k)$ be the colimit of the directed system indexed by $k \geq 0$

$$\ldots \xrightarrow{\Omega_n(\overline{i_{k-1}})} \Omega_n(\gamma_k) \xrightarrow{\Omega_n(\overline{i_k})} \Omega_n(\gamma_{k+1}) \xrightarrow{\Omega_n(\overline{i_{k+1}})} \ldots$$

Since $V_{k+1} \circ i_k = V_k$ holds for all $k \geq 0$, we obtain a map

$$V\colon\ \mathrm{colim}_{k\to\infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \Omega_n(X). \tag{3.27}$$

This map is bijective because of the classifying property of γ_k and the facts that for $k > n+1$ any closed manifold M of dimension n can be embedded into $\mathbb{R}^{n+k}$ and two such embeddings are isotopic.

We see a sequence of spaces $\mathrm{Th}(\gamma_k)$ together with maps

$$\mathrm{Th}(\overline{i_k})\colon \Sigma\,\mathrm{Th}(\gamma_k) = \mathrm{Th}(\gamma_k \oplus \underline{\mathbb{R}}) \to \mathrm{Th}(\gamma_{k+1}).$$

They induce homomorphisms

$$s_k\colon \pi_{n+k}(\mathrm{Th}(\gamma_k)) \to \pi_{n+k+1}(\Sigma\,\mathrm{Th}(\gamma_k)) \xrightarrow{\pi_{n+k}(\mathrm{Th}(\overline{i_k}))} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})),$$

where the first map is the suspension homomorphism. Let $\mathrm{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k))$ be the colimit of the directed system

$$\ldots \xrightarrow{s_{k-1}} \pi_{n+k}(\mathrm{Th}(\gamma_k)) \xrightarrow{s_k} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})) \xrightarrow{s_{k+1}} \ldots.$$

We get from Theorem 3.26 a bijection

$$P\colon\ \mathrm{colim}_{k\to\infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \mathrm{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

This implies

Theorem 3.28 (Pontrjagin Thom Construction and Oriented Bordism) *There is an isomorphism of abelian groups natural in X*

$$P\colon \Omega_n(X) \xrightarrow{\cong} \operatorname{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

Remark 3.29 Notice that this is the beginning of the theory of spectra and stable homotopy theory. A *spectrum* $\mathbf{E}$ consists of a sequence of spaces $(E_k)_{k\in\mathbb{Z}}$ together with so called structure maps $\sigma_k\colon \Sigma E_k \to E_{k+1}$. The n-th *stable homotopy group* $\pi_n(\mathbf{E})$ is the defined as the colimit $\operatorname{colim}_{k\to\infty} \pi_{n+k}(E_k)$ with respect to the directed system given by the composites

$$\pi_{n+k}(E_k) \to \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\pi_{n+k}(\sigma_k)} \pi_{n+k+1}(E_{k+1}).$$

Theorem 3.28 is a kind of mile stone in homotopy theory since it is the prototype of a result, where the computation of geometrically defined objects are translated into a computation of (stable) homotopy groups. It applies to all other kind of bordism groups, where one puts additional structures on the manifolds, for instance a Spin-structure. The bijection is always of the same type, but the sequence of bundles ξ_k depends on the additional structure. If we want to deal with the unoriented bordism ring we have to replace the bundle $\xi_k \to BSO(k)$ by the universal k-dimensional vector bundle over $BO(k)$.

3.2.2 Spherical Fibrations

A *spherical* $(k-1)$*-fibration* $p\colon E \to X$ is a fibration, i.e. a map having the homotopy lifting property, whose typical fiber is homotopy equivalent to S^{k-1}. Define its associated disk fibration $Dp\colon DE \to X$ by $DE = \mathrm{cyl}(p)$, where $\mathrm{cyl}(p)$ is the mapping cylinder of p and Dp the obvious map. Define its *Thom space* $\mathrm{Th}(p)$ to be the pointed space $\mathrm{cone}(p)$, where $\mathrm{cone}(p)$ is the mapping cone of p with its canonical base point. Notice that $\mathrm{Th}(p) = DE/E$.

If $\xi\colon E \to X$ is a k-dimensional vector bundle with Riemannian metric, its sphere bundle $p\colon SE \to X$ is an example of a $(k-1)$-spherical fibration. The role of the disk bundle $D\xi$ is now played by $DE = \mathrm{cyl}(p)$. Notice that the canonical inclusion of X in $\mathrm{cyl}(p)$ is a homotopy equivalence analogous to the fact that the inclusion of the zero-section of ξ into E is a homotopy equivalence. The canonical inclusion of E into $\mathrm{cyl}(p)$ corresponds to the inclusion of $SE \subset DE$. Hence it is clear that $\mathrm{Th}(p) = DE/E = \mathrm{cone}(p)$

for a $(k-1)$-spherical fibration corresponds to $\mathrm{Th}(\xi) = DE/SE$ for a k-dimensional vector bundle $\xi\colon E \to X$.

If one has two vector bundles $\xi_0\colon E_0 \to X$ and $\xi_1 : E_1 \to X$, one can form the Whitney sum $\xi_0 \oplus \xi_1$. Notice that $S(E_0 \oplus E_1)_x$ is homeomorphic to the join $S(E_0)_x * S(E_1)_x$. This allows us to rediscover $S(\xi_1 \oplus \xi_2)$ as a spherical fibration from $S\xi_0$ and $S\xi_1$ by the fiberwise join construction. Namely, given a $(k-1)$-spherical fibration $p_0\colon E_0 \to X$ and an $(l-1)$-spherical fibration $p_1\colon E_1 \to X$, define the $(k+l-1)$-spherical fibration $p_0 * p_1\colon E_0 * E_1 \to X$ called *fiberwise join* as follows. The total space $E_0 * E_1$ is the quotient of the space $\{(e_0, e_1, t) \in E_0 \times E_1 \times [0,1] \mid p_0(e_0) = p_1(e_1)\}$ under the equivalence relation generated by $(e_0, e_1, 1) \sim (e_0, e_1', 1)$ and $(e_0, e_1, 0) \sim (e_0', e_1, 0)$. The projection $p_0 * p_1$ sends the class of (e_0, e_1, t) to $p_0(e_0) = p_1(e_1)$. Given a spherical $(k-1)$-fibration $p\colon E \to X$, its l-fold suspension is the spherical $(k+l-1)$-fibration given by the fiberwise join of p and the trivial $(l-1)$-spherical fibration $\mathrm{pr}\colon X \times S^{l-1} \to X$.

There is natural homeomorphism for a spherical fibrations ξ

$$\mathrm{Th}(\xi * \underline{S^{k-1}}) \quad \cong \quad \Sigma^k\,\mathrm{Th}(\xi). \tag{3.30}$$

Given two fibrations $p_0\colon E_0 \to X$ and $p_1\colon E_1 \to X$, a fiber map $(\overline{f}, f)\colon p_0 \to p_1$ consists of maps $\overline{f}\colon E_0 \to E_1$ and $f\colon X_0 \to X_1$ satisfying $p_1 \circ \overline{f} = f \circ p_0$. There is an obvious notion of fiber homotopy $(\overline{h}, h)$. A fiber homotopy is called a strong fiber homotopy if $h_t\colon X_0 \to X_1$ is stationary for $t \in [0,1]$. Two fibrations p_0 and p_1 over the same base space are called *strongly fiber homotopy equivalent* if there are fiber maps $(\overline{f}, \mathrm{id})\colon p_0 \to p_1$ and $(\overline{g}, \mathrm{id})\colon p_1 \to p_0$ such that both composition are strongly fiber homotopy equivalent to the identity. Given a topological space F, let $G(F)$ be the monoid of selfhomotopy equivalences of F. One can associate to such a monoid a classifying space $BG(F)$ together with a fibration $p_F\colon EG(F) \to BG(F)$ with typical fiber F such that the pullback construction yields a bijection between the homotopy classes of maps from X to $BG(F)$ and the set of strong fiber homotopy classes of fibrations over X with typical fiber F [110]. We will abbreviate in the sequel $G(k) := G(S^{k-1})$.

Given a fibration $p\colon E \to X$, and $x \in X$, the fiber transport along loops at x defines a map of monoids $t_x\colon \pi_1(X,x) \to [p^{-1}(x), p^{-1}(x)]$. We call p *orientable* if t_x is trivial for any base point $x \in X$. In the case of $(k-1)$-spherical fibration this fiber transport is the same as a homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ called *the orientation homomorphism* since the degree defines a bijection $[S^{k-1}, S^{k-1}] \to \mathbb{Z}$ for $k \geq 2$. An *orientation* for an orientable spherical $(k-1)$-fibration p is the choice of fundamental class $[p^{-1}(x)] \in H_{k-1}(p^{-1}(x)) \cong \mathbb{Z}$ for any fiber $p^{-1}(x)$ for $x \in X$ such that for any path w in X the fiber transport along w yields a homotopy equivalence $p^{-1}(w(0)) \to p^{-1}(w(1))$ for which the induced map on H_{k-1} sends $[p^{-1}(w(0))]$ to $[p^{-1}(w(1))]$.

Theorem 3.31 (Thom Isomorphism) *Let $p\colon E \to X$ be a $(k-1)$-spherical fibration over a connected CW-complex X. Then there exists a group homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and a so called* Thom class $U_p \in H^k(DE, E; \mathbb{Z}^w)$ *such that the composites*

$$H_p(DE, E; \mathbb{Z}) \xrightarrow{U_p \cap ?} H_{p-k}(DE; \mathbb{Z}^w) \xrightarrow{H_{p-k}(p)} H_{p-k}(X; \mathbb{Z}^w); \quad (3.32)$$

$$H_p(DE, E; \mathbb{Z}^w) \xrightarrow{U_p \cap ?} H_{p-k}(DE; \mathbb{Z}) \xrightarrow{H_{p-k}(p)} H_{p-k}(X; \mathbb{Z}); \quad (3.33)$$

$$H^p(X; \mathbb{Z}) \xrightarrow{H^p(p)} H^p(DE; \mathbb{Z}) \xrightarrow{? \cup U_p} H^{p+k}(DE, SE; \mathbb{Z}^w); \quad (3.34)$$

$$H^p(X; \mathbb{Z}^w) \xrightarrow{H^p(p)} H^p(DE; \mathbb{Z}^w) \xrightarrow{? \cup U_p} H^{p+k}(DE, SE; \mathbb{Z}); \quad (3.35)$$

are bijective. These maps are called Thom isomorphisms.

There are precisely two possible choices for U_p because $H^k(DE, E; \mathbb{Z}^w) \cong H^0(X; \mathbb{Z}) \cong \mathbb{Z}$ is infinite cyclic. Moreover, w is uniquely determined by the spherical $(k-1)$-fibration. The proof is analogous to the one for Poincaré complexes.

If the spherical fibration p is orientable, then U_p is uniquely determined by the property that the composite

$$H^k(DE, E) \to H^k(p_{DE}^{-1}(x), p^{-1}(x); \mathbb{Z}) \xrightarrow{(\delta^{k-1})^{-1}} H^{k-1}(p^{-1}(x))$$

sends U_p to $[p^{-1}(x)]$. Moreover, a choice of orientation for p is the same as a choice of Thom class.

3.2.3 The Existence and Uniqueness of the Spivak Normal Fibration

Definition 3.36 *A* Spivak normal fibration *for an n-dimensional connected finite Poincaré complex X is a $(k-1)$-spherical fibration $p = p_X\colon E \to X$ together with a pointed map $c = c_X\colon S^{n+k} \to$* Th$(p)$ *such that X and p have the same orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and for some* *choice of Thom class* $U_p \in H^k(DE,E;\mathbb{Z}^w)$ *the fundamental class* $[X] \in H_n(X;\mathbb{Z}^w)$ *and the image* $h(c) \in H_{n+k}(\mathrm{Th}(p)) \cong H_{n+k}(DE,E;\mathbb{Z})$ *of* $[c]$ *under the Hurewicz homomorphism* $h\colon \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p),\mathbb{Z})$ *are related by the formula*

$$[X] = H_n(p)(U_p \cap h(c)).$$

Remark 3.37 A closed manifold M of dimension n admits a Spivak normal fibration. Namely, choose an embedding $i\colon M \to \mathbb{R}^{n+k}$ and take p to be the sphere bundle $S\nu(i) \to M$ and c to be the collaps map defined in (3.25).

Theorem 3.38 (Existence and Uniqueness of the Spivak Normal Fibration) *Let* X *be a connected finite* n*-dimensional Poincaré complex. Then:*

1. *For* $k > n$ *there exists a Spivak normal* $(k-1)$*-fibration for* X*;*
2. *Let* $p_i\colon E_i \to X$ *together with* $c_i\colon S^{n+k_i} \to \mathrm{Th}(p_i)$ *be a Spivak normal* (k_i-1)*-fibration of* X *for* $i = 0,1$*. Let* k *be an integer satisfying* $k \geq k_0, k_1$*. Then there is up to strong fiber homotopy precisely one strong fiber homotopy equivalence* $\overline{f}\colon \Sigma^{k-k_0}p_0 \to \Sigma^{k-k_1}p_1$ *for which*

$$\pi_{n+k}(\mathrm{Th}(\overline{f}))\colon \pi_{n+k}(\mathrm{Th}(p_0)) \to \pi_{n+k}(\mathrm{Th}(p_0))$$

   sends $[c_0]$ *to* $[c_1]$.

At least we want to give the idea of the proof of assertion (1) of Theorem 3.38 provided that the orientation homomorphism is trivial. For a detailed proof we refer for instance to [17, I.4], [108].

Consider a connected finite n-dimensional CW-complex X. One can always find an embedding $X \subset \mathbb{R}^{n+k}$ for $k > n$ together with a regular neighborhood $(N(X), \partial N(X))$. The regular neighboorhood is a compact manifold $N(X)$ with boundary $\partial N(K)$ such that $X \subset N(K)$ is a strong deformation retraction [100, Chapter 3]. This regular neighborhood corresponds in the case, where X is a closed manifold M, to a tubular neighborhood $(N(M), \partial N(M)) \cong (D\nu(M), S\nu(M))$. Let $[N(X), \partial N(X)] \in H_{n+k}(N(X), \partial N(X))$ be a fundamental class of the compact manifold $N(X)$ with boundary $\partial N(X)$ which corresponds to the orientation inherited from the standard orientation on $\mathbb{R}^{n+k}$. Let $i\colon X \to N(X)$ be the inclusion which is a homotopy equivalence. Then the following diagram commutes for any class $u \in H^k(N(X), \partial N(X))$

$$\begin{tikzcd}
H^{n-p}(X) \arrow[r, "?\cap\left(H_n(i)^{-1}(u\cap[N(X),\partial N(X)])\right)"] \arrow[d, "(u\cup ?)\circ H^{n-p}(i)^{-1}"'] & H_p(X) \arrow[d, "H_p(i)"] \\
H^{n+k-p}(N(X), \partial N(X)) \arrow[r, "?\cap[N(X),\partial N(X)]"'] & H_p(N(X))
\end{tikzcd}$$

The lower horizontal arrow and the right vertical arrow are bijective by Poincaré duality and homotopy invariance. Hence the upper horizontal arrow is bijective if and only if the left vertical arrow is bijective. Notice that bijectivity of the upper horizontal arrow corresponds to Poincaré duality for X with $H_n(i)^{-1}(u \cap [N(X), \partial N(X)])$ as fundamental class and the bijectivity of the left vertical arrow corresponds to the Thom isomorphism with u as Thom class.

Suppose that X is a connected finite Poincaré complex with fundamental class $[X] \in H_n(X)$. Let $u \in H^k(N(X), \partial N(X))$ be the class uniquely determined by the property that $H_n(i)^{-1}(u \cap [N(X), \partial N(X)]) = [X]$. Then the map

$$H^{n-p}(X) \to H^{n+k-p}(N(X), \partial N(X), \mathbb{Z}), \qquad v \mapsto u \cup H^{n-p}(i)^{-1}(v)$$

is bijective. Our first approximation of the Spivak normal $(k-1)$-fibration is the composition $f\colon \partial N(X) \xrightarrow{j} N(X) \xrightarrow{i^{-1}} X$, where j is the inclusion and i^{-1} a homotopy inverse of i. Of course f is not a fibration but possess a candidate for a Thom class, namely u. By a general construction we can turn f into a fibration. More precisely, for any map $g\colon X \to Y$ there is a functorial construction which yields a fibration $p_g\colon E_g \to Y$ together with a homotopy equivalence $i_g\colon X \to E_g$ satisfying $p_g \circ i_g = g$. Namely, define $E_g = \{(x, w) \in X \times \operatorname{map}([0,1], Y) \mid f(x) = w(0)\}$, $p_g(x, w) = w(1)$ and $i_g(x) = (x, c_{f(x)})$ for $c_{f(x)}$ the constant path at $f(x)$ (see [121, Theorem 7.30 on page 42]). We apply this to $f\colon \partial N(X) \to X$ and obtain a fibration $p_f\colon E_f \to X$ together with a homotopy equivalence $i_f\colon \partial N(X) \to E_f$ satisfying $p_f \circ i_f = f$. Since $j\colon \partial N(X) \to N(X)$ is a cofibration and $i^{-1}\colon N(X) \to X$ a homotopy equivalence, we can find an extension of i_f to a homotopy equivalence of pairs $(I_f, i_f)\colon (N(X), \partial N(X)) \to (DE_f, E_f)$ for $DE_f = \operatorname{cyl}(p_f)$. This extension is unique up to homotopy relative $\partial N(X)$.

Let $U_{p_f} \in H^k(DE_f, E_f)$ be the preimage of $u \in H^k(N(X), \partial N(X))$ under the isomorphism induces by (I_f, i_f). One easily checks that the map

$$H^p(X) \xrightarrow{H^p(p_f)} H^p(DE_f) \xrightarrow{? \cup U_{p_f}} H^{p+k}(DE_f, E)$$

is bijective. So U_{p_f} looks like a Thom class. We have already seen that a spherical fibration has a Thom class and it turns out that this does characterize the homotopy fiber of a fibration. Hence $p_f \colon \partial N(X) \to X$ is a spherical $(k-1)$-fibration. The collaps map $c \colon S^{n+k} \to N(X)/\partial N(X)$ can be composed with the map $N(X)/\partial N(X) \to \mathrm{Th}(p_f)$ induced by (I_g, i_g) and yields a pointed map $c_{p_f} \colon S^{n+k} \to \mathrm{Th}(p_f)$. It has the desired property $[X] = H_n(p_f)(U_{p_f} \cap h(c_{p_f}))$. Hence p_f and U_{p_f} yield a normal Spivak fibration for X.

Remark 3.39 Recall that we want to address Problem 3.1, whether a space X is homotopy equivalent to a connected closed n-dimensional manifold. We have already seen in Remark 3.8 that we only have to consider connected finite n-dimensional Poincaré complexes X. From Theorem 3.38 and Remark 3.37 we get the following new necessary condition. Namely, we must be able to find for $k > n$ a k-dimensional vector bundle $\xi \colon E \to X$ such that the associated sphere bundle $SE \to X$ is strongly fiber homotopy equivalent to the Spivak normal $(k-1)$-fibration of X.

Lemma 3.40 *Let $p_i \colon E_i \to X_i$ be a spherical $(k-1)$-fibration over a connected finite n-dimensional Poincaré complexes X_i for $i = 0, 1$. Let $(\overline{f}, f) \colon p_0 \to p_1$ be a fiber map which is fiberwise a homotopy equivalence. Then we get for the orientation homomorphisms $w(p_1) \circ \pi_1(f) = w(p_0)$. Consider a pointed map $c_0 \colon S^{n+k} \to \mathrm{Th}(p_0)$. Let $c_1 \colon S^{n+k} \to \mathrm{Th}(p_0)$ be the composition $\mathrm{Th}(\overline{f}) \circ c_0$. Then*

1. *Suppose that the degree of f is ± -1. Then (p_0, c_0) is the Spivak normal fibration of X_0 if and only if (p_1, c_1) is the Spivak normal fibration for X_1;*

2. *Suppose that (p_i, c_i) is the Spivak normal $(k-1)$-fibration of X_i for $i = 0, 1$. Then the degree of f is ± 1.*

Proof : The claim about the orientation homomorphisms follows from the fact that the fiber transport of p_0 and p_1 are compatible with $\pi_1(f)$. We can choose the Thom classes of p_0 and p_1 such that the map

$$H^k(\mathrm{Th}(\overline{f}))\colon H^{n+k}(DE_1, E_1; \mathbb{Z}^{w(p_1)}) \to H^k(DE_0, E_0; \mathbb{Z}^{w(p_0)})$$

sends U_{p_1} to U_{p_0}. Thus we get

$$H_n(f)\,(H_n(p_0)\,(U_{p_0} \cap h(c_0))) \;=\; H_n(p_1)\,(U_{p_1} \cap h(c_1))\,.$$

Now the claim follows from the definitions. ∎

3.3 Normal Maps

Motivated by Remark 3.39 we define

Definition 3.41 *Let X be a connected finite n-dimensional Poincaré complex. A* normal k-invariant (ξ, c) *consists of a k-dimensional vector bundle $\xi\colon E \to X$ together with an element $c \in \pi_{n+k}(\mathrm{Th}(\xi))$ such that for some choice of Thom class $U_p \in H^k(DE, SE; \mathbb{Z}^w)$ the equation $[X] = H_n(p)(U_p \cap h(c))$ holds. We call a normal k-invariant (ξ_0, c_0) and a normal k-invariant (ξ_1, c_1) equivalent if there is a bundle isomorphism $(\overline{f}, \mathrm{id})\colon \xi_0 \xrightarrow{\cong} \xi_1$ such that*

$$\pi_{n+k}(\mathrm{Th}(\overline{f}))\colon \pi_{n+k}(\mathrm{Th}(E_0)) \xrightarrow{\cong} \pi_{n+k}(\mathrm{Th}(\xi_1))$$

maps c_0 to c_1. The set of normal k-invariants $\mathcal{T}_n(X, k)$ *is the set of equivalence classes of normal k-invariants of X.*

Given a normal k-invariant (ξ, c), we obtain a normal $(k+1)$-invariant $(\xi \oplus \underline{\mathbb{R}}, \Sigma c)$, where $\Sigma\colon \pi_k(\mathrm{Th}(\xi)) \to \pi_{k+1}(\Sigma\,\mathrm{Th}(\xi)) = \pi_{k+1}(\mathrm{Th}(\xi \oplus \underline{\mathbb{R}}))$ is the suspension homomorphism. Define

Definition 3.42 *Let X be a connected finite n-dimensional Poincaré complex X. Define the* set of normal invariants $\mathcal{T}_n(X)$ *of X to be* $\mathrm{colim}_{k\to\infty}\,\mathcal{T}_n(X, k)$.

Let $J_k\colon BO(k) \to BG(k)$ be the classifying map for the universal k-dimensional vector bundle $\xi_k\colon E_k \to BO(k)$ viewed as a spherical fibration. Taking the Whitney sum with $\underline{\mathbb{R}}$ or the fiberwise join with $S\underline{\mathbb{R}}$ yields stabilization maps $BO(k) \to BO(k+1)$ and $BG(k) \to BG(k+1)$. This corresponds to the obvious stabilization maps $O(k) \to O(k+1)$ and $G(k) \to$

$G(k+1)$ given by direct sum with the identity map $\mathbb{R} \to \mathbb{R}$ and by suspending a selfhomotopy equivalence of S^k. One can arrange that these stabilization maps are cofibrations by a mapping cone construction and are compatible with the various maps J_k by a cofibration argument. Put

$$\begin{aligned} O &= \operatorname{colim}_{k\to\infty} O(k); \\ G &= \operatorname{colim}_{k\to\infty} G(k); \\ BO &= \operatorname{colim}_{k\to\infty} BO(k); \\ BG &= \operatorname{colim}_{k\to\infty} BG(k). \end{aligned}$$

Define $J\colon BO \to BG$ by $\operatorname{colim}_{k\to\infty} J_k$.

From Theorem 3.38 we get for a connected finite n-dimensional Poincaré complex X a map $s_X\colon X \to BG$ which is given by the classifying map of the Spivak normal $(k-1)$-bundle for large k. It is unique up to homotopy. We obtain from the universal properties of the classifying spaces

Theorem 3.43 *Let X be a connected finite n-dimensional Poincaré complex. Then $\mathcal{T}_n(X)$ is non-empty if and only there is a map $S\colon X \to BO$ such that $J \circ S$ is homotopic to s_X.*

Remark 3.44 In view of Remark 3.40 we see that a necessary condition for a connected finite n-dimensional Poincaré complex to be homotopy equivalent to a closed manifold is that the classifying map $s\colon X \xrightarrow{s_X} BG(k)$ lifts along $J\colon BO \to BG$. There is a fibration $BO \to BG \to BG/O$. Hence this condition is equivalent to the statement that the composition $X \xrightarrow{s_X} BG \to BG/O$ is homotopic to the constant map. There exists a finite Poincaré complex X for which composition $X \xrightarrow{s_X} BG \to BG/O$ is not nullhomotopic (see [78, page 32 f]). In particular X cannot be homotopy equivalent to a closed manifold.

Let G/O be the homotopy fiber of $J\colon BO \to BG$. This is the fiber of the fibration $\widehat{J}\colon E_J \to BG$ associated to J. Then the following holds

Theorem 3.45 *Let X be a connected finite n-dimensional Poincaré complex. Suppose that $\mathcal{T}_n(X)$ is non-empty. Then there is a canonical group structure on the set $[X, G/O]$ of homotopy classes of maps from X to G/O and a transitive free operation of this group on $\mathcal{T}_n(X)$.*

Proof : Define an abelian group $\mathcal{G}/\mathcal{O}(X)$ as follows. We consider pairs (ξ, t) consisting of a k-dimensional vector bundle ξ over X for some $k \geq 1$ and a strong fiber homotopy equivalence $t\colon S\xi \to \underline{S^{k-1}}$ from the associated spherical fibration given by the sphere bundle $S\xi$ and the trivial spherical $(k-1)$-fibration over X. We call two such pairs (ξ_0, t_0) and (ξ_1, t_1) equivalent, if for some k which is greater or equal to both $k_0 = \dim(\xi_0)$ and $k_1 = \dim(\xi_1)$ there is a bundle isomorphism $\overline{f}\colon \xi_0 \oplus \underline{\mathbb{R}^{k-k_0}} \xrightarrow{\cong} \xi_1 \oplus \underline{\mathbb{R}^{k-k_1}}$ covering the identity on X such that $\Sigma^{k-k_1} t_1 \circ S\overline{f}$ is fiber homotopic to $\Sigma^{k-k_0} t_0$. Let $\mathcal{G}/\mathcal{O}(X)$ be the set of equivalence classes $[\xi, t]$ of such pairs (ξ, t). Notice that $[\xi, t]$ for a pair (ξ, t) depends only on ξ and the fiber homotopy class of t. Addition is given by the Whitney sum. The neutral element is represented by $(\underline{\mathbb{R}^k}, \mathrm{id})$ for some $k \geq 1$. The existence of inverses follows from the fact that for a vector bundle ξ over X we can find another vector bundle η such that $\xi \oplus \eta$ is trivial and for a map $f\colon X \to G(k)$ we can find l and a map $f'\colon X \to G(l)$ such that the map given by the join $f * f'\colon X \to G(k+l)$ is homotopic to the identity.

Next we describe an action $\rho\colon \mathcal{G}/\mathcal{O}(X) \times \mathcal{T}_n(X) \to \mathcal{T}_n(X)$. Consider a pair (ξ, t) representing an element $[\xi, t] \in \mathcal{G}/\mathcal{O}(X)$ and a normal k_1-invariant (η, c) representing an element $[\eta, c] \in \mathcal{T}_n(X)$. If k_0 is the dimension of ξ, then $\xi \oplus \eta$ is a $(k_0 + k_1)$-dimensional vector bundle. Consider the composition

$$\mathrm{Th}(\xi \oplus \eta) \xrightarrow{\cong} \mathrm{Th}(S\xi * S\eta) \xrightarrow{t * \mathrm{id}} \mathrm{Th}(\underline{S^{k_0-1}} * S\eta) \xrightarrow{\cong} \Sigma^{k_0} \mathrm{Th}(\eta).$$

This together with the suspension $\Sigma^{k_0} c$ yields an element $d \in \pi_{n+k_0+k_1}(\mathrm{Th}(\xi \oplus \eta))$. One easily checks that $(\xi \oplus \eta, d)$ is a normal $(k_0 + k_1)$-invariant for X. Define

$$\rho([\xi, t], [\eta, c]) := [\xi \oplus \eta, d].$$

We leave it to the reader to check that this is a well-defined group action. It follows from Theorem 3.38 that this operation is transitive and free. Hence it remains to construct a bijection

$$\mu\colon [X, G/O] \xrightarrow{\cong} \mathcal{G}/\mathcal{O}(X).$$

Recall that G/O is the fiber of the fibration $\widehat{J}\colon E_J \to BG$ associated to $J\colon BO \to BG$ over some point $z \in BG$. By definition this is the space $\{(y, w) \in BO \times \mathrm{map}([0,1], BG) \mid J(y) = w(0), w(1) = z\}$. Hence a map $F\colon X \to G/O$ is the same as a pair (f, h) consisting of a map $f\colon X \to BO$ and a homotopy $h\colon X \times [0,1] \to BG$ such that $h_0 = J \circ f$ and h_1 is the constant map c_z. Since X is compact, the can find k such that the image of f and

h lie in $BO(k)$ and $BG(k)$. Recall that $J_k \colon BO(k) \to BG(k)$ is covered by a fiber map $\overline{J_k} \colon S\xi_k \to \eta_k$ which is fiberwise a homotopy equivalence, where ξ_k is the universal k-dimensional bundle over $BO(k)$ and η_k the universal spherical $(k-1)$-fibration. By the pullback construction applied to f and ξ_k and h and η_k, we get a vector bundle ξ over X and a spherical $(k-1)$-fibration η over $X \times [0,1]$ together with fiber homotopy equivalences $(u, \mathrm{id}) \colon S\xi \to \eta|_{X \times \{0\}}$ and $(v, \mathrm{id}) \colon \eta|_{X \times \{1\}} \to X \times S^{k-1}$. Up to fiber homotopy there is precisely one fiber homotopy equivalence $(\overline{g}, \mathrm{pr}) \colon \eta \to X \times [0,1] \times S^{k-1}$ whose restriction to $X \times \{1\} = X$ is v [113, Proposition 15.11 on page 342]. Thus we obtain a fiber homotopy equivalence $v|_{X \times \{0\}} \colon \eta_{X \times \{0\}} \to X \times S^{k-1}$ covering the identity $X \times \{0\} \to X$ which is unique up to fiber homotopy. Composing $v|_{X \times \{0\}}$ and u yields a fiber homotopy equivalence unique up to fiber homotopy $(w, \mathrm{id}) \colon S\xi \to \underline{S^{k-1}}$. Thus we can assign to map $F \colon X \to G/O$ an element $[\xi, w] \in \mathcal{G}/\mathcal{O}(X)$. We leave it to the reader to check that this induces the desired bijection $\mu \colon [X, G/O] \xrightarrow{\cong} \mathcal{G}/\mathcal{O}(X)$. This finishes the proof of Theorem 3.45. ∎

Notice that Theorem 3.45 yields after a choice of an element in $\mathcal{T}_n(X)$ a bijection of sets $[X, G/O] \xrightarrow{\cong} \mathcal{T}_n(X)$.

Definition 3.46 *Let X be a connected finite n-dimensional Poincaré complex together with a k-dimensional vector bundle $\xi \colon E \to X$. A* normal k-map $(M, i, f, \overline{f})$ *consists of a closed manifold M of dimension n together with an embedding $i \colon M \to \mathbb{R}^{n+k}$ and a bundle map $(\overline{f}, f) \colon \nu(M) \to \xi$. A* normal map of degree one *is a normal map such that the degree of $f \colon M \to X$ is one.*

Notice that this definition is the same as the definition of an element representing a class in $\Omega_n(\xi)$ except that we additionally require the map of degree 1. Analogously one requires in the definition of a bordism that the map $F \colon (W, \partial W) \to (X \times [0,1], X \times \partial[0,1])$ has degree one. Denote by $\mathcal{N}_n(X, k)$ the set of normal bordism classes of normal k-maps of degree one to X. Define $\mathcal{N}_n(X, k) \to \mathcal{N}_n(X, k+1)$ by sending the class of $(M, i, f, \overline{f})$ to the class of $(M, i', f, \overline{f}')$, where $i' \colon M \to \mathbb{R}^{n+k+1}$ is the composition of i with the standard inclusion $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$ and $\overline{f}'$ is $\nu(i') = \nu(i) \oplus \underline{\mathbb{R}} \xrightarrow{\nu(f) \oplus \mathrm{id}} \xi \oplus \underline{\mathbb{R}}$.

Definition 3.47 *Let X be a connected finite n-dimensional Poincaré complex. Define the* set of normal maps to X

$$\mathcal{N}_n(X) \;:=\; \operatorname{colim}_{k \to \infty} \mathcal{N}_n(X, k).$$

Notice that in the definition of $\mathcal{N}_n(X,k)$ and $\mathcal{N}_n(X)$ the bundle ξ over X may vary, we only fix X. The proof of the next result is similar to the one of Theorem 3.26.

Theorem 3.48 *Let X be a connected finite n-dimensional Poincaré complex. Then the Pontrjagin-Thom construction yields for each $k \geq 1$ a bijection*

$$P_k(X)\colon \mathcal{N}_n(X,k) \xrightarrow{\cong} \mathcal{T}_n(X,k).$$

This induces a bijection

$$P(X)\colon \mathcal{N}_n(X) \xrightarrow{\cong} \mathcal{T}_n(X).$$

Remark 3.49 In view of the Pontrjagin Thom construction it is convenient to work with the normal bundle. On the other hand one always needs an embedding and one would prefer an intrinsic definition. This is possible if one defines the normal map in terms of the tangent bundle which we will do below. Both approaches are equivalent. We will use in the sequel the one which is adequate for the concrete purpose. We mention that for a generalization to the equivariant setting the approach using the tangent bundle is more useful [73], [74].

Definition 3.50 *Let X be a connected finite n-dimensional Poincaré complex together with a vector bundle $\xi\colon E \to X$. A* normal map with respect to the tangent bundle $(\overline{f}, f)\colon M \to X$ *consists of an oriented closed manifold M of dimension n together with a bundle map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ for some integer $a \geq 0$. If f has degree one, we call $(\overline{f}, f)$ a* normal map of degree one with respect to the tangent bundle.

We define also a bordism relation as follows. Consider two normal maps of degree one with respect to the tangent bundle $(\overline{f}_i, f_i)\colon TM \oplus \underline{\mathbb{R}^{a_i}} \to \xi_i$ covering $f_i\colon M_i \to X$. A normal bordism is a normal map of degree one $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}}^b \to \eta$ covering $F\colon W \to X$ such that ∂W is a disjoint union $\partial_0 W \coprod \partial_1 W$ and we have the following data for $i = 0, 1$. We require diffeomorphisms $u_i\colon M_i \to \partial_i W$ with $F \circ u_i = f_i$. Moreover, we require bundle isomorphisms $v_i\colon \xi_i \oplus \underline{\mathbb{R}}^{b-a_i+1} \to \eta$ covering the identity on X such that following diagram commutes

$$\begin{tikzcd}
TM_i \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^b} \arrow[r, "\overline{f}_i \oplus \mathrm{id}_{\underline{\mathbb{R}^{b-a_i+1}}}"] \arrow[d, "Tu_i \oplus n_i \oplus \mathrm{id}_{\underline{\mathbb{R}^b}}"'] & \xi_i \oplus \underline{\mathbb{R}^{b-a_i+1}} \arrow[d, "v_i"] \\
TW|_{\partial_i W} \oplus \underline{\mathbb{R}^b} \arrow[r, "F|_{\partial_i W}"'] & \eta|_{\partial_i W}
\end{tikzcd}$$

Here $Tu_i \colon TM_i \to TW$ is given by the differential and $n_i \colon \underline{\mathbb{R}} \to TW$ is the bundle monomorphism given by an inward normal field of $TW|_{\partial_i W}$. Denote by $\mathcal{N}_n^T(X)$ the set of bordism classes of normal maps of degree one with respect to the tangent bundle.

Lemma 3.51 *Let X be a connected n-dimensional finite Poincaré complex. There is a natural bijection*

$$\mathcal{N}_n(X) \;\cong\; \mathcal{N}_n^T(X).$$

Proof : We define a map $\phi_n(k) \colon \mathcal{N}_n(X,k) \to \mathcal{N}_n^T(X)$ as follows. Consider a normal k-map $\overline{f} \colon \nu(M) \to \xi$ covering the map $f \colon M \to X$ of degree one for some closed oriented n-dimensional manifold M with an embedding $M \to \mathbb{R}^{n+k}$. Since X is compact, we can find a bundle η together with an isomorphism $u \colon \eta \oplus \xi \xrightarrow{\cong} \underline{\mathbb{R}^a}$. There is an explicit isomorphism $v \colon \nu(M) \oplus TM \cong \underline{\mathbb{R}^{n+k}}$. We get from $\overline{f}$, u and v an isomorphism of bundles covering the identity on M

$$f^*\eta \oplus \underline{\mathbb{R}^{n+k}} \;\cong\; f^*\eta \oplus TM \oplus \nu(M) \;\cong\; f^*\eta \oplus TM \oplus f^*\xi \cong TM \oplus \underline{\mathbb{R}^a}.$$

The inverse of this isomorphism is the same as a bundle map $\overline{g} \colon TM \oplus \underline{\mathbb{R}^a} \to \eta \oplus \underline{\mathbb{R}^{n+k}}$ covering f. Define the image under $\phi_n(k)$ of the class $[\overline{f}, f]$ to be the class $[\overline{g}, f]$. One easily checks that this is well-defined and that the $\phi_n(k)$-s fit together to a map $\phi_n \colon \mathcal{N}_n(X) \to \mathcal{N}_n^T(X)$. By the analogous construction one gets an inverse. ∎

Theorem 3.52 (Normal Maps and G/O) *Let M be a closed n-dimensional manifolds. Then the sets $\mathcal{T}_n(M)$, $\mathcal{N}_n(M)$ and $\mathcal{N}_n^T(M)$ are non-empty and come with preferred base points and there are canonical bijections*

$$[M, G/O] \cong \mathcal{T}_n(M) \cong \mathcal{N}_n(M) \cong \mathcal{N}_n^T(M).$$

Proof : follows from Theorem 3.45, Theorem 3.48 and Lemma 3.51. ■

Remark 3.53 Let X be a finite Poincaré complex of dimension n. Then there exists a closed manifold M, which is homotopy equivalent to X, only if there exists a normal map of degree one with target X. Namely, suppose $f\colon M \to X$ is a homotopy equivalence with a closed manifold as source. Choose a homotopy inverse $f^{-1}\colon X \to M$. Put $\xi = (f^{-1})^*TM$. Then we can cover f by a bundle map $\overline{f}\colon TM \to \xi$. Thus we get a normal map $(\overline{f}, f)$ of degree one with M as source and X as target. In order to solve Problem 3.1 we have to address the following problem.

Problem 3.54 *Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X. Can we change M and f leaving X fixed to get a normal map $(\overline{g}, g)$ such that g is a homotopy equivalence?*

3.4 The Surgery Step

In this section we explain the surgery step. We begin with a motivation.

3.4.1 Motivation for the Surgery Step

We first consider the CW-complex version of Problem 3.54.

Let $f\colon Y \to X$ be a map of CW-complexes. We want to find a procedure which changes Y and f leaving X fixed to map $f'\colon Y' \to X$ which is a homotopy equivalence. Of course this procedure should have the potential to carry over to the case, where Y is a manifold and the resulting source Y' is also a manifold. The Whitehead Theorem [121, Theorem V.3.1 and Theorem V.3.5 on page 230] says that f is a homotopy equivalence if and only if it is k-connected for all $k \geq 0$. Recall that *k-connected* means that $\pi_j(f, y)\colon \pi_j(Y, y) \to \pi_j(X, f(y))$ is bijective for $j < k$ and surjective for $j = k$ for all base points y in Y. Hence we would expect from the procedure to get $\pi_k(f)$ to be closer and closer to be trivial for all k. It is reasonable to try to work out an inductive procedure, where f is already k-connected and we would like to make it $(k+1)$-connected. Recall that there is a long exact homotopy sequence of a map $f\colon Y \to X$

$$\ldots \to \pi_{l+1}(Y) \to \pi_{l+1}(X) \to \pi_{l+1}(f) \to \pi_l(Y) \to \pi_l(X) \to \ldots,$$

where $\pi_{l+1}(f) \cong \pi_{l+1}(\mathrm{cyl}(f), Y)$ consists of homotopy classes of commutative squares with j the inclusion

$$\begin{tikzcd} S^l \arrow[r, "q"] \arrow[d, "j"'] & Y \arrow[d, "f"] \\ D^{l+1} \arrow[r, "Q"'] & X \end{tikzcd}$$

The map f is k-connected, if and only if $\pi_l(f) = 0$ for $l \leq k$. Suppose that f is k-connected. In order to achieve that f is $(k+1)$-connected, we must arrange that $\pi_{k+1}(f) = 0$ without changing $\pi_l(f)$ for $l \leq k$. Consider an element ω in $\pi_{k+1}(f)$ given by square as above. Define Y' to be the pushout

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d, "j"'] & Y \arrow[d, "j'"] \\ D^{k+1} \arrow[r, "\overline{q}"'] & Y' \end{tikzcd}$$

Then the universal property of the pushout gives a map $f' \colon Y' \to X$. We will say that f' is obtained from f by attaching a cell.

One easily checks that $\pi_l(f') = \pi_l(f)$ for $l \leq k$ and that there is a natural map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ which is surjective and whose kernel contains $\omega \in \pi_l(f)$. Recall that in general $\pi_l(f)$ is an abelian group for $l \geq 3$ and a group for $l = 2$ but carries no group structure for $l = 0, 1$. Moreover $\pi_1(X)$ acts on $\pi_l(f)$. Hence $\pi_l(f)$ is a $\mathbb{Z}\pi_1(X)$-module for $l \geq 3$. For $k \geq 3$ the kernel of the epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is the $\mathbb{Z}\pi_1(Y)$-submodule generated by ω [121, Section V.1]. We see that we can achieve by applying this construction that $\pi_{l+1}(f)$ becomes zero. Suppose that Y and X are finite CW-complexes. One can achieve $\pi_{k+1}(f) = 0$ in a finite number of steps by the following result.

Lemma 3.55 *Let $f \colon Y \to X$ be a map of finite CW-complexes. Suppose that f is $(k-1)$-connected for some integer $k \geq 0$.*

1. *Suppose that X is connected, $k \geq 2$ and $\pi_1(f) \colon \pi_1(Y) \to \pi_1(X)$ is bijective. Then $\pi_k(f)$ is a finitely generated $\mathbb{Z}\pi_1(X)$-module;*

2. *One can make f k-connected by attaching finitely many cells.*

Proof : (1) Notice that both Y and X are connected. We will identify $\pi := \pi_1(Y) = \pi_1(X)$. The map f lifts to π-map $\widetilde{f}\colon \widetilde{Y} \to \widetilde{X}$ between the universal coverings. The obvious map $\pi_k(\widetilde{f}) \to \pi_k(f)$ is bijective and compatible with the π-operations which are given by the π-actions on the universal covering and the operation of the fundamental group on the homotopy groups. The Hurewicz homomorphism induces an isomorphism of $\mathbb{Z}\pi$-modules $\pi_k(\widetilde{f}) \xrightarrow{\cong} H_k(\widetilde{f})$ since $\widetilde{Y}$ and $\widetilde{X}$ are simply-connected [121, Corollary IV.7.10 on page 181]. In particular we see that $\pi_k(f)$ is indeed an abelian group what is true in general only for $k \geq 3$. Since f and hence $\widetilde{f}$ is $(k-1)$-connected the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{f})$ which is defined to be the mapping cone $D_* := \operatorname{cone}_*(C_*(\widetilde{f}))$ of the $\mathbb{Z}\pi$-chain map $C_*(\widetilde{f})\colon C_*(\widetilde{Y}) \to C_*(\widetilde{X})$ is $(k-1)$-connected. Thus $C_*(\widetilde{f})$ yields an exact sequence of finitely generated free $\mathbb{Z}\pi$-modules

$$0 \to \ker(d_k) \to D_k \xrightarrow{d_k} D_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} D_0 \to 0.$$

Hence $\ker(d_k)$ is a finitely generated free projective $\mathbb{Z}\pi$-module which is stably free, i.e. after adding a finitely generated free $\mathbb{Z}\pi$-module it becomes free. Since $H_k(\widetilde{f})$ is a quotient of $\ker(d_k)$, it is finitely generated. Hence $\pi_k(f)$ is finitely generated free as $\mathbb{Z}\pi$-module.

(2) We begin with $k = 0$. Attaching zero cells means taking the disjoint union of Y with finitely many points. Obviously one can achieve in this way that $\pi_0(f)\colon \pi_0(Y) \to \pi_0(X)$ is surjective. In the sequel we assume that X is connected, otherwise we treat each component separately.

Next we treat the case $k = 1$. Since Y is by assumption a finite CW-complex, $\pi_0(Y)$ is finite. If two path components of Y are mapped to the same path component in X, one can attach a 1-cell in the obvious manner to connect these components. Thus we can achieve that $\pi_0(f)$ is bijective. Since X is finite, $\pi_1(X)$ is finitely generated. By attaching 1-cells trivially to Y which is the same as taking the one-point union of Y with S^1, we can achieve that $\pi_1(f)$ is an epimorphism.

Next we consider $k = 2$. Since both Y and X are finite, $\pi_1(f)\colon \pi_1(Y) \to \pi_1(X)$ is an epimorphism of a finitely generated group onto a finitely presented group. One easily checks that the kernel of such a group homomorphism is always finitely generated. For any element in a finite set of generators we can attach 2-cells to kill these elements. The resulting map induces an isomorphism on $\pi_1(Y)$. Now we can apply assertion (1). The cases $k \geq 3$ follow directly from assertion (1). This finishes the proof of Lemma 3.55. ■

Actually one can achieve in the world of CW-complexes the desired homotopy equivalence $f'\colon Y' \to X$ directly by the following construction. Namely consider the projection $\mathrm{pr}\colon \mathrm{cyl}(f) \to X$ of the mapping cylinder of f to X. Obviously the mapping cylinder is in general no manifold even if $f\colon Y \to X$ is a smooth map of closed manifolds. Neither there is a chance that the space Y' obtained from Y by attaching a cell is a manifold even if Y is a closed manifold and f smooth. But this single step of attaching a cell can be modified so that it applies to manifolds as source such that the resulting map has a manifold as source. This will be explained next.

Suppose that M is a compact manifold of dimension n and X is a CW-complex. Suppose that $f\colon M \to X$ is a k-connected map. Consider an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d, "j"'] & M \arrow[d, "f"] \\ D^{k+1} \arrow[r, "Q"'] & X \end{tikzcd}$$

We cannot attach a single cell to M without distroying the manifold structure. But one can glue two manifolds together along a common boundary such that the result is a manifold. Suppose that the map $q\colon S^k \to M$ extends to an embedding $\overline{q}\colon S^k \times D^{n-k} \to M$. Let $\mathrm{int}(\mathrm{im}(\overline{q}))$ be the interior of the image of $\overline{q}$. Then $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ is a manifold with boundary $\mathrm{im}(\overline{q}|_{S^k \times S^{n-k-1}})$. We can get rid of the boundary by attaching $D^{k+1} \times S^{n-k-1}$ along $\mathrm{im}(\overline{q}|_{S^k \times S^{n-k-1}})$. Call the result

$$M' \;:=\; D^{k+1} \times S^{n-k-1} \cup_{\mathrm{im}(\overline{q}|_{S^k \times S^{n-k-1}})} (M - \mathrm{int}(\mathrm{im}(\overline{q}))) \,.$$

Here and elsewhere we apply without further mentioning the technique of straightening the angle in order to get a well-defined smooth structure (see [14, Definition 13.11 on page 145 and (13.12) on page 148] and [54, Chapter 8, Section 2]). Choose a map $\overline{Q}\colon D^{k+1} \times D^{n-k} \to X$ which extends Q and $\overline{q}$. The restriction of f to $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ extends to a map $f'\colon M' \to X$ using $\overline{Q}|_{D^{k+1} \times S^{n-k}}$. Notice that the inclusion $M - \mathrm{int}(\mathrm{im}(\overline{q})) \to M$ is $(n-k-1)$-connected since $S^k \times S^{n-k-1} \to S^k \times D^{n-k}$ is $(n-k-1)$-connected. So the passage from M to $M - \mathrm{int}(\mathrm{im}(\overline{q}))$ will not affect $\pi_j(f)$ for $j < n-k-1$. All in all we see that $\pi_l(f) = \pi_l(f')$ for $l \leq k$ and that there is an epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ whose kernel contains ω, provided that $2(k+1) \leq n$. The condition $2(k+1) \leq n$ can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality forces also a change in dimension $(n-l)$. This phenomenon will cause surgery obstructions to appear.

It is important to notice that $f\colon M \to X$ and $f'\colon M' \to X$ are bordant. The relevant bordism is given by $W = D^{k+1} \times D^{n-k} \cup_{\overline{q}} M \times [0,1]$, where we think of $\overline{q}$ as an embedding $S^k \times D^{n-k} \to M \times \{1\}$. In other words, W is obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k}$ to $M \times \{1\}$. Then M appears in W as $M \times \{0\}$ and M' as other part of the boundary of W. Define $F\colon W \to X$ by $f \times \mathrm{id}_{[0,1]}$ and $\overline{Q}$. Then F restricted to M and M' is f and f'.

But before we come to surgery obstructions, we must figure out, whether we can arrange that q is an embedding and extends to an embedding $\overline{q}$. If $2k < n$, we can change q up to homotopy such that it becomes an embedding. In the case $2k = n$ we can change q up to homotopy to an immersion and the surgery obstruction will actually be the obstruction to change it up to homotopy to an embedding. Let us assume that q is an embedding. Then the existence of the extension $\overline{q}$ is equivalent to the triviality of the normal bundle of the embedding $q\colon S^k \to M$.

A priori there is no reason why this normal bundle should be trivial. At this point the bundle data attached to a normal map become useful. So far we have only considered them since it was a necessary condition to be able to solve Problem 3.1 although in Problem 3.1 no bundle maps appear. But now it will pay off that we have this bundle data available. Namely, if we want to kill an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow[Q]{} & X \end{array}$$

then $q^*TM \oplus \underline{\mathbb{R}^a}$ is isomorphic to $q^*f^*\xi = j^*Q^*\xi$ and hence is trivial since $Q^*\xi$ is a bundle over the contractible space D^{k+1}. Since $\nu(S^k, M) \oplus TS^k$ is isomorphic to q^*TM, the bundle $\nu(S^k, M) \oplus \underline{\mathbb{R}}^a$ is trivial for some $a \geq 0$. Suppose $2k \leq n-1$. Then the natural map $BO(n-k) \to BO(n-k+a)$ is $(k+1)$-connected. Hence $\nu(S^k, M)$ itself is trivial. So we are able to carry out one step. But we have to ensure that we can repeat this process. So we must arrange that the bundle data are also available for the resulting map $f'\colon M' \to X$. Therefore we must be more careful with the choice of embedding (resp. immersion) which is homotopy equivalent to g. The given bundle data should tell us which embedding we should choose. For this we need some information about embeddings and immersions which we will give next.

3.4.2 Immersions and Embeddings

Given two vector bundles $\xi\colon E \to M$ and $\eta\colon F \to N$, we have so far only considered bundle maps $(\overline{f}, f)\colon \xi \to \eta$ which are fiberwise isomorphisms. We need to consider now more generally bundle monomorphisms, i.e. we only will require that the map is fiberwise injective. Consider two bundle monomorphism $(\overline{f}_0, f_0), (\overline{f}_1, f_1)\colon \xi \to \eta$. Let $\xi \times [0,1]$ be the vector bundle $\xi \times \mathrm{id}\colon E \times [0,1] \to M \times [0,1]$. A homotopy of bundle monomorphisms $(\overline{h}, h)$ from $(\overline{f}_0, f_0)$ to $(\overline{f}_1, f_1)$ is a bundle monomorphism $(\overline{h}, h)\colon \xi \times [0,1] \to \eta$ whose restriction to $X \times \{j\}$ is $(\overline{f}_j, f_j)$ for $j = 0, 1$. Denote by $\pi_0(\mathrm{Mono}(\xi, \eta))$ the set of homotopy classes of bundle monomorphisms.

An immersion $f\colon M \to N$ is a map whose differential $Tf\colon TM \to TN$ is a bundle monomorphism. An immersion is locally an embedding but it is not isotopic to an embedding in general. A regular homotopy $h\colon M \times [0,1] \to N$ from an immersion $f_0\colon M \to N$ to an immersion $f_1\colon M \to N$ is a homotopy h such that $h_0 = f_0$, $h_1 = f_1$ and $h_t\colon M \to N$ is an immersion for each $t \in [0,1]$. Denote by $\pi_0(\mathrm{Imm}(M, N))$ the set of regular homotopy classes of immersions from M to N. The next result is due to Whitney [122], [123].

Theorem 3.56 *Let M and N be closed manifolds of dimensions m and n. Then any map $f\colon M \to N$ is arbitrarily close to an immersion provided that $2m \leq n$ and arbitrarily close to an embedding provided that $2m < n$.*

For a proof of the following result we refer to Haefliger-Poenaru [48], Hirsch [53] and Smale [107].

Theorem 3.57 (Immersions and Bundle Monomorphisms) *Let M be a m-dimensional and N be a n-dimensional closed manifold.*

1. *Suppose that $1 \leq m < n$. Then taking the differential of an immersion yields a bijection*

$$T\colon \pi_0(\mathrm{Imm}(M, N)) \xrightarrow{\cong} \pi_0(\mathrm{Mono}(TM, TN));$$

2. *Suppose that $1 \leq m \leq n$ and that M has a handlebody decomposition consisting of q-handles for $q \leq n-2$. Then taking the differential of an immersion yields a bijection*

$$T\colon \pi_0(\mathrm{Imm}(M,N)) \xrightarrow{\cong} \mathrm{colim}_{a\to\infty}\,\pi_0(\mathrm{Mono}(TM \oplus \underline{\mathbb{R}^a}, TN \oplus \underline{\mathbb{R}^a})),$$

where the colimit is given by stabilization.

Example 3.58 Theorem 3.57 (1) has the following remarkable consequence. We claim that $\pi_0(\mathrm{Imm}(S^2,\mathbb{R}^3))$ consists of one element. We have to show that $\pi_0(\mathrm{Mono}(TS^2,T\mathbb{R}^3))$ consists of one element. Consider bundle monomorphisms $(\overline{f}_i, f_i)\colon TS^2 \to T\mathbb{R}^3$ for $i=0,1$. Since $T_xf_i(T_xS^2) \subset T_{f_i(x)}\mathbb{R}^3$ is an oriented 2-dimensional subspace of the oriented Euclidean vector space $T_{f(x)}\mathbb{R}^3$, there is precisely one vector $v_i(x) \in T_{f_i(x)}\mathbb{R}^3$ whose norm is one and for which the orientation on $T_xf_i(T_xS^2) \oplus \mathbb{R}v(x) = T_{f_i(x)}\mathbb{R}^3$ induced by the one of T_xS^2 and $v(x)$ and the standard one on $T_{f_i(x)}\mathbb{R}^3$ agree. Hence we can find an orientation preserving bundle isomorphism $(\overline{g_i}, f_i)\colon TS^2 \oplus \underline{\mathbb{R}} \to T\mathbb{R}^3$ covering f_i and extending $\overline{f}_i$. Since f_i is homotopic to the constant map c with value 0, we can find a homotopy of bundle maps which are fiberwise orientation preserving isomorphisms from $(\overline{g}_i, f_i)$ to $(\overline{c}_i, c)$ for $i=0,1$. It suffices to show that $(\overline{c}_0, c)$ and $(\overline{c}_1, c)$ are strongly fiber homotopic as bundle maps (which are fiberwise isomorphisms) because then we get by restriction a homotopy of bundle monomorphism between $(\overline{f}_0, f_0)$ and $(\overline{f}_1, f_1)$. Now $(\overline{c}_0, c)$ and $(\overline{c}_1, c)$ differ by an orientation preserving bundle automorphism covering the identity $(\overline{u}, \mathrm{id})\colon S^2 \times T_0\mathbb{R}^3 \to S^2 \times T_0\mathbb{R}^3$. This is the same a map $\overline{u}\colon S^2 \to GL(3,\mathbb{R})^+$, where we identify $T_0\mathbb{R}^3 = \mathbb{R}^3$ and $GL(3,\mathbb{R})^+$ is the Lie group of orientation preserving linear automorphism of $\mathbb{R}^3$. The inclusion $SO(3) \to GL(3,\mathbb{R})^+$ is a homotopy equivalence by the polar decomposition. Since $\pi_2(SO(3))$ is known to be zero, there is a strong homotopy of bundle maps which are fiberwise isomorphisms and cover the identity from $(\overline{u}, id)$ to $(\mathrm{id},\mathrm{id})$. This proves that $\pi_0(\mathrm{Imm}(S^2,\mathbb{R}^3))$ consists of precisely one element.

Let $f_0\colon S^2 \to \mathbb{R}^3$ be the standard embedding. Let $f_1\colon S^2 \to \mathbb{R}^3$ be its composition with the involution $i\colon \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\}$ which sends x to $\frac{x}{||x||^2}$. By the argument above f_0 and f_1 are regular homotopic. Notice that i is the identity on $\mathrm{im}(f_0)$. Its differential at a point $x \in \mathrm{im}(f_0)$ sends the normal vector v pointing to the origin in $\mathbb{R}^3$ to $-v$. Therefore a regular homotopy from f_0 to f_1 will turn the inside of the standard sphere $f_0\colon S^2 \to \mathbb{R}^3$ to the outside.

3.4.3 The Surgery Step

Now we can carry out the surgery step.

Theorem 3.59 (The Surgery Step) *Consider a normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon M \to X$ and an element $\omega \in \pi_{k+1}(f)$ for $k \leq n-2$ for $n = \dim(M)$. Let $Tj \oplus n\colon T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}} \to T(D^{k+1} \times D^{n-k})$ be the bundle map covering the inclusion j which is given by the differential Tj of j and the inward normal field of the boundary of $D^{k+1} \times D^{n-k}$. Then*

1. *We can find a commutative diagram of vector bundles*

$$\begin{tikzcd} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b}} \arrow[r, "\overline{q}"] \arrow[d, "Tj \oplus n \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+b-1}}}"'] & TM \oplus \underline{\mathbb{R}^{a+b}} \arrow[d, "\overline{f}"] \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b-1}} \arrow[r, "\overline{Q}"'] & \xi \oplus \underline{\mathbb{R}^b} \end{tikzcd}$$

   covering a commutative diagram

$$\begin{tikzcd} S^k \times D^{n-k} \arrow[r, "q"] \arrow[d, "j"'] & M \arrow[d, "f"] \\ D^{k+1} \times D^{n-k} \arrow[r, "Q"'] & X \end{tikzcd}$$

   such that the restriction of the last diagram to $D^{k+1} \times \{0\}$ represents ω and $q\colon S^k \times D^{n-k} \to M$ is an immersion;

2. *The regular homotopy class of the immersion q appearing in assertion (1) is uniquely determined by the properties above and depends only on ω and $(\overline{f}, f)$;*

3. *Suppose that the regular homotopy class of the immersion q appearing in (1) contains an embedding. Then one can arrange q in assertion (1) to be an embedding. If $2k < n$, one can always find an embedding in the regular homotopy class of q;*

4. *Suppose that the map q appearing in assertion (1) is an embedding.*

   Let W be the manifold obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k}$ by $q\colon S^k \times D^{n-k} \to M = M \times \{1\}$. Let $F\colon W \to X$ be the map induced by $M \times [0,1] \xrightarrow{\mathrm{pr}} M \xrightarrow{f} X$ and $Q\colon D^k \times D^{k+1} \to X$.

After possibly stabilizing $\overline{f}$ the bundle maps $\overline{f}$ and $\overline{Q}$ induce a bundle map $\overline{F}\colon TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering $F\colon W \to X$. Thus we get a normal map $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ which extends $(\overline{f} \oplus (f \times \mathrm{id}_{\mathbb{R}^b}), f)\colon TM \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$. The normal map $(\overline{f}', f')\colon TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ obtained by restricting $(\overline{F}, F)$ to $\partial W - M \times \{0\} =: M'$ is a normal map of degree one which is normally bordant to $(\overline{f}, f)$ and has as underlying manifold $M' = M - \mathrm{int}(q(S^k \times D^{n-k})) \cup_q D^k \times S^{n-k-1}$.

Proof : (1) Choose a commutative diagram of smooth maps

$$\begin{tikzcd}
S^k \arrow[r, "q'"] \arrow[d, "j'"'] & M \arrow[d, "f"] \\
D^{k+1} \arrow[r, "Q'"'] & X
\end{tikzcd}$$

representing ω. Of course it can be extended to a diagram

$$\begin{tikzcd}
S^k \times D^{n-k} \arrow[r, "q"] \arrow[d, "j"'] & M \arrow[d, "f"] \\
D^{k+1} \times D^{n-k} \arrow[r, "Q"'] & X
\end{tikzcd}$$

Since $D^{k+1} \times D^{n-k+1}$ is contractible, we can find a bundle map $\overline{Q}\colon TD^{k+1} \times D^{n-k} \oplus \underline{\mathbb{R}^{a-1}} \to \xi$ covering Q. There is precisely one bundle map $\overline{q}$ covering q such that the following diagram commutes

$$\begin{tikzcd}
T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{q}"] \arrow[d, "Tj \oplus n \oplus \mathrm{id}_{\underline{\mathbb{R}^{a-1}}}"'] & TM \oplus \underline{\mathbb{R}^a} \arrow[d, "\overline{f}"] \\
T(D^{k+1} \times D^{n-k+1}) \oplus \underline{\mathbb{R}^{a-1}} \arrow[r, "\overline{Q}"'] & \xi
\end{tikzcd}$$

Suppose that $k \leq n-2$. From Theorem 3.57 (2) we get an immersion $q_0\colon S^k \times D^{n-k} \to M$, such that $(Tq_0, q_0)\colon T(S^k \times D^{n-k}) \to TM$ and $(\overline{q}, q)$ define the same element in

$$\mathrm{colim}_{c \to \infty} \pi_0(\mathrm{Mono}(TS^k \times D^{n-k} \oplus \underline{\mathbb{R}^c}, TM \oplus \underline{\mathbb{R}^s})).$$

Now after possibly stabilization $\overline{f}$ and thus enlarging a to $a+b$ we can achieve the desired diagram by a cofibration argument.

(2) The stable homotopy class of $T\overline{q}$ is uniquely determined by the commutativity of the diagram of vector bundles

$$\begin{tikzcd}
T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b}} \arrow[r, "T\overline{q} \oplus (q \times \mathrm{id}_{\mathbb{R}^{a+b}})"] \arrow[d, "Tj \oplus n \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+b-1}}}"'] & TM \oplus \underline{\mathbb{R}^{a+b}} \arrow[d, "\overline{f}"] \\
T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b-1}} \arrow[r, "\overline{Q}"'] & \xi
\end{tikzcd}$$

since $D^{k+1} \times D^{n+k}$ is contractible and hence $(\overline{Q}, Q)$ is unique up to fiber homotopy. Now apply Theorem 3.57 (2).

(3) By a cofibration argument we can use a regular homotopy from q to an embedding q_0 to change all the diagrams for q up to homotopy to diagrams for q_0. If $2k < n$, we can find an embedding q_0 which is arbitrary close to q by Theorem 3.56 and hence regular homotopic to q since the condition being an immersion is an open condition.

(4) We leave it to the reader to check that the construction of $(\overline{F}, F)$ makes sense. ■

Definition 3.60 *Consider a normal map $(\overline{f}, f)\colon M \to N$ and an element $\omega \in \pi_{k+1}(f)$ for $k \leq n-2$ for $n = \dim(M)$. We call the normal map $(\overline{f'}, f')\colon TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ appearing in Theorem 3.59* (4) *the result of surgery on $(\overline{f}, f)$ and ω if it exists. Sometimes we call the step from $(\overline{f}, f)$ to $(\overline{f'}, f')$ a* surgery step.

We conclude from Lemma 3.55, from the discussion of the effect of surgery after Lemma 3.55 and from Theorem 3.59

Theorem 3.61 *Let X be a connected finite n-dimensional Poincaré complex. Let $\overline{f}\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ be a normal map of degree one covering $f\colon M \to X$. Then we can carry out a finite sequence of surgery steps to obtain a normal map of degree one $\overline{g}\colon TN \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering $g\colon N \to X$ such that $(\overline{f}, f)$ and $(\overline{g}, g)$ are normally bordant and g is k-connected, where $n = 2k$ or $n = 2k+1$.*

Recall that we want to address Problem 3.54. The strategy we have developed so far is to do surgery to change our normal map into a homotopy equivalence. Theorem 3.61 gives us some hope to carry out this program successfully, at least we can get a highly connected map. So we can give the final version of the surgery problem.

Problem 3.62 (Surgery problem) *Suppose we have some normal map* $(\overline{f}, f)$ *from a closed manifold* M *to a finite Poincaré complex* X. *Can we change* M *and* f *leaving* X *fixed by finitely many surgery steps to get a normal map* $(\overline{g}, g)$ *from a closed manifold* N *to* X *such that* g *is a homotopy equivalence?*

Remark 3.63 Suppose that X appearing in Problem 3.62 is orientable and of dimension $n = 4k$. Notice that the surgery step does not change the normal bordism class. In particular the manifolds M and N are oriented bordant. Hence Lemma 3.18 implies that M and N have the same signature. This implies that $\operatorname{sign}(M) - \operatorname{sign}(X)$ is an invariant of the normal bordism class and is not changed by a surgery step, where for non-connected M we mean by $\operatorname{sign}(M)$ the sum of the signatures of the components of M. Since the signature is also an oriented homotopy invariant, we see an obstruction to solve the Surgery Problem 3.62, namely $\operatorname{sign}(M) - \operatorname{sign}(X)$ must be zero. We will see that this is the only obtruction if X is a simply connected orientable $4k$-dimensional Poincaré complex for $k \geq 2$. If X is not simply connected, the vanishing of $\operatorname{sign}(M) - \operatorname{sign}(X)$ will not be sufficient, more complicated surgery obstructions will occur.

3.5 Miscellaneous

We give more information about the space G/O in Section 6.6.

Chapter 4

The Algebraic Surgery Obstruction

Introduction

In this chapter we want to give the solution to the surgery Problem 3.62, whether we can change a normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X by finitely many surgery steps to get a normal map $(\overline{f'}, f')$ from a closed manifold N to X such that f' is a homotopy equivalence. We have already seen in Theorem 3.61 that we can make f k-connected if $n = 2k$ or $n = 2k + 1$. So it remains to achieve that f is $(k+1)$-connected because then f is a homotopy equivalence by Poincaré duality. Of course we want to do further surgery on elements in $\pi_{k+1}(f)$ to make f $(k+1)$-connected. It will turn out that this is not possible in general. We will encounter an obstruction, the so called surgery obstruction. It takes values in the so called L-groups which are defined in terms of forms and formations.

Let us start with the case $n = 2k$ for $k \geq 3$. Then the problem will be that we cannot do surgery on each element ω in $\pi_{k+1}(f)$. The main obstacle is that an immersion $f \colon S^k \to M$ associated to ω may not be regularly homotopic to an embedding. This assumption appears in Theorem 3.59 (3). If we put f in general position, we may encounter double points. We have to figure out whether we can get rid of these double points. The main tool will be the Whitney trick which allows to get rid of two of the double points under certain algebraic conditions. In Section 4.1 we introduce intersection numbers and selfintersection numbers for immersions $S^k \to M$ and show that the selfintersection number of f is trivial if and only f is regularly homotopic to an embedding provided that $k \geq 3$ (see Theorem 4.8).

We will explain in Section 4.2 that the intersection pairing λ and the selfintersection numbers $\mu(f)$ for pointed immersions $f \colon S^k \to M$ are linked to one another. They together define the structure of a non-degenerate $(-1)^k$-quadratic form on the surgery kernel $K_k(\widetilde{M})$ which is $\mathbb{Z}\pi_1(M)$-isomorphic to $\pi_{k+1}(f)$. We will show that we are able to kill the surgery kernel $K_k(\widetilde{M})$ by finitely many surgery steps if and only if this non-degenerate $(-1)^k$-quadratic form is isomorphic to a standard $(-1)^k$-hyperbolic form $H_{(-1)^k}(R^b)$ for some b after adding a standard $(-1)^k$-hyperbolic form $H_{(-1)^k}(R^a)$ for some a (see Theorem 4.27). This leads in a natural way to the definition of the even-dimensional L-groups in Section 4.3 and of the surgery obstruction in even dimensions in Section 4.4.

In the odd-dimensional case $n = 2k + 1$ the embedding question has always a positive answer because of $2k < n$. Hence one can always do surgery on an element in $\pi_{k+1}(f)$. But it is not clear that one can find appropriate elements so that after doing surgery on them the surgery kernel is trivial. The problem is that surgery on such an element simultaneously affects $K_k(\widetilde{M})$ and $K_{k+1}(\widetilde{M})$ since these are related by Poincaré duality. We will explain the definition of the relevant odd-dimensional L-groups in Section 4.5 and we will only sketch the definition and the proof of the main properties of the surgery obstruction in Section 4.6.

Our main concern is not the surgery Problem 3.62 but the question whether two closed manifolds are diffeomorphic. We have explained the surgery program in Remark 1.5. We will explain in Section 4.7 why this forces us to consider also normal maps whose underlying map $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ is a map from a compact manifold with boundary to a Poincaré pair such that ∂f is already a homotopy equivalence. In this situation the aim of surgery is to change f into a homotopy equivalence without changing ∂f. The relevant modification of the surgery obstruction will be introduced. Because of the appearance of the Whitehead torsion in the s-Cobordism Theorem 1.1 we are also forced to take Whitehead torsion into account. So we want to achieve that f is a simple homotopy equivalence provided that $(X, \partial X)$ is a simple pair and ∂f is a simple homotopy equivalence. This will lead to the definition of simple L-groups and the simple surgery obstruction.

4.1 Intersection and Selfintersection Pairings

4.1.1 Intersections of Immersions

We are facing the problem to decide whether we can change an immersion $f\colon S^k \to M$ within its regular homotopy class to an embedding, where M is a closed manifold of dimension $n = 2k$. This problem occurs when we want to carry out a surgery step in the middle dimension (see Theorem 3.59). We first deal with the necessary algebraic obstructions and then address the question whether their vanishing is also sufficient.

We fix base points $s \in S^k$ and $b \in M$ and assume that M is connected and $k \geq 2$. We will consider pointed immersions (f, w), i.e. an immersion $f\colon S^k \to M$ together with a path w from b to $f(s)$. A pointed regular homotopy from (f_0, w_0) to (f_1, w_1) is a regular homotopy $h\colon S^k \times [0,1] \to M$ from $h_0 = f_0$ to $h_1 = f_1$ such that $w_0 * h(s, ?)$ and w_1 are homotopic paths relative end points. Here $h(s, ?)$ is the path from $f_0(s)$ to $f_1(s)$ given by restricting h to $\{s\} \times [0,1]$. Denote by $I_k(M)$ the set of pointed homotopy classes of pointed immersions from S^k to M. We need the paths to define the structure of an abelian group on $I_k(M)$. The sum of $[(f_0, w_0)]$ and $[(f_1, w_1)]$ is given by the connected sum along the path $w_0^- * w_1$ from $f_0(s)$ to $f_1(s)$. The zero element is given by the composition of the standard embedding $S^k \to \mathbb{R}^{k+1} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{k-1} = \mathbb{R}^n$ with some embedding $\mathbb{R}^n \subset M$ and any path w from b to the image of s. The inverse of the class of (f, w) is the class of $(f \circ a, w)$ for any base point preserving diffeomorphism $a\colon S^k \to S^k$ of degree -1.

The fundamental group $\pi = \pi_1(M, b)$ operates on $I_k(M)$ by composing the path w with a loop at b. Thus $I_k(M)$ inherits the structure of a $\mathbb{Z}\pi$-module.

Next we want to define the *intersection pairing*

$$\lambda\colon I_k(M) \times I_k(M) \to \mathbb{Z}\pi. \tag{4.1}$$

For this purpose we will have to fix an orientation of T_bM at b. Consider $\alpha_0 = [(f_0, w_0)]$ and $\alpha_1 = [(f_1, w_1)]$ in $I_k(M)$. Choose representatives (f_0, w_0) and (f_1, w_1). We can arrange without changing the pointed regular homotopy class that $D = \operatorname{im}(f_0) \cap \operatorname{im}(f_1)$ is finite, for any $y \in D$ both the preimage $f_0^{-1}(y)$ and the preimage $f_1^{-1}(y)$ consists of precisely one point and for any two points x_0 and x_1 in S^k with $f_0(x_0) = f_1(x_1)$ we have $T_{x_0}f_0(T_{x_0}S^k) + T_{x_1}f_1(T_{x_1}S^k) = T_{f_0(x_0)}M$. Consider $d \in D$. Let x_0 and x_1 in S^k be the points uniquely determined by $f_0(x_0) = f_1(x_1) = d$. Let u_i be a path in S^k

from s to x_i. Then we obtain an element $g(d) \in \pi$ by the loop at b given by the composition $w_1 * f_1(u_1) * f_0(u_0)^- * w_0^-$. Recall that we have fixed an orientation of T_bM. The fiber transport along the path $w_0 * f(u_0)$ yields an isomorphism $T_bM \cong T_dM$ which is unique up to isotopy. Hence T_dM inherits an orientation from the given orientation of T_bM. The standard orientation of S^k determines an orientation on $T_{x_0}S^k$ and $T_{x_1}S^k$. We have the isomorphism of oriented vector spaces

$$T_{x_0}f_0 \oplus T_{x_1}f_1\colon T_{x_0}S^k \oplus T_{x_1}S^k \xrightarrow{\cong} T_dM.$$

Define $\epsilon(d) = 1$ if this isomorphism respects the orientations and $\epsilon(d) = -1$ otherwise. The elements $g(d) \in \pi$ and $\epsilon(d) \in \{\pm 1\}$ are independent of the choices of u_0 and u_1 since S^k is simply connected for $k \geq 2$. Define

$$\lambda(\alpha_0, \alpha_1) \ := \ \sum_{d \in D} \epsilon(d) \cdot g(d).$$

Lift $b \in M$ to a base point $\widetilde{b} \in \widetilde{M}$. Let $\widetilde{f_i}\colon S^k \to \widetilde{M}$ be the unique lift of f_i determined by w_i and $\widetilde{b}$ for $i = 0, 1$. Let $\lambda_{\mathbb{Z}}(\widetilde{f_0}, \widetilde{f_1})$ be the $\mathbb{Z}$-valued intersection number of $\widetilde{f_0}$ and $\widetilde{f_1}$. This is the same as the algebraic intersection number of the classes in the k-th homology with compact support defined by $\widetilde{f_0}$ and $\widetilde{f_1}$ which obviously depends only on the homotopy classes of $\widetilde{f_0}$ and $\widetilde{f_1}$. Then

$$\lambda(\alpha_0, \alpha_1) \ = \ \sum_{g \in \pi} \lambda_{\mathbb{Z}}(\widetilde{f_0}, l_{g^{-1}} \circ \widetilde{f_1}) \cdot g, \tag{4.2}$$

where $l_{g^{-1}}$ denotes left multiplication with g^{-1}. This shows that $\lambda(\alpha_0, \alpha_1)$ depends only on the pointed regular homotopy classes of (f_0, w_0) and (f_1, w_1).

In the sequel we use the $w = w_1(M)$-twisted involution on $\mathbb{Z}\pi$ which sends $\sum_{g \in \pi} a_g \cdot g$ to $\sum_{g \in \pi} w(g) \cdot a_g \cdot g^{-1}$. One easily checks

Lemma 4.3 *For $\alpha, \beta, \beta_1, \beta_2 \in I_k(M)$ and $u_1, u_2 \in \mathbb{Z}\pi$ we have*

$$\begin{aligned} \lambda(\alpha, \beta) &= (-1)^k \cdot \overline{\lambda(\beta, \alpha)}; \\ \lambda(\alpha, u_1 \cdot \beta_1 + u_2 \cdot \beta_2) &= u_1 \cdot \lambda(\alpha, \beta_1) + u_2 \cdot \lambda(\alpha, \beta_2). \end{aligned}$$

Remark 4.4 Suppose that the normal bundle of the immersion $f\colon S^k \to M$ has a nowhere vanishing section. (In our situation it actually will be trivial.) Suppose that f is regular homotopic to an embedding g. Then the normal bundle of g has a nowhere vanishing section σ. Let g' be the embedding obtained by moving g a little bit in the direction of this normal vector field σ. Choose a path w_f from $f(s)$ to b. Then for appropriate paths w_g and $w_{g'}$ we get pointed embeddings (g, w_g) and $(g', w_{g'})$ such that the pointed regular homotopy classes of (f, w), (g, w_g) and $(g', w_{g'})$ agree. Since g and g' have disjoint images, we conclude

$$\lambda([f,w],[f,w]) \;=\; 0.$$

Hence the vanishing of $\lambda([f,w],[f,w])$ is a necessary condition for finding an embedding in the regular homotopy class of f, provided that the normal bundle of f has a nowhere vanishing section. It is not a sufficient condition. To get a sufficient condition we have to consider self-intersections what we will do next.

4.1.2 Selfintersections of Immersions

Let $\alpha \in I_k(M)$ be an element. Let (f, w) be a pointed immersion representing α. Recall that we have fixed base points $s \in S^k$, $b \in M$ and an orientation of T_bM. Since we can find arbitrarily close to f an immersion which is in general position, we can assume without loss of generality that f itself is in general position. This means that there is a finite subset D of $\operatorname{im}(f)$ such that $f^{-1}(y)$ consists of precisely two points for $y \in D$ and of precisely one point for $y \in \operatorname{im}(f) - D$ and for two points x_0 and x_1 in S^k with $x_0 \neq x_1$ and $f(x_0) = f(x_1)$ we have $T_{x_0}f(T_{x_0}S^k) + T_{x_1}f(T_{x_1}S^k) = T_{f0(x_0)}M$. Now fix for any $d \in D$ an ordering $x_0(d), x_1(d)$ of $f^{-1}(d)$. Analogously to the construction above one defines $\epsilon(x_0(d), x_1(d)) \in \{\pm 1\}$ and $g(x_0(d), x_1(d)) \in \pi = \pi_1(M, b)$. Consider the element $\sum_{d \in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d))$ of $\mathbb{Z}\pi$. It does not only depend on f but also on the choice of the ordering of $f^{-1}(d)$ for $d \in D$. One easily checks that the change of ordering of $f^{-1}(d)$ has the following effect for $w = w_1(M) \colon \pi \to \{\pm 1\}$

$$\begin{aligned}
g(x_1(d), x_0(d)) &= g(x_0(d), x_1(d))^{-1}; \\
w(g(x_1(d), x_0(d))) &= w(g(x_0(d), x_1(d))); \\
\epsilon(x_1(d), x_0(d)) &= (-1)^k \cdot w(g(x_0(d), x_1(d))) \cdot \epsilon(x_0(d), x_1(d)); \\
\epsilon(x_1(d), x_0(d)) \cdot g(x_1(d), x_0(d)) &= (-1)^k \cdot \epsilon(x_0(d), x_1(d)) \cdot \overline{g(x_0(d), x_1(d))}.
\end{aligned}$$

Define an abelian group, where we use the w-twisted involution on $\mathbb{Z}\pi$

$$Q_{(-1)^k}(\mathbb{Z}\pi, w) \;:=\; \mathbb{Z}\pi/\{u - (-1)^k \cdot \overline{u} \mid u \in \mathbb{Z}\pi\}. \tag{4.5}$$

Define the *selfintersection element* for $\alpha \in I_k(M)$

$$\mu(\alpha) \quad := \left[\sum_{d \in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d))\right] \quad \in Q_{(-1)^k}(\mathbb{Z}\pi, w). \quad (4.6)$$

The passage from $\mathbb{Z}\pi$ to $Q_{(-1)^k}(\mathbb{Z}\pi, w)$ ensures that the definition is independent of the choice of the order on $f^{-1}(d)$ for $d \in D$. It remains to show that it depends only on the pointed regular homotopy class of (f, w). Let h be a pointed regular homotopy from (f, w) to (g, v). We can arrange that h is in general position. In particular each immersion h_t is in general position and comes with a set D_t. The collection of the D_t-s yields a bordism W from the finite set D_0 to the finite set D_1. Since W is a compact one-dimensional manifold, it consists of circles and arcs joining points in $D_0 \cup D_1$ to points in $D_0 \cup D_1$. Suppose that the point e and the point e' in $D_0 \cup D_1$ are joint by an arc. Then one easily checks that their contributions to

$$\begin{aligned}\mu(f, w) - \mu(g, w) \quad := \quad & \left[\sum_{d_0 \in D_0} \epsilon(x_0(d_0), x_1(d_0)) \cdot g(x_0(d_0), x_1(d_0))\right. \\ & \left. \qquad - \sum_{d_1 \in D_1} \epsilon(x_0(d_1), x_1(d_1)) \cdot g(x_0(d_1), x_1(d_1))\right]\end{aligned}$$

cancel out. This implies $\mu(f, w) = \mu(g, w)$.

Lemma 4.7 *Let $\mu\colon I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ be the map given by the selfintersection element (see* (4.6)*) and let $\lambda\colon I_k(M) \times I_k(M) \to \mathbb{Z}\pi$ be the intersection pairing (see* (4.1)*). Then*

1. *Let $(1 + (-1)^k \cdot T)\colon Q_{(-1)^k}(\mathbb{Z}\pi, w) \to \mathbb{Z}\pi$ be the homomorphism of abelian groups which sends $[u]$ to $u + (-1)^k \cdot \overline{u}$. Denote for $\alpha \in I_k(M)$ by $\chi(\alpha) \in \mathbb{Z}$ the Euler number of the normal bundle $\nu(f)$ for any representative (f, w) of α with respect to the orientation of $\nu(f)$ given by the standard orientation on S^k and the orientation on f^*TM given by the fixed orientation on T_bM and w. Then*

$$\lambda(\alpha, \alpha) = (1 + (-1)^k \cdot T)(\mu(\alpha)) + \chi(\alpha) \cdot 1;$$

2. *We get for $\mathrm{pr}\colon \mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ the canonical projection and $\alpha, \beta \in I_k(M)$*

$$\mu(\alpha + \beta) - \mu(\alpha) - \mu(\beta) = \mathrm{pr}(\lambda(\alpha, \beta));$$

3. *For $\alpha \in I_k(M)$ and $u \in \mathbb{Z}\pi$ we get with respect to the obvious $\mathbb{Z}\pi$-bimodule structure on $Q_{(-1)^k}(\mathbb{Z}\pi, w)$*
$$\mu(u \cdot \alpha) = u\mu(\alpha)\overline{u}.$$

Proof : (1) Represent $\alpha \in I_k(M)$ by a pointed immersion (f, w) which is in general position. Choose a section σ of $\nu(f)$ which meets the zero section transversally. Notice that then the Euler number satisfies

$$\chi(f) = \sum_{y \in N(\sigma)} \epsilon(y),$$

where $N(\sigma)$ is the (finite) set of zero points of σ and $\epsilon(y)$ is a sign which depends on the local orientations. We can arrange that no zero of σ is the preimage of an element in the set of double points D_f of f. Now move f a little bit in the direction of this normal field σ. We obtain a new immersion $g\colon S^k \to M$ with a path v from b to $g(s)$ such that (f, w) and (g, v) are pointed regularly homotopic.

We want to compute $\lambda(\alpha, \alpha)$ using the representatives (f, w) and (g, v). Notice that any point in $d \in D_f$ corresponds to two distinct points $x_0(d)$ and $x_1(d)$ in the set $D = \operatorname{im}(f) \cap \operatorname{im}(g)$ and any element $n \in N(\sigma)$ corresponds to one point $x(n)$ in D. Moreover any point in D occurs as $x_i(d)$ or $x(n)$ in a unique way. Now the contribution of d to $\lambda([(f, w)], [(g, v)])$ is $\epsilon(d) \cdot g(d) + (-1)^k \cdot \epsilon(d) \cdot \overline{g(d)}$ and the contribution of $n \in N(\sigma)$ is $\epsilon(n) \cdot 1$. Now assertion (1) follows. The elementary proof of assertions (2) and (3) is left to the reader. This finishes the proof of Lemma 4.7. ■

Theorem 4.8 *Let M be a compact connected manifold of dimension $n = 2k$. Fix base points $s \in S^k$ and $b \in M$ and an orientation of T_bM. Let (f, w) be a pointed immersion of S^k in M. Suppose that $k \geq 3$. Then (f, w) is pointed homotopic to a pointed immersion (g, v) for which $g\colon S^k \to M$ is an embedding, if and only $\mu(f, w) = 0$.*

Proof : If f is represented by an embedding, then $\mu(f, w) = 0$ by definition. Suppose that $\mu(f, w) = 0$. We can assume without loss of generality that f is in general position. Since $\mu(f, w) = 0$, we can find d and e in the set of double points D_f of f and a numeration $x_0(d), x_1(d)$ of $f^{-1}(d)$ and $x_0(e), x_1(e)$ of $f^{-1}(e)$ such that

$$\begin{aligned} \epsilon(x_0(d), x_1(d)) &= -\epsilon(x_0(e), x_1(e)); \\ g(x_0(d), x_1(d)) &= g(x_0(e), x_1(e)). \end{aligned}$$

Therefore we can find arcs u_0 and u_1 in S^k such that $u_0(0) = x_0(d)$, $u_0(1) = x_0(e)$, $u_1(0) = x_1(d)$ and $u_1(1) = x_1(e)$, the path u_0 and u_1 are disjoint from one another, $f(u_0((0,1)))$ and $f(u_1((0,1)))$ do not meet D_f and $f(u_0)$ and $f(u_1)$ are homotopic relative endpoints. We can change u_0 and u_1 without destroying the properties above and find a smooth map $U\colon D^2 \to M$ whose restriction to S^1 is an embedding (ignoring corners on the boundary) and is given on the upper hemisphere S^1_+ by u_0 and on the lower hemisphere S^1_- by u_1 and which meets $\operatorname{im}(f)$ transversally. There is a compact neighborhood K of S^1 such that $U|_K$ is an embedding. Since $k \geq 3$ we can find arbitrarily close to U an embedding which agrees with U on K. Hence we can assume without loss of generality that U itself is an embedding. The Whitney trick (see [83, Theorem 6.6 on page 71], [123]) allows to change f within its pointed regular homotopy class to a new pointed immersion (g, v) such that $D_g = D_f - \{d, e\}$ and $\mu(g, v) = 0$. By iterating this process we achieve $D_f = \emptyset$.

■

Remark 4.9 The condition $\dim(M) \geq 5$ which arises in high-dimensional manifold theory ensures in the proof of Theorem 4.8 that $k \geq 3$ and hence we can arrange U to be an embedding. If $k = 2$, one can achieve that U is an immersion but not necessarily an embedding. This is the technical reason why surgery in dimension 4 is much more complicated than in dimensions ≥ 5.

4.2 Kernels and Forms

4.2.1 Symmetric Forms and Surgery Kernels

For the rest of this section we fix a normal map of degree one $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon M \to X$, where M is a closed connected manifold of dimension n and X is a connected finite Poincaré complex of dimension n. Suppose that f induces an isomorphism on the fundamental groups. Fix a base point $b \in M$ together with lifts $\widetilde{b} \in \widetilde{M}$ of b and $\widetilde{f(b)} \in \widetilde{X}$ of $f(b)$. We identify $\pi = \pi_1(M, b) = \pi(X, f(b))$ by $\pi_1(f, b)$. The choices of $\widetilde{b}$ and $\widetilde{f(b)}$ determine π-operations on $\widetilde{M}$ and on $\widetilde{X}$ and a lift $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$ which is π-equivariant.

Definition 4.10 *Let $K_k(\widetilde{M})$ be the kernel of the $\mathbb{Z}\pi$-map $H_k(\widetilde{f})\colon H_k(\widetilde{M}) \to H_k(\widetilde{X})$. Denote by $K^k(\widetilde{M})$ the cokernel of the $\mathbb{Z}\pi$-map $H^k(\widetilde{f})\colon H^k(\widetilde{X}) \to H^k(\widetilde{M})$ which is the $\mathbb{Z}\pi$-map induced by $C^*(\widetilde{f})\colon C^*(\widetilde{X}) \to C^*(\widetilde{M})$.*

Lemma 4.11 *1. The cap product with $[M]$ induces isomorphisms*

$$? \cap [M] \colon K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

2. Suppose that f is k-connected. Then there is the composition of natural $\mathbb{Z}\pi$-isomorphisms

$$h_k \colon \pi_{k+1}(f) \xrightarrow{\cong} \pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M});$$

3. Suppose that f is k-connected and $n = 2k$. Then there is a natural $\mathbb{Z}\pi$-homomorphism

$$t_k \colon \pi_k(f) \to I_k(M).$$

Proof : (1) The following diagram commutes and has isomorphisms as vertical arrows

$$\begin{tikzcd} H^{n-k}(\widetilde{M}) \arrow[d, "?\cap[M]"', "\cong"] & H^{n-k}(\widetilde{X}) \arrow[l, "H^{n-k}(\widetilde{f})"'] \arrow[d, "\cong"', "?\cap[X]"] \\ H_k(\widetilde{M}) \arrow[r, "H_k(\widetilde{f})"'] & H_k(\widetilde{X}) \end{tikzcd} \tag{4.12}$$

Hence the composition $K_k(\widetilde{M}) \to H_k(\widetilde{M}) \xrightarrow{(?\cap[M])^{-1}} H^{n-k}(\widetilde{M}) \to K^{n-k}(\widetilde{M})$ is bijective.

(2) The commutative square (4.12) above implies that $H_l(\widetilde{f}) \colon H_l(\widetilde{M}) \to H_l(\widetilde{X})$ is split surjective for all l. We conclude from the long exact sequence of $C_*(\widetilde{f})$ that the boundary map

$$\partial \colon H_{k+1}(\widetilde{f}) := H_{k+1}(\mathrm{cone}(C_*(\widetilde{f}))) \to H_k(\widetilde{M})$$

induces an isomorphism

$$\partial_{k+1} \colon H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M}).$$

Since f and hence $\widetilde{f}$ is k-connected, the Hurewicz homomorphism

$$\pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f})$$

is bijective [121, Corollary IV.7.10 on on page 181]. The canonical map

$$\pi_{k+1}(\widetilde{f}) \to \pi_{k+1}(f)$$

is bijective. The composition of the maps above or their inverses yields a natural isomorphism $h_k \colon \pi_{k+1}(f) \to K_k(\widetilde{M})$.

(3) is analogous to the proof of Theorem 3.59 (2) which was based on Theorem 3.57 provided one takes the base loops into account. Notice that an element in $\pi_{k+1}(f, b)$ is given by a commutative diagram

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d] & M \arrow[d] \\ D^{k+1} \arrow[r, "Q"'] & X \end{tikzcd}$$

together with a path w from b to $f(s)$ for a fixed base point $s \in S^k$. ∎

Suppose that $n = 2k$. The Kronecker product $\langle \ , \ \rangle \colon H^k(\widetilde{M}) \times H_k(\widetilde{M}) \to \mathbb{Z}\pi$ is induced by the evaluation pairing $\hom_{\mathbb{Z}\pi}(C_p(\widetilde{M}), \mathbb{Z}\pi) \times C_p(\widetilde{M}) \to \mathbb{Z}\pi$ which sends (ϕ, x) to $\phi(x)$. It induces a pairing

$$\langle \ , \ \rangle \colon K^k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Together with the isomorphism

$$? \cap [M] \colon K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

of Theorem 4.11 (1) it yields the *intersection pairing*

$$s \colon K_k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi. \tag{4.13}$$

We get from Lemma 4.11 (2) and (3) a $\mathbb{Z}\pi$-homomorphism

$$\alpha \colon K_k(\widetilde{M}) \quad \to \quad I_k(\widetilde{M}). \tag{4.14}$$

We leave it to the reader to check

Lemma 4.15 *The following diagram commutes, where the upper pairing is defined in* (4.13)*, the lower pairing in* (4.1) *and the left vertical arrows in* (4.14)

$$\begin{tikzcd} K_k(\widetilde{M}) \times K_k(\widetilde{M}) \arrow[r, "s"] \arrow[d, "\alpha\times\alpha"'] & \mathbb{Z}\pi \arrow[d, "\mathrm{id}"] \\ I_k(M) \times I_k(M) \arrow[r, "\lambda"'] & \mathbb{Z}\pi \end{tikzcd}$$

The pairing s of (4.13) is the prototype of the following algebraic object which can be defined for any ring R with involution and $\epsilon \in \{\pm 1\}$.

Definition 4.16 *An* ϵ-symmetric form (P, ϕ) *over an associative ring R with unit and involution is a finitely generated projective R-module P together with an R-map $\phi\colon P \to P^*$ such that the composition $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{\phi^*} P^*$ agrees with $\epsilon \cdot \phi$. Here and elsewhere $e(P)$ is the canonical isomorphism sending $p \in P$ to the element in $(P^*)^*$ given by $P^* \to R, f \mapsto \overline{f(p)}$. We call* (P, ϕ) non-degenerate *if ϕ is an isomorphism.*

There are obvious notions of isomorphisms and direct sums of ϵ-symmetric forms.

We can rewrite (P, ϕ) as a pairing

$$\lambda\colon P \times P \to R, \qquad (p, q) \mapsto \phi(p)(q).$$

Then the condition that ϕ is R-linear becomes the condition

$$\begin{aligned} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}. \end{aligned}$$

The condition $\phi = \epsilon \cdot \phi^* \circ e(P)$ translates to $\lambda(q, p) \;=\; \epsilon \cdot \overline{\lambda(p, q)}$.

Example 4.17 Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-symmetric form $H^\epsilon(P)$* is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the R-isomorphism

$$\phi\colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* = (P \oplus P^*)^*.$$

If we write it as a pairing we obtain

$$(P \oplus P^*) \times (P \oplus P^*) \to R, \qquad ((p, \mu), (p', \mu')) \;\mapsto\; \mu(p') + \epsilon \cdot \mu'(p).$$

An example of a non-degenerate $(-1)^k$-symmetric form over $\mathbb{Z}\pi$ with the w-twisted involution is $K_k(\widetilde{M})$ with the pairing s of (4.13), provided that f is k-connected and $n = 2k$. We have to show that $K_k(\widetilde{M})$ is finitely generated projective.

Lemma 4.18 *Let D_* be a finite projective R-chain complex. Suppose for a fixed integer r that $H_i(D_*) = 0$ for $i < r$. Suppose that $H^{r+1}(\hom_R(D_*, V)) = 0$ for any R-module V. Then $\mathrm{im}(d_{r+1})$ is a direct summand in D_r and there are canonical exact sequences*

$$0 \to \ker(d_r) \to D_r \to D_{r-1} \to \ldots \to D_0 \to 0$$

and

$$0 \to \mathrm{im}(d_{r+1}) \to \ker(d_r) \to H_r(D_*) \to 0.$$

In particular $H_r(D_*)$ is finitely generated projective. If $H_i(D_*) = 0$ for $i > r$, we obtain an exact sequence

$$\ldots \to D_{r+3} \xrightarrow{d_{r+3}} D_{r+2} \xrightarrow{d_{r+2}} D_{r+1} \xrightarrow{d_{r+1}} \mathrm{im}(d_{r+1}) \to 0.$$

Proof : If we apply the assumption $H^{r+1}(\hom_R(D_*, V)) = 0$ in the case $V = \mathrm{im}(d_{r+1})$, we obtain an exact sequence

$$\hom_{\mathbb{Z}\pi}(D_r, \mathrm{im}(d_{r+1})) \xrightarrow{\hom_{\mathbb{Z}\pi}(d_{r+1},\mathrm{id})} \hom_{\mathbb{Z}\pi}(D_{r+1}, \mathrm{im}(d_{r+1}))$$
$$\xrightarrow{\hom_{\mathbb{Z}\pi}(d_{r+2},\mathrm{id})} \hom_{\mathbb{Z}\pi}(D_{r+2}, \mathrm{im}(d_{r+1})).$$

Since $d_{r+1} \in \hom_{\mathbb{Z}\pi}(D_{r+1}, \mathrm{im}(d_{r+1}))$ is mapped to zero under $\hom_{\mathbb{Z}\pi}(d_{r+2}, \mathrm{id})$, we can find an R-homomorphism $\rho\colon D_r \to \mathrm{im}(d_{r+1})$ with $\rho \circ d_{r+1} = d_{r+1}$. Hence $\mathrm{im}(d_{r+1})$ is a direct summand in D_r. The other claims are obvious. ∎

An R-module V is called *stably finitely generated free* if for some non-negative integer l the R-module $V \oplus R^l$ is a finitely generated free R-module.

Lemma 4.19 *If $f\colon X \to Y$ is k-connected for $n = 2k$ or $n = 2k+1$, then $K_k(\widetilde{M})$ is stably finitely generated free.*

Proof : We only give the proof for $n = 2k$. The proof for $n = 2k+1$ is along the same lines using Poincaré duality for the kernels (see Lemma 4.11 (1)).

Consider the finitely generated free $\mathbb{Z}\pi$-chain complex $D_* := \mathrm{cone}(C_*(\widetilde{f}))$. Its homology $H_p(D_*)$ is by definition $H_p(\widetilde{f})$. Since f is k-connected and D_* is projective, there is an R-subchain complex $E_* \subset D_*$ such that E_* is finite projective, $E_i = 0$ for $i \leq k$ and the inclusion $E_* \to D_*$ is a homology equivalence and hence an R-chain homotopy equivalence. Namely, take $E_i = D_i$ for

$i \geq k+2$, $E_{k+1} = \ker(d_{k+1})$ and $E_i = 0$ for $i \leq k$. We get from the commutative square (4.12) a $\mathbb{Z}\pi$-chain homotopy equivalence $D^{n+1-*} \to D_*$. This implies for any R-module V since $\hom_R(D_*, V)$ is chain homotopy equivalent to $\hom_R(E_*, V)$.

$$\begin{aligned} H^{n+1-i}(\hom_R(D_*, V)) &= 0 \qquad \text{for } i \leq k; \\ H_i(D) &= 0 \qquad \text{for } i \geq n+1-k; \end{aligned}$$

We conclude from Lemma 4.11.

$$\begin{aligned} H_{p+1}(D_*) &\cong \begin{cases} K_k(\widetilde{M}) & \text{, if } p = k; \\ 0 & \text{, if } p \neq k; \end{cases} \\ H^{k+2}(\hom_R(D_*, R)) &= 0. \end{aligned}$$

Now apply Lemma 4.18 to D_*. ■

Example 4.20 Consider the normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering the k-connected map of degree one $f\colon M \to N$ of closed n-dimensional manifolds for $n = 2k$. If we do surgery on the zero element in $\pi_{k+1}(f)$, then the effect on M is that M is replaced by the connected sum $M' = M\sharp(S^k \times S^k)$. The effect on $K_k(\widetilde{M})$ is that it is replaced by $K_k(\widetilde{M'}) = K_k(\widetilde{M}) \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)$. The intersection pairing on this new kernel is the orthogonal sum of the given intersection pairing on $K_k(\widetilde{M})$ together with the standard hyperbolic symmetric form $H^{(-1)^k}(\mathbb{Z}\pi)$. In particular we can arrange by finitely many surgery steps on the trivial element in $\pi_{k+1}(f)$ that $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module.

4.2.2 Quadratic Forms and Surgery Kernels

We have already seen that it will not be enough to study intersections of different immersions, we must also deal with selfintersections of one immersion. We have seen in Lemma 4.7 that we can enrich the intersection pairing by the self-intersection pairing. This leads to the following algebraic analog for an associative ring R with unit and involution and $\epsilon \in \{\pm 1\}$. For a finitely generated projective R-module P define an involution of R-modules

$$T\colon \hom_R(P, P^*) \to \hom(P, P^*), \qquad f \mapsto f^* \circ e(P) \tag{4.21}$$

where $e(P)\colon P \to (P^*)^*$ is the canonical isomorphism.

Definition 4.22 *Let P be a finitely generated projective R-module. Define*

$$\begin{aligned} Q^{\epsilon}(P) &:= \ker\left((1-\epsilon\cdot T)\colon \hom_R(P,P^*)\to\hom_R(P,P^*)\right); \\ Q_{\epsilon}(P) &:= \operatorname{coker}\left((1-\epsilon\cdot T)\colon \hom_R(P,P^*)\to\hom_R(P,P^*)\right). \end{aligned}$$

Let

$$(1+\epsilon\cdot T)\colon Q_{\epsilon}(P)\to Q^{\epsilon}(P)$$

be the homomorphism which sends the class represented by $f\colon P\to P^*$ to the element $f+\epsilon\cdot T(f)$.

An ϵ-quadratic form *(P,ψ) is a finitely generated projective R-module P together with an element $\psi\in Q_{\epsilon}(P)$. It is called* non-degenerate *if the associated ϵ-symmetric form $(P,(1+\epsilon\cdot T)(\psi))$ is non-degenerate, i.e. $(1+\epsilon\cdot T)(\psi)\colon P\to P^*$ is bijective.*

There is an obvious notion of direct sum of two ϵ-quadratic forms. An isomorphism $f\colon (P,\psi)\to(P',\psi')$ of two ϵ-quadratic forms is an R-isomorphism $f\colon P\xrightarrow{\cong}P'$ such that the induced map

$$Q_{\epsilon}(P')\to Q_{\epsilon}(P),\qquad [\phi\colon P'\to(P')^*]\mapsto[f^*\circ\phi\circ f\colon P\to P^*]$$

sends ψ' to ψ.

We can rewrite this as follows. An ϵ-quadratic form (P,ϕ) is the same as a triple (P,λ,μ) consisting of a pairing

$$\lambda\colon P\times P\to R$$

satisfying

$$\begin{aligned} \lambda(p,r_1\cdot q_1+r_2\cdot q_2,) &= r_1\cdot\lambda(p,q_1)+r_2\cdot\lambda(p,q_2); \\ \lambda(r_1\cdot p_1+r_2\cdot p_2,q) &= \lambda(p_1,q)\cdot\overline{r_1}+\lambda(p_2,q)\cdot\overline{r_2}; \\ \lambda(q,p) &= \epsilon\cdot\overline{\lambda(p,q)}. \end{aligned}$$

and a map

$$\mu\colon P\to Q_{\epsilon}(R)=R/\{r-\epsilon\cdot\overline{r}\mid r\in R\}$$

satisfying

$$\begin{aligned} \mu(rp) &= r\mu(p)\overline{r}; \\ \mu(p+q)-\mu(p)-\mu(q) &= \mathrm{pr}(\lambda(p,q)); \\ \lambda(p,p) &= (1+\epsilon\cdot T)(\mu(p)), \end{aligned}$$

where $\mathrm{pr}\colon R \to Q_\epsilon(R)$ is the projection and $(1+\epsilon\cdot T)\colon Q_\epsilon(R) \to R$ the map sending the class of r to $r+\epsilon\cdot\overline{r}$. Namely, put

$$\begin{aligned} \lambda(p,q) &= ((1+\epsilon\cdot T)(\psi))(p))(q); \\ \mu(p) &= \psi(p)(p). \end{aligned}$$

Example 4.23 Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-quadratic form* $H_\epsilon(P)$ is given by the $\mathbb{Z}\pi$-module $P\oplus P^*$ and the class in $Q_\epsilon(P\oplus P^*)$ of the R-homomorphism

$$\phi\colon (P\oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^*\oplus P \xrightarrow{\mathrm{id}\oplus e(P)} P^*\oplus (P^*)^* = (P\oplus P^*)^*.$$

The ϵ-symmetric form associated to $H_\epsilon(P)$ is $H^\epsilon(P)$.

Example 4.24 An example of a non-degenerate $(-1)^k$-quadratic form over $\mathbb{Z}\pi$ with the w-twisted involution is given as follows, provided that f is k-connected and $n=2k$. Namely, take $K_k(\widetilde{M})$ with the pairing s of (4.13) and the map

$$t\colon K_k(\widetilde{M}) \xrightarrow{\alpha} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w), \tag{4.25}$$

where $\mu\colon I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ is defined in (4.6) and α is defined in (4.14).

Remark 4.26 Suppose that $1/2\in R$. Then the homomorphism

$$(1+\epsilon\cdot T)\colon Q_\epsilon(P) \xrightarrow{\cong} Q^\epsilon(P), \qquad [\psi]\mapsto[\psi+\epsilon\cdot T(\psi)]$$

is bijective. The inverse sends $[u]$ to $[u/2]$. Hence any ϵ-symmetric form carries a unique ϵ-quadratic structure. Hence there is no difference between the symmetric and the quadratic setting if 2 is invertible in R.

The next result is the key step in translating the geometric question, whether we can change a normal map by a finite sequence of surgery steps into a homotopy equivalence to an algebraic question about quadratic forms. It will lead in a natural way to the definition of the surgery obstruction groups $L_{2k}(R)$.

Theorem 4.27 *Consider the normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}}^a \to \xi$ covering the k-connected map of degree one $f\colon M \to N$ of closed connected n-dimensional manifolds for $n = 2k$. Suppose that $k \geq 3$ and that for the non-degenerate $(-1)^k$-quadratic form $(K_k(\widetilde{M}), s, t)$ there are integers $u, v \geq 0$ together with an isomorphism of non-degenerate $(-1)^k$-quadratic forms*

$$(K_k(\widetilde{M}), s, t) \oplus H_{(-1)^k}(\mathbb{Z}\pi^u) \;\cong\; H_{(-1)^k}(\mathbb{Z}\pi^v).$$

Then we can perform a finite number of surgery steps resulting in a normal map of degree one $(\overline{g}, g)\colon TM' \oplus \underline{\mathbb{R}}^{a+b} \to \xi \oplus \underline{\mathbb{R}^b}$ such that $g\colon M' \to X$ is a homotopy equivalence.

Proof : If we do a surgery step on the trivial element in $\pi_{k+1}(f)$, we have explained the effect on $(K_k(\widetilde{M}), t)$ in Example 4.20. The effect on the quadratic form $(K_k(\widetilde{M}), s, t)$ is analogous, one adds a copy of $H_{(-1)^k}(\mathbb{Z}\pi)$. Hence we can assume without loss of generality that the non-degenerate quadratic form $(K_k(\widetilde{M}), s, t)$ is isomorphic to $H_{(-1)^k}(\mathbb{Z}\pi^v)$. Thus we can choose a $\mathbb{Z}\pi$-basis $\{b_1, b_2, \ldots, b_v, c_1, c_2, \ldots, c_v\}$ for $K_k(\widetilde{M})$ such that

$$\begin{array}{lcll}
s(b_i, c_i) & = & 1 & i \in \{1, 2, \ldots, v\};\\
s(b_i, c_j) & = & 0 & i, j \in \{1, 2, \ldots, v\},\; i \neq j;\\
s(b_i, b_j) & = & 0 & i, j \in \{1, 2, \ldots, v\};\\
s(c_i, c_j) & = & 0 & i, j \in \{1, 2, \ldots, v\};\\
t(b_i) & = & 0 & i \in \{1, 2, \ldots, v\}.
\end{array}$$

Notice that f is a homotopy equivalence if and only if the number v is zero. Hence it suffices to explain how we can lower the number v to $(v-1)$ by a surgery step on an element in $\pi_{k+1}(f)$. Of course our candidate is the element ω in $\pi_{k+1}(f)$ which corresponds under the isomorphism $h\colon \pi_{k+1}(f) \to K_k(\widetilde{M})$ (see Lemma 4.11 (2)) to the element b_v. By construction the composition

$$\pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w)$$

of the maps defined in (4.6) and in Lemma 4.11 (3) sends ω to zero. Now Theorem 3.59 and Theorem 4.8 ensure that we can perform surgery on ω. Notice that the assumption $k \geq 3$ and the quadratic structure on the kernel become relevant exactly at this point. Finally it remains to check whether the effect on $K_k(\widetilde{M})$ is the desired one, namely, that we get rid of one of the hyperbolic summands $H_\epsilon(\mathbb{Z}\pi)$, or equivalently, v is lowered to $v-1$.

We have explained earlier that doing surgery yields not only a new manifold M' but also a bordism from M to M'. Namely, take $W = M \times [0,1] \cup_{S^k \times D^{n-k}} D^{k+1} \times D^{n-k}$, where we attach $D^{k+1} \times D^{n-k}$ by an embedding $S^k \times D^{n-k} \to M \times \{1\}$, and $M' := \partial W - M$, where we identify $M = M \times \{0\}$. The manifold W comes with a map $F \colon W \to X \times [0,1]$ whose restriction to M is the given map $f \colon M = M \times \{0\} \to X = X \times \{0\}$ and whose restriction to M' is a map $f' \colon M' \to X \times \{1\}$. The definition of the kernels makes also sense for pair of maps. We obtain an exact braid combining the various long exact sequences of pairs

$$
\begin{tikzcd}[column sep=small]
0 \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(\widetilde{W},\widetilde{M}) \arrow[dr] \arrow[rr, bend left] & & K_k(\widetilde{M}) \arrow[dr] \arrow[rr, bend left] & & K_k(\widetilde{W},\widetilde{M'}) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & K_{k+1}(\widetilde{W},\widetilde{\partial W}) \arrow[ur] \arrow[dr] & & K_k(\widetilde{W}) \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
& & 0 \arrow[ur] \arrow[rr, bend right] & & K_k(\widetilde{M'}) \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] & &
\end{tikzcd}
$$

The $(k+1)$-handle $D^{k+1} \times D^{n-k}$ defines an element ϕ^{k+1} in $K_{k+1}(\widetilde{W},\widetilde{M})$ and the associated dual k-handle (see (1.25)) defines an element $\psi^k \in K_k(\widetilde{W},\widetilde{M'})$. These elements constitute a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{M}) \cong \mathbb{Z}\pi$ and $K_k(\widetilde{W},\widetilde{M'}) \cong \mathbb{Z}\pi$. The $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_k(\widetilde{M})$ maps ϕ to b_v. The $\mathbb{Z}\pi$-homomorphism $K_k(\widetilde{M}) \to K_k(\widetilde{W},\widetilde{M'})$ sends x to $s(b_v,x) \cdot \psi^k$. Hence we can find elements b'_1, b'_2 , … , b'_v and c'_1, c'_2 , … , c'_{v-1} in $K_{k+1}(\widetilde{W},\widetilde{\partial W})$ uniquely determined by the property that b'_i is mapped to b_i and c'_i to c_i under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M})$. Moreover, these elements form a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{\partial W})$ and the element ϕ^{k+1} is mapped to b'_v under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_{k+1}(\widetilde{W},\widetilde{\partial W})$. Define b''_i and c''_i for $i = 1,2,\ldots,(v-1)$ to be the image of b'_i and c'_i under under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M'})$. Then $\{b''_i \mid i = 1,2\ldots,(v-1)\} \coprod \{c''_i \mid i = 1,2\ldots,(v-1)\}$ is a $\mathbb{Z}\pi$-basis for $K_k(\widetilde{M'})$. One easily checks for the quadratic structure (s',t') on $K_k(\widetilde{M'})$

$$
\begin{array}{lclcll}
s'(b''_i,c''_i) & = & s(b_i,c_i) & = & 1 & i \in \{1,2,\ldots,(v-1)\}; \\
s'(b''_i,c''_j) & = & s(b_i,c_j) & = & 0 & i,j \in \{1,2,\ldots,(v-1)\},\ i \neq j; \\
s'(b''_i,b''_j) & = & s(b_i,b_j) & = & 0 & i,j \in \{1,2,\ldots,(v-1)\}; \\
s'(c''_i,c''_j) & = & s(c_i,c_j) & = & 0 & i,j \in \{1,2,\ldots,(v-1)\}; \\
t'(b''_i) & = & t(b_i) & = & 0 & i \in \{1,2,\ldots,(v-1)\}.
\end{array}
$$

This finishes the proof of Theorem 4.27. ■

4.3 Even Dimensional L-Groups

Next we define in even dimensions the abelian group, where our surgery obstruction will take values in.

Definition 4.28 *Let R be an associative ring with unit and involution. For an even integer $n = 2k$ define the abelian group $L_n(R)$ called the n-th* quadratic L-group *of R to be the abelian group of equivalence classes $[(F,\psi)]$ of non-degenerate $(-1)^k$-quadratic forms (F,ψ) whose underlying R-module F is a finitely generated free R-module with respect to the following equivalence relation. We call (F,ψ) and (F',ψ') equivalent if and only if there exists integers $u,u' \geq 0$ and an isomorphism of non-degenerate $(-1)^k$-quadratic forms*

$$(F,\psi) \oplus H_\epsilon(R)^u \cong (F',\psi') \oplus H_\epsilon(R)^{u'}.$$

Addition is given by the sum of two $(-1)^k$-quadratic forms. The zero element is represented by $[H_{(-1)^k}(R)^u]$ for any integer $u \geq 0$. The inverse of $[F,\psi]$ is given by $[F,-\psi]$.

A morphism $u\colon R \to S$ of rings with involution induces homomorphisms $u_*\colon L_k(R) \to L_k(S)$ for $k = 0,2$ by induction. One easily checks $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 0,2$.

Before we come to the surgery obstruction, we will present a criterion for an ϵ-quadratic form (P,ψ) to represent zero in $L_{1-\epsilon}(R)$ which we will later need at several places. Let (P,ψ) be a ϵ-quadratic form. A *subLagrangian* $L \subset P$ is an R-submodule such that the inclusion $i\colon L \to P$ is split injective, the image of ψ under the map $Q_\epsilon(i)\colon Q_\epsilon(P) \to Q_\epsilon(L)$ is zero and L is contained in its annihilator $L^\perp$ which is by definition the kernel of

$$P \xrightarrow{(1+\epsilon\cdot T)(\psi)} P^* \xrightarrow{i^*} L^*.$$

A subLagrangian $L \subset P$ is called *Lagrangian* if $L = L^\perp$. Equivalently, a Lagrangian $L \subset P$ is an R-submodule L with inclusion $i\colon L \to P$ such that the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^*\circ(1+\epsilon\cdot T)(\psi)} L^* \to 0.$$

is exact.

Lemma 4.29 *Let (P, ψ) be an ϵ-quadratic form. Let $L \subset P$ be a subLagrangian. Then L is a direct summand in $L^\perp$ and ψ induces the structure of a non-degenerate ϵ-quadratic form $(L^\perp/L, \psi^\perp/\psi)$. Moreover, the inclusion $i\colon L \to P$ extends to an isomorphism of ϵ-quadratic forms*

$$H_\epsilon(L) \oplus (L^\perp/L, \psi^\perp/\psi) \xrightarrow{\cong} (P, \psi).$$

In particular a non-degenerate ϵ-quadratic form (P, ψ) is isomorphic to $H_\epsilon(Q)$ if and only if it contains a Lagrangian $L \subset P$ which is isomorphic as R-module to Q. The analogous statement holds for ϵ-symmetric forms.

Proof: Choose an R-homomorphism $s\colon L^* \to P$ such that $i^* \circ (1+\epsilon \cdot T)(\psi) \circ s$ is the identity on L^*. Our first attempt is the obvious split injection $i \oplus s\colon L \oplus L^* \to P$. The problem is that it is not necessarily compatible with the ϵ-quadratic structure. To be compatible with the ϵ-quadratic structure it is necessary to be compatible with the ϵ-symmetric structure, i.e. the following diagram must commute

$$\begin{array}{ccc} L \oplus L^* & \xrightarrow{i \oplus s} & P \\ {\scriptstyle \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} \big\downarrow & & \big\downarrow {\scriptstyle \psi + \epsilon \cdot T(\psi)} \\ L^* \oplus L & \xleftarrow[(i^* \oplus s^*) \circ \Delta_{P^*}]{} & P^* \end{array}$$

where $\Delta_{P^*}\colon P^* \to P^* \oplus P^*$ is the diagonal map and we write s^* for the composition $P^* \xrightarrow{s^*} (L^*)^* \xrightarrow{e(L)^{-1}} L$. The diagram above commutes if and only if $s^* \circ (\psi + \epsilon \cdot T(\psi)) \circ s = 0$. Notice that s is not unique, we can replace s by $s' = s + i \circ v$ for any R-map $v\colon L^* \to L$. For this new section s' the diagram above commutes if and only if $s^* \circ (\psi + \epsilon \cdot T(\psi)) \circ s + v^* + \epsilon \cdot v = 0$. This suggests to take $v = -\epsilon s^* \circ \psi \circ s$. Now one easily checks that

$$g := i \oplus (s - \epsilon \cdot i \circ s^* \circ \psi \circ s)\colon L \oplus L^* \to P$$

is split injective and compatible with the ϵ-quadratic structures and hence induces a morphism $g\colon H_\epsilon(L) \to (P, \psi)$ of ϵ-quadratic forms.

Let $\operatorname{im}(g)^\perp$ be the annihilator of $\operatorname{im}(g)$. Denote by $j\colon \operatorname{im}(g) \to P$ the inclusion. We obtain an isomorphism of ϵ-quadratic forms

$$g \oplus j\colon H_\epsilon(L) \oplus (\operatorname{im}(g)^\perp, j^* \circ \psi \circ j) \to (P, \psi).$$

The inclusion $L^\perp \to \mathrm{im}(g)^\perp$ induces an isomorphism $h\colon L^\perp/L \to \mathrm{im}(g)^\perp$. Let $\psi^\perp/\psi$ be the ϵ-quadratic structure on $L^\perp/L$ for which h becomes an isomorphism of ϵ-quadratic forms. This finishes the proof in the quadratic case. The proof in the symmetric case is analogous. ■

Finally we state the computation of the even-dimensional L-groups of the integers. Consider an element (P,ϕ) in $L_0(\mathbb{Z})$. By tensoring over $\mathbb{Z}$ with $\mathbb{R}$ and only taking the symmetric structure into account we obtain a non-degenerate symmetric $\mathbb{R}$-bilinear pairing $\lambda\colon \mathbb{R}\otimes_{\mathbb{Z}} P \times \mathbb{R}\otimes_{\mathbb{Z}} P \to \mathbb{R}$. It turns out that its signature is always divisible by eight.

Theorem 4.30 *The signature defines an isomorphism*

$$\frac{1}{8}\cdot \mathrm{sign}\colon L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \qquad [P,\psi] \mapsto \frac{1}{8}\cdot \mathrm{sign}(\mathbb{R}\otimes_{\mathbb{Z}} P,\lambda).$$

Consider a non-degenerate quadratic form (P,ψ) over the field $\mathbb{F}_2$ of two elements. Write (P,ψ) as a triple (P,λ,μ) as explained above. Choose any symplectic basis $\{b_1,b_2,\dots,b_{2m}\}$ for P, where symplectic means that $\lambda(b_i,b_j)$ is 1 if $i-j=m$ and 0 otherwise. Define the *Arf invariant* of (P,ψ) by

$$\mathrm{Arf}(P,\psi) := \sum_{i=1}^{m} \mu(b_i)\cdot\mu(b_{i+m}) \qquad \in \mathbb{Z}/2. \tag{4.31}$$

The Arf invariant defines an isomorphism

$$\mathrm{Arf}\colon L_2(\mathbb{F}_2) \xrightarrow{\cong} \mathbb{Z}/2.$$

The change of rings homomorphism $\mathbb{Z}\to\mathbb{F}_2$ induces an isomorphism

$$L_2(\mathbb{Z}) \xrightarrow{\cong} L_2(\mathbb{F}_2).$$

Theorem 4.32 *The Arf invariant defines an isomorphism*

$$\mathrm{Arf}\colon L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2, \qquad [(P,\psi)] \mapsto \mathrm{Arf}(\mathbb{F}_2\otimes_{\mathbb{Z}}(P,\psi)).$$

For more information about forms over the integers and the Arf invariant we refer for instance to [17], [88]. Implicitly the computation of $L_n(\mathbb{Z})$ is already in [58]. Notice that Lemma 3.19, Remark 4.26 and Lemma 4.29 imply that the signature defines an isomorphism $L_n(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z}$.

4.4 The Surgery Obstruction in Even Dimensions

Consider a normal map of degree one $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon M \to X$, where M is a closed oriented manifold of dimension n and X is a connected finite Poincaré complex of dimension n for even $n = 2k$. To these data we want to assign an element $\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w)$ such that the following holds

Theorem 4.33 (Surgery Obstruction Theorem in Even Dimensions)

We get under the conditions above:

1. *Suppose $k \geq 3$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f)\colon TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ which covers a homotopy equivalence $f'\colon M' \to X$;*

2. *The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.*

We first explain the definition of the surgery obstruction $\sigma(\overline{f}, f)$. By finitely many surgery steps we can achieve that f is k-connected (see Theorem 3.61). By a finite number of surgery steps in the middle dimension we can achieve that the surgery kernel $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module (see Example 4.20). We have already explained in Example 4.24 that $K_k(\widetilde{M})$ carries the structure $(K_k(\widetilde{M}), t, s)$ of a non-degenerate $(-1)^k$-quadratic form. We want to define the *surgery obstruction* of $(\overline{f}, f)$

$$\sigma(\overline{f}, f) \quad := \quad [K_k(\widetilde{M}), t, s] \quad \in L_n(\mathbb{Z}\pi, w). \tag{4.34}$$

We have to show that this is independent of the surgery steps we have performed to make f k-connected and $K_k(\widetilde{M})$ finitely generated free. Notice that surgery does not change the normal bordism class. Hence it suffices to show that $\sigma(\overline{f}, f)$ and $\sigma(\overline{f'}, f')$ define the same element in $L_n(\mathbb{Z}\pi, w)$ if $f\colon M \to X$ and $f'\colon M' \to X$ are k-connected and normally bordant and their kernels $K_k(\widetilde{M})$ and $K_k(\widetilde{M'})$ are finitely generated free $\mathbb{Z}\pi$-modules. Notice that this also will prove Theorem 4.33 (2).

Consider a normal bordism $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^a} \to \eta$ covering a map $F\colon W \to X \times [0,1]$ such that ∂W is $M^- \coprod M'$, $F(M^-) \subset X \times \{0\}$, $F(M') \subset X \times \{1\}$, the restriction of $\overline{F}$ to M^- is $\overline{f}$ and the restriction of $\overline{F}$ to M' is $\overline{f'}$. By surgery on the interior of W we can change W and F leaving ∂W and

$F|_{\partial W}$ fixed such that F is k-connected. The proof of Theorem 3.61 carries directly over. By a handle subtraction argument (see [119, Theorem 1.4 on page 14, page 50]) we can achieve that $K_k(\widetilde{W}, \widetilde{\partial W}) = 0$. This handle subtraction leaves M fixed but may change M'. But the change on the surgery kernel of M' is adding a standard ϵ-hyperbolic form which does not change the class in the L-group. Moreover, f, f' and F remain k-connected. So we can assume without loss of generality that $K_i(\widetilde{W}, \widetilde{\partial W}) = 0$ for $i \leq k$ and $K_i(\widetilde{\partial W}) = 0$ for $i \leq k-1$. We know already that $K_k(\widetilde{\partial W})$ is stably finitely generated free by Lemma 4.18. A similar argument shows that $K_{k+1}(\widetilde{W}, \widetilde{\partial W})$ and $K_k(\widetilde{W})$ are stably finitely generated free.

We obtain an exact sequence of $\mathbb{Z}\pi$-modules

$$0 \to K_{k+1}(\widetilde{W}, \widetilde{\partial W}) \xrightarrow{\partial_{k+1}} K_k(\widetilde{\partial W}) \xrightarrow{K_k(\widetilde{i})} K_k(\widetilde{W}) \to 0. \tag{4.35}$$

where $\widetilde{\partial W}$ and $\widetilde{i}\colon \widetilde{\partial W} \to \widetilde{W}$ come from the pullback construction applied to the universal covering $\widetilde{W} \to W$ and the inclusion $i\colon \partial W \to W$.

The strategy of the proof is to show that the image of $\partial_{k+1}\colon K_{k+1}(\widetilde{W}, \widetilde{\partial W}) \to K_k(\widetilde{\partial W})$ is a Lagrangian for the non-degenerate ϵ-quadratic form $K_k(\widetilde{\partial W})$. Before we do this we explain how the claim follows then. Notice that $\partial W = M^- \coprod M'$. Since F, f and f' are k-connected and $k \geq 2$, we get identifications $\pi = \pi_1(X) = \pi_1(M) = \pi_1(M') = \pi_1(W)$ and $\widetilde{\partial W}$ is the disjoint union of the universal coverings $\widetilde{M}$ and $\widetilde{M'}$. Hence we get in $L_{1-\epsilon}(\mathbb{Z}\pi, w)$

$$\begin{aligned} [(K_k(\widetilde{\partial W}), t'', s'')] &= [(K_k(M), -t, -s)] + [(K_k(M'), s', t')] \\ &= -[(K_k(M), t, s)] + [(K_k(M'), s', t')]. \end{aligned}$$

If we can construct the Lagrangian above for $(K_k(\widetilde{\partial W}), t'', s'')$, we conclude $[(K_k(\widetilde{\partial W}), t'', s'')] = 0$ in $L_{1-\epsilon}(R)$ from Lemma 4.29. This implies the desired equation in $L_{1-\epsilon}(\mathbb{Z}\pi, w)$

$$[(K_k(M), -t, -s)] = [(K_k(M'), s', t')].$$

It remains to show that $\operatorname{im}(\partial_{k+1})$ is a Lagrangian.

This is rather easy for the symmetric structure. We first show for any $x, y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$ that $s(\partial_{k+1}(x), \partial_{k+1}(y)) = 0$. The following diagram commutes and has isomorphisms as vertical arrows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{k+1}(\widetilde{W}, \widetilde{\partial W}) & \xrightarrow{\partial_{k+1}} & K_k(\widetilde{\partial W}) & \xrightarrow{K_k(\widetilde{i})} & K_k(\widetilde{W}) & \longrightarrow & 0 \\ & & {\scriptstyle ?\cap[W,\partial W]}\uparrow & & {\scriptstyle ?\cap[\partial W]}\uparrow & & {\scriptstyle ?\cap[W,\partial W]}\uparrow & & \\ 0 & \longrightarrow & K^k(\widetilde{W}) & \xrightarrow{K^k(\widetilde{i})} & K^k(\widetilde{\partial W}) & \xrightarrow{\delta^k} & K^{k+1}(\widetilde{W}, \widetilde{\partial W}) & \longrightarrow & 0 \end{array}$$

We have for $x, y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$

$$\begin{aligned}
s(\partial_{k+1}(x), \partial_{k+1}(y)) &= \langle (? \cap [\partial W])^{-1} \circ \partial_{k+1}(x), \partial_{k+1}(y) \rangle \\
&= \langle (K^k(\widetilde{i}) \circ (? \cap [W, \partial W])^{-1}(x), \partial_{k+1}(y) \rangle \\
&= \langle (\delta^k \circ K^k(\widetilde{i}) \circ (? \cap [W, \partial W])^{-1}(x), y \rangle \\
&= \langle (0 \circ (? \cap [W, \partial W])^{-1}(x), y \rangle \\
&= 0.
\end{aligned}$$

Now suppose for $x \in K_k(\widetilde{M})$ that $s(x, \partial_{k+1}(y)) = 0$ for all $y \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$. We must show $x \in \operatorname{im}(\partial_{k+1})$. This is equivalent to $(? \cap [W, \partial W])^{-1} \circ K_k(\widetilde{i})(x) = 0$. Since $K_p(\widetilde{W}, \widetilde{\partial W}) = 0$ for $p \leq k$ and finitely generated projective for $p = k+1$, the canonical map

$$K^{k+1}(\widetilde{W}, \widetilde{\partial W}) \xrightarrow{\cong} \hom_{\mathbb{Z}\pi}(K_{k+1}(\widetilde{W}, \widetilde{\partial W}), \mathbb{Z}\pi), \qquad \alpha \mapsto \langle \alpha, ? \rangle$$

is bijective by an elementary chain complex argument or by the universal coefficient spectral sequence. Hence the claim follows from the following calculation for $y \in K_{k+1}(\widetilde{M})$

$$\begin{aligned}
\langle (? \cap [W, \partial W])^{-1} \circ K_k(\widetilde{i})(x), y \rangle &= \langle \delta^k \circ (? \cap [\partial W])^{-1}(x), y \rangle \\
&= \langle (? \cap [\partial W])^{-1}(x), \partial_{k+1}(y) \rangle \\
&= 0.
\end{aligned}$$

Thus we have shown that $\operatorname{im}(\partial_{k+1})$ is a Lagrangian for the non-degenerate ϵ-symmetric form $[K_k(\widetilde{\partial}), s)]$. It remains to show that it also is a Lagrangian for the non-degenerate ϵ-quadratic form $[K_k(\widetilde{\partial}), s]$. In other words, we must show that t vanishes on $\operatorname{im}(\partial_{k+1})$. We sketch the idea of the proof.

Consider $x \in K_{k+1}(\widetilde{W}, \widetilde{\partial W})$. We can find a smooth map $(g, \partial g) \colon (S, \partial S) \to (\widetilde{W}, \widetilde{\partial W})$ such that S is obtained from S^{k+1} by removing a finite number of open embedded disjoint discs D^{k+1} and the image of the fundamental class $[S, \partial S]$ under the map on homology induced by $(g, \partial g)$ is x. Moreover we can assume that $\partial g \colon \partial S \to \widetilde{\partial W}$ is an immersion and g is in general position. We have to show that $\mu(\partial g)$ is zero, where $\mu(\partial g)$ the sum of the self intersection numbers of $(\partial g)|_C \colon C \to \widetilde{W}$ for $C \in \pi_0(\partial S)$. Since g is in general position, the set of double points consists of circles which do not concern us and arcs whose end points are on $\widetilde{\partial W}$. Now one shows for each arc that the contributions of its two end points to the self intersection number $\mu(\partial g)$ cancel out. This proves $\mu(\partial_{k+1}(x)) = 0$. This finishes the proof that the surgery obstruction is well-defined and depends only on the normal bordism class. Thus we have proved Theorem 4.33 (2). Assertion (1) of Theorem 4.33 is a direct consequence of Theorem 4.27. This finishes the proof of Theorem 4.33. ■

Now we can give a complete answer to Problem 3.1 and Problem 3.62 for even-dimensions and in the simply connected case.

Theorem 4.36 *1. Let $(\overline{f}, f)$ be a normal map from a closed manifold M to a simply connected finite Poincaré complex X of dimension $n = 4k \geq 5$. Then we can change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence if and only if* $\operatorname{sign}(M) = \operatorname{sign}(X)$*;*

2. Let $(\overline{f}, f)$ be normal map from a closed manifold M to a simply connected finite Poincaré complex X of dimension $n = 4k+2 \geq 5$. Then we can change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence if and only if the Arf invariant taking values in $\mathbb{Z}/2$ vanishes;

3. Let X be a simply connected finite Poincaré complex of dimension $n = 4k \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi\colon E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty, such that

$$\langle \mathcal{L}(\xi)^{-1}, [X] \rangle \;=\; \operatorname{sign}(X);$$

4. Let X be a simply-connected finite connected Poincaré complex of dimension $n = 4k+2 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi\colon E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty such that the Arf invariant of the associated surgery problem, which takes values in $\mathbb{Z}/2$, vanishes.

Proof : (1) Because of Theorem 4.33 and Theorem 4.30 we have to show for a $2k$-connected normal map of degree one $f\colon M \to X$ from a closed simply connected oriented manifold M of dimension $n = 4k$ to a simply connected Poincaré complex X of dimension $n = 4k$ that for the non-degenerate symmetric bilinear form $\mathbb{R} \otimes_{\mathbb{Z}} (K_{2k}(M), \lambda)$ induced by the intersection pairing we get

$$\operatorname{sign}\left(\mathbb{R} \otimes_{\mathbb{Z}} (K_{2k}(M), \lambda)\right) \ = \operatorname{sign}(M) - \operatorname{sign}(X).$$

This follows from elementary considerations about signatures and from the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_{2k}(M) \arrow[r] & H_{2k}(M) \arrow[r, "H_{2k}(f)"] & H_{2k}(X) \arrow[r] & 0 \\
0 & K^{2k}(M) \arrow[l] \arrow[u, "? \cap [M]"] & H^{2k}(M) \arrow[l] \arrow[u, "? \cap [M]"] & H^{2k}(X) \arrow[l, "H^{2k}(f)"'] \arrow[u, "? \cap [X]"] & 0 \arrow[l]
\end{tikzcd}$$

(2) follows from Theorem 3.48, Theorem 4.33 and Theorem 4.32.

(3) follows from (1) and the Hirzebruch signature formula which implies for a surgery problem $(\overline{f}, f)\colon \nu(M) \to \xi$ covering $f\colon M \to X$ which is obtained from a reduction ξ of the Spivak normal fibration of X

$$\operatorname{sign}(M) \ = \ \langle \mathcal{L}(TM), [M] \rangle \ = \ \langle \mathcal{L}(\nu(M) \oplus \underline{\mathbb{R}^a})^{-1}, [M] \rangle \ = \ \langle \mathcal{L}(\xi)^{-1}, [X] \rangle.$$

(4) follows from Theorem 3.48 and (2). ■

4.5 Formations and Odd Dimensional L-Groups

In this subsection we explain the algebraic objects which describe the surgery obstruction and will be the typical elements in the surgery obstruction group in odd dimensions. Throughout this section R will be an associative ring with unit and involution and $\epsilon \in \{\pm 1\}$.

Definition 4.37 *An* ϵ-quadratic formation $(P, \psi; F, G)$ *consists of a non-degenerate ϵ-quadratic form (P, ψ) together with two Lagrangians F and G.*

An isomorphism $f\colon (P, \psi; F, G) \to (P', \psi'; F,', G')$ of ϵ-quadratic formations is an isomorphism $f\colon (P, \psi) \to (P', \psi')$ of non-degenerate ϵ-quadratic forms such that $f(F) = F'$ and $f(G) = G'$ holds.

Definition 4.38 *The* trivial ϵ-quadratic formation *associated to a finitely generated projective R-module P is the formation $(H_\epsilon(P); P, P^*)$. A formation $(P, \psi; F, G)$ is called* trivial *if it isomorphic to the trivial ϵ-quadratic formation associated to some finitely generated projective R-module. Two formations are* stably isomorphic *if they become isomorphic after taking the direct sum of trivial formations.*

Remark 4.39 We conclude from Lemma 4.29 that any formation is isomorphic to a formation of the type $(H_\epsilon(P); P, F)$ for some Lagrangian $F \subset P \oplus P^*$. Any automorphism $f\colon H_\epsilon(P) \xrightarrow{\cong} H_\epsilon(P)$ of the standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$ for some finitely generated projective R-module P defines a formation by $(H_\epsilon(P); P, f(P))$.

Consider a formation $(P, \psi; F, G)$ such that P, F and G are finitely generated free and suppose that R has the property that R^n and R^m are R-isomorphic if and only if $n = m$. Then $(P, \psi; F, G)$ is stably isomorphic to $(H_\epsilon(Q); Q, f(Q))$ for some finitely generated free R-module Q by the following argument. Because of Lemma 4.29 we can choose isomorphisms of non-degenerate ϵ-quadratic forms $f\colon H_\epsilon(F) \xrightarrow{\cong} (P, \psi)$ and $g\colon H_\epsilon(G) \xrightarrow{\cong} (P, \psi)$ such that $f(F) = F$ and $g(G) = G$. Since $F \cong R^a$ and $G \cong R^b$ by assumption and $R^{2a} \cong F \oplus F^* \cong P \cong G \oplus G^* \cong R^{2b}$, we conclude $a = b$. Hence we can choose an R-isomorphism $u\colon F \to G$. Then we obtain an isomorphism of non-degenerate ϵ-quadratic forms by the composition

$$v\colon H_\epsilon(F) \xrightarrow{H_\epsilon(u)} H_\epsilon(G) \xrightarrow{g} (P, \psi) \xrightarrow{f^{-1}} H_\epsilon(F)$$

and an isomorphism of ϵ-quadratic formations

$$f\colon (H_\epsilon(F); F, v(F)) \xrightarrow{\cong} (P, \psi; F, G).$$

Recall that $K_1(R)$ is defined in terms of automorphisms of finitely generated free R-modules. Hence it is plausible that the odd-dimensional L-groups will be defined in terms of formations which is essentially the same as in terms of automorphisms of the standard hyperbolic form over a finitely generated free R-module.

Definition 4.40 *Let (P, ψ) be a (not necessarily non-degenerate) $(-\epsilon)$-quadratic form. Define its* boundary $\partial(P, \psi)$ *to be the ϵ-quadratic formation $(H_\epsilon(P); P, \Gamma_\psi)$, where Γ_ψ is the Lagrangian given by the image of the R-homomorphism*

$$P \to P \oplus P^*, \qquad x \mapsto (x, (1 - \epsilon \cdot T)(\psi)(x)) .$$

One easily checks that Γ_ψ appearing in Definition 4.40 is indeed a Lagrangian. Two Lagrangians F, G of a non-degenerate ϵ-quadratic form (P, ψ) are called *complementary* if $F \cap G = \{0\}$ and $F + G = P$.

Lemma 4.41 *Let $(P,\psi;F,G)$ be an ϵ-quadratic formation. Then:*

1. *$(P,\psi;F,G)$ is trivial if and only F and G are complementary to one another;*

2. *$(P,\psi;F,G)$ is isomorphic to a boundary if and only if there is a Lagrangian $L\subset P$ such that L is a complement of both F and G;*

3. *There is an ϵ-quadratic formation $(P',\psi';F',G')$ such that $(P,\psi;F,G)\oplus(P',\psi';F',G')$ is a boundary;*

4. *An $(-\epsilon)$-quadratic form (Q,μ) is non-degenerate if and only if its boundary is trivial.*

Proof : (1) The inclusions of F and G in P induce an R-isomorphism $f\colon F\oplus G\to P$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : F\oplus G \to (F\oplus G)^* = F^*\oplus G^*$$

be $f^*\circ\psi\circ f$ for some representative $\psi\colon P\to P^*$ of $\psi\in Q_\epsilon(P)$ and let $\psi'\in Q_\epsilon(F\oplus G)$ be the associated class. Then f is an isomorphism of non-degenerate ϵ-quadratic forms $(F\oplus G,\psi')\to(P,\psi)$ and F and G are Lagrangians in $(F\oplus G,\psi')$. This implies that the isomorphism $\psi'+\epsilon\cdot T(\psi')$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \epsilon\cdot T\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} a+\epsilon\cdot a^* & b+\epsilon c^* \\ c+\epsilon b^* & d+\epsilon d^* \end{pmatrix} \;=\; \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix}$$

Hence $(b+\epsilon\cdot c^*)\colon G\to F^*$ is an isomorphism. Define an R-isomorphism

$$u\colon \begin{pmatrix} 1 & 0 \\ 0 & (b+\epsilon\cdot c^*)^{-1} \end{pmatrix} : F\oplus F^* \xrightarrow{\cong} P.$$

One easily checks that u defines an isomorphism of formations

$$u\colon (H_\epsilon(F);F,F^*) \xrightarrow{\cong} (F\oplus G,\psi';F,G).$$

(2) One easily checks that for an $(-\epsilon)$-quadratic form (P,ϕ) the Lagrangian P^* in its boundary $\partial(P,\psi) := (H_\epsilon(P);P,\Gamma_\psi)$ is complementary to both P and Γ_ψ. Conversely, suppose that $(P,\psi;F,G)$ is an ϵ-quadratic formation such that there exists a Lagrangian $L\subset P$ which is complementary to both

F and G. By the argument appearing in the proof of assertion (1) we find an isomorphism of ϵ-quadratic formations

$$f : (H_\epsilon(F); F, F^*) \xrightarrow{\cong} (P, \psi; F, L)$$

which is the identity on F. The preimage $G' := f^{-1}(G)$ is a Lagrangian in $H_\epsilon(F)$ which is complementary to F^*. Write the inclusion of G' into $F \oplus F^*$ as $(a, b) \colon G' \to F \oplus F^*$. Consider the $(-\epsilon)$-quadratic form (F, ψ'), where $\psi' \in Q_{-\epsilon}(F)$ is represented by $b \circ a^* \colon F \to F^*$. One easily checks that its boundary is precisely $(H_\epsilon(F); F, G')$ and f induces an isomorphism of ϵ-quadratic formations

$$\partial(F, \psi') \;=\; (H_\epsilon(F); F, G') \xrightarrow{\cong} (P, \psi; F, G).$$

(3) Because of Lemma 4.29 we can find Lagrangians F' and G' such that F and F' are complementary and G and G' are complementary. Put $(P', \psi'; F', G') = (P, -\psi, F', G')$. Then $M = \{(p, p) \mid p \in P\} \subset P \oplus P$ is a Lagrangian in the direct sum

$$(P, \psi; F, G) \oplus (P', \psi'; F', G') = (P \oplus P, \psi \oplus (-\psi), F \oplus F', G \oplus G')$$

which is complementary to both $F \oplus F'$ and $G \oplus G'$. Hence the direct sum is isomorphic to a boundary by assertion (2).

(4) The Lagrangian Γ_ψ in the boundary $\partial(Q, \mu) := H_\epsilon(P); P, \Gamma_\psi)$ is complementary to P if and only if $(1 - \epsilon \cdot T)(\mu) \colon P \to P^*$ is an isomorphism. This finishes the proof of Lemma 4.41. ■

Now we can define the odd-dimensional surgery groups.

Definition 4.42 *Let R be an associative ring with unit and involution. For an odd integer $n = 2k + 1$ define the abelian group $L_n(R)$ called the n-th* quadratic L-group *of R to be the abelian group of equivalence classes $[(P, \psi; F, G)]$ of $(-1)^k$-quadratic formations $(P, \psi; F, G)$ such that P, F and G are finitely generated free with respect to the following equivalence relation. We call $(P, \psi; F, G)$ and $(P', \psi'; F', G')$ equivalent if and only if there exist $(-(-1)^k)$-quadratic forms (Q, μ) and (Q', μ') for finitely generated free R-modules Q and Q' and finitely generated free R-modules S and S' together with an isomorphism of $(-1)^k$-quadratic formations*

$$\begin{aligned}(P, \psi; F, G) \oplus \partial(Q, \mu) \oplus (H_\epsilon(S); S, S^*) \\ \cong\;\; (P', \psi'; F', G') \oplus \partial(Q', \mu') \oplus (H_\epsilon(S'); S', (S')^*).\end{aligned}$$

Addition is given by the sum of two $(-1)^k$-quadratic formations. The zero element is represented by $\partial(Q,\mu) \oplus (H_{(-1)^k}(S); S, S^*)$ for any $(-(-1)^k)$-quadratic form (Q,μ) for any finitely generated free R-module Q and any finitely generated free R-module S. The inverse of $[(P,\psi;F,G)]$ is represented by $(P,-\psi;F',G')$ for any choice of Lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary.

A morphism $u\colon R \to S$ of rings with involution induces homomorphisms $u_*\colon L_k(R) \to L_k(S)$ for $k = 1,3$ by induction. One easily checks $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 1,3$.

The odd-dimensional L-groups of the ring of integers vanish.

Theorem 4.43 *We have $L_{2k+1}(\mathbb{Z}) = 0$.*

4.6 The Surgery Obstruction in Odd Dimensions

Consider a normal map of degree one $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon M \to X$, where M is a closed oriented manifold of dimension n and X is a connected finite Poincaré complex of dimension n for odd $n = 2k+1$. To these data we want to assign an element $\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w)$ such that the following holds

Theorem 4.44 (Surgery Obstruction Theorem in Odd Dimensions) *We get under the conditions above:*

1. *Suppose $k \geq 2$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f')\colon TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ covering a homotopy equivalence $f'\colon M' \to X$;*

2. *The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.*

We can arrange by finitely many surgery steps that f is k-connected (see Theorem 3.61). Consider a normal bordism $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^b} \to \eta$ covering a map $F\colon W \to X \times [0,1]$ of degree one from the given normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon M \to X$ to a new k-connected normal map $(\overline{f'}, f')\colon TM' \oplus \underline{\mathbb{R}^{a'}} \to \xi'$ covering $f'\colon M' \to X$. We finally want to arrange that f' is a homotopy equivalence. By applying Theorem 3.61 to the interior of W without changing ∂W we can arrange that W is $(k+1)$-connected. The kernels fit into an exact sequence

$$0 \to K_{k+1}(\widetilde{M'}) \xrightarrow{K_{k+1}(\widetilde{j})} K_{k+1}(\widetilde{W}) \xrightarrow{K_{k+1}(\widetilde{i'})} K_{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\partial_{k+1}} K_k(\widetilde{M'}) \to 0,$$

where $j\colon M' \to W$ and $i'\colon W \to (W, M')$ are the inclusions. Hence f' is a homotopy equivalence if and only if $K_{k+1}(\widetilde{i'})\colon K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is bijective. Therefore we must arrange that $K_{k+1}(\widetilde{i'})\colon K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is bijective.

We can associate to the normal bordism $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^b} \to \eta$ from the given k-connected normal map of degree one $(\overline{f}, f)$ to another k-connected normal map of degree one $(\overline{f'}, f')$ a $(-1)^k$-quadratic formation $(H_{(-1)^k}(F); F, G)$ as follows. The underlying non-degenerate $(-1)^k$-quadratic form is $H_{(-1)^k}(F)$ for $F := K_{k+1}(\widetilde{W}, \widetilde{M'})$. The first Lagrangian is F. The second Lagrangian G is given by the image of the map

$$\begin{aligned}(K_{k+1}(\widetilde{i'}), u \circ (? \cap [W, \partial W])^{-1} \circ K_{k+1}(\widetilde{i})\colon K_{k+1}(\widetilde{W})\\ \to F \oplus F^* = K_{k+1}(\widetilde{W}, \widetilde{M'}) \oplus K_{k+1}(\widetilde{W}, \widetilde{M'})^*,\end{aligned}$$

where $(? \cap [W, \partial W])\colon K^{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\cong} K_{k+1}(\widetilde{W}, \widetilde{M})$ is the Poincaré isomorphism, $i\colon W \to (W, M)$ is the inclusion and u is the canonical map

$$u\colon K^{k+1}(\widetilde{W}, \widetilde{M'}) \xrightarrow{\cong} K_{k+1}(\widetilde{W}, \widetilde{M'})^*, \qquad \alpha \mapsto \langle \alpha, ? \rangle$$

which is bijective by an elementary chain complex argument or by the universal coefficient spectral sequence.

What is the relation between this formation and the problem whether f' is a homotopy equivalence? Suppose that f' is a homotopy equivalence. Then $K_{k+1}(\widetilde{i'})\colon K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ is an isomorphism. Define a $(-1)^{k+1}$-quadratic form (H, ψ) by $H = K_{k+1}(\widetilde{W})$ and

$$\psi\colon H := K_{k+1}(\widetilde{W}) \xrightarrow{K_{k+1}(\widetilde{i})} K_{k+1}(\widetilde{W}, \widetilde{M}) \xrightarrow{(? \cap [W, \partial W])^{-1}} K^{k+1}(\widetilde{W}, \widetilde{M'})$$

$$\xrightarrow{u} K_{k+1}(\widetilde{W}, \widetilde{M'})^* \xrightarrow{K_{k+1}(\widetilde{i'})^*} K_{k+1}(\widetilde{W})^* = H^*.$$

One easily checks that the isomorphism $K_{k+1}(\widetilde{i'})\colon K_{k+1}(\widetilde{W}) \to K_{k+1}(\widetilde{W}, \widetilde{M'})$ induces an isomorphism of $(-1)^k$-quadratic formations

$$\partial(H, \psi) \xrightarrow{\cong} (H_{(-1)^k}(F); F, G).$$

Hence we see that f' is a homotopy equivalence only if $(H_{(-1)^k}(F); F, G)$ is isomorphic to a boundary.

One decisive step is to prove that the class of this $(-1)^k$-quadratic formation $(H_{(-1)^k}(F); F, G)$ in $L_{2k+1}(\mathbb{Z}\pi, w)$ is independent of the choice of the $(k+1)$-connected nullbordism. We will not give the proof of this fact. This fact enables us to define the *surgery obstruction* of $(\overline{f}, f)$

$$\sigma(\overline{f}, f) \quad := \quad [(H_\epsilon(F); F, G)] \qquad \in L_{2k+1}(\mathbb{Z}\pi, w), \tag{4.45}$$

where $(H_{(-1)^k}(F); F, G)$ is the $(-1)^k$-quadratic formation associated to any normal bordism of degree one $(\overline{F}, F)$ from a normal map $(f_0, \overline{f_0})$ to some normal map $(f', \overline{f'})$ such that $(f_0, \overline{f_0})$ and $(f', \overline{f'})$ are k-connected, F is $(k+1)$-connected and $(f_0, \overline{f_0})$ is obtained from the original normal map of degree one $(\overline{f}, f)$ by surgery below the middle dimension. From the discussion above it is clear that the vanishing of $\sigma(\overline{f}, f)$ is a necessary condition for the existence of a normal map $(\overline{f'}, f')$ which is normally bordant to f and whose underlying map is a homotopy equivalence. We omit the proof that for $k \geq 2$ this condition is also sufficient. This finishes the outline of the proof of Theorem 4.44.

More details can be found in [119, chapter 8]. The reason why we have been very brief in the odd-dimensional case is that the proofs are more complicated but not as illuminating as in the even-dimensional case and that a reader who is interested in details should directly read the approach using Poincaré complexes of Ranicki (see [95], [96], [97]). This approach is more conceptual and treats the even-dimensional and odd-dimensional case simultaneously.

Now we can give a rather complete answer to Problem 3.1 and Problem 3.62 for odd dimensions in the simply connected case.

Theorem 4.46 *1. Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a simply connected finite Poincaré complex X of odd dimension $n = 2k+1 \geq 5$. Then we can always change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence;*

2. Let X be a simply-connected finite connected Poincaré complex of odd dimension $n = 2k+1 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi \colon E \to X$, i.e. the set of normal invariants $\mathcal{T}_n(X)$ is non-empty.

Proof : (1) follows from Theorem 4.44 and Theorem 4.43.

(2) follows from assertion 1 together with Theorem 3.48. ∎

4.7 Variations of the Surgery Obstruction and the L-Groups

So far we have dealt with Problem 3.1 when a topological space is homotopy equivalent to a closed manifold. This question has motivated and led us to the notions of a finite Poincaré complex, of its Spivak normal fibration, of a normal map of degree one, of L-groups and of the surgery obstruction. This problem is certainly interesting but not our ultimate goal. We have already mentioned in Remark 1.5 our main goal which is to decide whether two closed manifolds are diffeomorphic and we have discussed the so called surgery program which is a strategy to attack it. The surgery program will lead to the surgery sequence in the next chapter (see Theorem 5.12). The surgery program suggests that we have to consider the following variations of the surgery obstruction and the L-groups which will play a role when establishing the surgery sequence.

Consider the third step (3) in the surgery program as explained in Remark 1.5. There are a cobordism $(W;M,N)$ and a map $(F,f,\mathrm{id})\colon (W;M,N) \to (N\times[0,1];N\times\{0\},N\times\{1\})$ for f a homotopy equivalence is given and we want to change F into a homotopy equivalence without changing W on the boundary. Thus we have to deal also with surgery problems $(\overline{g},g)$ where the underlying map $g\colon V\to X$ is a pair of maps $(g,\partial g)\colon (V,\partial V)\to(X,\partial X)$ from a compact oriented manifold V with boundary ∂V to a pair of finite Poincaré complexes $(X,\partial X)$ such that ∂f is already a homotopy equivalence. This will be done in Section 4.7.1. If this procedure is successful we obtain a h-cobordism W from M to N. We would like to know whether W is relative M diffeomorphic to $M\times[0,1]$. This would imply that M and N are diffeomorphic. Because of the s-Cobordism Theorem 1.1 this comes down to the problem to control the Whitehead torsion of the h-cobordism. The Whitehead torsion of the h-cobordism is trivial if and only if the Whitehead torsion of $\tau(F)$ and of $\tau(f)$ in $\mathrm{Wh}(\pi_1(X))$ agree (see Theorem 2.1). We may modify the first step (1) of the surgery program appearing in Remark 1.5 by requiring that f is a simple homotopy equivalence. The existence of a simple homotopy equivalence $f\colon M\to N$ is a necessary condition for M and N to be diffeomorphic. This means that we must modify our surgery obstruction so that its vanishing means that we get a simple homotopy equivalence $F\colon W \to N \times [0,1]$. This will be outlined in Section 4.7.2.

4.7.1 Surgery Obstructions for Manifolds with Boundary

We want to extend the notion of a normal map from closed manifolds to manifolds with boundary. The underlying map f is a map of pairs $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$, where M is a compact oriented manifold with boundary ∂M and $(X, \partial X)$ is a finite Poincaré pair, the degree of f is one and $\partial f\colon \partial M \to \partial X$ is required to be a homotopy equivalence. The bundle data are unchanged, they consist of a vector bundle ξ over X and a bundle map $\overline{f}\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$. Next we explain what a nullbordism in this setting means.

A *manifold triad* $(W; \partial_0 W, \partial_1 W)$ of dimension n consists of a compact manifold W of dimension n whose boundary decomposes as $\partial W = \partial_0 W \cup \partial_1 W$ for two compact submanifolds $\partial_0 W$ and $\partial_1 W$ with boundary such that $\partial_0 W \cap \partial_1 W = \partial(\partial_0 W) = \partial(\partial_1 W)$. A *Poincaré triad* $(X; \partial_0 X, \partial_1 X)$ of dimension n consists of a finite Poincaré pair $(X, \partial X)$ of dimension n together with $(n-1)$-dimensional finite subcomplexes $\partial_0 X$ and $\partial_1 X$ such that $\partial X = \partial_0 X \cup \partial_1 X$, the intersection $\partial_0 X \cap \partial_1 X$ is $(n-2)$-dimensional, ∂X is equipped with the structure of a Poincaré complex induced by Poincaré pair structure on $(X, \partial X)$ and both $(\partial_0 X, \partial_0 X \cap \partial_1 X)$ and $(\partial_1 X, \partial_0 X \cap \partial_1 X)$ are equipped with the structure of a Poincaré pair induced by the given structure of a Poincaré complex on ∂X. We use the convention that the fundamental class of $[X, \partial X]$ is sent to $[\partial X]$ under the boundary homomorphism $H_n(X, \partial X) \xrightarrow{\partial_n} H_{n-1}(\partial X)$ and $[\partial X]$ is sent to $([\partial_0 X, \partial_0 X \cap \partial_1 X], -[\partial_1 X, \partial_0 X \cap \partial_1 X])$ under the composite

$$\begin{aligned} H_{n-1}(\partial X) &\to H_{n-1}(\partial_0 X \cup \partial_1 X, \partial_0 X \cap \partial_1 X) \\ &\cong H_{n-1}(\partial_0 X, \partial_0 X \cap \partial_1 X) \oplus H_{n-1}(\partial_1 X, \partial_0 X \cap \partial_1 X). \end{aligned}$$

A manifold triad $(W; \partial_0 W, \partial_1 W)$ together with an orientation for W is a Poincaré triad. We allow that $\partial_0 X$ or $\partial_1 X$ or both are empty.

A normal nullbordism $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^b} \to \eta$ for a normal map of degree one $(\overline{f}, f)$ with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ consists of the following data. Put $n = \dim(M)$. We have a compact oriented $(n+1)$-dimensional manifold triad $(W; \partial_0 W, \partial_1 W)$, a finite $(n+1)$-dimensional Poincaré triad $(Y; \partial_0 Y, \partial Y)$, a map $(F; \partial_0 F, \partial_1 F)\colon (W; \partial_0 W, \partial_1 W) \to (Y; \partial_0 Y, \partial_1 Y)$, an orientation preserving diffeomorphism $(u, \partial u)\colon (M, \partial M) \to (\partial_0 W,$

$\partial(\partial_0 W)$) and an orientation preserving homeomorphism $(v, \partial v)\colon (X, \partial X) \to (\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$. We require that F has degree one, $\partial_1 F$ is a homotopy equivalence and $\partial_0 F \circ u = v \circ f$. Moreover, we have a bundle map $\overline{F}\colon TW \oplus \underline{\mathbb{R}^b} \to \eta$ covering F and a bundle map $\overline{v}\colon \xi \oplus \underline{\mathbb{R}^c} \to \eta$ covering v such that $\overline{F}$, Tu and $\overline{v}$ fit together.

We call two such normal maps $(\overline{f}, f)$ and $(\overline{f'}, f')$ normally bordant if the disjoint union of them after changing the orientation for M appearing in $(\overline{f}, f)$ possesses a normal nullbordism.

The definition and the main properties of the surgery obstruction carry over from normal maps for closed manifolds to normal maps for compact manifolds with boundary. The main reason is that we require $\partial f\colon \partial M \to \partial X$ to be a homotopy equivalence so that the surgery kernels "do not feel the boundary". All arguments such as making a map highly connected by surgery steps and intersection pairings and selfintersection can be carried out in the interior of M without affecting the boundary. Thus we get

Theorem 4.47 (Surgery Obstruction Theorem for Manifolds with Boundary).
Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ such that ∂f is a homotopy equivalence. Put $n = \dim(M)$. Then

1. *We can associate to it its surgery obstruction*
$$\sigma(\overline{f}, f) \quad \in \quad L_n(\mathbb{Z}\pi, w). \tag{4.48}$$
2. *The surgery obstruction depends only on the normal bordism class of $(\overline{f}, f)$;*
3. *Suppose $n \geq 5$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map $(\overline{f'}, f')\colon TM' \oplus \underline{\mathbb{R}^{a'}} \to \xi'$ which covers a homotopy equivalence of pairs $(f', \partial f')\colon (M', \partial M') \to (X, \partial X)$ with $\partial M' = \partial M$ and $\partial f' = \partial f$.*

4.7.2 Surgery Obstructions and Whitehead Torsion

Next we want to modify the L-groups and the surgery obstruction so that the surgery obstruction is the obstruction to achieve a simple homotopy equivalence.

We begin with the L-groups. It is clear that this requires to take equivalence classes of bases into account. Suppose that we have specified a subgroup $U \subset K_1(R)$ such that U is closed under the involution on $K_1(R)$ coming from the involution of R and contains the image of the change of ring homomorphism $K_1(\mathbb{Z}) \to K_1(R)$.

Two bases B and B' for the same finitely generated free R-module V are called *U-equivalent* if the change of basis matrix defines an element in $K_1(R)$ which belongs to U. Notice that the U-equivalence class of a basis B is unchanged if we permute the order of elements of B. We call an R-module V *U-based* if V is finitely generated free and we have chosen a U-equivalence class of bases.

Let V be a stably finitely generated free R-module. A *stable basis* for V is a basis B for $V \oplus R^u$ for some integer $u \geq 0$. Denote for any integer v the direct sum of the basis B and the standard basis S^a for R^a by $B \coprod S^a$ which is a basis for $V \oplus R^{u+a}$. Let C be a basis for $V \oplus R^v$. We call the stable basis B and C *stably U-equivalent* if and only if there is an integer $w \geq u, v$ such that $B \coprod S^{w-u}$ and $C \coprod S^{w-v}$ are U-equivalent basis. We call an R-module V *stably U-based* if V is stably finitely generated free and we have specified a stable U-equivalence class of stable basis for V.

Let V and W be stably U-based R-modules. Let $f\colon V \oplus R^a \xrightarrow{\cong} W \oplus R^b$ be an R-isomorphism. Choose a non-negative integer c together with basis for $V \oplus R^{a+c}$ and $W \oplus R^{b+c}$ which represent the given stable U-equivalence classes of basis for V and W. Let A be the matrix of $f \oplus \mathrm{id}_{R^c}\colon V \oplus R^{a+c} \xrightarrow{\cong} W \oplus R^{b+c}$ with respect to these bases. It defines an element [A] in $K_1(R)$. Define the *U-torsion*

$$\tau^U(f) \quad \in \quad K_1(R)/U \tag{4.49}$$

by the class represented by $[A]$. One easily checks that $\tau(f)$ is independent of the choices of c and the basis and depends only on f and the stable U-basis for V and W. Moreover, one easily checks

$$\begin{aligned}
\tau^U(g \circ f) &= \tau^U(g) + \tau^U(f);\\
\tau^U\begin{pmatrix} f & 0 \\ u & v \end{pmatrix} &= \tau^U(f) + \tau^U(v);\\
\tau^U(\mathrm{id}_V) &= 0
\end{aligned}$$

for R-isomorphisms $f\colon V_0 \xrightarrow{\cong} V_1$, $g\colon V_1 \xrightarrow{\cong} V_2$ and $v\colon V_3 \xrightarrow{\cong} V_4$ and an R-homomorphism $u\colon V_0 \to V_4$ of stably U-based R-modules V_i. Let C_* be a contractible stably U-based finite R-chain complex, i.e. a contractible R-chain complex C_* of stably U-based R-modules which satisfies $C_i = 0$ for $|i| > N$ for some integer N. The definition of Whitehead torsion in (2.7) carries over to the definition of the *U-torsion*

$$\tau^U(C_*) \quad = \quad [A] \qquad \in K_1(R)/U. \tag{4.50}$$

Analogously we can associate to an R-chain homotopy equivalence $f\colon C_* \to D_*$ of stably U-based finite R-chain complexes its *U-torsion* (cf (2.8))

$$\tau^U(f_*) \quad := \quad \tau(\mathrm{cone}_*(f_*)) \qquad \in K_1(R)/U. \tag{4.51}$$

Theorem 2.9 carries over to U-torsion in the obvious way.

We will consider U-based ϵ-quadratic forms (P, ψ), i.e. ϵ-quadratic forms whose underlying R-module P is a U-based finitely generated free R-module such that U-torsion of the isomorphism $(1 + \epsilon \cdot T)(\psi)\colon P \xrightarrow{\cong} P^*$ is zero in $K_1(R)/U$. An isomorphism $f\colon (P, \psi) \to (P', \psi')$ of U-based ϵ-quadratic forms is *U-simple* if the U-torsion of $f\colon P \to P'$ vanishes in $K_1(R)/U$. Notice that the ϵ-quadratic form $H_\epsilon(R)$ inherits a basis from the standard basis of R. The sum of two stably U-based ϵ-quadratic forms is again a stably U-based ϵ-quadratic form. It is worthwhile to mention the following U-simple version of Lemma 4.29.

Lemma 4.52 *Let (P, ψ) be a U-based ϵ-quadratic form. Let $L \subset P$ be a Lagrangian such that L is U-based and the U-torsion of the following 2-dimensional U-based finite R-chain complex*

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} L^* \to 0$$

vanishes in $K_1(R)/U$. Then the inclusion $i\colon L \to P$ extends to a U-simple isomorphism of ϵ-quadratic forms

$$H_\epsilon(L) \xrightarrow{\cong} (P, \psi).$$

Next we give the simple version of the even-dimensional L-groups.

Definition 4.53 *Let R be an associative ring with unit and involution. For $\epsilon \in \{\pm 1\}$ define $L^U_{1-\epsilon}(R)$ to be the abelian group of equivalence classes $[(F, \psi)]$ of U-based non-degenerate ϵ-quadratic forms (F, ψ) with respect to the following equivalence relation. We call (F, ψ) and (F', ψ') equivalent if and only if* *there exists integers $u, u' \geq 0$ and a U-simple isomorphism of non-degenerate ϵ-quadratic forms*

$$(F,\psi) \oplus H_\epsilon(R)^u \cong (F',\psi') \oplus H_\epsilon(R)^{u'}.$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $[H_\epsilon(R)^u]$ for any integer $u \geq 0$. The inverse of $[F,\psi]$ is given by $[F,-\psi]$.

For an even integer n define the abelian group $L_n^U(R)$ called the n-th U-decorated quadratic L-group *of R by*

$$L_n^U(R) \ := \ \begin{cases} L_0^U(R) & \textit{if } n \equiv 0 \mod 4; \\ L_2^U(R) & \textit{if } n \equiv 2 \mod 4. \end{cases}$$

A U-based ϵ-quadratic formation $(P,\psi;F,G)$ consists of an ϵ-quadratic formation $(P,\psi;F,G)$ such that (P,ψ) is a U-based non-degenerate ϵ-quadratic form, the Lagrangians F and G are U-based R-modules and the U-torsion of the following two contractible U-based finite R-chain complexes

$$0 \to F \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} F^* \to 0$$

and

$$0 \to G \xrightarrow{j} P \xrightarrow{j^* \circ (1+\epsilon \cdot T)(\psi)} G^* \to 0$$

vanish in $K_1(R)/U$, where $i\colon F \to P$ and $j\colon G \to P$ denote the inclusions. An isomorphism $f\colon (P,\psi;F,G) \to (P',\psi';F',G')$ of U-based ϵ-quadratic formations is U-simple if underlying the U-torsion of the induced R-isomorphisms $P \xrightarrow{\cong} P'$, $F \xrightarrow{\cong} F'$ and $G \xrightarrow{\cong} G'$ vanishes in $K_1(R)/U$. Notice that the trivial ϵ-quadratic formation $(H_\epsilon(R^u); R^u, (R^u)^*)$ inherits a U-basis from the standard basis on R^u. Given a U-based $(-\epsilon)$-quadratic form (Q,ψ), its boundary $\partial(Q,\psi)$ is a U-based ϵ-quadratic formation. Obviously the sum of two U-based ϵ-quadratic formations is again a U-based ϵ-quadratic formation. Next we give the simple version of the odd-dimensional L-groups.

Definition 4.54 *Let R be an associative ring with unit and involution. For $\epsilon \in \{\pm 1\}$ define $L_{2-\epsilon}^U(R)$ to be the abelian group of equivalence classes $[(P,\psi;F,G)]$ of U-based ϵ-quadratic formations $(P,\psi;F,G)$ with respect to the following equivalence relation. We call two U-based ϵ-quadratic formations $(P,\psi;F,G)$ and $(P',\psi';F',G')$ equivalent if and only if there exists*

U-based $(-\epsilon)$-quadratic forms (Q,μ) and (Q',μ') and non-negative integers u and u' together with a U-simple isomorphism of ϵ-quadratic formations

$$\begin{aligned} &(P,\psi;F,G)\oplus\partial(Q,\mu)\oplus(H_\epsilon(R^u);R^u,(R^u)^*) \\ &\qquad\cong\quad (P',\psi';F',G')\oplus\partial(Q',\mu')\oplus(H_\epsilon(R^{u'});R^{u'},(R^{u'})^*). \end{aligned}$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $\partial(Q,\mu)\oplus(H_\epsilon(R^u);R^u,(R^u)^*)$ for any U-based $(-\epsilon)$-quadratic form (Q,μ) and non-negative integer u. The inverse of $[(P,\psi;F,G)]$ is represented by $(P,-\psi;F',G')$ for any choice of stably U-based Lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary and the U-torsion of the the obvious isomorphism $F\oplus F'\xrightarrow{\cong}P$ and $F\oplus F'\xrightarrow{\cong}P$ vanishes in $K_1(R)/U$.

For an odd integer n define the abelian group $L_n^U(R)$ called the n-th U-decorated quadratic L-group *of R*

$$L_n^U(R)\;:=\;\begin{cases} L_1^U(R) & \text{if } n\equiv 1 \mod 4;\\ L_3^U(R) & \text{if } n\equiv 3 \mod 4.\end{cases}$$

Notation 4.55 *If $R=\mathbb{Z}\pi$ with the w-twisted involution and $U\subset K_1(\mathbb{Z}\pi)$ is the abelian group of elements of the shape $(\pm g)$ for $g\in\pi$, then we write*

$$\begin{aligned} L_n^s(\mathbb{Z}\pi,w) &:= L_n^U(\mathbb{Z}\pi,w);\\ L_n^h(\mathbb{Z}\pi,w) &:= L_n(\mathbb{Z}\pi,w); \end{aligned}$$

where $L_n(R)$ is the L-group introduced in Definitions 4.28 and 4.42 and $L_n^U(R)$ is the L-group introduced in Definitions 4.53 and 4.54. The L-groups $L_n^s(\mathbb{Z}\pi,w)$ are called simple quadratic L-groups.

Let $(\overline{f},f)$ be a normal map of degree one with $(f,\partial f)\colon(M,\partial M)\to(X,\partial X)$ as underlying map such that $(X,\partial X)$ is a simple finite Poincaré complex and ∂f is a simple homotopy equivalence. Next we want to explain how the definition of the surgery obstruction of (4.48) can be modified to the simple setting. Notice that the difference between the L-groups $L_n^h(\mathbb{Z}\pi,w)$ and the simple L-groups $L_n^s(\mathbb{Z}\pi,w)$ is the additional structure of a U-basis. The definition of the simple surgery obstruction

$$\sigma(\overline{f},f)\quad\in\quad L_n^s(\mathbb{Z}\pi,w). \tag{4.56}$$

is the same as the one in (4.48) except that we must explain how the various surgery kernels inherit a U-basis.

