Simulating a Strongly Driven JC Hamiltonian

Project for the PHY354 Course

S Shri Hari

Indian Institute of Science, Bengaluru

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About the Paper

- The Jaynes-Cummings (JC) Hamiltonian models a two-level system (Qubit) interacting with a quantized EM mode (in a Cavity).
- This paper has presented a study of the model subjected to the following parameters:
  1. Bad-Cavity Limit: Cavity Relaxation Rate κ larger than the qubit dephasing rates γ, γ_ϕ
  2. Strong Dispersive Regime: A non-negligible shift of the cavity frequency (greater than cavity-linewidth χ) due to the presence of qubit
  3. Presence of a Strong Drive: A sinusoidal drive $\xi(t) = \xi \cos(\omega_d t)$ where $\xi \gg \xi_1 = \kappa/\sqrt{2}$

Motivation behind the Project

- Open Quantum Systems: Quantum Mechanical Systems interacting with environment
- The paper was motivated by experiments done in the aforementioned systems and regimes prior to this paper.
- These experiments show a 'non-trivial' response in such conditions that cannot be examined analytically.
- Evolution of such systems cannot be described by Canonical QM (Unitary Dynamics) due to large degrees of freedom
- Theoretical studies limited to only small perturbation
- Numerical Simulations necessary for analysis outside such regimes

Theory I

Equations describing the system

- The Jaynes-Cummings (JC) Hamiltonian with a Drive

$$\begin{aligned} H &= H_{cavity} + H_{qubit} + H_{interaction} + H_{drive} \\ &= \omega_c a^\dagger a + \frac{\omega_q}{2}\sigma_z + g(a\sigma_+ + a^\dagger\sigma_-) + \frac{\xi(t)}{\sqrt{2}}(a + a^\dagger) \end{aligned} \tag{1}$$

- Decoupling Qubit and Cavity: Transformation from Bare States to "Dressed States" (Eigenkets of the non-driven JC Hamiltonian)

$$\begin{aligned} \tilde{H} &= UHU^\dagger \\ &= \omega_c a^\dagger a + (\omega_c - \Delta)\frac{\sigma_z}{2} + \frac{\xi}{\sqrt{2}}(a + a^\dagger)\cos(\omega_d t) \end{aligned} \tag{2}$$

where $N = a^\dagger a + \sigma_z/2 + 1/2$, $\delta = \omega_q - \omega_c$, and $\Delta = (\delta^2 + 4g^2 N)^{1/2}$

Theory II

The Quantum Master Equation and Key Simplifications

- The Quantum Master Equation

$$\dot{\rho} = -i[\tilde{H}, \rho] + \kappa([a\rho, a^\dagger] + [a, \rho a^\dagger])/2 \tag{3}$$

- Hierarchy of Scales

$$\gamma, \gamma_\phi \ll \kappa \ll g^2/\delta \ll g \ll \delta \ll \omega_c \tag{4}$$

- We can take the state of the Qubit to be a constant of motion as the time scale of the experiment/simulation is smaller than the qubit decoherence time $(\gamma^{-1}, \gamma_\phi^{-1})$
- The large ω_d means that number of drive oscillations in simulation time will be large.

Method of Quantum Trajectories

Monte-Carlo Approach to Wave Function Evolution

- Starting with the master equation, we construct a non-Hermitian effective Hamiltonian.

$$\dot{\rho} = -i[H, \rho] + \sum_n \frac{1}{2}([C_n\rho, C_n^\dagger] + [C_n, \rho C_n^\dagger])$$

$$H_{eff} = H - \frac{i}{2}\sum_n C_n^\dagger C_n \tag{5}$$

- If $\langle\psi(t)|\psi(t)\rangle = 1$, then $\langle\psi(t+\delta t)|\psi(t+\delta t)\rangle = 1 - \delta p$ where

$$\delta p = \delta t \sum_n \langle\psi(t)| C_n^\dagger C_n |\psi(t)\rangle \ll 1 \tag{6}$$

- There is a probability δp where the state 'jumps' to a new state.

$$|\psi(t+\delta t)\rangle = \frac{C_i |\psi(t)\rangle}{\langle\psi(t)| C_i^\dagger C_i |\psi(t)\rangle} \tag{7}$$

with the index i chosen with a probability of

$$P_i(t) = \langle\psi(t)| C_i^\dagger C_i |\psi(t)\rangle / \delta p \tag{8}$$

Implementing Method of Quantum Trajectories

Algorithm:

1. Choose a random number r between zero and one, representing the probability that a quantum jump occurs.
2. Integrate the Schrodinger Equation $i\frac{\partial}{\partial t}|\psi(t)\rangle = H_{eff}|\psi(t)\rangle$ until we reach a time τ where $\langle\psi(\tau)|\psi(\tau)\rangle = r$
3. After $t = \tau$, the state undergoes a quantum jump described previously. The index i is chosen such that i is the smallest number satisfying
$$\sum_{j=1}^{i} P_j(\tau) \geq r$$
4. The new state obtained will be now renormalized, another new random number r is drawn, and the process is repeated all over again till we reach the final simulation time.