The elementary proof of the following lemma is left to the reader. Notice that for any stably U-based R-module V and element $x \in K_1(R)/U$ we can find another stable U-basis C for V such that the U-torsion $\tau^U(\mathrm{id}\colon (V,B) \to (V,C)$ is x. This is not true in the unstable setting. For instance, there exists a ring R with an element $x \in K_1(R)/U$ for U the image of $K_1(\mathbb{Z}) \to K_1(R)$ such that x cannot be represented by a unit in R, in other words x is not the U-torsion of any R-automorphism of R.

Lemma 4.57 *Let C_* be a contractible finite stably free R-chain complex and r be an integer. Suppose that each chain module C_i with $i \neq r$ comes with a stable U-basis. Then C_r inherits a preferred stable U-basis which is uniquely defined by the property that the U-torsion of C_* vanishes in $K_1(R)/U$.*

Now Lemma 4.18 has the following version in the simple homotopy setting.

Lemma 4.58 *Let D_* be a stably U-based finite R-chain complex. Suppose for a fixed integer k that $H_i(D_*) = 0$ for $i \neq r$. Suppose that $H^{r+1}(\hom_R(D_*, V)) = 0$ for any R-module V. Then $H_r(D_*)$ is stably finitely generated free and inherits a preferred stable U-basis.*

Proof : combine Lemma 4.57 and Lemma 4.18. ■

Now we can prove the following version of Lemma 4.19

Lemma 4.59 *If $f\colon X \to Y$ is k-connected for $n = 2k$ or $n = 2k+1$, then $K_k(\widetilde{M})$ is stably finitely generated free and inherits a preferred stable U-basis.*

Proof : The proof is the same as the one of the proof of Lemma 4.19 except that we apply Lemma 4.58 instead of Lemma 4.18. ■

In the proof that the surgery obstruction is well-defined, one has to show that the U-equivalence classes of basis on the surgery kernels appearing in the short exact sequence (4.35) are compatible with the short exact sequence in the sense of Lemma 4.57. This follows from the following lemma whose elementary proof is left to the reader.

Lemma 4.60 *Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{q_*} E_* \to 0$ be a U-based exact sequence of U-based finite R-chain complexes. Here U-based exact means that for each*

$p \geq 0$ the U-torsion of the 2-dimensional U-based finite R-chain complex $0 \to C_p \xrightarrow{i_p} D_p \xrightarrow{q_p} E_p \to 0$ vanishes in $K_1(R)/U$. Let r be a fixed integer. Suppose that $H_i(C_*) = H_i(E_*) = 0$ for $i \neq r$ and $H^{r+1}(\hom_R(C_*, V)) = H^{r+1}(\hom_R(C_*, V)) = 0$ holds for any R-module V. Equip $H_r(C_*)$ and $H_r(E_*)$ with the U-equivalence class of stable basis defined in Lemma 4.58.

Then $H_i(D_*) = 0$ for $i \neq r$ and $H^{r+1}(\hom_R(D_*, V)) = 0$ holds for any R-module V. We obtain a short exact sequence

$$0 \to H_r(C_*) \xrightarrow{H_p(i_*)} H_r(D_*) \xrightarrow{H_r(q_*)} H_r(E_*) \to 0.$$

The U-equivalence class of stable basis on $H_r(D_*)$ obtained from Lemma 4.58 applied to this exact sequence and the U-equivalence class of stable basis on $H_r(D_*)$ obtained from Lemma 4.58 applied to D_* agree.

Next we can give the simple version of the surgery obstruction theorem. Notice that simple normal bordism class means that in the definition of normal nullbordisms the pairs $(Y, \partial Y)$, $(\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$ and $(\partial_1 Y, \partial_0 Y \cap \partial_1 Y)$ are required to be simple finite Poincaré pairs and the map $\partial_1 F \colon \partial_1 M \to \partial_1 Y$ is required to be a simple homotopy equivalence.

Theorem 4.61 (Simple Surgery Obstruction Theorem for Manifolds with Boundary) *Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ such that $(X, \partial X)$ is a simple finite Poincaré complex ∂f is a simple homotopy equivalence. Put $n =$ $\dim(M)$. Then:*

1. *The simple surgery obstruction depends only on the simple normal bordism class of $(\overline{f}, f)$;*
2. *Suppose $n \geq 5$. Then $\sigma(\overline{f}, f) = 0$ in $L_n^s(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map $(\overline{f'}, f')\colon TM' \oplus \underline{\mathbb{R}}^{a'} \to \xi'$ which covers a simple homotopy equivalence of pairs $(f', \partial f')\colon (M', \partial M') \to (X, \partial X)$ with $\partial M' = \partial M$ and $\partial f' = \partial f$.*

4.8 Miscellaneous

A guide for the calculation of the L-groups for finite groups is presented by Hambleton and Taylor [51], where further references are given.

Chapter 5

The Surgery Exact Sequence

Introduction

In this section we introduce the exact surgery sequence (see Theorem 5.12). It is the realization of the surgery program which we have explained in Remark 1.5. The surgery exact sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater than or equal to five.

5.1 The Structure Set

Definition 5.1 *Let X be a closed oriented manifold of dimension n. We call two orientation preserving simple homotopy equivalences $f_i\colon M_i \to X$ from closed oriented manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there exists an orientation preserving diffeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0. The* simple structure set *$\mathcal{S}_n^s(X)$ of X is the set of equivalence classes of orientation preserving simple homotopy equivalences $M \to X$ from closed oriented manifolds of dimension n to X. This set has a preferred base point, namely the class of the identity* $\mathrm{id}\colon X \to X$.

The simple structure set $\mathcal{S}_n^s(X)$ is the basic object in the study of manifolds which are diffeomorphic to X. Notice that a simple homotopy equivalence $f\colon M \to X$ is homotopic to a diffeomorphism if and only if it represents the base point in $\mathcal{S}_n^s(X)$. A manifold M is oriented diffeomorphic to N if and only if for some orientation preserving simple homotopy equivalence $f\colon M \to N$ the class of $[f]$ agrees with the preferred base point. Some care is necessary since it may be possible that a given orientation preserving simple homotopy equivalence $f\colon M \to N$ is not homotopic to a diffeomorphism although M and N are diffeomorphic. Hence it does not suffice to compute $\mathcal{S}_n^s(N)$, one also has to understand the operation of the group of homotopy classes of simple orientation preserving selfequivalences of N on $\mathcal{S}_n^s(N)$. This can be rather complicated in general. But it will be no problem in the case $N = S^n$, because any orientation preserving selfhomotopy equivalence $S^n \to S^n$ is homotopic to the identity.

There is also a version of the structure set which does not take Whitehead torsion into account.

Definition 5.2 *Let X be a closed oriented manifold of dimension n. We call two orientation preserving homotopy equivalences $f_i\colon M_i \to X$ from closed oriented manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there is a manifold triad $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W \cap \partial_1 W = \emptyset$ and an orientation preserving homotopy equivalence of triads $(F; \partial_0 F, \partial_1 F)\colon (W; \partial_0 W, \partial_1 W) \to (X \times [0,1]; X \times \{0\}, X \times \{1\})$ together with orientation preserving diffeomorphisms $g_0\colon M_0 \to \partial_0 W$ and $g_1\colon M_1^- \to \partial_1 W$ satisfying $\partial_i F \circ g_i = f_i$ for $i = 0, 1$. (Here M_1^- is obtained from M_1 by reversing the orientation.) The* structure set *$\mathcal{S}_n^h(X)$ of X is the set of equivalence classes of orientation preserving homotopy equivalences $M \to X$ from a closed oriented manifold M of dimension n to X. This set has a preferred base point, namely the class of the identity* $\mathrm{id}\colon X \to X$.

Remark 5.3 If we require in Definition 5.2 the homotopy equivalences F, f_0 and f_1 to be simple homotopy equivalences, we get the simple structure set $\mathcal{S}_n^s(X)$ of Definition 5.1, provided that $n \geq 5$. We have to show that the two equivalence relations are the same. This follows from the s-Cobordism Theorem 1.1. Namely, W appearing in Definition 5.2 is an h-cobordism and is even an s-cobordism if we require F, f_0 and f_1 to be simple homotopy equivalences (see Theorem 2.1). Hence there is a diffeomorphism $\Phi\colon \partial_0 W \times [0,1] \to W$ inducing the obvious identification $\partial_0 W \times \{0\} \to \partial_0 W$ and some orientation preserving diffeomorphism $\phi_1\colon (\partial_0 W)^- = (\partial_0 W \times \{1\})^- \to \partial_1 W$. Then $\phi\colon M_0 \to M_1$ given by $g_1^{-1} \circ \phi_1 \circ g_0$ is an orientation preserving diffeomorphism such that $f_1 \circ \phi$ is homotopic to f_0. The other implication is obvious.

Remark 5.4 As long as we are dealing with smooth manifolds, there is in general no canonical group structure on the structure set.

5.2 Realizability of Surgery Obstructions

In this section we explain that any element in the L-groups can be realized as the surgery obstruction of a normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f\colon (M, \partial M) \to (Y, \partial Y)$ if we allow M to have non-empty boundary ∂M.

Theorem 5.5 *Suppose $n \geq 5$. Consider a connected compact oriented manifold X possibly with boundary ∂X. Let π be its fundamental group and let $w\colon \pi \to \{\pm 1\}$ be its orientation homomorphism. Consider an element $x \in L_n(\mathbb{Z}\pi, w)$.*

Then we can find a normal map of degree one

$$(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to TX \times [0,1] \oplus \underline{\mathbb{R}^a}$$

covering a map of triads

$$f = (f; \partial_0 f, \partial_1 f)\colon (M; \partial_0 M, \partial_1 M) \to (X \times [0,1], X \times \{0\} \cup \partial X \times [0,1], X \times \{1\})$$

with the following properties:

1. *$\partial_0 f$ is a diffeomorphism and $\overline{f}|_{\partial_0 M}$ is given by $T(\partial f_0) \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+1}}}$;*
2. *$\partial_1 f$ is a homotopy equivalence;*
3. *The surgery obstruction $\sigma(\overline{f}, f)$ in $L_n(\mathbb{Z}\pi, w)$ (see Theorem 4.48) is the given element x.*

The analogous statement holds for $x \in L_n^s(\mathbb{Z}\pi, w)$ if we require $\partial_1 f$ to be a simple homotopy equivalence and we consider the simple surgery obstruction (see (4.56)*).*

Proof : We give at least the idea of the proof in the case $n = 2k$, more details can be found in [119, Theorem 5.3 on page 53, Theorem 6.5 on page 66].

Recall that the element $x \in L_n(\mathbb{Z}\pi, w)$ is represented by a non-degenerate $(-1)^k$-quadratic form (P, ψ) with P a finitely generated free $\mathbb{Z}\pi$-module. We fix such a representative and write it as a triple (P, μ, λ) as explained in Subsection 4.2.2. Let $\{b_1, b_r, \dots, b_r\}$ be a $\mathbb{Z}\pi$-basis for P.

Choose r disjoint embeddings $j_i\colon D^{2k-1} \to X - \partial X$ into the interior of X for $i = 1, 2, \dots, r$. Let $F^0\colon S^{k-1} \times D^k \to D^{2k-1}$ be the standard embedding. Define embeddings $F_i^0\colon S^{k-1}\times D^k\to X$ by the composition of the standard embedding F^0 and j_i for $i=1,2,\dots,r$. Let $f_i^0\colon S^{k-1}\to X$ be the restriction of F_i to $S^{k-1}\times\{0\}\subset S^{k-1}\times D^k$. Fix base points $b\in X-\partial X$ and $s\in S^{k-1}$ and paths w_i from b to $f_i^0(s)$ for $i=1,2,\dots,r$. Thus each f_i^0 is a pointed immersion. Next we construct regular homotopies $\eta_i\colon S^{k-1}\to X$ from f_i^0 to a new embedding f_i^1 which are modelled upon (P,μ,λ). These regular homotopies define associated immersions $\eta_i'\colon S^{k-1}\times[0,1]\to X\times[0,1]$, $(x,t)\mapsto(\eta_i(x),t)$. Since these immersions η_i' are embeddings on the boundary, we can define their intersection and selfintersection number as in (4.1) and (4.6).

We want to achieve that the intersection number of η_i' and η_j' for $i<j$ is $\lambda(b_i,b_j)$ and the selfintersection number $\mu(\eta_i')$ is $\mu(b_i)$ for $i=1,2,\dots,r$. Notice that these numbers are additive under stacking regular homotopies together. Hence it suffices to explain how to introduce a single intersection with value $\pm g$ between η_i' and η_j' for $i<j$ or a selfintersection for η_i' with value $\pm g$ for $i\in\{1,2,\dots,r\}$ without introducing other intersections or selfintersections. We explain the construction for introducing an intersection between η_i' and η_j' for $i<j$, the construction for a selfintersection is analogous. This construction will only change η_i, the other regular homotopies η_j for $j\neq i$ will be unchanged.

Join $f_i^0(s)$ and $f_j^0(s)$ by a path v such that v is an embedding, does not meet any of the f_i^0-s and the composition $w_j * v * w_i^-$ represents the given element $g\in\pi$. Now there is an obvious regular homotopy from f_i^0 to another embedding along the path v inside a small tubular neighborhood $N(v)$ of v which leaves f_i^0 fixed outside a small neighborhood $U(s)$ of s and moves $f_i^0|_{U(s)}$ within this small neighborhood $N(v)$ along v very close to $f_i^1(s)$. Now use a disc D^k which meets f_j^0 transversally at the origin to move f_i^0 further, thus introducing the desired intersection with value $\pm g$. Inside the disc the move looks like pushing the upper hemisphere of the boundary down to the lower hemisphere of the boundary, thus having no intersection with the origin at any point of time with one exception. One has to check that there is enough room to realize both possible signs.

Since f_i^1 is regularly homotopic to f_i^0 which is obtained from the trivial embeddings F_i^0, we can extend f_i^1 to an embedding $F_i^1\colon S^{k-1}\times D^k\to X-\partial X$. Now attach for each $i\in\{1,2\dots,r\}$ a handle $(F_i^1)=D^k\times D^k$ to $X\times[0,1]$ by $F_i^1\times\{1\}\colon S^{k-1}\times D^k\to X\times\{1\}$. Let M be the resulting manifold. We obtain a manifold triad $(M;\partial_0M,\partial_1M)$ if we put $\partial_0M=X\times 0\cup\partial X\times[0,1]$ and $\partial_1M=\partial M-\operatorname{int}(\partial_0X)$. There is an extension of the identity id: $X \times [0,1] \to X \times [0,1]$ to a map $f\colon M \to X \times [0,1]$ which induces a map of triads $(f; \partial_0 f, \partial_1 f)$ such that $\partial_0 f$ is a diffeomorphism. This map is covered by a bundle map $\overline{f}\colon TM \oplus \underline{\mathbb{R}^a} \to TX \oplus \mathbb{R}^a$ which is given on $\partial_0 M$ by the differential of $\partial_0 f$. These claims follow from the fact that f_i^1 is regularly homotopic to f_i^0 and the f_i^0 were trivial embeddings so that we can regard this construction as surgery on the identity map $X \to X$. The kernel $K_k(\widetilde{M}) = K_k(\widetilde{M}, \widetilde{X \times [0,1]})$ has a preferred basis corresponding to the cores $D^k \times \{0\} \subset D^k \times D^k$ of the new handles (F_i^1). These cores can be completed to immersed spheres S_i by adjoining the images of the η_i' in $X \times [0,1]$ and finally the obvious discs D^k in $X \times \{0\}$ whose boundaries are given by $f_i^0(S^{k-1})$. Then the intersection of S_i with S_j is the same as the intersection of η_i' with η_j' and the selfintersection of S_i is the selfintersection of η_i' with itself. Hence by construction $\lambda(S_i, S_j) = \lambda(b_i, b_j)$ for $i < j$ and the selfintersection number $\mu(S_i')$ is $\mu(b_i)$ for $i = 1, 2, \ldots, r$. This implies $\lambda(S_i, S_j) = \lambda(b_i, b_j)$ for $i, j \in \{1, 2, \ldots, r\}$ (see Subsection 4.2.2). The isomorphism $P \to P^*$ associated to the given non-degenerate $(-1)^k$-quadratic form can be identified with $K_k(\widetilde{M}) \to K_k(\widetilde{M}, \widetilde{\partial_1 M})$ if we use for $K_k(\widetilde{M}, \widetilde{\partial_1 M})$ the basis given by the cocores of the handles (F_i^1). Hence $K_k(\widetilde{\partial_1 M}) = 0$ and we conclude that $\partial_1 f$ is a homotopy equivalence. By definition and construction $\sigma(\overline{f}, f)$ in $L_n(\mathbb{Z}\pi, w)$ is the class of the non-degenerated $(-1)^k$-quadratic form (P, μ, λ). This finishes the outline of the proof of Theorem 5.5. ■

Remark 5.6 It is not true that for any closed oriented manifold N of dimension n with fundamental group π and orientation homomorphism $w\colon \pi \to \{\pm 1\}$ and any element $x \in L_n(\mathbb{Z}\pi, w)$ there is a normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of closed oriented manifolds $f\colon M \to N$ of degree one such that $\sigma(\overline{f}, f) = x$. Notice that in Theorem 5.5 the target manifold $X \times [0,1]$ is not closed. The same remark holds for $L_n^s(\mathbb{Z}\pi, w)$.

5.3 The Surgery Exact Sequence

Now we can establish one of the main tools in the classification of manifolds, the surgery exact sequence. For this purpose we have to extend the Definition 3.50 of a normal map for closed manifolds to manifolds with boundary. Let $(X, \partial X)$ be a compact oriented manifold of dimension n with boundary ∂X. We consider normal maps of degree one $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ for which $\partial f\colon \partial M \to \partial X$ is a diffeomorphism. A normal nullbordism for $(\overline{f}, f)$ consists of a bundle map

$$(\overline{F}, F) \colon TW \oplus \underline{\mathbb{R}^b} \to \eta$$

with underlying map of manifold triads of degree one

$$(F; \partial_0 F, \partial_1 F) \colon (W; \partial_0 W, \partial_1 W) \to (X \times [0,1]; X \times \{0\}, \partial X \times [0,1] \cup X \times \{1\})$$

together with a diffeomorphism

$$g \colon (M, \partial M) \to (\partial_0 W, \partial_0 W \cap \partial_1 W)$$

covered by a bundle map $\overline{g} \colon TM \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^{a+b}}$ and a bundle isomorphism

$$(\overline{i_0}, i_0) \colon \xi \oplus \underline{\mathbb{R}^{b+1}} \to \eta$$

covering the inclusion $i_0 \colon X \to X \times [0,1]$, $x \mapsto (x, 0)$ such that the composition $(\overline{F}, F) \circ (\overline{g}, g)$ agrees with $(\overline{i_0}, i_0) \circ (\overline{f} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}}, f)$ and $\partial_1 F \colon \partial_1 W \to \partial X \times [0,1] \cup X \times \{1\}$ is a diffeomorphism. Notice that here the target of the bordism is $X \times [0,1]$, whereas we have allowed in Theorem 4.47 (2) a more general notion of normal bordism, where the target manifold also could vary.

Definition 5.7 *Let $(X, \partial X)$ be a compact oriented manifold of dimension n with boundary ∂X. Define the* set of normal maps to $(X, \partial X)$

$$\mathcal{N}_n(X, \partial X)$$

to be the set of normal bordism classes of normal maps of degree one $(\overline{f}, f) \colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ with underlying map $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ for which $\partial f \colon \partial M \to \partial X$ is a diffeomorphism.

Notice that this definition uses tangential bundle data. One could also use normal bundle data (see Lemma 3.51).

Let X be a closed oriented connected manifold of dimension $n \geq 5$. Denote by π its fundamental group and by $w \colon \pi \to \{\pm 1\}$ its orientation homomorphism. Let $\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ and $\mathcal{N}_n(X)$ be the set of normal maps of degree one as introduced in Definition 3.47. Let $\mathcal{S}_n^s(X)$ be the structure set of Definition 5.1. Denote by $L_n^s(\mathbb{Z}\pi)$ the simple surgery obstruction group (see Notation 4.55). Denote by

$$\sigma \colon \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \to L_{n+1}^s(\mathbb{Z}\pi, w); \qquad (5.8)$$

$$\sigma \colon \mathcal{N}_n(X) \to L_n^s(\mathbb{Z}\pi, w) \qquad (5.9)$$

the maps which assign to the normal bordism class of a normal map of degree one its simple surgery obstruction (see (4.56)). This is well-defined by Theorem 4.61 (1). Let

$$\eta\colon \mathcal{S}_n^s(X) \quad \to \quad \mathcal{N}_n(X) \tag{5.10}$$

be the map which sends the class $[f] \in \mathcal{S}_n^s(X)$ represented by a simple homotopy equivalence $f\colon M \to X$ to the normal bordism class of the following normal map of degree one. Choose a homotopy inverse $f^{-1}\colon X \to M$ and a homotopy $h\colon \mathrm{id}_M \simeq f^{-1} \circ f$. Put $\xi = (f^{-1})^* TM$. Up to isotopy of bundle maps there is precisely one bundle map $(\overline{h}, h)\colon TM \times [0,1] \to TM$ covering $h\colon M \times [0,1] \to M$ whose restriction to $TM \times \{0\}$ is the identity map $TM \times \{0\} \to TM$. The restriction of $\overline{h}$ to $X \times \{1\}$ induces a bundle map $\overline{f}\colon TM \to \xi$ covering $f\colon M \to X$. Put $\eta([f]) := [(\overline{f}, f)]$. One easily checks that the normal bordism class of $(\overline{f}, f)$ depends only on $[f] \in \mathcal{N}_n^s(X)$ and hence that η is well-defined. Next we define an action of the abelian group $L_{n+1}^s(\mathbb{Z}\pi, w)$ on the structure set $\mathcal{S}_n^s(X)$

$$\rho\colon L_{n+1}^s(\mathbb{Z}\pi, w) \times \mathcal{S}_n^s(X) \quad \to \quad \mathcal{S}_n^s(X). \tag{5.11}$$

Fix $x \in L_{n+1}^s(\mathbb{Z}\pi, w)$ and $[f] \in \mathcal{N}_n^s(X)$ represented by a simple homotopy equivalence $f\colon M \to X$. By Theorem 5.5 we can find a normal map $(\overline{F}, F)$ covering a map of triads $(F; \partial_0 F, \partial_1 F)\colon (W; \partial_0 W, \partial_1 W) \to (M \times [0,1], M \times \{0\}, M \times \{1\})$ such that $\partial_0 F$ is a diffeomorphism and $\partial_1 F$ is a simple homotopy equivalence and $\sigma(\overline{F}, F) = x$. Then define $\rho(x, [f])$ by the class $[f \circ \partial_1 F\colon \partial_1 W \to X]$. We have to show that this is independent of the choice of $(\overline{F}, F)$. Let $(\overline{F'}, F')$ be a second choice. We can glue W' and W^- together along the diffeomorphism $(\partial_0 F)^{-1} \circ \partial_0 F'\colon \partial_1 W' \to \partial_1 W$ and obtain a normal bordism from $(\overline{F}|_{\partial_1 W}, \partial_1 F)$ to $(\overline{F'}|_{\partial_1 W'}, \partial_1 F')$. The simple obstruction of this normal bordism is

$$\sigma(\overline{F'}, F') - \sigma(\overline{F}, F) \;=\; x - x \;=\; 0.$$

Because of Theorem 4.61 (2) we can perform surgery relative boundary on this normal bordism to arrange that the reference map from it to $X \times [0,1]$ is a simple homotopy equivalence. In view of Remark 5.3 this shows that $f \circ \partial_1 F$ and $f \circ \partial_1 F'$ define the same element in $\mathcal{S}_n^s(X)$. One easily checks that this defines a group action since the surgery obstruction is additive under stacking normal bordisms together. The next result is the main result of this chapter and follows from the definitions and Theorem 4.61 (2)

Theorem 5.12 (The surgery Exact Sequence) *Under the conditions and in the notation above the so called* surgery sequence

$$\mathcal{N}_{n+1}(X\times[0,1],X\times\{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi,w) \xrightarrow{\partial} \mathcal{S}^s_n(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi,w)$$

is exact for $n \geq 5$ in the following sense. An element $z \in \mathcal{N}_n(X)$ lies in the image of η if and only if $\sigma(z) = 0$. Two elements $y_1, y_2 \in \mathcal{S}^s_n(X)$ have the same image under η if and only if there exists an element $x \in L^s_{n+1}(\mathbb{Z}\pi,w)$ with $\rho(x,y_1) = y_2$. For two elements x_1, x_2 in $L^s_{n+1}(\mathbb{Z}\pi)$ we have $\rho(x_1,[\mathrm{id}\colon X \to X]) = \rho(x_2,[\mathrm{id}\colon X \to X])$ if and only if there is $u \in \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ with $\sigma(u) = x_1 - x_2$.

There is an analogous surgery exact sequence

$$\mathcal{N}^h_{n+1}(X\times[0,1],X\times\{0,1\}) \xrightarrow{\sigma} L^h_{n+1}(\mathbb{Z}\pi,w) \xrightarrow{\partial} \mathcal{S}^h(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi,w)$$

where $\mathcal{S}^h(X)$ is the structure set of Definition 5.2 and $L^h_n(\mathbb{Z}\pi,w) := L_n(\mathbb{Z}\pi,w)$ have been introduced in Definitions 4.28 and 4.42.

Remark 5.13 There is no group structure known for $\mathcal{S}^h_s(X)$ and $\mathcal{N}_n(X)$ such that $\mathcal{S}^h_s(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi,w)$ is a sequence of groups (and analogous for the simple version). Notice that the composition (see Theorem 3.52)

$$[X;G/O] \cong \mathcal{N}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi,w)$$

is a map whose source and target come with canonical group structures but it is not a homomorphism of abelian groups in general. These difficulties are connected to the existence of homotopy spheres and can be overcome in the topological category.

The surgery sequence of Theorem 5.12 can be extended to infinity to the left. In the range far enough to the left it is a sequence of abelian groups.

5.4 Miscellaneous

One can also develop surgery theory in the PL-category or in the topological category [59]. This requires to carry over the notions of vector bundles and tangent bundles to these categories. There are analogs of the sets of normal invariants $\mathcal{N}^{PL}_n(X)$ and $\mathcal{N}^{TOP}_n(X)$ and the structure sets $\mathcal{S}^{PL,h}_n(X)$, $\mathcal{S}^{PL,s}_n(X)$ $\mathcal{S}^{TOP,h}_n(X)$ and $\mathcal{S}^{TOP,s}_n(X)$. There are analogs PL and TOP of the group O. Theorem 3.52 and Theorem 3.48 (see also Remark 3.49) carry over to the PL-category and the topological category.

Theorem 5.14 *Let X be a connected finite n-dimensional Poincaré complex. Suppose that $\mathcal{N}_n^{PL}(X)$ is non-empty. Then there is a canonical group structure on the set $[X, G/PL]$ of homotopy classes of maps from X to G/PL and a transitive free operation of this group on $\mathcal{N}_n^{PL}(X)$. The analogous statement holds for TOP instead of PL.*

There are analogs of the surgery exact sequence (see Theorem 5.12) for the PL-category and the topological category.

Theorem 5.15 (The surgery Exact Sequence) *There is a surgery sequence*

$$\mathcal{N}_{n+1}^{PL}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}_n^{PL,s}(X) \xrightarrow{\eta} \mathcal{N}_n^{PL}(X) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi, w)$$

which is exact for $n \geq 5$ in the sense of Theorem 5.12. There is an analogous surgery exact sequence

$$\mathcal{N}_{n+1}^{PL}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}^h(\mathbb{Z}\pi, w) \xrightarrow{\partial} \mathcal{S}_n^{PL,h}(X) \xrightarrow{\eta} \mathcal{N}_n^{PL}(X) \xrightarrow{\sigma} L_n^h(\mathbb{Z}\pi, w)$$

The analogous sequences exists in the topological category.

Notice that the surgery obstruction groups are the same in the smooth category, PL-category and in the topological category. Only the set of normal invariants and the structure sets are different. The set of normal invariants in the smooth category, PL-category or topological category do not depend on the decoration h and s, whereas the structure sets and the surgery obstruction groups depend on the decoration h and s. In particular the structure set depends on both the choice of category and choice of decoration.

As in the smooth setting the surgery sequence above can be extended to infinity to the left.

Some interesting constructions can be carried out in the topological category which do not have smooth counterparts. An algebraic surgery sequence is constructed in [98, §14, §18] and identified with the geometric surgery sequence above in the topological category. Moreover, in the topological situation one can find abelian group structures on $\mathcal{S}_s^h(X)$ and $\mathcal{N}_n(X)$ such that the surgery sequence becomes a sequence of abelian groups. The main point is to find the right addition on G/TOP. Given a finite Poincaré complex X of dimension ≥ 5, a single obstruction, the so called total surgery obstruction, is constructed in [98, §17]. It vanishes if and only if X is homotopy equivalent to a closed topological manifold. It combines the two stages of the obstruction we have seen before, namely, the problem whether the Spivak normal fibration has a reduction to a TOP-bundle and whether the surgery obstruction of the associated normal map is trivial.

The computation of the homotopy type of the space G/O and G/PL due to Sullivan is explained in detail in [78, Chapter 4]. One obtains homotopy equivalences

$$\begin{aligned} G/TOP\left[\frac{1}{2}\right] &\simeq BO\left[\frac{1}{2}\right]; \\ G/TOP_{(2)} &\simeq \prod_{j\geq 1} K(\mathbb{Z}_{(2)},4j) \times \prod_{j\geq 1} K(\mathbb{Z}/2,4j-2), \end{aligned}$$

where $K(A;l)$ denotes the *Eilenberg-MacLane space* of type (A,l), i.e. a CW-complex such that $\pi_n(K(A,l))$ is trivial for $n \neq l$ and is isomorphic to A if $n = 2l$, $_{(2)}$ stands for localizing at (2), i.e. all primes except 2 are inverted, and $\left[\frac{1}{2}\right]$ stands for localization of 2, i.e. 2 is inverted. In particular we get for a space X isomorphisms

$$\begin{aligned} [X,G/TOP]\left[\frac{1}{2}\right] &\cong \widetilde{KO}^0(X)\left[\frac{1}{2}\right]; \\ [X,G/TOP]_{(2)} &\cong \prod_{j\geq 1} H^{4j}(X;\mathbb{Z}_{(2)}) \times \prod_{j\geq 1} H^{4j-2}(M;\mathbb{Z}/2), \end{aligned}$$

where KO^* is K-theory of real vector bundles.

Kirby and Siebenmann [59, Theorem 5.5 in Essay V on page 251] (see also [101]) have proved

Theorem 5.16 *The space TOP/PL is an Eilenberg MacLane space of type $(\mathbb{Z}/2,3)$.*

More information about the the spaces O, PL, G, G/O and G/PL will be given in Section 6.6.

Finally we mention that a different approach to surgery has been developed by Kreck. A survey about his approach is given in [60]. Its advantage is that one does not have to get a complete homotopy classification first. The price to pay is that the L-groups are much more complicated, they are not necessarily abelian groups any more. This approach is in particular successful when the manifolds under consideration are already highly connected. See for instance [61], [62], [112].

Chapter 6

Homotopy Spheres

Introduction

Recall that S^n is the standard sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. We will equip it with the structure of a smooth manifold for which the canonical inclusion $S^n \to \mathbb{R}^{n+1}$ is an embedding of smooth manifolds. We use the orientation on S^n which is compatible with the isomorphism $T_xS^n \oplus \nu(S^n, \mathbb{R}^n) \xrightarrow{\cong} T_x\mathbb{R}^n$, where we use on $\nu_x(S^n, \mathbb{R}^n)$ the orientation coming from the normal vector field pointing to the origin and on $T_x\mathbb{R}^n$ the standard orientation. This agrees with the convention that $S^n = \partial D^{n+1}$ inherits its orientation from D^{n+1}. A *homotopy n-sphere* Σ is a closed oriented n-dimensional smooth manifold which is homotopy equivalent S^n. The Poincaré Conjecture says that any homotopy n-sphere Σ is oriented homeomorphic to S^n and is known to be true for all dimensions except $n = 3$. In this chapter we want to solve the problem how many oriented diffeomorphism classes of homotopy n-spheres exist for given n. For $n \neq 3$ this is the same as determining how many different oriented smooth structures exist on S^n. This is a beautiful and illuminating example. It shows how the general surgery methods which we have developed so far apply to a concrete problem. It illustrates what kind of input from homotopy theory and algebra is needed for the final solution.

The following theorem summarizes what we will prove in this chapter. Here Θ^n denotes the abelian group of oriented h-cobordism classes of oriented homotopy n-spheres, $bP^{n+1} \subset \Theta^n$ is the subgroup of those homotopy n-spheres which bound a stably parallelizable compact manifold, $J_n\colon \pi_n(SO) \to \pi_n^s$ denotes the J-homomorphism and B_n is the n-th Bernoulli number. These notions and the proof of the next result will be presented in this chapter (see Theorem 6.39, Corollary 6.43, Theorem 6.44, Theorem 6.46 and Theorem 6.56).

Theorem 6.1 (Classification of Homotopy Spheres) *1. Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\frac{3-(-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1}-1) \cdot \text{numerator}(B_k/(4k));$$

2. Let $k \geq 1$ be an integer. Then bP^{4k+2} is trivial or isomorphic to $\mathbb{Z}/2$. We have

$$bP^{4k+2} = \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l-2, k \geq 1; \\ 0 & 4k+2 \in \{6,14,30,62\}. \end{cases}$$

3. If $n = 4k+2$ for $k \geq 2$, then there is an exact sequence

$$0 \to \Theta^n \to \text{coker}(J_n) \to \mathbb{Z}/2.$$

If $n = 4k$ for $k \geq 2$ or $n = 4k+2$ with $4k+2 \neq 2^l-2$, then

$$\Theta^n \cong \text{coker}(J_n);$$

4. Let $n \geq 5$ be odd. Then there is an exact sequence

$$0 \to bP^{n+1} \to \Theta^n \to \text{coker}(J_n) \to 0.$$

If $n \neq 2^l-3$, the sequence splits.

The following table taken from [58, pages 504 and 512] gives the orders of the finite groups Θ^n, bP^{n+1} and Θ^n/bP^{n+1} as far as they are known in low dimensions. The values in dimension 1 and 2 come from obvious adhoc computations. The computation of Θ^4 requires some additional analysis which we will not present here. Notice that $\Theta^4 = 1$ does *not* mean that any homotopy 4-sphere is diffeomorphic to S^4 (see Lemma 6.2).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Θ^n	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256
bP^{n+1}	1	1	?	1	1	1	28	1	2	1	992	1	1	1	8128
Θ^n/bP^{n+1}	1	1	1	1	1	1	1	2	4	6	1	1	3	2	2

The basic paper about homotopy spheres is the one of Kervaire and Milnor [58] which contains a systematic study and can be viewed as the beginning of surgery theory. Nearly all the results presented here are taken from this paper. Another survey article about homotopy spheres has been written by Lance [66] and Levine [68].

6.1 The Group of Homotopy Spheres

Define the *n-th group of homotopy spheres* Θ^n as follows. Elements are oriented h-cobordism classes $[\Sigma]$ of oriented homotopy n-spheres Σ, where Σ and Σ' are called oriented h-cobordant if there is an oriented h-cobordism $(W, \partial_0 W, \partial_1 W)$ together with orientation preserving diffeomorphisms $\Sigma \to \partial_0 W$ and $(\Sigma')^- \to \partial_1 W$. The addition is given by the connected sum. The zero element is represented by S^n. The inverse of $[\Sigma]$ is given by $[\Sigma^-]$, where Σ^- is obtained from Σ by reversing the orientation.

Obviously Θ^n becomes an abelian semi-group by the connected sum. It remains to check that $[\Sigma^-]$ is an inverse of $[\Sigma]$. It is easy to see that for a homotopy n-sphere Σ that there is an h-cobordism W from the connected sum $\Sigma \sharp \Sigma^-$ to S^n. Let $D^n \subset \Sigma$ be an embedded disk. Let $\Sigma - \mathrm{int}(D^n)$ be the manifold with boundary obtained by removing this disk. Delete the interior of an embedded disk $D^{n+1} \subset (\Sigma - \mathrm{int}(D^n)) \times [0,1]$. The result is the desired h-cobordism W.

The fundamental group π of a homotopy n-sphere is trivial for $n \geq 2$. This implies that the orientation homomorphism w is always trivial and $\mathrm{Wh}(\pi) = \{0\}$. Hence it does not matter whether we work with simple homotopy equivalences or homotopy equivalence since $\mathcal{S}^s_n(S^n) = \mathcal{S}^h_n(S^n)$ for $n \geq 5$ and $L^s_n(\mathbb{Z}\pi) = L^h_n(\mathbb{Z}\pi) = L_n(\mathbb{Z})$ holds. Therefore we will omit the decoration h or s for the remainder of this chapter.

Lemma 6.2 *Let $\overline{\Theta}^n$ be the set of oriented diffeomorphism classes $[\Sigma]$ of oriented homotopy n-spheres Σ. The forgetful map*

$$f\colon \overline{\Theta^n} \to \Theta^n$$

is bijective for $n \neq 3, 4$.

Proof : This follows from the s-cobordism Theorem 1.1 for $n \geq 5$ and by obvious adhoc computations for $n \leq 2$. ∎

Lemma 6.3 *There is a natural bijection*

$$\alpha\colon \mathcal{S}^s_n(S^n) \xrightarrow{\cong} \overline{\Theta}^n \qquad [f\colon M \to S^n] \mapsto [M].$$

If $n \neq 3$, there is an obvious bijection

$$\{\textit{smooth oriented structures on } S^n\}/\textit{oriented diffeomorphic} \xrightarrow{\cong} \overline{\Theta}^n.$$

Proof : For any homotopy n-sphere Σ there is up to homotopy precisely one map $f\colon \Sigma \to S^n$ of degree one. For $n \neq 3$ the Poincaré Conjecture is true which says that any homotopy n-sphere is homeomorphic to S^n. ■

In dimension 3 there is no difference between the topological and smooth category. Any closed topological 3-manifold M carries a unique smooth structure. Hence any manifold which is homeomorphic to S^3 is automatically diffeomorphic to S^3. The Poincaré Conjecture is at the time of writing open in dimension 3. It is not known whether any closed 3-manifold which is homotopy equivalent to S^3 is homeomorphic to S^3.

Definition 6.4 *A manifold M is called* stably parallelizable *if $TM \oplus \underline{\mathbb{R}^a}$ is trivial for some $a \geq 0$.*

Definition 6.5 *Let $bP^{n+1} \subset \Theta^n$ be the subset of elements $[\Sigma]$ for which Σ is oriented diffeomorphic to the boundary ∂M of a stably parallelizable compact manifold M.*

Lemma 6.6 *The subset $bP^{n+1} \subset \Theta^n$ is a subgroup of Θ^n. It is the preimage under the composition*

$$\Theta^n \xrightarrow{(f\circ\alpha)^{-1}} \mathcal{S}_n(S^n) \xrightarrow{\eta} \mathcal{N}_n(S^n)$$

of the base point $[\mathrm{id}\colon TS^n \to TS^n]$ in $\mathcal{N}_n(S^n)$, where f is the bijection of Lemma 6.2 and α is the bijection of Lemma 6.3.

Proof : Suppose that Σ bounds W and Σ' bounds W' for stably parallelizable manifolds W and W'. Then the boundary connected sum $W\sharp W'$ is stably parallelizable and has $\Sigma\sharp\Sigma'$ as boundary. This shows that $bP^{n+1} \subset \Theta^{n+1}$ is a subgroup.

Consider an element $[f\colon \Sigma \to S^n]$ in $\eta^{-1}([\mathrm{id}\colon TS^n \to TS^n])$. Then there is a normal bordism from a normal map $(\overline{f}, f)\colon T\Sigma \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f\colon \Sigma \to S^n$ to the normal map $\mathrm{id}\colon TS^n \to TS^n$. This normal bordism is given by a bundle map $(\overline{F}, F)\colon TW \oplus \underline{\mathbb{R}^{a+b}} \to \eta$ covering a map of triads $(F; \partial_0 F, \partial_1 F)\colon (W; \partial_0 W, \partial_1 W) \to (S^n \times [0,1]; S^n \times \{0\}, S^n \times \{1\})$, and bundle isomorphisms $(\overline{u}, u)\colon T\Sigma \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^b}$ covering an orientation preserving diffeomorphism $u\colon \Sigma \to \partial_0 W$, $(\overline{u'}, u')\colon TS^n \oplus \underline{\mathbb{R}^{a+b+1}} \to TW \oplus \underline{\mathbb{R}^b}$ covering an orientation preserving diffeomorphism $u'\colon (S^n)^- \to$

$\partial_1 W$, $(\overline{v}, v)\colon \xi \oplus \underline{\mathbb{R}^{b+1}} \to \eta$ covering the obvious map $v\colon S^n \to S^n \times \{0\}$ and $(\overline{v'}, v')\colon TS^n \oplus \underline{\mathbb{R}^{a+b+1}} \to \eta$ covering the obvious map $v'\colon S^n \to S^n \times \{1\}$ such that $(\overline{F}, F) \circ (\overline{u}, u) = (\overline{v}, v) \circ (\overline{f} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}})$ and $(\overline{F}, F) \circ (\overline{u'}, u') = (\overline{v'}, v')$ holds. Then $D^{n+1} \cup_{u'} W$ is a manifold whose boundary is oriented diffeomorphic to Σ by u and for which the bundle data above yield a stable isomorphism $TW \oplus \underline{\mathbb{R}^{a+b}} \to \underline{\mathbb{R}^{n+1+a+b}}$. Hence $[\Sigma]$ lies in bP^{n+1}.

Conversely, consider $[\Sigma]$ such that there exists a stably parallelizable manifold W together with an orientation preserving diffeomorphism $u\colon \Sigma \to \partial W$. We can assume without loss of generality that W is connected. Choose an orientation preserving homotopy equivalence $f\colon \Sigma \to S^n$. We can extend $\partial_0 F := f \circ u^{-1}\colon \partial W \to S^n$ to a smooth map $F\colon W \to D^{n+1}$. Since f has degree one, the map $(F, \partial F)\colon (W, \partial W) \to (D^{n+1}, S^n)$ has degree one. Let $y \in D^{n+1} - S^n$ be a regular value. Then the degree of F is the finite sum $\sum_{x \in F^{-1}(y)} \epsilon(x)$, where $\epsilon(x) = 1$, if $T_x F\colon T_x W \to T_y D^n$ preserves the induced orientations and $\epsilon(x) = -1$ otherwise. If two points x_1 and x_2 in $F^{-1}(y)$ satisfy $\epsilon(x_1) \neq \epsilon(x_2)$, one can change F up to homotopy relative ∂W such that $F^{-1}(y)$ contains two points less than before. Thus one can arrange that $F^{-1}(y)$ consists of precisely one point x and that $T_x F\colon T_x W \to T_x D^{n+1}$ is orientation preserving. Then one can change F up to homotopy in a small neigborhood of x such that there is an embedded disk $D_o^{n+1} \subset W - \partial W$ such that F induces a diffeomorphism $D_0^{n+1} \to F(D_0^{n+1})$ and no point outside D_0^{n+1} is mapped to $F(D_0^{n+1})$. Define $V = W - \mathrm{int}(D_0^{n+1})$. Then F induces a map also denoted by $F\colon V \to D^{n+1} - F(D_0^{n+1})$. If we identify $D^{n+1} - F(D_0^{n+1})$ with $S^n \times [0,1]$ by an orientation preserving diffeomorphism, we obtain a map of triads $(F; \partial_0 F, \partial_1 F)\colon (V; \partial_0 V, \partial_1 V) \to (S^{n+1} \times [0,1], S^n \times \{0\}, S^n \times \{1\})$ together with diffeomorphisms $u\colon \Sigma \to \partial_0 V$, $v\colon S^n \to S^n \times \{0\}$, $u'\colon S^n \to \partial_0 V$, $v'\colon S^n \to S^n \times \{1\}$ such that $\partial_1 F$ is an orientation preserving diffeomorphism and $F \circ u = v \circ f$ and $F \circ u' = v'$. Now one covers everything with appropriate bundle data to obtain a normal bordism from $(\overline{f}, f)\colon T\Sigma \oplus \underline{\mathbb{R}^a} \to \xi$ to $\mathrm{id}\colon TS^n \to TS^n$. This shows $\eta \circ \alpha^{-1} \circ f^{-1}([\Sigma]) = [\mathrm{id}\colon TS^n \to TS^n]$. ■

6.2 The Surgery Sequence for Homotopy Spheres

In this section we examine the surgery sequence (see Theorem 5.12) in the case of the sphere. In contrast to the general case we will obtain a long exact sequence of abelian groups. We have to introduce the following bordism groups.

Definition 6.7 *A* stable framing *of a closed oriented manifold M of dimension n is a strong bundle isomorphism $\overline{u}\colon TM \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation. (Recall that strong means that $\overline{u}$ covers the identity.) An* almost stable framing *of a closed oriented manifold M of dimension n is a choice of a point $x \in M$ together with a strong bundle isomorphism $\overline{u}\colon TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation on $M - \{x\}$.*

Of course any stably framed manifold is in particular an almost stably framed manifold. A homotopy n-sphere Σ is an almost stably framed manifold since for any point $x \in \Sigma$ the complement $\Sigma - \{x\}$ is contractible and hence $T\Sigma|_{\Sigma-\{x\}}$ is trivial. We will later show the non-trivial fact that any homotopy n-sphere is stably parallelizable, i.e. admits a stable framing. The standard sphere S^n inherits its standard stable framing from its embedding to $\mathbb{R}^{n+1}$.

A stably framed nullbordism for a stably framed manifold $(M, \overline{u})$ is a compact manifold W with a stable framing $\overline{U}\colon TW \oplus \underline{\mathbb{R}^{a+b}} \xrightarrow{\cong} \underline{\mathbb{R}^{n+1+a+b}}$ and a bundle isomorphism $(\overline{v}, v)\colon TM \oplus \underline{\mathbb{R}^{a+1+b}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^{a+b}}$ coming from the differential of an orientation preserving diffeomorphism $v\colon M \to \partial W$ such that $\overline{U} \circ \overline{v} = \overline{u} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}}$. Now define the notion of a stably framed bordism from a stably framed manifold $(M, \overline{u})$ to another stably framed manifold $(M', \overline{u'})$ to be a stably framed nullbordism for the disjoint union of $(M^-, \overline{u}^-)$ and $(M', \overline{u'})$, where M^- is obtained from M by reversing the orientation and $\overline{u}^-$ is the composition

$$\overline{u}^-\colon TM \oplus \underline{\mathbb{R}^a} \xrightarrow{\overline{u}} \underline{\mathbb{R}^{a+n}} = \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^{a+n-1}} \xrightarrow{-\mathrm{id}_{\underline{\mathbb{R}}} \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+n-1}}}} \underline{\mathbb{R}} \oplus \underline{\mathbb{R}^{a+n-1}} = \underline{\mathbb{R}^{a+n}}.$$

Consider two almost stably framed manifolds $(M, x, \overline{u}\colon TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}})$ and $(M', x', \overline{u'}\colon TM'|_{M'-\{x'\}} \oplus \underline{\mathbb{R}^{a'}} \xrightarrow{\cong} \underline{\mathbb{R}^{a'+n}})$. An almost stably framed bordism from the first to the second consists of the following data. There is a compact oriented $(n+1)$-dimensional manifold triad $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W \cap \partial_1 W = \emptyset$ together with an embedding $j\colon ([0,1]; \{0\}, \{1\}) \to (W; \partial_0 W, \partial_1 W)$ such that j is transversal at the boundary. We also need a strong bundle isomorphism $\overline{U}\colon TW_{W-\mathrm{im}(j)} \oplus \underline{\mathbb{R}^b} \xrightarrow{\cong} \underline{\mathbb{R}^{n+1+b}}$ for some $b \geq a, a'$. Furthermore we require the existence of a bundle isomorphism $(\overline{v}, v)\colon TM \oplus \underline{\mathbb{R}^{b+1}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^b}$ coming from the differential of an orientation preserving diffeomorphism $v\colon M^- \to \partial_0 W$ with $v(x) = j(0)$ and of a bundle isomorphism $(\overline{v'}, v')\colon TM \oplus \underline{\mathbb{R}^{b+1}} \xrightarrow{\cong} TW \oplus \underline{\mathbb{R}^b}$ coming from the differential of an orientation preserving diffeomorphism $v'\colon M' \to \partial_1 W$ with $v'(x') = j(1)$ such that $\overline{U} \circ \overline{v} = \overline{u^-} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b-a+1}}}$ and $\overline{U} \circ \overline{v'} = \overline{u'} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1-a'}}}$ holds.

Definition 6.8 *Let Ω_n^{fr} be the abelian group of stably framed bordism classes of stably framed closed oriented manifolds of dimension n. This becomes an abelian group by the disjoint union. The zero element is represented by S^n with its standard stable framing. The inverse of the class of $(M, \overline{u})$ is represented by the class of $(M^-, \overline{u} \oplus (-\mathrm{id}_{\underline{\mathbb{R}}}))$.*

Let Ω_n^{alm} be the abelian group of almost stably framed bordism classes of almost stably framed closed oriented manifolds of dimension n. This becomes an abelian group by the connected sum at the preferred base points. The zero element is represented by S^n with the base point $s = (1, 0, \ldots, 0)$ with its standard stable framing restricted to $S^n - \{s\}$. The inverse of the class of $(M, x, \overline{u})$ is represented by the class of $(M^-, x, \overline{u} \oplus (-\mathrm{id}_{\underline{\mathbb{R}}}))$.

Lemma 6.9 *There are canonical bijections of pointed sets*

$$\begin{aligned} \beta\colon \mathcal{N}_n(S^n) &\xrightarrow{\cong} \Omega_n^{\mathrm{alm}}; \\ \gamma\colon \mathcal{N}_{n+1}(S^n \times [0,1], S^n \times \{0,1\}) &\xrightarrow{\cong} \mathcal{N}_{n+1}(S^{n+1}). \end{aligned}$$

Proof : Consider an element $r \in \mathcal{N}_n(S^n)$ represented by a normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of degree one $f\colon M \to S^n$. Since f has degree one, one can change $(\overline{f}, f)$ by a homotopy such that $f^{-1}(s)$ consists of one point $x \in M$ for a fixed point $s \in S^n$. Since $S^n - \{s\}$ is contractible, $\xi_{S^n - \{s\}}$ admits a trivialization which is unique up to isotopy. It induces together with $\overline{f}$ an almost stable framing on (M, x). The class of (M, x) with this stable framing in Ω_n^{alm} is defined to be the image of r under β.

The inverse β^{-1} of β is defined as follows. Let $r \in \Omega_n^{\mathrm{alm}}$ be represented by the almost framed manifold $(M, x, \overline{u})$. Let $c\colon M \to S^n$ be the collaps map for a small embedded disk $D^n \subset M$ with origin x. By construction c induces a diffeomorphism $c|_{\mathrm{int}(D^n)}\colon \mathrm{int}(D^n) \to S^n - \{s\}$ and maps $M - \mathrm{int}(D^n)$ to $\{s\}$ for fixed $s \in S^n$. The almost stable framing $\overline{u}$ yields a bundle map $\overline{c'}\colon TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ covering $c|_{M-\{x\}}\colon M - \{x\} \to S^n - \{c(x)\}$. Since D^n is contractible, we obtain a bundle map unique up to isotopy $\overline{c''}\colon TD^n \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ covering $c|_{D^n}\colon D^n \to S^n$. The composition of the inverse of the restriction of $\overline{c''}$ to $\mathrm{int}(D^n) - \{x\}$ and of the restriction of $\overline{c'}$

to $\mathrm{int}(D^n) - \{x\}$ yields a strong bundle automorphism of the trivial bundle $\underline{\mathbb{R}^{n+a}}$ over $S^n - \{s, c(x)\}$. Let ξ be the bundle obtained by glueing the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over $S^n - \{s\}$ and the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over $S^n - \{c(x)\}$ together using this bundle automorphism over $S^n - \{s, c(x)\}$. Then $\overline{c'}$ and $\overline{c''}$ fit together to a bundle map $\overline{c}\colon TM \oplus \underline{\mathbb{R}^{n+a}} \to \xi$ covering c. Define the image of r under β^{-1} to be the class of $(\overline{c}, c)$.

Consider $r \in \mathcal{N}_{n+1}(S^n \times [0,1], S^n \times \{0,1\})$ represented by a normal map $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $(f, \partial f)\colon (M, \partial M) \to (S^n \times [0,1], S^n \times \{0,1\})$. Recall that ∂f is a diffeomorphism. Hence one can form the closed manifold $N = M \cup_{\partial f\colon \partial M \to S^n \times \{0,1\}} D^{n+1} \times \{0,1\}$. The map f and the identity on $D^{n+1} \times \{0,1\}$ induce a map of degree one $g\colon N \to S^n \times [0,1] \cup_{S^n \times \{0,1\}} D^{n+1} \times \{0,1\} \cong S^{n+1}$. Define the bundle η over $S^n \times [0,1] \cup_{S^n \times \{0,1\}} D^{n+1} \times \{0,1\} \cong S^{n+1}$ by glueing ξ and $T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^{a-1}}$ together over $S^n \times \{0,1\}$ by the strong bundle isomorphism

$$\xi|_{S^{n-1} \times \{0,1\}} \xrightarrow{\left(\overline{f}|_{S^{n-1} \times \{0,1\}}\right)^{-1}} TM|_{\partial M} \oplus \underline{\mathbb{R}^a} = T(\partial M) \oplus \underline{\mathbb{R}^{a+1}}$$

$$\xrightarrow{T\partial f} T(S^{n-1} \times \{0,1\}) \oplus \underline{\mathbb{R}^{a+1}} = T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a}.$$

Then $\overline{f}$ and $\mathrm{id}\colon T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a} \to T(D^{n+1} \times \{0,1\}) \oplus \underline{\mathbb{R}^a}$ fit together yielding a bundle map $\overline{g}\colon TN \oplus \underline{\mathbb{R}^a} \to \eta$ covering g. Define the image of r under γ by the class of $(\overline{g}, g)$. We leave it to the reader to construct the inverse of γ which is similar to the construction in Lemma 6.6 but now two embedded discs instead of one embedded disc are removed. ■

Next we want to construct a long exact sequence of abelian groups

$$\ldots \to \Omega^{\mathrm{alm}}_{n+1} \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}) \xrightarrow{\partial} \Theta^n \xrightarrow{\eta} \Omega^{\mathrm{alm}}_n \xrightarrow{\sigma} L_n(\mathbb{Z}) \to \ldots$$

The map

$$\sigma\colon \Omega^{\mathrm{alm}}_{n+1} \to L_{n+1}(\mathbb{Z})$$

is given by the composition

$$\Omega^{\mathrm{alm}}_{n+1} \xrightarrow{\beta^{-1}} \mathcal{N}_{n+1}(S^{n+1}) \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}),$$

where β is the bijection of Lemma 6.9 and $\sigma\colon \mathcal{N}_n(S^n) \to L_n(\mathbb{Z})$ is given by the surgery obstruction and has already appeared in the surgery sequence (see Theorem 5.12). The map

$$\partial\colon L_{n+1}(\mathbb{Z}) \to \Theta^n$$

is the composition of the inverse of the bijection $\alpha \circ f \colon \mathcal{S}_n(S^n) \xrightarrow{\cong} \Theta^n$ coming from Lemma 6.2 and Lemma 6.3 and the map $\partial : L_{n+1}(\mathbb{Z}) \to \mathcal{S}_n(S^n)$ of the surgery sequence (see Theorem 5.12). The map

$$\eta \colon \Theta^n \quad \to \quad \Omega_n^{\mathrm{alm}} \tag{6.10}$$

sends the class of a homotopy sphere Σ to the class of $(\Sigma, x, \overline{u})$, where x is any point in Σ and the stable framing of $T\Sigma|_{\Sigma - \{x\}}$ comes from the fact that $\Sigma - \{x\}$ is contractible. This map η corresponds to the map η appearing in the surgery exact sequence (see Theorem 5.12) under the identification $\alpha \circ f \colon \mathcal{S}_n(S^n) \xrightarrow{\cong} \Theta^n$ coming from Lemma 6.2 and Lemma 6.3.

We leave it to the reader to check that all these maps are homomorphisms of abelian groups. The surgery sequence (see Theorem 5.12) implies

Theorem 6.11 *The long sequence of abelian groups which extends infinitely to the left*

$$\ldots \to \Omega_{n+1}^{\mathrm{alm}} \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}) \xrightarrow{\partial} \Theta^n \xrightarrow{\eta} \Omega_n^{\mathrm{alm}} \xrightarrow{\sigma} L_n(\mathbb{Z}) \xrightarrow{\partial} \ldots \xrightarrow{\eta} \Omega_5^{\mathrm{alm}} \xrightarrow{\sigma} L_5(\mathbb{Z})$$

is exact.

Recall that we have shown in Theorem 4.30, Theorem 4.32 and Theorem 4.43 that there are isomorphisms

$$\frac{1}{8} \cdot \mathrm{sign} \colon L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$$

and

$$\mathrm{Arf} \colon L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$$

and that $L_{2i+1}(\mathbb{Z}) = 0$ for $i \in \mathbb{Z}$. Consider a normal map $(\overline{f}, f) \colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map of degree one $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ of oriented compact $4i$-dimensional manifolds such that X is simply connected and ∂f a homotopy equivalence. Then the isomorphism $\frac{1}{8} \cdot \mathrm{sign} \colon L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ sends $\sigma(\overline{f}, f)$ to $\frac{1}{8} \cdot \mathrm{sign}(M \cup_{\partial f} X^-)$. Hence Theorem 6.11 and Lemma 6.6 imply

Corollary 6.12 *There are for $i \geq 2$ and $j \geq 3$ short exact sequences of abelian groups*

$$0 \to \Theta^{4i} \xrightarrow{\eta} \Omega_{4i}^{\mathrm{alm}} \xrightarrow{\frac{\mathrm{sign}}{8}} \mathbb{Z} \xrightarrow{\partial} bP^{4i} \to 0$$

and

$$0 \to \Theta^{4i-2} \xrightarrow{\eta} \Omega_{4i-2}^{\mathrm{alm}} \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2 \xrightarrow{\partial} bP^{4i-2} \to 0$$

and

$$0 \to bP^{2j} \to \Theta^{2j-1} \xrightarrow{\eta} \Omega^{\mathrm{alm}}_{2j-1} \to 0.$$

Here the map

$$\frac{\mathrm{sign}}{8} \colon \Omega^{\mathrm{alm}}_{4i} \to \mathbb{Z}$$

sends $[M]$ *to* $\frac{1}{8} \cdot \mathrm{sign}(M)$ *and* $\mathrm{Arf} \colon \Omega^{\mathrm{alm}} \to \mathbb{Z}/2$ *sends* $[M]$ *to the Arf invariant of the normal map* $\beta^{-1}([M]) \in \mathcal{N}_n(S^n)$ *for* β *the bijection appearing in Lemma 6.9.*

6.3 The J-Homomorphism and Stably Framed Bordism

By Theorem 6.11 we have reduced the computation of Θ^n to computations about Ω^{alm}_n and certain maps to $\mathbb{Z}$ and $\mathbb{Z}/2$ given by the signature divided by 8 and the Arf invariant. This reduction is essentially due to the surgery machinery. The rest of the computation will essentially be homotopy theory. First we try to understand Ω^{fr}_* geometrically. There is an obvious forgetful map

$$f \colon \Omega^{\mathrm{fr}}_n \quad \to \quad \Omega^{\mathrm{alm}}_n. \tag{6.13}$$

Define the group homomorphism

$$\partial \colon \Omega^{\mathrm{alm}}_n \quad \to \quad \pi_{n-1}(SO) \tag{6.14}$$

as follows. Given $r \in \Omega^{alm}_n$ choose a representative $(M, x, \overline{u} \colon TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}})$. Let $D^n \subset M$ be an embedded disk with origin x. Since D^n is contractible, we obtain a strong bundle isomorphism unique up to isotopy $\overline{v} \colon TM|_{D^n} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$. The composition of the inverse of the restriction of $\overline{u}$ to $S^{n-1} = \partial D^n$ and of the restriction of $\overline{v}$ to S^{n-1} is an orientation preserving bundle automorphism of the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over S^{n-1}. This is the same as a map $S^{n-1} \to SO(n+a)$. Its composition with the canonical map $SO(n+a) \to SO$ represents an element in $\pi_{n-1}(SO)$ which is defined to be the image of r under $\partial \colon \Omega^{\mathrm{alm}}_n \to \pi_{n-1}(SO)$. Let

$$\overline{J} \colon \pi_n(SO) \quad \to \quad \Omega^{\mathrm{fr}}_n \tag{6.15}$$

be the group homomorphism which assigns to the element $r \in \pi_n(SO)$ represented by a map $\overline{u} \colon S^n \to SO(n+a)$ the class of S^n with the stable framing

$TS^n \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$ which is induced by the standard trivialization

$$TS^n \oplus \underline{\mathbb{R}} \cong T(\partial D^{n+1}) \oplus \nu(\partial D^{n+1}, D^{n+1}) \cong TD^{n+1}|_{\partial D^{n+1}} \cong \underline{\mathbb{R}^{n+1}}$$

and the strong bundle automorphism of the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over S^n given by $\overline{u}$. One easily checks

Lemma 6.16 *The following sequence is a long exact sequence of abelian groups*

$$\ldots \xrightarrow{\partial} \pi_n(SO) \xrightarrow{\overline{J}} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{\mathrm{alm}} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\overline{J}} \Omega_{n-1}^{fr} \xrightarrow{f} \ldots .$$

Next we want to interpret the exact sequence appearing in Lemma 6.16 homotopy theoretically. We begin with the homomorphism $\overline{J}$. Notice that there is a natural bijection

$$\tau' \colon \operatorname{colim}_{k\to\infty} \Omega_n(\underline{\mathbb{R}^k} \to \{*\}) \quad \xrightarrow{\cong} \quad \Omega_n^{\mathrm{fr}} \tag{6.17}$$

which is defined as follows. Consider an element

$$x = [(M, i \colon M \to \mathbb{R}^{n+k}, \mathrm{pr} \colon M \to \{*\}, \overline{u} \colon \nu(i) \to \underline{\mathbb{R}^k})] \ \in \ \Omega_n(\underline{\mathbb{R}^k} \to \{*\})$$

There is a canonical strong isomorphism $\overline{u'} \colon TM \oplus \nu(i) \xrightarrow{\cong} \underline{\mathbb{R}^{n+k}}$. From $\overline{u}$ and $\overline{u'}$ we get an isomorphism $\overline{v} \colon TM \oplus \underline{\mathbb{R}^k} \to \underline{\mathbb{R}^{n+k}}$ covering the projection $M \to \{*\}$. Define $\tau'(x)$ by the class of $(M, \overline{v})$. The Thom space $\mathrm{Th}(\underline{\mathbb{R}^{n+k}})$ is S^{n+k}. Hence the Pontrjagin-Thom construction (see Theorem 3.26) induces an isomorphism

$$P \colon \operatorname{colim}_{k\to\infty} \Omega_n(\underline{\mathbb{R}^k} \to \{*\}) \ \cong \ \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k). \tag{6.18}$$

Notice that the *stable n-th homotopy group* of a space X is defined by

$$\pi_n^s(X) \ := \ \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k \wedge X). \tag{6.19}$$

The stable *n-stem* is defined by

$$\pi_n^s \ := \pi_n^s(S^0) \ := \ \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k). \tag{6.20}$$

Thus the isomorphism τ' of (6.17) and the isomorphism P of (6.18) yield an isomorphism

$$\tau \colon \Omega_n^{\mathrm{fr}} \ \xrightarrow{\cong} \ \pi_n^s. \tag{6.21}$$

Next we explain the so called *Hopf construction* which defines for spaces X, Y and Z a map

$$H\colon [X \times Y, Z] \quad\to\quad [X * Y, \Sigma Z] \tag{6.22}$$

as follows. Recall that the join $X * Y$ is defined by $X \times Y \times [0,1]/\sim$, where $\sim$ is given by $(x,y,0) \sim (x',y,0)$ and $(x,y,1) \sim (x,y',1)$, and that the (unreduced) suspension ΣZ is defined by $Z \times [0,1]/\sim$, where $\sim$ is given by $(z,0) \sim (z',0)$ and $(z,1) \sim (z',1)$. Given $f\colon X \times Y \to Z$, let $H(f)\colon X*Y \to \Sigma Z$ be the map induced by $f \times \mathrm{id}\colon X \times Y \times [0,1] \to Z \times [0,1]$. Consider the following composition

$$\begin{aligned}[S^n, SO(k)] \to [S^n, \mathrm{aut}(S^{k-1})] \to [S^n \times S^{k-1}, S^{k-1}]\\ \xrightarrow{H} [S^n * S^{k-1}, \Sigma S^{k-1}] = [S^{n+k}, S^k].\end{aligned}$$

Notice that $\pi_1(SO(k))$ acts trivially on $\pi_n(SO(k))$ and $\pi_1(S^k)$ acts trivially on $\pi_{n+k}(S^k)$ for $k, n \geq 1$. Hence no base point questions arise in the next definition.

Definition 6.23 *The composition above induces for $n, k \geq 1$ homomorphisms of abelian groups*

$$J_{n,k}\colon \pi_n(SO(k)) \to \pi_{n+k}(S^k).$$

Taking the colimit for $k \to \infty$ induces the so called J-homomorphism

$$J_n\colon \pi_n(SO) \to \pi_n^s.$$

One easily checks

Lemma 6.24 *The composition of the homomorphism $\overline{J}\colon \pi_n(SO) \to \Omega_n^{\mathrm{fr}}$ of* (6.15) *with the isomorphism $\tau\colon \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s$ of* (6.21) *is the J-homomorphism $J_n\colon \pi_n(SO) \to \pi_n^s$ of Definition 6.23.*

The homotopy groups of O are 8-periodic and given by

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Notice that $\pi_i(SO) = \pi_i(O)$ for $i \geq 1$ and $\pi_0(SO) = 1$. The first stable stems are given by

n	0	1	2	3	4	5	6	7	8	9
π_n^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

The *Bernoulli numbers* B_n for $n \geq 1$ are defined by

$$\frac{z}{e^z - 1} \;=\; 1 - \frac{z}{2} + \sum_{n \geq 1} \frac{(-1)^{n+1} \cdot B_n}{(2n)!} \cdot (z)^{2n}. \tag{6.25}$$

The first values are given by

n	1	2	3	4	5	6	7	8
B_n	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

The next result is a deep theorem due to Adams [2, Theorem 1.1, Theorem 1.3 and Theorem 1.5].

Theorem 6.26 *1. If $n \neq 3 \mod 4$, then the J-homomorphism $J_n\colon \pi_n(SO) \to \pi_n^s$ is injective;*

2. The order of the image of the J-homomorphism $J_{4k-1}\colon \pi_{4k-1}(SO) \to \pi_{4k-1}^s$ is denominator$(B_k/4k)$, *where B_k is the k-th Bernoulli number.*

6.4 Computation of bP^{n+1}

In this section we want to compute the subgroups $bP^{n+1} \subset \Theta^n$ (see Definition 6.5).

We have introduced the bijection $\beta\colon \mathcal{N}_n(S^n) \xrightarrow{\cong} \Omega_n^{\mathrm{alm}}$ in Lemma 6.9 and the map $\partial\colon \Omega_n^{alm} \to \pi_{n-1}(SO)$ in (6.14). Let

$$\delta(k)\colon \pi_n(BSO(k)) \;\xrightarrow{\cong}\; \pi_{n-1}(SO(k)) \tag{6.27}$$

be the boundary map in the long exact homotopy sequence associated to the fibration $SO(k) \to ESO(k) \to BSO(k)$. It is an isomorphism since $ESO(k)$ is contractible. It can be described as follows. Consider $x \in \pi_n(BSO(k))$. Choose a representative $f\colon S^n \to BSO(k)$ for some k. If $\gamma_k \to BSO(k)$ is the universal k-dimensional oriented vector bundle, $f^*\gamma_k$ is a k-dimensional oriented vector bundle over S^n. Let S^n_- be the lower and S^n_+ be the upper hemisphere and $S^{n-1} = S^n_- \cap S^n_+$. Since the hemispheres are contractible, we obtain up to isotopy unique strong bundle isomorphisms $\overline{u}_-\colon f^*\gamma_k|_{S^n_-} \xrightarrow{\cong} \underline{\mathbb{R}^k}$ and $\overline{u}_+\colon f^*\gamma_k|_{S^n_+} \xrightarrow{\cong} \underline{\mathbb{R}^k}$. The composition of the inverse of the restriction of $\overline{u}_-$ to S^{n-1} with the restriction of $\overline{u}_+$ to S^{n-1} is a bundle automorphism of the trivial bundle $\underline{\mathbb{R}^k}$ over S^{n-1} which is the same as map

$S^{n-1} \to SO(k)$. Define its class in $\pi_{n-1}(SO(k))$ to be the image of x under $\delta(k)^{-1}\colon \pi_n(BSO(k)) \to \pi_{n-1}(SO(k))$. Analogously we get an isomorphism

$$\delta\colon \pi_n(BSO) \xrightarrow{\cong} \pi_{n-1}(SO). \tag{6.28}$$

Define a map

$$\gamma\colon \mathcal{N}_n(S^n) \to \pi_n(BSO) \tag{6.29}$$

by sending the class of the normal map of degree one $(\overline{f}, f)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f\colon M \to S^n$ to the the class represented by the classifying map $f_\xi\colon S^n \to BSO(n+k)$ of ξ. One easily checks

Lemma 6.30 *The following diagram commutes*

$$\begin{tikzcd} \Omega_n^{alm} \arrow[r, "\partial"] \arrow[d, "\beta^{-1}"'] & \pi_{n-1}(SO) \arrow[d, "\delta^{-1}"] \\ \mathcal{N}_n(S^n) \arrow[r, "\gamma"'] & \pi_n(BSO) \end{tikzcd}$$

The next ingredient is the Hirzebruch signature formula. It says for a closed oriented manifold M of dimension $n = 4k$ that its signature can be computed in terms of the L-class $\mathcal{L}(M)$ by

$$\operatorname{sign}(M) = \langle \mathcal{L}(M), [M] \rangle. \tag{6.31}$$

The L-class is a cohomology class which is obtained from inserting the Pontrjagin classes $p_i(TM)$ into a certain polynomial $L(x_1, x_2, \ldots, x_k)$. The L-polynomial $L(x_1, x_2, \ldots, x_k)$ is the sum of $s_k \cdot x_k$ and terms which do not involve x_k, where s_k is given in terms of the Bernoulli numbers B_k by

$$s_k := \frac{2^{2k} \cdot (2^{2k-1} - 1) \cdot B_k}{(2k)!}. \tag{6.32}$$

Assume that M is almost stably parallelizable. Then for some point $x \in M$ the restriction of the tangent bundle TM to $M - \{x\}$ is stably trivial and hence has trivial Pontrjagin classes. Since the inclusion induces an isomorphism $H^p(M) \xrightarrow{\cong} H^p(M - \{x\})$ for $p \leq n - 2$, we get $p_i(M) = 0$ for $i \leq k - 1$. Hence (6.31) implies for a closed oriented almost stably parallelizable manifold M of dimension $4k$

$$\operatorname{sign}(M) = s_k \cdot \langle p_k(TM), [M] \rangle. \tag{6.33}$$

We omit the proof of the next lemma which is based on certain homotopy theoretical computations (see for instance [68, Theorem 3.8 on page 76]).

Lemma 6.34 *Let $n = 4k$. Then there is an isomorphism*

$$\phi\colon \pi_{n-1}(SO) \xrightarrow{\cong} \mathbb{Z}.$$

Define a map

$$p_k\colon \pi_n(BSO) \to \mathbb{Z}$$

by sending the element $x \in \pi_n(BSO)$ represented by a map $f: S^n \to BSO(m)$ to $\langle p_k(f^*\gamma_m), [S^n]\rangle$ for $\gamma_m \to BSO(m)$ the universal bundle. Let $\delta\colon \pi_n(BSO) \to \pi_{n-1}(SO)$ be the isomorphism of (6.27). Put

$$t_k \quad := \tfrac{3-(-1)^k}{2} \cdot (2k-1)! \tag{6.35}$$

Then

$$t_k \cdot \phi = p_k \circ \delta^{-1}.$$

Lemma 6.36 *The following diagram commutes for $n = 4k$*

$$\begin{array}{ccc} \Omega_n^{\mathrm{alm}} & \xrightarrow{\frac{\mathrm{sign}}{8}} & \mathbb{Z} \\ \partial \big\downarrow & & \big\uparrow \frac{s_k \cdot t_k}{8}\cdot \mathrm{id} \\ \pi_{n-1}(SO) & \xrightarrow[\cong]{\phi} & \mathbb{Z} \end{array}$$

where $\frac{\mathrm{sign}}{8}$ is the homomorphism appearing in Corollary 6.12, the homomorphism ∂ has been defined in in (6.14) and the isomorphism ϕ is taken from (6.34).

Proof : Consider an almost stably parallelizable manifold M of dimension $n = 4k$. We conclude from Lemma 6.30 and Lemma 6.34

$$\begin{aligned} \frac{s_k \cdot t_k}{8} \cdot \phi \circ \partial([M]) &= \frac{s_k}{8} \cdot p_k \circ \delta^{-1} \circ \partial([M]) \\ &= \frac{s_k}{8} \cdot p_k \circ \gamma \circ \beta^{-1}([M]). \end{aligned} \tag{6.37}$$

By definition the composition

$$\Omega_n^{\mathrm{alm}} \xrightarrow{\beta^{-1}} \mathcal{N}_n(S^n) \xrightarrow{\gamma} \pi_n(SO) \xrightarrow{p_k} \mathbb{Z}$$

sends the class of M to $\langle p_k(\xi), [S^n]\rangle$ for a bundle ξ over S^n for which there exists a bundle map $(\overline{c}, c)\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $c\colon M \to S^n$ of degree one. This implies

$$\begin{aligned} \frac{s_k}{8} \cdot p_k \circ \gamma \circ \beta^{-1}([M]) &= \frac{s_k}{8} \cdot \langle p_k(\xi), [S^n]\rangle \\ &= \frac{s_k}{8} \cdot \langle p_k(\xi), c_*([M])\rangle \\ &= \frac{s_k}{8} \cdot \langle c^*(p_k(\xi)), [M]\rangle. \\ &= \frac{s_k}{8} \cdot \langle p_k(TM), [M]\rangle \end{aligned} \tag{6.38}$$

Now the claim follows from (6.33), (6.37) and (6.38). ■

Theorem 6.39 *Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\begin{aligned} &\frac{s_k \cdot t_k}{8} \cdot \left|\operatorname{im}\left(J_{4k-1}\colon \pi_{4k-1}(SO) \to \pi_{4k-1}^s\right)\right| \\ &= \frac{1}{8} \cdot \frac{2^{2k} \cdot (2^{2k-1}-1) \cdot B_k}{(2k)!} \cdot \frac{3-(-1)^k}{2} \cdot (2k-1)! \cdot \operatorname{denominator}(B_k/4k) \\ &= \frac{3-(-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1}-1) \cdot \operatorname{numerator}(B_k/(4k)). \end{aligned}$$

Proof : This follows from Theorem 6.11, Theorem 6.26 (2) and Lemma 6.36. ■

Next we treat the case $n = 4k+2$ for $k \geq 1$. Let

$$\operatorname{Arf}\colon \pi_{4k+2}^s \to \mathbb{Z}/2 \tag{6.40}$$

be the composition of the inverse of the Pontrjagin-Thom isomorphism $\tau\colon \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s$ of (6.21), the forgetful homomorphism $f\colon \Omega_{4k+2}^{\mathrm{fr}} \to \Omega_{4k+2}^{\mathrm{alm}}$ of (6.13) and the map $\operatorname{Arf}\colon \Omega_{4k+2}^{\mathrm{alm}} \to \mathbb{Z}/2$ appearing in Corollary 6.12.

Theorem 6.41 *Let $k \geq 3$. Then bP^{4k+2} is a trivial group if the homomorphism $\operatorname{Arf}\colon \pi_{4k+2}^s \to \mathbb{Z}/2$ of (6.40) is surjective and is $\mathbb{Z}/2$ if the homomorphism $\operatorname{Arf}\colon \pi_{4k+2}^s \to \mathbb{Z}/2$ of (6.40) is trivial.*

Proof : We conclude from Lemma 6.16, Lemma 6.24 and and Theorem 6.26 (1) that the forgetful map $f\colon \Omega_{4k+2}^{\mathrm{fr}} \to \Omega_{4k+2}^{\mathrm{alm}}$ is surjective. Now the claim follows from Corollary 6.12. ■

The next result is due to Browder [16].

Theorem 6.42 *The homomorphism* $\mathrm{Arf}\colon \pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (6.40) *is trivial if* $2k+1 \neq 2^l - 1$.

The homomorphism $\mathrm{Arf}\colon \pi^s_{4k+2} \to \mathbb{Z}/2$ of (6.40) is also known to be non-trivial for $4k+2 \in \{6, 14, 30, 62\}$ (combine [16], [6] and [79]). Hence Theorem 6.41 and Theorem 6.42 imply

Corollary 6.43 *The group* bP^{4k+2} *is trivial or isomorphic to* $\mathbb{Z}/2$. *We have*

$$bP^{4k+2} \quad = \quad \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l - 2, k \geq 1; \\ 0 & 4k+2 \in \{6, 14, 30, 62\}. \end{cases}$$

We conclude from Corollary 6.12

Theorem 6.44 *We have for* $k \geq 3$

$$bP^{2k+1} = 0.$$

Theorem 6.45 *For* $n \geq 1$ *any homotopy* n*-sphere* Σ *is stably parallelizable.*

For an almost parallelizable manifold M the image of its class $[M] \in \Omega_n^{\mathrm{alm}}$ under the homomorphism $\partial\colon \Omega_n^{\mathrm{alm}} \to \pi_n(SO(n-1))$ is exactly the obstruction to extend the almost stable framing to a stable framing. Recall that any homotopy n-sphere is almost stably parallelizable. The map ∂ is trivial for $n \neq 0 \mod 4$ by Lemma 6.16, Lemma 6.24, Theorem 6.26 (1). If $n = 0 \mod 4$, the claim follows from Lemma 6.16, Lemma 6.24 and and Lemma 6.36 since the signature of a homotopy n-sphere is trivial. ■

6.5 Computation of Θ^n/bP^{n+1}

In this section we compute Θ^n/bP^{n+1}.

Theorem 6.46 *1. If* $n = 4k+2$, *then there is an exact sequence*

$$0 \to \Theta^n/bP^{n+1} \to \operatorname{coker}\left(J_n\colon \pi_n(SO) \to \pi_n^s\right) \to \mathbb{Z}/2;$$

2. If $n \neq 2 \mod 4$ *or if* $n = 4k+2$ *with* $2k+1 \neq 2^l - 1$, *then*

$$\Theta^n/bP^{n+1} \ \cong \ \operatorname{coker}\left(J_n\colon \pi_n(SO) \to \pi_n^s\right).$$

Proof : Lemma 6.16, Lemma 6.24, Theorem 6.26 (1) and Lemma 6.36 imply

$$\begin{array}{lcll} \ker\left(\partial\colon \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \Omega_n^{\mathrm{alm}} & n \neq 0 \mod 4; \\ \ker\left(\partial\colon \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \ker\left(\frac{\mathrm{sign}}{8}\colon \Omega_n^{\mathrm{alm}} \to \mathbb{Z}\right) & n = 0 \mod 4; \\ \ker\left(\partial\colon \Omega_n^{\mathrm{alm}} \to \pi_{n-1}(SO)\right) & = & \operatorname{coker}\left(J_n\colon \pi_n(SO) \to \pi_n^{\mathrm{fr}}\right). & \end{array}$$

Now the claim follows from Corollary 6.12 and Theorem 6.42. ■

6.6 The Kervaire-Milnor Braid

We have established in Lemma 6.16 the long exact sequence

$$\ldots \xrightarrow{\partial} \pi_n(SO) \xrightarrow{\overline{J}} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{\mathrm{alm}} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\overline{J}} \Omega_{n-1}^{fr} \xrightarrow{f} \ldots .$$

The long exact sequence

$$\ldots \to L_{n+1}(\mathbb{Z}) \to \Theta^n \to \Omega_n^{\mathrm{alm}} \to L_n(\mathbb{Z}) \to \Theta^{n-1} \to \ldots$$

is taken from Theorem 6.11. Denote by Θ^n_{fr} the abelian group of stably framed h-cobordism classes of stably framed homotopy n-spheres. There is a long exact sequence

$$\ldots \to L_{n+1}(\mathbb{Z}) \to \Theta^n_{\mathrm{fr}} \to \Omega_n^{\mathrm{fr}} \to L_n(\mathbb{Z}) \to \Theta^{n-1}_{\mathrm{fr}} \to \ldots$$

which is defined as follows. The map $\Theta^n_{\mathrm{fr}} \to \Omega_n^{\mathrm{fr}}$ assign to the class of a framed homotopy sphere its class in Ω_n^{fr}. The map $\Omega_n^{\mathrm{fr}} \to L_n(\mathbb{Z})$ assigns to a class $[M]$ in Ω_n^{fr} the surgery obstruction of the normal map $(\overline{c}, c)\colon TM \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}}$ for any map of degree one $c\colon M \to S^n$, where $\overline{c}$ is given by the framing on M. The map $L_{n+1}(\mathbb{Z}) \to \Theta^n_{\mathrm{fr}}$ assigns to $x \in L_{n+1}(\mathbb{Z})$ the class of the framed homotopy n-sphere $(\Sigma, \overline{v}\colon T\Sigma \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{a+n}})$, for which there is a normal map of degree one $(\overline{U}, U)\colon TW \oplus \underline{\mathbb{R}^{a+b}} \to \underline{\mathbb{R}^{n+a+b}}$ covering a map of triads $U\colon (W; \partial_0 W, \partial_1 W) \to (S^n \times [0,1]; S^n \times \{0\}, S^n \times \{1\})$ and a bundle map $(\overline{u_0}, u_0)\colon T\Sigma \oplus \underline{\mathbb{R}^{a+b+1}} \to TW_0 \oplus \underline{\mathbb{R}^{a+b}}$ covering the orientation preserving diffeomorphism $u_0\colon \Sigma \to \partial_0 W$ such that $\overline{U} \circ \overline{u_0} = \overline{v} \oplus \mathrm{id}_{\underline{\mathbb{R}^{b+1}}}$, U induces a diffeomorphism $\partial_1 W \to S^n$ and the surgery obstruction associated to $(\overline{U}, U)$ is the given element $x \in L_{n+1}(\mathbb{Z})$.

Theorem 6.47 (The Kervaire-Milnor Braid) *The long exact sequences above fit together to an exact braid for $n \geq 5$*

$$\begin{tikzcd}
\pi_n(SO) \arrow[dr] \arrow[rr, bend left] & & \Omega_n^{\mathrm{fr}} \arrow[dr] \arrow[rr, bend left] & & L_n(\mathbb{Z}) \arrow[dr] \arrow[rr, bend left] & & \Theta^{n-1} \\
& \Theta^n_{\mathrm{fr}} \arrow[ur] \arrow[dr] & & \Omega_n^{\mathrm{alm}} \arrow[ur] \arrow[dr] & & \Theta^{n-1}_{\mathrm{fr}} \arrow[ur] \arrow[dr] & \\
L_{n+1}(\mathbb{Z}) \arrow[ur] \arrow[rr, bend right] & & \Theta^n \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(SO) \arrow[ur] \arrow[rr, bend right] & & \Omega_{n-1}^{\mathrm{fr}}
\end{tikzcd}$$

We have already introduced BO and BG and have defined G/O as the homotopy fiber of this map. There is also a PL-version of BO called BPL . We can define analogously spaces G/PL and PL/O and fibrations $G/PL \to BPL \to BG$ and $PL/O \to BPL \to BO$. Since $\Omega BO \simeq O$, $\Omega BPL \simeq BPL$ and $\Omega BG \simeq G$ holds, we get fibrations $O \to G \to G/O$, $PL \to G \to G/PL$ and $O \to PL \to PL/O$. Notice that for an inclusion of topological groups $H \subset K$ there is an obvious fibration $H \to K \to K/H$ and the fibrations above are in this spirit. But we have to use the classifying spaces since for instance G is not a group and we cannot talk about the homogeneous space G/PL. More information about these spaces and their homotopy theoretic properties can be found for instance in [78].

Theorem 6.48 (Homotopy Theoretic Interpretation of the Kervaire-Milnor Braid) *The long exact homotopy sequences of these three fibrations above yield an exact braid*

$$\begin{tikzcd}
\pi_n(O) \arrow[dr] \arrow[rr, bend left] & & \pi_n(G) \arrow[dr] \arrow[rr, bend left] & & \pi_n(G/PL) \arrow[dr] \arrow[rr, bend left] & & \pi_{n-1}(PL/O) \\
& \pi_n(PL) \arrow[ur] \arrow[dr] & & \pi_n(G/O) \arrow[ur] \arrow[dr] & & \pi_{n-1}(PL) \arrow[ur] \arrow[dr] & \\
\pi_{n+1}(G/PL) \arrow[ur] \arrow[rr, bend right] & & \pi_n(PL/O) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(O) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(G)
\end{tikzcd}$$

which is for $n \geq 5$ isomorphic to the Kervaire-Milnor braid of Theorem 6.47

Proof : At least we explain how the two braids are related by isomorphisms. Since O/SO is the discrete group $\{\pm 1\}$, the inclusion induces an isomorphism

$$\pi_n(SO) \xrightarrow{\cong} \pi_n(O) \qquad \text{for } n \geq 1.$$

The Pontrjagin-Thom isomorphism $\tau\colon \Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n^s$ of (6.21) and the canonical isomorphism $\pi_n(G) \cong \pi_n^s$ yield an isomorphism

$$\Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n(G).$$

The isomorphism $\beta\colon \mathcal{N}_n(S^n) \xrightarrow{\cong} \Omega_n^{\text{alm}}$ of Lemma 6.9 and the bijection $\pi_n(G/O) \xrightarrow{\cong} \mathcal{N}_n(S^n)$ (see Theorem 3.52) induce a bijection

$$\Omega_n^{\text{alm}} \quad \xrightarrow{\cong} \quad \pi_n(G/O).$$

One can also develop surgery theory in the PL-category instead of the smooth category. We have the surgery exact sequence in the PL-category (see Theorem 5.15.

$$\ldots \to \pi_{n+1}(G/PL) \to L_{n+1}(\mathbb{Z}) \to \mathcal{S}_n^{PL}(S^{n+1}) \to \pi_n(G/PL) \to L_n(\mathbb{Z}) \to \ldots$$

Since the Poincare Conjecture is true for $n \geq 5$ and any PL-homeomorphism $f\colon S^n \to S^n$ can be extended to a PL-homeomorphism $F\colon D^{n+1} \to D^{n+1}$ by coning off (what is not possible in the smooth category), any homotopy n-sphere is PL-homeomorphic to S^n for $n \geq 5$. Hence $\mathcal{S}_n^{PL}(S^n) = \{*\}$ for $n \geq 5$. This is the main ingredient in the proof for $n \geq 5$, the low dimensional cases follow from direct computations, that we obtain isomorphisms of abelian groups

$$\pi_n(G/PL) \xrightarrow{\cong} L_n(\mathbb{Z}) \qquad n \geq 1$$

There is an isomorphism

$$\Theta^n \xrightarrow{\cong} \pi_n(PL/O) \qquad n \geq 0, n \neq 3$$

which is defined as follows. Let Σ be a homotopy n-sphere. As explained above, there is an orientation preserving PL-homeomorphism $h\colon \Sigma \to S^n$. Fix a classifying map $f_{S^n}\colon S^n \to BPL$ for the PL-tangent bundle of S^n. We obtain a classifying map $f_\Sigma\colon \Sigma \to BO$ of the smooth tangent bundle of Σ together with a homotopy $h\colon Bi \circ f_\Sigma \simeq f_{S^n}$, where $Bi\colon BO \to BPL$ is the canonical map. The pair (f_Σ, h) yields a map $S^n \to PL/O$, since PL/O is the homotopy fiber of $Bi\colon BO \to BPL$. The bijectivity of this map for $n \geq 5$ follows from the five lemma and the comparision of the surgery exact sequence with the long homotopy sequence associated to the fibration $PL/O \to G/O \to G/PL$. There is an isomorphism

$$\Theta_{\text{fr}}^n \xrightarrow{\cong} \pi_n(PL) \qquad n \geq 5$$

which is defined as follows. As above we get a pair (f_Σ, h). The framing yields also a homotopy $g\colon f_\Sigma \simeq c$ for c the constant map. Since PL can be viewed as the homotopy fiber of the obvious map $PL/O \to BO$, these data yield a map $S^n \to PL$. ■

6.7 Miscellaneous

Remark 6.49 We have shown in Theorem 6.39 that bP^{4k} for $k \geq 2$ is a finite cyclic group. If we go through the construction again, an explicit generator can be constructed as follows. Recall that $\frac{\text{sign}}{8}\colon L_{4k}(\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism. Let W be any oriented stably parallelizable manifold of dimension $4k$ whose boundary is a homotopy sphere and whose intersection pairing on $H_{2k}(W)$ has signature 8. Then $\partial_1 W$ is an exotic sphere representing a generator of bP^{4k}. Such manifolds W can be explicitly constructed by a plumbing construction (see for instance [17, Theorem V.2.9 on page 122]).

Example 6.50 The first example of an exotic sphere, i.e. a closed manifold which is homeomorphic but not diffeomorphic to S^n, was constructed by Milnor [81]. See also [87]. The construction and the detection that it is an exotic sphere is summarized below.

There is an isomorphism

$$\begin{array}{rcl} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\cong} & \pi_3(SO(4)) \\ (h,j) & \mapsto & \omega(h,j)\colon S^3 \to SO(4) \quad x \mapsto \left(y \mapsto x^h \cdot y \cdot x^j\right) \end{array}$$

where we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ by $(a,b,c,d) \mapsto a + bi + cj + dk$ and $x^h \cdot y \cdot x^j$ is to be understood with respect to the multiplication in $\mathbb{H}$. Since $\pi_3(SO(4)) \cong \pi_4(BSO(4))$, each pair $(h,j) \in \mathbb{Z} \oplus \mathbb{Z}$ determines an oriented vector bundle $E(h,j)$ with Riemannian metric over S^4 unique up to orientation and Riemannian metric preserving isomorphism. The Euler number and the first Pontrjagin class of $E(j,h)$ are given by

$$\begin{array}{rcl} \chi(E(h,j)) & = & h + j; \\ \langle p_1(E(h,j)), [S^4] \rangle & = & 2(h-j). \end{array}$$

The Gysin sequence shows that the sphere bundle $SE(h,j)$ is a homotopy 7-sphere if and only if $\chi(E(h,j)) = h + j = 1$.

Let k be any odd integer. Let $W(k)$ be the disk bundle $D(E((1+k)/2,(1-k)/2))$ and $\Sigma(k)$ be $\partial W(k) = S(E((1+k)/2,(1-k)/2))$. Then $\Sigma(k)$ is a homotopy 7-sphere. Next we recall Milnor's argument why $\Sigma(k)$ cannot be diffeomorphic to S^7.

The obvious embedding $i\colon S^4 \to W(k)$ given by the zero section is a homotopy equivalence and $i^*TW(k)$ is isomorphic to $TS^4 \oplus E((1+k)/2,(1-$

$k)/2)$. Hence
$$\langle i^* p_1(TW(k)), [S^4]\rangle = 2k.$$
Suppose that there exists a diffeomorphism $\Sigma(k) \to S^7$. Then we can form the closed oriented smooth 8-dimensional manifold $M(k) = W(k) \cup_f D^8$. Let $j\colon W(k) \to M(k)$ be the inclusion. Since the inclusion $j \circ i\colon S^4 \to M(k)$ induces an isomorphism on H_4, the signature of $M(k)$ is one. The Hirzebruch signature Theorem says $1 = \operatorname{sign}(M) = \langle \mathcal{L}(M), [M]\rangle$. Since
$$\mathcal{L}(M) = \frac{7}{45} \cdot p_2(TM) - \frac{1}{45} \cdot p_1(TM)^2$$
we conclude
$$1 = \langle \frac{7}{45} \cdot p_2(TM) - \frac{1}{45} p_1(TM)^2, [M]\rangle = \frac{7}{45}\langle p_2(TM), [M]\rangle - \frac{1}{45} \cdot \langle p_1(TM)^2, [M]\rangle$$
$$= \frac{7}{45}\langle p_2(TM), [M]\rangle - \frac{1}{45} \cdot \langle i^* j^* p_1(TM), [S^4]\rangle^2 = \frac{7}{45}\langle p_2(TM), [M]\rangle - \frac{4k^2}{45}.$$
Since $\langle p_2(TM), [M]\rangle$ is an integer, we conclude $k^2 = 1 \mod 7$. Hence $\Sigma(k)$ is an exotic homotopy 7-sphere if $k^2 \neq 1 \mod 7$.

Remark 6.51 Milnor's example 6.50 above fits into the general context as follows. Recall that $bP^8 = \mathbb{Z}/28$ and that we have an isomorphism
$$\frac{\operatorname{sign}}{8}\colon \Theta^7 \to \mathbb{Z}/28$$
which sends $[\Sigma]$ to $\operatorname{sign}(W)/8$ for any stably parallelizable manifold W whose boundary is oriented diffeomorphic to Σ. If $\Sigma(k)$ is the oriented homotopy 8-sphere of Example 6.50, then the isomorphism above sends $[\Sigma(k)]$ to $(1 - k^2) \in \mathbb{Z}/28$.

Example 6.52 Let $W^{2n-1}(d)$ be the subset of $\mathbb{C}^{n+1}$ consisting of those points $(z_0, z_1, \ldots, z_n)$ which satisfy the equations $z_0^d + z_1^2 + \ldots + z_n^2 = 0$ and $||z_0||^2 + ||z_1||^2 + \ldots + ||z_n||^2 = 1$. These turns out to be smooth submanifolds and are called *Brieskorn varieties* (see [13], [55]). Suppose that d and n are odd, Then $W^{2n-1}(d)$ is a homotopy $(2n-1)$-sphere. It is diffeomorphic to the standard sphere S^{2n-1} if $d = \pm 1 \mod 8$ and it is an exotic sphere representing the generator of bP^{2n} if $d = \pm 3 \mod 8$ [13, page 11]. In general one can study the intersection $K = f^{-1}(0) \cap \{z \in \mathbb{C}^{n+1} \mid ||z|| = \epsilon\}$ for a polynomial $f(z_0, z_1, , \ldots, z_n)$ with an isolated singularity at the origin and examine when K is a homotopy sphere and when K is an exotic sphere [85, §8, §9].

Remark 6.53 Let Σ be a homotopy n-sphere for $n \geq 5$. Let $D_0^n \to \Sigma$ and $D_1^n \to \Sigma$ be two disjoint embedded discs. Then $W = \Sigma - (\operatorname{int}(D_0^n) \coprod \operatorname{int}(D_1^n))$ is a simply-connected h-cobordism. By the h-Cobordism Theorem 1.2 there is a diffeomorphism $(F, \mathrm{id}, f)\colon \partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$. Hence Σ is oriented diffeomorphic to $D^n \cup_{f\colon S^{n-1} \to S^{n-1}} (D^n)^-$ for some orientation preserving diffeomorphism $f\colon S^{n-1} \to S^{n-1}$. If f is isotopic to the identity, Σ is oriented diffeomorphic to S^n. Hence the existence of exotic spheres shows the existence of selfdiffeomorphisms of S^{n-1} which are homotopic but not isotopic to the identity.

The next result is due to Berger and Klingenberg. Its proof and the proof of the following theorem can be found for instance in [27, Theorem 6.1 on page 106, Theorem 7.16 on page 126].

Theorem 6.54 (Sphere Theorem) *Let M be a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \frac{1}{4}$. Then M is homeomorphic to the standard sphere.*

Theorem 6.55 (Differentiable Sphere Theorem) *There exists a constant δ with $1 > \delta \geq \frac{1}{4}$ with the following property: if M is a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \delta$. then M is diffeomorphic to the standard sphere.*

Brumfield and Frank [19] have shown

Theorem 6.56 *For $n \neq 2^k - 2$ or $n \neq 2^k - 3$ the sequence*

$$0 \to bP^{n+1} \to \Theta^n \to \operatorname{im}\left((\Theta^n \to \operatorname{coker}(J_n)\right) \to 0$$

splits where the map $\Theta^n \to \operatorname{im}\left(\Theta^n \to \operatorname{coker}(J_n)\right)$ comes from the map $\Theta^n \to \operatorname{coker}(J_n)$ appearing in Theorem 6.46.

Fake real projective spaces, lens spaces and and fake tori have been investigated by Wall [119, 14D, 14E, 15A]. These are other examples of manifolds which look close to standard models but are not equal to them.

Chapter 7

The Farrell-Jones and Baum-Connes Conjectures

In this chapter we explain the Farrell-Jones Isomorphism Conjecture for the algebraic K- and L-theory of group rings and the Baum-Connes Conjecture for the topological K-theory of reduced group C^*-algebras. These conjectures say that certain assembly maps are isomorphisms. The targets of the assembly maps are the groups we are interested in, namely, $K_n(RG)$, $L_n(RG)$ or $K_n^{\mathrm{top}}(C_*^r(G))$.

The sources of the assembly maps are certain G-homology theories evaluated at classifying spaces of G with respect to the family of finite subgroups $\mathcal{FIN}$ or virtually cyclic subgroups $\mathcal{VC}$. The construction of the sources and of the assembly maps is presented in Section 7.1.

In Section 7.2 we give a unified formulation of these conjectures. We illustrate their importance by explaining that they imply other well-known conjectures like the Novikov Conjecture about the homotopy invariance of higher signatures, the Borel Conjecture about the topological rigidity of closed aspherical topological manifolds and the stable Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive sectional curvature.

In Section 7.3 we give an abstract characterization of assembly maps as the best approximation of a homotopy invariant functor from the category of G-CW-complexes to the category of spectra from the left by an excisive functor, i.e. a functor which yields a G-homology theory after taking homotopy groups. This is useful for identifying assembly maps.

In Section 7.4 we discuss why the family $\mathcal{FIN}$ or $\mathcal{VC}$ appear in the

Isomorphism Conjectures and under which circumstances one can replace $\mathcal{VC}$ by $\mathcal{FIN}$. This is important for explicit calculations which are much easier in the context with $\mathcal{FIN}$ as explained in Chapter 8.

7.1 G-Homology Theories

Let $\mathcal{C}$ be a small category. Our main example will be

Definition 7.1 *The* orbit category $\operatorname{Or}(G)$ *of a group G has as objects homogeneous G-spaces G/H and as morphisms G-maps.*

We will always work in the category SPACES of compactly generated spaces (see [111] and [121, I.4]). Denote by SPACES_+ the category of pointed spaces. We require that the inclusion of the base point is a cofibration.

We define the category SPECTRA of spectra as follows. A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$. A *map* of spectra (sometimes also called function in the literature) $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps of pointed spaces $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \operatorname{id}_{S^1})$ for all $n \in \mathbb{Z}$. This should not be confused with the notion of map of spectra in the stable category (see [3, III.2.]). Recall that the homotopy groups of a spectrum are defined by

$$\pi_p(\mathbf{E}) \quad := \quad \operatorname{colim}_{k \to \infty} \pi_{p+k}(E(k)), \tag{7.2}$$

where the system $\pi_{p+k}(E(k))$ is given by the composition

$$\pi_{p+k}(E(k)) \xrightarrow{S} \pi_{p+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{p+k+1}(E(k+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map. A *weak homotopy equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups. A spectrum $\mathbf{E}$ is called Ω-*spectrum* if for each structure map its adjoint

$$E(n) \to \Omega E(n+1) = \operatorname{map}(S^1, E(n+1))$$

is a weak homotopy equivalence of spaces. We denote by Ω-SPECTRA the corresponding full subcategory of SPECTRA.

Definition 7.3 A *covariant $\mathcal{C}$-space* is a covariant functor from $\mathcal{C}$ to the category of topological spaces. Morphisms are natural transformations. Define analogously *covariant pointed space*, *covariant spectrum*, *covariant Ω-spectrum* and the contravariant notions.

Notice that the notions of coproduct, product, pushout, pullback, colimit, and limit exist in the category of $\mathcal{C}$-spaces. They are constructed by applying these notions in the category SPACES objectwise. We mention that the terms direct limit and inverse limit are sometimes used in the literature instead of colimit and limit.

Example 7.4 Let G be a group. Let $\mathcal{G}$ be the category with a single object whose morphism set is G. Composition is given by the group multiplication. A covariant (contravariant) $\mathcal{G}$-space is the same as a left (right) G-space. Maps of $\mathcal{G}$-spaces correspond to G-maps.

Example 7.5 Let $\mathbb{Z}/p$ be the cyclic group of order p for a prime number p. A contravariant $\mathrm{Or}(\mathbb{Z}/p)$-space Y is specified by a $\mathbb{Z}/p$-space $Y((\mathbb{Z}/p)/\{1\})$, a space $Y((\mathbb{Z}/p)/(\mathbb{Z}/p))$, and a map $Y((\mathbb{Z}/p)/(\mathbb{Z}/p)) \to Y((\mathbb{Z}/p)/\{1\})^{\mathbb{Z}/p}$.

Definition 7.6 *Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their* tensor product *to be the space*

$$X \otimes_{\mathcal{C}} Y \;=\; \coprod_{c \in \mathrm{ob}(\mathcal{C})} X(c) \times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi\colon c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. ■

Given $\mathcal{C}$-spaces X and Y, denote by $\hom_{\mathcal{C}}(X, Y)$ the space of maps of $\mathcal{C}$-spaces from X to Y with the subspace topology coming from the obvious inclusion into $\prod_{c \in \mathrm{ob}(\mathcal{C})} \mathrm{map}(X(c), Y(c))$.

The main property of the tensor product is that it is the left adjoint of $\hom_{\mathcal{C}}$ as explained in

Lemma 7.7 *Let X be a contravariant $\mathcal{C}$-space, Y be a covariant $\mathcal{C}$-space and Z be a space. Denote by* $\mathrm{map}(Y, Z)$ *the obvious contravariant $\mathcal{C}$-space* *whose value at an object c is the mapping space* $\operatorname{map}(Y(c), Z)$. *Then there is a homeomorphism natural in X, Y and Z*

$$T = T(X,Y,Z)\colon \operatorname{map}(X \otimes_{\mathcal{C}} Y, Z) \xrightarrow{\cong} \hom_{\mathcal{C}}(X, \operatorname{map}(Y,Z)).$$

Proof : We only indicate the definition of T. Given a map $g\colon X \otimes_{\mathcal{C}} Y \to Z$, we have to specify for each object c in $\mathcal{C}$ a map $T(g)(c)\colon X(c) \to \operatorname{map}(Y(c), Z)$. This is the same as specifying a map $X(c) \times Y(c) \to Z$ which is defined to be the composition of g with the obvious map from $X(c) \times Y(c)$ to $X \otimes_{\mathcal{C}} Y$. ∎

Definition 7.8 *Let Y be a (left) G-space. Define the* associated contravariant $\operatorname{Or}(G)$-space $\operatorname{map}_G(?, Y)$ *by*

$$\operatorname{Or}(G) \to \text{SPACES}, \qquad\qquad G/H \mapsto \operatorname{map}_G(G/H, Y) = Y^H.$$

A $\mathcal{C}$-spectrum $\mathbf{E}$ can also be thought of as a sequence $\{E(n) \mid n \in \mathbb{Z}\}$ of pointed $\mathcal{C}$-spaces and the structure maps as maps of pointed $\mathcal{C}$-spaces. With this interpretation it is obvious what the *tensor product spectrum* $X \otimes_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$-space and a covariant $\mathcal{C}$-spectrum means. The canonical associativity homeomorphism

$$(X \otimes_{\mathcal{C}} E(n)) \wedge S^1 \xrightarrow{\cong} X \otimes_{\mathcal{C}} (E(n) \wedge S^1)$$

is used in order to define the structure maps. It is given on representatives by sending $(x \otimes_{\mathcal{C}} e) \wedge z$ to $x \otimes_{\mathcal{C}} (e \wedge z)$.

Definition 7.9 *A G-CW-*complex *X is a G-space X together with a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

such that $X = \operatorname{colim}_{n\to\infty} X_n$ and for any $n \geq 0$ the n-skeleton *X_n is obtained from the $(n-1)$-skeleton X_{n-1} by attaching equivariant cells, i.e. there exists a pushout of G-spaces of the form*

$$\begin{array}{ccc}
\coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I_n} G/H_i \times D^n & \longrightarrow & X_n
\end{array}$$

A G-CW-subcomplex $A \subset X$ is a G-CW-complex such that its n-th skeleton A_n is the intersection $A \cap X_n$. A G-map $f\colon X \to Y$ of G-CW-complexes is called cellular if $f(X_n) \subset Y_n$. More information about G-CW-complexes can be found for instance in [115, II.1 and II.2] and [69, section 1 and 2].

Definition 7.10 *A G-*homology theory *$\mathcal{H}_*^G$ consists of a covariant functor from the category of pairs of G-CW-complexes to the category of $\mathbb{Z}$-graded abelian groups $(X, A) \mapsto (\mathcal{H}_p^G(X, A))_{p \in \mathbb{Z}}$ with the following properties.*

1. *Two G-homotopic maps induce the same homomorphisms;*

2. *There are long natural exact sequences associated to pairs of G-CW-complexes;*

3. *Consider a cellular G-pushout*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

i.e. a G-pushout such that $i_1\colon X_0 \to X_1$ is an inclusion of G-CW-complexes, the map $i_2\colon X_0 \to X_1$ is cellular and the G-CW-structure on X is induced by the ones on X_i for $i = 0, 1, 2$ in the obvious way. Then we get for all $p \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_p^G(j_1, i_2)\colon \mathcal{H}_p^G(X_1, X_0) \xrightarrow{\cong} \mathcal{H}_n^G(X, X_2).$$

It satisfies the disjoint union axiom *if for any collection $\{X_i \mid i \in I\}$ of G-CW-complexes the canonical map induced by the various inclusions is bijective for all $n \in \mathbb{Z}$*

$$\oplus_{i \in I} \mathcal{H}_p^G(X_i) \ \xrightarrow{\cong}\ \mathcal{H}_p^G(\coprod_{i \in I} X_i).$$

Example 7.11 Let $\mathcal{H}_*$ be a $\mathbb{Z}$-graded (non-equivariant) homology theory such as singular homology, cobordism or K-homology. Then we obtain two G-homology theories

$$\begin{array}{rcl} X & \mapsto & \mathcal{H}_*(G\backslash X); \\ X & \mapsto & \mathcal{H}_*(G\backslash(EG \times X)). \end{array}$$

The second one is called the associated *Borel homology* associated to $\mathcal{H}_*$.

Example 7.12 Equivariant cobordism defines a G-homology theory on the category of proper G-CW-complexes. A G-CW-complex is *proper* if and only if all its isotropy groups are finite. Namely, one assigns to a proper G-CW-complex the equivariant cobordism group $\Omega_p^G(X)$ of cocompact proper smooth G-manifolds M of dimension p with a G-map $M \to X$.

Definition 7.13 *Let* $\mathbf{E}$ *be a covariant* $\mathrm{Or}(G)$*-spectrum. Define for a pair of* G*-*CW*-complexes* (X, A) *its homology with coefficients in* $\mathbf{E}$

$$H_p^G(X, A; \mathbf{E}) \ := \ \pi_p((X \cup_A \mathrm{cone}(A)) \otimes_{\mathrm{Or}(G)} \mathbf{E}),$$

where $\mathrm{cone}(A)$ *is the cone over* A.

As usual $H_p^G(X;\mathbf{E})$ means $H_p(X, \emptyset; \mathbf{E})$. Notice that the cone over the empty set consists of one point so that $X \cup_A \mathrm{cone}(A)$ is X_+ for $A = \emptyset$. The proof of the next result is presented in [30, Lemma 4.4].

Theorem 7.14 $H_*^G(-, \mathbf{E})$ *is a* G*-homology theory for* G*-*CW*-complexes which satisfies the disjoint union axiom. We have canonical isomorphisms*

$$H_p^G(G/H; \mathbf{E}) = \pi_p(\mathbf{E}(G/H)).$$

Our main examples of $\mathrm{Or}(G)$-spectra are presented in the next theorem.

Theorem 7.15 *There are specific constructions of covariant* $\mathrm{Or}(G)$*-spectra*

$$\begin{aligned}
\mathbf{K}\colon \mathrm{Or}(G) &\to \Omega - \mathrm{SPECTRA};\\
\mathbf{L}\colon \mathrm{Or}(G) &\to \Omega - \mathrm{SPECTRA};\\
\mathbf{K}^{\mathrm{top}}\colon \mathrm{Or}(G) &\to \Omega - \mathrm{SPECTRA}
\end{aligned}$$

satisfying for all $p \in \mathbb{Z}$

$$\begin{aligned}
\pi_p(\mathbf{K}(G/H)) &\cong K_p(RH);\\
\pi_p(\mathbf{L}(G/H)) &\cong L_p^{\langle -\infty\rangle}(RH);\\
\pi_p(\mathbf{K}^{\mathrm{top}}(G/H)) &\cong K_p^{\mathrm{top}}(C_r^*(H)).
\end{aligned}$$

Here $K_p(RH)$ is the algebraic (non-connective) K-theory of the group ring RH with coefficients in the associative ring with unit R, $L_p^{\langle -\infty\rangle}(RH)$ is the algebraic L-theory with respect to the decoration $\langle -\infty\rangle$ of RH with respect to the involution sending $\sum_{h\in H}\lambda_h\cdot h$ to $\sum_{h\in H}\overline{\lambda_h}\cdot h^{-1}$ for fixed involution $R\to R,\ r\mapsto\overline{r}$ and $K_p^{\mathrm{top}}(C_r^*(H))$ is the topological K-theory of the reduced complex C_r^*-algebra $C_r^*(H)$ of H. An example for the involution on R is the identity for commutative R or $R=\mathbb{C}$ with complex conjugation.

All the constructions first send G/H to the groupoid $\mathcal{G}^G(G/H)$, where for any G-set S the groupoid $\mathcal{G}^G(S)$ has as objects the elements of S and as morphism from s_1 to s_2 the set of elements $\{g\in G\mid g\cdot s_1=s_2\}$. Composition is given by group multiplication. Then certain standard constructions of spectra for modules over the group ring RH are extended to the category of contravariant functors from $\mathcal{G}^G(G/H)$ to the category of R-modules. In the C^*-case one must use C^*-categories. Notice that one cannot assign directly $\mathbf{K}(RH)$, $\mathbf{L}(RH)$ or $\mathbf{K}^{\mathrm{top}}(C_r^*(H))$ to G/H since this conflicts functoriality. Namely, an inner automorphism of H acts trivially on the homotopy groups of the spectrum but it does not act trivially on the spectrum itself and for $h\in H$ the automorphism $G/H\to G/H,\ gH\mapsto ghH$ is the identity. More details of the construction can be found in [30, Section 2].

7.2 Isomorphisms Conjectures and Assembly Maps

In this section we formulate an abstract isomorphism conjecture and specialize it to the two most important examples, the Farrell-Jones Isomorphism Conjecture and the Baum-Connes Conjecture.

7.2.1 Classifying Spaces of Families

The next definition is due to tom Dieck [114], [115, I.6].

Definition 7.16 *Let G be a group. A family of subgroups $\mathcal{F}$ is a set of subgroups closed under conjugation and finite intersection. A* classifying space of G with respect to $\mathcal{F}$ *is a G-CW-complex $E(G;\mathcal{F})$ such that $E(G,\mathcal{F})^H$ is contractible for $H\in\mathcal{F}$ and any isotropy group belongs to $\mathcal{F}$.*

Theorem 7.17

1. *A model for $E(G;\mathcal{F})$ exists. There are functorial constructions;*
2. *For any G-CW-complex X whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $X\to E(G;\mathcal{F})$. In particular two models for $E(G;\mathcal{F})$ are G-homotopic.*

Proof : see [114], [115, I.6]. A functorial "bar-type" construction is given in [30, section 7]. ∎

A group H is called *virtually cyclic* if it contains a cyclic group of finite index. Denote by $\mathcal{VC}$ the family of virtually cyclic subgroups and by $\mathcal{FIN}$ the family of finite subgroups.

Remark 7.18 If $\mathcal{F}$ consists only of the trivial subgroup, $E(G;\mathcal{F})$ is the same as the total space EG of the universal G-principal bundle $EG \to BG$. If $\mathcal{F}$ consists of all groups, then G/G is a model for $E(G;\mathcal{F})$. Sometimes the space $E(G;\mathcal{FIN})$ is denoted by $\underline{E}G$ in the literature. For discrete groups G the G-CW-complex $E(G;\mathcal{FIN})$ is a classifying space for proper G-actions which is studied for instance in [9, section 2].

Sometimes geometry yields small models for $E(G;\mathcal{FIN})$. Let G be a discrete subgroup of a Lie group L with a finite number of components. If K is a maximal compact subgroup of L, then L/K is homeomorphic to $\mathbb{R}^n$ and L/K with the obvious left G-action can be taken as a model for $E(G,\mathcal{FIN})$ (see [1, Corollary 4.14]). If G is a group of finite virtual cohomological dimension, then there is a finite-dimensional classifying space $E(G,\mathcal{FIN})$ (see [104, Proposition 12]) and hence a finite dimensional model for $E(G,\mathcal{FIN})$. Many examples of such groups are discussed by Serre in [104]. The Rips complex of a word hyperbolic group is a model for $E(G,\mathcal{FIN})$ [80]. More information about $E(G;\mathcal{FIN})$ can be found for instance in [63], [71], [75].

7.2.2 The Formulation of the Isomorphism Conjectures

Now we can formulate the general Isomorphism Conjecture.

Definition 7.19 *Let* $\mathbf{E}$ *be a covariant* $\mathrm{Or}(G)$*-spectrum and let* $\mathcal{F}$ *be a family of subgroups. The* assembly map *for* $\mathbf{E}$ *with respect to* $\mathcal{F}$ *is the map*

$$\mathrm{asmb}\colon H_p^G(E(G;\mathcal{F});\mathbf{E}) \to H_p^G(G/G;\mathbf{E}) = \pi_p(\mathbf{E}(G/G)),$$

which is induced by the canonical projection $E(G;\mathcal{F}) \to G/G$ *and the canonical identification* $\mathrm{mor}_{\mathrm{Or}(G)}(?,G/G) \otimes_{\mathrm{Or}(G)} \mathbf{E} = \mathbf{E}(G/G)$. *The* Isomorphism Conjecture *for* $\mathbf{E}$ *with respect to* $\mathcal{F}$ *says that the assembly map is bijective for all* $p \in \mathbb{Z}$.

Remark 7.20 The point is to find for given $\mathbf{E}$ the family $\mathcal{F}$ as small as possible. If we take $\mathcal{F}$ to be the family of all subgroups, the map above is an isomorphism but this is a trivial and useless fact. The philosophy is to express $\pi_p(\mathbf{E}(G/G))$, which is the group we are interested in, by the groups $\pi_q(\mathbf{E}(G/H))$ for $q \leq p$ and $H \in \mathcal{F}$ and all their relations coming from inclusion of subgroups of $\mathcal{F}$, which we hopefully understand. Notice that in principle $H_p^G(E(G;\mathcal{F});\mathbf{E})$ can be computed from the groups $\pi_q(\mathbf{E}(G/H))$ by an equivariant Atiyah-Hirzebruch spectral sequence. In this sense one may think of the Isomorphism Conjecture as a kind of induction theorem such as Artin's Theorem about representations of finite groups.

Remark 7.21 The assembly map appearing in Definition 7.19

$$\mathrm{asmb}\colon H_p^G(E(G;\mathcal{F});\mathbf{E}) \to H_p^G(G/G;\mathbf{E}) = \pi_p(\mathbf{E}(G/G))$$

can be identified with the homomorphism induced on homotopy groups by the canonical map

$$\operatorname{hocolim}_{\mathrm{Or}(G;\mathcal{F})} \mathbf{E}|_{\mathrm{Or}(G;\mathcal{F})} \to \left(\operatorname{hocolim}_{\mathrm{Or}(G)} \mathbf{E}\right) \ = \ \mathbf{E}(G/G)$$

as explained in [30, Section 5.2], where $\mathrm{Or}(G;\mathcal{F})$ is the full subcategory of $\mathrm{Or}(G)$ consisting of objects G/H for $H \in \mathcal{F}$.

Definition 7.22 *The* Farrell-Jones Isomorphism Conjecture *for K-theory or L-theory respectively for the group G and the associative ring with unit R is the Isomorphism Conjecture of Definition 7.19 for $\mathbf{E} = \mathbf{K}$ or $\mathbf{E} = \mathbf{L}$ respectively and the family $\mathcal{F} = \mathcal{VC}$ of virtually cyclic subgroups, i.e. the assembly map*

$$H_p^G(E(G;\mathcal{VC});\mathbf{K}) \xrightarrow{\cong} K_p(RG)$$

or

$$H_p^G(E(G;\mathcal{VC});\mathbf{L}) \xrightarrow{\cong} L_p^{\langle -\infty \rangle}(RG)$$

respectively is bijective for all $p \in \mathbb{Z}$.

Definition 7.23 *The* Baum-Connes Conjecture *is the Isomorphism Conjecture of Definition 7.19 for $\mathbf{E} = \mathbf{K}^{\mathrm{top}}$ and the family $\mathcal{F} = \mathcal{FIN}$ of finite subgroups, i.e. the assembly map*

$$H_p^G(E(G;\mathcal{FIN});\mathbf{K}^{\mathrm{top}}) \xrightarrow{\cong} K_p^{\mathrm{top}}(C_r^*(G))$$

is bijective for all $p \in \mathbb{Z}$.

Remark 7.24 The Farrell-Jones Isomorphism Conjecture 7.22 for $L_p^{\langle -\infty \rangle}(\mathbb{Z}G)$ for all $p \in \mathbb{Z}$ and the Farrell-Jones Isomorphism Conjecture 7.22 for K-theory for $K_p(\mathbb{Z}G)$ for all $p \leq 1$ have been proved for groups G, for which there exists a Lie group L with finitely many components and a discrete cocompact subgroup $H \subset L$ such that G occurs as subgroup of H [39, Theorem 2.1 and Remark 2.1.3.]. If G is torsionfree and $R = \mathbb{Z}$, then the Farrell-Jones Isomorphism Conjecture for K-theory for $K_p(\mathbb{Z}G)$ with $p \leq 1$ is equivalent to the vanishing of $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_i(\mathbb{Z}G)$ for $i \leq -1$ and has been proved by Farrel and Jones for all groups G which occur as fundamental groups of closed manifold with non-positive sectional curvature or of complete Riemannian manifolds with pinched negative sectional curvature [42]. The Farrell-Jones Isomorphism Conjecture 7.22 for K-theory for $K_p(RG)$ with $p \leq 1$ and any associative ring with unit R has been proved for groups G which occur as fundamental groups of closed Riemannian manifolds with negative sectional curvature by Bartels, Farrell, Jones and Reich [7].

The Baum-Connes Conjecture 7.23 has been proved for groups which satisfy the Haagerup property (= a-T-menable group) by Higson and Kasparov [52]. For information about groups having the Haagerup property we refer to [10], where it is shown that this class of groups includes all amenable groups but no infinite group which has Kazhdan's property T. For information about groups with property T we refer to [31]. Lafforgue proved the Baum-Connes Conjecture 7.23 also for some groups which do have property T (see [64], [65]). The Baum-Connes-Conjecture 7.23 is known for all word-hyperbolic groups [89].

We refer for instance to [11], [37], [38], [39], [40], [41], [42] and [44] for more information about the Farrell-Jones Isomorphism Conjecture 7.22 and for instance to [9], [56], [64] and [116] for more information about the Baum-Connes Conjecture 7.23.

7.2.3 Conclusions from the Isomorphism Conjectures

Next we draw some conclusions from the Farrell-Jones Isomorphism Conjecture 7.22 and the Baum-Connes Conjecture 7.23 in order to illustrate their meaning.

Recall the Hirzebruch signature formula for a closed oriented manifold M

$$\operatorname{sign}(M) \quad = \quad \langle \mathcal{L}(M), [M] \rangle$$

and the obvious fact that $\operatorname{sign}(M)$ is an oriented homotopy invariant.

Conjecture 7.25 (Novikov Conjecture for G) *Given a closed oriented manifold M and a map $f\colon M \to BG$ and $x \in H^*(BG)$, define the* higher signature *by*

$$\operatorname{sign}_x(M, f) \;=\; \langle f^*(x) \cup \mathcal{L}(M), [M] \rangle.$$

The Novikov Conjecture for G *says that these are oriented homotopy invariants, i.e. for $f\colon M \to BG$, $g\colon N \to BG$ and an orientation preserving homotopy equivalence $u\colon M \to N$ with $g \circ u \simeq f$ we have*

$$\operatorname{sign}_x(M, f) \;=\; \operatorname{sign}_x(N, g).$$

The next result is discussed for instance in [45].

Theorem 7.26 *The Farrell-Jones Isomorphism Conjecture 7.22 for algebraic L-theory for G and $R = \mathbb{Z}$ as well as the Baum-Connes Conjecture 7.23 for G imply the Novikov Conjecture for G.*

Conjecture 7.27 (Borel Conjecture) *Let M and N be closed aspherical manifolds. Then any homotopy equivalence $f\colon M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic if and only if they have isomorphic fundamental groups.*

Theorem 7.28 *If the Farrell-Jones Isomorphism Conjecture 7.22 holds for G and $R = \mathbb{Z}$ for both K-theory and L-theory, then the Borel Conjecture holds for closed aspherical manifolds M and N of dimension ≥ 5 with $\pi_1(M) \cong \pi_1(N) \cong G$.*

Proof : We give a sketch of the proof. The Borel Conjecture is equivalent to the claim that the structure set of M in the topological category is trivial

$$\mathcal{S}_n^{\mathrm{top}}(M) = \{\mathrm{id}\colon M \to M\}.$$

We have the surgery exact sequence

$$[\Sigma M, G/TOP] \;\to\; L_{p+1}^s(\mathbb{Z}\pi) \;\to\; \mathcal{S}_n^{\mathrm{top}}(M) \;\to\; [M, G/O] \;\to\; L_p^s(\mathbb{Z}\pi).$$

The K-theory part of the Farrell-Jones Isomorphism Conjecture implies that $\mathrm{Wh}(\pi)$, $\widetilde{K}_0(\mathbb{Z}\pi)$ and $K_i(\mathbb{Z}\pi)$ for $i \leq -1$ vanish. Hence the Rothenberg sequence implies that we do not have to take care of the decorations for the L-groups and structure set. The assembly map in the L-theory part in dimension p and $p+1$ can be identified with the first map and last map appearing in the part of surgery sequence above using Poincaré duality for the sources. ■

More information about the Novikov-Conjecture and the Borel Conjecture can be found for instance in [29], [45].

Conjecture 7.29 (Stable Gromov-Lawson-Rosenberg Conjecture for G) *Let G be a group. Let M be a closed Spin-manifold with fundamental group G. Let B be some closed simply connected manifold with vanishing $\widehat{A}$-genus. Then $M \times B^k$ carries a Riemannian metric of positive scalar curvature for some $k \geq 0$ if and only if*

$$\operatorname{index}_{C_r^*(G)}(\widetilde{M}, \widetilde{D}) \;=\; 0.$$

Here D is the Dirac operator and $\widetilde{D}$ its lift to $\widetilde{M}$ and $\operatorname{index}_{C_r^*(G)}(\widetilde{M}, \widetilde{D})$ *is the $C_r^*(G)$-index defined by Mishchenko and Fomenko [90].*

Remark 7.30 Notice that by Bott periodicity

$$\operatorname{index}_{C_r^*(G)}(\widetilde{M}, \widetilde{D}) \;=\; \operatorname{index}_{C_r^*(G)}(\widetilde{M \times B^k}, \widetilde{D}).$$

Hence the index theory cannot distinguish between M and $M \times B^k$. Indeed, Schick [103] has constructed counterexamples to the unstable version of the Gromov-Lawson-Rosenberg Conjecture following ideas of Stolz, where one requires a positive scalar curvature metric on M itself instead of $M \times B^k$ for some $k \geq 0$.

The next result is due to Stolz.

Theorem 7.31 *If the group G satisfies the Baum-Connes Conjecture 7.23, then the stable Gromov-Lawson Conjecture 7.29 holds for G.*

Conjecture 7.32 (Kadison Conjecture for G) *Let G be torsionfree. Let $p \in C_r^*(G)$ be an idempotent, i.e. $p^2 = p$. Then $p = 0, 1$.*

Remark 7.33 The Baum-Connes Conjecture 7.23 for G implies the Kadison Conjecture for G. This is a consequence of the L^2-index theorem of Atiyah. For a discussion of this proof and the relation of the Baum-Connes Conjecture 7.23 to the (modified) Trace Conjecture we refer to [72].

7.3 Characterizing Assembly Maps

Next we give an abstract characterization of assembly maps. It generalizes the non-equivariant characterization of Weiss-Williams [120] to the equivariant setting.

A covariant functor

$$\mathbf{E}\colon G - CW - \text{COMPLEXES} \to \text{SPECTRA}$$

is called *(weakly) homotopy invariant* if it sends G-homotopy equivalences to (weak) homotopy equivalences of spectra. The functor $\mathbf{E}$ is *(weakly) excisive* if it has the following four properties:

1. it is (weakly) homotopy invariant;
2. $\mathbf{E}(\emptyset)$ is contractible;
3. it respects homotopy pushouts up to (weak) homotopy equivalence;
4. $\mathbf{E}$ sends countable disjoint unions to countable wedges up to (weak) homotopy;

Remark 7.34 $\mathbf{E}$ is weakly excisive if and only if $\pi_q(\mathbf{E}(X))$ defines a G-homology theory on the category of G-CW-complexes satisfying the disjoint union axiom for countable disjoint unions.

Definition 7.35 *Let* $\mathbf{E}$ *be a covariant* $\mathrm{Or}(G)$*-spectrum, i.e. a covariant functor*

$$\mathbf{E}\colon \ \mathrm{Or}(G) \to \text{SPECTRA}\,.$$

Define an extension

$$\mathbf{E}_{\%}\colon G{-}CW{-}\text{COMPLEXES} \to \text{SPECTRA}, \qquad X \mapsto \mathrm{map}_G(?;X)_+ \otimes_{\mathrm{Or}(G)} \mathbf{E}.$$

We conclude from Theorem 7.14.

Lemma 7.36 $\mathbf{E}_{\%}$ *is excisive.*

The proof of the next two results can be found in [30, Theorem 6.3].

Lemma 7.37 *Let* $\mathbf{T}\colon \mathbf{E} \to \mathbf{F}$ *be a transformation of (weakly) excisive functors*

$$\mathbf{E}, \mathbf{F}\colon G - CW - \text{COMPLEXES} \to \text{SPECTRA}$$

so that $\mathbf{T}(G/H)$ *is a (weak) homotopy equivalence of spectra for all* $H \subset G$. *Then* $\mathbf{T}(X)$ *is a (weak) homotopy equivalence of spectra for all* G-CW-*complexes* X.

Theorem 7.38 *For any (weakly) homotopy invariant functor*

$$\mathbf{E}\colon G - CW - \text{COMPLEXES} \to \text{SPECTRA}$$

there is a (weakly) excisive functor

$$\mathbf{E}^{\%}\colon G - CW - \text{COMPLEXES} \to \text{SPECTRA}$$

and natural transformations

$$\begin{array}{rcl} \mathbf{A}_{\mathbf{E}}\colon \mathbf{E}^{\%} & \to & \mathbf{E}; \\ \mathbf{B}_{\mathbf{E}}\colon \mathbf{E}^{\%} & \to & (\mathbf{E}\,|_{\mathrm{Or}(G)})\%, \end{array}$$

which induce (weak) homotopy equivalences of spectra $\mathbf{A}_{\mathbf{E}}(G/H)$ *for all* $H \subset G$ *and (weak) homotopy equivalences of spectra* $\mathbf{B}_{\mathbf{E}}(X)$ *for all* G-CW-*complexes* X.

$\mathbf{E}$ *is (weakly) excisive if and only if* $\mathbf{A}_{\mathbf{E}}(X)$ *is a (weak) homotopy equivalence of spectra for all* G-CW-*complexes* X.

Remark 7.39 Theorem 7.38 characterizes the assembly map in the sense that

$$\mathbf{A}_{\mathbf{E}}\colon \mathbf{E}^{\%} \longrightarrow \mathbf{E}$$

is the universal approximation from the left by a (weakly) excisive functor of a (weakly) homotopy invariant functor $\mathbf{E}$ from $G - CW - \text{COMPLEXES}$ to SPECTRA which agrees with $\mathbf{E}$ on homogeneous spaces G/H. Namely, let

$$\mathbf{T}\colon \mathbf{F} \to \mathbf{E}$$

be a transformation of functors from $G - CW - \text{COMPLEXES}$ to SPECTRA such that $\mathbf{F}$ is (weakly) excisive and $\mathbf{T}(G/H)$ is a (weak) homotopy equivalence for all $H \subset G$. Then for any G-CW-complex X the following diagram commutes

$$\begin{array}{ccc}
\mathbf{F}^{\%}(X) & \xrightarrow[\simeq]{\mathbf{A_F}(X)} & \mathbf{F}(X) \\
{\scriptstyle \mathbf{T}^{\%}(X)}\downarrow\simeq & & \downarrow{\scriptstyle \mathbf{T}(X)} \\
\mathbf{E}^{\%}(X) & \xrightarrow{\mathbf{A_E}(X)} & \mathbf{E}(X)
\end{array}$$

and $\mathbf{A_F}(X)$ and $\mathbf{T}^{\%}(X)$ are (weak) homotopy equivalences. Hence one may say that $\mathbf{T}(X)$ factorizes over $\mathbf{A_E}(\mathbf{X})$.

Remark 7.40 We can apply the construction above to the homotopy invariant functor

$$\mathbf{E}\colon G - CW - \text{COMPLEXES} \to \text{SPECTRA}$$

which sends X to

$$\begin{gathered}
\mathbf{K}(\pi(EG \times_G X)); \\
\mathbf{L}(\pi(EG \times_G X)); \\
\mathbf{K}^{\mathrm{top}}(C_r^*(\pi(EG \times_G X)).
\end{gathered}$$

Then the assembly map appearing in the Isomorphism Conjectures above is given by

$$\pi_p(\mathbf{A_E}(X))\colon \pi_p(\mathbf{E}^{\%}(X)) \to \pi_p(\mathbf{E}(X))$$

if one puts $X = E(G; \mathcal{VC})$ or $X = E(G, \mathcal{FIN})$. More generally, if Z is a free G-CW-complex and one replaces $EG \times_G X$ by $Z \times_G X$ above, one gets the fibered Farrell-Jones Isomorphism Conjecture [39, 1.7 on page 262].

Remark 7.41 The assembly maps appearing in the Farrell-Jones Isomorphism Conjecture 7.22 and in the Baum-Connes Conjecture 7.23 were defined differently, in terms of index theory or as forget control maps. The identification of the original definitions with the one presented here is non-trivial and carried out by Hambleton-Pedersen [50] using the universal characterization of assembly maps of Section 7.3.

7.4 The Choice of the Families

In this subsection we want to illustrate why one uses in the Farrell-Jones Isomorphism Conjecture 7.22 the family $\mathcal{VC}$ and in the Baum-Connes Conjecture 7.23 the family $\mathcal{FIN}$ and under which circumstances one may replace $\mathcal{VC}$ by $\mathcal{FIN}$ also in the Farrell-Jones Isomorphism Conjecture 7.22.

Example 7.42 Consider a semi-direct product $G \rtimes_\phi \mathbb{Z}$. The automorphism ϕ induces a $\phi\colon G \to G$-equivariant homotopy equivalence $E\phi\colon E(G;\mathcal{FIN}) \to E(G;\mathcal{FIN})$. The to both sides infinite mapping telescope $T_{E\phi}$ with the obvious $G \rtimes_\phi \mathbb{Z}$-action is a model for $E(G \rtimes_\phi \mathbb{Z};\mathcal{FIN})$. This implies that we get for any covariant $\mathrm{Or}(G \rtimes_\phi \mathbb{Z};\mathcal{FIN})$-spectrum $\mathbf{E}$ a Wang sequence

$$\begin{aligned} \ldots \to H_p^G(E(G;\mathcal{FIN});\mathbf{E}|_G) \xrightarrow{\phi_*-\mathrm{id}} H_p^G(E(G;\mathcal{FIN});\mathbf{E}|_G) \\ \to H_p^{G\rtimes_\phi\mathbb{Z}}(E(G \rtimes_\phi \mathbb{Z};\mathcal{FIN});\mathbf{E}) \\ \to H_{p-1}^G(E(G;\mathcal{FIN});\mathbf{E}|_G) \xrightarrow{\phi_*-\mathrm{id}} H_{p-1}^G(E(G;\mathcal{FIN});\mathbf{E}|_G) \to \ldots \end{aligned}$$

In particular we get for $\phi = \mathrm{id}$ an isomorphism

$$H_p^{G\times\mathbb{Z}}(E(G\times\mathbb{Z};\mathcal{FIN});\mathbf{E}) \;\cong\; H_p^G(E(G;\mathcal{FIN});\mathbf{E}|_G)\oplus H_{p-1}^G(E(G;\mathcal{FIN});\mathbf{E}|_G).$$

Hence the Baum-Connes Conjecture 7.23 can only be true if there is an isomorphism

$$K_p^{\mathrm{top}}(C_r^*(G\times\mathbb{Z})) \;\cong\; K_p^{\mathrm{top}}(G) \oplus K_{p-1}^{\mathrm{top}}(G).$$

This is indeed the case. More generally, a Wang sequence exists by results of Pimsner-Voiculescu [93, Theorem 3.1 on page 151].

Remark 7.43 The computation of algebraic K-groups involves $\mathcal{VC}$ and not only $\mathcal{FIN}$ since in the Bass-Heller-Swan decomposition

$$K_i(R[\mathbb{Z}]) \;\cong\; K_{i-1}(R) \oplus K_i(R) \oplus \mathrm{Nil}_{i-1}(R) \oplus \mathrm{Nil}_{i-1}(R).$$

Nil-terms occur. These Nil-terms are very hard to compute (see e.g. [36]). At least they vanish if R is regular, for instance if R is $\mathbb{Z}$ or a field. The difficulties about the Nil-terms are absorbed by taking the family $\mathcal{VC}$ instead of $\mathcal{FIN}$. Notice that enlarging the family $\mathcal{FIN}$ to $\mathcal{VC}$ is a necessary step to take Nil-terms into account but it is a priori not clear that this takes care of this complication in general.

The corresponding result for L-theory is the Shaneson splitting [106] which involves a change of decoration. For instance

$$L_p^s(\mathbb{Z}[G\times\mathbb{Z}]) \;=\; L_p^s(\mathbb{Z}G) \oplus L_{p-1}^h(\mathbb{Z}G).$$

Therefore one must pass to the L-groups $L^{\langle -\infty\rangle}$ which do satisfy

$$L_p^{\langle -\infty\rangle}(\mathbb{Z}[G\times\mathbb{Z}]) \;=\; L_p^{\langle -\infty\rangle}(\mathbb{Z}G)\oplus L_{p-1}^{\langle -\infty\rangle}(\mathbb{Z}G).$$

The version of the Farrell-Jones Isomorphism Conjecture 7.22 for L^p, L^h and L^s is definitely false by a result of Farrell-Jones-Lück [43].

Example 7.44 Let G be the amalgamated product $G = G_1 *_{G_0} G_2$ over a common subgroup G_0 of G_1 and G_2. Then one can construct a cellular G-pushout

$$\begin{array}{ccc} G\times_{G_0} E(G_0;\mathcal{FIN}) & \longrightarrow & G\times_{G_1} E(G_1;\mathcal{FIN}) \\ \downarrow & & \downarrow \\ G\times_{G_2} E(G_2;\mathcal{FIN}) & \longrightarrow & E(G;\mathcal{FIN}) \end{array}$$

Hence for any covariant $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ there is a Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to\, & H_p^{G_0}(E(G_0;\mathcal{FIN});\mathbf{E}|_{G_0}) \\ & \to H_p^{G_1}(E(G_1;\mathcal{FIN});\mathbf{E}|_{G_1})\oplus H_p^{G_2}(E(G_2;\mathcal{FIN});\mathbf{E}|_{G_2}) \\ & \to H_p^{G}(E(G;\mathcal{FIN});\mathbf{E}) \to H_{p-1}^{G_0}(E(G_0;\mathcal{FIN});\mathbf{E}|_{G_0}) \\ & \to H_{p-1}^{G_1}(E(G_1;\mathcal{FIN});\mathbf{E}|_{G_1})\oplus H_{p-1}^{G_2}(E(G_2;\mathcal{FIN});\mathbf{E}|_{G_2}) \to \ldots \end{aligned}$$

Hence the Baum-Connes Conjecture 7.23 predicts the existence of a Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to K_p^{\mathrm{top}}(C_r^*(G_0)) \to K_p^{\mathrm{top}}(C_r^*(G_1))\oplus K_p^{\mathrm{top}}(C_r^*(G_2)) \to K_p^{\mathrm{top}}(C_r^*(G)) \\ \to K_{p-1}^{\mathrm{top}}(C_r^*(G_0)) \to K_{p-1}^{\mathrm{top}}(C_r^*(G_1))\oplus K_{p-1}^{\mathrm{top}}(C_r^*(G_2)) \to \ldots \end{aligned}$$

Such a sequence exists by results of Pimsner [94, Theorem 18 on page 632].

Notice that such a cellular G-pushout does not exist if one replaces $\mathcal{FIN}$ by $\mathcal{VC}$.

Remark 7.45 Waldhausen [117] has shown that the corresponding sequence for K-theory does not exists in this from, again certain Nil-terms arise. Again this shows that one needs to deal with $\mathcal{VC}$ for algebraic K-theory.

Remark 7.46 A similar complication occurs for the L-groups by the UNil-groups which appear in the formula for free products due to Cappell [20]

$$\widetilde{L}_p^{\langle -\infty \rangle}(\mathbb{Z}[G * H]) \;\cong\; \widetilde{L}_p^{\langle -\infty \rangle}(\mathbb{Z}G) \oplus \widetilde{L}_p^{\langle -\infty \rangle}(\mathbb{Z}H) \oplus \mathrm{UNil}\,.$$

The UNil-groups are infinitely generated 2-torsion groups and have not been computed completely. Therefore one has to use the family $\mathcal{VC}$ in the Isomorphism Conjecture for L-theory.

The following theorem is useful if one wants to decrease the family $\mathcal{F}$ appearing in the isomorphism conjecture.

Theorem 7.47 *Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of the group G. Let n be an integer and $\mathcal{P}$ be a set of prime numbers. Let $\mathbf{E}$ be $\mathbf{K}$, $\mathbf{L}$ or $\mathbf{K}^{\mathrm{top}}$. Suppose for every $H \in \mathcal{G}$ that the assembly map induces an isomorphism*

$$H_p^H(E(H; H \cap \mathcal{F}); \mathbf{E}) \left[\frac{1}{\mathcal{P}}\right] \;\to\; H_p^H(H/H; \mathbf{E}) \left[\frac{1}{\mathcal{P}}\right]$$

for $p \leq n$, where $H \cap \mathcal{F}$ is the family of subgroups of H given by $\{H \cap K \mid K \in \mathcal{F}\}$. Then the map

$$H_p^G(E(G; \mathcal{F}); \mathbf{E}) \left[\frac{1}{\mathcal{P}}\right] \to H_p^G(E(G; \mathcal{G}); \mathbf{E}) \left[\frac{1}{\mathcal{P}}\right]$$

is an isomorphism for $p \leq n$.

Proof : This follows from [76, Theorem 2.3]. ■

Remark 7.48 In general it is much harder to deal with the family $\mathcal{VC}$ instead of the family $\mathcal{FIN}$. Suppose that $R = \mathbb{Z}$. In L-theory there is no difference if one inverts 2. Then the decorations do not matter by the Rothenberg sequences and all UNil-terms vanish. Moreover, for any group G we conclude from Theorem 7.47 and [124]

$$H_p^G(E(G, \mathcal{FIN}); \mathbf{L}) \left[\frac{1}{2}\right] \;\cong\; H_p^G(E(G, \mathcal{VC}); \mathbf{L}) \left[\frac{1}{2}\right]$$

and from [97, page 376] or [98, Proposition 22.34 on page 252]

$$L_p(\mathbb{Z}G) \left[\frac{1}{2}\right] \;\cong\; L_p(\mathbb{Q}G) \left[\frac{1}{2}\right].$$

This allows for computations of $L_p(\mathbb{Z}G)[1/2]$ to pass to rational coefficients and to use $\mathcal{FIN}$ instead of $\mathcal{VC}$. This much easier than to work with $\mathbb{Z}$-coefficients and $\mathcal{VC}$.

Remark 7.49 Notice that $K_q(\mathbb{Z}G) \to K_q(\mathbb{Q}G)$ is in general not bijective rationally as the example $G = \{1\}$ and $q = 1$ shows. So for computations of $K_q(\mathbb{Z}G)$ we have to stick to integral coefficients and to the family $\mathcal{VC}$ even if we only want to get rational information.

If F is a field of characteristic zero, then the obvious map

$$H_p^G(E(G;\mathcal{FIN});\mathbf{K}) \xrightarrow{\cong} H_p^G(E(G;\mathcal{VC});\mathbf{K})$$

is bijective, where $\mathbf{K}$ is the covariant $\mathrm{Or}(G)$-spectrum with $\pi_n(\mathbf{K}(G/H)) = K_n(FH)$. This follows from Theorem 7.47 and [117] since any virtually cyclic group maps to $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$ with finite kernel and hence is a semidirect product of a finite group with $\mathbb{Z}$ or can be written as an amalgam of two finite groups over a finite subgroup and the relevant Nil-terms vanish because for any finite group (actually poly-cyclic suffices) group H and field F of characteristic zero the ring FH is regular. Moreover, the Farrell-Jones Isomorphism Conjecture 7.22 predicts that $K_i(FG) = 0$ for $i \leq -1$ and the canonical map

$$\operatorname{colim}_{G/H \in \mathrm{Or}(G;\mathcal{FIN})} K_0(FH) \to K_0(FG)$$

is bijective, where $\mathrm{Or}(G;\mathcal{FIN})$ is the full subcategory of $\mathrm{Or}(G)$ consisting of objects G/H with finite H.

It seems to be the case that in general the computation of $K_p^{\mathrm{top}}(C_r^*(G))$ is easier than the one of $L_p(\mathbb{Z}G)$ and the computation of $K_p(\mathbb{Z}G)$ is the hardest.

7.5 Miscellaneous

One can construct the following commutative diagram

$$\begin{array}{ccc}
H_n^G(E(G;\mathcal{FIN});\mathbf{L}(\mathbb{Z}?)[1/2]) & \xrightarrow{\text{asmb}_1} & L_n(\mathbb{Z}G)[1/2] \\
i_1 \downarrow \cong & & j_1 \downarrow \cong \\
H_n^G(E(G;\mathcal{FIN});\mathbf{L}(\mathbb{Q}?)[1/2]) & \xrightarrow{\text{asmb}_2} & L_n(\mathbb{Q}G)[1/2] \\
i_2 \downarrow \cong & & j_2 \downarrow \\
H_n^G(E(G;\mathcal{FIN});\mathbf{L}(\mathbb{R}?)[1/2]) & \xrightarrow{\text{asmb}_3} & L_n(\mathbb{R}G)[1/2] \\
i_3 \downarrow \cong & & j_3 \downarrow \\
H_n^G(E(G;\mathcal{FIN});\mathbf{L}(C_r^*(?;\mathbb{R}))[1/2]) & \xrightarrow{\text{asmb}_4} & L_n(C_r^*(?;\mathbb{R}))[1/2] \\
i_4 \downarrow \cong & & j_4 \downarrow \cong \\
H_n^G(E(G;\mathcal{FIN});\mathbf{K}^{\text{top}}(C_r^*(?;\mathbb{R}))[1/2]) & \xrightarrow{\text{asmb}_5} & K_n^{\text{top}}(C_r^*(?;\mathbb{R}))[1/2]
\end{array}$$

where $\mathbf{L}(\mathbb{Z}?)[1/2]$, $\mathbf{L}(\mathbb{Q}?)[1/2]$, $\mathbf{L}(\mathbb{R}?)[1/2]$, $\mathbf{L}(C_r^*(?))[1/2]$ and $\mathbf{K}^{\text{top}}(C_r^*(?;\mathbb{R}))[1/2]$ are the variants of the covariant $\text{Or}(G)$-spectra such that the n-th homotopy group of its evaluation at G/H is $L_n(\mathbb{Z}H)[1/2]$, $L_n(\mathbb{Q}H)[1/2]$, $L_n(\mathbb{R}H)[1/2]$, $L_n(C_r^*(H;\mathbb{R}))[1/2]$, $K_n^{\text{top}}(C_r^*(H))[1/2]$ for $C_r^*(H;\mathbb{R})$ the reduced real group C^*-algebra of H. All horizontal maps are assembly maps as appearing in Definition 7.19. The maps i_k and j_k for $k = 1, 2, 3$ are given by change of rings map. The isomorphisms i_4 and j_4 come from the general isomorphism for any real C^*-algebra A

$$L_n^p(A)[1/2] \xrightarrow{\cong} K_n^{\text{top}}(A)[1/2]$$

and its spectrum version [99, Theorem 1.11 on page 350]. The maps i_1, j_1, i_2 are isomorphisms by [97, page 376] and [98, Proposition 22.34 on page 252]. The map i_3 is bijective since for a finite group H we have $\mathbb{R}H = C_r^*(H)$.

We see that the injectivity of asmb_5 which is the assembly map in the Baum-Connes Conjecture 7.23 after inverting 2 implies the injectivity of asmb_1 which is the assembly map in the Farrell-Jones Isomorphism Conjecture 7.22 for L-theory and the family $\mathcal{FIN}$ after inverting 2. One may conjecture that all right vertical maps are isomorphisms and try to prove this directly. Then the Baum-Connes Conjecture 7.23 after inverting 2 becomes equivalent to the Farrell-Jones Isomorphism Conjecture 7.22 for the family $\mathcal{FIN}$ after inverting 2 because of Remark 7.48.

Chapter 8

Computations of K- and L-Groups for Infinite Groups

In this chapter we introduce equivariant Chern characters which compute rationally the G-homology of a proper G-CW-complex in terms of ordinary singular homology of various fixed point sets provided some additional structures (induction structure, Mackey structure on the coefficients) are available on the G-homology theory. These additional structures exist for the various G-homology theories appearing as the source of the assembly map in the Farrell-Jones Isomorphism Conjecture 7.22 and the Baum-Connes Conjecture 7.23. This is used to give general formulas for the rationalized algebraic K- and L-theory of group rings and the topological K-theory of reduced group C^*-algebras provided the Isomorphism Conjectures hold. We also present some integral computations of K- and L-groups for some special groups. The integral calculations are much harder and there seems to be no general formula as in the rationalized situation.

8.1 Equivariant Chern Characters

As an illustration we recall a construction of a non-equivariant Chern character due to Dold [34]. The actual construction of the equivariant Chern character is technical much more elaborate and will therefore not be presented here but Dold's construction is an important ingredient.

Example 8.1 Consider a (non-equivariant) homology theory $\mathcal{H}_*$ with values in R-modules for $\mathbb{Q} \subset R$. Then a (non-equivariant) Chern character for a CW-complex X is given by the following composite

$$\begin{aligned}\mathrm{ch}_n\colon \bigoplus_{p+q=n} H_p(X;\mathcal{H}_q(*)) &\xrightarrow{\bigoplus_{p+q=n}\alpha_{p,q}^{-1}} \bigoplus_{p+q=n} H_p(X;R)\otimes_R \mathcal{H}_q(*)\\ &\xrightarrow{\bigoplus_{p+q=n}(\mathrm{hur}\otimes\mathrm{id})^{-1}} \bigoplus_{p+q=n} \pi_p^s(X_+,*)\otimes_{\mathbb{Z}} R\otimes_R \mathcal{H}_q(*)\\ &\xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X).\end{aligned}$$

Here the canonical map $\alpha_{p,q}\colon H_p(X;R)\otimes_R \mathcal{H}_q(*) \to H_p(X;\mathcal{H}_q(*))$ is bijective, since any R-module is flat over $\mathbb{Z}$ because of the assumption $\mathbb{Q}\subset R$. The second bijective map comes from the Hurewicz homomorphism $\pi_p^s(X_+,*)\otimes_{\mathbb{Z}} R \to H_p(X;R)$ which is bijective because of $\mathbb{Q}\subset R$. The map $D_{p,q}$ is defined as follows. For an element $a\otimes b\in \pi_p^s(X_+,*)\otimes_{\mathbb{Z}} \mathcal{H}_q(*)$ choose a representative $f\colon S^{p+k}\to S^k\wedge X_+$ of a. Define $D_{p,q}(a\otimes b)$ to be the image of b under the composite

$$\mathcal{H}_q(*) \xrightarrow{\sigma} \mathcal{H}_{p+q+k}(S^{p+k},*) \xrightarrow{\mathcal{H}_{p+q+k}(f)} \mathcal{H}_{p+q+k}(S^k\wedge X_+,*) \xrightarrow{\sigma^{-1}} \mathcal{H}_{p+q}(X),$$

where σ denotes the suspension isomorphism. This map turns out to be a transformation of homology theories and induces an isomorphism for $X=*$. Hence it is a natural equivalence of homology theories.

We have introduced the notion of a G-homology theory in Definition 7.10. A *proper G-homology theory* is a G-homology theory which is only defined for proper G-CW-complexes. To construct equivariant Chern characters we need the following additional structure called induction structure which does exist in the examples we are interested in.

Definition 8.2 *A* (proper) equivariant homology theory $\mathcal{H}_*^?$ *with values in R-modules consists of a (proper) G-homology theory $\mathcal{H}_*^G$ with values in R-modules for each group G together with the following so called* induction structure*: given a group homomorphism $\alpha\colon H\to G$ and a (proper) H-CW-pair (X,A) such that* $\ker(\alpha)$ *acts freely on X, there are for each $n\in\mathbb{Z}$ natural isomorphisms*

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(\mathrm{ind}_\alpha(X,A))$$

satisfying

1. *Compatibility with the boundary homomorphisms*
   $\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H$;

2. *Functoriality*
   Let $\beta\colon G \to K$ *be another group homomorphism such that* $\ker(\beta \circ \alpha)$ *acts freely on* X. *Then we have for* $n \in \mathbb{Z}$

   $$\mathrm{ind}_{\beta\circ\alpha} \;=\; \mathcal{H}_n^K(f_1) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X,A)),$$

   where $f_1\colon \mathrm{ind}_\beta \mathrm{ind}_\alpha(X,A) \xrightarrow{\cong} \mathrm{ind}_{\beta\circ\alpha}(X,A), \quad (k,g,x) \mapsto (k\beta(g),x)$ *is the natural* K*-homeomorphism;*

3. *Compatibility with conjugation*
   For $n \in \mathbb{Z}$, $g \in G$ *and a (proper)* G*-CW-pair* (X,A) *the homomorphism* $\mathrm{ind}_{c(g)\colon G\to G}\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_n^G(\mathrm{ind}_{c(g)\colon G\to G}(X,A))$ *agrees with* $\mathcal{H}_n^G(f_2)$ *for* $c(g)$ *the conjugation homomorphism which sends* g' *to* $gg'g^{-1}$ *and the* G*-homeomorphism* $f_2\colon (X,A) \to \mathrm{ind}_{c(g)\colon G\to G}(X,A)$ *which sends* x *to* $(1,g^{-1}x)$ *in* $G \times_{c(g)} (X,A)$.

Example 8.3 Equivariant cobordism yields an equivariant proper homology theory. The G-homology theory Ω_*^G has been explained in Example 7.12. The induction structure comes from induction of spaces. Notice that for a group homomorphism $\alpha\colon H \to G$ and a cocompact smooth H-manifold M such that $\ker(\alpha)$ acts freely on M, the G-space $\mathrm{ind}_\alpha(M) = G \times_\alpha M$ is a cocompact G-manifold.

Example 8.4 Let

$$\mathbf{E}\colon \mathrm{GROUPOIDS}^{\mathrm{inj}} \to \mathrm{SPECTRA}$$

be a functor from the category of groupoids with injective functors as morphisms to spectra, where a functor $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ of groupoids is called injective if for each object x of $\mathcal{G}_1$ the group homomorphism $\mathrm{aut}_{\mathcal{G}_1}(x) \to \mathrm{aut}_{\mathcal{G}_2}(F(x))$ induced by F is injective. For any group G and G-set S let $\mathcal{G}^G(S)$ be the groupoid whose set of objects is S and whose set of morphisms from s_1 to s_2 is $\{g \in G \mid gs_1 = s_2\}$. Composition in this groupoid is given by the multiplication in G. We obtain a functor

$$\mathcal{G}^G\colon \mathrm{Or}(G) \to \mathrm{GROUPOIDS}^{\mathrm{inj}}$$

Define for each group G a covariant $\mathrm{Or}(G)$-spectrum

$$\mathbf{E}^G\colon \mathrm{Or}(G) \to \mathrm{SPECTRA}, \quad G/H \mapsto \mathbf{E}(\mathcal{G}^G(G(H))$$

Then we obtain an equivariant homology theory $\mathcal{H}_*^?$ by assigning to a group G the G-homology theory $\mathcal{H}_*^G(-;\mathbf{E}^G)$. We leave it to the reader to figure out the induction structure. The key observation is that the obvious functor of groupoids

$$\mathcal{G}^H(H/L) \to \mathcal{G}^G(\mathrm{ind}_\alpha G/L)$$

is an equivalence of categories if $L \cap \ker(\alpha) = \{1\}$. This implies that the sources of the assembly maps appearing in the Farrell-Jones Isomorphism Conjecture 7.22 and the Baum-Connes Conjecture 7.23 are equivariant homology theories and the induction structures on the sources are compatible with the obvious induction homomorphism on the target since the relevant spectra are constructed as mentioned above.

One easily checks (see [70, Lemma 1.2])

Lemma 8.5 *Consider finite subgroups $H, K \subset G$ and an element $g \in G$ with $gHg^{-1} \subset K$. Let $R_{g^{-1}}\colon G/H \to G/K$ be the G-map sending $g'H$ to $g'g^{-1}K$ and let $c(g)\colon H \to K$ be the homomorphism sending h to ghg^{-1}. Denote by $\mathrm{pr}\colon (\mathrm{ind}_{c(g)\colon H\to K} *) \to *$ the projection. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{H}_n^H(*) \arrow[r, "\mathcal{H}_n^K(\mathrm{pr})\circ \mathrm{ind}_{c(g)}"] \arrow[d, "\mathrm{ind}_H^G"', "\cong"] & \mathcal{H}_n^K(*) \arrow[d, "\mathrm{ind}_K^G"', "\cong"] \\
\mathcal{H}_n^G(G/H) \arrow[r, "\mathcal{H}_n^G(R_{g^{-1}})"] & \mathcal{H}_n^G(G/K)
\end{tikzcd}$$

Let R be an associative commutative ring with unit. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let $M\colon$ FGINJ $\to R-$ MODULES be a bifunctor, i.e. a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from FGINJ to $R-$MODULES which agree on objects. We will often denote for an injective group homomorphism $f\colon H \to G$ the map $M_*(f)\colon M(H) \to M(G)$ by ind_f and the map $M^*(f)\colon M(G) \to M(H)$ by res_f and write $\mathrm{ind}_H^G = \mathrm{ind}_f$ and $\mathrm{res}_G^H = \mathrm{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in R-modules if

1. For an inner automorphism $c(g)\colon G \to G$ we have $M_*(c(g)) = \mathrm{id}\colon M(G) \to M(G)$;

2. For an isomorphism of groups $f\colon G \xrightarrow{\cong} H$ the composites $\mathrm{res}_f \circ \mathrm{ind}_f$ and $\mathrm{ind}_f \circ \mathrm{res}_f$ are the identity;

3. Double coset formula
   We have for two subgroups $H, K \subset G$
$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G \;=\; \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H\cap g^{-1}Kg},$$
   where $c(g)$ is conjugation with g, i.e. $c(g)(h) = ghg^{-1}$.

Example 8.6 Our main examples of Mackey functors will be $R_{\mathbb{Q}}(H)$, $K_q(RH)$, $L_q(RH)$ and $K_q^{\mathrm{top}}(C_*^r(H))$. The Mackey structure comes from induction and restriction of modules.

Define for a finite group H
$$S_H\left(\mathcal{H}_q^H(*)\right) := \mathrm{coker}\left(\bigoplus_{\substack{K\subset H\\ K\neq H}} \mathrm{ind}_K^H\colon \bigoplus_{\substack{K\subset H\\ K\neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right). \tag{8.7}$$
For a subgroup $H \subset G$ we denote by $N_GH := \{g \in G \mid gHg^{-1} = H\}$ the *normalizer* and by $C_GH = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ the *centralizer* of H in G. Let $H \cdot C_GH$ be the subgroup of N_GH consisting of elements of the form hc for $h \in H$ and $c \in C_GH$. Denote by W_GH the quotient $N_GH/H{\cdot}C_GH$. Notice that W_GH is finite if H is finite. It acts from the right on $C_GH\backslash X^H$ and from the left on $\mathcal{H}_q^H(*) = \mathcal{H}_q^G(G/H)$ and $S_H\left(\mathcal{H}_q^H(*)\right)$. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. The proof of the following result can be found in [70, Theorem 0.3].

Theorem 8.8 *Let R be a commutative ring with $\mathbb{Q} \subset R$. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in R-modules. Suppose that the covariant functor*
$$\mathcal{H}_q^?(*)\colon\ \mathrm{FGINJ} \to R-\mathrm{MODULES}, \qquad H \;\mapsto\; \mathcal{H}_q^H(*)$$
extends to a Mackey functor for all $q \in \mathbb{Z}$. Then there is for any group G and any proper G-CW-complex X a natural isomorphism
$$\mathrm{ch}_n^G\colon\quad \bigoplus_{p+q=n}\ \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H; R) \otimes_{R[W_GH]} S_H\left(\mathcal{H}_q^H(*)\right) \;\xrightarrow{\cong}\; \mathcal{H}_n^G(X),$$
where I is the set of conjugacy classes (H) of finite subgroups H of G.

Remark 8.9 If $\mathcal{H}_*^?$ comes with additional structures such as a restriction structure or a multiplicative structure, then one can define the corresponding structure on the source of the equivariant Chern character and the equivariant Chern character respect these. [70, Section 6].

In the next step we want to specialize further and to express the terms $S_H\left(\mathcal{H}_q^H(*)\right)$ by easier terms. This can be done in the cases we are interested in by using further structure.

Let $\phi\colon R \to S$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in R-modules and let N and P be Mackey functors with values in S-modules. A *pairing* $M \times N \to P$ with respect to ϕ is a family of maps

$$m(H)\colon M(H) \times N(H) \to P(H), \qquad (x,y) \mapsto m(H)(x,y) =: x \cdot y,$$

where H runs through the finite groups and we require the following properties for all injective group homomorphisms $f\colon H \to K$ of finite groups:

$$\begin{array}{lll}
(x_1 + x_2)\cdot y = x_1 \cdot y + x_2 \cdot y & \text{for} & x_1, x_2 \in M(H), y \in N(H);\\
x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2 & \text{for} & x \in M(H), y_1, y_2 \in N(H);\\
(rx)\cdot y = \phi(r)(x \cdot y) & \text{for} & r \in R, x \in M(H), y \in N(H);\\
x \cdot sy = s(x \cdot y) & \text{for} & s \in S, x \in M(H), y \in N(H);\\
\operatorname{res}_f(x \cdot y) = \operatorname{res}_f(x)\cdot \operatorname{res}_f(y) & \text{for} & x \in M(K), y \in N(K);\\
\operatorname{ind}_f(x)\cdot y = \operatorname{ind}_f\left(x \cdot \operatorname{res}_f(y)\right) & \text{for} & x \in M(H), y \in N(K);\\
x \cdot \operatorname{ind}_f(y) = \operatorname{ind}_f\left(\operatorname{res}_f(x)\cdot y\right) & \text{for} & x \in M(K), y \in N(H).
\end{array}$$

A *Green functor* with values in R-modules is a Mackey functor U together with a pairing $U \times U \to U$ with respect to $\mathrm{id}\colon R \to R$ and elements $1_H \in U(H)$ for each finite group H such that for each finite group H the pairing $U(H) \times U(H) \to U(H)$ induces the structure of an R-algebra on $U(H)$ with unit 1_H and for any morphism $f\colon H \to K$ in FGINJ the map $U^*(f)\colon U(K) \to U(H)$ is a homomorphism of R-algebras with unit. Let U be a Green functor with values in R-modules and M be a Mackey functor with values in S-modules. A (left) U-module structure on M with respect to the ring homomorphism $\phi\colon R \to S$ is a pairing $U \times M \to M$ such that any of the maps $U(H) \times M(H) \to M(H)$ induces the structure of a (left) module over the R-algebra $U(H)$ on the R-module $\phi^*M(H)$ which is obtained from the S-module $M(H)$ by $rx := \phi(r)x$ for $r \in R$ and $x \in M(H)$.

Example 8.10 Our main examples of a Green functor will be $R_{\mathbb{Q}}(?)$. Our main example of modules over the Green functor $R_{\mathbb{Q}}(H)$ will be $K_q(R?)$,

$L_q(R?)$ and $K_q^{\mathrm{top}}(C_*^r(?))$ for a commutative ring R with $\mathbb{Q} \subset R$. The Mackey structure comes from induction and restriction of modules and the multiplicative structure of a Green functor from the tensor product with the diagonal action (see [70, Section 8]).

Let $\mathrm{class}_{\mathbb{Q}}(H)$ be the $\mathbb{Q}$-vector space of functions $H \to \mathbb{Q}$ which are invariant under $\mathbb{Q}$-conjugation, i.e. we have $f(h_1) = f(h_2)$ for two elements $h_1, h_2 \in H$ if the cyclic subgroups $\langle h_1 \rangle$ and $\langle h_2 \rangle$ generated by h_1 and h_2 are conjugate in H. Elementwise multiplication defines the structure of a $\mathbb{Q}$-algebra on $\mathrm{class}_{\mathbb{Q}}$ with the function which is constant 1 as unit element. Taking the character of a rational representation yields an isomorphism of $\mathbb{Q}$-algebras [105, Theorem 29 on page 102]

$$\chi^H \colon \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) \xrightarrow{\cong} \mathrm{class}_{\mathbb{Q}}(H). \tag{8.11}$$

For a finite cyclic group C define

$$\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(C) \cong \mathrm{class}_{\mathbb{Q}}(C) \tag{8.12}$$

to be the function which sends $h \in C$ to 1 if h is a generator and to 0 otherwise. Obviously θ_C is an idempotent.

The proof of the next theorem can be found in [70, Theorem 0.4].

Theorem 8.13 *Let R be a commutative ring with $\mathbb{Q} \subset R$. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in R-modules. Suppose that the covariant functor* FGINJ $\to R-$MODULES *sending H to $\mathcal{H}_q^H(*)$ extends to a Mackey functor for all $q \in \mathbb{Z}$, which is a module over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ with respect to the inclusion $\mathbb{Q} \to R$. Let J be the set of conjugacy classes (C) of finite cyclic subgroups C of G.*

Then for any group G and any proper G-CW-complex X there is a natural isomorphism

$$\mathrm{ch}_n^G \colon \bigoplus_{p+q=n} \bigoplus_{(C) \in J} H_p(C_G C \backslash X^C; R) \otimes_{R[W_G C]} \left(\theta_C \cdot \mathcal{H}_q^C(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X).$$

where $\theta_C \cdot \mathcal{H}_q^C(*)$ is the direct summand of $\mathcal{H}_q^C(*)$ given by the $\mathrm{Rep}_{\mathbb{Q}}(C)$-*module structure on $\mathcal{H}_q^C(*)$ and the idempotent $\theta_C \in \mathrm{Rep}_{\mathbb{Q}}(C)$.*

8.2 Rational Computations

Next we apply the results about equivariant Chern characters to the homology groups $H_p^G(E(G;\mathcal{FIN});\mathbf{K})$, $H_p^G(E(G;\mathcal{FIN});\mathbf{L})$, and $H_p^G(E(G;\mathcal{FIN});\mathbf{K}^{\mathrm{top}})$ and discuss what these computations together with the Farrell-Jones Isomorphism Conjecture 7.22 and the Baum-Connes Conjecture 7.23 imply for $K_n(RG)$, $L_n(RG)$ and $K^{\mathrm{top}}(C_*^r(G))$. Notice that we can only handle $E(G;\mathcal{FIN})$ and not $E(G;\mathcal{VC})$ since our equivariant Chern character only works for proper G-CW-complexes.

We begin with a survey about some computations of K- and L-groups for finite groups. Let $q(G)$, $r(G)$ and $c(G)$ be the number of irreducible rational, real and complex representations of a finite group G. Denote by $r_{\mathbb{C}}(G)$ the number of irreducible real representations which are of complex type.

Theorem 8.14 *For a finite group G, we have the following:*

1. *There are isomorphisms*

$$\begin{aligned}\mathrm{Rep}_{\mathbb{Q}}(G) &\cong \mathbb{Z}^{q(G)};\\ \mathrm{Rep}_{\mathbb{R}}(G) &\cong \mathbb{Z}^{r(G)};\\ \mathrm{Rep}_{\mathbb{C}}(G) &\cong \mathbb{Z}^{c(G)}.\end{aligned}$$

The number $q(G)$ is the number of conjugacy classes of cyclic subgroups in G, the number $c(G)$ is the number of conjugacy classes of elements in G and the number $r(G)$ is the number of $\mathbb{R}$-conjugacy classes of elements in G, where g_1 and g_2 in G are $\mathbb{R}$-conjugated if g_1 and g_2 or g_1^{-1} and g_2 are conjugated;

2.

$$K_q^{\mathrm{top}}(C_r^*(G)) \cong \begin{cases} \mathrm{Rep}_{\mathbb{C}}(G) \cong \mathbb{Z}^{c(G)} & q=0;\\ 0 & q=1;\end{cases}$$

3.

$$\begin{aligned}L_q(\mathbb{Z}G)[1/2] \cong L_q(\mathbb{Q}G)[1/2] \cong L_q(\mathbb{R}G)[1/2]&\\ \cong \begin{cases} \mathbb{Z}[1/2]^{r(G)} & q\equiv 0\ (4);\\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & q\equiv 2\ (4);\\ 0 & q \equiv 1,3\ (4);\end{cases}&\end{aligned}$$

4. *$K_q(\mathbb{Z}G) = 0$ for $q \leq -2$;*

5. *$\widetilde{K}_0(\mathbb{Z}G)$ is finite. For a prime number p $K_0(\mathbb{Z}[\mathbb{Z}/p])$ is isomorphic to the ideal class group of the Dedekind domain $\mathbb{Z}[\exp(2\pi i/p)]$ (which is unknown);*

6. *The rank of $\mathrm{Wh}(G)$ as an abelian group is $r(G) - q(G)$. The torsion of $\mathrm{Wh}(G)$ is isomorphic to the kernel of the change of rings homomorphism $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$. In particular we have*

$$\mathrm{Wh}(\mathbb{Z}/m) \quad \cong \quad \mathbb{Z}^{[m/2]+1-\delta(m)},$$

where $\delta(m)$ is the number of divisors of m and $[m/2]$ is the largest integer less or equal to $m/2$;

7. *Let D_{2m} be the dihedral group of order $2m$. We have*

$$\begin{aligned} \mathrm{Wh}_2(\pi) &= 0, \text{ for } \pi = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4; \\ |\,\mathrm{Wh}_2(\mathbb{Z}/6)| &\leq 2; \\ \mathrm{Wh}_2(D_6) &= \mathbb{Z}/2; \\ \mathrm{Wh}_2((\mathbb{Z}/2)^2) &\geq (\mathbb{Z}/2)^2. \end{aligned}$$

(1) is proved in [105, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page 106].

(2) follows from Morita equivalence applied to $\mathbb{C}[\pi] \cong \prod_{i=1}^{c(\pi)} M(n_i, n_i, \mathbb{C})$ and the computation $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1^{\mathrm{top}}(\mathbb{C}) = 0$.

(3) follows from [98, Proposition 22.34 on page 253].

(4) is proved in [24].

(5) see [86, page 29-32].

(6) For information about $\mathrm{Wh}(\pi)$ we refer to [91].

(7) is proved in [76, Theorem 3.2]. ■

Theorem 8.15 *Let R be an associative ring with unit satisfying $\mathbb{Q} \subset R$. Let G be a (discrete) group. Let J be the set of conjugacy classes (C) of finite cyclic subgroups C of G. Then the rationalized assembly map in the Farrell-Jones Isomorphism Conjecture 7.22 with respect to $\mathcal{FIN}$ for the algebraic*

K-groups $K_n(RG)$ and the algebraic L-groups $L_n(RG)$ and in the Baum Connes Conjecture 7.23 for the topological K-groups $K_n^{\mathrm{top}}(C_r^*(G))$

$$\begin{aligned} \mathrm{asmb}\colon \mathbb{Q}\otimes_{\mathbb{Z}} H_n^G(E(G;\mathcal{FIN});\mathbf{K}) &\to \mathbb{Q}\otimes_{\mathbb{Z}} K_n(RG);\\ \mathrm{asmb}\colon \mathbb{Q}\otimes_{\mathbb{Z}} H_n^G(E(G;\mathcal{FIN});\mathbf{L}) &\to \mathbb{Q}\otimes_{\mathbb{Z}} L_n(RG);\\ \mathrm{asmb}\colon \mathbb{Q}\otimes_{\mathbb{Z}} H_n^G(E(G;\mathcal{FIN});\mathbf{K}^{\mathrm{top}}) &\to \mathbb{Q}\otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G)) \end{aligned}$$

can be identified with the homomorphisms

$$\begin{aligned} \bigoplus_{p+q=n}\bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot(\mathbb{Q}\otimes_{\mathbb{Z}} K_q(RC)) &\to \mathbb{Q}\otimes_{\mathbb{Z}} K_n(RG);\\ \bigoplus_{p+q=n}\bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot(\mathbb{Q}\otimes_{\mathbb{Z}} L_q(RC)) &\to \mathbb{Q}\otimes_{\mathbb{Z}} L_n(RG);\\ \bigoplus_{p+q=n}\bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot\big(\mathbb{Q}\otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(C))\big) &\to \mathbb{Q}\otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(G))). \end{aligned}$$

In the L-theory case we assume that R comes with an involution $R\to R,\ r\mapsto \overline{r}$ and that we use on RG the involution which sends $\sum_{g\in G} r_g\cdot g$ to $\sum_{g\in G}\overline{r_g}\cdot g^{-1}$.

Remark 8.16 We conclude from Theorem 8.14 (2) and Theorem 8.15 that the Baum-Connes Conjecture 7.23 for a group G implies for $q\in\mathbb{Z}$

$$\bigoplus_{\substack{p\in\mathbb{Z}\\ p+q=0 \mod 2}}\ \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot(\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{Rep}_{\mathbb{C}}(C)) \cong \mathbb{Q}\otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(G)).$$

Remark 8.17 Next we want to compute $\mathbb{Q}\otimes_{\mathbb{Z}} L_n(\mathbb{Z}G)$. Because of Remark 7.48 and Theorem 8.15 the Farrell-Jones Isomorphism Conjecture 7.22 implies

$$\bigoplus_{p+q=n}\bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot(\mathbb{Q}\otimes_{\mathbb{Z}} L_q(\mathbb{Q}C)) \cong \mathbb{Q}\otimes_{\mathbb{Z}} L_n(\mathbb{Z}G).$$

Remark 8.18 In the algebraic K-theory case we face the problem that in the Farrell-Jones Isomorphism Conjecture 7.22 we have to deal with $\mathcal{VC}$ instead of $\mathcal{FIN}$ (see Remark 7.43 and Remark 7.45) even if we only want to carry out rational computations. If R is a field F of characteristic zero, we can replace $\mathcal{VC}$ by $\mathcal{FIN}$ as explained in Remark 7.49. Hence the Farrell-Jones Isomorphism Conjecture 7.22 and Theorem 8.15 imply for a field F of characteristic zero

$$\bigoplus_{p+q=n}\bigoplus_{(C)\in J} H_p(C_GC;F)\otimes_{\mathbb{Q}[W_GC]}\theta_C^C\cdot(\mathbb{Q}\otimes_{\mathbb{Z}}K_q(FC)) \;\cong\; \mathbb{Q}\otimes_{\mathbb{Z}}K_n(FG).$$

Further reductions in special cases can be found for instance in [70, Section 8].

Remark 8.19 The rationally computations above show all the same pattern which one may call a separation of variables. Namely, the relevant group G occurs in all cases as group homology terms involving the group homology of the centralizers of finite cyclic subgroups and its contribution is independent of the relevant K- or L-theory and the ring R. The choice of the K- and L-theory and of R appear in a different contribution in terms of the values at RC for all finite cyclic subgroups. This contribution is only affected by the group G in a weak manner, only the possible finite cyclic subgroups C and the finite groups W_GC together with their actions on the group homology of C_GC and the K-and L-theory of RC matter.

8.3 Some Special Cases

There does not seem to be such general formulas as in Theorem 8.15 for the algebraic K- and L-theory of group rings and of the topological K-theory of reduced group C^*-algebras without rationalizing. We state as illustration several special cases. The methods of these computations are essentially Mayer-Vietoris sequence or spectral sequence arguments applied to the sources of the assembly maps and use the fact that for these special cases the Farrell-Jones Isomorphism Conjecture 7.22 and the Baum-Connes Conjecture 7.23 are known to be true.

The next result is a special case of computations in [76, Theorem 4.4], where more generally compact planar groups are treated. The case of cocompact Fuchsian groups has also been carried out for K-theory by Berkhove, Juan-Pineda and Pearson [12].

Theorem 8.20 *Let F be a cocompact Fuchsian group with presentation*

$$F=\langle a_1,b_1,\ldots,a_g,b_g,c_1,\ldots,c_t \mid c_1^{\gamma_1}=\ldots=c_t^{\gamma_t}=c_1^{-1}\cdots c_t^{-1}[a_1,b_1]\cdots[a_g,b_g]=1\rangle$$

for integers $g,t\geq 0$ and $\gamma_i>1$. Then

1. *The inclusions of the maximal subgroups* $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ *induce an isomorphism*
$$\bigoplus_{i=1}^{t} \mathrm{Wh}_q(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_q(F)$$
for $q \leq 1$*;*

2. *There are isomorphisms*
$$L_q(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1 + \sum_{i=1}^{t} \left[\frac{\gamma_i}{2}\right]\right) \cdot \mathbb{Z}[1/2] & q \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z}[1/2] & q \equiv 1 \ (4); \\ \left(1 + \sum_{i=1}^{t} \left[\frac{\gamma_i - 1}{2}\right]\right) \cdot \mathbb{Z}[1/2] & q \equiv 2 \ (4); \\ 0 & q \equiv 3 \ (4), \end{cases}$$
where $[r]$ *for* $r \in \mathbb{R}$ *denotes the largest integer less than or equal to* r*;*

3. *There are isomorphisms*
$$K_q^{\mathrm{top}}(C_r^*(F)) \cong \begin{cases} \left(2 + \sum_{i=1}^{t} (\gamma_i - 1)\right) \cdot \mathbb{Z} & q = 0; \\ (2g) \cdot \mathbb{Z} & q = 1. \end{cases}$$

The next result is taken from [76, Theorem 0.2].

Theorem 8.21 *Let* $1 \to \mathbb{Z}^n \to G \to \pi \to 1$ *be a group extension for a finite group* π *such that the conjugation action of* π *on* $\mathbb{Z}^n$ *is free, i.e. the only element in* π *with a fixed point in* $\mathbb{Z}^n$ *different from zero is the identity element in* π*. Let* $\{M_\alpha \mid \alpha \in A\}$ *be a complete system of representatives of conjugacy classes of maximal finite subgroups of* G*. Then*

1. *The natural map induced by the inclusions of subgroups*
$$\bigoplus_{\alpha \in A} \mathrm{Wh}_q(M_\alpha) \quad \to \quad \mathrm{Wh}_q(G)$$
is an isomorphism for $q \leq 1$*, and* $K_q(\mathbb{Z}G)$ *is trivial for* $q \leq -2$*;*

2. *There are short exact sequences*
$$0 \to \bigoplus_{\alpha \in A} \widetilde{L}_q(\mathbb{Z}M_\alpha)[1/2] \to L_q(\mathbb{Z}G)[1/2] \\ \to H_q(G\backslash E(G, \mathcal{FIN}); \mathbf{L}(\mathbb{Z}))[1/2] \to 0,$$

where $\mathbf{L}(\mathbb{Z})$ *is the* L*-theory spectrum associated to the ring* $\mathbb{Z}$, $H_*(-;\mathbf{L}(\mathbb{Z}))$ *is the associated homology theory and the first map is induced by the various inclusions* $M_\alpha \to G$.

If we invert $2|\pi|$, *this sequence splits and we obtain isomorphisms*

$$\left(\bigoplus_{\alpha\in A} \widetilde{L}_q(\mathbb{Z}M_\alpha)\left[\frac{1}{2|\pi|}\right]\right) \oplus H_q(\pi\backslash T^n;\mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right] \xrightarrow{\cong} L_q(\mathbb{Z}G)\left[\frac{1}{2|\pi|}\right],$$

where the π*-action on* T^n *is induced by the conjugation action of* π *on* $\mathbb{Z}^n$*;*

3. *If* $|\pi|$ *is odd, then we obtain a short exact sequence which splits after inverting* $|\pi|$

$$0 \to \bigoplus_{\alpha} \widetilde{L}_q^\epsilon(\mathbb{Z}M_\alpha) \to L_q^\epsilon(\mathbb{Z}G) \to H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \to 0;$$

4. *There are short exact sequences*

$$0 \to \bigoplus_{\alpha\in A} \widetilde{K}_q^{\mathrm{top}}(C_r^*(M_\alpha)) \to K_q^{\mathrm{top}}(C_r^*(G)) \to K_q^{\mathrm{top}}(G\backslash E(G;\mathcal{FIN})) \to 0,$$

where $K_q^{\mathrm{top}}(G\backslash E(G;\mathcal{FIN}))$ *is the topological complex* K*-homology of the orbit space* $G\backslash E(G;\mathcal{FIN})$ *and the first map is induced by the various inclusions* $M_\alpha \to G$.

If we invert $|\pi|$, *this sequence splits and we obtain isomorphisms*

$$\left(\bigoplus_{\alpha\in A} \widetilde{K}_q^{\mathrm{top}}(C_r^*(M_\alpha))\left[\frac{1}{|\pi|}\right]\right) \oplus K_q^{\mathrm{top}}(\pi\backslash T^n)\left[\frac{1}{|\pi|}\right] \xrightarrow{\cong} K_q^{\mathrm{top}}(C_r^*(G))\left[\frac{1}{|\pi|}\right].$$

Next we give a description of the algebraic K- and L-groups of the integral group ring and the topological K-theory of the reduced C^*-algebra of all 2-dimensional crystallographic groups. The algebraic K-theory in dimension ≤ 1 has been determined in [92], and the topological K-theory has been computed in [125].

Our notation for the two-dimensional crystallographic groups follows that of [77]. The so called signatures of crystallographic groups which encode their presentations have been listed in [77, page 1204].

For the computations of Wh_2 we assume that the Farrell-Jones Isomorphism Conjecture 7.22 for algebraic K-theory holds also for $q = 2$. In some cases we can drop this assumption for some special reasons.

We denote by D_{2n} the dihedral group of order $2n$.

Group	Signature	$\mathrm{Wh}_q \neq 0, q \leq 2$	$L_q(\mathbb{Z}G)$	$K_q^{\mathrm{top}}(C_r^*(G))$
$P1$	$(1, +, [\,], \{\,\})$		$L_0 = \mathbb{Z} \oplus \mathbb{Z}/2$ $L_1 = \mathbb{Z}^2$ $L_2 = \mathbb{Z} \oplus \mathbb{Z}/2$ $L_3 = (\mathbb{Z}/2)^2$	$K_0 = \mathbb{Z}^2$ $K_1 = \mathbb{Z}^2$
$P2$	$(0, +, [2, 2, 2, 2], \{\,\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^6$ $K_1 = 0$
$P3$	$(0, +, [3, 3, 3], \{\,\})$		$L_0 = \mathbb{Z}^4 \oplus \mathbb{Z}/2$ $L_1 = 0$ $L_2 = \mathbb{Z}^4 \oplus \mathbb{Z}/2$ $L_3 = 0$	$K_0 = \mathbb{Z}^8$ $K_1 = 0$
$P4$	$(0, +, [2, 4, 4], \{\,\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^9$ $K_1 = 0$
$P6$	$(0, +, [2, 3, 6], \{\,\})$	$K_{-1} = \mathbb{Z}$ $\mathrm{Wh}_2 = \mathrm{Wh}_2(\mathbb{Z}/6)$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^{10}$ $K_1 = 0$
Cm	$(1, -, [\,], \{(\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^2$ $L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^2$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^2$ $K_1 = \mathbb{Z}^2$

Group	Signature	$\mathrm{Wh}_q \neq 0, q \leq 2$	$L_q(\mathbb{Z}G)$	$K_q^{\mathrm{top}}(C_r^*(G))$
Pm	$(0,+,[\,],\{(\,),(\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$ $L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^3$ $K_1 = \mathbb{Z}^3$
Pg	$(2,-,[\,],\{\,\})$		$L_0 = \mathbb{Z}\oplus\mathbb{Z}/2$ $L_1 = \mathbb{Z}\oplus\mathbb{Z}/2$ $L_2 = \mathbb{Z}/2$ $L_3 = (\mathbb{Z}/2)^2$	$K_0 = \mathbb{Z}$ $K_1 = \mathbb{Z}\oplus\mathbb{Z}/2$
Cmm	$(0,+,[2],\{(2,2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)^2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^6$ $K_1 = 0$
Pmm	$(0,+,[\,],\{(2,2,2,2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)^4$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^9$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^9$ $K_1 = 0$
Pmg	$(0,+,[2,2],\{(\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$ $L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^4$ $K_1 = \mathbb{Z}$
Pgg	$(1,-,[2,2],\{\,\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^3$ $K_1 = \mathbb{Z}/2$
$P3m1$	$(0,+,[3],\{(3)\})$	$\mathrm{Wh}_2 = \mathbb{Z}/2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$ $L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$ $L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^5$ $K_1 = \mathbb{Z}$
$P31m$	$(0,+,[\,],\{(3,3,3)\})$	$\mathrm{Wh}_2 = (\mathbb{Z}/2)^3$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$ $L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^5$ $K_1 = \mathbb{Z}$
$P6m$	$(0,+,[\,],\{(2,3,6)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)\oplus\mathbb{Z}/2$ $\oplus\,\mathrm{Wh}_2(D_{12})$ $K_{-1} = \mathbb{Z}$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^8$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^8$ $K_1 = 0$
$P4m$	$(1,+,[\,],\{(2,4,4)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)\oplus\mathrm{Wh}_2(D_8)^2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^9$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = 0$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^9$ $K_1 = 0$
$P4g$	$(0,+,[4],\{(2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$ $L_1\left[\frac{1}{2}\right] = 0$ $L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^1$ $L_3\left[\frac{1}{2}\right] = 0$	$K_0 = \mathbb{Z}^6$ $K_1 = 0$

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Notation

The Borel Conjecture

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Contents

0 Preface

These lectures are about showing that homotopy equivalence implies homeomorphism for a large class of manifolds. About 50 years ago Borel conjectured that this class includes all closed manifolds with contractible universal covers. A more precise statement of his conjecture is the following.

Borel Conjecture. Let $f : M \to N$ be a homotopy equivalence where both M and N are closed aspherical manifolds. Then f is homotopic to a homeomorphism.

We explain in lectures 2-5 why this conjecture is true in the following special cases (due to Farrell and Jones);

1. M is a non-positively curved Riemannian manifold and $\dim(M) \neq 3, 4$.
2. Both M and N are complete affine flat manifolds.
3. $\pi_1(M)$ is isomorphic to a discrete subgroup of $GL_n(\mathbb{R})$ for some n, and $\dim(M) \neq 3, 4$.

The Borel Conjecture has the following (slightly weaker when $n \neq 3$) group theoretic interpretation in which $\mathrm{Top}(\mathbb{R}^n)$ denotes the group of all self-homeomorphisms of $\mathbb{R}^n$ equipped with the compact open topology.

(Topological) Strong Rigidity Conjecture. Let Γ_1 and Γ_2 be any pair of isomorphic subgroups of $\mathrm{Top}(\mathbb{R}^n)$. Suppose that the two naturally induced actions on $\mathbb{R}^n$ are free, properly discontinuous, and have compact fundamental domains. Then Γ_1 and Γ_2 are conjugate subgroups inside $\mathrm{Top}(\mathbb{R}^n)$ (with the isomorphism induced by the conjugation).

Special case 3 of the Borel Conjecture yields that the (Topological) Strong Rigidity Conjecture is true under the extra assumption that there exists a linear (virtually connected) Lie group G containing Γ_1 and contained in $\mathrm{Top}(\mathbb{R}^n)$, and $n \neq 3, 4$. This partial result is an analogue of Mostow's Strong Rigidity Theorem [57] in Lie group theory. In fact it was motivated by Mostow's result although the technique of proof, surgery theory, is very different from Mostow's.

Lecture 1 is an introduction to the general problem of classifying, up to homeomorphism, all manifolds homotopically equivalent to a given manifold M. This is the topic of surgery theory. After reading lecture 1, we recommend perusing W. Lueck's lectures on surgery theory before looking at lectures 2-5 which also depend on L.E. Jones' lectures and on the first two of A. Ranicki.

1 Introduction to high dimensional manifold topology

Throughout this talk M and N will denote (connected) closed n-manifolds by which I mean (as usual) compact Hausdorff spaces which are locally homeomorphic to $\mathbb{R}^n$.

Basic Problem. Find calculable invariants which imply that M and N are homeomorphic.

This problem is easy to solve when dim $M \leq 2$. For example the circle is the only such 1-manifold. And the following is a complete list of the orientable 2-manifolds:

Figure 1.1 genus = # of holes

To study this problem when dim $M > 2$, it is helpful to use the weaker notion of *homotopy equivalence* which is easier to study than homeomorphism because of the following result.

Theorem. *(J.H.C. Whitehead) A continuous map $f : M \to N$ is a homotopy equivalence iff it induces an isomorphism on π_n for all n.*

Caveat. There are examples where $\pi_n(M) \simeq \pi_n(N)$ for all n; but M is *not* homotopically equivalent to N. Whitehead requires that the isomorphism is induced by a continuous map! Here is an explicit example.

Let $M = S^2 \times S^2$ and $N = S(\eta^2 \oplus \theta^1)$ where η^2 is the canonical $\mathbb{C}$-line bundle over $\mathbb{C}P^1 = S^2$, θ^1 is the trivial $\mathbb{R}$-line bundle and $S(\eta^2 \oplus \theta^1)$ denotes the sphere bundle associated to the Whitney sum $\eta^2 \oplus \theta^1$. Since the fibration

$$S^2 \to N \to S^2$$

has a cross section

$$\pi_n(N) = \pi_n(S^2) \oplus \pi_n(S^2)$$

but the 2nd Stiefel-Whitney class $w_2(N) \neq 0$. And recall that w_2 is an invariant of homotopy equivalences but

$$w_2(S^2 \times S^2) = 0.$$

Remark. Note that $N = \mathbb{C}P^2 \# - \mathbb{C}P^2$ and hence its cup product pairing is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

while the pairing for M is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And these are inequivalent bilinear forms over $\mathbb{Z}$.

However there is an important special case where this worry is unnecessary.

Definition. M is *aspherical* if $\pi_n(M) = 0$ for all $n \neq 1$. (This is equivalent to requiring that the universal cover $\tilde{M}$ of M is contractible; but not necessarily that $\tilde{M} = \mathbb{R}^m$ as M. Davis will show in his lecture today.)

Corollary. *(Hurewicz) If $\pi_1(M) \simeq \pi_1(N)$ and both M and N are aspherical, then M and N are homotopically equivalent.*

Historical Remark. Hurewicz proved this result before Whitehead proved his theorem.

Examples. Every orientable (connected) 2-manifold except the sphere is aspherical (as is the circle).

More generally every non-positively curved closed Riemannian m-manifold is aspherical because of Cartan's theorem that the universal cover of such a manifold is diffeomorphic to $\mathbb{R}^m$. On the other hand, the following homogeneous space G/Γ is an example of a closed aspherical 3-manifold which does not support a non-positively curved Riemannian metric. Here G is the matrix group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

and Γ is the discrete subgroup where x, y, z are all integers. Note that the universal cover of G/Γ is G which is diffeomorphic to $\mathbb{R}^3$; hence G/Γ is aspherical and is easily seen to be compact. Also

$$\pi_1(G/\Gamma) = \Gamma$$

which is *nilpotent* but *not abelian.* However Gromoll and Wolf [40] and Yau [71] independently proved that if M is a (closed) non-positively curved Riemannian manifold and $\pi_1(M)$ is nilpotent, then $\pi_1(M)$ is abelian (and if solvable, then virtually abelian).

Basic Question. *Are homotopically equivalent closed manifolds M and N homeomorphic?* More precisely*: Is every homotopy equivalence $f : M \to N$ homotopic to a homeomorphism?*

The answer is *Yes* for 1 and 2 dimensional manifolds. But Moise [55] showed that in general the answer is *No.* In fact it is No for 3-manifolds. Let me explain. Lens spaces were studied extensively in the 1930's. These are 3-manifolds whose universal covers are the 3-sphere and have cyclic π_1's of order > 2 (and all deck transformations in $SO(4)$).

Reidemeister gave examples of pairs of Lens spaces which are homotopically equivalent but *not diffeomorphic.* (See W. Lueck's lecture 2.4 for a detailed discussion of Lens spaces which includes this result.) In particular $M = L(7;1,1)$ and $N = L(7;1,2)$ are such a pair. Here the subgroups $\pi_1(M)$ and $\pi_1(N)$ of $SO(4)$ are described as follows. Note that $SU(2) \subseteq SO(4)$. Then $\pi_1(M)$ and $\pi_1(N)$ are the cyclic subgroups of order 7 in $SU(2)$ generated by the two diagonal matrices

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}$$

respectively, where $\theta = e^{2\pi i/7}$. Since Moise [55] showed (1952) that *homeomorphic 3-manifolds are diffeomorphic*, it follows that *No* is the answer to the *Basic Question.*

In light of Moise's result, the Basic Question was refined at this time into 2 disjoint conjectures.

1. *Hurewicz Conjecture.* Homotopically equivalent closed manifolds with $\pi_1 = 0$ are homeomorphic.

2. *Borel Conjecture.* Homotopically equivalent aspherical manifolds (i.e. with $\pi_n = 0$ for $n \neq 1$) are homeomorphic.

Remark. In Moise's example both π_1 and $\pi_3 \neq 0$.

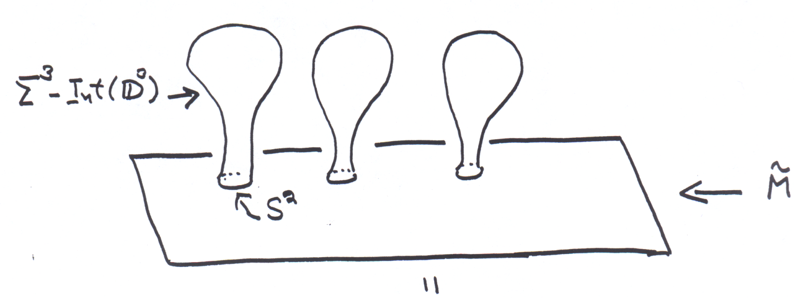
Remark. The Poincaré Conjecture is a special case of the Hurewicz Conjecture where $M = S^3$. It is still open; but the Hurewicz Conjecture has been proven when $M = S^m$, $m \neq 3$, by Smale ($m \geq 6$), Stallings ($m = 5$) and Freedman ($m = 4$).

Remark. Although S^3 is not aspherical, the Borel Conjecture also (indirectly) implies the Poincaré Conjecture. To see this we use the following two results.

Generalized Schoenflies Theorem. *(M. Brown [11]). Let $f : S^2 \to S^3$ be a bicollared embedding, then $f(S^2)$ bounds closed (topological) balls on both sides.*

Alexander Trick. *Let $h : S^n \to S^n$ be any homeomorphism. Then h extends to a homeomorphism $\bar{h} : \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ where $\mathbb{D}^{n+1}$ denotes the closed ball in $\mathbb{R}^{n+1}$ which bounds S^n.*

Now let Σ^3 be a closed 3-manifold homotopically equivalent to S^3 and consider the connected sum $M = T^3 \# \Sigma^3$, where T^3 denotes the 3-torus $S^1 \times S^1 \times S^1$. Since T^3 and M^3 have isomorphic fundamental groups and M is easily seen to be aspherical, T^3 and M^3 are homotopically equivalent because of the above mentioned Corollary due to Hurewicz. Hence Borel's Conjecture implies that $T^3 \# \Sigma^3$ is homeomorphic to T^3. And consequently $\tilde{M}$ = the universal cover of $T^3 \# \Sigma^3$ is homeomorphic to $\mathbb{R}^3$. Therefore the Generalized Schoenflies theorem shows that $\Sigma^3 - \mathrm{Int}(\mathbb{D}^3)$ is homeomorphic to $\mathbb{D}^3$.



$\mathbb{R}^3$

Figure 1.2

(This Int($\mathbb{D}^3$) is the interior of the 3-ball removed from Σ^3 in forming the connected sum with T^3.) Now applying the Alexander Trick we get that Σ^3 is homeomorphic to S^3. In this way, the Borel Conjecture implies the Poincaré Conjecture.

The Borel Conjecture is still open; but Novikov [58], [59] in 1966 showed that the Hurewicz Conjecture is *false*, in fact "generically" false.

Theorem. *(Novikov) Let M^m ($m \geq 5$ and $\not\equiv 2 \bmod 4$) be any smooth (closed) manifold such that*

1. $\pi_1(M) = 0$
2. $H_4(M, \mathbb{Q}) \neq 0$
3. *$M - pt$ is parallelizable*

(e.g. $M = S^4 \times S^5$). Then there exists a homotopically equivalent smooth (closed) manifold N which is not homeomorphic to M.

To understand this result, I need to recall the notions of tangent bundle and Pontryagin classes. Due to Whitney every smooth manifold M^m embeds in $\mathbb{R}^{2m+2}$.

$$M^m \quad\subseteq\quad \mathbb{R}^{2m+2}$$

Figure 1.3

To each point $x \in M$, let T_xM be the tangent space to M at x parallel translated to the origin $0 \in \mathbb{R}^{2m+2}$. This defines the *Gauss map* $TM : M \to G_m$ = the Grassman manifold of all m-planes (containing 0) in $\mathbb{R}^{2m+2}$. The Gauss map is continuous and well defined up to homotopy since any pair of embeddings are isotopic. Also if $f : M \to N$ is a diffeomorphism, then $TN \circ f$ is homotopic to TM.

The cohomology groups of G_m can be computed. In particular there are classes $p_i \in H^{4i}(G_m, \mathbb{Q})$ and their pullbacks under

$$(TM)^* : H^*(G_m, \mathbb{Q}) \to H^*(M, \mathbb{Q})$$

are called the (rational) *Pontryagin classes* of M and denoted by $p_i(M)$.

So if $f : M \to N$ is a diffeomorphism, then $f^*(p_i(N)) = p_i(M)$. But one can construct a smooth manifold N^9 which is homotopically equivalent to $M^9 = S^4 \times S^5$ where $p_1(N) \neq 0$. In fact N^9 is the total space of a fiber bundle over S^4 whose fibre is S^5 constructed using 24 times the generator of $\pi_3(O(6)) = \mathbb{Z}$. But $p_1(S^4 \times S^5) = 0$ since $S^4 \times S^5 \subseteq \mathbb{R}^{10}$ and hence

$$T(S^4 \times S^5) : S^4 \times S^5 \to G_9$$

factors through $\mathbb{R}P^9$ and $H^4(\mathbb{R}P^9, \mathbb{Q}) = 0$. So M and N are *not diffeomorphic.*

But, on the other hand, Milnor [53] had shown in 1956 that (in high dimensions) homeomorphism does *not* imply diffeomorphism as it does in dimensions 1, 2 and 3. Still Novikov [59] proved the following.

Theorem. *(Novikov 1966) If $f : M \to N$ is a homeomorphism between smooth manifolds, then $f^*(p_iN) = p_iM$.*

Corollary. *$S^4 \times S^5$ and N^9 are not homeomorphic thus disproving the Hurewicz Conjecture.*

Novikov had used the strong advances in Algebraic Topology made during the 1950's (e.g. Serre's mod-$\mathcal{C}$ Hurewicz Theorem) to reduce the proof of his Theorem to the following key lemma.

Lemma. *(Novikov) Let E be the total space of a real vector bundle η whose base space is S^m. If E is homeomorphic to $S^m \times \mathbb{R}^n$ and both $m \geq 5$ and $n \geq m + 2$, then η is the trivial bundle.*

We sketch a proof of Novikov's Key Lemma since it contains new ideas which allow the tools of differential topology to be used under a topological assumption. The strategy is to construct a smooth embedding $\sigma : S^m \to E$ homotopic to the 0-section embedding $\sigma_0 : S^m \to E$ and such that $v_\sigma =$ normal bundle of σ is trivial. This implies that η is trivial since $\eta = v_{\sigma_0}$ and $v_{\sigma_0} \simeq v_\sigma$ because of the Whitney Embedding Theorem which shows that σ is isotopic to σ_0. To construct σ, one first builds a sequence

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = E$$

of smooth submanifolds satisfying:

1. $\dim(M_i) = m + i$.

2. M_i is 2-sided in M_{i+1} $(i < n)$.

3. M_0 is homeomorphic to S^m.

4. The composite embedding $\tau : M_0 \to E$ is a homotopy equivalence.

Let us assume for the moment that this has been done. Now v_τ is clearly trivial. So if M_0 were diffeomorphic to S^m, we could set $\sigma = \tau$ and we'd be done. However M_0 may be an exotic sphere; i.e. a smooth manifold homeomorphic but *not* diffeomorphic to S^m. (See W. Lueck's lecture 6 for a detailed discussion of exotic spheres.) But Kervaire and Milnor [48], using a deep result of Adams [3], showed (for any exotic sphere M_0) that $M_0 \times \mathbb{R}^n$ is diffeomorphic to $S^m \times \mathbb{R}^n$. Identifying $M_0 \times \mathbb{R}^n$ with a tubular neighborhood of M_0 in E, we can set σ to be the composite

$$S^m = S^m \times 0 \subseteq S^m \times \mathbb{R}^n = M_0 \times \mathbb{R}^n \subseteq E.$$

Therefore it remains to indicate how the sequence of submanifolds M_i is constructed. Start by putting an "anchor ring" $T^{n-1} \times \mathbb{R}$ into $\mathbb{R}^n$ where T^s denotes the s-torus; i.e. $T^s = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{s-\text{copies}}$. This is easy to do; the case $n = 2$ is pictured below. The general construction of anchor rings proceeds by induction on n. In particular, cross

$$T^{n-1} \times \mathbb{R} \subseteq \mathbb{R}^n$$

with $\mathbb{R}$ to obtain

$$T^{n-1} \times \mathbb{R}^2 \subseteq \mathbb{R}^{n+1}$$

and note that

$$T^n \times \mathbb{R} = T^{n-1} \times (S^1 \times \mathbb{R}) \subseteq T^{n-1} \times \mathbb{R}^2$$

where $S^1 \times \mathbb{R} \subseteq \mathbb{R}^2$ is the case $n = 2$.

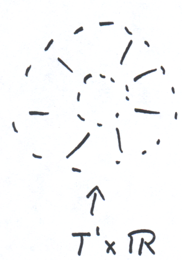


Figure 1.4

Consider the following diagram $*$ in which $f : E \to S^m \times \mathbb{R}^n$ is the given homeomorphism and $V = f^{-1}(S^m \times T^{n-1} \times \mathbb{R})$ which is a smooth manifold since it is an open subset of E.

$$
\begin{array}{ccc}
E \xrightarrow{\quad f \quad} & S^m \times \mathbb{R}^n & \\
\cup & \cup & \\
V \xrightarrow{\ f|_V\ } & S^m \times T^{n-1} \times \mathbb{R} & \\
\cup & \cup & \\
M_{n-1} \xrightarrow{f_{n-1}} & S^m \times T^{n-1} \times 0 & = S^m \times T^{n-1} \\
\cup & & \cup \\
M_{n-2} & \xrightarrow{\qquad\qquad f_{n-2} \qquad\qquad} & S^m \times T^{n-2} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
M_1 & \xrightarrow{\qquad\qquad f_1 \qquad\qquad} & S^m \times S^1 \\
\cup & & \cup \\
M_0 & \xrightarrow{\qquad\qquad f_0 \qquad\qquad} & S^m \times 1 = S^m
\end{array}
$$

The manifolds M_i and homotopy equivalences $f_i : M_i \to S^m \times T^i$ are constructed (downwards) inductively by applying a codimension-one splitting theorem to $f_{i+1} : M_{i+1} \to S^m \times T^{i+1}$, when $i \leq n-2$, and to $f|_V : V \to S^m \times T^{n-1} \times \mathbb{R}$ when $i = n-1$.

The setup for such a theorem is the following:

Let $\phi : \mathcal{W} \to W$ be a (proper) homotopy equivalence between smooth manifolds and $T \subset W$ be a closed 2-sided codimension-one smooth submanifold.

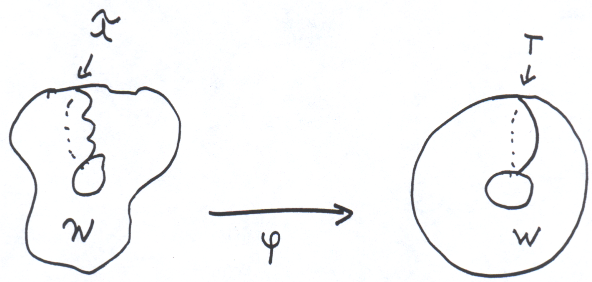


Figure 1.5

Question. Can $T \subset W$ be modeled in $\mathcal{W}$; i.e., can ϕ be (properly) homotoped to a map ψ which is transverse to T and such that

$$\psi|_{\mathcal{T}} : \mathcal{T} \to T$$

is a homotopy equivalence where $\mathcal{T} = \psi^{-1}(T)$.

In the situations occurring in Diagram $*$, the answer is Yes. Codimension-one splitting theorems of this sort have been proved by Browder, Novikov, Levine, Livesay, Siebenmann, Farrell, Hsiang, and Cappell. The two cases occuring in diagram $(*)$ are $W = T \times \mathbb{R}$ and $W = T \times S^1$. If $W = T \times \mathbb{R}$, this splitting theorem is due to Browder [9] when $\pi_1(T) = 0$, Novikov [59] when $\pi_1(T)$ is free abelian, and Siebenmann [67] in general. When $W = T \times S^1$ it is due to Browder and Levine [10] when $\pi_1(T) = 0$ and to Farrell [20] in general.

Notice that Novikov's Theorem also bears on the Borel Conjecture. In particular if the Borel Conjecture is true, then any homotopy equivalence $f : M \to N$ between smooth (closed) aspherical manifolds must preserve (rational) Pontryagin classes. Novikov formulated this explicitly as a conjecture and proved some partial results on it. Further partial results were obtained by Farrell, Hsiang, Kasparov, and Cappell using codimension-one splitting theorems. But perhaps the most interesting early result on this conjecture is due to Mishchenko [56] extending work of Lusztig [52]. (Their work uses a different technique; namely the extension of the Hirzebruck Index Theorem due to Atiyah and Singer.)

Theorem. *(Mischenko [56] 1974). Let $f : M \to N$ be a homotopy equivalence where M is a closed non-positive curved Riemannian manifold (and N is also closed), then $f^*(p_i(N)) = p_i(M)$.*

Let me finish this talk by briefly describing the three steps needed to replace a homotopy equivalence $f : M \to N$ by a homeomorphism. (This process is called surgery theory and is discussed in detail in W. Lueck's lectures.) For this purpose I now make a dimension assumption; namely, I assume that dim $M \geq 5$.

Step 1. Construct a *normal cobordism* W from M to N; i.e. a compact manifold W with boundary $\partial W = M \cup N$ together with a *tangential map* $F : (W, \partial W) \to (N \times [0,1], \partial)$ such that $F|_N = \mathrm{id}_N$ and $F|_M = f$.

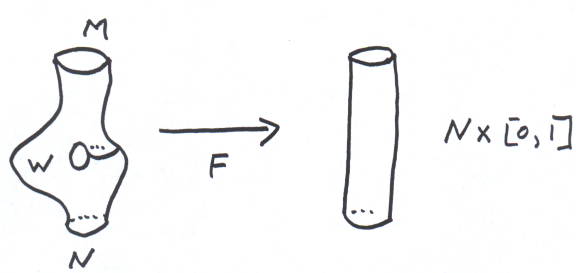


Figure 1.6

(A *tangential map* is a continuous function covered by a map of tangent bundles.) Notice that Step 1 implies that $f^*(p_i(N)) = p_i(M)$. In particular if the construction in Step 1 can be done whenever M is aspherical, then the above conjecture of Novikov is true.

Step 2. Modify some normal cobordism W from M to N, by cutting out (surgering) excess homology, to form a new normal cobordism $\bar{F} : \bar{W} \to N \times [0,1]$ so that $\bar{F}$ is a homotopy equivalence; i.e., $\bar{W}$ is a *h-cobordism.*

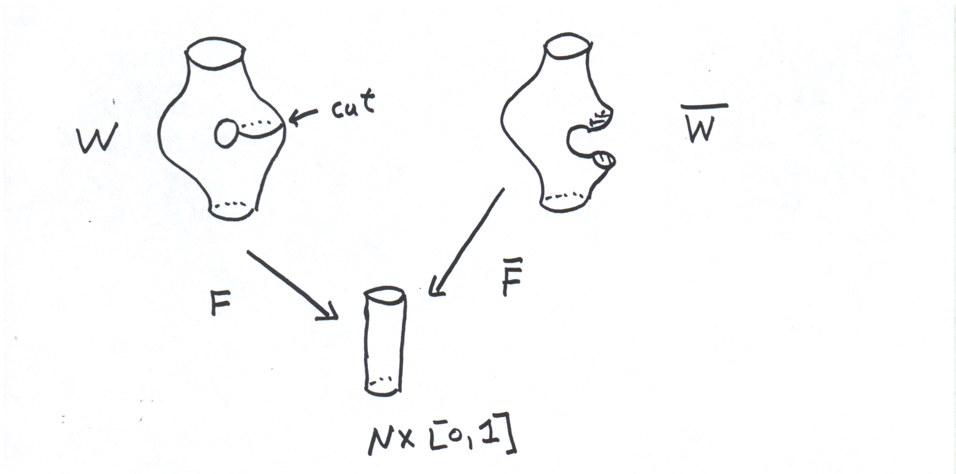


Figure 1.7

Note that the map F from Step 1 has degree one; hence it induces a split epimorphism on homology groups (even with twisted coefficients) because of Lefschetz duality. When there is one, there is usually many normal cobordisms W between M and N. To each of these is associated an element

$$\omega(W) \in L_{m+1}(\pi_1 M^m)$$

– an abelian group defined by C.T.C. Wall, cf. [70]. And $\omega(W) = 0$ iff the desired surgery can be done. Steps 1 and 2 involve calculating $L_{m+1}(\pi_1 M^m)$.
Step 3. Show that the h-cobordism $\bar{W}$ is a cylinder; i.e. $\bar{W} = N \times [0,1]$. Because of the (topological) s-cobordism Theorem, this step involves calculating J.H.C. Whitehead's group $\mathrm{Wh}(\pi_1 M)$. (See lecture 1 of W. Lueck for a discussion of Whitehead groups and the s-cobordism Theorem.)

In particular if the Borel Conjecture is true, then $\mathrm{Wh}(\pi_1 M) = 0$ for every compact aspherical manifold M. This is so even if $\partial M \neq \emptyset$ because Mike Davis has shown how to reduce this more general case to the special case where $\partial M = \emptyset$. (See M. Davis' lectures.) The Borel Conjecture also implies that $L_{m+1}(\pi_1 M^m)$ is finitely generated.

2 Splitting the surgery map under a geometric assumption

Throughout this lecture (unless otherwise stated) M (and N) will denote complete (connected) Riemannian manifolds. Furthermore Γ will denote the group of all deck transformations of the universal cover $\tilde{M} \to M$ and we identify Γ with $\pi_1(M)$. If v is a vector tangent to M (i.e. $v \in TM$ = tangent bundle of M) then

$$\alpha_v : \mathbb{R} \to M$$

denotes the unique geodesic such that $\dot{\alpha}_v(0) = v$.

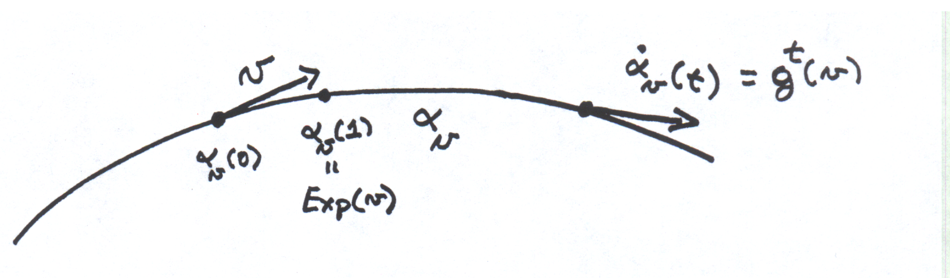


Figure 2.1

The function $\mathbb{R} \times TM \to TM$ defined by

$$g^t(v) = \dot{\alpha}_v(t)$$

for $t \in \mathbb{R}$ and $v \in TM$, is a flow on TM; i.e. it is smooth and satisfies the equation

$$g^s(g^t(v)) = g^{s+t}(v)$$

for all $s, t \in \mathbb{R}$ and $v \in TM$. This flow leaves invariant SM = unit sphere bundle of M and its restriction to SM is called the *geodesic flow*. Closely related to the geodesic flow is the *exponential function* Exp : $TM \to M$ defined by

$$\mathrm{Exp}(v) = \alpha_v(1).$$

It is also a smooth function. If we fix a base point $x_0 \in M$, then the restriction of Exp to $T_{x_0}M$ = tangent space to M at x_0 is also called the exponential function and denoted by

$$\exp_{x_0} : T_{x_0}M \to M.$$

(Or more simply by exp when no ambiguity is possible.) Note that the vector space $T_{x_0}M$ considered as a smooth manifold $N = T_{x_0}M$ has a natural complete Riemannian metric; namely, if $u \in TN$, then $|u| = \sqrt{U \cdot U}$ where U is the parallel translate of u to 0.

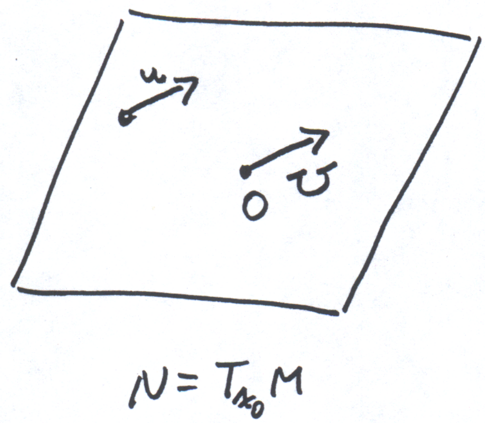


Figure 2.2

We say that M is *non-positively curved* (resp. *negatively curved*) if all its sectional curvatures are ≤ 0 (resp. < 0). And a negatively curved manifold is *pinched negatively curved* if its sectional curvatures are bounded away from 0 and $-\infty$. Note that a closed negatively curved manifold is pinched negatively curved.

Definition. A smooth map $f : M \to N$ is called (weakly) *expanding* if

$$|df(v)| \geq |v|$$

for all vectors $v \in TM$.

There is the following important result relative to these definitions.

Theorem. *(Cartan) Let M be non-positively curved and $x_0 \in M$ be a base point. Then* exp $: T_{x_0}M \to M$ *is an expanding map. Furthermore it is a covering projection and hence a diffeomorphism when $\pi_1(M) = 0$.*

Because of Cartan's theorem a non-positively curved (Riemannian) manifold M^m is aspherical since its universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^m$.

(See [42, p. 172] and [54, p. 102] for a discussion of Cartan's theorem.) It also leads to the following useful alternate description of TM as the bundle with fiber $\tilde{M}$ associated to the principal Γ-bundle $\tilde{M} \to M$; namely

$$\tilde{M} \times_\Gamma \tilde{M} \to M.$$

In fact this bundle is indentified with $TM \to M$ as $\mathrm{Diff}(\mathbb{R}^m)$-bundles via the Γ-equivariant diffeomorphism

$$T\tilde{M} \to \tilde{M} \times \tilde{M}$$

which sends $v \in T\tilde{M}$ to $(\alpha_v(0), \alpha_v(1))$. The 0-section of TM corresponds (under this identification) with the image of the diagonal Δ of $\tilde{M} \times \tilde{M}$ in $\tilde{M} \times_\Gamma \tilde{M}$.

There is also a natural *geodesic ray compactification* $\bar{M}$ of $\tilde{M}$ due to Eberlein and O'Neill [17] such that $(\bar{M}, \tilde{M})$ is homeomorphic to $(\mathbb{D}^m, \mathrm{Int}\ \mathbb{D}^m)$ where

$$\mathbb{D}^m = \{v \in \mathbb{R}^m \mid |v| \leq 1\}.$$

Let $M(\infty) = \bar{M} - \tilde{M}$ denote the points added; called *ideal points.* Each ideal point is an asymptoty class of geodesic rays in $\tilde{M}$. A *geodesic ray* is a subset of $\tilde{M}$ of the form

$$\{\alpha_v(t) \mid t \in [0, +\infty)\}$$

for some $v \in S\tilde{M}$. Two rays R_1 and R_2 are *asymptotic* if there exists a positive number b such that each point of R_1 is within distance b of some point of R_2 and vice-versa.

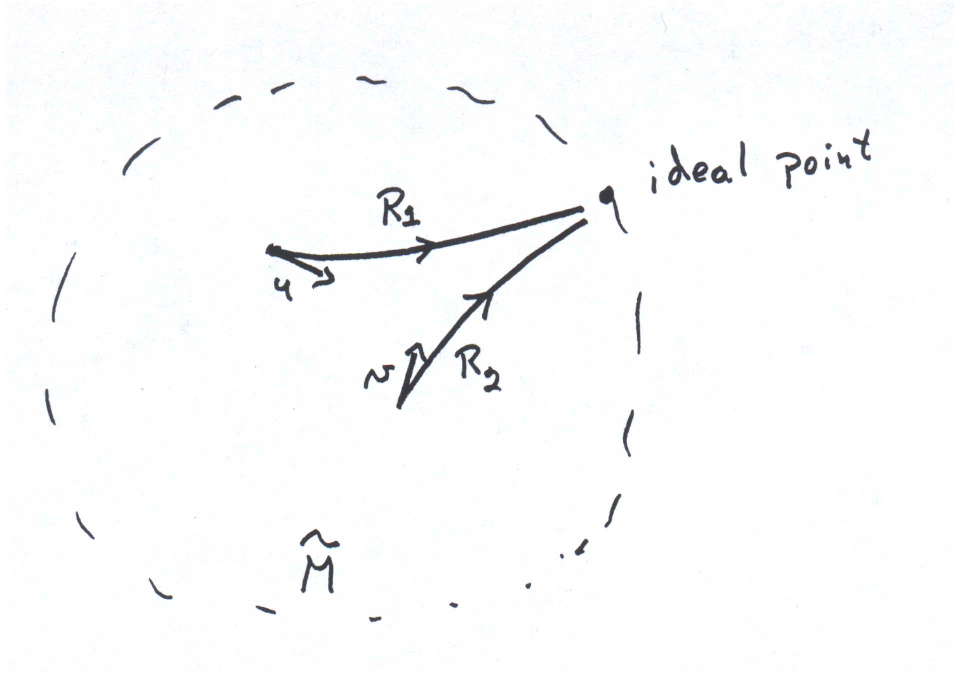


Figure 2.3

The deck transformation action of Γ on $\tilde{M}$ extends to an action on $\bar{M}$ since Γ acts via isometries on $\tilde{M}$ and isometries preserve both geodesic rays and the relation of being asymptotic.

W.C. Hsiang and I abstracted an additional key property possessed by the geodesic ray compactification in the following definition [24], see also lectures 6, 7, 8 in [21]. (For the rest of this lecture M denotes a closed topological manifold and *not necessarily a Riemannian manifold.*)

Definition. A closed manifold M^m satisfies condition $(*)$ provided there exists an action of $\Gamma = \pi_1(M^m)$ on $\mathbb{D}^m$ with the following two properties.

1. The restriction of this action to $\mathrm{Int}(\mathbb{D}^m)$ is equivalent via a Γ-equivariant homeomorphism to the action of Γ by deck transformations on the universal cover $\tilde{M}$ of M^m.

2. Given any compact subset K of $\mathrm{Int}(\mathbb{D}^m)$ and any $\epsilon > 0$, there exists a real number $\delta > 0$ such that the following is true for every $\gamma \in \Gamma$. If the distance between γK and $S^{m-1} = \partial\mathbb{D}^m$ is less than δ, then the diameter of γK is less than ϵ.

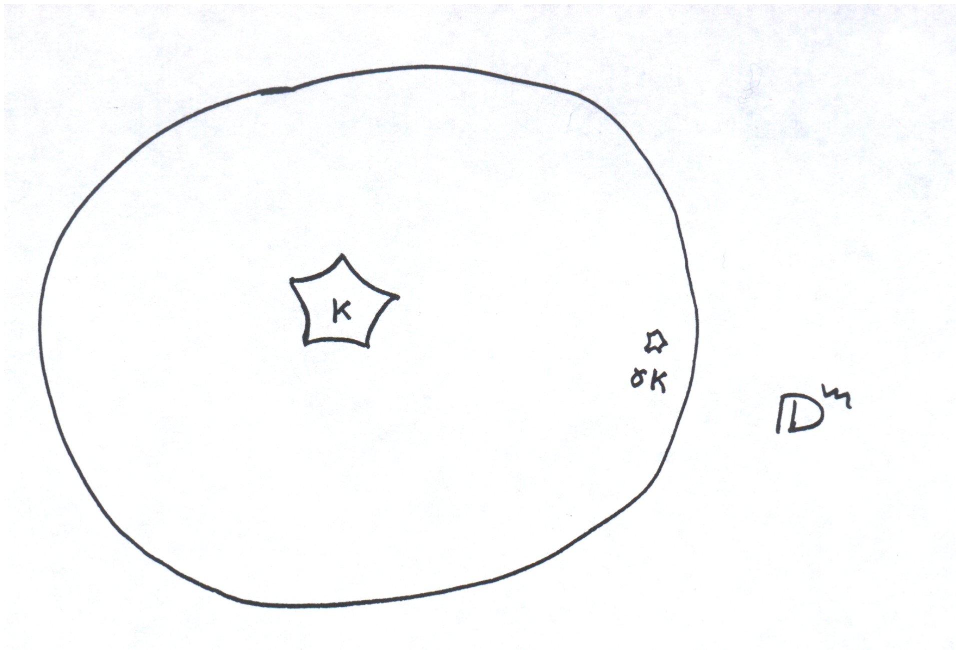


Figure 2.4

The above picture illustrates property 2 of condition $(*)$.

Remark. Hsiang and I showed that every closed (connected) non-positively curved Riemannian manifold M satisfies condition $(*)$ by using its geodesic ray compactification.

Remark. Any manifold satisfying condition $(*)$ is obviously aspherical. It was conceivable 20 years ago, when this condition was formulated, that every closed aspherical manifold M^m satisfies condition $(*)$. But then Mike Davis [14] constructed closed aspherical manifolds M^m where $\tilde{M} \neq \mathbb{R}^m$ contradicting property 1 of condition $(*)$.

On the other hand, $M^m \times S^1$ satisfies property 1 of condition $(*)$ whenever $\tilde{M} = \mathbb{R}^m$. This is seen as follows. Let $\mathbb{Z}$ denote the additive group of integers. Its natural action by translations on $\mathbb{R}$ extends to an action on $[-\infty, +\infty)$ where each group element fixes $-\infty$. We hence have a product action of $\pi_1(M \times S^1) = \pi_1(M) \times \mathbb{Z}$ on

$$\tilde{M} \times [-\infty, +\infty) = \mathbb{R}^m \times [0, +\infty)$$

which extends to its one point compactification $\mathbb{D}^{m+1}$. If we let this be the action posited in the above Definition, then it satisfies property 1 of condition $(*)$ but *not* property 2.

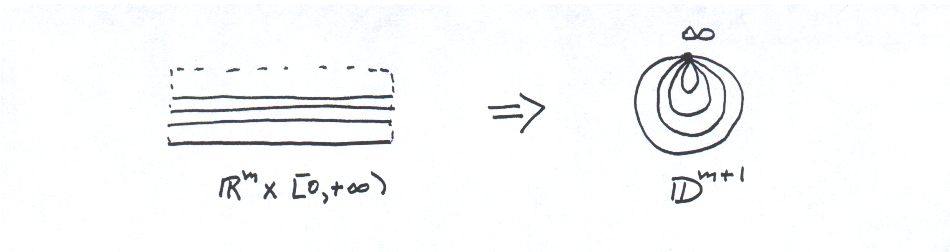


Figure 2.5

We also note that the universal cover X of $M^m \times S^1$ is $\mathbb{R}^{m+1}$ for any closed aspherical manifold M^m where $m \geq 5$ because X is contractible and simply connected at ∞. This is a result of Newman (1966).

Theorem. *(Farrell-Hsiang [24] 1981) Let M^m be a closed manifold satisfying condition $(*)$. Then the map in the (simple) surgery sequence*

$$\mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) \to [M^m \times \mathbb{D}^n, \partial; G/\mathrm{Top}]$$

is identically zero when $n \geq 1$ and $n + m \geq 6$.

So as not to obscure the argument, we sketch the proof of this Theorem under the extra assumptions that M is triangulable and $n = 1$. (See also lectures 6, 7, 8 in [21].) Set

$$E^{2m} = \tilde{M} \times_\Gamma \tilde{M}$$

and let $p : E^{2m} \to M$ denote the bundle projection. Then the following square commutes:

$$\begin{array}{ccc} \mathcal{S}^s(\mathbb{D}^1 \times M, \partial) & \longrightarrow & [\mathbb{D}^1 \times M, \partial; G/\mathrm{Top}] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (\mathrm{id}\times p)^*} \\ \mathcal{S}(\mathbb{D}^1 \times E, \partial) & \longrightarrow & [\mathbb{D}^1 \times E, \partial; G/\mathrm{Top}] \end{array}$$

where α is the transfer map defined as follows. Let the simple homotopy equivalence

$$h : (W, \partial W) \to (\mathbb{D}^1 \times M, \partial)$$

represent an element $b \in \mathcal{S}^s(\mathbb{D}^1 \times M, \partial)$. Then the proper homotopy equivalence

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

represents $\alpha(b) \in \mathcal{S}(\mathbb{D}^1 \times E, \partial)$ where

$$\mathcal{W} = \{(x, y) \in W \times (\mathbb{D}^1 \times E) \mid h(x) = \mathrm{id} \times p(y)\}$$

and $\hat{h}(x, y) = y$. Since p is a homotopy equivalence, $(\mathrm{id} \times p)^*$ is an isomorphism. Hence the Theorem is a consequence of the following:

Assertion. *The map α is identically zero.*

We proceed to verify this. Note first that W is an s-cobordism and hence a cylinder because of the s-cobordism theorem. We may therefore assume that $W = [0, 1] \times M$ and that h is a homotopy between id_M and a self-homeomorphism $f : M \to M$. Furthermore, if f is pseudo-isotopic to id_M via a pseudo-isotopy homotopic to h rel ∂, then $b = 0$.

Let $\tilde{h}$ be the unique lift of h to $[0, 1] \times \tilde{M}$ such that $\tilde{h}$ is a proper homotopy between $\mathrm{id}_{\tilde{M}}$ and a self-homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$, which is a lift of f. Then $\tilde{h} \times \mathrm{id}_{\tilde{M}}$ determines a proper homotopy

$$k : [0, 1] \times E \to [0, 1] \times E$$

between id_E and a self-homeomorphism $g : E \to E$ (which is also determined by $\tilde{f} \times \mathrm{id}_{\tilde{M}}$). Since

$$\hat{h} : (\mathcal{W}, \partial W) \to (\mathbb{D}^1 \times E, \partial)$$

can be identified with

$$k : ([0, 1] \times E, \partial) \to ([0, 1] \times E, \partial),$$

the Assertion is an immediate consequence of the following.

Lemma. *g is pseudo-isotopic to id_E via a pseudo-isotopy which is properly homotopic to k rel ∂.*

We now use our assumption that M^m satisfies condition $(*)$ to prove this lemma. Identify $\tilde{M}$ with $\mathbb{D}^m$ and define a manifold $\bar{E}$ by

$$\bar{E} = \mathbb{D}^m \times_\Gamma \tilde{M}.$$

Then $E = \mathrm{Int}(\bar{E})$ and property 2 of condition $(*)$ implies that $\tilde{f}$ extends to a Γ-equivariant homeomorphism

$$\bar{f} : \mathbb{D}^m \to \mathbb{D}^m$$

by setting $\bar{f}|_{S^{m-1}} = \mathrm{id}_{S^{m-1}}$. Consequently $\bar{f} \times \mathrm{id}_{\tilde{M}}$ determines a self-homeomorphism

$$\bar{g} : \bar{E} \to \bar{E}$$

which extends $g : E \to E$ and satisfies $\bar{g}|_{\partial\bar{E}} = \mathrm{id}_{\partial\bar{E}}$. We proceed to construct a pseudo-isotopy

$$\phi : \bar{E} \times [0,1] \to \bar{E} \times [0,1]$$

satisfying

1. $\phi|_{\bar{E}\times 0} = \bar{g}$;

2. $\phi|_{\bar{E}\times 1} = \mathrm{id}_{\bar{E}\times 1}$;

3. $\phi|_{(\partial\bar{E})\times[0,1]} = \mathrm{id}_{(\partial\bar{E})\times[0,1]}$.

Properties (1-3) define ϕ on $\partial(\bar{E} \times [0,1])$. To construct ϕ over $\mathrm{Int}(\bar{E} \times [0,1])$ consider the natural fiber bundle

$$\bar{E} \times [0,1] \xrightarrow[q]{} M$$

with fiber $\mathbb{D}^m \times [0,1]$. And note the following. If Δ is an n-simplex in M, then $q^{-1}(\Delta)$ can be identified with $\mathbb{D}^{n+m+1}$.

The construction of ϕ proceeds by induction over the skeleta of M via a standard obstruction theory argument. And the obstructions encountered in extending ϕ from over the $(n-1)$-skeleton to over the n-skeleton are the problem of extending a self-homeomorphism of S^{n+m} to one of $\mathbb{D}^{n+m+1}$. But these obstructions all vanish because of the Alexander Trick. Recall that this Trick asserts that any self-homeomorphism η of S^n extends to a self-homeomorphism $\bar{\eta}$ of $\mathbb{D}^{n+1}$. In fact

$$\bar{\eta}(tx) = t\eta(x)$$

where $x \in S^n$ and $t \in [0,1]$ is an explicit extension.

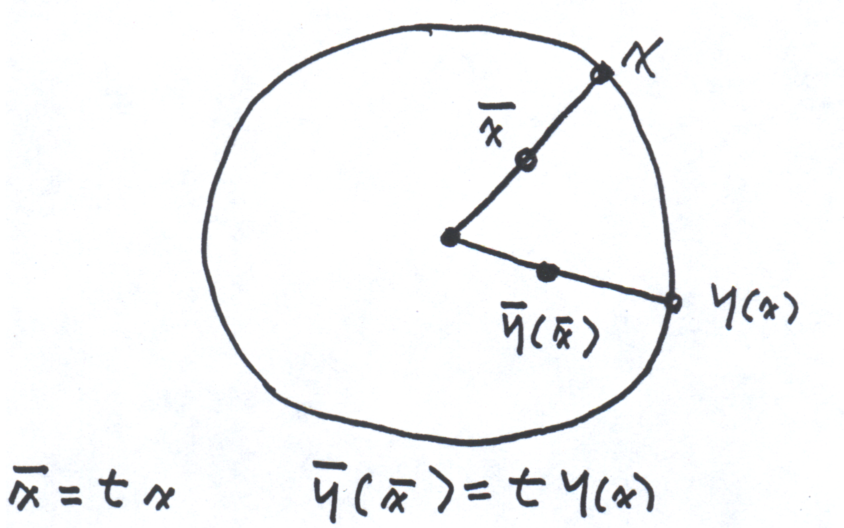


Figure 2.6

Now $\psi = \phi|_{E\times[0,1]}$ is the pseudo-isotopy from g to id_E posited in the Lemma. And a similar argument, which we omit, shows that ψ is properly homotopic to k rel ∂. Q.E.D.

Remark. It follows from results of Davis and Januszkiewicz [15] that PL non-positively curved closed manifolds also satisfy condition $(*)$. And Bizhong Hu showed that every non-positive curved finite complex K is a retract of such a manifold. Hu [45] (1995) deduced from this, using Ranicki's algebraic formulation of surgery theory, that the assembly map is split monic for such a K. Ferry-Weinberger [36], [12] and Carlsson-Pedersen [13] also obtained this in addition to many further results on the split injectivity of σ.

Corollary. *Let $f : N \to M$ be a homotopy equivalence between closed smooth manifolds such that M supports a non-positively curved Riemannian metric. Then N and M are stably homeomorphic; i.e.*

$$f \times \mathrm{id} : N \times \mathbb{R}^{m+4} \to M \times \mathbb{R}^{m+4}$$

is homotopic to a homeomorphism where $m = \dim(M)$.

Proof. Let $\phi : N \times S^1 \to M \times S^1 \times \mathbb{R}^{m+3}$ be an embedding homotopic to the composition

$$N \times S^1 \xrightarrow{f \times \mathrm{id}_{S^1}} M \times S^1 \times 0 \subseteq M \times S^1 \times \mathbb{R}^{m+3}.$$

Note that ϕ exists because of the Whitney Embedding Theorem. And let v denote the normal bundle to ϕ. We proceed to show that v is topologically trivial. Now Kwan and Szczarba [50] showed that $f \times \mathrm{id}_{S^1}$ is a simple homotopy equivalence and hence represents an element in $\mathcal{S}^s(M \times S^1)$. This element maps to 0 in $[M \times S^1; G/\mathrm{Top}]$ because of the Theorem and the 4-fold (semi) periodicity of the topological surgery exact sequence. But v (equipped with a specific homotopy trivialization) is this image element; in particular, v is topologically trivial.

Since the region outside an open tubular neighborhood of image(ϕ) is a (half open) h-cobordism, we can use the h-cobordism theorem to show that the total space E of v is diffeomorphic to $M \times S^1 \times \mathbb{R}^{m+3}$. But E can also be topologically identified with $N \times S^1 \times \mathbb{R}^{m+3}$ since v is topologically trivial. Hence there is a homeomorphism

$$\psi : N \times S^1 \times \mathbb{R}^{m+3} \to M \times S^1 \times \mathbb{R}^{m+3}$$

such that $\psi_{\#}(\pi_1 N) = \pi_1(M)$. The homeomorphism posited to exist in the Corollary is obtained by lifting ψ to the infinite cyclic covering spaces corresponding to $\pi_1(N)$ and $\pi_1(M)$, respectively. $\square$

3 The vanishing of $\mathbf{Wh}(\pi_1 M)$ for non-positively curved manifolds M

In my last lecture, I showed that step 1 in the program to replace a homotopy equivalence $f : N \to M$ between closed manifolds with a homeomorphism can be accomplished when M satisfies a certain geometric condition $(*)$. In particular, this can be done when M is a non-positively curved Riemannian manifold.

This lecture is about step 3 of the program; i.e., analyzing h-cobordisms with base M. Because of the s-cobordism theorem, this is equivalent to calculating $Wh(\pi_1 M)$ when $\dim(M) \geq 5$. The discussion will focus on the following vanishing result.

Vanishing Theorem. *(Farrell and Jones [31]) Let M be a closed non-positively curved Riemannian manifold. Then*

$$Wh(\pi_1 M) = 0.$$

Remark. The special cases of this theorem where M is the m-torus T^m was proven by Bass-Heller-Swan [6] (1964) and for arbitrary flat Riemannian manifolds M by Farrell-Hsiang [22] (1978).

We need to develop a few more geometric ideas before discussing the proof of the Vanishing Theorem. See [26] and [30, §3, §4] for more details. Throughout this lecture M will denote a closed (connected) non-positively curved Riemannian manifold and $\tilde{M}$ is its universal cover. And we keep the geometric notation from our last lecture; in particular

$$\begin{array}{rcl} \Gamma & = & \pi_1(M). \\ \bar{M} & & \text{is the geodesic ray compactification of } \tilde{M}. \\ M(\infty) & = & \bar{M} - \tilde{M}. \\ \alpha_v & & \text{is the geodesic with } \dot{\alpha}_v(0) = v. \end{array}$$

We call a pair of vectors $u, v \in S\tilde{M}$ *asymptotic* if the two rays

$$\{\alpha_u(t) \mid t \geq 0\} \text{ and } \{\alpha_v(t) \mid t \geq 0\}$$

are asymptotic.

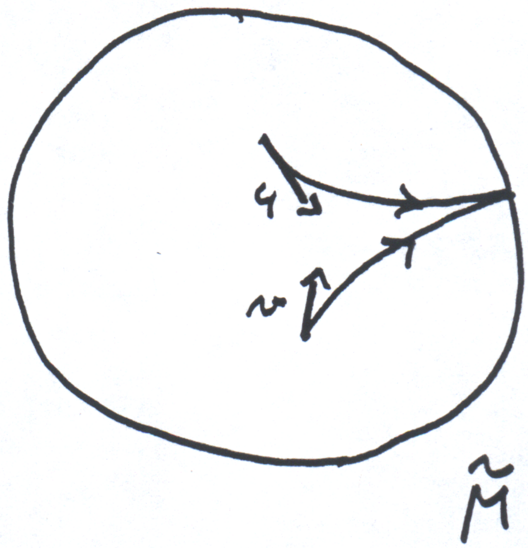


Figure 3.1
For each pair $v \in S\tilde{M}$ and $x \in \tilde{M}$, there is a unique asymptotic vector $v(x) \in S_x\tilde{M}$. ($S_x\tilde{M}$ = unit sphere in $T_x\tilde{M}$.)

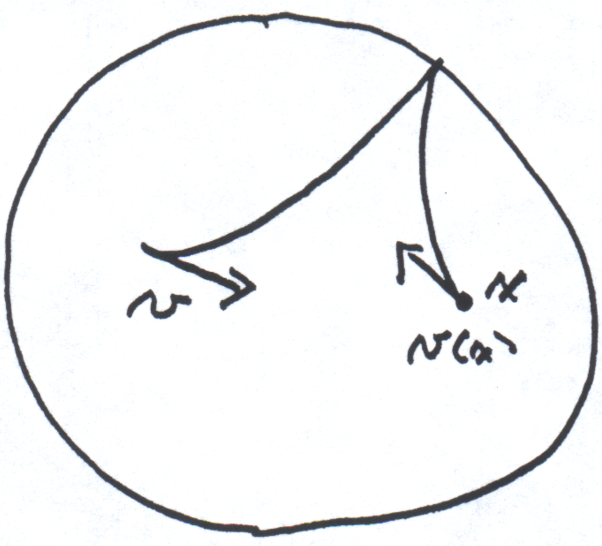


Figure 3.2

Furthermore the function $S\tilde{M} \times \tilde{M} \to S\tilde{M}$ defined by $(v, x) \to v(x)$ is continuous, C^1 in x, and its differential (in x) depends continuously on v. The (weakly) stable foliation of $S\tilde{M}$ has for its leaves the asymptoty classes of vectors. Note that under the bundle projection $S\tilde{M} \to \tilde{M}$ each leaf of this foliation maps diffeomorphically onto $\tilde{M}$. Since an isometry of $\tilde{M}$ sends asymptotic vectors to asymptotic vectors, this foliation induces a foliation of SM called its (*weakly*) *stable foliation*. Restriction of the bundle projection $SM \to M$ to any leaf L of this foliation is a covering space projection

$$L \to M.$$

And the geodesic flow $g^t : SM \to SM$ preserves the leaves of the (weakly) stable foliation.

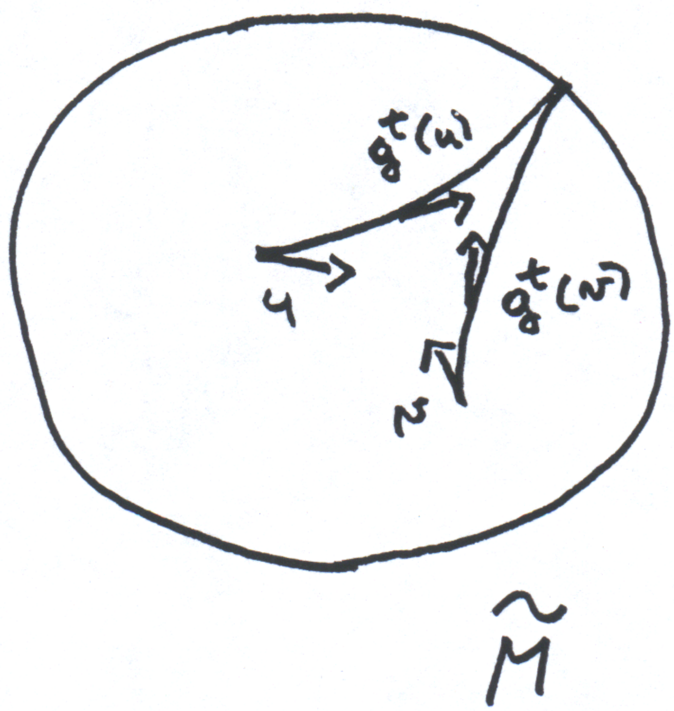


Figure 3.3

The total space SN of the unit sphere bundle of a Riemannian manifold N has a natural Riemannian metric defined as follows. Let $v(t)$ be a smooth curve in SN representing a tangent vector η to SN at $v(0)$; i.e., $v(t)$ is a unit length vector field along a smooth curve $\gamma(t)$ in N. Then

$$|\eta| = \sqrt{|\dot{\gamma}(0)|^2 + |u|^2}$$

where u is the covariant derivative of $v(t)$ at $t = 0$.

We next describe the *asymptotic transfer* of a path $\gamma : [0,1] \to M$ to a path $v\gamma$ in SM where $v \in S_{\gamma(0)}M$. The asymptotic transfer sits on top of γ in the sense that the composite path $p \circ (v\gamma)$ is γ; where

$$p : SM \to M$$

denotes the bundle projection. Let L be the leaf of the (weakly) stable foliation of SM containing v. Recall that

$$p|_L : L \to M$$

is a covering space. Then $v\gamma$ is defined to be the unique lift of γ starting at v.

The following are some of the properties of the asymptotic transfer.

1. If γ is a null homotopic loop, then so is $v\gamma$.

2. If γ is a constant loop, so is $v\gamma$.

3. If γ is a C^1-curve, so is $v\gamma$.

Furthermore, if $-a^2$ is any lower bound for the sectional curvatures of M, then

$$|\dot{v\gamma}(t)| \leq \sqrt{1 + a^2}\ |\dot{\gamma}(t)|$$

for each $t \in [0,1]$.

Let W be a smooth h-cobordism with base M equipped with a smooth deformation retraction h_t of W^{m+1} onto M^m. In particular $h_0 = \mathrm{id}_W$, and $r = h_1$ is a retraction of W onto M. Let $\mathcal{W}^{2m}$ be the total space of the pullback of $p : SM \to M$ via r; i.e.,

$$\mathcal{W} = \{(y,v) \in W \times SM \mid r(y) = p(v)\}.$$

Then $\mathcal{W}$ is an h-cobordism with base SM and the asymptotic transfer can be used to equip $\mathcal{W}$ with a useful C^1 deformation retraction k_t of $\mathcal{W}$ onto

SM defined as follows. First associate to h_t a family of paths $\{\gamma_y \mid y \in W\}$ in M called the *tracks* of h_t. These are given by the equation

$$\gamma_y(t) = r(h_t(y)).$$

Note that each track γ_y is a smooth null homotopic loop in M based at $r(x)$. Hence, for each vector $v \in S_{r(y)}M$, the asymptotic transfer $v\gamma_y$ of γ_y to SM is a C^1 null homotopic loop based at v. Now k_t is defined by the formula

$$k_t(y, v) = (h_t(y), v\gamma_y(t))$$

where $t \in [0, 1]$, $y \in W$ and $v \in S_{r(y)}M$. And notice that the retraction

$$k_1 : \mathcal{W} \to SM$$

is given by the formula

$$k_1(y, v) = v;$$

this follows from properties 1 and 2 of the asymptotic transfer together with the fact that each γ_y is a null homotopic loop. Consequently the tracks of k_t are

$$\{v\gamma_y \mid (y, v) \in \mathcal{W}\};$$

namely, they are all the asymptotic transfers of the tracks of h_t. Furthermore given a self-diffeomorphism $f : SM \to SM$ homotopic to id_{SM}, we can change k_t to a new C^1 deformation retraction of $\mathcal{W}$ onto SM whose tracks are

$$\{f \circ (v\gamma_y) \mid (y, v) \in \mathcal{W}\}.$$

This comment applies in particular when $f = g^{t_0}$ where g^t is the geodesic flow on SM and t_0 is a fixed (large) positive real number. Which is useful because of the following consequence of Anosov's analysis of the geodesic flow.

Key Property of $v\gamma$. The following is true when M is *negatively curved.* Given numbers β and ϵ in $(0, +\infty)$, there exists a number $t_0 \in (0, +\infty)$ satisfying the following. Let γ be any smooth path in M whose arc length is $\leq \beta$, and v be any vector in $S_{\gamma(0)}M$. Then, for any $t \geq t_0$, the composite path $g^t \circ (v\gamma)$ is (β, ϵ)-controlled in SM with respect to the 1-dimensional foliation by the orbits of the geodesic flow.

Figure 1.4 indicates why this property is true. In it $\tilde{\gamma}$ is a lift of γ to $\tilde{M}$; $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ is the vector lying over v; $u\tilde{\gamma}$ is the lift of $v\gamma$ to $S\tilde{M}$ starting at u, and $u(\infty) \in M(\infty)$ is the ideal point corresponding to the ray $\{\alpha_u(t) \mid t \geq 0\}$. Also $\tilde{M}$ is identified with the (weakly) stable leaf L of $S\tilde{M}$ containing u. And the lines converging to $u(\infty)$ are the flow lines of the geodesic flow which are inside of L; while the $\perp$ codimension-one submanifolds abuting to $u(\infty)$ are the horospheres inside of L; i.e. the *strongly stable leaves.*

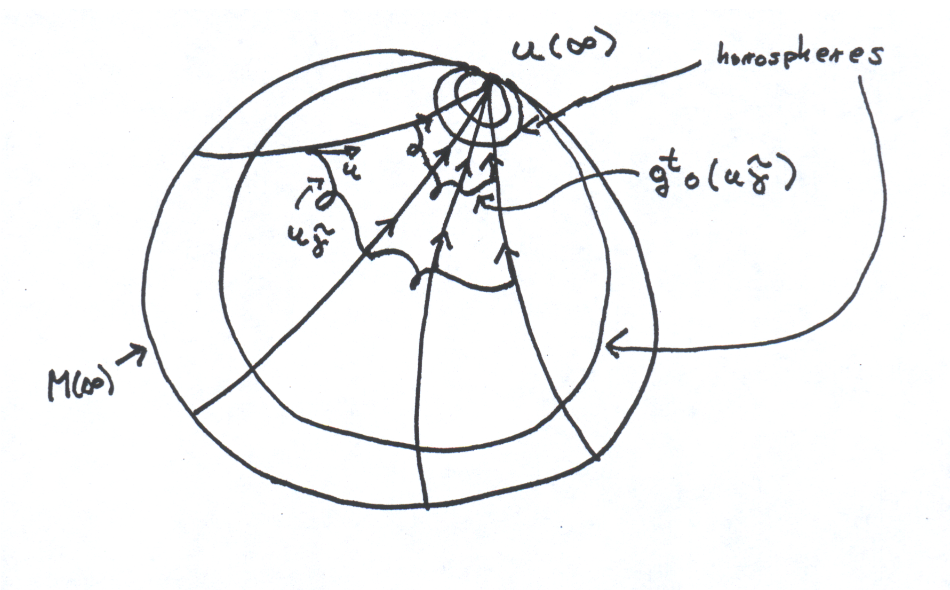


Figure 3.4

Each diffeomorphism g^t, $t > 0$, of the geodesic flow preserves the family of horospheres as well as the flow lines. It is (strongly) contracting on horospheres and is an isometry on flow lines.

Remark. This Key Property of the asymptotic transfer is *not* true (in general) when M is only non-positively curved. For example it doesn't hold when M is flat since asymptotic rays are parallel in Euclidean space.

Using the above construction of a deformation retraction of $\mathcal{W}$ onto SM relative to g^{t_0}, we see that $\mathcal{W}$ is a (β, ϵ)-controlled h-cobordism over SM for a fixed positive real number β but arbitrarily small positive numbers ϵ when M is negatively curved because of the Key Property of the asymptotic transfer. Hence the Foliated Control Theorem, Theorem 1.8 of L.E. Jones' lectures, shows that the Whitehead torsion $\tau(\mathcal{W}) = 0$.

Codicil. We must make the following minor addition to our setup in order to apply the Foliated Control Theorem. Let M^+ denote the "top" of the h-cobordism W; i.e., $\partial W = M^+ \coprod M$. And fix a second smooth deformation retraction h_t^+ of W onto M^+. Associate to h_t^+ a second family of tracks γ_y^+, $y \in W$, defined by the equation

$$\gamma_y^+(t) = r(h_t^+(y)).$$

Use these new tracks to define a second C^1 deformation retraction k_t^+ of the transferred h-cobordism $\mathcal{W}$ onto its "top" $\mathcal{M}^+$ where

$$\mathcal{M}^+ = \partial \mathcal{W} - SM.$$

Define k_t^+ by the formula

$$k_t^+(y, v) = (h_t^+(y), v\gamma_y^+(t))$$

where $t \in [0,1]$, $y \in W$ and $v \in S_{r(y)}M$. And note that the tracks of k_t^+ are

$$\{v\gamma_y^+ \mid (y, v) \in \mathcal{W}\};$$

namely, all the asymptotic transfers of the tracks of h_t^+. (Notice that the tracks γ_y^+ are *not* loops; but this is irrelevant since the retraction k_1 rather than k_1^+ is used in defining the tracks of k_t^+.) Furthermore we can change k_t^+ to a new C^1 deformation retraction of $\mathcal{W}$ onto $\mathcal{M}^+$ whose tracks are

$$\{g^{t_0} \circ (v\gamma_y^+) \mid (y, v) \in \mathcal{W}\}.$$

Clearly these tracks are also (β, ϵ)-controlled provided t_0 is sufficiently large, and hence the Foliated Control Theorem is applicable.

Since every element $x \in Wh(\pi_1 M)$ is the torsion $\tau(W)$ of some smooth h-cobordism with base M, the fact that $\tau(\mathcal{W}) = 0$ would show that $Wh(\pi_1 M)$ vanishes, when M is negatively curved, provided

$$\tau(\mathcal{W}) = \tau(W).$$

Unfortunately this equation is not true in general. In fact the following formula calculates $\tau(\mathcal{W})$ in terms of $\tau(W)$.

Theorem. *(D.R. Anderson [4] 1972). Let W and $\mathcal{W}$ be h-cobordisms with bases M and $\mathcal{M}$, respectively. And let $p : \mathcal{W} \to W$ be a smooth fiber bundle with $p^{-1}(M) = \mathcal{M}$ and $\dim M > 4$. Assume that $\pi_1(W)$ acts trivially on the integral homology groups of the fiber F of p, then*

$$p_*(\tau(\mathcal{W})) = \chi(F)\tau(W)$$

where $\chi(F)$ denotes the Euler characteristic of F and

$$p_* : Wh(\pi_1\mathcal{M}) \to Wh(\pi_1 M)$$

is the homomorphism induced by p.

Applying Anderson's theorem to the h-cobordism $\mathcal{W}$ constructed above, we see that

$$\tau(\mathcal{W}) = \begin{cases} 2\tau(W) & \text{if } m \text{ is odd} \\ 0\tau(W) = 0 & \text{if } m \text{ is even} \end{cases}$$

(provided M^m is orientable) since the fiber of $\mathcal{W} \to W$ is S^{m-1}.

To get around this difficulty we need a sub-bundle E of SM with fiber F satisfying

1. $\chi(F) = 1$;
2. E is invariant under g^t;
3. for each path γ in M and each vector $v \in E$ lying over $\gamma(0)$, $v\gamma$ is a path in E.

It unfortunately is impossible to find such a sub-bundle when M is closed because every orbit of the action of Γ on $M(\infty)$ is then dense. We are thus forced to consider a certain non-compact but complete and pinched negatively curved Riemannian manifold N^{m+1} called the *enlargement* of M^m. It is diffeomorphic to $\mathbb{R} \times M^m$ and contains M^m as a totally geodesic codimension-one subspace. In fact N is the warped product (defined by Bishop and O'Neill in [7])

$$N = \mathbb{R} \times_{\cosh(t)} M$$

and $0 \times M$ is the totally geodesic subspace identified with M.

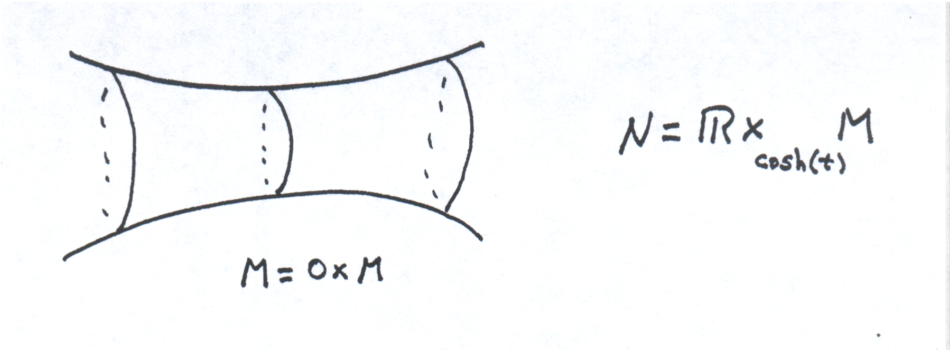


Figure 3.5

The Riemannian metric $|| \ ||$ on N is determined from the Riemannian metrics $| \ |$ on M and $| \ |$ on $\mathbb{R}$ by the properties

1. $\mathbb{R} \times x \perp t \times M$ for all $x \in M$, $t \in \mathbb{R}$.
2. $||v|| = \cosh(t)|v|$ if $v \in T(t \times \tilde{M})$.
3. $||v|| = |v|$ if $v \in T(\mathbb{R} \times x)$.

Let $q : N = \mathbb{R} \times M \to \mathbb{R}$ denote projection onto the first factor. Inside of SN is an *upper hemisphere* sub-bundle defined by $v \in S^+N$ iff the following set of real numbers is bounded below

$$\{q(\alpha_v(t)) \mid t \in [0, +\infty)\}.$$

(This lower bound depends on v.) That is $v \notin S^+N$ iff the geodesic $\alpha_v(t) \to$ "$-\infty$" as $t \to +\infty$. This sub-bundle satisfies the three conditions listed above; in particular its fiber is $\mathbb{D}^m$.

Now an arbitrary element $x \in Wh(\Gamma)$ can be realized as the Whitehead torsion $\tau(W)$ of a compactly supported h-cobordism with base N. And the associated h-cobordism $\mathcal{W}$ with base S^+N is (β, ϵ)-controlled for a fixed positive number β but arbitrarily small positive ϵ. Hence the Foliated Control Theorem (in one of its more sophisticated forms) together with Anderson's Theorem shows that

$$x = \tau(W) = \tau(\mathcal{W}) = 0$$

proving that $Wh(\pi_1 M) = 0$ when M is negatively curved.

To prove the general case of the Vanishing Theorem, where M is allowed to have some zero sectional curvature, we must replace the asymptotic transfer with a new *focal transfer*. It associates to each path $\gamma : [0,1] \to M$, each vector $v \in S_{\gamma(0)}M$, and every (large) positive number $d \in \mathbb{R}$ (called the *focal length* of the transfer) a path

$$v(\gamma, d) : [0,1] \to M.$$

The focal transfer satisfies properties 1-3 of the asymptotic transfer. And it satisfies the following analogue of the Key Property of $v\gamma$.

Key Property of $v(\gamma, d)$. Given M as well as numbers $\beta, \epsilon \in (0, +\infty)$, there exists a positive number t_0 ($t_0 > \beta$) satisfying the following statement for every smooth path γ in M whose arc length is $\leq \beta$ and every vector $v \in S_{\gamma(0)}M$. The composite path

$$g^d \circ v(\gamma, d)$$

is (β, ϵ)-controlled in SM with respect to the foliation given by the orbits of the geodesic flow provided $d \geq t_0$.

Remark. The focal transfer $v(\gamma, d)$ focuses when flowed a distance equal to its focal length d. When flowed farther, it gets out of focus.

To construct $v(\gamma, d)$ pick a lift $\tilde{\gamma}$ of γ to $\tilde{M}$ and let $u \in S_{\tilde{\gamma}(0)}\tilde{M}$ be the unique vector which maps to v via $d\rho$ where

$$\rho : \tilde{M} \to M$$

denotes the covering projection. Figure 6 illustrates the construction of the path $u(\tilde{\gamma}, d)$ in $S\tilde{M}$.

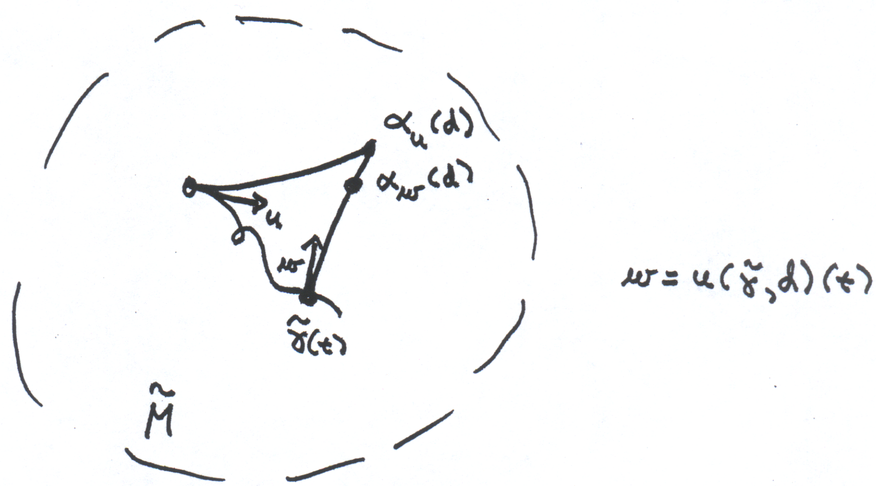


Figure 3.6

If w denotes the vector $u(\tilde{\gamma}, d)(t) \in S_{\tilde{\gamma}(t)}\tilde{M}$, then w is the unique vector such that the geodesic ray

$$\{\alpha_w(s) \mid s \geq 0\}$$

contains the point $\alpha_u(d)$. Note we must have that

$$d \geq \operatorname{diam}\{\gamma(t) \mid t \in [0,1]\}$$

for w to be necessarily defined. Since this construction is equivariant with respect to Γ, we can (and do) define the focal transfer $v(\gamma, d)$ by the equation

$$v(\gamma, d) = d\rho \circ u(\tilde{\gamma}, d).$$

The only problem with the focal transfer is that the bundle $S^+N \to N$ does not satisfy property 3 (see nine lines above Figure 1.5) with respect to it. But it does except near $\partial(S^+N)$ and so the construction is slightly modified near $\partial(S^+N)$. When this is done, then the argument given above proving the Vanishing Theorem in the special case where M is negatively curved works in general after the asymptotic transfer is replaced with the focal transfer. In fact a simplification can be made in the earlier argument by using N equal to the Riemannian product

$$\mathbb{R} \times M$$

instead of the warped product

$$\mathbb{R} \times_{\cosh(t)} M.$$

We can even set N equal to the Riemannian product $S^1 \times M$ and proceed as outlined in L.E. Jones' lecture 1. The advantage to this is that the basic Foliated Control Theorem (Theorem 1.8 of L.E. Jones' lectures) can then be used since $S^+(S^1 \times M)$ is compact. (See lecture 1 of L.E. Jones for the precise definition of $S^+(S^1 \times M)$.)

We end this lecture by discussing a generalization of the Vanishing Theorem to the case where M is complete but *not* necessarily compact. Needed for this purpose is an extra geometric condition on M; namely, that M is *A-regular*.

Definition. A Riemannian manifold N is A-regular if there exists a sequence of positive real numbers $A_0, A_1, A_2, \cdots$ with $|D^n(K)| \leq A_n$. Here K is the curvature tensor and D is covariant differentiation.

Remark. Every closed Riemannian manifold N is A-regular. This is a consequence of an elementary continuity argument.

Remark. Every locally symmetric space is A-regular since $DK \equiv 0$ is one of the definitions of a locally symmetric space.

Addendum. *(Farrell and Jones [35] 1998) Let N be any complete Riemannian manifold which is both non-positively curved and A-regular. Then $Wh(\pi_1 N) = 0$.*

Corollary 1. *$Wh(\Gamma) = 0$ for every discrete torsion-free subgroup Γ of $GL_n(\mathbb{R})$.*

Reason. Note that $\Gamma = \pi_1(N)$ where N is the double coset space

$$\Gamma \backslash GL_n(\mathbb{R})/O_n$$

which is a complete non-positively curved locally symmetric space and hence A-regular by Remark 2.

Corollary 2. *Let N be any complete and pinched negatively curved Riemannian manifold, then*

$$Wh(\pi_1 N) = 0.$$

Reason. Shi [66] and Abresch [2] show that the given Riemannian metric can be deformed to an A-regular one while keeping it negatively curved and complete.

The proof of the Addendum follows the same pattern as the proof of the Vanishing Theorem except that it uses the more difficult Foliated Control Theorem which Lowell Jones will discuss in his last lecture.

Let me also mention that Jones' former Ph.D. student B. Hu showed how to adapt the proof of the Vanishing Theorem to the language of Alexandroff PL-geometry thus obtaining the following result.

Theorem. *(Hu [44] 1993) Let K be a non-positively curved finite complex, then $Wh(\pi_1 K) = 0$.*

Remark. Hu's result does not obviously include the Vanishing Theorem since Davis, Okun and Zheng [16] have shown that *no* rank ≥ 2, irreducible, closed, non-positively curved locally symmetric space is also a non-positively curved PL-manifold.

4 The Borel Conjecture for non-positively curved manifolds

The focus of this lecture is Borel's Conjecture for closed non-positively curved Riemannian manifolds of dimension $\neq 3, 4$. It is an immediate consequence of the following result "TRT".

Topological Rigidity Theorem. *(Farrell and Jones [32]) Let M^m be a closed non-positively curved Riemannian manifold. Then the homotopy-topological structure set $\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ contains only one element when $m + n \geq 5$.*

Remark. TRT was proven for T^m ($m \geq 5$) by Hsiang-Wall [43] (1969). And it was proven for all closed flat Riemannian manifolds M^m ($m \geq 5$) by Farrell-Hsiang [25] (1983).

Corollary. *Let $f : N^m \to M^m$ be a homotopy equivalence between closed manifolds where $m \neq 3, 4$. If M^m is a non-positively curved Riemannian manifold, then f is homotopic to a homeomorphism.*

Proof. This result is classical when $m = 1$ or 2. When $m \geq 5$ set $n = 0$ in TRT to conclude that N and M are h-cobordant and hence homeomorphic by the s-cobordism since $Wh(\pi_1 M) = 0$ because of the Vanishing Theorem. □

Remark. Gabai [39] has a program for showing that the Borel Conjecture for closed hyperbolic 3-manifolds is equivalent to the Poincaré Conjecture.

Remark. The Borel Conjecture for closed non-positively curved 4-manifolds M^4 is an interesting open problem which is perhaps more accessible than the 3-dimensional case. The 5-dimensional s-cobordism Theorem of Freedman and Quinn [37] combined with TRT shows it is true when M^4 is a closed flat Riemannian manifold.

We now discuss the proof of the TRT. Throughout this lecture M^m denotes a closed (connected) non-positively curved m-dimensional Riemannian manifold. We also keep the notation from our last lecture; in particular

$\tilde{M}$	is the universal cover of M;
Γ	$= \pi_1(M)$;
α_v	is the geodesic with $\dot{\alpha}_v(0) = v$.

And we make the simplifying assumption that M^m is orientable so that our discussion is as transparent as possible. Note there are the following two identifications since $Wh(\Gamma) = 0$:

$$\begin{aligned} L_k^s(\Gamma) &= L_k(\Gamma) \quad \text{and} \\ \mathcal{S}^s(M^m \times \mathbb{D}^n, \partial) &= \mathcal{S}(M^m \times \mathbb{D}^n, \partial) \end{aligned}$$

where $\mathcal{S}^s(\ \ )$ denotes the simple homotopy-topological structure set.

The following result, used to reduce TRT to a special case, is a consequence of the codimension-one splitting theorems mentioned in my first lecture.

Lemma 0. *$\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ can be identified with a subset of $\mathcal{S}(M^m \times T^n)$ provided $m + n \geq 5$; and $\mathcal{S}(M^m)$ with a subset of $\mathcal{S}(M^m \times S^1)$ provided $m \geq 5$.*

Remark. Note that $\mathcal{S}^s(N \times [0,1], \partial)$ maps to $\mathcal{S}^s(N \times S^1)$ by sending the structure

$$f : (W, \partial_0 W \amalg \partial_1 W) \to (N \times [0,1], N \times 0 \amalg N \times 1)$$

to the structure

$$\mathcal{W} \to N \times S^1$$

where $\mathcal{W}$ results from W by glueing $\partial_0 W$ to $\partial_1 W$ via the composite homeomorphism $(f|_{\partial_1 W})^{-1} \circ (f|_{\partial_0 W})$. The first identification in Lemma 0 is a n-fold elaboration of this map using that $\mathbb{D}^n = \mathbb{D}^{n-1} \times [0,1]$. The second identification sends the structure $f : N \to M$ to the structure $f \times \mathrm{id} : N \times S^1 \to M \times S^1$; which is shown in Lemma 3 (below) to be monic.

Lemma 0 together with the fact that $M^m \times T^n$ is also non-positively curved reduces the TRT to the special case where $n = 0$ and m is an *odd integer.*

Note that the main result of our second lecture, together with the (semi)-periodicity of the surgery exact sequence, yields the following short exact sequence of pointed sets

$$0 \to [M^m \times [0,1], \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_{m+1}(\Gamma) \to \mathcal{S}(M^m) \to 0.$$

Remark. The techniques developed in this lecture (and the last) give an independent proof (via the focal transfer and the geodesic flow) that the surgery sequence is short exact for non-positively curved closed manifolds M^m. This alternate proof does not use the (semi)-periodicity of the surgery sequence.

Hence it remains to show that σ is an epimorphism; which is Step 2 in the program from Lecture 1 for replacing a homotopy equivalence $f : N \to M$ with a homeomorphism. This is the most complicated step in the program and was the last to be solved. The argument accomplishing it is modeled on the one used to solve Step 3 given in the last lecture. The s-cobordism theorem was used in that argument. It's surgery analogue is the algebraic classification of normal cobordisms over M due to Wall. Given a group π, Wall [70] algebraically defined a sequence of abelian groups $L_n(\pi)$ with $L_{n+4}(\pi) = L_n(\pi)$ for all $n \in \mathbb{Z}$. He then showed that there is a natural bijection between the equivalence classes of normal cobordisms W over $M^m \times \mathbb{D}^{n-1}$ and $L_{m+n}(\Gamma)$ with the trivial normal cobordism corresponding to 0. Denote this correspondence by

$$W \mapsto \omega(W) \in L_{m+n}(\Gamma).$$

Wall also proved the following product formula.

Let N^{4k} be a simply connected closed oriented manifold and W be a normal cobordism over $M^m \times \mathbb{D}^{n-1}$. Form a new normal cobordism $W \times N$ over $M^m \times \mathbb{D}^{n-1} \times N^{4k}$ by producting W with N, then

$$\omega(W \times N) = \mathrm{Index}(N)\omega(W).$$

Remark. Anderson's Theorem is an analogue of this result where $\chi(N)$ replaces $\mathrm{Index}(N)$.

This product formula has the following geometric consequence.

Proposition. *Let K^{4k} be a closed oriented simply connected manifold with* $\mathrm{Index}(K) = 1$. *Let $f : N \to M$ be a homotopy equivalence where N is also a closed manifold. If*

$$f \times \mathrm{id} : N \times K \to M \times K$$

is homotopic to a homeomorphism, then f is also homotopic to a homeomorphism.

Sketch of Proof. Arguing as in the proof of the main result of Lecture 2, we compare the surgery exact sequence for $\mathcal{S}(M)$ with that for $\mathcal{S}(M \times K)$. If $x \in \mathcal{S}(M)$ denotes the homotopy-topological structure $f : N \to M$, it goes to 0 in $\mathcal{S}(M \times K)$. And since the map $[M, G/\mathrm{Top}] \to [M \times K, G/\mathrm{Top}]$ is monic, x is the image of an element $\bar{x} \in L_{m+1}(\Gamma)$ which maps to an element

$\hat{x} \in L_{m+1+4k}(\Gamma)$ by producting the normal cobordism with K^{4k}. But the image of $\hat{x}$ in $\mathcal{S}(M \times K)$ is represented by

$$f \times \mathrm{id} : N \times K \to M \times K$$

and is hence zero. Therefore $\hat{x}$ is in the image of the Quinn assembly map in the surgery sequence for $M \times K$. But this map factors through the assembly map

$$[M^m \times \mathbb{D}^{4k+1}, \partial; G/\mathrm{Top}] \to L_{m+4k+1}(\Gamma)$$

which is periodic of period $4k$ with $\bar{x}$ going to $\hat{x}$. This factoring can be seen using Quinn's Δ-set description of the surgery sequence [60], [61] (cf. [70, §17A]) or Ranicki's algebraic formulation of it. (See Ranicki's 2nd lecture.) Hence $\bar{x}$ is in the image of σ, and therefore $x = 0$. □

The complex projective plane $\mathbb{C}P^2$ is the natural candidate for K when applying this Proposition. It is important for this purpose to have the following alternate description of $\mathbb{C}P^2$. Let C_2 denote the cyclic group of order 2. It has a natural action on $S^n \times S^n$ determined by the involution $(x, y) \mapsto (y, x)$ where $x, y \in S^n$. Denote the orbit space of this action by F_n; i.e.

$$F_n = S^n \times S^n / C_2.$$

Lemma 1. $\mathbb{C}P^2 = F_2$.

Proof. Let $sl_2(\mathbb{C})$ be the set of all 2×2 matrices with complex number entries and trace zero. Since $sl_2(\mathbb{C})$ is a 3-dimensional $\mathbb{C}$-vector space, $\mathbb{C}P^2$ can be identified as the set of all equivalence classes $[A]$ of non-zero matrices $A \in sl_2(\mathbb{C})$ where A is equivalent to B iff $A = zB$ for some $z \in \mathbb{C}$. The characteristic polynomial of $A \in sl_2(\mathbb{C})$ is $\lambda^2 + \det(A)$. Consequently, A has two distinct 1-dimensional eigenspaces if $\det(A) \neq 0$, and a single 1-dimensional eigenspace if $\det(A) = 0$ and $A \neq 0$. Also, A and zA have the same eigenspaces provided $z \neq 0$. These eigenspaces correspond to points in S^2 under the identification $S^2 = \mathbb{C}P^1$. The assignment

$$[A] \mapsto \text{ the eigenspaces of } A$$

determines a homeomorphism of $\mathbb{C}P^2$ to F_2. □

Remark. The TRT was first proved in the case where M^m is a hyperbolic 3-dimensional manifold by making use of Lemma 1. It was then realized that the general result for m odd could be proven using F_{m-1} once one could handle the technical complications arising from the fact that F_k is *not* a manifold when $k > 2$. The following result is used in overcoming these complications. It shows that F_k is "very close" to being a manifold of index equal to 1 when k is even.

Lemma 2. *Let n be an even positive integer. Then F_n has the following properties.*

1. *F_n is orientable $2n$-dimensional $\mathbb{Z}[\frac{1}{2}]$-homology manifold.*

2. *F_n is simply connected*

3. $H_i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n < i < 2n \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

4. $H^i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, 2n \\ \mathbb{Z}_2 & \text{if } n+2 < i < 2n \text{ and } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

5. *The cup product pairing*

$$H^n(F_n) \otimes H^n(F_n) \to H^{2n}(F_n)$$

   is unimodular and its signature is either 1 *or* -1.

Proof. There is a natural stratification of F_n consisting of two strata B and T. The bottom stratum B consists of all (agreeing) unordered pairs $\langle u, v\rangle$ where $u = v$; while the top stratum T consists of all (disagreeing) pairs $\langle u, v\rangle$ where $u \neq v$.

Note that B can be identified with S^n. Also real projective n-space $\mathbb{R}P^n$ can be identified with the set of all unordered pairs $\langle u, -u\rangle$ in F_n. It is seen that F_n is the union of "tubular neighborhoods" of S^n and $\mathbb{R}P^n$ intersecting in their boundaries. The first tubular neighborhood is a bundle over S^n with fiber the cone on $\mathbb{R}P^{n-1}$. The second tubular neighborhood is a bundle over $\mathbb{R}P^n$ with fiber $\mathbb{D}^n$. Furthermore, they intersect in the total space of the $\mathbb{R}P^{n-1}$-bundle associated to the tangent bundle of S^n. This description of F_n can be used to verify Lemma 2. See [29, p. 299] for more details. □

Caveat. The fundamental class of B represents twice a generator of $H_n(F_n)$. On the other hand, if we fix a point $y_0 \in S^n$, then the map $x \mapsto \langle x, y_0 \rangle$ is an embedding of S^n in F_n which represents a generator of $H_n(F_n)$.

Let $f : N \to M$ represent an element in $\mathcal{S}(M)$. Then $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ represents an element in $\mathcal{S}(M \times S^1)$. This defines a map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$.

Lemma 3. *The map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$ is monic.*

Proof. Suppose $f \times \mathrm{id}$ is homotopic to a homeomorphism g via a homotopy

$$h : N \times S^1 \times [0,1] \to M \times S^1 \times [0,1]$$

where $h|_{N \times S^1 \times 0} = f \times \mathrm{id}$ and $h|_{N \times S^1 \times 1} = g$. By one of the codimension-one splitting theorems mentioned in my first lecture [20], we can split h along $M \times 1 \times [0,1]$.

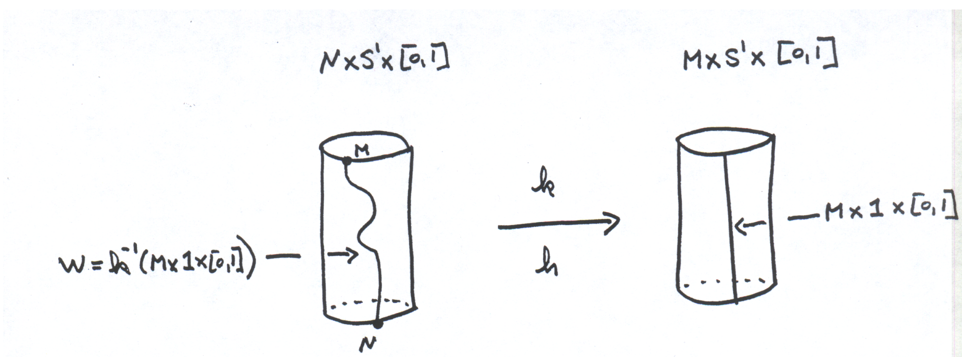


Figure 4.1

That is h is homotopic rel ∂ to a map k such that

$$k|_W : W \to M \times 1 \times [0,1]$$

is a homotopy equivalence where

$$W = k^{-1}(M \times 1 \times [0,1]).$$

We use that $Wh(\Gamma) = 0$ to get this. Now note that W is an h-cobordism between M and N. But W is a cylinder; again since $Wh(\Gamma) = 0$. $\square$

Remark. In order to prove the TRT, it suffices to show that $f \times \mathrm{id}$ is homotopic to a homeomorphism because of Lemma 3.

We now formulate a variant of the Proposition. This variant is used in showing that

$$f \times \mathrm{id} : N \times S^1 \to M \times S^1$$

is homotopic to a homeomorphism. There is a bundle

$$p : \mathcal{F}M \to M \times S^1$$

whose fiber over a point $(x, \theta) \in M \times S^1$ consists of all unordered pairs of unit length vectors $\langle u, v\rangle$ tangent to $M \times S^1$ at (x, θ) and satisfying the following two constraints.

1. If $u \neq v$, then both u and v are tangent to the level surface $M \times \theta$.

2. If $u = v$, then the projection $\bar{u}$ of u onto $T_\theta S^1$ points in the counter-clockwise direction (or is 0).

The total space $\mathcal{F}M$ is stratified with three strata:

$$\begin{array}{rcl} \mathbb{B} & = & \{\langle u, u\rangle \mid \bar{u} = 0\} \\ \mathbb{A} & = & \{\langle u, u\rangle \mid \bar{u} \neq 0\} \\ \mathbb{T} & = & \{\langle u, v\rangle \mid u \neq v\}. \end{array}$$

Note that $\mathbb{B}$ is the bottom stratum and that $\mathcal{F}M - \mathbb{B}$ is the union of the two open sets $\mathbb{A}$ (auxiliary stratum) and $\mathbb{T}$ (top stratum). The restriction of p to each stratum is a sub-bundle. Let $\mathcal{F}_x$, B_x, A_x and T_x denote the fibers of these bundles over $x \in M \times S^1$; i.e.,

$$\mathcal{F}_x = p^{-1}(x),\ B_x = \mathcal{F}_x \cap \mathbb{B},\ A_x = \mathcal{F}_x \cap \mathbb{A},\ T_x = \mathcal{F}_x \cap \mathbb{T}.$$

Note that $B_x = S^{m-1}$, $A_x = \mathbb{D}^m$, $T_x \cup B_x = F_{m-1}$ and the bundle $p : \mathbb{B} \to M \times S^1$ is the pullback of the tangent unit sphere bundle of M under the projection $M \times S^1 \to M$.

The space F_{m-1} will play the role of the index one manifold K in our variant of the Proposition. Since it is unfortunately not a manifold when $m > 3$, we need to introduce the auxiliary fibers A_x. Hence the total fiber is homeomorphic to $F_{m-1} \cup \mathbb{D}^m$ where the subspace B in F_{m-1} is identified with $S^{m-1} = \partial\mathbb{D}^m$. Let

$$\mathcal{F}_f \to N \times S^1$$

denote the pullback of

$$\mathcal{F}M \to M \times S^1$$

along $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ and let

$$\hat{f} : \mathcal{F}_f \to \mathcal{F}M$$

be the induced bundle map. Note that the stratification of $\mathcal{F}M$ induces one on $\mathcal{F}_f$ and that $\hat{f}$ preserves strata.

We say that $\hat{f}$ is *admissibly homotopic to a split map* provided there exists a homotopy h_t, $t \in [0,1]$, with $h_0 = \hat{f}$ and satisfying the following four conditions.

1. Each h_t is strata preserving.

2. Over some closed "tubular neighborhood" $\mathcal{N}_0$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$, each h_t is a bundle map; in particular, h_t maps fibers homeomorphically to fibers.

3. There is a larger closed "tubular neighborhood" $\mathcal{N}_1$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$ such that h_1 is a homeomorphism over $\mathbb{B} \cup \mathbb{T} - \mathrm{Int}(\mathcal{N}_1)$ and over $\mathbb{B} \cup \mathbb{A}$.

4. Let $\rho : \mathcal{N}_1 \to M \times S^1$ denote the composition of the two bundle projections $\mathcal{N}_1 \to \mathbb{B}$ and $\mathbb{B} \to M \times S^1$. Then there is a triangulation K for $M \times S^1$ such that h_1 is transverse to $\rho^{-1}(\sigma)$ for each simplex σ of K. Furthermore
$$h_1 : h_1^{-1}(\rho^{-1}(\sigma)) \to \rho^{-1}(\sigma)$$
is a homotopy equivalence.

Remark. Conditions 3 and 4 should be heuristically replaced by the simpler and stronger condition that "h_1 is a homeomorphism". But for technical reasons we need to work instead with conditions 3 and 4.

The variant of the Proposition needed to prove the TRT is the following.

Proposition $(*)$**.** *The map $f : N \to M$ is homotopic to a homeomorphism provided $\hat{f} : \mathcal{F}_f \to \mathcal{F}M$ is admissibly homotopic to a split map.*

The proof of Proposition $(*)$ is basically an elaboration of the one sketched above for Proposition. (See [29, §4 and §9].) It in particular uses again Quinn's Δ-set approach to the surgery exact sequence and generalizes Wall's product formula to the stratified setting above by using Lemma 2.

Proposition $(*)$ is the surgery theory part of the proof of the TRT. The geometry of M (in particular, its non-positive curvature) is used to show that the hypothesis of Proposition $(*)$ is satisfied; i.e, that $\hat{f}$ is admissibly homotopic to a split map. We now proceed to discuss how this is done.

It is a consequence of several applications of both ordinary and foliated topological control theory as discussed in Lowell Jones' lectures. Let $g : M \to N$ be a (strong) homotopy inverse to f and let h_t and k_t be (strong) homotopies of the composite $f \circ g$ to id_M and $g \circ f$ to id_N, respectively. Strong means base point preserving. It implies the following useful property.

Property $(*)$**.** For each point $x \in N$, the two paths

$$t \mapsto h_t(f(x)) \quad \text{and} \quad t \mapsto f(k_t(x))$$

are homotopic rel end points.

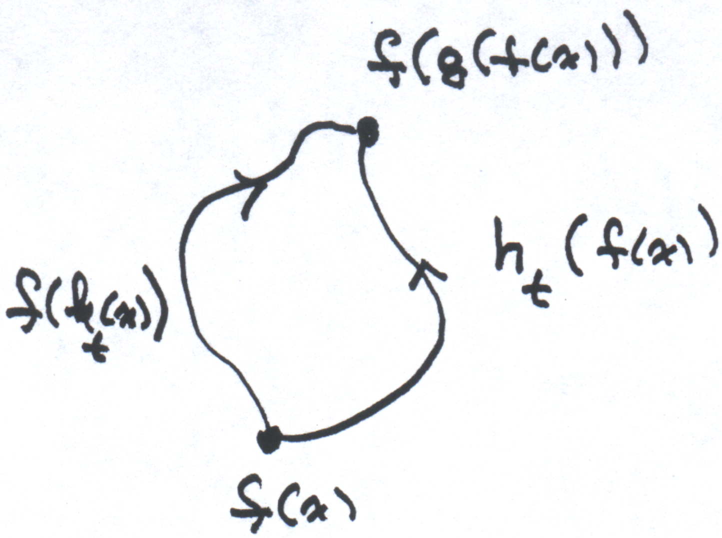


Figure 4.2

We may assume that N is a smooth manifold by using Kirby-Siebenmann smoothing theory [49]. For this we need only observe that the stable topological tangent bundle of N has a real vector bundle structure since it is the pull back of TM stabilized via f because $f : M \to N$ maps to 0 in $[M, G/\mathrm{Top}]$. Therefore we may also assume that f and g are smooth maps and that both h_t and k_t are smooth homotopies.

The crucial point is to construct "good" transfers of the map g and the homotopies h_t, k_t to a map $\hat{g} : \mathcal{F}M \to \mathcal{F}_f$ and homotopies $\hat{h}_t$, $\hat{k}_t$ from $\hat{f} \circ \hat{g}$ to $\mathrm{id}_{\mathcal{F}M}$, and $\hat{g} \circ \hat{f}$ to $\mathrm{id}_{\mathcal{F}_f}$, respectively, so that control theory can be applied to admissibly homotope $\hat{f}$ to a split map. We proceed to describe what a good transfer is and then indicate how to construct one. The first requirement is that $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ be bundle maps covering $g \times \mathrm{id}$, $h_t \times \mathrm{id}$, $k_t \times \mathrm{id}$, respectively. (Here id is the identity map on S^1.) Second, each map $\hat{g}$, $\hat{h}_t$ and $\hat{k}_t$ should preserve strata. Finally, it is necessary that a certain family $\mathcal{T}$ of paths determined by the lift is sufficiently "shrinkable". A path $\alpha : [0,1] \to \mathcal{F}M$ is in $\mathcal{T}$ if either

$$\begin{aligned} \alpha(t) &= \hat{h}_t(\omega) \qquad \text{for some } \ \omega \in \mathcal{F}M, \quad \text{or} \\ \alpha(t) &= \hat{f}(\hat{k}_t(\omega)) \qquad \text{for some } \ \omega \in \mathcal{F}_f. \end{aligned}$$

(The family $\mathcal{T}$ is called the tracks of the transfer.) Note that each track is contained in a single stratum of $\mathcal{F}M$.

We construct good transfers by constructing their tracks $\mathcal{T}$. Since this is easier to explain when M *is negatively curved, we now make this assumption.* The construction of $\mathcal{T}$ uses (mainly) the asymptotic transfer of paths discussed in lecture 3. (The general case uses the focal transfer which, although more elementary, requires greater technical details.) Let $\mathcal{T}_1$ be the tracks determined by f, g, h_t and k_t; i.e. a curve $\alpha : [0,1] \to M$ is in $\mathcal{T}_1$ if for all $t \in [0,1]$ either

$$\begin{aligned} \alpha(t) &= h_t(x) \qquad \text{for some } \ x \in M; \quad \text{or} \\ \alpha(t) &= f(k_t(y)) \qquad \text{for some } \ y \in N. \end{aligned}$$

Given $\gamma \in \mathcal{T}_1$ and $\omega = \langle u, v \rangle \in \mathcal{F}M$ with foot $(\gamma(0), \theta) \in M \times S^1$, we associate a lift $\omega\gamma$ of γ to a path in $\mathcal{F}M$ covering γ_θ which is the path in $M \times S^1$ defined by

$$\gamma_\theta(t) = (\gamma(t), \theta).$$

When $\omega \in \mathbb{B} \cup \mathbb{T}$, $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u\gamma_\theta(t), v\gamma_\theta(t) \rangle$$

where $u\gamma_\theta$ and $v\gamma_\theta$ are the asymptotic transfers defined in Lecture 3. When $\omega \in \mathbb{A}$ (and hence $u = v$) $\omega\gamma$ is defined by

$$\omega\gamma(t) = \langle u(\gamma_\theta, d)(t), u(\gamma_\theta, d)(t) \rangle$$

where $u(\gamma_\theta, d)$ is the focal transfer with focal length d and chosen so that $d \to \infty$ as the angle between u and the level surface $M \times \theta$ approaches 0. Using that the asymptotic and focal transfers both satisfy properties 1-3 of lecture 3 and that property $(*)$ is satisfied by g, f, h_t, k_t; there is a natural construction of a good transfer $\hat{g}$, $\hat{h}_t$, $\hat{k}_t$ whose tracks

$$\mathcal{T} = \{\omega\gamma \mid \gamma \in \mathcal{T}_1, \omega \in \mathcal{F}M\}.$$

We now address the problem of "shrinking" the paths $\omega\gamma \in \mathcal{T}$. Since the geodesic flow g^t is defined on $\mathbb{A} \cup \mathbb{B}$, applying it to $\omega\gamma$ gives a method for making $\omega\gamma$ skinny when $\omega \in \mathbb{A} \cup \mathbb{B}$; i.e. $g^t \circ (\omega\gamma)$ is (β, ϵ)-controlled with respect to the 1-dimensional foliation of the manifold $\mathbb{A} \cup \mathbb{B}$ by the flow lines of the geodesic flow.

But the situation is different when $\omega = \langle u, v\rangle \in \mathbb{T}$. We are tempted then to "flow ω" in the direction of its arithmetic average $\frac{u+v}{2}$. But this does nothing when $u = -v$. Fortunately a different method can be used on the top stratum $\mathbb{T}$. But to describe it we need some more geometric preliminaries. We start by defining the *core* $\mathbb{P}$ of $\mathbb{T}$ by

$$\mathbb{P} = \{\langle u, -u\rangle \in \mathbb{T}\}.$$

The core is naturally identified with the total space of the projective line bundle associated to $(TM) \times S^1$. In particular there is a natural 2-sheeted covering space

$$\mathbb{B} = (SM) \times S^1 \to (\mathbb{R}P^{m-1}M) \times S^1 = \mathbb{P}$$

and the image of the geodesic line foliation of $\mathbb{B}$ gives $\mathbb{P}$ a canonical 1-dimensional foliation denoted by $\mathcal{G}$. The top strata $\mathbb{T}$ also has an *asymptotic foliation* $\mathcal{A}$ by m-dimensional leaves where each leaf of $\mathcal{A}$ is an *asymptoty class* of elements in $\mathbb{T}$. We say that elements $\omega_1 = \langle u_1, v_1\rangle$, $\omega_2 = \langle u_2, v_2\rangle \in \mathbb{T}$ lying over $M \times \theta$ (for some $\theta \in S^1$) are *asymptotic* provided (up to interchanging u_1 and v_1) there exist points $x, y \in \tilde{M}$ together with vectors $\tilde{u}_1, \tilde{v}_1 \in S_{(x,\theta)}(\tilde{M} \times S^1)$ and $\tilde{u}_2, \tilde{v}_2 \in S_{(y,\theta)}(\tilde{M} \times S^1)$ lying over u_1, v_1, u_2, v_2, respectively, and satisfying:

$$\begin{aligned} &\tilde{u}_1 \text{ is asymptotic to } \tilde{u}_2, \text{ and} \\ &\tilde{v}_1 \text{ is asymptotic to } \tilde{v}_2. \end{aligned}$$

Note that the restriction of the bundle map

$$\mathbb{T} \xrightarrow[p]{} M \times S^1 \xrightarrow[\text{proj}]{} M$$

to any leaf L of $\mathcal{A}$ is a covering space. This puts a flat structure on this bundle. And each leaf L of $\mathcal{A}$ inherits a negatively curved Riemannian metric from M via this covering projection. We call it the natural metric and note that it is compatible with the leaf topology on L.

The foliation $\mathcal{A}$ intersects the core $\mathbb{P}$ in its $\mathcal{G}$ foliation; i.e., there is a bijective correspondence between the leaves of $\mathcal{A}$ and $\mathcal{G}$ given by

$$L \mapsto L \cap \mathbb{P}, \quad L \in \mathcal{A}.$$

Also $L \cap \mathbb{P}$ is a closed subset of L in its leaf topology and is a (simple) geodesic of its natural metric. This geodesic $\mathbb{P} \cap L$ is called the *marking* of L. Furthermore, the inclusion map of $\mathbb{P} \cap L$ into L is a homotopy equivalence when L is given the leaf topology and $\mathbb{P} \cap L$ is given the subspace of L topology.

Now there is a bundle with fiber $\mathbb{R}^{m-1}$

$$\rho : \mathbb{T} \to \mathbb{P}$$

defined as follows.

For each $\omega \in \mathbb{T}$ let $L \in \mathcal{A}$ be the leaf containing ω and g be its marking. Then $\rho(\omega)$ denotes the (unique) closest point to ω on g measured inside L.

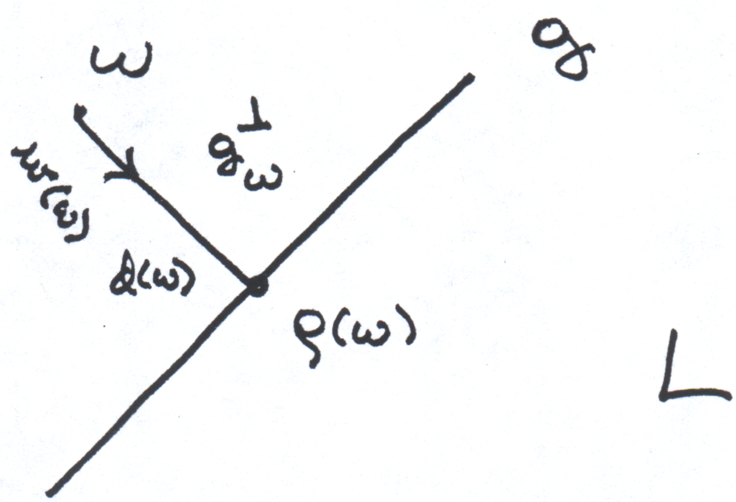


Figure 4.3

When $\omega \notin \mathbb{P}$, there is a unique geodesic segment $g_\omega^\perp$ in L connecting ω to $\rho(\omega)$. The unit length vector tangent to $g_\omega^\perp$ at ω which points towards $\rho(\omega)$ is denoted by $w(\omega)$. This defines a continuous vector field on $\mathbb{T} - \mathbb{P}$.

We denote the length of $g_\omega^\perp$ by $d(\omega)$. This extends to a continuous function $d : \mathbb{T} \cup \mathbb{B} \to [0, +\infty]$ when we set

$$d(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathbb{P} \\ +\infty & \text{if } \omega \in \mathbb{B}. \end{cases}$$

There is also a bundle with fiber the open cone in $\mathbb{R}P^{m-1}$

$$\eta : (\mathbb{T} - \mathbb{P}) \cup \mathbb{B} \to \mathbb{B} = SM \times S^1$$

defined by

$$\eta(\omega) = \begin{cases} \omega & \text{if } \omega \in \mathbb{B} \\ dp(w(\omega)) & \text{if } \omega \in \mathbb{T} - \mathbb{P}. \end{cases}$$

Remark. We think of $\eta(\omega)$ as the *asymptotic average* of the two vectors u and v where $\omega = \langle u, v \rangle$ as opposed to their *arithmetic* average $\frac{1}{2}(u + v)$.

The vector field $w(\ \ )$ integrates to give an incomplete *radial flow* r^t on $\mathbb{T}$. In particular $r^t(\omega)$ is only defined for $t \in [0, d(w)]$. And there is the following important relation between r^t and g^t.

Intertwining Equation. $\eta(r^t(\omega)) = g^t(\eta(\omega))$ for all $\omega \in \mathbb{T} - \mathbb{P}$ and $t \in [0, d(\omega)]$.

We associate to each closed interval $J \subseteq [0, +\infty]$ a compact subspace W_J of $\mathbb{T} \cup \mathbb{B}$ defined by

$$W_J = d^{-1}(J).$$

If $+\infty \notin J$, then W_J is a codimension-0 submanifold of $\mathbb{T}$ with

$$\partial W_J = \begin{cases} d^{-1}(\partial J) & \text{if } 0 \notin J \\ d^{-1}(b) & \text{if } J = [0, b]. \end{cases}$$

Furthermore, we have the following:

1. If $0 \in J$ and $+\infty \notin J$, then $\rho : W_J \to \mathbb{P}$ is a fiber bundle with fiber $\mathbb{D}^m$.

2. If $+\infty \in J$ and $0 \notin J$, then $\eta : W_J \to \mathbb{B}$ is a fiber bundle with fiber the (closed) cone on $\mathbb{R}P^{m-1}$.

3. If neither 0 nor $+\infty$ is in J, then $\eta \times d : W_J \to \mathbb{B} \times J$ is a fiber bundle with fiber $\mathbb{R}P^{m-1}$.

Now fix a closed interval $I \subseteq (0, +\infty)$ containing 1 in its interior, and a very large positive real number σ together with a second closed interval R which contains $+\infty$ and is disjoint from σI. Then $[0, +\infty) - (\mathrm{Int}(R) \cup \mathrm{Int}(\sigma I))$ is the disjoint union of 2 closed intervals A and B denoted so that $0 \in A$.

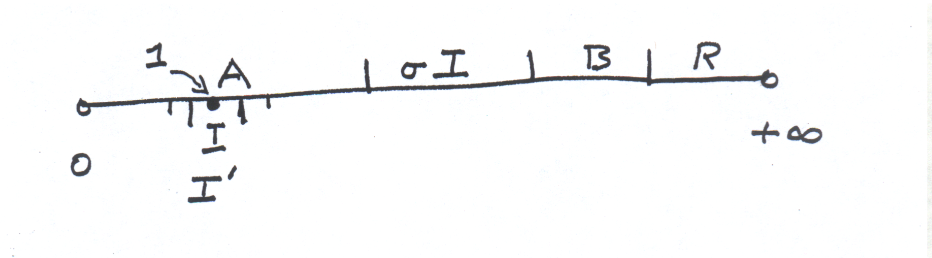


Figure 4.4

Fix another closed interval $I' \subseteq (0, +\infty)$ which contains I in its interior but is slightly larger and define a homeomorphism

$$\phi : W_{\sigma I'} \to W_{I'},$$

using the radial flow, by the formula

$$\phi(\omega) = r^t(\omega)$$

where $t = d(\omega) - \frac{1}{\sigma} d(\omega)$.

Note that ϕ becomes arbitrarily strongly contracting as we let $\sigma \to +\infty$. In particular if $\omega\gamma \in \mathcal{T}$ with $\omega \in W_{\sigma I}$, then $\phi \circ \omega\gamma$ is uniformly pointwise ϵ_σ-controlled in W_I with $\lim\limits_{\sigma \to \infty} \epsilon_\sigma = 0$.

Therefore we can use the ordinary control theorem to homotope $\hat{f}$ over $W_{\sigma I}$ (i.e. homotope $\hat{f}|_{\hat{f}^{-1}(W_{\sigma I})}$), in a controlled way, to a homeomorphism provided σ is large enough. This begins our construction of the admissible homotopy of $\hat{f}$ to a split map. And it gives a slightly different collection of tracks $\mathcal{T}_1$. These new tracks differ only for some of those $\gamma \in \mathcal{T}_1$ which start in $W_{\sigma I}$. And for them $\phi \circ \gamma$ is pointwise close to $\phi \circ \bar{\gamma}$ where $\bar{\gamma} \in \mathcal{T}$ is the corresponding track.

We next extend this homotopy to a homotopy of $\hat{f}$ over W_A to a homeomorphism. This is done by using the fibered and foliated version of the control theorem with respect to the fiber bundle

$$\rho : W_A \to \mathbb{P}$$

and the foliation $\mathcal{G}$ of $\mathbb{P}$. It is applicable since the fiber of ρ is $\mathbb{D}^m$ and the structure set $\mathcal{S}(\mathbb{D}^m \times \mathbb{D}^k, \partial)$ contains only one element for each $k \geq 0$. We need only check the control condition. For this, note that there exists a positive number β such that

$$\rho \circ \omega\gamma$$

is $(\beta, 0)$-controlled for each $\omega\gamma \in \mathcal{T}$ such that $\omega \in A$. And hence the tracks of $\mathcal{T}_1$ which start in A are $(\beta + \epsilon, \epsilon)$-controlled where $\epsilon \to 0$ as $\sigma \to \infty$.

Independent of these two steps, we use the foliated control theorem with respect to the foliation of $\mathbb{B} \cup \mathbb{A}$ by the orbits of the geodesic flow and the control map

$$g^\sigma : \mathbb{B} \cup \mathbb{A} \to \mathbb{B} \cup \mathbb{A}$$

to homotope $\hat{f}$ over $\mathbb{B} \cup \mathbb{A}$ to a homeomorphism. And then use the covering homotopy theorem to extend this to a homotopy of bundle maps over W_R relative to the fiber bundle

$$\eta : W_R \to \mathbb{B}$$

to a homeomorphism over W_R.

Let $\tau : B \to [0,1]$ denote the (unique) increasing linear homeomorphism, and fix a continuous function $\phi : B \to [\sigma, +\infty)$ such that

$$\phi(x) = \begin{cases} \sigma & \text{for all } x \text{ close to } B \cap R \\ (1 - \frac{1}{\sigma})x & \text{for } x \in \sigma I' \cap B. \end{cases}$$

Consider the fiber bundle

$$\xi : W_B \to \mathbb{B} \times [0,1]$$

where ξ is the composite

$$W_B \xrightarrow{\eta \times d} \mathbb{B} \times B \xrightarrow{g^{\phi \circ d} \times \tau} \mathbb{B} \times [0,1]$$

i.e.,

$$\xi(x) = (g^{\phi(d(x))}(\eta(x)), \tau(d(x))).$$

Finally, we use a foliated and fibered version of the control theorem with respect to the fiber bundle $\xi : W_B \to \mathbb{B} \times [0,1]$ and the foliation of $\mathbb{B} \times [0,1]$ by the flow lines of the geodesic flow in order to extend over W_B the homotopy defined in steps 1, 2 and 3 given above. And thus complete the construction of an admissible homotopy of $\hat{f}$ to a split map. The control condition is met provided σ is sufficiently large and R is contained in a sufficiently small neighborhood of $+\infty$. The intertwining equation is used to see this. But there is one extra point to observe. The fiber of ξ is $\mathbb{R}P^{m-1}$ and $\mathcal{S}(\mathbb{R}P^{m-1} \times \mathbb{D}^k, \partial)$ usually contains more than a single element. Consequently, the control theorem only yields the weaker conclusion that the result of the homotopy is a split map rather than a homeomorphism.

5 Some calculations of $\pi_n(\mathbf{Top}\ M)$, $\pi_n(\mathbf{Diff}\ M)$ and other applications

Recall (Lecture 3) that the Vanishing Theorem showing $Wh(\pi_1 M) = 0$ extends to complete, A-regular, non-positively curved Riemannian manifolds M. Likewise there is a version of the Topological Rigidity Theorem (TRT) valid for such manifolds which we proceed to formulate.

Let M be an arbitrary manifold; i.e. it can be non-compact and can have non-empty boundary. We say that M is *topologically rigid* if it has the following property. Let

$$h : (N, \partial N) \to (M, \partial M)$$

be any proper homotopy equivalence where N is another manifold. Suppose there exists a compact subset $C \subseteq N$ such that the restriction of h to $\partial N \cup (N - C)$ is a homeomorphism. Then there exists a proper homotopy

$$h_t : (N, \partial N) \to (M, \partial M)$$

from h to a homeomorphism and a perhaps larger compact subset K of N such that the restrictions of h_t and h to $\partial N \cup (N - K)$ agree for all $t \in [0, 1]$. (When M and N are closed, this just says that a homotopy equivalence $h : N \to M$ is homotopic to a homeomorphism.)

Addendum to TRT. *(Farrell and Jones [35] 1998). Let M^m be an arbitrary aspherical manifold with $m \geq 5$. Suppose $\pi_1(M)$ is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. (This happens for example when $\pi_1(M)$ is isomorphic to a torsion-free discrete subgroup of $GL_n(\mathbb{R})$.) Then M is topologically rigid. In particular, every A-regular complete non-positively curved Riemannian manifold of* $\dim \geq 5$ *is topologically rigid.*

The special case of this Addendum where M is an A-regular complete non-positively curved Riemannian manifold is proved by an argument very close to that made in Lecture 4 for TRT. But stronger control theorems are needed when M is not closed; in particular when the injectivity radius at a point $x \in M$ goes to 0 as $x \to \infty$. These control theorems were discussed by Lowell Jones in his last lecture. The general case of the Addendum follows from this special case and the version of the surgery sequence for arbitrary spaces developed by Andrew Ranicki in his lectures; in particular that the assembly map in homology

$$A_* : H_*(M;\mathcal{L}) \to L_*(\pi_1 M, w)$$

is uniquely determined by the homotopy type of M and the orientation data $w : \pi_1(M) \to \mathbb{Z}_2$.

This Addendum even has (perhaps unexpectedly) consequences beyond, what follows from TRT, for closed manifolds. We now discuss some of these.

Corollary 1. *Let N and M be a pair of closed complete affine flat manifolds. If $\pi_1(N) \simeq \pi_1(M)$, then N and M are homeomorphic (via a homeomorphism inducing this isomorphism).*

Corollary 1 is an affine analogue of the classical Bieberbach Theorem valid for Riemannian flat manifolds. We note that Corollary 1 (when dim $(M) \geq 5$) does *not* follow from the TRT proved in Lecture 4 since there are closed complete affine flat manifolds M which *cannot* support a Riemannian metric of non-positive curvature. For example $M^3 = \mathbb{R}^3/\Gamma$ does *not* where Γ is the group generated by the three affine motions α, β and γ of $\mathbb{R}^3$ with

$$\begin{aligned} \alpha(x,y,z) &= (x+1,y,z) \\ \beta(x,\gamma,z) &= (x,y+1,z) \\ \gamma(x,y,z) &= (x+y,2x+3y,z+1). \end{aligned}$$

Since Γ is solvable but not virtually abelian, the result of Gromoll-Wolf [40] and Yau [71], quoted in lecture 1, shows that M cannot support a non-positively curved Riemannian metric. But Corollary 1 (when dim $(M) \geq 5$) does follow from the Addendum to TRT since M is aspherical and $\pi_1(M)$ is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^m)$ which is a closed subgroup of $GL_{m+1}(\mathbb{R})$; namely

$$\mathrm{Aff}(\mathbb{R}^m) = \left\{ A \in GL_{m+1}(\mathbb{R}) \,\middle|\, A_{m+1,i} = \begin{cases} 0 & i \leq m \\ 1 & i = m+1 \end{cases} \right\}$$

Corollary 1 is a classical result when dim $(M) \leq 2$. And, when dim $(M) = 3$, Corollary 1 was proven by D. Fried and W.M. Goldman in [38]. Hence it remains to discuss the case when dim $(M) = 4$. In this case (in fact more generally when dim $(M) \leq 6$) H. Abels, G.A. Margulis and G.A. Soifer [1] proved that $\pi_1(M)$ is virtually polycyclic. And hence Corollary 1 follows from Farrell and Jones [28] when dim $(M) = 4$. A key ingredient in [28] is that M. Freedman and F. Quinn [37] have shown that topological surgery works in dimension 4 for manifolds with virtually poly-cyclic fundamental groups.

Corollary 1 suggests the following question.

Question. Are compact complete affine flat manifolds with isomorphic fundamental groups diffeomorphic?

Compare [34] where the analogous question for infrasolvmanifolds was affirmatively answered except in dimension 4.

We next use this Addendum to verify a special case of a well known conjecture of C.T.C. Wall; cf. [69].

Conjecture. (Wall) Let Γ be a torsion-free group which contains a subgroup of finite index isomorphic to the fundamental group of a closed aspherical manifold. Then Γ is the fundamental group of a closed aspherical manifold.

Corollary 2. *Let M^m be a closed (connected) non-positively curved Riemannian manifold and Γ be a torsion-free group which contains $\pi_1(M)$ as a subgroup with finite index. Assume that $m \neq 3, 4$, then the deck transformation action of $\pi_1(M)$ on the universal cover $\tilde{M}$ extends to a topological action of Γ on $\tilde{M}$. Consequently Wall's Conjecture is true in this case since $\tilde{M}/\Gamma$ is a closed apsherical manifold with $\pi_1(\tilde{M}/\Gamma) = \Gamma$.*

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional factors, Γ embeds in its isometry group $\text{Iso}(\tilde{M})$ extending $\pi_1(M) \subseteq \text{Iso}(\tilde{M})$; this is a consequence of Mostow's Strong Rigidity Theorem [57]. When $m = 2$, Corollary 2 is a consequence of a result due to Eckmann, Linnell and Muller [18], [19]; our proof only applies to the situation $m \geq 5$.

In proving Corollary 2 we can clearly make the simplifying assumptions that M is orientable and $\pi_1(M)$ is normal in Γ. We now use an important trick due to Serre [65] where he constructs a natural, properly discontinuous action of Γ via isometries on the Riemannian product $\mathcal{M}^{sm}$ of s-copies of $\tilde{M}$

$$\mathcal{M} = \tilde{M} \times \tilde{M} \times \cdots \times \tilde{M}$$

where $s = [\pi_1 M : \Gamma]$. (Serre's construction is a kind of geometric co-induced representation.) Note that $\mathcal{M}^{sm}$ is A-regular and non-positively curved since

M^m is. Hence $N^{sm} = \mathcal{M}/\Gamma$ is a complete (but *not* closed) A-regular non-positively curved manifold with $\pi_1(N) = \Gamma$. Thus the Addendum to TRT applies to $N^{sm} \times \mathbb{D}^k$ for all $k \geq 0$. From this we conclude that Ranicki's periodic assembly map

$$A_* : H_*(B\Gamma, \mathcal{L}) \to L_*(\Gamma)$$

is an isomorphism. Also the Vanishing Theorem applies showing that

$$Wh(\Gamma) = 0 = \tilde{K}_0(\mathbb{Z}\Gamma).$$

And Ranicki, by reworking the existence part of surgery theory, has shown that when this happens $B\Gamma$ is homotopically equivalent to a closed manifold K^m provided $B\pi$ is for some subgroup π of finite index in Γ; cf. [63, §13]. In this case, we can take $\pi = \pi_1(M)$.

Let $\hat{K}$ be the cover of K corresponding to $\pi_1(M)$. And note that $\hat{K}$ is homotopically equivalent to M since both are aspherical and have the same fundamental group. Therefore $\hat{K}$ is homeomorphic to M by the TRT. Consequently $\tilde{K} = \tilde{M}$ and the deck transformation action of $\Gamma = \pi_1(K)$ on $\tilde{M}$ is the desired extension of the action by $\pi_1(M)$. Q.E.D.

Corollary 2 can be applied to obtain positive information about the following generalization of the classical Nielsen Problem. Let $\text{Top}(M)$ denote the group of all homeomorphisms of a manifold M and denote the group of all outer automorphisms of $\pi_1(M)$ by $\text{Out}(\pi_1 M)$.

Generalized Nielsen Problem. (GNP) Let M be a closed aspherical manifold and F be a finite subgroup of $\text{Out}(\pi_1 M)$. Does F split back to $\text{Top}(M)$; i.e., does there exist a finite subgroup $\bar{F}$ of $\text{Top}(M)$ which maps isomorphically onto F under the natural homomorphism

$$\text{Top}(M) \to \text{Out}(\pi_1 M)?$$

Remark. There are cases where this is impossible. One necessary extra condition is that there exist an extension

$$1 \to \pi_1(M) \to \Gamma \to F \to 1$$

inducing the embedding $F \subseteq \text{Out}(\pi_1 M)$. F. Raymond and L. Scott [64] gave an example where this condition is not satisfied. In their example M is a nilmanifold. There is a natural exact sequence

$$1 \to \text{Center}(\Gamma) \to \Gamma \underset{\phi}{\longrightarrow} \text{Aut}(\Gamma) \underset{\psi}{\longrightarrow} \text{Out}(\Gamma) \to 1$$

where $\phi(\gamma)$ is conjugation by γ. Let $\Gamma_F = \psi^{-1}(F)$. When $\mathrm{Center}(\Gamma) = 1$

$$1 \to \Gamma \to \Gamma_F \to F \to 1$$

is the necessary extension mentioned in this Remark.

Corollary 3. *The finite group F of the GNP splits back to* $\mathrm{Top}(M)$ *under the following extra assumptions:*

1. $\mathrm{Center}(\pi_1 M) = 1$.
2. *M is a non-positively curved Riemannian manifold.*
3. $\dim(M) \neq 3, 4$.
4. *Γ_F is torsion-free.*

Remark. Conditions 1 and 2 are satisfied when M is negatively curved.

Remark. When $\dim(M) = 2$, this result is due to Eckmann, Linnell and Muller (1981).

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, this result, due to Mostow [57], is true even with conditions 1, 3 and 4 dropped.

Remark. Corollary 3 remains true when condition 2 is replaced by the weaker condition that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular non-positively curved Riemannian manifold. This is because Corollary 2 is also true under the same weakening of its hypotheses.

To prove Corollary 3, note that Γ_F satisfies the hypotheses for the group Γ in Corollary 2. Hence Γ_F acts on $\tilde{M}$ extending the action of $\pi_1(M)$ by deck transformations. The image of this action in $\mathrm{Top}(M)$ is the subgroup $\bar{F}$ asked for in GNP. Q.E.D.

There is also the related question of whether the natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$$

is onto?

Corollary 4. *Let M^m be a closed aspherical manifold. Assume that $m \neq 3, 4$ and that $\pi_1(M)$ is isomorphic to the fundamental group of a complete, A-regular, non-positively curved Riemannian manifold. Then the natural homomorphism* $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$ *is a surjection.*

Corollary 4 is classical for $m = 2$ or 1. And for $m \geq 5$, it follows immediately from the Addendum to TRT since every outer automorphism of $\pi_1(M)$ is induced by a self homotopy equivalence of M; cf. Hurewicz's result mentioned in Lecture 1. Q.E.D.

Remark. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, Corollary 4 is due to Mostow [57].

Give the group $\mathrm{Top}(M)$ the compact open topology and let its closed subgroup $\mathrm{Top}_0(M)$ be the kernel of the natural continuous homomorphism (analyzed in Corollary 4) to the discrete group $\mathrm{Out}(\pi_1 M)$. $\mathrm{Top}_0(M)$ is *not* in general the connected component of the identity element in $\mathrm{Top}(M)$. However the following was proved in [32].

Corollary 5. *Let M^m be a closed (connected) non-positively curved Riemannian manifold with $m > 10$. Then*

$$\begin{aligned} \pi_0(\mathrm{Top}_0\ M) &= \mathbb{Z}_2^\infty, \\ \pi_1(\mathrm{Top}\ M) \otimes \mathbb{Q} &= \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}, \quad \textit{and} \\ \pi_n(\mathrm{Top}\ M) \otimes \mathbb{Q} &= 0 \quad \textit{if} \ \ 1 < n \leq \frac{(m-7)}{3}. \end{aligned}$$

Remark. There is in particular the following exact sequence

$$1 \to \mathbb{Z}_2^\infty \to \pi_0(\mathrm{Top}\ M) \to \mathrm{Out}(\pi_1 M) \to 1.$$

And $\mathbb{Z}_2^\infty$ denotes the direct sum of a countably infinite number of copies of $\mathbb{Z}_2$.

The proof of Corollary 5 depends not only on the Addendum to TRT but also on the following result "PIT" concerning the stable topological pseudo-isotopy functor $\mathcal{P}(\ \ )$. Recall that this functor was defined and discussed earlier in lectures by Tom Goodwillie, Lowell Jones and Frank Quinn.

Pseudo Isotopy Theorem. *(Farrell and Jones [31]) Let M be a closed (connected) non-positively curved Riemannian manifold. Then, for all n,*

$$\begin{aligned} \pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} &= 0 \quad \textit{and} \\ \pi_0(\mathcal{P}(M)) &= \mathbb{Z}_2^\infty. \end{aligned}$$

We will discuss the ideas behind the proof of PIT after first using it in proving Corollary 5.

For this we need to introduce the auxiliary spaces $G(M)$ and $\overline{\text{Top}}(M)$. Let $G(M)$ denote the H-space of all self-homotopy equivalences of M; note that $\text{Top}(M)$ is a subspace of $G(M)$. The semisimplicial group $\overline{\text{Top}}(M)$ of blocked homeomorphisms of M can be interpolated between $\text{Top}(M)$ and $G(M)$. A typical k-simplex of $\overline{\text{Top}}(M)$ consists of a homeomorphism

$$h : \Delta^k \times M \to \Delta^k \times M$$

such that $h(\Delta \times M) = \Delta \times M$ for each face Δ of Δ^k, where Δ^k is the standard k-simplex.

Let $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ denote the homotopy fiber of the map

$$B\ \text{Top}(M) \to BG(M) \quad \text{and} \quad B\ \text{Top}(M) \to B\ \overline{\text{Top}}(M),$$

respectively. Because of Frank Quinn's function space interpretation of the surgery exact sequence [60], [61], cf. [70, §17A]; the relative homotopy groups of the map

$$\overline{\text{Top}}(M) \to G(M)$$

can be identified with the groups

$$\mathcal{S}(M \times \mathbb{D}^n, \partial).$$

And these all vanish because of the TRT; consequently the following is true.

Fact 1. $G(M)/\text{Top}(M)$ and $\overline{\text{Top}}(M)/\text{Top}(M)$ have the same weak homotopy type.

Now the homotopy groups of $G(M)$ are easy to calculate. They are

Fact 2.

$$\pi_n(G(M)) = \begin{cases} \text{Out}(\pi_1 M) & \text{if } n = 0 \\ \text{Center}(\pi_1 M) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Since the calculation for $n \geq 2$ is particularly easy to do, we sketch it. Let

$$f : S^n \times M \to M$$

represent an element in $\pi_n(G(M))$. To show this element is zero, we need to extend f to a map

$$\hat{f} : \mathbb{D}^{n+1} \times M \to M.$$

The construction of $\hat{f}$ is by an elementary obstruction theory argument. Fix a triangulation of M and assume $\hat{f}$ has already been defined over $\mathbb{D}^{n+1} \times \sigma$ for all simplices σ with $\dim(\sigma) < k$. Let σ be a k-simplex and identify $\mathbb{D}^{n+1} \times \sigma$ with $\mathbb{D}^{n+k+1}$. Then $\hat{f}|_{\partial \mathbb{D}^{n+k+1}}$ has already been defined and represents an element of $\pi_{n+k}(M)$ which vanishes since M is aspherical. Therefore $\hat{f}$ extends over $\mathbb{D}^{n+1} \times \sigma$. It is shown in this way that $\pi_n(G(M)) = 0$ when $n \geq 2$.

It therefore remains to analyze $\overline{\mathrm{Top}}(M)/\mathrm{Top}(M)$. Which can be done in terms of $\mathcal{P}(M)$ by using the following result of Hatcher [41].

Theorem. *(Hatcher) When $m > 10$ ($m = \dim M$) there is a spectral sequence converging to*

$$\pi_{p+q+1}(\overline{\mathrm{Top}}(M)/\mathrm{Top}(M))$$

with

$$E^2_{pq} = H_p(\mathbb{Z}_2; \pi_q(\mathcal{P}(M)))$$

in the stable range $q \leq \frac{(m+p-7)}{3}$.

Remark. This result depends on Igusa's Stability Theorem [47] for pseudo-isotopy spaces which Tom Goodwillie discussed in an earlier lecture.

Combining Hatcher's Theorem and PIT together with Facts 1 and 2 yields that

Fact 3.

$$\pi_n(\mathrm{Top}(M)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } 1 \leq n \leq \frac{m-7}{3} \\ \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q} & \text{if } m = 1 \end{cases}$$

and the following exact sequence:

$$\mathrm{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi_1 M).$$

Since the kernel of $\pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi_1 M)$ is $\pi_0(\mathrm{Top}_0(M))$, this exact sequence can be rewritten as

Fact 4.

$$\mathrm{Center}(\pi_1 M) \to H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \to \pi_0(\mathrm{Top}_0(M)) \to 0.$$

Define a homomorphism $d : \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty$ by

$$d(x) = x + \bar{x}$$

where $x \mapsto \bar{x}$ denotes the action of the generator of $\mathbb{Z}_2$ on $\mathbb{Z}_2^\infty$. Then the formula

$$H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty) = \mathbb{Z}_2^\infty/\mathrm{image}(d)$$

is the definition of $H_0(\mathbb{Z}_2, \mathbb{Z}_2^\infty)$. We claim that $\mathbb{Z}_2^\infty/\mathrm{image}(d)$ cannot be a finite group. If it were, then $\mathbb{Z}_2^\infty/\ker(d)$ would also be finite since

$$\ker(d) \supseteq \mathrm{image}(d).$$

(Note that $d^2 = 0$ since $\mathbb{Z}_2^\infty$ has exponent 2.) But image(d) is isomorphic to $\mathbb{Z}_2^\infty/\ker(d)$. And the finiteness of both image(d) and $\mathbb{Z}_2^\infty/\mathrm{image}(d)$ would imply that $\mathbb{Z}_2^\infty$ is also finite, which is a contradiction. Since $H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty)$ is thus a countable infinite group of exponent 2, it must be isomorphic to $\mathbb{Z}_2^\infty$. We therefore rewrite the sequence in Fact 4 as

Fact 5.

$$\mathrm{Center}(\pi_1 M) \to \mathbb{Z}_2^\infty \to \pi_0(\mathrm{Top}_0(M)) \to 0.$$

Now Lawson and Yau [51] showed that Center$(\pi_1 M^m)$ is finitely generated. (In fact it is isomorphic to $\mathbb{Z}^n$ where $n \leq m$.) Hence Fact 5 implies that $\pi_0(\mathrm{Top}_0(M))$ is a countably infinite group of exponent 2, and therefore it is isomorphic to $\mathbb{Z}_2^\infty$. This result together with Fact 3 proves Corollary 5.

We now return to a discussion of PIT. Its proof follows the pattern established in proving the Vanishing Theorem (cf. Lecture 3). The main difference is that the corresponding foliated control theorem is obstructed since $\mathcal{P}(S^1)$ is *not* contractible. So we get a calculation instead of a vanishing theorem. Key ingredients for this calculation are ideas developed by Frank Quinn which were discussed in his and Lowell Jones' lectures.

We formulate a more precise result than PIT; namely a weak version of the Isomorphism Conjectures which Wolfgang Lueck talked about in one of his lectures. For the rest of this lecture M denotes a closed (connected) non-positively curved Riemannian manifold, $\tilde{M}$ its universal cover, and $\Gamma = \pi_1(M)$ its group of deck transformations. Fix a universal space $\mathcal{E}$ for Γ relative to the class $\mathcal{C}$ of all virtually cyclic subgroups of Γ.

Theorem. *(Farrell and Jones [31], also [33]) There exists a spectral sequence converging to $\pi_{p+q}(\underline{\mathcal{P}}(M))$ with $E^2_{pq} = H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma)))$.*

Remark. In this theorem Γ_σ denotes the subgroup of Γ fixing a cell σ of $\mathcal{E}$. And
$H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma)))$ is the p-th homology group of a chain complex whose p-th chain group is the direct sum of the groups $\pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_\sigma))$ where σ varies over a set S_p of p-cells of $\mathcal{E}$. The set S_p contains exactly one p-cell from each Γ-orbit of p-cells. Here $\underline{\mathcal{P}}(X)$ denotes the stable topological pseudo-isotopy Ω-spectrum of a space X; cf. [33, §1.1] for more details.

To deduce PIT from this result, we must analyze the spectral sequence. Note first that Γ_σ is either infinite cyclic or trivial since Γ is torsion-free. Therefore $\tilde{M}/\Gamma_\sigma$ is homotopically equivalent to either the circle S^1 or a point $*$ since $\tilde{M}/\Gamma_\sigma$ is aspherical. And there is the following important calculation:

Calculation 1. (a) $\pi_n(\mathcal{P}(*)) = 0$ for all n,

(b) $\pi_n(\mathcal{P}(S^1)) \otimes \mathbb{Q} = 0$ for all n,

(c) $\pi_0(\mathcal{P}(S^1)) = \mathbb{Z}_2^\infty$.

Calculation (a) is a consequence of Alexander's Trick discussed in Lecture 1. Calculation (b) is due to Waldhausen [68], and (c) is due to Waldhausen and Igusa [46]. Calculations (b) and (c) are deep results related to Tom Goodwillie's Lectures 1 and 2. Because of (a) and (b), $E^2_{pq} \otimes \mathbb{Q} = 0$. Hence the Theorem yields that $\pi_n(\mathcal{P}(M)) \otimes \mathbb{Q} = 0$; which is the first assertion of PIT.

Our Theorem also yields that

$$\pi_0(\mathcal{P}(M)) = H_0(\mathcal{E}/\Gamma; \pi_0(\mathcal{P}(\tilde{M}/\Gamma_\sigma))).$$

Since we can pick $\mathcal{E}$ to be a countable CW-complex (because Γ is countable) this equation together with Calculations (a) and (c) imply that $\pi_0(\mathcal{P}(M))$ is a quotient group of $\mathbb{Z}_2^\infty$; i.e. is a countable abelian group of exponent 2. To complete the proof of PIT, it remains to show that $\pi_0(\mathcal{P}(M))$ is an infinite group. We will only show this when M is negatively curved, since the general case depends on constructing a universal space $\mathcal{E}$ for Γ with better properties than the abstract construction. This geometric construction of $\mathcal{E}$ uses strongly the assumption that M is closed and non-positively curved. But, when M is negatively curved, the fact that $\pi_0(\mathcal{P}(M))$ is infinite is an immediate consequence of the following Assertion.

Assertion. *Assume M is negatively curved and let $\gamma : S^1 \to M$ represent a non-trivial element $[\gamma] \in \pi_1(M)$. Then*

$$\mathcal{P}(\gamma)_\# : \pi_0(\mathcal{P}(S^1)) \to \pi_0(\mathcal{P}(M))$$

is monic where $\mathcal{P}(\gamma) : \mathcal{P}(S^1) \to \mathcal{P}(M)$ is the functorially induced map.

We indicate the proof of this Assertion under the simplifying assumption that M is orientable. To do this we construct a transfer map

$$\tau : \mathcal{P}(M) \to \mathcal{P}(S^1)$$

such that $\tau \circ \mathcal{P}(\gamma)$ is homotopic to $\mathrm{id}_{\mathcal{P}(S^1)}$. The Assertion is clearly a consequence of this. Our construction uses ideas from Lecture 2. We first define a map

$$P(M) \to P(\bar{M}).$$

(Recall that $\bar{M} = \tilde{M} \cup M(\infty)$ is homeomorphic to $\mathbb{D}^m$.) This is done by sending the pseudo-isotopy f to the pseudo-isotopy $\bar{f}$ where

$$\bar{f}(x) = \begin{cases} x & \text{if } x \in M(\infty) \times [0,1] \\ \tilde{f}(x) & \text{if } x \in \tilde{M} \times [0,1] \end{cases}$$

and $\tilde{f}$ is the unique lift of f such that $\tilde{f}|_{\tilde{M}\times 0} = \mathrm{id}_{\tilde{M}\times 0}$. This pseudo-isotopy $\bar{f}$ is "well-defined" because Cartan's Theorem shows that property 2 of Condition $(*)$ holds (cf. Lecture 2). To be precise, $\bar{f}$ is only well defined after we collapse $x \times [0,1]$, $x \in M(\infty)$, to the single point x. But this quotient space can be identified with $\bar{M} \times [0,1]$. Note that $\bar{f}$ is Γ-equivariant. Let S be the infinite cyclic subgroup of Γ generated by $[\gamma]$. There are exactly two points S^+ and S^- on $M(\infty)$ fixed by S since $[\gamma]$ can be represented by a closed geodesic (because M is compact). Furthermore S acts freely and properly discontinuously on $\bar{M} - \{S^+, S^-\} = M_S$ and hence $\bar{f}$ induces a pseudo-isotopy

$$\hat{f} \in P(M_S/S).$$

But M_S/S is homeomorphic to $S^1 \times \mathbb{D}^{m-1}$ since M is orientable. The function $f \mapsto \hat{f}$ mapping

$$P(M) \to P(S^1 \times \mathbb{D}^{m-1})$$

stabilizes to give the desired transfer τ. (See [27, §2] for more details.) Q.E.D.

We end our lectures by giving an analogue of Corollary 5 true for $\mathrm{Diff}(M)$.

Corollary 6. *Suppose that M^m is orientable, $m > 10$ and $1 < n \leq \frac{(m-7)}{3}$. Then*

$$\pi_n(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } m \text{ is even} \\ \displaystyle\bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(M, \mathbb{Q}) & \text{if } m \text{ is odd.} \end{cases}$$

Furthermore, $\pi_1(\mathrm{Diff}(M)) \otimes \mathbb{Q} = \mathrm{Center}(\pi_1 M) \otimes \mathbb{Q}$.

Corollary 6 is an immediate consequence of the following result combined with TRT, PIT and the Vanishing Theorem.

Theorem. *(Farrell and Hsiang [23]) Let N^m be a closed aspherical manifold such that*

$$\begin{aligned} \mathcal{S}(N^m \times \mathbb{D}^k, \partial) &= 0 \qquad \text{for all } k \geq 0, \\ \pi_k(\mathcal{P}(N)) \otimes \mathbb{Q} &= 0 \qquad \text{for all } k \geq 0, \\ Wh(\pi_1(N) \times \mathbb{Z}^k) &= 0 \qquad \text{for all } k \geq 0. \end{aligned}$$

Then for $1 \leq n \leq (\frac{m-7}{3})$

$$\pi_n(\mathrm{Diff}(N)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } n > 1,\ n \text{ even} \\ \displaystyle\bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(N, \mathbb{Q}) & \text{if } n > 1,\ n \text{ odd} \\ \mathrm{Center}(\pi_1 N) \otimes \mathbb{Q} & \text{if } n = 1. \end{cases}$$

6 Conclusion

Recall from lecture 1 that a basic problem in topology since the time of Poincaré has been to classify manifolds. Surgery theory gives an approach to doing this for all manifolds homotopy equivalent to a given manifold M. And this theory is effective provided the Wall L-groups $L_n(\pi_1 M)$ and the Whitehead group $Wh(\pi_1 M)$ can be calculated. Furthermore the calculation of $\pi_n(\mathrm{Top}(M))$ and $\pi_n(\mathrm{Diff}(M))$, through a stable range of dimensions n, can also be reduced to classical algebraic topology problems if additionally $\pi_n(\mathcal{P}(M))$ can be calculated.

In the above lectures, we've examined these problems for the case where M is aspherical. Let us now discuss the modifications needed to handle the general case; i.e., we no longer assume that M is aspherical. For this purpose, L.E. Jones and myself have formulated in [33] the "Isomorphism Conjectures" which can be thought of as a generalization of the Borel Conjecture (when dim $M \neq 3, 4$). There are separate conjectures for K-theory, L-theory and the stable topology pseudo-isotopy spectrum $\underline{\mathcal{P}}(X)$. For concreteness we focus attention here on the conjecture for $\underline{\mathcal{P}}(X)$.

F. Quinn in [62, appendix] constructed a covariant functor which associates to a continuous surjection $p : X \to B$ (of topological spaces) an Ω-spectrum $\underline{\mathcal{P}}(X, p)$–called the stable topological pseudo-isotopy spectrum of X with control relative to p. When p is the constant map to a point, $\underline{\mathcal{P}}(X, p)$ is the ordinary stable topological pseudo-isotopy spectrum $\underline{\mathcal{P}}(X)$. And there is a functorially induced forget control map from $\underline{\mathcal{P}}(X, p)$ to $\mathcal{P}(X)$, defined for arbitrary p, called the *Quinn assembly map.* The isomorphism conjecture for $\underline{\mathcal{P}}$ (and more generally the fibered isomorphism conjecture for $\underline{\mathcal{P}}$) is formulated in terms of the Quinn assembly map.

Let Γ be an arbitrary (discrete) group and $\mathcal{E}$ be a universal Γ-space for the class consisting of all virtually cyclic subgroups of Γ. (See [33, appendix] and W. Lueck's lectures for a detailed description of $\mathcal{E}$.) And let $\tilde{X} \to X$ be any regular covering space of a CW complex X with Γ for its group of deck transformations. Note that $\tilde{X} \to X$ is a principal Γ-bundle and form the associated bundle with fiber $\mathcal{E}$

$$\tilde{X} \times_\Gamma \mathcal{E} \underset{q}{\to} X.$$

Note that q is a homotopy equivalence, since $\mathcal{E}$ is contractible, and hence functorially induces an equivalence of the spectrum $\underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E})$ with the spectrum $\underline{\mathcal{P}}(X)$. Recall that $\tilde{X} \times_\Gamma \mathcal{E}$ is the quotient space of $\tilde{X} \times \mathcal{E}$ under the natural diagonal action of Γ. Let $\rho : \tilde{X} \times_\Gamma \mathcal{E} \to \mathcal{E}/\Gamma$ be the continuous map induced by projection of $\tilde{X} \times \mathcal{E}$ onto its second factor. The *fibered isomorphism conjecture* (FIC) *for* Γ states that the Quinn assembly map

$$\underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E}, \rho) \to \underline{\mathcal{P}}(\tilde{X} \times_\Gamma \mathcal{E}) = \mathcal{P}(X)$$

is an equivalence of Ω-spectra for every $\tilde{X} \to X$. And the *isomorphism conjecture* (IC) *for* Γ is the same statement made under the extra assumption that $\tilde{X} \to X$ is an arbitrary universal covering space, with $\pi_1(X) = \Gamma$. Two reasons why the FIC (and its special case the IC) is interesting are:

1. Quinn constructed in [62, appendix] a spectral sequence E^n_{pq} converging to $\pi_{p+q}(\underline{\mathcal{P}}(X))$ with

$$E^2_{pq} = H_p(\mathcal{E}/\Gamma; \pi_q(\underline{\mathcal{P}}(\tilde{X}/\Gamma_x)))$$

   where Γ_x denotes the virtually cyclic subgroup of Γ fixing $x \in \mathcal{E}$.

2. Anderson and Hsiang [5] showed that

$$\pi_q(\underline{\mathcal{P}}(X)) = \begin{cases} Wh(\pi_1(X)) & \text{if } q = -1 \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)) & \text{if } q = -2 \\ K_{q+2}(\mathbb{Z}\pi_1(X)) & \text{if } q \leq -3. \end{cases}$$

Evidence for the FIC is the following result proved by L.E. Jones and myself (1993) in [33, Th. 2.1 and Th. A.8].

Theorem. *The* FIC *is true for every discrete cocompact subgroup* Γ *of any (virtually connected) Lie group* G*, and (more generally) for every subgroup of such a group* Γ.

Final Remark. In order to study $\underline{\mathcal{P}}(M)$, let $\Gamma = \pi_1(M)$ and consider the universal covering space $\tilde{M} \to M$. If the IC is true for Γ, then the Ω-spectrum $\underline{\mathcal{P}}(M)$ is equivalent to $\underline{\mathcal{P}}(\tilde{M} \times_\Gamma \mathcal{E}, \rho)$ which can be analyzed using Quinn's spectral sequence. This analysis requires being able to calculate $\pi_q(\underline{\mathcal{P}}(\tilde{M}/\Gamma_x))$. For this note that $\tilde{M}/\Gamma_x$ is a manifold whose fundamental group is virtually cyclic. Now the Anderson-Hsiang result [5], mentioned above, is useful for $q < 0$ and [8] is useful for $q \geq 0$, at least when $\Gamma_x = 1$.

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Algebraic K- and L-Theory and Applications to the Topology of Manifolds

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Contents

The development of geometric topology has led to the identification of specific algebraic structures of great richness and usefulness. A common theme in this area is the study of algebraic invariants of discrete groups or rings by topological methods. The resulting subject is now called algebraic K-theory.

The purpose of these lecture notes is to survey some of the main constructions and techniques in algebraic K-theory, together with an indication of the topological background and applications. More details about proofs can be found in the references. The material is organized into some introductory sections, concerning linear and unitary K-theory, followed by descriptions of four important geometric problems and their related algebraic methods. Good general sources for much of the preliminary material are the books of K. Brown [6], Curtis-Reiner [15], [14], Milnor [44], Milnor-Husemoller [45], and Swan-Evans [61]. Some of the material in Section 7 is based on my DMV lecture notes [63].

1 Homology of coverings

Let X be a CW-complex with fundamental group $\pi = \pi_1(X, x_0)$, and denote by $\Lambda := \mathbf{Z}\pi$ the integral group ring of π. The standard involution $\lambda \mapsto \bar{\lambda}$ on Λ is induced by the formula

$$\sum n_g g \mapsto \sum n_g g^{-1}$$

for $n_g \in \mathbf{Z}$ and $g \in \pi$. Notice that this gives an anti-automorphism of the group ring since $\overline{uv} = \bar{v}\bar{u}$ for all u, $v \in \Lambda$.

If $\widetilde{X}$ denotes the universal covering of X, then π acts cellularly on $\widetilde{X}$ by deck transformations, and the cellular chain complex $C_*(\widetilde{X}; \mathbf{Z})$ becomes a free Λ-chain complex of right Λ-modules. We define $C_*(X) := C_*(\widetilde{X}; \mathbf{Z})$ with this right Λ-module structure, and

$$H_*(X; \Lambda) := H(C_*(X)) \ . \tag{1.1}$$

These are just the homology groups of the universal covering together with the π-action. More generally, if M is any right Λ-module, we define

$$H_*(X; M) := H(C_*(X) \otimes_\Lambda M) \tag{1.2}$$

where to define the tensor product we convert M into a left Λ-module by the rule $\lambda m = m\bar{\lambda}$.

Lemma 1.3. *Let $\rho \triangleleft \pi$ and $X(\rho)$ be the orbit space of $\widetilde{X}$ under the action of ρ. Then $H_*(X; \mathbf{Z}[\pi/\rho]) = H_*(X(\rho); \mathbf{Z})$.*

For $\rho = \pi$ acting trivially on $\mathbf{Z}$ this agrees with the ordinary homology of $X = X(\pi)$. The homology of the coverings $X(\rho)$ are related to $H_*(X; \mathbf{Z})$ via the projection maps $p\colon X(\rho) \to X$ and the transfer $trf\colon \Sigma^\infty X_+ \to \Sigma^\infty X(\rho)_+$.

Proposition 1.4. *If $\pi = \pi_1(X, x_0)$ is a finite group, the composition $p_* \circ trf_*\colon H_i(X; \mathbf{Z}) \to H_i(X; \mathbf{Z})$ is multiplication by $|\pi|$.*

We can also define the cohomology of X with coefficients in a right Λ-module M by

$$H^*(X; M) := H(\mathrm{Hom}_\Lambda(C_*(X), M)) \ .$$

Lemma 1.5. *For X a finite CW-complex, the groups $H^*(X; \Lambda)$ are isomorphic to the cohomology groups $H^*_{cp}(\widetilde{X}; \mathbf{Z})$ of $\widetilde{X}$ with compact support.*

A finite connected CW-complex X which homologically resembles a manifold of dimension n is called a finite Poincaré complex of formal dimension n. More precisely, we start with a pair (X, w) where $w\colon \pi_1(X, x_0) \to \{\pm 1\}$ is a homomorphism (in the case of an actual manifold, this is the orientation data dual to the first Stiefel-Whitney class). If w is trivial we suppress it from the notation. For such a pair (X, w) we define a new involution on $\Lambda = \mathbf{Z}\pi_1(X, x_0)$ by the formula

$$\sum n_g g \mapsto \sum w(g) n_g g^{-1}$$

taking into account the values of the orientation homomorphism. Then w-twisted homology groups $H^w_*(X; M) = H(C_*(X) \otimes_\Lambda M)$ are defined as above, using the w-twisted involution to convert M from a right Λ-module into a left Λ-module.

The Poincaré duality between the chain complex $C_*(X)$ and the co-chain complex $C^*(X) := \mathrm{Hom}_\Lambda(C(X), \Lambda)$ is defined with respect to a fundamental class $[X] \in H^w_n(X; \mathbf{Z})$. Notice that co-chain groups have an obvious left Λ-module structure, which we convert to a right Λ-module structure using the involution again. More generally, the right Λ-module structure on the Λ-dual $M^* := \mathrm{Hom}_\Lambda(M, \Lambda)$ of any right Λ-module M is defined by the formula $(\phi \cdot \lambda)(m) := \bar{\lambda}\phi(m)$ for all $\phi \in M^*$, $\lambda \in \Lambda$, and $m \in M$.

Let $\xi \in C_n(X) \otimes_\Lambda \mathbf{Z}$ be a representative cycle for $[X]$, so the transfer $trf\,\xi \in C_n(X)$ is a locally finite chain on $\widetilde{X}$. Then (X, w) is a Poincaré complex of formal dimension n and orientation class w if the chain map

$$\xi \cap \colon C^*(X) \to C_*(X)$$

defined by the cap product with $trf\,\xi$ is a chain homotopy equivalence. Since the choice of representative ξ for $[X]$ is unique up to chain homotopy, the Poincaré duality condition just says that the cap product induces isomorphisms

$$[X] \cap \colon H^r(X; M) \to H^w_{n-r}(X; M)$$

for all r, and any coefficient module M. In the special case where $M = \Lambda$ this is the usual duality between homology and cohomology with compact supports on $\widetilde{X}$.

2 Homology of groups

Let G be a discrete group and recall that $K(G, 1)$ denotes any CW-complex X with $\pi_1(X, x_0) = G$ and $\pi_i(X) = 0$ for $i > 1$. Such a space is uniquely determined up to homotopy equivalence by G. The homology of the group G with coefficients in a right G-module (i.e. a right $\mathbf{Z}G$-module) is defined to be

$$H_*(G; M) := H_*(K(G, 1); M) \ . \tag{2.1}$$

Similarly, we can define the cohomology of G with coefficients in a G-module as the cohomology of $K(G, 1)$.