Representing Wave Functions and Operators

- Basis of the Hilbert Space: We will be using the Fock Basis of the Hilbert Space corresponding to the Cavity
- Dimensions of the Hilbert Space: In Theory, the Hilbert Space and the associated operators like a and $a^\dagger$ are infinite dimensional. For our purpose, we will be working with a 'truncated' Hilbert Space, limiting the maximum number of photons in the cavity possible to a finite value $n \approx 1000$.

Coding Routines

- Running Multiple Trajectories Simultaneously: Wave Functions for multiple trials/trajectories are simulated simultaneously across time. The number of trajectories are set beforehand
- Time Dependent Hamiltonian: Hamiltonians are separated into time-independent and time-dependent parts, where the dependence in time is restricted to only coefficients of operators

$$H = H_0 + \sum_{i=1} c_i(t) \cdot H_i \tag{9}$$

- Time Evolution of States computed via two methods:
  1. Using RK4 routine: Viable for Time Independent Hamiltonian
  2. Using Spectral Decomposition: For Time Dependent Hamiltonians

Spectral Decomposition of Hamiltonian

Speeding up Simulation

- Time Evolution of Eigenkets of (Time-Independent) Hamiltonian:
$$i\frac{\partial}{\partial t}|\lambda\rangle = H|\lambda\rangle$$
$$H|\lambda\rangle = E_\lambda|\lambda\rangle \rightarrow |\lambda(t)\rangle = e^{-iE_\lambda t}|\lambda(0)\rangle \tag{10}$$
- Spectral Decomposition of Hamiltonian:
$$H = Q\Lambda Q^{-1} \tag{11}$$
  - Q - $n \times n$ matrix whose ith column corresponds to the ith Eigenvector of H. Invertible as Eigenvectors are linearly independent (unless zero Eigenvalue exists)
  - Λ - $n \times n$ Diagonal Matrix whose diagonal elements are corresponding Eigenvectors

Spectral Decomposition of Hamiltonian

Continued

- Time Evolution of Wavefunction

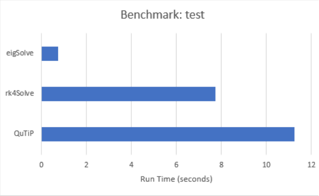
$$|\psi(t)\rangle = Qe^{-it\Lambda}Q^{-1}|\psi(0)\rangle \tag{12}$$

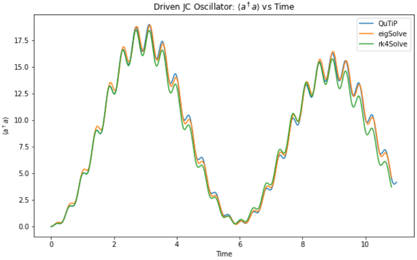
- For Time Dependent Hamiltonian $H(t)$, for each step in time δt, we can assume the Hamiltonian $H(\tau)$ to be constant and use the above to evolve the wavefunction from $t = \tau$ to $t = \tau + \delta t$
- This is especially useful for a periodic Hamiltonian as the required matrices can be computed beforehand, reducing number of function calls and computations. Very useful for long simulation times

Results: Comparison with QuTiP

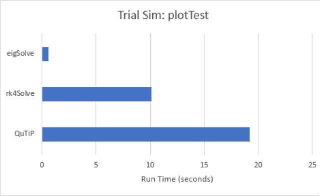
Benchmarking with Base Parameters

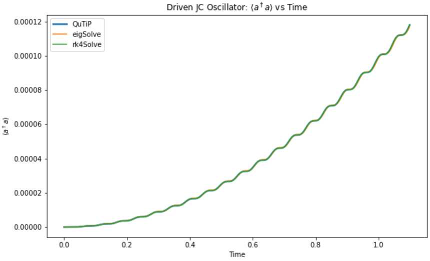
- Comparing two different solving routines against QuTiP (uses ZVODE).
- The values are plotted against time and the execution time is recorded (Wall Time)

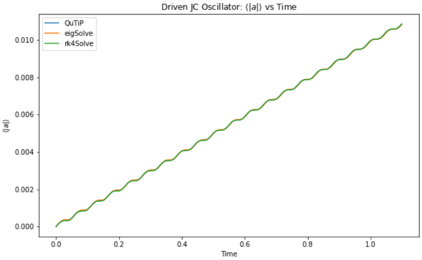




- Testing with simulation parameters for a short simulation time
- Rotating Wave Approximation made for QuTiP implementation for faster runtime







Results: Comparison with Literature

Figure: Plot from Literature: Intercavity Amplitude vs Drive detuning for fixed drive $\xi = 6.3\xi_1$

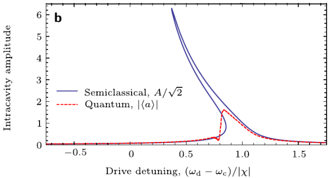


Figure: Plot Generated from Multiple Routines Implemented (incl. QuTiP)