Example 2.2. For $G = \mathbf{Z}/2$, the space $X = K(\mathbf{Z}/2, 1) = RP^\infty$ is the union of all the real projective spaces RP^n as $n \to \infty$. Then $\widetilde{X} = S^\infty$ and

$$\Lambda = \mathbf{Z}[\mathbf{Z}/2] = \mathbf{Z}[T]/(T^2 - 1) \cong \mathbf{Z} + \mathbf{Z}T$$

where T denotes the deck transformation given by the antipodal map on S^∞. The chain complex $C_*(RP^\infty; \Lambda)$ in this case is

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial"] \arrow[d, equal] & C_2 \arrow[r, "\partial"] \arrow[d, equal] & C_1 \arrow[r, "\partial"] \arrow[d, equal] & C_0 \arrow[r, "\epsilon"] \arrow[d, equal] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \Lambda \arrow[r, "1-T"] & \Lambda \arrow[r, "1+T"] & \Lambda \arrow[r, "1-T"] & \Lambda \arrow[r, "\epsilon"] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

where $\epsilon\colon \Lambda \to \mathbf{Z}$ is the augmentation map $\sum n_g g \mapsto \sum n_g$. Since S^∞ is contractible (this always holds for the universal covering of a $K(G,1)$), the sequence above is exact and we get a resolution of $\mathbf{Z}$ by free Λ-modules. In order to compute the homology groups $H_*(\mathbf{Z}/2; M)$ for a $\mathbf{Z}/2$-module M, we tensor the complex above with M to obtain

$$\cdots \qquad \longrightarrow M \xrightarrow{1-T} M \xrightarrow{1+T} M \xrightarrow{1-T} M \xrightarrow{1\otimes\epsilon} M$$

The differentials $\Lambda \otimes M \to \Lambda \otimes M$ are given by $\lambda \otimes m \mapsto \partial\lambda \otimes m$, which is just multiplication by $1 \pm T$. Therefore

$$H_{2k}(\mathbf{Z}/2; M) = \{m \in M \mid m = -Tm\}/\{m - Tm \mid m \in M\}$$

and

$$H_{2k+1}(\mathbf{Z}/2; M) = \{m \in M \mid m = Tm\}/\{m + Tm \mid m \in M\}$$

for $k > 0$, and these groups are all 2-torsion. □

Proposition 2.3. *For G a finite group, the homology $H_*(G; M)$, $* > 0$, is torsion of exponent $|G|$ for any G-module M.*

If $M = \mathbf{Z}$ with trivial G-action, we write $H_*(G) = H_*(G; \mathbf{Z})$. Here are some useful properties:

(i) $H_1(G) = G/[G, G]$.

(ii) If $G = F/R$ where F is a free group and R is a normal subgroup of F, then $H_2(G) = R \cap [F, F]/[F, R]$ (Hopf's formula).

(iii) If $G = B_1 *_A B_2$ is the amalgamated free product of B_1 and B_2 over a common subgroup A, then there is an exact sequence

$$\cdots \to H_i(A) \to H_i(B_1) \oplus H_i(B_2) \to H_i(G) \to H_{i-1}(A) \to \ldots$$

(iv) $H_i(G \times \mathbf{Z}) = H_i(G) \oplus H_{i-1}(G)$.

(v) If $f\colon G_1 \to G_2$ is a group homomorphism, there is a long exact sequence

$$\ldots H_i(G_1) \to H_i(G_2) \to H_i(f) \to H_{i-1}(G_1) \ldots$$

where $H_*(f)$ is the homology of the mapping cylinder of the induced map $f\colon K(G_1, 1) \to K(G_2, 1)$.

Sometimes the space $K(G,1)$ is homotopy equivalent to a finite Poincaré complex, or even a closed manifold. These groups are of great interest in geometric topology (see [20] for a comprehensive survey of progress on the Novikov and Borel conjectures concerning the topology of aspherical manifolds). The basic example is $G = \mathbf{Z}^k$, where $K(G,1) = T^k$ is a k-dimensional torus.

Lemma 2.4. *Suppose that $K(G,1)$ is homotopy equivalent to a finite complex. Then G contains no elements of finite order except the identity.*

This is the first necessary condition for G to be the fundamental group of an aspherical manifold.

3 Projective Modules

Let R be a ring with unit element. An R-module P is *projective* if it is a direct summand of a free R-module. The projective class group $K_0(R)$ is the Grothendieck group of the category $\mathcal{P}(R)$ of *finitely-generated* projective R-modules. More explicitly, the generators of $K_0(R)$ are isomorphism classes $[P]$, for each $P \in \mathcal{P}(R)$, and relations $[P \oplus Q] = [P] + [Q]$ for all P, Q in $\mathcal{P}(R)$. Then $K_0(R)$ is an abelian group, and $[P] = [Q]$ in $K_0(R)$ if and only if $P \oplus R^k \cong Q \oplus R^k$ for some integer $k \geq 0$. This relation is called *stable* isomorphism. In many cases (e.g. for R a field or skew field), stable isomorphism implies isomorphism. We will discuss this "cancellation" problem more below.

Proposition 3.1. *If R is a field, skew field, local ring, or a principal ideal domain, then $K_0(R) \cong \mathbf{Z}$, where the isomorphism is given by the rank.*

For R a Dedekind domain, such as the ring of integers in an algebraic number field, the group $K_0(R)$ is difficult to calculate since it involves the ideal class group of R. If K denotes the field of fractions of R, then a fractional R-ideal is a finitely-generated R-submodule of K. The product $J_1 J_2$ of two fractional ideals is the R-ideal consisting of all finite sums $\sum x_i y_i$ with $x_i \in J_1$ and $y_i \in J_2$. The inverse ideal $J^{-1} = \{x \in K | xJ \subseteq R\}$, and the product $JJ^{-1} = R$. Two fractional ideals are called equivalent if they are isomorphic as R-modules, and the equivalence class of J is denoted $[J]$. The ideals equivalent to R are called principal ideals, and the abelian group (under multiplication) of ideal classes modulo principal ideals is the ideal class group $Cl(R)$.

Theorem 3.2. *Let R be a Dedekind domain whose quotient field is an algebraic number field or a function field. Then the ideal class group $Cl(R)$ is finite.*

Even for $R = \mathbf{Z}[e^{2\pi i/p}]$, where p is an odd prime, the structure of the ideal class group $Cl(R)$ is generally unknown (see [44, p. 30]), although there is an explicit formula for its order, called the *ideal class number*.

Theorem 3.3. *Let R be a Dedekind domain. Then $K_0(R) = \mathbf{Z} \oplus Cl(R)$.*

The ideal class number is often non-trivial (e.g. for $p = 23, 29$), so these rings have projective modules which are not stably isomorphic to free modules.

The projective class group respects products of rings

$$K_0(R \times S) \cong K_0(R) \oplus K_0(S)$$

and if $f\colon R \to S$ is a ring homomorphism, there is an induced map

$$f_*\colon K_0(R) \to K_0(S)$$

induced by $f_*(P) = P \otimes_R S$, with the usual functorial properties. In addition, K_0 is Morita invariant so that

$$K_0(M_n(R)) \cong K_0(R) \ .$$

We define $\widetilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the subgroup generated by the free modules $[R^k]$. In other words, if $i\colon \mathbf{Z} \to R$ maps $1 \in \mathbf{Z}$ to the unit element of R, then

$$\widetilde{K}_0(R) := K_0(R) / \operatorname{Im} i_*$$

The finiteness result above for the class group of Dedekind domains has a striking generalization due to R. Swan. A module is called *locally free* if it becomes free after tensoring with $\mathbf{Z}_{(p)}$ for all primes p.

Theorem 3.4 (Swan [59]). *Let R be a Dedekind domain of characteristic 0, and G be a finite group such that no rational prime dividing the order of G is invertible in R. Then every finitely generated projective RG module is locally free. Moreover, $K_0(RG) = \mathbf{Z} \oplus \widetilde{K}_0(RG)$ and $\widetilde{K}_0(RG)$ is finite.*

If R can be embedded into a (skew) field F, then we define $\operatorname{rank}(P) = \dim_F(P \otimes_R F)$. It follows that $\widetilde{K}_0(R) \cong \ker\{r\colon K_0(R) \to K_0(F)\}$, where $r([P]) = \operatorname{rank} P$.

Lemma 3.5. *If R can be embedded in a field or skew field, then* $K_0(R) \cong \mathbf{Z} \oplus \widetilde{K}_0(R)$.

This direct sum splitting doesn't always hold. For example, if $R = M_n(F)$ is the ring of $n \times n$ matrices over a field F, then $K_0(R) \cong \mathbf{Z}$ generated by the simple module, but $\widetilde{K}_0(R) = \mathbf{Z}/n$.

The primary methods for computing $K_0(R)$ are localization sequences and the Mayer-Vietoris type exact sequences arising from fibre squares of rings. The general idea is to reduce the study of projective R-modules to the same problem over simpler rings, such as full matrix rings over (skew) fields, or complete rings. Recall that R with a 2-sided ideal I is a complete in the I-adic topology if $R = \varprojlim R/I^k$.

Lemma 3.6. *If R be a complete in the I-adic topology, then* $K_0(R) = K_0(R/I)$ *via the projection map.*

For example, this holds if R is either a left artinian ring, or a finitely generated algebra over a complete noetherian local ring, and I is contained in $\operatorname{Rad} R$. We will be particularly interested in group rings RG, where G is a finite group and $R = \widehat{\mathbf{Z}}_p$. In this case, $K_0(\widehat{\mathbf{Z}}_p G) \cong K_0(\mathbf{F}_p G) \cong K_0(\widehat{\mathbf{Z}}_p G/Rad)$.

To reconstruct projectives over R out of projectives over simpler rings, we need the technique of fibre squares. Suppose that a commutative square of rings

$$\begin{array}{ccc} R & \xrightarrow{i} & S_1 \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} \\ S_2 & \xrightarrow{j} & T \end{array} \tag{3.7}$$

has the following properties

(i) The maps i, j, k, and l are ring homomorphisms.

(ii) R is the fibre product of S_1 and S_2 over T.

(iii) At least one of j and l is surjective.

Given left modules P_1 and P_2 over S_1 and S_2 respectively, together with an isomorphism

$$h\colon j_*(P_1) = P_1 \otimes_{S_1} T \cong P_2 \otimes_{S_2} T = l_*(P_2)$$

let $M(P_1, P_2, h) := \{(p_1, p_2) \in P_1 \times P_2 \mid h(j_*(p_1)) = l_*(p_2)\}$. This has an R-module structure by the formula $r \cdot (p_1, p_2) := (i(r)p_1, k(r)p_2)$.

Proposition 3.8 (Milnor [44]). *If P_1 and P_2 are finitely generated or projective, then so is $M(P_1, P_2, h)$. Every projective R-module is isomorphic to some $M(P_1, P_2, h)$ for suitable choices of P_1, P_2 and h.*

The choice of isomorphism h does change the isomorphism class of $M(P_1, P_2, h)$ in general, but the information so far tells us that the sequence

$$K_0(R) \to K_0(S_1) \oplus K_0(S_2) \to K_0(T)$$

is exact. Extending this Mayer-Vietoris type sequence to the left or right will involve the definition of new K-theory functors.

Example 3.9. Let $G = \mathbf{Z}/p$ for p a prime, and let $R = \mathbf{Z}G$. Then there is a fibre square of the kind just considered

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "k"] & \mathbf{Z} \arrow[d, "l"] \\ \mathbf{Z}[\zeta_p] \arrow[r, "j"] & \mathbf{F}_p \end{tikzcd}$$

where $\zeta_p = e^{2\pi i/p}$. The sequence above shows that the new ingredient in the calculation of $\widetilde{K}_0(\mathbf{Z}G)$ is the kernel

$$D(\mathbf{Z}G) := \ker\{K_0(\mathbf{Z}G) \to K_0(\mathbf{Z}[\zeta_p] \oplus K_0(\mathbf{Z})\} \ .$$

By analysing this fibre square one can show:

Theorem 3.10 (Reiner). *Let $G = \mathbf{Z}/p$ for p a prime. Then $D(\mathbf{Z}G) = 0$.*

Using this result, Reiner [15] was able to completely classify the integral representation of $\mathbf{Z}/p$, or in other words, the finitely-generated modules over $\mathbf{Z}[\mathbf{Z}/p]$ which are torsion-free as abelian groups. For most finite groups such a classification is not available. $\square$

For G any finite group, there exists a maximal $\mathbf{Z}$-order $\mathcal{M} \subset \mathbf{Q}G$ containing $\mathbf{Z}G$. In particular, $\mathcal{M}$ is a subring of $\mathbf{Q}G$ which is finitely generated as a $\mathbf{Z}$-module, and such that $\mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}G$. In the example above, $\mathcal{M} = \mathbf{Z}[\zeta_p] \oplus \mathbf{Z}$. For any finite group G, we define

$$D(\mathbf{Z}G) := \ker\{K_0(\mathbf{Z}G) \to K_0(\mathcal{M})\} \subseteq \widetilde{K}_0(\mathbf{Z}G) \ .$$

The calculation of this group has been a major research goal in the algebraic K-theory of finite groups (see [47] or [14] for references). Note that $D(\mathbf{Z}G)$ has finite order by Swan's theorem.

4 Finiteness obstructions

For X a finite CW-complex and $\Lambda = \mathbf{Z}\pi_1(X, x_0)$, the chain complex $C_*(X; \Lambda)$ is a complex of finitely generated free Λ-modules. We say that a CW-complex X is *finitely dominated* if there exists a finite CW-complex Y and continuous maps $r\colon Y \to X$ and $i\colon X \to Y$ such that $r \circ i \simeq id_X$. Here is a nice result of Mather and Ferry [19]:

Theorem 4.1. *Let X be a finitely dominated CW-complex. Then the product space $X \times S^1$ has a canonical finite CW-structure (independent of the finite domination).*

The chain complex $C_*(X; \Lambda)$ of a finitely dominated space is chain homotopy equivalent to a finite length complex of finitely generated projective Λ-modules. In this situation, C. T. C. Wall defined the finiteness obstruction

$$\theta_W(X) = \sum (-1)^i [C_i(\widetilde{X}; \mathbf{Z})] \in \widetilde{K}_0(\mathbf{Z}\pi_1(X, x_0))$$

and proved:

Theorem 4.2 (Wall [64], [65]). *If X is a finitely dominated CW-complex, then $\theta_W(X)$ is a homotopy invariant. Moreover $\theta_W(X) = 0$ if and only if X is homotopy equivalent to a finite complex.*

Wall also proved that any element of $\widetilde{K}_0(\mathbf{Z}G)$ could arise as the finiteness obstruction of some finitely dominated complex. In some interesting cases there are restrictions on the allowable finiteness obstructions.

Theorem 4.3 (Mislin-Varadarajan [46]). *Suppose that X is a finitely dominated nilpotent space with finite fundamental group G. Then $\theta_W(X) \in D(\mathbf{Z}G)$.*

One example of a nilpotent space is the quotient of sphere $X = S^n/G$, where G is a nilpotent group acting freely.

Another assumption which restricts the possible finiteness obstructions is Poincaré duality. If (X, w) is a finitely dominated Poincaré complex, the w-twisted involution $\lambda \mapsto \bar{\lambda}$ on Λ induces an $\mathbf{Z}/2$-module structure on $\widetilde{K}_0(\mathbf{Z}G)$ by the formula $[P] \mapsto -[P^*]$, where P is a projective right Λ-module and $P^* = \mathrm{Hom}_\Lambda(P, \Lambda)$ is converted from a left to a right Λ-module by the involution.

Lemma 4.4. *Let (X, w) be a finitely dominated Poincaré complex with fundamental group G and with formal dimension n. Then $\overline{\theta_W(X)} = (-1)^{n+1}\theta_W(X)$.*

This shows that the finiteness obstruction of a Poincaré n-complex gives a well-defined element in $H^{n+1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}G))$.

5 Whitehead torsion

In this section we consider automorphisms of finitely generated free R-modules. See [42] for a more detailed exposition and references. If the free module has rank k, the group of all automorphisms is $GL_k(R)$. Identifying each $A \in GL_k(R)$ with the matrix

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in GL_{k+1}(R)$$

we obtain the inclusions

$$GL_1(R) \subset GL_2(R) \subset GL_3(R) \ldots$$

and the union is the infinite general linear group $GL(R)$. A matrix is called elementary if its entries coincide with those of the identity matrix except for one off-diagonal entry.

Lemma 5.1 (Whitehead). *The subgroup $E(R) \subset GL(R)$ generated by all elementary matrices is just the commutator subgroup of $GL(R)$.*

Proof. Let aE_{ij} denote the matrix which has at most one non-zero entry a in the (i,j) position. Then the relation

$$(I + aE_{ij})(I + E_{jk})(I - aE_{ij})(IE_{jk}) = (I + aE_{ik})$$

for i, j, k all distinct, shows that every elementary matrix of $GL_n(R)$ is a commutator for $n \geqq 3$. On the other hand, Whitehead's identities below show that any commutator $ABA^{-1}B^{-1}$ in $GL_n(R)$ can be written as a product of elementary matrices in $GL_{2n}(R)$.

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A^{-1} & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A & I \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=n+1}^{n} (I + x_{ij} E_{ij}) \ .$$

□

The abelian quotient group

$$K_1(R) = GL(R)/E(R)$$

was defined by J. H. C. Whitehead in order to compare homotopy equivalent complexes. Notice that a ring homomorphism $f\colon R \to S$ induces a map $f_*\colon K_1(R) \to K_1(S)$ with the usual functorial properties. In addition the functor K_1 respects products of rings

$$K_1(R \times S) \cong K_1(R) \oplus K_1(S)$$

and is Morita invariant

$$K_1(M_n(R)) \cong K_1(R) \ .$$

For calculations in the case where R is commutative we have a homomorphism

$$\det\colon K_1(R) \to R^\times$$

given by the determinant. If R is a field, or $R = \mathbf{Z}$ then $K_1(R) \cong R^\times$. For R a skew field, then $K_1(R) = R^\times/[R^\times, R^\times]$ via the "non-commutative determinant". If R is a $\mathbf{Z}$-order in a semi-simple $\mathbf{Q}$-algebra S, then $S \otimes_{\mathbf{Q}} \mathbb{C}$ is a product of full matrix rings over the complex numbers by Wedderburn's theorem. Composing the inclusion $R \subset S$ with projection onto one of these factors $M_n(\mathbb{C})$ gives a homomorphism

$$\mathrm{nr}\colon K_1(R) \to K_1(S) \to K_1(M_n(\mathbb{C})) \xrightarrow{\det} \mathbb{C}^\times$$

detecting the torsion-free part of $K_1(R)$. The image of this homomorphism lies in the ring of integers of the centre field of the associated factor of S.

Theorem 5.2 (Bass). *Let R be a $\mathbf{Z}$-order in a semisimple $\mathbf{Q}$-algebra S. Then $K_1(R)$ is a finitely generated abelian group of rank $r - q$, where q denotes the number of simple factors in S and r denotes the number of simple factors in $S \otimes_{\mathbf{Q}} \mathbb{R}$.*

For more precise calculations, the method of pull-back squares is available. If R is the pullback of S_1 and S_2 over T as in (3.7), then we have a six term exact sequence

$$K_1(R) \to K_1(S_1) \oplus K_1(S_2) \to K_1(T) \xrightarrow{\partial} K_0(R) \to K_0(S_1) \oplus K_0(S_2) \to K_0(T) .$$

For example, this sequence explains the role of the isomorphism h in the pullback construction $M(P_1, P_2, h)$ of projectives over R from projectives over S_1 and S_2.

Remark 5.3. There is an interesting connection between the following three questions:

(i) given $a \in K_1(R)$, what is the minimum n such that $a = [A]$ for some matrix $A \in GL_n(R)$?

(ii) given an ideal $I \subset R$, what is the minimum number of generators $I = \langle r_1, \dots r_k \rangle$ among all generating sets for I ?

(iii) given an isomorphism $M \oplus N \cong M' \oplus N$, does it follow that $M \cong M'$?

The last question is the cancellation problem for modules over R. The unifying idea linking these three questions is transitivity of elementary matrices on the set of unimodular elements (i.e. those generating a free direct summand) in a given R-module M.

Theorem 5.4 (Bass [4]). *Suppose that R is a ring with Krull dimension d, and M, M' and N are right R-modules, with N projective, such that $M \oplus N \cong M' \oplus N$. If N contains a free direct summand R^k of rank $k \geq d+2$, then $M \cong M'$.*

Proof. We may assume that $N = R^k$. Under the given stability condition, the elementary linear automorphisms of $M \oplus R^k$ act transitively on the set of unimodular elements. Therefore, any isomorphism $M' \oplus R^k \cong M \oplus R^k$ can be composed with an elementary automorphism to ensure that the standard basis of R^k is mapped by the identity. It follows that $M \cong M'$. □

A similar method gives stability bounds for the other two questions.

We turn now to the original geometric motivation for introducing the K_1 functor. If X is a finite CW-complex, its fundamental group $\pi := \pi_1(X, x_0)$

acts on the cells of $\widetilde{X}$ to give $C_*(X)$ the structure of a free Λ-module chain complex. To obtain a basis for this chain complex, order the cells of X (of a given dimension r), orient each one, and then choose a lifting of each cell to an r-cell of $\widetilde{X}$. This gives a free Λ-base for $C_r(\widetilde{X})$, unique up to order, sign, and multiplication on the right by elements of $\pi_1(X, x_0)$. Now if $f\colon X \to Y$ is a homotopy equivalence of finite CW-complexes, we have a short exact exact sequence of chain complexes

$$0 \to C_*(X) \to C_*(Y) \to C_*(f) \to 0$$

where the chain complex of the mapping cylinder of f has chain groups $C_i(f) := C_{i-1}(X) \oplus C_i(Y)$ and its differential is given by the formula

$$\partial_i := \begin{pmatrix} -\partial^X_{i-1} & 0 \\ f & \partial^Y_i \end{pmatrix} .$$

The chosen bases of $C_*(X)$ and $C_*(Y)$ induce a basis for $C_*(f)$, so the above is an exact sequence of free, based, Λ-module chain complexes. Moreover, since f is asssumed to be a homotopy equivalence, the homology $H(C_*(f))$ of the mapping cylinder is zero. In this situation, one can define a K_1-invariant called the Whitehead torsion of f.

To explain the process, we will consider any acyclic (i.e. zero homology) chain complex

$$C_n \to C_{n-1} \to \cdots \to C_2 \to C_1 \to C_0$$

of free Λ-modules, and assume that each group C_i has a given Λ-basis $\{c_i\}$. Let B_i denote the image of $\partial_{i+1}\colon C_{i+1} \to C_i$, and note that we have exact sequences

$$0 \to B_i \to C_i \to B_{i-1} \to 0$$

for each i, together with the equality $B_0 = C_0$. Inductively we see that all of these sequences split, so the modules B_i are all stably free. By taking the direct sum of the complex with elementary based complexes of the form

$$0 \to \Lambda^r \to \Lambda^r \oplus \Lambda^s \to \Lambda^s \to 0$$

we may assume that all the modules B_i are free to begin with. Choose a basis $\{b_i\}$ for each B_i, and notice that we now have two different bases, namely c_i and $\{b_i, b_{i-1}\}$ for each C_i. Let $[c_i/b_ib_{i-1}] \in K_1(\Lambda)$ denote the element given by the change of basis isomorphism on C_i.

Lemma 5.5. *The element*

$$\tau(C) = \sum (-1)^i [c_i/b_i b_{i-1}] \in K_1(\Lambda)$$

is independent of the choice of bases $\{b_i\}$ *for the* B_i.

Now in the geometric situation, we have made choices of the bases for $C_*(X)$ and $C_*(Y)$. To allow for the effect of these choices, we define the Whitehead group

$$\mathrm{Wh}(\mathbf{Z}G) := K_1(\mathbf{Z}G)/\{\pm g \mid g \in G\}$$

for any group G. Then we have

Theorem 5.6 (Whitehead). *Let* $f\colon X \to Y$ *be a homotopy equivalence of finite CW-complexes with fundamental group* π. *Then the element* $\tau(f) := \tau(C_*(f)) \in \mathrm{Wh}(\mathbf{Z}\pi)$ *is a homotopy invariant.*

Whitehead went on to show that $\tau(f) = 0$ if and only if X and Y were related by a sequence of cellular operations called "elementary expansions and collapses". In addition, if G is a finitely presented group, Whitehead proved that any element of $\mathrm{Wh}(\mathbf{Z}G)$ can be realized by some homotopy equivalence of finite CW-complexes with fundamental group G.

A homotopy equivalence $f\colon X \to Y$ with $\tau(f) = 0$ is called a *simple* homotopy equivalence. The study of simple homotopy types is now an important subject within homotopy theory. Here is another result of Whitehead which opened up an active research area. Let $X \vee rS^2$ denote the wedge of X with r copies of S^2.

Theorem 5.7 (Whitehead [70]). *Let* X *and* Y *be finite* 2*-complexes with the same Euler characteristic, and let* $\alpha\colon \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ *be an isomorphism between their fundamental groups. Then there is a simple homotopy equivalence* $f\colon X \vee rS^2 \simeq Y \vee rS^2$ *realizing the given isomorphism* α *on fundamental groups.*

There is also a geometric analogue of the cancellation problem for modules, namely to remove as many S^2 wedge summands as possible from a stable homotopy equivalence. For complexes with finite fundamental groups, we can remove all but one S^2.

Theorem 5.8 ([29]). *Let* X *and* Y *be finite* 2*-complexes with the same Euler characteristic and finite fundamental group. Let* $\alpha\colon \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ *be a given isomorphism and suppose that* $X \simeq X_0 \vee S^2$. *Then there is a simple homotopy equivalence* $f\colon X \to Y$ *inducing* α *on the fundamental groups.*

If (X, w) is a finite Poincaré complex of formal dimension n, then the mapping cylinder of the duality map $[X]\cap\colon C^* \to C_*$ is acylic. We call $\tau(X,w) := \tau([X]\cap) \in \mathrm{Wh}(\mathbf{Z}\pi_1(X,x_0))$ the *torsion* of (X,w) and say that (X,w) is a *simple* Poincaré complex if $\tau(X,w)=0$. The w-twisted involution on Λ induces an involution $A \mapsto (\overline{A^t})$ on $GL(\Lambda)$ and hence an involution on $\mathrm{Wh}(\Lambda)$. Any closed manifold is a simple Poincaré complex.

Theorem 5.9. *Let (X,w) be a finite Poincaré complex of formal dimension n. Then $\overline{\tau(X,w)} = (-1)^n\tau(X,w)$. If (X,w) is homotopy equivalent to a closed n-manifold with orientation class w, then $\tau(X,w)=0$.*

One of the most famous results about Whitehead torsion is that $\tau(f)$ is an obstruction for f to be homotopic to a homemorphism.

Theorem 5.10 (Chapman [13]). *Let $f\colon X \to Y$ be a homeomorphism of finite CW-complexes. Then $\tau(f)=0$.*

For the geometric applications of Whitehead torsion we must develop methods to compute $\mathrm{Wh}(\mathbf{Z}G)$ for finitely presented groups G. In this problem, there are two sharply different approaches depending on whether G is finite or infinite. If G is infinite and torsion-free, then the main conjecture is that $\mathrm{Wh}(\mathbf{Z}G)=0$ and the methods are geometric (see [20]). On the other hand, if G is finite the Whitehead group is generally non-trivial and there are extensive calculations available using algebraic methods (see [47]). Of course this summary leaves open what to do about infinite groups which have non-trivial elements of finite order, for example $\mathbf{Z}\times G$ where G is finite. More generally it would clearly be useful to have some idea how the Whitehead groups change under Laurent polynomial extension (i.e. direct product with $\mathbf{Z}$), amalgamated free products and HNN extension. We mention only the result on polynomial extensions, involving new K-theory functors $\widetilde{Nil}(\mathbf{Z}G)$ based on nilpotent matrices over the group ring.

Theorem 5.11 (Bass-Heller-Swan [5]). *For any group G,*

$$\mathrm{Wh}(\mathbf{Z}[\mathbf{Z}\times G]) \cong \mathrm{Wh}(\mathbf{Z}G) \oplus \widetilde{K}_0(\mathbf{Z}G) \oplus \widetilde{Nil}(\mathbf{Z}G) \oplus \widetilde{Nil}(\mathbf{Z}G)\ .$$

For G a finite group, we define the *arithmetic square*

$$\begin{tikzcd} \mathbf{Z}G \arrow[r] \arrow[d] & \mathbf{Q}G \arrow[d] \\ \widehat{\mathbf{Z}}G \arrow[r] & \widehat{\mathbf{Q}}G \end{tikzcd}$$

where $\widehat{\mathbf{Z}}$ is the direct product of all the rings $\widehat{\mathbf{Z}}_p$ and $\widehat{\mathbf{Q}}$ is the restricted product of the rings $\widehat{\mathbf{Q}}_p$. An element of the direct product $\prod \widehat{\mathbf{Q}}_p$ is in the restricted product if all but finitely many of its entries are in $\widehat{\mathbf{Z}}_p$. Although the arithmetic square is not a pullback in the sense of (3.7), the strong approximation theorem in algebraic number theory gives an exact sequence (due to H. Bass and A. Bak)

$$K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G) \oplus K_1(\widehat{\mathbf{Z}}G) \to K_1(\widehat{\mathbf{Q}}G) \xrightarrow{\partial} K_0(\mathbf{Z}G) \to$$
$$\to K_0(\mathbf{Q}G) \oplus K_0(\widehat{\mathbf{Z}}G) \to K_0(\widehat{\mathbf{Q}}G)$$

which is very effective for calculations. For example, Swan's theorem show that the map $K_1(\widehat{\mathbf{Q}}G) \xrightarrow{\partial} \widetilde{K}_0(\mathbf{Z}G)$ is surjective, and this suggests that the finiteness obstruction $\theta_W(X)$ for a finitely dominated space X should have a lifting to $K_1(\widehat{\mathbf{Q}}G)$. This is indeed the case: after choosing bases $\mathbf{h}$ for the homology of $C_*(X) \otimes_{\mathbf{Z}} \widehat{\mathbf{Q}}$, one can define the *idelic* Reidemeister torsion $\hat{\Delta}(X, \mathbf{h}) \in K_1(\widehat{\mathbf{Q}}G)$ so that $\partial\hat{\Delta}(X, \mathbf{h}) = \theta_W(X)$. This invariant plays an important role in the solution of the spherical space form problem.

Let

$$SK_1(\mathbf{Z}G) := \ker\{K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G)\} \; .$$

Then

Theorem 5.12 (Wall [66]). *For G a finite group, the torsion subgroup of $K_1(\mathbf{Z}G)$ is just $\{\pm G^{ab}\} \oplus SK_1(\mathbf{Z}G)$. The standard oriented involution induces the identity on the torsion-free quotient* $\mathrm{Wh}'(\mathbf{Z}G) := \mathrm{Wh}(\mathbf{Z}G)/SK_1(\mathbf{Z}G)$.

For G finite cyclic, $SK_1(\mathbf{Z}G) = 0$ and the Whitehead group is torsion-free (the rank was given above). In general however, the groups $SK_1(\mathbf{Z}G)$ are highly non-trivial. A homotopy equivalence $f\colon X \to Y$ with $\tau(f) \in SK_1(\mathbf{Z}G) \oplus \{\pm G^{ab}\}$ is called a *weakly simple homotopy equivalence* . Similarly, a Poincaré complex X is weakly simple if its duality map has zero torsion in $\mathrm{Wh}'(\mathbf{Z}\pi_1(X, x_0))$.

Corollary 5.13 (Wall [66]). *An orientable finite Poincaré complex of odd formal dimension, with finite fundamental group, is weakly simple. A homotopy equivalence between (weakly) simple oriented Poincaré complexes of even formal dimension, with finite fundamental group, is weakly simple.*

6 Hermitian forms

A ring with antistructure (R, α, u) consists of an "involution" $\alpha(r) = \bar{r}$ on the ring R, and a unit $u \in R^{\times}$ with the properties

(i) $\alpha(r+s) = \alpha(r) + \alpha(s)$ for all r, $s \in R$.

(ii) $\alpha(rs) = \alpha(s)\alpha(r)$ for all r, $s \in R$.

(iii) $\alpha^2(r) = uru^{-1}$ for all $r \in R$.

(iv) $\alpha(u) = u^{-1}$.

In the special case when $u \in R^{\times}$ lies in the centre of R, we have $\alpha^2 = id$ and say that R is a ring with involution.

Let (R, α, u) be a ring with antistructure. A u-symmetric hermitian form on a right R-module M is a map $h\colon M \times M \to R$ such that

(i) $h(x+y, z) = h(x, z) + h(y, z)$ for all $x, y, z \in M$.

(ii) $h(x, yr) = h(x, y)r$ for all $x, y \in M$ and all $r \in R$.

(iii) $h(y, x) = \overline{h(x, y)}u$ for all $x, y \in M$.

The adjoint map $ad(h)\colon M \to M^* = \mathrm{Hom}_R(M, R)$ is defined by $ad(h)(x)(y) = h(x, y)$ for all $x, y \in M$. The form (M, h) is non-degenerate if the adjoint map $ad(h)$ is injective, and non-singular if $ad(h)$ is an isomorphism of right R-modules. As usual, we convert M^* from a left R-module to a right R-module by using the involution. The form can be described either by h or by $ad(h)$, whichever is most convenient. We usually take $u = \pm 1$ and refer to symmetric or skew-symmetric forms. Two u-symmetric hermitian forms (M, h) and (N, k) are isometric if there exists an R-module isomorphism $\varphi\colon M \to N$ such that $h(x, y) = k(\varphi(x), \varphi(y))$ for all $x, y \in M$. There is an obvious notion of orthogonal direct sum $(M, h) \perp (N, k) = (M \oplus N, h \perp k)$, where $h \perp k = \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$. We can then define $K_0(\mathcal{H}(R, \alpha))$ to be the Grothendieck group of the category of hermitian forms on finitely generated projective R-modules. This is a hermitian version of $K_0(R)$ and we have a forgetful map

$$K_0(\mathcal{H}(R, \alpha)) \to K_0(R),$$

taking a form (M, h) to its underlying module M.

Example 6.1. Let $M = N \oplus N^*$ and $h((x,f),(y,g)) = f(y) + \overline{g(x)}u$, for all $x, y \in N$ and $f, g \in N^*$. This defines the hyperbolic form $\mathbf{H}(N)$ on $N \oplus N^*$. Applying this to projective modules N, we get a homomorphism

$$\mathbf{H}\colon K_0(R) \to K_0(\mathcal{H}(R,\alpha))$$

called the hyperbolic map.

If $M \cong R^n$ is a free R-module of *even* rank $n = 2k$, and $\{e_1, \dots, e_{2k}\}$ is a basis, then a non-singular hermitian form (M,h) has a K_1-valued "discriminant" invariant $d(M,h) = [ad(h)] \in K_1(R)$, generalizing and refining the determinant. The dual module M^* is given the basis $\{e_1^*, e_2^* u^{-1}, \dots, e_{2k-1}^*, e_{2k}^* u^{-1}\}$, where e_i^* is the dual basis element to e_i. The element of K_1 arising from the isomorphism $ad(h)\colon M \to M^*$ of based free R-modules gives an invariant which is additive under orthogonal direct sums of based forms.

It is not hard to check that $\overline{d(M,h)} = d(M,h)$, and that the descriminant of a standard based hyperbolic form on a free module $M = R^k \oplus R^k$ is trivial. Changing the basis of M changes the matrix for h by the usual formula $A \mapsto \overline{P}^t AP$, so the K_1-invariant changes by an element of the form $a + \bar{a}$. We get a well-defined invariant $[d(M,h)] \in H^0(\mathbf{Z}/(2); K_1(R))$, additive under orthogonal direct sums.

If $R = K$ is a field with fixed field F under the involution, and $u = 1$, this invariant is just the usual determinant for a symmetric hermitian form taking values in $F^\times$ with indeterminancy from choice of basis in the image of the norm map $N_{K/F}(K^\times)$. □

Example 6.2. Let $R = \mathbf{Z}$, $u = \pm 1$, and $M = \mathbf{Z}^n$ be a free abelian group of rank n. Then a non-singular u-symmetric hermitian form on M is just a symmetric or skew-symmetric unimodular form on $\mathbf{Z}^n$. In the symmetric case, we say that a form h is *even* if $h(x,x) \equiv 0 \bmod 2$ for all $x \in M$, and otherwise h is *odd* . A form is called *definite* if $h(x,x) \neq 0$ whenever $x \neq 0$, and otherwise indefinite.

Theorem 6.3. *Indefinite unimodular symmetric forms are classified by the rank, type (odd or even), and the signature.*

The classification of definite symmetric unimodular forms over $\mathbf{Z}$ is a fascinating subject (see [45]). The number of distinct isometry classes grows rapidly with the rank n. In contrast, the classification of skew-symmetric unimodular forms over $\mathbf{Z}$ is trivial: there is just one such form (the hyperbolic form) for each even rank. Similarly, we often encounter (skew) symmetric forms on vector spaces over fields with trivial involution (e.g. $\mathbf{Q}$, $\mathbb{R}$, or finite fields $\mathbf{F}_p$. If the characteristic of the field is not 2, every symmetric form can be diagonalized and every non-singular skew-symmetric form is hyperbolic. For symmetric forms (M, h) over $\mathbf{Q}$ or $\mathbb{R}$, the *signature* $\sigma(M, h) \in \mathbf{Z}$ is defined to be the number of positive entries minus the number of negative entries in any diagonalization of (M, h). This integer is well-defined, and together the rank, determinant, and signature classify the forms over $\mathbf{Q}$ or $\mathbb{R}$ up to isometry. Over $R = \mathbf{F}_p$, for p odd, the rank and determinant classify the forms.

An interesting contrast is the case $R = \mathbf{F}_2$, or any finite field with characteristic 2. In this case there is no difference between symmetric and skew-symmetric forms, but there are non-isometric forms, for example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, with the same rank and determinant. Any non-singular form over $\mathbf{F}_2$ is isometric to orthogonal direct sums of these with the rank 1 form $\langle 1 \rangle$. □

Example 6.4. Let $R = \mathbb{C}$ with the involution given by complex conjugation, and $M = \mathbb{C}^n$. Then the form $h(z, w) = \sum \bar{z}_i w_i$ is a non-singular hermitian form. □

7 Normal maps and surgery obstructions

We now describe a geometrical setting for the algebra of hermitian forms. This is the Browder-Novikov-Sullivan-Wall theory of surgery, which has had such a decisive impact on geometric topology.

Suppose that W^{n+1} is a smooth compact manifold with two boundary components M_0 and M_1. Let $f\colon W \to [0, 1]$ denote a Morse function, namely a smooth function with $f(M_0) = 0$, $f(M_1) = 1$, non-degenerate critical points and distinct critical values $0 < c_1 < c_2 < \ldots < c_r < 1$. By the Morse lemma, in a neighborhood U of critical point $p_0 \in W$ with $f(p_0) = c$, there exists a co-ordinate system $x_i = x_i(p)$, $1 \leq i \leq n + 1$, so that

$$f(p) = f(p_0) - x_1^2 \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2$$

for all $p \in U$. The integer k, $0 \leq k \leq n + 1$ is the index of the critical point. If $\epsilon > 0$ is so small that $W = f^{-1}(c - \epsilon, c + \epsilon)$ has no critical points other than p_0, then $M_{c+e} = f^{-1}(c + \epsilon)$ is obtained from $M_{c-\epsilon}$ by an elementary surgery of type $(k, n-k)$:

$$M_{c+\epsilon} = (M_{c-\epsilon} - \varphi(S^{k-1} \times D^{n-k+1})) \cup_\varphi (D^k \times S^{n-k})$$

where $\varphi\colon S^{k-1} \times D^{n-k+1} \to M_{c-\epsilon}$ is an embedding. The manifold $W_0 = (M_{c-\epsilon} \times I) \cup D^k \times D^{n-k+1}$ is usually called the *trace* of the surgery. This is the basic construction in surgery.

The discussion above shows that the equivalence relation cobordism of manifolds is generated by elementary surgeries. To reverse this point of view, and produce a scheme for the classification of manifolds requires a way to keep track of the effect of elementary surgeries. First we define, for any space X, the n-dimensional structure set $\mathcal{S}_n(X)$. This is the set of equivalence classes of pairs (M^n, f), where M^n is a closed n-manifold and $f\colon M \to X$ is a simple homotopy equivalence. Two such pairs (M_0, f_0), (M_1, f_1) are equivalent if there is a diffeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g \simeq f_0$. One can now ask for a "computation" of $\mathcal{S}_n(X)$ given X. Of course it would be reasonable to start with X a closed n-manifold, or at least a finite Poincaré complex of formal dimension n, and then $\mathcal{S}_n(X)$ would measure the manifolds in the same homotopy type.

A Poincaré space (X, w) resembles a manifold in another way. Let $X \to \mathbb{R}^{n+k}$ be an embedding (for k large) and N a regular neighborhood. Then it turns out that the composite $i\colon \partial N \to N \to X$ is (up to homotopy) a spherical fibration, with each fibre homotopy equivalent to S^{k-1}. If k is sufficiently large, this fibration ν_X is unique up to fibre homotopy equivalence and is called the Spivak normal fibre space of X. By construction, the collapse map

$$c\colon S^{n+k} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - N) := T(\nu_X)$$

together with the Thom isomorphism Φ induces a degree 1 map

$$H_{n+k}(S^{n+k}; \mathbf{Z}) \xrightarrow{c_*} H_{n+k}(T(\nu_X); \mathbf{Z}) \xleftarrow[\approx]{\Phi} H_n^w(X; \mathbf{Z})$$

taking a generator $[S^{n+k}]$ onto $[X]$. Conversely, the Spivak normal fibre space is characterized, up to stable fibre homotopy equivalence, as a spherical fibration ν over X such that $\pi_{n+k}(T(\nu))$ contains a map of degree 1.

We now define a degree 1 normal map with target (X, w). This consists of a degree 1 map $f\colon M^n \to X$ where M is a closed n-manifold and $f^*w = w_1(M)$, together with a bundle map $b\colon \nu_M \to \xi$ covering f, for some vector bundle ξ over X. Two normal maps (M_i, f_i, b_i), $i = 0, 1$ are normally cobordant if there is a cobordism W^{n+1} from M_0 to M_1 and maps

$F\colon W \to X \subset I$, and $B\colon \nu_N \to \xi \oplus 1$, extending (f_i, b_i). The set of normal maps with target (X, w) is denoted $T(X, w)$. Note that from the discussion above, each bundle ξ occurring in a degree 1 normal map must be fibre homotopy equivalent to ν_X (such a ξ is called a vector bundle reduction of ν_X). The elements of $T(X, w)$ are in bijection with the union of all elements of degree 1 in $\pi_{n+k}(T(\xi))$ as ξ varies over all vector bundle reductions of ν_X.

A primary obstruction for the existence of any manifold homotopy equivalent to X is therefore the existence of some reduction of ν_X. For arbitrary Poincaré complexes X, these need not exist. Assuming that $T(X, w)$ is nonempty, we seek a procedure for determining when a normal map is normally cobordant to a homotopy equivalence.

We first notice that the set $T(X, w)$ provides a good way to keep track of the effect of surgeries. If $f\colon M \to X$ is a degree 1 map, the main observation is that the diagram

$$\begin{tikzcd} H^{n-k}(M) \arrow[d, "\cap [M]"] & H^{n-k}(X) \arrow[l, "f^*"'] \arrow[d, "\cap [X]"] \\ H^w_k(M) \arrow[r, "f_*"] & H^w_k(X) \end{tikzcd}$$

commutes. Therefore, in each dimension, f_* is split surjective and f^* is split injective. Let $K_i(f)$, (respectively $K^i(f)$) denote the i-dimensional kernel (respectively cokernel) of f_* (respectively f^*). Then $\cap[M]$ induces an isomorphism of $K^{n-i}(f)$ onto $K_i(f)$ for all $i \geq 0$. Now f is a homotopy equivalence if and only if it induces an isomorphism on π_1 and $K_i(f) = 0$ for all $i \geq 0$.

Furthermore, if $b\colon \nu_M \to \xi$ is a bundle map covering f and $\phi\colon S^i \to M$ is an embedding of a sphere in M with $f \circ \phi \simeq *$, then $\phi^*\nu_M = \phi^* f^*(\xi)$ is a trivial bundle. Since the tangent bundle of a sphere is trivial after stabilizing once, we see that $\phi(S^i)$ has trivial normal bundle in M if $i < [n/2]$. Therefore, starting with a degree 1 normal map, we can simplify it by elementary surgeries, to obtain:

Proposition 7.1. *A degree 1 normal map $(f, b)\colon M^n \to X$ is normally cobordant to an $[n/2]$-connected normal map.*

Proof. By elementary surgeries on 0 and 1 spheres we can assume that f induces an isomorphism on π_0 and π_1. By induction we assume that f is i-connected for $i + 1 \leq [n/2]$. Then $\pi_{i+1}(f) \cong K_i(f)$ and any element is represented by an embedded i-sphere with trivial normal bundle. We perform an elementary surgery on this class. Since we have used the normal bundle trivialization arising from an extension of $f \circ \phi$ over D^{i+1}, the bundle map b extends over the trace of the surgery. □

When we do surgery on an i-sphere, the homology class in $K_i(f)$ carried by this sphere is eliminated, but a dual class in dimension $(n - i - 1)$ is introduced. If $i < [n/2]$ the new class is in dimension $\geq [n/2]$, so progress can be made easily. It remains to discuss the middle dimensions.

Note that if $n = 2k$ and we do surgery on a trivial $S^{k-1} \times D^{i+1}$ (i.e. one contained in a 2k-disk in M), the result is to replace M by $M \# S^k \times S^k$. Similarly, if $n = 2k+1$ and we surger $S^k \times D^{k+1} \subset D^{2k+1}$, we get $M \# S^k \times S^{k+1}$.

If $n = 2k$, it is no longer true that every class in $K_k(f)$ is represented by an embedded sphere with trivial normal bundle. Since $L = K_k(f)$ is the single non-trivial homology group of the chain complex $C_*(f)$ of the mapping case, it follows that L is a stably-free finitely generated Λ-module. By surgering on some trivial $(k-1)$-spheres, we may assume L is a free Λ-module. So is $K^k(f) \cong \mathrm{Hom}_\Lambda(K_k(f), \Lambda)$, where the isomorphism is given by Poincaré duality. This gives a $(-1)^k$-hermitian pairing

$$\lambda \colon L \times L \to \Lambda$$

induced by intersection numbers, which will now be described more geometrically following [69, Chap. 5]. From the discussion, a new algebraic structure emerges - the notion of a *quadratic refinement* for the intersection pairing.

According to results of Hirsch, Poenaru and Haefliger [69, p.10], regular homotopy classes of immersions $\phi \colon S^k \to M^{2k}$ correspond bijectively (by the tangent map) to stable homotopy classes of stable bundle monomorphisms $\tau_{S^k} \to \phi^* \tau_M$. We represent elements of $K_k(f)$ by immersions equipped with a path in M joining a fixed base point $x_0 \in M$ to $\phi(p_0)$, where $p_0 \in S^k$ is a base point. These immersions may be chosen so that the Euler class of the normal bundle is trivial. Note that $\pi_1(M, x_0)$ acts on such an immersed sphere by composing the path with a loop at x_0.

Suppose that S_1 and S_2 are two immersed k-spheres in M, meeting transversely in a finite set of points $\{p\}$. To each point P we assign a fundamental group element g_P and an orientation $\epsilon_P = \pm 1$. The Λ-valued intersection form is defined by

$$\lambda(S_1, S_2) = \sum_p \epsilon_P g_P.$$

This is related to the ordinary intersection form $\lambda_0 \colon L \times L \to \mathbf{Z}$ by the formula

$$\lambda(x, y) = \sum_{g \in \pi_1} \lambda_0(x, yg^{-1})g$$

The same procedure can be used to define the self-intersection of an immersed sphere S_1 (in general position) with *trivial* normal bundle. At each intersection point P, after an order of the branches is chosen, the quantities ϵ_P and g_P are defined as before. If the order is interchanged, $\epsilon_P g_P$ becomes $(-1)^k w(g_P)\epsilon_P g_P^{-1} = (-1)^k \epsilon_P \overline{g_P}$ (using the notation introduced before for the anti-involution). Therefore, the self-intersection defines a map

$$\mu \colon L \to \Lambda / I_k \ .$$

where $I_k := \{\nu - (-1)^k \bar{\nu} \mid \nu \in \Lambda\}$.

Theorem 7.2 (Wall [69, Chap. 5]). *The quadratic form (L, λ, μ) has the following properties:*

(i) *For $x \in L$ fixed, $y \to \lambda(x, y)$ is a Λ-homomorphism $L \to \Lambda$.*

(ii) $\lambda(y, x) = (-1)^k \overline{\lambda(x, y)}$, *for $x, y \in L$.*

(iii) $\lambda(x, x) = \mu(x) + (-1)^k \overline{\mu(x)}$, *for $x \in L$.*

(iv) $\mu(x + y) - \mu(x)) - \mu(y) = \lambda(x, y)$, *for $x, y \in L$.*

(v) $\mu(xa) = \bar{a}\mu(x)a$, *for $x \in L$, $a \in \Lambda$.*

(vi) *If $k \geq 3$, the class x is represented by an embedding if and only if $\mu(x) = 0$.*

The assumption that $k \geq 3$ in the last property (a result of H. Whitney) is critical for the whole theory. M. Freedman's celebrated Field's Medal work on the Disk Theorem for topological 4-manifolds [21], [22] deals with the case $k = 2$ for special fundamental groups (including finite fundamental groups). Notice that the first two properties just say that (L, λ) is a $(-1)^k$-hermitian form. The new algebraic ingredient is the quadratic refinement μ. By property (iv) the quadratic form determines the associated hermitian form. Note that μ takes values in an abelian group Λ/I_k, and that the action $a \mapsto \bar{a}\mu a$ is well-defined for $a \in \Lambda$, $\mu \in \Lambda/I_k$ (independent of the choice of lift for μ. Also the map $\Lambda/I_k \to \Lambda$ as in (iii) given by $\mu \mapsto \mu + (-1)^k\bar{\mu}$ is computed by taking any lift of μ to Λ.

The definition of quadratic form (L, λ, μ) makes sense for modules over arbitrary rings R with involution. We say that two quadratic forms (L, λ, μ) and (L', λ', μ') are isomorphic if there is an isometry $f\colon L \to L'$ of the hermitian forms λ, λ' such that $\mu' \circ f = \mu$. A hyperbolic form in this setting is one that is isomorphic to $\mathbf{H}(\Lambda^n) = \Lambda^n \oplus \Lambda^n$ with μ vanishing on the direct summands $\Lambda^n \oplus 0$ and $0 \oplus \Lambda^n$.

Note that if R is a field of characteristic not 2, with trivial involution, the relation $\lambda(x,x) = \mu(x) + (-1)^k\overline{\mu(x)}$ for k even shows that $\frac{1}{2}\lambda(x,x) = \mu(x)$. Therefore, in the symmetric case the quadratic form is determined by the associated hermitian form. On the other hand, if R has characteristic 2, there is a difference between quadratic and hermitian forms.

Example 7.3. Let $R = \mathbf{F}_2$ and (L, λ, μ) a non-singular quadratic form on a free $\mathbf{F}$-modules L. The associated hermitian form λ is always hyperbolic: let $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ be a hyperbolic basis for L with $\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$ and $\lambda(e_i, f_j) = \delta_{ij}$. The Arf invariant $c(L, \lambda, \mu) \in \mathbf{Z}/2$ is defined by the formula

$$c(L, \lambda, \mu) = \sum \mu(e_i)\mu(f_i)$$

Theorem 7.4 (Arf [45]). *The Arf invariant is an isometry invariant of the quadratic forms over $\mathbf{F}_2$, and additive under orthogonal direct sums. A form (L, λ, μ) is hyperbolic if and only if $c(L, \lambda, \mu) = 0$.*

□

To relate the algebra of quadratic forms to the problem of eliminating $K_k(f)$, we make the following two geometric observations.

(i) If $(f, b)\colon M \to X$ is normally cobordant to a homotopy equivalence, then (L, λ, μ) contains a free-direct summand L_0 such that $L_0 = L_0^\perp$ and $\mu(L_0) = 0$. This is called a subkernel or *lagrangian*. An easy algebraic argument implies that a quadratic form contains a subkernel if and only if it is isomorphic to an orthogonal direct sum of hyperbolic planes (these are free Λ-modules of rank 2 with base $\{x, y\}$, $\mu(x) = \mu(y) = 0$ and $\lambda(x, y) = 1$).

(ii) A hyperbolic plane can be removed from (L, λ, μ) by surgery on one of the basis elements if $k \geq 3$. The picture to keep in mind here is the "plumbing" of two copies of $S^k \times D^k$, which just $S^k \times S^k - D^{2k}$, and has boundary S^{2k-1}.

These points motivate the definition of the even-dimensional surgery obstruction group $L_{2k}(\mathbf{Z}[\pi_1 X], w)$: the stable isomorphism classes of $(-1)^k$-quadratic forms (L, λ, μ) on free Λ-modules L, modulo hyperbolic forms. Here "stable isomorphism" means that the forms become isomorphic after adding hyperbolics. Similarly we can define $L_{2k}(R, \alpha)$ for any ring R with involution α.

The odd-dimensional case leads to a more complicated situation. Suppose that $(f, b)\colon M^{2k+1} \to X$ is a degree 1 normal map with $K_i(f) = 0$ for $i < k$. Choose a set of generators $\phi_j\colon S^k \times D^{k+1} \to M$ (each joined by a path to the base-point) for $K_k(f)$ as a Λ-module. These may be assumed to have disjoint images in M, so let U be the union of the images and $M_0 = M - U$. We assume further that $f(U) = x_0 \in X$, and $X = X_0 \cup D^{2k+1}$ where $(X_0, \partial X_0)$ is a finite Poincaré pair. We can then obtain a map of triads

$$f\colon (M, M_0, U) \to (X, X_0, D^{2k+1}),$$

leading to the diagram (see [69, Chap. 6]):

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(U,\partial U) \arrow[dr] \arrow[rr, bend left] & & K_k(M_0) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_{k+1}(M) \arrow[ur] \arrow[dr] & & K_k(\partial U) \arrow[ur] \arrow[dr] & & K_k(M) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & K_{k+1}(M_0,\partial U) \arrow[ur] \arrow[rr, bend right, "\tau"'] & & K_k(U) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

Now $\partial U \approx \#\,(S^k \times S^k)_i$, so the term $K_k(\partial U)$ supports a hyperbolic form with two standard subkernels $K_{k+1}(U, \partial U)$ and $K_k(U)$. Furthermore $K_{k+1}(M_0, \partial U)$ is also a subkernel in $K_k(\partial U)$. The main observation is that $K_k(M)$ and $K_{k+1}(M)$ are trivial if and only if $K_{k+1}(M_0, \partial U)$ is a complementary subkernel to $K_{k+1}(U, \partial U)$ for some choice of the $\{\phi_i\}$. In the diagram above, this is equivalent to τ being an isomorphism.

The discussion so far suggests that the relevant data is $(\mathbf{H}(\Lambda^r), L_0, L_1)$ where $\mathbf{H}(\Lambda^r)$ is the hyperbolic form $(\Lambda^r \oplus \Lambda^t, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and L_0, L_1 are two subkernels. This is correct and the precise definition of this "formation" structure are due to Ranicki, following earlier work of Novikov. For our purposes, the original definition of Wall for $L_{2k+1}(\mathbf{Z}[\pi_1 X], w)$ is more convenient. It rests on an algebraic fact:

Lemma 7.5. *If L_0, L_1 are subkernels in a quadratic form (L, λ, μ), then any Λ-module isomorphism $\theta\colon L_0 \to L_1$ extends to an isometry of (L, λ, μ).*

Let $SU_r(\Lambda)$ denote the group of isometries of the standard hyperbolic form $\mathbf{H}(\Lambda^r)$, and $TU_r(\Lambda)$ the subgroup leaving the subkernel $\Lambda^r \oplus 0$ invariant. A detailed analysis of the construction above, shows that there is a well-defined invariant after allowing for

(i) stabilization: $SU_r(\Lambda) \subset SU_{r+1}(\Lambda) \subset \cdots \subset SU(\Lambda)$.

(ii) the action of $TU_r(\Lambda) \subset TU_{r+1}(\Lambda) \subset \cdots \subset TU(\Lambda)$.

(iii) interchanging $\Lambda^r \oplus 0$ and $0 \oplus \Lambda^r$.

Let $\sigma = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} \in SU_1(\Lambda)$ and let $RU(\Lambda)$ be the subgroup of $SU(\Lambda)$ generated by σ and $TU(\Lambda)$. Then surgery to a homotopy equivalence is possible, if and only if the automorphism relating $K_{k+1}(U, \partial U)$ to $K_{k+1}(M_0, \partial U)$ is equivalent to $\sigma \oplus \sigma \oplus \cdots \oplus \sigma$ (the automorphism in (iii) above) under the 2-sided action of $RU(\Lambda)$. Wall finally proves (with the aid of a remarkable identity) that $RU(\Lambda) \supset [SU(\Lambda), SU(\Lambda)]$ and so

$$L_{2k+1}(\mathbf{Z}[\pi_1 X], w) := SU(\Lambda)/RU(\Lambda)$$

is an abelian group.

The main outcome of this analysis is the surgery exact sequence:

Theorem 7.6 (Browder-Novikov-Sullivan-Wall). *If (X, w) is a finite Poincaré complex of formal dimension $n \geq 5$, there is an exact sequence (of groups and pointed sets)*

$$L_{n+1}(\mathbf{Z}[\pi_1 X], w) \to \mathcal{S}_n(X) \xrightarrow{\eta} T(X, w) \xrightarrow{\lambda} L_n(\mathbf{Z}[\pi_1 X], w).$$

In the further development of geometric surgery, one shows that $L_n \cong L_{n+4}$ (geometrically this is just crossing a surgery problem with $\mathbb{C}P^2$ in domain and range) and studies the maps in the surgery exact sequence. The set $T(X,w)$ is related to classifying spaces for topological bundles and spherical fibrations. The "assembly map" description of the surgery obstruction map λ will be described in the next section.

One variation of the whole setup which is important for the applications is to take account of Whitehead torsion. This idea is due to S. Cappell. The definition for $\mathcal{S}_n(X)$ is given in terms of simple homotopy equivalences $f\colon M \to X$. Since an arbitrary homotopy equivalence has a torsion $\tau(f) \in Wh(\mathbf{Z}[\pi_1 X])$, we could define $\mathcal{S}_n^U(X)$ for subgroups $U \subseteq Wh(\mathbf{Z}[\pi_1 X])$ by requiring that all torsions lie in U. Notice that Poincaré duality imposes the condition $\tau(f) = (-1)^n \overline{\tau(f)}$ so it is natural to suppose that U is an involution-invariant subgroup. If two homotopy equivalences f_0, f_1 are normally cobordant, then $\tau(f_0) - \tau(f_1) = v + (-1)^n \bar{v}$, for some $v \in Wh(\mathbf{Z}[\pi_1(x)]$.

The definition of the surgery obstruction group must be modified by choosing bases for our free modules, and then requiring that any isomorphisms which occur have torsions in U. The special choices $U = \{0\}$ and $U = Wh(\mathbf{Z}\pi)$ are denoted L^s and L^h respectively. We define the structure set $\mathcal{S}_n^h(X)$ as the equivalence classes of pairs (M^n, f), with $f\colon M \to X$ a homotopy equivalence, modulo the equivalence relation given by h-cobordisms. This structure set fits into a surgery exact sequence with the same normal invariant term, and the L^h-groups as the surgery obstruction groups.

If $U \subseteq V$ are involution-invariant subgroups of $Wh(\mathbf{Z}[\pi])$ then there is a long exact sequence

$$\cdots \to H^{n+1}(\mathbf{Z}/2, V/U) \to L_n^U(\mathbf{Z}\pi, w) \to L_n^V(\mathbf{Z}\pi, w) \to H^n(\mathbf{Z}/2; V/U) \to$$

8 Computation of L-groups

In order to compute the surgery obstruction groups $L_*(\mathbf{Z}G, w)$ for finite groups G, we want to take advantage of the fact that the L-groups are algebraically defined, so we have groups $L_*(R, \alpha)$ for any ring with involution. To use this generality effectively, we would first like to establish the methods already described for K-theory, namely reducing to more tractable rings via exact sequences arising from pullback squares or the arithmetic square. However, operations such as change of rings, which are natural algebraically have no geometric analogue, so it isn't clear that any purely algebraic calculation can give usable geometric information. The algebraic theory of surgery developed by Ranicki [52], [53], based on the work of Mischenko, Novikov and Wall, addresses both of these objectives.

The algebraic theory of surgery starts from the notion of an (ϵ-symmetric) algebraic Poincaré complex. This is a chain complex (C, d) of finitely-generated projective modules over a ring (R, α, ϵ) with anti-structure

$$C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_1 \xrightarrow{d} C_0,$$

together with a collection of R-module maps

$$\varphi_s\colon C^{n-r+s} \to C_r \ (s \geq 0)$$

such that

$$d\varphi_s + (-1)^r \varphi_s d^* + (-1)^{n+s-1}(\varphi_{s-1} + (-1)^s T_\epsilon \varphi_{s-1}) = 0$$

and such that the chain map

$$\varphi_0\colon C^{n-*} \to C_*$$

is a chain equivalence. Here C^{n-*} is the dual complex (shifted by n) and T_ϵ is the duality involution

$$\begin{array}{rccl} T_\epsilon\colon & \mathrm{Hom}_R(C^p, C_q) & \to & \mathrm{Hom}_R(C^q, C_p) \\ & \varphi & \mapsto & (-1)^{pq}\varphi^* \end{array}$$

The map φ_0 induces the Poincaré duality isomorphisms $H^{n-r}(C) \to H_r(C)$, φ_1 is a chain homotopy between φ_0 and $T_\epsilon\varphi_0$, and so on.

If $(f, b)\colon M^n \to X$ is a degree 1 normal map, then the kernel complex $C(f)_{*+1}$ has the structure of an algebraic Poincaré complex. Furthermore, the bundle map b gives in a natural way, a quadratic refinement of this structure (a "quadratic Poincaré complex") which determines the surgery obstruction. For these examples, $\epsilon = (-1)^k$ where $k = [n/2]$.

An algebraic Poincaré complex (C, d) as above is *quadratic* if there exists a collection of R-module maps

$$\psi_s\colon C^{n-r-s} \to C_r \ (s \geq 0)$$

such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1}(\psi_{s-1} + (-1)^{s+1} T_\epsilon \psi_{s-1}) = 0$$

and such that the chain map

$$\varphi_0 := (1 + T_\epsilon)\psi_0 \colon C^{n-*} \to C_*$$

is a chain equivalence. The symmetrization of a quadratic Poincaré complex gives a symmetric Poincaré complex.

These definitions generalize those of forms and formations. For example, an ϵ-symmetric algebraic Poincaré complex of dimension zero is just a non-singular ϵ-symmetric hermitian form on the projective module C_0, and a quadratic Poincaré complex of dimension zero is just a quadratic form on C_0 whose symmetrization is non-singular.

One of the main results of the algebraic theory is the description of the surgery obstruction groups in terms of a variant of these cobordism groups $L_n(R, \alpha, \epsilon)$ of algebraic n-dimensional quadratic Poincaré complexes. We will take $\epsilon = +1$ and use the identification

$$L_n(R, \alpha, \epsilon) \cong L_{n+2}(R, \alpha, -\epsilon)$$

to recover (-1)-quadratic forms.

There is no difficulty in replacing projective R-module chain complexes by free chain complexes, but we apparently lose the possibility of Whitehead torsion variant L-groups since the Whitehead group is only defined for group rings. However if $\widetilde{U} \subseteq \widetilde{K}_1(R) := K_1(R)/\{\pm 1\}$ is an involution-invariant subgroup, the groups $L_*^{\widetilde{U}}(R, \alpha)$ are defined as the cobordism groups of complexes with $\tau(\varphi_0) \in \widetilde{U}$. This is consistent with our previous definitions for group rings. For example, if $R = \mathbf{Z}G$ and $\widetilde{U} = \{\pm G^{ab}\}$, then

$$L_n^s(\mathbf{Z}G) = L_n^{\widetilde{U}}(R)$$

since $Wh(\mathbf{Z}G) = K_1(\mathbf{Z}G)/\{\pm G^{ab}\}$.

If we add to our chain complexes the requirement that the Euler characteristic $\chi(C) = 0$, then we can define variant L-groups $L_n^U(R, \alpha)$ based on involution-invariant subgroups $U \subseteq K_1(R)$. The extreme cases $U = \{0\}$ and $U = K_1(R)$ are denoted L^S and L^K respectively. These L-groups are well-behaved under products and Morita equivalence.

They are related to the previous groups by an exact sequence,

$$0 \to L_{2k}^U(R, \alpha) \to L_{2k}^{\widetilde{U}}(R, \alpha) \to \mathbf{Z}/2 \to L_{2k-1}^U(R, \alpha) \to L_{2k-1}^{\widetilde{U}}(R, \alpha) \to 0$$

When $R = \mathbf{Z}G$ and $U = K_1(R)$, $L^K_{2k}(\mathbf{Z}G) \cong L^h_{2k}(\mathbf{Z}G)$ and

$$L^h_{2k+1}(\mathbf{Z}G) = L^K_{2k+1}(\mathbf{Z}G)/ < \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} > .$$

In terms of our original discussion of L^h this means: define L^K_n using forms of even rank if $n = 2k$, and let $L^K_{2k+1}(\mathbf{Z}G) = SU(\Lambda)/TU(\Lambda)$.

The cobordism description provides a uniform way to derive exact sequences, which can then be used for calculations. For example, if $R \to S$ is a map of rings with involution, there is a long exact sequence

$$\cdots \to L_n(R) \to L_n(S) \to L_n(R \to S) \to L_{n-1}(R) \to \dots$$

The most important of these is the "Main Exact Sequence" of Wall, which is obtained from the arithmetic square.

Theorem 8.1 (Wall [67]). *Let G be a finite group and $U = SK_1(\mathbf{Z}G) \subset K_1(\mathbf{Z}G)$ or its image in $K_1(\widehat{\mathbf{Z}}G)$. Then there is a long exact sequence*

$$\cdots \to L^S_{n+1}(\widehat{\mathbf{Q}}G) \to L^U_n(\mathbf{Z}G) \to L^U_n(\widehat{\mathbf{Z}}G) \oplus L^S_n(\mathbf{Q}G) \to L^S_n(\widehat{\mathbf{Q}}G) \dots$$

For geometric surgery problems, we must have the freedom to change our Λ-bases for $C_i(f)$ by elements $g \in G$. This means that the smallest geometrically relevant torsion decoration containing $U = SK_1(\mathbf{Z}G)$ is

$$V = SK_1(\mathbf{Z}G) \oplus \{\pm G^{ab}\} .$$

Then there are natural maps,

$$L^s_n(\mathbf{Z}G) \to L^{\widetilde{V}}_n(\mathbf{Z}G) \to L^h_n(\mathbf{Z}G),$$

so that $L^{\widetilde{Y}}_n(\mathbf{Z}G)$ is "intermediate", between the two L-groups of most geometric significance.

It is worth remarking that the L-groups $L^p_n(\mathbf{Z}G)$ based on projective Λ-module chain complexes also have some geometric use. In fact, if $(f,b)\colon M^n \to X$ is a degree 1 normal map and X is a finitely dominated (but not necessarily finite) Poincaré complex, then a surgery obstruction $\lambda(f,b)$ is defined in $L^p_n(\mathbf{Z}[\pi_1 X])$. Moreover when $n \geq 5$, $\lambda(f,b) = 0$ if and only if the product normal map $(f,b) \times 1\colon M \times S^1 \to X \times S^1$ is normally cobordant to a homotopy equivalence (see [50]). In addition, the projective L-groups (and their generalizations) are the natural obstruction groups for surgery on non-compact manifolds. The version of this setting which incorporates bounded or controlled surgery problems has been particularly useful (see [51], [20]).

The projective L-groups can also be studied by an arithmetic sequence. If $L_n^P(\mathbf{Z}G)$ denotes the L-groups with the added condition $\chi = 0$, then there is an exact sequence

$$\cdots \to L_{n+1}^K(\widehat{\mathbf{Q}}G) \to L_n^P(\mathbf{Z}G) \to L_n^K(\widehat{\mathbf{Z}}G) \oplus L_n^K(\mathbf{Q}G) \to L_n^K(\widehat{\mathbf{Q}}G) \to \dots$$

The arithmetic exact sequences relate the computation of surgery obstruction groups to the L-theory of rings with much better algebraic properties. For example, $\mathbf{Q}G = \prod M_{n_i}(D_i)$ where the D_i are skew fields (Wedderburn's theorem) and

$$L_n^K(\mathbf{Q}G) = \prod L_n^K(D_i, \alpha_i),$$

by invariance under products and Morita equivalence. The terms $L_n^K(D_i, \alpha_i)$ must be interpreted with some care: our involution α on $\mathbf{Q}G$, induces an involution on the centre of each invariant factor $A_i = M_{n_i}(D_i)$, however in the transition from forms over A_i to forms over D_i a change of symmetry can occur. Nevertheless the product decomposition formula suggests that we should use the rational representation theory of G in a systematic way to organize and simplify the calculation.

The basic building blocks for character theory are the p-hyperelementary groups: extensions

$$1 \to C \to G \to P \to 1$$

where C is cyclic of order prime to p and P is a p-group.

Theorem 8.2 (Dress Induction). *Let G be a finite group and $U \subseteq K_1(\mathbf{Z}G)$ an involution-invariant subgroup. Then $L_n^U(\mathbf{Z}G)$ can be computed in terms of $\{L_n^U(\mathbf{Z}H) \mid H \subseteq G$ is 2-hyperelementary$\}$.*

This result means in particular that the sum of all the restricton maps $L_n^U(\mathbf{Z}G) \to L_n^U(\mathbf{Z}H)$ to the 2-hyperelementary subgroups is an injection. Therefore to decide whether a surgery obstruction is zero it is sufficient to restrict to these groups. Notice that for a normal map, restriction to a proper subgroup is given geometrically by taking a finite covering of the normal map. Dress induction exploits the Mackey functor structure on the L-groups modelled on classical induction and restriction of representations. This additional structure is a powerful tool for calculations.

Even with the help of character theory, the group $L_n^K(\mathbf{Q}G)$ is not easy to study. For example, $L_0^K(\mathbf{Q})$ is not finitely-generated [45, p.87] ! A classical remedy is the "local-global" comparison or "Hasse principle". This can be incorporated into our formulation by setting

$$CL_n^U(\mathbf{Q}G) = L_n^U(\mathbf{Q}G \to \widehat{\mathbf{Q}}G \oplus \mathbb{R}G)$$

for $U \subseteq K_1(\mathbf{Q}G)$, and rewriting the arithmetic sequences for example as

$$\cdots \to CL_{n+1}^K(\mathbf{Q}G) \to L_n^P(\mathbf{Z}G) \to L_n^K(\widehat{\mathbf{Z}}G) \oplus L_n^K(\mathbb{R}G) \to CL_n^K(\mathbf{Q}G) \ldots$$
$$\cdots \to CL_{n+1}^S(\mathbf{Q}G) \to L_n^X(\mathbf{Z}G) \to L_n^X(\widehat{\mathbf{Z}}G) \oplus L_n^S(\mathbb{R}G) \to CL_n^S(\mathbf{Q}G) \ldots$$

The computation of the $CL_n^U(D)$ for D a division algebra (with involution) is the deepest part of the theory, and involves methods from Galois cohomology (see Kneser's Tata Institute notes [35]).

Let us consider now the other terms in the arithmetic sequence. For $L_n^K(\mathbb{R}G)$ we have an immediate expression (via character theory) in terms of the most classical calculations in quadratic forms, namely forms over $\mathbb{R}$, $\mathbb{C}$ and (the quaternions) $\mathbf{H}$. For these cases, the signature, discriminant (and Pfaffian for L^S) give a complete list of invariants.

The term $L_n^K(\widehat{\mathbf{Z}}_pG)$ also reduces to quadratic forms over fields, since the L^K-groups have the property that

$$L_n^K(\widehat{\mathbf{Z}}_pG) = L_n^K(\widehat{\mathbf{Z}}_pG/J_pG)$$

where $J_pG \subseteq \widehat{\mathbf{Z}}_pG$ is the Jacobson radical. The quotient ring is finite and semi-simple, so we reduce via Morita equivalence to the L^K-groups of finite fields. In odd characteristic, the discriminant and Pfaffian are sufficient invariants; in characteristic 2 we must add in the Arf invariant. We remark that for finite fields with non-trivial involution, the L^K-groups are zero in characteristic 2 and the L^S-groups are all zero.

The corresponding term $L_n^X(\widehat{\mathbf{Z}}_pG)$ is also easy when $p \neq 2$. For p odd,

$$L_n^X(\widehat{\mathbf{Z}}_pG) \cong L_n^S(\widehat{\mathbf{Z}}_pG) \xrightarrow{\approx} L_n^S(\widehat{\mathbf{Z}}_pG/J_pG)$$

and we have L^S-groups of finite fields. If $p = 2$, there is an exact sequence

$$\to H^{n+1}(K_1(\widehat{\mathbf{Z}}_2G)/X) \to L_n^X(\widehat{\mathbf{Z}}_2G) \to L_n^K(\widehat{\mathbf{Z}}_2G) \to \ldots$$

so the new difficulty is the left-hand term and determining the maps in the exact sequence (see [31] for more information).

Even if we completely understand the terms $L_n^K(\hat{\mathbf{Z}}G)$ or $L_n^X(\hat{\mathbf{Z}}G)$, a problem still remains. If p divides $|G|$, the map

$$L_n^K(\widehat{\mathbf{Z}}_pG) \to L_n^K(\widehat{\mathbf{Q}}_pG)$$

in the arithmetic sequence is badly behaved, since $\widehat{\mathbf{Q}}_pG$ splits into more factors than $\widehat{\mathbf{Z}}_pG$ and the image spreads over these factors in a complicated way. To control this problem, we introduce a final improvement in the arithmetic sequences. Let $G = \mathbf{Z}/m \rtimes \sigma$ be a 2-hyperelementary group, where m is odd and σ is a 2-group. The extension is given by a homomorphism $t\colon \sigma \to Aut(\mathbf{Z}/m)$. For each $d \mid m$, let

$$R(d) = \mathbf{Z}[\zeta_d]^t\sigma$$

and $S(d) = R(d) \oplus \mathbf{Q}$, $T(d) = R(d) \otimes \mathbb{R}$.

Theorem 8.3 ([30]). *There is a natural direct sum splitting*

$$L_n^p(\mathbf{Z}G) = \bigoplus_{d|m} L_n^p(\mathbf{Z}G)(d)$$

such that

(i) *$L_i^p(\mathbf{Z}G)(d)$ is mapped isomorphically to $L_i^p(\mathbf{Z}[\mathbf{Z}/d \rtimes \sigma])(d)$ by the restriction map, and $L_n^p(\mathbf{Z}G)(d) = L_n^P(\mathbf{Z}G)(d)$ for $d > 1$.*

(ii) *There is an exact sequence for each $d \mid m$*

$$\to CL_{n+1}^K(S(d)) \to L_n^p(\mathbf{Z}G)(d) \to \prod_{p\nmid d} L_n^K(\hat{R}_p(d)) \oplus L_i^K(T(d)) \to CL_i^K(S(d)) \to \dots$$

The improvement that has been made here is in the local term. Now if $p \nmid 2d$ and $G = \mathbf{Z}/d \rtimes \sigma$, the map

$$L_n^k(\hat{R}_p(d)) \to L_n^K(\hat{S}_p(d))$$

splits according to the rational representations of G which are faithful on $\mathbf{Z}/d$. The remaining problem occurs for $p = 2$, in determining the map

$$L_n^k(\widehat{\mathbf{Z}}_2 \otimes \mathbf{Z}[\zeta_d]^t\sigma) \to L_n^k(\widehat{\mathbf{Q}}_2 \otimes \mathbf{Z}[\zeta_d]^t\sigma),$$

but we refer to [31] for further details.

There is also an analogue of this splitting theorem for $L_n^X(\mathbf{Z}G)(d)$, and again the only remaining "spreading" occurs at $p = 2$.

This concludes our brief outline of the techniques for the calculation of $L_*(\mathbf{Z}G)$, developed (for the most part) by C. T. C. Wall over a 10 year period [67]. The answers for specific groups G are likely to be complicated. Here are two nice cases.

Example 8.4. Kervaire and Milnor calculated the L-groups of the trivial group

$$L_n^s(\mathbf{Z}) = 8\mathbf{Z}, 0, \mathbf{Z}/2, 0 \quad \text{for } n = 0, 1, 2, 3 \bmod 4 ,$$

where the non–zero groups are detected by the signature or Arf invariant, and the notation $8\mathbf{Z}$ means that the signature can take on any value $\equiv 0 \bmod 8$. More geometrically, the generator in dimension $4k \geq 8$ is represented by the Milnor manifold surgery problem $(f, b)\colon M^{4k} \to S^{4k}$. Here $M^{4k} = W^{4k}(E_8) \cup D^{4k}$ is the closed topological manifold obtained by adjoining a disk D^{4k} to the boundary of the smooth plumbing manifold $W^{4k}(E_8)$ (see [32]). The boundary $\partial W^{4k}(E_8)$ is a homotopy sphere of dimension $4k - 1$, and is homeomorphic to S^{4k-1} by the generalized Poincaré conjecture (proved by S. Smale). Alternately, $W^{4k}(E_8)$ is the Brieskorn variety given by the points $z = (z_0, z_1, \ldots, z_{2k})$ in $\mathbb{C}^{2k+1}$ satisfying the two equations

$$z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2k}^2 = \epsilon$$

for a fixed small $\epsilon > 0$, and

$$|z_0|^2 + |z_1|^2 + \cdots + |z_{2k}|^2 = 1 .$$

The generator of $L_2(\mathbf{Z}) = \mathbf{Z}/2$ in dimension $4k + 2 \geq 6$ is represented by the Kervaire manifold surgery problem $(f, b)\colon K^{4k+2} \to S^{4k+2}$, where $K^{4k+2} = W^{4k+2}(A_2) \cup D^{4k+2}$ and $W^{4k+2}(A_2)$ also has a plumbing description, or as the Brieskorn variety

$$z_0^2 + z_1^2 + z_2^2 + \cdots + z_{2k+1}^2 = \epsilon$$

for a fixed small $\epsilon > 0$, and

$$|z_0|^2 + |z_1|^2 + \cdots + |z_{2k+1}|^2 = 1 .$$

The simply connected surgery obstruction in $L_{4k}(\mathbf{Z}) \cong 8\mathbf{Z}$ is defined for any degree 1 normal map $(f, b)\colon M^{4k} \to X^{4k}$, by the formula

$$\operatorname{Index}(f) = \operatorname{Index}(M) - \operatorname{Index}(X),$$

where $\operatorname{Index}(M)$ is the signature of the intersection form on $H_{2k}(M; \mathbf{Z})$. $\square$

A. Bak made extensive computations of L-groups. One result which has been very useful in topological applications is:

Theorem 8.5 (Bak [3]). *Let G be a finite group of odd order. Then $L^?_{2k+1}(\mathbf{Z}G) = 0$ for $? = s,\ h$ or p.*

A much more complete survey, with references and more results in particular cases can be found in [31].

9 Topological 4-manifolds with finite fundamental group

M. Freedman [21] proved that the surgery exact sequence is valid for topological 4-manifolds with finite fundamental groups (and more generally for polycyclic-by-finite fundamental groups). In particular, Freedman proved the 5-dimensional s-cobordism theorem in this setting. At the same time, S. Donaldson showed how the Yang-Mills gauge theory could give new information about smooth 4-manifolds, and demonstrated that smooth 5-dimensional s-cobordisms need not be products [17]. The combination of these two dramatic developments led to an exciting period of discovery, in which many of the long-standing open problems in the topology of 4-manifolds were settled. In this section, we will stick to applications of surgery theory and topological 4-manifolds.

Theorem 9.1 (Freedman). *Let X be a closed, simply connected, topological 4-manifold. Then X is classified up to homeomorphism by the intersection form on $H_2(X;\mathbf{Z})$ and the Kirby-Siebenmann invariant.*

The Kirby-Siebenmann invariant (in $\mathbf{Z}/2$) is the obstruction to finding a PL-structure on a closed topological 4-manifold. Freedman's classification can be generalized by geometric "cancellation" techniques, based on the unitary analogue of cancellation for modules, for 4-manifolds with finite fundamental groups.

We say that two closed topological 4-manifolds X and Y are *stably homeomorphic* if there exists a homeomorphism

$$h\colon X \,\sharp\, r(S^2 \times S^2) \approx Y \,\sharp\, r(S^2 \times S^2)$$

for some integer r. The cancellation problem is to remove copies of $S^2 \times S^2$, or in other words to determine the minimum r for which the two sides are homeomorphic. There is also a version of this problem for smooth manifolds (about which almost nothing is known !).

Theorem 9.2 ([28]). *Let X and Y be closed, oriented topological 4–manifolds with finite fundamental group. Suppose that the connected sum $X \sharp r(S^2 \times S^2)$ is homeomorphic to $Y \sharp r(S^2 \times S^2)$. If $X = X_0 \sharp (S^2 \times S^2)$, then X is homeomorphic to Y.*

Note that the assumption that X splits off one $S^2 \times S^2$ cannot be omitted in general. There are, for example, even simply–connected closed topological 4-manifolds which are stably homeomorphic but not homeomorphic because they have non–isometric intersection forms.

There is now a fairly clear two-part strategy for classifying 4-manifolds. First we try to classify up to stable homeomorphism, and then we apply cancellation. For the first part, there are fairly explicit results, but complete answers for the second part are available only for special fundamental groups (e.g. π_1 cyclic).

The following notation is useful for keeping track of the second Stiefel-Whitney data of a manifold X. We say that X has *w_2--type*:

(I) if $w_2(\widetilde{X}) \neq 0$.

(II) if $w_2(X) = 0$.

(III) if $w_2(\widetilde{X}) = 0$ and $w_2(X) \neq 0$.

Theorem 9.3 ([27]). *Let X be a closed, oriented 4-manifold with finite cyclic fundamental group. Then X is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(X; \mathbf{Z})/Tors$, the w_2-type, and the Kirby-Siebenmann invariant. Moreover, any isometry of the intersection form can be realized by a homeomorphism. The invariants can all be realized independently, except in the case of w_2-type II, where the Kirby-Siebenmann invariant is determined by the intersection form.*

Note that we do not assume any stability condition here, so the proof requires a sharper version of our cancellation theorem. In the remainder of the section, we will describe M. Kreck's approach to the stable classification [36] which is used in the proofs of the above results.

There is a close analogy between the stable classification of homotopy types of 2-complexes (as discussed in Section 5) and stable homeomorphism types of 4-manifolds. Consider the thickening functor from finite 2-complexes to closed 4-manifolds, obtained by embedding a 2-complex K as polyhedron in $\mathbb{R}^5$ and taking the boundary of a smooth regular neighborhood. If two 2-complexes are simply homotopy equivalent the corresponding 4-manifolds are s-cobordant (implying homeomorphic, if the fundamental groups are poly-(finite or cyclic) [22]) and we denote the corresponding s-cobordism class by $M(K)$. If we replace the 2-complex by its 1-point union with S^2, the corresponding 4-manifold changes by connected sum with $S^2 \times S^2$. This indicates the analogy of stable equivalence classes of 2-complexes with the following notation for 4-manifolds.

Since the smooth stable s-cobordism theorem (implying that two s-cobordant 4-manifolds are stably diffeomorphic) holds, the stable diffeomorphism class of $M(K)$ is determined by the stable simple homotopy class of K and so, (see §1) by $\pi_1(K)$.

However, the stable classification of 4-manifolds (in contrast to the situation for 2-complexes) needs more invariants than the fundamental group and the Euler characteristic. For example, we must include the orientation, signature, and existence of a spin-structure. To obtain a complete answer we express the stable classification as a bordism problem and compute the bordism groups by the Atiyah-Hirzebruch spectral sequence.

Let $c\colon X \to K(\pi,1)$ be the classifying map of the universal covering $\widetilde{X}$, where $\pi := \pi_1(X,x_0)$. The Kirby-Siebenmann invariant of X will be denoted $KS(X)$. There is an isomorphism $c^*\colon H^1(\pi;\mathbf{Z}/2) \to H^1(X;\mathbf{Z}/2)$ and an exact sequence

$$0 \to H^2(\pi;\mathbf{Z}/2) \xrightarrow{c^*} H^2(X;\mathbf{Z}/2) \longrightarrow H^2(\tilde{X};\mathbf{Z}/2)\ .$$

Thus we can always pull back $w_1(X)$ by c^* from a class denoted $w_1 \in H^1(\pi;\mathbf{Z}/2)$, and we can pull back $w_2(X)$ from a class denoted $w_2 \in H^2(\pi;\mathbf{Z}/2)$, if $w_2(\tilde{X}) = 0$.

For a smooth 4-manifold X, the normal 1-type of X is a fibration

$$p\colon B(\pi,w_1,w_2) \to BO\ .$$

If $w_2(\widetilde{X}) \neq 0$, then $B(\pi,w_1,w_2) = K(\pi,1) \times BSO$ and p is given by the composition

$$p\colon K(\pi,1)\times BSO \xrightarrow{E\times i} BO \times BO \xrightarrow{\oplus} BO\ ,$$

wherer $E\colon K(\pi,1) \to BO$ is the classifying map of the stable line bundle given by w_1, $i\colon BSO \to BO$ is the inclusion, and $\oplus$ is the H-space structure on BO given by the Whitney sum.

If $w_2 \neq \infty$ we define the normal 1-type as the fibration $p : B(\pi, w_1, w_2) \longrightarrow BO$ given by the following pullback square

$$\begin{tikzcd} B(\pi, w_1, w_2) \arrow[r] \arrow[d, "p"] & K(\pi, 1) \arrow[d, "w_1 \times w_2"] \\ BO \arrow[r, "\widehat{w}_1 \times \widehat{w}_2"] & K(\mathbf{Z}/2, 1) \times K(\mathbf{Z}/2, 2) \end{tikzcd}$$

where $\widehat{w}_i := w_i(EO)$ are the Stiefel-Whitney classes of the universal bundle and we interpret w_i as maps to $K(\mathbf{Z}/2, i)$.

If $w_1 = 0$, $B(\pi, 0, w_2)$ factorizes over BSO and we choose one of the possible lifts. To deal with the oriented case ($w_1 = 0$) and the non-oriented case simultaneously we write $p\colon B(\pi, w_1, w_2) \longrightarrow B(S)O$.

For topological manifolds one can make the obvious changes. In the notation replace the linear normal bundle by the topological normal bundle given by a map $\nu\colon X \to B(S)Top)$, and define the normal 1-type $p\colon B(\pi, w_1, w_2) \longrightarrow B(S)Top$.

Given any fibration $B \to B(S)O$, abbreviated for short as B, we consider the B-bordism group $\Omega_n(B)$ consisting of bordism classes of closed smooth n-manifolds, which are oriented if the fibration is over BSO, together with a lift $\bar{\nu}$ over B of the classifying map $\nu\colon X \to B(S)O$ for the stable normal bundle of X. Such a lift is called a *normal 1-smoothing* if $\bar{\nu}$ is a 2-equivalence. By construction, X admits a normal 1-smoothing in $B(\pi, w_1, w_2)$. Similarly for topological manifolds one starts with a fibration $B \to B(S)Top$, and introduces the analogous bordism group of topological manifolds denoted $\Omega_n^{Top}(B)$.

Theorem 9.4 (Kreck [36]). *Two smooth (topological) 4-manifolds X_0 and X_1 with the same normal 1-type $B(\pi, w_1, w_2)$ are stably diffeomorphic (homeomorphic) if and only if:*

(i) they have the same Euler characteristic, and

(ii) they admit normal 1-smoothings $\bar{\nu}_0$ and $\bar{\nu}_1$ such that $(X_0, \bar{\nu}_0)$ and $(X_1, \bar{\nu}_1)$ represent the same bordism class in $\Omega_4(B(\pi, w_1, w_2))$ (in $\Omega_4(B^{Top}(\pi, w_1, w_2))$).

A computation of the bordism groups now gives:

Theorem 9.5 (Kreck [36]). *Two oriented smooth (topological) 4-manifolds X_0 and X_1 with the same fundamental group and with $w_2(\widetilde{X}_i) \neq 0$ are stably* *diffeomorphic (homeomorphic), if and only if they have the same Euler characteristic and signature, if $c_*[X_0] = c_*[X_1] \in H_4(K(\pi,1);\mathbf{Z})/Out(\pi)$ and, in the topological case, $KS(X_0) = KS(X_1)$.*

10 Surgery obstructions on closed manifolds

The surgery exact sequence

$$L_{n+1}(\mathbf{Z}[\pi_1 X], w) \to \mathcal{S}_n(X) \xrightarrow{\eta} T(X,w) \xrightarrow{\lambda} L_n(\mathbf{Z}[\pi_1 X], w)$$

provides a good framework for classifying manifolds, but to obtain concrete results in particular cases we must know how to compute the maps in the sequence. In this section we will consider only the *oriented* case ($w \equiv 1$) and suppose that X is a closed oriented topological manifold of dimension $n \geq 5$, with *finite* fundamental group $\pi := \pi_1(X, x_0)$. Then $\mathcal{S}_n(X)$ is just the set of manifolds homotopy equivalent to X modulo h-cobordism.

Suppose that $(f,b)\colon M \to N$ is a degree 1 normal map of closed manifolds, or in other words, a closed manifold surgery problem. It turns out that the closed manifold surgery obstructions are very restricted, related to the low dimensional group homology $H_*(\mathbf{Z}\pi;\mathbf{Z}/2)$, while $L_n^h(\mathbf{Z}\pi)$ depends on representations of π and the number theory of their centre fields. We will actually obtain results about the *weakly simple* surgery obstructions in $L'_n(\mathbf{Z}\pi) := L_n^U(\mathbf{Z}\pi)$, where $U = SK_1(\mathbf{Z}\pi) \oplus \{\pm\pi^{ab}\}$. The natural map $L_n^U(\mathbf{Z}\pi) \to L_n^h(\mathbf{Z}\pi)$ shows that these results also hold for L^h.

One way to obtain closed manifold surgery problems form the cartesian product of a simply connected surgery problem with a closed manifold P in domain and range. The standard simply connected connected surgery problems are the Milnor problem and the Kervaire problem (8.4).

Theorem 10.1 ([23]). *Let P^k be a closed, oriented, topological manifold with $\pi_1(P)$ finite, and let $(f,b)\colon M^n \to N^n$ be a simply connected closed manifold surgery problem, with $n+k \geq 5$. Then the product normal map*

$$(f \times id, b \times id)\colon M \times P \to N \times P$$

is normally cobordant to a weakly simple homotopy equivalence either

(i) *for $n \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 4$, if the Euler characteristic of P is even, or* *(ii) for* $n \equiv 0 (\mathrm{mod}\, 4)$ *if Index*$(P) = 0$.

The most complete result is for odd dimensional surgery problems. Recall that if $\bar{\pi}$ is a subquotient of π (that is, $\bar{\pi} = \rho/\rho_0$ where $\rho_0 \triangleleft \rho \subseteq \pi$) there is a "transfer-projection" homomorphism $L_n^{'}(\mathbf{Z}\pi) \to L_n^{'}(\mathbf{Z}\bar{\pi})$ induced geometrically by surgery on a covering normal map. Let $C(2)$ denote the cyclic group of order 2, and $Q(2^k)$ the generalized quaternion group of order 2^k.

Theorem 10.2 ([23]). *Let* N^n *be a closed oriented topological manifold with* $\pi_1(N)$ *finite and* $n \geq 5$ *odd. Then a closed manifold surgery problem* $(f, b)\colon M \to N$ *is normally cobordant to a weakly simple homotopy equivalence if and only if:*

(i) $n \equiv 1 (\mathrm{mod}\, 4)$ *and* $\lambda(f, b)$ *maps to zero under transfer-projection to all quaternionic subquotients* $Q(2^k)$ *of* $\pi_1(N)$*, or*

(ii) $n \equiv 3 (\mathrm{mod}\, 4)$ *and* $\lambda(f, b)$ *maps to zero under transfer-projection to all* $C(2)$ *quotients of* $\pi_1(N)$*.*

A closed manifold X provides a base point for the homotopy theoretic description $T(X) \cong [X, G/TOP]$, due to Sullivan and Kirby-Siebenmann, for the set of degree 1 normal maps. The surgery obstruction gives a map

$$\sigma_X\colon [X, G/TOP] \to L_n^U(\mathbf{Z}\pi)$$

which can be understood in terms of a "universal" family of homomorphisms

$$\kappa_j^U\colon H_j(\pi, \mathbf{Z}/2) \to L_{j+2}^U(\mathbf{Z}\pi)_{(2)}$$

depending only on the fundamental group.

The definition of the $\{\kappa_j^U\}$ depends on the 2-local splitting $G/TOP_{(2)} = \prod_{k>0} K(\mathbf{Z}_{(2)}, 4k) \times K(\mathbf{Z}/2, 4k-2)$ given by the cohomology classes $\ell = \{\ell_{4*}\} \in H^{4*}(G/TOP, \mathbf{Z}_{(2)})$ and $k = \{k_{4*+2} \in H^{4*+2}(G/TOP, \mathbf{Z}/2)$ of Morgan-Sullivan, Rourke-Sullivan, and Milgram.

The next major ingredient is the fact that a closed topological n-manifold satisfies an enriched form of Poincaré duality, namely,

$$\cap [X]_{\mathbb{L}_0}\colon [X, G/TOP] \cong H^0(X; \mathbb{L}_0) \xrightarrow{\approx} H_n(X; \mathbb{L}_0)$$

where $\mathbb{L}_0$ is the Quinn-Ranicki connective L-spectrum with 0-th space G/TOP (see [55]). The surgery obstruction map σ_X is a homomorphism with respect to this H-space structure (not the one induced by Whitney sum of bundles). Note that the generalized homology functor $H_n(X; \mathbb{L}_0)$ on the right-hand side can be applied to any space, not just to n-manifolds. Let $c\colon X \to K(\pi, 1)$ be the classifying map of the universal covering space $\widetilde{X}$.

Theorem 10.3 (Quinn, Ranicki [55]). *For any group π, and any integer n, there exists an assembly map*

$$A_\pi \colon H_n(\pi; \mathbb{L}_0) \to L_n^U(\mathbf{Z}\pi),$$

functorial in π. Furthermore, if X is a closed, oriented topological n-manifold, the surgery obstruction homomorphism $\sigma_X(f) = A_\pi \circ c_*(\sigma_X(f) \cap [X]_{\mathbb{L}_0})$, for all maps $f\colon X \to G/TOP$.

The L-spectrum has a 2-local splitting as above into Eilenberg-Maclane spectra [62], so that

$$H_n(\pi; \mathbb{L}_0) = \bigoplus_{k>0} H_{n-4k}(\pi; \mathbf{Z}_{(2)}) \times H_{n-4k-2}(\pi; \mathbf{Z}/2)$$

and the corresponding splitting of the assembly map A_π restricted to one of the $\mathbf{Z}/2$-homology summands gives κ_j^U for $j = n - 4k - 2$. It isn't obvious (but true) that another pair (n', k') with the same value $j = n' - 4k' - 2$ leads to the same homomorphism κ_j^U, after identifying $L_n \cong L_{n+4(k-k')}$ by the periodicity isomorphism (note that $n' = n + 4(k - k')$).

Let V_X denote the total Wu class of the stable normal bundle ν_X, and for any map $f\colon X \to G/TOP$ let

$$\mathrm{ARF}_j(f) = \{(V_X^2 \cup f^*(k)) \cap [X]\} \in H_j(X; \mathbf{Z}/2)$$

be the j-dimensional component of the indicated homology class. We let $\mathrm{ARF}(f)$ and $\mathrm{Index}(f)$ denote the ordinary (simply-connected) Arf invariant and index of the surgery problem given by f (considered as elements in $L_*(\mathbf{Z})$). Finally, let

$$s_r\colon H_{2^r+2}(X; \mathbf{Z}/2) \to H_4(X; \mathbf{Z}/2)$$

for $r \geq 0$ be the Hom-dual of the iterated squaring maps in cohomology.

Theorem 10.4 ([23]). *Let X be a closed, oriented topological n-manifold with finite fundamental group π. Let $U \subset \mathrm{Wh}(\mathbf{Z}\pi)$ be an involution invariant subgroup containing $Im(SK_1(\mathbf{Z}\rho) \to SK_1(\mathbf{Z}\pi))$, where $\rho \subseteq \pi$ is a 2-Sylow subgroup. For any surgery problem $f\colon X \to G/TOP$ of closed manifolds, the surgery obstruction $\sigma_X(f) \in L_n^U(\mathbf{Z}\pi)$ is equal to:*

(i) *Index* $(f) + \kappa_2^U\{c_*(ARF_2(f))\}$ *for* $n \equiv 0 (\mathrm{mod}\, 4)$.

(ii) $\kappa_3^U\{c_*(ARF_3(f))\}$ *for* $n \equiv 1 (\mathrm{mod}\, 4)$.

(iii) $ARF(f) + \kappa_4^U\{c_*(\sum_{r\geq 0} s_r(ARF_{2^{r+2}}(f)))\}$ *for* $n \equiv 2 (\mathrm{mod}\, 4)$.

(iv) $\kappa_1^U\{c_*(ARF_1(f))\}$ *for* $n \equiv 3 (\mathrm{mod}\, 4)$.

This result and the applications above are proved by factoring the κ-homomorphisms through a more computable form of L-theory, and then using the arithmetic square techniques (see [23] for more details).

11 The spherical space form problem

The classification of orthogonal spherical space forms up to isometry [71] was first proposed by Killing in 1891, and the problem attracted the attention of famous mathematicians of the time, such as Clifford, Hopf, Klein, and Poincaré. In 1925, H. Hopf proved:

Theorem 11.1 (Hopf [33]). *The following is a list of all finite fixed-point free subgroups of* $SO(4)$*:*

(a) *The cyclic group* $C(n)$*, the generalized quaternion group* $Q(4n)$*, the binary tetrahedral group* $T^*(24)$*, the binary octahedral group* $O^*(48)$*, and the binary icosahedral group* $I^*(120)$*.*

(b) *The semidirect product* $C(2n+1) \rtimes C(2^k)$ *of an odd order cyclic group with a cyclic 2-group. More explicitly* $C(2n+1) \rtimes C(2^k)$ *is given by the presentation* $\{A, B : A^{2^k} = B^{2n+1} = 1, ABA^{-1} = B^{-1}\}$ *where* $k \geq 2, n \geq 1$*.*

(c) *A semidirect product* $Q(8) \rtimes C(3^k)$ *of the quaternion group* $Q(8)$ *with a cyclic 3-group. More explicitly,* $Q(8) \rtimes C(3^k)$ *is given by the presentation* $\{P, Q, X : P^2 = (PQ)^2 = Q^2,\ X^{3^k} = 1,\ XPX^{-1} = Q,\ XQX^{-1} = PQ\}$ *where* $k \geq 1$*. For* $k = 1$*, this is the binary tetrahedral group* $T^*(24)$*.*

(d) *The product of any of the above groups with a cyclic group of coprime order.*

At first glance, the above list may appear to be random. In the forties and fifties, efforts were made to interpret Hopf's list using group cohomology [12] and it was discovered that all these groups have periodic Tate cohomology of period four. In general, a finite group has periodic cohomology if and only if it satisfies the p^2-conditions ("any subgroup of order p^2 is cyclic") for all primes p. From the viewpoint of group theory, this condition means that the odd Sylow subgroup is cyclic and the 2-Sylow subgroup is cyclic or generalized quaternion. If the cohomology has period four then, in addition, the pq-conditions hold ("every subgroup of order pq is cyclic") for p and q distinct odd primes.

The necessity of the $2q$-conditions was established by J. Milnor [43] in 1957, when he showed that the dihedral group of order $2q$ cannot operate freely on any $\mathbf{Z}/2$-homology sphere despite the fact that it has periodic cohomology of period 4. In [43] Milnor also compiled the following list of all finite groups, not in Hopf's list (11.1.a)-(11.1.d), but satisfying the restrictions known at the time on fundamental groups of 3-manifolds.

Theorem 11.2 (Milnor). *The following are the finite groups with periodic cohomology of period* 4, *containing no dihedral subgroups.*

(a) The semidirect product $Q(8n,k,l)$ of the odd cyclic group $C(kl)$ with the generalized quaternion group $Q(8n)$. More explicitly, $Q(8n,k,l)$ has the presentation: $\{X,Y,Z : X^2 = Y^{2n} = (XY)^2,\ Z^{kl} = 1,\ XZX^{-1} = Z^r,\ YZY = Z^{-1}\}$. Here n,k,l are all odd integers and relatively prime to each other, $n > k > l \geq 1$, and r satisfies $r \equiv -1 \pmod{k}$, $r \equiv 1 \pmod{l}$. If $l = 1$, we set $Q(8n,k) \equiv Q(n,k,1)$.

(b) The group $Q(8n,k,l)$ with the same presentation as (1.5), but with n even.

(c) An extension $O(48;3^{k-1},l)$ of the odd order cyclic group $C(3^{k-1}l)$, $3 \nmid l$, by the binary octahedral group $O^*(48)$. More precisely, $O(48;3^{k-1},l)$ has five generators X,P,Q,R,A and the following relations:

$$X^{3^k} = P^4 = A^l = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1}$$
$$XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP$$
$$RQR^{-1} = Q^{-1},\ AP = PA,\ AQ = QA,\ RAR^{-1} = A^{-1}.$$

(d) The product of any of the above groups with a cyclic group of coprime order.

Thus Hopf's problem is to prove that groups in the above list (11.2.a)-(11.2.d) do not act freely on homotopy 3-spheres.

In the late sixties, C. T. C. Wall asked whether Milnor's result could be interpreted using the new theory of nonsimply connected surgery. Ronnie Lee [37] answered this question in 1973 by defining a "semicharacteristic" obstruction for the problem. As well as recovering the previous result of Milnor, the semicharacteristic rules out the family of groups $Q(8n, k, l)$, n even, in (11.2.b). Later C. B. Thomas observed that this also eliminates the family of groups $O(48, 3^{k-1}, l)$ in (11.2.c) because groups of this type always contain a subgroup isomorphic to $Q(16, 3^{k-1}, 1)$. These results leave undecided only the groups $Q(8n, k, l)$, n odd, in (11.2.a) and their products with cyclic groups of coprime order in (11.2.d) from Milnor's original list.

The remaining part of Hopf's problem is to prove that for any distinct odd primes p, q, the group $Q(8p, q)$ does not operate freely on any homotopy 3-sphere. Notice that a group $Q(8n, k, l)$ in the family (11.2.a) always contains a subgroup of the form $Q(8p, q)$. Hence ruling out the groups $Q(8p, q)$ would also eliminate the family (11.2.c) in Milnor's list and the corresponding products in (11.2.d).

In contrast with the 3-dimensional case, the analogous spherical space form problem in higher dimensions has been almost completely settled. The goal is the topological (smooth) classification of finite group actions (Σ^{2n-1}, G) on (homotopy) spheres Σ^{2n-1} of dimension $2n-1$, $n \geq 3$. This problem was both a motivation and an important test case for the techniques of algebraic and geometric topology developed in the period 1960–1985. P. A. Smith had already shown in 1944 that the p^2 conditions were necessary for a G-action on any homology sphere. Conversely, Swan proved:

Theorem 11.3 (Swan [60]). *Every group with periodic cohomology acts freely and simplicially on a finite-dimensional CW complex homotopy equivalent to a sphere.*

Given a group G with periodic cohomology of period $2d$, Swan's contruction produces finitely dominated Poincaré complexes X with $\pi_1(X, x_0) = G$, and $\widetilde{X} \simeq S^{2n-1}$, for some multiple n of d. We call these Swan complexes for short. The chain complex $C_*(X)$ gives an exact sequence (or periodic resolution) of the form

$$0 \to \mathbf{Z} \to P_{2n-1} \to \cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

where the C_i are finitely generated projective $\mathbf{Z}G$-modules. Two such sequences C_* and C'_* are isomorphic if there is a chain map $f\colon C \to C'$ inducing the identity on the homology groups $H_0 = H_{2n-1} = \mathbf{Z}$, and the homotopy types of X are in bijective correspondence with the isomorphism classes of the periodic resolutions. The Wall finiteness obstruction is just

$$\theta_W(X) = \sum(-1)^i[P_i] \in \widetilde{K}_0(\mathbf{Z}G)$$

and Swan discovered a beautiful formula for the difference $\theta_W(X) - \theta_W(X')$ if X, X' are two Swan complexes of the same dimension.

Any two periodic resolutions can be compared by a chain map:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, "r"] & P_{2n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & P'_{2n-1} \arrow[r] & \cdots \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

inducing a map of degree r. Equivalently, if X and Y are Swan complexes of the same dimension, there is a map $f\colon X \to Y$ of degree r between them. We let $\langle r, N\rangle \subset \mathbf{Z}G$ denote the ideal generated by the integer r and the group ring element $N = \sum\{g \mid g \in G\}$

Theorem 11.4 (Swan [60]). *Let X and Y be Swan complexes for G of the same dimension. Then $\theta_W(Y) = \theta_W(X) + [\langle r, N\rangle] \in \widetilde{K}_0(\mathbf{Z}G)$.*

The existence and classification of Swan complexes opened the way for a systematic attack on the problem using surgery theory. Throughout the 1970's remarkable progress was made on the higher dimensional space form problem, culminating in the paper of Madsen, Thomas and Wall [39].

Theorem 11.5 ([39]). *Any finite group G satisfying the p^2 and $2p$ conditions (for all primes p) acts freely and smoothly on a homotopy sphere of some odd dimension $2n - 1 > 3$.*

The precise dimensional bounds were not determined, although for G of period $2d$ they show that $n = 2d$ is always realizable ($n = d$ is best possible).

The next big step forward was the explicit calculation by Milgram [41] in 1979 of the finiteness obstruction for some of the period 4 groups $G = Q(8p, q)$, following the method of [68]. Tensoring a periodic resolution for G with the adele ring $\widehat{\mathbf{Q}}$ allows one to define an "idelic" Reidemeister torsion invariant

$$\widehat{\Delta}(X) \in K_1(\widehat{\mathbf{Q}}G)$$

whose image under the boundary map $\partial\colon K_1(\widehat{\mathbf{Q}}G) \to \widetilde{K}_0(\mathbf{Z}G)$ gives the formula

$$\partial\widehat{\Delta}(X) = \theta_W(X)$$

Now the arithmetic square techniques can be applied to compute the finiteness obstruction in terms of units in algebraic number fields. In particular, Milgram showed that some of the groups in Milnor's list are not fundamental groups of spherical space forms in any dimension (including dimension 3).

After this followed a sequence of papers by Milgram (see the survey in [16]), and independently by Madsen [40], aiming at the calculation of the relevant surgery obstruction. Here the problem is to determine which of the groups $Q(8p, q)$ act freely on Σ^{8k+3}, for $k > 0$, since they act linearly on S^{8k+7} for all $k \geq 0$. It turned out that the answer is computable in principle, but depends sporadically on the number theory of the primes p, q. Note that the vanishing of the high-dimensional obstruction is equivalent to the existence of a free action of the corresponding group $Q(8p, q)$ on an integral homology 3-sphere.

12 Bounded K and L-theory

The next two sections give an introduction to "bounded" topology, and generalize algebraic K-theory to this setting. This algebra has many applications in topology, including the problem discussed in Section 14 of these notes.

Let M be a metric space. Assume there is a group G acting on M by eventual Lipschitz maps [49]. Recall that an eventual Lipschitz map $g : M \longrightarrow M$ is a map so there exists $k, l \in \mathbb{R}_+$ so that $d(gx, gy) \leq k \cdot d(x, y) + l$. We want k and l to be independent of g.

Example 12.1. Let M be a finitely generated group exhibited with the word metric, and $G \subseteq M$ a subgroup. Then the action of G on M by conjugation is by eventual Lipschitz equivalences. Specifically if $g \in G$ has length l then $d(gxg^{-1}, gyg^{-1}) = d(gxy^{-1}g^{-1}, e) \leq 2l + d(x, y)$

Example 12.2. Let (V, G) be an orthogonal representation. Then G acts by isometries on V hence clearly by eventual Lipschitz maps.

Given M and G as above, and a commutative ring with unit R, we define a category $\mathcal{G}_{M,G}(R)$ as follows:

Definition 12.3 ([24]). An object A is a right RG-module together with a map $f : A \to F(M)$, where $F(M)$ is the set of finite subsets of M, satisfying

(i) f is G-equivariant.

(ii) $A_x = \{a \in A \mid f(a) \subseteq \{x\}\}$ is a finitely generated RG_x-module, free as an R-module.

(iii) As an R-module $A = \bigoplus_{x \in M} A_x$.

(iv) $f(a+b) \subseteq f(a) \cup f(b)$.

(v) For each ball $B \subset M$, the subset $\{x \in B \mid A_x \neq 0\}$ is finite.

A morphism $\phi : A \to B$ is a morphism of RG-modules, satisfying the following condition: there exists k so that the components $\phi_n^m : A_m \to B_n$ (which are R-module morphisms) are zero when $d(m,n) > k$. The category $\mathcal{G}_{M,G}(R)$ is an additive category in an obvious way.

Remark 12.4. When M has more than one point it follows from these conditions that $f(a) = \emptyset$ if and only if $a = 0$. When M is precisely one point, this has to be added as an extra assumption. It follows easily from the conditions that f measures exactly where an element has components. In other words, if $x_1, \ldots x_n \in M$ are different points and $a_i \in A_{x_i}$, $a_i \neq 0$ then $f(a_1 + \ldots + a_n) = \{x_1, \ldots, x_n\}$.

Given an object A, an R-module homomorphism $\phi : A \to R$ is said to be locally finite if the set of $x \in M$ for which $\phi(A_x) \neq 0$ is finite. Define $A^* = \mathrm{Hom}_R^{l.f.}(A, R)$, as the set of locally finite R-homomorphisms. We want to make $*$ a functor from $\mathcal{G}_{M,G}(R)$ to itself to make $\mathcal{G}_{M,G}(R)$ a category with involution. We define $f^* : A^* \to FM$ by $f^*(\phi) = \{x \mid \phi(A_x) \neq 0\}$ which is finite by assumption. The dual module A^* has an obvious left action of G, turning it into a left RG module via the formula $\phi g(a) = \phi(ga)$, and f^* is equivariant with respect to the left action on M given by $xg = g^{-1}x$. To make $*$ an endofunctor of $\mathcal{G}_{M,G}(R)$ we need to replace the left action by a right action. As is usual in surgery theory, this may be done in various ways, the standard one being to let g act on the right by letting g^{-1} act on the left. However given a homomorphism $w : G \to \{\pm 1\}$, we may let g act on the right of A^* by $w(g) \cdot g^{-1}$ on the left.

Proposition 12.5. *$(\mathcal{G}_{M,G}(R), *)$ is an additive category with involution.*

For many purposes we are more interested in the subcategory of $\mathcal{G}_{M,G}(R)$ for which all objects are free RG modules.

Definition 12.6. The subcategory of $\mathcal{G}_{M,G}(R)$ where the modules are required to be free RG modules is denoted by $\mathcal{C}_{M,G}(R)$.

It is easy to see that $*$ induces a functor on $\mathcal{C}_{M,G}(R)$, so that $\mathcal{C}_{M,G}(R)$ is a subcategory with involution.

Example 12.7. If G acts trivially on M and G is finite, then $\mathcal{C}_{M,G}(R)$ is naturally equivalent to $\mathcal{C}_M(RG)$ where RG is the category of free finitely generated based RG modules. There is also a nice expression due to Pedersen and Weibel [48] for the lower algebraic K-theory functors defined by Bass [4]:

$$K_{-i}(\mathbf{Z}G) = K_1(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathbf{Z}G))$$

for $i > 0$.

Example 12.8. If G is finitely generated and $|G|$ denotes the metric space with the same underlying set as G, and the word metric, then $\mathcal{C}_{|G|,G}(R)$ is naturally equivalent to $\mathcal{C}_{pt}(RG)$ (as categories with involution). Notice it does not matter which generating set we choose for G since 2 different generating sets will give eventual Lipschitz equivalent metrics. In case G is finite, this means $\mathcal{C}_{|G|,G}(R)$ is equivalent to $\mathcal{C}_{pt,G}(R)$ which is equivalent to $\mathcal{C}_{pt}(RG)$.

Using the algebraic L-theory of additive categories with involution, as developed by Ranicki [54], we can define the functors $L_n^K(\mathcal{C}_{M,G}(R))$ where K is some $*$ invariant subgroup of $\widetilde{K}_i(\mathcal{C}_{M,G}(R))$, $i = 0, 1$. Here $\widetilde{K}_1(\mathcal{C}_{M,G}(R)) = K_1(\mathcal{C}_{M,G}(R))/\{\pm 1\}$ and $\widetilde{K}_0(\mathcal{C}_{M,G}(R)) = (K_0(\mathcal{C}_{M,G}(R))^\wedge)/K_0(\mathcal{C}_{M,G}(R))$, where $^\wedge$ denotes idempotent completion.

Let N be a sub–metric space of the metric space M. In the equivariant case, we suppose that N is an invariant subspace.

Definition 12.9. The category $\mathcal{C}_{M,G}^{>N}(R)$ of germs away from N, has the same objects as $\mathcal{C}_{M,G}(R)$, and morphisms are germs of morphisms away from N: two morphisms are identified if there exists k so that they only differ in a k-neighborhood of N.

Consider the metric space $M \times \mathbb{R}$ where G acts trivially on the $\mathbb{R}$-factor. Inside we have the metric space $M \cup N \times [0, \infty)$. It follows immediately from the methods of [49], see also [1] for a more formalized description, that the natural functor

$$\mathcal{C}_{M\cup N\times[0,\infty),G}(R) \longrightarrow \mathcal{C}^{>N}_{M,G}(R)$$

induces an isomorphism on K-theory, and it follows from the proofs of [56], that it induces an isomorphism in L-theory (Eilenberg swindle is allowed in L-theory).

Theorem 12.10 ([18], [24]). *There is a long exact sequence*

$$\dots K_*(\mathcal{C}_{N,G}(R)) \longrightarrow K_*(\mathcal{C}_{M,G}(R)) \longrightarrow K_*(\mathcal{C}^{>N}_{M,G}(R)) \longrightarrow K_{*-1}(\mathcal{C}_{N,G}(R))\dots$$

Here it should be noted that we are using the non-connective deloopings of [49] to define K-theory in negative dimensions.

Theorem 12.11 ([18], [24]). *There is a 4-periodic long exact sequence*

$$\dots L^h_n(\mathcal{C}_{(N,G)}(R)) \to L^h_n(\mathcal{C}_{(M,G)}(R)) \to L^K_n(\mathcal{C}^{>N}_{(M,G)}(R)) \to L^h_{n-1}(\mathcal{C}_{(N,G)}(R))\dots$$

where $K = \operatorname{Im}(\widetilde{K}_1(\mathcal{C}_{(M,G)}(R)) \longrightarrow \widetilde{K}_1(\mathcal{C}^{>N}_{(M,G)}(R)))$.

The formulation in [56] uses $\mathcal{C}_{M\cup N\times[0,\infty)}$ instead of $\mathcal{C}^{>N}_{M}$. We saw in example 12.7 that trivial group action corresponds to RG coefficients. This is part of a more general phenomenon motivating the following definition

Definition 12.12. Suppose G is acting on the metric space M with invariant subspace N. We say that the set of subgroups $\{H_\alpha\}$ of G is the effective fundamental group for (M,G) away from N if the following is satisfied: For every $k > 0$ the set $\{x \in M \mid \operatorname{diam}(H_\alpha \cdot x) < k\}$ is not contained in a bounded neighborhood of N.

Example 12.13. Let (V,G) be a representation. Then the effective fundamental group away from 0 is the set of isotropy subgroups of the representation.

We also need a geometric formulation for bounded surgery. A map $X \longrightarrow M$ from a space to a metric space is eventually continuous if there exist a covering $\{U_\alpha\}$ of X so that $\operatorname{diam}(pU_\alpha)$ is uniformly bounded, and the inverse image of a bounded set is precompact. When the metric space is a cone, an eventually continuous map may always be replaced by a continuous map which is only a bounded distance away.

Theorem 12.14 (Ferry-Pedersen [18]). *Let X be a free $G - CW$ complex together with a G-equivariant, eventually continuous map $X \to M$ such that $X \to M$ is boundedly simply connected, and X satisfies Poincaré duality with respect to some homomorphism $w\colon G \to \mathbf{Z}/2$, in the category $\mathcal{C}_{M,G}(\mathbb{Z})$, dim(X) ≥ 5. Let $W \to X$ be a degree one normal map. Then W is normally cobordant to a bounded homotopy equivalence if and only if an invariant in $L_n(\mathcal{C}_{M,G}(\mathbb{Z}))$ vanishes.*

The concept boundedly simply connected is defined in [18, 2.7] . As in standard surgery theory, normal invariants corresponds to lifts of the Spivak normal fibre space $X \to BF$ to $BTOP$. If we fix a lift (defining a basepoint) then:

Theorem 12.15 (Ferry-Pedersen [18]). *There is a bounded surgery exact sequence*

$$\ldots \to L^h_{n+1}\mathcal{C}_{M,G}(\mathbb{Z})) \to \mathcal{S}^b \begin{pmatrix} X/G \\ \downarrow \\ M/G \end{pmatrix} \to [X/G, F/TOP] \to L^h_n(\mathcal{C}_{M,G}(\mathbb{Z}))$$

Tensor product defines a pairing

$$\mathcal{C}_{|G|,G}(R) \times \mathcal{G}_{M,G}(R) \longrightarrow \mathcal{C}_{|G|\times M,G}(R)$$

whenever R is a commutative ring with unit. When $|G|$ is finite, this means we may replace $\mathcal{C}_{|G|,G}(R)$ by $\mathcal{C}_{pt}(RG)$ and $\mathcal{C}_{|G|\times M,G}(R)$ by $\mathcal{C}_{M,G}(R)$, so for finite G we have a pairing

$$\mathcal{C}_{pt}(RG) \times \mathcal{G}_{M,G}(R) \longrightarrow \mathcal{C}_{M,G}(R)$$

Using the fact that $(A \otimes B)^* = A^* \otimes B^*$ for finitely generated R-modules, it follows that this commutes with the pairings, so it follows from [56] that there is a pairing

$$L_n(RG) \otimes L^k(\mathcal{G}_{M,G}(R)) \longrightarrow L_{n+k}(\mathcal{C}_{M,G}(R))$$

between quadratic and symmetric bounded L-groups, geometrically corresponding to the tensor product of chain complexes.

13 Mackey properties

Let M be a metric space and G a finite group acting on M by eventual Lipschitz maps, R a commutative ring with unit. Consider the category

$\mathcal{C}_{M,G}(R)$. Given two subgroups $G_1 \subset G_2 \subset G$ we have G_1 and G_2 acting on M by restriction and there are restriction functors $\mathcal{C}_{M,G_2}(R) \to \mathcal{C}_{M,G_1}(R)$ and induction functors $\mathcal{C}_{M,G_1}(R) \to \mathcal{C}_{M,G_2}(R)$. The restriction functor is obtained just by restriction of the group action, and the induction functor sends an object A to $RG_2 \otimes_{RG_1} A$. The required map from $RG_2 \otimes_{RG_1} A$ to the finite subsets of M is extended from the map of A to the finite subsets of M by equivariance: let $f(g \otimes a) = g \cdot f(a)$. Clearly restriction and induction are functors. We need

Lemma 13.1. *Restriction and induction are functors of categories with involution.*

Proof. The involution is given by $A^* = \mathrm{Hom}^{l.f.}(A, R)$ turned into a right RG-module as described above, and it does not matter whether we restrict before or after applying $\mathrm{Hom}^{l.f.}$. Also

$$\begin{aligned}\mathrm{Hom}^{l.f.}(RG_2 \otimes_{RG_1} A, R) &= \mathrm{Hom}(RG_2, \mathrm{Hom}^{l.f.}(A, R)) \\ &= RG_2 \otimes_{RG_1} \mathrm{Hom}^{l.f.}(A, R)\end{aligned}$$

□

Given two functors between additive categories with involution, we may form a new functor, the direct sum of the two functors. It is easy to see that

Lemma 13.2. *A functor between additive categories with involution induces a map of L-groups. The sum of two functors induces the sum of the two maps.*

Proof. Direct from the definitions since L-groups are defined as a bordism theory where direct sum is turned into addition [54]. □

Consider the category $A(G)$ defined as follows. The objects are the subgroups of G, and the $\mathrm{Hom}(H_1, H_2)$ is the Grothendieck construction applied to the collection of finite "free bi-sets" (these are just finite sets Z with free left H_2 action and free right H_1-action) where the addition is disjoint union. The balanced product

$$({}_{H_3}Z_{H_2}) \times_{H_2} ({}_{H_2}Y_{H_1})$$

is a free biset and can be easily shown to induce a composition $\mathrm{Hom}(H_1, H_2) \times \mathrm{Hom}(H_2, H_3) \to \mathrm{Hom}(H_1, H_3)$ which is bilinear. The set H as an $H - H$

biset is the identity element for $\mathrm{Hom}(H, H)$.
There is a functor $Gr(G) \to A(G)$ from the category of subgroups of G (morphisms are $\mathrm{Maps}(H_1, H_2) = \{g \in G \,|\, gH_1g^{-1} \subset H_2\}$). It is the identity on the objects and sends $g \in \mathrm{Maps}(H_1, H_2)$ to the equivalence class of H_2 considered as a left H_2 set in the obvious manner, and $h_2h_1 = h_2gh_1g^{-1}$ for all $h_1 \in H_1$, and all $h_2 \in H_2$. This is a Mackey functor, and any functor out of $A(G)$ to an additive category yields a Mackey functor by composition. It follows that

Theorem 13.3. *Given a finite group G and a metric space M as above, then $\mathcal{C}_{M,?}(R)$ is a Mackey functor, and hence $L_n(\mathcal{C}_{M,?}(R))$ is a Mackey functor.*

Remark 13.4. We suppress the upper index in the L-groups in the above statement. The point is that the upper index has to be a subgroup of a K-theoretic group which is in itself a Mackey functor e.g. the whole group or the trivial subgroup, but also naturally defined image groups will work.

Proof. Given an $H_1 - H_2$ biset Z then sending A to $RZ \otimes_{RH_2} A$ and extending the reference map by equivariance defines a functor from $\mathcal{C}_{X,H_2}(R)$ to $\mathcal{C}_{X,H_1}(R)$. □

14 Non-linear similarity

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically equivalent* to V' (denoted $V \sim_t V'$) if there exists a homeomorphism $h\colon V \to V'$ which is G-equivariant. If V, V' are topologically equivalent, but not linearly isomorphic, then such a homeomorphism is called a non-linear similarity. These notions were introduced and studied by de Rham [57], [58], and developed extensively in [7], [8], [34], [38], and [11].

Recently, Erik Pedersen and I have completed de Rham's program by showing that Reidemeister torsion invariants and number theory determine non-linear similarity for finite cyclic groups. I will describe some of our results in this section. The new ingredient is the use of "bounded surgery" techniques.

A G-representation is called *free* if each element $1 \neq g \in G$ fixes only the zero vector. Every representation of a finite cyclic group has a unique maximal free subrepresentation.

Theorem 14.1 ([26]). *Let G be a finite cyclic group and V_1, V_2 be free G–represen- tations. For any G–representation W, the existence of a non-linear similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is entirely determined by explicit congruences in the weights of the free summands V_1, V_2, and the ratio $\Delta(V_1)/\Delta(V_2)$ of their Reidemeister torsions, up to an algebraically described indeterminacy.*

This is just a general formulation, intended to give an overview of the answer. Precise statements of our results are given in [26]. For example, for cyclic groups of 2-power order, we obtain a complete classification of non-linear similarities.

Two fundamental result on the problem were proved in the 1980's by Cappell-Shaneson [7], Hsiang-Pardon [34], and Madsen-Rothenberg [38].

Theorem 14.2 (Cappell-Shaneson). *Non-linear similarities $V \sim_t V'$ exist for cyclic groups $G = C(4q)$ of every order $4q \geqq 8$.*

Theorem 14.3 (Hsiang-Pardon, Madsen-Rothenberg). *If $G = C(q)$, or $G = C(2q)$, for q odd, topological equivalence of G–representations implies linear equivalence.*

This is called the Odd Order Theorem (the same conclusion also holds for $G = C(4)$, but this is easy). Since linear G-equivalence for general finite groups G is detected by restriction to cyclic subgroups, it is reasonable to study this case first. For the rest of this section, unless otherwise mentioned, G denotes a finite cyclic group.

Further positive results can be obtained by imposing assumptions on the isotropy subgroups allowed in V and V'. For example, de Rham [57] proved in 1935 that piecewise linear similarity implies linear equivalence for free G-representations, by using Reidemeister torsion and the Franz Independence Lemma. Topological invariance of Whitehead torsion shows that his method also rules out non–linear similarity in this case. In [25, Thm.A] we studied "first–time" similarities, where $\operatorname{Res}_K V \cong \operatorname{Res}_K V'$ for all proper subgroups $K \subsetneqq G$, and showed that topological equivalence implies linear equivalence if V, V' have no isotropy subgroup of index 2. This result is an application of bounded surgery theory (see [24], [25, §4]), and provides a more conceptual proof of the Odd Order Theorem. These techniques are extended in [26] to provide a necessary and sufficient condition for non-linear similarity in terms of the vanishing of a bounded transfer map. This gives a new approach to de Rham's problem.

An interesting question in non–linear similarity concerns the minimum possible dimension for examples. It is easy to see that the existence of a non–linear similarity $V \sim_t V'$ implies $\dim V = \dim V' \geqq 5$. Cappell, Shaneson, Steinberger and West proved:

Theorem 14.4 ([11]). *Non-linear similarity starts in dimension* 6 *for* $G = C(2^r)$*, with* $r \geqq 4$.

A 1981 Cappell-Shaneson preprint [10] shows that 5–dimensional similarities do not exist for any finite group.

In [8], Cappell and Shaneson initiated the study of *stable* topological equvalence for G–representations. We say that V_1 and V_2 are stably topologically similar ($V_1 \approx_t V_2$) if there exists a G–representation W such that $V_1 \oplus W \sim_t V_2 \oplus W$. Let $R_{\mathrm{Top}}(G) = R(G)/R_t(G)$ denote the quotient group of the real representation ring of G by the subgroup $R_t(G) = \{[V_1] - [V_2] \mid V_1 \approx_t V_2\}$. In [8], $R_{\mathrm{Top}}(G) \otimes \mathbf{Z}[1/2]$ was computed, and the torsion subgroup was shown to be 2–primary. As an application of our general results, we determine the structure of the torsion in $R_{\mathrm{Top}}(G)$, for G any cyclic group. In Theorem 14.11 we give the calculation of $R_{\mathrm{Top}}(G)$ for $G = C(2^r)$. This is the first complete calculation of $R_{\mathrm{Top}}(G)$ for any group that admits non–linear similarities.

In order to state a sample of the results from [26] precisely, we need some notation. Let $G = C(4q)$, where $q > 1$, and let $H = C(2q)$ denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t \rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbf{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbf{R}_-$.

A free G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$, denote the complex numbers $\mathbf{C}$ with action $t \cdot z = \zeta^a z$ for all $z \in \mathbf{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \bmod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The Reidemeister torsion invariant of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k} (t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\} \ .$$

Let $V_2 = t^{b_1} + \cdots + t^{b_k}$ be another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent. This just means that the products of the weights $\prod a_i \equiv \prod b_i \bmod 4q$. Then the Whitehead torsion of any G-homotopy equivalence is determined by the element

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$$

since $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Q}G)$ is monic [47, p.14]. When there exists a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ which is normally cobordant to the identity map on $S(V_1)$, we say that $S(V_1)$ and $S(V_2)$ are *normally cobordant.* More generally, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant* if $S(V_1 \oplus U)$ and $S(V_2 \oplus U)$ are normally cobordant for all free G–representations U. This is a necessary condition for non-linear similarity, which can be decided by explicit congruences in the weights.

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. It represents a unit in the group ring $\mathbf{Z}G$, explicitly described for $G = C(2^r)$ by Cappell and Shaneson in [9, §1] using a pull-back square of rings. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbf{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbf{Z}/2; A)$ of $\mathbf{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbf{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbf{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod(t^{a_i}-1)\prod((-t)^{a_i}-1)}{\prod(t^{b_i}-1)\prod((-t)^{b_i}-1)} = \prod \frac{(t^2)^{a_i}-1}{((t^2)^{b_i}-1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbf{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))\ .$$

This calculation takes place over the ring $\Lambda_{2q} = \mathbf{Z}[t]/(1 + t^2 + \cdots + t^{4q-2})$, but the result holds over $\mathbf{Z}G$ via the involution–invariant pull-back square

$$\begin{array}{ccc} \mathbf{Z}G & \to & \Lambda_{2q} \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \to & \mathbf{Z}/2q[\mathbf{Z}/2] \end{array}$$

Consider the exact sequence of modules with involution:

(14.5) $$K_1(\mathbf{Z}H) \to K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G)$$

and define $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = K_1(\mathbf{Z}H \to \mathbf{Z}G)/\{\pm G\}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\,\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. Such an exact sequence of $\mathbf{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \ .$$

Our first result deals with isotropy groups of index 2, as is the case for the non–linear similarities constructed in [7].

Theorem 14.6 ([26, Thm. A]). *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, with $a_i \equiv b_i \equiv 1 \bmod 4$. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,

(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*

(iii) the element $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$ *is in the image of the coboundary* $\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$.

Remark 14.7. More general isotropy is handled in the other results of [26]. Theorem 14.6 should be compared with [7, Cor.1], where more explicit conditions are given for "first-time" similarities of this kind under the assumption that q is odd, or a 2-power, or $4q$ is a "tempered" number.

The case $\dim V_1 = \dim V_2 = 4$ gives a reduction to number theory for the existence of 5-dimensional similarities.

We turn now to results on the structure of $R_{\mathrm{Top}}(G)$. There is a filtration

(14.8) $$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G)$$

on the real representation ring $R(G)$, inducing a filtration on

$$R_{\mathrm{Top}}(G) = R(G)/R_t(G) \ .$$

Here $R_h(G)$ consists of those virtual elements with no homotopy obstruction to similarity, and $R_n(G)$ the virtual elements with no normal invariant obstruction to similarity Note that $R(G)$ has the nice basis $\{t^i, \delta, \epsilon \mid 1 \leqq i \leqq 2q-1\}$, where $\delta = [\mathbf{R}_-]$ and $\epsilon = [\mathbf{R}_+]$.

Let $R^{free}(G) = \{t^a \mid (a, 4q) = 1\} \subset R(G)$ be the subgroup generated by the free representations. To complete the definition, we let $R^{free}(C(2)) = \{\mathbf{R}_-\}$ and $R^{free}(e) = \{\mathbf{R}_+\}$. Then

$$R(G) = \bigoplus_{K \subseteq G} R^{free}(G/K)$$

and this direct sum splitting can be intersected with the filtration above to define $R_h^{free}(G)$, $R_n^{free}(G)$ and $R_t^{free}(G)$. In addition, we can divide out $R_t^{free}(G)$ and obtain subgroups $R_{h,\mathrm{Top}}^{free}(G)$ and $R_{n,\mathrm{Top}}^{free}(G)$ of $R_{\mathrm{Top}}^{free}(G) = R^{free}(G)/R_t^{free}(G)$. By induction on the order of G, we see that it suffices to study the summand $R_{\mathrm{Top}}^{free}(G)$.

Let $\widetilde{R}^{free}(G) = \ker(\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$, and then project into $R_{\mathrm{Top}}(G)$ to define

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \widetilde{R}^{free}(G)/R_t^{free}(G)\ .$$

Theorem 14.9. *The torsion subgroup of $R_{\mathrm{Top}}^{free}(G)$ is precisely $\widetilde{R}_{\mathrm{Top}}^{free}(G)$, and the subquotient $\widetilde{R}_{n,\mathrm{Top}}^{free}(G) = \widetilde{R}_n^{free}(G)/R_t^{free}(G)$ always has exponent two.*

Here is a specific computation.

Theorem 14.10 ([26, Thm. D]). *Let $G = C(4q)$, with $q > 1$ odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then $\widetilde{R}_{\mathrm{Top}}^{free}(G) = \mathbf{Z}/4$ generated by $(t - t^{1+2q})$.*

For any cyclic group G, both $R^{free}(G)/R_h^{free}(G)$ and $R_h^{free}(G)/R_n^{free}(G)$ are torsion groups which can be explicitly determined by congruences in the weights.

We conclude this list of sample results with a calculation of $R_{\mathrm{Top}}(G)$ for cyclic 2-groups.

Theorem 14.11 ([26, Thm. E]). *Let $G = C(2^r)$, with $r \geqq 4$. Then*

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \big\langle \alpha_1, \alpha_2, \ldots, \alpha_{r-2}, \beta_1, \beta_2, \ldots, \beta_{r-3} \big\rangle$$

subject to the relations $2^s\alpha_s = 0$ for $1 \leqq s \leqq r-2$, and $2^{s-1}(\alpha_s + \beta_s) = 0$ for $2 \leqq s \leqq r-3$, together with $2(\alpha_1 + \beta_1) = 0$.

The generators for $r \geqq 4$ are given by the elements

$$\alpha_s = t - t^{5^{2^{r-s-2}}} \quad \text{and} \quad \beta_s = t^5 - t^{5^{2^{r-s-2}}+1}.$$

We remark that $\widetilde{R}^{free}_{\mathrm{Top}}(C(8)) = \mathbf{Z}/4$ generated by $t - t^5$.

Our approach to the non-linear similarity problem starts with an elementary observation about topological equivalences for cyclic groups.

Lemma 14.12. *If $V_1 \oplus W \sim_t V_2 \oplus W'$, where V_1, V_2 are free G–representations, and W and W' have no free summands, then there is a G-homeomorphism $h\colon V_1 \oplus W \to V_2 \oplus W$ such that*

$$h| \bigcup_{1 \neq H \leq G} W^H$$

is the identity.

Proof. Let h be the homeomorphism. We will successively change h, stratum by stratum. For every subgroup K of G, consider the homeomorphism of K-fixed sets

$$f^K\colon W^K \to W'^K.$$

This is a homeomorphism of G/K, hence of G–representations. As G–representations we can split

$$V_2 \oplus W' = U \oplus W'^K \sim_t U \oplus W^K = V_2 \oplus W''$$

where the similarity uses the product of the identity and $(f^K)^{-1}$. Notice that the composition of f with this similarity is the *identity* on the K-fixed set. Rename W'' as W' and repeat this successively for all subgroups. We end up with $W = W'$ and a homeomorphism as claimed. □

One consequence is

Lemma 14.13. *If $V_1 \oplus W \sim_t V_2 \oplus W$ then there exists a G-homotopy equivalence $S(V_2) \to S(V_1)$.*

Proof. If we 1-point compactify h we obtain a G-homeomorphism

$$h^+\colon S(V_1 \oplus W \oplus \mathbf{R}) \to S(V_2 \oplus W \oplus \mathbf{R}).$$

After an isotopy, the image of the free G-sphere $S(V_1)$ may be assumed to lie in the complement $S(V_2 \oplus W \oplus \mathbf{R}) - S(W \oplus \mathbf{R})$ of $S(W \oplus \mathbf{R})$ which is G-homotopy equivalent to $S(V_2)$. □

Any homotopy equivalence $f\colon S(V_2)/G \to S(V_1)/G$ defines an element $[f]$ in the structure set $\mathcal{S}^h(S(V_1)/G)$. We may assume that $\dim V_i \geqq 4$. This element must be non-trivial: otherwise $S(V_2)/G$ would be topologically h-cobordant to $S(V_1)/G$, and Stallings infinite repetition of h-cobordisms trick would produce a homeomorphism $V_1 \to V_2$ contradicting [2], since V_1 and V_2 are free representations. More precisely, we use Wall's extension of the Atiyah–Singer equivariant index formula to the topological locally linear case [69]. If $\dim V_i = 4$, we can cross with $\mathbf{CP}^2$ to avoid low–dimensional difficulties. Crossing with W and parameterising by projection on W defines a map from the classical surgery sequence to the bounded surgery exact sequence

(14.14)

$$\begin{tikzcd}
L^h_n(\mathbf{Z}G) \arrow[r] \arrow[d] & \mathcal{S}^h(S(V_1)/G) \arrow[r] \arrow[d] & {[S(V_1)/G, \mathrm{F/Top}]} \arrow[d] \\
L^h_{n+k}(\mathcal{C}_{W,G}(\mathbf{Z})) \arrow[r] & \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times W/G \\ \downarrow \\ W/G \end{pmatrix} \arrow[r] & {[S(V_1)\times_G W, \mathrm{F/Top}]}
\end{tikzcd}$$

The L–groups in the upper row are the ordinary surgery obstruction groups for oriented manifolds and surgery up to homotopy equivalence. In the lower row, we have bounded L–groups corresponding to an orthogonal action $\rho_W\colon G \to O(W)$, with orientation character given by $\det(\rho_W)$. Our main criterion for non–linear similarities is:

Theorem 14.15. *Let V_1 and V_2 be free G–representations with $\dim V_i \geqq 2$, and suppose that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if the element $[f] \in \mathcal{S}^h(S(V_1)/G)$ is in the kernel of the bounded transfer map*

$$trf_W\colon \mathcal{S}^h(S(V_1)/G) \to \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times_G W \\ \downarrow \\ W/G \end{pmatrix}.$$

Proof. For necessity, we refer the reader to [25] where this is proved using a version of equivariant engulfing. For sufficiency, we notice that crossing with $\mathbf{R}$ gives an isomorphism of the bounded surgery exact sequences parameterized by W to simple bounded surgery exact sequence parameterized by $W \times \mathbf{R}$. By the bounded s-cobordism theorem, this means that the vanishing of the bounded transfer implies that

$$\begin{tikzcd} S(V_2) \times W \times \mathbf{R} \arrow[r, "f\times 1"] & S(V_1) \times W \times \mathbf{R} \arrow[d] \\ & W \times \mathbf{R} \end{tikzcd}$$

is within a bounded distance of an equivariant homeomorphism h, where distances are measured in $W \times \mathbf{R}$. We can obviously complete $f \times 1$ to the map

$$f * 1\colon S(V_2) * S(W \times \mathbf{R}) \to S(V_1) * S(W \times \mathbf{R})$$

and since bounded in $W \times \mathbf{R}$ means small near the subset

$$S(W \times \mathbf{R}) \subset S(V_i) * S(W \times \mathbf{R}) = S(V_i \oplus W \oplus \mathbf{R}),$$

we can complete h by the identity to get a homeomorphism

$$S(V_2 \oplus W \oplus \mathbf{R}) \to S(V_1 \oplus W \oplus \mathbf{R})$$

and taking a point out we have a homeomorphisms $V_2 \times W \to V_2 \times W$ $\square$

By comparing the ordinary and bounded surgery exact sequences (14.14), and noting that the bounded transfer induces the identity on the normal invariant term, we see that a necessary condition for the existence of any stable similarity $f\colon V_2 \approx_t V_1$ is that $f\colon S(V_2) \to S(V_1)$ has s–normal invariant zero. Assuming this, under the natural map

$$L_n^h(\mathbf{Z}G) \to \mathcal{S}^h(S(V_1)/G),$$

where $n = \dim V_1$, the element $[f]$ is the image of $\sigma(f) \in L_n^h(\mathbf{Z}G)$, obtained as the surgery obstruction (relative to the boundary) of a normal cobordism from f to the identity. The element $\sigma(f)$ is well-defined in $\tilde{L}_n^h(\mathbf{Z}G) = \mathrm{Coker}(L_n^h(\mathbf{Z}) \to L_n^h(\mathbf{Z}G))$. Since the image of the normal invariants

$$[S(V_1)/G \times I, S(V_1)/G \times \partial I, \mathrm{F/Top}] \to L_n^h(\mathbf{Z}G)$$

factors through $L_n^h(\mathbf{Z})$ (see [23, Thm.A, 7.4] for the image of the assembly map), we may apply the criterion of 14.15 to any lift $\sigma(f)$ of $[f]$. This reduces the evaluation of the bounded transfer on structure sets to a bounded L–theory calculation.

Theorem 14.16. *Let V_1 and V_2 be free G–representations with* $\dim V_i \geqq 2$, *and suppose that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence which is G–normally cobordant to the identity. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if $trf_W(\sigma(f)) = 0$, where $trf_W\colon L^h_n(\mathbf{Z}G) \to L^h_{n+k}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is the bounded transfer.*

The main work of [26] is to establish methods for effective calculation of the bounded transfer in the presence of isotropy groups of arbitrary index.

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Exotic Aspherical Manifolds

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Contents

1 The geometric realization of a simplicial complex

A *simplicial complex* L consists of a set I (called the *vertex set*) and a collection of finite subsets $\mathcal{S}(L)$ of I such that

- $\emptyset \in \mathcal{S}(L)$
- for each $i \in I$, $\{i\} \in \mathcal{S}(L)$, and
- if $\sigma \in \mathcal{S}(L)$ and $\tau < \sigma$, then $\tau \in \mathcal{S}(L)$.

An element σ of $\mathcal{S}(L)$ is called a *simplex*; its *dimension* is defined by $\dim \sigma = \mathrm{Card}(\sigma) - 1$.

Let us assume that $I = \{1, 2, \dots, m\}$. The *standard* $(m-1)$*-simplex* on I, denoted by Δ^{m-1}, is the convex polytope in $\mathbb{R}^m$ defined by intersecting the positive quadrant (defined by $x_i \geq 0$ for all $i \in I$) with the hyperplane $\Sigma x_i = 1$. A vertex of Δ^{m-1} is an element e_i of the standard basis for $\mathbb{R}^m$. The poset of faces of Δ^{m-1}, denoted $\mathcal{F}(\Delta^{m-1})$, is isomorphic to the poset of all nonempty subsets of I. This gives us a simplicial complex which we will also denote by Δ^{m-1}. (Usual practice is to blur the distinction between a simplicial complex and its geometric realization.)

If σ is a nonempty subset of I $(= \{1, \dots m\})$, then let Δ_σ denote the face of Δ^{m-1} spanned by $\{e_i\}_{i \in \sigma}$.

If L is a simplicial complex with vertex set I, then its *geometric realization* is defined to be the union of all subspaces of Δ^{m-1} of the form Δ_σ for some $\sigma \in \mathcal{S}(L)$. The geometric realization will also be denoted L.

2 Cubical cell complexes

As before, $I = \{1, \dots, m\}$. The *standard* m*-dimensional cube* is the convex polytope $[-1,1]^m \subset \mathbb{R}^m$. For each subset σ of I let $\mathbb{R}^\sigma$ denote the linear subspace spanned by $\{e_i\}_{i\in\sigma}$ and let $\square_\sigma$ denote the standard cube in $\mathbb{R}^\sigma$. (If $\sigma = \emptyset$, then $\mathbb{R}^\emptyset = \square_\emptyset = \{0\}$.) The faces of $[-1,1]^m$ which are parallel to $\square_\sigma$ have the form $v + \square_\sigma$ for some vertex v of $[-1,1]^m$.

Next we want to describe the poset of nonempty faces of $[-1,1]^m$. For each $i \in I$, let $r_i : [-1,1]^m \to [-1,1]^m$ denote the orthogonal reflection across the hyperplane $x_i = 0$. Let J be the group of symmetries of $[-1,1]^m$ generated by $\{r_i\}_{i\in I}$. Then J is isomorphic to $(\mathbb{Z}/2)^m$. The group J acts simply transitively on the vertex set of $[-1,1]^m$. The stabilizer of a face $v + \square_\sigma$ is the subgroup J_σ generated by $\{r_i\}_{i\in\sigma}$. Hence, the poset of nonempty faces of [-1,1]^m is isomorphic to the poset of cosets

$$\coprod_{\sigma\subset I} J/J_\sigma.$$

Roughly, a cubical cell complex P is a regular cell complex in which each cell is combinatorially isomorphic to a standard cube of some dimension. More precisely, P consists of a poset $\mathcal{F}(=\mathcal{F}(P))$ such that for each $c\in\mathcal{F}$ the subposet $\mathcal{F}_{\leq c}$ is isomorphic to the poset of nonempty faces of [-1,1]^k; k is the *dimension* of c. (Here $\mathcal{F}_{\leq c}=\{x\in\mathcal{F}|x\leq c\}$.) The elements of $\mathcal{F}$ are called *cells*. A *vertex* of P is synonymous with a 0-dimensional cell. By definition, the *link* of a vertex v in P, denoted by $Lk(v,P)$, is the subposet $\mathcal{F}_{>v}$ of all cells which are strictly greater than v (i.e., which have v as a vertex.) For example, if v is a vertex of [-1,1]^m, then $Lk(v,$ [-1,1]$^m)$ is the simplicial complex Δ^{m-1}. It follows that the link of a vertex in any cubical cell complex is a simplicial complex.

The *geometric realization* of a cubical complex P can be defined by pasting together standard cubes, one for each element of $\mathcal{F}$. A neighborhood of a vertex v in (the geometric realization of) P is homeomorphic to the cone on $Lk(v,P)$.

3 The cubical complex P_L

Given a simplicial complex L with vertex set I, we shall now define a subcomplex P_L of [-1,1]^m. The vertex set of P_L will be the same as that of [-1,1]^m. The main property of P_L will be that the link of each of its vertices is isomorphic to L. The construction is very similar to the way in which we realized L as a subcomplex of Δ^{m-1}.

By definition, P_L is the union of all faces of [-1,1]^m which are parallel to $\square_\sigma$ for some $\sigma\in\mathcal{S}(L)$. Hence, the poset of cells of P_L can be identified with

$$\coprod_{\sigma\in\mathcal{S}(L)} J/J_\sigma.$$

(This construction is also described in [2] as well as in [12].)

Examples 3.1.

- If $L=\Delta^{m-1}$, then $P_L=$[-1,1]^m.
- If $L=\partial(\Delta^{m-1})$, then P_L is the boundary of an m-cube i.e., P_L is homeomorphic to S^{m-1}.

- If L is the disjoint union of m points, then P_L is the 1-skeleton of an m-cube.
- If $m = 3$ and L is the disjoint union of a 1-simplex and a point then P_L is the subcomplex of the 3-cube consisting of the top and bottom faces and the 4 vertical edges.

From the fact that a neighborhood of a vertex in P_L is homeomorphic to the cone on L we get the following.

Proposition 3.2. *If L is homeomorphic to S^{n-1}, then P_L is an n-manifold.*

Proof. The cone on S^{n-1} is homeomorphic to an n-disk. □

4 The universal cover of P_L and the group W_L

Let $\tilde{P}_L$ denote the universal cover of P_L. The cubical cell structure on P_L lifts to a cubical structure on $\tilde{P}_L$. The group J ($\cong (\mathbb{Z}/2)^m$) acts on P_L. Let W_L denote the group of all lifts of elements of J to $\tilde{P}_L$ and let $\varphi : W_L \to J$ be the homomorphism induced by the projection $\tilde{P}_L \to P_L$. We have a short exact sequence,

$$1 \to \pi_1(P_L) \to W_L \xrightarrow{\varphi} J \to 1.$$

We will use the notation: $\Gamma_L = \pi_1(P_L)$.

Since J acts simply transitively on the vertex set of P_L, the group W_L acts simply transitively on the vertex set of $\tilde{P}_L$. It follows that the 2-skeleton of $\tilde{P}_L$ is the Cayley 2-complex associated to a presentation of W_L. (In particular, the 1-skeleton is the Cayley graph associated to a set of generators.) Next, we use this observation to write down a presentation for W_L.

The vertex set of P_L can be identified with J. Fix a vertex v of P_L (corresponding to the identity element in J). Let $\tilde{v}$ be a lift of v in $\tilde{P}_L$. The 1-cells at v or at $\tilde{v}$ correspond to vertices of L, i.e., to elements of $\{1, \dots, m\}$. The reflection r_i stabilizes the i^{th} 1-cell at v. Let s_i denote the unique lift of r_i which stabilizes the i^{th} 1-cell at $\tilde{v}$. Since s_i^2 fixes $\tilde{v}$ and covers the identity on P_L, it follows that $s_i^2 = 1$. Suppose σ is a 1-simplex of L connecting vertices i and j. The corresponding 2-cell at $\tilde{v}$ is then a square with edges labelled successively by s_i, s_j, s_i, s_j. Hence, we get a relation $(s_i s_j)^2 = 1$ for each 1-simplex $\{i, j\}$ of L.

Let $\hat{W}_L$ denote the group defined by this presentation, i.e., a set of generators for $\hat{W}_L$ is $\{s_i\}_{i \in I}$ and the relations are given by: $s_i^2 = 1$, for all $i \in I$, and $(s_i s_j)^2 = 1$ whenever $\{i, j\}$ is a 1-simplex of L. The Cayley 2-complex associated to this presentation maps to the 2-skeleton of $\tilde{P}_L$ by a covering projection. Since $\tilde{P}_L$ is simply connected, this covering projection is a homeomorphism and the natural homomorphism $\hat{W}_L \to W_L$ is an isomorphism. Therefore, we have proved the following.

Proposition 4.1. *W_L has the presentation described above.*

Remarks 4.2.

- A group with a presentation of the above form is a Coxeter group, in fact, it is a "right-angled" Coxeter group. (In a general Coxeter group we allow relations of the form $(s_i s_j)^{m_{ij}} = 1$, where the integers m_{ij} can be > 2.)
- Examining the presentation, we see that the abelianization of W_L is J. Thus, Γ_L is the commutator subgroup of W_L.

For each subset σ of I, let W_σ denote the subgroup generated by $\{s_i\}_{i\in\sigma}$. If $\sigma \in \mathcal{S}(L)$, then W_σ is the stabilizer of the corresponding cube in $\tilde{P}_L$ which contains $\tilde{v}$ (and so, for $\sigma \in \mathcal{S}(L)$, $W_\sigma \cong (\mathbb{Z}/2)^{\mathrm{Card}(\sigma)}$). It follows that the poset of cells of $\tilde{P}_L$ is isomorphic to the poset of cosets,

$$\coprod_{\sigma\in\mathcal{S}(L)} W_L/W_\sigma .$$

5 When is $\tilde{P}_L$ contractible?

Definition 5.1. A simplicial complex L is a *flag complex* if any finite set of vertices which are pairwise connected by edges spans a simplex of L.

The following theorem is proved in [5]. Proofs can also be found in [8] or [12].

Theorem 5.2. *$\tilde{P}_L$ is contractible if and only if L is a flag complex.*

Definition 5.3. A path connected space X is *aspherical* if $\pi_i(X) = 0$, for all $i \geq 2$.

Suppose that a space X has the homotopy type of a CW complex. A basic fact of covering space theory is that the higher homotopy groups of X (that is, the groups $\pi_i(X)$, for $i \geq 2$) are isomorphic to those of its universal cover $\tilde{X}$. Thus, X is aspherical if and only if $\tilde{X}$ is contractible. Therefore, we have the following corollary to Theorem 5.2.

Corollary 5.4. *P_L is aspherical if and only if L is a flag complex.*

Before sketching the proof of Theorem 5.2 in Section 7, we make a few comments on the nature of flag complexes.

An *incidence relation* on a set I is a symmetric and reflexive relation R. A *flag* in I is defined to be a finite subset of I of pairwise incident elements. The poset $Flag(R)$ of nonempty flags in I is then a simplicial complex with vertex set I. It is obviously a flag complex. Conversely, any flag complex arises from this construction. (Indeed, given a flag complex L define two vertices to be incident if they are connected by an edge; if R denotes this incidence relation, then $L = Flag(R)$.)

Given a poset, we can symmetrize the partial order relation to get an incidence relation. A flag is then a finite chain of elements in the poset. If $\mathcal{P}$ is a poset, then let $Flag(\mathcal{P})$ denote the flag complex of chains in $\mathcal{P}$. If $\mathcal{F}$ is the poset of nonempty cells in a regular convex cell complex P, then $Flag(\mathcal{F})$ can be identified the poset of simplices in the barycentric subdivision of P. It follows that the condition of being a flag complex does not restrict the topological type of L: it can be any polyhedron.

Example 5.5. A *polygon* is a simplicial complex homeomorphic to S^1. It is a *k-gon* if it has k edges. A k-gon is a flag complex if and only if $k > 3$.

Remarks 5.6.

- Gromov in [16] has used the terminology that a simplicial complex L satisfies the "no Δ condition" to mean that it is a flag complex. The idea is that L is a flag complex if and only if it has no "missing simplices".
- A flag complex is a simplicial complex which is, in a certain sense, "determined by its 1-skeleton". Indeed, suppose Λ is a 1-dimensional simplicial complex. Then Λ determines an incidence relation on its vertex set. The associated flag complex L is constructed by filling in the missing simplices corresponding to the complete subgraphs of Λ. (So Λ is the 1-skeleton of L.)
- The graph Λ also provides the data for a presentation of a right-angled Coxeter group W $(= W_\Lambda)$. In this case, the associated flag complex L is called the *nerve* of the Coxeter group.

6 Nonpositive curvature

The notion of "nonpositive curvature" makes sense for a more general class of metric spaces than Riemannian manifolds. A *geodesic* in a metric space

X is a path $\gamma : [a,b] \to X$ which is an isometric embedding. X is called a *geodesic space* if any two points can be connected by a geodesic segment. A *triangle* in a geodesic space X is the image of three geodesic segments meeting at their endpoints. Given a triangle T in X, there is a triangle T^* in $\mathbb{R}^2$ with the same edge lengths. T^* is called a *comparison triangle* for T. To each point $x \in T$ there is a corresponding point $x^* \in T^*$. The triangle T is said to *satisfy the* $CAT(0)$*-inequality*, if given any two points $x, y \in T$ we have $d(x,y) \leq d(x^*, y^*)$. The space X is *nonpositively* curved if the $CAT(0)$-inequality holds for all sufficiently small triangles. X is a $CAT(0)$*-space* (or a *Hadamard space*) if it is complete and if the $CAT(0)$-inequality holds for all triangles in X. It follows immediately from the definitions that there is a unique geodesic between any two points in a $CAT(0)$-space and from this that any $CAT(0)$-space is contractible. Gromov observed that the universal cover of a complete nonpositively curved geodesic space is $CAT(0)$. Hence, any such nonpositively curved space is aspherical. (An excellent reference for this material is [2].)

Next, suppose that P is a connected cubical cell complex. There is a natural piecewise Euclidean metric on P. Roughly speaking, it is defined by declaring each cell of P to be (locally) isometric to a standard Euclidean cube (of edge length 2). More precisely, the distance between two points $x, y \in P$ is defined to be the infimum of the lengths of all piecewise linear curves connecting x to y. It then can be shown that P is a complete geodesic space. Gromov [16, p. 122] proved the following.

Lemma 6.1 (Gromov's Lemma). *A cubical cell complex P is nonpositively curved if and only if the link of each of its vertices is a flag complex.*

A corollary is that the previously constructed cubical complex $\tilde{P}_L$ is $CAT(0)$ if and only if L is a flag complex. In particular, this gives a proof in one direction of Theorem 5.2: if L is a flag complex, then $\tilde{P}_L$ is contractible. (A proof of Gromov's Lemma can be found in [2], [8], or [12].)

7 Another construction of P_L and $\tilde{P}_L$

The group J $(= (\mathbb{Z}/2)^m)$ acts as a group generated by reflection on $[-1,1]^m$. The subspace $[0,1]^m$ is a fundamental domain. (Also, $[0,1]^m$ can be identified with the orbit space of the J-action.) Let e be the vertex $(1, \ldots, 1) \in [0,1]^m$ and for each subset σ of I $(= \{1, \ldots, m\})$ let $\square_\sigma^+ = (e + \square_\sigma) \cap [0,1]^m$.

The corresponding fundamental domain K for the J-action on P_L is given by

$$K = P_L \bigcap [0,1]^m.$$

Thus,

$$K = \bigcup_{\sigma \in \mathcal{S}(L)} \square_\sigma^+$$

For each $i \in I$ let $[0,1]_i^m$ denote the intersection of the fixed point set of r_i with $[0,1]^m$, i.e., $[0,1]_i^m = [0,1]^m \cap \{x_i = 0\}$. Set $K_i = K \cap [0,1]_i^m$ and call it a *mirror* of K. It is not difficult to see that $\bigcup K_i$ can be identified with the barycentric subdivision of L, that K_i is the closed star of the vertex i in this barycentric subdivision and that K is homeomorphic to the cone on L.

For each $x \in K$, let $\sigma(x) = \{i \in I | x \in K_i\}$. The space P_L can be constructed by pasting together copies of K, one for each element of J. More precisely, define an equivalence relation $\sim$ on $J \times K$ by $(g, x) \sim (g', x')$ if and only if $x = x'$ and $g^{-1}g' \in J_{\sigma(x)}$. (In other words, $\sim$ is the equivalence relation generated by identifying $g \times K_i$ with $gr_i \times K_i$, for all $i \in I$ and $g \in J$.) Then

$$P_L \cong (J \times K)/\sim .$$

Similarly,

$$\tilde{P}_L \cong (W_L \times K)/\sim$$

where the equivalence relation on $W_L \times K$ is defined in an analogous fashion.

Let $\hat{L}$ denote the flag complex determined by the 1-skeleton of L. Thus, $\hat{L}$ is the nerve of $W_L : \sigma \in \mathcal{S}(\hat{L})$ if and only if W_σ is finite. For each $\sigma \in \mathcal{S}(\hat{L})$, set

$$K^\sigma = \bigcup_{i \in \sigma} K_i$$

We are now in position to prove two lemmas which imply Theorem 5.2.

Lemma 7.1. *The following conditions are equivalent.*

(i) L is a flag complex.

(ii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is contractible.

(iii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is acyclic.

Proof. If L is a flag complex, then $L = \hat{L}$ and K^σ can be identified with a closed regular neighborhood of σ in the barycentric subdivision of L. Hence, (i)$\Rightarrow$ (ii). The implication (ii)$\Rightarrow$(iii) is obvious. If $\sigma \in \mathcal{S}(\hat{L})$ is a k-simplex such that $\partial\sigma \subset L$ but $\sigma \not\subset L$, then K^σ has the homology of a $(k-1)$-sphere (since each K_i, $i \in \sigma$, as well as each proper family of intersections of the K_i is contractible). Hence, (iii)$\Rightarrow$(i). $\square$

For each $w \in W$, let $\sigma(w) = \{i \in I | \ell(ws_i) < \ell(w)\}$. Here $\ell : W \to \mathbb{N}$ denotes word length. Thus, $\sigma(w)$ is the set of letters with which a minimal word for w can end. Geometrically, it indexes the set of mirrors of wK such that the adjacent chamber ws_iK is one chamber closer to the base chamber K. The following is a basic fact about Coxeter groups (its proof, which can be found in [5], is omitted).

Lemma 7.2. *For each $w \in W_L$, $\sigma(w) \in \mathcal{S}(\hat{L})$. (In other words, for any w, $\sigma(w)$ generates a finite subgroup of W_L.)*

Now we can sketch the proof of Theorem 5.2: that $\tilde{P}_L$ is contractible if and only if L is a flag complex.

Proof of Theorem 5.2. First note that $\tilde{P}_L$ is simply connected. Hence, it suffices to compute the homology of $\tilde{P}_L$. Order the elements of $W_L : w_1, w_2, \dots,$ so that $\ell(w_{k+1}) \geq \ell(w_k)$. Set $Y_k = w_1K \cup \dots \cup w_kK$. Then $w_kK \cap Y_{k-1} \cong K^{\sigma(w_k)}$. Hence, $H_*(Y_k, Y_{k-1}) \cong H_*(K, K^{\sigma(w_k)}) \cong \tilde{H}_{*-1}(K^{\sigma(w_k)})$, where the second isomorphism holds since K is contractible. The exact sequence of the pair (Y_k, Y_{k-1}) gives

$$\to H_*(Y_{k-1}) \to H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$$

The map $H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$ is a split surjection. Indeed, a splitting $\varphi_* : H_*(K, K^{\sigma(w_k)}) \to H_*(Y_k)$ can be defined by the formula:

$$\varphi(\alpha) = \sum_{u \in W_{\sigma(w_k)}} (-1)^{\ell(u)} w_k u \alpha$$

where α is a relative cycle in $C_*(K, K^{\sigma(w_k)})$. It is then easy to see that $\varphi(\alpha)$ is a cycle and that the induced map on homology is a splitting. Thus, $H_*(Y_k) \cong H_*(Y_{k-1}) \oplus H_*(K, K^{\sigma(w_k)})$, and consequently,

$$\tilde{H}_*(\tilde{P}_L) \cong \bigoplus_{i=1}^{\infty} \tilde{H}_{*-1}(K^{\sigma(w_k)}).$$

Thus, $\tilde{P}_L$ is acyclic if and only if each $K^{\sigma(w_k)}$ is acyclic. The theorem follows from Lemmas 7.1 and 7.2. $\square$

Henceforth, we shall always assume that L is a flag complex. Moreover, we shall use the notation Σ_L instead of $\tilde{P}_L$.

8 The reflection group trick

Next we modify the construction of the previous section.

Suppose X is a space and that ∂X is a subspace which is homeomorphic to a polyhedron. Let L be a triangulation of ∂X as a flag complex with vertex set I. For each $i \in I$, put $X_i = K_i$ where K_i is the previously defined subcomplex of the barycentric subdivision of L. In other words, X_i is the closed star of the vertex i in the barycentric subdivision of L $(= \partial X)$. Let J, W_L and Γ_L be the groups defined previously. Set

$$P_L(X) = (J \times X)/\sim$$

and

$$\Sigma_L(X) = (W_L \times X)/\sim,$$

where the equivalence relations are defined exactly as before.

We record a few elementary properties of this construction. The orbit space of the J-action on $P_L(X)$ is X. Let $r : P_L(X) \to X$ be the orbit map. Since X can also be regarded as a subspace of $P_L(X)$ (namely as the image of $1 \times X$), we have the following theorem.

Theorem 8.1. *The map $r : P_L(X) \to X$ is a retraction.*

Corollary 8.2. *$\pi_1(X)$ is a retract of $\pi_1(P_L(X))$.*

Theorem 8.3. *If X is a compact n-dimensional manifold with boundary $(= \partial X)$, then $P_L(X)$ is a closed n-manifold.*

Proof. For each $x \in X$, let $\sigma(x) = \{i \in I | x \in X_I\}$. A neighborhood of x in ∂X has the form $\mathbb{R}^{n-k} \times \mathbb{R}_+^{\sigma(x)}$ where $k = \mathrm{Card}(\sigma(x))$, where $\mathbb{R}_+^{\sigma(x)}$ denotes the positive quadrant in $\mathbb{R}^{\sigma(x)}$ where all coordinates are nonnegative. It follows that a neighborhood of $(1, x)$ in $P_L(X)$ has the form $\mathbb{R}^{n-k} \times (J_{\sigma(x)} \times \mathbb{R}_+^{\sigma(x)})/\sim$ which is homeomorphic to $\mathbb{R}^n$. Thus, $P_L(X)$ is an n-manifold. $\square$

We also have that $\Sigma_L(X) \to P_L(X)$ is a regular covering with group of deck transformation Γ_L. The proof of the next proposition is the same as the proof of Theorem 5.2 given in Section 7.

Proposition 8.4. *(i) If X is simply connected, then $\Sigma_L(X)$ is the universal cover of $P_L(X)$ and hence, $\pi_1(P_L(X)) = \Gamma_L$.*

(ii) If X is contractible, then $\Sigma_L(X)$ is contractible.

Further discussion of the reflection group trick can be found in [5] and [7].

9 Aspherical manifolds not covered by Euclidean space

Suppose Y is a reasonable space (for example, suppose Y is a locally compact, locally path connected, second countable Hausdorff space). Also, suppose Y is not compact. A *neighborhood* of *infinity* in Y is the complement of a compact set. Y is *one-ended* if every neighborhood of infinity contains a connected neighborhood of infinity. A one-ended space Y is *simply connected at infinity* if for any compact subset $C \subset Y$ there is a larger compact subset $C' \supset C$ such that for any loop γ in $Y - C'$, γ is null-homotopic in $Y - C$.

For example, $\mathbb{R}^n$ is one-ended for $n \geq 2$ and simply connected at infinity for $n \geq 3$. The following characterization of Euclidean space was proved by Stallings for $n \geq 5$ and by Freedman for $n = 4$. For $n = 3$, the corresponding result is not known (it is a version of the 3-dimensional Poincaré Conjecture).

Theorem 9.1. (Stallings [26] and Freedman [15]) *Let M^n be a contractible n-manifold, $n \geq 4$. Then M^n is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity.*

In certain circumstances it is possible to define a "fundamental group at infinity" for a one-ended space Y. Suppose that $C_1 \subset C_2 \subset \dots$ is an exhaustive sequence of compact subsets (i.e., $Y = \bigcup C_i$). This gives an inverse system of fundamental groups, $\pi_1(Y - C_1) \leftarrow \pi_1,(Y - C_2) \leftarrow \dots$. In general, an inverse sequence of groups, $G_1 \leftarrow G_2 \leftarrow \dots$, is *Mittag-Leffler* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that the image of G_k in G_n is the same for all $k \geq f(n)$. The space Y is *semistable* if there is such an inverse system of fundamental groups which is Mittag-Leffler. If this condition holds for one such choice of inverse system, then it holds for all such choices and the resulting inverse limit is independent of the choice of base points. Hence, when Y is semistable, we can define its *fundamental group at infinity*, denoted by $\pi_1^\infty(Y)$, to be the inverse limit, $\varprojlim \pi_1(Y - C_k)$.

Definition 9.2. A closed manifold N^n is a *homology n-sphere* if $H_*(N^n) \cong H_*(S^n)$.

Poincaré originally conjectured that any homology 3-sphere was homeomorphic to S^3. However, he soon discovered a counterexample. It was S^3/G, where S^3 is regarded as the Lie group $SU(2)$ and where G is the binary icosahedral group (a subgroup of order 120). (In order to prove that S^3/G was not homeomorphic to the 3-sphere, Poincaré invented the concept of the fundamental group and proved that it was a topological invariant.) It turns out that, for $n \geq 3$, there are many other examples of homology spheres N^n which are not simply connected (however, note that $\pi_1(N^n)$ must be a perfect group).

For $n \geq 5$, the next result is an easy consequence of surgery theory. For $n = 4$, it was proved by Freedman [15]. (For $n = 4$, the result holds only in the category of topological manifolds.)

Theorem 9.3. *Let N^{n-1} be a homology $(n-1)$-sphere. Then there is a compact contractible n-manifold with boundary X such that $\partial X = N^{n-1}$.*

Now suppose that a flag complex L is a homology $(n-1)$-sphere and that X^n is a contractible n-manifold with $\partial X = L$. By Theorem 8.3 and Proposition 8.4, $P_L(X)$ is an aspherical n-manifold and its universal cover $\Sigma_L(X)$ is a contractible n-manifold.

Proposition 9.4. *Suppose, as above, that L is a homology $(n-1)$-sphere bounding a contractible n-manifold X. If L is not simply connected, then $\Sigma_L(X)$ is not simply connected at infinity.*

Proof. As in the proof of Theorem 5.2 at the end of Section 7, order the elements of W_L: $w_1, w_2, \ldots$ so that $\ell(w_{k+1}) \geq \ell(w_k)$ and put

$$Y_k = w_1 X \cup \cdots \cup w_k X.$$

Then Y_k is a contractible manifold with boundary. Let $\overline{Y}_k$ denote the complement of an open collared neighborhood of ∂Y_k in Y_k. Then it is easy to see that $\Sigma_L(X) - \overline{Y}_k$ is homotopy equivalent to ∂Y_k. Moreover, since $w_k X \cap Y_{k-1} \cong K^{\sigma(w_k)}$, which is an $(n-1)$-disk, it follows that ∂Y_k is the connected sum of k copies of ∂X $(= L)$. Hence, if $n \geq 3$, π_1 (∂Y_k) is the free product of k copies of $\pi_1(L)$ and the inverse system of fundamental groups is,

$$\pi_1(L) \leftarrow \pi_1(L) * \pi_1(L) \leftarrow \ldots$$

Thus, if $\pi_1(L)$ is not trivial, then neither is $\pi_1^\infty(\Sigma_L(X))$. □

As a corollary of Proposition 9.4 and the fact that there exist homology $(n-1)$-spheres with nontrivial fundamental groups for all $n \geq 4$, we have proved the following result of [5].

Theorem 9.5. *For each $n \geq 4$, there is a closed aspherical n-manifold (of the form $P_L(X)$) such that its universal cover is not homeomorphic to $\mathbb{R}^n$.*

10 The reflection group trick, continued

As before, suppose $(X, \partial X)$ is a pair of spaces and L is a triangulation of ∂X as a flag complex.

Theorem 10.1. *If X is aspherical, then so is $P_L(X)$.*

Proof. It suffices to show that the covering space $\Sigma_L(X)$ of $P_L(X)$ is aspherical. Order the elements of W_L as before and set $Y_k = w_1 X \cup \cdots \cup w_k X$. Then, Y_k is formed by gluing on a copy of X to Y_{k-1}. Since $w_k X \cap Y_{k-1}$ is contractible, Y_k is homotopy equivalent to $Y_{k-1} \vee X$ and hence, to $X \vee \cdots \vee X$ (k times). Since X is aspherical, so is Y_k. Since $\Sigma_L(X)$ is the increasing union of the Y_k, $\Sigma_L(X)$ is also aspherical. $\square$

Definition 10.2. A group π is of *type* F if its classifying space $B\pi$ has the homotopy type of a finite complex. ($B\pi$ is also called the "$K(\pi, 1)$-complex.")

If B is a finite CW complex, then we can "thicken" it to a compact manifold with boundary. This means that we can find a compact manifold with boundary X which is homotopy equivalent to B. The proof goes as follows. First, up to homotopy, we can assume that B is a finite simplicial complex. The next step is to piecewise linearly embed B in some Euclidean space $\mathbb{R}^n$. So, we can assume B is a subcomplex of some triangulation of $\mathbb{R}^n$. Finally, possibly after taking barycentric subdivisions, we can replace B by a regular neighborhood X in $\mathbb{R}^n$.

The "reflection group trick" can then be summarized as follows. Start with a group π of type F. After thickening we may assume that $B\pi$ is a compact manifold with boundary X. Triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold which retracts onto $B\pi$.

11 The Assembly Map Conjecture

Conjecture 11.1 (The Borel Conjecture). *Let $(M, \partial M)$ and $(M', \partial M')$ be aspherical manifolds with boundary and $f : (M, \partial M) \to (M', \partial M')$ a homotopy equivalence such that $f|_{\partial M}$ is a homeomorphism. Then f is homotopic rel ∂M to a homeomorphism.*

In the case where $\partial M = \emptyset$, the Borel Conjecture asserts that two closed aspherical manifolds with the same fundamental group are homeomorphic.

Suppose $(X, \partial X)$ is a pair of finite complexes with $\pi_1(X) = \pi$. Then $(X, \partial X)$ is a *Poincaré pair* of dimension n if there is a $\mathbb{Z}\pi$-module D which is isomorphic to $\mathbb{Z}$ as an abelian group and a homology class $\mu \in H_n(X, \partial X; D)$ so that for any $\mathbb{Z}\pi$-module A, cap product with μ defines an isomorphism: $H^i(X; A) \cong H_{n-i}(X, \partial X; D \otimes A)$. If $\partial X = \emptyset$, then X is a *Poincaré complex.* A group π of type F is a *Poincaré duality group of dimension* n (or a PD^n-*group* for short) if $B\pi$ is a Poincaré complex.

Conjecture 11.2 (The PD^n-group Conjecture). *Suppose π is a group of type F and that $(X, \partial X)$ is a Poincaré pair with X homotopy equivalent to $B\pi$ and with ∂X a manifold. Then $(X, \partial X)$ is homotopy equivalent rel ∂X to a compact manifold with boundary.*

A weak version of this conjecture replaces the word manifold by "ANR homology manifold."

In the absolute case, where $\partial X = \emptyset$, the PD^n-group Conjecture asserts that for any PD^n-group π of type F, $B\pi$ is homotopy equivalent to a closed manifold.

As is explained elsewhere in this volume, there is a surgery exact sequence:

$$\dots L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(X, \partial X) \to [(X, \partial X), (G/TOP, *)] \to L_n(\mathbb{Z}\pi)$$

where $\mathcal{S}(X, \partial X)$ denotes the structure set, G/TOP is a certain space, $\pi = \pi_1(X)$ and where $L_n(\mathbb{Z}\pi)$ denotes Wall's surgery group for $\mathbb{Z}\pi$. The surgery groups are 4-periodic. The homotopy groups of G/TOP are also 4-periodic and are given by the formula:

$$\pi_i(G/TOP) = \begin{cases} \mathbb{Z}, & \text{if } i \equiv 0 \pmod 4; \\ \mathbb{Z}/2, & \text{if } i \equiv 2 \pmod 4; \\ 0, & \text{otherwise} \end{cases}$$

Moreover, the 4-fold loop space, $\Omega_4(\mathbb{Z} \times G/TOP)$, is homotopy equivalent to $\mathbb{Z} \times G/TOP$. It follows that $\mathbb{Z} \times G/TOP$ defines a spectrum $\mathbb{L}$ and a generalized homology theory $H_*(X; \mathbb{L})$. It is almost, but not quite, true that

$$H_n(X; \mathbb{L}) \cong \bigoplus_k H_{n-4k}(X; \mathbb{Z}) \oplus H_{n-4k-2}(X; \mathbb{Z}/2).$$

Quinn has defined an "assembly map" $A_n : H_n(X; \mathbb{L}) \to L_n(\mathbb{Z}\pi)$ so that if we use Poincaré duality to identify $H_n(X; \mathbb{L})$ with $[(X, \partial X), (G/TOP, *)] \cong H^0(X, \partial X; \mathbb{L})$, then A_n is identified with the surgery obstruction map.

Conjecture 11.3 (The Assembly Map Conjecture). *Suppose π is a group of type F. Then the assembly map $A_* : H_*(B\pi; \mathbb{L}) \to L_*(\mathbb{Z}\pi)$ is an isomorphism.*

In dimensions ≥ 5, for a given group π of type F, it follows from the surgery exact sequence that the truth the Assembly Map Conjecture and the conjecture that the Whitehead group of π vanishes is equivalent to the truth of both the Borel Conjecture and (the weak version of) the PD^n-group Conjecture.

Conceivably, the Assembly Map Conjecture could be true for any torsion-free group π.

Theorem 11.4. *The Assembly Map Conjecture is true for the fundamental groups of all closed aspherical manifolds if and only if it is true for all groups of type F.*

Proof. We use the reflection group trick. Suppose π is a group of type F. Thicken $B\pi$ to a manifold with boundary X and triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold. Let $r : P_L(X) \to X$ be the retraction from Theorem 8.1 and let $G = \pi_1(P_L(X))$. We have the following commutative diagram:

$$\begin{array}{ccc} H_n(P_L(X);\mathbb{L}) & \to & L_n(\mathbb{Z}G) \\ i_* \uparrow\downarrow r_* & & i_* \uparrow\downarrow r_* \\ H_n(X;\mathbb{L}) & \to & L_n(\mathbb{Z}\pi). \end{array}$$

Hence, if the arrow on the top row is an isomorphism, so is the arrow on the bottom. $\square$

Remark 11.5. A similar argument shows that the Whitehead group vanishes for the fundamental groups of all closed aspherical manifolds if and only if it vanishes for all groups of type F.

Remark 11.6. What these arguments show is that, in dimensions ≥ 5, if we have a counterexample to the relative version of the Borel Conjecture or to the relative version of the PD^n-group Conjecture, then the reflection group trick will provide us with a counterexample in the absolute case. For example, suppose $f : (M, \partial M) \to (M', \partial M')$ is a counterexample to the Borel Conjecture. We might as well assume that $\partial M = \partial M' = L$ and that $f|_{\partial M} = \mathrm{id}$. Then f induces a homotopy equivalence $P_L(M) \to P_L(M')$ which is not homotopy equivalent to a homeomorphism. Similarly, if $(X, \partial X)$ is a counterexample to the PD^n-group Conjecture and $L = \partial X$, then $G = \pi_1(P_L(X))$ is a PD^n-group which is not the fundamental group of a closed aspherical manifold.

12 Aspherical manifolds which cannot be smoothed

Suppose $(X, \partial X)$ is a compact aspherical n-manifold with boundary and that ∂X is triangulable. Suppose further that the Spivak normal fibration of X does not reduce to a linear vector bundle. (In other words, a certain map $X \to BG$ does not lift to BO.) Apply the reflection group trick with $L = \partial X$. Since X is codimension 0 in $P_L(X)$ and since X is a retract of $P_L(X)$, the Spivak normal fibration of $P_L(X)$ does not reduce to linear vector bundle. Hence, $P_L(X)$ cannot be homotopy equivalent to a smooth manifold. In [9] J-C. Hausmann and I showed that there exist examples of such X for each $n \geq 13$, thereby proving the following result. (We will prove a stronger result in Section 16.)

Theorem 12.1. ([9]) *In each dimension ≥ 13, there is a closed aspherical manifold not homotopy equivalent to a smooth manifold.*

13 Further applications of the reflection group trick

The next two results were proved by G. Mess.

Theorem 13.1. (Mess [21]). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which is not residually finite.*

(Recall that a group Γ is *residually finite* if given any two elements $\gamma_1, \gamma_2 \in \Gamma$ there is a homomorphism ϕ to some finite group F such that $\phi(\gamma_1) \neq \phi(\gamma_2)$.)

Theorem 13.2. (Mess [21]). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which contains an infinitely divisible abelian group.*

On the other hand, it is known that there are no such examples in dimension 3.

Remark 13.3. With regard to the first theorem, Tom Farrell pointed out to me that Raganuthan has observed that there are cocompact lattices in (nonlinear) Lie groups which are not virtually torsion-free and not residually finite. (See [23].) On the other hand, it follows from Selberg's Lemma that any finitely generated subgroup of a linear Lie group is residually finite.

The proofs of both theorems are similar. By a theorem of R. Lyndon [20], if π is a finitely generated 1-relator group and if the relation cannot be written as a proper power of another word, then the presentation 2-complex for π is aspherical. In particular, any such π is of type F with a 2-dimensional $B\pi$. This 2-complex can then be thickened to a compact 4-manifold. For

Theorem 13.1 take π to be the Baumslag-Solitar group $< a, b | ab^2a^{-1} = b^3 >$. It is known that π is not residually finite; hence, neither is any group which contains it. For Theorem 13.2 take π to be the Baumslag-Solitar group $< a, b | aba^{-1} = b^2 >$. The centralizer of b in this group is isomorphic to a copy of the dyadic rationals.

S. Weinberger has noted that there are examples of finitely presented groups π with unsolvable word problem such that $B\pi$ is a finite 2-complex. (See [22, Theorem 4.12].) Since any group which retracts onto such a group also has unsolvable word problem, the reflection group trick gives the following result.

Theorem 13.4. (Weinberger). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which has unsolvable word problem.*

14 Hyperbolization

"Hyperbolization" refers to certain constructions, invented by Gromov [16], for converting any cell complex into an aspherical polyhedron (in fact, into a nonpositively curved polyhedron). One of the key features of these constructions is that they preserve local structure. Thus, hyperbolization will convert an n-manifold into an aspherical n-manifold. In this section we describe one such construction, Gromov's "Möbius band hyperbolization procedure," which given a cubical cell complex P produces an aspherical cubical cell complex $h(P)$. Before giving the definition, we list some properties of the construction:

1) The procedure is functorial in the following sense: if $f : P \to Q$ is a cellular embedding onto a subcomplex, then there is an induced embedding $h(f) : h(P) \to h(Q)$ onto a subcomplex.

2) The cubical complexes P and $h(P)$ have the same vertex set. Moreover, for each vertex v, $Lk(v, h(P))$ is the barycentric subdivision of $Lk(v, P)$.

In particular, it follows from 2) that if P is an n-manifold, then so is $h(P)$. Property 2) also shows that the link of each vertex in $h(P)$ is a flag complex. Hence, by Gromov's Lemma 6.1, the piecewise Euclidean metric on $h(P)$ is nonpositively curved. Consequently, we have the following property:

3) $h(P)$ is aspherical.

By functoriality, each cube $\square$ in P is converted into a subspace $h(\square)$ of $h(P)$ (called a "hyperbolized cell"). Furthermore, there is a map $c : h(P) \to P$, unique up to homotopy, which is the identity on the vertex set and which takes each hyperbolized cell to the corresponding cell of P. The map c has the following two properties:

4) c induces a surjection on homology groups with coefficients in $\mathbb{Z}/2$.

5) $c_* : \pi_1(h(P)) \to \pi_1(P)$ is surjective.

We also list one final property:

6) If P is an n-manifold, then there is an (unoriented) cobordism between P and $h(P)$.

A corollary of 6) is the following.

Theorem 14.1. *if two closed aspherical manifolds M_1 and M_2 were cobordant, then one could choose the cobordism to be an aspherical manifold with boundary. Although I don't know how to prove Gromov's assertion, the construction in Remark 14.2 does prove the following result. Every triangulable manifold is cobordant to an aspherical manifold.*

The definition of $h(P)$ is by induction on dim P. If dim $P \leq 1$, then $h(P) = P$. Suppose that the construction has been defined for all cubical complexes of dimension $< n$ and that properties 1) and 2) hold. Let $\square^n$ denote the standard n-cube and let $a : \square^n \to \square^n$ denote the antipodal map (a is also called the "central symmetry" of $\square^n$). By induction, $h(\partial\square^n)$ has been defined and by 1) the isomorphism a induces an involution $h(a) : h(\partial\square^n) \to h(\partial\square^n)$. The quotient space $h(\partial\square^n)/(\mathbb{Z}/2)$ is also a cubical complex. We define $h(\square^n)$ to be the canonical interval bundle over $h(\partial\square^n)/(\mathbb{Z}/2)$, i.e.,

$$h(\square^n) = [\text{-}1,1] \times_{\mathbb{Z}/2} h(\partial\square^n)$$

where $\mathbb{Z}/2$ acts on the first factor via $t \to -t$ and on the second via $h(a)$.

Since the restriction of this interval bundle to each cell of $h(\partial\square^n)/\mathbb{Z}/2$ is a trivial bundle, $h(\square^n)$ naturally has the structure of a cubical complex: each new cell is the product of [-1,1] with a cell of $h(\partial\square^n)/(\mathbb{Z}/2)$. It follows that if v is a vertex of $\square^n$, then there is a $(k+1)$-simplex in $Lk(v, h(\square^n))$ for each k-simplex in $Lk(v, h(\partial\square^n))$. Thus, $Lk(v, h(\square^n))$ is the cone on $Lk(v, h(\partial\square^n))$. Using induction, this implies that $Lk(v, h(\square^n))$ is the barycentric subdivision of Δ^{n-1}.

We note that the boundary of $h(\square^n)$ is canonically identified with $h(\partial\square^n)$. (The identification is canonical because a lies in the center of the automorphism group of the cube.)

Let $P^{(k)}$ denote the k-skeleton of a cubical complex P. If dim $P = n$, then $h(P)$ is defined by attaching, for each n-cell $\Box^n$ in P, a copy of $h(\Box^n)$ to the subcomplex $h(\partial\Box^n)$ of $h(P^{(n-1)})$ via the canonical identification.

Suppose that dim $P = n$, that $f : P \to Q$ is a cellular embedding, that $f^{(n-1)} : P^{(n-1)} \to Q^{(n-1)}$ denotes the restriction to the $(n-1)$-skeleton and that $h(f^{(n-1)})$ has been defined. The map $h(f) : h(P) \to h(Q)$ is induced by the map on each new cell which is the product of the identity map of [-1,1] with $h(f^{(n-1)})$. Property 1) follows. The new links are clearly the barycentric subdivisions of the old ones (so property 2) holds). The proof of properties 3) and 4) are straightforward and are left to the reader.

To check property 5), let $\tilde{P}$ denote the universal cover of P and let $\pi = \pi_1(P)$. By functorality, π acts freely on $h(\tilde{P})$ and $h(\tilde{P}) \to h(\tilde{P})/\pi$ is a covering projection. In fact, it is clear that $h(\tilde{P})/\pi \cong h(P)$. This defines an epimorphism $\varphi : \pi_1(h(P)) \to \pi$. It is not hard to see that φ is just the homomorphism induced by the canonical map $c : h(P) \to P$.

To check 6), suppose that M is triangulated manifold. Let CM denote the cone on M, i.e., $CM = (M \times [0,1])/\sim$ where $(x,0) \sim (x',0)$ for all $x, x' \in M$. Let c denote the cone point. The cone on the barycentric subdivision of a k-simplex is a standard subdivision of a $(k+1)$-cube. Regarding each simplex as the cone on its boundary, this gives a standard method for subdividing each simplex of CM into cubes and gives CM the structure of a cubical complex. The link of c in $h(CM)$ is isomorphic to the barycentric subdivision of M. Hence, removing the open star of c from $h(CM)$ we obtain a cobordism between M and $h(M)$ $(= h(M \times 1))$.

Remark 14.2. There is a slight variation of the above which provides a sort of "relative hyperbolization procedure." Suppose Q is a subcomplex of P. Let $p : \tilde{P} \to P$ be the universal cover and let $\hat{P}$ be the cubical complex formed by attaching a cone to each component of $p^{-1}(Q)$ in $\tilde{P}$. Let $h(\hat{P})_0$ denote the complement of small regular neighborhoods of the cone points in $h(\hat{P})$. By functorality, the fundamental group $\pi = \pi_1(P)$ acts on $h(\hat{P})_0$. Define the *hyperbolization of* P *relative to* Q by

$$h(P,Q) = h(\hat{P})_0/\pi,$$

In general, $h(P,Q)$ will not be aspherical. However, if Q is aspherical and if $\pi_1(Q) \to \pi_1(P)$ is injective, then the link of each cone point in $h(\hat{P})$ is contractible (since it is a copy of the universal cover of Q). Hence, $h(\hat{P})$ and $h(\hat{P})_0$ will be homotopy equivalent. So, in this case, $h(P,Q)$ is aspherical.

Not only did Gromov state Theorem 14.1, he also asserted that if an aspherical manifold M is a boundary, then it bounds an aspherical manifold

N so that $\pi_1(M) \to \pi_1(N)$ is injective. We shall give a proof of this below. First we need the following result of [17].

Theorem 14.3. (Hausmann [17]). *Suppose that manifolds M_1 and M_2 are cobordant. Then they are cobordant by a cobordism N such that for $i = 1, 2$, $\pi_1(M_i) \to \pi_1(N)$ is injective.*

Shmuel Weinberger explained the following version of Hausmann's proof to me.

Proof. It is proved in [1, Theorem 5.5] that for any finitely generated group G one can find a finitely generated acyclic group A such that G injects into A. For $i = 1, 2$ choose an acyclic group A_i so that $\pi_1(M_i)$ injects into A_i. Then $\pi_1(M_i)$ injects into the acyclic group $A = A_1 \times A_2$. Let $\Omega_*(-)$ be any bordism theory. Since A is acyclic and since $\Omega_*(-)$ is a generalized homology theory, it follows from the Atiyah–Hirzebruch spectral sequence that $\Omega_*(BA) \to \Omega_*(\text{point})$ is an isomorphism. Hence, if M_1 and M_2 are bordant over a point, then they are bordant over BA. In other words, the fundamental group of the cobordism N maps to A by a map extending the maps on $\pi_1(M_1)$ and $\pi_1(M_2)$. In particular, both of these fundamental groups must inject into $\pi_1(N)$. $\square$

Theorem 14.4. *Suppose that there is a triangulable cobordism between two closed aspherical manifolds M_1^n and M_2^n. Then there is an aspherical cobordism N^{n+1} between M_1^n and M_2^n so that for $i = 1, 2$, $\pi_1(M_i^n) \to \pi_1(N^{n+1})$ is injective.*

Proof. Let P be the cobordism between M_1 and M_2 provided by Theorem 14.3 and let $Q = M_1 \coprod M_2$. Set $N = h(P, Q)$. By Remark 14.2, N is the desired aspherical cobordism. $\square$

Remark 14.5. Let $M^n = h(\partial\square^{n+1})/(\mathbb{Z}/2)$. The manifolds M^n occur in nature. Indeed, consider real projective space $\mathbb{R}P^n$ and the collection of all its coordinate hyperplanes, defined by the equations $x_i = 0$, for $1 \leq i \leq n+1$. Then M^n is the manifold resulting from blowing up (in the sense of algebraic geometry) all projective subspaces which are intersections of such coordinate hyperplanes. It follows from this that M^n can also be described as the "closure of a generic torus orbit on a real flag manifold." More precisely, the flag manifold is the homogeneous space $SL(n+1, \mathbb{R})/B$ where B denotes the Borel subgroup of upper triangular matrices. The real "torus" H is the subgroup of diagonal matrices in $SL(n+1, \mathbb{R})$. Thus, $H \cong (\mathbb{R}^*)^n$. An H-orbit on the homogeneous space is "generic" if it is the orbit of a flag which is in general position with respect to the coordinate hyperplanes in $\mathbb{R}^{n+1}$

(in other words, it is generic if its stabilizer in H is trivial). It can then be shown that the closure of a generic H-orbit is homeomorphic to M^n, cf. [11].

15 An orientable hyperbolization procedure

The trouble with the Möbius band procedure is that the hyperbolized cells are not orientable. Gromov [16] gave a second construction which remedied this. We explain it below. (Further expositions of this construction can be found in [10] and [4].)

The rough idea behind any hyperbolization procedure is this. We first give some functorial procedure for hyperbolizing cells. Then, given a cell complex Λ, we define its hyperbolization by gluing together the hyperbolized cells in the same combinatorial pattern as the cells of Λ.

Gromov's second procedure can be applied to any finite dimensional simplicial complex Λ. The result will be an aspherical (in fact, nonpositively curved) cubical cell complex $h(\Lambda)$. The construction will have analogous properties to properties 1) through 6) in the previous section. In addition, it will have the following two properties:

7) The natural map $c : h(\Lambda) \to \Lambda$ induces on surjection on integral homology groups.

8) If Λ is a manifold, then c pulls back the stable tangent bundle of Λ to the stable tangent bundle of $h(\Lambda)$.

The definition of the construction again is by induction on dimension. In order to define the hyperbolization of an n-dimensional simplicial complex Λ, we first need to define the hyperbolization of an n-simplex, $h(\Delta^n)$. Then, to complete the definition of $h(\Lambda)$, we need to have some fixed identification of each n-simplex in Λ with the standard n-simplex. One way to insure this is to assume that Λ admits a "folding map" $p : \Lambda \to \Delta^n$, that is, a simplicial map p which restricts to an injection on each simplex. If we replace Λ with its barycentric subdivision Λ', then it admits such a folding map. Once we have such a p, $h(\Lambda)$ is defined to be the fiber product of $p : \Lambda \to \Delta^n$ and $c : h(\Delta^n) \to \Delta^n$. In other words, $h(\Lambda) = \{(x, y) \in \Lambda \times h(\Delta^n) | p(x) = c(y)\}$.

If dim $\Lambda \leq 1$, then, by definition, $h(\Lambda) = \Lambda$. Suppose that $h(\Lambda)$ has been defined for simplicial complexes of dimension $< n$.

We turn now to the definition on a hyperbolized n-simplex. Each transposition of two vertices of Δ^n induces a reflection on Δ^n. In order to make such a reflection into a simplicial isomorphism it is necessary to pass to the barycentric subdivision $(\Delta^n)'$. Choose a reflection $r : (\partial\Delta^n)' \to (\partial\Delta^n)'$. By functorality, we have an induced involution $h(r) : h((\partial\Delta^n)') \to h((\partial\Delta^n)')$.

The involution $h(r)$ acts as a reflection on the $(n-1)$-manifold $h((\partial\Delta^n)')$ and its fixed point set separates $h((\partial\Delta^n)')$ into two "half-spaces" which we denote by H_+ and H_-. Then $h(\Delta^n)$ is defined by

$$h(\Delta^n) = (h((\partial\Delta)') \times \text{ [-1,1]})/\sim$$

where the equivalence relation $\sim$ identifies $H_- \times \{-1\}$ with $H_- \times \{+1\}$. The boundary of $h(\Delta^n)$ consists of two copies of H_+ glued together along the fixed point set of $h(r)$, i.e., it can be identified with $h((\partial\Delta^n)')$. Another way to describe this procedure is to form the manifold $h((\partial\Delta^n)') \times S^1$ and then cut it open along $H_+ \times \{1\}$. It follows from this that $h(\Delta^n)$ is an orientable manifold with boundary (assuming, by induction, that $h((\partial\Delta^n)')$ is orientable). Assuming further that $h((\partial\Delta^n)')$ is a cubical cell complex, we see that $h(\Delta^n)$ inherits the structure of a cubical cell complex (possibly after subdividing the [-1,1] factor). Moreover, the link of a vertex v in $h(\Delta^n)$ is the cone on $Lk(v, h((\partial\Delta^n)'))$. It follows that the link of any vertex in $h(\Delta^n)$ is a flag complex and hence, that $h(\Delta^n)$ is nonpositively curved. As was explained earlier, the hyperbolization $h(\Lambda)$ of an arbitrary n-dimensional simplicial complex Λ is then defined by the fiber product construction.

Using the fact that $H_n(h(\Delta^n), \partial h(\Delta^n)) \cong \mathbb{Z}$, it is not hard to verify property 7). Property 8) follows from the observation that when Λ is a manifold, $h(\Lambda)$ can be identified with a submanifold of $\Lambda \times \Delta^n$ with trivial normal bundle and from the fact that the stable tangent bundle of $h(\Delta^n)$ is trivial.

Remarks 15.1.

- One consequence of 8) is that for any (triangulable) manifold M, the map $c : h(M) \to M$ pulls back the stable characteristic classes of M (e.g., its Pontryagin classes and Stiefel-Whitney classes) to those of $h(M)$. Thus, the characteristic numbers of $h(M)$ are the same as those of M. This shows that the condition of being aspherical does not impose any restrictions on the characteristic numbers of a manifold.

- Property 8) can be rephrased as saying that $c : h(M) \to M$ is covered by a normal map.

- The proof of Theorem 14.1 shows that $c : h(M) \to M$ to normally cobordant to the identity map, $\mathrm{id} : M \to M$.

- The proofs of Theorems 14.3 and 14.4 show that if aspherical manifolds M_1 and M_2 are cobordant in any bordism theory, then they are cobordant, in the same theory, by an aspherical cobordism.

16 A nontriangulable aspherical 4-manifold

The E_8-form is a certain positive definite, unimodular, even, symmetric bilinear form on $\mathbb{Z}^8$ which is associated to the Dynkin diagram E_8. It is represented by the following matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

(*Unimodular* means that the determinant of this matrix is 1; *even* means that the diagonal entries are even integers.) Since the E_8 is positive definite, its signature is 8.

The plumbing construction of Kervaire–Milnor gives a smooth, simply connected 4-manifold with boundary, N^4, with intersection form the E_8-form. (One "plumbs" together 8 copies of the tangent disk bundle of S^2 according to the diagram E_8.) It follows from the exact sequence of the pair $(N^4, \partial N^4)$ and from the fact that the E_8-form is unimodular that the boundary of N^4 is a homology 3-sphere. It turns out that it is not simply connected. In fact, ∂N^4 is Poincaré's homology 3-sphere. (The same construction was used in higher dimensions, by Kervaire–Milnor, to construct exotic spheres of dimension $4k-1$, $k \geq 2$.)

Let Λ^4 be a simplicial complex formed by triangulating N^4 and then attaching the cone on ∂N^4. Of course, Λ^4 is not a 4-manifold since there is no Euclidean neighborhood of the cone point. On the other hand, it is a polyhedral homology manifold in the sense that the link of each vertex is a homology sphere. It is called the E_8 *homology* 4-*manifold*. Stiefel-Whitney classes make sense for homology manifolds and it follows from the fact that N^4 is stably parallelizable that the Stiefel-Whitney classes of Λ^4 all vanish. Its signature is 8.

It is a theorem from algebra that any unimodular, even, symmetric bilinear form over the integers has signature divisible by 8. For a closed orientable 4-manifold, the condition that its intersection form is even is equivalent to its second Stiefel–Whitney class, w_2, being 0. The following is a famous theorem of Rohlin [24].

Theorem 16.1 (Rohlin's Theorem). *For any smooth or PL closed 4-manifold M^4 with $w_1(M^4) = 0 = w_2(M^4)$, the signature of its intersection* *form must be divisible by 16.*

It follows that Λ^4 is not homotopy equivalent to a smooth or PL 4-manifold. On the other hand, by Freedman's result, Theorem 9.3, there is a contractible manifold N' with $\partial N' = \partial N$. Hence, Λ^4 is homotopy equivalent to the topological manifold $M^4 = N \cup N'$. (We have replaced the cone on a homology 3-sphere by the contractible manifold N'.) By Rohlin's Theorem, M^4 cannot be homotopy equivalent to a PL 4-manifold.

The condition that a manifold admit a PL structure is stronger than the condition that it be homeorphic to a simplicial complex: to have a PL structure it must, in addition, be locally piecewise linearly (PL) homeomorphic to Euclidean space (with its standard PL structure). In particular, the link of any vertex in a PL manifold must be PL homeomorphic to a sphere. It is easy to see that in a general triangulated n-manifold the link of any vertex must be a simply connected (provided $n \geq 3$) homology manifold with the same homology as S^{n-1}, but there is no reason for it to be PL homeomorhic to S^{n-1}. When $n = 4$, since any polyhedral homology 3-manifold is a 3-manifold, each such link must be a homotopy 3-sphere.

Thus, for a few years after Freedman's result [15] was proved, it still seemed possible that Λ^4 could be homotopy equivalent to a triangulated manifold M^4 (since the 3-dimensional Poincaré Conjecture is not known). However, it follows from Casson's work on the Casson invariant that this is also not the case. Indeed, the link of any vertex in a triangulation of M^4 must be a homotopy 3-sphere (and at least one such link must be a fake 3-sphere since the triangulation cannot be PL). One can then arrange that the connected sum of all fake 3-spheres which arise as links bounds a PL submanifold of M^4 of signature 8. This implies that the Casson invariant of such a fake 3-sphere must be an odd integer. On the other hand, Casson showed that the Casson invariant of homology 3-sphere depends only on its fundamental group. Hence, this integer is 0. This contradiction shows that M^4 is not triangulable.

The hyperbolization technique of the previous section allows us to promote this result to aspherical 4-manifolds. Consider $h(\Lambda^4)$, the result of applying Gromov's oriented hyperbolization procedure to Λ^4. It is a polyhedral homology manifold with only one non-manifold point (namely, the hyperbolization of the cone point). By property 8) of the previous section, its Stiefel-Whitney classes vanish. Since the complement of a regular neighborhood of the cone point is oriented cobordant rel ∂N to N^4 and since the signature is an oriented cobordism invariant, it follows that the signature of $h(\Lambda^4)$ is 8. Now let M^4 denote the result of replacing the regular neighborhood of the cone point in $h(\Lambda^4)$ by the contractible manifold N'. Since M^4

is homotopy equivalent to $h(\Lambda^4)$, it is aspherical. The previous arguments now prove the following result of [10].

Theorem 16.2. *There is an aspherical 4-manifold M^4 which is not homotopy equivalent to any triangulable 4-manifold.*

In particular, M^4 is not homotopy equivalent to a PL manifold. Standard arguments show that this property is preserved when we take the product with a k-torus; hence, we also have the following result of [10].

Theorem 16.3. *For each $n \geq 4$, there is a closed aspherical n-manifold which is not homotopy equivalent to a PL manifold.*

Remark 16.4. For another application of this orientable hyperbolization procedure, see [25].

17 Relative hyperbolization

As is explained elsewhere in this volume, Farrell and Jones have proved the Assembly Map Conjecture (cf. Section 11) for the fundamental group of any nonpositively curved, closed Riemannian manifold. It seems likely (or at least plausible) that the Farrell-Jones program can be adapted to prove the Assembly Map Conjecture for the fundamental group of any closed PL manifold M^n equipped with a nonpositively curved, piecewise Euclidean metric (cf. Section 6). (Here we want to assume that the piecewise Euclidean structure on M^n is compatible with its structure of a PL manifold. The reason for this hypothesis is that a theorem of [10] then asserts that the compactification of its universal cover is homeomorphic to an n-disk.)

In this section we describe a variant of the reflection group trick (it is also a variant of hyperbolization) which can be used to prove the following result.

Theorem 17.1. *Suppose that the Assembly Map Conjecture is true for the fundamental group of any closed PL manifold with a nonpositively curved, piecewise Euclidean metric. Then it is also true for the fundamental group of any finite polyhedron with a nonpositively curved, piecewise Euclidean metric.*

This program has been partially carried out by B. Hu in [18] and [19]. Hu [18] first showed that the Farrell–Jones arguments work to prove the vanishing the Whitehead group of the fundamental group of any closed PL manifold with a nonpositively curved, polyhedral metric. He then used a construction similar to the one described below to derive the following theorem.

Theorem 17.2. (Hu [18]) *For any finite polyhedron with a nonpositively curved piecewise Euclidean metric, the Whitehead group of its fundamental group vanishes.*

As Farrell [13] pointed out in his lectures, Hu [19, p. 146] also observed that the Farrell–Hsiang [14] proof of the Novikov Conjecture [14] works for closed PL manifolds with nonpositively curved polyhedral metrics. (This uses the fact that the compactification of its universal cover is homeomorphic to a disk.) Hence, the construction described below yields the following theorem.

Theorem 17.3. (Hu [19]) *The Novikov Conjecture holds for the fundamental group of any finite polyhedron with a nonpositively curved piecewise Euclidean metric.*

Suppose B is a cell complex equipped with a piecewise Euclidean metric. Subdividing if necessary, we may assume that B is a simplicial complex. Suppose further that B is a subcomplex of another simplicial complex X. Possibly after another subdivision, we may assume that B is a full subcomplex of X. This means that if a simplex σ of X has nonempty intersection with B, then the intersection is a simplex of B (and a face of σ). We note that while each simplex of B is given a Euclidean metric, no metric is assumed on the simplices of X which are not in B. In practice, X will be a PL manifold.

We will define a new cell complex $D(X,B)$ equipped with a polyhedral metric. We will also define a finite-sheeted covering space $\tilde{D}(X,B)$ of $D(X,B)$. Each cell of $D(X,B)$ (or of $\tilde{D}(X,B)$) will have the form $\alpha\times$ [-1,1]^k, for some integer $k \geq 0$, where α is a simplex of B and where $\alpha\times$[-1,1]^k is equipped with the product metric. (Usually we will use α to stand for a simplex of B and σ for a simplex in X which is not in B.) Define $\dim(X,B)$ to be the maximum dimension of any simplex of X which intersects B but which is not contained in B. Here are some properties which the construction will have:

i) For $n = \dim(X,B)$, there will be 2^n disjoint copies of B in $D(X,B)$.

ii) For each such copy and for each vertex v in B, the link of v in $D(X,B)$ will be isomorphic to a subdivision of $Lk(v,X)$. In particular, if X is a manifold, then $D(X,B)$ will be a manifold.

iii) If the metric on B is nonpositively curved, then the metric on $D(X,B)$ will be nonpositively curved and each copy of B will be a totally geodesic subspace of $D(X,B)$.

iv) The group $(\mathbb{Z}/2)^n$ will act as a reflection group on $D(X,B)$. A fundamental chamber for this action will be denoted by $K(X,B)$. It will be homeomorphic to a regular neighborhood of B in X. Thus, $K(X,B)$ will be a retract of $D(X,B)$ and B will be a deformation retract of $K(X,B)$.

In fact, the entire construction depends only on a regular neighborhood of B in X. More precisely, it depends only on the set of simplices of X which intersect B.

Let $\mathcal{P}$ denote the poset of simplices σ in X such that $\sigma \cap B \neq \emptyset$ and such that σ is not a simplex of B. For each simplex α of B, let $\mathcal{P}_{>\alpha}$ denote the subposet of $\mathcal{P}$ consisting of all σ which have α as a face. Let $\mathcal{F}$ =Flag($\mathcal{P}$) denote the poset of chains in $\mathcal{P}$ (an element of $\mathcal{F}$ is a nonempty, finite, totally ordered subset of $\mathcal{P}$).

Given a chain $f = \{\sigma_0 < \cdots < \sigma_k\} \in \mathcal{F}$, let σ_f denote its least element, i.e., $\sigma_f = \sigma_0$. Given a simplex α of B, let $\mathcal{F}_{>\{\alpha\}}$ denote the set of chains f with $\sigma_f > \alpha$.

We begin by defining the fundamental chamber K $(= K(X,B))$. Each cell of K will have the form $\alpha\times$ $[0,1]^f$, for some $f \in \mathcal{F}_{>\{\alpha\}}$. Thus, the number of interval factors of $\alpha\times[0,1]^f$ is the number of elements of f. If $f \leq f'$, then we identify $[0,1]^f$ with the face of $[0,1]^{f'}$ defined by setting the coordinates $x_\sigma = 1$, for all $\sigma \in f' - f$.

We define an incidence relation on the set of such cells as follows: $\alpha\times[0,1]^f \leq \alpha'\times[0,1]^{f'}$ if and only if $\alpha \leq \alpha'$ and $f \leq f'$. (Notice that if $\alpha \leq \alpha'$, then $\mathcal{F}_{>\{\alpha'\}} \subset \mathcal{F}_{>\{\alpha\}}$.) K is defined to be the cell complex formed from the disjoint union $\coprod \alpha\times[0,1]^f$ by gluing together two such cells whenever they are incident. It is clear that K is homeomorphic to a regular neighborhood of B in X.

Next we define the mirrors of K. For each $\sigma \in \mathcal{P}_{>\alpha}$ and for each chain f with $\sigma_f = \sigma$ define $\delta_\sigma(\alpha\times[0,1]^f)$ to be the face of $\alpha\times[0,1]^f$ defined by setting $x_\sigma = 0$, i.e.,

$$\delta_\sigma(\alpha \times [0,1]^f) = \alpha \times 0 \times [0,1]^{f-\{\sigma\}}.$$

The mirror $\delta_\sigma K$ is the subcomplex of K consisting of all such cells. In other words, $\delta_\sigma K = \alpha \times S_\sigma$, where $\alpha = B \cap \sigma$ and where S_σ denotes the star of the barycenter of σ in the simplicial complex $\mathcal{F}$ (=Flag($\mathcal{P}$)).

Next we apply the reflection group trick. Set $J = (\mathbb{Z}/2)^{\mathcal{P}}$. Define $\tilde{D}$ $(= \tilde{D}(X,B))$ by

$$\tilde{D} = (J \times K)/\sim$$

where the equivalence relation $\sim$ is defined as in Section 7.

Remark 17.4. Suppose X is the cone on a simplicial complex ∂X and that B is the cone point. Then $\tilde{D}(X, B)$ coincides with the cubical complex P_L defined in Section 3 (where L is the barycentric subdivision of ∂X).

The definition of the space D $(= D(X, B))$ is similar to that of $\tilde{D}$ only one uses the smaller group $(\mathbb{Z}/2)^n$, $n = \dim(X, B)$, instead of J. If $\{r_1, \dots, r_n\}$ is the standard set of generators for $(\mathbb{Z}/2)^n$, then we identify the points (gr_i, x) and (g, x) whenever x belongs to a mirror $\delta_\sigma K$ with $i = \dim \sigma$.

There is another definition of $D(X, B)$ which is similar to the definitions of the hyperbolization constructions given in the previous two sections. We shall define a space $D^{(k)}(X, B)$ for any pair (X, B) with $\dim(X, B) \leq k$. The definition is by induction on $\dim(X, B)$. First of all, $D^{(0)}(X, B)$ is defined to be B. Assume that $D^{(n-1)}$ has been defined and that $\dim(X, B) = n$. Set

$$D^{(n)}(X^{(n-1)} \cup B, B) = D^{(n-1)}(X^{(n-1)} \cup B, B) \times \{-1, 1\}.$$

If σ is an n-simplex such that $\sigma \cap B \neq \emptyset$ and σ is not in B, then define

$$D^{(n)}(\sigma, \sigma \cap B) = D^{(n-1)}(\partial\sigma, \partial\sigma \cap B) \times [-1, 1].$$

We note that the boundary of $D^{(n)}(\sigma, \sigma \cap B)$ is naturally a subcomplex of $D^{(n)}(X^{(n-1)} \cup B, B)$. Hence, we can glue in each hyperbolized simplex $D^{(n)}(\sigma, \sigma \cap B)$ to obtain $D^{(n)}(X, B) = D(X, B)$.

The advantage of this definition is that it makes it easier to prove property iii) (that if B is nonpositively curved, then so is $D(X, B)$.) The proof is based on a Gluing Lemma of Reshetnyak [3, p. 316] or [16, p. 124]. This lemma asserts that if we glue together two nonpositively curved spaces via an isometry of a common totally geodesic subspace, then the new metric space is nonpositively curved. The inductive hypothesis gives that the spaces $D^{(n-1)}(X^{(n-1)} \cup B, B)$ and $D^{(n-1)}(\partial\sigma, \partial\sigma \cap B)$ are nonpositively curved. Using the Gluing Lemma, we get that $D^{(n)}(X, B)$ is also nonpositively curved.

Remark 17.5. The above construction of $D(X, B)$ was explained to me about eight years ago by Lowell Jones. It is a variation of the "cross with interval" hyperbolization procedure which had been described previously by Tadeusz Januszkiewicz and me in [10]. Relative versions of this were described by Hu [19] and by Ruth Charney and me in [4]. In these earlier versions of relative hyperbolization the 1-skeleton of X was not changed. Jones realized that the construction is nicer if, as in this section, we also hyperbolize the 1-simplices.

18 Comments on the references

My first work on the complexes P_L and Σ_L was in my paper [5] on reflection groups. This paper was inspired by lectures of Thurston on Andreev's Theorem. This theorem concerned the realization of convex polytopes in hyperbolic 3-space with prescribed dihedral angles, thereby producing many examples of reflection groups on hyperbolic 3-space. (Andreev's Theorem provided the first large class of manifolds for which Thurston's Geometrization Conjecture was verified.) In the right-angled case, one of Andreev's conditions was the flag complex condition of Section 5. Thurston's explanation of the relationship of this condition to the asphericity of the corresponding 3-manifolds led me to guess (and then prove in [5]) that a similar result held in all dimensions.

In [5], I defined the complexes P_L and Σ_L as in Section 7, I gave the examples in Section 9 of aspherical manifolds not covered by Euclidean space and I discussed the reflection group trick. In [6], a version of Theorem 11.4 appeared for the first time. (However, since I was using $[(X, \partial X), (G/TOP, *)]$ instead of $H_n(X; \mathbb{L})$, I only made a statement about the injectivity of the assembly map.)

Subsequently, in a seminal paper, Gromov [16] described the cubical structure on P_L and proved that the resulting piecewise Euclidean metric was nonpositively curved. (For an exposition of Gromov's ideas on nonpositive curvature, the reader is referred to the excellent book of Bridson and Haefliger [2] as well as to the new textbook [3].) In the same paper Gromov introduced the hyperbolization techniques, described in Sections 14 and 15, for producing nonpositively curved polyhedra. (At least in the case of the Möbius band procedure, he also claimed that the hyperbolized space admitted a metric of strict negative curvature. This stronger assertion was in error, but later Charney and I in [4] showed how Gromov's constructions could be modified so that it would be true.)

In [10] Januszkiewicz and I used Gromov's hyperbolization construction to produce the example in Section 16 of a nontriangulable aspherical 4-manifold.

I have written three other survey papers [7], [8] and [12] which the reader is referred to for further details on the material discussed here.

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Foliated Control Theory and its Applications

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Abstract

The control theorems and fibered control theorems due to Chapman, Ferry and Quinn, concerning controlled h-cobordisms and controlled homotopy equivalences, are reviewed. Some foliated control theorems, due to Farrell and Jones, are formulated and deduced from the fibered control theorems. The role that foliated control theory plays in proving the Borel conjecture for closed Riemannian manifolds having non-positive sectional curvature, and in calculating Whitehead groups for the fundamental group of such manifolds, is described.

Contents

1 Introduction

Recall that the **Borel Conjecture** states that any homotopy equivalence $f : M \longrightarrow N$ from a closed aspherical manifold M to a closed aspherical manifold N must be homotopic to a homeomorphism. There is a related conjecture, which in these notes will be referred to as the **Whitehead Group Conjecture**, which states that the Whitehead group $Wh(\pi)$ of any torsion free group π vanishes. In particular the Whitehead Group Conjecture claims that $Wh(\pi) = 0$ for π the fundamental group of any aspherical manifold.

Any closed Riemannian manifold N whose sectional curvature values satisfy $K \leq 0$ everywhere must be an aspherical manifold; infact, by a theorem due to Hadamard (cf. reference [1]), the universal covering space of N must be diffeomorphic to Euclidean space. Thus the following two theorems, due to Farrell and Jones (cf. [13] and [14]), represent a partial verification of the Whitehead Group Conjecture and the Borel Conjecture.

Theorem 1.1. *Let N denote a smooth closed Riemannian manifold whose sectional curvature values satisfy $K \leq 0$ everywhere; and let π denote the fundamental group of N. Then $Wh(\pi) = 0$.*

Theorem 1.2. *Let N denote a smooth closed Riemannian manifold whose sectional curvature values satisfy $K \leq 0$ everywhere. Also suppose that $dim(N) \geq 5$. Then any homotopy equivalence*

$$M \xrightarrow{f} N$$

from a closed aspherical manifold M is homotopic to a homeomorphism.

The main object in the rest of this paper is to show how "control theory" in topology plays a role in proving Theorems 1.1 and 1.2.

In section 2 we formulate some control theorems for h-cobordisms which are of use in proving Theorem 1.1, and we outline a proof for Theorem 1.1. In sections 4-7 the "foliated control" theorems for h-cobordisms due to Farrell and Jones [10] are deduced from the "fibered control"theorems for h-cobordisms of Ferry [17], Chapmann [3] and [4], Chapman and Ferry [5], Quinn [19]. A more detailed description of the contents of sections 4-7 is given at the end of section 2.

In section 3 we formulate some control theorems for "splitting problems" which are of use in proving Theorem 1.2, and we give a very sketchy outline of a proof for Theorem 1.2. In sections 8-10 the "foliated control" theorems for splitting problems due to Farrell and Jones [10] are deduced from the "fibered control" theorems for splitting problems of Quinn [19]. A more detailed description of the contents of sections 8-10 is given at the end of section 3.

2 Outline of a proof for Theorem 1.1

An h-**cobordism** of the compact Riemannian manifold pair $(X, \partial X)$ consists of a cobordism (W, W_∂) from $(X, \partial X) = \partial_-(W, W_\partial)$ to $\partial_+(W, W_\partial)$, such that (W, W_∂) may be equipped with deformation retracts of the pair (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$; these deformation retracts are denoted by

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial) \quad \text{and} \quad r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial),$$

for $t \in [0, 1]$. If dimX ≥ 5 then any elemenent $\omega \in Wh(\pi_1(X))$ can be represented by an h-cobordism (W, W_∂) of $(X, \partial X)$; and α=0 iff $(W, \partial_- W)$ is diffeomorphic to $(X \times [0, 1], X \times 0)$. Thus one approach to proving Theorem 1.1 would be to show that any h-cobordism of the manifold N in Theorem 1.1 is a product cobordism. To show this we need some tools which allow us to conclude that under certain circumstances an h-cobordism is a product cobordism; the tools that suffice are the controlled h-cobordism theorems of Ferry, Chapmann and Quinn, and their foliated versions.

Control theory for h-cobordisms.

An h-cobordism W of the closed Riemannian manifold X is said to be ε-**controlled** if there are deformation retracts r_t^- and r_t^+ of W onto $\partial_- W$ and $\partial_+ W$ respectively, such that for each $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \text{ and } r_1^- \circ r_t^+(w), t \in [0, 1]$$

have diameter less than ε in X. S. Ferry has proven the following control theorem (cf. reference [17]).

Theorem 2.1. *Suppose that dim$X \geq 5$. There is an $\varepsilon > 0$ which depends only on the compact Riemannian manifold X. Any ε-controlled h-cobordism W of X is diffeomorphic to $X \times [0, 1]$.*

Fibered control theory for h-cobordisms.

Fibered versions of Theorem 2.1 have also been proven. In these versions we need a smooth fiber bundle projection

$$\rho : (X, \partial X) \longrightarrow (Y, \partial Y)$$

from the compact smooth manifold pair $(X, \partial X)$ onto the compact Riemannian manifold pair $(Y, \partial Y)$ which has a closed manifold F for fiber. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be $(\varepsilon; \rho)$-**controlled** if there are deformation retracts of r_t^- and r_t^+ of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for each $w \in W$ both of the paths

$$\rho \circ r_1^- \circ r_t^-(w) \text{ and } \rho \circ r_1^- \circ r_t^+(w), t \in [0, 1]$$

have diameter less than ε in Y. The following theorem was proven by F. Quinn [19] and by and by T.A. Chapman [4].

Theorem 2.2. *Suppose that $dim X \geq 6$. Suppose that the fiber of*

$$\rho : (X, \partial X) \longrightarrow (Y, \partial Y),$$

denoted by F, satisfies $Wh(\pi_1(F) \oplus G)$=0 for any finitely generated free abelian group G. Then there is an $\varepsilon > 0$ which depends only on the compact Riemannian manifold pair $(Y, \partial Y)$. Any $(\varepsilon; \rho)$-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ is diffeomorphic to the product $(X, \partial X) \times [0, 1]$.

Remark 2.3. If $\partial X = \emptyset$ in Theorem 2.2 then we may weaken the dimension hypothesis of Theorem 2.2 to $dim X \geq 5$.

Remark 2.4. The hypothesis of Theorem 2.2, that $Wh(\pi_1(F) \oplus G)$=0 for any finitely generated free abelian group G, is known to be satisfied for the following types of manifolds F.

(a) F equal an n-dimensional torus T^n (cf. reference [2]).

(b) F equal an aspherical manifold whose fundamental group is a torsion free virtually poly-$\mathbb{Z}$ group (cf. reference [7]).

Remark 2.5. Note that it follows from Theorem 1.1 that the hypothesis of Theorem 2.2 is also satisfied for F equal a closed Riemannian manifold having non-positive sectional curvature values everywhere. In fact, the product $F \times T^n$, when equipped with the product of the given Riemannian structure on F with a standard flat Riemannian structure on the n-dimensional torus T^n, is a closed Riemannian manifold having non-positive sectional curvature values everywhere. Moreover $\pi_1(F \times T^n) = \pi_1(F) \oplus G$, where G is a free abelian group of rank n. So by applying Theorem 1.1 to $F \times T^n$ for all $n = 0, 1, 2, ...$ we get that F satisfies the hypothesis of Theorem 2.2.

Foliated control theory for h-cobordisms.

Let Ξ denote a smooth one-dimensional foliation for the compact Riemannian manifold X which also foliates the boundary ∂X. A path

$$p : [0, 1] \longrightarrow X$$

in X is said to have **Ξ-diameter less than** (α, ε) if the following hold: there is a connected subset A of a leaf of Ξ which has length less than α; any point in image(p) is a distance less than ε from A. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be **$(\alpha, \varepsilon; \Xi)$-controlled** if there are deformation retracts

r_t^- and r_t^+ of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for any fixed $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \quad \text{and} \quad r_1^- \circ r_t^+(w), t \in [0,1]$$

have Ξ-diameter less than (α, ε) . The following theorem has been proven by Farrell and Jones (cf. references [9] and [10]).

Theorem 2.6. *Suppose that $dimX \geq 6$. Given any number $\alpha > 0$ there is a number $\varepsilon > 0$ which depends only on α and Ξ. Any h-cobordism (W, W_∂) of $(X, \partial X)$ which is $(\alpha, \varepsilon; \Xi)$-controlled is diffeomorphic to $(X, \partial X) \times [0,1]$.*

Remark 2.7. If $\partial X = \emptyset$ in Theorem 2.6 then the dimension hypothesis may be weakened to $dimX \geq 5$.

Note that Theorem 2.6 generalizes Theorem 2.2 in the special case that the fiber F of Theorem 2.2 is of dimension one; and Theorem 2.6 is an immediate corollary of Theorem 2.2 in the special case that the the leaves of Ξ are equal to the fibers of a smooth fiber bundle projection

$$\rho : (X, \partial X) \longrightarrow (Y, \partial Y)$$

onto a compact Riemannian manifold pair $(Y, \partial Y)$. A complete proof for Theorem 2.6, which is carried out in section 7 below, is based upon some "local, relative " versions of Theorem 2.2. These "local, relative" versions of Theorem 2.2 are formlated in sections 4 and 5 below.

There is also a fibered version of Theorem 2.6. This fibered version requires a smooth fiber bundle projection

$$\rho : (X, \partial X) \longrightarrow (Y, \partial Y)$$

onto a compact Riemannian manifold pair $(Y, \partial Y)$ which has a closed manifold F for fiber. Now we let Ξ denote a one-dimensional smooth foliation for Y which also foliates ∂Y. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be $(\alpha, \varepsilon; \Xi, \rho)$**-controlled** if there are deformation retracts r_t^- and r_t^+ of (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for any fixed $w \in W$ both of the paths

$$\rho \circ r_1^- \circ r_t^-(w) \quad \text{and} \quad \rho \circ r_1^- \circ r_t^+(w), \quad t \in [0,1]$$

have Ξ-diameter less (α, ε). The following theorem is also due to Farrell and Jones (cf. references [9] and [10]).

Theorem 2.8. *Suppose $dimX \geq 6$. Suppose also that $Wh(\pi_1(E) \oplus G) = 0$ for any finitely generated free abelian group G, and for any space E of the form $\rho^{-1}(L)$* or *$\rho^{-1}(y)$ where L denotes any leaf of Ξ and y denotes any point in Y. For any number $\alpha > 0$ there is a number $\varepsilon > 0$ which depends only on α and Ξ. Any h-cobordism (W, W_∂) of $(X, \partial X)$ which is $(\alpha, \varepsilon; \Xi, \rho)$-controlled is diffeomorphic to $(X, \partial X) \times [0, 1]$.*

Remark 2.9. If $\partial X = \emptyset$ in Theorem 2.8 then we may weaken the dimension hypothesis of Theorem 2.8 to $dimX \geq 5$.

Sketch of the proof of Theorem 1.1.

Let N denote the closed Riemannian manifold of Theorem 1.1, and let $N \times S^1$ denote the product of N with the circle S^1 equipped with the product metric. Note that $N \times S^1$ also has non-positive sectional curvature every where; and note also that $Wh(\pi_1(N))$ is a direct summand of $Wh(\pi_1(N \times S^1))$. Thus in proving Theorem 1.1 we may assume that $dimN \geq 3$. Let $S(N \times S^1)$ denote the unit sphere bundle for $N \times S^1$, and let

$$u : N \times S^1 \longrightarrow S(N \times S^1)$$

denote a unit vector field which is always tangent to the second factor of $N \times S^1$. Let X denote the subset of all $v \in S(N \times S^1)$ such that $\langle v, u(p) \rangle \geq 0$, where $p \in N \times S^1$ is the base point of v; note that X is the total space of a smooth disc bundle over $N \times S^1$. Thus X is a compact smooth manifold with boundary ∂X. Choose any Riemannian metric for X. Denote by

$$g_t : S(N \times S^1) \longrightarrow S(N \times S^1), \quad t \in (-\infty, \infty)$$

the geodesic flow for $N \times S^1$; note that this flow leaves invariant the subsets X and ∂X. Thus there is the one-dimensional smooth foliation Ξ for X whose leaves are just the orbits of the geodesic flow which intersect X; note that Ξ also foliates ∂X. The following theorem, due to Farrell and Jones, is the key result which allows one to apply foliated control theory to prove Theorem 1.1.

Theorem 2.10. *Suppose $dimN \geq 3$. Given $\omega \in Wh(\pi_1(N \times S^1))$ there is an $\alpha > 0$ with the following property. For each $\varepsilon > 0$, if the number $s > 0$ is choosen sufficiently large, there is an $(\alpha, \varepsilon; \Xi, g_s)$-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ which represents α in the following sense: $\alpha = 0$ iff (W, W_∂) is diffeomorphic to $(X, \partial X) \times [0, 1]$.*

Remark 2.11. The h-cobordism (W, W_∂) in Theorem 2.10 is gotten as follows. Choose an h-cobordism V of $N \times S^1$ which represents $\omega \in Wh(\pi_1(N \times S^1))$. Choose deformation retracts s_t^- and s_t^+ for V onto $\partial_- V = N \times S^1$ and onto $\partial_+ V$ respectively. Define W to be the total space of the fiber bundle obtained by pulling back the disc bundle $X \longrightarrow N \times S^1$ along the map $s_1^- : V \longrightarrow N \times S^1$. Define the subset $W_\partial \subset W$ to be the pull back along the map $s_1^- : V \longrightarrow N \times S^1$ of the subbundle of $S(N \times S^1)$ consisting of all unit vectors tangent to the first factor of $N \times S^1$. The deformation retracts s_t^- and s_t^+ for V can be "lifted" to deformation retracts r_t^- and r_t^+ for (W, W_∂) by using the homotopy lifting property for fiber bundles. There are many such liftings, most of which are of no use in verifying the control property claimed for (W, W_∂) in Theorem 2.10. Liftings which are useful in this regard are the "focal transfers" which are described in detail in Tom Farrell's lecture notes for this "school".

Now we will complete the sketch of our proof of Theorem 1.1. First recall that the group $Wh(\pi_1(N))$ is a direct summand of the group $Wh(\pi_1(N \times S^1))$. So to complete the proof of Theorem 1.1 it will suffice to show that $Wh(\pi_1(N \times S^1)) = 0$. This last equality follows by applying Theorems 2.8 and 2.10. In more detail we represent any $\omega \in Wh(\pi_1(N \times S^1))$ by an h-cobordism (W, W_∂) as in Theorem 2.10. We choose the bundle projection map

$$\rho : X \longrightarrow Y$$

of Theorem 2.8 to be equal to the map

$$g_s : X \longrightarrow X$$

of Theorem 2.10, where s is choosen sufficiently large so that (W, W_∂) is $(\alpha, \varepsilon; \Xi, g_s)$-contolled. Now we just apply Theorem 2.8 to see that (W, W_∂) is diffeomophic to $(X, \partial X) \times [0,1]$ (recall that $dimN \geq 3$, and hence $dimX \geq 6$ as is required in Theorem 2.8). Hence, by Theorem 2.10, $\alpha = 0$ as desired.

This completes our sketch of the proof for Theorem 1.1.

Contents of sections 4-7.

Section 4: We discuss a "local-relative" version of the fibered control Theorem 2.2. The word "local" here refers to the fact that the fiber bundles involved all project to open subsets of Euclidean space; and the word "relative" here refers to the fact that fibered control results discussed yield product structures which extend previously existing product structures.

Section 5: We formulate a very general "Foliated Control Conjecture" for h-cobordisms which would imply Theorem 2.2 and Theorem 2.8 as corollaries. Some supporting evidence is given for this conjecture (in addition to Theorems 2.2 and 2.8). A "local-relative" version of Theorem 2.8 is discussed.

Section 6: We consider "long and thin" pieces of a closed Riemannian manifold Y which comes equipped with a smooth one-dimensional foliation Ξ. By a "long and thin" piece of Y we simply mean a small neighborhood in Y of either a closed leaf of Ξ or a small neighborhood of a long arc within a leaf of Ξ. The main result of section 6 states that there is is a covering of Y by "long and thin pieces" which satisfies a number of useful properties. For example each piece in the covering looks like either a long thin open rectangle (i.e like $(0, a) \times (0, b)^n$, where a is a large positive real number and b is a smaller real positive number), or it looks like the total space of a vector bundle over the product $S^1 \times B$, where B is a ball of small radius in Euclidean space.

Section 7: We complete the proof of Theorem 2.8. This is done by "restricting" the h-cobordism (W, W_∂) of Theorem 2.8 to those parts of X which lie over the "long and thin" pieces of Y discussed in section 6. We then apply to each of these pieces of (W, W_∂) the "local-relative" versions of Theorem 2.8 which are proven in sections 4 and 5 below.

3 Outline of a proof for Theorem 1.2

We say that a closed manifold N is **topologically rigid** if any homotopy equivalence

$$h : M \longrightarrow N$$

from another closed manifold M is homotopic to a homeomorphism. The Borel Conjecture can be restated as follows: every aspherical closed manifold N is topologically rigid. We say that a closed manifold N is **strongly topologically rigid** if for each $k = 0, 1, 2, 3...$, and for each homotopy equivalence

$$h : (M, \partial M) \longrightarrow (N \times B^k, N \times \partial B^k)$$

which restricts to a homeomorphism

$$h : \partial M \longrightarrow N \times \partial B^k \quad ,$$

h is homotopic rel ∂ to a homeomorphism. (Here B^k denotes the closed unit ball centered at the origin of Euclidean space R^k; and to be "homotopic rel ∂" means that the homotopy is constant on the boundary.) Now we can formulate the **Strong Borel Conjecture**: any closed aspherical manifold N is strongly topologically rigid. In the remainder of this section I will outline the proof for the following theorem (which is a generalization of Theorem 1.2).

Theorem 3.1. *Let N denote a smooth closed Riemannian manifold whose sectional curvature values satisfy $K \leq 0$. Also suppose that $dim(N) \geq 5$. Then N is strongly topologically rigid.*

To prove Theorem 3.1 we need some tools from surgery theory. The tools that suffice are: the assembly map for surgery theory, and the long exact surgery sequence; some splitting results of Farrell and Hsiang; and the controlled splitting results of Quinn and their foliated versions.

The long exact surgery sequence

We assume for the sake of simplicity that N is an oriented closed manifold. Let π denote the fundamental group for N and let $L_i(\pi)$ denote the surgery group (for surgery up to homotopy equivalence) in dimension i defined by C.T.C.Wall [24]. Let $H_*(N, \mathcal{L}_*)$ denote the homology groups for N with coefficients in the simply connected surgery spectrum $\mathcal{L}_*$. Let $\mathcal{S}(N \times B^k, rel\partial)$ denote the **structure set** for $N \times B^k$ relative to its boundary: thus each element in this structure set is represented by a homotopy equivalence

$$h : (M, \partial M) \longrightarrow (N \times B^k, N \times \partial B^k)$$

which is a homomeomorphism on the boundary; and the homotopy equivalence h represents the zero element in $\mathcal{S}(N \times B^k, rel\partial)$ iff h is homotopic rel ∂ to a homeomorphism. Note that N is strongly topologically rigid iff each structure set $\mathcal{S}(N \times B^k, rel\partial)$ contains only the zero element. A proof of the following theorem involves techniques developed by D. Sullivan, C.T.C Wall [24], F. Quinn [21], Kirby-Siebenmann [18], A. Ranicki [22].

Theorem 3.2. *Suppose that $n \geq 5$, where n=$dimN$; and that $Wh(\pi \oplus G) = 0$ for any finitely generated free abelian group G. Then there is a long exact "surgery sequence"*

$$.... \longrightarrow L_{n+k+1}(\pi) \longrightarrow \mathcal{S}(N \times B^k, rel\partial) \longrightarrow H_{n+k}(N, \mathcal{L}_*) \longrightarrow L_{n+k}(\pi) \longrightarrow ......,$$

which ends on the right at the term $L_n(\pi)$, and which continues indefinitely to the left.

The mapping denoted by

$$H_{n+k}(N, \mathcal{L}_*) \longrightarrow L_{n+k}(\pi)$$

in Theorem 3.2 is the **assembly map** in dimension $n + k$. The assembly map in all dimensions will be denoted by

$$A_* : H_*(N, \mathcal{L}_*) \longrightarrow L_*(\pi).$$

The following two theorems are easily deduced from Theorem 3.2, and from the "four fold periodicity" for the assembly map and surgery theory.

Theorem 3.3. *Suppose that N is as in Theorem 3.2. Then N is strongly toplogically rigid iff the assembly map*

$$A_* : H_*(N, \mathcal{L}_*) \longrightarrow L_*$$

is an isomorphism.

Theorem 3.4. *Suppose that N is as in Theorem 3.2. Then N is strongly topolically rigid iff there is a positive integer j such that for any $k = j, j+1, j+2, j+3, j+4$, and for any homotopy equivalence*

$$h : (M, \partial M) \longrightarrow (N \times B^k, N \times \partial B^k)$$

which is homeomorphism on the boundary, we have that h is homotopic rel ∂ to a homeomorphism.

The dimension of N may be altered in Theorem 3.1.

Let N denote a closed manifold with fundamental group π, and let

$$h : (M, \partial M) \longrightarrow (N \times S^1 \times B^k, N \times S^1 \times \partial B^k)$$

denote a homotopy equivalence between compact manifold pairs which is a homeomorphism on the boundary. We say that h can be **split rel ∂ along the submanifold** $N \times s_0 \times B^k \subset N \times S^1 \times B^k$, where $s_0 \in S^1$, if after a homotopy of h which is constant on the boundary we have that h is in transverse position to $N \times s_0 \times B^k$ and

$$h : h^{-1}(N \times s_0 \times B^k) \longrightarrow N \times s_0 \times B^k$$

is a homotopy equivalence. The following theorem is proven by F.T. Farrell and W.c Hsiang [8].

Theorem 3.5. *Suppose that $dimN \geq 5$ and $Wh(\pi \oplus G) = 0$ for any finitely generated free abelian group G. Let*

$$h : (M, \partial M) \longrightarrow (N \times S^1 \times B^k, N \times S^1 \times \partial B^k)$$

be a homotopy equivalence of manifold pairs which is a homeomorphism of the boundaries. Then h can be split along $N \times s_0 \times B^k$.

It is easy to deduce the following theorem from Theorem 3.5.

Theorem 3.6. *Let N be as in 3.5. Then N is strongly topologically rigid iff $N \times S^1$ is strongly topologically rigid.*

Reducing the proof of Theorem 3.1 to a splitting problem.
Let

$$\rho : E \longrightarrow N$$

denote a fiber bundle projection onto the closed Riemannian manifold N from the manifold E which has a compact manifold F for fiber; F is allowed to have a nonempty boundary. Let K denote a triangulation for N and let

$$h : (M, \partial M) \longrightarrow (E, \partial E)$$

denote a homotopy equivalence from the compact manifold pair $(M, \partial M)$. We say that h is **split over the triangulation** K if the following hold: $h \mid \partial M$ is in transverse position to $\rho^{-1}(\triangle) \cap \partial E$ for each triangle $\triangle \in K$, and each restricted map

$$h : (\rho \circ h)^{-1}(\triangle) \cap \partial M \longrightarrow \rho^{-1}(\triangle) \cap \partial E$$

is a homotopy equivalence; h is in transverse position to $\rho^{-1}(\triangle)$ for each triangle $\triangle \in K$ and each of the restricted maps

$$h : (\rho \circ h)^{-1}(\triangle) \longrightarrow \rho^{-1}(\triangle)$$

is a homotopy equivalence. If the restricted mapping

$$h : \partial M \longrightarrow \partial E$$

is split over K we will say that **the splitting for $h \mid \partial M$ over K extends to a splitting for all of h over K** if there is a homotopy

$$h_t : M \longrightarrow E \quad , \quad t \in [0, 1],$$

of h rel ∂ such that h_1 is split over a K.

In the rest of this subsection we will be concerned with the special fiber bundle projection

$$\rho_k : E_k \longrightarrow N \quad ,$$

where $E_k = N \times B^k$ and B^k denotes the closed unit ball centered at the origin of Euclidean space R^k, and where ρ_k denotes the standard projection of E_k onto its first factor. Let

$$h : (M, \partial M) \longrightarrow (E_k, \partial E_k)$$

denote a homotopy equivalence of compact manifold pairs which restricts to a homeomorphism of the boundaries; note that in particular $h \mid \partial M$ is split over any triangulation K for N. A proof of the following theorem is outlined below.

Theorem 3.7. *Suppose that $k \geq 6$. Then the given splitting for $h \mid \partial M$ over K extends to a splitting for all of h over K iff h is homotopic rel ∂ to a homeomorphism*

$$h' : (M, \partial M) \longrightarrow (E_k, \partial E_k) \quad .$$

The following theorem is a direct consequence of Theorems 3.4 and 3.7.

Theorem 3.8. *The closed Riemannian manifold N is strongly topologically rigid if the following hold. There is an integer $j \geq 6$ such that for each integer $k = j, j+1, j+2, j+3, j+4$, and for each homotopy equivalence*

$$h : (M, \partial M) \longrightarrow (E_k, \partial E_k),$$

any given splitting for $h \mid \partial M$ over a triangualtion K for N extends to a splitting for all of h over K.

Here is a brief outline for the proof of Theorem 3.7. Clearly the existence of the map h' as described in Theorem 3.7 implies that the given splitting for $h \mid \partial M$ extends to a splitting for all of h over K. Now we must show the converse. That is we suppose that h is homotopic rel ∂ to a new mapping (this new mapping is also denoted by h) which is split over the triangulation K; and we must conclude that there is a homotopy of this new h rel ∂ to a homeomorphism

$$h' : (M, \partial M) \longrightarrow (E_k, \partial E_k).$$

We shall construct a homotopy h_t from our new h (at $t = 0$) to h' (at $t = 1$) which satisfies the following properties:

3.7.1 $h_t \mid \partial M = h \mid \partial M$ for all $t \in [0, 1]$.

3.7.2 $h_t(h^{-1}(\rho_k^{-1}(\triangle)) \subset \rho_k^{-1}(\triangle)$ for all $t \in [0, 1]$ and for all $\triangle \in K$.

3.7.3 h_1 is the desired homeomorphism h'.

To do this we proceed by induction. Let $E_{k,j}$ denote the union $\partial E_k \bigcup \rho_k^{-1}(K^j)$, where K^j denotes the j-dimensional skeleton of the triangulation K. Our induction hypothesis (for the r'th step in our induction argument) is that a homotopy $h_{t,r}$ of h has been constructed satisfying properties 3.7.1 and 3.7.2 above, and also satisfying the following weak version of property 3.7.3 above:

$$h_{1,r} : h^{-1}(E_{k,r}) \longrightarrow E_{k,r}$$

is a homeomorphism. Now we consider each $\triangle \in K$ with $dim(\triangle) = r+1$. Note that $\rho_k^{-1}(\triangle)$ is homeomorpic to the closed $(r+1+k)$-dimensional ball B^{r+1+k}, the map

$$h_{1,r} : h^{-1}(\partial\rho_k^{-1}(\triangle)) \longrightarrow \partial\rho_k^{-1}(\triangle)$$

is a homeomorphism, and the space $h^{-1}(\rho_k^{-1}(\triangle))$ is a homotopy ball with dimension equal $r+1+k$. Applying the h-cobordism Theorem (this requires the hypothesis of Theorem 3.7 that $k \geq 6$) we conclude that $h^{-1}(\rho_k^{-1}(\triangle))$ is homeomorphic to the $(r+1+k)$-dimensional ball. Thus (by Alexander's trick) there is a homotopy $h_{t,\triangle}$ of each of the restricted maps

$$h_{1,r} : h^{-1}(\rho_k^{-1}(\triangle)) \longrightarrow \rho_k^{-1}(\triangle)$$

to a homeomorphism which is the constant homotopy on the boundary of $h^{-1}(\rho_k^{-1}(\triangle))$. Taking the union of all the homotopies $h_{t,\triangle}$ gives a homotopy of the restricted map $h_{1,r} \mid h^{-1}(E_{k,r+1})$ which extends to a homotopy for all of $h_{1,r}$ that we denote by $h_{1,r,t}$. Now define the homotopy $h_{t,r+1}$ of h by $h_{t,r+1} = h_{2t,r}$ if $0 \leq t \leq 1/2$ and by $h_{t,r+1} = h_{1,r,2t-1}$ if $1/2 \leq t \leq 1$.

Fibered-controlled splitting theory.

Let

$$\rho : X \longrightarrow Y$$

denote a fiber bundle projection onto the closed Riemannian manifold Y having a compact manifold F for fiber; ∂F is not required to be empty. Let

$$h : (M, \partial M) \longrightarrow (X, \partial X)$$

denote a homotopy equivalence of pairs from the compact manifold pair $(M, \partial M)$. We say that h is **$(\varepsilon; \rho)$-controlled** if there is a homotopy inverse

$$g : (X, \partial X) \longrightarrow (M, \partial M)$$

for h, and homotopies

$$(h \circ g)_t : (X, \partial X) \longrightarrow (X, \partial X)$$

from $h \circ g$ to the identity map 1_X and

$$(g \circ h)_t : (M, \partial M) \longrightarrow (M, \partial M)$$

from $g \circ h$ to the identity map 1_M, such that for each $x \in X$ and $y \in M$ we have that each of the paths

$$\rho((h \circ g)_t(x)) \quad , \quad \rho \circ h((g \circ h)_t(y)), \quad t \in [0,1]$$

has diameter less than ε in Y.

The following theorem is due to F. Quinn [19].

Theorem 3.9. *Suppose that the fiber F satifies $Wh(\pi_1(F) \oplus G) = 0$ for all finitely generated free abelian groups; and $dim(F) \geq 6$. There is $\varepsilon > 0$ depending only on Y such that the following holds. Suppose that the restricted homotopy equivalence*

$$h : \partial M \longrightarrow \partial X$$

is already split over the triangulation K for Y; and that h is (ε, ρ)-controlled. Then the given splitting for $h \mid \partial M$ over K extends to a splitting for all of h over K.

Foliated-fibered-controlled splitting theory.

We continue to use the same notation as in Theorem 3.9 and the preceeding subsection; in addition we let Ξ denote a smooth foliation for the closed Riemannian manifold Y. We say that the homotopy equivalence

$$h : (M, \partial M) \longrightarrow (X, \partial X)$$

is $(\alpha, \varepsilon; \Xi, \rho)$-**controlled** if there is a homotopy inverse for h

$$g : (X, \partial X) \longrightarrow (M, \partial M) \ \ ,$$

and homotopies

$$(h \circ g)_t : (X, \partial X) \longrightarrow (X, \partial X)$$

from $h \circ g$ to the identity map 1_X and

$$(g \circ h)_t : (M, \partial M) \longrightarrow (M, \partial M)$$

from $g \circ h$ to the identity map 1_M, such that for each $x \in X$ and each $y \in M$ the paths

$$\rho((h \circ g)_t(x)) \quad (\rho \circ h)((g \circ h)_t(y))$$

has Ξ-diameter less than (α, ε).

The following theorem is a foliated version of Theorem 3.9 due to Farrell and Jones [10].

Theorem 3.10. *Suppose that the fiber F satisfies $dim(F) \geq 6$, and that $Wh(\pi_1(\rho^{-1}(x)) \oplus G) = 0$ and $Wh(\pi_1(\rho^{-1}(L)) \oplus G) = 0$ for each $x \in Y$ and for each leaf $L \in \Xi$, where G denotes any finitely generated free abelian group G. Given any $\alpha > 0$ there is an $\varepsilon > 0$ which depends only on α and on Ξ. Suppose that the restricted homotopy equivalence*

$$h : \partial M \longrightarrow \partial X$$

is already split over the triangulation K for Y; and suppose that h is $(\alpha, \epsilon; \Xi, \rho)$-controlled. Then the given splitting for $h \mid \partial M$ over K extends to a splitting for all of h over K.

The transfer of a splitting problem.

We continue with the notation used in Theorems 3.9 and 3.10 above. Let

$$\rho' : X' \longrightarrow X$$

denote a fiber bundle projection map onto the space X of 3.9 and 3.10 which satisfies the following properties:

(1) this fiber bundle is a product bundle over the one-dimensional skeleton of a triangulation for X;

(2) the fiber of ρ' is a closed oriented manifold with dimension $= 4l$ for some integer l, and with signature equal to 1.

We denote by

$$\rho^* : X^* \longrightarrow M$$

the pull back of the bundle $p' : X' \longrightarrow X$ along the homotopy equivalence $h : M \longrightarrow X$ of 3.9 and 3.10. Note that there is a canonical lifting of h

$$h^\wedge : (X^*, \partial X^*) \longrightarrow (X', \partial X');$$

this lifting is called the **transfer** of h. If the restricted homotopy equivalence

$$h : \partial M \longrightarrow \partial X$$

is already split over the triangulation K of Y, then this splitting "transfers" to a splitting for the restricted map

$$h^\wedge : \partial X^* \longrightarrow \partial X',$$

over K: note that X' is also the total space of a fiber bundle over Y with projection map equal the composition $\rho \circ \rho'$; thus we may speak of $h^\wedge$ or of $h^\wedge \mid \partial X^*$ being split over K. The following theorem is a consequence of the "four fold periodicity" which surgery theory satisfies (cf. [24]), and of properties (1) and (2) in the preceeding paragraph.

Theorem 3.11. *Suppose that the fiber F for $\rho : X \longrightarrow Y$ satisfies $dimF \geq 6$; and suppose that the restricted homotopy equivalence*

$$h : \partial M \longrightarrow \partial X$$

is already split over the triangulation K for Y. Then this splitting extends to a splitting for all of h over K iff the transferred splitting for

$$h^\wedge : \partial X^* \longrightarrow \partial X'$$

over K extends to a splitting for all of $h^\wedge$ over K.

Sketch of the proof of Theorem 3.1.

By Theorem 3.8 it will suffice to show that for any k=6,7,8,9,10, and for any homotopy equivalence

$$h : (M, \partial M) \longrightarrow (E_k, \partial E_k),$$

any given splitting for the restricted map $h \mid \partial M$ over the triangulation K for N extends to a splitting for all of h over K.

By Theorem 3.6 we may assume that $dimN$ is equal an odd integer. In addition (for the sake of simplicity in our exposition) we assume that N is an oriented manifold. We construct a fiber bundle

$$\tau \longrightarrow N$$

over N as follows: the total space τ is the set of all unordered pairs (u, v) of unit vectors tangent to N at the same point $x \in N$; the bundle projection sends the unordered pair (u, v) to the point x. Note, because of the assumptions just made concerning $dimN$ and the orientability of N, this fiber bundle satisfies the following properties.

(3.1.1) The fiber is is equal the orbit space for the group action

$$\mathbf{Z_2} \times (S^{n-1} \times S^{n-1}) \longrightarrow S^{n-1} \times S^{n-1}$$

which permutes the coordinates, where $n = dimN$; this equals complex projective space CP^2 if $dimN = 3$, and is a closed, oriented $Z(1/2)$-homology manifold with $dim = 4l$ (with $l = (n-1)/2$) and signature equal 1 if $dimN = 5, 7, 9, \ldots$.

(3.1.2) The bundle is a trivial bundle over the one-skeleton K^1.

We denote by
$$\rho_k' : E_k' \longrightarrow E_k$$
the pull back of the bundle $\tau \longrightarrow N$ along the bundle projection $\rho_k : E_k \longrightarrow N$; and we denote by
$$\rho_k^* : E_k^* \longrightarrow M$$
the pull back of the bundle $\rho_k' : E_k' \longrightarrow E_k$ along the map $h : M \longrightarrow E_k$. Note that there is a canonical lifting

$$h^\wedge : (E_k^*, \partial E_k^*) \longrightarrow (E_k', \partial E_k')$$

of the map h. Farrell and Jones having proven a variant of Theorem 3.11 in [16] which implies that the given splitting for $h \mid \partial M$ over the triangulation K for N extends to a splitting for all of h over K iff the transfered splitting for $h^\wedge \mid \partial E_k^*$ over K extends to a splitting of all of $h^\wedge$ over K. Note that $(E_k^*, \partial E_k^*)$ and $(E_k', \partial E_k')$ are stratified spaces with two stratra; and $(h^\wedge, h^\wedge \mid \partial E_k^*)$ preserves the strata. In the Farrell-Jones variant of Theorem 3.11 all splittings for $h^\wedge$ or for $h^\wedge \mid \partial E_k^*$ are required to be "stratified".

Now choose a homotopy inverse

$$g : (E_k, \partial E_k) \longrightarrow (M, \partial M)$$

for h such that $g \mid \partial M$ equals the inverse homeomorphism for $h \mid \partial M$; and choose homotopies

$$(g \circ h)_t : (M, \partial M) \longrightarrow (M, \partial M)$$

and
$$(h \circ g)_t : (E_k, \partial E_k) \longrightarrow (E_k, \partial E_k)$$
from $g \circ h$ to 1_M and from $h \circ g$ to 1_{E_k} respectively; note we may assume that both $(g \circ h)_t \mid \partial M$ and $(h \circ g)_t \mid \partial E_k$ are the constant homotopies.

We need to "transfer" (or "lift") g to get a homotopy inverse $g^\wedge$ for $h^\wedge$; and we need to "transfer" (or "lift") the homotopies $(g \circ h)_t$ and $(h \circ g)_t$ to get homotopies $(g^\wedge \circ h^\wedge)_t$ and $(h^\wedge \circ g^\wedge)_t$ from $g^\wedge \circ h^\wedge$ to $1_{E_k^*}$ and from $h^\wedge \circ g^\wedge$ to $1_{E_k'}$ respectively; so that when the "control" for $h^\wedge$ is measured with respect to these transfered homotopies we will see that $h^\wedge$ is "well controlled" (somewhere, perhaps in τ, or in one of the strata of τ, or in N). If this were accomplised than we could use (relative versions of) Theorems 3.9 and 3.10 to extend the transferred splitting for $h^\wedge \mid \partial E_k^*$ over K to a splitting of all of $h^\wedge$ over K; this would complete our outline for the proof of Theorem 1.2. (The analogous "gaining control" step in the outline of the proof for Theorem 1.1 occured in Theorem 2.10 and in Remark 2.11 above.)

There are many transfers $g^\wedge, (g^\wedge \circ h^\wedge)_t, (h^\wedge \circ g^\wedge)_t$ of the $g, (g \circ h)_t, (h \circ g)_t$; they can be constructed by appealing to the homotopy lifting property. However $h^\wedge$ is rarely "well controlled" with respect to such transfers. (Recall that the analogous problem for transfers involving h-cobordism control data was discussed in Remark 2.11 above.) Transfers which make $h^\wedge$ "well controlled" are described in detail in Tom Farrell's lecture notes for this "school". Unfortunately there is no simple theorem analogous to Theorem 2.10 which will describe the type of control acheived for $h^\wedge$ by cleverly choosing transfers of the $g, (g \circ h)_t, (h \circ g)_t$; the control for $h^\wedge$ occurs piecemeal, and it is measured in a different space for each piece.

Contents of sections 8, 9, 10.

Section 8: We discuss a "local-relative" version of the fibered control splitting Theorem 3.9. The word "local" here refers to the fact that the fiber bundles involved all project to open subsets of Euclidean space; and the word "relative" here refers to the fact that the fiber control result discussed in this section yields a splitting which extends previously existing partial splittings. This section establishes for controlled splitting theory what section 4 established for controlled h-cobordism theory.

Section 9: We formulate a "local-relative" version of the foliated-fibered splitting Theorem 3.10, and deduce it from the results of section 8.

Section 10: We complete the proof of Theorem 3.10. This is done by "restricting" the the mapping h of 3.10 to those parts of M which are the pre-images (under h) of those portions of X that lie over the "long and thin" pieces of Y which are discussed in section 6.

4 Fibered control theory for h-cobordisms.

In this section we formulate "relative" and "local-relative" versions of Theorem 2.2. The word "local" here means that instead of considering the entire h-cobordism of the manifold X which is given in Theorem 2.2 we consider only a piece of it which lies over an open subset of the manifold Y of Theorem 2.2 ; these pieces are the "partial h-cobordisms" which are defined below. The word "relative" here means that the hypothesis of Theorem 2.2 is now strengthened to include a product structure for part of the the "partial h-cobordism", and the conclusion of Theorem 2.2 is now strenthened to extending most of the given product structure to a product structure for most of the "partial h-cobordism".

Relative fibered control theory for h-cobordisms.
Let F denote a smooth closed manifold and let

$$\rho : X \longrightarrow Y$$

denote a smooth fiber bundle projection from the Riemannian manifold X to the Riemannian manifold Y having F for fiber; X and Y need not be compact, but are assumed to have empty manifold boundaries. S denotes a compact subset of Y.

A **partial h-cobordism of X over S** consists of the following objects: two open subsets A, B of Y which satisfy $S \subset B \subset A$; a smooth cobordism pair (W, V) of the pair $\rho^{-1}(A, B)$, having boundary components $(\partial_- W, \partial_- V) = \rho^{-1}(A, B)$ and $(\partial_+ W, \partial_+ V)$; a retraction map

$$H : W \longrightarrow \rho^{-1}(A),$$

and a family of maps

$$h_t^- : V \longrightarrow W \quad \text{and} \quad h_t^+ : V \longrightarrow W, \quad t \in [0,1],$$

which depend continuously on the variable t. The cobordism V satisfies $H^{-1}(\rho^{-1}(S)) \subset V$. In addition the mappings H, h_t^- and h_t^- must satisfy the following properties: $h_1^- = H \mid V$ and $h_1^+(V) \subset \partial_+ W$; h_0^- and h_0^+ are both equal to the inclusion map $V \subset W$; $h_t^- \mid (\partial_- V)$ and $h_t^+ \mid (\partial_+ V)$ are equal to the inclusion maps

$$\partial_- V \subset W \quad \text{and} \quad \partial_+ V \subset W$$

respectively for all $t \in [0,1]$.

Let T denote any subset of S. We say that the partial h-cobordism (W, V, H, h_t^-, h_t^+) of X over S is **$(\varepsilon;\rho)$-controlled over T** if for any fixed $z \in H^{-1}(\rho^{-1}(T))$ both of the paths

$$\rho \circ H \circ h_t^-(z) \quad \text{and} \quad \rho \circ H \circ h_t^+(z), \quad t \in [0,1],$$

have diameter less than ε in Y.

A **product structure over T** for the partial h-cobordism (W, V, H, h_t^-, h_t^+) of X over S consists of an embedding

$$P : \rho^{-1}(T) \times [0,1] \longrightarrow V$$

which satisfies $P(z, 0) = z$ and $P(z, 1) \in \partial_+ V$ for all $z \in \rho^{-1}(T)$. This product structure is said to be **$(\varepsilon;\rho)$-controlled over U**, for some subset $U \subset T$, if for each $z \in \rho^{-1}(U)$ the path

$$p(t) = \rho \circ H \circ P(z, t), \quad t \in [0,1]$$

has diameter less than ε in Y.

We will need the following notation in this next theorem. For any subset Z of Y and any number $\delta > 0$ we let Z^{δ} denote the subset of all points in Y which are a distance less than or equal to δ from Z, and we let $Z^{-\delta}$ denote the subset of all points in Z which are a distance greater than or equal to δ from $Y \setminus Z$. This next theorem, which generalizes Theorem 2.2, is also due to T.A. Chapman [4] and F. Quinn [19].

Theorem 4.1. *Suppose that $dim X \geq 5$. Suppose also that $Wh(\pi_1(F) \oplus G) = 0$ for any finitely generated free abelian group G. Given compact subsets T and S of Y, with $T \subset S$, there is a number $\kappa > 0$ which depends only on the isometry type of the triple (Y, S, T); ε will denote any number in $(0, \kappa)$. Let (W, V, H, h_t^-, h_t^+) denote a partial h-cobordism of X over S which is $(\varepsilon; \rho)$-controlled over S; and let*

$$P : \rho^{-1}(T) \times [0,1] \longrightarrow V$$

denote a product structure for this partial h-cobordism over the subset $T \subset Y$ which is $(\varepsilon; \rho)$-controlled over T.

(a) There is $\varepsilon' > \varepsilon$, which depends only on ε and on the isometry type of (Y, S, T), and which satisfies

$$\lim_{\varepsilon \to 0} \varepsilon' = 0\,.$$

(b) There is another product structure

$$P' : \rho^{-1}(S^{-\varepsilon'}) \times [0,1] \longrightarrow V$$

(c) P' is $(\varepsilon'; \rho)$-controlled over $S^{-\varepsilon'}$.

(d) P and P' are equal over $T^{-\varepsilon'}$.

Local-relative fibered control theory for h-cobordisms.

When Y of Theorem 4.1 is equal to an open subset of Euclidean space $\mathbb{R}^n$ then we can strengthen the conclusions of Theorem 4.1 as follows.

Theorem 4.2. *If Y is equal an open subset of $\mathbb{R}^n$ in Theorem 4.1 then there is a number $\lambda > 0$, which depends only on n, such that ε and ε' of Theorem 4.1 are related by*

$$\varepsilon' = \lambda \varepsilon \quad .$$

Moreover the number κ of Theorem 4.1 may be taken to be any positive number.

It is not difficult to deduce Theorem 4.2 from Theorem 4.1. To do so we will need the following lemma. We will first use this lemma to complete the proof of Theorem 4.2, and then we will deduce the lemma from Theorem 4.1.

It is also true that Theorem 4.1 can be deduced from Theorem 4.2. This is the subject of homework problem 4.8 below. It would be a good idea for the reader to work through this homework problem as preparation to reading the proof for Theorem 2.8 in section 7 below.

Before stating the next lemma we need some notation. Let C denote a cell structure for $\mathbb{R}^n$ defined as follows. Let J denote the standard unit n-cube in $\mathbb{R}^n$, i.e. $(x_1, x_2, ..., x_n) \in J$ iff $0 \leq x_i \leq 1$ for all i. The n-dimensional cells of C are just the translates of J by vectors which have all integer valued coordinates; the lower dimensional cells of C are the translates of all the faces of J by these same vectors. Let K and L denote subcomplexes of C with $L \subset K$ and with $K^{1/3} \subset Y$; let (W, V, H, h_t^-, h_t^+) denote a partial h-cobordism over $K^{1/3}$; and let P denote a product structure for (W, V, H, h_t^-, h_t^+) over $L^{1/3}$

Lemma 4.3. *There is a number $\delta \in (0, 1/3)$ which depends only on $n = dimX$. Suppose that (W, V, H, h_t^-, h_t^+) is $(\delta; \rho)$-controlled over $K^{1/3}$ and that P is $(\delta; \rho)$-controlled over $L^{1/3}$. Then there is a another product structure Q for (W, V, H, h_t^-, h_t^+) over K which is $(1/3; \rho)$-controlled over K and is equal to P over L.*

Proof of Theorem 4.2

We will prove this for $Y = \mathbb{R}^n$; the proof for the general situation is handled in the same way.

Define a number $\alpha > 0$ by

4.2.1 $\alpha = \delta\varepsilon^{-1}$,

and define a map

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

to be multiplication by α. Let K and L denote the maximal subcomplexes of C which satisfy $K^{1/3} \subset f(S)$ and $L^{1/3} \subset f(T)$ respectively.

We now apply Lemma 4.3 to the partial h-cobordism (W, V, H, h_t^-, h_t^+) and to the product structure P of Theorem 4.1 and Theorem 4.2; in this application we use the bundle projection map $\rho' = f \circ \rho$ instead of the projection map ρ. Note that the control hypothesis of Theorem 4.2 and property 4.2.1 together imply that (W, V, H, h_t^-, h_t^+) and P satisfy the control hypothesis of Lemma 4.3 with respect to the projection ρ'. Thus Lemma 4.3 applies to yield a another product structure

$$Q : \rho'^{-1}(K) \times [0,1] \longrightarrow W$$

which is equal to P on $(\rho')^{-1}(L) \times [0,1]$. Now we define the desired product structure P' and number $\lambda > 0$ of Theorem 4.2 by

4.2.2.

(a) $\lambda = 3n\delta^{-1}$.

(b) $P' = Q$ on $\rho^{-1}(S^{-\lambda\varepsilon}) \times [0,1]$.

Now the conclusions of Theorem 4.2 follow from 4.2.1, 4.2.2 and from the conclusions of Lemma 4.3. This completes the proof of Theorem 4.2.

Proof of Lemma 4.3

Let $c_i, i \in I$, denote all the cells of K. Note that there is a sequence

$$I_1 \subset I_2 \subset I_3 \subset .... \subset I_{\mu_n} = I$$

of subsets of I which satisfy the following properties.

4.3.1.

(a) μ_n depends only on the number n.

(b) For any r and any $i, j \in I_{r+1} \setminus I_r$ we have that $c_i \cap c_j = \emptyset$.

(c) For any r and any $i \in I_r$ and any $j \in I_{r+1} \setminus I_r$ we have $dim(c_i) \leq dim(c_j)$.

For each $r = 1, 2, ..., \mu_n$ we set

$$K_r = \bigcup_{i \in I_r} c_i,$$

and we set K_0 equal the empty set. Note that each K_r is a subcomplex of K, and that $K_r \subset K_{r+1}$. The proof of Lemma 4.3 will be carried out by induction over the increasing sequence

$$\emptyset = K_0 \subset K_1 \subset K_2 \subset .... \subset K_{\mu_n} = K$$

of subsets of K.

Induction Hypothesis 4.3.2(r). Set $\sigma_r = 1/(3r+3)$. If the number δ is choosen sufficiently small, where how small is sufficient depends only on the numbers r and n, then there will be another number $\delta_r > \delta$, and there will be another product structure Q_r for (W, V, H, h_t^-, h_t^+) over $(L \bigcup K_r)^{\sigma_r}$, such that the following properties hold.

(a) δ_r depends only on the numbers δ, r and n.

(b) $\lim_{\delta \to 0} \delta_r = 0$.

(c) Q_r is equal to P over L^{σ_r}.

(d) Q_r is $(\delta_r; \rho)$-controlled over $(L \bigcup K_r)^{\sigma_r}$.

Note that $4.3.2(\mu_n)$ implies all the conclusions of Lemma 4.3, provided δ is choosen sufficiently small in $4.3.2(\mu_n)$ so as to assure that $\delta_{\mu_n} \leq 1/3$. Thus to complete the proof for Lemma 4.3 it will suffice to carry out the induction step

$$4.3.2(r) \to 4.3.2(r+1).$$

For each $i \in I_{r+1} \setminus I_r$ define S_i and T_i by

4.3.3.

(a) $S_i = (c_i)^{\sigma_r}$

(b) $T_i = (c_i)^{\sigma_r} \bigcap (L \bigcup K_r)^{\sigma_r}$.

We may regard (W, V, H, h_t^-, h_t^+) as a partial h-cobordism over S_i; define a product structure P_i for (W, V, H, h_t^-, h_t^+) over T_i by letting P_i equal the restriction of Q_r to $\rho^{-1}(T_i) \times [0,1]$. Note that (W, V, H, h_t^-, h_t^+) is $(\delta_r; \rho)$-controlled over S_i, and P_i is $(\delta_r; \rho)$-controlled over T_i. Thus we may apply Theorem 4.1 to this situation to conclude that there is a number $\delta_{r,i} > \delta_r$, and that there is another product structure P_i' for (W, V, h_t^-, h_t^+) over $(S_i)^{-\delta_{r,i}}$, which satisfies the following properties.

4.3.4.

(a) $\delta_{r,i}$ depends only on δ_r and on the isometry type of the triple (Y, S_i, T_i).

(b) $\lim_{\delta_r \to 0} \delta_{r,i} = 0$.

(c) P_i' is equal to Q_r over $T_i^{-\delta_{r,i}}$.

(d) P_i' is $(\delta_{r,i};\rho)$-controlled over $(S_i)^{-\delta_{r,i}}$.

Note that there is an upper bound, which depends only on the numbers r and n, to the number of distinct isometry types of the triples (Y, S_i, T_i), $i \in I_{r+1} \setminus I_r$. So, by 4.3.4(a) above, there is the same upper bound for the number of distinct numbers $\delta_{r,i}$, $i \in I_{r+1} \setminus I_r$. Thus we may define δ_{r+1} by

4.3.4.

(e)
$$\delta_{r+1} = max\{\delta_{r,i} : i \in I_{r+1} \setminus I_r\}.$$

The following properties are then deduced immediately from 4.3.4.

4.3.5.

(a) δ_{r+1} depends only on δ_r and on the numbers r and n.

(b) $\lim_{\delta_r \to 0} \delta_{r+1} = 0$; and $\delta_{r+1} > \delta_r$.

(c) P_i' is defined over $(S_i)^{-\delta_{r+1}}$ and is equal to Q_r over $T_i^{-\delta_{r+1}}$.

(d) P_i' is $(\delta_{r+1};\rho)$-controlled over $(S_i)^{-\delta_{r+1}}$.

Now we can complete the induction step as follows. Note that it follows from the definition of the numbers σ_r and σ_{r+1} given in 4.3.2(r) and 4.3.2(r+1), and from 4.3.2(r)(b), 4.3.3(a)(b) and 4.3.5(b), that if δ is choosen sufficiently small then we will have

4.3.6.

(a) $(c_i)^{\sigma_{r+1}} \subset (S_i)^{-\delta_{r+1}}$,

(b) $(c_i)^{\sigma_{r+1}} \bigcap (L \cup K_r)^{\sigma_{r+1}} \quad \subset \quad T_i^{-\delta_{r+1}}$;

and that how small is "sufficient" here depends only on the numbers r and n. Note also that it follows from 4.3.1(b) that

4.3.6.

(c) $(c_i)^{\sigma_{r+1}} \bigcap (c_j)^{\sigma_{r+1}} = \emptyset$ for any $i, j \in I_{r+1} \setminus I_r$ with $i \neq j$.

Now it follows from 4.3.6 and 4.3.5(c) that the product structure Q_{r+1} of 4.3.2(r+1) is well defined by

4.3.7.

(a) $Q_{r+1} = Q_r$ over $(L \cup K_r)^{\sigma_{r+1}}$,

(b) Q_{r+1} is equal to P_i' over $(c_i)^{\sigma_{r+1}}$ for each $i \in I_{r+1} \setminus I_r$.

This completes the proof for Lemma 4.3.

Stratified fibered version of Theorem 4.2.

In proving the foliated control Theorem 2.8 in section 7 below we shall also need a generalization of Theorem 4.2 for a stratified fiber bundle projection map

$$\rho' : X' \longrightarrow Y'$$

described as follows. Let $\rho : X \longrightarrow Y$ be as in Theorem 4.2 and we let

$$\pi : X' \longrightarrow X$$

denote bundle projection for a vector bundle over X with total space X'; we assume that this vector bundle is equipped with an inner product which gives rise to the norm $\| \quad \|$. Set $Y' = Y \times [0, \infty)$ equipped with the product metric, and define the projection ρ' by

$$\rho'(v) = (\rho \circ \pi(v), \|v\|)$$

for each $v \in X'$. We can define as before the following notions: a partial h-cobordism of X' over the subset S' of Y'; such a partial h-cobordism over S' being $(\varepsilon; \rho')$-controlled over S'; a product structure over a subset $T' \subset S'$ for such a partial h-cobordism; a product structure being $(\varepsilon; \rho')$-controlled over T'.

There is a "stratified fibered" version of Theorem 4.2 in which we simply replace the projection map $\rho : X \longrightarrow Y$ of Theorem 4.2 by the projection map $\rho' : X' \longrightarrow Y'$ just described. Unfortunately this version is not sufficiently general for the applications we have in mind in section 7 below. What is needed in section 7 is a type of "stratified fibered" version of Theorem 4.2 where the control is of a bit more exotic nature then the "$(\varepsilon; \rho')$-control" just discussed.

A partial h-cobordism (W, V, H, h_t^-, h_t^+) of X' over the subset $S' \subset Y'$ is said to be $(\varepsilon; \lambda; \rho')$-**controlled over** S' if it is $(\varepsilon + \lambda s; \rho')$-controlled over $S' \cap (Y \times [0, s])$ for all $s > 0$. A product structure P for (W, V, H, h_t^-, h_t^+) over a subset $T' \subset S'$ is said to be $(\varepsilon; \lambda; \rho')$-**controlled over** T' if P is $(\varepsilon + \lambda s; \rho')$-controlled over $T' \cap (Y \times [0, s])$ for all $s > 0$

Theorem 4.4. *Given any number $\lambda_1 > 1$ there is another number $\lambda_2 > \lambda_1$ which depends only on λ_1 and $n = \dim(Y)$. Let a, ε be positive numbers which satisfy $\lambda_2\varepsilon < a$; let T, S be compact subsets of Y with $T \subset S$; let (W, V, H, h_t^-, h_t^+) denote a partial h-cobordism of X' over*

$$S' = S \times [0, a]$$

which is $(\lambda_1\varepsilon; \lambda_1; \rho')$-controlled over S'; and let P denote a product structure for this partial h-cobordism defined over the subset

$$T' = (T \times [0, a]) \cup (S \times [\varepsilon, a])$$

and which is $(\lambda_1\varepsilon; \lambda_1; \rho')$-controlled over T'.

(a) *There is another product structure P' for (W, V, H, h_t^-, h_t^+) over $S^{-\lambda_2\varepsilon} \times [0, a]$.*

(b) *P' is $(\lambda_2\varepsilon; \lambda_2; \rho')$-controlled over $S^{-\lambda_2\varepsilon} \times [0, a]$.*

(c) *P' is equal to P over $(T^{-\lambda_2\varepsilon} \times [0, a]) \bigcup (S^{-\lambda_2\varepsilon} \times [\lambda_2\varepsilon, a])$.*

The proof of Theorem 4.4 is outlined in homeworks 4.5, 4.6 and 4.7 below.

Homework for section 4.

Homework 4.5. There is a more general version of Theorem 4.1 (cf. [19]) for fiber bundle projections $\rho : X \longrightarrow Y$ between Riemannian manifolds X and Y where the fiber of ρ is compact but not necessarily closed, and where Y has no boundary. Note that if $\partial F \neq \emptyset$ then this forces X to have a boundary ∂X; in fact the restricted projection $\rho : \partial X \longrightarrow Y$ is a fiber bundle projection having the closed manifold ∂F for fiber. In this version of Theorem 4.1 the partial h-cobordism (W, V, H, h_t^-, h_t^+) of Theorem 4.1 must contain a subcobordism (W_∂, V_∂) of the pair $(\rho^{-1}(A) \cap \partial X, \rho^{-1}(B) \cap \partial X)$; and V_∂ comes equipped with a product structure

$$P_\partial : (\rho^{-1}(B) \cap \partial X) \times [0, 1] \longrightarrow V_\partial$$

which satisfies

$$h_t^- \circ P_\partial(v, s) = P_\partial(v, (1-t)s) \quad \text{and} \quad h_t^+ \circ P_\partial(v, s) = P_\partial(v, (1-t)s + t)$$

for all $v \in \rho^{-1}(B) \cap \partial X$ and for all $t \in [0, 1]$. Also in this version of Theorem 4.1 both of the product structures P and $P\prime$ must be extensions of the product structure P_∂.

In this homework the reader is asked to give a precise formulation of the generalization of Theorem 4.1 that has just been sketched. Then deduce this generalization of Theorem 4.1 from the original Theorem 4.1.

Homework 4.6. There is also a more general version of Theorem 4.2 where the fiber F of the projection map $\rho : X \longrightarrow Y$ in Theorem 4.2 is a compact manifold but not necessarily closed. Give a precise formulation of this version of Theorem 4.2 and use the more general version of Theorem 4.1 discussed in the the preceeding homework to prove this more general version of Theorem 4.2.

Homework 4.7. Use the general version of Theorem 4.2 discussed in the preceeding homework to prove Theorem 4.4. Hint: Let

$$\pi : X' \longrightarrow X$$

denote the vector bundle over X introduced just prior to the statement of Theorem 4.4; and let $X'(\varepsilon)$ denote the subset of all vectors $v \in X'$ satisfying $\|v\| \leq \varepsilon$. Note that the restricted map

$$\rho \circ \pi : X'(\varepsilon) \longrightarrow Y$$

is a fiber bundle projection having for fiber a compact (but not necessarily closed) smooth manifold. One can construct from the partial h-cobordism (W, V, H, h_t^-, h_t^+) of X' over the subset $S' \subset Y \times [0, \infty)$ — which is given in Theorem 4.4 — a partial h-cobordism $(W(\varepsilon), V(\varepsilon), G, g_t^-, g_t^+)$ of $X'(\varepsilon)$ over a subset of Y slightly smaller than $S \subset Y$: this new partial h-cobordism is of the type discussed in 4.5 and 4.6, where the projection mapping $\rho : X \longrightarrow Y$ of 4.5 and 4.6 is now taken to be our map $\rho \circ \pi : X'(\varepsilon) \longrightarrow Y$. One can also construct from the product structure P' of Theorem 4.4 a product structure $P'(\varepsilon)$ for the partial h-cobordism $(W(\varepsilon), V(\varepsilon), G, g_t^-, g_t^+)$ over a subset slightly smaller than $T \subset Y$. The control data for (W, V, H, h_t^-, h_t^+) and for P' translates to control data for $(W(\varepsilon), V(\varepsilon), G, g_t^-, g_t^+)$ and for $P'(\varepsilon)$; thus we may apply the more general theorems discussed in 4.5 and 4.6 to complete the proof of Theorem 4.4.

Homework 4.8. Use Theorem 4.2 to prove Theorem 2.2 in the special case that $\partial Y = \emptyset$ in 2.2. Carrying out this homework, which is outlined below, will prepare the reader for the proof of the foliated control Theorem 2.8 given in section 7 below. The first step towards completing this homework is to choose a sufficiently small number $\epsilon > 0$ and a finite number of smooth charts

$$g_i : \mathbb{B}^n_\epsilon \longrightarrow Y, i = 1, 2, ..., m,$$

for Y, from the open ball of radius ϵ centered at the origin of $\mathbb{R}^n$ (with $n = dimY$), such that the following properties hold.

(a) The images $g_i(\mathbb{B}^n_{\epsilon/2})$, i=1,2,3,...,m, cover all of Y.

(b) The derivatives dg_i for the g_i satisfy

$$\|2v\| \geq \|dg_i(v)\| \geq \|v/2\|$$

for all vectors v which are tangent to $\mathbb{B}^n_\epsilon$.

(c) There is an increasing sequence

$$J^1 \subset J^2 \subset ..... \subset J^{\pi(n)} = \{1, 2, ..., m\},$$

where $\pi(n)$ depends only on n . If $k, j \in J^i \setminus J^{i-1}$ for some integers i, j, k, then $g_j(\mathbb{B}^n_\epsilon)$ and $g_k(\mathbb{B}^n_\epsilon)$ are disjoint.

To complete this homework problem one proceeds by induction over the sequence of subsets $J^1 \subset J^2 \subset ... \subset J^{\pi(n)}$. Choose a decreasing sequence of small numbers $3/4 > \epsilon_1 > \epsilon_2 > .... > \epsilon_{\pi(n)} > 1/2$. Our induction hypothesis is that a product structure P_r for the h-cobordism of 2.2 has been constructed over the subset $\bigcup_{i \in J^r} g_i(\mathbb{B}^n_{\epsilon_r})$ with good control over this same subset. To construct P_{r+1} one applies Theorem 4.2 to the "pieces" of the h-cobordism which "lie over" each subset $g_i(\mathbb{B}^n_\epsilon)$ with $i \in (J^{r+1} \setminus J^r)$; note (by (c) above) that Theorem 4.2 may be applied independently over each such subset. The relevant projections for these applications of Theorem 4.2 are maps

$$\rho_i : X_i \longrightarrow \mathbb{B}^n_\epsilon,$$

where $X_i = \rho^{-1}(g_i((B^n))$ and $\rho_i = g_i^{-1} \circ \rho$.

5 Foliated control theory for h-cobordisms.

In this section we formulate the general notion of foliated control theory, formulate a general conjecture concerning foliated control, and prove a theorem which will be used in the proof of Theorem 2.8 in section 7 below.

Let X denote a compact Riemannian manifold with boundary ∂X, and let Ξ denote a smooth foliation for X which also foliates ∂X. A path

$$p : [0, 1] \longrightarrow X$$

is said to have **Ξ-diameter less than (α, ε) in (X, Ξ)** if there is a connected subset A in some leaf L of Ξ such that the following hold: the diameter of

A in L, with respect to the Riemannian structure that L inherits from X, is less than α; any point $p(t)$, $t \in [0,1]$, is a distance in X less than ε from the subset A.

An h-cobordism (W, W_∂) of $(X, \partial X) = \partial_-(W, W_\partial)$ is said to be $(\alpha, \varepsilon; \Xi)$-**controlled** if there are deformation retracts

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

and

$$r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

for (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, such that for each $w \in W$ both of the paths

$$r_1^- \circ r_t^-(w) \quad \text{and} \quad r_1^- \circ r_t^+(w), \quad t \in [0,1]$$

have Ξ-diameters less than (α, ε) in (X, Ξ). (We say that the deformation retracts r_t^- and r_t^+ are $(\alpha, \varepsilon; \Xi)$-**controlled** if they satisfy the preceeding properties.)

The following conjecture is an attempt to generalize to the foliated setting the fibered controlled h-cobordism Theorem 2.2.

Conjecture 5.1. *Suppose each leaf L of the foliation Ξ satisfies $Wh(\pi_1(L) \oplus G) = 0$ for every finitely generated free abelian group G. Suppose also that $dim(X) \geq 6$. Then for any number $\alpha > 0$ there is a number $\varepsilon > 0$. If (W, W_∂) is an h-cobordism of X which is $(\alpha, \varepsilon; \Xi)$-controlled, then (W, W_∂) is a product cobordism.*

Special cases of Conjecture 5.1.

In the next few paragraphs we remark upon some special cases of this conjecture.

Case I: dim(Ξ)=1. In this case Theorem 2.6 above states that Conjecture 5.1 is true.

Case II: dim(Ξ)=2. Whether this special case of Conjecture 5.1 is true is not known. However in this case the hypothesis of 5.1, that $Wh(\pi_1(L) \oplus G) = 0$ for every finitely generated free abelian group G, may be dropped from Conjecture 5.1 because this equality holds automatically for any leaf L of a smooth two-dimensional foliation Ξ. To see this we first note that any such leaf L is a connected 2-dimensional manifold without boundary. Next recall that each such manifold is homeomorphic to one of the following four types of surfaces (cf. reference [23]).

(a) L is homeomorphic to the 2-sphere.

(b) L is homeomorphic to the 2-dimensional real projective space.

(c) L is homeomorphic to a closed manifold with constant sectional curvature equal 0.

(d) L is homeomorphic to a complete Riemannian manifold with constant sectional curvature equal -1.

Note that if L is of type (a) then $Wh(\pi_1(L) \oplus G) = 0$ follows from Remark 2.4; if L is of type (b) then $Wh(\pi_1(L) \oplus G) = 0$ is proven in reference [15]; if L is of type (c) or (d) then $Wh(\pi_1(L) \oplus G) = 0$ is proven in reference [12].

Case III: the leaves of Ξ have bounded diameter. The diameter refered to here is the diameter of each leaf L of Ξ computed with respect to the Riemannian metric inherited by L from X. Our assumption is that there is a finite upper bound for the diameters of all the leaves of Ξ. Note, that since X is compact, this implies that each leaf of Ξ is compact. Let Y denote the quotient space obtained from X by collapsing each leaf of Ξ to a point and let

$$\rho : X \longrightarrow Y$$

denote the quotient map. Note that Y can be equipped with a finite triangulation (in fact it inherits a PL structure from the smooth structure on X); and the map $\rho : X \longrightarrow Y$, although not in general a fiber bundle projection, is the projection map of a "stratified fibration" in the sense of Quinn (cf. reference [20]). Quinn has proven in reference [20] a more general "stratified" version of Theorem 2.2 in which the fiber bundle projection of Theorem 2.2 may be replaced by the projection map for any "stratified fibration". In our present context, Quinn's "stratified fibration" version of Theorem 2.2 allows us to apply the conclusions of Theorem 2.2 to the quotient map $\rho : X \longrightarrow Y$. Note that if W is an h-cobordism of X which is $(\alpha, \varepsilon; \Xi)$-controlled, then W is also $(\varepsilon'; \rho)$-controlled with respect to some metric on Y, where

$$\lim_{\varepsilon \to 0} \varepsilon' = 0.$$

Thus Quinn's "stratified fibration" version of Theorem 2.2 may be applied to conclude that W is a product. Thus we have proven (or more precisely Frank Quinn has proven) the following theorem.

Theorem 5.2. *If the leaves of Ξ have bounded diameter then Conjecture 5.1 is true.*

Case IV: each leaf of Ξ has non-postive sectional curvature. Each leaf L inherits a Riemannian structure from X; we require that L has non-positive sectional curvature values everywhere with respect to this structure. In this case Conjecture 5.1 is known to be true, provided we replace the word "controlled" in 5.1 by the phrase "simply-controlled", which we define now.

Let X^{cov} denote the universal covering space for X and let Ξ^{cov} denote the lifting to X^{cov} of the foliation Ξ; X^{cov} is equipped with a Riemannian structure lifted from that on X. Note that any h-cobordism (W, W_∂) of $(X, \partial X)$ lifts to an h-cobordism $(W^{cov}, W_\partial^{cov})$ of $(X^{cov}, \partial X^{cov})$. An h-cobordism (W, W_∂) of $(X, \partial X)$ is said to be $(\alpha, \varepsilon; \Xi)$-**simply-controlled** if there are deformation retracts

$$r_t^- : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1],$$

and

$$r_t^+ : (W, W_\partial) \longrightarrow (W, W_\partial), \quad t \in [0,1]$$

for (W, W_∂) onto $\partial_-(W, W_\partial)$ and $\partial_+(W, W_\partial)$ respectively, which lift to $(\alpha, \varepsilon; \Xi^{cov})$-controlled deformation retracts retracts for $(W^{cov}, W_\partial^{cov})$ onto $\partial_-(W^{cov}, W_\partial^{cov})$ and $\partial_+(W^{cov}, W_\partial^{cov})$ respectively.

Theorem 5.3. *Suppose that each leaf of Ξ has non-positive sectional curvature values everywhere. For any number $\alpha > 0$ there is a number $\varepsilon > 0$ which depends only on α and Ξ. Any $(\alpha, \varepsilon; \Xi)$-simply-controlled h-cobordism (W, W_∂) of $(X, \partial X)$ is a product cobordism.*

Note that Theorem 5.3 is a corollary of Theorem 1.1 and the s-cobordism theorem in the special case that the there is only the one leaf X in the foliation Ξ. Theorem 5.3 is proven in [10] for the special case that each leaf L of Ξ has constant sectional curvature -1; to prove Theorem 5.3 in general one should follow the ideas in the proof of [10], replacing the "asymptotic transfer" constructions used in that proof by the "focal transfer" constructions used in references [13] and [14]. (As of yet no one has worked out the details of this proof.) For more information about the "asymptotic and focal transfer" constructions the reader is referred to Tom Farrell's lecture notes for this school.

Foliated-control over Euclidean space.

In this subsection we deduce a very simple foliated control result from the the results of section 4. Let

$$\rho : X \longrightarrow Y$$

denote the fiber bundle projection of Theorem 4.2 above; thus Y is an open subset of Euclidean space $\mathbb{R}^n$. We let Γ denote a foliation of $\mathbb{R}^n$ whose leaves are gotten by choosing a plane in $\mathbb{R}^n$ and translating this plane to every point of $\mathbb{R}^n$. A foliation Ξ for Y is gotten by just restricting Γ to Y.

We say that a path

$$p : [0,1] \longrightarrow Y$$

has **Ξ-diameter less than** (α, ε), for positive numbers α and ε, if there is a connected subset $A \subset L$ lying in a leaf $L \in \Xi$ such that the diameter of A in L — computed with respect to the Riemannian structure which L inherits from Y — is less than α, and such that each point $p(t)$ on the path is a distance less than ε from A. For any subset $S \subset Y$ and for any positive numbers α and ε we denote by $S^{\alpha,\varepsilon}$ the set of all points $y \in Y$ for which there is a path p in Y which starts in S and ends at y and which has Ξ-diameter less than (α, ε). We denote by $S^{-\alpha,-\varepsilon}$ the difference subset $S \setminus (Y \setminus S)^{\alpha,\varepsilon}$.

In the next theorem we consider a partial h-cobordism $(W, V, , H, h_t^-, h_t^+)$ over the compact subset $S \subset Y$ (as defined in section 4), and a product structure

$$P : \rho^{-1}(T) \times [0,1] \longrightarrow W$$

for (W, V, H, h_t^-, h_t^+) over the compact subset $T \subset S$ (also as described in section 4). We say that this partial h-cobordism is **$(\alpha, \varepsilon; \Xi, \rho)$-controlled** over S if for each $w \in W$ both of the paths

$$\rho \circ H \circ h_t^-(w) \quad \text{and} \quad \rho \circ H \circ h_t^+(w), \quad t \in [0,1],$$

have Ξ-diameter less than (α, ε) in (Y, Ξ). The product structure P is said to be **$(\alpha, \varepsilon; \Xi, \rho)$-controlled** over T if for each $x \in \rho^{-1}(T)$ the path

$$\rho \circ H \circ P(x,t) \quad , \quad t \in [0,1],$$

has Ξ-diameter less than (α, ε) in (Y, Ξ).

Theorem 5.4. *Suppose that the fiber F of $\rho : X \longrightarrow Y$ satisfies $Wh(\pi_1(F) \oplus G) = 0$ for all finitely generated free abelian groups G; suppose also that dimension$(X) \geq 5$. There is positive number λ which depends only on the integer n. Suppose that the partial h-cobordism $(W, V, , H, h_t^-, h_t^+)$ is $(\alpha, \varepsilon; \Xi, \rho)$-controlled over S, and that the product structure P is also $(\alpha, \varepsilon; \Xi, \rho)$-controlled over T, for some positive numbers α and ε satisfying $\alpha \geq \varepsilon$. Then the following properties hold.*

(a) There is another product structure P' for the partial h-cobordism defined over $S^{-\lambda\alpha,-\lambda\varepsilon}$.

(b) P' is equal to P over $T^{-\lambda\alpha,-\lambda\varepsilon}$.

(c) P' is $(\lambda\alpha,\lambda\varepsilon;\Xi,\rho)$-controlled over $S^{-\lambda\alpha,-\lambda\varepsilon}$.

Proof of Theorem 5.4

This is really an immediate corollary of Theorem 4.2 in section 4. To see this we let $f:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the linear map that multiplies a vector v by the number $\alpha^{-1}\varepsilon$ if v is tangent to the foliation Γ, and which multiplies v by 1 if v is perpendicular to the foliation Γ. Note that the partial h-cobordism $(W,V,,H,h_t^-,h_t^+)$ is $(2\varepsilon; f\circ\rho)$-controlled over $f(S)$, and that P is $(2\varepsilon; f\circ\rho)$-controlled over $f(T)$. Thus we may apply Theorem 4.2 to conclude that there there is another product structure P' over $f(S)^{-\lambda(2\varepsilon)}$ which is $(\lambda(2\varepsilon); f\circ\rho)$-controlled over this same set, where λ is the number in Theorem 4.2. This translates into the desired conclusions for Theorem 5.4 provided the λ of 5.4 is chosen to be equal to four times the value of λ in 4.2.

This completes the proof of Theorem 5.4.

6 Long and thin charts for a one-dimensional foliation.

In this section we consider a smooth one-dimensional foliation Ξ of a closed Riemannian manifold Y. We formulate two lemmas which describe a nice covering of Y by "long and thin" open subsets of Y (long in the foliation direction and thin in the direction perpendicular to the foliation). In section 7 we will complete the proof of Theorem 2.8 by applying fiber control results of sections 4 and 5 to the pieces of the h-cobordism in 2.8 which lie over each long and thin set in the open covering of Y

Let γ denote a given positive number and let Ξ^γ denote all the closed leaves of Ξ which have length less than or equal to γ. We define a sequence of subsets

6.0.

$$\emptyset = \Xi_0^\gamma \subset \Xi_1^\gamma \subset \Xi_2^\gamma \subset \ldots \subset \Xi_z^\gamma = \Xi^\gamma$$

as follows: each Ξ_i^γ is a union of closed leaves in Ξ; a leaf L in Ξ_i^γ is also a leaf of Ξ_{i-1}^γ iff there is a tubular neighborhood A for L in Y, with projection

$$A \longrightarrow L,$$

and a sequence of leaves $L_j, j=1,2,3,\ldots$, in Ξ_i^γ which converge to L, such that each composite map $L_j \subset A \longrightarrow L$ is an n_j-fold covering map for L with $n_j > 1$. The following lemma is easy to verify.

Lemma 6.1. *Let ν denote the length of the shortest closed leaf of Ξ. Then the number z in 6.0 depends only on ν and γ.*

The next lemma, which is more difficult to verify, is proven in sections 7 and 8 of reference [9]. We shall need some notation and a definition before stating this next lemma.

Let n denote the dimension of Y, and let U denote an open subset of Euclidean space $\mathbb{R}^{n-1}$, and let $(-r, r)$ denote the interval of length $2r$ centered at 0 in the real numbers. A smooth map

$$g : (-r, r) \times U \longrightarrow Y$$

is a called a **rectangular foliation chart** for Ξ if the following properties hold: g is a one-one immersion; for each $u \in U$ the path $g(t, u)$ has unit speed and is contained in a leaf of Ξ.

Lemma 6.2. *Let C denote a compact subset of $Y \setminus \Xi^\gamma$. There are numbers $\sigma_1, \sigma_2, \epsilon_1$, $\epsilon_2 \in (0, 1)$; the σ_i depend only on $n = dim(Y)$; ϵ_1 depends only on γ and Ξ; ϵ_2 depends only on C, γ and Ξ. There is a finite collection*

$$g_i : (-r_i, r_i) \times U_i \longrightarrow Y \quad , i \in I,$$

of rectangular foliation charts for Ξ such that the following properties hold.

(a) We have that $\sigma_1\gamma < r_i$ for all $i \in I$.

(b) The images $g_i((-\sigma_2 r_i, \sigma_2 r_i) \times U_i^{-\epsilon_2})$, $i \in I$, cover C.

(c) For each $i \in I$ and for each vector v which is tangent to $(-r_i, r_i) \times U_i$, we have that $\epsilon_1\|v\| \leq \|dg_i(v)\| \leq (\epsilon_1)^{-1}\|v\|$.

(d) The index set I is a disjoint union $I = I_1 \cup I_2 \cup ... \cup I_{n+1}$ of subsets. For any integer k satisfying $1 \leq k \leq n+1$, and for any $i, j \in I_k$, the images of g_i and of g_j are disjoint.

The following lemma is an easy corollary of Lemma 6.2.

Lemma 6.3. *There are numbers $\lambda, \mu > 1$; with λ depending only on $n = \dim(Y)$; and with μ depending only on γ and Ξ. For any of the rectangular charts g_i of Lemma 6.2, and for any path*

$$g : [0, 1] \longrightarrow \text{image}(g_i)$$

the following are true.

(a) *If the path g has Ξ-diameter less than (α, ε) then the path $g_i^{-1} \circ g$ has Ω-diameter less then $(\lambda\alpha, \mu\varepsilon)$, where Ω denotes for foliation of $(-r_i, r_i) \times U_i$ by the lines $(-r_i, r_i) \times x$ with $x \in U_i$.*

(b) *If the path $g_i^{-1} \circ g$ has Ω-diameter less than (α, ε) then the path g has Ξ-diameter less then $(\lambda\alpha, \mu\varepsilon)$.*

We will need some more notation and definitions before stating the final lemma of this section.

Let $\mathbb{R}^k \times S^1$ denote the product of Euclidean space $\mathbb{R}^k$ with the unit circle S^1. Let

$$p : \tau \longrightarrow \mathbb{R}^k \times S^1$$

denote the projection map for a vector bundle over $\mathbb{R}^n \times S^1$ which is equipped with an inner product structure: for each $v \in \tau$ let $\|v\|$ denote the length of v. We denote the first factor map of p by

$$p_1 : \tau \longrightarrow \mathbb{R}^k$$

Define another map

$$q : \tau \longrightarrow \mathbb{R}^k \times [0, \infty)$$

by $q(v) = (p_1(v), \|v\|)$. For any numbers $r, s > 0$ set

$$\tau(r, s) = q^{-1}(\mathbb{B}_r^k \times [0, s)),$$

where $\mathbb{B}_r^k$ denotes the open ball of radius r centered at the origin of $\mathbb{R}^k$, and set

$$\tau(r) = \tau(r, r).$$

A mapping

$$f : \tau(r) \longrightarrow Y$$

is called a **circular chart for** Y if the following properties hold: $\dim(\tau) = \dim(Y)$; f is a one-one smooth immersion. We are interested in circular charts that are "good approximations" to those parts of the foliation Ξ which contain leaves which are closed or "almost closed".

One could discuss what this means analytically as follows. Note that the zero section of $\tau(r)$ is equal to $\mathbb{B}_r^k \times S^1$. One could insist that for each $b \in \mathbb{B}_r^k$ the restricted map

$$f : b \times S^1 \longrightarrow Y$$

parametize a path in Y which has constant speed and is "very close" in some analytical sense to a leaf of Ξ (not necessarily a closed leaf) – this would be analougous to the requirement for rectangular foliation charts

$$g : (-r, r) \times U \longrightarrow Y$$

that for each fixed $u \in U$ the path $g(t,u)$ has unit speed and lies in a leaf of Ξ. Furthemore one could place "curvature bounds" on the submanifold $f(\mathbb{B}^k_r \times S^1) \subset Y$; insist that $f(\tau(r))$ is equal to the standard tubular neighborhood for $f(\mathbb{B}^k_r \times S^1)$ in Y of radius r; and insist that there are bounds on the derivative of f which are analogous to the bounds for rectangular foliation charts given in Lemma 6.2(c).

We prefer not to persue here in detail this analytic definition of what it means for a circular chart to be a "good approximation" to Ξ. But rather we take, as our definition of a "good approximation" to Ξ, what would be some of the metric consequences of a viable analytic definition for this phrase. To get into the spirit of this alternate definition the reader should note that in our applications (in section 7) of the rectangular foliation charts of Lemma 6.2 we shall use only the metric and set theoretic properties of these charts recorded in Lemma 6.2(a)(b)(d) and in Lemma 6.3(a)(b).

A circular chart $f : \tau(r) \longrightarrow Y$ is called a $(\alpha;\beta;\mu)$ **− approximation to** Ξ, for postive real numbers α, β and μ, if the following properties hold for any path

$$g : [0,1] \longrightarrow f(\tau(r)) \quad .$$

(a) If g has Ξ-diameter less than (α, ε), for $\varepsilon > \beta$, then the path $q \circ f^{-1} \circ g$ must have diameter less than $\mu \| f^{-1}(g(0)) \| + \mu\varepsilon$ in $\mathbb{R}^n \times [0,\infty)$.

(b) If the path $q \circ f^{-1} \circ g$ has diameter less than ε in $\mathbb{R}^n \times [0,\infty)$, for $\varepsilon > \beta$, then the path g must have Ξ-diameter less than $(\alpha, \mu \| f^{-1}(g(0)) \| + \mu\varepsilon)$ in (Y, Ξ).

A proof for the next lemma can be found in sections 1 and 2 of reference [10].

Lemma 6.4. *There is a real number $\mu > 1$, and a continuous function*

$$h : [0,\infty) \longrightarrow [0,\infty)$$

satisfying $h(0) = 0$, both of which depend only on the number γ and on the foliation Ξ. Let the number z be as in 6.0, and for $j \in \{1,2,3,...,z\}$ let $C(j)$ denote any compact subset of $\Xi^\gamma_j \setminus \Xi^\gamma_{j-1}$. There is a number $\delta_j > 0$ which depends on $C(j)$, γ and on Ξ. For each $\epsilon_j \in (0,\delta_j)$ there is a finite collection of circular charts $f_{j,i} : \tau_{j,i}(\epsilon_j) \longrightarrow Y$, $i \in I_j$, for Y, all of which satisfy the following properties.

(a) Each $f_{j,i}$ is a $(\gamma; h(\epsilon_j)\epsilon_j; \mu)$-approximation to Ξ.

(b) Each set $f_{j,i}(\tau_{j,i}(\epsilon_j)) \setminus f_{j,i}(\tau_{j,i}(\epsilon_j, h(\epsilon_j)\epsilon_j))$ is disjoint from Ξ^γ_j.

(c) *The sets* $f_{j,i}(\tau_{j,i}(\epsilon_j/2, h(\epsilon_j)\epsilon_j))$*, for* $i \in I_j$*, cover* $C(j)$*.*

(d) *The index set* I_j *is a disjoint union*

$$I_j = I_{j,1} \cup I_{j,2} \cup ... \cup I_{j,s},$$

where s *is some positive integer which depends only on* λ *and* Ξ*. For each* $t = 1, 2, 3, ..., s$*, and for each* $i, i' \in I_{j,t}$*, we have that the images of* $f_{j,i}$ *and of* $f_{j,i'}$ *are disjoint.*

7 Proof of Theorem 2.8.

We will prove Theorem 2.8 for the special case when $\partial Y = \emptyset$. It is not difficult to deduce the general version of Theorem 2.8 from this special case (cf. Homework 7.5).

Let $\rho : X \longrightarrow Y$ denote the smooth fiber bundle map of Theorem 2.8, and let

$$\emptyset = \Xi_0 \subset \Xi_1^\gamma \subset \Xi_2^\gamma \subset .... \subset \Xi_z^\gamma = \Xi^\gamma$$

be the increasing sequence of subsets of Y described in 6.0 above. The proof of Theorem 2.8 proceedes by induction over the increasing sequence of subspaces

$$Y \setminus \Xi_z^\gamma \subset Y \setminus \Xi_{z-1}^\gamma \subset Y \setminus \Xi_{z-2}^\gamma \subset .... \subset Y \setminus \Xi_1^\gamma \subset Y \setminus \Xi_0^\gamma = Y \quad .$$

The first step in our induction argument is carried out in Lemma 7.1 below. The induction hypothesis is the same as the hypothesis of Lemma 7.3 below; the induction step is carried out in Lemma 7.3.

For any subset $C \subset Y$ and for any numbers $\alpha, \varepsilon > 0$ we denote by $C^{\alpha,\varepsilon}$ the subset of all $y \in Y$ for which there is a path $g : [0,1] \longrightarrow Y$ which begins in C and ends at y and has Ξ-diameter less than (α, ε) in (Y, Ξ). We define the subset $C^{-\alpha,-\varepsilon}$ by

$$C^{-\alpha,-\varepsilon} = C \setminus (Y \setminus C)^{\alpha,\varepsilon}.$$

Let W denote an h-cobordism of X and let

$$P : \rho^{-1}(C) \times [0,1] \longrightarrow W$$

denote a product structure for W over the subset $C \subset Y$. We say that P is $(\alpha, \varepsilon; \Xi, \rho)$**-controlled over** C if there is a deformation retraction

$$r_t^- : W \longrightarrow W \ , \ t \in [0,1]$$

of W into $\partial_- W$ such that for each $x \in \rho^{-1}(C)$ the path

$$\rho(r_1^-(P(x,t)))\ ,\ t \in [0,1]$$

has Ξ-diameter less than (α, ε).

Lemma 7.1. *There is number $1 > \lambda > 0$, which depends only on the number $n = dimY$; there is a number $\omega > 1$, which depends only on the the number γ and on the foliation Ξ. Given an arbitrarily large compact subset C of $Y \setminus \Xi^\gamma$, there is a number $\delta \in (0,1)$, which depends only on C, γ and Ξ. For any numbers $\alpha \in (0, \lambda\gamma)$ and $\varepsilon \in (0,\delta)$, and for any h-cobordism W of X which is $(\alpha, \varepsilon; \Xi, \rho)$-controlled, there is a product structure P for W defined over $C^{-\gamma,-\omega\varepsilon}$, which is $(\gamma, \omega\varepsilon; \Xi, \rho)$-controlled over $C^{-\gamma,-\omega\varepsilon}$.*

Remark 7.2. In Lemma 7.1 if a pair of deformation retractions $r_t^\pm$ for the h-cobordism are given with respect to which the control data for W is as stated in the lemma, then the product structure P may be choosen so that its control data with respect to the same deformation retractions $r_t^\pm$ is also as stated in 7.1.

Lemma 7.3. *Choose a number j satisfing $1 \le j \le z$ and let C_j denote an arbitrarily large compact subset of $Y \setminus \Xi_j^\gamma$. There is a number $\omega_j > 1$ which depends only on j,γ and on the foliation Ξ; and there is another number $\delta_j \in (0,1)$ which depends only on j, γ, C_j and on Ξ. For any number $\varepsilon_j \in (0, \delta_j)$ let W denote an h-cobordism of X which is $(\gamma, \varepsilon_j; \Xi, \rho)$-controlled, and let P_j denote a product structure for W defined over C_j which is $(\gamma, \varepsilon_j; \Xi, \rho)$-controlled over C_j. Then there is an arbitrarily large compact subset C_{j-1} of $Y \setminus \Xi_{j-1}^\gamma$, and a product strucuture P_{j-1} for W over C_{j-1}, which satisfy the following properties.*

(a) $C_j^{-\gamma,-\omega_j\varepsilon_j} \subset C_{j-1}$.

(b) P_{j-1} is equal to P_j over the subset $C_j^{-\gamma,-\omega_j\varepsilon_j}$ of Y.

(c) P_{j-1} is $(\gamma, \omega_j\varepsilon_j; \Xi, \rho)$-controlled over the subset C_{j-1} of Y.

Remark 7.4. In Lemma 7.3 if a pair of deformation retractions $r_t^\pm$ for the h-cobordism are given with respect to which the control data for W and P_j are as stated in the lemma, then the product structure P_{j-1} may be choosen so that its control data with respect to the same deformation retractions $r_t^\pm$ is also as stated in 7.3.

Proof of Lemma 7.1.

We cover the set C by a finite number of foliation charts $\{g_i : i \in I\}$ as provided in Lemmas 6.2 and 6.3. Let

$$I = I_1 \cup I_2 \cup \ldots \cup I_{n+1}$$

denote the disjoint union given in Lemma 6.2(d). The proof proceedes by induction over the sequence $I_1, I_2, ..., I_{n+1}$. The induction hypothesis consists of assuming that the product structure has already been constructed over the union

$$D(r) \;=\; \bigcup_{i\in I(r)} (\text{image}(g_i))^{-\lambda_{r,1}\alpha, -\omega_r\varepsilon} \quad ,$$

where

$$I(r) = I_1 \cup I_2 \cup ... \cup I_r \quad ,$$

and where $\lambda_{r,1}, \omega_r, \alpha, \varepsilon$ are the numbers given in 7.1.1(r) below. The induction step consists of applying Theorem 5.4, and Lemmas 6.2 and 6.3, "over" each each of the subsets image(g_i) with $i \in I_{r+1}$.

Now we will fill in some of the metric "control" details for this proof. We begin with a careful statement of the induction hypothesis for this argument.

7.1.1(r) Induction Hypothesis. There are numbers $\lambda_{r,1} > 1 > \lambda_{r,2} > 0$ which depend only on the numbers $n = dimY$ and r. There is a number $\omega_r > 1$ which depends only on the numbers γ, r and on the foliation Ξ. There is also a number $\delta_r \in (0, 1)$ which depends on C, Ξ and on the numbers r, γ. For any numbers $\alpha \in (0, \lambda_{r,2}\gamma)$ and $\varepsilon \in (0, \delta_r)$, and for any $(\alpha, \varepsilon; \Xi, \rho)$-controlled h-cobordism W of X, there is a product structure P_r for W defined over $D(r)$ which is $(\lambda_{r,1}\alpha, \omega_r\varepsilon; \Xi, \rho)$-controlled over $D(r)$. Moreover, if the control data for W is measured with respect to deformation retractions $r_t^{\pm}$, then P_r may be choosen so that its control data is also measured with respect to these same deformation retractions $r_t^{\pm}$.

Before carrying out the induction step

$$7.1.1(r) \implies 7.1.1(r+1)$$

we must first introduce the following notation.

For each $i \in I_{r+1}$ we consider the rectangular foliation chart

$$g_i : (-r_i, r_i) \times U_i \longrightarrow Y$$

of Lemma 6.2. We denote by

$$\rho_i : X_i \longrightarrow Y_i$$

the composition of the projection map $\rho : X \longrightarrow Y$ of Theorem 2.8 with the inverse of the map $g_i : (-r_i, r_i) \times U_i \longrightarrow Y$. We define a partial h-cobordism

$$(W_i, V_i, H_i, (h_i^-)_t, (h_i^+)_t)$$

as follows. Set $W_i = (r_1^-)^{-1}(X_i)$; and set $H_i = (r_1^-) \mid (W_i)$. Define V_i to be the maximal open subset of W_i of the form $V_i = (\rho_i \circ H_i)^{-1}(B_i)$ — with $B_i \subset Y_i$ — which satisfies $r_t^-(V_i) \subset W_i$ and $r_t^+(V_i) \subset W_i$ for all $t \in [0,1]$; and set $(h_i^-)_t = r_t^- \mid V_i$ and set $(h_i^+)_t = r_t^+ \mid V_i$. We set $D(r,i) = (g_i)^{-1}(D(r))$. Finally we choose S_i to be an arbitrarily large compact subset of $B_i = (\rho \circ H_i)(V_i)$, and we choose T_i to be an arbitrarily large compact subset of $D(r,i) \cap S_i$. Note that $(W_i, V_i, H_i, (h_i^-)_t, (h_i^+)_t)$ is a partial h-cobordism over the subset $S_i \subset Y_i$; and $P_{r,i}$, defined by

$$P_{r,i} \;=\; P_r \mid \rho_i^{-1}(T_i) \times [0,1],$$

is a product structure for the partial h-cobordism over T_i.

Our induction step

$$7.1.1(r) \implies 7.1.1(r+1)$$

consists of applying Theorem 5.4 to each of the the partial h-cobordisms $(W_i, V_i, H_i, (h_i^-)_t, (h_i^+)_t)$ and to each of the product structures $P_{r,i}$. Note that Y_i is open subset of Euclidean space as is required in 5.4. Note that the lines $(-r_i, r_i) \times u$, for $u \in U_i$, define a smooth foliation for Y_i which we denote by Ξ_i. Note that Ξ_i is a foliation of Y_i by parallel planes (one-dimensional planes in this case) as is required in Theorem 5.4. It follows from Lemma 6.3 that the control data for the h-cobordsim (W, r_t^-, r_t^+) given in 7.1.1(r) translate into control data for the partial h-cobordisms $(W_i, V_i, H_i, (h_i^-)_t, h_i^+)_t)$ over the subsets $S_i \subset Y_i$. It also follows from Lemma 6.3 that the control data for the partial product structure P_r given in 7.1.1(r) translate into control data for the partial product structures $P_{r,i}$ over the subsets $T_i \subset Y_i$. Thus, by Theorem 5.4, there are slightly smaller subsets T_i^- and S_i^- of T_i and S_i respectively (these subsets are given in 5.4(a)(b) respectively), and there is another controlled product structure $P'_{r,i}$ for $(W_i, V_i, H_i, (h_i^-)_t, (h_i^+)_t)$ over S_i^- which is equal to $P_{r,i}$ over T_i^-. (The control data for $P'_{r,i}$ is given in 5.4.)

We can now complete our induction step as follows. We define P_{r+1} in 7.1.1(r+1) by

7.1.2

(a) $P_{r+1} = P_r$ over $\bigcup_{i \in I_r} \text{image}(g_i)^{-\lambda_{r+1,1}\alpha, -\omega_{r+1}\varepsilon}$.

(b) $P_{r+1} = P'_{r,i}$ over $(\text{image}(g_i))^{-\lambda_{r+1,1}\alpha, -\omega_{r+1}\varepsilon}$ for each $i \in I_{r+1}$.

It follows from the information given concerning $\lambda_{r,1}, \lambda_{r,2}$ and ω_r in 7.1.1(r), from the information given about the number λ in Theorem 5.4, and from the information given about the numbers λ and μ in Lemma 6.3, that there are numbers $\lambda_{r+1,1}, \lambda_{r+1,2}, \omega_{r+1}$ which statisfy all the properties listed in 7.1.1(r+1) when the product structure P_{r+1} of 7.1.1(r+1) is given by 7.1.2. (Note in particular we must require that $\lambda_{r+1,1} > \lambda_{r,1}$ and $\omega_{r+1} > \omega_r$ in order that the description of P_{r+1} given in 7.1.2 make sense.)

Note that Lemma 7.1 now follows from 7.1.1(n+1) provided we choose λ and ω of Lemma 7.1 as follows.

7.1.3
(a) $\lambda = \text{minimum}\{\lambda_{n+1,2}, (\lambda_{n+1,1})^{-1}\}$.
(b) $\omega = \omega_{n+1}$.

For further details about this proof see reference [10].
This completes the proof of Lemma 7.1.

Proof of Lemma 7.3.
We note that if $j = z$ in Lemma 7.3, then Lemma 7.3 is an immediate consequence of Lemma 7.1. So in the rest of this proof we assume that $0 < j < z$.

Let $C(j)$ denote an arbitrarly large compact subset of $\Xi_j^\gamma \setminus \Xi_{j-1}^\gamma$. We cover the set $C(j)$ by a finite number of circular charts $\{f_{j,i} : i \in I_j\}$ for Y as indicated in Lemma 6.4. Let

$$I_j = I_{j,1} \cup I_{j,2} \cup ... \cup I_{j,s}$$

denote the disjoint union given in Lemma 6.4(d). The proof proceedes by induction over the sequence $I_{j,1}, I_{j,2}, ..., I_{j,s}$. The induction hypothesis consists of assuming that the restriction of the given product structure P_j to $\rho^{-1}(C(j;r)) \times [0,1]$ — where $C(j;r)$ is a slightly smaller set than $C(j)$ — has been extended to a product structure $P_{j,r}$ over the union

$$\bigcup_{i \in I_j(r)} E(i;r),$$

where

$$I_j(r) = I_{j,1} \cup I_{j,2} \cup ... \cup I_{j,r} \ ,$$

and where $E(i;r)$ is a large compact subset of image$(f_{j,i})$. The induction step for this proof (which is analogous to the induction step in the preceeding proof) consists of applying Theorem 4.4 “over” large compact subsets of each of the subsets image$(f_{j,i})$ with $i \in I_{j,r+1}$.

Further details can be found in reference [10].

This completes the proof for Lemma 7.3.

Homework 7.5. Use the special case of Theorem 2.8 when $\partial Y = \emptyset$ to prove the more general case of Theorem 2.8 when $\partial Y \neq \emptyset$. (Hint: If $\partial Y \neq \emptyset$ then we denote $Y\sharp Y$ the union of two copies of Y with their boundaries identified. There is the foliation $\Xi\sharp\Xi$ for $Y\sharp Y$ and a fiber bundle projection $\rho\sharp\rho : X\sharp X \longrightarrow Y\sharp Y$. Without loss of generality we may assume that the h-cobordism $(W, W_\partial, r_t^-, r_t^+)$ of $(X, \partial X)$ satisfies the following properties: W_∂ is equal to the product $\partial X \times [0,1]$; for each $(x,s) \in \partial X \times [0,1]$ and for each $t \in [0,1]$ we have that $r_t^-(x,s) = (x,(1-t)s)$ and $r_t^+(x,s) = (x,(1-t)s+t)$. Thus we may extend the h-cobordism (W, r_t^-, r_t^+) of X to an h-cobordism (V, s_t^-, s_t^+) of $X\sharp X$ by setting

$$V = W\sharp(X \times [0,1]) \ ,$$

$$s_t^- \mid W = r_t^- \ , \ s_t^+ \mid W = r_t^+ \ ,$$

$$s_t^-(x,s) = (x,(1-t)s) \text{ and } s_t^+(x,s) = (x,(1-t)s+t)$$

for all $(x,s) \in X \times [0,1]$. Note that if $(W, W_\partial, r_t^-, r_t^+)$ is $(\alpha, \varepsilon; \Xi, \rho)$-controlled, then (V, s_t^-, s_t^+) is also $(\alpha, \varepsilon; \Xi\sharp\Xi, \rho\sharp\rho)$-controlled. Thus we may apply the special case of Theorem 2.8 just proven — note that the base space $Y\sharp Y$ of $\rho\sharp\rho$ is a closed manifold — to show that V is a product cobordism. It is not difficult now to conclude that W must also be a product cobordism.)

Homework 7.6. In the proof given above for Lemma 7.1 find explicit formulae for the constants $\lambda_{r+1,1}, \lambda_{r+1,2}, \omega_{r+1}$ of 7.1.1(r+1) in terms of the numbers $\lambda_{r,1}, \lambda_{r,2}, \omega_r$ of 7.1.1(r) and the constants described in Theorem 5.4 and Lemma 6.3.

Homework 7.7. Carefully formulate the induction hypothesis and the induction step referred to in the proof of Lemma 7.3 above. Also formulate the desired relation between the sets C_j, C_{j-1} and $C(j)$ in the proof of Lemma 7.3.

8 Fibered-controlled splitting theory.

In this section we formulate "relative" and "relative-local" version of Theorem 3.9, which are analogous to the "relative" and "relative-local" versions of Theorem 2.2 formulated in section 4 above (cf. Theorems 4.1 and 4.2).

We also formulate a stratified version of Theorem 3.9 which is analogous to the stratified version of Theorem 2.2 formulated in Theorem 4.4.

The word "local" here means instead of considering the entire homotopy equivalence

$$h : (M, \partial M) \longrightarrow (X, \partial X)$$

which is given in Theorem 3.9 we consider only a piece of it which lies over an open subset of the manifold Y of 3.9; these pieces are the "partial homotopy equivalences" which are defined below. The word "relative" here means that the hypothesis of Theorem 3.9 is now strengthened to include a splitting for part of the "partial homotopy equivalence", and the conclusion of Theorem 3.9 is also now strenthened to extending most of this given splitting to a splitting of most of the "partial homotopy equivalence".

Relative fibered-controlled splitting theory.

For the time being we let F denote a closed manifold and we let

$$\rho : X \longrightarrow Y$$

denote a smooth fiber bundle projection from the smooth manifold X to the Riemannian manifold Y having F for fiber; X and Y need not be compact but are assumed to have empty boundaries. Let S denote a compact subset of Y.

A **partial homotopy equivalence into X over S** consists of the following objects: open subsets A, B of Y which satisfy $S \subset B \subset A$; a proper continuous map

$$h : M \longrightarrow \rho^{-1}(A)$$

from the manifold M (with $\partial M = \emptyset$); a continous map

$$g : \rho^{-1}(B) \longrightarrow M \quad ;$$

and homotopies

$$(h \circ g)_t : \rho^{-1}(B) \longrightarrow \rho^{-1}(A)$$

from $h \circ g$ to the identity map $1_{\rho^{-1}(B)}$, and

$$(g \circ h)_t : (\rho \circ h)^{-1}(B) \longrightarrow M$$

from $g \circ h$ to the identity map $1_{(\rho \circ h)^{-1}(B)}$. We shall denote this partial homotopy equivalence by $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$.

The partial homotopy equivalence $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ into X over the subset $S \subset Y$ is said to be $(\varepsilon; \rho)$**-controlled over a subset** $T \subset S$ if for every $x \in (\rho \circ h)^{-1}(T)$ and for every $y \in \rho^{-1}(T)$ both of the paths

$$\rho(h((g \circ h)_t(x))) \quad \textit{and} \quad \rho((h \circ g)_t(y)) \; , t \in [0,1]$$

have diameter less than ε in Y.

Let K denote a triangulation for Y and let T denote a subcomplex of K satisfying $T \subset S$. We say that the partial homotopy equivalence $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ into X over S is **split over** T if for every triangle $\triangle \in T$ we have that h is in transverse position to $\rho^{-1}(\triangle)$ and the restricted map

$$h : (\rho \circ h)^{-1}(\triangle) \longrightarrow \rho^{-1}(\triangle)$$

is a homotopy equivalence. More generally if T is merely a subset of S (but not necessarily a subcomplex of K) we say that the partial homotopy equivalence $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ is **split over** T if it is split over the maximal subcomplex of K which is contained in T.

A homotopy of h

$$h_t : M \longrightarrow \rho^{-1}(A)$$

is said to be **constant over a subset** $T \subset A$ if for each $x \in (\rho \circ h)^{-1}(T)$ the path

$$h_t(x) \quad , t \in [0,1]$$

is the constant path. The homotopy h_t is said to be $(\varepsilon; \rho)$**-controlled over a subset** $T \subset A$ if for each $x \in (\rho \circ h)^{-1}(T)$ the path

$$\rho(h_t(x)) \quad , t \in [0,1]$$

has diameter less than ε in Y.

The following theorem is due to F.Quinn [19].

Theorem 8.1. *Suppose that the fiber F satisfies $Wh(\pi_1(F) \oplus G) = 0$ for any finitely generated free abelian group G; and $dimF \geq 6$. Given compact subsets T and S of Y, with $T \subset S$, there is a number $\kappa > 0$ which depends only on the isometry type of the triple (Y, S, T); ε will denote any number in $(0, \kappa)$. Let $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ denote a partial homotopy equivalence into X over the subset S which is $(\varepsilon; \rho)$-controlled over S and which is already split over T. Suppose that the diameter of each triangle in K is less than ε. Then there is a number $\varepsilon' \geq \varepsilon$, and there is a homotopy of h*

$$h_t : M \longrightarrow \rho^{-1}(A)$$

such that the following properties hold.

(a) *ε' depends only ε and on the isometry type of (Y, S, T).*

(b) $\lim_{\varepsilon \to 0} \varepsilon' \quad = \quad 0 \quad .$

(c) *h_1 is split over $S^{-\varepsilon'}$.*

(d) The homotopy h_t is $(\varepsilon'; \rho)$-controlled over S, and is the constant homotopy over $T^{-\varepsilon'}$.

Remark 8.2. There is also a version of Theorem 8.1 in which the fiber F is compact but not necessarily closed. In this version (also due to Quinn [19]) we require that all of the maps $h, g, (h \circ g)_t, (g \circ h)_t$ map boundaries to boundaries. We also require that the restricted map

$$h : \partial M \longrightarrow \partial \rho^{-1}(A)$$

is already split over S; and we require that the restricted homotopy $h_t \mid \partial M$ be the constant homotopy.

Local-relative fibered-controlled splitting theory.

In the special case that the Riemannian manifold Y of Theorem 8.1 is equal to an open subset of Euclidean space $\mathbb{R}^n$ we can strengthen the conclusions of Theorem 8.1 as follows.

Theorem 8.3. *If Y of Theorem 8.1 is equal to an open subset of $\mathbb{R}^n$ then there is a number $\lambda > 1$ which depends only on n, such that ε and ε' of Theorem 8.1 are related by*

$$\varepsilon' = \lambda \varepsilon \quad .$$

Morever the number κ of Theorem 8.1 may be chosen to be any positive number.

Remark 8.4. If Y is equal to an open subset of $\mathbb{R}^n$, and the fiber F is a compact manifold but not necessarily closed, then in the generalization of Theorem 8.1 described in Remark 8.2 the numbers ε and ε' are related as in Theorem 8.3 and the number κ may be taken to be any positive number.

The proof of Theorem 8.3, and the proof of its generalization described in Remark 8.4, are the subject of homework 8.7 below.

Stratified version of Theorem 8.3.

In this subsection we describe a controlled splitting theorem analogous to the controlled h-cobordism theorem formulated in Theorem 4.4 above.

Let

$$\rho : X \longrightarrow Y$$

denote the fiber bundle projection of Theorem 8.3, and let

$$\pi : X' \longrightarrow X$$

denote the projection map for a vector bundle with total space X'; we assume that this vector bundle is equipped with an inner product structure which gives rise to the norm $\| \quad \|$. Set $Y' = Y \times [0, \infty)$, and equip this product space with the product Riemannian metric. Define a map

$$\rho' : X' \longrightarrow Y'$$

by

$$\rho'(v) = (\rho \circ \pi(v), \| v \|)$$

for each $v \in X'$. We can define (as in the paragraphs preceeding the statement of Theorem 8.1) the following notions: a "partial homotopy equivalence into X' over the subset $S' \subset Y'$"— denoted by $(A', B', h, g, (h \circ g)_t, (g \circ h)_t)$ where A' and B' are open subsets of Y' with $S' \subset B' \subset A'$; a partial homotopy equivalence into X' over S' being "$(\varepsilon; \rho)$-controlled" over another subset $T' \subset S'$; the partial homotopy equivalence $(A', B', h, g, (h \circ g)_t, (g \circ h)_t)$ being "split" over a subset $T' \subset S'$ with respect to a given triangulation K' for Y'; a homotopy h_t of h being "constant over a subset $T' \subset A'$"; a homotopy h_t of h being "$(\varepsilon; \rho)$-controlled" over the subset $T' \subset A'$.

There is a stratified version of Theorem 8.3 in which we simply replace the projection map $\rho : X \longrightarrow Y$ of Theorem 8.3 by the projection map $\rho' : X' \longrightarrow Y'$ just described in the preceeding paragraph. Unfortunately this result is not sufficiently general to help with the completion of the proof for Theorem 3.10. What is needed for this purpose is a stratified version of Theorem 8.3 where the notion of "$(\varepsilon; \rho')$-control" is replaced by the notion of "$(\varepsilon; \lambda; \rho')$-control". A partial homotopy equivalence $(A', B', h, g, (h \circ g)_t, (g \circ h)_t)$ into X' over S' is said to be $(\varepsilon; \lambda; \rho')$**-controlled over the subset** $T' \subset S'$ if it is $(\varepsilon + \lambda s; \rho')$-controlled over $T' \cap (Y \times [0, s])$ for all $s > 0$. Likewise a homotopy h_t of h is said to be $(\varepsilon; \lambda; \rho')$**-controlled over a subset** $T' \subset S'$ if it is $(\varepsilon + \lambda s; \rho')$-controlled over $T' \cap (Y \times [0, s])$ for all $s > 0$.

Theorem 8.5. *Suppose that the fiber F of the projection map $\rho : X \longrightarrow Y$ satisfies $Wh(\pi_1(F) \oplus G) = 0$ for any finitely generated free abelian group G; suppose also that $dimF \geq 6$. Given any number $\lambda_1 > 1$ there is another number $\lambda_2 > \lambda_1$ which depends only on λ_1 and on $n = dimY$. Let a, ε be any positive numbers which satisfy $\lambda_2 \varepsilon < a$; let T, S denote compact subsets of Y with $T \subset S$; let $(A', B', h, g, (h \circ g)_t, (g \circ h)_t)$ denote a partial homotopy equivalence into X' over*

$$S' \quad = \quad S \times [0, a]$$

which is $(\lambda_1 \varepsilon; \lambda_1; \rho')$-controlled over S'. Suppose that each triangle in the triangulation K' for Y' has diameter less than ε. Also suppose that h is *already split over over the subset*

$$T' \quad = \quad (T \times [0,a]) \cup (S \times [\varepsilon, a]) \quad .$$

Then there is a homotopy of h

$$h_t : M \longrightarrow (\rho')^{-1}(A')$$

which satisfies the following properties.

(a) h_t *is* $(\lambda_2\varepsilon; \lambda_2; \rho')$*- controlled over* S'.

(b) h_t *is constant over* $(T^{-\lambda_2\varepsilon} \times [0,a]) \bigcup (S^{-\lambda_2\varepsilon} \times [\lambda_2\varepsilon, a])$.

(c) h_1 *is split over* $S^{-\lambda_2\varepsilon} \times [0,a]$.

Remark 8.6. If the fiber F is a compact manifold (but not necessarily closed), then there is a generalization of Theorem 8.5. In this generalization we require that all the maps $h, g, (h \circ g)_t, (g \circ h)_t$ map boundaries to boundaries; we also require that the restricted map $h \mid \partial M$ is already split over the subset S'. Finally we require that the restricted homotopy $h_t \mid \partial M$ is constant.

A proof of Theorem 8.5 , and of its generalization described in Remark 8.6, is the subject of homework 8.8.

Homework for section 8.

Homework 8.7. Theorem 8.3 can be deduced from Theorem 8.1 by using the same argument that was used in section 4 above to deduce Theorem 4.2 from Theorem 4.1 (cf. the proof of Theorem 4.2). Similarly the generalization of Theorem 8.3 given in Remark 8.4 can also be deduced from the generalization of Theorem 8.1 given in Remark 8.2. Carry out the details of this argument.

Homework 8.8. Use the generalization of Theorem 8.3 given in Remark 8.4 to prove Theorem 8.5 and its generalization described in Remark 8.6. Hint: Review the hint given for homework 4.7 and try to utilize it here.

Homework 8.9. Use the generalization for Theorem 8.3 given in Remark 8.4 to prove Theorem 3.9. Hint: Review the outline of a proof given in homework 4.8.

9 Foliated-controlled splitting over Euclidean space.

In this section we formulate the analogue for splitting theory of the controlled h-cobordism Theorem 5.4.

Let

$$\rho : X \longrightarrow Y$$

denote the projection map of Theorem 8.3 having the closed manifold F for fiber and having the open subset Y of Euclidean space $\mathbb{R}^n$ for base space; Γ denotes a foliation of Euclidean space $\mathbb{R}^n$ by parallel planes; and Ξ denotes the restriction to Y of Γ.

A partial homotopy equivalence $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ into X over the compact subset $S \subset Y$ is said to be $(\alpha, \varepsilon; \Xi, \rho)$**-controlled over a subset** $T \subset S$ if for each $x \in (\rho \circ h)^{-1}(T)$ and each $y \in \rho^{-1}(T)$ both of the paths

$$\rho(h((g \circ h)_t(x))) \quad \text{and} \quad \rho((h \circ g)_t(y))\ , t \in [0, 1]$$

have Ξ-diameter less than (α, ε). Likewise we say that a homotopy of h

$$h_t : M \longrightarrow \rho^{-1}(A)$$

is $(\alpha, \varepsilon; \Xi; \rho)$**-controlled over a subset** $T \subset A$ if for each $x \in (\rho \circ h)^{-1}(T)$ the path

$$\rho(h_t(x))$$

has Ξ-diameter less than (α, ε).

Theorem 9.1. *Suppose that the fiber F satisfies $Wh(\pi_1(F) \oplus G) = 0$ for any finitely generated free abelian group G; and that $dimF \geq 6$. There is a positive number λ which depends only on $dimY$. Let $(A, B, h, g, (h \circ g)_t, (g \circ h)_t)$ denote a partial homotopy equivalence into X over a compact subset $S \subset Y$ which is $(\alpha, \varepsilon; \Xi; \rho)$-controlled over S. Let K denote a triangulation for Y for which diameter($\triangle$) $\leq \varepsilon$ holds for each triangle $\triangle \in K$. Suppose that h is already split over a compact subset $T \subset S$ (with respect to K). Then there is a homotopy of h*

$$h_t : M \longrightarrow \rho^{-1}(A)$$

which satisfies the following properties.

(a) h_t is constant over $T^{-\lambda\alpha, -\lambda\varepsilon}$.

(b) h_1 is split over $S^{-\lambda\alpha, -\lambda\varepsilon}$.

(c) h_t is $(\lambda\alpha, \lambda\varepsilon; \Xi; \rho)$-controlled over S.

Remark 9.2. There is a generalization of Theorem 9.1 which requires that the fiber F be a compact manifold but not necessarily closed. In this generalization all the maps $h, g, (h \circ g)_t, (g \circ h)_t$ must send boundaries to boundaries; and the restricted map $h \mid \partial M$ is assumed to already be split over the subset S. We require that the homotopy h_t — in addition to satisfying properties 9.1(a)-(c) — be constant on the boundary ∂M.

Proof of Theorem 9.1 and its generalization 9.2.
We deduce Theorem 9.1 from Theorem 8.3 by using the same argument used in section 5 to deduce Theorem 5.4 from Theorem 4.2 (c.f. the proof of Theorem 5.4). A similar argument allows us to deduce the generalization of Theorem 9.1 given in Remark 9.2 from the generalization of Theorem 8.3 given in Remark 8.4.

Homework for section 9.

Homework 9.3. Carry out the details for the proof of Theorem 9.1; and of the proof of the generalization of Theorem 9.1 given in Remark 9.2.

10 Proof of Theorem 3.10.

We use the same notation as in sections 6 and 7 and as in Theorem 3.10. Thus $\rho : X \longrightarrow Y$ is the fiber bundle projection of Theorem 3.10, Ξ is the one-dimensional foliation for Y referred to in Theorem 3.10, and $\Xi_i^\gamma \subset Y$ are the subsets of 6.0. To prove Theorem 3.10 we use an argument which is similar to (infact almost a repeat of) the one used in section 7 to prove Theorem 2.8. Our proof for Theorem 3.10 proceeds (just as did our proof for Theorem 2.8) by induction over the increasing sequence of subspaces

$$Y \setminus \Xi^\gamma = Y \setminus \Xi_z^\gamma \subset Y \setminus \Xi_{z-1}^\gamma \subset Y \setminus \Xi_{z-2}^\gamma \subset .... \subset Y \setminus \Xi_1^\gamma \subset Y \setminus \Xi_0^\gamma = Y \quad .$$

The first step in this induction argument is carried out in Lemma 10.1 below, provided the fiber F of the projection ρ is a closed manifold; if the fiber F is compact but not closed than the first step in our induction argument is carried out in the generalization of Lemma 10.1 given in Remark 10.2 below. Note that Lemma 10.1 is for controlled splitting theory what Lemma 7.1 is for controlled h-cobordism theory.

If the fiber F is a closed manifold then the induction hypothesis for this induction argument is the same as the hypothesis of Lemma 10.3 below; and the induction step is carried out in Lemma 10.3. If the fiber F is compact but not closed then the induction hypothsis is the same as the hypothesis for the generalization of Lemma 10.3 described in Remark 10.4 below; and the induction step is carried out in the generalization to Lemma 10.3 described in Remark 10.4 below.

Lemma 10.1. *Suppose the fiber F of the projection map $\rho : X \longrightarrow Y$ in Theorem 3.10 is a closed manifold satisfying the hypothesis of Theorem 3.10. There is a number $1 > \lambda > 0$ which depends only on the number $n = dimY$; there is a number $\omega > 1$ which depends on the number γ and on the foliation Ξ. Given an arbitrarily large compact subset $C \subset Y \setminus \Xi^\gamma$ there is a number $\delta \in (0,1)$ which depends only on C, γ and on Ξ. Choose $\alpha \in (0, \lambda\gamma)$ and $\varepsilon \in (0, \delta)$. Let K denote a triangulation for Y such that each triangle in K has diameter less than ε; and let*

$$h : M \longrightarrow X$$

denote a $(\alpha, \varepsilon; \Xi, \rho)$-controlled homotopy equivalence from a closed manifold M. Then there is a homotopy of h

$$h_t : M \longrightarrow X$$

which satisfies the following properties.

(a) *h_t is $(\gamma, \lambda\varepsilon; \Xi, \rho)$-controlled.*

(b) *h_1 is split over $C^{-\gamma, -\omega\varepsilon}$.*

Remark 10.2. If the fiber F is a compact manifold (but not necessarily closed) and satisfies the hypothesis of Theorem 3.10 then there is a generalization of Lemma 10.1. In this generalization we must assume that the map

$$h : (M, \partial M) \longrightarrow (X, \partial X)$$

is a homotopy equivalence between compact manifold pairs such that the restricted map $h \mid \partial M$ is already split over the triangulation K. We must also require that the restricted homotopy $h_t \mid \partial M$ is constant.

Lemma 10.3. *Suppose that the fiber F of the projection map $\rho : X \longrightarrow Y$ is a closed manifold which satisfies the hypothesis of Theorem 3.10. Choose a number j which satisfies $1 \leq j \leq z$ and let C_j denote an arbitrarily large compact subset of $Y \setminus \Xi_j^\gamma$. There is a number $\omega_j > 1$ which depends only on j, γ and on the foliation Ξ; there is another number $\delta_j \in (0,1)$ which depends only on j, γ, Ξ and on C_j. Choose $\varepsilon_j \in (0, \delta_j)$. Let K denote a triangulation for Y such that each triangle of K has diameter less than ε_j; and let*

$$h : M \longrightarrow X$$

denote a $(\gamma, \varepsilon_j; \Xi, \rho)$-controlled homotopy equivalence from a closed manifold M, which is already split over C_j. Then there is an arbitrarily large compact subset C_{j-1} of $Y \setminus \Xi^{\gamma}_{j-1}$, and there is a homotopy of h

$$h_t : M \longrightarrow X$$

which satisfies the following properties.

(a) h_t is $(\gamma, \omega_j \varepsilon_j; \Xi, \rho)$-controlled.

(b) h_t is constant over $(C_j)^{-\gamma, -\omega_j \varepsilon_j}$.

(c) h_1 is split over C_{j-1}.

Remark 10.4. If the fiber F is a compact (but not necessarily closed) manifold there is a generalization of Lemma 10.3. In this generalization we must assume that the map

$$h : (M, \partial M) \longrightarrow (X, \partial X)$$

is a homotopy equivalence between compact manifold pairs such that the restricted map $h \mid \partial M$ is already split over all of the triangulation K. We also must require that the restricted homotopy $h_t \mid \partial M$ is constant.

Homework for section 10.

Homework 10.5. Prove Lemma 10.1 and its generalization given in Remark 10.2. Hint: The proof of Lemma 10.1 is basically just a repeat of the proof given in section 7 for Lemma 7.1. Recall that the proof of Lemma 7.1 combined Lemmas 6.2 and 6.3 with Theorem 5.4 and an induction argument. Modify that proof by substituting Theorem 9.1 for Theorem 5.4, to get a proof of Lemma 10.1. To get a proof of the generalization of Lemma 10.1 which is described in Remark 10.2 just modify the proof for Lemma 7.1 by substituting the generalization of Theorem 9.1 described in Remark 9.2 for Theorem 5.4.

Homework 10.6. Prove Lemma 10.3 and its generalization given in Remark 10.4. Hint: The proof of Lemma 10.3 is basically a repeat of the proof given in section 7 for Lemma 7.3. Recall that the proof of Lemma 7.3 combined Lemma 6.4 with Theorem 4.4 and an induction argument. Modify that proof by substituting Theorem 8.5 for Theorem 4.4, to get a proof for Lemma 10.3. To get a proof of the generalization of Lemma 10.3 which is described in Remark 10.4 just modify the proof for Lemma 7.3 by substituting the generalization for Theorem 8.5 which is described in Remark 8.6 for Theorem 4.4.

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Lectures on Controlled Topology: Mapping Cylinder Neighborhoods

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Abstract

The existence theorem for mapping cylinder neighborhoods is discussed as a prototypical example of controlled topology and its applications. The first of a projected series developed from lectures at the Summer School on High-Dimensional Topology, Trieste, Italy 2001.

Contents

1 Introduction

Controlled topology has the hallmarks of a mature mathematical subject: powerful results, sophisticated interactions with, and applications to, other subjects, difficult and unexpectedly beautiful conjectures. It is not very accessible, however. Partly this is because complete results are difficult and there is not a large enough community to sustain interest in partial answers. Another problem is that it blossomed rapidly, so lacks the more-accessible historical development and expositions of most mature subjects. This paper is the first in a projected series to try to address this. Here we outline the setting and applications of the existence theorem for mapping cylinder neighborhoods (originally, "completions" of ends of maps). This illustrates most of the ingredients of the subject: local homotopy theory, local fundamental groups, elaborate algebraic obstructions, interesting applications. The focus is on what all these things mean and how they fit together, and most details are omitted.

This paper is an expansion of the first third of a series of lectures given at the Summer School on High-Dimensional Topology in Trieste, Italy, in the summer of 2001. Other topics were the controlled h-cobordism theorem, illustrating some of the geometric and algebraic techniques; and homology manifolds, illustrating the still-incomplete theory of controlled surgery.

1.1 Locating the subject. In the first half of the 20th century topology had two main branches: point-set topology, concerned with local properties (separation, connectedness, dimension theory etc); and algebraic topology, concerned with definition and detection of global structure (homology, characteristic classes, etc.). In the 50s and 60s the algebraic branch split into homotopy theory and geometric topology. Homotopy theory was still largely descriptive, but in the geometric area the emphasis changed from description to construction. For instance rather than computing homology of examples of manifolds, the objective was to construct or classify manifolds with given homological structure. This development was mainly restricted to spaces with uniform local structure, i.e. manifolds. Some of the descriptive techinques had extensions to spaces with symmetries (group actions) and stratified spaces such as algebraic varieties. Extensions of constructive methods were very limited due to complexity of interactions between different levels in group actions or stratified sets.

Controlled topology began in the late 1970s and 80s as a way to apply the constructive techniques of geometric topology to local questions more typical of point-set topology. For example, which subspaces of a space have a neighborhood homeomorphic to a mapping cylinder? Mapping cylinders must be constructed rather than simply detected, so although this is a local question it requires constructive techniques.

Controlled topology has had striking successes in elucidating geometric structure. Unexpectedly, it has also had striking applications in algebra. Geometric problems have obstructions related to linear or quadratic algebra (K- and L-theory). Controlled geometric problems have obstructions in controlled-algebra analogs, essentially homology with coefficients in spectra related to the uncontrolled obstructions. This turns out to be a two-way street: results about ordinary obstructions give information about control and local geometric structure, and conversely direct controlled constructions can give information about ordinary obstructions. For instance the famous "strong Novikov conjecture" asserts that some assembly maps from homology to ordinary obstruction groups are isomorphisms, at least rationally. The homology corresponds to controlled problems, ordinary groups correspond to uncontrolled problems, and the assembly map corresponds to "forgetting control." If the assembly map is an isomorphism then solvability of an uncontrolled problem determines solvability of a more delicate controlled one. Conversely if there is a geometric construction that "gains control" — produces a controlled solution from an uncontrolled one — then the assembly map must be an isomorphism. Most geometric proofs of cases of the Novikov conjecture rely on this principle, and the most delicate (especially work of Farrell and Jones) use it explicitly.

Controlled topology thus lies at the juncture between geometric and point-set topology, homotopy theory, and stable algebra. Constructions and proofs tend to be elaborate, but outcomes can be deep and powerful.

1.2 Plan. True mastery of a subject requires understanding the details. However to get started, or for those looking for application rather than mastery, an overview can be helpful. The goal here is to give such an overview: definitions and enough explanation for good understanding of the statements of theorems, sketches of proofs in enough detail to show how the hypotheses are used and what the difficult points are. Finally in this paper we focus on the construction and application of mapping cylinders. This illustrates most of the techniques and issues of controlled topology.

The central result in the paper is the existence theorem for mapping cylinder neighborhoods, 3.1. However the hypotheses are quite elaborate, so Section 2 is devoted to developing them. Specifically, 2.1 gives the definition, 2.2 describes the use of "control spaces", and 2.3 describes the simplest ("uncontrolled") case of neighborhoods of points. Section 2.4 defines tameness and describes some of the results. Tameness often does not appear in statements of applications because it follows from other hypotheses, but it is central to proofs. Section 2.5 discusses homotopy links, which provide homotopy models and play an important role in controlling local fundamental groups, as explained in 2.6. Stratified systems of fibrations are introduced in 2.7. These are needed to impose some regularity on local fundamental groups, and appear prominently in the structure of stratified sets. Section 2.8 begins development of the "spectral sheaf homology" used to describe the obstruction groups. A key feature of this theory is the assembly map defined in 2.9. The most elaborate part of the development is in section 2.10, where the controlled K-theory (more precisely, pseudoisotopy) spectrum is discussed. It is in large part the controlled assembly isomorphism theorem for this spectrum that makes the theory accessible and useful.

With the setting and hypotheses explained, the existence theorem for mapping cylinder neighborhoods is stated in 3.1. The proof is outlined in sections 3.2 and 3.3. Some useful refinements are given in 3.4. The first concerns smooth and PL structures, and the second gives a recognition criterion for mapping cylinders.

Applications are given in section 4. The first three are straightforward: the special case of manifolds (4.1) with its corollary the finiteness of compact finite dimensional ANRs (the Borsuk conjecture), and collaring in homology manifolds (4.3). Then we consider mapping cylinders in cases where the tameness and local fundamental group structure are more elaborate, namely stratified spaces (4.4), and a special case where the obstructions can be made relatively explicit, topological actions of finite groups (4.5). The final application (in 4.6) is to define topological regular neighborhoods. These are mapping cylinders in a product with $[0, \infty)$, and generalize the "approximate tubular neighborhoods" in stratified spaces developed by Hughes and others.

2 The setting

To illustrate the basic ideas of control we investigate the existence of mapping cylinder neighborhoods. Fix a space X and a closed subset Y. We suppose the complement $X - Y$ is a manifold since we use manifold techniques there. Finally we suppose X and Y are finite-dimensional ANRs (absolute neighborhood retracts), to avoid local point-set problems.

2.1 Definition. A *mapping cylinder neighborhood* of Y in X is a closed neighborhood $N \supset Y$ with frontier ∂N a submanifold of $X - Y$, a map $f : \partial N \to Y$, and a homeomorphism of the mapping cylinder of f with N which is the identity on ∂N and Y.

More explicitly the mapping cylinder is the identification space $\partial N \times$

$I \cup_f Y$, where "$\cup_f$" indicates that points $(x,0) \in \partial N \times I$ are identified with $f(x) \in Y$, and the homeomorphism $\partial N \times I \cup_f Y \to N$ restricts to the identities on $\partial N \times \{1\} \to \partial N$ and $Y \to Y$.

2.2 The control space. There is a canonical projection of a mapping cylinder to the subspace Y. In fact we have assumed Y is an ANR so there is a projection of a neighborhood to Y whether there is a mapping cylinder or not. Denote this by $p : N - Y \to Y$. We refer to Y as the control space and the projection as the control map. To explain the terminology we observe that a mapping cylinder is equivalent to a product structure on the complement $N - Y \simeq \partial N \times (0,1]$ so that the images of open arcs $p(\{x\} \times (0,1])$ converge. The map $f : \partial N \to Y$ can be recovered as the limit $f(x) = \lim_{t\to 0} p(x,t)$. Convergence is arranged using the Cauchy criterion: constructions are done so that images of subintervals $p(\{x\} \times [\frac{1}{n+1}, \frac{1}{n}])$ have preassigned small diameter. The crucial issue is control of sizes of images in Y, hence the "control" terminology.

2.3 The uncontrolled case. Most controlled theorems have older "uncontrolled" versions in which the control space is implicitly taken to be a point. This version of the mapping cylinder question is: when does a point have a neighborhood homeomorphic to a cone? Recalling that we have assumed the complement is a manifold, this can be reformulated as: when is a noncompact manifold the interior of a compact manifold with boundary? The 1-point compactification then plays the role of X, and Y is the point at infinity. Theorems of Browder, Livesay and Levine [2] and Siebenmann [17] answer this: there is a necessary homotopy-theoretical "tameness" condition, and then an obstruction in algebraic K-theory.

We expand on the K-theory part. When the tameness condition is satisfied we can predict the fundamental group of the boundary of the neighborhood: the groups $\pi_1(U - Y)$ indexed by the inverse system of neighborhoods U of Y converges nicely to a finitely presented group π. In the course of the construction of actual boundaries a finitely presented projective module over the group ring $\mathbb{Z}[\pi]$ is encountered. If this projective module is stably free the construction can be continued to give the desired structure. The obstruction is therefore essentially the class of this module in the group of stable equivalence classes of projective modules, $K_0(\mathbb{Z}[\pi])$. This is a little too big: K_0 records the rank of the module, which is irrelevant to the topology. The actual obstruction group is the reduced group $\tilde{K}_0$, defined to be either the cokernel of the inclusion of the trivial group, $K_0(\mathbb{Z}[\{1\}]) \to K_0(\mathbb{Z}[\pi])$, or the kernel of the rank homomorphism $K_0(\mathbb{Z}[\pi]) \to Z$.

2.4 Tameness. Mapping cylinder neighborhoods have special homotopy properties. The eventual result is that certain of these actually characterize mapping cylinders, modulo K-theory problems. We describe these.

The first property is that the neighborhood deformation retracts to Y, by pushing toward the 0 end of the mapping cylinder. A key feature of this deformation is that the complement of Y stays in the complement until the last instant when everything collapses into Y. We formalize this as:

Definition. An embedding $Y \subset X$ is *forward tame* if there is a map $f : X \times I \to X$ satisfying

(1) $f(x,t) = x$ if $t = 0$ or $x \in Y$;

(2) $f^{-1}(Y) = Y \times I \cup U \times \{1\}$, where U is some neighborhood of Y.

On the other hand we could pull the complement of Y away from Y by pushing toward the other end of the mapping cylinder. This formalizes to:

Definition. An embedding $Y \subset X$ is *backwards tame* if there is a map $b : (X - Y) \times I \to X - Y$ satisfying

(1) $b(x,t) = x$ if $t = 0$; and

(2) for every $t > 0$ the closure in X of $b((X-Y) \times \{t\})$ is disjoint from Y.

Putting these together we say:

Definition. An embedding $Y \subset X$ is *tame* if it is both forward and backward tame.

Quite a bit is known about tameness. For instance if the embedding has finitely presented constant local fundamental groups (see below) then there are homological characterizations, and forward and backward tameness are equivalent because their homological formulations are Poincaré dual [15, 2.14]. If the embedding has *trivial* local fundamental groups then it is always tame because the homological conditions are implied by the ANR hypotheses and excision [15, 2.12]. See [7] for a treatment in the nonmanifold case.

2.5 Homotopy links. One of the main applications of tameness is to give a comparison of the embedding with a "universal" mapping cylinder constructed using the homotopy link.

Definition. The *homotopy link* of $Y \subset X$, denoted $\operatorname{holink}(X, Y)$, is a subset of the space of paths in X with the compact-open topology. Specifically it consists of the paths $s : [0,1] \to X$ with $s^{-1}(Y) = \{0\}$. Evaluation at 0 gives a map $\mathrm{ev}_0 : \mathrm{holink}(X,Y) \to Y$. The whole evaluation map is $\mathrm{holink}(X,Y) \times I \to X$, and continuity implies this factors through a map on the mapping cylinder; $\mathrm{ev} : \mathrm{cylinder}(\mathrm{ev}_0) \to X$. This preserves complements and is the identity on Y, and in fact is the universal such map from a mapping cylinder.

Now suppose Y has a mapping cylinder neighborhood $N \simeq \mathrm{cylinder}(q)$, with map $q : \partial N \to Y$. Since the homotopy link cylinder is universal, the geometric one factors through it. More explicitly, each point in ∂N determines a cylinder arc in N. These arcs are points in the homotopy link so define a map $\partial N \to \mathrm{holink}(X,Y)$. This extends to a map of mapping cylinders. Further, $\partial N \to \mathrm{holink}(X,Y)$ turns out to be a controlled homotopy equivalence over Y, so the homotopy link provides a homotopy model for any geometric mapping cylinder neighborhood.

Some of this last construction can be done using forward tameness in place of an actual mapping cylinder. Suppose $f : X \times I \to X$ is a forward-tameness deformation, and U is a neighborhood of Y with $f(U \times \{1\}) \subset Y$. Then the arcs $f : \{x\} \times I \to X$ for $x \in U - Y$ define points in the homotopy link. This defines a map $U - Y \to \mathrm{holink}(X,Y)$. Using this in the first coordinate and distance from Y in the second gives a map to the universal mapping cylinder, $U \to \mathrm{cylinder}(\mathrm{ev}_0)$. When Y is also backwards tame this map is in an appropriate sense a controlled local equivalence near Y. Tameness therefore encodes essentially the same local homotopy information as a mapping cylinder neighborhood.

2.6 Controlling local fundamental groups. In standard (uncontrolled) geometric topology the fundamental group plays a central role. Roughly this is because algebraic topology is effective with 1-connected spaces, and general spaces are made 1-connected by taking universal covers. In controlled topology the same principle applies, but fundamental groups cannot be used directly because (among other problems) their definition depends on choices of basepoints. Instead we use comparisons with reference spaces.

The general setting is a reference map $p : E \to Y$, the controlled thing being studied, $W \to Y$, and a map $f : W \to E$ that is required to commute with maps to Y up to some error δ. f is said to be $(\delta, 1)$-connected if given a relative 2-complex (K, L) and a δ-commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & W \\ \big\downarrow{\scriptstyle \subset} & & \big\downarrow{\scriptstyle f} \\ K & \longrightarrow & E \end{array}$$

then there is an extension $K \to W$ whose composition into Y is within δ

of $K \to E \to Y$. When this is satisfied W and E have the same local fundamental group behavior over Y, even if "local fundamental groups" do not make sense.

The importance of using reference maps to control π_1 increases with increased complexity of local π_1 behavior. If the geometric situation is locally 1-connected over Y then no π_1 control is needed. If the local fundamental groups are constant then a locally 1-connected covering space can be used. If local fundamental groups are locally constant over Y then we can use covering spaces of inverse images of open sets in Y. But now the situation starts getting complicated: in geometric constructions we are controlling sizes, so we need to know these open sets are fairly large. In fact we need a priori estimates on these sizes so geometric data can be chosen small in comparison. The simplest way to do this is to control local π_1 using a fixed reference map $E \to Y$. In this way whatever size data we need is determined by the reference map, and doesn't have to be made explicit to be controlled. In the most general situation local fundamental groups change from place to place. This is easily encoded using reference maps and awful to do with groups.

2.7 Stratified systems of fibrations. In the previous section we described reference maps as a way to avoid the awkwardness of group formulations of local π_1 structure. However geometric constructions do use group formulations. Core steps of proofs are usually done assuming constant local fundamental groups and using locally 1-connected covers. General cases are obtained from this by fitting together locally constant pieces. Thus the general π_1 control apparatus is not intended to feed directly into core proofs, but to formulate general hypotheses that in proofs inductively reduce to constant cases. "Stratified systems of fibrations" [13] work well for this.

Definition. Suppose $p : E \to Y$ is a map, and $Y = Y^n \supset Y^{n-1} \supset \cdots \supset Y^0$ is a filtration by closed subsets. p is a *stratified system of fibrations* (with filtration Y^*) if

(1) the restriction to each of the strata,

$$p^{-1}(Y^i - Y^{i-1}) \xrightarrow{p} Y^i - Y^{i-1}$$

is a fibration, and

(2) each term in the filtration is a p-NDR. This means there is a neighborhood of Y^i, a deformation of it into Y^i in Y that preserves strata until the very end, and this deformation is covered by a deformation of the inverse image in E.

Lots is known about these. There are many examples, reductions to apparently weaker data, constructions, etc., see [15, 3, 6].

In the mapping cylinder context the tameness hypothesis provides us with a canonical reference map, the homotopy link. Local π_1 hypotheses are formulated in terms of this. Standard procedure (see the statement in 2.10) is to assume there is a stratified system of fibrations $E \to Y$ and a map $\operatorname{holink}(X, Y) \to E$ that is locally 1-connected over Y. In many applications the homotopy link itself is a stratified system of fibrations.

2.8 Homology. The mapping cylinder problem has obstructions lying in locally finite homology with coefficients in a spectral cosheaf. This sounds complicated but is actually good news: nothing simpler could work; it is reasonably accessible to calculation; and the formal properties alone have important applications. In this section we outline the general setup developed in [13]. We assume general familiarity with the use of spectra to construct homology theories.

The basic setting is a spectrum-valued functor of maps with locally-compact target. In more detail, the domain of this functor is the category with objects maps $p : E \to B$, with B a locally compact metric space. Morphisms are pairs of maps (F, f) forming a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

and so that f is proper (inverse images of compact sets are compact). In the application B is the control space where sizes are measured, and E serves to control local fundamental groups as in Section 2.5.

We explain the local compactness hypothesis. The technical work concerns manifolds mapping to E. We work over small open sets in B, and the inverse image must have compact closure in the manifold. To get this we assume the map from the manifold to B is proper. But then we have to restrict to proper maps of B to preserve this property. We cannot simply require the manifold to be compact because we need a restriction operation that destroys compactness. If $U \subset B$ is an open set then restriction to inverse images of U gives a map from manifold gadgets over B to ones over U. Even if we start with a compact manifold over B the result will usually be noncompact (but proper) over U.

The homology of X with coefficients in a spectrum $\mathbf{J}$ is the spectrum $X \wedge \mathbf{J}$. In more detail, $\mathbf{J}$ is a seqence of spaces J_n with various maps. Start with the sequence of spaces $X \times J_n$, divide out X times the basepoint in J_n. The maps for J_* then give this sequence of spaces the stucture of a suspension spectrum. The associated spectrum is $X \wedge \mathbf{J}$. We also denote this by $\mathbf{H}(X; \mathbf{J})$. Note this is a spectrum; the homology *groups* are the homotopy groups of this spectrum

$$H_i(X; \mathbf{J}) = \pi_i(\mathbf{H}(X; \mathbf{J}).$$

Note also that (unlike ordinary homology) these groups may be nontrivial for $i < 0$.

The locally-compact wrinkle in the theory requires us to work with locally finite homology. This is essentially the relative homology of the 1-point compactification.

The "Atiyah-Hirzebruch" spectral sequence (due originally to G. Whitehead) is a spectral sequence of the form

$$E^2_{i,j} = H_i(X; \pi_j \mathbf{J}) \implies H_{i+j}(X; \mathbf{J}).$$

From this one sees, for example, that $H_j(X; \mathbf{J})$ always vanishes for $j < j_{\min}$ exactly when $\pi_j(\mathbf{J}) = 0$ for $j < j_{\min}$, and that groups near the vanishing line are quite accessible. This turns out to be very useful in applications.

We now return to the context of a spectrum-valued functor $\mathbf{J}(p)$, defined on the category of maps p with locally compact metric range spaces. In this case we can define a "sheaf" generalization of the homology construction. Suppose $p : E \to Y$ is a map in the category. We can apply the functor fiber-wise to get a spectrum $\mathbf{J}(p^{-1}(y) \to y)$ over each $y \in Y$. With mild additional assumptions we can fit these together to get a "spectral cosheaf" over Y. This is a sequence of spaces $J_n(p^{-1}(\#))$ with maps to Y and maps to each other making the fibers over Y into spectra. In the constant-coefficient case $F \times Y \to Y$ this just gives $J_n(F \to \mathrm{pt}) \times Y$. By analogy with the constant-coefficient case we define *homology* with coefficients in this cosheaf by first dividing out the 0-section of each $J_n(p^{-1}(\#)) \to Y$, then taking the spectrum associated to the resulting suspension spectrum. We use the notation $\mathbf{H}(Y; \mathbf{J}(p^{-1}(\#)))$ for this spectrum, and H_* for its homotopy groups.

Again we actually need locally-finite homology. The spectrum for this is obtained by adding a point at infinity to the spectral cosheaf, over the point at infinity in the 1-point compactification of Y. Then divide out the 0-section and proceed as before. If Y is already compact this does not change the homology.

There is a generalization of the Atiyah-Hirzebruch spectral sequence to the non-constant coefficient case. Namely in the situation of the previous paragraph we get

$$E^2_{i,j} = H_i(X; \pi_j \mathbf{J}(p^{-1}(\#))) \implies H_{i+j}(X; \mathbf{J}(p^{-1}(\#))),$$

where the groups on the right are "ordinary" cosheaf homology groups. Again these are reasonably accessible near the vanishing line for the coefficient spectra.

There is a useful extension of the spectral cosheaf construction. Suppose, as before, that $E \to Y$ is a map in the domain of the functor, but now assume also that $f: Y \to Z$ is a proper map. Then we can construct a spectral cosheaf over Z by applying the functor to inverses under f. More explicitly, over a point $z \in Z$ we put the spectrum $\mathbf{J}(p^{-1}(f^{-1}(z)) \to f^{-1}(z))$. As before we can define a homology spectrum by dividing out 0-sections and taking the associated spectrum. The output of this construction is denoted by $\mathbf{H}(Z; \mathbf{J}(p^{-1}(f^{-1}(\#))))$. The notation is a bit tricky. Note we can do the previous construction to the composition fp and get a spectral cosheaf denoted $(fp)^{-1}(\#)$. This is different from the cosheaf just constructed, though in some cases they have the same homology.

2.9 Assembly maps. We continue with the terminology of the previous section. Suppose $p: E \to Y$ is an object in the category of "proper maps to locally compact spaces". Then for each $y \in Y$ the inclusion

$$\begin{array}{ccc} p^{-1}(y) & \longrightarrow & E \\ \big\downarrow p & & \big\downarrow p \\ y & \xrightarrow{\subset} & Y \end{array}$$

is a morphism in the category. Applying $\mathbf{J}$ gives maps from fibers of the spectral cosheaf into $\mathbf{J}(p)$. Under mild continuity hypotheses these fit together to give a map on the total space of the cosheaf. Since the target of this map is a spectrum the map factors through the associated spectrum of the total space to define a map of spectra

$$\mathbf{H}(Y; \mathbf{J}(p^{-1})) \to \mathbf{J}(p).$$

This is the "general nonsense" description of the assembly map. In special cases there are other descriptions that may give better understanding.

We will make use of the functoriality of assembly maps. Suppose there is a morphism

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \big\downarrow p_1 & & \big\downarrow p_2 \\ Y_1 & \xrightarrow{f} & Y_2 \end{array}$$

in the domain category of $\mathbf{J}$. Then following diagram of spectra commutes:

$$\begin{array}{ccc} \mathbf{H}(Y_1;\mathbf{J}(p_1^{-1})) & \longrightarrow & \mathbf{H}(Y_2;\mathbf{J}(p_2^{-1})) \\ \downarrow & & \downarrow \\ \mathbf{J}(p_1) & \longrightarrow & \mathbf{J}(p_2) \end{array}$$

The top map also factors through the mixed homology $\mathbf{H}(Y_2;\mathbf{J}(f(p_2^{-1}))$.

2.10 K-theory. The previous section gives the context for homological obstructions. In this section we discuss particular functors used to make contact with the topological problems.

The logical context for the next theorem is that geometric-topological techniques can be used to formulate obstruction groups for controlled problems (see Section 3). This tells us what they are good for, but says very little about their nature. The next theorem provides another description that displays global properties. This is incorporated in the final statement in §3.1.

Controlled assembly isomorphism theorem. *There is a spectrum-valued functor $\mathcal{S}$ defined on maps to locally compact metric spaces, such that if $p: E \to Y$ is a stratified system of fibrations over a locally compact finite-dimensional ANR then*

(1) $\pi_0\mathcal{S}(p)$ is the obstruction group for mapping cylinder neighborhoods of Y with local fundamental groups modeled on p, and

(2) the assembly map $\mathbf{H}^{lf}(Y;\mathcal{S}(p^{-1}(\#)) \to \mathcal{S}(p)$ is an equivalence of spectra.

First we explore the significance of conclusion (1). Uncontrolled work determines some of the homotopy of the "coefficient" spectra (Y a point):

$$\pi_i(\mathcal{S}(F \to pt)) = \begin{cases} Wh(\pi_1 F) & \text{when } i = 1; \\ \tilde{K}_0(\mathbb{Z}[\pi_1 F]) & \text{when } i = 0, \text{ and} \\ K_{-i}(\mathbb{Z}[\pi_1 F]) & \text{when } -i < 0. \end{cases}$$

The $i = 1$ case comes requiring the same spectrum to work for h-cobordisms, $i = 0$ is the uncontrolled end theorem (Siebenmann, see §2.3), and $-i < 0$ comes from seeing Bass' definition of lower K-theory [1] come out of tinkering with Euclidean spaces $Y = \mathbb{R}^i$ ([11]). One can also require a connection to pseudoisotopy and get π_2 to be Wh_2, [14]. The spectral sequence shows the higher homotopy plays no role in the obstructions of interest, so we don't particularly care what it is.

There are many constructions of spectra encoding lower K-theory, and many of these extend to spectrum-valued functors satisfying condition (1) of the theorem. Conclusion (2), which is the source of the real power of the theory, is much more delicate.

The proof of (2) follows the proof of uniqueness of homology, i.e. a morphism of homology theories that induces an isomorphism on homology of a point is an isomorphism on finite-dimensional ANRs. The proof proceeds by induction using exact sequences, first establishing isomorphism for spheres, then finite CW complexes, then (by a trick) ANRs. To prove (2) this way we (i) show the right side ($\mathcal{S}(p)$) satisfies appropriate versions of the axioms of homology in the Y variable; (ii) observe that the map gives isomorphisms over points by definition; and (iii) make minor adjustments to incorporate the "coefficient system" (reference map p). The hard part of this is (i). The axioms are homotopy; excision; and a fibration condition for pairs. The fibration hypothesis is the spectrum version of the long exact sequence of homology *groups* of a pair: the long exact sequence is the homotopy sequence of the fibration. Technically since we are working with locally finite homology the pair axiom is replaced by a condition on restrictions to open sets, but it amounts to the same thing.

Several conclusions can be drawn from this outline. First, we may not care about the higher homotopy of the spectrum but Nature does. The inductive plan of the proof only works if *all* the groups line up correctly, so we have to get them right whether we want to use them or not. Second, the proof is all-or-nothing. Again because it depends on an induction it either works or fails, and there are no interesting partial results when it fails. Finally, the bottleneck in such arguments is usually the excision axiom. Recall that this is the one that fails for homotopy groups, and thus enables homotopy to be so much more complicated than homology.

The theorem is proved in [13] with a spectrum $\mathcal{S}$ constructed using pseudoisotopy of manifolds. This is a version of what is now known as A-theory or Waldhausen K-theory. For current topological applications one such spectrum is enough. However the proof in [13] is complicated and not too clear, so there have been efforts to find other proofs of this key step. Also, a version extending algebraic K-theory would have significant applications to algebra. So far this has been not been done: none of the other formulations of Waldhausen K-theory and none of the algebraic K constructions (Quillen, Volodin and so on) have been acceptable to the methods of the proof. The author thinks he has a construction for algebraic K-theory, but he has thought this before (cf.[16]) so skepticism is appropriate until details appear.

We offer a philosophical explanation of why the controlled assembly isomorphism theorem is so hard to prove. Frequently a complicated proof is "explained" by the existence of a false similar statement. For example the "reason" Freedman's topological embedding theorem for 4-manifolds is so hard is that the analogous statement for smooth embeddings is false. The proof must have a topological construction so bizarre that it cannot possibly give a smooth outcome, and must depend on it so essentially that it cannot possibly be avoided. What then is the false thing forcing the isomorphism theorem proof to be so intolerant? The problem is probably in *quadratic* stable algebra (surgery, L-theory). There the geometrically significant lower homotopy groups of the spectrum do have some imprint of strange behavior in the higher groups. As a result the surgery spectrum constructions now known *cannot* satisfy the isomorphism theorem. The proof must be delicate enough to reject these impostors. Apparently K-theory doesn't do anything strange enough to deserve the complexity; it is just an innocent victum of problems in surgery.

3 The theorem

Collecting the hypotheses developed in §2, we suppose X is a locally compact finite dimensional metric ANR, $Y \subset X$ is tame, $X - Y$ is a manifold, $p : E \to Y$ is a stratified system of fibrations, and there is a controlled 1-connected map $\operatorname{holink}(X, Y) \to E$.

3.1 Mapping cylinder existence theorem. *Under these conditions there is an invariant* $q_0(X, Y) \in H_0^{lf}(Y; \mathcal{S}(p^{-1}(\#)))$. *This vanishes if* Y *has a mapping cylinder neighborhood, and conversely if the invariant vanishes and* $\dim X - Y \geq 6$ *then there is a mapping cylinder neighborhood.*

Dimension 5. This is still true in dimension 5 when the local fundamental groups of p have subexponential growth [5, 9].

We outline the proof only well enough to show the major features.

3.2 Nice neighborhoods and the obstruction. The key objective is to find neighborhoods N with the right controlled homotopy type: closed manifold neighborhoods so that $\partial N \to N - Y$ is an ϵ homotopy equivalence over Y. Tameness is the main ingredient. Choose any small manifold neighborhood N, and choose handlebody structures. The tameness deformations provide homotopy data to show how to swap handles to make $\partial N \to N - Y$

highly connected. The final step, which would make it a homotopy equivalence, is obstructed. In the uncontrolled case we see a single nonvanishing relative homology group. If it is stably free over the group ring then we can stabilize and swap handles corresponding to a basis to get a good N. This module is a direct summand of a finitely generated free chain group, so is finitely generated projective. The obstruction is the equivalence class of the projective module, modulo stably free modules. In other words, its image in $\tilde{K}_0(\mathbb{Z}[\pi_1])$. We indicate modifications needed in the controlled setting. We can't use homology because this is a quotient and quotients destroy size estimates. Instead we directly use the projection on controlled chain groups. We define $\tilde{K}_0(Y; p, \epsilon, \delta)$ to be free modules over Y with "$\mathbb{Z}[\pi_1(p^{-1}(\#))]$" coefficients (this is clarified in §3), with projections of radius $< \delta$, modulo ones with basis-preserving projections of radius $< \epsilon$. Adding estimates to the uncontrolled argument gives an element of this set, and shows that if it is trivial then the argument can be completed to get a nice N.

We pause the proof to expand on the obstructons. The main point is that although we have an "obstruction", and can arrange for the set $\tilde{K}_0(Y; p, \epsilon, \delta)$ in which it lies to be a group, we know nothing about it. This is where the characterization theorem of §9 takes over. This shows:

(1) these groups are stable in the sense that for every $\epsilon > 0$ there is $\delta > 0$ so that the map from the inverse limit

$$\tilde{K}_0(Y; p, \epsilon, \delta) \leftarrow \lim_{\leftarrow} \tilde{K}_0(Y; p, *, *)$$

is an isomorphism; and

(2) the inverse limit is the spectral sheaf homology group $H_0^{lf}(Y; \mathcal{S}(p^{-1}(\#)))$.

The stability in (1) is subtle and actually harder to prove than the description of the limit in (2). For instance in the controlled algebra used here, sizes grow when morphisms are composed. This means morphisms of fixed size do not form a category, and in place of the homological and categorical techniques of the uncontrolled theory we have to work with chain complexes and constantly estimate sizes. In contrast it is possible to set up the inverse limit theory directly so the work takes place in a category [10]. The setup is more elaborate, but no estimates are needed and the group $\lim_{\leftarrow} \tilde{K}_0(Y; p, *, *)$ appears as ordinary K-theory of a category. In some applications the stability property is essential (see §5). However for mapping cylinders it is not. We have extracted an invariant from a single sufficiently small neighborhood N. But one could repeat the construction at smaller scales to get a sequence of neighborhoods N_i with estimates going to 0. From this we could extract a sequence of related algebraic objects with estimates going to 0, or in other words an element of the inverse limit. This approach may yet have significant applications. However so far the benefits (convenience for the categorically sophisticated) do not seem to outweigh the drawbacks (weaker theorems, more elaborate setups).

3.3 Getting mapping cylinders. Returning to the proof, we suppose the obstruction vanishes so we can find nice neighborhoods N. Repeat at smaller scales to get a decreasing sequence $N_i \supset N_{i+1} \dots$ which are "nice" with decreasing size estimates. Recall that "nice" meant roughly that the inclusion $\partial N_i \to N_i - Y$ is a controlled homotopy equivalence. It follows that the regions between these are controlled h-cobordisms. Explicitly, the inclusions of ∂N_i and ∂N_{i+1} in $N_i - \text{interior}(N_{i+1})$ are controlled homotopy equivalences. If these h-cobordisms are all products then we can fit together product structures $\partial N_i \times [\frac{1}{i+1}, \frac{1}{i}] \simeq N_i - \text{interior}(N_{i+1})$ to get a product structure $\partial N_1 \times (0,1] \simeq N_1$. The control on the size of the product structures shows the images of the arcs converge in Y, so this gives a mapping cylinder.

The intermediate regions originally constructed may not be products, but we can use a "swindle" to make them so. If we factor each $N_i - \text{interior}(N_{i+1})$ as a composition of h-cobordisms $U_i \cup V_i$, N_1 becomes an infinite union $(U_1 \cup V_1) \cup (U_2 \cup V_2) \cup \dots$. Reassociating expresses it as $U_1 \cup (V_1 \cup U_2) \cup \dots$. The idea is choose the decompositions so the new pieces, $V_i \cup U_{i+1}$ are all products. In the uncontrolled setting this is a simple consequence of the invertibility of h-cobordisms. The controlled version is not so simple. We want to inductively choose the decomposition $U_{i+1} \cup V_{i+1}$ so the union $V_i \cup U_{i+1}$ is a product. But we must maintain finer control on U_{i+1} than is available on V_i. The argument thus uses stability of h-cobordism obstructions: we need not only that V_i has some inverse, but that it has one with arbitrarily finer control. This is a deep fact, so this "swindle" is not just a formal argument. As explained above this can be avoided by working with a sequence $\{N_i\}$ to formulate the obstruction directly as an element of the inverse limit. When this vanishes the intermediate regions are automatically already products.

This completes the sketch of the proof.

3.4 Refinements. We give two refinements that follow from the proof. The first concerns smooth or PL structures, and the second provides a way to recognize mapping cylinders themselves, not just existence of neighborhoods.

Smooth and PL cylinders. *If the manifold in Theorem 3.1 has a smooth or PL structure then the mapping cylinder can be chosen to be smooth or PL,* *in the sense that the submanifold $N-Y$ is, and the map $\partial N \times (0,1] \to N-Y$ is a diffeomorphism or PL isomorphism.*

The proof of the Theorem uses handlebody theory, which works in any category of manifolds (dim > 4 in the topological case). Thus the argument and obstructions are category-independent: if $X-Y$ has a smooth structure we get a smooth N, etc.

There is also a smoothing and triangulation theory that shows a topological mapping cylinder in a PL manifold can be made PL, and similarly for smoothing. Using this we could deduce the structure refinement from the topological case. The point of observing it directly from the proof is that eventually it is possible to run the argument backwards and *derive* the smoothing and triangulation structure theory from controlled theorems. In such ways the controlled theory unifies as well as extends the older work.

The structure refinement is *not* true in dimension 5, no matter how nice the local fundamental groups are.

The second result of the section gives a criterion for X itself to be a mapping cylinder over Y. As in the discussion of tameness in 2.4 we extract properties of the radial deformation of a mapping cylinder. If $X = \mathrm{cyl}(g)$ for some map $N \to Y$ then the radial deformation to Y is a map $f : X \times I \to X$ satisfying:

(1) $f^{-1}(Y) = Y \times I \cup X \times \{1\}$

(2) $f(x,t) = x$ if $t = 0$ or $x \in Y$, and

(3) $f(f(x,t),1) = f(x,1)$

The last condition means that if we use the time-1 retraction $f_1 : X \to Y$ as a control map then the deformation f has radius 0 in Y. The criterion relaxes this, requiring only that f has radius less than some appropriate δ. Note that to be useful this "appropriate δ" must be known in advance, before X and f are chosen. Note also that if Y is not compact then this sort of control uses a *function* $\delta : Y \to (0,\infty)$. Typically these functions go to 0, so provide progressively finer control near the ends of Y.

Mapping cylinder recognition. *Suppose Y is a locally compact finite dimensional ANR, $p : E \to Y$ is a stratified system of fibrations, and a dimension $n \geq 6$ is given. Then there is $\delta > 0$ so that if*

(1) $X \supset Y$ with $X - Y$ a manifold (with boundary) of dimension n;

(2) there is a map $\mathrm{holink}(X,Y) \to E$ *that is $(\delta,1)$-connected over Y;*

(3) *$f : X \times I \to X$ is a proper deformation retraction of X to Y that preserves the complement of Y when $t < 1$, and $f_1 f$ has radius $< \delta$ in Y.*

(4) *the inclusion $\partial X \subset X - Y$ is $(\delta, 1)$-connected over Y, using f_1 as control map.*

Then there is a map $g : \partial X \to Y$ and a homeomorphism $\mathrm{cyl}(g) \to X$ *that is the identity on the boundary and Y.*

We can further arrange for the radial deformation in $\mathrm{cyl}(g)$ to be close to the deformation f. More specifically choose $\epsilon > 0$ along with Y, p, and n. Then there is a choice of δ so we can get g and homeomorphism $h : \mathrm{cyl}(g) \to X$ with the diagram

$$\begin{array}{ccc} \mathrm{cyl}(g) \times I & \xrightarrow{h \times \mathrm{id}} & X \times I \\ \Big\downarrow \text{cyl. projection} & & \Big\downarrow f_1 f \\ Y & \xrightarrow{\mathrm{id}} & Y \end{array}$$

commutative within ϵ.

The hypotheses collected in this theorem encode the properties of the "nice neighborhoods" used in the proof of the theorem of 3.1. The part of the proof outlined in 3.2 shows the obstruction vanishes if and only if nice neighborhoods exist. The argument in 3.3 then proves the theorem stated here, that nice neighborhoods are mapping cylinders.

We remark on the role of stability. The argument in 3.3 requires finding a descending sequence of nice neighborhoods with size parameters going to 0. These exist because the obstructions are stable (the same at all sufficiently small scales), and the initial nice neighborhood is chosen to have size in the stable range so it's existence shows the obstructions are trivial. The abstract existence of mapping cylinder neighborhoods can be formulated directly in terms of the inverse limit, avoiding stability. However the recognition theorem is not accessible to this approach because the existence of a single nice neighborhood does not show the vanishing of the obstruction.

4 Applications

The applications detailed here are briefly described in the introduction.

4.1 Mapping cylinders in manifolds [12]. *Suppose $Y \subset M$ is a closed embedding with locally 1-connected complement, of an ANR in the interior of* *a manifold of dimension ≥ 5. Then Y has a mapping cylinder neighborhood.*

To derive this from the main theorem we must show tameness and vanishing of the obstruction. The homological characterization of forward tameness follows from excision and triviality of local fundamental groups. Backwards tameness follows from this. Since the local fundamental groups are trivial we can use the identity $Y \to Y$ as the control map. The main theorem identifies the obstruction as lying in $H_0^{lf}(Y; \mathcal{S}(id))$. The spectral sequence for this has E^2 terms $H_0^{lf}(Y; \tilde{K}_0(Z))$ and $H_i^{lf}(Y; K_-i(Z))$ for $i > 0$. But $\tilde{K}_0$ and the lower K-theory of Z is all trivial, so the obstruction group is trivial. Therefore a mapping cylinder exists.

4.2 Finiteness of ANRs. *A compact finite-dimensional ANR is homotopy equivalent to a finite complex.*

This statement is the "Borsuk conjecture", and was proved for all compact ANRs (not necessarily finite-dimensional) by J. West. The finite-dimensional case follows from the previous result as follows: a compact finite-dimensional ANR has an embedding in some Euclidean space. If this is has locally 1-connected complement then there is a mapping cylinder neighborhood. The neighborhood is a manifold (smooth, actually) so is a finite complex. The mapping cylinder projection is a homotopy equivalence. What if the embedding does not have locally 1-connected complement? The inclusion into R^{n+1} has locally 0-connected complement, and if an embedding has locally 0-connected complement then the inclusion into R^{n+1} has locally 1-connected complement. Thus increasing dimension by two always makes the complement locally 1-connected.

4.3 Collaring in homology manifolds. *Suppose M is an ANR homology manifold of dimension ≥ 5, the embedding $\partial M \subset M$ has locally 1-connected complement, and $M - \partial M$ is a manifold. Then there is a mapping cylinder neighborhood of ∂M. There is a collar (neighborhood homeomorphic to $(\partial M) \times I$) if and only if $(\partial M) \times R$ is a manifold.*

The inclusion of the boundary of a homology manifold is homologically locally infinitely-connected, but local fundamental groups may be nontrivial. For instance closure of the strange component of the complement of the Alexander horned sphere in S^3 is a homology manifold with non-locally 1-connected complement of the boundary. With the 1-connected hypothesis the proof of mapping cylinders is the same as the previous theorem with a little modification of the proof of forward tameness.

We give some context for the collaring statement. First, the map in the mapping cylinder must be a resolution (map from a manifold to a homology manifold that is a local homotopy equivalence, see §4). Edwards' resolution theorem is that when the dimension is ≥ 5 this map can be approximated by a homeomorphism if and only if the homology manifold has the "disjoint 2-disk property." If N is a homology manifold then an easy argument shows $N \times R^2$ has the disjoint 2-disk property so N resolvable implies $N \times R^2$ is resolvable, and therefore a manifold. It is one of the outstanding conjectures in the area that $N \times R$ is already a manifold. The theorem shows this conjecture is equivalent to existence of collars of boundaries in certain homology manifolds.

The proof of the collaring statement follows from the fact that the mapping cylinder map is a resolution, so Edwards' theorem shows the product with R can be approximated by a homeomorphism if and only if $(\partial M) \times I$ is a manifold.

4.4 Stratified spaces. Many interesting spaces are not manifolds, but are built of manifold pieces. These include algebraic varieties, stratifications coming from singularities, and polyhedra. Generally a "stratified space" has a closed filtration $X = X_n \supset X_{n-1} \cdots \supset X_0$ and the strata $X_i - X_{i-1}$ are required to be manifolds. There are several versions that differ in the way the strata fit together. The geometric versions (Whitney, Thom, and PL) have mapping cylinder neighborhoods and complicated relations among them as part of their structure. The most successful topological version, homotopy, or "Quinn" stratified spaces [15], were identified as an outgrowth of controlled topology and have local homotopy conditions relating the strata. In these the strata may not have mapping cylinder neighborhoods, and the obstruction is exactly the one identified in 2.10.

More specifically, suppose X is a homotopy stratified space in the sense of [15]. Then more-or-less by definition

- the embedding $X_{i-1} \subset X_i$ is tame;
- the projection of the homotopy link $ev_0 : \mathrm{holink}(X_i, X_{i-1}) \to X_{i-1}$ is a stratified system of fibrations; and
- $X_i - X_{i-1}$ is a manifold.

Thus we conclude there is an obstruction in $H_0^{lf}(X_{i-1}; \mathcal{S}(ev_0^{-1}(\#)))$ to the existence of a mapping cylinder neighborhood of X_{i-1} in X_i. Vanishing of the obstruction implies existence of such a neighborhood if either $\dim X_i \geq 6$

or $\dim X_i = 4$ and fundamental groups of point-inverses in the homotopy link are "good."

We note that the fact that existence is obstructed means that mapping cylinders are not the natural local structure in these spaces. A weaker version developed by Hughes and others seems to be correct, see §3.5.

4.5 Topological actions of finite groups. Suppose a finite group G acts on a manifold M. We can filter the quotient M/G by orbit types: images of points lie in the same stratum of the quotient if their isotropy subgroups are conjugate. If the action is smooth or PL then the quotient is a smooth or PL stratified space, though the stratification may not be exactly the orbit type stratification. If the action is just topological then really awful point-set things can happen in the quotient. A nice compromise is the class of "homotopically stratified" actions [15], where the quotient with orbit type filtration is assumed to be homotopically stratified in the sense discussed above. This rules out weird point-set behavior but allows many other things. For instance these can have mapping cylinder problems.

An "equivariant mapping cylinder neighborhood" of a G-invariant subset of M is just what it sounds like: a mapping cylinder structure invariant under the action of G. The quotient is an ordinary mapping cylinder neighborhood of the quotient subset. Therefore Theorem 3.1 can be applied in the quotient to determine the existence of equivariant mapping cylinder neighborhoods. We discuss the easiest case, neighborhoods of the non-free points.

Theorem. *Suppose the finite group G acts in a homotopically-stratified way on a compact manifold, let $Y \subset M$ be the points not moved freely by G, and suppose the codimension of Y is ≥ 3. Then*

(1) there is a stratified system of fibrations $p : B_{G_\#} \to Y/G$ whose fiber over $x \in Y$ is the classifying space of the subgroup $G_x \subset G$ fixing x;

(2) there is an obstruction in $H_0(Y/G; \mathcal{S}(p^{-1}(\#)))$ to the existence of an equivariant mapping cylinder neighborhood of Y; and

(3) if there is an equivariant cylinder neighborhood of $Y \cap \partial M$ in ∂M and $\dim M \geq 5$, then the obstruction vanishes if and only if there is an extension of the boundary structure to an equivariant mapping cylinder neighborhood of all of Y.

The hypothesis that Y has codimension is at least 3 implies the embedding $Y \subset M$ has locally 1-connected complement. Therefore local fundamental groups in the quotient come from the the group action, and are modeled by the isotropy groups described in the theorem. The obstructions are often quite accessible:

(1) $K_{-i}(\mathbb{Z}[H]) = 0$ for finite groups H and $-i \leq -2$ [4]. Therefore the spectral sequence for the obstruction group has E^2 terms only $H_0(Y/G; \tilde{K}_0(\mathbb{Z}[G_\#]))$ and $H_1(Y/G; K_{-1}(\mathbb{Z}[G_\#]))$. (these are group, not spectral, cosheaf homology groups);

(2) if Y/G is connected and there is a point fixed by G then the H_0 term reduces to $\tilde{K}_0(\mathbb{Z}[G])$; and

(3) there is an action of a finite group on a disk that is smooth on the boundary and locally linear (in fact can be smoothed in the complement of any fixed point), but the non-free set does not have an equivariant mapping cylinder neighborhood because the $\tilde{K}_0(\mathbb{Z}[G])$ part of the obstruction is nontrivial [13].

4.6 Topological regular neighborhoods. Mapping cylinders are wonderful, but since they do not always exist they are not satisfactory objects for a topological theory of "regular neighborhoods." The appropriate notion seems to be a skewed mapping cylinder neighborhood in $X \times [0, \infty)$.

Definition. Suppose X is locally compact and $Y \subset X$ is closed. A *topological regular neighborhood* of Y consists of

(1) an open neighborhood U of Y,

(2) a proper map $q : U \to Y \times [0, 1)$ that is the identity $Y \to Y \times \{0\}$ and preserves complements of these sets,

(3) a homeomorphism of the relative mapping cylinder $\mathrm{cyl}(q, \mathrm{id}_Y)$ with a neighborhood of $Y \times \{0\}$ in $Y \times [0, \infty)$ that is the identity on $Y \times [0, 1)$ and $U \times \{0\}$.

The relative mapping cylinder is the ordinary mapping cylinder with the cylinder arcs in the subset $Y \times I$ identified to points. The result contains a copy of Y, and the complement of this is the mapping cylinder of the restriction of q to $U - Y \to Y \times (0, 1)$. The homeomorphism in (3) takes the cyinder arcs to arcs that start on $U \times \{0\}$ and go diagonally to $Y \times [0, 1)$, see the figure.

The idea is that we may not be able to find mapping cylinder neighborhoods because there is an obstruction to finding an appropriate domain for the

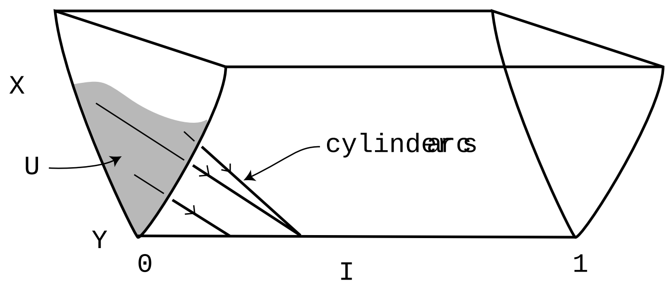

Figure 1: Topological Regular Neighborhood

map. So we use the neighborhood itself as the domain for a map, and get a mapping cylinder in the next higher dimension.

Existence of regular neighborhoods. *Suppose X is a locally compact ANR, $Y \subset X$ is tame, and there is a map, controlled 1-connected over Y, from the homotopy link of Y in X to a stratified system of fibrations over Y. Finally suppose $X - Y$ is a manifold of dimension ≥ 5. Then there is a topological regular neighborhood of Y in X.*

As usual this also holds for X of dimension 4, provided the local fundamental groups have subexponential growth.

Before indicating the proof we discuss some of the structure of these neighborhoods. Suppose $B \subset A$ has a mapping cylinder neighborhood with map $q : U \to B$. The homotopy link is in a sense universal for mapping cylinders mapping to (A, B), so the cylinder structure defines a map $U \to \operatorname{holink}(A, B)$. This is an "approximate fiber homotopy equivalence" over Y [15, 2.7]. In the regular neighborhood situation the homotopy link of $Y \times (0,1) \subset X \times (0,1)$ is the pullback of the homotopy link of $Y \subset X$. Thus the regular neighborhood structure gives an approximate fiber homotopy equivalence over $Y \times (0,1)$, $U \to \operatorname{holink}(X, Y) \times (0,1)$.

In the important special case where X is a homotopically stratified space as in §4.4 and Y is the union of the lower strata, these regular neighborhoods are the same as the "approximate tubular neighborhoods" developed by Hughes [6], and earlier in special cases by Hughes, Taylor, Weinberger and Williams [8]. In a stratified set $\operatorname{holink}(X, Y) \to Y$ is itself a stratified system of fibrations. Thus $q : U - Y \to Y \times (0,1)$ is approximately fiber homotopy equivalent to a stratified system of fibrations. This identifies q as a "manifold stratified approximate fibration" over $Y \times (0,1)$.

Finally we outline how the theorem follows from a version of the mapping cylinder recognition theorem of 3.4. Let $h : X \times [0,1] \to X$ be a forward-tame deformation, and let V be a neighborhood of Y such that $h(V \times \{1\}) \subset Y$. Define $f : X \times [0, \infty) \times [0, 1] \to X \times [0, \infty)$ by

$$f(x, s, t) = (h(x, t), s + td(x, Y))$$

where $d(x, Y)$ is the distance from the point to the subspace Y. Then properties of h imply

(1) f is a homotopy from the identity at $t = 0$ to a map at $t = 1$ that takes $V \times [0, \infty)$ into $Y \times [0, \infty)$;

(2) when $t < 1$ $f(\#, t)$ takes the complement of $Y \times [0, \infty)$ into itself;

(3) $f(\#, t)$ is the identity on $Y \times [0, \infty)$;

(4) $f(\#, t)$ takes the complement of $Y \times \{0\}$ into itself for all t.

Delete $Y \times \{0\}$, then this is a homotopy of $(X - Y) \times \{0\} \cup X \times (0, \infty)$. We would like to arrange it to satisfy the conditions of the mapping cylinder recognition theorem, to get a mapping cylinder over $Y \times (0, \infty)$. This cannot be done completely: the end near $Y \times \{0\}$ is ok, but none of the conditions hold near ∞. Instead we use a relative version: if the conditions hold over $Y \times (0, 1)$ then some neighborhood of $Y \times \{0\}$ is a mapping cylinder over $Y \times (0, 1 - \epsilon)$. Such relative versions are standard parts of controlled theory (see e.g. the remarks before Theorem 1.3 in [13]). They are often not stated explicitly because the statements are so complicated, but follow from the proofs. The actual goal is thus to arrange the conditions of the recognition theorem to hold over $Y \times (0, 1)$.

Recall that we want to use f_1 as the control map. The first problem is that f_1 does not even map all of the space into $Y \times (0, \infty)$. However this does work near 0. Suppose the neighborhood V taken into Y by h contains the points within ϵ of Y. Then f does deform all of $f_1^{-1}(Y \times [0, \epsilon))$ into $Y \times (0, \infty)$. Restrict to this, in the sense that we consider the space $f_1^{-1}(Y \times [0, \epsilon))$ with control map f_1 over $Y \times [0, \epsilon)$. Reparameterize $[0, \epsilon)$ as $[0, \infty)$. The situation is now that f_1 can serve as a control map. It is also proper. We have lost something: since the original deformation did not preserve $f_1^{-1}(Y \times [0, \epsilon)$, the restriction does not define a deformation of the space. However since $Y \times \{0\}$ is left fixed, by continuity f keeps some $f_1^{-1}(Y \times [0, \tau)$ inside the new space. Reparameterize the interval again to arrange the deformation to be defined on $f_1^{-1}(Y \times [0, 2))$. We now have the control map and deformation defined over $Y \times [0, 2)$

The last step is to arrange arbitrarily good size control, at least over $Y \times (0, 1)$. The deformation f is the identity on $Y \times \{0\}$, or in other words the composition $f_1 f$ has radius 0 as a homotopy of $Y \times \{0\}$ in itself. It follows that $f_1 f$ has very small radius over $Y \times [0, \epsilon)$, for small ϵ. By reparameterizing $[0, \epsilon)$ to $[0, 2)$ we can arrange that $f_1 f$ has arbitrarily small radius in the Y coordinate over $Y \times [0, 2)$. We need to do a little better. We are controlling over the 0 end of $Y \times (0, \infty)$, which is non-compact even if Y is compact. Assume Y is compact to simplify the argument, then the control objective is a continuous function $\delta : (0, \infty) \to (0\infty)$, not a constant $\delta > 0$. Elaborate the previous argument: since $f_1 f$ has radius 0 over $Y \times \{0\}$, there is a continuous increasing function $\epsilon : [0, \infty) \to [0, \infty)$ taking 0 to 0, so that $f_1 f$ has radius $< \epsilon$ in the Y coordinate, over $Y \times [0, 2)$. Now reparameterize by a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ so that $\epsilon\theta < \delta$. The result is δ controlled in the Y coordinate. It remains to get control in the $[0, \infty)$ coordinate. This is again a standard argument using continuity and reparameterization.

The outcome of all this is a neighborhood of $Y \times \{0\}$ in $X \times [0, \infty)$, a control map to $Y \times [0, \infty)$, and a deformation defined over $Y \times [0, 2)$ and satisfying the control needed for the Recognition Theorem over $Y \times [0, 1)$. The theorem then asserts that there is a mapping cylinder structure provided the dimension of $X \times [0, \infty)$ is at least 6, or in other words if X has dimension at least 5, or X has dimension 4 and the local fundamental groups are small.

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Foundations of Algebraic Surgery

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Abstract

The algebraic theory of surgery on chain complexes C with Poincaré duality

$$H^*(C) \;\cong\; H_{n-*}(C)$$

describes geometric surgeries on the chain level. The algebraic effect of a geometric surgery on an n-dimensional manifold M is an algebraic surgery on the n-dimensional symmetric Poincaré complex (C,ϕ) over $\mathbb{Z}[\pi_1(M)]$ with the homology of the universal cover $\widetilde{M}$

$$H_*(C) \;=\; H_*(\widetilde{M}) \ .$$

The algebraic effect of a geometric surgery on an n-dimensional normal map $(f,b) : M \to X$ is an algebraic surgery on the kernel n-dimensional quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi_1(X)]$ with homology

$$H_*(C) \;=\; K_*(M) \;=\; \ker(f_* : H_*(\widetilde{M}) \to H_*(\widetilde{X})) \ .$$

For $n > 4$ and i-connected (f,b) with $2i \leqslant n$ there is a one-one correspondence between geometric surgeries on (f,b) killing elements $x \in K_i(M)$ and algebraic surgeries on (C,ψ) killing $x \in H_i(C)$. The Wall surgery obstruction of an n-dimensional normal map $(f,b) : M \to X$

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

was originally defined by first making (f,b) $[n/2]$-connected by geometric surgery below the middle dimension, using forms for even n and automorphisms of forms for odd n. The algebraic theory of surgery identifies $\sigma_*(f,b)$ with the cobordism class of the kernel quadratic Poincaré complex (C,ψ), so the algebraic surgery obstruction has the same formulation for odd and even n. The identification is used for $n = 2i$ (resp. $2i+1$) to find a representative form (resp. automorphism) without preliminary geometric surgeries.

Contents

1 Introduction

We compare the homology and chain level descriptions of surgery on a manifold, using a minimum of algebraic development.

Manifolds M are to be finite-dimensional, compact, and oriented (unless stated otherwise), with $C(M)$ denoting the cellular chain complex for some CW structure on M.

Cobordisms $(W; M, M')$ are to be oriented (unless stated otherwise) with $\partial W = M \cup -M'$, where $-M'$ denotes M' with the opposite orientation.

1.1 Background

The surgery method of classifying manifolds within a homotopy type was first applied by Kervaire and Milnor [2] to exotic spheres, using exact sequences to describe the homology effect of geometric surgery. Homology was quite adequate for the subsequent development of surgery theory on simply-connected manifolds (Browder [1], Novikov). Wall [11] used a combination of topology and homology to describe the effect of surgery on non-simply-connected manifolds. In general, the homology $\mathbb{Z}[\pi_1(M)]$-modules $H_*(\widetilde{M})$ of the universal cover $\widetilde{M}$ of a compact manifold M are not finitely generated, so a chain level approach is indicated. The algebraic theory of surgery of Ranicki [4],[5] provided a model for surgery using chain complexes with Poincaré duality.

Surgery was originally developed for differentiable manifolds, but has since been extended to PL and topological manifolds. The algebraic theory of surgery applies to all categories of manifolds.

1.2 The algebraic effect of a geometric surgery

Let M be an n-dimensional manifold. Surgery on $S^i \times D^{n-i} \subset M$ results in an n-dimensional manifold

$$M' = (M \backslash S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1} .$$

The trace of the surgery is the cobordism $(W; M, M')$ given by attaching a $(i+1)$-handle at $S^i \times D^{n-i} \subset M$

$$W = M \times I \cup D^{i+1} \times D^{n-i} .$$

The trace of the surgery on $D^{i+1}\times S^{n-i-1} \subset M'$ is the cobordism $(W'; M', M)$ with

$$W' \;=\; -W \;=\; M'\times I \cup D^{i+1}\times D^{n-i}\ .$$

In fact, every cobordism of manifolds is a union of the traces of surgeries.

In terms of homotopy theory the trace W is obtained from M by attaching an $(i+1)$-cell, and M' is then obtained from W by detaching an $(n-i)$-cell, with homotopy equivalences

$$W \;\simeq\; M\cup_x D^{i+1} \;\simeq\; M'\cup_{x'} D^{n-i}\ ,$$

with $x: S^i \to M$ the inclusion $S^i\times\{0\}\subset S^i\times D^{n-i}\subset M$, and similarly for $x': S^{n-i-1}\to M'$. The immediate homology effect of the surgery is to kill $x\in H_i(M)$,

$$H_i(W) \;=\; H_i(M)/\langle x\rangle$$

with $\langle x\rangle \subseteq H_i(M)$ the subgroup generated by x. On the chain level

(i) $C(W)$ is chain equivalent to the algebraic mapping cone $\mathcal{C}(x)$ of a chain map $x: S^i\mathbb{Z}\to C(M)$ representing $x\in H_i(M)$, where

$$S^i\mathbb{Z}\ :\ \cdots\to 0\to\mathbb{Z}\to 0\to\dots\quad\text{(concentrated in degree } i)\ ,$$

and similarly for $C(W)\simeq\mathcal{C}(x': S^{n-i-1}\mathbb{Z}\to C(M'))$,

(ii) there is defined a commutative braid of chain homotopy exact sequences of chain complexes

$$\begin{tikzcd}
S^i\mathbb{Z} \arrow[dr] \arrow[rr, bend left, "x"] & & C(M) \arrow[dr] \arrow[rr, bend left, "x^*"] & & S^{n-i}\mathbb{Z} \\
& C(W, M\cup M')_{*+1} \arrow[ur] \arrow[dr] & & C(W) \arrow[ur, "y"] \arrow[dr, "y'"'] & \\
S^{n-i-1}\mathbb{Z} \arrow[ur] \arrow[rr, bend right, "x'"'] & & C(M') \arrow[ur] \arrow[rr, bend right, "x'^*"'] & & S^{i+1}\mathbb{Z}
\end{tikzcd}$$

with $x^*: C(M)\to S^{n-i}\mathbb{Z}$ a chain map representing the Poincaré dual $x^*\in H^{n-i}(M)$ of $x\in H_i(M)$, and similarly for x'^*,

(iii) $C(M')$ is chain equivalent to the dimension shifted algebraic mapping cone $\mathcal{C}(y)_{*+1}$ of a chain map $y : C(W) \to S^{n-i}\mathbb{Z}$ representing a cohomology class $y \in H^{n-i}(W)$ with image the Poincaré dual $x^* \in H^{n-i}(M)$ of $x \in H_i(M)$, and similarly for $C(M)$.

Algebraic surgery gives a precise algebraic model for a chain complex in the chain homotopy type of $C(M')$, which is obtained from $C(M)$ by attaching x and detaching y.

The homology groups $H_*(M), H_*(M'), H_*(W)$ are related by the long exact sequences

$$\begin{aligned} \dots \to H_r(M) \to H_r(W) \to H_r(W,M) \to H_{r-1}(M) \to \dots\ , \\ \dots \to H_r(M') \to H_r(W) \to H_r(W,M') \to H_{r-1}(M') \to \dots\ . \end{aligned}$$

It now follows from the excision isomorphisms

$$\begin{aligned} H_r(W,M) &= H_r(D^{i+1},S^i) = \begin{cases} \mathbb{Z} & \text{for } r=i+1 \\ 0 & \text{otherwise}\ , \end{cases} \\ H_r(W,M') &= H_r(D^{n-i},S^{n-i-1}) = \begin{cases} \mathbb{Z} & \text{for } r=n-i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

that

$$H_r(M) = H_r(W) = H_r(M') \text{ for } r \neq i, i+1, n-i-1, n-i\ .$$

The relationship between $H_r(M), H_r(M'), H_r(W)$ for $r = i, i+1, n-i-1, n-i$ is more complicated, especially in the middle dimensional cases $n = 2i, 2i+1$.

Here are some of the advantages of chain complexes over homology in describing the algebraic effects of surgery on manifolds. The chain complex method :

(•) makes it easier to follow the passage from the embedding $S^i \times D^{n-i} \subset M$ to the homology $H_*(M')$ on the chain level;

(•) provides a uniform description for all n, i ;

(•) avoids the indeterminacies inherent in exact sequences;

(•) works just as well in the non-simply connected case;

(•) keeps track of the effect of successive surgeries.

Surgery on manifolds is described algebraically by surgery on chain complexes with symmetric Poincaré duality. The applications of surgery to the classification of manifolds involve a normal map $(f,b) : M \to X$, and only surgeries with an extension of (f,b) to a normal map on the trace

$$((g,c);(f,b),(f',b')) \; : \; (W;M,M') \to X \times ([0,1];\{0\},\{1\})$$

are considered. Surgery on normal maps is described algebraically by surgery on chain complexes with quadratic Poincaré duality. The quadratic refinement corresponds to the additional information carried by the bundle map $b : \nu_M \to \nu_X$. The formulae for algebraic surgery on symmetric Poincaré complexes are entirely analogous to the formulae for quadratic Poincaré complexes.

1.3 The Principle of algebraic surgery

In its simplest form, the Principle states that for a cobordism of n-dimensional manifolds $(W;M,M')$ the chain homotopy type of $C(M')$ and the Poincaré duality chain equivalence

$$[M'] \cap - \; : \; C(M')^{n-*} \; \simeq \; C(M')$$

can be obtained from

(i) the chain homotopy type of $C(M)$,

(ii) the Poincaré duality chain equivalence

$$\phi_0 \; = \; [M] \cap - \; : \; C(M)^{n-*} \; \simeq \; C(M)$$

and the chain homotopy

$$\phi_1 \; : \; (\phi_0)^* \; \simeq \; \phi_0 \; : \; C(M)^{n-*} \to C(M)$$

determined up to higher chain homotopies by topology,

(iii) the chain homotopy class of the chain map $j : C(M) \to C(W,M')$ induced by the inclusion $M \subset W$,

(iv) the chain homotopy

$$\delta\phi_0 \; : \; j\phi_0 j^* \; \simeq \; 0 \; : \; C(W,M')^{n-*} \to C(W,M')$$

determined up to higher chain homotopies by topology.

The chain complex $C(M')$ is chain equivalent to the chain complex C' obtained from $C(M)$ by algebraic surgery, with

$$C'_r \ = \ C_r(M) \oplus C_{r+1}(W,M') \oplus C^{n-r-1}(W,M') \ .$$

See §3 for formulae for the differentials and Poincaré duality of C'.

In particular, if $(W;M,M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ as in §1.2 then $C(W,M')$ is chain equivalent to $S^{n-i}\mathbb{Z}$, and replacing $C(W,M')$ by $S^{n-i}\mathbb{Z}$ in the formula for C'_r gives a smaller chain complex (also denoted by C')

$$\begin{aligned} C' \ : \ C_n(M) \to \dots \to C_{n-i}(M) \xrightarrow{d\oplus y} C_{n-i-1}(M) \oplus \mathbb{Z} \xrightarrow{d\oplus 0} C_{n-i-2}(M) \to \dots \\ \to C_{i+2}(M) \xrightarrow{d\oplus 0} C_{i+1}(M) \oplus \mathbb{Z} \xrightarrow{d\oplus x} C_i(M) \to \dots \to C_0(M) \end{aligned}$$

chain equivalent to $C(M')$. The attaching chain map $x : S^i\mathbb{Z} \to C(M)$ and the chain map $j : C(M) \to C(W,M') \simeq S^{n-i}\mathbb{Z}$ in (iii) are determined by the homotopy class of the core embedding $S^i \times \{0\} \subset M$. The detaching chain map $y : \mathcal{C}(x) \to S^{n-i}\mathbb{Z}$ and the chain homotopy $\delta\phi_0$ in (iv) are determined by the framing of the core, and are much more subtle. (See the Examples below). In this case the algebraic surgery kills the homology class $x \in H_i(M)$. In the general algebraic context surgery kills entire subcomplexes rather than just individual homology classes.

Example. The effect of surgery on $S^0 \times D^1 \subset M = S^1$ is a double cover of S^1. There are two possibilities:

(i) If the two paths $S^0 \times D^1 \subset S^1$ move in opposite senses the effect of the surgery is the trivial double cover $M' = S^1 \cup S^1$ of S^1, and the trace $(W;M,M')$ is given by the orientable

$$W \ = \ \mathrm{cl.}(S^2\backslash(D^2 \cup D^2 \cup D^2)) \ .$$

(ii) If the two paths $S^0 \times D^1 \subset S^1$ move in the same sense the effect of the surgery is the nontrivial double cover $M'' = S^1$ of S^1, and the trace $(W';M,M'')$ is given by the nonorientable

$$W' \ = \ \mathrm{cl.}(\text{Möbius band}\backslash D^2) \ . \qquad \square$$

More generally:

Example. As usual, let $O(j)$ be the orthogonal group of $\mathbb{R}^j$. For any map $\omega : S^i \to O(j)$ write $n = i + j$, and define an embedding

$$e_\omega \ : \ S^i \times D^j \to S^n = S^i \times D^j \cup D^{i+1} \times S^{j-1} \ ; \ (x,y) \mapsto (x,\omega(x)(y)) \ .$$

Surgery on $M = S^n$ killing e_ω has effect the $(j-1)$-sphere bundle over $S^{i+1} = D^{i+1} \cup D^{i+1}$

$$M' \; = \; S(\omega) \; = \; D^{i+1} \times S^{j-1} \cup_\omega D^{i+1} \times S^{j-1}$$

of the j-plane vector bundle over S^{i+1}

$$E(\omega) \; = \; D^{i+1} \times \mathbb{R}^j \cup_\omega D^{i+1} \times \mathbb{R}^j$$

classified by $\omega \in \pi_i(O(j)) = \pi_{i+1}(BO(j))$. The trace of the surgery is

$$W \; = \; \mathrm{cl.}(D(\omega)\backslash D^{n+1}) \; ,$$

with

$$D(\omega) \; = \; D^{i+1} \times D^j \cup_\omega D^{i+1} \times D^j$$

the j-disk bundle of ω, which fits into a fibre bundle

$$(D^j, S^{j-1}) \to (D(\omega), S(\omega)) \to S^{i+1} \; .$$

Exercise: work out the algebraic effect of the surgery! $\square$

2 Forms and formations

The quadratic L-groups $L_*(A)$ were originally defined by Wall, with $L_{2i}(A)$ a Witt-type group of stable isomorphism classes of nonsingular $(-)^i$-quadratic forms over a ring with involution A, and $L_{2i+1}(A)$ a Whitehead-type group of automorphisms of $(-)^i$-quadratic forms over A (now replaced by formations). The surgery obstruction of a normal map $(f,b) : M \to X$ from an n-dimensional manifold M to an n-dimensional Poincaré complex X

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

was defined by first making (f,b) i-connected for $n = 2i$ (resp. $2i+1$) by surgery below the middle dimension. The surgery obstruction is such that $\sigma_*(f,b) = 0$ if (and for $n > 4$ only if) (f,b) is normal bordant to a homotopy equivalence.

Let A be an associative ring with 1, and with an involution $A \to A; a \mapsto \overline{a}$ satisfying

$$\overline{a+b} \; = \; \overline{a} + \overline{b} \; , \; \overline{ab} \; = \; \overline{b}\,\overline{a} \; , \; \overline{\overline{a}} \; = \; a \; , \; \overline{1} \; = \; 1 \; .$$

In the applications to topology $A = \mathbb{Z}[\pi]$ is a group ring with the involution

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \; ; \; x \; = \; \sum_{g \in \pi} n_g g \mapsto \overline{x} \; = \; \sum_{g \in \pi} n_g g^{-1} \; .$$

The dual of a left A-module is the left A-module

$$K^* = \mathrm{Hom}_A(K, A)\ ,\ A \times K^* \to K^*\ ;\ (a, f) \mapsto (x \mapsto f(x)\overline{a})\ .$$

The dual of an A-module morphism $f : K \to L$ is the A-module morphism

$$f^* : L^* \to K^*\ ;\ g \mapsto (x \mapsto g(f(x)))\ .$$

For f.g. free K, L identify

$$f^{**} = f : K^{**} = K \to L^{**} = L\ ,$$

using the isomorphism $K \to K^{**}; x \mapsto (f \mapsto \overline{f(x)})$ to identify $K = K^{**}$, and similarly for L.

A $(-)^i$-*quadratic form* (K, λ, μ) is a f.g. free A-module K together with a $(-)^i$-symmetric form

$$\lambda = (-)^i\lambda^* : K \to K^*$$

and a function

$$\mu : K \to Q_{(-)^i}(A) = A/\{a - (-)^i\overline{a} \,|\, a \in A\}$$

such that

$$\lambda(x)(x) = \mu(x)+(-)^i\overline{\mu(x)}\ ,\ \mu(ax) = a\mu(x)\overline{a}\ ,\ \mu(x+y) = \mu(x)+\mu(y)+\lambda(x, y)\ .$$

The form is *nonsingular* if $\lambda : K \to K^*$ is an isomorphism.

A *lagrangian* for a nonsingular $(-)^i$-quadratic form (K, λ, μ) is a f.g. free direct summand $L \subset K$ such that $\lambda(L)(L) = 0$, $\mu(L) = 0$, and $L = L^\perp$, where

$$L^\perp = \{x \in K \,|\, \lambda(x)(L) = 0\}\ .$$

A nonsingular form admits a lagrangian if and only if it is isomorphic to the hyperbolic form

$$H_{(-)^i}(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ (-)^i & 0 \end{pmatrix}, \mu)$$

with $\mu(x, f) = f(x)$.

The $2i$-*dimensional quadratic* L-*group* $L_{2i}(A)$ is the Witt group of stable isomorphism classes of nonsingular $(-)^i$-quadratic forms (K, λ, μ) over A, where stability is with respect to the hyperbolic forms.

As in Chapter 5 of Wall [11] the surgery obstruction of an i-connected $2i$-dimensional normal map $(f,b) : M \to X$ is the Witt class

$$\sigma_*(f,b) \;=\; (K_i(M),\lambda,\mu) \in L_{2i}(\mathbb{Z}[\pi_1(X)])$$

of the kernel nonsingular $(-)^i$-quadratic form $(K_i(M),\lambda,\mu)$ over $\mathbb{Z}[\pi_1(X)]$, with λ,μ defined by geometric intersection numbers. An algebraic surgery on (f,b) removing $S^j \times D^{2i-j} \subset M$ for $j = i-1$ (resp. i) correspond to the algebraic surgery of the addition (resp. subtraction) of the hyperbolic form $H_{(-)^i}(\mathbb{Z}[\pi_1(X)])$ to (resp. from) the kernel form.

A *nonsingular $(-)^i$-quadratic formation* $(K,\lambda,\mu;F,G)$ is a nonsingular $(-)^i$-quadratic form (K,λ,μ) together with an ordered pair of lagrangians F,G.

The $(2i+1)$*-dimensional quadratic L-group* $L_{2i+1}(A)$ is the group of stable isomorphism classes of nonsingular $(-)^i$-quadratic formations $(K,\lambda,\mu;F,G)$ over A, where stability is with respect to the formations such that either F,G are direct complements in K or share a common lagrangian complement in K.

The surgery obstruction of an i-connected $(2i+1)$-dimensional normal map $(f,b) : M \to X$

$$\sigma_*(f,b) \;=\; (K,\lambda,\mu;F,G) \in L_{2i+1}(\mathbb{Z}[\pi_1(X)])$$

is the Witt-type equivalence class of a kernel $(-)^i$-quadratic formation over $\mathbb{Z}[\pi_1(X)]$ with

$$F \cap G \;=\; K_{i+1}(M) \quad , \quad K/(F+G) \;=\; K_i(M)\ .$$

As in Chapter 6 of Wall [11] such a kernel formation $(K,\lambda,\mu;F,G)$ is obtained by realizing any finite set $\{x_1,x_2,\dots,x_k\} \subset K_i(M)$ of $\mathbb{Z}[\pi_1(X)]$-module generators by a high-dimensional Heegaard-type decomposition of (f,b) as a union of normal maps

$$(f,b) \;=\; (f_0,b_0)\cup(g,c) \;:\; M \;=\; M_0 \cup U \to X \;=\; X_0 \cup D^{2i+1}$$

with

$$\begin{aligned}
(g,c) \;&:\; (U,\partial U) \;=\; (\#_k S^i \times D^{i+1}, \#_k S^i \times S^i) \to (D^{2i+1},S^{2i})\ ,\\
F \;&=\; \mathrm{im}(K_{i+1}(U,\partial U) \to K_i(\partial U)) \;=\; \mathbb{Z}[\pi_1(X)]^k\ ,\\
G \;&=\; \mathrm{im}(K_{i+1}(M_0,\partial U) \to K_i(\partial U)) \;\cong\; \mathbb{Z}[\pi_1(X)]^k\ ,\\
K \;&=\; K_i(\partial U) \;=\; F \oplus F^*\ ,\ (\lambda,\mu) \;=\; \text{hyperbolic } (-)^i\text{-quadratic form}\ .
\end{aligned}$$

3 Surgery on symmetric Poincaré complexes

Symmetric Poincaré complexes are chain complexes with the Poincaré duality properties of manifolds. A manifold M determines a symmetric Poincaré complex (C,ϕ), such that a surgery on M determines an algebraic surgery on (C,ϕ). However, not every algebraic surgery on (C,ϕ) can be realized by a surgery on M.

Given a f.g. free A-module chain complex

$$C\ :\ \cdots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \dots$$

write the dual f.g. free A-modules as

$$C^r\ =\ (C_r)^*\ .$$

For any $n \geqslant 0$ let C^{n-*} be the f.g. free A-module chain complex with

$$d_{C^{n-*}}\ =\ (-)^r d_C^*\ :\ (C^{n-*})_r\ =\ C^{n-r} \to (C^{n-*})_{r-1}\ =\ C^{n-r+1}\ .$$

The duality isomorphisms

$$T\ :\ \mathrm{Hom}_A(C^p,C_q) \to \mathrm{Hom}_A(C^q,C_p)\ ;\ \phi \mapsto (-)^{pq}\phi^*$$

are involutions with the property that the dual of a chain map $f : C^{n-*} \to C$ is a chain map $Tf : C^{n-*} \to C$, with $T(Tf) = f$.

The algebraic mapping cone $\mathcal{C}(f)$ of a chain map $f : C \to D$ is the chain complex with

$$d_{C(f)}\ =\ \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix}\ :\ C(f)_r\ =\ D_r \oplus C_{r-1} \to C(f)_{r-1}\ =\ D_{r-1} \oplus C_{r-2}\ .$$

An *n-dimensional symmetric complex* (C,ϕ) over A is a f.g. free A-module chain complex

$$C\ :\ C_n \xrightarrow{d_C} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_C} C_0$$

together with a collection of A-module morphisms

$$\phi\ =\ \{\phi_s : C^{n-r+s} \to C_r \mid s \geqslant 0\}$$

such that

$$d_C\phi_s + (-1)^r\phi_s d_C^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1})\ =\ 0\ :\ C^{n-r+s-1} \to C_r$$
$$(s \geqslant 0, \phi_{-1} = 0)\ .$$

Thus $\phi_0 : C^{n-*} \to C$ is a chain map, ϕ_1 is a chain homotopy $\phi_1 : \phi_0 \simeq T\phi_0 : C^{n-*} \to C$, and so on ... . More intrinsically, ϕ is an n-dimensional cycle in the $\mathbb{Z}$-module chain complex

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))$$

with

$$W \ : \ \cdots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$. The symmetric complex (C, ϕ) is *Poincaré* if the chain map $\phi_0 : C^{n-*} \to C$ is a chain equivalence.

Example. (Mishchenko [3]) An n-dimensional manifold M and a regular covering $\widetilde{M}$ with group of covering translations π determine an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ $(C(\widetilde{M}), \phi)$ with

$$\phi_0 \ = \ [M] \cap - \ : \ C(\widetilde{M})^{n-*} \to C(\widetilde{M})$$

the Poincaré duality chain equivalence. The higher chain homotopies $\phi_1, \phi_2, \ldots$ are determined by an equivariant analogue of the construction of the Steenrod squares.

□

An $(n+1)$-*dimensional symmetric pair* $(j : C \to D, (\delta\phi, \phi))$ is an n-dimensional symmetric complex (C, ϕ) together with a chain map $j : C \to D$ to an $(n+1)$-dimensional f.g. free A-module chain complex D and A-module morphisms $\delta\phi = \{\delta\phi_s : D^{n+1-r+s} \to D_r \mid s \geqslant 0\}$ such that

$$\begin{aligned} j\phi_s j^* \ &= \ d_D \delta\phi_s + (-)^r \delta\phi_s d_D^* + (-)^{n+s+1} \\ &\quad (\delta\phi_{s-1} + (-)^s T\delta\phi_{s-1}) \ : \ D^{n+1-r-s} \to D_r (s \geqslant 0, \delta\phi_{-1} = 0) \ . \end{aligned}$$

The pair is *Poincaré* if the chain map

$$\begin{pmatrix} \delta\phi_0 \\ \phi_0 j^* \end{pmatrix} \ : \ D^{n+1-*} \to \mathcal{C}(j)$$

is a chain equivalence, in which case (C, ϕ) is a n-dimensional symmetric Poincaré complex.

A *cobordism* of n-dimensional symmetric Poincaré complexes (C, ϕ), (C', ϕ') is an $(n+1)$-dimensional symmetric Poincaré pair of the type $(C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi'))$. Symmetric complexes (C, ϕ), (C', ϕ') are *homotopy equivalent* if there exists a cobordism with $C \to D$, $C' \to D$ chain equivalences.

Example. A 0-dimensional symmetric complex (C,ϕ) is a f.g. free A-module C_0 together with a symmetric form ϕ_0 on C^0. The complex is Poincaré if and only if the form is nonsingular. Two 0-dimensional symmetric Poincaré complexes (C,ϕ), (C',ϕ') are cobordant if and only if the forms (C^0,ϕ_0), (C'^0,ϕ'_0) are Witt-equivalent, i.e. become isomorphic after stabilization with metabolic forms $(L\oplus L^*, \begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix})$.

□

Example. An $(n+1)$-dimensional manifold with boundary $(W,\partial W)$ and cover $(\widetilde{W},\partial\widetilde{W})$ determines an $(n+1)$-dimensional symmetric Poincaré pair $(j: C(\partial\widetilde{W})\to C(\widetilde{W}),(\delta\phi,\phi))$ over $\mathbb{Z}[\pi]$ with

$$\begin{pmatrix}\delta\phi_0\\ \phi_0 j^*\end{pmatrix} = [W]\cap - \ :\ D^{n+1-*} = C(\widetilde{W})^{n-*}\to \mathcal{C}(j) = C(\widetilde{W},\partial\widetilde{W})$$

the Poincaré-Lefschetz duality chain equivalence. □

The *data* for *algebraic surgery* on an n-dimensional symmetric Poincaré complex (C,ϕ) is an $(n+1)$-dimensional symmetric pair $(j:C\to D,(\delta\phi,\phi))$. The *effect* of the algebraic surgery is the n-dimensional symmetric Poincaré complex (C',ϕ') with

$$\begin{aligned}
d_{C'} &= \begin{pmatrix} d_C & 0 & (-)^{n+1}\phi_0 j^*\\ (-)^r j & d_D & (-)^r\delta\phi_0\\ 0 & 0 & d_D^*\end{pmatrix}\ :\\
&\quad C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\to C'_{r-1} = C_{r-1}\oplus D_r\oplus D^{n-r+2}\ ,\\
\phi'_0 &= \begin{pmatrix} \phi_0 & 0 & 0\\ (-)^{n-r}jT\phi_1 & (-)^{n-r}T\delta\phi_1 & 0\\ 0 & 1 & 0\end{pmatrix}\ :\\
&\quad C'^{n-r} = C^{n-r}\oplus D^{n-r+1}\oplus D_{r+1}\to C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\ ,\\
\phi'_s &= \begin{pmatrix} \phi_s & 0 & 0\\ (-)^{n-r}jT\phi_{s+1} & (-)^{n-r+s}T\delta\phi_{s+1} & 0\\ 0 & 0 & 0\end{pmatrix}\ :\\
&\quad C'^{n-r+s} = C^{n-r+s}\oplus D^{n-r+s+1}\oplus D_{r-s+1}\to C'_r\\
&\quad = C_r\oplus D_{r+1}\oplus D^{n-r+1}\ \ (s\geqslant 1)\ .
\end{aligned}$$

Remark. The appearance of the chain homotopy $\phi_1:\phi_0\simeq T\phi_0$ in the formula for the Poincaré duality chain equivalence ϕ'_0 is a reason for taking account of ϕ_1. The appearance of the higher chain homotopy $\phi_2 : \phi_1 \simeq T\phi_1$ in the formula for ϕ'_1 is a reason for taking account of ϕ_2. And so on .... □

The *trace* of an algebraic surgery is the $(n+1)$-dimensional symmetric Poincaré cobordism between (C,ϕ) and (C',ϕ')

$$((f\ f') : C \oplus C' \to D', (0, \phi \oplus -\phi'))$$

defined by

$$\begin{aligned} d_{D'} \ &= \ \begin{pmatrix} d_C & (-)^{n+1}\phi_0 j^* \\ 0 & d_D^* \end{pmatrix} \ : \ D'_r \ = \ C_r \oplus D^{n-r+1} \to D'_{r-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \ C_{r-1} \oplus D^{n-r+2} \ , \\ f \ &= \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ : \ C_r \to D'_r \ = \ C_r \oplus D^{n-r+1} \ , \\ f' \ &= \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ : \ C'_r \ = \ C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r \ = \ C_r \oplus D^{n-r+1} \ . \end{aligned}$$

Theorem. (Ranicki [4]) *The cobordism of symmetric Poincaré complexes is the equivalence relation generated by homotopy equivalence and algebraic surgery.*

Proof. Homotopy equivalent complexes are cobordant, by definition. Surgery equivalent complexes are cobordant by the trace construction.

Conversely, suppose given a cobordism of n-dimensional symmetric Poincaré complexes

$$\Gamma \ = \ ((f\ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi')) \ .$$

Let

$$\overline{\Gamma} \ = \ ((\overline{f}\ \overline{f}') : C \oplus \overline{C}' \to \overline{D}, (0, \phi \oplus -\phi'))$$

be the trace of the algebraic surgery on (C,ϕ) with data $(j : C \to \mathcal{C}(f'), (\delta\phi/\phi', \phi))$ given by

$$\begin{aligned} j \ &= \ \begin{pmatrix} f \\ 0 \end{pmatrix} \ : \ C_r \to \mathcal{C}(f')_r \ = \ D_r \oplus C'_{r-1} \ , \\ (\delta\phi/\phi')_s \ &= \ \begin{pmatrix} \delta\phi_s & (-)^s f'\phi'_s \\ 0 & (-)^{n-r+s} T\phi'_{s-1} \end{pmatrix} \ : \\ &\qquad \mathcal{C}(f')^{n-r+s+1} \ = \ D^{n-r+s+1} \oplus C'^{n-r+s} \to \mathcal{C}(f')_r \ = \ D_r \oplus C'_{r-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (s \geqslant 0, \phi'_{-1} = 0) \ . \end{aligned}$$

The A-module morphisms

$$\begin{aligned} g \ &= \ (0\ \ 0\ \ 1\ \ 0\ \ 0) \ : \ \overline{C}'_r \ = \ C_r \oplus D_{r+1} \oplus C'_r \oplus D^{n-r+1} \oplus C'^{n-r} \to C'_r \ , \\ h \ &= \ (f\ \ \delta\phi_0\ \ f'\phi'_0) \ : \ \overline{D}_r \ = \ C_r \oplus D^{n-r+1} \oplus C'^{n-r} \to D_r \end{aligned}$$

define a homotopy equivalence $(h, 1_C \oplus g) : \overline{\Gamma} \to \Gamma$. □

Symmetric Surgery Principle. For any $(n{+}1)$-dimensional cobordism $(W; M, M')$ and regular cover $(\widetilde{W}; \widetilde{M}, \widetilde{M'})$ with group π the symmetric Poincaré complex $(C(\widetilde{M'}), \phi')$ is homotopy equivalent to the effect of algebraic surgery on $(C(\widetilde{M}), \phi)$ with data

$$(j : C(\widetilde{M}) \to C(\widetilde{W}, \widetilde{M'}), (\delta\phi', \phi)) \ .$$

Proof. The manifold cobordism determines a cobordism of n-dimensional symmetric Poincaré complexes over $\mathbb{Z}[\pi]$

$$\Gamma \ = \ (C(\widetilde{M}) \oplus C(\widetilde{M'}) \to C(\widetilde{W}), (\delta\phi, \phi \oplus -\phi')) \ .$$

Now apply the Theorem to Γ, with $\delta\phi' = \delta\phi/\phi'$. □

Example. If $(W; M, M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ then

$$C(\widetilde{W}, \widetilde{M'}) \ \simeq \ S^{n-i}\mathbb{Z}[\pi]$$

is concentrated in dimension $(n-i)$, and the effect is to kill the spherical (co)homology class

$$j \ = \ [S^i] \in H^{n-i}(\widetilde{M}) \ \cong \ H_i(\widetilde{M}) \ .$$

The embedding $S^i \subset M$ determines $j : C(\widetilde{M}) \to S^{n-i}\mathbb{Z}[\pi]$, and the choice of extension to an embedding $S^i \times D^{n-i} \subset M$ determines $\delta\phi'$. □

The cobordism groups $L^n(A)$ $(n \geqslant 0)$ start with the symmetric Witt group $L^0(A)$. The symmetric signature map from manifold bordism to symmetric Poincaré bordism

$$\sigma^* \ : \ \Omega_n(X) \to L^n(\mathbb{Z}[\pi_1(X)]) \ ; \ M \mapsto (C(\widetilde{M}), \phi)$$

is a generalization of the signature map

$$\sigma^* \ : \ \Omega_{4k} \to L^{4k}(\mathbb{Z}) \ = \ \mathbb{Z} \ ; \ M \mapsto \text{signature}(H^{2k}(M), \text{intersection form}) \ .$$

Although the symmetric signature maps σ^* are not isomorphisms in general, they do provide many invariants. The symmetric and quadratic L-groups only differ in 8-torsion :

(i) the symmetrization maps

$$1 + T \ : \ L_n(A) \to L^n(A) \ ; \ (C, \psi) \mapsto (C, (1+T)\psi)$$

are isomorphisms modulo 8-torsion,

(ii) if $1/2 \in A$ the symmetrization maps are isomorphisms.

4 Surgery on quadratic Poincaré complexes

Quadratic Poincaré complexes are chain complexes with the Poincaré duality properties of kernels of normal maps. The quadratic Poincaré analogues of cobordism and surgery are defined by analogy with the symmetric case. Although there are many similarities between the quadratic and symmetric theories, there is one essential difference : the quadratic Poincaré cobordism groups are the Wall surgery obstruction groups $L_*(A)$, so for $A = \mathbb{Z}[\pi]$ every element is geometrically significant.

An *n-dimensional quadratic complex* (C, ψ) over A is a f.g. free A-module chain complex C together with a collection of A-module morphisms

$$\psi \;=\; \{\psi_s : C^{n-r-s} \to C_r \,|\, s \geqslant 0\}$$

such that

$$d_C\psi_s + (-1)^r\psi_s d_C^* + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1}T\psi_{s+1}) = 0 : C^{n-r-s-1} \to C_r \qquad (s \geqslant 0) \;.$$

More intrinsically, ψ is an n-dimensional cycle in the $\mathbb{Z}$-module chain complex

$$W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C)$$

with W the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ (as above). The quadratic complex (C, ψ) is *Poincaré* if the chain map $(1+T)\psi_0 : C^{n-*} \to C$ is a chain equivalence. A quadratic complex (C, ψ) determines the symmetric complex (C, ϕ) with $\phi_0 = (1+T)\psi_0$, $\phi_s = 0$ $(s \geqslant 1)$.

Example. ([5]) An n-dimensional normal map $(f, b) : M \to X$ and a regular covering $\widetilde{X}$ of X with group of covering translations π determine a *kernel* n-dimensional quadratic Poincaré complex (C, ψ) over $\mathbb{Z}[\pi]$ with $C = \mathcal{C}(f^!)$ the algebraic mapping cone of the Umkehr chain map

$$f^! \;:\; C(\widetilde{X}) \;\simeq\; C(\widetilde{X})^{n-*} \xrightarrow{f^*} C(\widetilde{M})^{n-*} \;\simeq\; C(\widetilde{M})$$

and $(1+T)\psi_0 = [M] \cap - : C^{n-*} \to C$ the Poincaré duality chain equivalence. It follows from $f_*[M] = [X] \in H_m(X)$ (f is degree 1) that there exists a chain homotopy $ff^! \simeq 1 : C(\widetilde{X}) \to C(\widetilde{X})$. The homology $\mathbb{Z}[\pi]$-modules of C are thus the kernels of f

$$H_*(C) \;=\; K_*(M) \;=\; \ker(f_* : H_*(\widetilde{M}) \to H_*(\widetilde{X})) \;,$$

such that

$$H_*(\widetilde{M}) \;=\; K_*(M) \oplus H_*(\widetilde{X}) \;. \qquad \square$$

An $(n+1)$-*dimensional quadratic pair* $(j : C \to D, (\delta\psi, \psi))$ is an n-dimensional quadratic complex (C, ψ) together with a chain map $j : C \to D$ to an $(n+1)$-dimensional f.g. free A-module chain complex D and A-module morphisms

$$\delta\psi \;=\; \{\delta\psi_s : D^{n+1-r-s} \to D_r \,|\, s \geqslant 0\}$$

such that

$$\begin{aligned} j\psi_s j^* \;&=\; d_D\delta\psi_s + (-)^r\delta\psi_s d_D^* + (-)^{n+s+1} \\ &\quad (\delta\psi_{s+1} + (-)^s T\delta\psi_{s+1}) \;:\; D^{n+1-r-s} \to D_r \quad (s \geqslant 0)\ . \end{aligned}$$

The pair is *Poincaré* if the chain map

$$\begin{pmatrix} (1+T)\delta\psi_0 \\ (1+T)\psi_0 j^* \end{pmatrix} \;:\; D^{n+1-*} \to \mathcal{C}(j)$$

is a chain equivalence, in which case (C, ψ) is a n-dimensional quadratic Poincaré complex.

A *cobordism* of n-dimensional quadratic Poincaré complexes (C, ψ), (C', ψ') is an $(n+1)$-dimensional quadratic Poincaré pair of the type $(C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$. Quadratic complexes (C, ψ), (C', ψ') are *homotopy equivalent* if there exists a cobordism with $C \to D$, $C' \to D$ chain equivalences.

Example. An $(n+1)$-dimensional normal map of pairs $(g, c) : (W, \partial W) \to (Y, \partial Y)$ determines a kernel $(n+1)$-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi]$ $(j : C(\partial g^!) \to C(g^!), (\delta\psi, \psi))$ with

$$\begin{pmatrix} (1+T)\delta\psi_0 j^* \\ (1+T)\psi_0 \end{pmatrix} \;=\; [W] \cap - \;:\; C(g^!)^{n+1-*} \to \mathcal{C}(j)$$

the Poincaré-Lefschetz duality chain equivalence. □

The *data* for *algebraic surgery* on an n-dimensional quadratic Poincaré complex (C, ψ) is an $(n+1)$-dimensional quadratic pair $(j : C \to D, (\delta\psi, \psi))$. The *effect* of the algebraic surgery is the n-dimensional quadratic Poincaré complex (C',ψ') with

$$\begin{aligned}
d_{C'} \;&=\; \begin{pmatrix} d_C & 0 & (-)^{n+1}(1+T)\psi_0 j^* \\ (-)^r j & d_D & (-)^r(1+T)\delta\psi_0 \\ 0 & 0 & d_D^* \end{pmatrix} \;: \\
&\qquad C'_r \;=\; C_r \oplus D_{r+1} \oplus D^{n-r+1} \to C'_{r-1} \;=\; C_{r-1} \oplus D_r \oplus D^{n-r+2}\ , \\
\psi'_0 \;&=\; \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;: \\
&\qquad C'^{n-r} \;=\; C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \to C'_r \;=\; C_r \oplus D_{r+1} \oplus D^{n-r+1}\ , \\
\psi'_s \;&=\; \begin{pmatrix} \psi_s & (-)^{r+s}T\psi_{s-1}j^* & 0 \\ 0 & (-)^{n-r-s+1}T\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \;: \\
&\qquad C'^{n-r-s} \;=\; C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \to C'_r \\
&\qquad =\; C_r \oplus D_{r+1} \oplus D^{n-r+1} \quad (s \geqslant 1)\ .
\end{aligned}$$

The trace of the algebraic surgery is an $(n+1)$-dimensional quadratic Poincaré cobordism $((f\ f') : C \oplus C' \to D', (\delta\psi', \psi \oplus -\psi'))$ defined by analogy with the symmetric case. As in the symmetric case :

Theorem. (Ranicki [4]) *The cobordism of quadratic Poincaré complexes is the equivalence relation generated by homotopy equivalence and algebraic surgery.* □

Quadratic Surgery Principle. For a bordism of n-dimensional normal maps

$$((g,c);(f,b),(f',b'))\ :\ (W;M,M') \to X \times ([0,1];\{0\},\{1\})$$

the quadratic Poincaré complex $(C(f'^{!}),\psi')$ is homotopy equivalent to the effect of algebraic surgery on $(C(f^{!}),\psi)$ with data $(C(f^{!}) \to C(g^{!},f'^{!}),(\delta\psi,\psi))$. □

Example. If $((g,c);(f,b),(f',b'))$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ then

$$C(g^{!},f'^{!})\ \simeq\ S^{n-i}\mathbb{Z}[\pi]$$

is concentrated in dimension $(n-i)$. □

A n-dimensional quadratic Poincaré complex (C,ψ) is *highly-connected* if it is homotopy equivalent to a complex (also denoted (C,ψ)) with

$$\begin{aligned}
C\ &:\ \dots \to 0 \to C_i \to 0 \to \dots && \text{if } n = 2i \\
C\ &:\ \dots \to 0 \to C_{i+1} \to C_i \to 0 \to \dots && \text{if } n = 2i+1.
\end{aligned}$$

Example. (i) The quadratic kernel (C,ψ) of an n-dimensional normal map $(f,b) : M \to X$ is highly-connected if and only if $f : M \to X$ is i-connected, that is $\pi_r(f) = 0$ for $r \leqslant i$.
(ii) The quadratic Poincaré kernel (C,ψ) of an i-connected $2i$-dimensional normal map $(f,b) : M \to X$ is essentially the same as the geometric $(-)^i$-quadratic intersection form $(K_i(M),\lambda,\mu)$ of Wall [11], with

$$\begin{aligned} \lambda \ &= \ (1+T)\psi_0 \ : \ H^i(C) \ = \ K^i(M) \to H_i(C) \ = \ K_i(M) \ \cong \ H^i(C)^* \ , \\ \mu(x) \ &= \ \psi_0(x)(x) \in Q_{(-)^i}(\mathbb{Z}[\pi_1(X)]) \ . \end{aligned}$$

For $i \geqslant 3$ an element $x \in K_i(M)$ can be killed by a geometric surgery if and only if $\mu(x) = 0$, if and only if there exists algebraic surgery data $(x : C \to S^i\mathbb{Z}[\pi_1(X)], (\delta\psi,\psi))$. The effect of the surgery is a normal map $(f',b') : M' \to X$ with quadratic Poincaré kernel (C',ψ') such that

$$C' \ : \ \cdots \to 0 \to \mathbb{Z}[\pi_1(X)] \xrightarrow{x} K_i(M) \xrightarrow{x^*\lambda} \mathbb{Z}[\pi_1(X)] \to 0 \to \dots \ . \qquad \square$$

Theorem. (Ranicki [4])
(i) *Every n-dimensional quadratic Poincaré complex (C,ψ) is cobordant to a highly-connected complex.*
(ii) *The cobordism group of n-dimensional quadratic complexes over A is isomorphic to $L_n(A)$, with the 4-periodicity isomorphisms given by*

$$L_n(A) \to L_{n+4}(A) \ ; \ (C,\psi) \mapsto (C_{*-2},\psi) \ .$$

Proof: (i) Let $n = 2i$ or $2i+1$. Let D be the quotient complex of C with $D_r = C_r$ for $r > n-i$, and let $j : C \to D$ be the projection. The effect of algebraic surgery on (C,ψ) with data $(j : C \to D, (0,\psi))$ is homotopy equivalent to a highly-connected complex (C',ψ').
(ii) $(n = 2i)$ A highly-connected $2i$-dimensional quadratic Poincaré complex (C,ψ) is essentially the same as a nonsingular $(-)^i$-quadratic form (C^0,ψ_0). The relative version of (i) shows that a null-cobordism of (C,ψ) is essentially the same as an isomorphism of forms

$$(C^0,\psi_0) \oplus \text{hyperbolic} \ \cong \ \text{hyperbolic} \ ,$$

which is precisely the condition for $(C^0,\psi_0) = 0 \in L_{2i}(A)$.
(ii) $(n = 2i+1)$ A highly-connected $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) is essentially the same as a nonsingular $(-)^i$-quadratic formation. See [4] for further details.

$\square$

Instant surgery obstruction for $n = 2i$. A $2i$-dimensional quadratic Poincaré complex (C, ψ) is cobordant to the highly-connected complex (C', ψ') with

$$(C'^i, \psi'_0) = \left(\mathrm{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T)\psi_0 & d \end{pmatrix} : C^{i-1} \oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{pmatrix} \psi_0 & d \\ 0 & 0 \end{pmatrix}\right).$$

Thus if (C, ψ) is the quadratic Poincaré kernel of a $2i$-dimensional normal map $(f, b) : M \to X$ then (C'^i, ψ'_0) is a nonsingular $(-)^i$-quadratic form representing the surgery obstruction $\sigma_*(f, b) \in L_{2i}(\mathbb{Z}[\pi_1(X)])$ (without preliminary geometric surgeries below the middle dimension). $\square$

See §I.4 of [4] for the instant surgery obstruction formation in the case $n = 2i + 1$.

5 The localization exact sequence

For any morphism of rings with involution $f : A \to B$ there is defined an exact sequence of L-groups

$$\cdots \to L_n(A) \xrightarrow{f} L_n(B) \to L_n(f) \to L_{n-1}(A) \to \ldots$$

with the relative L-group $L_n(f)$ the cobordism groups of pairs

$$\begin{aligned}&((n-1)\text{-dimensional quadratic Poincaré complex } (C, \psi) \text{ over } A, \\ &\ n\text{-dimensional quadratic Poincaré pair } (B \otimes_A C \to D, (\delta\psi, 1 \otimes \psi)) \text{ over } B)\ .\end{aligned}$$

Algebraic surgery provides a particularly useful expression for the relative L-groups $L_*(A \to S^{-1}A)$ of the localization map $A \to S^{-1}A$ inverting a multiplicatively closed subset $S \subset A$ of central non-zero divisors.

Localization exact sequence. (Ranicki [6]) The relative L-group $L_n(A \to S^{-1}A)$ is isomorphic to the cobordism group $L_n(A, S)$ of $(n-1)$-dimensional quadratic Poincaré complexes over A which are $S^{-1}A$-acyclic.

Proof Clearing denominators it is possible to lift every quadratic Poincaré pair over $S^{-1}A$ as above to an n-dimensional quadratic pair $(C \to D', (\delta\psi', \psi))$ over A. This is the data for algebraic surgery on (C, ψ) with effect a cobordant $(n-1)$-dimensional quadratic Poincaré complex (C', ψ') over A which is $S^{-1}A$-acyclic (i.e. $H_*(S^{-1}A \otimes_A C') = 0$). $\square$

The localization exact sequence

$$\cdots \to L_n(A) \to L_n(S^{-1}A) \to L_n(A, S) \to L_{n-1}(A) \to \ldots$$

and its extensions to noncommutative localization and to symmetric L-theory have many applications to the computation of L-groups, as well as to surgery on submanifolds (cf. Ranicki [7]).

Example. The localization of $A = \mathbb{Z}$ inverting $S = \mathbb{Z}\backslash\{0\} \subset A$ is $S^{-1}A = \mathbb{Q}$. See Chapter 4 of [6] for detailed accounts of the way in which the classification of quadratic forms over $\mathbb{Q}$ is combined with the localization exact sequence

$$\begin{aligned}
&\cdots \to L^n(\mathbb{Z}) \to L^n(\mathbb{Q}) \to L^n(\mathbb{Z}, S) \to L^{n-1}(\mathbb{Z}) \to \dots \ , \\
&\cdots \to L_n(\mathbb{Z}) \to L_n(\mathbb{Q}) \to L_n(\mathbb{Z}, S) \to L_{n-1}(\mathbb{Z}) \to \dots
\end{aligned}$$

to give

$$\begin{aligned}
L^n(\mathbb{Z}) \ &= \ \begin{cases} \mathbb{Z} & \text{(signature)} \\ \mathbb{Z}_2 & \text{(deRham invariant)} \\ 0 & \\ 0 & \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 \ , \\
L_n(\mathbb{Z}) \ &= \ \begin{cases} \mathbb{Z} & \text{(signature)/8} \\ 0 & \\ \mathbb{Z}_2 & \text{(Arf invariant)} \\ 0 & \end{cases} \qquad\quad \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 \ .
\end{aligned}$$

□

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The Structure Set of an Arbitrary Space, the Algebraic Surgery Exact Sequence and the Total Surgery Obstruction

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Abstract

The algebraic theory of surgery gives a necessary and sufficient chain level condition for a space with n-dimensional Poincaré duality to be homotopy equivalent to an n-dimensional topological manifold. A relative version gives a necessary and sufficient chain level condition for a simple homotopy equivalence of n-dimensional topological manifolds to be homotopic to a homeomorphism. The chain level obstructions come from a chain level interpretation of the fibre of the assembly map in surgery.

The assembly map $A : H_n(X;\mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi_1(X)])$ is a natural transformation from the generalized homology groups of a space X with coefficients in the 1-connective simply-connected surgery spectrum $\mathbb{L}_\bullet$ to the non-simply-connected surgery obstruction groups $L_*(\mathbb{Z}[\pi_1(X)])$. The $(\mathbb{Z}, X)$-category has objects based f.g. free $\mathbb{Z}$-modules with an X-local structure. The assembly maps A are induced by a functor from the $(\mathbb{Z}, X)$-category to the category of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules. The generalized homology groups $H_*(X;\mathbb{L}_\bullet)$ are the cobordism groups of quadratic Poincaré complexes over $(\mathbb{Z}, X)$. The relative groups $\mathbb{S}_*(X)$ in the algebraic surgery exact sequence of X

$$\cdots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

are the cobordism groups of quadratic Poincaré complexes over $(\mathbb{Z}, X)$ which assemble to contractible quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X)]$.

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional simple Poincaré complex X is the cobordism class of a quadratic Poincaré complex over $(\mathbb{Z}, X)$ with contractible assembly over $\mathbb{Z}[\pi_1(X)]$, which measures the homotopy invariant part of the failure of the link of each simplex in X to be a homology sphere. The total surgery obstruction is $s(X) = 0$ if (and for $n \geqslant 5$ only if) X is simple homotopy equivalent to an n-dimensional topological manifold.

The Browder-Novikov-Sullivan-Wall surgery exact sequence for an n-dimensional topological manifold M with $n \geqslant 5$

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)])$$

is identified with the corresponding portion of the algebraic surgery exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}_{n+1}(M) \to H_n(M;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)]) \ .$$

The structure invariant $s(h) \in \mathbb{S}^{TOP}(M) = \mathbb{S}_{n+1}(M)$ of a simple homotopy equivalence of n-dimensional topological manifolds $h : N \to M$ is the cobordism class of an n-dimensional quadratic Poincaré complex in $(\mathbb{Z}, M)$ with contractible assembly over $\mathbb{Z}[\pi_1(M)]$, which measures the homotopy invariant part of the failure of the point inverses $h^{-1}(x)$ $(x \in M)$ to be acyclic. The structure invariant is $s(h) = 0$ if (and for $n \geqslant 5$ only if) h is homotopic to a homeomorphism.

Contents

1 Introduction

The structure set of a differentiable n-dimensional manifold M is the set $\mathbb{S}^O(M)$ of equivalence classes of pairs (N, h) with N a differentiable manifold and $h : N \to M$ a simple homotopy equivalence, subject to $(N, h) \sim (N', h')$ if there exist a diffeomorphism $f : N \to N'$ and a homotopy $f \simeq h'f : N \to M$. The differentiable structure set was first computed for $N = S^n$ $(n \geqslant 5)$, with $\mathbb{S}^O(S^n) = \Theta^n$ the Kervaire-Milnor group of exotic spheres. In this case the structure set is an abelian group, since the connected sum of homotopy equivalences $h_1 : N_1 \to S^n$, $h_2 : N_2 \to S^n$ is a homotopy equivalence

$$h_1 \# h_2 \ : \ N_1 \# N_2 \to S^n \# S^n \ = \ S^n \ .$$

The Browder-Novikov-Sullivan-Wall theory for the classification of manifold structures within the simple homotopy type of an n-dimensional differentiable manifold M with $n \geqslant 5$ fits $\mathbb{S}^O(M)$ into an exact sequence of pointed sets

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^O(M) \to [M, G/O] \to L_n(\mathbb{Z}[\pi_1(M)])$$

corresponding to the two stages of the obstruction theory for deciding if a simple homotopy equivalence $h : N \to M$ is homotopic to a diffeomorphism:

(i) The primary obstruction in $[M, G/O]$ to the extension of h to a normal bordism $(f, b) : (W; M, N) \to M \times ([0,1]; \{0\}, \{1\})$ with $f| = 1 : M \to M$. Here G/O is the classifying space for fibre homotopy trivialized vector bundles, and $[M, G/O]$ is identified with the bordism of normal maps $M' \to M$ by the Browder-Novikov transversality construction.

(ii) The secondary obstruction $\sigma_*(f, b) \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$ to performing surgery on (f, b) to make (f, b) a simple homotopy equivalence, which depends on the choice of solution in (i). Here, it is necessary to use the version of the L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ in which modules are based and isomorphisms are simple, in order to take advantage of the s-cobordism theorem.

The Whitney sum of vector bundles makes G/O an H-space (in fact an infinite loop space), so that $[M, G/O]$ is an abelian group. However, the surgery obstruction function $[M, G/O] \to L_n(\mathbb{Z}[\pi_1(M)])$ is not a morphism of groups, and in general the differentiable structure set $\mathbb{S}^O(M)$ does not have a group structure (or at least is not known to have), abelian or otherwise.

The 1960's development of surgery theory culminated in the work of Kirby and Siebenmann [4] on high-dimensional topological manifolds, which revealed both a striking similarity and a striking difference between the differentiable and topological catgeories. Define the structure set of a topological n-dimensional manifold M exactly as before, to be the set $\mathbb{S}^{TOP}(M)$ of equivalence classes of pairs (N,h) with N a topological manifold and $h : N \to M$ a simple homotopy equivalence, subject to $(N,h) \sim (N',h')$ if there exist a homeomorphism $f : N \to N'$. The similarity is that again there is a surgery exact sequence for $n \geqslant 5$

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)]) \quad (*)$$

corresponding to a two-stage obstruction theory for deciding if a simple homotopy equivalence is homotopic to a homeomorphism, with G/TOP the classifying space for fibre homotopy trivialized topological block bundles. The difference is that the topological structure set $\mathbb{S}^{TOP}(M)$ has an abelian group structure and G/TOP has an infinite loop space structure with respect to which $(*)$ is an exact sequence of abelian groups. Another difference is given by the computation $\mathbb{S}^{TOP}(S^n) = 0$, which is just a restatement of the generalized Poincaré conjecture in the topological category : for $n \geqslant 5$ every homotopy equivalence $h : M^n \to S^n$ is homotopic to a homeomorphism.

Originally, the abelian group structure on $(*)$ was suggested by the characteristic variety theorem of Sullivan [15] on the homotopy type of G/TOP, including the computation $\pi_*(G/TOP) = L_*(\mathbb{Z})$. Next, Quinn [6] proposed that the surgery obstruction function should be factored as the composite

$$[M, G/TOP] \;=\; H^0(M; \underline{G/TOP}) \;\cong\; H_n(M; \underline{G/TOP}) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])$$

with $\underline{G/TOP}$ the simply-connected surgery spectrum with 0th space G/TOP, identifying the topological structure sequence with the homotopy exact sequence of a geometrically defined spectrum-level assembly map. This was all done in Ranicki [8], [9], but with algebra instead of geometry.

The algebraic theory of surgery was used in [9] to define the algebraic surgery exact sequence of abelian groups for any space X

$$\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \ldots \;. \quad (**)$$

The expression of the L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ as the cobordism groups of quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X)]$ (recalled in the notes on the foundations of algebraic surgery) was extended to $H_*(X; \mathbb{L}_\bullet)$ and $\mathbb{S}_*(X)$, using quadratic Poincaré complexes in categories containing much more of the topology of X than just the fundamental group $\pi_1(X)$. The topological surgery exact sequence of an n-dimensional manifold M with $n \geqslant 5$ was shown to be in bijective correspondence with the corresponding portion of the algebraic surgery sequence, including an explicit bijection

$$s \ : \ \mathbb{S}^{TOP}(M) \to \mathbb{S}_{n+1}(M) \ ; \ h \mapsto s(h) \ .$$

The structure invariant $s(h) \in \mathbb{S}_{n+1}(M)$ of a simple homotopy equivalance $h : N \to M$ measures the chain level cobordism failure of the point inverses $h^{-1}(x) \subset N$ $(x \in M)$ to be points.

The Browder-Novikov-Sulivan-Wall surgery theory deals both with the existence and uniqueness of manifolds in the simple homotopy type of a geometric simple n-dimensional Poincaré complex X with $n \geq 5$. Again, this was first done for differentiable manifolds, and then extended to topological manifolds, with a two-stage obstruction :

(i) The primary obstruction in $[X, B(G/TOP)]$ to the existence of a normal map $(f, b) : M \to X$, with $b : \nu_M \to \widetilde{\nu}_X$ a bundle map from the stable normal bundle ν_M of M to a topological reduction $\widetilde{\nu}_X : X \to BTOP$ of the Spivak normal fibration $\nu_X : X \to BG$.

(ii) The secondary obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)])$ to performing surgery to make (f, b) a simple homotopy equivalence, which depends on the choice of solution in (i).

For $n \geqslant 5$ X is simple homotopy equivalent to a topological manifold if and only if there exists a topological reduction $\widetilde{\nu}_X : X \to BTOP$ for which the corresponding normal map $(f, b) : M \to X$ has surgery obstruction $\sigma_*(f, b) = 0$. The algebraic surgery exact sequence $(**)$ unites the two stages into a single invariant, the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$, which measures the chain level cobordism failure of the points $x \in X$ to have Euclidean neighbourhoods. For $\pi_1(X) = \{1\}$, $n = 4k$ the condition $s(X) = 0 \in \mathbb{S}_{4k}(X)$ is precisely the Browder condition that there exist a topological reduction $\widetilde{\nu}_X$ for which the signature of X is given by the Hirzebruch formula

$$\text{signature}(X) \ = \ \langle \mathcal{L}(-\widetilde{\nu}_X), [X] \rangle \in \mathbb{Z} \ .$$

2 Geometric Poincaré assembly

This section describes the assembly for geometric Poincaré bordism, setting the scene for the use of quadratic Poincaré bordism in the assembly map in algebraic L-theory. In both cases assembly is the passage from objects with local Poincaré duality to objects with global Poincaré duality.

Given a space X let $\Omega^P_n(X)$ be the bordism group of maps $f : Q \to X$ from n-dimensional geometric Poincaré complexes Q. The functor $X \mapsto \Omega^P_*(X)$ is homotopy invariant. If $X = X_1 \cup_Y X_2$ it is not in general possible to make $f : Q \to X$ Poincaré transverse at $Y \subset X$, i.e. $f^{-1}(Y) \subset Q$ will not be an $(n-1)$-dimensional geometric Poincaré complex. Thus $X \mapsto \Omega^P_*(X)$ does not have Mayer-Vietoris sequences, and is not a generalized homology theory. The general theory of Weiss and Williams [19] provides a generalized homology theory $X \mapsto H_*(X;\Omega^P_\bullet)$ with an assembly map $A : H_*(X;\Omega^P_\bullet) \to \Omega^P_*(X)$. However, it is possible to obtain A by a direct geometric construction : $H_n(X;\Omega^P_\bullet)$ is the bordism group of Poincaré transverse maps $f : Q \to X$ from n-dimensional Poincaré complexes Q, and A forgets the transversality. The coefficient spectrum $\Omega^P_\bullet$ is such that

$$\pi_*(\Omega^P_\bullet) \;=\; \Omega^P_*(\{\text{pt.}\}) \;,$$

and may be constructed using geometric Poincaré n-ads.

In order to give a precise geometric description of $H_n(X;\Omega^P_\bullet)$ it is convenient to assume that X is the polyhedron of a finite simplicial complex (also denoted X). The *dual cell* of a simplex $\sigma \in X$ is the subcomplex of the barycentric subdivision X'

$$D(\sigma,X) \;=\; \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \dots < \sigma_n\} \subset X' \;,$$

with boundary the subcomplex

$$\partial D(\sigma,X) \;=\; \bigcup_{\tau>\sigma} D(\tau,X) \;=\; \{\widehat{\tau}_0\widehat{\tau}_1\dots\widehat{\tau}_n \,|\, \sigma < \tau_0 < \tau_1 < \dots < \tau_n\} \subset D(\sigma,X) \;.$$

Every map $f : M \to X$ from an n-manifold M can be made transverse across the dual cells, meaning that for each $\sigma \in X$

$$(M(\sigma),\partial M(\sigma)) \;=\; f^{-1}(D(\sigma,X),\partial D(\sigma,X))$$

is an $(n-|\sigma|)$-dimensional manifold with boundary. Better still, for an n-dimensional PL manifold M every simplicial map $f : M \to X'$ is already transverse in this sense, by a result of Marshall Cohen.

A map $f : Q \to X$ is *n-dimensional Poincaré transverse* if for each $\sigma \in X$

$$(Q(\sigma),\partial Q(\sigma)) \;=\; f^{-1}(D(\sigma,X),\partial D(\sigma,X))$$

is an $(n-|\sigma|)$-dimensional geometric Poincaré pair.

Proposition. $H_n(X;\Omega^P_\bullet)$ is the bordism group of Poincaré transverse maps $Q \to X$ from n-dimensional geometric Poincaré complexes. $\square$

It is worth noting that

(i) The identity $1 : X \to X$ is n-dimensional Poincaré transverse if and only if X is an n-dimensional homology manifold.

(ii) If a map $f : Q \to X$ is n-dimensional Poincaré transverse then Q is an n-dimensional geometric Poincaré complex. The global Poincaré duality of Q is assembled from the local Poincaré dualities of $(Q(\sigma), \partial Q(\sigma))$. For $f = 1 : Q = X \to X$ this is the essence of Poincaré's original proof of his duality for a homology manifold.

The *Poincaré structure group* $\mathbb{S}^P_n(X)$ is the relative group in the *geometric Poincaré surgery exact sequence*

$$\cdots \to H_n(X;\Omega^P_\bullet) \xrightarrow{A} \Omega^P_n(X) \to \mathbb{S}^P_n(X) \to H_{n-1}(X;\Omega^P_\bullet) \to \dots ,$$

which is the cobordism group of maps $(f, \partial f) : (Q, \partial Q) \to X$ from n-dimensional Poincaré pairs $(Q, \partial Q)$ with $\partial f : \partial Q \to X$ Poincaré transverse. The *total Poincaré surgery obstruction* of an n-dimensional geometric Poincaré complex X is the image $s^P(X) \in \mathbb{S}^P_n(X)$ of $(1 : X \to X) \in \Omega^P_n(X)$, with $s^P(X) = 0$ if and only if there exists an $\Omega^P_\bullet$-coefficient fundamental class $[X]_P \in H_n(X;\Omega^P_\bullet)$ with $A([X]_P) = (1 : X \to X) \in \Omega^P_n(X)$.

In fact, it follows from the Levitt-Jones-Quinn-Hausmann-Vogel Poincaré bordism theory that $\mathbb{S}^P_n(X) = \mathbb{S}_n(X)$ for $n \geqslant 5$, and that $s^P(X) = 0$ if and only if X is homotopy equivalent to an n-dimensional topological manifold. The geometric Poincaré bordism approach to the structure sets and total surgery obstruction is intuitive, and has the virtue(?) of dispensing with the algebra altogether. Maybe it even applies in the low dimensions $n = 3, 4$. However, at present our understanding of the Poincaré bordism theory is not good enough to use it for foundational purposes. So back to the algebra!

3 The algebraic surgery exact sequence

This section constructs the quadratic L-theory assembly map A and the algebraic surgery exact sequence

$$\cdots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots \quad (**)$$

for a finite simplicial complex X. A '$(\mathbb{Z}, X)$-module' is a based f.g. free $\mathbb{Z}$-module in which every basis element is associated to a simplex of X. The construction of $(**)$ makes use of a chain complex duality on the $(\mathbb{Z}, X)$-module category $\mathbb{A}(\mathbb{Z}, X)$.

The quadratic L-spectrum $\mathbb{L}_\bullet$ is 1-connective, with connected 0th space $\mathbb{L}_0 \simeq G/TOP$ and

$$\pi_n(\mathbb{L}_\bullet) \ = \ \pi_n(\mathbb{L}_0) \ = \ L_n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\bmod 4) \text{ (signature)}/8 \\ \mathbb{Z}_2 & \text{if } n \equiv 2 (\bmod 4) \text{ (Arf invariant)} \\ 0 & \text{otherwise} \end{cases}$$

for $n \geqslant 1$. From the algebraic point of view it is easier to start with the 0-connective quadratic L-spectrum $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})$, such that

$$\pi_n(\overline{\mathbb{L}}_\bullet) \ = \ \begin{cases} L_n(\mathbb{Z}) & \text{if } n \geqslant 0 \\ 0 & \text{if } n \leqslant -1 \end{cases}$$

with disconnected 0th space $\overline{\mathbb{L}}_0 \simeq L_0(\mathbb{Z}) \times G/TOP$. The two spectra are related by a fibration sequence $\mathbb{L}_\bullet \to \overline{\mathbb{L}}_\bullet \to \mathbb{K}(L_0(\mathbb{Z}))$ with $\mathbb{K}(L_0(\mathbb{Z}))$ the Eilenberg-MacLane spectrum of $L_0(\mathbb{Z})$.

The algebraic surgery exact sequence was constructed in Ranicki [9] using the $(\mathbb{Z}, X)$-module category of Ranicki and Weiss [12]. (This is a rudimentary version of controlled topology, cf. Ranicki [11]).

A $(\mathbb{Z}, X)$-*module* is a direct sum of based f.g. free $\mathbb{Z}$-modules

$$B \ = \ \sum_{\sigma \in X} B(\sigma) \ .$$

A $(\mathbb{Z}, X)$-module morphism $f : B \to C$ is a $\mathbb{Z}$-module morphism such that

$$f(B(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} C(\tau) \ ,$$

so that the matrix of f is upper triangular. A $(\mathbb{Z}, X)$-module chain map $f : B \to C$ is a chain equivalence if and only if each $f(\sigma, \sigma) : B(\sigma) \to C(\sigma)$ $(\sigma \in X)$ is a $\mathbb{Z}$-module chain equivalence. The universal covering projection $p : \widetilde{X} \to X$ is used to define the $(\mathbb{Z}, X)$-module assembly functor

$$A \ : \ \mathbb{A}(\mathbb{Z}, X) \to \mathbb{A}(\mathbb{Z}[\pi_1(X)]) \ ; \ B \mapsto \sum_{\widetilde{\sigma} \in \widetilde{X}} B(p\widetilde{\sigma})$$

with $\mathbb{A}(\mathbb{Z},X)$ the category of $(\mathbb{Z},X)$-modules and $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$ the category of based f.g. free $\mathbb{Z}[\pi_1(X)]$-modules. In the language of sheaf theory $A = q_! p^!$ (cf. Verdier [16]), with $q : \widetilde{X} \to \{\text{pt.}\}$.

The involution $g \mapsto \overline{g} = g^{-1}$ on $\mathbb{Z}[\pi_1(X)]$ extends in the usual way to a duality involution on $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$, sending a based f.g. free $\mathbb{Z}[\pi_1(X)]$-module F to the dual f.g. free $\mathbb{Z}[\pi_1(X)]$-module $F^* = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(F, \mathbb{Z}[\pi_1(X)])$. Unfortunately, it is not possible to define a duality involution on $\mathbb{A}(\mathbb{Z},X)$ (since the transpose of an upper triangular matrix is a lower triangular matrix). See Chapter 5 of Ranicki [9] for the construction of a 'chain duality' on $\mathbb{A}(\mathbb{Z},X)$ and of the L-groups $L_*(\mathbb{A}(\mathbb{Z},X))$. The chain duality associates to a chain complex C in $\mathbb{A}(\mathbb{Z},X)$ a chain complex TC in $\mathbb{A}(\mathbb{Z},X)$ with

$$TC(\sigma)_r \;=\; \sum_{\tau \geqslant \sigma} \mathrm{Hom}_{\mathbb{Z}}(C_{-|\sigma|-r}(\tau), \mathbb{Z}) \;.$$

Example. The simplicial chain complex $C(X')$ is a $(\mathbb{Z},X)$-module chain complex, with assembly $A(C(X'))$ $\mathbb{Z}[\pi_1(X)]$-module chain equivalent to $C(\widetilde{X})$. The chain dual $TC(X')$ is $(\mathbb{Z},X)$-module chain equivalent to the simplicial cochain complex $D = \mathrm{Hom}_{\mathbb{Z}}(C(X),\mathbb{Z})^{-*}$, with assembly $A(D)$ which is $\mathbb{Z}[\pi_1(X)]$-module chain equivalent to $C(\widetilde{X})^{-*}$. □

The quadratic L-group $L_n(\mathbb{A}(\mathbb{Z},X))$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C,ψ) in $\mathbb{A}(\mathbb{Z},X)$.

Proposition. ([9], 14.5) The functor $X \mapsto L_*(\mathbb{A}(\mathbb{Z},X))$ is the generalized homology theory with $\mathbb{L}_\bullet(\mathbb{Z})$-coefficients

$$L_*(A(\mathbb{Z},X)) \;=\; H_*(X;\mathbb{L}_\bullet(\mathbb{Z})) \;. \qquad \square$$

The coefficient spectrum $\overline{\mathbb{L}}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})$ is the special case $R = \mathbb{Z}$ of a general construction. For any ring with involution R there is a 0-connective spectrum $\mathbb{L}_\bullet(R)$ such that

$$\pi_*(\overline{\mathbb{L}}_\bullet(R)) \;=\; L_*(R) \;,$$

which may be constructed using quadratic Poincaré n-ads over R.

The assembly functor $A : \mathbb{A}(\mathbb{Z},X) \to \mathbb{A}(\mathbb{Z}[\pi_1(X)])$ induces assembly maps in the quadratic L-groups, which fit into the *4-periodic algebraic surgery exact sequence*

$$\dots \to H_n(X;\overline{\mathbb{L}}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \overline{\mathbb{S}}_n(X) \to H_{n-1}(X;\overline{\mathbb{L}}_\bullet) \to \dots$$

with the *4-periodic algebraic structure set* $\overline{\mathbb{S}}_n(X)$ the cobordism group of $(n-1)$-dimensional quadratic Poincaré complexes (C,ψ) in $\mathbb{A}(\mathbb{Z},X)$ such that the assembly $A(C)$ is a simple contractible based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex. (See section 4.5 for the geometric interpretation). A priori, an element of the relative group $\overline{\mathbb{S}}_n(X) = \pi_n(A)$ is an n-dimensional quadratic $\mathbb{Z}[\pi_1(X)]$-Poincaré pair $(C \to D, (\delta\psi, \psi))$ in $\mathbb{A}(\mathbb{Z}, X)$. Using this as data for algebraic surgery results in an $(n-1)$-dimensional quadratic Poincaré complex (C', ψ') in $\mathbb{A}(\mathbb{Z}, X)$ such that the assembly $A(C')$ is a simple contractible based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex.

Killing $\pi_0(\overline{\mathbb{L}}_\bullet) = L_0(\mathbb{Z})$ in $\overline{\mathbb{L}}_\bullet$ results in the 1-connective spectrum $\mathbb{L}_\bullet$, and the *algebraic surgery exact sequence*

$$\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \dots \qquad (**)$$

with $\mathbb{S}_n(X)$ the *algebraic structure set.* The two types of structure set are related by an exact sequence

$$\cdots \to H_n(X; L_0(\mathbb{Z})) \to \mathbb{S}_n(X) \to \overline{\mathbb{S}}_n(X) \to H_{n-1}(X; L_0(\mathbb{Z})) \to \dots \ .$$

4 The structure set and the total surgery obstruction

This chapter states the results in Chapters 16,17,18 of Ranicki [9] on the L-theory orientation of topology, the total surgery obstruction and the structure set.

The algebraic theory of surgery fits the homotopy category of topological manifolds of dimension $\geqslant 5$ into a pullback square

$$\begin{array}{ccc}
\{\text{topological manifolds}\} & \longrightarrow & \{\text{local algebraic Poincaré complexes}\} \\
\downarrow & & \downarrow \\
\{\text{geometric Poincaré complexes}\} & \longrightarrow & \{\text{global algebraic Poincaré complexes}\}
\end{array}$$

where local means $\mathbb{A}(\mathbb{Z}, X)$ and global means $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$. In words : the homotopy type of a topological manifold is the homotopy type of a geometric Poincaré complex with a local algebraic Poincaré structure.

4.1 The L-theory orientation of topological block bundles

The topological k-block bundles of Rourke and Sanderson [13] are topological analogues of vector bundles. By analogy with the classifying spaces $BO(k)$,

BO for vector bundles there are classifying spaces $B\widetilde{TOP}(k)$ for topological block bundles, and a stable classifying space $BTOP$. It is known from the work of Sullivan [15] and Kirby-Siebenmann [4] that the classifying space for fibre homotopy trivialized topological block bundles

$$G/TOP \;=\; \text{homotopy fibre}(BTOP \to BG)$$

has homotopy groups $\pi_*(G/TOP) = L_*(\mathbb{Z})$. A map $S^n \to G/TOP$ classifies a topological block bundle $\eta : S^n \to B\widetilde{TOP}(k)$ with a fibre homotopy trivialization $J\eta \simeq \{*\} : S^n \to BG(k)$ $(k \geqslant 3)$. The isomorphism $\pi_n(G/TOP) \to L_n(\mathbb{Z})$ is defined by sending $S^n \to G/TOP$ to the surgery obstruction $\sigma_*(f,b)$ of the corresponding normal map $(f,b) : M \to S^n$ from a topological n-dimensional manifold M, with $b : \nu_M \to \nu_{S^n} \oplus \eta$. Sullivan [15] proved that G/TOP and BO have the same homotopy type localized away from 2

$$G/TOP[1/2] \;\simeq\; BO[1/2] \; .$$

(The localization $\mathbb{Z}[1/2]$ is the subring $\{\ell/2^m \,|\, \ell \in \mathbb{Z}, m \geqslant 0\} \subset \mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting 2. The localization $X[1/2]$ of a space X is a space such that

$$\pi_*(X[1/2]) \;=\; \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \; .$$

Thus $X \mapsto X[1/2]$ kills all the 2-primary torsion in $\pi_*(X)$.)

Let $\mathbb{L}^\bullet = \mathbb{L}(\mathbb{Z})^\bullet$ be the symmetric L-spectrum of $\mathbb{Z}$, with homotopy groups

$$\pi_n(\mathbb{L}^\bullet) \;=\; L^n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } n \equiv 0(\text{mod } 4) \text{ (signature)} \\ \mathbb{Z}_2 & \text{if } n \equiv 1(\text{mod } 4) \text{ (deRham invariant)} \\ 0 & \text{otherwise} \; . \end{cases}$$

The hyperquadratic $\mathbb{L}$-spectrum of $\mathbb{Z}$ is defined by

$$\widehat{\mathbb{L}}^\bullet \;=\; \text{cofibre}(1+T : \mathbb{L}_\bullet \to \mathbb{L}^\bullet) \; .$$

It is 0-connective, fits into a (co)fibration sequence of spectra

$$\dots \to \mathbb{L}_\bullet \xrightarrow{1+T} \mathbb{L}^\bullet \to \widehat{\mathbb{L}}^\bullet \to \Sigma\mathbb{L}_\bullet \to \dots \; ,$$

and has homotopy groups

$$\pi_n(\widehat{\mathbb{L}}^\bullet) \;=\; \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_8 & \text{if } n \equiv 0(\text{mod } 4) \text{ and } n > 0 \\ \mathbb{Z}_2 & \text{if } n \equiv 2,3(\text{mod } 4) \\ 0 & \text{if } n \equiv 1(\text{mod } 4) \; . \end{cases}$$

An *h-orientation* of a spherical fibration $\nu : X \to BG(k)$ with respect to a ring spectrum h is an h-coefficient Thom class in the reduced h-cohomology $U \in \dot{h}^k(T(\nu))$ of the Thom space $T(\nu)$, i.e. a $\dot{h}$-cohomology class which restricts to $1 \in \dot{h}^k(S^k) = \pi_0(h)$ over each $x \in X$.

Theorem ([9], 16.1) (i) The 0th space $\mathbb{L}_0$ of $\mathbb{L}_\bullet$ is homotopy equivalent to G/TOP

$$\mathbb{L}_0 \ \simeq\ G/TOP\ .$$

(ii) Every topological k-block bundle $\nu : X \to B\widetilde{TOP}(k)$ has a canonical $\mathbb{L}^\bullet$-orientation

$$U_\nu \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet)\ .$$

(iii) Every $(k-1)$-spherical fibration $\nu : X \to BG(k)$ has a canonical $\widehat{\mathbb{L}}^\bullet$-orientation

$$\widehat{U}_\nu \in \dot{H}^k(T(\nu);\widehat{\mathbb{L}}^\bullet)\ ,$$

with $\dot{H}$ denoting reduced cohomology. The *topological reducibility obstruction*

$$t(\nu)\ =\ \delta(\widehat{U}_\nu) \in \dot{H}^{k+1}(T(\nu);\mathbb{L}_\bullet)$$

is such that $t(\nu) = 0$ if and only if ν admits a topological block bundle reduction $\widetilde{\nu} : X \to B\widetilde{TOP}(k)$. Here, δ is the connecting map in the exact sequence

$$\cdots \to \dot{H}^k(T(\nu);\mathbb{L}_\bullet) \to \dot{H}^k(T(\nu);\mathbb{L}^\bullet) \to \dot{H}^k(T(\nu);\widehat{\mathbb{L}}^\bullet) \xrightarrow{\delta} \dot{H}^{k+1}(T(\nu);\mathbb{L}_\bullet) \to \dots\ .$$

The topological block bundle reductions of ν are in one-one correspondence with lifts of $\widehat{U}_\nu$ to a $\mathbb{L}^\bullet$-orientation $U_\nu \in H^k(T(\nu);\mathbb{L}^\bullet)$. □

Example. Rationally, the symmetric L-theory orientation of $\nu : X \to B\widetilde{TOP}(k)$ is the $\mathcal{L}$-genus

$$U_\nu \otimes \mathbb{Q}\ =\ \mathcal{L}(\nu) \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet)\otimes\mathbb{Q}\ =\ H^{4*}(X;\mathbb{Q})\ . \qquad \square$$

Example. Localized away from 2, the symmetric L-theory orientation of $\nu : X \to B\widetilde{TOP}(k)$ is the $KO[1/2]$-orientation of Sullivan [15]

$$U_\nu[1/2]\ =\ \Delta_\nu \in \dot{H}^k(T(\nu);\mathbb{L}^\bullet)[1/2]\ =\ \widetilde{KO}^k(T(\nu))[1/2]\ . \qquad \square$$

4.2 The total surgery obstruction

The *total surgery obstruction* $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional geometric Poincaré complex X is the cobordism class of the $\mathbb{Z}[\pi_1(X)]$-contractible $(n-1)$-dimensional quadratic Poincaré complex (C, ψ) in $\mathbb{A}(\mathbb{Z}, X)$ with $C = C([X] \cap - : C(X)^{n-*} \to C(X'))_{*+1}$, using the dual cells in the barycentric subdivision X' to regard the simplicial chain complex $C(X')$ as a chain complex in $\mathbb{A}(\mathbb{Z}, X)$.

Theorem ([9], 17.4) The total surgery obstruction is such that $s(X) = 0 \in \mathbb{S}_n(X)$ if (and for $n \geqslant 5$ only if) X is homotopy equivalent to an n-dimensional topological manifold.
Proof A regular neighbourhood $(W, \partial W)$ of an embedding $X \subset S^{n+k}$ (k large) gives a Spivak normal fibration

$$S^{k-1} \to \partial W \to W \ \simeq \ X$$

with Thom space $T(\nu) = W/\partial W$ S-dual to $X_+ = X \cup \{\text{pt.}\}$. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ has image the topological reducibility obstruction

$$t(\nu) \in \dot{H}^{k+1}(T(\nu); \mathbb{L}_\bullet) \ \cong \ H_{n-1}(X; \mathbb{L}_\bullet) \ .$$

Thus $s(X)$ has image $t(\nu) = 0 \in H_{n-1}(X; \mathbb{L}_\bullet)$ if and only if ν admits a topological block bundle reduction $\widetilde{\nu} : X \to \widetilde{BTOP}(k)$, in which case the topological version of the Browder-Novikov transversality construction applied to the degree 1 map $\rho : S^{n+k} \to T(\nu)$ gives a normal map $(f, b) = \rho| : M = f^{-1}(X) \to X$. The surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ has image

$$[\sigma_*(f,b)] \ = \ s(X) \in \operatorname{im}(L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X)) \ = \ \ker(\mathbb{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet))$$

The total surgery obstruction is $s(X) = 0$ if and only if there exists a reduction $\widetilde{\nu}$ with $\sigma_*(f, b) = 0$. □

Example. For a simply-connected space X the assembly map $A : H_*(X; \mathbb{L}_\bullet) \to L_*(\mathbb{Z})$ is onto, so that

$$\mathbb{S}_n(X) \ = \ \ker(A : H_{n-1}(X; \mathbb{L}_\bullet) \to L_{n-1}(\mathbb{Z})) \ = \ \dot{H}_{n-1}(X; \mathbb{L}_\bullet) \ ,$$

with $\dot{H}$ denoting reduced homology. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of a simply-connected n-dimensional geometric Poincaré complex X is just the obstruction to the topological reducibility of the Spivak normal fibration $\nu_X : X \to BG$. □

There are also relative and rel ∂ versions of the total surgery obstruction.

For any pair of spaces $(X, Y \subseteq X)$ let $\mathbb{S}_n(X, Y)$ be the relative groups in the exact sequence

$$\cdots \to H_n(X,Y;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(Y)] \to \mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X,Y) \to H_{n-1}(X,Y;\mathbb{L}_\bullet) \to \ldots .$$

The *relative total surgery obstruction* $s(X,Y) \in \mathbb{S}_n(X,Y)$ of an n-dimensional geometric Poincaré pair is such that $s(X,Y) = 0$ if (and for $n \geqslant 6$ only if) (X,Y) is homotopy equivalent to an n-dimensional topological manifold with boundary $(M, \partial M)$. In the special case $\pi_1(X) = \pi_1(Y)$

$$s(X,Y) \in \mathbb{S}_n(X,Y) = H_{n-1}(X,Y;\mathbb{L}_\bullet)$$

is just the obstruction to the topological reducibility of the Spivak normal fibration $\nu_X : X \to BG$, which is the π-π theorem of Chapter 4 of Wall [17].

The *rel ∂ total surgery obstruction* $s_\partial(X,Y) \in \mathbb{S}_n(X)$ of an n-dimensional geometric Poincaré pair (X,Y) with manifold boundary Y is such that $s_\partial(X,Y) = 0$ if (and for $n \geqslant 5$ only if) (X,Y) is homotopy equivalent rel ∂ to an n-dimensional manifold with boundary.

4.3 The L-theory orientation of topological manifolds

An n-dimensional geometric Poincaré complex X determines a symmetric $\mathbb{Z}[\pi_1(X)]$-Poincaré complex $(C(X'), \phi)$ in $\mathbb{A}(\mathbb{Z}, X)$, with assembly the usual symmetric Poinacaré complex $(C(\widetilde{X}), \phi(\widetilde{X}))$ representing the symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])$.

Example. For $n = 4k$ $\sigma^*(M) \in L^{4k}(\mathbb{Z}[\pi_1(M)])$ has image

$$\text{signature}(X) \;=\; \text{signature}(H^{2k}(X;\mathbb{Q}), \cup) \in L^{4k}(\mathbb{Z}) = \mathbb{Z} . \qquad \square$$

A triangulated n-dimensional manifold M determines a symmetric Poincaré complex $(C(M'), \phi)$ in $\mathbb{A}(\mathbb{Z}, M)$. The *symmetric L-theory orientation* of M is the $\mathbb{L}^\bullet$-coefficient class

$$[M]_{\mathbb{L}} \;=\; (C(M'), \phi) \in L^n(\mathbb{A}(\mathbb{Z}, M)) \;=\; H_n(M;\mathbb{L}^\bullet)$$

with assembly

$$A([M]_{\mathbb{L}}) \;=\; \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(M)]) .$$

Example. Rationally, the symmetric L-theory orientation is the Poincaré dual of the $\mathcal{L}$-genus

$$[M]_{\mathbb{L}} \;=\; \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \in H_n(M;\mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H_{n-4*}(M;\mathbb{Q}) \;=\; H^{4*}(M;\mathbb{Q}) .$$

Thus $A([M]_{\mathbb{L}}) = \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(M)])$ is a $\pi_1(M)$-equivariant generalization of the Hirzebruch signature theorem for a $4k$-dimensional manifold

$$\mathrm{signature}(M) \;=\; \langle \mathcal{L}(-\widetilde{\nu}_M), [M] \rangle \in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}\ . \qquad \square$$

Example. Localized away from 2, the symmetric L-theory orientation is the $KO[1/2]$-orientation $\Delta(M)$ of Sullivan [15]

$$[M]_{\mathbb{L}} \otimes \mathbb{Z}[1/2] \;=\; \Delta(M) \in H_n(M;\mathbb{L}^\bullet)[1/2] \;=\; KO_n(M)[1/2]\ . \qquad \square$$

See Chapter 16 of [9] for the detailed definition of the *visible symmetric L-groups* $VL^*(X)$ of a space X, with the following properties :

(i) $VL^n(X)$ is the cobordism group of n-dimensional symmetric complexes (C,ϕ) in $\mathbb{A}(\mathbb{Z},X)$ such that the assembly $A(C,\phi)$ is an n-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z}[\pi_1(X)])$, and such that each $(C(\sigma),\phi(\sigma))$ $(\sigma \in X^{(n)})$ is a 0-dimensional symmetric Poincaré complex in $\mathbb{A}(\mathbb{Z})$.

(ii) The (covariant) functor $X \mapsto VL^*(X)$ is homotopy invariant.

(iii) The visible symmetric L-groups $VL^*(K(\pi,1))$ of an Eilenberg-MacLane space $K(\pi,1)$ of a group π are the visible symmetric L-groups $VL^*(\mathbb{Z}[\pi])$ of Weiss [18].

(iv) The VL-groups fit into a commutative braid of exact sequences

$$\begin{tikzcd}[column sep=small]
\mathbb{S}_{n+1}(X) \arrow[dr] \arrow[rr, bend left] & & H_n(X;\mathbb{L}^\bullet) \arrow[dr, "A"] \arrow[rr, bend left] & & H_n(X;\widehat{\mathbb{L}}^\bullet) \\
& H_n(X;\mathbb{L}_\bullet) \arrow[ur, "1+T"] \arrow[dr, "A"] & & VL^n(X) \arrow[ur] \arrow[dr] & \\
H_{n+1}(X;\widehat{\mathbb{L}}^\bullet) \arrow[ur] \arrow[rr, bend right] & & L_n(\mathbb{Z}[\pi_1(X)]) \arrow[ur, "1+T"] \arrow[rr, bend right] & & \mathbb{S}_n(X)
\end{tikzcd}$$

(v) Every n-dimensional simple Poincaré complex X has a visible symmetric signature $\sigma^*(X) \in VL^n(X)$ with image the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$.

An *h-orientation* of an n-dimensional Poincaré complex X with respect to ring spectrum h is an h-homology class $[X]_h \in h_n(X)$ which corresponds under the S-duality isomorphism $h_n(X) \cong \dot{h}^{k+1}(T(\nu))$ to an h-coefficient Thom class $U_h \in \dot{h}^k(T(\nu))$ of the Spivak normal fibration $\nu : X \to BG(k)$ (k large, $X \subset S^{n+k}$).

Theorem ([9], 16.16) Every n-dimensional topological manifold M has a canonical $\mathbb{L}^\bullet$-orientation $[M]_{\mathbb{L}} \in H_n(M;\mathbb{L}^\bullet)$ with assembly

$$A([M]_{\mathbb{L}}) \;=\; \sigma^*(M) \in VL^n(M) \;. \qquad \square$$

If M is triangulated by a simplicial complex K then

$$[M]_{\mathbb{L}} \;=\; (C,\phi) \in H_n(M;\mathbb{L}^\bullet) \;=\; L^n(\mathbb{Z},K)$$

is the cobordism class of an n-dimensional symmetric Poincaré complex (C,ϕ) in $\mathbb{A}(\mathbb{Z},K)$ with $C = C(K')$.

Example. The canonical $\mathbb{L}^\bullet$-homology class of an n-dimensional manifold M is given rationally by the Poincaré dual of the $\mathcal{L}(M)$-genus $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \in H_n(M;\mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H_{n-4*}(M;\mathbb{Q}) \;. \qquad \square$$

Theorem ([9], pp. 190–191) For $n \geqslant 5$ an n-dimensional simple Poincaré complex X is simple homotopy equivalent to an n-dimensional topological manifold if and only if there exists a symmetric L-theory fundamental class $[X]_{\mathbb{L}} \in H_n(X;\mathbb{L}^\bullet)$ with assembly

$$A([X]_{\mathbb{L}}) \;=\; \sigma^*(X) \in VL^n(X) \;. \qquad \square$$

In the simply-connected case $\pi_1(X) = \{1\}$ with $n = 4k$ this is just :

Example. For $k \geqslant 2$ a simply-connected $4k$-dimensional Poincaré complex X is homotopy equivalent to a $4k$-dimensional topological manifold if and only if the Spivak normal fibration $\nu_X : X \to BG$ admits a topological reduction $\widetilde{\nu}_X : X \to BTOP$ for which the Hirzebruch signature formula

$$\text{signature}(X) \;=\; \langle \mathcal{L}(-\widetilde{\nu}_X), [X] \rangle \in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}$$

holds. The if part is the topological version of the original result of Browder [1] on the converse of the Hirzebruch signature theorem for the homotopy types of differentiable manifolds.

$\square$

4.4 The structure set

The *structure invariant* of a homotopy equivalence $h : N \to M$ of n-dimensional topological manifolds is is the rel ∂ total surgery obstruction

$$s(h) \;=\; s_\partial(W, M\cup N) \in \mathbb{S}_{n+1}(W) \;=\; \mathbb{S}_{n+1}(M)$$

of the $(n+1)$-dimensional geometric Poincaré pair with manifold boundary $(W, M\cup N)$ defined by the mapping cylinder W of h.

Here is a more direct description of the structure invariant, in terms of the point inverses $h^{-1}(x) \subset N$ $(x \in M)$. Choose a simplicial complex K with a homotopy equivalence $g : M \to K$ such that g and $gh : N \to K$ are topologically transverse across the dual cells $D(\sigma, K) \subset K'$. (For triangulated M take $K = M$). Then $s(h)$ is the cobordism class

$$s(h) \;=\; (C,\psi) \in \mathbb{S}_{n+1}(K) \;=\; \mathbb{S}_{n+1}(M)$$

of a $\mathbb{Z}[\pi_1(M)]$-contractible n-dimensional quadratic Poincaré complex (C,ψ) in $\mathbb{A}(\mathbb{Z}, K)$ with

$$C \;=\; C(h : C(N) \to C(K'))_{*+1} \;.$$

Theorem ([9], 18.3, 18.5) (i) The structure invariant is such that $s(h) = 0 \in \mathbb{S}_{n+1}(M)$ if (and for $n \geqslant 5$ only if) h is homotopic to a homeomorphism.
(ii) The Sullivan-Wall surgery sequence of an n-dimensional topological manifold M with $n \geqslant 5$ is in one-one correspondence with a portion of the algebraic surgery exact sequence, by a bijection

$$\begin{tikzcd}
\cdots \arrow[r] & L_{n+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r] \arrow[d, equal] & \mathbb{S}^{TOP}(M) \arrow[r] \arrow[d, "\cong"', "s"] & {[M, G/TOP]} \arrow[r] \arrow[d, "\cong"', "t"] & L_n(\mathbb{Z}[\pi_1(M)]) \arrow[d, equal] \\
\cdots \arrow[r] & L_{n+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r] & \mathbb{S}_{n+1}(M) \arrow[r] & H_n(M;\mathbb{L}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}[\pi_1(M)])
\end{tikzcd}$$

The higher structure groups are the rel ∂ structure sets

$$\mathbb{S}_{n+k+1}(M) \;=\; \mathbb{S}^{TOP}_\partial(M\times D^k, M\times S^{k-1}) \;\; (k \geqslant 1)$$

of homotopy equivalences $(h, \partial h) : (N, \partial N) \to (M\times D^k, M\times S^{k-1})$ with $\partial h : \partial N \to M \times S^{k-1}$ a homeomorphism. □

Example. For a simply-connected space M the assembly maps $A : H_*(M;\mathbb{L}_\bullet) \to L_*(\mathbb{Z})$ are onto. Thus for a simply-connected n-dimensional manifold M

$$\begin{aligned}
\mathbb{S}^{TOP}(M) &= \mathbb{S}_{n+1}(M) \\
&= \ker(A : H_n(M;\mathbb{L}_\bullet) \to L_n(\mathbb{Z})) \;=\; \dot{H}_n(M;\mathbb{L}_\bullet) \\
&= \ker(\sigma_* : [M, G/TOP] \to L_n(\mathbb{Z}))
\end{aligned}$$

with σ_* the surgery obstruction map. The structure invariant $s(h) \in \mathbb{S}^{TOP}(M)$ of a homotopy equivalence $h : N \to M$ is given modulo 2-primary torsion by the difference of the canonical $\mathbb{L}^\bullet$-orientations

$$s(h)[1/2] \ = \ (h_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}}, 0) \in \dot{H}_n(M;\mathbb{L}^\bullet)[1/2] \ = \ \dot{H}_n(M;\mathbb{L}_\bullet)[1/2] \oplus H_n(M)[1/2] \ .$$

Rationally, this is just the difference of the Poincaré duals of the $\mathcal{L}$-genera

$$\begin{aligned} s(h) \otimes \mathbb{Q} \ &= \ h_*(\mathcal{L}(N) \cap [N]_{\mathbb{Q}}) - \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \\ &\in \mathbb{S}_n(M) \otimes \mathbb{Q} \ = \ \dot{H}_n(M;\mathbb{L}_\bullet) \otimes \mathbb{Q} \ = \ \sum_{4k \neq n} H_{n-4k}(M;\mathbb{Q}) \ . \ \square \end{aligned}$$

Example. Smale [14] proved the generalized Poincaré conjecture: if N is a differentiable n-dimensional manifold with a homotopy equivalence $h : N \to S^n$ and $n \geqslant 5$ then h is homotopic to a homeomorphism. Stallings and Newman then proved the topological version: if N is a topological n-dimensional manifold with a homotopy equivalence $h : N \to S^n$ and $n \geqslant 5$ then h is homotopic to a homeomorphism. This is the geometric content of the computation of the structure set of S^n

$$\mathbb{S}^{TOP}(S^n) \ = \ \mathbb{S}_{n+1}(S^n) \ = \ 0 \ \ (n \geqslant 5) \ . \qquad \square$$

Here are three consequences of the Theorem in the non-simply-connected case, subject to the canonical restriction $n \geqslant 5$:

(i) For any finitely presented group π the image of the assembly map

$$A \ : \ H_n(K(\pi,1);\mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi])$$

is the subgroup consisting of the surgery obstructions $\sigma_*(f,b)$ of normal maps $(f,b) : N \to M$ of closed n-dimensional manifolds with $\pi_1(M) = \pi$.

(ii) The Novikov conjecture for a group π is that the higher signatures for any manifold M with $\pi_1(M) = \pi$

$$\sigma_x(M) \ = \ \langle x \cup \mathcal{L}(M), [M] \rangle \in \mathbb{Q} \ \ (x \in H^*(K(\pi,1);\mathbb{Q}))$$

are homotopy invariant. The conjecture holds for π if and only if the rational assembly maps

$$A \ : \ H_n(K(\pi,1);\mathbb{L}_\bullet) \otimes \mathbb{Q} \ = \ H_{n-4*}(K(\pi,1);\mathbb{Q}) \to L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

are injective.

(iii) The topological Borel rigidity conjecture for an n-dimensional aspherical manifold $M = K(\pi, 1)$ is that every simple homotopy equivalence of manifolds $h : N \to M$ is homotopic to a homeomeorphism, i.e. $\mathbb{S}^{TOP}(M) = \{*\}$, and more generally that

$$\mathbb{S}^{TOP}_{\partial}(M \times D^k, M \times S^{k-1}) \;=\; \{*\} \;\; (k \geqslant 1) \; .$$

The conjecture holds for π if and only if the assembly map

$$A \;:\; H_{n+k}(K(\pi,1); \mathbb{L}_\bullet) \to L_{n+k}(\mathbb{Z}[\pi])$$

is injective for $k = 0$ and an isomorphism for $k \geqslant 1$.

See Chapter 23 of Ranicki [9] and Chapter 8 of Ranicki [10] for the algebraic Poincaré transversality treatment of the splitting obstruction theory for homotopy equivalences of manifolds along codimension q submanifolds, involving natural morphisms $\mathbb{S}_*(X) \to LS_{*-q-1}$ to the LS-groups defined geometrically in Chapter 11 of Wall [17]. The case $q = 1$ is particularly important : a homotopy invariant functor is a homology theory if and only if it has excision, and excision is a codimension 1 transversality property.

4.5 Homology manifolds

An n-dimensional Poincaré complex X has a *4-periodic total surgery obstruction* $\overline{s}(X) \in \overline{\mathbb{S}}_n(X)$ such that $\overline{s}(X) = 0$ if (and for $n \geqslant 6$ only if) X is simple homotopy equivalent to a compact ANR homology manifold (Bryant, Ferry, Mio and Weinberger [2]). The $\mathbb{S}$- and $\overline{\mathbb{S}}$-groups are related by an exact sequence

$$0 \to \mathbb{S}_{n+1}(X) \to \overline{\mathbb{S}}_{n+1}(X) \to H_n(X; L_0(\mathbb{Z})) \to \mathbb{S}_n(X) \to \overline{\mathbb{S}}_n(X) \to 0 \; .$$

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional homology manifold X is the image of the Quinn [7] resolution obstruction $i(X) \in H_n(X; L_0(\mathbb{Z}))$, such that $i(X) = 0$ if (and for $n \geqslant 6$ only if) there exists a map $M \to X$ from an n-dimensional topological manifold M with contractible point inverses. The homology manifold surgery sequence of X with $n \geqslant 6$ is in one-one correspondence with a portion of the 4-periodic algebraic surgery exact sequence, by a bijection

$$\begin{tikzcd}
\cdots \arrow[r] & L_{n+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d, equal] & \mathbb{S}^H(X) \arrow[r] \arrow[d, "\overline{s}", "\cong"'] & {[X, L_0(\mathbb{Z}) \times G/TOP]} \arrow[r] \arrow[d, "\overline{t}", "\cong"'] & L_n(\mathbb{Z}[\pi_1(X)]) \arrow[d, equal] \\
\cdots \arrow[r] & L_{n+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] & \overline{\mathbb{S}}_{n+1}(X) \arrow[r] & H_n(X; \overline{\mathbb{L}}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}[\pi_1(X)])
\end{tikzcd}$$

with $\mathbb{S}^H(X)$ the structure set of simple homotopy equivalences $h : Y \to X$ of n-dimensional homology manifolds, up to s-cobordism.

Example. The homology manifold structure set of S^n $(n \geqslant 6)$ is

$$\mathbb{S}^H(S^n) \ = \ \overline{\mathbb{S}}_{n+1}(S^n) \ = \ L_0(\mathbb{Z}) \ ,$$

detected by the resolution obstruction. □

See Chapter 25 of Ranicki [9] and Johnston and Ranicki [3] for more detailed accounts of the algebraic surgery classification of homology manifolds.

The homology manifold surgery exact sequence of [2] required the controlled algebraic surgery exact sequence

$$H_{n+1}(B;\overline{\mathbb{L}}_\bullet) \to \mathbb{S}_{\epsilon,\delta}(N,f) \to [N,\partial N;G/TOP,*] \to H_n(B;\overline{\mathbb{L}}_\bullet)$$

which has now been established by Pedersen, Quinn and Ranicki [5].

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Circle Valued Morse Theory and Novikov Homology

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Abstract

Traditional Morse theory deals with real valued functions $f : M \to \mathbb{R}$ and ordinary homology $H_*(M)$. The critical points of a Morse function f generate the Morse-Smale complex $C^{MS}(f)$ over $\mathbb{Z}$, using the gradient flow to define the differentials. The isomorphism $H_*(C^{MS}(f)) \cong H_*(M)$ imposes homological restrictions on real valued Morse functions. There is also a universal coefficient version of the Morse-Smale complex, involving the universal cover $\widetilde{M}$ and the fundamental group ring $\mathbb{Z}[\pi_1(M)]$.

The more recent Morse theory of circle valued functions $f : M \to S^1$ is more complicated, but shares many features of the real valued theory. The critical points of a Morse function f generate the Novikov complex $C^{Nov}(f)$ over the Novikov ring $\mathbb{Z}((z))$ of formal power series with integer coefficients, using the gradient flow of the real valued Morse function $\overline{f} : \overline{M} = f^*\mathbb{R} \to \mathbb{R}$ on the infinite cyclic cover to define the differentials. The Novikov homology $H_*^{Nov}(M)$ is the $\mathbb{Z}((z))$-coefficient homology of $\overline{M}$. The isomorphism $H_*(C^{Nov}(f)) \cong H_*^{Nov}(M)$ imposes homological restrictions on circle valued Morse functions.

Chapter 1 reviews real valued Morse theory. Chapters 2,3,4 introduce circle valued Morse theory and the universal coefficient versions of the Novikov complex and Novikov homology, which involve the universal cover $\widetilde{M}$ and a completion $\mathbb{Z}\widehat{[\pi_1(M)]}$ of $\mathbb{Z}[\pi_1(M)]$. Chapter 5 formulates an algebraic chain complex model (in the universal coefficient version) for the relationship between the $\mathbb{Z}((z))$-module Novikov complex $C^{Nov}(f)$ of a circle valued Morse function $f : M \to S^1$ and the $\mathbb{Z}$-module Morse-Smale complex $C^{MS}(f_N)$ of the real valued Morse function $f_N = \overline{f}| : M_N = \overline{f}^{-1}[0,1] \to [0,1]$ on a fundamental domain of the infinite cyclic cover $\overline{M}$.

Contents

1 Introduction

The Morse theory of circle valued functions $f : M \to S^1$ relates the topology of a manifold M to the critical points of f, generalizing the traditional theory of real valued Morse functions $M \to \mathbb{R}$. However, the relationship is somewhat more complicated in the circle valued case than in the real valued case, and the roles of the fundamental group $\pi_1(M)$ and of the choice of gradient-like vector field v are more significant (and less well understood).

The Morse-Smale complex $C = C^{MS}(M, f, v)$ is defined geometrically for a real valued Morse function $f : M^m \to \mathbb{R}$ and a suitable choice of gradient-like vector field $v : M \to \tau_M$. In general, there is a $\mathbb{Z}[\pi]$-coefficient Morse-Smale complex for each group morphism $\pi_1(M) \to \pi$, with

$$C_i \;=\; \mathbb{Z}[\pi]^{c_i(f)}$$

if there are $c_i(f)$ critical points of index i. The differentials $d : C_i \to C_{i-1}$ are defined by counting the $\widetilde{v}$-gradient flow lines in the cover $\widetilde{M}$ of M classified by $\pi_1(M) \to \pi$. In the simplest case $\pi = \{1\}$ this is just $\widetilde{M} = M$, and if $p \in M$ is a critical point of index i and $q \in M$ is a critical point of index $i-1$ the (p,q)-coefficient in d is the number $n(p,q)$ of lines from p to q, with sign chosen according to orientations. The homology of the Morse-Smale complex is isomorphic to the ordinary homology of M

$$H_*(C^{MS}(M, f, v)) \;\cong\; H_*(M)$$

so that

(a) the critical points of f can be used to compute $H_*(M)$,

(b) $H_*(M)$ provides lower bounds on the number of critical points in any Morse function $f : M \to \mathbb{R}$, which must have at least as many critical points of index i as there are $\mathbb{Z}$-module generators for $H_i(M)$ (Morse inequalities).

Basic real valued Morse theory is reviewed in Chapter 2.

In the last 40 years there has been much interest in the Morse theory of circle valued functions $f : M^m \to S^1$, starting with the work of Stallings [35], Browder and Levine [3], Farrell [8] and Siebenmann [34] on the characterization of the maps f which are homotopic to the projections of fibre bundles over S^1 : these are the circle valued Morse functions without any critical points.

About 20 years ago, Novikov ([17],[18],[19],[20] (pp. 194–199)) was motivated by problems in physics and dynamical systems to initiate the general Morse theory of closed 1-forms, including circle valued functions $f : M \to S^1$ as the most important special case. The new idea was to use the *Novikov ring* of formal power series with an infinite number of positive coefficients and a finite number of negative coefficients

$$\mathbb{Z}((z)) \;=\; \mathbb{Z}[[z]][z^{-1}] \;=\{\sum_{j=-\infty}^{\infty} n_j z^j \,|\, n_j \in \mathbb{Z},\, n_j = 0 \text{ for all } j < k, \text{ for some } k\}$$

as a counting device for the gradient flow lines of the real valued Morse function $\overline{f} : \overline{M} = f^*\mathbb{R} \to \mathbb{R}$ on the (non-compact) infinite cyclic cover $\overline{M}$ of M, with the indeterminate z corresponding to the generating covering translation $z : \overline{M} \to \overline{M}$. For f the number of gradient flow lines starting at a critical point $p \in M$ is finite in the generic case. On the other hand, for $\overline{f}$ the number of gradient flow lines starting at a critical point $\overline{p} \in \overline{M}$ may be infinite in the generic case, so the counting methods for real and circle valued Morse theory are necessarily different.

The *Novikov complex* $\widehat{C} = C^{Nov}(M, f, v)$ is defined for a circle valued Morse function $f : M^m \to S^1$ and a suitable choice of gradient-like vector field $v : M \to \tau_M$. In general, there is a $\widehat{\mathbb{Z}[\Pi]}$-coefficient Novikov complex for each factorization of $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ as $\pi_1(M) \to \Pi \to \mathbb{Z}$, with $\widehat{\mathbb{Z}[\Pi]}$ a completion of $\mathbb{Z}[\Pi]$, with

$$\widehat{C}_i \;=\; \widehat{\mathbb{Z}[\Pi]}^{c_i(f)}$$

if there are $c_i(f)$ critical points of index i. The differentials $d : C_i \to C_{i-1}$ are defined by counting the $\widetilde{v}$-gradient flow lines in the cover $\widetilde{M}$ of M classified by $\pi_1(M) \to \Pi$. The construction of the Novikov complex for arbitrary $\widehat{Z[\Pi]}$ is described in Chapter 3. In the simplest case

$$\Pi \;=\; \mathbb{Z} \;,\; \mathbb{Z}[\Pi] \;=\; \mathbb{Z}[z, z^{-1}] \;,\; \widehat{\mathbb{Z}[\Pi]} \;=\; \mathbb{Z}((z)) \;,\; \widetilde{M} \;=\; \overline{M} \;=\; f^*\mathbb{R} \;.$$

For a critical point $\overline{p} \in \overline{M}$ of index i and a critical point $\overline{q} \in \overline{M}$ of an index $i-1$ the $(\overline{p}, \overline{q})$-coefficients in $\widehat{d}$ is

$$\widehat{n}(\overline{p}, \overline{q}) \;=\; \sum_{j=k}^{\infty} n(\overline{p}, z^j\overline{q}) z^j \in \mathbb{Z}((z))$$

with $n(\overline{p}, z^j\overline{q})$ the signed number of $\overline{v}$-gradient flow lines of the real valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ from $\overline{p}$ to the translate $z^j\overline{q}$ of $\overline{q}$, and $k = [\overline{f}(\overline{p}) -$

$\overline{f}(\overline{q})]$. The convention is that the generating covering translation $z : \overline{M} \to \overline{M}$ is to be chosen parallel to the downward gradient flow $v : M \to \tau_M$, with

$$\overline{f}(zx) \;=\; \overline{f}(x) - 1 \in \mathbb{R} \;\; (x \in \overline{M}) \;.$$

In particular, this means that for $f = 1 : M = S^1 \to S^1$

$$z \;:\; \overline{M} \;=\; \mathbb{R} \to \overline{M} \;=\; \mathbb{R}\;;\; x \mapsto x - 1 \;.$$

Circle valued Morse theory is necessarily more complicated than real valued Morse theory. The Morse-Smale complex $C^{MS}(M, f : M \to \mathbb{R}, v)$ is an absolute object, describing M on the chain level, with $c_0(f) > 0$, $c_m(f) > 0$. This is the algebraic analogue of the fact that every continuous function $f : M \to \mathbb{R}$ on a compact space attains an absolute minimum and an absolute maximum. By contrast, the Novikov complex $C^{Nov}(M, f : M \to S^1, v)$ is a relative object, measuring the chain level difference between f and the projection of a fibre bundle (= Morse function with no critical points). A continuous function $f : M \to S^1$ can just go round and round! The connection between the geometric properties of f and the algebraic topology of M is still not yet completely understood, although there has been much progress in the work of Pajitnov, Farber, the author and others.

The *Novikov homology groups* of a space M with respect to a cohomology class $f \in [M, S^1] = H^1(M)$ are defined by

$$H^{Nov}_*(M, f) \;=\; H_*\big(\mathbb{Z}((z)) \otimes_{\mathbb{Z}[z,z^{-1}]} C(\overline{M})\big) \;.$$

The homology groups of the Novikov complex are isomorphic to the Novikov homology groups

$$H_*(C^{Nov}(M, f, v)) \;\cong\; H^{Nov}_*(M, f) \;.$$

By analogy with the real valued case :

(a) the critical points of f can be used to compute $H^{Nov}_*(M, f)$,

(b) $H^{Nov}_*(M, f)$ provides lower bounds on the number of critical points in any Morse function $f : M \to S^1$, which must have at least as many critical points of index i as there are $\mathbb{Z}((z))$-module generators for $H^{Nov}_i(M, f)$ (Morse-Novikov inequalities).

Novikov homology is constructed in Chapter 4, for arbitrary $\widehat{\mathbb{Z}[\Pi]}$-coefficients..

Novikov conjectured ([1]) that for a generic class of gradient-like vector fields $v \in \mathcal{GT}(f)$ the functions $j \mapsto n(\overline{p}, z^j\overline{q})$ have subexponential growth.

Let $S \subset \mathbb{Z}[z]$ be the subring of the polynomials $s(z)$ such that $s(0) = 1$, which (up to sign) are precisely the polynomials which are invertible in the power series ring $\mathbb{Z}[[z]]$. The localization $S^{-1}\mathbb{Z}[z, z^{-1}]$ of $\mathbb{Z}[z, z^{-1}]$ is identified with the subring of $\mathbb{Z}((z))$ consisting of the quotients $\dfrac{r(z)}{s(z)}$ with $r(z) \in \mathbb{Z}[z, z^{-1}]$, $s(z) \in S$.

Pajitnov [23],[24] constructed a C^0-dense subspace $\mathcal{GCCT}(f) \subset \mathcal{GT}(f)$ of gradient-like vector fields v for which the differentials in the Novikov complex $C^{Nov}(M, f, v)$ are rational

$$\widehat{n}(\overline{p}, \overline{q}) \;=\; \sum_{j=-\infty}^{\infty} n(\overline{p}, z^j \overline{q}) z^j \in S^{-1}\mathbb{Z}[z, z^{-1}] \subset \mathbb{Z}((z))$$

and the functions $j \mapsto n(\overline{p}, z^j \overline{q})$ have polynomial growth. The idea is to cut M along the inverse image $N = f^{-1}(0)$ (assuming $0 \in S^1$ is a regular value of f), giving a fundamental domain

$$(M_N; N, z^{-1}N) \;=\; \overline{f}^{-1}([0,1]; \{0\}, \{1\})$$

for $\overline{f} : \overline{M} \to \mathbb{R}$, and to then use a kind of cellular approximation theorem to give a chain level approximation to the gradient flow in

$$(f_N, v_N) \;=\; (\overline{f}, \overline{v})| \;: (M_N, f_N, v_N) \to ([0,1]; \{0\}, \{1\}) \;.$$

The mechanism described in Chapter 5 below then gives a chain complex over $S^{-1}\mathbb{Z}[z, z^{-1}]$ inducing $C^{Nov}(M, f, v)$. Hutchings and Lee [10],[11] used a similar method to get enough information from $C^{Nov}(M, f, v)$ for generic v to obtain an estimate on the number of closed v-gradient flow lines $\gamma : S^1 \to M$.

Farber and Ranicki [7] and Ranicki [30] constructed an 'algebraic Novikov complex' in $S^{-1}\mathbb{Z}[z, z^{-1}]$ for any circle Morse valued function $f : M \to S^1$, using any CW structure on $N = f^{-1}(0)$, the extension to a CW structure on M_N, and a cellular approximation to the inclusion $z^{-1}N \to M_N$. The construction is recalled in Chapter 5, including the non simply connected version. In many cases (e.g. for $v \in \mathcal{GCCT}(f)$) this algebraic model does actually coincide with the geometric Novikov complex $C^{Nov}(M, f, v)$.

The Morse-Novikov theory of circle valued functions on finite-dimensional manifolds and Novikov homology have many applications to symplectic topology, Floer homology, and Seiberg-Witten theory (Poźniak [26], Le and Ono [14], Hutchings and Lee [10], [11], ...). Also, circle valued Morse theory on infinite-dimensional manifolds features in the work of Taubes on Casson's homology 3-sphere invariant and gauge theory. However, these notes are not a survey of all the applications of circle valued Morse theory and Novikov homology! They deal exclusively with the basic development in the finite-dimensional case and some of the applications to the classification of manifolds.

2 Real valued Morse theory

This section reviews the real valued Morse theory, which is a prerequisite for circle valued Morse theory. The traditional references Milnor [15], [16] remain the best introductions to real valued Morse theory. Bott [2] gives a beautiful account of the history of Morse theory, including the development of the modern chain complex point of view inspired by Witten.

Let M be a compact differentiable m-dimensional manifold. The *critical points* of a differentiable function $f : M \to \mathbb{R}$ are the zeros $p \in M$ of the differential $\nabla f : \tau_M \to \tau_{\mathbb{R}}$. A *Morse function* $f : M \to \mathbb{R}$ is a differentiable function in which every critical point $p \in M$ is required to be isolated and nondegenerate, meaning that in local coordinates

$$f(p + (x_1, x_2, \dots, x_m)) \;=\; f(p) - \sum_{j=1}^{i} (x_j)^2 + \sum_{j=i+1}^{m} (x_j)^2$$

with i the index of p. The subspace of Morse functions is C^2-dense in the space of all differentiable functions $f : M \to \mathbb{R}$.

A vector field $v : M \to \tau_M$ is *gradient-like* for f if there exists a Riemannian metric $\langle\ ,\ \rangle$ on M such that

$$\langle v, w\rangle \;=\; \nabla f(w) \in \mathbb{R} \quad (w \in \tau_M)\ .$$

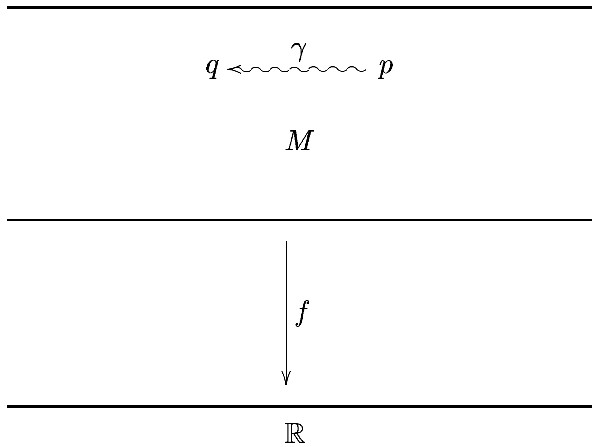
Note that $\langle\ ,\ \rangle$ and ∇f determine v, and that the zeros of v are the critical points of f.

A *v-gradient flow line* $\gamma : \mathbb{R} \to M$ satisfies

$$\gamma'(t) \;=\; -\,v(\gamma(t)) \in \tau_M(\gamma(t)) \quad (t \in \mathbb{R})\ .$$

The minus sign here gives the *downward* gradient flow, so that

$$f(\gamma(s)) > f(\gamma(t)) \text{ if } s < t\ .$$



The limits

$$\lim_{t \to -\infty} \gamma(t) \;=\; p \quad , \quad \lim_{t \to \infty} \gamma(t) \;=\; q \in M$$

are critical points of f with $f(q) < f(p)$, and if γ is isolated then

$$\mathrm{index}(q) \;=\; \mathrm{index}(p) - 1 \ .$$

For every point $x \in M$ there is a v-gradient flow line $\gamma_x : \mathbb{R} \to M$ (which is unique up to scaling) such that $\gamma_x(0) = x \in M$. If x is a critical point take γ_x to be the constant path at x.

The *unstable* and *stable* manifolds of a critical point $p \in M$ of index i are the open manifolds

$$W^u(p,v) \;=\; \{x \in M \mid \lim_{t \to -\infty} \gamma_x(t) = p\} \ , \ W^s(p,v) \;=\; \{x \in M \mid \lim_{t \to \infty} \gamma_x(t) = p\} \ .$$

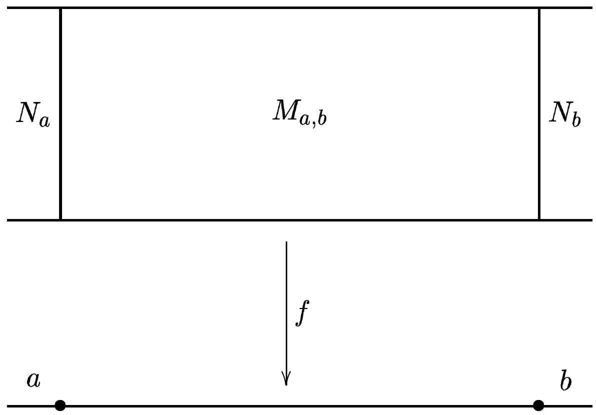
The unstable and stable manifolds are images of immersions $\mathbb{R}^i \to M$, $\mathbb{R}^{m-i} \to M$ respectively, which are embeddings near $p \in M$.

The basic results relating a Morse function $f : M^m \to \mathbb{R}$ to the topology of M concern the inverse images

$$N_a \;=\; f^{-1}(a)$$

of the regular values $a \in \mathbb{R}$, which are closed $(m-1)$-dimensional manifolds, and the cobordisms

$$(M_{a,b}; N_a, N_b) \;=\; f^{-1}([a,b]; \{a\}, \{b\}) \quad (a < b) \ .$$



The results are:

(i) if $[a,b] \subset \mathbb{R}$ contains no critical values the v-gradient flow determines a diffeomorphism

$$N_b \to N_a \ ; \ x \mapsto \gamma_x((f\gamma_x)^{-1}(a)) \ ,$$

(ii) if $[a,b] \subset \mathbb{R}$ contains a unique critical value $f(p) \in (a,b)$, and $p \in M$ is a critical point of index i, then N_b is obtained from N_a by surgery on a tubular neighbourhood $S^{i-1} \times D^{m-i} \subset N_a$ of $S^{i-1} = W^u(p,v) \cap N_a$

$$N_b \ = \ N_a \backslash (S^{i-1} \times D^{m-i}) \cup D^i \times S^{m-i-1}$$

with $D^i \times S^{m-i-1} \subset N_b$ a tubular neighbourhood of $S^{m-i-1} = W^s(p,v) \cap N_b$, and $(M_{a,b}; N_a, N_b)$ the trace of the surgery

$$M_{a,b} \ = \ N_a \times [0,1] \cup D^i \times D^{m-i} \ .$$

Let $\mathcal{GT}(f)$ denote the set of gradient-like vector fields v on M which satisfy the Morse-Smale transversality condition that for any critical points $p, q \in M$ with $\mathrm{index}(p) = i$, $\mathrm{index}(q) = j$ the submanifolds $W^u(p,v)^i$, $W^s(q,v)^{m-j} \subset M^m$ intersect transversely in an $(i-j)$-dimensional submanifold $W^u(p,v) \cap W^s(q,v) \subset M$. The subspace $\mathcal{GT}(f)$ is dense in the space of gradient-like vector fields for f.

Suppose that the Morse function $f : M \to \mathbb{R}$ has $c_i(f)$ critical points of f of index i, and that the critical points $p_0, p_1, p_2, \cdots \in M$ are arranged to satisfy

$$\mathrm{index}(p_0) \leqslant \mathrm{index}(p_1) \leqslant \mathrm{index}(p_2) \leqslant \dots \ , \ f(p_0) < f(p_1) < f(p_2) < \dots \ .$$

A choice of $v \in \mathcal{GT}(f)$ determines a handle decomposition of M

$$M \;=\; \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i}$$

with one i-handle $h^i = D^i \times D^{m-i}$ for each critical point of index i.

The *Morse-Smale complex* $C^{MS}(M,f,v)$ is defined for a Morse-Smale pair $(f : M \to \mathbb{R}, v \in \mathcal{GT}(f))$ and a regular cover $\widetilde{M}$ of M with group of covering translations π, to be the based f.g. free $\mathbb{Z}[\pi]$-module chain complex with

$$d \;:\; C^{MS}(M,f,v)_i \;=\; \mathbb{Z}[\pi]^{c_i(f)} \to C^{MS}(M,f,v)_{i-1} \;=\; \mathbb{Z}[\pi]^{c_{i-1}(f)} \;;\; \widetilde{p} \mapsto \sum_{\widetilde{q}} n(\widetilde{p},\widetilde{q})\widetilde{q}$$

with $n(\widetilde{p},\widetilde{q}) \in \mathbb{Z}$ the finite signed number of $\widetilde{v}$-gradient flow lines $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ which start at a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f} : \widetilde{M} \to \mathbb{R}$ with index i and terminate at a critical point $\widetilde{q} \in \widetilde{M}$ of index $i-1$. Choose an arbitrary lift of each critical point $p \in M$ of f to a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f}$, obtaining a basis for $C^{MS}(M,f,v)$. The Morse-Smale complex is the cellular chain complex

$$C^{MS}(M,f,v) \;=\; C(\widetilde{M})$$

of the CW structure on $\widetilde{M}$ in which the i-cells are the lifts of the i-handles h^i. In particular, the homology of the Morse-Smale complex is the ordinary homology of $\widetilde{M}$

$$H_*(C^{MS}(M,f,v)) \;=\; H_*(\widetilde{M}) \;.$$

If $(f,v) : M \to \mathbb{R}$ is modified to $(f',v') : M \to \mathbb{R}$ by adding a pair of critical points p,q of index $i,i-1$ with $n(p,q,v) = 1$ the Morse-Smale complex $C^{MS}(M,f',v')$ is obtained from $C^{MS}(M,f,v)$ by attaching an elementary chain complex

$$E \;:\; \dots \to 0 \to E_i \;=\; \mathbb{Z}[\pi] \xrightarrow{1} E_{i-1} \;=\; \mathbb{Z}[\pi] \to 0 \to \dots \;,$$

with an exact sequence

$$0 \to C^{MS}(M,f,v) \to C^{MS}(M,f',v') \to E \to 0 \;.$$

Conversely, if $m \geqslant 5$ then the Whitney trick applies to realize the elementary moves of Whitehead torsion theory by cancellation of pairs of critical points (or equivalently, handles). This cancellation is the basis of the proofs of the h- and s-cobordism theorems.

The identity $C^{MS}(M,f,v) = C(M)$ (for $\widetilde{M} = M$) gives the *Morse inequalities*

$$c_i(f) \geqslant b_i(M) + q_i(M) + q_{i-1}(M)$$

with

$$b_i(M) \;=\; \dim_{\mathbb{Z}}\big(H_i(M)/T_i(M)\big) \;,\; q_i(M) \;=\; \# T_i(M)$$

the Betti numbers of M, where

$$T_i(M) \;=\; \{x \in H_i(M) \,|\, nx = 0 \text{ for some } n \neq 0 \in \mathbb{Z}\}$$

is the torsion subgroup of $H_i(M)$ and $\#$ denotes the minimum number of generators. Smale used the cancellation of critical points to prove that these inequalities are sharp for $\pi_1(M) = \{1\}$, $m \geqslant 5$: there exists $(f,v) : M \to \mathbb{R}$ with the minimum possible number of critical points

$$c_i(f) \;=\; b_i(M) + q_i(M) + q_{i-1}(M) \;.$$

The method is to start with an arbitrary Morse function $f : M \to \mathbb{R}$, and to systematically cancel pairs of critical points until this is no longer possible.

The Morse-Smale complex $C^{MS}(M,f,v)$ is also defined for a Morse function on an m-dimensional cobordism $f : (M;N,N') \to ([0,1];\{0\},\{1\})$ with $v \in \mathcal{GT}(f)$. In this case there is a relative handle decomposition

$$M \;=\; N \times [0,1] \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i}$$

and $C^{MS}(M,f,v) = C(\widetilde{M},\widetilde{N})$. The s-cobordism theorem states that for a Morse function f on an h-cobordism $\tau(C^{MS}(M,f,v)) = 0 \in Wh(\pi_1(M))$ if (and for $m \geqslant 6$ only if) the critical points of f can be stably cancelled in pairs.

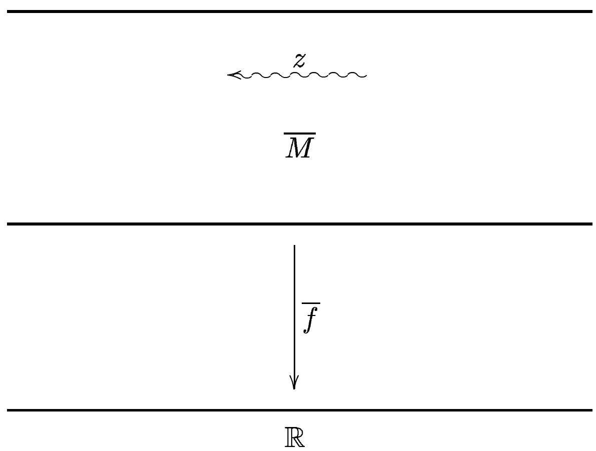
3 The Novikov complex

Morse functions $f : M \to S^1$, gradient-like vector field v, critical points, index, $c_i(f)$, are defined in the same way as for the real valued case in Chapter 1. Again, the subspace of Morse functions is C^2-dense in the space of all functions $f : M \to S^1$. But it is harder to decide which pairs of critical points can be cancelled.

A Morse function $f : M \to S^1$ lifts to a $\mathbb{Z}$-equivariant Morse function $\overline{f} : \overline{M} = f^*\mathbb{R} \to \mathbb{R}$ on the infinite cyclic cover

$$\begin{tikzcd} \overline{M} \arrow[r] \arrow[d, "\overline{f}"'] & M \arrow[d, "f"] \\ \mathbb{R} \arrow[r] & S^1 \end{tikzcd}$$

Let $z : \overline{M} \to \overline{M}$ be the downward generating covering translation.



Let $\mathcal{GT}(f)$ be the space of gradient-like vector fields $v : M \to \tau_M$ such that a lift $\overline{v} : \overline{M} \to \tau_{\overline{M}}$ satisfies the Morse-Smale transversality condition. The Novikov complex of a circle valued Morse function is defined by analogy with the Morse-Smale complex of a real valued function, as follows.

Given a ring A and an automorphism $\alpha : A \to A$ let z be an indeterminate over A with

$$az \;=\; z\alpha(a) \quad (a \in A)\ .$$

The *α-twisted Laurent polynomial extension* of A is the localization of the α-twisted polynomial extension $A_\alpha[z]$ inverting z

$$A_\alpha[z,z^{-1}] \;=\; A_\alpha[z][z^{-1}]\ ,$$

the ring of polynomials $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ $(a_j \in A)$ such that $\{j \in \mathbb{Z} \,|\, a_j \neq 0\}$ is finite.

The *α-twisted Novikov ring* of A is the localization of the completion of $A_\alpha[z]$

$$A_\alpha((z)) \;=\; A_\alpha[[z]][z^{-1}] \;,$$

the ring of power series $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ $(a_j \in A)$ such that $\{j \leqslant 0 \,|\, a_j \neq 0\}$ is finite.

Given $f : M \to S^1$ let $\widetilde{M}$ be a regular cover of $\overline{M}$, with group of covering translations π. Only the case of connected $M, \overline{M}, \widetilde{M}$ will be considered. Let Π be the group of covering translations of $\widetilde{M}$ over M, so that there is defined a group extension

$$\{1\} \to \pi \to \Pi \to \mathbb{Z} \to \{1\}$$

with a lift of $1 \in \mathbb{Z}$ to an element $z \in \Pi$ such that the covering translation $z : \widetilde{M} \to \widetilde{M}$ induces $z : \overline{M} \to \overline{M}$ on $\overline{M} = \widetilde{M}/\pi$. Thus

$$\Pi \;=\; \pi \times_\alpha \mathbb{Z} \quad , \quad \mathbb{Z}[\Pi] \;=\; \mathbb{Z}[\pi]_\alpha[z, z^{-1}] \;.$$

Write the α-twisted Novikov ring as

$$\widehat{\mathbb{Z}[\Pi]} \;=\; \mathbb{Z}[\pi]_\alpha((z)) \;.$$

Choose a lift of each critical point $p \in M$ of f to a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f}$.

The *Novikov complex* $C^{Nov}(M, f, v)$ of $(f : M \to S^1, v \in \mathcal{GT}(f))$ is the based f.g. free $\widehat{\mathbb{Z}[\Pi]}$-module chain complex with

$$\begin{aligned} d \;:\; C^{Nov}(M,f,v)_i \;=\; \mathbb{Z}[\pi]_\alpha((z))^{c_i(f)} &\to C^{Nov}(M,f,v)_{i-1} \;=\; \mathbb{Z}[\pi]_\alpha((z))^{c_{i-1}(f)} \;; \\ \widetilde{p} &\mapsto \sum_{j=-\infty}^{\infty} \sum_{\widetilde{q}} n(\widetilde{p}, z^j \widetilde{q}) z^j \widetilde{q} \end{aligned}$$

with $n(\widetilde{p}, \widetilde{q}) \in \mathbb{Z}$ the finite signed number of $\widetilde{v}$-gradient flow lines $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ which start at a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f} : \widetilde{M} \to \mathbb{R}$ with index i and terminate at a critical point $\widetilde{q} \in \widetilde{M}$ of index $i-1$.

Exercise. Work out $C^{Nov}(S^1, f, v)$ for

$$f \;:\; S^1 \to S^1 \;;\; [t] \mapsto [4t - 9t^2 + 6t^3] \;\; (0 \leqslant t \leqslant 1) \;. \qquad \square$$

The original definition of Novikov [17],[18] was in the special case

$$\widetilde{M} \;=\; \overline{M} \;,\; \pi \;=\; \{1\} \;,\; \Pi \;=\; \mathbb{Z} \;,\; \alpha \;=\; 1 \;,\; \widehat{\mathbb{Z}[\Pi]} \;=\; \mathbb{Z}((z))$$

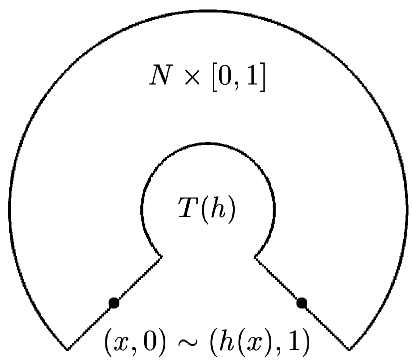
when $C^{Nov}(M,f,v)$ is a based f.g. free $\mathbb{Z}((z))$-module chain complex (as in the Exercise).

Take $\widetilde{M}$ to be the universal cover of M and $\pi = \pi_1(\overline{M})$, $\alpha : \pi \to \pi$ the automorphism induced by a generating covering translation $z : \overline{M} \to \overline{M}$, $\Pi = \pi_1(M) = \pi \times_\alpha \mathbb{Z}$. This case gives the based f.g. free $\widehat{\mathbb{Z}[\pi_1(M)]}$-module Novikov complex $C^{Nov}(M,f,v)$ of Pajitnov [22].

There is only one class of Morse functions $f : M \to S^1$ for which the Novikov complex is easy to compute:

Example. Let M be the mapping torus of a diffeomorphism $h : N \to N$ of a closed $(m-1)$-dimensional manifold

$$M \ = \ T(h) \ = \ (N \times [0,1])/\{(x,0) \sim (h(x),1)\} \ .$$



The fibre bundle projection

$$f \ : \ M \ = \ T(h) \to S^1 \ = \ [0,1]/\{0 \sim 1\} \ ; \ [x,t] \mapsto [t]$$

has no critical points, so that $C^{Nov}(M,f,v) = 0$ for any $v \in \mathcal{GT}(f)$. $\square$

4 Novikov homology

The Novikov homology $H_*^{Nov}(M,f;\widehat{\mathbb{Z}[\Pi]})$ is defined for a space M with a map $f : M \to S^1$ and a factorization of $f_* : \pi_1(M) \to \pi_1(S^1)$ through a group Π. The relevance of the Novikov complex $C^{Nov}(M,f,v)$ to the Morse theory of a Morse map $f : M \to S^1$ is immediately obvious. The relevance of the Novikov homology is rather less obvious, even though there are isomorphisms $H_*(C^{Nov}(M,f,v)) \cong H_*^{Nov}(M,f;\widehat{\mathbb{Z}[\Pi]})$!

The R-coefficient homology of a space M is defined for any ring morphism $\mathbb{Z}[\pi_1(M)] \to R$

$$H_*(M;R) \;=\; H_*(C(M;R))$$

using any free $\mathbb{Z}[\pi_1(M)]$-module chain complex $C(\widetilde{M})$ (e.g. cellular, if M is a CW complex) and $C(M;R) = R \otimes_{\mathbb{Z}[\pi_1(M)]} C(\widetilde{M})$.

Given a group π and an automorphism $\alpha : \pi \to \pi$ let $\pi \times_\alpha \mathbb{Z}$ be the group with elements gz^j ($g \in \pi$, $j \in \mathbb{Z}$), and multiplication by $gz = \alpha(g)z$, so that

$$\mathbb{Z}[\pi \times_\alpha \mathbb{Z}] \;=\; \mathbb{Z}[\pi]_\alpha[z, z^{-1}] \;.$$

For any map $f : M \to S^1$ with M connected the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ is connected if and only if $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ is onto, in which case

$$\pi_1(M) \;=\; \pi_1(\overline{M}) \times_{\alpha_M} \mathbb{Z}$$

with $\alpha_M : \pi_1(\overline{M}) \to \pi_1(\overline{M})$ the automorphism induced by a generating covering translation $z : \overline{M} \to \overline{M}$.

Suppose given a connected space M with a cohomology class $f \in [M, S^1] = H^1(M)$ such that $\overline{M} = f^*\mathbb{R}$ is connected. Given a factorization of the surjection $f_* : \pi_1(M) \to \pi_1(S^1)$

$$f_* \;:\; \pi_1(M) \;=\; \pi_1(\overline{M}) \times_{\alpha_M} \mathbb{Z} \to \Pi \to \mathbb{Z}$$

let $\pi = \ker(\Pi \to \mathbb{Z})$ and let $z \in \Pi$ be the image of $z = (0,1) \in \pi_1(M)$, so that $\Pi = \pi \times_\alpha \mathbb{Z}$ with

$$\alpha \;:\; \pi \to \pi \;;\; g \mapsto z^{-1}gz \;.$$

The $\widehat{\mathbb{Z}[\Pi]}$-coefficient *Novikov homology* of (M, f) is

$$H_*^{Nov}(M, f; \widehat{\mathbb{Z}[\Pi]}) \;=\; H_*(M; \widehat{\mathbb{Z}[\Pi]}) \;,$$

with $\widehat{\mathbb{Z}[\Pi]} = \mathbb{Z}[\pi]_\alpha((z))$.

In the original case

$$\widetilde{M} \;=\; \overline{M} \;,\; \pi \;=\; \{1\} \;,\; \Pi \;=\; \mathbb{Z} \;,\; \widehat{\mathbb{Z}[\Pi]} \;=\; \mathbb{Z}((z)) \;,$$

and $H_*^{Nov}(M, f; \widehat{\mathbb{Z}[\Pi]})$ may be written as $H_*^{Nov}(M, f)$, or even just $H_*^{Nov}(M)$.

Example 1. The $\mathbb{Z}((z))$-coefficient cellular chain complex of S^1 is

$$C\big(S^1; \mathbb{Z}((z))\big) \;:\; \cdots \to 0 \to \mathbb{Z}((z)) \xrightarrow{1-z} \mathbb{Z}((z))$$

and $1-z \in \mathbb{Z}((z))$ is a unit, so $H_*^{Nov}(S^1) = 0$. □

Example 2. Let N be a connected finite CW complex with cellular $\mathbb{Z}$-module chain complex $C(N)$, and let $h : N \to N$ be a self-map with induced $\mathbb{Z}$-module chain map $h : C(N) \to C(N)$. The $\mathbb{Z}((z))$-coefficient cellular chain complex of the mapping torus $T(h)$ with respect to the canonical projection

$$f \ : \ T(h) \to S^1 \ ; \ [x,t] \mapsto [t]$$

is the algebraic mapping cone

$$C^+\big(T(h);\mathbb{Z}((z))\big) \ = \ \mathcal{C}\big(1-zh : C(N)((z)) \to C(N)((z))\big) \ .$$

Now $1-zh$ is a $\mathbb{Z}((z))$-module chain equivalence, so that

$$H_*^{Nov}(T(h),f) \ = \ 0 \ .$$

The $\mathbb{Z}((z))$-coefficient cellular chain complex of the mapping torus $T(h)$ with respect to the other projection

$$-f \ : \ T(h) \to S^1 \ ; \ [x,t] \mapsto [1-t]$$

is the algebraic mapping cone

$$C^-\big(T(h);\mathbb{Z}((z))\big) \ = \ \mathcal{C}\big(z-h : C(N)((z)) \to C(N)((z))\big) \ .$$

If $h : N \to N$ is a homotopy equivalence then $z-h$ is a $\mathbb{Z}((z))$-module chain equivalence, so that

$$H_*^{Nov}(T(h),-f) \ = \ 0 \ ,$$

but in general $H_*^{Nov}(T(h),-f) \neq 0$ – see Example 3 below for an explicit non-zero example.

□

Example 3. The Novikov homology groups of the mapping torus $T(2)$ of the double covering map $2 : S^1 \to S^1$ are

$$\begin{aligned} H_1^{Nov}(T(2),f) \ &= \ \mathbb{Z}((z))/(1-2z) \ = \ 0 \ , \\ H_1^{Nov}(T(2),-f) \ &= \ \mathbb{Z}((z))/(z-2) \ = \ \widehat{\mathbb{Z}}_2[1/2] \ = \ \widehat{\mathbb{Q}}_2 \neq 0 \end{aligned}$$

with $\widehat{\mathbb{Q}}_2$ the 2-adic field (Example 23.25 of Hughes and Ranicki [9]). The inverse of

$$n \ = \ 2^a(2b+1) \in \mathbb{Z}$$

is

$$n^{-1} \ = \ z^{-a}(1-zb+z^2b^2-z^3b^3+\dots) \in \mathbb{Z}((z))/(2-z) \ = \ \widehat{\mathbb{Q}}_2 \ . \qquad □$$

Theorem. (Novikov [17], [18] for $\pi = \{1\}$, Pajitnov [21])
The Novikov complex $C^{Nov}(M,f,v)$ *is* $\widehat{\mathbb{Z}[\Pi]}$*-module chain equivalent to* $C(M;\widehat{\mathbb{Z}[\Pi]})$*, with isomorphisms*

$$H_*(C^{Nov}(M,f,v)) \;\cong\; H_*^{Nov}(M,f;\widehat{\mathbb{Z}[\Pi]}) \;. \qquad \square$$

The chain equivalence $C^{Nov}(M,f,v) \simeq C(M;\widehat{\mathbb{Z}[\Pi]})$ will be described in Chapter 4 below.

The Novikov ring $\mathbb{Z}((z))$ is a principal ideal domain, and $H_*^{Nov}(M,f)$ is the homology of a f.g. free $\mathbb{Z}((z))$-module chain complex. Thus each $H_i^{Nov}(M,f)$ is a f.g. $\mathbb{Z}((z))$-module, which splits as free$\oplus$torsion, by the structure theorem for f.g. modules over a principal ideal domain.

The *Novikov numbers* of any finite *CW* complex M and $f \in H^1(M)$ are the Betti numbers of Novikov homology

$$b_i^{Nov}(M,f) \;=\; \dim_{\mathbb{Z}((z))}\big(H_i^{Nov}(M,f)/T_i^{Nov}(M,f)\big)\;,\; q_i^{Nov}(M,f) \;=\; \# T_i^{Nov}(M,f)$$

where

$$T_i^{Nov}(M,f) \;=\; \{x \in H_i^{Nov}(M,f) \,|\, ax = 0 \text{ for some } a \neq 0 \in \mathbb{Z}((z))\}$$

is the torsion $\mathbb{Z}((z))$-submodule of $H_i^{Nov}(M,f)$, and $\#$ denotes the minimum number of generators.

The *Morse-Novikov inequalities* ([17])

$$c_i(f) \geqslant b_i^{Nov}(M,f) + q_i^{Nov}(M,f) + q_{i-1}^{Nov}(M,f)$$

are an immediate consequence of the isomorphisms $H_*(C^{Nov}(M,f,v)) \cong H_*^{Nov}(M,f)$, since for any f.g. free chain complex C over a principal ideal domain R

$$\dim_R(C_i) \geqslant b_i(C) + q_i(C) + q_{i-1}(C)$$

where

$$b_i(C) \;=\; \dim_R\big(H_i(C)/T_i(C)\big)\;,\; q_i(C) \;=\; \# T_i(C)$$

with

$$T_i(C) \;=\; \{x \in H_i(C) \,|\, rx = 0 \text{ for some } r \neq 0 \in R\}$$

the R-torsion submodule of $H_i(C)$, and $\#$ denoting the minimal number of R-module generators.

Farber [5] proved that the Morse-Novikov inequalities are sharp for $\pi_1(M) = \mathbb{Z}$, $m \geqslant 6$: for any such manifold there exists a Morse function $f : M \to S^1$

representing $1 \in [M, S^1] = H^1(M)$ with the minimum possible numbers of critical points

$$c_i(f) \;=\; b_i^{Nov}(M,f) + q_i^{Nov}(M,f) + q_{i-1}^{Nov}(M,f)\ .$$

Again, the method is to start with an arbitrary Morse function $f : M \to S^1$ in the homotopy class, and to systematically cancel pairs of critical points until this is no longer possible.

When does the Novikov homology vanish?

Proposition (Ranicki [28]) Let A be a ring with an automorphism $\alpha : A \to A$. A finite f.g. free $A_\alpha[z, z^{-1}]$-module chain complex C is such that

$$H_*(A_\alpha((z)) \otimes_{A_\alpha[z,z^{-1}]} C) \;=\; H_*(A_\alpha((z^{-1})) \otimes_{A_\alpha[z,z^{-1}]} C) \;=\; 0$$

if and only if C is A-module chain equivalent to a finite f.g. projective A-module chain complex. □

Note that for an algebraic Poincaré complex (C, ϕ)

$$H_*(A_\alpha((z))\otimes_{A_\alpha[z,z^{-1}]}C) \;=\; 0 \text{ if and only if } H_*(A_\alpha((z^{-1}))\otimes_{A_\alpha[z,z^{-1}]}C) \;=\; 0\,,$$

so the two Novikov homology vanishing conditions can be replaced by just one.

Recall that a space X is *finitely dominated* if there exist a finite CW complex and maps $i : X \to K$, $j : K \to X$ such that $ji \simeq 1 : X \to X$. Wall [36] proved that a CW complex X is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the cellular chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex.

In the simply-connected case $\pi_1(\overline{M}) = \{1\}$ the following conditions on a map $f : M \to S^1$ from an m-dimensional manifold M are equivalent :

(i) $\overline{M}$ is finitely dominated,

(ii) $\overline{M}$ is homotopy equivalent to a finite CW complex,

(iii) $H_*^{Nov}(M,f) = 0$,

(iv) $b_i^{Nov}(M,f) = q_i^{Nov}(M,f) = 0$,

(v) $C(\overline{M})$ is chain equivalent to a finite f.g. free $\mathbb{Z}$-module chain complex,

(vi) the homology groups $H_*(\overline{M})$ are f.g. $\mathbb{Z}$-modules.

Browder and Levine [3] used handle exchanges (= the ambient surgery version of the cancellation of adjacent critical points) to prove that (vi) holds if (and for $m \geqslant 6$ only if) $f : M \to S^1$ is homotopic to the projection of a fibre bundle.

Farrell [8] and Siebenmann [34] defined a Whitehead torsion obstruction $\Phi(M,f) \in Wh(\pi_1(M))$ for a map $f : M^m \to S^1$ with finitely dominated $\overline{M} = f^*\mathbb{R}$, such that $\Phi(M,f) = 0$ if (and for $m \geqslant 6$ only if) f is homotopic to the projection of a fibre bundle.

Theorem (Ranicki [28])
(i) *For any finite CW complex M and map $f : M \to S^1$ the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M is finitely dominated if and only if $\pi_1(\overline{M})$ is finitely presented and*

$$H_*^{Nov}(M, f; \mathbb{Z}\widehat{[\pi_1(M)]}) \;=\; 0 \;.$$

(ii) *For any Morse map $f : M \to S^1$ on an m-dimensional manifold M with finitely dominated $\overline{M}$ the torsion of the Novikov complex $\tau(C^{Nov}(M,f,v)) \in K_1(\mathbb{Z}\widehat{[\pi_1(M)]})/I$ determines and is determined by the Farrell-Siebenmann fibering obstruction $\Phi(M,f) \in Wh(\pi_1(M))$, where $I \subseteq K_1(\mathbb{Z}\widehat{[\pi_1(M)]})$ is the subgroup generated by $\pm\pi_1(M)$ and $\tau(1 - zh)$ for square matrices h over $\mathbb{Z}[\pi_1(\overline{M})]$. Thus $\tau(C^{Nov}(M,f,v)) \in I$ if (and for $m \geqslant 6$ only if) f is homotopic to a fibre bundle.* □

See Chapter 22 of Hughes and Ranicki [9] and Chapter 15 of Ranicki [29] for more detailed accounts of the relationship between the torsion of the Novikov complex and the Farrell-Siebenmann fibering obstruction.

See Latour [12] and Pajitnov [22] for circle Morse-theoretic proofs that if $m \geqslant 6$, $\overline{M}$ is finitely dominated and $\tau(C^{Nov}(M,f,v)) \in I$ then it is possible to pairwise cancel all the critical points of f.

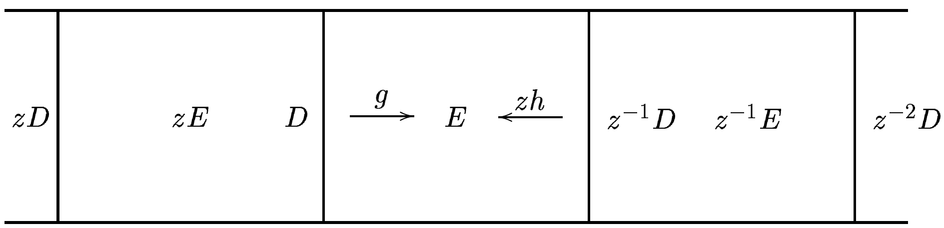
5 The algebraic model for circle valued Morse theory

In many cases the Novikov complex $C^{Nov}(M,f,\alpha)$ of a circle valued Morse function $f : M \to S^1$ can be constructed from an algebraic model for the $\overline{v}$-gradient flow in a fundamental domain of the infinite cyclic cover $\overline{M}$.

An *algebraic fundamental domain* (D, E, F, g, h) consists of finite based f.g. free A-module chain complexes D, E and chain maps $g : D \to E$,

$h : z^{-1}D \to E$ of the form

$$\begin{aligned}
d_E &= \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \to E_{i-1} = D_{i-1} \oplus F_{i-1} , \\
g &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \to E_i = D_i \oplus F_i , \\
h &= \begin{pmatrix} h_D \\ h_F \end{pmatrix} : z^{-1}D_i \to E_i = D_i \oplus F_i .
\end{aligned}$$



Define the *algebraic Novikov complex* $\widehat{F}$ to be the based f.g. free $A_\alpha((z))$-module chain complex with

$$\begin{aligned}
d_{\widehat{F}} &= d_F + zh_F(1 - zh_D)^{-1}c \\
&= d_F + \sum_{j=1}^{\infty} z^j h_F(h_D)^{j-1}c : \widehat{F}_i = (F_i)_\alpha((z)) \to \widehat{F}_{i-1} = (F_{i-1})_\alpha((z)) ,
\end{aligned}$$

as in Farber and Ranicki [7] and Ranicki [30]. The $A_\alpha((z))$-module chain map

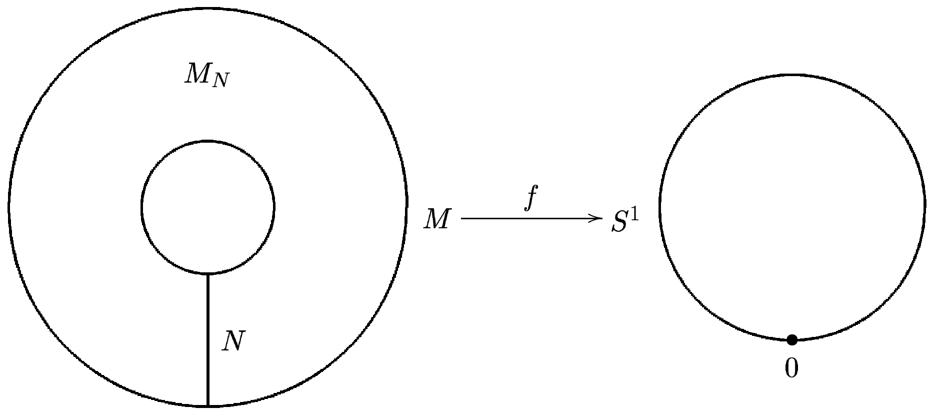
$$\phi = g - zh = \begin{pmatrix} 1 - zh_D \\ -zh_F \end{pmatrix} : D_\alpha((z)) \to E_\alpha((z))$$

is a split injection in each degree (since $1 - zh_D$ is an isomorphism), and the inclusions $F_i \to E_i$ determine a canonical isomorphism of based f.g. free $A_\alpha[z, z^{-1}]$-module chain complexes

$$\widehat{F} \cong \operatorname{coker}(\phi) .$$

Here is how algebraic fundamental domains and the algebraic Novikov complex arise in topology.

Let $f : M \to S^1$ be a Morse function with regular value $0 \in S^1$.



Cut M along the inverse image

$$N^{m-1} \;=\; f^{-1}(0) \subset M$$

to obtain a geometric fundamental domain

$$(M_N; N, z^{-1}N) \;=\; \overline{f}^{-1}([0,1];\{0\},\{1\})$$

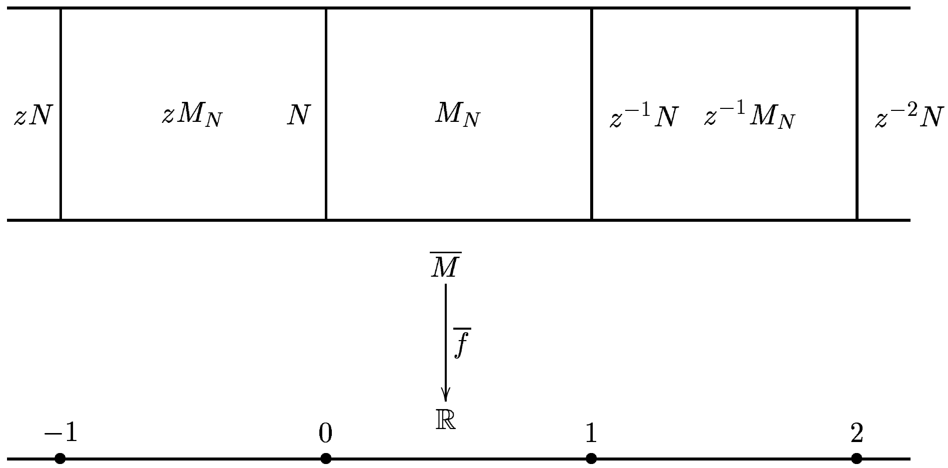
for the infinite cyclic cover

$$\overline{M} \;=\; f^*\mathbb{R} \;=\; \bigcup_{j=-\infty}^{\infty} z^j M_N \ .$$

The restriction

$$f_N \;=\; \overline{f}| \;:\; (M_N; N, z^{-1}N) \to ([0,1];\{0\},\{1\})$$

is a real valued Morse function with $v_N = \overline{v}| \in \mathcal{GT}(f_N)$.



The cobordism $(M_N; N, z^{-1}N)$ has a handlebody decomposition

$$M_N \;=\; N \times [0,1] \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i}$$

with one i-handle for each index i critical point of f. Given a CW structure on N with $c_i(N)$ i-cells use this handlebody decomposition to define a CW structure on M_N with $c_i(N) + c_i(f)$ i-cells. A regular cover $\widetilde{M}$ of $\overline{M}$ with group of covering translations π is a regular cover of M with group of covering translations $\Pi = \pi \times_\alpha \mathbb{Z}$ (as before), with

$$\mathbb{Z}[\Pi] \;=\; \mathbb{Z}[\pi]_\alpha[z, z^{-1}] \;\;,\;\; \widehat{\mathbb{Z}[\Pi]} \;=\; \mathbb{Z}[\Pi]_\alpha((z)) \;.$$

Use a cellular approximation $h : z^{-1}N \to M_N$ to the inclusion to define an algebraic fundamental domain (D, E, F, g, h) over $A = \mathbb{Z}[\pi]$

$$D \;=\; C(\widetilde{N}) \;\;,\;\; E \;=\; C(\widetilde{M}_N) \;\;,\;\; F \;=\; C^{MS}(M_N, f_N, v_N) \;=\; C(\widetilde{M}_N, \widetilde{N}) \;.$$

The mapping cylinder of $h : N \to M_N$ is a CW complex M'_N with two copies of N as subcomplexes. Identifying these copies there is obtained a CW complex structure on M with $\widehat{\mathbb{Z}[\Pi]}$-coefficient cellular chain complex

$$C(M; \widehat{\mathbb{Z}[\Pi]}) \;=\; \mathcal{C}(\phi)$$

the algebraic mapping cone of the $\widehat{\mathbb{Z}[\Pi]}$-module chain map

$$\phi \ = \ g - zh \ : \ D_\alpha((z)) \to E_\alpha((z)) \ ,$$

with

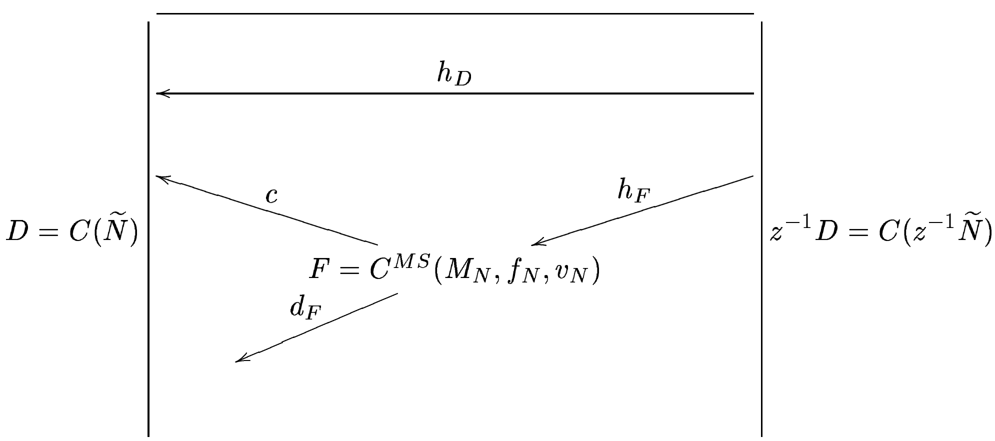
$$d_{\mathcal{C}(\phi)} \ = \ \begin{pmatrix} -d_D & 0 & 0 \\ 1 - zh_D & d_D & c \\ -zh_F & 0 & d_D \end{pmatrix} \ :$$

$$\mathcal{C}(\phi)_i \ = \ (D_{i-1} \oplus D_i \oplus F_i)_\alpha[z, z^{-1}] \to \mathcal{C}(\phi)_{i-1} \ = \ (D_{i-2} \oplus D_{i-1} \oplus F_{i-1})_\alpha[z, z^{-1}] \ .$$

The algebraic Novikov complex $\widehat{F} = \operatorname{coker}(\phi)$ is a based f.g. free $\widehat{\mathbb{Z}[\Pi]}$-module chain complex such that

$$\dim_{\widehat{\mathbb{Z}[\Pi]}} \widehat{F}_i \ = \ c_i(f) \ .$$

In many cases $\widehat{F} = C^{Nov}(M, f, v)$, and in even more cases $\widehat{F}$ is simple isomorphic to $C^{Nov}(M, f, v)$.



The philosophy here is that $\mathcal{C}(\phi)$ counts the $\overline{v}$-gradient flow lines of $\overline{f} : \overline{M} \to \mathbb{R}$, as follows :

(i) the $(z^{-1}p, q)$-coefficient of $h_D : z^{-1}D_i \to D_i$ counts the number of portions in M_N of the $\overline{v}$-gradient flow lines which start in $z^{-1}M_N$, enter M_N at $z^{-1}p \in z^{-1}N$, exit at $q \in N$ and end in zM_N,

(ii) the $(z^{-1}p, q)$-coefficient of $h_F : z^{-1}D_i \to F_i$ counts the number of portions in M_N of the $\overline{v}$-gradient flow lines which start in $z^{-1}M_N$, enter M_N at $z^{-1}p \in z^{-1}N$ and end at $q \in M_N$,

(iii) the (p,q)-coefficient of $c : F_i \to D_{i-1}$ counts the number of portions in M_N of the $\overline{v}$-gradient flow lines which start at $p \in M_N$, exit at $q \in N$, and end in zM_N.

Then for $j = 1,2,3,\ldots$ the $(p, z^j q)$-coefficient of $h_F(h_D)^{j-1}c : F_i \to z^j F_i$ is the number of the $\overline{v}$-gradient flow lines which start at $p \in M_N$ and end at $z^j q \in z^j M_N$, crossing the walls $N, zN, \ldots, z^{j-1}N$. If such is the case, i.e. if the chain map h is *gradient-like* in the terminology of Ranicki [30], this is just the $(p, z^j q)$-coefficient of $d_{C^{Nov}(M,f,v)}$, so $\widehat{F} = C^{Nov}(M,f,\alpha)$. Pajitnov [24] constructed a C^0-dense subspace $\mathcal{GCCT}(f) \subset \mathcal{GT}(f)$ of gradient-like vector fields v for which there exist a CW structure N and a gradient-like chain map h. Cornea and Ranicki [4] construct for any $v \in GT(f)$ a Morse function $f' : M \to S^1$ arbitrarily close to f with $v' \in \mathcal{GT}(f')$ such that

$$C^{Nov}(M, f', v') \;=\; \mathcal{C}(\phi) \;.$$

The projection

$$p \;:\; C(M; \widehat{\mathbb{Z}[\Pi]}) \;=\; \mathcal{C}(\phi) \to \mathrm{coker}(\phi) \;\cong\; \widehat{F}$$

is a chain equivalence of based f.g. free $\widehat{\mathbb{Z}[\Pi]}$-module chain complexes, with torsion

$$\tau(p) \;=\; \sum_{i=0}^{\infty} (-)^i \tau\big(1 - zh_D : (D_i)_\alpha((z)) \to (D_i)_\alpha((z))\big) \in K_1(\widehat{\mathbb{Z}[\Pi]}) \;.$$

If h is a gradient-like chain map the torsion of p is a measure of the number of closed orbits of the v-gradient flow in M, i.e. the closed flow lines $\gamma : S^1 \to M$ (Hutchings and Lee [10],[11], Pajitnov [24],[25], Schütz [31],[32]).

The algebraic surgery treatment of high-dimensional knot theory in Ranicki [29] gives the following algebraic model for the circle valued Morse function on a knot complement.

Example. Let $k : S^n \subset S^{n+2}$ be a knot with $\pi_1(S^{n+2} \backslash k(S^n)) = \mathbb{Z}$. The complement of a tubular neighbourhood $k(S^n) \times D^2 \subset S^{n+2}$ is an $(n+2)$-dimensional manifold with boundary

$$(M, \partial M) \;=\; (\mathrm{cl.}\big(S^{n+2} \backslash (k(S^n) \times D^2)\big), k(S^n) \times S^1)$$

with

$$\pi_1(M) \;=\; \mathbb{Z} \;,\; \pi_1(\overline{M}) \;=\; \{1\} \;,\; H_*(M) \;=\; H_*(S^1) \;.$$

Let $f : (M, \partial M) \to S^1$ be a map representing $1 \in H^1(M) = \mathbb{Z}$, with $f| : \partial M \to S^1$ the projection. Making f transverse regular at $0 \in S^1$ there is obtained a Seifert surface $N^{n+1} = f^{-1}(0) \subset M$ for k, with $\partial N = k(S^n)$. As before, cut M along N to obtain a fundamental domain $(M_N; N, z^{-1}N)$ for the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M. For any CW structures on N, M_N write the reduced chain complexes as

$$\dot{C}(N) \;=\; C(N,\{\mathrm{pt.}\}) \;\;,\;\; \dot{C}(M_N) \;=\; C(M_N,\{\mathrm{pt.}\}) \;.$$

The inclusions $G : N \to M_N$, $H : z^{-1}N \to M_N$ induce $\mathbb{Z}$-module chain maps

$$G \;:\; \dot{C}(N) \to \dot{C}(M_N) \;,\; H \;:\; z^{-1}\dot{C}(N) \to \dot{C}(M_N)$$

such that $G - H : \dot{C}(N) \to \dot{C}(M_N)$ is a chain equivalence. The chain map

$$e \;=\; (G-H)^{-1}G \;:\; \dot{C}(N) \to \dot{C}(N)$$

is a generalization of the Seifert matrix, such that up to $\mathbb{Z}$-module chain homotopy

$$1 - e \;=\; -(G-H)^{-1}H \;:\; \dot{C}(N) \to \dot{C}(N)$$

and such that there is defined a $\mathbb{Z}[z,z^{-1}]$-module chain equivalence

$$C(\overline{M},\mathbb{R}) \;\simeq\; \mathcal{C}\big(e + z(1-e) : \dot{C}(N)[z,z^{-1}] \to \dot{C}(N)[z,z^{-1}]\big) \;.$$

Let $\dot{N} = \mathrm{cl.}(N \backslash D^{n+1})$, for any embedding $D^{n+1} \subset N \backslash \partial N$. For any handlebody decomposition of the $(n+1)$-dimensional cobordism $(\dot{N}; k(S^n), S^n)$ with $c_i(N)$ i-handles

$$\dot{N} \;=\; k(S^n) \times [0,1] \cup \bigcup_{i=1}^{n} \bigcup_{c_i(N)} D^i \times D^{n+1-i}$$

there exists a Morse function $f : M \to S^1$ in the homotopy class $1 \in [M,S^1] = H^1(M) = \mathbb{Z}$ with

$$c_i(f) \;=\; c_i(N) + c_{i-1}(N)$$

critical points of index i. In this case the algebraic model for $C^{Nov}(M,f,v)$ has

$$\begin{aligned}
D \;&=\; C(N) \;=\; \mathbb{Z} \oplus \dot{D} \;,\; \dot{D}_i \;=\; \mathbb{Z}^{c_i(N)} \;, \\
F \;&=\; C^{MS}(M_N, f_N, v_N) \;=\; \mathcal{C}(e : \dot{D} \to \dot{D}) \;, \\
d_F \;&=\; \begin{pmatrix} d_{\dot{D}} & e \\ 0 & -d_{\dot{D}} \end{pmatrix} \;:\; F_i \;=\; \dot{D}_i \oplus \dot{D}_{i-1} \to F_{i-1} \;=\; \dot{D}_{i-1} \oplus \dot{D}_{i-2} \;, \\
c \;&=\; \begin{pmatrix} 0 & 1 \end{pmatrix} \;:\; F_i \;=\; \dot{D}_i \oplus \dot{D}_{i-1} \to D_{i-1} \;, \\
h_D \;&=\; 0 \;:\; z^{-1}D_i \to D_i \;, \\
h_F \;&=\; \begin{pmatrix} 1-e \\ 0 \end{pmatrix} \;:\; z^{-1}D_i \to F_i \;=\; \dot{D}_i \oplus \dot{D}_{i-1}
\end{aligned}$$

with algebraic Novikov complex

$$\begin{aligned} d_{\widehat{F}} &= d_F + \sum_{j=1}^{\infty} z^j h_F (h_D)^{j-1} c \\ &= \begin{pmatrix} d_{\dot{D}} & e + z(1-e) \\ 0 & -d_{\dot{D}} \end{pmatrix} : \widehat{F}_i = (\dot{D}_i \oplus \dot{D}_{i-1})((z)) \to \widehat{F}_{i-1} \\ &\qquad\qquad\qquad\qquad\qquad = (\dot{D}_{i-1} \oplus \dot{D}_{i-2})((z)) \end{aligned}$$

such that $H_*(\widehat{F}) = H_*^{Nov}(M, f)$. The short exact sequences of $\mathbb{Z}((z))$-modules

$$0 \to H_i(N)((z)) \xrightarrow{e+z(1-e)} H_i(N)((z)) \to H_i^{Nov}(M, f) \to 0$$

can be used to express the Novikov numbers $b_i^{Nov}(M, f), q_i^{Nov}(M, f)$ of the knot complement in terms of the Alexander polynomials

$$\Delta_i(z) = \det(e+z(1-e) : H_i(N)[z, z^{-1}] \to H_i(N)[z, z^{-1}]) \in \mathbb{Z}[z, z^{-1}]\ (1 \leqslant i \leqslant n)\ ,$$

generalizing the case $n = 1$ due to Lazarev [13]. For $n \geqslant 4$ and $\pi_1(M) = \mathbb{Z}$ the following conditions are equivalent :

(i) the knot fibres, i.e. $f : M \to S^1$ is homotopic to the projection of a fibre bundle, with no critical points,

(ii) $b_*^{Nov}(M, f) = q_*^{Nov}(M, f) = 0$, i.e. $H_*^{Nov}(M, f) = 0$,

(iii) the constant and leading coefficients of $\Delta_*(z)$ are $\pm 1 \in \mathbb{Z}$. □

There is also a more refined version of the algebraic model for circle valued Morse theory, using the noncommutative Cohn localization $\Sigma^{-1}A_\alpha[z, z^{-1}]$ of $A_\alpha[z, z^{-1}]$ inverting the set Σ of square matrices of the form $1 - zh$ for a square matrix h over A. Indeed, the formula for the differentials in the algebraic Novikov complex

$$d_{\widehat{F}} = d_F + zh_F(1 - zh_D)^{-1}c$$

is already defined in $\Sigma^{-1}A_\alpha[z, z^{-1}]$. See Farber and Ranicki [7] and Ranicki [30] for further details of the construction. Farber [6] applied the refinement to obtain improvements of the Morse-Novikov inequalities, using homology with coefficients in flat line bundles instead of Novikov homology. It should be noted that the natural morphism $\Sigma^{-1}A_\alpha[z, z^{-1}] \to A_\alpha((z))$ is injective for commutative A with $\alpha = 1$, but it is not injective in general (Sheiham [33]).

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Operator Algebras and Topology

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Lectures given at the
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Abstract

These notes, based on three lectures on operator algebras and topology at the "School on High Dimensional Manifold Theory" at the ICTP in Trieste, introduce a new set of tools to high dimensional manifold theory, namely techniques coming from the theory of operator algebras, in particular C^*-algebras. These are extensively studied in their own right. We will focus on the basic definitions and properties, and on their relevance to the geometry and topology of manifolds.

A central pillar of work in the theory of C^*-algebras is the Baum-Connes conjecture. This is an isomorphism conjecture, as discussed in the talks of Lück, but with a certain special flavor. Nevertheless, it has important direct applications to the topology of manifolds, it implies e.g. the Novikov conjecture. In the first chapter, the Baum-Connes conjecture will be explained and put into our context.

Another application of the Baum-Connes conjecture is to the positive scalar curvature question. This will be discussed by Stephan Stolz. It implies the so-called "stable Gromov-Lawson-Rosenberg conjecture". The unstable version of this conjecture said that, given a closed spin manifold M, a certain obstruction, living in a certain (topological) K-theory group, vanishes if and only M admits a Riemannian metric with positive scalar curvature. It turns out that this is wrong, and counterexamples will be presented in the second chapter.

The third chapter introduces another set of invariants, also using operator algebra techniques, namely L^2-cohomology, L^2-Betti numbers and other L^2-invariants. These invariants, their basic properties, and the central questions about them, are introduced in the third chapter.

Several people contributed to these notes by reading preliminary parts and suggesting improvements, in particular Marc Johnson, Roman Sauer, Marco Varisco und Guido Mislin. I am very indebted to all of them.

Contents

Chapter 1

Index theory and the Baum-Connes conjecture

1.1 Index theory

The Atiyah-Singer index theorem is one of the great achievements of modern mathematics. It gives a formula for the index of a differential operator (the index is by definition the dimension of the space of its solutions minus the dimension of the solution space for its adjoint operator) in terms only of topological data associated to the operator and the underlying space. There are many good treatments of this subject available, apart from the original literature (most found in [2]). Much more detailed than the present notes can be, because of constraints of length and time, are e.g. [44, 7, 32].

1.1.1 Elliptic operators and their index

We quickly review what type of operators we are looking at.

1.1.1. Definition. Let M be a smooth manifold of dimension m; E, F smooth (complex) vector bundles on M. A *differential operator* (of order d) from E to F is a $\mathbb{C}$-linear map from the space of smooth sections $C^\infty(E)$ of E to the space of smooth sections of F:

$$D\colon C^\infty(E) \to C^\infty(F),$$

such that in local coordinates and with local trivializations of the bundles it can be written in the form

$$D = \sum_{|\alpha| \le d} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here $A_\alpha(x)$ is a matrix of smooth complex valued functions, $\alpha = (\alpha_1, \dots, \alpha_m)$ is an m-tuple of non-negative integers and $|\alpha| = \alpha_1 + \dots + \alpha_m$. $\partial^{|\alpha|}/\partial x^\alpha$ is an abbreviation for $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}$. We require that $A_\alpha(x) \neq 0$ for some α with $|\alpha| = d$ (else, the operator is of order strictly smaller than d).

Let $\pi \colon T^*M \to M$ be the bundle projection of the cotangent bundle of M. We get pull-backs π^*E and π^*F of the bundles E and F, respectively, to T^*M.

The *symbol* $\sigma(D)$ of the differential operator D is the section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ on T^*M defined as follows:

In the above local coordinates, using $\xi = (\xi_1, \dots, \xi_m)$ as coordinate for the cotangent vectors in T^*M, in the fiber of (x, ξ), the symbol $\sigma(D)$ is given by multiplication with

$$\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}$.

The operator D is called *elliptic*, if $\sigma(D)_{(x,\xi)} \colon \pi^*E_{(x,\xi)} \to \pi^*F_{(x,\xi)}$ is invertible outside the zero section of T^*M, i.e. in each fiber over $(x,\xi) \in T^*M$ with $\xi \neq 0$. Observe that elliptic operators can only exist if the fiber dimensions of E and F coincide.

In other words, the symbol of an elliptic operator gives us two vector bundles over T^*M, namely π^*E and π^*F, together with a choice of an isomorphism of the fibers of these two bundles outside the zero section. If M is compact, this gives an element of the relative K-theory group $K^0(DT^*M, ST^*M)$, where DT^*M and ST^*M are the disc bundle and sphere bundle of T^*M, respectively (with respect to some arbitrary Riemannian metric).

Recall the following definition:

1.1.2. Definition. Let X be a compact topological space. We define the K-theory of X, $K^0(X)$, to be the Grothendieck group of (isomorphism classes of) complex vector bundles over X (with finite fiber dimension). More precisely, $K^0(X)$ consists of equivalence classes of pairs (E, F) of (isomorphism classes of) vector bundles over X, where $(E, F) \sim (E', F')$ if and only if there exists another vector bundle G on X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. One often writes $[E] - [F]$ for the element of $K^0(X)$ represented by (E, F).

Let Y now be a closed subspace of X. The *relative K-theory* $K^0(X, Y)$ is given by equivalence classes of triples (E, F, ϕ), where E and F are complex vector bundles over X, and $\phi \colon E|_Y \to F|_Y$ is a given isomorphism between the restrictions of E and F to Y. Then (E, F, ϕ) is isomorphic to (E', F', ϕ')

if we find isomorphisms $\alpha\colon E \to E'$ and $\beta\colon F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} E|_Y & \xrightarrow{\phi} & F|_Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ E'|_Y & \xrightarrow{\phi'} & F'|_Y \end{array}$$

Two pairs (E, F, ϕ) and (E', F', ϕ') are equivalent, if there is a bundle G on X such that $(E \oplus G, F \oplus G, \phi \oplus \mathrm{id})$ is isomorphic to $(E' \oplus G, F' \oplus G, \phi' \oplus \mathrm{id})$.

1.1.3. Example. The element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic differential operator D mentioned above is represented by the restriction of the bundles π^*E and π^*F to the disc bundle DT^*M, together with the isomorphism $\sigma(D)_{(x,\xi)}\colon E_{(x,\xi)} \to F_{(x,\xi)}$ for $(x, \xi) \in ST^*M$.

1.1.4. Example. Let $M = \mathbb{R}^m$ and $D = \sum_{i=1}^m (\partial/\partial_i)^2$ be the Laplace operator on functions. This is an elliptic differential operator, with symbol $\sigma(D) = \sum_{i=1}^m \xi_i^2$.

More generally, a second-order differential operator $D\colon C^\infty(E) \to C^\infty(E)$ on a Riemannian manifold M is a *generalized Laplacian*, if $\sigma(D)_{(x,\xi)} = |\xi|^2 \cdot \mathrm{id}_{E_x}$ (the norm of the cotangent vector $|\xi|$ is given by the Riemannian metric).

Notice that all generalized Laplacians are elliptic.

1.1.5. Definition. *(Adjoint operator)*
Assume that we have a differential operator $D\colon C^\infty(E) \to C^\infty(F)$ between two Hermitian bundles E and F on a Riemannian manifold (M, g). We define an L^2-inner product on $C^\infty(E)$ by the formula

$$\langle f, g \rangle_{L^2(E)} := \int_M \langle f(x), g(x) \rangle_{E_x} \, d\mu(x) \qquad \forall f, g \in C_0^\infty(E),$$

where $\langle \cdot, \cdot \rangle_{E_x}$ is the fiber-wise inner product given by the Hermitian metric, and $d\mu$ is the measure on M induced from the Riemannian metric. Here C_0^∞ is the space of smooth section with compact support. The Hilbert space completion of $C_0^\infty(E)$ with respect to this inner product is called $L^2(E)$.

The *formal adjoint* D^* of D is then defined by

$$\langle Df, g \rangle_{L^2(F)} = \langle f, D^*g \rangle_{L^2(E)} \qquad \forall f \in C_0^\infty(E),\ g \in C_0^\infty(F).$$

It turns out that exactly one operator with this property exists, which is another differential operator, and which is elliptic if and only if D is elliptic.

1.1.6. Remark. The class of differential operators is quite restricted. Many constructions one would like to carry out with differential operators automatically lead out of this class. Therefore, one often has to use *pseudodifferential operators*. Pseudodifferential operators are defined as a generalization of differential operators. There are many well written sources dealing with the theory of pseudodifferential operators. Since we will not discuss them in detail here, we omit even their precise definition and refer e.g. to [44] and [78]. What we have done so far with elliptic operators can all be extended to pseudodifferential operators. In particular, they have a symbol, and the concept of ellipticity is defined for them. When studying elliptic differential operators, pseudodifferential operators naturally appear and play a very important role. An pseudodifferential operator P (which could e.g. be a differential operator) is elliptic if and only if a pseudodifferential operator Q exists such that $PQ - \mathrm{id}$ and $QP - \mathrm{id}$ are so called *smoothing* operators, a particularly nice class of pseudodifferential operators. For many purposes, Q can be considered to act like an inverse of P, and this kind of invertibility is frequently used in the theory of elliptic operators. However, if P happens to be an elliptic differential operator of positive order, then Q necessarily is not a differential operator, but only a pseudodifferential operator.

It should be noted that almost all of the results we present here for differential operators hold also for pseudodifferential operators, and often the proof is best given using them.

We now want to state several important properties of elliptic operators.

1.1.7. Theorem. *Let M be a smooth manifold, E and F smooth finite dimensional vector bundles over M. Let $P\colon C^\infty(E) \to C^\infty(F)$ be an elliptic operator.*

Then the following holds.

(1) *Elliptic regularity:*
If $f \in L^2(E)$ is weakly in the null space of P, i.e. $\langle f, P^*g\rangle_{L^2(E)} = 0$ for all $g \in C_0^\infty(F)$, then $f \in C^\infty(E)$.

(2) *Decomposition into finite dimensional eigenspaces:*
Assume M is compact and $P = P^*$ (in particular, $E = F$). Then the set $s(P)$ of eigenvalues of P (P acting on $C^\infty(E)$) is a discrete subset of $\mathbb{R}$, each eigenspace e_λ ($\lambda \in s(P)$) is finite dimensional, and $L^2(E) = \oplus_{\lambda\in s(P)} e_\lambda$ (here we use the completed direct sum in the sense of Hilbert spaces, which means by definition that the algebraic direct sum is dense in $L^2(E)$).

(3) *If M is compact, then* $\ker(P)$ *and* $\ker(P^*)$ *are finite dimensional, and*

then we define the index of P

$$\operatorname{ind}(P) := \dim_{\mathbb{C}} \ker(P) - \dim_{\mathbb{C}} \ker(P^*).$$

(Here, we could replace $\ker(P^*)$ *by* $\operatorname{coker}(P)$*, because these two vector spaces are isomorphic).*

1.1.2 Statement of the Atiyah-Singer index theorem

There are different variants of the Atiyah-Singer index theorem. We start with a cohomological formula for the index.

1.1.8. Theorem. *Let M be a compact oriented manifold of dimension m, and $D\colon C^\infty(E) \to C^\infty(F)$ an elliptic operator with symbol $\sigma(D)$. There is a characteristic (inhomogeneous) cohomology class $\operatorname{Td}(M) \in H^*(M;\mathbb{Q})$ of the tangent bundle of M (called the complex Todd class of the complexified tangent bundle). Moreover, to the symbol is associated a certain (inhomogeneous) cohomology class $\pi_! \operatorname{ch}(\sigma(D)) \in H^*(M;\mathbb{Q})$ such that*

$$\operatorname{ind}(D) = (-1)^{m(m+1)/2} \langle \pi_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(M), [M] \rangle.$$

The class $[M] \in H_m(M;\mathbb{Q})$ is the fundamental class of the oriented manifold M, and $\langle \cdot, \cdot \rangle$ is the usual pairing between homology and cohomology.

If we start with specific operators given by the geometry, explicit calculation usually give more familiar terms on the right hand side.

For example, for the signature operator we obtain Hirzebruch's signature formula expressing the signature in terms of the L-class, for the Euler characteristic operator we obtain the Gauss-Bonnet formula expressing the Euler characteristic in terms of the Pfaffian, and for the spin or spin^c Dirac operator we obtain an $\hat{A}$-formula. For applications, these formulas prove to be particularly useful.

We give some more details about the signature operator, which we are going to use later again. To define the signature operator, fix a Riemannian metric g on M. Assume $\dim M = 4k$ is divisible by four.

The signature operator maps from a certain subspace Ω^+ of the space of differential forms to another subspace Ω^-. These subspaces are defined as follows. Define, on p-forms, the operator $\tau := i^{p(p-1)+2k} *$, where $*$ is the Hodge-$*$ operator given by the Riemannian metric, and $i^2 = -1$. Since $\dim M$ is divisible by 4, an easy calculation shows that $\tau^2 = \mathrm{id}$. We then define $\Omega^\pm$ to be the ± 1 eigenspaces of τ.

The signature operator D_{sig} is now simply defined to by $D_{sig} := d + d^*$, where d is the exterior derivative on differential forms, and $d^* = \pm * d *$ is its formal adjoint. We restrict this operator to Ω^+, and another easy calculation shows that Ω^+ is mapped to Ω^-. D_{sig} is elliptic, and a classical calculation shows that its index is the signature of M given by the intersection form in middle homology.

1.1.9. Definition. The *Hirzebruch L-class* as normalized by Atiyah and Singer is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space X a cohomology class $L(E) \in H^*(X;\mathbb{Q})$. It is characterized by the following properties:

(1) Naturality: for any map $f\colon Y \to X$ we have $L(f^*E) = f^*L(E)$.

(2) Normalization: If L is a complex line bundle with first Chern class x, then

$$L(E) = \frac{x/2}{\tanh(x/2)} = 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \in H^*(X;\mathbb{Q}).$$

(3) Multiplicativity: $L(E \oplus F) = L(E)L(F)$.

It turns out that L is a *stable* characteristic class, i.e. $L(E) = 1$ if E is a trivial bundle. This implies that L defines a map from the K-theory $K^0(X) \to H^*(X;\mathbb{Q})$.

The Atiyah-Singer index theorem now specializes to

$$\operatorname{sign}(M) = \operatorname{ind}(D_{sig}) = \langle 2^{2k}L(TM), [M]\rangle,$$

with $\dim M = 4k$ as above.

1.1.10. Remark. One direction to generalize the Atiyah-Singer index theorem is to give an index formula for manifolds with boundary. Indeed, this is achieved in the Atiyah-Patodi-Singer index theorem. However, these results are much less topological than the results for manifolds without boundary. They are not discussed in these notes.

Next, we explain the K-theoretic version of the Atiyah-Singer index theorem. It starts with the element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic operator. Given any compact manifold M, there is a well defined homomorphism

$$K^0(DT^*M, ST^*M) \to K^0(*) = \mathbb{Z},$$

constructed by embedding T^*M into high dimensional Euclidean space, then using a transfer map and Bott periodicity. The image of the symbol element under this homomorphism is denoted the *topological index* $\operatorname{ind}_t(D) \in K^0(*) = \mathbb{Z}$. The reason for the terminology is that it is obtained from the symbol only, using purely topological constructions. Now, the Atiyah-Singer index theorem states

1.1.11. Theorem. $\mathrm{ind}_t(D) = \mathrm{ind}(D)$.

1.1.3 The G-index

Let G be a finite group, or more generally a compact Lie group. The representation ring RG of G is defined to be the Grothendieck group of all finite dimensional complex representations of G, i.e. an element of RG is a formal difference $[V] - [W]$ of two finite dimensional G-representations V and W, and we have $[V] - [W] = [X] - [Y]$ if and only if $V \oplus Y \cong W \oplus X$ (strictly speaking, we have to pass to isomorphism classes of representations to avoid set theoretical problems). The direct sum of representations induces the structure of an abelian group on RG, and the tensor product makes it a commutative unital ring (the unit given by the trivial one-dimensional representation). More about this representation ring can be found e.g. in [11].

Assume now that the manifold M is a compact smooth manifold with a smooth G-action, and let E, F be complex G-vector bundles on M (this means that G acts on E and F by vector bundle automorphisms (i.e. carries fibers to fibers linearly), and the bundle projection maps are G-equivariant).

Let $D\colon C^\infty(E) \to C^\infty(F)$ be a G-equivariant elliptic differential operator.

This implies that $\ker(D)$ and $\mathrm{coker}(D)$ inherit a G-action by restriction, i.e. are finite dimensional G-representations. We define the (analytic) G-index of D to be

$$\mathrm{ind}^G(D) := [\ker(D)] - [\mathrm{coker}(D)] \in RG.$$

If G is the trivial group then $RG \cong \mathbb{Z}$ in a canonical way, and then $\mathrm{ind}^G(D)$ coincides with the usual index of D.

We can also define a *topological* equivariant index similar to the non-equivariant topological index, using transfer maps and Bott periodicity. This topological index lives in the G-equivariant K-theory of a point, which is canonically isomorphic to the representation ring RG. Again, the Atiyah-Singer index theorem says

1.1.12. Theorem. $\mathrm{ind}^G(D) = \mathrm{ind}_t^G(D) \in K_G^0(*) = RG$.

1.1.4 Families of operators and their index

Another generalization is given if we don't look at one operator on one manifold, but a family of operators on a family of manifolds. More precisely, let X be any compact topological space, $Y \to X$ a locally trivial fiber bundle with fiber M a smooth compact manifold, and structure group the diffeomorphisms of M. Let E, F be families of smooth vector bundles on Y (i.e. vector bundles which are fiber-wise smooth), and $C^\infty(E)$, $C^\infty(F)$ the continuous sections which are smooth along the fibers. Assume that $D\colon C^\infty(E) \to C^\infty(F)$ is a family $\{D_x\}$ of elliptic differential operator along the fiber $Y_x \cong M$ $(x \in X)$, i.e., in local coordinates D becomes

$$\sum_{|\alpha|\le m} A_\alpha(y,x)\frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

with $y \in M$ and $x \in X$ such that $A_\alpha(y,x)$ depends continuously on x, and each D_x is an elliptic differential operator on Y_x.

If $\dim_{\mathbb{C}} \ker(D_x)$ is independent of $x \in X$, then all of these vector spaces patch together to give a vector bundle called $\ker(D)$ on X, and similarly for the (fiber-wise) adjoint D^*. This then gives a K-theory element $[\ker(D)] - [\ker(D^*)] \in K^0(X)$.

Unfortunately, it does sometimes happen that these dimensions jump. However, using appropriate perturbations, one can always define the K-theory element

$$\operatorname{ind}(D) := [\ker(D)] - [\ker(D^*)] \in K^0(X),$$

the analytic index of the family of elliptic operators D.

There is also a family version of the construction of the topological index, giving $\operatorname{ind}_t(D) \in K^0(X)$. The Atiyah-Singer index theorem for families states:

1.1.13. Theorem. $\operatorname{ind}(D) = \operatorname{ind}_t(D) \in K^0(X)$.

The upshot of the discussion of this and the last section (for the details the reader is referred to the literature) is that the natural receptacle for the index of differential operators in various situations are appropriate K-theory groups, and much of todays index theory deals with investigating these K-theory groups.

1.2 Survey on C^*-algebras and their K-theory

More detailed references for this section are, among others, [88], [32], and [8].

1.2.1 C^*-algebras

1.2.1. Definition. A *Banach algebra* A is a complex algebra which is a complete normed space, and such that $|ab| \le |a|\,|b|$ for each $a, b \in A$.

A $*$-*algebra* A is a complex algebra with an anti-linear involution $*\colon A \to A$ (i.e. $(\lambda a)^* = \overline{\lambda} a^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$ for all $a, b \in A$).

A *Banach* $*$-*algebra* A is a Banach algebra which is a $*$-algebra such that $|a^*| = |a|$ for all $a \in A$.

A C^*-*algebra* A is a Banach $*$-algebra which satisfies $|a^* a| = |a|^2$ for all $a \in A$.

Alternatively, a C^*-algebra is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed subalgebra of the algebra of bounded operators on some Hilbert space H (this is the Gelfand-Naimark representation theorem, compare e.g. [32, 1.6.2]).

A C^*-algebra A is called separable if there exists a countable dense subset of A.

1.2.2. Example. If X is a compact topological space, then $C(X)$, the algebra of complex valued continuous functions on X, is a commutative C^*-algebra (with unit). The adjoint is given by complex conjugation: $f^*(x) = \overline{f(x)}$, the norm is the supremum-norm.

Conversely, it is a theorem that every abelian unital C^*-algebra is isomorphic to $C(X)$ for a suitable compact topological space X [32, Theorem 1.3.12].

Assume X is locally compact, and set

$$C_0(X) := \{f\colon X \to \mathbb{C} \mid f \text{ continuous}, f(x) \xrightarrow{x\to\infty} 0\}.$$

Here, we say $f(x) \to 0$ for $x \to \infty$, or$f$ *vanishes at infinity*, if for all $\epsilon > 0$ there is a compact subset K of X with $|f(x)| < \epsilon$ whenever $x \in X - K$. This is again a commutative C^*-algebra (we use the supremum norm on $C_0(X)$), and it is unital if and only if X is compact (in this case, $C_0(X) = C(X)$).

1.2.2 K_0 of a ring

Suppose R is an arbitrary ring with 1 (not necessarily commutative). A module M over R is called finitely generated projective, if there is another R-module N and a number $n \ge 0$ such that

$$M \oplus N \cong R^n.$$

This is equivalent to the assertion that the matrix ring $M_n(R) = End_R(R^n)$ contains an idempotent e, i.e. with $e^2 = e$, such that M is isomorphic to the image of e, i.e. $M \cong eR^n$.

1.2.3. Example. Description of projective modules.

(1) If R is a field, the finitely generated projective R-modules are exactly the finite dimensional vector spaces. (In this case, every module is projective).

(2) If $R = \mathbb{Z}$, the finitely generated projective modules are the free abelian groups of finite rank

(3) Assume X is a compact topological space and $A = C(X)$. Then, by the Swan-Serre theorem [84], M is a finitely generated projective A-module if and only if M is isomorphic to the space $\Gamma(E)$ of continuous sections of some complex vector bundle E over X.

1.2.4. Definition. Let R be any ring with unit. $K_0(R)$ is defined to be the Grothendieck group of finitely generated projective modules over R, i.e. the group of equivalence classes $[(M,N)]$ of pairs of (isomorphism classes of) finitely generated projective R-modules M, N, where $(M,N) \equiv (M',N')$ if and only if there is an $n \geq 0$ with

$$M \oplus N' \oplus R^n \cong M' \oplus N \oplus R^n.$$

The group composition is given by

$$[(M,N)] + [(M',N')] := [(M \oplus M', N \oplus N')].$$

We can think of (M,N) as the formal difference of modules $M - N$.

Any unital ring homomorphism $f\colon R \to S$ induces a map

$$f_*\colon K_0(R) \to K_0(S)\colon [M] \mapsto [S \otimes_R M],$$

where S becomes a right R-module via f. We obtain that K_0 is a covariant functor from the category of unital rings to the category of abelian groups.

1.2.5. Example. Calculation of K_0.

- If R is a field, then $K_0(R) \cong \mathbb{Z}$, the isomorphism given by the dimension: $\dim_R(M,N) := \dim_R(M) - \dim_R(N)$.
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$, given by the rank.
- If X is a compact topological space, then $K_0(C(X)) \cong K^0(X)$, the topological K-theory given in terms of complex vector bundles. To each vector bundle E one associates the $C(X)$-module $\Gamma(E)$ of continuous sections of E.

- Let G be a discrete group. The group algebra $\mathbb{C}G$ is a vector space with basis G, and with multiplication coming from the group structure, i.e. given by $g \cdot h = (gh)$.

  If G is a finite group, then $K_0(\mathbb{C}G)$ is the complex representation ring of G.

1.2.3 K-Theory of C^*-algebras

1.2.6. Definition. Let A be a unital C^*-algebra. Then $K_0(A)$ is defined as in Definition 1.2.4, i.e. by forgetting the topology of A.

1.2.3.1 K-theory for non-unital C^*-algebras

When studying (the K-theory of) C^*-algebras, one has to understand morphisms $f\colon A \to B$. This necessarily involves studying the kernel of f, which is a closed ideal of A, and hence a *non-unital* C^*-algebra. Therefore, we proceed by defining the K-theory of C^*-algebras without unit.

1.2.7. Definition. To any C^*-algebra A, with or without unit, we assign in a functorial way a new, unital C^*-algebra A_+ as follows. As $\mathbb{C}$-vector space, $A_+ := A \oplus \mathbb{C}$, with product

$$(a, \lambda)(b, \mu) := (ab + \lambda a + \mu b, \lambda\mu) \qquad \text{for } (a, \lambda), (b, \mu) \in A \oplus \mathbb{C}.$$

The unit is given by $(0, 1)$. The star-operation is defined as $(a, \lambda)^* := (a^*, \overline{\lambda})$, and the new norm is given by

$$|(a, \lambda)| = \sup\{|ax + \lambda x| \mid x \in A \text{ with } |x| = 1\}$$

1.2.8. Remark. A is a closed ideal of A_+, the kernel of the canonical projection $A_+ \twoheadrightarrow \mathbb{C}$ onto the second factor. If A itself is unital, the unit of A is of course different from the unit of A_+.

1.2.9. Example. Assume X is a locally compact space, and let $X_+ := X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X)_+ \cong C(X_+).$$

The ideal $C_0(X)$ of $C_0(X)_+$ is identified with the ideal of those functions $f \in C(X_+)$ such that $f(\infty) = 0$.

1.2.10. Definition. For an arbitrary C^*-algebra A (not necessarily unital) define

$$K_0(A) := \ker(K_0(A_+) \to K_0(\mathbb{C})).$$

Any C^*-algebra homomorphisms $f\colon A \to B$ (not necessarily unital) induces a unital homomorphism $f_+\colon A_+ \to B_+$. The induced map

$$(f_+)_*\colon K_0(A_+) \to K_0(B_+)$$

maps the kernel of the map $K_0(A_+) \to K_0(\mathbb{C})$ to the kernel of $K_0(B_+) \to K_0(\mathbb{C})$. This means it restricts to a map $f_*\colon K_0(A) \to K_0(B)$. We obtain a covariant functor from the category of (not necessarily unital) C^*-algebras to abelian groups.

Of course, we need the following result.

1.2.11. Proposition. *If A is a unital C^*-algebra, the new and the old definition of $K_0(A)$ are canonically isomorphic.*

1.2.3.2 Higher topological K-groups

We also want to define higher topological K-theory groups. We have an ad hoc definition using suspensions (this is similar to the corresponding idea in topological K-theory of spaces). For this we need the following.

1.2.12. Definition. Let A be a C^*-algebra. We define the cone CA and the suspension SA as follows.

$$\begin{aligned} CA &:= \{f\colon [0,1] \to A \mid f(0) = 0\} \\ SA &:= \{f\colon [0,1] \to A \mid f(0) = 0 = f(1)\}. \end{aligned}$$

These are again C^*-algebras, using pointwise operations and the supremum norm.

Inductively, we define

$$S^0A := A \qquad S^nA := S(S^{n-1}A) \quad \text{for } n \geq 1.$$

1.2.13. Definition. Assume A is a C^*-algebra. For $n \geq 0$, define

$$K_n(A) := K_0(S^nA).$$

These are the *topological K-theory groups of A*. For each $n \geq 0$, we obtain a functor from the category of C^*-algebras to the category of abelian groups.

For unital C^*-algebras, we can also give a more direct definition of higher K-groups (in particular useful for K_1, which is then defined in terms of (classes of) invertible matrices). This is done as follows:

1.2.14. Definition. Let A be a unital C^*-algebra. Then $Gl_n(A)$ becomes a topological group, and we have continuous embeddings

$$Gl_n(A) \hookrightarrow Gl_{n+1}(A) \colon X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We set $Gl_\infty(A) := \lim_{n\to\infty} Gl_n(A)$, and we equip $Gl_\infty(A)$ with the direct limit topology.

1.2.15. Proposition. *Let A be a unital C^*-algebra. If $k \geq 1$, then*

$$K_k(A) = \pi_{k-1}(Gl_\infty(A))(\cong \pi_k(BGl_\infty(A))).$$

Observe that any unital morphism $f\colon A \to B$ of unital C^*-algebras induces a map $Gl_n(A) \to Gl_n(B)$ and therefore also between $\pi_k(Gl_\infty(A))$ and $\pi_k(Gl_\infty(B))$. This map coincides with the previously defined induced map in topological K-theory.

1.2.16. Remark. Note that the topology of the C^*-algebra enters the definition of the higher topological K-theory of A, and in general the topological K-theory of A will be vastly different from the algebraic K-theory of the algebra underlying A. For connections in special cases, compare [83].

1.2.17. Example. It is well known that $Gl_n(\mathbb{C})$ is connected for each $n \in \mathbb{N}$. Therefore

$$K_1(\mathbb{C}) = \pi_0(Gl_\infty(\mathbb{C})) = 0.$$

A very important result about K-theory of C^*-algebras is the following long exact sequence. A proof can be found e.g. in [32, Proposition 4.5.9].

1.2.18. Theorem. *Assume I is a closed ideal of a C^*-algebra A. Then, we get a short exact sequence of C^*-algebras $0 \to I \to A \to A/I \to 0$, which induces a long exact sequence in K-theory*

$$\to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(I) \to \cdots \to K_0(A/I).$$

1.2.4 Bott periodicity and the cyclic exact sequence

One of the most important and remarkable results about the K-theory of C^*-algebras is Bott periodicity, which can be stated as follows.

1.2.19. Theorem. *Assume A is a C^*-algebra. There is a natural isomorphism, called the Bott map*

$$K_0(A) \to K_0(S^2 A),$$

which implies immediately that there are natural isomorphism

$$K_n(A) \cong K_{n+2}(A) \qquad \forall n \geq 0.$$

1.2.20. Remark. Bott periodicity allows us to define $K_n(A)$ for each $n \in \mathbb{Z}$, or to regard the K-theory of C^*-algebras as a $\mathbb{Z}/2$-graded theory, i.e. to talk of $K_n(A)$ with $n \in \mathbb{Z}/2$. This way, the long exact sequence of Theorem 1.2.18 becomes a (six-term) cyclic exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow{\scriptstyle \mu_*} \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

The connecting homomorphism μ_* is the composition of the Bott periodicity isomorphism and the connecting homomorphism of Theorem 1.2.18.

1.2.5 The C^*-algebra of a group

Let Γ be a discrete group. Define $l^2(\Gamma)$ to be the Hilbert space of square summable complex valued functions on Γ. We can write an element $f \in l^2(\Gamma)$ as a sum $\sum_{g\in\Gamma} \lambda_g g$ with $\lambda_g \in \mathbb{C}$ and $\sum_{g\in\Gamma} |\lambda_g|^2 < \infty$.

We defined the *complex group algebra* (often also called the *complex group ring*) $\mathbb{C}\Gamma$ to be the complex vector space with basis the elements of Γ (this can also be considered as the space of complex valued functions on Γ with finite support, and as such is a subspace of $l^2(\Gamma)$). The product in $\mathbb{C}\Gamma$ is induced by the multiplication in Γ, namely, if $f = \sum_{g\in\Gamma} \lambda_g g, u = \sum_{g\in\Gamma} \mu_g g \in \mathbb{C}\Gamma$, then

$$(\sum_{g\in\Gamma} \lambda_g g)(\sum_{g\in\Gamma} \mu_g g) := \sum_{g,h\in\Gamma} \lambda_g \mu_h (gh) = \sum_{g\in\Gamma} \left(\sum_{h\in\Gamma} \lambda_h \mu_{h^{-1}g} \right) g.$$

This is a convolution product.

We have the *left regular representation* λ_Γ of Γ on $l^2(\Gamma)$, given by

$$\lambda_\Gamma(g) \cdot (\sum_{h \in \Gamma} \lambda_h h) := \sum_{h \in \Gamma} \lambda_h gh$$

for $g \in \Gamma$ and $\sum_{h \in \Gamma} \lambda_h h \in l^2(\Gamma)$.

This unitary representation extends linearly to $\mathbb{C}\Gamma$.

The *reduced C^*-algebra* $C_r^*\Gamma$ of Γ is defined to be the norm closure of the image $\lambda_\Gamma(\mathbb{C}\Gamma)$ in the C^*-algebra of bounded operators on $l^2(\Gamma)$.

1.2.21. Remark. It's no surprise that there is also a *maximal C^*-algebra* $C^*_{max}\Gamma$ of a group Γ. It is defined using not only the left regular representation of Γ, but simultaneously all of its representations. We will not make use of $C^*_{max}\Gamma$ in these notes, and therefore will not define it here.

Given a topological group G, one can define C^*-algebras C_r^*G and $C^*_{max}G$ which take the topology of G into account. They actually play an important role in the study of the Baum-Connes conjecture, which can be defined for (almost arbitrary) topological groups, but again we will not cover this subject here. Instead, we will throughout stick to discrete groups.

1.2.22. Example. If Γ is finite, then $C_r^*\Gamma = \mathbb{C}\Gamma$ is the complex group ring of Γ.

In particular, in this case $K_0(C_r^*\Gamma) \cong R(\Gamma)$ coincides with the (additive group of) the complex representation ring of Γ.

1.3 The Baum-Connes conjecture

The Baum-Connes conjecture relates an object from algebraic topology, namely the K-homology of the classifying space of a given group Γ, to representation theory and the world of C^*-algebras, namely to the K-theory of the reduced C^*-algebra of Γ.

Unfortunately, the material is very technical. Because of lack of space and time we can not go into the details (even of some of the definitions). We recommend the sources [86], [87], [32], [4], [58] and [8].

1.3.1 The Baum-Connes conjecture for torsion-free groups

1.3.1. Definition. Let X be any CW-complex. $K_*(X)$ is the K-homology of X, where K-homology is the homology theory dual to topological K-theory. If BU is the spectrum of topological K-theory, and X_+ is X with a disjoint basepoint added, then

$$K_n(X) := \pi_n(X_+ \wedge BU).$$

1.3.2. Definition. Let Γ be a discrete group. A classifying space $B\Gamma$ for Γ is a CW-complex with the property that $\pi_1(B\Gamma) \cong \Gamma$, and $\pi_k(B\Gamma) = 0$ if $k \neq 1$. A classifying space always exists, and is unique up to homotopy equivalence. Its universal covering $E\Gamma$ is a contractible CW-complex with a free cellular Γ-action, the so called *universal space for* Γ*-actions.*

1.3.3. Remark. In the literature about the Baum-Connes conjecture, one will often find the definition

$$RK_n(X) := \varinjlim K_n(Y),$$

where the limit is taken over all finite subcomplexes Y of X. Note, however, that K-homology (like any homology theory in algebraic topology) is compatible with direct limits, which implies $RK_n(X) = K_n(X)$ as defined above. The confusion comes from the fact that operator algebraists often use Kasparov's bivariant KK-theory to define $K_*(X)$, and this coincides with the homotopy theoretic definition only if X is compact.

Recall that a group Γ is called torsion-free, if $g^n = 1$ for $g \in \Gamma$ and $n > 0$ implies that $g = 1$.

We can now formulate the Baum-Connes conjecture for torsion-free discrete groups.

1.3.4. Conjecture. *Assume* Γ *is a torsion-free discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\overline{\mu}_*\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma) \tag{1.3.5}$$

(which will be defined later). The Baum-Connes conjecture *says that this map is an isomorphism.*

1.3.6. Example. The map $\overline{\mu}_*$ of Equation (1.3.5) is also defined if Γ is not torsion-free. However, in this situation it will in general not be an isomorphism. This can already be seen if $\Gamma = \mathbb{Z}/2$. Then $C_r^*\Gamma = \mathbb{C}\Gamma \cong \mathbb{C}\oplus\mathbb{C}$ as a $\mathbb{C}$-algebra. Consequently,

$$K_0(C_r^*\Gamma) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{1.3.7}$$

On the other hand, using the homological Chern character,

$$K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{n=0}^{\infty} H_{2n}(B\Gamma; \mathbb{Q}) \cong \mathbb{Q}. \tag{1.3.8}$$

(Here we use the fact that the rational homology of every finite group is zero in positive degrees, which follows from the fact that the transfer homomorphism $H_k(B\Gamma;\mathbb{Q}) \to H_k(\{1\};\mathbb{Q})$ is (with rational coefficients) up to a factor

$|\Gamma|$ a left inverse to the map induced from the inclusion, and therefore is injective.)

The calculations (1.3.7) and (1.3.8) prevent μ_0 of (1.3.5) from being an isomorphism.

1.3.2 The Baum-Connes conjecture in general

To account for the problem visible in Example 1.3.6 if we are dealing with groups with torsion, one replaces the left hand side by a more complicated gadget, the equivariant K-homology of a certain Γ-space $E(\Gamma, fin)$, the classifying space for proper actions. We will define all of this later. Then, the Baum-Connes conjecture says the following.

1.3.9. Conjecture. *Assume Γ is a discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\mu_*\colon K_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma) \tag{1.3.10}$$

(we will define it later). The conjecture says that this map is an isomorphism.

1.3.11. Remark. If Γ is torsion-free, then $K_*(B\Gamma) = K_*^\Gamma(E(\Gamma, fin))$, and the assembly maps $\overline{\mu}$ of Conjectures 1.3.4 and μ of 1.3.9 coincide (see Proposition 1.3.29).

Last, we want to mention that there is also a *real version* of the Baum-Connes conjecture, where on the left hand side the K-homology is replaced by KO-homology, i.e. the homology dual to the K-theory of real vector spaces (or an equivariant version hereof), and on the right hand side $C_r^*\Gamma$ is replaced by the real reduced C^*-algebra $C_{r,\mathbb{R}}^*\Gamma$.

1.3.3 Consequences of the Baum-Connes conjecture

1.3.3.1 Idempotents in $C_r^*\Gamma$

The connection between the Baum-Connes conjecture and idempotents is best shown via Atiyah's L^2-index theorem, which we discuss first.

Given a closed manifold M with an elliptic differential operator $D\colon C^\infty(E) \to C^\infty(F)$ between two bundles on M, and a normal covering $\tilde{M} \to M$ (with deck transformation group Γ, normal means that $M = \tilde{M}/\Gamma$), we can lift E, F and D to $\tilde{M}$, and get an elliptic Γ-equivariant differential operator $\tilde{D}\colon C^\infty(\tilde{E}) \to C^\infty(\tilde{F})$. If Γ is not finite, we can not use the equivariant index of Section 1.1.3. However, because the action is free, it is possible to define an equivariant analytic index

$$\mathrm{ind}_\Gamma(\tilde{D}) \in K_{\dim M}(C_r^*\Gamma).$$

This is described in Example 1.3.37.

Atiyah used a certain real valued homomorphism, the Γ-dimension

$$\dim_\Gamma \colon K_0(C_r^*\Gamma) \to \mathbb{R},$$

to define the L^2-index of $\tilde{D}$ (on an even dimensional manifold):

$$L^2\text{-}\,\mathrm{ind}(\tilde{D}) := \dim_\Gamma(\mathrm{ind}_\Gamma(\tilde{D})).$$

The L^2-index theorem says

$$L^2\text{-}\,\mathrm{ind}(\tilde{D}) = \mathrm{ind}(D),$$

in particular, it follows that the L^2-index is an integer. For a different point of view of the L^2-index theorem, compare Section 3.1.

An alternative description of the left hand side of (1.3.5) and (1.3.10) shows that, as long as Γ is torsion-free, the image of μ_0 coincides with the subset of $K_0(C_r^*\Gamma)$ consisting of $\mathrm{ind}_\Gamma(\tilde{D})$, where $\tilde{D}$ is as above. In particular, if μ_0 is surjective (and Γ is torsion-free), for each $x \in K_0(C_r^*\Gamma)$ we find a differential operator D such that $x = \mathrm{ind}_\Gamma(\tilde{D})$. As a consequence, $\dim_\Gamma(x) \in \mathbb{Z}$, i.e. the range of $\dim_\Gamma$ is contained in $\mathbb{Z}$. This is the statement of the so called *trace conjecture.*

1.3.12. Conjecture. *Assume Γ is a torsion-free discrete group. Then*

$$\dim_\Gamma(K_0(C_r^*\Gamma)) \subset \mathbb{Z}.$$

On the other hand, if $x \in K_0(C_r^*\Gamma)$ is represented by a projection $p = p^2 \in C_r^*\Gamma$, then elementary properties of $\dim_\Gamma$ (monotonicity and faithfulness) imply that $0 \le \dim_\Gamma(p) \le 1$, and $\dim_\Gamma(p) \notin \{0,1\}$ if $p \neq 0, 1$.

Therefore, we have the following consequence of the Baum-Connes conjecture. If Γ is torsion-free and the Baum-Connes map μ_0 is surjective, then $C_r^*\Gamma$ does not contain any projection different from 0 or 1.

This is the assertion of the Kadison-Kaplansky conjecture:

1.3.13. Conjecture. *Assume Γ is torsion-free. Then $C_r^*\Gamma$ does not contain any non-trivial projections.*

The following consequence of the Kadison-Kaplansky conjecture deserves to be mentioned:

1.3.14. Proposition. *If the Kadison-Kaplansky conjecture is true for a group Γ, then the spectrum $s(x)$ of every self adjoint element $x \in C_r^*\Gamma$ is connected. Recall that the spectrum is defined in the following way:*

$$s(x) := \{\lambda \in \mathbb{C} \mid (x - \lambda \cdot 1) \text{ not invertible}\}.$$

If Γ is not torsion-free, it is easy to construct non-trivial projections, and it is clear that the range of ind_Γ is not contained in $\mathbb{Z}$. Baum and Connes originally conjectured that it is contained in the abelian subgroup $Fin^{-1}(\Gamma)$ of $\mathbb{Q}$ generated by $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$. This conjecture is not correct, as is shown by an example of Roy [67]. In [52], Lück proves that the Baum-Connes conjecture implies that the range of $\dim_\Gamma$ is contained in the *subring* of $\mathbb{Q}$ generated by $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$.

1.3.3.2 Obstructions to positive scalar curvature

The Baum-Connes conjecture implies the so called "stable Gromov-Lawson-Rosenberg" conjecture. This implication is a theorem due to Stephan Stolz. The details of this will be discussed in the lectures of Stephan Stolz, therefore we can be very brief. We just mention the result.

1.3.15. Theorem. *Fix a group Γ. Assume that μ in the real version of* (1.3.10) *discussed in Section 1.4 is injective (which follows e.g. if μ in* (1.3.10) *is an isomorphism), and assume that M is a closed spin manifold with $\pi_1(M) = \Gamma$. Assume that a certain (index theoretic) invariant $\alpha(M) \in K_{\dim M}(C^*_{\mathbb{R},r}\Gamma)$ vanishes. Then there is an $n \geq 0$ such that $M \times B^n$ admits a metric with positive scalar curvature.*

Here, B is any simply connected 8-dimensional spin manifold with $\hat{A}(M) = 1$. Such a manifold is called a *Bott manifold*.

The converse of Theorem 1.3.15, i.e. positive scalar curvature implies vanishing of $\alpha(M)$, is true for arbitrary groups and without knowing anything about the Baum-Connes conjecture.

1.3.3.3 The Novikov conjecture about higher signatures

Direct approach The original form of the Novikov conjecture states that higher signatures are homotopy invariant.

More precisely, let M be an (even dimensional) closed oriented manifold with fundamental group Γ. Let $B\Gamma$ be a classifying space for Γ. There is a unique (up to homotopy) *classifying map* $u\colon M \to B\Gamma$ which is defined by the property that it induces an isomorphism on π_1. Equivalently, u classifies a universal covering of M.

Let $L(M) \in H^*(M;\mathbb{Q})$ be the Hirzebruch L-class (as normalized by Atiyah and Singer). Given any cohomology class $a \in H^*(B\Gamma,\mathbb{Q})$, we define the higher signature

$$\sigma_a(M) := \langle L(M) \cup u^* a, [M]\rangle \in \mathbb{Q}.$$

Here $[M] \in H_{\dim M}(M;\mathbb{Q})$ is the fundamental class of the oriented manifold M, and $\langle \cdot,\cdot \rangle$ is the usual pairing between cohomology and homology.

Recall that the Hirzebruch signature theorem states that $\sigma_1(M)$ is the signature of M, which evidently is an oriented homotopy invariant.

The Novikov conjecture generalizes this as follows.

1.3.16. Conjecture. *Assume $f\colon M \to M'$ is an oriented homotopy equivalence between two even dimensional closed oriented manifolds, with (common) fundamental group π. "Oriented" means that $f_*[M] = [M']$. Then all higher signatures of M and M' are equal, i.e.*

$$\sigma_a(M) = \sigma_a(M') \qquad \forall a \in H^*(B\Gamma,\mathbb{Q}).$$

There is an equivalent reformulation of this conjecture in terms of K-homology. To see this, let D be the signature operator of M. (We assume here that M is smooth, and we choose a Riemannian metric on M to define this operator. It is an elliptic differential operator on M.) The operator D defines an element in the K-homology of M, $[D] \in K_{\dim M}(M)$. Using the map u, we can push $[D]$ to $K_{\dim M}(B\Gamma)$. We define the higher signature $\sigma(M) := u_*[D] \in K_{\dim M}(B\Gamma)\otimes\mathbb{Q}$. It turns out that

$$2^{\dim M/2}\sigma_a(M) = \langle a, ch(\sigma(M))\rangle \qquad \forall a \in H^*(B\Gamma;\mathbb{Q}),$$

where $ch\colon K_*(B\Gamma)\otimes\mathbb{Q} \to H_*(B\Gamma,\mathbb{Q})$ is the homological Chern character (an isomorphism).

Therefore, the Novikov conjecture translates to the statement that $\sigma(M) = \sigma(M')$ if M and M' are oriented homotopy equivalent.

Now one can show *directly* that

$$\overline{\mu}(\sigma(M)) = \overline{\mu}(\sigma(M')) \in K_*(C_r^*\Gamma),$$

if M and M' are oriented homotopy equivalent. Consequently, rational injectivity of the Baum-Connes map $\overline{\mu}$ immediately implies the Novikov conjecture. If Γ is torsion-free, this is part of the assertion of the Baum-Connes conjecture. Because of this relation, injectivity of the Baum-Connes map μ is often called the "analytic Novikov conjecture".

L-theory approach There is a more obvious connection between the Baum-Connes isomorphism conjecture and the L-theory isomorphism conjecture (discussed in other lectures).

Namely, the L-theory isomorphism conjecture is concerned with a certain assembly map

$$A_\Gamma \colon H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma]).$$

Here, the left hand side is the homology of $B\Gamma$ with coefficients the algebraic surgery spectrum of $\mathbb{Z}$, and the right hand side is the free quadratic L-group of the ring with involution $\mathbb{Z}[\Gamma]$.

The Novikov conjecture is equivalent to the statement that this map is rationally injective, i.e. that

$$A_\Gamma \otimes \mathrm{id}_\mathbb{Q} \colon H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}$$

is an injection. This formulation has the advantage that, tensored with $\mathbb{Q}$, all the different flavors of L-theory are isomorphic (therefore, we don't have to and we won't discuss these distinctions here).

Now, we get a commutative diagram

$$\begin{tikzcd}
H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \arrow[r, "A_\Gamma"] \arrow[d] & L_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q} \arrow[d] \\
H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{C})) \otimes \mathbb{Q} \arrow[r, "A_{\Gamma,\mathbb{C}}"] \arrow[d] & L_*(\mathbb{C}[\Gamma]) \otimes \mathbb{Q} \arrow[d] \\
K_*(B\Gamma) \otimes \mathbb{Q} \arrow[r, "\overline{\mu}"] & K_*(C_r^*\Gamma) \otimes \mathbb{Q} = L_*(C_r^*\Gamma) \otimes \mathbb{Q}.
\end{tikzcd} \tag{1.3.17}$$

The maps on the left hand side are given by natural transformations of homology theories with values in rational vector spaces. These transformations are easily seen to be injective for the coefficients. Since we deal with *rational* homology theories, they are injective in general.

The maps on the right hand side are the maps in L-theory induced by the obvious ring homomorphisms $\mathbb{Z}\Gamma \to \mathbb{C}\Gamma \to C_r^*\Gamma$. Then we use the "folk theorem" that, for C^*-algebras, K-theory and L-theory are canonically isomorphic (even non-rationally). Of course, it remains to establish commutativity of the diagram (1.3.17). For more details, we refer to [66]. Using all these facts and the diagram (1.3.17), we see that for torsion-free groups, rational injectivity of the Baum-Connes map μ implies rational injectivity of the L-theory assembly A_Γ, i.e. the Novikov conjecture.

Groups with torsion For an arbitrary group Γ, we have a factorization of $\overline{\mu}$ as follows:

$$K_*(B\Gamma) \xrightarrow{f} K_*^\Gamma(E(\Gamma, fin)) \xrightarrow{\mu} K_*(C_r^*\Gamma).$$

One can show that f is rationally injective, so that rational injectivity of the Baum-Connes map μ implies the Novikov conjecture also in general.

1.3.4 The universal space for proper actions

1.3.18. Definition. Let Γ be a discrete group and X a Hausdorff space with an action of Γ. We say that the action is *proper*, if for all $x, y \in X$ there are open neighborhood $U_x \ni x$ and $U_y \ni y$ such that $gU_x \cap U_y$ is non-empty only for finitely many $g \in \Gamma$ (the number depending on x and y).

The action is said to be *cocompact*, if X/Γ is compact.

1.3.19. Lemma. *If the action of* Γ *on* X *is proper, then for each* $x \in X$ *the* isotropy group $\Gamma_x := \{g \in \Gamma \mid gx = x\}$ *is finite.*

1.3.20. Definition. Let Γ be a discrete group. A CW-complex X is a Γ-*CW-complex*, if X is a CW-complex with a cellular action of Γ with the additional property that, whenever $g(D) \subset D$ for a cell D of X and some $g \in \Gamma$, then $g|_D = \mathrm{id}_D$, i.e. g doesn't move D at all.

1.3.21. Remark. There exists also the notion of G-CW-complex for topological groups G (taking the topology of G into account). These have to be defined in a different way, namely by gluing together G-equivariant cells $D^n \times G/H$. In general, such a G-CW-complex is not an ordinary CW-complex.

1.3.22. Lemma. *The action of a discrete group* Γ *on a* Γ-*CW-complex is proper if and only if every isotropy group is finite.*

1.3.23. Definition. A proper Γ-CW-complex X is called *universal*, or more precisely *universal for proper actions*, if for every proper Γ-CW-complex Y there is a Γ-equivariant map $f\colon Y \to X$ which is unique up to Γ-equivariant homotopy. Any such space is denoted $E(\Gamma, fin)$ or $\underline{E}\Gamma$.

1.3.24. Proposition. *A* Γ-*CW-complex* X *is universal for proper actions if and only if the fixed point set*

$$X^H := \{x \in X \mid hx = x \quad \forall h \in H\}$$

is empty whenever H *is an infinite subgroup of* Γ, *and is contractible (and in particular non-empty) if* H *is a finite subgroup of* Γ.

1.3.25. Proposition. *If Γ is a discrete group, then $E(\Gamma, fin)$ exists and is unique up to Γ-homotopy equivalence.*

1.3.26. Remark. The general context for this discussion are actions of a group Γ where the isotropy belongs to a fixed family of subgroups of Γ (in our case, the family of all finite subgroups). For more information, compare [85].

1.3.27. Example.

- If Γ is torsion-free, then $E(\Gamma, fin) = E\Gamma$, the universal covering of the classifying space $B\Gamma$. Indeed, Γ acts freely on $E\Gamma$, and $E\Gamma$ is contractible.
- If Γ is finite, then $E(\Gamma, fin) = \{*\}$.
- If G is a connected Lie group with maximal compact subgroup K, and Γ is a discrete subgroup of G, then $E(\Gamma, fin) = G/K$ [4, Section 2].

1.3.28. Remark. In the literature (in particular, in [4]), also a slightly different notion of universal spaces is discussed. One allows X to be any proper metrizable Γ-space, and requires the universal property for all proper metrizable Γ-spaces Y. For discrete groups (which are the only groups we are discussing here), a universal space in the sense of Definition 1.3.23 is universal in this sense.

However, for some of the proofs of the Baum-Connes conjecture (for special groups) it is useful to use certain models of $E(\Gamma, fin)$ (in the broader sense) coming from the geometry of the group, which are not Γ-CW-complexes.

1.3.5 Equivariant K-homology

Let Γ be a discrete group. We have seen that, if Γ is not torsion-free, the assembly map (1.3.5) is not an isomorphism. To account for that, we replace $K_*(B\Gamma)$ by the equivariant K-theory of $E(\Gamma, fin)$. Let X be any proper Γ-CW complex. The original definition of equivariant K-homology is due to Kasparov, making ideas of Atiyah precise. In this definition, elements of $K_*^\Gamma(X)$ are equivalence classes of generalized elliptic operators. In [14], a more homotopy theoretic definition of $K_*^\Gamma(X)$ is given, which puts the Baum-Connes conjecture in the context of other isomorphism conjectures.

1.3.5.1 Homotopy theoretic definition of equivariant K-homology

The details of this definition are quite technical, using spaces and spectra over the orbit category of the discrete group Γ. The objects of the orbit category are the orbits Γ/H, H any subgroup of Γ. The morphisms from Γ/H to

Γ/K are simply the Γ-equivariant maps. In this setting, any spectrum over the orbit category gives rise to an equivariant homology theory. The decisive step is then the construction of a (periodic) topological K-theory spectrum $\mathbf{K}^\Gamma$ over the orbit category of Γ. This gives us then a functor from the category of (arbitrary) Γ-CW-complexes to the category of (graded) abelian groups, the *equivariant K-homology* $K_*^\Gamma(X)$ (X any Γ-CW-complex).

The important property (which justifies the name "topological K-theory spectrum) is that

$$K_k^\Gamma(\Gamma/H) = \pi_k(\mathbf{K}^\Gamma(\Gamma/H)) \cong K_k(C_r^*H)$$

for every subgroup H of Γ. In particular,

$$K_k^\Gamma(\{*\}) \cong K_k(C_r^*\Gamma).$$

Moreover, we have the following properties:

1.3.29. Proposition. *(1) Assume Γ is the trivial group. Then*

$$K_*^\Gamma(X) = K_*(X),$$

i.e. we get back the ordinary K-homology introduced above.

(2) If $H \leq \Gamma$ and X is an H-CW-complex, then there is a natural isomorphism

$$K_*^H(X) \cong K_*^\Gamma(\Gamma \times_H X).$$

Here $\Gamma \times_H X = \Gamma \times H/\sim$, where we divide out the equivalence relation generated by $(gh, x) \sim (g, hx)$ for $g \in \Gamma$, $h \in H$ and $x \in X$. This is in the obvious way a left Γ-space.

(3) Assume X is a free Γ-CW-complex. Then there is a natural isomorphism

$$K_*(\Gamma\backslash X) \to K_*^\Gamma(X).$$

In particular, using the canonical Γ-equivariant map $E\Gamma \to E(\Gamma, fin)$, we get a natural homomorphism

$$K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(E\Gamma) \to K_*^\Gamma(E(\Gamma, fin)).$$

1.3.5.2 Analytic definition of equivariant K-homology

Here we will give the original definition, which embeds into the powerful framework of equivariant KK-theory, and which is used for almost all proofs of special cases of the Baum-Connes conjecture. However, to derive some of the consequences of the Baum-Connes conjecture, most notably about the positive scalar curvature question —this is discussed in one of the lectures of Stephan Stolz— the homotopy theoretic definition is used.

1.3.30. Definition. A Hilbert space H is called *($\mathbb{Z}/2$)-graded*, if H comes with an orthogonal sum decomposition $H = H_0 \oplus H_1$. Equivalently, a unitary operator ϵ with $\epsilon^2 = 1$ is given on H. The subspaces H_0 and H_1 can be recovered as the $+1$ and -1 eigenspaces of ϵ, respectively.

A bounded operator $T\colon H \to H$ is called *even* (with respect to the given grading), if T commutes with ϵ, and *odd*, if ϵ and T anti-commute, i.e. if $T\epsilon = -\epsilon T$. An even operator decomposes as $T = \left(\begin{smallmatrix} T_0 & 0 \\ 0 & T_1 \end{smallmatrix}\right)$, an odd one as $T = \left(\begin{smallmatrix} 0 & T_0 \\ T_1 & 0 \end{smallmatrix}\right)$ in the given decomposition $H = H_0 \oplus H_1$.

1.3.31. Definition. A *generalized elliptic* Γ*-operator on* X, or a *cycle for* Γ*-K-homology of the* Γ*-space* X, simply a *cycle* for short, is a triple (H, π, F), where

- $H = H_0 \oplus H_1$ is a $\mathbb{Z}/2$-graded Γ-Hilbert space (i.e. the direct sum of two Hilbert spaces with unitary Γ-action)
- π is a Γ-equivariant $*$-representation of $C_0(X)$ on even bounded operators of H (equivariant means that $\pi(fg^{-1}) = g\pi(f)g^{-1}$ for all $f \in C_0(X)$ and all $g \in \Gamma$.
- $F\colon H \to H$ is a bounded, Γ-equivariant, self adjoint operator such that $\pi(f)(F^2 - 1)$ and $[\pi(f), F] := \pi(f)F - F\pi(f)$ are compact operators for all $f \in C_0(X)$. Moreover, we require that F is odd, i.e. $F = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right)$ in the decomposition $H = H_0 \oplus H_1$.

1.3.32. Remark. There are many different definitions of cycles, slightly weakening or strengthening some of the conditions. Of course, this does not effect the equivariant K-homology groups which are eventually defined using them.

1.3.33. Definition. We define the direct sum of two cycles in the obvious way.

1.3.34. Definition. Assume $\alpha = (H, \pi, F)$ and $\alpha' = (H', \pi', F')$ are two cycles.

(1) They are called (isometrically) isomorphic, if there is a Γ-equivariant grading preserving isometry $\Psi\colon H \to H'$ such that $\Psi \circ \pi(f) = \pi'(f) \circ \Psi$ for all $f \in C_0(X)$ and $\Psi \circ F = F' \circ \Psi$.

(2) They are called *homotopic* (or *operator homotopic*) if $H = H'$, $\pi = \pi'$, and there is a norm continuous path $(F_t)_{t\in[0,1]}$ of operators with $F_0 = F$ and $F_1 = F'$ and such that (H, π, F_t) is a cycle for each $t \in [0, 1]$.

(3) (H, π, F) is called *degenerate*, if $[\pi(f), F] = 0$ and $\pi(f)(F^2 - 1) = 0$ for each $f \in C_0(X)$.

(4) The two cycles are called equivalent if there are degenerate cycles β and β' such that $\alpha \oplus \beta$ is operator homotopic to a cycle isometrically isomorphic to $\alpha' \oplus \beta'$.

The set of equivalence classes of cycles is denoted $KK_0^\Gamma(X)$. (Caution, this is slightly unusual, mostly one will find the notation $K^\Gamma(X)$ instead of $KK^\Gamma(X)$).

1.3.35. Proposition. *Direct sum induces the structure of an abelian group on $KK_0^\Gamma(X)$.*

1.3.36. Proposition. *Any proper Γ-equivariant map $\phi\colon X \to Y$ between two proper Γ-CW-complexes induces a homomorphism*

$$KK_0^\Gamma(X) \to KK_0^\Gamma(Y)$$

by $(H, \pi, F) \mapsto (H, \pi \circ \phi^*, F)$, where $\phi^*\colon C_0(Y) \to C_0(X)\colon f \mapsto f \circ \phi$ is defined since ϕ is a proper map (else $f \circ \phi$ does not necessarily vanish at infinity).

Recall that a continuous map $\phi\colon X \to Y$ is called *proper* if the inverse image of every compact subset of Y is compact .

It turns out that the analytic definition of equivariant K-homology is quite flexible. It is designed to make it easy to construct elements of these groups —in many geometric situations they automatically show up. We give one of the most typical examples of such a situation, which we will need later.

1.3.37. Example. Assume that M is a compact even dimensional Riemannian manifold. Let $X = \overline{M}$ be a normal covering of M with deck transformation group Γ (normal means that $X/\Gamma = M$). Of course, the action is free, in particular, proper. Let $E = E_0 \oplus E_1$ be a graded Hermitian vector bundle on M, and

$$D\colon C^\infty(E) \to C^\infty(E)$$

an odd elliptic self adjoint differential operator (odd means that D maps the subspace $C^\infty(E_0)$ to $C^\infty(E_1)$, and vice versa). If M is oriented, the signature operator on M is such an operator, if M is a spin-manifold, the same is true for its Dirac operator.

Now we can pull back E to a bundle $\overline{E}$ on $\overline{M}$, and lift D to an operator $\overline{D}$ on $\overline{E}$. The assumptions imply that $\overline{D}$ extends to an unbounded self adjoint operator on $L^2(\overline{E})$, the space of square integrable sections of $\overline{E}$. This space is the completion of $C_c^\infty(\overline{E})$ with respect to the canonical inner product (compare Definition 3.1.1). (The subscript c denotes sections with compact support). Using the functional calculus, we can replace $\overline{D}$ by

$$F := (\overline{D}^2 + 1)^{-1/2}\overline{D}\colon L^2(\overline{E}) \to L^2(\overline{E}).$$

Observe that

$$L^2(\overline{E}) = L^2(\overline{E}_0) \oplus L^2(\overline{E}_1)$$

is a $\mathbb{Z}/2$-graded Hilbert space with a unitary Γ-action, which admits an (equivariant) action π of $C_0(\overline{M}) = C_0(X)$ by fiber-wise multiplication. This action preserves the grading. Moreover, $\overline{D}$ as well as F are odd, Γ-equivariant, self adjoint operators on $L^2(\overline{E})$ and F is a bounded operator. From ellipticity it follows that

$$\pi(f)(F^2 - 1) = -\pi(f)(\overline{D}^2 + 1)^{-1}$$

is compact for each $f \in C_0(\overline{M})$ (observe that this is not true for $(\overline{D}^2 + 1)^{-1}$ itself, if $\overline{M}$ is not compact). Consequently, $(L^2(E), \pi, F)$ defines an (even) cycle for Γ-K-homology, i.e. it represents an element in $KK_0^\Gamma(X)$.

One can slightly reformulate the construction as follows: $\overline{M}$ is a principal Γ-bundle over M, and $l^2(\Gamma)$ has a (unitary) left Γ-action. We therefore can construct the associated flat bundle

$$L := l^2(\Gamma) \times_\Gamma \overline{M}$$

on M with fiber $l^2(\Gamma)$. Now we can twist D with this bundle L, i.e. define

$$\overline{D} := \nabla_L \otimes \mathrm{id} + \mathrm{id} \otimes D\colon C^\infty(L \otimes E) \to C^\infty(L \otimes E),$$

using the given flat connection ∇_L on L. Again, we can complete to $L^2(L \otimes E)$ and define

$$F := (\overline{D}^2 + 1)^{-1/2}\overline{D}.$$

The left action of Γ on $l^2\Gamma$ induces an action of Γ on L and then a unitary action on $L^2(L \otimes E)$. Since ∇_L preserves the Γ-action, $\overline{D}$ is Γ-equivariant. There is a canonical Γ-isometry between $L^2(L \otimes E)$ and $L^2(\overline{E})$ which identifies the two versions of $\overline{D}$ and F. The action of $C_0(\overline{M})$ on $L^2(L \otimes E)$ can be described by identifying $C_0(\overline{M})$ with the continuous sections of M on the associated bundle

$$C_0(\Gamma) \times_\Gamma \overline{M},$$

where $C_0(\Gamma)$ is the C^*-algebra of functions on Γ vanishing at infinity, and then using the obvious action of $C_0(\Gamma)$ on $l^2(\Gamma)$.

It is easy to see how this examples generalizes to Γ-equivariant elliptic differential operators on manifolds with a proper, but not necessarily free, Γ-action (with the exception of the last part, of course).

Work in progress of Baum and Schick [5] suggests the (somewhat surprising) fact that, given any proper Γ-CW-complex Y, we can, for each element $y \in KK_0^\Gamma(Y)$, find such a proper Γ-manifold X, together with a Γ-equivariant map $f\colon X \to Y$ and an elliptic differential operator on X giving an element $x \in KK_0^\Gamma(X)$ as in the example, such that $y = f_*(x)$.

Analytic K-homology is homotopy invariant, a proof can be found in [8].

1.3.38. Theorem. *If $\phi_1, \phi_2\colon X \to Y$ are proper Γ-equivariant maps which are homotopic through proper Γ-equivariant maps, then*

$$(\phi_1)_* = (\phi_2)_*\colon KK_*^\Gamma(X) \to KK_*^\Gamma(Y).$$

1.3.39. Theorem. *If Γ acts freely on X, then*

$$KK_*^\Gamma(X) \cong K_*(\Gamma\backslash X),$$

where the right hand side is the ordinary K-homology of $\Gamma\backslash X$.

1.3.40. Definition. Assume Y is an arbitrary proper Γ-CW-complex. Set

$$RK_*^\Gamma(Y) := \varinjlim KK_*^\Gamma(X),$$

where we take the direct limit over the direct system of Γ-invariant subcomplexes of Y with compact quotient (by the action of Γ).

1.3.41. Definition. To define higher (analytic) equivariant K-homology, there are two ways. The short one only works for complex K-homology. One considers cycles and an equivalence relation exactly as above — with the notable exception that one does not require any grading! This way, one defines $KK_1^\Gamma(X)$. Because of Bott periodicity (which has period 2), this is enough to define all K-homology groups ($KK_n^\Gamma(X) = KK_{n+2k}^\Gamma(X)$ for any $k \in \mathbb{Z}$).

A perhaps more conceptual approach is the following. Here, one generalizes the notion of a graded Hilbert space by the notion of a p-multigraded Hilbert space ($p \geq 0$). This means that the graded Hilbert space comes with p unitary operators $\epsilon_1, \dots, \epsilon_p$ which are odd with respect to the grading, which satisfy $\epsilon_i^2 = -1$ and $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 0$ for all i and j with $i \neq j$. An operator $T\colon H \to H$ on a p-multigraded Hilbert space is called *multigraded* if it commutes with $\epsilon_1, \dots, \epsilon_p$. Such operators can (in addition) be even or odd.

This definition can be reformulated as saying that a multigraded Hilbert space is a (right) module over the Clifford algebra Cl_p, and a multigraded operator is a module map.

We now define $KK_p^\Gamma(X)$ using cycles as above, with the additional assumption that the Hilbert space is p-graded, that the representation π takes values in π-multigraded even operators, and that the operator F is an odd p-multigraded operator. Isomorphism and equivalence of these multigraded cycles is defined as above, requiring that the multigradings are preserved throughout.

This definition gives an equivariant homology theory if we restrict to *proper* maps. Moreover, it satisfies Bott periodicity. The period is two for the (complex) K-homology we have considered so far. All results mentioned in this section generalize to higher equivariant K-homology.

If X is a proper Γ-CW-complex, the analytically defined representable equivariant K-homology groups $RK_p^\Gamma(X)$ are canonically isomorphic to the equivariant K-homology groups $K_p^\Gamma(X)$ defined by Davis and Lück in [14] as described in Section 1.3.5.1.

1.3.6 The assembly map

Here, we will use the homotopy theoretic description of equivariant K-homology due to Davis and Lück [14] described in Section 1.3.5.1. The assembly map then becomes particularly convenient to describe. From the present point of view, the main virtue is that they define a functor from *arbitrary*, not necessarily proper, Γ-CW-complexes to abelian groups.

The Baum-Connes assembly map is now simply defined using the equivariant collapse $E(\Gamma, fin) \to *$:

$$\mu\colon K_k^\Gamma(E(\Gamma, fin)) \to K_k^\Gamma(*) = K_k(C_r^*\Gamma). \tag{1.3.42}$$

If Γ is torsion-free, then $E\Gamma = E(\Gamma, fin)$, and the assembly map of (1.3.5) is defined as the composition of (1.3.42) with the appropriate isomorphism in Proposition 1.3.29.

1.3.7 Survey of KK-theory

The analytic definition of Γ-equivariant K-homology can be extended to a bivariant functor on Γ-C^*-algebras. Here, a Γ-C^*-algebra is a C^*-algebra A with an action (by C^*-algebra automorphisms) of Γ. If X is a proper Γ-space, $C_0(X)$ is such a Γ-C^*-algebra.

Given two Γ-C^*-algebras A and B, Kasparov defines the bivariant KK-groups $KK_*^\Gamma(A,B)$. The most important property of this bivariant KK-theory is that it comes with a (composition) product, the *Kasparov product*. This can be stated most conveniently as follows:

Given a discrete group Γ, we have a category KK^Γ whose objects are Γ-C^*-algebras (we restrict here to separable C^*-algebras). The morphisms in this category between two Γ-C^*-algebras A and B are called $KK_*^\Gamma(A,B)$. They are $\mathbb{Z}/2$-graded abelian groups, and the composition preserves the grading, i.e. if $\phi \in KK_i^\Gamma(A,B)$ and $\psi \in KK_j^\Gamma(B,C)$ then $\psi\phi \in KK_{i+j}^\Gamma(A,C)$.

There is a functor from the category of separable Γ-C^*-algebras (where morphisms are Γ-equivariant $*$-homomorphisms) to the category KK_*^Γ which maps an object A to A, and such that the image of a morphism $\phi\colon A \to B$ is contained in $KK_0^\Gamma(A,B)$.

If X is a proper cocompact Γ-CW-complex then (by definition)

$$KK_p^\Gamma(C_0(X), \mathbb{C}) = KK_{-p}^\Gamma(X).$$

Here, $\mathbb{C}$ has the trivial Γ-action.

On the other hand, for any C^*-algebra A without a group action (i.e. with trivial action of hte trivial group $\{1\}$), $KK_*^{\{1\}}(\mathbb{C}, A) = K_*(A)$.

There is a functor from KK^Γ to $KK^{\{1\}}$, called *descent*, which assigns to every Γ-C^*-algebra A the *reduced crossed product* $C_r^*(\Gamma, A)$. The crossed product has the property that $C_r^*(\Gamma, \mathbb{C}) = C_r^*\Gamma$.

1.3.8 KK assembly

We now want to give an account of the analytic definition of the assembly map, which was the original definition. The basic idea is that the assembly map is given by taking an index. To start with, assume that we have an even generalized elliptic Γ-operator (H, π, F), representing an element in $K_0^\Gamma(X)$, where X is a proper Γ-space such that $\Gamma\backslash X$ is compact. The index of this operator should give us an element in $K_0(C_r^*\Gamma)$. Since the cycle is even, H

split as $H = H_0 \oplus H_1$, and $F = \left(\begin{smallmatrix} 0 & P \\ P^* & 0 \end{smallmatrix}\right)$ with respect to this splitting. Indeed, now, the kernel and cokernel of P are modules over $\mathbb{C}\Gamma$, and should, in most cases, give modules over $C_r^*\Gamma$.

If Γ is finite, the latter is indeed the case (since $C_r^*\Gamma = \mathbb{C}\Gamma$). Moreover, since $\Gamma \backslash X$ is compact and Γ is finite, X is compact, which implies that $C_0(X)$ is unital. We may then assume that π is unital (switching to an equivalent cycle with Hilbert space $\pi(1)H$, if necessary). But then the axioms for a cycle imply that $F^2 - 1$ is compact, i.e. that F is invertible modulo compact operators, or that F is Fredholm, which means that $\ker(P)$ and $\ker(P^*)$ are finite dimensional. Since Γ acts on them, $[\ker(P)] - [\ker(P^*)]$ defines an element of the representation ring $R\Gamma = K_0(C_r^*\Gamma)$ for the finite group Γ. It remains to show that this map respects the equivalence relation defining $K_0^\Gamma(X)$.

However, if Γ is not finite, the modules $\ker(P)$ and $\ker(P^*)$, even if they are $C_r^*\Gamma$-modules, are in general not finitely generated projective.

To grasp the difficulty, consider Example 1.3.37. Using the description where F acts on a bundle over the base space M with infinite dimensional fiber $L \otimes E$, we see that loosely speaking, the null space of F should rather "contain" certain copies of $l^2\Gamma$ than copies of $C_r^*\Gamma$ (for finite groups, "accidentally" these two are the same!). However, in general $l^2\Gamma$ is not projective over $C_r^*\Gamma$ (although it is a module over this algebra). To be specific, assume that M is a point, $E_0 = \mathbb{C}$ and $E_1 = 0$, and $D = 0$. Here we obtain, $L^2(E_0) = l^2\Gamma$, $L^2(E_1) = 0$, $F = 0$, and indeed, $\ker(P) = l^2\Gamma$.

In the situation of our example, there is a way around this problem: Instead of twisting the operator D with the flat bundle $l^2(\Gamma) \times_\Gamma \overline{M}$, we twist with $C_r^*(\Gamma) \times_\Gamma \overline{M}$, to obtain an operator D' acting on a bundle with fiber $C_r^*\Gamma \otimes \mathbb{C}^{\dim E}$. This way, we replace $l^2\Gamma$ by $C_r^*\Gamma$ throughout. Still, it is not true in general that the kernels we get in this way are finitely generated projective modules over $C_r^*\Gamma$. However, it is a fact that one can always add to the new F' an appropriate compact operator such that this is the case. Then the obvious definition gives an element

$$\operatorname{ind}(D') \in K_0(C_r^*\Gamma).$$

This is the Mishchenko-Fomenko index of D' which does not depend on the chosen compact perturbation. Mishchenko and Fomenko give a formula for this index extending the Atiyah-Singer index formula.

One way to get around the difficulty in the general situation (not necessarily studying a lifted differential operator) is to deform (H, π, F) to an equivalent (H, π, F') which is better behaved (reminiscent to the compact perturbation above). This allows to proceeds with a rather elaborate generalization of the Mishchenko-Fomenko example we just considered, essentially replacing $l^2(\Gamma)$ by $C_r^*\Gamma$ again. In this way, one defines an index as an element of $K_*(C_r^*\Gamma)$.

This gives a homomorphism $\mu^\Gamma\colon KK_*^\Gamma(C_0(X)) \to K_*(C_r^*\Gamma)$ for each proper Γ-CW-complex X where $\Gamma\backslash X$ is compact. This passes to direct limits and defines, in particular,

$$\mu_*\colon RK_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma).$$

Next, we proceed with an alternative definition of the Baum-Connes map using KK-theory and the Kasparov product. The basic observation here is that, given any proper Γ-CW-space X, there is a specific projection $p \in C_r^*(\Gamma, C_0(X))$ (unique up to an appropriate type of homotopy) which gives rise to a canonical element $[L_X] \in K_0(C_r^*(\Gamma, C_0(X))) = KK_0(\mathbb{C}, C_r^*(\Gamma, C_0(X)))$. This defines by composition the homomorphism

$$\begin{aligned} KK_*^\Gamma(X) = KK_*^\Gamma(C_0(X), \mathbb{C}) &\xrightarrow{\text{descent}} KK_*(C_r^*(\Gamma, C_0(X)), C_r^*\Gamma) \\ &\xrightarrow{[L_X]\circ\cdot} KK_*(\mathbb{C}, C_r^*\Gamma) = K_*(C_r^*\Gamma). \end{aligned}$$

Again, this passes to direct limits and defines as a special case the Baum-Connes assembly map

$$\mu\colon RK_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma).$$

1.3.43. Remark. It is a non-trivial fact (due to Hambleton and Pedersen [28]) that this assembly map coincides with the map μ of (1.3.10).

Almost all positive results about the Baum-Connes have been obtained using the powerful methods of KK-theory, in particular the so called Dirac-dual Dirac method, compare e.g. [86].

1.3.9 The status of the conjecture

The Baum-Connes conjecture is known to be true for the following classes of groups.

(1) discrete subgroups of $SO(n,1)$ and $SU(n,1)$ [37]

(2) Groups with the *Haagerup property*, sometimes called *a-T-menable groups*, i.e. which admit an isometric action on some affine Hilbert H space which is proper, i.e. such that $g_n v \xrightarrow{n\to\infty} \infty$ for every $v \in H$ whenever $g_n \xrightarrow{n\to\infty} \infty$ in G [29]. Examples of groups with the Haagerup property are amenable groups, Coxeter groups, groups acting properly on trees, and groups acting properly on simply connected CAT(0) cubical complexes

(3) One-relator groups, i.e. groups with a presentation $G = \langle g_1, \dots, g_n \mid r \rangle$ with only one defining relation r [6].

(4) Cocompact lattices in $Sl_3(\mathbb{R})$, $Sl_3(\mathbb{C})$ and $Sl_3(\mathbb{Q}_p)$ ($\mathbb{Q}_p$ denotes the p-adic numbers) [43]

(5) Word hyperbolic groups in the sense of Gromov [57].

(6) Artin's full braid groups B_n [73].

Since we will encounter amenability later on, we recall the definition here.

1.3.44. Definition. A finitely generated discrete group Γ is called amenable, if for any given finite set of generators S (where we require $1 \in S$ and require that $s \in S$ implies $s^{-1} \in S$) there exists a sequence of finite subsets X_k of Γ such that

$$\frac{|SX_k := \{sx \mid s \in S, x \in X_k\}|}{|X_k|} \xrightarrow{k\to\infty} 1.$$

$|Y|$ denotes the number of elements of the set Y.

An arbitrary discrete group is called amenable, if each finitely generated subgroup is amenable.

Examples of amenable groups are all finite groups, all abelian, nilpotent and solvable groups. Moreover, the class of amenable groups is closed under taking subgroups, quotients, extensions, and directed unions.

The free group on two generators is not amenable. "Most" examples of non-amenable groups do contain a non-abelian free group.

There is a certain stronger variant of the Baum-Connes conjecture, the *Baum-Connes conjecture with coefficients.* It has the following stability properties:

(1) If a group Γ acts on a tree such that the stabilizer of every edge and every vertex satisfies the Baum-Connes conjecture with coefficients, the same is true for Γ [61].

(2) If a group Γ satisfies the Baum-Connes conjecture with coefficients, then so does every subgroup of Γ [61]

(3) If we have an extension $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$, Γ_3 is torsion-free and Γ_1 as well as Γ_3 satisfy the Baum-Connes conjecture with coefficients, then so does Γ_2.

It should be remarked that in the above list, all groups except for word hyperbolic groups, and cocompact subgroups of Sl_3 actually satisfy the Baum-Connes conjecture with coefficients.

The Baum-Connes assembly map μ of (1.3.10) is known to be rationally injective for considerably larger classes of groups, in particular the following.

(1) Discrete subgroups of connected Lie groups [38]

(2) Discrete subgroups of p-adic groups [39]

(3) Bolic groups (a certain generalization of word hyperbolic groups) [40].

(4) Groups which admit an amenable action on some compact space [31].

Last, it should be mentioned that recent constructions of Gromov show that certain variants of the Baum-Connes conjecture, among them the Baum-Connes conjecture with coefficients, and an extension called the *Baum-Connes conjecture for groupoids*, are false [30]. At the moment, no counterexample to the Baum-Connes conjecture 1.3.9 seems to be known. However, there are many experts in the field who think that such a counterexample eventually will be constructed [30].

1.4 Real C^*-algebras and K-theory

1.4.1 Real C^*-algebras

The applications of the theory of C^*-algebras to geometry and topology we present here require at some point that we work with real C^*-algebras. Most of the theory is parallel to the theory of complex C^*-algebras.

1.4.1. Definition. A unital real C^*-algebra is a Banach-algebra A with unit over the real numbers, with an isometric involution $*\colon A \to A$, such that

$$|x|^2 = |x^*x| \qquad \text{and } 1 + x^*x \text{ is invertible} \quad \forall x \in A.$$

It turns out that this is equivalent to the existence of a $*$-isometric embedding of A as a closed subalgebra into $\mathcal{B}H_{\mathbb{R}}$, the bounded operators on a suitable real Hilbert space (compare [62]).

1.4.2. Example. If X is a compact topological space, then $C(X;\mathbb{R})$, the algebra of real valued continuous function on X, is a real C^*-algebra with unit (and with trivial involution).

More generally, if X comes with an involution $\tau\colon X \to X$ (i.e. $\tau^2 = \mathrm{id}_X$), then $C_\tau(X) := \{f\colon X \to \mathbb{C} \mid f(\tau x) = \overline{f(x)}\}$ is a real C^*-algebra with involution $f^*(x) = \overline{f(\tau x)}$.

Conversely, every commutative unital real C^*-algebra is isomorphic to some $C_\tau(X)$.

If X is only locally compact, we can produce examples of non-unital real C^*-algebras as in Example 1.2.2.

Essentially everything we have done for (complex) C^*-algebras carries over to real C^*-algebras, substituting $\mathbb{R}$ for $\mathbb{C}$ throughout. In particular, the definition of the K-theory of real C^*-algebras is literally the same as for complex C^*-algebras (actually, the definitions make sense for even more general topological algebras), and a short exact sequence of real C^*-algebras gives rise to a long exact K-theory sequence.

The notable exception is Bott periodicity. We don't get the period 2, but the period 8.

1.4.3. Theorem. *Assume that A is a real C^*-algebra. Then we have a Bott periodicity isomorphism*

$$K_0(A) \cong K_0(S^8 A).$$

This implies

$$K_n(A) \cong K_{n+8}(A) \qquad \text{for } n \geq 0.$$

1.4.4. Remark. Again, we can use Bott periodicity to define $K_n(A)$ for arbitrary $n \in \mathbb{Z}$, or we may view $K_n(A)$ as an 8-periodic theory, i.e. with $n \in \mathbb{Z}/8$.

The long exact sequence of Theorem 1.2.18 becomes a 24-term cyclic exact sequence.

The *real reduced C^*-algebra* of a group Γ, denoted $C^*_{\mathbb{R},r}\Gamma$, is the norm closure of $\mathbb{R}\Gamma$ in the bounded operators on $l^2\Gamma$.

1.4.2 Real K-homology and Baum-Connes

A variant of the cohomology theory given by complex vector bundles is KO-theory, which is given by real vector bundles. The homology theory dual to this is KO-homology. If KO is the spectrum of topological KO-theory, then $KO_n(X) = \pi_n(X_+ \wedge KO)$.

The homotopy theoretic definition of equivariant K-homology can be varied easily to define equivariant KO-homology. The analytic definition can also be adapted easily, replacing $\mathbb{C}$ by $\mathbb{R}$ throughout, using in particular real Hilbert spaces. However, we have to stick to n-multigraded cycles to define $KK_n^\Gamma(X)$, it is not sufficient to consider only even and odd cycles.

All the constructions and properties translate appropriately from the complex to the real situation, again with the notable exception that Bott periodicity does not give the period 2, but 8. The upshot of all of this is that we get a real version of the Baum-Connes conjecture, namely

1.4.5. Conjecture. *The real Baum-Connes assembly map*

$$\mu_n\colon KO_n^\Gamma(E(\Gamma, fin)) \to KO_n(C^*_{\mathbb{R},r}\Gamma),$$

is an isomorphism.

It should be remarked that all known results about injectivity or surjectivity of the Baum-Connes map can be proved for the real version as well as for the complex version, since each proof translates without too much difficulty. Moreover, it is known that the complex version of the Baum-Connes conjecture for a group Γ implies the real version (for this abstract result, the isomorphism is needed as input, since this is based on the use of the five-lemma at a certain point).

Chapter 2

A counterexample to the Gromov-Lawson-Rosenberg conjecture

The Gromov-Lawson-Rosenberg conjecture is discussed in the notes by Stephan Stolz. As a reminder, we quickly recall the problem:

2.0.1. Question. Given a compact smooth spin-manifold M without boundary, when does M admit a Riemannian metric with positive scalar curvature?

Recall that a spin-manifold is a manifold for which the first and second Stiefel-Whitney class of the tangent bundle vanish. The spin condition can be compared to the condition that a manifold is orientable. Indeed, every spin-manifold is orientable. But the spin condition is considerably stronger (it is like orientability "squared").

The reason that we concentrate on spin-manifolds is that powerful obstructions to the existence of a metric with positive scalar curvature have been developed for them.

2.1 Obstructions to positive scalar curvature

2.1.1 Index theoretic obstructions

We start with a discussion of the index obstruction for spin manifolds to admit a metric with scal > 0, constructed by Lichnerowicz [45], Hitchin [33] and in the following refined version due to Rosenberg [64].

2.1.1. Theorem. *One can construct a homomorphism, called index, from the singular spin bordism group $\Omega_*^{spin}(B\pi)$ to the (real) KO-theory of the reduced real C^*-algebra of π:*

$$\mathrm{ind}\colon \Omega_*^{spin}(B\pi) \to KO_*(C^*_{\mathbb{R},r}\pi)$$

(this homomorphism is often called α instead of ind*). Assume $f\colon N \to B\pi$ represents an element of $\Omega_m^{spin}(B\pi)$. If N admits a metric with positive scalar curvature, then*

$$\mathrm{ind}([f\colon N \to B\pi]) = 0 \in KO_m(C^*_{\mathbb{R},r}\pi)$$

The converse of this theorem is the content of the following *Gromov-Lawson-Rosenberg conjecture.*

2.1.2. Conjecture. *Let M be a compact spin-manifold without boundary, $\pi = \pi_1(M)$, and $u\colon M \to B\pi$ be the classifying map for a universal covering of M. Assume that $m = \dim(M) \geq 5$.*

Then M admits a metric with $\mathrm{scal} > 0$ *if and only if*

$$\mathrm{ind}[u\colon M \to B\pi] = 0 \in KO_m(C^*_{\mathbb{R},r}\pi).$$

This conjecture was developed in [27] and [63].

The restriction to dimensions ≥ 5 comes from the observation that in these dimensions (and not below) the question of existence of metrics with $\mathrm{scal} > 0$ in a certain sense is a bordism invariant, which of course fits with the structure of the obstruction described in Theorem 2.1.1. Failure of this bordism invariance in dimension 4 is also reflected by the fact that for 4-dimensional manifolds, the *Seiberg-Witten invariants* provide additional obstructions to the existence of a metric with positive scalar curvature, which show in particular that the conjecture is not true if $m = 4$.

The conjecture was proved by Stefan Stolz [80] for $\pi = 1$, and subsequently by him and other authors also for some other groups [63, 42, 10, 65, 34].

2.1.2 Minimal surface obstructions

In dimension ≥ 5 there is only one known additional obstruction for positive scalar curvature metrics, the minimal surface method of Schoen and Yau, which we will recall now. (In dimension 4, the Seiberg-Witten theory yields additional obstructions).

The first theorem is the differential geometrical backbone for the application of minimal surfaces to the positive scalar curvature problem:

2.1.3. Theorem. *Let (M^m, g) be a manifold with* $\operatorname{scal} > 0$, $\dim M = m \geq 3$. *If V is a smooth $(m-1)$-dimensional submanifold of M with trivial normal bundle, and if V is a local minimum of the volume functional, then V admits a metric of positive scalar curvature, too. "Local minimum" means that for any small deformation of the hypersurface, the $(m-1)$-volume of the surface increases.*

Actually, V can be a "minimal hypersurfaces" in the sense of differential geometry, defined in terms of curvature and second fundamental form of the hypersurface. Every local minimum for the $(m-1)$-volume is such a minimal hypersurface; the converse is not true.

Proof. Schoen/Yau: [75, 5.1] for $m = 3$, [76, proof of Theorem 1] for $m > 3$. We outline the proof, following closely [76, Theorem 1].

Given V, since its normal bundle is trivial, any smooth function ϕ on V gives rise to a variation (if ν is the unit normal vector field, pushing V in normal direction $\phi\nu$). Lt $\operatorname{Ric} \in \Gamma(\operatorname{End}(TM))$ be the Ricci curvature, considered as an operator on each fiber of TM. Let l be the second fundamental form of V. It is well known that minimality implies $\operatorname{tr}(l) = 0$. Moreover, the second variation of the area is non-negative. It is given by (see [13])

$$-\int_V \left(\langle \operatorname{Ric}(\nu), \nu\rangle + |l|^2\right)\phi^2 + \int_V |\nabla\phi|^2 \geq 0. \tag{2.1.4}$$

We now use the Gauss curvature equation (the "theorema egregium") to relate this to the scalar curvature of the submanifold. Taking appropriate traces of the Gauss equations, we obtain

$$\operatorname{scal}_V = \operatorname{scal}_M - 2\langle \operatorname{Ric}(\nu), \nu\rangle + (\operatorname{tr} l)^2 - |l|^2\,, \tag{2.1.5}$$

where scal_V is the scalar curvature of V with the induced Riemannian metric and scal_M the scalar curvature of M. Putting Equation (2.1.5) into Inequality (2.1.4), we have

$$\int_V \operatorname{scal}_M \phi^2 - \int_V \operatorname{scal}_V \phi^2 + \int_V |l|^2\,\phi^2 \leq 2\int_V |\nabla\phi|^2 \tag{2.1.6}$$

for all smooth functions $\phi\colon V \to \mathbb{R}$. We assume that the scalar curvature of M is everywhere strictly positive. Hence (2.1.6) implies

$$-\int_V \operatorname{scal}_V \phi^2 < 2\int_V |\nabla\phi|^2\,,$$

as long as ϕ is not identically zero.

Consider the *conformal Laplacian* $\Delta_c := \Delta + \frac{n-3}{4(n-2)} \operatorname{scal}_V$ on V (where Δ is the positive Laplacian on functions). Then all eigenvalues of Δ_c are strictly positive. Assume, otherwise, that ϕ is an eigenfunction to the eigenvalue $\lambda \leq 0$, i.e.

$$\Delta\phi = -\frac{m-3}{4(m-2)} \operatorname{scal}_V \phi + \lambda\phi.$$

Taking the L^2-inner product of this equation with ϕ (and integration by parts) gives

$$\int_V |\nabla\phi|^2 = -\frac{m-3}{4(m-2)} \int_V \operatorname{scal}_V \phi^2 + \lambda \int_V \phi^2 < \frac{m-3}{2(m-2)} \int_V |\nabla\phi|^2,$$

which is a contradiction. Now it's a standard fact in conformal geometry that, if the conformal Laplacian has only positive eigenvalues, then one can conformally deform the metric to a metric with positive scalar curvature (compare [41]). This is done in two steps: a generalized maximum principle implies that we can find an eigenfunction f to the first eigenvalues of Δ_c which is strictly positive everywhere. Then, explicit formulas for the scalar curvature of a conformally changed metric show that $f^{4/(m-3)}g$ indeed has a metric with $\operatorname{scal} > 0$.

Hence, on V there exists a metric with $\operatorname{scal} > 0$ (observe, however, that it is not necessarily the metric induced from M, but only conformally equivalent to this metric). □

The next statement due to Simons and Smale (special cases due to Fleming and Almgren) from geometric measure theory implies applicability of the previous theorem if $\dim(M) \leq 8$.

2.1.7. Theorem. *Suppose M is an orientable Riemannian manifold of dimension $\dim M = m \leq 8$. Furthermore let $\alpha \in H^1(M, \mathbb{Z})$. Then*

$$x := \alpha \cap [M] \in H_{m-1}(M, \mathbb{Z})$$

can be represented by an embedded hypersurface V with trivial normal bundle which is a local minimum for $(m-1)$-volume (if $m = 8$ with respect to suitable metrics arbitrarily close in C^3 to the metric we started with).

Proof. For $m \leq 7$ this is a classical result of geometric measure theory (cf. [59, Chapter 8]) and references therein, in particular [23, 5.4.18].

The case $m = 8$ follows from the following result of Nathan Smale [79]: the set of C^k-metrics for which the regularity statement holds is open and dense in the set of all C^k-metrics ($k \geq 3$ and with the usual Banach-space topology). We are only interested in C^∞-metrics. But these are dense in the set of C^k-metrics, which concludes the proof. $\square$

Unfortunately, the proofs of the theorems we have cited are very involved and require a lot of technical work. Therefore, we don't attempt to indicate the arguments here.

Recall that if we are given a class $\alpha \in H^1(M, \mathbb{Z})$ we may represent it by a map $f\colon M \to S^1$ being transverse to $1 \in S^1$. Then $V = f^{-1}(1) \subset M$ represents $\alpha \cap [M]$ (and conversely, every hypersurface representing $\alpha \cap [M]$ is obtained in this way). Furthermore, if $f'\colon M \to S^1$ is a second map as above and $V' = f'^{-1}(1)$ then f and f' are homotopic, and a homotopy $H\colon f \simeq f'$ being transverse to $1 \in S^1$ provides a bordism $W = H^{-1}(1)\colon V \sim V'$ embedded in $M \times [0,1]$. Since the normal bundle of $V \subset M$ and $W \subset M \times [0,1]$,respectively, is trivial, the manifolds and bordisms we construct this way belong to the spin-category, if we start with a spin-manifold M.

We want to use these ideas to construct, for an arbitrary space X, a map

$$\cap\colon H^1(X, \mathbb{Z}) \times \Omega_m^{spin}(X) \to \Omega_{m-1}^{spin}(X). \tag{2.1.8}$$

To do this, let $\phi\colon M \to X$ be a singular spin manifold for X, representing an element in $\Omega_m^{spin}(X)$. If $f\colon X \to S^1$ represents an element $\alpha \in H^1(X, \mathbb{Z})$, then $f \circ \phi$ is homotopic to a map $\psi\colon M \to S^1$ which is transverse to $1 \in S^1$.

Restricting ϕ to $V := \psi^{-1}(1)$ then gives a singular spin manifold $\phi|_V\colon V \to X$, which by definition represents $\alpha \cap [\phi\colon M \to X] \in \Omega_{m-1}^{spin}(X)$.

We have to check that this is well defined. To do this, let $\Phi\colon W \to X$ be a spin-bordism between $\phi\colon M \to X$ and $\phi'\colon M' \to X$. Then $f \circ \Phi$ is homotopic to a map $\Psi\colon W \to S^1$ with Ψ and $\Psi|\partial W$ being transverse to $1 \in S^1$; moreover, the map $\Psi|\partial W$ with the corresponding properties may be given in advance. Then $\Psi^{-1}(1) \subset W$ is a spin-bordism between the hypersurfaces $V = (\Psi)^{-1}(1) \cap M \subset M$ and $\Psi^{-1}(1) \cap M' \subset M'$, and restricting Φ to $\Psi^{-1}(1)$ now yields a singular spin-bordism between singular spin-hypersurfaces into X.

If $f'\colon X \to S^1$ is homotopic to f, a similar construction gives a singular spin-bordism between the resulting singular spin-hypersurfaces into X. Together, this implies that our map indeed is well defined.

The two theorems above now imply

2.1.9. Theorem. *Let X be a space and let $3 \leq m \leq 8$. Then* (2.1.8) *restricts to a homomorphism:*

$$\cap\colon H^1(X, \mathbb{Z}) \times \Omega_m^{spin,+}(X) \to \Omega_{m-1}^{spin,+}(X) \tag{2.1.10}$$

where $\Omega_*^{spin,+}(X) \subset \Omega_*^{spin}(X)$ *is the subgroup of bordism classes which can be represented by singular manifolds which admit a metric with* scal > 0 *(observe, only one representative with* scal > 0 *is required).*

Proof. Theorem 2.1.7 implies that, given $f\colon M \to S^1$ (dual to a given class in $H_{m-1}(M,\mathbb{Z})$) we find a homotopic map $g\colon M \to S^1$ which is transverse to 1 and such that the hypersurface $V = g^{-1}(1)$ is minimal for the $(m-1)$-volume (in dimension 8 we replace the given metric by one which is C^3-close). In any case, since the scalar curvature is continuous with respect to the C^3-topology on the space of all Riemannian metrics, V is volume minimizing with respect to a metric with positive scalar curvature whenever we start with such a metric. By Theorem 2.1.3, it admits a metric with scal > 0. □

To be honest, this does not quite give an obstruction, but rather a method to produce counterexamples. Namely, if we know $\Omega_{n-1}^{spin,+}(B\pi)$, and also the cap-product of (2.1.8) well enough, we can get information about $\Omega_n^{spin,+}(B\pi)$ (with $n \leq 8$).

Obviously, one does need some information to start with. This can be obtained in dimension 2 using the Gauss-Bonnet theorem.

2.1.3 Gauss-Bonnet obstruction in dimension 2

2.1.11. Theorem. *Let G be a discrete group. Then*

$$\Omega_2^{spin,+}(BG) := \left\{ \begin{array}{c} \textit{bordism classes } [M \to BG] \in \Omega_2^{spin}(BG), \\ \textit{where } M \textit{ admits a metric with } \mathrm{scal} > 0 \end{array} \right\} = 0. \tag{2.1.12}$$

Proof. By the Gauss-Bonnet theorem there is only one orientable 2-manifold with positive (scalar) curvature, namely S^2. On the other hand, S^2 is a spin-manifold with a unique spin-structure, and is spin-bordant to zero, being the boundary of D^3. Since $\pi_2(BG)$ is trivial, up to homotopy only the trivial map from S^2 to BG exists. Therefore only the trivial element in $\Omega_2^{spin}(BG)$ can be represented by a manifold with positive scalar curvature. □

2.2 Construction of the counterexample

2.2.1 Application of the minimal hypersurface obstruction

Now, we will construct a particular example of a manifold which does not admit a metric with positive scalar curvature, using the minimal hypersurface obstruction.

Let $p\colon S^1 \to B\mathbb{Z}/3$ be a map so that $\pi_1(p)$ is surjective and equip S^1 with the spin structure induced from D^2. Consider the singular manifold

$$f = \mathrm{id} \times p\colon T^5 = S^1 \overbrace{\times \cdots \times}^{4} S^1 \times S^1 \to S^1 \overbrace{\times \cdots \times}^{4} S^1 \times B\mathbb{Z}/3 = B\pi,$$

where $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$. This represents a certain element $x \in \Omega_5^{spin}(B\pi)$.

We have four distinguished maps from $B\pi$ to S^1, given by the projections $p_i\colon B\pi \to S^1$ onto each of the first four factors. Let $a_1, \dots, a_4 \in H^1(B\pi)$ be the corresponding elements in cohomology. Using the description of the cap-product given before (2.1.8), in our situation it is easy to find a representative for

$$z := a_1 \cap (a_2 \cap (a_3 \cap w)) \in \Omega_2^{spin}(B\pi).$$

Namely, taking inverse images of the base point, this z is given by

$$g = * \times * \times * \times \mathrm{id} \times p\colon T^2 = * \times * \times * \times S^1 \times S^1 \to S^1 \times S^1 \times S^1 \times S^1 \times B\mathbb{Z}/3 = B\pi.$$

We want to show that $z \notin \Omega_2^{spin,+}(B\pi)$, because then, by (2.1.10), $x \notin \Omega_5^{spin,+}(B\pi)$, i.e. whenever we find a representative $[f\colon M \to B\pi] = x$, then M does not admit a Riemannian metric with scal > 0 (in particular, this follows then for T^5, which, however, is not the manifold we are interested in here).

Because of Theorem 2.1.11 we only have to show that z is a non-trivial element of $\Omega_2^{spin}(B\pi)$. We have the natural homomorphism $\Omega_*^{spin}(B\pi) \to H_*(B\pi, \mathbb{Z})$, which maps $[f\colon M \to B\pi]$ to $f_*[M]$, i.e. to the image of the fundamental class of M, and the Künneth theorem implies immediately that the image of z under this homomorphism in $H_2(B\pi)$ is non-trivial, therefore the same is true for z.

2.2.2 Calculation of the index obstruction

We proceed by proving that the index obstruction 2.1.1 does vanish for the example constructed in Subsection 2.2.1.

This index obstruction is an element of $KO_5(C^*_{\mathbb{R},r}\pi)$, where $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$. First, we compute this K-theory group to the extent needed here. By a Künneth theorem for the K-theory of C^*-algebras, the K-theory of $C^*_{\mathbb{R},r}(G \times \mathbb{Z})$ can easily be computed from the K-theory of $C^*_{\mathbb{R},r}G$. Namely, by [77, p. 14–15 and 1.5.4]

$$KO_n(C^*_{\mathbb{R},r}(\mathbb{Z}^4 \times \mathbb{Z}/3)) \cong \bigoplus_{i=1}^{16} KO_{n-n_i}(C^*_{\mathbb{R},r}(\mathbb{Z}/3)); \qquad \text{for suitable } n_i \in \mathbb{N}.$$

For a finite group G, it is well known that $KO_*(C^*_{\mathbb{R},r}(G))$ is a direct sum of copies of the (known) KO-theories of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In particular, it is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. Therefore, the same is true for π. This implies the following Proposition.

2.2.1. Proposition. *$KO_*(C^*_{\mathbb{R},r}\pi)$ is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. In particular, its torsion is only 2-torsion.*

Let $p\colon S^1 \to B\mathbb{Z}/3$ be the map of Subsection 2.2.1 so that $\pi_1(p)$ is surjective (and S^1 is equipped with the spin structure induced from D^2). This represents a 3-torsion element y in $\Omega_1^{spin}(B\mathbb{Z}/3)$ since

$$\tilde{\Omega}_1^{spin}(B\mathbb{Z}/3) \cong H_1(B\mathbb{Z}/3, \mathbb{Z}) \cong \mathbb{Z}/3$$

(using e.g. the Atiyah-Hirzebruch spectral sequence).

It follows that

$$x = [\mathrm{id}_{(S^1)^4} \times p\colon T^5 \to B(\mathbb{Z}^4 \times \mathbb{Z}/3)]$$

is also 3-torsion (a zero bordism for $3x$ is obtained as the product of a zero bordism for $3y$ with $\mathrm{id}_{(S^1)^4}$).

Since $\mathrm{ind}\colon \Omega_5^{spin}(B\pi) \to KO_5(C^*_{\mathbb{R},r}\pi)$ is a group homomorphism,

$$3 \cdot \mathrm{ind}(x) = 0 \in KO_5(C^*_{\mathbb{R},r}\pi).$$

But for $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$, by Proposition 2.2.1 this implies that $\mathrm{ind}(x) = 0$, i.e. the index obstruction vanishes.

2.2.3 Surgery to produce the counterexample

So far, we have found a bordism class $x \in \Omega_5^{spin}(B\pi)$ ($\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$) such that the index obstruction 2.1.1 vanishes for x, but on the other hand no representative $[f\colon M \to B\pi]$ can be found such that M has a metric with positive scalar curvature. To give a counterexample to Conjecture 2.1.2, we have to find a representative such that f induces an isomorphism on fundamental groups (i.e. is the classifying map for the universal covering of M). This is not the case for the tori we have explicitly constructed so far (and indeed, for tori one can use the *index method* to show that they do not admit a metric with positive scalar curvature).

But adjusting the fundamental group is easy. We only have to perform surgery on our explicitly given torus T^5. That is, we have to choose an embedded $S^1 \to T^5$ which represents the kernel of $\pi_1(f)\colon \pi_1(T^5) \to \pi_1(B\pi)$

(observe that in this situation, the kernel actually is cyclic) and which has a trivial normal bundle. Then a tubular neighborhood of S^1 is diffeomorphic to $S^1 \times D^4$, with boundary $S^1 \times S^3$. We can now cut away this tubular neighborhood and glue in $D^2 \times S^3$ (also with boundary $S^1 \times S^3$) instead. The fundamental group of the new manifold M' is the quotient of the fundamental group of the old manifold by the (normal) subgroup generated by the loop we started with, i.e. is isomorphic to π. Let $u\colon M' \to B\pi$ be the classifying map for the universal covering. Using classical "surgery below the middle dimension", we can arrange all this in such a way that

$$[f : T^5 \to B\pi] = [u\colon M' \to B\pi] \in \Omega_5^{spin}(B\pi)$$

(compare [82, Lemma 5.6]). Consequently, M' is a counterexample to the Gromov-Lawson-Rosenberg conjecture 2.1.2.

2.3 Other questions, other examples

The index map of Theorem 2.1.1 admits a factorization

$$\mathrm{ind}\colon \Omega_*^{spin}(B\pi) \xrightarrow{D} ko_*(B\pi) \xrightarrow{p} KO_*(B\pi) \xrightarrow{\mu} KO_*(C^*_{r,\mathbb{R}}\pi).$$

Here, ko_* is connective real K-homology, KO_* the periodic real K-homology we have considered so far, D is the ko-theoretic orientation, p the canonical map between the connective and the periodic theory, and μ the assembly map in topological K-theory. Note that for torsion free groups, this μ is the Baum-Connes map, and the Baum-Connes conjecture states that this map is an isomorphism.

The original conjecture of Gromov and Lawson asserted that the vanishing of the image of $[u\colon M \to B\pi_1(M)]$ in $KO_m(B\pi)$ decides whether M admits a metric with positive scalar curvature. Rosenberg observed that there are manifolds with $\mathrm{scal} > 0$ for which this element does not vanish, and proposed to modify the conjecture as stated in Conjecture 2.1.2. We adopt the convention that $u\colon M \to B\pi_1(M)$ denotes the classifying map for the universal covering of M.

However, the following question remains.

2.3.1. Question. Is the stronger vanishing condition that

$$pD[u\colon M \to B\pi_1(M)] = 0$$

sufficient for the existence of metrics with positive scalar curvature?

If even $D[u\colon M \to B\pi_1(M)] = 0$, then M admits a metric with positive scalar curvature by a result of Stephan Stolz [81] (as usual, we have to assume that $\dim(M) \geq 5$).

In [34], a counterexample to question 2.3.1 is given. The first step to construct the counterexample is to find a group such that

$$p\colon ko_*(B\pi) \to KO_*(B\pi)$$

has a kernel, and since we want to use the minimal surface method, this kernel should be given for $* = 2$. In [34], this is done using explicit K-homology calculations for finite groups. The remaining proof is very much along the lines of the proof we have given above.

One of the virtues of the example we have given is that we avoid the calculation of the index. This is replaced by some (easy) considerations about torsion. To be able to do this, we used a fundamental group π with torsion. Dwyer and Stolz (unpublished) have constructed a counterexample to the Gromov-Lawson-Rosenberg conjecture with torsion-free fundamental group. In [74] a refinement of this is given where the classifying space $B\pi$ is a manifold with negative curvature. The first key idea is the same as in the example in [34] just described, namely to find an element in the kernel of $ko_2(B\pi) \to KO_2(B\pi)$. To find a π such that $B\pi$ is particularly nice (e.g. finite dimensional, which implies that π is torsion-free, or even a manifold of negative curvature) one uses asphericalization procedures of Baumslag, Dyer and Heller, or Charney, Davis, and Januszkiewicz, which produce nice $B\pi$ with certain prescribed homological properties (starting with (worse) spaces which have these same homological properties). More constructions of this kind are described in the lectures of Mike Davis.

The positive scalar curvature question makes sense also for manifolds which are not spin manifolds. There are "twisted" index obstructions as long as the universal covering is a spin manifold, and one can formulate an appropriate "twisted Gromov-Lawson-Rosenberg" conjecture. In [34], counterexamples to this twisted conjecture are given, as well.

Chapter 3

L^2-cohomology and the conjectures of Atiyah, Singer, and Hopf

L^2-cohomology and L^2-Betti numbers are certain "higher invariants" of manifolds and more general spaces. They were introduced 1976 by Michael Atiyah in [1], and since then have proved to be useful invariants with connections and applications in many other mathematical fields, from differential geometry to group theory and algebra. Apart from the original literature, there exists Lücks informative survey article [50], and also Eckmann lecture notes [20]. Moreover, at the time of writing of this article, Lück's very comprehensive textbook/research monograph [51] is almost finished, and the current version is available from the author's homepage. This chapter is a survey style article which focuses on the main points of the very extensive subject, leaving out many of the less illuminating details, which can be found e.g. in [51].

3.1 Analytic L^2-Betti numbers

3.1.1. Definition. Let $\overline{M}$ be a (not necessarily compact) Riemannian manifold without boundary, which is complete as a metric space. Define

$$L^2\Omega^p(\overline{M}) := \{\omega \text{ measurable } p\text{-form on } M \mid \int_{\overline{M}} |\omega(x)|_x^2 \; d\mu(x) < \infty\}.$$

Here, $|\omega(x)|_x$ is the pointwise norm (at $x \in \overline{M}$) of $\omega(x)$, which is given by the Riemannian metric, and $d\mu(x)$ is the measure induced by the Riemannian metric.

$L^2\Omega^p(\overline{M})$ can be considered as the Hilbert space completion of the space of compactly supported p-forms on $\overline{M}$. The inner product is given by integrating the pointwise inner product, i.e.

$$\langle \omega, \eta \rangle_{L^2} := \int_{\overline{M}} \langle \omega(x), \eta(x) \rangle_x \; d\mu(x).$$

3.1.2. Definition. Let M be a smooth compact Riemannian manifold without boundary, with Riemannian metric g. Let $\overline{M}$ be a normal covering of M, i.e. if Γ is the deck transformation group, then $M = \overline{M}/\Gamma$. Lift the metric g to $\overline{M}$. Then Γ acts isometrically on $\overline{M}$. Let $\overline{\Delta}_p$ be the Laplacian on p-forms on $\overline{M}$. This gives rise to an unbounded operator

$$\overline{\Delta}_p \colon L^2\Omega^p(\overline{M}) \to L^2\Omega^p(\overline{M}).$$

This operator is an elliptic differential operator (but on the not necessarily compact manifold $\overline{M}$) Let

$$\mathrm{pr}_p \colon L^2\Omega^p(\overline{M}) \to L^2\Omega^p(\overline{M})$$

be the orthogonal projection onto $\ker(\overline{\Delta}_p)$. Ellipticity of $\overline{\Delta}_p$ implies that pr_p has a smooth integral kernel, i.e. that there is a smooth section $\mathrm{pr}_p(x,y)$ over $\overline{M} \times \overline{M}$ (of the bundle with fiber $\mathrm{Hom}(\Lambda^p T_y^* M, \Lambda^p T_x^* M)$ over $(x,y) \in \overline{M} \times \overline{M}$), such that

$$\mathrm{pr}_p\, \omega(x) = \int_{\overline{M}} \mathrm{pr}_p(x,y)\omega(y) \; d\mu(y)$$

for every L^2-p-form on $\overline{M}$.

We define the L^2-cohomology of $\overline{M}$ by

$$H^p_{(2)}(M) := \ker(\overline{\Delta}^p) = \mathrm{im}(\mathrm{pr}_p).$$

It is an easy observation that, given a projection $P \colon V \to V$ on a finite dimensional vector space V, then $\dim(\mathrm{im}(P)) = \mathrm{tr}(P)$. On the other hand, the trace of an operator with a smooth integral kernel can be computed by integration over the diagonal.

We want to use these ideas to define a useful dimension for $\ker(\Delta_p)$. Note, however, that the Laplacian $\overline{\Delta}_p$ is defined using the Riemannian metric on

$\overline{M}$. It follows that it commutes with the induced action of Γ on $L^2\Omega^p(\overline{M})$. Consequently, $\ker(\overline{\Delta}_p)$ is Γ-invariant, and

$$\operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x}) = \operatorname{tr}_{g\overline{x}} \operatorname{pr}_p(g\overline{x},g\overline{x}) \qquad \forall \overline{x} \in \overline{M}, g \in \Gamma.$$

Observe that $\operatorname{pr}_p(\overline{x},\overline{x}) \in \operatorname{End}(\Lambda^p T_x^* M)$ is an endomorphism of a finite dimensional vector space for each $\overline{x} \in \overline{M}$, and $\operatorname{tr}_{\overline{x}}$ is the usual trace of such endomorphisms. If $\overline{M}$ is not compact (i.e. if Γ is infinite), it follows that

$$\int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})\; d\mu(x)$$

does not converge, and indeed, in general $\ker(\overline{\Delta}_p)$ is not a finite dimensional $\mathbb{C}$-vector space.

On the other hand, because of the Γ-invariance, the function

$$\overline{x} \mapsto \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})$$

"contains the same information many times", and it doesn't make sense to try to compute the integral over all of $\overline{M}$. We therefore adopt the notion of "dimension per (unit) volume".

More concretely, because of Γ-invariance, the function $\overline{x} \mapsto \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})$ descents to a smooth function on the quotient $\overline{M}/\Gamma = M$. We now define the L^2-Betti numbers

$$b^p_{(2)}(\overline{M},\Gamma) := \dim_\Gamma \ker(\overline{\Delta}_p) := \int_M \operatorname{tr}_x \operatorname{pr}_p(x,x)\; d\mu(x) \in [0,\infty).$$

This number is a non-negative real number. However, a priori no further restrictions appear for these values.

Extensions of all these definitions to manifolds with boundary are possible, compare e.g. [69].

The definition given here is the original definition of L^2-Betti numbers as given by Atiyah. Of course, the same construction can be applied to any elliptic differential operator D on M. If D is such an elliptic differential operator, and D^* its formal adjoint, Atiyah defined in this way the Γ-index of the lift $\overline{D}$ of D to $\overline{M}$ by

$$\operatorname{ind}_\Gamma(\overline{D}) := \dim_\Gamma(\ker \overline{D}) - \dim_\Gamma(\ker \overline{D^*}).$$

Atiyah's celebrated L^2-index theorem now states

3.1.3. Theorem.

$$\operatorname{ind}_\Gamma(\overline{D}) = \operatorname{ind}(D).$$

Here, recall that ellipticity of D and compactness of M imply that $\ker(D)$ and $\ker(D^*)$ are finite dimensional $\mathbb{C}$-vector spaces, and

$$\operatorname{ind}(D) = \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\ker D^*).$$

It should be observed that it is far from true in general that $\dim_\Gamma(\ker \overline{D}) = \dim_{\mathbb{C}}(\ker D)$. In particular, the L^2-Betti numbers and the ordinary Betti numbers usually are quite different from each other. However, Atiyah's L^2-index theorem has the following consequence for the L^2-Betti numbers:

$$\chi(M) = \sum_{p=0}^{\dim M} (-1)^p b^p_{(2)}(\overline{M}, \Gamma). \tag{3.1.4}$$

An extension of Atiyah's L^2-index theorem to manifolds with boundary can be found in [72], which provides one way to prove a corresponding result for the Euler characteristic of manifolds with boundary.

3.1.5. Example. Assume Γ is finite. Then $\overline{M}$ itself is a compact manifold and, by the above considerations, we get for the ordinary Betti numbers of $\overline{M}$:

$$b^p(\overline{M}) = \int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x}, \overline{x})\ d\mu(\overline{x}).$$

Because of Γ-invariance,

$$\int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x}, \overline{x})\ d\mu(\overline{x}) = |\Gamma| \cdot \int_M \operatorname{tr}_x \operatorname{pr}_p(x, x)\ d\mu(x),$$

in other words,

$$b^p_{(2)}(\overline{M}, \Gamma) = \frac{b^p(\overline{M})}{|\Gamma|}$$

3.1.6. Example. Let $p = 0$, and assume that $\overline{M}$ is connected. Integration by parts shows that $f \in L^2\Omega^0(\overline{M})$ belongs to $\ker(\overline{\Delta}_0)$ if and only if f is constant (recall that $L^2\Omega^0(\overline{M}) = L^2(\overline{M})$ is the space of L^2-functions on $\overline{M}$). If $\operatorname{vol}(\overline{M}) = \infty$, or equivalently $|\Gamma| = \infty$ then an L^2-function f is constant if and only if it is zero, i.e. $\ker(\overline{\Delta}_0) = 0$. Therefore, $\operatorname{pr}_p(x, y) = 0$ for all $x, y \in \overline{M}$ and $b^0_{(2)}(\overline{M}, \Gamma) = 0$.

Note that the 0-th ordinary Betti number never vanishes. Many of the applications of L^2-Betti numbers rely on such vanishing results, which don't hold for ordinary Betti numbers.

3.1.7. Theorem. *Assume $\overline{M}$ is orientable. Then, the Hodge-$*$ operator is defined and intertwines p-forms and $(\dim M - p)$-forms on $\overline{M}$. Since this is an isometry which commutes with the Laplace operators, it induces an isometry between $H^p_{(2)}(\overline{M})$ and $H^{\dim M-p}_{(2)}(\overline{M})$. Moreover, this isometry is compatible with the action of Γ and, in particular extends to the integral kernel of pr_p. As a consequence, we have Poincaré duality for L^2-Betti numbers:*

$$b^p_{(2)}(\overline{M},\Gamma) = b^{\dim M-p}_{(2)}(\overline{M},\Gamma).$$

3.1.1 The conjectures of Hopf and Singer

3.1.8. Example. In general, it will be almost impossible to compute the L^2-Betti numbers using the Riemannian metric and the integral kernel of pr_p. For very nice metrics, however, this is can be done, in particular if $(\overline{M},\overline{g})$ is a symmetric space. One obtains e.g.

(1) If $M = T^n$ is a flat torus, $\overline{M} = \mathbb{R}^n$ is flat Euclidean space, then

$$b^p_{(2)}(\mathbb{R}^n,\mathbb{Z}^n) = 0 \qquad \forall p \in \mathbb{Z}.$$

(2) If (M,g) has constant sectional curvature $K = -1$, $\Gamma = \pi_1(M)$ and $\overline{M} = \mathbb{H}^m$ is the hyperbolic m-plane, then

$$b^p_{(2)}(\overline{M},\Gamma) = 0 \qquad \text{if } p \neq m/2,$$

and if m is even and $p = m/2$, then

$$b^{m/2}_{(2)}(\overline{M},\Gamma) > 0.$$

In particular, we conclude, using (3.1.4), that in this situation

$$(-1)^{m/2}\chi(M) > 0.$$

(3) If, more generally, (M,g) is a connected, locally symmetric space with strictly negative sectional curvature, $\overline{M}$ is its universal covering (a symmetric space) and $\Gamma = \pi_1(M)$, then

$$b^p_{(2)}(\overline{M},\Gamma) = 0 \qquad \text{if } p \neq \dim M/2,$$

and if $\dim M/2$ is an integer, then

$$b^{\dim M/2}_{(2)}(\overline{M},\Gamma) > 0.$$

In particular, if $\dim M$ is even we have again

$$(-1)^{\dim M/2}\chi(M) > 0.$$

(4) If (M,g) is a connected locally symmetric space with non-positive, but not strictly negative, sectional curvature, then

$$b^p_{(2)}(\overline{M},\Gamma)=0 \qquad \forall p\in\mathbb{Z},$$

with $\overline{M}$ and Γ as above. In particular $\chi(M)=0$.

Proof. These calculations are carried out in [9], another account can be found in [60], using the "representation theory of symmetric spaces". A more geometric proof of the hyperbolic case (i.e. constant curvature -1) is given in [17]. ☐

Given any compact Riemannian manifold without boundary, we can compute the Euler characteristic using the Pfaffian and the Gauss-Bonnet formula in higher dimensions. This gives rise to another argument for the inequality

$$(-1)^{\dim M/2}\chi(M)>0,$$

if M is an even dimensional manifold with constant negative sectional curvature, and this has been known for a long time. It has lead to the following conjecture, which is attributed to Hopf.

3.1.9. Conjecture. *Assume (M,g) is a compact Riemannian $2n$-dimensional manifold without boundary, and with strictly negative sectional curvature. Then*

$$(-1)^n\chi(M)>0.$$

If the sectional curvature is non-positive, then

$$(-1)^n\chi(M)\geq 0.$$

3.1.10. Remark. The flat torus, or the product of any negatively curved manifold with a flat torus, shows that $\chi(M)=0$ is possible if M is a manifold with non-positive sectional curvature.

In view of Atiyah's formula (3.1.4), and because of the calculations of Jozef Dodziuk and the others in Example 3.1.8, Singer proposed to use the L^2-Betti numbers to prove the Hopf conjecture 3.1.9. More precisely, he made the following stronger conjecture.

3.1.11. Conjecture. *If (M,g) is a compact Riemannian manifold without boundary and with non-positive sectional curvature, then*

$$b^p_{(2)}(\tilde{M},\pi_1(M))=0, \qquad \text{if } p\neq \dim M/2.$$

If $\dim M = 2n$ *is even and the sectional curvature is strictly negative, then*

$$b_{(2)}^{\dim M/2}(\tilde{M}, \pi_1(M)) > 0.$$

Because of (3.1.4), the Singer conjecture 3.1.11 implies the Hopf conjecture 3.1.9.

Using estimates for the Laplacians and their spectrum, Ballmann and Brüning [3] prove (improving earlier results of Donnelly and Xavier [19] and Jost and Xin [35]) part of the Singer conjecture for very negative sectional curvatures:

3.1.12. Theorem. *If* (M, g) *is a closed Riemannian manifold of even dimension* $2n$*, such that its sectional curvature* K *satisfies* $-1 \le K \le -(a_n)^2$*, with* $1 \ge a_n > 1 - 1/n$*, then*

$$b_{(2)}^{p}(\tilde{M}, \pi_1(M)) = 0; \qquad \text{if } p \neq \dim M/2.$$

Therefore

$$(-1)^{\dim M/2}\chi(M) \ge 0.$$

Proof. This result is not explicitly stated in [3]. However, it follows from their [3, Theorem 5.3] in the same way in which [19, Theorem 3.2] of Donnelly and Xavier follows from the corresponding [19, Theorem 2.2]. □

3.1.13. Remark. Classical estimates of the Gauss-Bonnet integrand imply that
$(-1)^{\dim M/2}\chi(M) > 0$ if $-1 \le K \le b_n$ with $1 \ge b_n > 1 - 3/(\dim M + 1)$.

This gives strict positivity, but the curvature bound of Theorem 3.1.12 is weaker.

3.1.2 Hodge decomposition

In the classical situation, (de Rham) cohomology is of course not defined as the kernel of the Laplacian, as is suggested at the beginning of Section 3.1, but as the cohomology of the de Rham cochain complex. Only afterwards, the Hodge de Rham theorem shows that these de Rham cohomology groups are canonically isomorphic to the space of harmonic forms. The picture is parallel for L^2-cohomology. Whenever $(\overline{M}, g)$ is a complete Riemannian manifold, we can define the L^2-de Rham complex

$$\to L^2\Omega^{p-1}(\overline{M}) \xrightarrow{d} L^2\Omega^{p}(\overline{M}) \xrightarrow{d} L^2\Omega^{p+1}(\overline{M}) \to$$

where d is the exterior differential considered as an unbounded operator on the Hilbert space $L^2\Omega^p(\overline{M})$. (The fact that $(\overline{M}, g)$ is complete implies that d has a unique self adjoint extension. Usually, we work with this self adjoint extension instead of d itself).

We then define the L^2-cohomology as

$$H^p_{(2)}(\overline{M}) := \ker(d)/\overline{\operatorname{im}(d)}.$$

Observe that we divide through the *closure* of the image of d. This way, we stay in the category of Hilbert spaces. Sometimes, the L^2-cohomology groups obtained this way are called the *reduced* L^2-cohomology groups.

We have to check that this definition coincides with the one given in Definition 3.1.2. Now, if $(\overline{M}, g)$ is a complete Riemannian manifold, then we have the following Hodge decomposition:

$$L^2\Omega^p(M) = \ker(\overline{\Delta}_p) \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*}, \tag{3.1.14}$$

where d^* is the formal adjoint of d, and where the sum is an orthogonal direct sum. This implies that we also have an orthogonal decomposition

$$\ker(d|_{L^2\Omega^p(\overline{M})}) = \ker(\overline{\Delta}_p) \oplus \overline{\operatorname{im} d}.$$

Of course this implies immediately that the inclusion of $\ker(\overline{\Delta}_p)$ into $L^2\Omega^p(\overline{M})$ induces an isomorphism between $\ker(\overline{\Delta}_p)$ and $\ker(d)/\overline{\operatorname{im} d}$.

3.1.3 The Singer conjecture and Kähler manifolds

The use of the Singer conjecture to prove the Hopf conjecture in the examples presented so far is probably not very impressive. In this section we will discuss a much more striking result, which makes use of additional structure, namely the presence of a Kähler metric. The idea to do this is due to Gromov [26].

3.1.15. Definition. A Riemannian manifold (M, g) is called a Kähler manifold, if the (real) tangent bundle TM comes with the structure of a complex vector bundle (i.e. M is an almost complex manifold) with a Hermitian metric $h\colon TM \times TM \to \mathbb{C}$ (for the given complex structure on TM) such that the following conditions are satisfied:

- g is the real part of the Hermitian metric h.
- The 2-form ω associated to the Hermitian metric by

$$\omega(v, w) = -\frac{1}{2} Im(h(v, w)),$$

the so called *Kähler form*, is closed, i.e. $d\omega = 0$.

3.1.16. Remark. Under these conditions, the almost complex structure on M is integrable, i.e. M is a complex manifold. Moreover, ω is non-degenerate, i.e. $\omega^{\dim M/2}$ is a nowhere vanishing multiple of the volume form of (M, g).

3.1.17. Definition. A compact Kähler manifold M is called *Kähler hyperbolic*, if we find a 1-form η on the universal covering $\tilde{M}$ such that $\tilde{\omega} = d\eta$, where $\tilde{\omega}$ is the pullback of the Kähler form ω to $\tilde{M}$, and such that $|\eta|_\infty := \sup_{x\in\tilde{M}} |\eta(x)|_x < \infty$.

3.1.18. Example. The following manifolds are Kähler hyperbolic:

(1) closed Kähler manifold which are homotopy equivalent to a Riemannian manifold with negative sectional curvature.

(2) closed Kähler manifolds with word-hyperbolic fundamental group, provided the second homotopy group vanishes

(3) Complex submanifolds of Kähler hyperbolic manifolds, or the product of two Kähler hyperbolic manifolds.

Proof. Compare [26, Example 0.3]. □

From the point of view of the Hopf conjecture and the Singer conjecture, the first example is the most relevant: manifolds with negative sectional curvature, which also admit a Kähler structure, are covered by the results of this section.

Gromov proved the Singer conjecture for Kähler hyperbolic manifolds. More precisely, he proved in [26] the following.

3.1.19. Theorem. *Let M be a closed Kähler hyperbolic manifold of real dimension $2n$. Then*

$$b^p_{(2)}(\tilde{M}, \pi_1(M)) = 0 \text{ if } p \neq n$$
$$b^n_{(2)}(\tilde{M}, \pi_1(M)) > 0.$$

In particular, because of (3.1.4), $(-1)^n\chi(M) > 0$.

Proof. The proof splits into two rather different parts. On one side, we have to show vanishing outside the middle dimension. This is done using a Lefschetz theorem, which in particular implies that the cup-product with the lift of the Kähler form $\tilde{\omega}$ induces a bounded injective map

$$L\colon \ker(\tilde{\Delta}_r) \to \ker(\tilde{\Delta}_{r+2}) \qquad \text{for } r < n.$$

This is a classical fact from complex geometry in the compact case, and it extends rather easily to our non-compact situation.

But, by assumption, $\tilde{\omega} = d\eta$ where η is an L^∞-bounded one form. The cup product of a closed form f with a boundary $d\eta$ is always a boundary, which shows that (on a compact manifold) cup product with a boundary $d\eta$ induces the zero map on de Rham cohomology.

The extra boundedness condition on η allows us to deduce the same for the L^2-de Rham cohomology groups, i.e. cup product with $\tilde{\omega}$ induces the zero map on $H^p_{(2)}(\tilde{M})$ if M is Kähler hyperbolic. Since the Lefschetz theorem implies that this map is also injective, as long as $p < n$, $H^p_{(2)}(\tilde{M}) = 0$ for $p < n$. Because of the Poincaré duality theorem 3.1.7, the same holds for $p > n$, thus establishing the vanishing part of the theorem.

The first step to prove the non-vanishing of the L^2-cohomology in the middle dimension is the following result: Using similar, but more delicate, arguments as the one above, one shows that $\tilde{\Delta}_r\colon L^2\Omega^r(\tilde{M}) \to L^2\Omega^r(\tilde{M})$ is invertible with a bounded inverse when restricted to the orthogonal complement of its null space. (If $r \neq n$, this kernel is 0, but we later have to prove that $\tilde{\Delta}_n$ has a non-trivial null space.)

The key point now is that one can construct a continuous family of twisted L^2-de Rham complexes, indexed by $\lambda \in \mathbb{R}$, such that for $\lambda = 0$ we obtain the original L^2-de Rham complex we are interested in. An extension of Atiyah's L^2-index theorem 3.1.3 (a proof can be found e.g. in [56, Theorem 3.6]) implies that the L^2-Euler characteristics of these twisted complexes are a non-trivial polynomial $p(\lambda)$. Here, the twisted L^2-Euler characteristic is defined as the alternating sum of the L^2-dimension of $\ker(\Delta_p(\lambda))$, where the twisted Laplacian $\Delta_p(\lambda)$ is obtained from the twisted de Rham complex. Of course, $\Delta_p(0) = \tilde{\Delta}_0$, and $p(0) = \chi(M)$.

On the other hand, if $\ker(\tilde{\Delta}_n)$ was zero, then all the operators $\overline{\Delta}_r$ would be invertible, with bounded inverse. Because of the properties of the perturbation, the twisted Laplacian $\Delta_r(\lambda)$ would also be invertible for λ sufficiently close to zero, which would imply that $p(\lambda) = 0$ for λ sufficiently close to 0. This is a contradiction to the result that $p(\lambda)$ is a non-trivial polynomial. □

3.1.20. Remark. The vanishing part of Theorem 3.1.19 definitely is the easier part. To obtain the corresponding statement, the assumptions on the Kähler form can be weakened a little bit, namely, it suffices that the lift $\tilde{\omega}$ satisfies $\tilde{\omega} = d\eta$ for a one form η which has at most linear growth (such manifolds are called *Kähler non-elliptic*). This is carried out independently in [36] and [12]. The most important example of Kähler non-elliptic manifolds are closed Riemannian manifolds with non-positive sectional curvature which admit a Kähler structure. In particular, for such a manifold the assertions of the Singer conjecture and of the Hopf conjecture are true.

3.2 Combinatorial L^2-Betti numbers

So far, we have defined L^2-cohomology and L^2-Betti numbers only for coverings of Riemannian manifolds, and we have not even shown that they do not depend on the chosen Riemannian metric. Here, we will extend the definition to coverings of arbitrary CW-complexes. Moreover, we will show that L^2-cohomology is an equivariant homotopy invariant.

From now on, therefore, assume that X is a compact CW-complex, and that $\overline{X}$ is a regular covering of X, with deck transformation group Γ (in particular, Γ acts freely on $\overline{X}$, and $X = \overline{X}/\Gamma$). We use the induced structure of a CW-complex on $\overline{X}$, and Γ acts by cellular homeomorphisms.

Then, the cellular chain complex of $\overline{X}$ is a chain complex of finitely generated free $\mathbb{Z}\Gamma$-modules, with $\mathbb{Z}\Gamma$-basis given by lifts of cells of X . This way, the basis is unique up to permutation and multiplication with $\pm g \in \mathbb{Z}\Gamma$ for $g \in \Gamma$.

The cellular L^2-cochain complex is defined by

$$C^*_{(2)}(\overline{X}, \Gamma) := \mathrm{Hom}_{\mathbb{Z}\Gamma}(C^{cell}_*(\overline{X}), l^2(\Gamma)).$$

We assume that Γ acts on X from the right, and therefore consider homomorphisms of right $\mathbb{Z}\Gamma$-modules. Γ still acts on $C^*_{(2)}(\overline{\Gamma})$ by isometries, with

$$(fg)(x) := g^{-1} \cdot (f(x)) \text{ for } \in \Gamma,\ f \in C^*_{(2)}(\overline{X}) \text{ and } x \in C^{cell}_*(\overline{X}).$$

The choice of a cellular $\mathbb{Z}\Gamma$-basis for $C^{cell}_*(\overline{X})$ identifies $C^p_{(2)}(\overline{X})$ with $(l^2\Gamma)^n$ (n being the number of p-cells in X), which induces in particular the structure of a Hilbert space on the cochain groups (which does not depend on the particular cellular basis).

The L^2-cochain maps

$$d_p\colon C^p_{(2)}(\overline{X}, \Gamma) \to C^{p+1}_{(2)}(\overline{X}, \Gamma)$$

are bounded equivariant linear maps. Let d_p^* be the adjoint operator, and set

$$\Delta_p := d_p^* d_p + d_{p-1} d_{p-1}^*.$$

This is the *cellular Laplacian*, a bounded self adjoint equivariant operator on $C^p_{(2)}(\overline{X})$.

3.2.1. Remark. We can also define the cellular L^2-chain complex by

$$C_*^{(2)}(\overline{X}) := C_*^{cell}(\overline{X}) \otimes_{\mathbb{Z}\Gamma} l^2(\Gamma).$$

The self duality of the Hilbert space $l^2(\Gamma)$ induces a duality between $C_*^{(2)}$ and $C^*_{(2)}$ which gives a chain isometry between the L^2-cochain and the L^2-chain complex. In particular, in all the definitions we are going to make, there will be a canonical isomorphism between the cohomological and homological version. Because for manifolds de Rham cohomology seems to be more natural to use, we will stick to the cohomological version throughout.

3.2.1 Hilbert modules

3.2.2. Definition. A (finitely generated) Hilbert $\mathcal{N}\Gamma$-module is a Hilbert space V with a right Γ-action which admits an equivariant isometric embedding into $l^2(\Gamma)^n$ for some n.

3.2.3. Remark. To explain this notation, we remark that an isometric action of Γ on a (complex) Hilbert space by linearity extends to an "action" of the integral group ring $\mathbb{Z}[\Gamma]$ and the complex group ring $\mathbb{C}[\Gamma]$ (recall that, given a ring R, the group ring $R\Gamma$ is defined to consist of finite formal linear combination $\sum_{g\in G} r_g g$ with $r_g \in R$, with component-wise addition, and multiplication defined by $(r_g g)(r_h h) = (r_g r_h)(gh)$). We can embed $\mathbb{C}\Gamma$ into a certain completion, the reduced C^*-algebra $C_r^*\Gamma$.

3.2.4. Definition. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is an even bigger completion of $\mathbb{C}\Gamma$. It can be defined to consists of those bounded operators on $l^2\Gamma$ which commute with the *right* action of Γ on $l^2\Gamma$, i.e.

$$\mathcal{N}\Gamma := \mathcal{B}(l^2\Gamma)^\Gamma.$$

The right action is given by

$$(\sum_{g\in\Gamma} \lambda_g g) \cdot v := \sum_{g\in\Gamma} \lambda_g (gv) \text{ for } v \in G \text{ and } \sum_{g\in\Gamma} \lambda_g g \in l^2\Gamma.$$

Then $\mathcal{N}\Gamma$ is a ring which acts on the left on $l^2\Gamma$, and $\mathbb{C}\Gamma$ (actually $C_r^*\Gamma$) is contained in $\mathcal{N}\Gamma$.

Equivalently, one can define $\mathcal{N}\Gamma$ to be the closure of $\mathbb{C}\Gamma$ (with the left action) in $\mathcal{B}\Gamma$ with respect to the weak topology. This is a consequence of von Neumann's bicommutant theorem.

The action of Γ on a Hilbert $\mathcal{N}\Gamma$-module V extends to $\mathbb{C}\Gamma$ and then to $\mathcal{N}\Gamma$, making V indeed a module over $\mathcal{N}\Gamma$. Observe, however, that we don't get arbitrary algebraic modules, but modules with a topology and of a rather special kind. This additional $\mathcal{N}\Gamma$-module structure is underlying many of the definitions and proofs in L^2-cohomology. However, in the sequel we will omit explicitly using this, and instead work with the (for our purposes equivalent) unitary action of Γ and existence of the embedding into $(l^2\Gamma)^n$.

Given a Hilbert $\mathcal{N}\Gamma$-module V, let $\mathrm{pr}\colon l^2(\Gamma)^n \to l^2(\Gamma)^n$ be the orthogonal projection onto the image of any such embedding. We define the Γ-dimension of V by

$$\dim_\Gamma(V) := \mathrm{tr}_\Gamma(\mathrm{pr}) := \sum_{i=1}^n \langle \mathrm{pr}(e_i), e_i\rangle_{l^2(\Gamma)^n}. \tag{3.2.5}$$

Here, $e_i = (0, \dots, \delta_1, \dots, 0)$ is the standard basis vector of $l^2(\Gamma)^n$ with i-th entry being the characteristic function of the unit of Γ, and all other entries being zero.

Observe that Γ-invariance of V implies that pr is Γ-equivariant, i.e.

$$\langle \mathrm{pr}(e_i g), e_i g\rangle = \langle \mathrm{pr}(e_i), e_i\rangle$$

for all $g \in \Gamma$. If we would like to compute the $\mathbb{C}$-dimension of V, and therefore take the ordinary trace of pr (as endomorphism of $\mathbb{C}$-vector spaces) we would have to sum over $\langle \mathrm{pr}(e_i g), e_i g\rangle$ for all $g \in \Gamma$. Of course, in general this doesn't make sense since pr is not of trace class. As in the Definition 3.1.2, we pick the relevant part of this trace in (3.2.5), summing over a "fundamental domain" for the Γ-action on $l^2(\Gamma)^n$.

It is not hard to check that the above definition is independent of the choice of the embedding of V into $l^2(\Gamma)^n$.

3.2.6. Example. If Γ is finite, then every finitely generated Hilbert $\mathcal{N}\Gamma$-module V is a finite dimensional vector space over $\mathbb{C}$, and

$$\dim_\Gamma(V) = \frac{1}{|\Gamma|}\dim_{\mathbb{C}}(V).$$

3.2.7. Example. A more interesting example is given by free abelian groups. Assume that $\Gamma = \mathbb{Z}$. Then Fourier transform provides an isometric isomorphism between $l^2(\Gamma)$ and $L^2(S^1)$. Under this isomorphism, the subspace $\mathbb{C}[\Gamma]$ corresponds to the space of trigonometric polynomials in $L^2(S^1)$, which act by pointwise multiplication. The reduced C^*-algebra $C_r^*\mathbb{Z}$ becomes $C(S^1)$, and the von Neumann algebra $\mathcal{N}\mathbb{Z}$ becomes $L^\infty(S^1)$, also acting by pointwise multiplication. A projection P of $L^2(\Gamma)$ which commutes with all these trigonometric polynomials is itself given by multiplication with a measurable function f, and being a projection translates to the fact that f only takes the values 0 and 1 (up to a set of measure zero).

The image of P is the set of functions in $L^2(S^1)$ which vanish on the zero set of f. The Γ-trace of P is the constant term in the Fourier expansion of f, which can be computed by integration over S^1, i.e.

$$\operatorname{tr}_\Gamma(P) = \int_{S^1} f = \operatorname{vol}(\operatorname{supp}(f)),$$

which here is of course just the volume of the support of f, i.e. the set of all $x \in S^1$ with $f(x) = 1$ We use the standard measure on S^1, normalized in such a way that $\operatorname{vol}(S^1) = 1$.

It should be observed that one can obtain any real number between 0 and 1 in this way.

The Γ-dimension has the following useful properties, which in particular justify the term "dimension".

3.2.8. Proposition. *Let U, V, W be finitely generated Hilbert $\mathcal{N}\Gamma$-modules.*

(1) Faithfulness: $\dim_\Gamma(U) = 0$ *if and only if* $U = 0$.

(2) Additivity: If we have a weakly exact sequence of Hilbert $\mathcal{N}\Gamma$-modules

$$0 \to U \to W \to V \to 0$$

then

$$\dim_\Gamma(W) = \dim_\Gamma(U) + \dim_\Gamma(V).$$

Weakly exact means that the kernel of the outgoing map coincides with the closure *of the image of the incoming map, i.e.*

$$\cdots \xrightarrow{\phi_1} X \xrightarrow{\phi_2} \cdots$$

is weakly exact at X if and only if $\ker(\phi_2) = \overline{\operatorname{im}(\phi_1)}$.

(3) Monotonicity: If $U \subset V$ then $\dim_\Gamma(U) \leq \dim_\Gamma(V)$, *and* $\dim_\Gamma(U) = \dim_\Gamma(V)$ *if and only if* $U = V$.

(4) Normalization: $\dim_\Gamma(l^2(\Gamma)) = 1$.

(5) If H is a subgroup of finite index d in Γ, then every finitely generated Hilbert $\mathcal{N}\Gamma$-module V becomes by restriction of the action a finitely generated Hilbert $\mathcal{N}H$-module. Then

$$\dim_H(V) = d \cdot \dim_\Gamma(V).$$

(Note that Γ finite and H trivial is a special case of this situation.)

3.2.2 Cellular L^2-cohomology

3.2.9. Definition. We define the cellular L^2-cohomology by

$$H^p_{(2)}(\overline{X},\Gamma) := \ker(d^p)/\overline{\mathrm{im}}(d^{p-1}).$$

We have a Hodge decomposition

$$C^p_{(2)}(\overline{M},\Gamma) = \ker(\Delta_p) \oplus \overline{\mathrm{im}\, d} \oplus \overline{\mathrm{im}\, d^*}.$$

This is similar to Hodge decomposition for differential forms on Riemannian manifolds, but, since all operators involved here are bounded, is a much more elementary result. From this it follows that we have an isometric Γ-isomorphism

$$H^p_{(2)}(\overline{X},\Gamma) \cong \ker(\Delta_p)$$

We define the L^2-Betti numbers

$$b^p_{(2)}(\overline{X},\Gamma) := \dim_\Gamma(\ker(\Delta_p)).$$

Observe again the important fact that we divide by the closure of the image of the differential, such as to remain in the category of Hilbert spaces. This is the decisive difference to the equivariant cohomology with values in the $\mathbb{Z}\Gamma$-module $l^2\Gamma$. However, in [54, 55], Lück generalized the concept of L^2-Betti numbers from normal coverings of finite CW-complexes to arbitrary spaces with group action, using the usual twisted cohomology (with coefficients the group von Neumann algebra $\mathcal{N}\Gamma$). The starting point, however, is also in this treatment the theory of Hilbert $\mathcal{N}\Gamma$-modules.

3.2.2.1 Matrices over the group ring

Given a compact CW-complex, we can explicitly compute it's cohomology using a cellular basis and solving certain systems of linear equations. A similar approach is possible here (leading to more complicated, "non-commutative", equations in this situation).

In the compact case, the choice of an orientation for each cell identifies $C^{cell}_p(X)$ with $\mathbb{Z}^{c_p}$, where c_p is the number of p-cells of X, and this identification is well defined up to permutation of the basis, and multiplication of basis elements with ± 1. The boundary map, in this representation, is given by multiplication with an appropriate matrix with integral entries.

To proceed in the L^2-case, observe that each cell of the finitely many cells of X has as inverse image a free Γ-orbit of cells in $\tilde{X}$. We can choose one cell in each orbit. Together with the choice of an orientation, this identifies the $\mathbb{Z}\Gamma$-module $C_p^{cell}(\overline{X})$ with $(\mathbb{Z}\Gamma)^{c_p}$, and this identification is unique up to multiplication of the basis elements with $\pm g$ ($g \in \Gamma$) and permutation. In this realization, the boundary maps of $C_p^{cell}(\overline{X})$ are given by multiplication with matrices A_p over the integral group ring.

The chosen $\mathbb{Z}\Gamma$-module isomorphism of $C_p^{cell}(\overline{X})$ with $(\mathbb{Z}\Gamma)^{c_p}$ induces an isomorphism of $C^*_{(2)}(\overline{X}, \Gamma)$ with $(l^2\Gamma)^{c_p}$. An easy calculation shows that the coboundary maps are given by multiplication with the adjoint matrices A_p^* (extending the multiplication of elements of $\mathbb{C}\Gamma$ with elements of $l^2\Gamma$ used before). Here, if

$$u = \sum_{g\in\Gamma} \lambda_g g \in \mathbb{C}\Gamma \text{ then } u^* := \sum_{g\in\Gamma} \overline{\lambda_g} g^{-1},$$

and obviously $u^* \in \mathbb{Z}\Gamma$ if $u \in \mathbb{Z}\Gamma$. If $A = (A_{ij}) \in M(d_1 \times d_2, \mathbb{C}\Gamma)$, then

$$A^* := (A_{ji}^*) \in M(d_2 \times d_1, \mathbb{C}\Gamma),$$

and again this restricts to an operation on matrices over $\mathbb{Z}\Gamma$.

To finish the picture, the combinatorial Laplacian $\Delta_p = d_p^* d_p + d_{p-1} d_{p-1}^*$ is given by the matrix

$$\Delta := A_p^* A_p + A_{p-1}^* A_{p-1} \in M(c_p \times c_p, \mathbb{Z}\Gamma).$$

To understand the L^2-cohomology we therefore have to understand the kernel of such matrices, acting on $(l^2\Gamma)^{c_p}$.

This gives an algebraic way of studying questions about L^2-cohomology —they translate to questions about matrices over $\mathbb{Z}\Gamma$.

Actually, if Γ is finitely presented (i.e. has a presentation with finitely many generators and finitely many relations), given any matrix $A \in M(d \times d, \mathbb{Z}\Gamma)$, a standard construction provides us with a compact CW-complex X with $\pi_1(X) = \Gamma$ such that the kernel of the combinatorial Laplacian Δ_3 becomes the kernel of A, acting on $l^2(\Gamma)^d$. Therefore, we can also translate questions about matrices over $\mathbb{Z}\Gamma$ to questions in L^2-cohomology.

3.2.2.2 Properties of L^2-Betti numbers

3.2.10. Theorem. *L^2-cohomology and in particular L^2-Betti numbers have the following basic properties. Here, let $\overline{X}$ be a normal covering of a finite CW-complex X with covering group Γ.*

(1) Let (M, g) be a closed Riemannian manifold, and equip it with the CW-structure coming from a smooth triangulation. Let $(\overline{M}, \overline{g})$ be a normal

covering of M with covering group Γ. Then integration of forms over simplices (the de Rham map) defines a Γ-isomorphism

$$\ker(\overline{\Delta}_p(\overline{g})) \to H^p_{(2),cell}(\overline{M},\Gamma).$$

In particular, the L^2-Betti numbers defined using the Riemannian metric and using the Γ-CW-structure coincide.

(2) *Let Y be another finite CW-complexes and $f\colon Y \to X$ a homotopy equivalence. Let $\overline{Y}$ be the pullback of $\overline{X}$ along f (this means that the deck transformation group for $\overline{Y}$ is also Γ). Then*

$$b^p_{(2)}(\overline{X},\Gamma) = b^p_{(2)}(\overline{Y},\Gamma) \qquad \forall p \geq 0.$$

(3) *For the Euler characteristic of X, we get*

$$\chi(X) = \sum_{p=0}^{\infty} (-1)^p b^p_{(2)}(\overline{X},\Gamma). \tag{3.2.11}$$

(4) *Assume Γ is finite. Then $\overline{X}$ is itself a finite CW-complex, and its (ordinary) Betti number $b^p(\overline{X})$ are defined. They satisfy*

$$b^p_{(2)}(\overline{X},\Gamma) = \frac{1}{|\Gamma|} b^p(\overline{X}).$$

(5) *If Γ is infinite, then for the zeroth L^2-Betti number we get*

$$b^0_{(2)}(\overline{X},\Gamma) = 0.$$

(6) *Let $H \leq \Gamma$ be a subgroup of finite index d, and set $X_1 := \overline{X}/H$. This is a finite d-sheeted covering of X, and $\overline{X}$ can be considered to be a normal covering of X_1 with covering group H. Then*

$$b^p_{(2)}(\overline{X},H) = d \cdot b^p_{(2)}(\overline{X},\Gamma) \qquad \forall p \geq 0.$$

Note that the Euler characteristic is multiplicative under finite covering, in the situation of (6) this means that $\chi(X_1) = d \cdot \chi(X)$. In view of (6) and the Euler characteristic formula (3), the L^2-Betti numbers are the appropriate refinement of the Euler characteristic which (unlike the ordinary Betti numbers) remain multiplicative under finite coverings.

Proof of Theorem 3.2.10. We only indicate reference and the main points.

(1) This was a classical question of Atiyah, proved by Dodziuk in [16].

(2) f is covered by a Γ-homotopy equivalence $\overline{f}\colon \overline{Y} \to \overline{X}$. One easily checks that such a map induces a map on L^2-cohomology, and that two Γ-homotopic maps induce the same map. The claim follows.

(3) The Euler characteristic formula follows exactly as in the classical situation, using additivity of the Γ-dimension and the normalization

$$\dim_\Gamma(l^2\Gamma) = 1.$$

(4) If Γ is finite, all the Hilbert spaces in question are finite dimensional. Consequently, $\operatorname{im}(d)$ is automatically closed, and there is no difference between L^2-cohomology and ordinary cohomology with complex coefficients of $\overline{X}$. In particular,

$$b^p(\overline{X}) = \dim_{\mathbb{C}} H^p_{(2)}(\overline{X},\Gamma) = |\Gamma| \cdot \dim_\Gamma(H^p_{(2)}(\overline{X},\Gamma)) = |\Gamma| \cdot b^p_{(2)}(\overline{X},\Gamma).$$

(6) This follows immediately from the corresponding formula in Proposition 3.2.8.

□

3.3 Approximating L^2-Betti numbers

As mentioned in Section 3.1, there are almost no relations between the ordinary Betti numbers of a space X and the L^2-Betti numbers of a covering $\tilde{X}$ of X. However, if we have a whole sequence of nested coverings $X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$, "converging" to $\tilde{X}$, in many cases we can approximate the L^2-Betti numbers of $\tilde{X}$ in terms of this sequence. More precisely, let $\tilde{X}$ be a Γ-covering of X. Assume that there is a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ (each Γ_k normal in Γ) such that $\bigcap_{k\geq 1}\Gamma_k = \{1\}$. Then $X_k := \Gamma_k\backslash\tilde{X}$ is a normal covering of X, with covering group Γ/Γ_k.

3.3.1. Conjecture. *In this situation,*

$$b^p_{(2)}(\tilde{X},\Gamma) = \lim_{k\to\infty} b^p_{(2)}(X_k,\Gamma/\Gamma_k). \tag{3.3.2}$$

Observe that convergence is not clear, but part of the statement.

This question was first asked by Gromov if all the groups Γ/Γ_k are finite. In this case, the conjecture is true, as proved by Lück:

3.3.3. Theorem. *Equation* (3.3.2) *is correct if* Γ/Γ_k *is finite for each* $k \in \mathbb{N}$.

Observe that, in this setting, X_k *is a finite covering of* X. *Consequently, we can express the* L^2*-Betti numbers in terms of ordinary Betti numbers and obtain, in the setting of Conjecture 3.3.1,*

$$b^p_{(2)}(\tilde{X},\Gamma) = \lim_{k\to\infty} \frac{b^p(X_k)}{|\Gamma/\Gamma_k|}.$$

In general, Conjecture 3.3.1 is still open. However, in [71] (with an improvement in [18]) a quite large class $\mathcal{G}$ of groups is constructed for which the conjecture is true. $\mathcal{G}$ contains all amenable and all free groups, and is closed under taking subgroups, extensions with amenable quotients, directed unions, and inverse limits (therefore, it contains e.g. all residually finite groups).

More precisely

3.3.4. Definition. Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following processes:

- If $H \in \mathcal{G}$ and G is a generalized amenable extension of H, then $G \in \mathcal{G}$.
- If $G = \lim_{i\in I} G_i$ is the direct or inverse limit of a directed system of groups $G_i \in \mathcal{G}$, then $G \in \mathcal{G}$.
- If $H \in \mathcal{G}$ and $U \leq H$, then $U \in \mathcal{G}$.

Here, the notion of *generalized amenable extension* is defined as follows:

3.3.5. Definition. Assume that G is a finitely generated discrete group with a finite symmetric set of generators S (i.e. $s \in S$ implies $s^{-1} \in S$), and let H be an arbitrary discrete group. We say that G *is a generalized amenable extension of* H, if there is a set X with a free G-action (from the left) and a commuting free H-action (from the right), such that a sequence of H-subsets $X_1 \subset X_2 \subset X_3 \subset \cdots \subset X$ exists with $\bigcup_{k\in\mathbb{N}} X_k = X$, and with $|X_k/H| < \infty$ for every $k \in \mathbb{N}$, and such that

$$\frac{|(S\cdot X_k - X_k)/H|}{|X_k/H|} \xrightarrow{k\to\infty} 0.$$

In [71] and [18], the following theorem is proved.

3.3.6. Theorem. *Equation* (3.3.2) *is correct if* Γ/Γ_k *belongs to* $\mathcal{G}$ *for each* $k \in \mathbb{N}$.

In particular, we obtain the following corollary which we will use later.

3.3.7. Corollary. *Equation* (3.3.2) *is true if* Γ/Γ_k *is amenable, e.g. solvable or nilpotent, or virtually solvable, for each* $k \in \mathbb{N}$. *Recall that a group* G *has* virtually *a certain property* P, *if it contains a subgroup of finite index which has property* P.

3.3.8. Remark. There are generalizations of the above approximation results to other L^2-invariants, in particular to the L^2-signature, compare [53].

3.4 The Atiyah conjecture

Fix a discrete group Γ. The L^2-Betti numbers $b^p_{(2)}(\overline{X}, \Gamma)$ of a Γ-covering of a finite CW-complex X are the Γ-dimensions of certain Hilbert $\mathcal{N}\Gamma$-modules. In Example 3.2.7 we have seen that a priori arbitrary non-negative real numbers could occur, even for groups as nice as $\mathbb{Z}$. However, the Euler characteristic formula (3.2.11) shows that certain combinations ofL^2-Betti numbers are always integers.

The Atiyah conjecture predicts a certain amount of integrality for the individual L^2-Betti numbers.

3.4.1. Conjecture. *Fix a discrete group* Γ. *Let* $Fin^{-1}(\Gamma)$ *be the additive subgroup of* $\mathbb{Q}$ *generated by*

$$\{\frac{1}{|F|} \mid F \text{ finite subgroup of } \Gamma\}.$$

Let X *be a finite CW-complex or a compact manifold,* $\overline{X}$ *a* Γ*-covering of* X.

(1) *If* Γ *is torsion-free, then*

$$b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Z}.$$

(2) *Assume there is a bound on the finite subgroups of* Γ *(observe that this is equivalent to* $Fin^{-1}(\Gamma)$ *being a discrete subset of* $\mathbb{R}$*). Then*

$$b^p_{(2)}(\overline{X}, \Gamma) \in Fin^{-1}(\Gamma).$$

(3) *Without any assumption on* Γ,

$$b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Q}.$$

For a while, also the following conjecture was around:

3.4.2. Conjecture. *Without any assumption on* Γ, $b^p_{(2)}(\overline{X},\Gamma) \in Fin^{-1}(\Gamma)$.

This last conjecture is singled out here because it is wrong. In [24], a smooth 7-dimensional Riemannian manifold M is constructed such that every finite subgroup of $\pi_1(M)$ is an elementary abelian 2-group, but $b^3_{(2)}(\tilde{M},\pi_1(M)) = \frac{1}{3}$. This example is based on the explicit calculation of the eigenspaces and their L^2-dimensions of a certain operator in [25], using in particular the methods of the proof of Theorem 3.3.3. A more direct and slightly more general computation for such eigenspaces is carried out in [15].

It should be remarked that none of the above conjectures were formulated by Atiyah as stated here, although he makes some remarks which show that he was interested in the question of the possible values of L^2-Betti numbers.

Statement (3) of Conjecture 3.4.1, which is the oldest version of the Atiyah conjecture, is also quite unlikely to hold in general. In [15], for each $r,s \in \mathbb{N}$ with $r,s \geq 2$, a manifold $M_{r,s}$ is constructed such that

$$b^3_{(2)}(\tilde{M},\pi_1(M)) = \alpha_{r,s} := (r-1)^2(s-1)^2 \cdot \sum_{n=2}^{\infty} \frac{\phi(n)}{(r^n-1)(s^n-1)},$$

where $\phi(n)$ is Euler's phi-function, i.e. the number of primitive n-th roots of unity. At the moment, it is unknown whether any of the numbers $\alpha_{r,s}$ is irrational. However, the use of computer algebra shows e.g. that, if $\alpha_{2,2}$ is rational, both the numerator and denominator exceed 10^{100}. It seems reasonable to assert that $\alpha_{2,2}$ is not obviously rational.

3.4.1 Combinatorial reformulation of the Atiyah Conjecture

The following assertion is equivalent to Conjecture 3.4.1.

3.4.3. Conjecture. *Let* Γ *be a discrete group, and assume that* $A \in M(d\times d,\mathbb{Z}\Gamma)$. *Consider* A *to be a bounded operator on* $l^2(\Gamma)^d$, *as in Section 3.2.2.1.*

(1) If Γ *is torsion-free, then* $\dim_\Gamma(\ker(A)) \in \mathbb{Z}$.

(2) If $Fin^{-1}(\Gamma)$ *is a discrete subset of* $\mathbb{R}$, *then*

$$\dim_\Gamma(\ker(A)) \in Fin^{-1}(\Gamma).$$

Without any assumption on Γ, $\dim_\Gamma(\ker(A)) \in \mathbb{Q}$.

3.4.4. Remark. It is equivalent to require the assertions of Conjecture 3.4.3 for all matrices over $\mathbb{Z}\Gamma$, or for all square matrices, or for all self-adjoint matrices (these are automatically square matrices), or for all matrices of the form $A = B^*B$ (these are automatically self-adjoint). This is the case since $\ker(A) = \ker(A^*A)$. Moreover, we have the weakly exact sequence

$$0 \to \ker(A) \hookrightarrow l^2(\Gamma)^d \xrightarrow{A} \overline{\operatorname{im} A} \to 0.$$

Because of additivity and normalization of the Γ-dimension (Proposition 3.2.8), we could replace the kernel of A by the closure of the image, throughout.

The equivalence of Conjecture 3.4.1 and Conjecture 3.4.3 follows immediately from the principle described at the end of Section 3.2.2.1.

3.4.2 Atiyah conjecture and non-commutative algebraic geometry — Generalizations

As an illustration, we now want to study the Atiyah conjecture for the group $\Gamma = \mathbb{Z}$, which we understand particularly well because of Example 3.2.7.We look at the algebraic reformulation. For simplicity, assume first that $d = 1$. Then $A \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ is a Laurent polynomial with $\mathbb{Z}$-coefficients. Under Fourier transform we get the commutative diagram

$$\begin{array}{ccc} l^2(\mathbb{Z}) & \xrightarrow{A} & l^2(\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong \\ L^2(S^1) & \xrightarrow{A(z)} & L^2(S^1), \end{array}$$

i.e. the action of A translates to multiplication with the function $A(z)$, ($z \in S^1 \subset \mathbb{C}$). Now,

$$\dim_{\mathbb{Z}}(\ker(A)) = \mu(\{z \in S^1 \mid A(z) = 0\})$$

is the volume of the set of zeros of the Laurant polynomial $A(z)$ on S^1. But the Laurant polynomial $A(z)$ has, if it is not identically zero, only finitely many zero. Therefore

$$\dim_{\mathbb{Z}}(\ker(A)) = 0 \text{ if } A \neq 0,$$

and of course

$$\dim_{\mathbb{Z}}(\ker(A)) = 1 \text{ if } A = 0.$$

Since $\mathbb{Z}$ is commutative, every matrix A can be replaced by a diagonal matrix without changing the dimension of the kernel, and this way the above calculation proves the Atiyah conjecture for $\mathbb{Z}$.

Similar considerations for $\Gamma = \mathbb{Z}^n$ show that the Atiyah conjecture here amounts to understanding the zeros of polynomials in several variables. This created the slogan that the Atiyah conjecture (and more generally L^2-cohomology) in a certain sense is non-commutative algebraic geometry.

Exactly the same proof works if we replace the coefficient ring $\mathbb{Z}$ by $\mathbb{C}$, or by any subring of $\mathbb{C}$.

This leads to the following algebraic generalization of the Atiyah conjecture.

3.4.5. Conjecture. *Fix a discrete group Γ. Let K be any subring of $\mathbb{C}$ which is closed under complex conjugation. Let Γ be a discrete group, and assume that $a \in M(d \times d, K\Gamma)$.*

If $Fin^{-1}(\Gamma)$ is discrete, then $\dim_\Gamma(\ker(A)) \in Fin^{-1}(\Gamma)$.

Since we can multiply any matrix with a non-zero constant (e.g. a common denominator of the finitely many non-zero coefficients), the assertion of Conjecture 3.4.5 for a ring $K \subset \mathbb{C}$ and its field of fractions is equivalent. In the sequel, we will therefore usually assume that the subring K is a field.

It is not clear, however, whether Conjecture 3.4.5 is equivalent to the original geometric Atiyah conjecture 3.4.1 if $K \not\subset \mathbb{Q}$.

Observe that the proof of the Atiyah conjecture for $\mathbb{Z}$ extends from the ring of Laurent polynomials (with complex coefficients) to the ring of meromorphic functions $\mathbb{C}$ without poles on S^1, which is also a subring of $L^\infty(S^1) \cong \mathcal{N}\mathbb{Z}$. The question arises whether there are reasonable generalizations similar to this ring for other groups. One possibility would be to look at infinite sums $a = \sum_{g \in \Gamma} \lambda_g g$, where the coefficients λ_g very rapidly tend to zero as $g \to \infty$ (with respect to a suitable word length metric). (Observe that this is the case for the coefficients of the Laurent expansion of a meromorphic function on $\mathbb{C}$). Under suitable circumstances, (convolution) multiplication with such an a will indeed give rise to a (very special) Γ-equivariant operator on $l^2\Gamma$, and the question arises whether for the dimension of its kernel the statement of the Atiyah conjecture holds. This was suggested by Nigel Higson. It is quite distinct from Conjecture 3.4.5 in that it is analytic in flavor, and no longer algebraic.

3.4.3 Atiyah conjecture and zero divisors

Among the most interesting observations about the Atiyah conjecture are its strong connections to questions in algebra, in particular to group rings.

Here, we address the following conjecture, the so called *zero divisor conjecture.*

3.4.6. Conjecture. *Let Γ be a torsion-free discrete group and K a subring of the complex numbers. Then, there are no non-trivial zero divisors in the group ring $K\Gamma$, i.e. if $a, b \in K\Gamma$ with $ab = 0$ then either $a = 0$ or $b = 0$.*

This is one of the longstanding questions in the theory of group rings (which of course makes also sense for other coefficients rings K and is studied also in this broader generality by ring theorists).

Observe that, if $g \in \Gamma$ is a torsion element, i.e. $g \neq 1$ but $g^n = 1$ for some $n > 0$, then $a = (1 - g)$ and $b = 1 + g + \cdots + g^{n-1}$ are two non-zero elements of $\mathbb{Z}\Gamma$ with $ab = 0$.

It now turns out that the Atiyah conjecture implies the zero divisor conjecture. More precisely:

3.4.7. Theorem. *Assume Γ is a torsion-free discrete group, and $K \subset \mathbb{C}$ is a ring (closed under complex conjugation).*

If the statement of the algebraic Atiyah conjecture 3.4.5 is true for every $A \in M(1 \times 1, K\Gamma)$, then there are no non-trivial zero divisors in $K\Gamma$.

Proof. Fix $a, b \in K\Gamma$ with $ab = 0$. We have to show that either $a = 0$ or $b = 0$. Now observe that

$$a \in K\Gamma = M(1 \times 1, K\Gamma)$$

is a 1-by-1 matrix over $K\Gamma$. On the other hand,

$$b \in K\Gamma \subset l^2\Gamma$$

can be considered to be an element of $l^2\Gamma$. And $0 = ab$ is just the result of the action of the matrix a on the l^2-function b. Therefore, $b \in \ker(a)$. Now we know that $\dim_\Gamma(\ker(a)) \in \mathbb{Z}$ because the Atiyah conjecture is true for the torsion-free group Γ. Evidently,

$$\{0\} \subset \ker(a) \subset l^2\Gamma,$$

with

$$0 = \dim_\Gamma(\{0\}) \quad \text{and} \quad 1 = \dim_\Gamma(l^2\Gamma).$$

Because of monotonicity, either $\dim_\Gamma(\ker(a)) = 0$ or $\dim_\Gamma(\ker(a)) = 1$. In the first case, because of faithfulness $\ker(a) = \{0\}$ which implies $b = 0$. In the second case, $\ker(a) = l^2\Gamma$ (again because of faithfulness) which means that $a = 0$. This proves the statement.

Here, we used some of the properties of $\dim_\Gamma$ developed in Proposition 3.2.8. □

In Theorem 3.4.18, we will see that there is actually an even stronger relation between the Atiyah conjecture and the zero divisor conjecture for a torsion free group Γ.

3.4.4 Atiyah conjecture and calculations

A second possible application of the Atiyah conjecture could be the explicit calculation of L^2-Betti numbers.

Here one would use that by now there are several approximation formulas for L^2-Betti numbers. In particular, we have discussed one of these in Section 3.3. Obviously, if we know in advance that the limit has to be an integer, this can make it much easier to exactly compute the limit, in particular if (as is the case for some of the approximation results) error bounds are available. Up to now, however, to the authors knowledge this idea has not been used anywhere.

3.4.5 The status of the Atiyah conjecture

By now, the Atiyah conjecture is known for a reasonably large class of groups. We use the following definitions.

3.4.8. Definition. The class of *elementary amenable groups* is the smallest class of groups which contains all abelian and all finite groups and is closed under extensions and directed unions. It is denoted $\mathcal{Y}$. Obviously, every elementary amenable group is amenable. Moreover, every nilpotent and every solvable group is elementary amenable, as well as every group which is virtually solvable. (A group has *virtually* a property P, if it contains a subgroup of finite index which actually has property P).

3.4.9. Definition. Let $\mathcal{D}$ be the smallest non-empty class of groups such that:

(1) If G is torsion-free and A is elementary amenable, and we have a projection $p\colon G \to A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A, then $G \in \mathcal{D}$.

(2) $\mathcal{D}$ is subgroup closed.

(3) Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

3.4.10. Definition. A *directed system* of groups is a system of groups G_i, indexed by an index set I with a partial ordering $<$, and either with a homomorphism $\phi_{ij}\colon G_i \to G_j$ if $i < j$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ if $i < j <$

k (then we take the direct limit), or with homomorphisms the other way around, i.e. $\phi_{ij}\colon G_j \to G_i$ if $i < j$, with the corresponding compatibility condition (then we take the inverse limit).

Directed means that to each $i, j \in I$ exists $k \in I$ with $i < k$ and $j < k$. The most obvious examples are systems of groups indexed by $\mathbb{N}$ with its usual ordering.

Observe that $\mathcal{D}$ contains only torsion-free groups.

3.4.11. Example. The class $\mathcal{D}$ contains all torsion-free elementary amenable groups. It also contains all free groups and all braid groups (compare Section 3.4.5.4). Moreover, $\mathcal{D}$ is closed under direct sums, direct products, and free products.

To see this, observe that clearly $\mathcal{D}$ contains all elementary amenable groups, as long as they are torsion-free. Moreover, if Γ contains a sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ with $\bigcap_{k\in\mathbb{N}} \Gamma_k = \{1\}$ and such that Γ/Γ_k is torsion-free elementary amenable, then Γ is a subgroup of the inverse limit of the sequence of quotients, and consequently belongs to $\mathcal{D}$.

In particular, every free group admits such a sequence in such a way that the quotients are torsion-free nilpotent, and every braid group admits such a sequence where the quotients are torsion-free and virtually nilpotent.

The following theorem is proved in [70] and [18].

3.4.12. Theorem. *Set $K := \overline{\mathbb{Q}}$, the field of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{C}$.*

If $\Gamma \in \mathcal{D}$, then the Atiyah conjecture 3.4.5 is true for $K\Gamma$.

To prove this, one only has to prove that the Atiyah conjecture is preserved when passing to subgroups, doing extensions with torsion-free elementary amenable quotients, and under direct and inverse limits of directed systems of groups in $\mathcal{D}$. The former (almost) translates to directed unions of groups, and the latter to the case where a group Γ has the nested sequence of normal subgroups Γ_k with trivial intersection we have discussed previously.

We want to start with two rather elementary observations, which are not very useful without examples of groups for which the Atiyah conjecture is true, but which give rise to part of the statements of Theorem 3.4.12.

3.4.13. Proposition. *If G fulfills the Atiyah conjecture, and if U is a subgroup of G with $Fin^{-1}(U) = Fin^{-1}(G)$, then U fulfills the Atiyah conjecture.*

Proof. This follows from the fact that the U-dimension of the kernel of a matrix over KU acting on $l^2(U)^n$ coincides with the G-dimension of the same matrix, considered as an operator on $l^2(G)^n$ (compare e.g. [71, 3.1]), which follows from a simply diagonal decomposition argument. $\square$

3.4.14. Proposition. *Let G be the directed union of groups $\{G_i\}_{i\in I}$ and assume that each G_i fulfills the Atiyah conjecture. Then G fulfills the Atiyah conjecture.*

Proof. A matrix over KG, having only finitely many non-trivial coefficients, already is a matrix over KG_i for some i. The G_i-dimension and the G-dimension of the kernel of the matrix coincide, as in Proposition 3.4.13, compare e.g. [71, 3.1]. Note that $Fin^{-1}(G_i)$ is contained in $Fin^{-1}(G)$ for each $i \in I$. $\square$

Besides of these two results, at the moment three rather different methods are known to prove the Atiyah conjecture for certain groups.

3.4.5.1 Fredholm modules and the Atiyah conjecture

One of these methods, which one could call the method of the "finite rank Fredholm module", was developed by Peter Linnell in [48] to prove the Atiyah conjecture for free groups. It extends in an ingenious way the method used by Connes to prove the trace conjecture 1.3.12 (compare e.g. [21]) for the free group, and therefore is related to the Baum-Connes conjecture. However, whereas the KK-theory methods for Baum-Connes turn out to be quite flexible and generalize to many other groups, compare e.g. [73], nobody so far was able to find a corresponding generalized approach to the Atiyah conjecture. Since the Atiyah conjecture for free groups follows also from one of the other methods to be described later, we don't discuss Linnell's original approach but refer instead to the original article [48] and the review article [49]. (Note, however, that the generalization to products of free groups does not work as described in [49], since the proof of the basic Lemma [49, Lemma 11.7] has a gap. One has to rely on a different method for the proof, instead.)

3.4.5.2 Atiyah conjecture and algebra

We have shown in Theorem 3.4.7 that the Atiyah conjecture for a torsion-free group implies the zero divisor conjecture. The ring $K\Gamma$ evidently has no zero divisors, if it can be embedded into a skew field, i.e. a (not necessary commutative) ring where every non-zero element has a multiplicative inverse. The optimal solution to the zero divisor question therefore is to construct exactly this. It turns out that the Atiyah conjecture provides us with such an embedding. We need the following definitions.

3.4.15. Definition. Given the group von Neumann algebra $\mathcal{N}\Gamma$, we define $\mathcal{U}\Gamma$ to be the algebra of all unbounded operators affiliated to $\mathcal{N}\Gamma$ (compare e.g. [49, Section 8]). That means a densely defined unbounded operator D on $l^2\Gamma$ belongs to $\mathcal{U}\Gamma$ if and only if all its spectral projections belong to $\mathcal{N}\Gamma$. It is a classical fact that the ring $\mathcal{U}\Gamma$ is a (non-commutative) localization of $\mathcal{N}\Gamma$, which here means it is obtained from $\mathcal{N}\Gamma$ by inverting all non-zero divisors of $\mathcal{N}\Gamma$.

Fix a subfield $K \subset \mathbb{C}$ which is closed under complex conjugation. We have to consider $K\Gamma \subset \mathcal{N}\Gamma \subset \mathcal{U}\Gamma$. Define $D_K\Gamma$ as the *division closure* of $K\Gamma$ in $\mathcal{U}\Gamma$. By definition, this is the smallest subring of $\mathcal{U}\Gamma$ which contains $K\Gamma$ and which has the property that, whenever $x \in D_K\Gamma$ is invertible in $\mathcal{U}\Gamma$, then $x^{-1} \in D_K\Gamma$.

3.4.16. Remark. Since the operators which belong to $\mathcal{U}\Gamma$ are only densely defined, one has to be careful when defining the sum or product of two such operators. This is done by first defining these operators on the obvious (common) domain, but then taking there closure, i.e. to extend the domain of definition as far as possible. One has to check that this indeed gives a reasonable ring. This is a classical result which uses the Γ-dimension.

3.4.17. Example. Again, we turn to the example $\Gamma = \mathbb{Z}$. We have seen that, via Fourier transform, $\mathcal{N}\Gamma$ becomes $L^\infty(S^1)$ acting on $L^2(S^1)$ by pointwise multiplication. The ring $\mathcal{U}\mathbb{Z}$ becomes the ring of *all* measurable functions on S^1, still acting by pointwise multiplication. These operators are in general unbounded and not defined on all of $L^2(S^1)$, because the product of an L^2-function with an arbitrary measurable function belongs not necessarily to $L^2(S^1)$. It is not hard to show that every measurable function f on S^1 is the quotient of two bounded functions $g, h \in L^\infty(S^1)$, $f = g/h$, where the set of zeros of h has measure zero. This reflects the fact that $\mathcal{U}\Gamma$ is a localization of $\mathcal{N}\Gamma$.

If $K \subset \mathbb{C}$, then $K\mathbb{Z}$ are the Laurent polynomials $K[z, z^{-1}]$ identified with functions on S^1 (by substituting $z \in S^1$ for the variable). In the same way, $D_K\Gamma$ is the field of rational functions $K(z)$, identifies with functions on S^1 by substituting $z \in S^1$ for the variable.

The (very strong) connection between the Atiyah conjecture and ring theoretic properties of $D_K\Gamma$ is given by the following theorem.

3.4.18. Theorem. *Let Γ be a torsion-free group, and let K be a subfield of $\mathbb{C}$ which is closed under complex conjugation.*

$K\Gamma$ fulfills the strong Atiyah conjecture in the sense of Conjecture 3.4.5 if and only if the division closure $D_k\Gamma$ of $K\Gamma$ in $\mathcal{U}\Gamma$ is a skew field.

In other words, we have a canonical candidate $D_K\Gamma$ for a skew field, into which $K\Gamma$ embeds, and this ring is a skew field if and only if $K\Gamma$ satisfies the Atiyah conjecture.

Proof of Theorem 3.4.18. If $D_K\Gamma$ is a skew field, then each matrix $A \in M(d \times d, K\Gamma)$ acts on $(D_K\Gamma)^d$, and its kernel is a finite dimensional vector space over the field $D_K\Gamma$. In particular, its $D_K\Gamma$-dimension of course is an integer. Then, one can establish that the Γ-dimension of $\ker(A\colon (l^2\Gamma)^d \to (l^2\Gamma)^d)$ coincides with this dimension. Details are given in [70, Lemma 3].

For the converse, given an element $0 \neq a \in D_K\Gamma$ one can, using a matrix trick for division closures due to Cohn, produce a $d \times d$-matrix A over $K\Gamma$ such that the Γ-dimension of $\ker(A)$ is evidently strictly smaller than 1, and which is non-zero if and only if a is invertible in $\mathcal{U}\Gamma$ (slogan: "a non-trivial kernel of a gives rise to a non-trivial kernel of A"). Because $\dim_\Gamma(\ker(A)) \in \mathbb{Z}$ by assumption, $\dim_\Gamma(\ker(A)) = 0$, i.e. a is invertible in $\mathcal{U}\Gamma$. Since $D_K\Gamma$ is division closed, a is invertible in $D_K\Gamma$, as well. □

This property allows to prove the Atiyah conjecture for the first interesting class of groups (containing non-abelian groups), namely the class of elementary amenable groups.

3.4.19. Theorem. *Fix a subfield $K \subset \mathbb{C}$ which is closed under complex conjugation.*

Let $1 \to H \to G \to A \to 1$ be an exact sequence of groups. Assume that G is torsion free and A is elementary amenable. For every finite subgroup $E \leq A$ let H_E be the inverse image of E in G. Assume for all finite subgroups $E \leq G$ that KH_E fulfills the Atiyah conjecture 3.4.5. Then KG fulfills also the Atiyah conjecture.

Proof. The proof is given in [48] for $K = \mathbb{Q}$. Essentially the same proof works for arbitrary K, compare [70, Proposition 3.1]. □

3.4.20. Corollary. *Fix a subfield $K = \overline{K} \subset \mathbb{C}$.*

Suppose H is torsion-free and KH fulfills the Atiyah conjecture. If G is an extension of H with elementary amenable torsion-free quotient then KG fulfills the Atiyah conjecture.

In particular (with $H = 1$) if G is a torsion-free elementary amenable group then KG satisfies the Atiyah conjecture.

Proof. By assumption, the only finite subgroup of G/H is the trivial group and the Atiyah conjecture is true for its inverse image H. □

3.4.5.3 Atiyah conjecture and approximation

Here, we describe the last method of proof for the Atiyah conjecture. It is based on the approximation results of Section 3.3.

3.4.21. Theorem. *Assume Γ is a torsion-free discrete group with a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ such that $\bigcap_{k\in\mathbb{N}} \Gamma/\Gamma_k = \{1\}$ and such that $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$. Moreover, assume that all the quotient groups Γ/Γ_k are torsion-free and satisfy the Atiyah conjecture 3.4.1.*

For example, all the groups Γ/Γ_k might be torsion-free elementary amenable.

Then Γ also satisfies the Atiyah conjecture 3.4.1.

Proof. Given any Γ-covering $\overline{X} \to X$ (with a finite CW-complex X), we have to prove that $b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Z}$ for each p.

Now, the sequence of normal subgroups Γ_k provides us with a sequences of normal coverings $X_k := \overline{X}/\Gamma_k$ of X, with covering group Γ/Γ_k. Since $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$, by Theorem 3.3.6

$$b^p_{(2)}(\overline{X}, \Gamma) = \lim_{k\to\infty} b^p_{(2)}(X_k, \Gamma/\Gamma_k).$$

By assumption, each term on the right hand side is an integer, since the Atiyah conjecture holds for Γ/Γ_k. Since $\mathbb{Z}$ is discrete in $\mathbb{R}$, the same will be true for its limit, and this is exactly what we have to prove. □

As observed above, this translates to a statement about the integral group ring of Γ. To extend this result from the integral group ring to more general coefficient rings, which is interesting because of the algebraic consequences, we have to generalize the approximation results of Section 3.3 to algebraic approximating results for more general coefficient rings. In fact, we have the following result of [18].

3.4.22. Theorem. *Assume Γ is a discrete group with a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ such that $\bigcap_{k\in\mathbb{N}} \Gamma/\Gamma_k = \{1\}$ and such that $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$.*

For example, all the groups Γ/Γ_k might be elementary amenable, e.g. finite.

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{C}$. Assume $A \in M(d \times d, \overline{\mathbb{Q}}\Gamma)$.

The projection $\Gamma \to \Gamma/\Gamma_k$ extends canonically to the group rings and to matrix rings over the group rings. Let $A_k \in M(d \times d, \overline{\mathbb{Q}}\Gamma/\Gamma_k)$ be the image of A under this induced homomorphism.

Then A acts on $(l^2\Gamma)^d$ and A_k acts on $(l^2\Gamma/\Gamma_k)^d$.

The following approximation result for the kernels of these operators holds:

$$\dim_\Gamma(\ker(A)) = \lim_{k\to\infty} \dim_{\Gamma/\Gamma_k}(\ker(A_k)).$$

In particular, if all the groups Γ/Γ_k are torsion-free and $\overline{\mathbb{Q}}[\Gamma/\Gamma_k]$ satisfies the algebraic Atiyah conjecture 3.4.5, then, using the same argument as above, $\overline{\mathbb{Q}}\Gamma$ also satisfies the Atiyah conjecture 3.4.5.

3.4.23. Remark. The approximation result 3.3.6 is a special case, since if Δ is a matrix representative for a combinatorial Laplacian for $\overline{X}$, then Δ_k, constructed as in Theorem 3.4.22, is a matrix representative for the corresponding combinatorial Laplacian of X_k.

The proof uses the fact that one is working with algebraic coefficients. So far, no generalization to $\mathbb{C}\Gamma$ has been obtained.

However, it is a well known fact that, if $\overline{\mathbb{Q}}\Gamma$ has no non-trivial zero divisors, then the same is true for $\mathbb{C}\Gamma$ (compare e.g. [18]). Therefore, from the point of view of the zero divisor conjecture 3.4.6, there is no need to generalize Theorem 3.4.22.

3.4.24. Remark. It is not hard to see that $\mathcal{D}$ is contained in $\mathcal{G}$. Therefore, Theorem 3.4.22 provides the last step for the proof of Theorem 3.4.12.

3.4.5.4 Atiyah conjecture for braid groups

3.4.25. Definition. Fix $n \in \mathbb{N}$. A *braid with n-strings* is an embedding

$$\phi\colon \{1,\dots,n\} \times [0,1] \to \mathbb{C} \times [0,1]$$

such that $\phi(p,0) = (p,0)$ and $\phi(p,1) \in \{1,\dots,n\} \times \{1\}$ for $p = 1,\dots,n$. Two braid are considered equal if they are isotopic where the isotopy fixes the top and the bottom.

Isotopy classes of braids from a group by stacking two braids together, the so called *Artin braid group* B_n. Note that the p-th string is not necessarily stacked on the p-th string, since the p-th string might lead from $(p,0)$ to $(\sigma(p),1)$ for some permutation σ of $\{1,\dots,n\}$. We have to account for this when we define the "stacked" map $\{1,\dots,n\} \times [0,1] \to \mathbb{C} \times [0,1]$.

The braid group B_n contains a normal subgroup P_n, the *pure braid group*, where we require $\phi(p,1) = (p,1)$ for each $p \in \{1,\dots,n\}$. The quotient B_n/P_n is the symmetric group S_n of permutations of $\{1,\dots,n\}$, where the image permutation is given as above.

In Example 3.4.11, we assert that all the braid groups belong to $\mathcal{D}$. Indeed, every braid group B_n has a nested sequence of normal subgroups $B_n \geq P_{1,n} \geq P_{2,n} \geq \cdots$ with $B_n/P_{k,n}$ torsion-free elementary amenable for each k and such that $\bigcap_{k\in\mathbb{Z}} P_{k,n} = \{1\}$. The corresponding result for the pure braid groups is proved in [22, Theorem 2.6]. However, to extend such a result from a subgroup of finite index to a bigger group is highly non-trivial and in general not possible. Indeed, for the full braid group it was conjectured for a while that it has no non-trivial torsion-free quotients at all, opposite to what we need. Using certain totally disconnected completions of the groups involved, and cohomology of these completions (Galois cohomology, which takes the topology of the completions into account) the above result is proved in [46]. Actually, it is proved there that every torsion-free finite extension of P_n has a sequence of subgroups as above, and therefore belongs to $\mathcal{D}$. This paper also contains generalizations, where the pure braid groups are replaced by other kinds of groups, still with the result that the property to belong to $\mathcal{D}$ passes to finite extensions (as long as they are torsion-free). In [47], it is shown how this applies to fundamental groups of certain complements of links in $\mathbb{R}^3$ (a link is an embedding of the disjoint union of finitely many circles).

The proof of the Baum-Connes conjecture for the full braid group [68] mentioned in Section 1.3.9 is based on the same results.

3.4.6 Atiyah conjecture for groups with torsion

The Atiyah conjecture has also been obtained for many groups Γ with torsion, as long as $Fin^{-1}(\Gamma)$ is a discrete subset of $\mathbb{R}$. We are not discussing them here because of lack of space and time, and because there is no zero divisor conjecture for groups with torsion. The ring $D_K\Gamma$ also exists for groups with torsion. It can not be a skew field but, under the assumption that $Fin^{-1}(\Gamma)$ is discrete, it often turns out to be a semi-simple Artinian ring. More details can be found e.g. in the original sources [49, 48, 70].

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Manifolds of Positive Scalar Curvature

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Contents

1 Survey on the problem of finding a positive scalar curvature metric on a closed manifold

The basic question that we want to address in the first three lectures is the following.

1.1. Question. Which manifolds admit a Riemannian metric of positive scalar curvature?

We will consider this question for smooth, compact manifolds without boundary, and will assume that all manifolds mentioned in these lectures are of this type unless otherwise indicated. The above is just one of the many questions that one might ask relating the geometry and topology of manifolds. Here the 'geometry' of a manifold M is determined by a Riemannian metric on M and its curvature. There are various 'flavors' of curvature (sectional, Ricci, scalar, curvature operator, e.t.c., see [Be]), of which the scalar curvature is the simplest in the sense that the scalar curvature is just real valued function $s\colon M \to \mathbb{R}$ (the other flavors of curvature are described as 'tensors' on M). One might ask finer questions concerning the scalar curvature; for example: Given a manifold M, which smooth functions on M can be realized as the scalar curvature functions of Riemannian metrics on M? However, thanks to results of Kazdan-Warner [KW1], [KW2] (see also [Fu]), it turns out that this question essentially boils down to answering Question 1.1. Unlike most other curvature problems, we know quite a bit about the answer to Question 1.1 as we shall see.

1.1 Scalar curvature

We begin by recalling the definition of the scalar curvature function. Usually, the scalar curvature is defined in terms of the 'curvature tensor', which is the right thing to do if the goal is to do calculations. However, for our purposes the following geometric/conceptual definition seems more adequate. Let M be a manifold of dimension n equipped with a Riemannian metric g. Then the scalar curvature $s(p) \in \mathbb{R}$ at a point $p \in M$ is determined by the volume growth of geodesic balls around p. More precisely, let $B_r(p, M)$ be the geodesic ball of radius $r > 0$ around the point p (consisting of all points $x \in M$ whose distance from p is $\leq r$), and let us write $\operatorname{vol} B_r(p, M)$ for the volume of this ball. Then $s(p)$ is determined by the power series expansion (see [Be, 0.60])

$$\frac{\operatorname{vol} B_r(p, M)}{\operatorname{vol} B_r(0, \mathbb{R}^n)} = 1 - \frac{s(p)}{6(n+2)} r^2 + \dots . \tag{1.2}$$

In particular, $s(p) > 0$ means that geodesic balls around p of sufficiently small radius have *smaller volume* than the balls of the same radius in $\mathbb{R}^n$.

1.3. Examples of manifolds with $s > 0$. The *round sphere* (the geometer's slang for the sphere with its standard metric) has positive scalar curvature (provided of course $n \geq 2$, where n is the dimension of the sphere). Since curvature is local it follows that any manifold that has the round sphere as its universal covering, like the real projective space, or more generally any lens space, has positive scalar curvature.

The complex projective space $\mathbb{C}\mathrm{P}^{\mathrm{n}}$ or the quaternionic projective space $\mathbb{H}\mathrm{P}^{\mathrm{n}}$ also inherit a Riemannian metric from the round metric on the sphere whose quotients they are (these are referred to as the *Fubini-Study metric* in the geometric literature). It can be shown that their scalar curvature is positive as well.

1.2 Constructions of positive scalar curvature metrics

In order to answer Question 1.1, we have to do two things:

- construct positive scalar curvature metrics on some manifolds, and
- show that other manifolds do not admit such metrics.

We will begin with the 'constructive' part. The following observation allows us to construct a lot of new manifolds with positive scalar curvature metrics from a given manifold with positive scalar curvature metric.

1.4. Observation. Let g be a positive scalar curvature metric on a manifold M. Then the following manifolds admit metrics of positive scalar curvature:

1. The product $M \times N$ with any manifold N.
2. The total space of any fiber bundle $E \to N$ with fiber M, provided the transition functions are isometries of (M, g).

To prove the first claim, pick any metric h (not necessarily with positive scalar curvature) on N. Then g and h combine to determine a Riemannian metric $g \times h$ (called the product metric) on $M \times N$. The scalar curvature of $g \times h$ can be expressed in terms of the scalar curvature of g and h via the formula

$$s((x,y); g \times h) = s(x; g) + s(y; h) \qquad \text{for } (x,y) \in M \times N.$$

Here we write $s(x; g)$ for the scalar curvature at $x \in M$ with respect to the metric g; the meaning of $s(y; h)$ and $s((x,y); g \times h)$ is analogous. Assuming no further information about the metric h, of course this quantity could be positive or negative, and that doesn't seem to bode well for our goal of producing a positive scalar curvature metric on $M \times N$. Here is the trick: shrink M! Phrased more mathematically: replace g by tg where t is a positive real number, and let t approach 0. We see that

$$s((x,y); tg \times h) = s(x; tg) + s(y; h) = \frac{1}{t} s(x; g) + s(y; h).$$

This is *positive* for t sufficiently small thanks to the positivity of $s(x; g)$. Of course, *how* small we need to choose t will depend on (x, y); however, compactness of M and N implies that there is a t that will do the job for all $(x, y) \in M \times N$.

The same trick still works in the case of the 'twisted product' $E \to N$. In addition to a metric on the base N, we need to choose here a 'connection' on this bundle; assuming that the transition functions are isometries, these data determine a 'twisted product metric' on E (which makes $E \to M$ a Riemannian submersion with totally geodesic fibers; see [Be, Ch. 9]); the *O'Neill formulas* express the curvature of this metric in terms of the curvatures of g, h and the curvature of the connection (see [Be, Proposition 9.70]). Replacing g by tg, the dominant term of the scalar curvature of E for small t is again $\frac{1}{t}$ times the scalar curvature of g (see [Be, Formula (9.70d)]). Hence the scalar curvature on E is positive for sufficiently small t.

In differential topology important ways of modifying a given manifold is *surgery* and *attaching a handle*. Of course these modifications are closely related: if W is a manifold with boundary $\partial W = M$, and $\widehat{W}$ is obtained from W by attaching a handle $D^{k+1} \times D^{n-k}$ via an embedding $S^k \times D^{n-k} \subset M^n$, then $\widehat{M} \overset{\text{def}}{=} \partial \widehat{W}$ is obtained from M by a surgery (i.e., by removing $S^k \times D^{n-k}$ and replacing it by $D^{k+1} \times S^{n-k-1}$). Independently Gromov-Lawson [GL] and Schoen-Yau [SY] showed that if M admits a positive scalar curvature metric, and $n-k$ (the *codimension of the surgery/handle*) is greater than 2, then $\widehat{M}$ also admits such a metric. Based on their techniques, Gajer [Gaj] later proved the following result:

Theorem 1.5. *Let W be a manifold with boundary and let g be a positive scalar curvature metric on W. Assume that $\widehat{W}$ is obtained from W by attaching a handle of codimension ≥ 3. Then g extends to a positive scalar curvature metric on $\widehat{W}$.*

Here we make the convention that all metrics considered on manifolds with boundary are *product metrics near the boundary.*

Gromov and Lawson made the fundamental observation that this result implies that the answer to our Question 1.1 whether a manifold M admits a positive scalar curvature metric depends only on the *bordism class of M* in a suitable bordism group [GL]. We recall that two closed n-manifolds M, N are called *bordant* if there is a manifold W of dimension $n+1$ whose boundary ∂W is the disjoint union $M \coprod N$.

1.6. Spin structures. (see [LaM, Chap. II, §1]) Let M be an oriented Riemannian manifold of dimension n, and let $SO(M) \to M$ be its *oriented frame bundle*; i.e., the principal $SO(n)$-bundle whose fiber over $x \in M$ consists of all orientation preserving isometries $\mathbb{R}^n \to T_xM$ of $\mathbb{R}^n$ to the tangent space at x (the image of the standard base element in $\mathbb{R}^n$ then gives a 'frame' of T_xM; $SO(n)$ acts on these isometries by precomposition). A *spin structure* on M consists of a double covering of $SO(M)$, whose restriction to each fiber $SO(M)_x$ is a *non-trivial* double covering (the universal covering if $n \geq 3$). A *spin manifold* is a manifold equipped with a spin structure. We note that a spin structure implicitly involves the choice of an orientation; for each spin structure on M there is an 'opposite' spin structure whose underlying orientation is the opposite of the previous one. If M is a spin manifold (resp. oriented manifold), we denote by $-M$ the manifold equipped with the opposite spin structure (resp. orientation).

1.7. Bordism groups. We recall that two n-manifolds M, N are called *bordant* if there is a $(n+1)$-manifold W whose boundary ∂W is the disjoint union $M \coprod N$. If M and N are oriented manifolds (resp. spin manifolds), the requirement is that W is equipped with an orientation (resp. spin structure) such that $\partial W = M \coprod -N$, where the orientation or spin structure on ∂W is induced by that on W. To treat both – orientations and spin structures – on the same footing, it is convenient to refer to them as G-structures, where $G = SO$ if we talk about orientations, and $G = Spin$ for spin structures. More generally, if M, N are n-manifolds with G-structures and $f\colon M \to X$, $g\colon N \to X$ are maps to a topological space X, then the pairs (M, f), (N, g) are *bordant* if there is a $n+1$-manifold with G-structure W with $\partial W = M \coprod -N$, and a map $F\colon W \to X$, which restricts to f on $M \subset W$ and to g on $N \subset W$. We write $[M, f]$ for the bordism class of the pair (M, f), and denote by $\Omega_n^G(X)$ the set consisting of the bordism classes of such pairs. The disjoint union of pairs gives $\Omega_n^G(X)$ the structure of an abelian group; the neutral element is represented by the empty n-dimensional manifold; the inverse of $[M, f]$ is given by $[-M, f]$.

The following result was proved by Gromov-Lawson for simply connected manifolds [GL]; a proof in the general case can be found in [RS1]. In the statement of this result the following subgroup of $\Omega_n^G(X)$ plays a crucial role:

$$\Omega_n^{G,+}(X) \overset{\text{def}}{=} \left\{ [N,f] \in \Omega_n^G(X) \;\middle|\; \begin{matrix} N \text{ admits a metric of} \\ \text{positive scalar curvature} \end{matrix} \right\}.$$

Theorem 1.8. *Let M be a manifold of dimension $n \geq 5$ with fundamental group π. Let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Assume that*

(a) M admits a Spin-structure, or that

(b) M admits a SO-structure and $\widetilde{M}$ does not admit a Spin-structure,

and let $[M,u] \in \Omega_n^G(B\pi)$ be the element represented by the pair (M,u), where $G = Spin$ in case (a) and $G = SO$ in case (b). Then M admits a positive scalar curvature metric if and only if $[M,u]$ is in $\Omega_n^{G,+}(B\pi)$.

1.9. Remark. There are closed manifolds of dimension $n \geq 5$ which don't satisfy the assumptions of the above theorem; for example non-orientable manifolds, or manifolds without a spin structure whose universal cover admits a spin structure, like the real projective space $\mathbb{RP}^n$ for $n \equiv 1 \bmod 4$. There is a more general version of this theorem that applies to *all* closed manifolds of dimension $n \geq 5$, the proof of which is no harder than the proof of Theorem 1.8; it is just more technical to define the relevant bordism groups, which are spin (resp. oriented) bordism groups of $B\pi$ with 'twisted coefficients' if $\widetilde{M}$ admits a spin structure (if $\widetilde{M}$ does not admit a spin structure) (see [RS1], [St4]).

Outline of the proof of Theorem 1.8. At first glance the statement of the theorem might appear to be tautological. Of course, the existence of a positive scalar curvature metric on M implies that $[M,u]$ is in $\Omega_n^{G,+}(B\pi)$ by definition of that subgroup, but the converse statement is *not* obvious: $[M,u] \in \Omega_n^{G,+}(B\pi)$ means that (M,u) is *bordant* to a pair (N,f), where N admits a positive scalar curvature metric, and the claim is that M itself admits such a metric. To prove this, we would like to argue that a bordism W between M and N is obtained from $N \times [0,1]$ by attaching handles of codimension ≥ 3; then Theorem 1.5 would imply that the positive scalar curvature metric on $N \times [0,1]$ extends to a positive scalar curvature metric on all of W. In particular, its restriction to $M \subset \partial W$ gives a positive scalar curvature metric on M (we recall that all metrics considered on manifolds with boundary are required to be product metrics near the boundary). While not *every* bordism between M and N can be obtained by attaching handles of codimension ≥ 3, it turns out that the conditions of the theorem are carefully chosen in such a way that W can always be modified by surgeries in the interior in order to obtain a bordism for which this *is* the case. $\square$

A striking application of the 'Bordism Theorem' 1.8 is the following result.

Theorem 1.10 (Gromov-Lawson [GL]). *Every simply connected closed non-spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric.*

Proof. The cartesian product of manifolds gives $\Omega_*^G = \bigoplus_{n=0}^\infty \Omega_n^G$ the structure of a graded ring. C.T.C. Wall constructed explicitly manifolds which are multiplicative generators for Ω_*^{SO}. These manifolds are either projective spaces or total spaces of fiber bundles with projective spaces as fibers whose transition functions are isometries of the Fubini-Study metric (cf. 1.3) on projective space. By part 2 of Observation 1.4 then all of these manifolds admit positive scalar curvature metrics. Since these manifolds multiplicatively generate Ω_*^{SO} it follows from part 1 of Observation 1.4 that the subgroup $\Omega_n^{SO,+}$ is *equal* to Ω_n^{SO} for $n > 0$. Hence Theorem 1.8 implies the corollary. $\square$

1.3 Obstructions to positive scalar curvature metrics

The discussion of the preceding subsection, notably Corollary 1.10 might leave the impression that most manifolds admit metrics of positive scalar curvature. This is not so; in fact, currently, there are three known methods to show that some manifolds do not admits such metrics. These methods are – in chronological order – the following:

Index obstructions. This method, pioneered by Lichnerowicz in the early sixties [Li] and developed since then by many mathematicians is based on the 'Bochner-Lichnerowicz-Weitzenböck formula' (see 1.17) which provides a relationship between positive scalar curvature and the 'Dirac operator' (see 1.15) defined by Atiyah-Singer on any Riemannian manifold equipped with a spin-structure. This method – described below – is the most powerful of the three methods available. Its limitations come from the fact that it requires a spin structure (this can actually be weakened to requiring a spin-structure on the universal cover, see [St4]).

Minimal hypersurface method. Schoen and Yau proved in 1980 [SY] that if M is a Riemannian manifold of dimension n of positive scalar curvature then any stable minimal hypersurface $N \subset M$ (i.e., N is a local minimum of the area functional) admits a positive scalar curvature metric (the induced metric might *not* have positive scalar curvature, but a conformal change produces a positive scalar curvature metric on N). This can lead to interesting restrictions if N represents a non-trivial element in $H_{n-1}(M;\mathbb{Z})$: As Thomas Schick will explain in his second lecture, there is a 5-dimensional manifold for which the index obstruction is zero, but for which the minimal hypersurface method can be used to show that it cannot admit a positive scalar curvature metric. A limitation of this method is that it doesn't give restrictions if $H_{n-1}(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is trivial, for example if the fundamental group $\pi_1(M)$ is finite. Moreover, even if this group is non-trivial, there might only be a stable minimal hypersurface *with singularities* representing a given homology class for $n \geq 8$. It has been claimed [Y] that the technical difficulties associated with the possible singularities can be overcome to prove non-existence of positive scalar curvature metrics for manifolds of arbitrary dimension; however, the author is not aware of a published account of this.

Seiberg-Witten invariants. This is a diffeomorphism invariant of 4-dimensional manifolds, which vanishes if the manifold admits a positive scalar curvature metric. For example, the manifold

$$X^2(d) = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; \mathrm{z}_0^{\mathrm{d}} + \cdots + \mathrm{z}_3^{\mathrm{d}} = 0 \right\}$$

is simply connected of real dimension 4; its Seiberg-Witten invariant is non-zero for $d \geq 3$ [Ta2] (the restriction $d \geq 3$ guarantees $b_2^+ > 1$, where b_2^+ is the number of positive eigenvalues of the intersection form, which is a necessary restriction for the definition of the Seiberg-Witten invariant. A 'fancier' version of the invariant is defined for $b_2^+ = 1$, but that doesn't seem to lead to obstructions for positive scalar curvature metrics as the example $X^2(d) = \mathbb{CP}^2$ shows). We note that $X^2(d)$ is a simply connected manifold, which is non-spin for d odd. This shows that Theorem 1.10 does not hold in dimension $n = 4$. We also observe that the non-existence of a positive scalar curvature metric on $X^2(d)$ for d odd cannot be proved by the other methods (the minimal hypersurface method doesn't apply, since $X^2(d)$ is simply connected and there are no index obstructions coming from the Dirac operator since $X^2(d)$ doesn't admit a spin structure). The obvious limitation of this method is that it applies only to 4-dimensional manifolds.

Now we explain the 'Index obstruction' method in more detail. The first result in this direction is the following.

Theorem 1.11 (Lichnerowicz, [Li]). *Let M be a spin manifold of dimension $n = 4k$ which admits a positive scalar curvature metric. Then the $\widehat{A}$-genus $\widehat{A}(M)$ vanishes.*

1.12. Definition of the $\widehat{A}$-genus. (see [LaM, Chap. III, §11]) To define the $\widehat{A}$-genus $\widehat{A}(M)$ of a manifold M, we first recall the definition of the 'characteristic class' $\widehat{A}(E) \in H^*(X;\mathbb{Q})$ associated to any vector bundle $E \to X$. It is characterized by the following properties:

(naturality) for any map $f\colon X' \to X$ we have $\widehat{A}(f^*E) = f^*(\widehat{A}(E))$

(exponential property) $\widehat{A}(E \oplus F) = \widehat{A}(E) \cdot \widehat{A}(F)$.

(normalization) If $L \to X$ is a complex line bundle with Euler class (which equals the first Chern class) $x \in H^2(X;\mathbb{Q})$, then

$$\widehat{A}(L) = \frac{x/2}{\sinh(x/2)} = 1 - \frac{1}{2^3 \cdot 3}\, x^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}\, x^4 + \cdots \in H^*(X;\mathbb{Q}).$$

If M is a oriented manifold of dimension $n = 4k$, its $\widehat{A}$-genus is defined as

$$\widehat{A}(M) = \langle \widehat{A}(TM), [M] \rangle \in \mathbb{Q},$$

where TM is the tangent bundle of M, and $\langle\ \ , [M]\rangle$ is the evaluation on the fundamental class $[M] \in H_n(M;\mathbb{Z})$.

Example 1.13. $\widehat{A}(X^2(d)) = \frac{d(d-2)(d+2)}{24}$ (see [LaM, Ch. IV, Formula 4.4]), which implies via Lichnerowicz' Theorem 1.11 that $X^2(d)$ does not admit a positive scalar curvature metric for $d \geq 4$ even (d even guarantees that $X^2(d)$ has a spin structure).

1.14. The complex spinor bundle. (see [LaM, Chap. II, §§3–4]). Let M be a Riemannian manifold of dimension $n = 2k$ equipped with a spin structure $Spin(M) \to SO(M)$ (cf. 1.6). We recall that $Spin(M) \to SO(M)$ is a principal $Spin(n)$-bundle, where $Spin(n)$ is the connected Lie group obtained as the non-trivial double covering of $SO(n)$ (the universal covering for $n \geq 3$). The Spinor bundle $S \to M$ is the vector bundle associated to a certain representation Δ of $Spin(n)$ called the *spinor representation.* To construct Δ, the group $Spin(n)$ is identified with a subgroup of units of the *Clifford algebra* $C\ell_n$, and then Δ is a certain $C\ell_n$-module considered as a representation of $Spin(n) \subset C\ell_n^\times$. We recall that the Clifford algebra $C\ell_n = C\ell_n^+ \oplus C\ell_n^-$ is the $\mathbb{Z}/2$-graded $\mathbb{R}$-algebra with unit generated by all vectors $v \in \mathbb{R}^n \subset C\ell_n^-$ subject to the relations $v \cdot v = -|v|^2 \cdot 1$. The subgroup

$Pin(n)$ of $C\ell_n^\times$ generated by all unit vectors $v \in \mathbb{R}^n \subset C\ell_n$ is a double covering group of the orthogonal group $O(n)$ (the double covering map is given by sending v to the reflection at the hyperplane perpendicular to v). The identity component of $Pin(n)$ can then be identified with $Spin(n)$.

It can be shown (see [LaM, Ch. I, §4]) that the complexification $C\ell_{2k} \otimes \mathbb{C}$ is the algebra $\mathbb{C}(2^k)$ of $2^k \times 2^k$-matrices over $\mathbb{C}$. Let Δ be $\mathbb{C}^{2^k}$ with the $\mathbb{C}(2^k)$-module structure given by multiplying a $2^k \times 2^k$-matrix by a 2^k-vector. We consider Δ as a module over $C\ell_{2k} \otimes \mathbb{C}$ and define a $\mathbb{Z}/2$-grading $\Delta \overset{\text{def}}{=} \Delta^+ \oplus \Delta^-$ by letting $\Delta^\pm$ be the ± 1-eigenspace of the involution given by multiplication by the *complex volume element* $\omega_{\mathbb{C}} = i^k e_1 \cdots e_{2k} \in C\ell_{2k} \otimes \mathbb{C}$. Then the *complex spinor bundle* $S \to M$ is defined by

$$S = Spin(M) \times_{Spin(n)} \Delta.$$

The crucial feature of the spinor bundle is that there is a *Clifford multiplication*, a vector bundle map

$$TM \otimes S \longrightarrow S.$$

It is induced by the module multiplication map $\mathbb{R}^n \otimes \Delta \subset C\ell_n \otimes \Delta \to \Delta$. The $\mathbb{Z}/2$-grading $\Delta = \Delta^+ \oplus \Delta^-$ induces a corresponding $\mathbb{Z}/2$-grading $S = S^+ \oplus S^-$ on the vector bundle S. In particular, Clifford multiplication by a tangent vector maps S^+ to S^- and vice versa. The Levi-Civita connection on TM induces a principal connection on the frame bundle $SO(M)$, which lifts to a principal connection on $Spin(M)$, which in turn induces a connection on the associated vector bundle $S \to M$ (see [LaM, Ch. II, §4]).

1.15. The Dirac operator. (see [LaM, Ch. II, §5]) The Dirac operator $D\colon C^\infty(S) \to C^\infty(S)$ is the first order elliptic differential operator defined by

$$(D\psi)(x) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi. \tag{1.16}$$

Here $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_xM, $\nabla_{e_i}\psi \in S_x$ is the covariant derivative of ψ in the direction of e_i, and $e_i\cdot$ is Clifford multiplication by e_i. We note that D is an *odd* operator in the sense that if ψ is a section of S^+, then $D\psi$ is a section of S^- and vice versa: if ψ is a section of S^+, then $\nabla_{e_i}\psi \in S_x^+$, and hence $e_i \cdot \nabla_{e_i}\psi \in S_x^-$). In particular, restricting D gives operators $D^\pm\colon C^\infty(S^\pm) \to C^\infty(S^\mp)$.

1.17. The Bochner-Lichnerowicz-Weitzenböck Formula (see [LaM, Ch. II, §8]) is the equation

$$D^2 = \nabla^*\nabla + \frac{1}{4}s, \tag{1.18}$$

where ∇ is the connection on the spinor bundle S, considered as a homomorphism $\nabla\colon C^\infty(S) \to C^\infty(T^*M \otimes S)$, and $\nabla^*\colon C^\infty(T^*M \otimes S) \to C^\infty(S)$ is its adjoint with repect to the inner product on these spaces of sections induced by the Riemannian metric on M (T^*M is the cotangent bundle of M). All the terms in the formula above are considered as linear maps $C^\infty(S) \to C^\infty(S)$; the term $\frac{1}{4}s$ is multiplication by $\frac{1}{4}$ times the scalar curvature function.

Proof of Lichnerowicz' Theorem 1.11. Let $\langle\ ,\ \rangle$ be the inner product on each fiber of the spinor bundle $S \to M$ (which is induced by the Riemannian metric on M), and let $(\phi, \psi) \in \mathbb{R}$ be the inner product of sections $\phi, \psi \in C^\infty(S)$ defined by

$$(\phi, \psi) = \int_M \langle \phi(x), \psi(x) \rangle \, dvol(x),$$

where *dvol* is the volume element determined by the Riemannian metric on M. We observe that the Weitzenböck Formula 1.18 has the following consequence: if ψ is in the kernel of the Dirac operator D, then

$$0 = (D^2\psi, \psi) = (\nabla^*\nabla\psi + s\psi, \psi) = \|\nabla\psi\|^2 + (s\,\psi, \psi) \geq (s\,\psi, \psi).$$

Assuming that the scalar curvature function s is everywhere positive, a nonzero ψ would force $(s\,\psi, \psi)$ to be strictly positive in contradiction to the inequality above.

Besides the Weitzenböck formula, the other main input in the proof of Theorem 1.11 is the Atiyah-Singer Index Theorem. Specializing to the Dirac operator on a spin manifold M of dimension $n = 4k$ (see [LaM, Ch. III, Thm. 13.10]), it says

$$\operatorname{index}(D^+) = \widehat{A}(M),$$

where $\operatorname{index}(D_+) = \dim\ker(D^+) - \dim\operatorname{coker}(D^+)$. It can be shown that D^+ is the adjoint operator of D^-, which allows us to identify $\operatorname{coker}(D_+)$ with $\ker(D^-)$. This shows that if the scalar curvature function is strictly positive, then both, $\dim\ker(D^+)$ and $\dim\operatorname{coker}(D^+)$ vanish, and hence so does $\widehat{A}(M)$. $\square$

Lichnerowicz' result 1.11 has been refined by Hitchin [Hit] and later Rosenberg [Ro2]. The idea is to construct a version of the Dirac operator D which commutes with the action of a C^*-algebra A. In the simplest case, the relevant algebra is the Clifford algebra, and the construction is the following.

1.19. The $C\ell_n$-linear Dirac operator (see [LaM, Ch. II, §7]). Let M be a n-dimensional spin manifold. The *$C\ell_n$-linear spinor bundle* is the vector bundle

$$\mathfrak{S} \stackrel{\text{def}}{=} Spin(M) \times_{Spin(n)} C\ell_n.$$

This is a variation of the spinor bundle described in 1.14. We note that the Clifford algebra $C\ell_n$ acts by right multiplication on $\mathfrak{S}$. This action is fiber preserving and hence gives the space of sections $C^\infty(\mathfrak{S})$ the structure of a $\mathbb{Z}/2$-graded right $C\ell_n$-module. As in 1.15 we can define the Dirac operator $\mathfrak{D}\colon C^\infty(\mathfrak{S}) \to C^\infty(\mathfrak{S})$. It commutes with the $C\ell_n$-action, and it is therefore referred to as the *$C\ell_n$-linear Dirac operator*. The kernel of $\mathfrak{D}$ is then a $\mathbb{Z}/2$-graded module over $C\ell_n$ and so represents an element the Grothendieck group $\widehat{\mathfrak{M}}_n$ of $\mathbb{Z}/2$-graded $C\ell_n$-modules. This element might depend on the Riemannian metric on M; however, the class $[\ker \mathfrak{D}] \in \widehat{\mathfrak{M}}_n / i^* \widehat{\mathfrak{M}}_{n+1}$ is independent of the metric, where $i^*\colon \widehat{\mathfrak{M}}_{n+1} \to \widehat{\mathfrak{M}}_n$ is induced by the inclusion $i\colon C\ell_n \to C\ell_{n+1}$. The element

$$\alpha(M) \stackrel{\text{def}}{=} [\ker \mathfrak{D}] \in KO_n(\mathbb{R}) \stackrel{\text{def}}{=} \widehat{\mathfrak{M}}_n / i^* \widehat{\mathfrak{M}}_{n+1} \tag{1.20}$$

is the *Clifford index* of $\mathfrak{D}$ [LaM, Ch. III. Def. 10.4.].

The same argument as for Lichnerowicz' Theorem 1.11 leads to the following result due to Hitchin (with a somewhat different proof).

Theorem 1.21 (Hitchin [Hit]). *Let M be a spin manifold of dimension n. If M admits a positive scalar curvature metric, then $\alpha(M) \in KO_n(\mathbb{R})$ is zero.*

Remark 1.22. The groups $KO_n(\mathbb{R})$ depend only on n modulo 8, and are given by the following table.

n mod 8	0	1	2	3	4	5	6	7
$KO_n(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

Moreover, if M is a spin manifold of dimension $n \equiv 0 \bmod 4$, then $\widehat{A}(M) = \alpha(M)$ for $n \equiv 0 \bmod 8$, and $\widehat{A}(M) = 2\alpha(M)$ for $n \equiv 4 \bmod 8$ (if we identify $KO_n(\mathbb{R})$ with $\mathbb{Z}$ by choosing as generator of $KO_n(\mathbb{R})$ the $\mathbb{Z}/2$-graded

$C\ell_n$-module Δ used in the construction of the spinor bundle in those dimensions, see 1.14). This shows that Hitchin's result is a generalization of Lichnerowicz' Theorem 1.11.

In dimensions $n \equiv 1, 2 \bmod 8$, $n \geq 9$, there are smooth manifolds Σ homeomorphic, but not diffeomorphic to the n-dimensional sphere with $\alpha(\Sigma) \neq 0 \in KO_n(\mathbb{R}) \cong \mathbb{Z}/2$. This is interesting since it shows that the answer to the question "Does a given manifold M admit a positive scalar curvature metric?" might depend on quite subtle things like the differentiable structure of M.

Hitchin's result can be generalized by 'twisting' the Dirac operator as follows. Suppose $E \to M$ is a real vector bundle with connection. Then we can define the *twisted Dirac operator*

$$\mathfrak{D}_E \colon C^\infty(\mathfrak{S} \otimes E) \longrightarrow C^\infty(\mathfrak{S} \otimes E)$$

by the same Formula 1.16 defining the usual Dirac operator, where now ∇ is the product connection on $\mathfrak{S} \otimes E$ (of the usual connection on $\mathfrak{S}$ induced by the Levi-Civita connection and the given connection on E). The Bochner-Lichnerowicz-Weitzenböck formula 1.18 continues to hold, provided the connection on E is *flat*. In that case, the $C\ell_n$-module $\ker \mathfrak{D}_E$ gives an element

$$[\ker \mathfrak{D}_E] \in KO_n(\mathbb{R}),$$

which must be zero if M admits a positive scalar curvature metric. All flat vector bundles over M are obtained in the following way: Given an orthogonal representation $\rho \colon \pi \to O(V)$ of a discrete group π and a map $f \colon M \to B\pi$, we can form the flat vector bundle $E(\rho) = E\pi \times_\pi V$ over $B\pi = E\pi/\pi$ and pull it back via f to get a flat vector bundle $f^*E(\rho)$ over M.

We can do better: all the obstructions $[\ker \mathfrak{D}_{f^*E(\rho)}] \in KO_n(\mathbb{R})$ corresponding to various orthogonal representations ρ of π can be obtained as the images of a single obstruction

$$[\ker \mathfrak{D}_{f^*\mathcal{V}(\pi)}] \in KO_n(C^*\pi)$$

under homomorphisms $\rho_* \colon KO_n(C^*\pi) \to KO_n(\mathbb{R})$. Here $C^*\pi$ is the *group C^*-algebra* of π, and $\mathcal{V}(\pi) = E\pi \times_\pi C^*\pi$ is the *Miščenko-Fomenko line bundle*. Let us recall the relevant definitions.

1.23. The group C^*-algebra of a discrete group π. We recall that a *C^*-algebra* is an algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ (the more usual case considered is $\mathbb{C}$; the C^*-algebras considered in these lectures are all over $\mathbb{R}$) equipped with an anti-involution $*\colon A \to A$ which is $*$-isomorphic to a closed subalgebra of the algebra of bounded operators $\mathcal{B}(H)$ on a $\mathbb{F}$-Hilbert space H. Here the involution on $\mathcal{B}(H)$ is given by sending an operator to its adjoint, and the subalgebra is required to be closed with respect to the norm topology on $\mathcal{B}(H)$. It should be mentioned that there are more 'intrinsic' definitions (see [WO, 1.1]), but the above suffices for our purposes.

The main example of a C^*-algebra of interest to us is the (real) C^*-algebra $C^*\pi$ of a discrete group π. It is a norm completion of the real group ring $\mathbb{R}\pi = \{\sum_{g\in\pi} r_g g \mid r_g \in \mathbb{R}\}$ (these sums are *finite* sums and they are multiplied using the product in π) equipped with the anti-involution induced by $g \mapsto g^{-1}$ for $g \in \pi \subset \mathbb{R}\pi$. We note that there is a one-to-one correspondence between orthogonal representations of π on a real Hilbert space H and $*$-homomorphisms from $\mathbb{R}\pi$ to $\mathfrak{B}(H)$ (by extending $\rho\colon \pi \to O(H)$ linearly to an algebra homomorphism $\rho\colon \mathbb{R}\pi \to \mathfrak{B}(H)$). The (maximal, real) *group C^*-algebra* $C^*\pi$ is the completion of $\mathbb{R}\pi$ with respect to the norm on $\mathbb{R}\pi$ defined by

$$\|\sigma\|_{max} = \sup_{\rho} \{\|\rho(\sigma)\|\} \qquad \text{for} \quad \sigma \in \mathbb{R}\pi,$$

where the sup is taken over all $*$-homomorphisms ρ from $\mathbb{R}\pi$ to the bounded operators on some Hilbert space.

A variant of $C^*\pi$ is the *reduced* C^*-algebra $C^*_r\pi$ of a group π; as $C^*\pi$, the C^*-algebra $C^*_r\pi$ is a norm completion of $\mathbb{R}\pi$, but with respect to the norm $\|\sigma\| = \|\rho(\sigma)\|$, where $\rho\colon \mathbb{R}\pi \to \mathfrak{B}(l^2(H))$ is the *regular representation of* $\mathbb{R}\pi$, corresponding to the orthogonal representation of π via translations on the Hilbert space $l^2(\pi) = \{f\colon \pi \to \mathbb{R} | \sum_{g\in\pi} |f(g)|^2 < \infty\}$.

Other C^*-algebras of interest to us are the Clifford algebra $C\ell_n$ with anti-involution given by $v \mapsto -v$ for a generator $v \in \mathbb{R}^n$, and the tensor product $C\ell_n \otimes C^*\pi$.

1.24. A Dirac type operator commuting with an action of $C\ell_n \otimes C^*\pi$. Given a discrete group π, the bundle $\mathcal{V}(\pi) \stackrel{\text{def}}{=} E\pi \times_\pi C^*\pi$ over $B\pi$ with fiber $C^*\pi$ is called the *Miščenko-Fomenko line bundle.* We note that $\mathcal{V}(\pi)$ is a flat vector bundle, which is infinite dimensional if the group π is infinite. The reason that $\mathcal{V}(\pi)$ is called a *line* bundle is that $C^*\pi$ acts on $\mathcal{V}(\pi)$ by right multiplication, making the fibers 1-dimensional free modules over $C^*\pi$.

As above, we can defined the twisted Dirac operator

$$\mathfrak{D}_{f^*\mathcal{V}(\pi)}\colon C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi)) \longrightarrow C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi))$$

for any map $f\colon M \to B\pi$ from a spin manifold M to $B\pi$. We note that the C^*-algebra $C\ell_n \otimes C^*\pi$ acts fiber preserving on $S \otimes f^*\mathcal{V}(\pi)$ (via the $C\ell_n$-action on $\mathfrak{S}$, and the $C^*\pi$-action on $\mathcal{V}(\pi)$), thus making $C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi))$ a $\mathbb{Z}/2$-graded module over it. Moreover, this action commutes with $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$, giving in particular $\ker \mathfrak{D}_{f^*\mathcal{V}(\pi)}$ the structure of a $\mathbb{Z}/2$-graded module over $C\ell_n \otimes C^*\pi$.

1.25. The K-theory of C^*-algebras. For a real $Z/2$-graded C^*-algebra A its K-theory is defined by

$$KO_0(A) = \left\{ \begin{matrix} \text{equivalence classes of finitely} \\ \text{generated projective } A\text{-modules} \end{matrix} \right\}$$

and $KO_n(A) = KO_0(C\ell_n \otimes A)$. It is tempting to try to define a $KO_n(C^*\pi)$-valued index for the operator $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$ as the class of $KO_n(C^*\pi)$ represented by the kernel of $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$. But this module over $C\ell_n \otimes C^*\pi$ is in general neither finitely generated nor projective. Luckily one can always find a 'compact perturbation' $\mathfrak{D}'_{f^*\mathcal{V}(\pi)}$ of $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$ whose kernel is finitely generated projective (the difference of $\mathfrak{D}$ and $\mathfrak{D}'$ is an operator that is compact in the sense of Hilbert modules over the C^*-algebra $C\ell_n \otimes C^*\pi$; see [WO, Ch. 17]). It turns out that the element in $KO_n(C^*\pi)$ represented by the kernel of $\mathfrak{D}'$ is independent of the choice of the perturbation, and that allows one to define

$$\alpha(M, f) \stackrel{\text{def}}{=} \left[\ker \mathfrak{D}'_{f^*\mathcal{V}(\pi)}\right] \in KO_n(C^*\pi).$$

If $\rho\colon \pi \to O(\mathbb{R}^d)$ is a finite dimensional orthogonal representation of π, the corresponding $*$-homomorphism $\rho\colon C^*\pi \to \mathbb{R}(d)$ (where $\mathbb{R}(d)$ is the C^*-algebra of $d \times d$-matrices with entries in $\mathbb{R}$) induces a homomorphism $\rho_*\colon KO_n(C^*\pi) \to KO_n(\mathbb{R}(d)) \cong KO_n(\mathbb{R})$, under which $\alpha(M, f)$ maps to $[\ker \mathfrak{D}_{f^*E(\rho)}] \in KO_n(\mathbb{R})$.

Theorem 1.26 (Rosenberg [Ro2]). *Let M be a spin manifold of dimension n and let $f\colon M \to B\pi$ be a map to the classifying space of a discrete group π. If M admits a Riemannian metric with positive scalar curvature then $\alpha(M, f) \in KO_n(C^*\pi)$ vanishes.*

2 The Gromov-Lawson-Rosenberg Conjecture

In the last lecture we defined the invariant $\alpha(M,f) \in KO_n(C^*\pi)$ for any spin manifold M of dimension n equipped with a map $f\colon M \to B\pi$ to the classifying space of a discrete group π. We will refer to $\alpha(M,f)$ as the *index obstruction*, since it is the $KO_n(C^*\pi)$-valued index of a twisted Dirac operator on M, and the vanishing of $\alpha(M,f)$ is a necessary condition for the existence of a positive scalar curvature metric on M by Theorem 1.26. Optimistically, one might hope that the vanishing of the α-invariant is not only necessary, but also *sufficient* for the existence of a positive scalar curvature metric, and this is what the following conjecture asserts.

2.1. The Gromov-Lawson-Rosenberg Conjecture. Let M be a connected spin manifold of dimension $n \geq 5$. Then M admits a metric of positive scalar curvature if and only if all index obstructions $\alpha(M,f) \in KO_n(C^*\pi)$ vanish.

Although Thomas Schick has found a counterexample to this conjecture [Sch], which he described in his lectures at this summer school, we state this conjecture, since

1. it has been a very influential conjecture in the field;
2. it is true for manifolds with certain fundamental groups (cf. Thm. 2.13), as we will discuss below, in particular, for simply connected manifolds (cf. Thm. 2.4);
3. a weaker form of the conjecture is true for large classes of fundamental groups (cf. Thm. 3.3 and Thm. 3.10).

In this lecture we will outline the proofs of the known cases of the Gromov-Lawson-Rosenberg Conjecture. We would like to begin by giving a reformulation of the Conjecture. We observe that any map $f\colon M \to B\pi$ to the classifying space of a discrete group π will factor in the form

$$M \xrightarrow{u} B\pi_1(M) \xrightarrow{B\rho} B\pi,$$

where u is the classifying map of the universal covering of M, and $B\rho$ is a map of classifying spaces induced by a homomorphism ρ from the fundamental group $\pi_1(M)$ to π. Then $\alpha(M,u) \in KO_n(C^*\pi_1(M))$ maps to $\alpha(M,f) \in KO_n(C^*\pi)$ via the map $KO_n(C^*\pi_1(M)) \to KO_n(C^*\pi)$ induced by ρ. In particular, the vanishing of $\alpha(M,u)$ implies the vanishing of $\alpha(M,f)$ for all f.

The motivation for defining $\alpha(M,f)$ for a general map f rather than just $\alpha(M,u)$ is that we get a well-defined homomorphism

$$\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi) \qquad \text{by} \quad [M,f] \mapsto \alpha(M,f), \tag{2.2}$$

since it can be shown that the index obstruction $\alpha(M,f)$ depends only on the bordism class of $[M,f] \in \Omega_n^{Spin}(B\pi)$.

In view of the Bordism Theorem 1.8 the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following conjecture:

2.3. Conjecture. $\Omega_n^{Spin,+}(B\pi) = \ker\Big(\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi)\Big)$.

Of course, according to Theorem 1.26 we have the inclusion $\Omega_n^{Spin,+}(B\pi) \subset \ker\alpha$, while the converse inclusion is conjectural.

2.1 The simply connected case

Theorem 2.4 (Stolz [St1]). *The Gromov-Lawson-Rosenberg Conjecture holds for simply connected manifolds.*

We want to mention that in dimensions $n \leq 23$ this was proved by Rosenberg [Ro3, Thm. 3.6] in a spirit similar to the proof of Theorem 1.10 by producing explicit spin manifolds whose bordism classes are generators of $\ker\alpha$. For example, the kernel of $\alpha\colon \Omega_n^{Spin} \to KO_n(\mathbb{R})$ is trivial in dimensions $n < 8$. For $n = 8$, it is infinite cyclic and a generator is given by the bordism class of the quaternionic projective plane $\mathbb{HP}^2$. Since this manifold admits a positive scalar curvature metric this proves the Gromov-Lawson-Rosenberg Conjecture in the simply connected case for $n \leq 8$.

The difficulty with this line of argument is that although the bordism groups Ω_n^{Spin} have been computed [ABP], we do *not* know explicit spin manifolds whose bordism classes generate the spin bordism ring (unlike the oriented bordism ring, for which Wall has given explicit generators, which was the key for the proof of Theorem 1.10). So the dilemma when trying to prove Theorem 2.4 is to try to represent bordism classes by manifolds admitting positive scalar curvature metrics *without knowing how to represent these classes by explicit manifolds.* This is not as impossible as it sounds: according to part 2 of Observation 1.4 if M is a Riemannian manifold with positive scalar curvature, then the total space of *any* fiber bundle with fiber M whose transition functions are isometries of M admits a positive scalar curvature metric. This suggests to analyze which bordism classes are represented by such total spaces for some fixed M, e.g. $M = \mathbb{HP}^2$ equipped with the Fubini-Study metric. For simplicity we will call these bundles $\mathbb{HP}^2$-bundles. The answer is this:

Theorem 2.5 (Stolz [St1]). *The subgroup of Ω_n^{Spin} represented by total spaces of $\mathbb{HP}^2$-bundles is equal to the kernel of $\alpha\colon \Omega_n^{Spin} \to KO_n$.*

By our discussion above, this implies Theorem 2.4.

Idea of proof of Theorem 2.5. The isometry group of $\mathbb{HP}^2$ is the projective symplectic group $G = PSp(3) = Sp(3)/\pm 1$. It acts transitively on $\mathbb{HP}^2$ with isotropy group $H = (Sp(2)\times Sp(1))/\pm 1$, allowing us to identify $\mathbb{HP}^2$ with the homogeneous space G/H. The map of classifying spaces $BH \to BG$ induced by the inclusion map $H \to G$ is then a fiber bundle with fiber $\mathbb{HP}^2 = \mathrm{G/H}$. This is the *universal* $\mathbb{HP}^2$-bundle in the sense that *any* $\mathbb{HP}^2$-bundle over a manifold N is the pull-back $\widehat{N} = f^*BH \to N$ of $BH \to BG$ via some map $f\colon N \to BG$. This discussion shows that the subgroup $T_n \subset \Omega_n^{Spin}$ represented by total spaces of $\mathbb{HP}^2$-bundles is the image of the following *transfer map*:

$$\Psi\colon \Omega_{n-8}^{Spin}(BG) \longrightarrow \Omega_n^{Spin} \qquad [N,f] \mapsto [\widehat{N}].$$

Hence the claim of the theorem is equivalent to the exactness of the following sequence at the middle group:

$$\Omega_{n-8}^{Spin}(BG) \xrightarrow{\Psi} \Omega_n^{Spin} \xrightarrow{\alpha} KO_n(\mathbb{R}).$$

It is well-known that the Pontryagin-Thom construction allows us to identify the bordism group Ω_n^{Spin} with the n-th homotopy group of the 'Thom spectrum' $MSpin$ [Sto]. In fact, it turns out that the whole sequence above is isomorphic (for $n \geq 0$) to the following sequence of homotopy groups:

$$\pi_n(MSpin \wedge \Sigma^8 BG_+) \xrightarrow{T_*} \pi_n(MSpin) \xrightarrow{D_*} \pi_n(ko). \tag{2.6}$$

Here $\Sigma^8 BG_+$ is the 8-th suspension of BG furnished with a disjoint base point, and ko is the *connective real K-theory spectrum.* There is a closely related spectrum KO, the *periodic real K-theory spectrum* whose homotopy groups $\pi_n(KO)$ are isomorphic to $KO_n(\mathbb{R})$ for all n. The spectrum ko is the *connective cover* of KO in the sense that $\pi_n(ko)$ is trivial for $n < 0$, and that there is a map per: $ko \to KO$ which induces an isomorphism on π_n for $n \geq 0$.

It can be shown that the composition

$$MSpin \wedge \Sigma^8 BG_+ \xrightarrow{T} MSpin \xrightarrow{D} ko$$

is homotopic to the constant map (by interpreting it as a family index). This implies that T can be factored in the form

$$\begin{tikzcd} & \widehat{MSpin} \arrow[d] \\ MSpin \wedge \Sigma^8 BG_+ \arrow[ur, "\widehat{T}"] \arrow[r, "T"] & MSpin \arrow[d, "D"] \\ & ko \end{tikzcd}$$

where $\widehat{MSpin}$ is the homotopy fiber of D. The exactness of the sequence above is equivalent to the surjectivity of the map induced by $\widehat{T}$ on homotopy groups.

It turns out to be convenient to break the proof that $\widehat{T}_*$ is surjective into two steps: surjectivity of $\widehat{T}_*$ *localized at the prime* 2 (i.e., after tensoring with $\mathbb{Z}_{(2)} = \{\frac{a}{b} \mid b \text{ is prime to } 2\}$) and surjectivity *away from* 2 (i.e., after tensoring with $\mathbb{Z}[\frac{1}{2}]$). Away from 2, the bordism ring $\pi_*(MSpin) \cong \Omega_*^{Spin}$ is a polynomial ring with generators x_n in degrees $n = 4, 8, 12, \ldots$, while $\pi_*(ko)$ is the polynomial ring generated by $D_*(x_4)$ (both $MSpin$ and ko are *ring spectra* and D is compatible with this structure, which implies that the homotopy groups of $MSpin$ and ko form a graded ring, and that D_* is a ring homomorphism). In particular, D_* is surjective, and , $\pi_*(\widehat{MSpin})$ can be identified with the ideal generated by $x_8, x_{12}, \ldots$. Hence it suffices to show that T_* is onto modulo decomposable elements in degrees $8, 12, \ldots$, which is proved by a calculation of characteristic numbers for certain $\mathbb{HP}^2$-bundles (see [KS, §4]).

The proof of surjectivity localized at 2 is technically more involved due to the existence of 2-torsion in $\pi_*(MSpin)$. It is proved using the mod 2 Adams spectral sequence, whose E_2-term depends only on the mod 2 cohomology of the spectrum in question as a module over the Steenrod algebra A; it converges to the homotopy groups of the spectrum localized at 2. A detailed analysis of the action of the Steenrod algebra on $H^*(MSpin \wedge \Sigma^8 BG_+; \mathbb{Z}/2)$ and $H^*(\widehat{MSpin}; \mathbb{Z}/2)$ shows that as A-module the latter can be identified as a direct summand of the former via the map $\widehat{T}^*$. It follows that the map of Adams spectral sequences induced by $\widehat{T}$ is surjective on E_2-terms. The vanishing of all differentials in the Adams spectral sequence of $MSpin \wedge \Sigma^8 BG_+$ then implies that $\widehat{T}$ induces a surjection on E_∞-terms and hence on homotopy groups localized at 2. $\square$

2.2 Positive scalar curvature metrics on non-simply connected spin manifolds

It is tempting to believe that a manifold M with finite fundamental group admits a positive scalar curvature metric if and only if its universal covering $\widetilde{M}$ does using the following line of reasoning. Suppose $\widetilde{g}$ is a positive scalar curvature metric on $\widetilde{M}$. Then the metric $\widetilde{g}$ might not be invariant under the action of $\pi_1(M)$ via deck transformations, but the space of Riemannian metrics on $\widetilde{M}$ is a convex subspace of the vector space of 2-tensors on M, and averaging over the orbit through $\widetilde{g}$, we obtain an *invariant* Riemannian metric $\widetilde{g}'$ on $\widetilde{M}$, which then descents to a Riemannian metric g' on M. It seems reasonable to expect $\widetilde{g}'$ (and hence g') to have positive scalar curvature – after all, it is obtained by averaging positive scalar curvature metrics. However, the following example shows that the average of positive scalar curvature metrics might not have positive scalar curvature.

Example 2.7. Let M be the connected sum of $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ and a 9-dimensional homotopy sphere Σ^9 with $\alpha(\Sigma^9) \neq 0$. We note that $\mathbb{R}\mathrm{P}^7$ is a spin manifold and hence so is $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$. We note that $\alpha(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2)$ is zero since $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ is zero bordant. Since the connected sum $\left(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2\right) \# \Sigma^9$ is spin bordant to the disjoint union of $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ and Σ^9, we see that

$$\alpha(M) = \alpha(\left(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2\right) \# \Sigma^9) = \alpha(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2) + \alpha(\Sigma^9) = \alpha(\Sigma^9) \neq 0.$$

Hence by Theorem 1.21 the manifold M does not admit a positive scalar curvature metric. However, the universal covering $\widetilde{M}$ does admit a positive scalar curvature metric, since $\widetilde{M} = (S^7 \times S^2)\#\Sigma^9\#\Sigma^9$, which is diffeomeorphic to $S^7 \times S^2$, since $\Sigma^9\#\Sigma^9$ is diffeomorphic to S^9.

While the above example shows that the question whether a spin manifold with finite fundamental group π admits a positive scalar curvature metric cannot be reduced to the *universal* covering, Kwasik and Schultz observed that it can be reduced to the coverings corresponding to the Sylow subgroups of π.

Theorem 2.8 (Kwasik-Schultz [KwS]). *Let M be a spin manifold of dimension $n \geq 5$ with finite fundamental group π. Then M admits a positive scalar curvature metric if and only if all coverings of M corresponding to the p-Sylow groups of π admit such a metric.*

In particular, if the Gromov-Lawson-Rosenberg Conjecture is true for all p-Sylow groups of a finite group π, then it holds for π.

Proof. To prove the non-trivial implication of this theorem, assume that all coverings of M corresponding to p-Sylow subgroups of π admit positive scalar curvature metrics. To show that M also admits such a metric, it suffices by the Bordism Theorem 1.8 to show $[M,u] \in \Omega_n^{Spin,+}(B\pi)$, where $u\colon M \to B\pi$ is the map classifying the universal covering of M.

In order to relate $[M,u]$ to the corresponding bordism class for a covering of M corresponding to a subgroup $H \subset \pi$, we consider the *transfer homomorphism*

$$\Omega_n^{Spin}(B\pi) \xrightarrow{(Bi)^!} \Omega_n^{Spin}(BH) \qquad \text{defined by} \quad [N,f] \mapsto [\widehat{N}, \widehat{f}],$$

where $\widehat{N} \overset{\text{def}}{=} f^*BH \to N$ is the covering obtained by pulling back the covering $BH \to B\pi$ via f, and $\widehat{f}$ is the map making the diagram

$$\begin{array}{ccc} \widehat{N} = f^*BH & \xrightarrow{\widehat{f}} & BH \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & B\pi \end{array}$$

commutative. Now consider the composition

$$\Omega_n^{Spin}(B\pi) \xrightarrow{(Bi)^!} \Omega_n^{Spin}(BH) \xrightarrow{Bi_*} \Omega_n^{Spin}(B\pi)$$

assuming that H is a p-Sylow group of π. Then

- The image of $[M,u]$ under the transfer map is in $\Omega_n^{Spin,+}(BH)$ due to the assumption that p-Sylow coverings of M admit a positive scalar curvature metrics. Hence the image of $[M,u]$ under the composition is in $\Omega_n^{Spin,+}(B\pi)$.

- The composition is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$ (replacing spin bordism by homology, the composition above is multiplication by the index of the subgroup H; for a Sylow subgroup, this index is prime to p and consequently the composition on homology is an isomorphism after localizing at p; the Atiyah-Hirzebruch spectral sequence then shows that the same holds for spin bordism).

This implies $[M,u] \in \Omega_n^{Spin,+}(B\pi) \otimes \mathbb{Z}_{(p)}$ for all primes p and hence $[M,u] \in \Omega_n^{Spin,+}(B\pi)$. □

We recall that the Gromov-Lawson-Rosenberg Conjecture for a group π is equivalent to $\Omega_n^{Spin,+}(B\pi) = \ker(\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi))$. This shows that when attempting to prove the conjecture, it is useful to understand the kernel of α; this is facilitated by factoring α as follows:

$$\Omega_n^{Spin}(B\pi) \xrightarrow{D} ko_n(B\pi) \xrightarrow{\text{per}} KO_n(B\pi) \xrightarrow{A} KO_n(C^*\pi). \tag{2.9}$$

Here $KO_n(X) = \pi_n(KO \wedge X_+)$ and $ko_n(X) = \pi_n(ko \wedge X_+)$ are the *generalized homology theories* associated to the periodic (resp. connective) real K-theory spectrum KO (resp. ko), which we mentioned earlier in this lecture (in the proof of Theorem 2.5). The maps D and per are natural transformations between homology theories which are induced by the corresponding maps between spectra $D\colon MSpin \to ko$ and $\text{per}\colon ko \to KO$ (abusing notation we use the same letter for the map between spectra and the natural transformation). The homomorphism A is the *assembly map* which has been mentioned in the talks by Lück and Schick. We will use the following notation.

2.10. Notation. For a spin manifold M, the bordism class $[M, \mathrm{id}_M] \in \Omega_n^{Spin}(M)$ is the *fundamental class* of M in Ω^{Spin}-theory. We define

$$[M]_{ko} \stackrel{\text{def}}{=} D([M, \mathrm{id}_M]) \in ko_n(M) \quad \text{and} \quad [M]_{KO} \stackrel{\text{def}}{=} \text{per}([M]_{ko}) \in KO_n(M).$$

We call $[M]_{ko}$ (resp. $[M]_{KO}$) the *fundamental class of* M in connective (resp. periodic) KO-homology. Given a map $f\colon M \to B\pi$, and the corresponding bordism class $[M, f] \in \Omega_n^{Spin}(B\pi)$, we have

$$D([M, f]) = f_*[M]_{ko} \in ko_n(B\pi) \quad \text{per}(D([M, f])) = f_*[M]_{KO} \in KO_n(B\pi).$$

Finally, we define $ko_n^+(X) \stackrel{\text{def}}{=} D\left(\Omega_n^{Spin,+}(X)\right) \subset ko_n(X)$ and define $KO_n^+(X) \subset KO_n(X)$ to be the subgroup generated by $\text{per}(ko_{n+8k}^+(X)) \subset KO_{n+8k}(X) \cong KO_n(X)$ for all k.

The following result is similar to the Bordism Theorem 1.8.

Theorem 2.11 (Stolz, Jung). *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π and let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Then M admits a positive scalar curvature metric if and only if $u_*[M]_{ko}$ is in $ko_n^+(B\pi)$.*

In particular, by this result the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following conjecture.

2.12. Conjecture. $ko_n^{Spin,+}(B\pi) = \ker\left(A \circ \text{per}\colon ko_n(B\pi) \to KO_n(C^*\pi)\right)$.

The advantage of this formulation of the Gromov-Lawson-Rosenberg Conjecture over that used in Conjecture 2.3 is that the groups $ko_* B\pi)$ are *a lot* smaller than $\Omega_*(B\pi)$. For example in joint work with Botvinnik and Gilkey [BGS], the author could calculate the groups $ko_n(B\pi)$ and the kernel of $A \circ \mathrm{per}$ in the case of 2-groups which are cyclic or generalized quaternion. It turns out that the kernel of $A \circ \mathrm{per}$ is generated by ko-fundamental classes of lens spaces and lens space bundles over S^2 (in the cyclic case) resp. by lens spaces and quaternionic space forms (for the quaternionic groups). This proves the Gromov-Lawson-Rosenberg Conjecture for these 2-groups.

Previously Kwasik and Schultz [KwS] had proved the Gromov-Lawson-Rosenberg Conjecture 2.1 for cyclic groups of prime order $p \neq 2$, and Rosenberg [Ro2] for cyclic groups odd order. This implies by the Kwasik-Schultz induction result 2.8 that the Conjecture is true for all finite groups whose p-Sylow groups are cyclic or generalized quaternion (for $p = 2$). These are precisely the finite groups with periodic cohomology and so these arguments prove:

Theorem 2.13 (Botvinnik-Gilkey-Stolz [BGS]). *The Gromov-Lawson-Rosenberg Conjecture holds for finite groups with periodic cohomology.*

Next we want to outline the proof of Theorem 2.11. It suffices to show $\ker\left(D\colon \Omega_n^{Spin}(X) \to ko_n(X)\right) \subset \Omega_n^{Spin,+}(X)$. As in the proof of the Gromov-Lawson-Rosenberg Conjecture in the simply connected case, there are two somewhat different arguments, showing that the required inclusion holds after tensoring with $\mathbb{Z}_{(2)}$ and $\mathbb{Z}[\frac{1}{2}]$, respectively (and this of course suffices to prove the inclusion).

To prove the inclusion localized at 2 (i.e., after tensoring with $\mathbb{Z}_{(2)}$), we consider the subgroup $T_n(X) \subset \Omega_n^{Spin}(X)$ consisting of bordism classes of the form $[M \xrightarrow{p} N \xrightarrow{f} X]$, where $p\colon M \to N$ is a $\mathbb{HP}^2$-bundle over a spin manifold N. The following result is proved by strengthening the homotopy theoretic techniques used in the proof of Theorem 2.5.

Theorem 2.14 ([St2]). *The map D induces a 2-local isomorphism*

$$ko_n(X) \cong \Omega_n^{Spin}(X)/T_n(X).$$

This implies in particular that 2-locally the kernel of $D\colon \Omega_n^{Spin}(X) \to ko_n(X)$ is equal to $T_n(X)$. Since total spaces of $\mathbb{HP}^2$-bundles admit positive scalar curvature metrics by Observation 1.4, this implies the desired inclusion 2-locally.

Proof of Theorem 2.14. We recall from the proof of Theorem 2.5 that the map $\widehat{T}\colon MSpin \wedge \Sigma^8 BG_+ \to \widehat{MSpin}$ induces a split injection of A-modules; an fact, more is true and proved in [St2], Proposition 8.3: $\widehat{T}$ is a split surjection of spectra. In particular, for any space X, the map

$$MSpin \wedge \Sigma^8 BG_+ \wedge X_+ \xrightarrow{\widehat{T}\wedge 1} \widehat{MSpin} \wedge X_+$$

induces 2-locally a surjection on homotopy groups. Since $\widehat{MSpin} \wedge X_+$ is the homotopy fiber of $D \wedge 1\colon MSpin \wedge X_+ \to ko \wedge X_+$, this implies that the following sequence is 2-locally exact:

$$\pi_n(MSpin \wedge \Sigma^8 BG_+ \wedge X_+) \xrightarrow{(T\wedge 1)_*} \pi_n(MSpin \wedge X_+) \xrightarrow{(D\wedge 1)_*} \pi_n(ko \wedge X_+).$$

Identifying $\pi_n(MSpin \wedge X_+)$ with $\Omega_n^{Spin}(X)$ via the Pontryagin-Thom construction, the image of $(T \wedge 1)_*$ can be identified as the subgroup $T_n(X) \subset \Omega_n^{Spin}(X)$. It is well-known that the map $(D \wedge 1)_*$ is 2-locally surjective, since $MSpin \to ko$ is a split surjection of spectra (cf. [ABP]). This shows that $ko_n(X)$ is isomorphic to $\Omega_n^{Spin}(X)/\ker D = \Omega_n^{Spin}(X)/T_n(X)$. $\square$

The proof of the inclusion $\ker\Big(D\colon \Omega_n^{Spin}(X) \to ko_n(X)\Big) \subset \Omega_n^{Spin,+}(X)$ after inverting 2 (i.e., tensoring with $\mathbb{Z}[\frac{1}{2}]$), is again a consequence of a *geometric* description of ko-homology, this time as a bordism group of manifolds with singularities a la Baas-Sullivan. This then implies that if $[M, f]$ is in the kernel of D, then M is *zero bordant as a manifold with singularities.* Equivalently, after removing a neighborhood of the singularities from the zero bordism, we obtain a bordism W between M and another manifold N which is constructed inductively from the set Σ of manifolds describing the possible types of the admissible singularities. The essential observation is that in the Baas-Sullivan type description of $ko_n(X)$, the set Σ *can be chosen to consist of manifolds admitting positive scalar curvature metrics.* An inductive argument then shows that N admits a positive scalar curvature metric and then so does M by the Bordism Theorem 1.8.

At the end of this lecture we would like to mention the following result that can be proved quite similarly to Theorem 2.11 above (replacing e.g. $\mathbb{HP}^2$-bundles by $\mathbb{CP}^2$-bundles). To state it, we need the following notation. For an oriented manifold N, let $[N] \in H_n(N;\mathbb{Z})$ be the usual homology fundamental class of N. For any space X, let $H_n^+(X;\mathbb{Z}) \subset H_n(X;\mathbb{Z})$ be the subgroup consisting of all homology classes of the form $f_*[N]$, where N is a manifold with positive scalar curvature metric and $f\colon N \to X$.

Theorem 2.15 (Stolz, Jung). *Let M be an oriented manifold of dimension $n \geq 5$ with fundamental group π and let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Assume that $\widetilde{M}$ does not admit a spin-structure. Then M admits a positive scalar curvature metric if and only if $u_*[M]$ is in $H_n^+(B\pi)$.*

3 The Gromov-Lawson-Rosenberg Conjecture and its relation to the Baum-Connes Conjecture

An important feature of the K-theory of C^*-algebras is the periodicity of these groups; they are 2-periodic for complex C^*-algebras, and 8-periodic for real C^*-algebras. With the definition of K-theory we have adopted this is not a deep fact, but rather reflects the algebraic periodicity of the Clifford algebras. If M is a spin manifold of dimension n with fundamental group π, the periodicity isomorphism

$$KO_n(C^*\pi) \xrightarrow{\cong} KO_{n+8}(C^*\pi)$$

maps the index obstruction $\alpha(M,u)$ to $\alpha(M \times B, u)$, where B is any simply connected spin manifold of dimension 8 with $\widehat{A}(B) = 8$ (we use the letter u for the classifying map of the universal covering of whatever manifold we are talking about) . We pick such a manifold B – the particular choice of it is immaterial for our purposes – and refer to it as 'Bott manifold', since the cartesian product with B corresponds to Bott-periodicity. It should be mentioned that Joyce [J] has constructed manifolds with these properties and metrics thereon with particular interesting geometric properties: they are Ricci flat and their holonomy group reduces to $Spin(7)$.

This shows that the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following two conjectures.

3.1. Cancellation Conjecture. *Let M be a spin manifold of dimension $n \geq 5$. Then M admits a positive scalar curvature metric if and only if $M \times B$ does.*

3.2. Stable Conjecture *Let M be a spin manifold. Then M admits* stably a positive scalar curvature metric *(i.e., the product of M with sufficiently many copies of B has a positive scalar curvature metric) if and only if $\alpha(M,u) = 0$.*

Theorem 3.3 (Rosenberg-Stolz [RS1]). *The stable Gromov-Lawson-Rosenberg Conjecture is true for manifolds with finite fundamental group.*

For the proof of this result we need a K-theoretic reformulation of the Stable Conjecture, which is based on the following geometric description of KO-homology. Let $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}]$ be the direct limit of the homomorphisms

$$\Omega_n^{Spin}(X)/T_n(X) \xrightarrow{\times B} \Omega_{n+8}^{Spin}(X)/T_{n+8}(X) \xrightarrow{\times B} \dots,$$

given by the cartesian product with the Bott manifold B. We note that the groups $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}]$ are 8-periodic; in fact, multiplication by B provides an isomorphism. In other words, we made the non-periodic groups $\Omega_n^{Spin}(X)/T_n(X)$ periodic by *inverting the Bott-element*, which motivates the notation.

Theorem 3.4 (Kreck-Stolz [KS]). *The map* $\mathrm{per}\circ D\colon \Omega_n^{Spin}(X)\to KO_n(X)$ *induces an isomorphism* $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}]\cong KO_n(X)$.

We remark that this is a direct consequence of Theorem 2.14 at the prime 2; an additional argument is needed localized away from 2.

Theorem 3.4 shows in particular that if $[M,u]\in\Omega_n^{Spin}(B\pi_1(M))$, is in the kernel of $\mathrm{per}\circ D$, then the product $M\times B\times\cdots\times B$ with sufficiently many copies of B represents an element in $T_n(B\pi_1(M))$, and hence carries a positive scalar curvature metric. This implies the following result.

Corollary 3.5. *Let M be a spin manifold of dimension $n\geq 5$ with fundamental group π and let $u\colon M\to B\pi$ be the classifying map of the universal covering $\widetilde{M}\to M$. Then M admits stably a positive scalar curvature metric if and only if $u_*[M]_{KO}$ is in $KO_n^+(B\pi)$.*

We recall from (2.10) that $KO_n^+(X)\subset KO_n(X)$ is the subgroup consisting of all elements of the form $f_*[N]_{KO}$ for some manifold N equipped with a positive scalar curvature metric and $\dim N\equiv n \bmod 8$.

Corollary 3.6. *The Stable Conjecture holds for spin manifolds with fundamental group π if and only if $KO_*^+(B\pi)$ is the kernel of the assembly map.*

For finite fundamental groups π the group C^*-algebra $C^*\pi$ is just the real group ring $\mathbb{R}\pi$ which is isomorphic to a product of matrix rings over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. It follows that the $KO_*(C^*\pi)$ is a sum of copies of the real K-theory of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ with as many summands of each kind as the corresponding matrix factors of $\mathbb{R}\pi$.

It is a well-known result of Atiyah that for a finite group π the complex K-theory $K^0(B\pi)$ can be identified with a completion of the representation ring of π. This was later refined by Atiyah and Segal to give a description of $KO^*(B\pi)$ in terms of the representation theory of π (see [AS]). The following result is obtained by dualizing their theorem.

Proposition 3.7 (Rosenberg-Stolz). *If π is a finite p-group, then the assembly map*

$$KO_n(B\pi;\mathbb{Z}/p^\infty)\xrightarrow{A}KO_n(C^*\pi;\mathbb{Z}/p^\infty)$$

with coefficients in $\mathbb{Z}/p^\infty$ is an isomorphism.

Corollary 3.8. *If π is a finite p-group, then there is a long exact sequence*

$$\to KO_n(B\pi) \xrightarrow{A} KO_n(C^*\pi) \to \widetilde{KO}_n(C^*\pi) \otimes \mathbb{Q} \xrightarrow{\partial} KO_{n-1}(B\pi) \to$$

Proof of corollary. The groups $KO_n(B\pi)$ and $KO_n(C^*\pi)$ are the n-th homotopy groups of certain spectra; moreover, the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$ is induced by a map between these spectra. This implies that KO-groups of $B\pi$ (resp. $C^*\pi$) with *coefficients* in some abelian group can be defined, and that associated to short exact sequences of coefficients we obtain long exact sequences of KO-groups. Moreover, there is an assembly map for KO-theory with coefficients compatible with these long exact sequences. In particular, the short exact sequence

$$\mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty$$

gives rise to the following commutative diagram, whose rows are long exact sequences.

$$\begin{tikzcd}
{} \arrow[r] & KO_{n+1}(B\pi;\mathbb{Z}/p^\infty) \arrow[r, "\partial"] \arrow[d, "A"', "\cong"] & KO_n(B\pi;\mathbb{Z}_{(p)}) \arrow[r] \arrow[d, "A"'] & KO_n(B\pi;\mathbb{Q}) \arrow[r] \arrow[d, "A"'] & {} \\
{} \arrow[r] & KO_{n+1}(C^*\pi;\mathbb{Z}/p^\infty) \arrow[r, "\partial"] & KO_n(C^*\pi;\mathbb{Z}_{(p)}) \arrow[r] & KO_n(C^*\pi;\mathbb{Q}) \arrow[r] & {}
\end{tikzcd}$$

It is well-known and proved by a quick diagram chase in the above diagram, that this leads to a Meyer-Vietoris type long exact sequence

$$\to KO_n(B\pi;\mathbb{Z}_{(p)}) \to KO_n(C^*\pi;\mathbb{Z}_{(p)}) \oplus KO_n(B\pi;\mathbb{Q}) \to KO_n(C^*\pi;\mathbb{Q}) \xrightarrow{\partial},$$

which reduces to the long exact sequence of the corollary. □

Proof of Theorem 3.3. By Theorem 2.8 it is enough to prove the Stable Conjecture for finite p-groups. For *cyclic* groups, the Gromov-Lawson-Rosenberg Conjecture and hence also the weaker Stable Conjecture hold. So the idea of the proof is to compare the assembly map A for a finite p-group π with the assembly maps A_H for its cyclic subgroups $H \subset \pi$ by means of the following diagram.

$$\begin{tikzcd}
{} \arrow[r] & \bigoplus_H \widetilde{KO}_{n+1}(C^*H;\mathbb{Q}) \arrow[r, "\oplus \partial_H"] \arrow[d, "\mathrm{Ind}"'] & \bigoplus_H KO_n(BH) \arrow[r, "\oplus A_H"] \arrow[d, "\mathrm{Ind}"'] & \bigoplus_H KO_n(C^*H) \arrow[r] \arrow[d, "\mathrm{Ind}"'] & {} \\
{} \arrow[r] & \widetilde{KO}_{n+1}(C^*\pi) \otimes \mathbb{Q} \arrow[r, "\partial"] & KO_n(B\pi)) \arrow[r, "A"] & KO_n(C^*\pi) \arrow[r] & {}
\end{tikzcd}$$

Here the vertical maps $\mathrm{Ind} = \bigoplus \mathrm{Ind}_H^\pi$ are sums of induction maps for cyclic subgroups $H \subset \pi$; we sum over representatives H of all conjugacy classes of cyclic subgroups of π. The rows are exact by Corollary 3.8.

By Artin induction, the left vertical map is surjective. This implies by a diagram chase that an element in the kernel of A is in the image of $\mathrm{Ind} \circ \bigoplus \partial_H$. Since the Stable Conjecture holds for cyclic groups, the group $\mathrm{image}(\partial_H) = \ker(A_H)$ is equal to $KO_n^+(BH)$. It follows that the image of $\mathrm{Ind} \circ \bigoplus \partial_H$ is contained in $KO_n^+(B\pi)$ which proves the theorem. $\square$

Now we want to discuss groups π which are not necessarily finite. If π is *torsion free*, then according to (a form of) the Novikov-Conjecture, the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$ is injective. If this is true for π, then obviously the kernel of A is contained in $KO_n^+(B\pi)$ and the Stable Conjecture holds for π. In general the assembly map is *not* injective, for example for finite groups. However, the Novikov Conjecture can be generalized to any discrete group π in the following way. A s explained in Schick's lectures, the assembly map can be expressed in terms of the equivariant KO-homology and then factored as follows:

$$\begin{tikzcd}
KO_n(B\pi) \arrow[rr, "A"] \arrow[d, "\cong"'] & & KO_n(C^*\pi) \arrow[d, "\cong"] \\
KO_n^\pi(E\pi) \arrow[r] & KO_n^\pi(E(\pi,\mathcal{F})) \arrow[r, "\mu"] & KO_n^\pi(pt)
\end{tikzcd}$$

Here $E\pi$ (resp. $E(\pi,\mathcal{F})$) is the universal π-space with trivial (resp. finite) isotropy groups and the map μ is induced by the projection of $E(\pi,\mathcal{F})$ to the point. We note that if π is torsionfree, then $E(\pi,\mathcal{F}) = E\pi$, and hence μ can be identified with the assembly map. The map μ is called the *Baum-Connes map.*

Conjecture 3.9 (Baum-Connes [BCH]). *The map μ is an isomorphism for any discrete group π.*

Theorem 3.10 (Stolz [St5]). *If the Baum-Connes map μ is injective for a group π, then the Stable Conjecture holds for π.*

We recall that the Stable Conjecture for a group π is equivalent to the statement $\mathfrak{K}O_n(B\pi) = KO_n^+(B\pi)$ (cf. 3.6), where from now on we write $\mathfrak{K}O_n(B\pi)$ for the kernel of the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$. To prove this equality, it suffices to prove that it holds localized at p for all primes p (localizing an abelian group at p means tensoring it with $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ is prime to } p\}$). So throughout this section we will fix a prime p and localize all abelian groups and spectra at that prime.

Our strategy to prove $\mathfrak{K}O_n(B\pi) = KO_n^+(B\pi)$ is to use the injectivity of the Baum-Connes map and induction techniques to show that that every element in $\mathfrak{K}O_n(B\pi)$ comes from some finite cyclic p-subgroup of π in a sense made precise by the following theorem.

By Theorem 3.4 any element of $KO_*(X)$ is represented by some spin manifold N and a map $f\colon N \to X$. We will write $[N, f] \in KO_*(X)$ for this element (or $[N, f]_{KO} \in KO_*(X)$ if there might be danger of confusion with the bordism class $[N, f] \in \Omega_*^{Spin}(X)$). Let H be a subgroup of π, and let $C(H) = \{g \in \pi \mid gh = hg \text{ for all } h \in H\}$ be its centralizer. Then there is a pairing

$$KO_*(BH) \otimes KO_*(BC(H)) \longrightarrow KO_*(B\pi)$$

given by sending $[M, f] \otimes [N, g]$ to the class represented by

$$M \times N \xrightarrow{f \times g} BH \times BC(H) = B(H \times C(H)) \xrightarrow{Bj} B\pi,$$

where $j\colon H \times C(H) \to \pi$ maps (h, c) to the product hc (note that this is a homomorphism since the elements of $C(H)$ commute with the elements of H). The multiplicative properties of the assembly map imply that the above pairing restricts to a pairing $\mathfrak{K}O_*(BH) \otimes KO_*(BC(H)) \to \mathfrak{K}O_*(B\pi)$.

Theorem 3.11. *Assume that the Baum-Connes map for the group π is injective. Then the homomorphism*

$$\bigoplus_H \mathfrak{K}O_*(BH) \otimes KO_*(BC(H)) \longrightarrow \mathfrak{K}O_*(B\pi),$$

is p-locally surjective, where H runs through representatives of all conjugacy classes of cyclic p-subgroups of π.

Proof of Theorem 3.10 assuming Theorem 3.11. We have

$$\mathfrak{K}O_n(BH) = KO_n^+(BH)$$

for every cyclic p-subgroup $H \subseteq \pi$, since the Stable Conjecture holds for cyclic groups. We observe that the cartesian product of manifolds $M \times N$ admits a positive scalar curvature metric if M does, which implies that the image of the above pairing restricted to $KO_*^+(BH) \otimes KO_*(BC(H))$ is contained in $KO_*^+(B\pi)$. This proves Theorem 3.10. $\square$

Outline of the proof of Theorem 3.11. For the proof it is necessary to express the groups $KO_n(B\pi) = KO_n^\pi(E\pi)$, $KO_n^\pi(E(\pi, \mathcal{F}))$ and $KO_n(C^*\pi) =$

$KO^{\pi}_n(pt)$ as the n-th homotopy group of homotopy limits as explained in the lectures by Thomas Schick. More precisely, let $\mathrm{Or}(\pi)$ be the *orbit category* of π, whose objects are orbits of π (i.e., transitive π-sets); morphisms from an orbit V to an orbit U are the π-equivariant maps $U \to V$. There is a functor

$$KO\colon \mathrm{Or}(\pi) \to SPECTRA$$

to the category of spectra with the property that for a π-orbit π/H we have $\pi_n(KO(\pi/H)) \cong KO_n(C^*H)$ [DL]. Let

$$\mathrm{Or}(\pi,\mathcal{T}) \subset \mathrm{Or}(\pi,\mathcal{F}(p)) \subset \mathrm{Or}(\pi,\mathcal{F}) \subset \mathrm{Or}(\pi)$$

be the the full subcategories of the orbit category consisting of those orbits whose isotropy groups are trivial (resp. finite p-groups, resp. finite groups). Restricting the functor KO to these subcategories, and abusing notation by writing KO again for the restricted functor, the inclusions above induce the following maps of homotopy colimits:

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO \xrightarrow{f} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} KO \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F})} KO \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(\pi)} KO.$$

On the n-th homotopy group, this sequence of maps induces the following sequence of homomorphisms whose composition is the assembly map

$$KO_n(B\pi) \to KO^{\pi}_n(E(\pi,\mathcal{F}(p))) \to KO^{\pi}_n(E(\pi,\mathcal{F})) \to KO_n(C^*\pi).$$

By assumption the right map is injective, and it can be shown that the middle map is p-locally injective by constructing a p-local splitting for the corresponding map of homotopy colimits. Hence the elements in the kernel of A are in the image of the map

$$\pi_n(\mathrm{fiber}(f)) \to \pi_n(\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO) = KO_n(B\pi).$$

To identify the homotopy fiber $\mathrm{fiber}(f)$, it is useful to rewrite

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO \qquad \text{in the form} \qquad \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} i_*KO,$$

where the functor $i_*KO\colon \mathrm{Or}(\pi,\mathcal{F}(p)) \to SPECTRA$ is the *Kan extension* of the functor $KO\colon \mathrm{Or}(\pi,\mathcal{T}) \to SPECTRA$ via the inclusion $i\colon \mathrm{Or}(\pi,\mathcal{T}) \to \mathrm{Or}(\pi,\mathcal{F}(p))$. More explicitly, $i_*KO(\pi/H)$ can be identified with $KO \wedge BH_+$ (the domain of the assembly map for the finite p-subgroup $H \subset \pi$), and the natural transformation $i_*KO(\pi/H) \to KO(\pi/H) = KO(C^*H)$ is just the (spectrum level) assembly map for H. Corollary 3.8 implies that the homotopy fiber of this map is $\Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}$, where the subscript $\mathbb{Q}$ indicates the rationalization of the spectrum in question. Thus we obtain the following homotopy fibration

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{g} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} i_*KO \xrightarrow{f} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} KO.$$

Let $\mathrm{Or}(\pi,\mathcal{C}(p)) \subset \mathrm{Or}(\pi,\mathcal{F}(p))$ be the subcategory consisting of all orbits whose isotropy subgroups are cyclic p-groups. Artin induction implies that the map

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{h} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}$$

induced by the inclusion of categories is a homotopy equivalence.

There is a simpler category $\mathrm{Or}(\pi,\mathcal{C}(p))'$, whose objects are cyclic p-subgroups $H \subset \pi$, one for each conjugacy class of such subgroups. The endomorphisms of the object H is the centralizer $C(H)$ of H in π, and there are no other morphisms in $\mathrm{Or}(\pi,\mathcal{C}(p))'$. Let $F\colon\ \mathrm{Or}(\pi,\mathcal{C}(p))' \to \mathrm{Or}(\pi,\mathcal{C}(p))$ be the functor which sends a subgroup H to the orbit π/H and an element $c \in C(H)$ to the π-map $\pi/H \to \pi/H$ given by $gH \mapsto cgH$. It can be shown that the functor F induces a surjection on homotopy groups of the corresponding homotopy colimits

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))'} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{k} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}.$$

(here the fact that $KO_{\mathbb{Q}}$ is *rational* is of central importance).

It is easy to obtain the isomorphism

$$\pi_n\left(\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))'} \Sigma^{-1}KO_{\mathbb{Q}}\right) \cong \bigoplus_H \widetilde{KO}_{n+1}(C^*H;\mathbb{Q}),$$

where we sum over representatives of the conjugacy classes of cyclic p-subgroups of π. Moreover, the image of the pairing of Theorem 3.11 can be identified with the image of the composition $g\,h\,k$ on homotopy groups. However, the induced map $(g\,h\,k)_*$ surjects onto the kernel of f_*, which agrees with the kernel of the assembly map if we assume the injectivity of the Baum-Connes map. This finishes the outline of the proof of Theorem 3.11. $\square$

4 Survey on the problem of finding a positive Ricci curvature metric on a closed manifold

4.1 Ricci curvature

In this lecture we will talk about Ricci curvature, which is a finer curvature invariant of a Riemannian manifold than the scalar curvature $s(p)$: For each point $p \in M$ the Ricci curvature is a quadratic form

$$Ric\colon T_pM \longrightarrow \mathbb{R}$$

on the tangent space at this point. Averaging $Ric(v)$ over all unit tangent vector $v \in T_pM$ gives the scalar curvature $s(p)$ at the point p (up to a factor). More precisely,

$$s(p) = \sum_{i=1}^{n} Ric(e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis for T_pM.

It is interesting that the geometric interpretation of scalar curvature in terms of volume measurement can be refined to a description of the Ricci curvature as follows. We recall that the geometric description of $s(p)$ is based on comparing the volume of the ball $B_r(0, \mathbb{R}^n)$ of radius r in euclidean space with the volume of the ball $B_r(p, M)$ of radius r around p. We note that we might identify $B_r(0, \mathbb{R}^n)$ with the ball of radius r around the origin in the tangent space T_pM, and $B_r(p, M)$ with its image under the exponential map $\exp_p\colon T_pM \to M$. This suggests to refine $s(p)$ by comparing the standard volume form on T_pM with $\exp_p^*(dvol)$, where $dvol$ is the volume form on M determined by the Riemannian metric. To do this, it is convenient to work with polar coordinates on T_pM; in other words, we identify $T_pM \setminus 0$ with $U_p \times \mathbb{R}_+$, where $U_p \subset T_pM$ denotes the unit tangent bundle, and define a function $f(v, t)$ on $U_p \times \mathbb{R}_+$ by the equation

$$\exp_p^*(dvol) = \theta(v, t)\, \mu \wedge dt,$$

where μ is the canonical volume form on U_p. Then the Ricci curvature $Ric(v)$ shows up when expanding $\theta(v, t)$ in t (see [Be, 0.61]):

$$\theta(v, t) = t^{n-1}(1 - \frac{1}{3} Ric(v) t^2 + \dots).$$

By integrating over U_p we obtain from this the geometric description of scalar curvature 1.2.

In this lecture which is based on [St6], we want to adress the following question:

4.1. Question. Which manifolds admit metrics of positive Ricci curvature (i.e., $Ric(v) > 0$ for all non-zero tangent vectors v)?

There is a classical result that restricts the topology of manifolds with $Ric > 0$ (see [Be, Cor. 6.52]):

Theorem 4.2 (Myers). *If M is a Riemannian manifold with $Ric > 0$, then the fundamental group $\pi_1(M)$ is finite.*

One would expect that there are many manifolds with positive scalar curvature metric, which do not admit a Riemannian metric with $Ric > 0$, since the scalar curvature of a manifold M is the trace over the Ricci curvature, and since the requirement that a quadratic form is positive definite is so much stronger than the requirement that its trace is positive. So it is surprising that in some sense Myer's Theorem is the *only* known restriction for simply connected manifolds with positive scalar curvature of dimension $n \geq 5$ to admit a metric with $Ric > 0$: There are no known examples of simply connected manifolds with positive scalar curvature metrics, which don't admit metrics with $Ric > 0$. The following conjecture would imply in particular the existence of such examples (see part 2 of Remark 4.5).

4.2 A Conjecture concerning Ricci curvature

4.3. Conjecture [St6]. Let M be a spin manifold of dimension $n = 4k$ and assume that $\frac{p_1}{2}(M) \in H^4(M;\mathbb{Z})$ vanishes. If M carries a Riemannian metric with $Ric > 0$, then the Witten genus $\phi_W(M)$ vanishes.

Explanation of $\frac{p_1}{2}(M)$: for vector bundles with spin structure, the first Pontryagin class p_1 is divisible by 2 (in fact canonically: do it for the universal spin bundle over $BSpin(n)$). Short of a better name for it, this class is denoted $\frac{p_1}{2}$. We should stress that due to the possible torsion in $H^4(M;\mathbb{Z})$, the condition $\frac{p_1}{2}(M) = 0$ might be *stronger* than the requirement $p_1(M) = 0$.

4.4. The Witten genus (see [Wi]) If M is an oriented manifold of dimension $n = 4k$, its *Witten genus* is the power series $\phi_W(M) \in \mathbb{Q}[[q]]$ defined by

$$\phi_W(M) = \left(\prod_{l=1}^{\infty}(1-q^l)\right)^n \sum_{l=1}^{\infty} a_l q^l \qquad a_l = \langle \widehat{A}(TM)ch(R_l), [M]\rangle,$$

where the complex vector bundles $R_l \to M$ are constructed from the complexified tangent bundle $E = TM_{\mathbb{C}}$ and its symmetric powers $S^k E$ in the following way: Combine all symmetric powers of E to the *total symmetric power*

$$S_t E \overset{\text{def}}{=} 1 + Et + (S^2E)t^2 + (S^3E)t^3 + \dots,$$

where t is a formal variable. Then expand the following expression as a powerseries of q whose coefficients are vector bundles over M:

$$\bigotimes_{l=1}^{\infty} S_{q^l} E = R_0 + R_1 \cdot q + R_2 \cdot q^2 + \dots$$

To illustrate the procedure, we calculate R_l for $0 \leq l \leq 3$ explicitly by expanding $\bigotimes_{l=1}^{\infty} S_{q^l} E$ while ignoring all terms involving q^l for $l > 3$:

$$\begin{aligned}(1 + Eq + S^2Eq^2 + S^3Eq^3 + \dots)&(1 + Eq^2 + \dots)(1 + Eq^3 + \dots) \\ &= 1 + Eq + (S^2E + E)q^2 + (S^3E + E \otimes E + E)q^3 + \dots\end{aligned}$$

This shows

$$R_0 = 1 \qquad R_1 = E \qquad R_2 = S^2E \oplus E \qquad R_3 = S^3E \oplus (E \otimes E) \oplus E$$

where 1 is the trivial complex line bundle.

4.5. Remarks.

1. The conjecture is true for formal reasons for manifolds of dimension $n < 24$ by the following argument. It can be shown that the Witten genus of any n-manifold M with $p_1(M) = 0$ is the *q-expansion of a modular form of weight* $n/2$ (for a definition of these terms, we refer to [HBJ, §6.3]). If we denote by $\mathfrak{M}_k$ the $\mathbb{C}$-vector space of modular forms of weight k, the direct sum $\mathfrak{M}_* = \bigoplus_{k=0}^{\infty} \mathfrak{M}_k$ is a graded $\mathbb{C}$-algebra, which turns out to be isomorphic to the polynomial algebra generated by the modular forms

$$C_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad \text{and} \quad C_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

of weight 4 and 6, respectively. In particular, $\mathfrak{M}_k$ has dimension ≤ 1 for $k < 12$, which implies that the Witten genus of a manifold M of dimension $n < 24$ (assuming $p_1(M) = 0$) is determined by its constant term, which equals $\widehat{A}(M)$. This of course vanishes by Lichnerowiz' result Theorem 1.11 if we assume $Ric > 0$, which proves the Conjecture for $n < 24$.

2. There are simply connected spin manifolds M of dimension $= 24$ and $n = 4k \geq 30$ with $\frac{p_1}{2}(M) = 0$, $\widehat{A}(M) = 0$ and $\phi_W(M) \neq 0$. By Theorem 2.5 any such manifold admits a positive scalar curvature metric, but – if the Conjecture holds – M does not admit a metric with $Ric > 0$.

3. The standard Fubini-Study metric on the quaternionic projective plane $\mathbb{HP}^2$ has $Ric > 0$ (even the sectional curvature is positive), however $\phi_W(\mathbb{HP}^2) \neq 0$. This is not a counterexample to Conjecture 2.3, but it shows that the condition $\frac{p_1}{2}(M) = 0$ cannot be dropped. It is interesting to compare this to what happens in the case of the complex projective plane $\mathbb{CP}^2$: the standard metric has positive scalar curvature (even positive sectional curvature), yet $\widehat{A}(\mathbb{CP}^2) \neq 0$. Of course this is not a contradiction to Lichnerowicz' Theorem, since $\mathbb{CP}^2$ is not a spin manifold.

4.3 Evidence for the Conjecture

Supporting evidence for the Conjecture 2.3 is provided by the fact that it has been checked for some classes of manifolds, including the following:

4.6. Examples of manifolds with $Ric > 0$:

1. Homogeneous spaces G/H with G compact and semi-simple. To obtain a Ricci positive Riemannian metric on G/H, pick a bi-invariant metric on G; this induces a metric on G/H, referred to as the *normal homogeneous metric* which is characterized by the property that $G \to G/H$ is a Riemannian submersion [Be, Def. 9.8].

2. Complete intersections $X^n \subset \mathbb{CP}^{n+r}$ with positive first Chern class (i.e., transverse intersections of r hyperplanes in $\mathbb{CP}^{n+r}$). The standard Fubini-Study metric on $\mathbb{CP}^{n+r}$ induces a Kähler metric on X, which might not have positive Ricci curvature. However, thanks to the positivity of the first Chern class and the Calabi-Yau Theorem [Be, Thm. 11.15, 11.16(ii)], X admits a Kähler metric with positive Ricci curvature.

In the case of homogeneous spaces, the vanishing of the Witten genus is a consequence of the following result which was proved independently by Dessai [De] and Höhn (unpublished), based on work of Liu [Liu]. This result is analogous to the classical result of Atiyah and Hirzebruch saying that the $\widehat{A}$-genus of a spin manifold with non-trivial S^1-action vanishes.

Theorem 4.7 (Dessai, Höhn). *Let M be a spin manifold of dimension $n = 4k$ with vanishing first Pontryagin class $p_1(M)$. If M admits a non-trivial S^3-action, then its Witten genus $\phi_W(M)$ is zero.*

The Witten genus of complete intersections was calculated by Landweber and Stong (see [HBJ], the last Example in section 6.3). Their result is:

Theorem 4.8 (Landweber-Stong). *If $X^n \subset \mathbb{C}\mathrm{P}^{\mathrm{n+r}}$ is a complete intersection with vanishing first Pontryagin class, then its Witten genus is zero.*

4.4 Towards a proof of the Conjecture

At first sight, one hope for proving the vanishing of the Witten genus for a manifold M or – equivalently – the vanishing of all the numbers a_l is to interpret the latter as indices of 'twisted' Dirac operators.

4.9. Twisted Dirac operators. Let M be a spin manifold with spinor bundle S and let E be a complex vector bundle over M equipped with a connection. Then we can define an elliptic first order differential operator

$$D_E \colon C^\infty(S \otimes E) \longrightarrow C^\infty(S \otimes E)$$

by the *same* formula 1.16 as the Dirac operator; the only difference is that now ∇ is the product connection on $S \otimes E$ induced by the usual connection on S and the given connection on E. The operator D_E is called the *Dirac operator twisted by E*. There is a Bochner-Lichnerowicz-Weitzenböck formula for D_E (cf. [LaM, Ch. II, Thm. 8.17]) of the form

$$D^2 = \nabla^*\nabla + \frac{s}{4} + \mathfrak{R}^E,$$

where $\mathfrak{R}^E \colon S \otimes E \to S \otimes E$ is a vector bundle homomorphism determined by the curvature tensor of the connection on E.

As in the untwisted case, we denote by $D_E^+ \colon C^\infty(S^+ \otimes E) \longrightarrow C^\infty(S^- \otimes E)$ the restriction of D_E. According to the Atiyah-Singer Index Theorem (cf. [LaM, Ch. III, Thm. 13.10])

$$\mathrm{index}(D_E^+) = \langle \widehat{A}(TM)ch(E), [M] \rangle.$$

Coming back to the case of interest to us, we note that for a spin manifold M we have $a_l = \mathrm{index}(D_{R_l})$. Moreover, since R_l is build from symmetric powers of $TM_{\mathbb{C}}$, the Levi-Civita connection on TM induces a connection on R_l. The curvature term $\mathfrak{R}^{R_l}$ can then (at least in principle) be expressed in terms of the curvature tensor of M. One might hope to prove the conjectured vanishing of a_l by arguing that $Ric > 0$ implies that $F = \frac{s}{4} + \mathfrak{R}^{R_l}$ is a *positive* endomorphism of each fiber of $S \otimes R_l$ (i.e., $\langle Fv, v\rangle > 0$ for each non-zero $v \in S \otimes R_l$), which by the same argument as in the proof of Lichnerowicz' Theorem would imply the vanishing of D_{R_l}).

Alas, this strategy can't work, since nowhere in this line of argument did we use the assumption $\frac{p_1}{2}(M) = 0$, without which the conjecture is false as we've seen in part 3 of Remark 4.5.

4.5 Relation with the loop space

What we have said so far in this lecture is bound to appear quite mysterious, and the reader might have wondered about some of the following questions:

(i) Where does the Witten genus come from?

(ii) Why the assumption $\frac{p_1}{2}(M) = 0$?

(iii) Even if the conjecture happens to hold for homogeneous spaces and complete intersections, is there some heuristic argument that should let us expect it to be true in general?

Thinking of the conjecture as analogous to Lichnerowicz' Theorem, let us imagine we go back in time to the late fifties after Hirzebruch defined the $\widehat{A}$-genus, but before Atiyah and Singer constructed the 'Dirac' operator on a general spin manifold and before Lichnerowicz proved his formula (1.17). Imagine being lectured to about the definition of the $\widehat{A}$-genus (as in our first lecture, cf. 1.12), and being presented with the conjecture that the $\widehat{A}$-genus vanishes for manifolds with $w_2(M) = 0$ which admit a positive scalar curvature metric. As supporting evidence for this 'conjecture' it is observed that it is true for homogeneous spaces and complete intersections. Then the following questions might come to mind:

(i') Where does the $\widehat{A}$-genus come from?

(ii') Why the assumption $w_2(M) = 0$?

(iii') Even if the conjecture happens to hold for homogeneous spaces and complete intersections, is there some heuristic argument that should let us expect it to be true in general?

Questions (i') and (ii') are basically answered by the construction of the Dirac operator by Atiyah and Singer: the condition $w_2(M) = 0$ is needed to construct the spinor bundle M and hence the Dirac operator which acts on the sections of this bundle. The index of the Dirac operator is the $\widehat{A}$-genus.

Similarly, the questions (i) and (ii) are essentially answered by Witten's construction of the 'Dirac operator' on the free loop space LM consisting of all 'loops' $\gamma\colon S^1 \to M$: the condition $\frac{p_1}{2}(M) = 0$ is needed to construct the 'spinor bundle' over LM, on whose sections the 'Dirac operator' D_{LM} acts. The S^1-equivariant index of D_{LM} can be identified with the Witten genus of M. Unfortunately, Witten's considerations in [Wi] with regard to D_{LM} are very much on a formal/heuristic level and to date there is no mathematically rigorous construction of D_{LM} (except for homogeneous spaces [La], as we discuss below). It should be mentioned that Taubes [Ta1] has constructed an operator with the correct equivariant index, which can be interpreted as the 'Dirac operator on small loops'; unfortunatly, one cannot hope to use this operator to prove the conjecture, since its construction does not need the condition $\frac{p_1}{2} = 0$. In his paper, Witten does not discuss the question of how to construct D_{LM}, but rather *assumes* it has been constructed, that its S^1-equivariant index can be defined, and that the fixed point formula which expresses the equivariant index of an elliptic differential operator on a finite dimensional compact manifold continues to hold for the infinite dimensional manifold LM.

4.10. Digression on the fixed point formula. Let M be a spin manifold of dimension $n = 4k$, on which S^1 acts by isometries. We further assume that the S^1-action is compatible with the spin structure in the sense that the induced S^1-action on the oriented frame bundle $SO(M)$ lifts to an S^1-action on the double covering $Spin(M) \to SO(M)$ given by the spin structure. The induced action on $C^\infty(S)$ commutes with the Dirac operator D; in particular, $\ker D^+$ and $\operatorname{coker} D^+$ are representation of S^1,and we define the *S^1-equivariant index*

$$\operatorname{index}^{S^1}(D^+) \overset{\text{def}}{=} \sum_{l\in\mathbb{Z}} \left(\dim\,[\ker D^+]_l - \dim\,[\operatorname{coker} D^+]_l\right)\, q^l,$$

where for any representation of S^1, we denote by $V_l \subset V$ is the subspace where $z \in S^1$ acts by multiplication by z^l.

According to the *Fixed Point Formula*, also referred to as the *equivariant Atiyah-Singer Index Theorem*, the equivariant index can be computed in terms of the fixed point set of the S^1-action and the equivariant normal bundle of the fixed point set. To describe the explicit formula, let F be a component of the fixed point set. Then S^1 acts on the normal bundle $N \to F$, and the real vector bundle N can be written uniquely in the form $N = \bigoplus_{l>0} N_l$, where $N_l \to F$ is a *complex* vector bundle over F, on which

$z \in S^1$ acts by multiplication by z^l. Then

$$\mathrm{index}^{S^1}(D) = \sum_F \pm \langle \widehat{A}(TF)\, ch\left(\bigotimes_{l=1}^{\infty} (\Lambda^{n_l} q^l N_l)^{1/2} S_{q^l} N_l\right), [F]\rangle,$$

where $n_l = \dim_{\mathbb{C}} N_l$, and we sum over the connected components of the fixed point set. The determination of the sign for each component is quite involved and we refer to Atiyah-Bott for details [AB1], [AB2].

4.11. Witten's 'Index Theorem' for the Dirac operator on the loop space [Wi], [HBJ, §6.1]. Now we will 'apply' the Fixed Point Formula to the Dirac operator on the free loop space LM of a manifold M, equipped with the S^1-action given by rotating the parametrization of the loops. It should be stressed that this is a 'formal' calculation, since neither has a Dirac operator on LM be constructed, nor is the Fixed Point Formula a priory valid when applied to an infinite dimensional manifold like the free loop space.

We note that the fixed point set of the S^1-action on LM consists of the constant loops. Moreover, if γ is a constant loop, say $\gamma(t) = x_0 \in M$ for all $t \in [0,1]$, then the tangent space $T_\gamma LM$ consists of all loops $\{s\colon [0,1] \to T_{x_0}M \mid \gamma(0) = \gamma(1)\}$. Such a loop s has a Fourier decomposition

$$s(t) = \frac{a_0}{2} + \sum_{l>0} (a_l \cos 2\pi lt + b_l \sin 2\pi lt) \qquad \text{with} \quad a_l, b_l \in T_{x_0}M.$$

This implies that we have an isomorphism

$$T_\gamma LM \cong T_{x_0}M \oplus \bigoplus_{l>0} T_{x_0}M_{\mathbb{C}}$$

given by sending a loop s to its Fourier components (thinking of $a_l + ib_l$ as an element of the l-th copy of $T_{x_0}M_{\mathbb{C}} = T_{x_0}M \otimes_{\mathbb{R}} \mathbb{C}$). It is easy to check (cf. [HBJ, §6.1]) that with this identification $z \in S^1$ acts trivially on $T_{x_0}M$ and by multiplication by z^l on the l-th copy of $T_{x_0}M_{\mathbb{C}}$. This shows that the normal bundle N of the fixed point set $M \subset LM$ decomposes equivariantly as $N = \bigoplus_l N_l$, where $N_l = TM_{\mathbb{C}}$. It follows that $(\Lambda^{n_l} q^l N_l)^{1/2} = q^{\frac{ln}{2}}$, since $\Lambda^n TM_{\mathbb{C}}$ is the complexification of $\Lambda^n TM$, which is trivial since M is assumed orientable. After pulling out all these factors we obtain

$$\mathrm{index}^{S^1}(D_{LM}) = \left(\prod_{l=1}^{\infty} q^l\right)^{n/2} \langle \widehat{A}(TM)\, ch\left(\bigotimes_{l=1}^{\infty} S_{q^l}(TM_{\mathbb{C}})\right), [M]\rangle. \quad (4.12)$$

To the earthbound eyes of a mathematician, the factor

$$\left(\prod_{l=1}^{\infty} q^l\right) = q^{\sum_{l=1}^{\infty} l}$$

appears to make no sense. However a physicist, used to dealing with ugly infinities showing up when trying to sum certain series, would proceed to 'regularize' the sum $\sum_{l=1}^{\infty} l$ by considering the Riemann ζ-function $\zeta(s) = \sum_{l=1}^{\infty} l^{-s}$ which converges to a holomorphic function if the real part of the complex number s is sufficiently large. Then this function can be extended to a meromorphic function $\zeta(s)$ for $s \in \mathbb{C}$. It turns out that $\zeta(s)$ has no pole at $s = -1$, and $\zeta(-1) = -\frac{1}{12}$. Note that if we formally substitute -1 for s in the sum defining $\zeta(s)$ for s with large real part, we obtain $\sum_{l=1}^{\infty} l$; this is the motivation behind considering $-\frac{1}{12}$ as the 'regularized' value of this sum.

We recall from Definition 4.4 that

$$\phi_W(M) = \left(\prod_{l=1}^{\infty}(1-q^l)\right)^n \langle \widehat{A}(TM)\, ch\left(\bigotimes_{l=1}^{\infty} S_{q^l}(TM_{\mathbb{C}})\right), [M]\rangle.$$

Comparison with formula 4.12 shows that

$$\operatorname{index}^{S^1}(D_{LM}) = \frac{\phi_W(M)}{\eta(q)^n}, \tag{4.13}$$

where $\eta(q) = q^{1/24}\prod_{l=1}^{\infty}(1-q^l)$ is Dedekind's η-function.

This Index 'Theorem' of Witten shows that heuristically the Witten genus $\phi_W(M)$ should be thought of as the equivariant index of the Dirac operator on the free loop space LM. As an optimist, one might believe that it should be possible to imitate the argument used in the proof of Lichnerowicz' Theorem 1.11. In other words, the hope is to prove a 'Bochner-Lichnerowicz-Weitzenböck Formula' for D_{LM} which implies that if M has positive Ricci curvature, then D_{LM} is positive which in conjunction with Witten's 'Index Theorem' (Formula 4.13) would imply the vanishing of the Witten genus for manifolds with $\frac{p_1}{2}(M) = 0$ and $Ric > 0$.

In analogy with the finite dimensional case, one might suspect that D_{LM} is invertible if the scalar curvature of LM is positive, and might hope that the scalar curvature of LM at a loop γ is given by integrating the Ricci curvature $Ric(\dot\gamma)$ applied to the tangent vector $\dot\gamma$ to the loop γ over S^1. However, this is too naive for various reasons: first, D_{LM} is more analogous to Dirac operator associated to a $Spin^c$-structure than the Dirac operator of a spin manifold, which produces an extra term in the Weitzenböck formula. Secondly, there are many possible Riemannian metrics on LM induced by a fixed Riemannian metric on M; for the simplest one (the L^2-metric), the sectional curvature of the loop space is easy to calculate, but the sum describing the scalar curvature is divergent. For the other 'Sobolev-metrics', it seems that the (very complicated) expression for the scalar curvature of LM depends not only on the curvature tensor along γ, but also its covariant derivative, thus making it seem unlikely to be able to prove positivity when given only control over the Ricci tensor (but not its derivatives).

An interesting test case for this line of argument was provided by the recent construction of a 'Dirac operator' D_{LM} for the loop space of a homogeneous space $M = G/H$ of a compact semi-simple Lie group G by G. Landweber [La]. Of course, we know that Conjecture 4.3 is true for homogeneous spaces by Theorem 4.7; however, with the proposed line of argument, not just the *index of* D^+_{LM}, but the *kernel of* D_{LM} should be trivial. This is indeed the case for Landweber's operator.

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