Home integrals.
Calculus.
Solutions.

Gleb Minaev @ 102 (20.Б02-мкн)

Problem 4039. The given domain $\Omega := \{(x;y) \mid x \geqslant 0 \wedge y \geqslant 0 \wedge x + y = 1\}$. The surface is defined as $z^2 = 2xy$, but it means that upper part ($z = \sqrt{2xy}$) and lower part ($z = -\sqrt{2xy}$) of it are symmetrical, hence the area of upper part $z := \sqrt{2xy}$ is a half of the total area. Then the area of the surface is

$$A = 2\int_\Omega \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dydx.$$

But $\frac{\partial z}{\partial x} = \frac{\partial\sqrt{2xy}}{\partial x} = \frac{\sqrt{2y}}{2\sqrt{x}} = \sqrt{\frac{y}{2x}}$; similarly $\frac{\partial z}{\partial y} = \sqrt{\frac{x}{2y}}$. Then

$$A = 2\int_\Omega \sqrt{1 + \frac{x}{2y} + \frac{y}{2x}} dxdy.$$

Let $u := \sqrt{x/y}$ and $v := x + y$. Then $v = \frac{x}{y}y + y = y(u^2+1)$, so $y = \frac{v}{u^2+1}$ and $x = \frac{vu^2}{u^2+1}$. Also v goes from 0 to 1 and u goes from 0 to $+\infty$, Jacobian of the substitution is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2vu}{(u^2+1)^2} & \frac{u^2}{(u^2+1)} \\ \frac{-2vu}{(u^2+1)^2} & \frac{1}{(u^2+1)} \end{vmatrix} = \frac{2vu + 2vu^3}{(u^2+1)^3} = \frac{2vu}{(u^2+1)^2},$$

and $(x;y) \in \Omega$ iff $u \in [0;+\infty] \wedge v \in [0;1]$. Hence

$$\begin{aligned}
A &= 2\int_0^1 \int_0^{+\infty} \sqrt{1 + \frac{u^2}{2} + \frac{1}{2u^2}} \frac{2vu}{(u^2+1)^2} dudv \\
&= \int_0^1 2vdv \int_0^{+\infty} \sqrt{1 + \frac{u^2}{2} + \frac{1}{2u^2}} \frac{2u}{(u^2+1)^2} du \\
&= v^2\Big|_0^1 \int_0^{+\infty} \sqrt{\frac{u^4 + 2u^2 + 1}{2u^2}} \frac{2u}{(u^2+1)^2} du \\
&= \int_0^{+\infty} \frac{u^2+1}{\sqrt{2}u} \frac{2u}{(u^2+1)^2} du \\
&= \int_0^{+\infty} \frac{\sqrt{2}du}{u^2+1} du \\
&= \sqrt{2}\tan^{-1}(u)\Big|_0^{+\infty} \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Problem 4044. Let $p = x/a$, $q = y/a$, $r = z/a$. Then we are looking for area of surface $r = (p^2+q^2)/2$ inside cylinder $(p^2+q^2)^2 = 2pq$.

Let $p = l\cos(\varphi)$, $q = l\sin(\varphi)$. Then $p^2+q^2 = l^2$, $2pq = l^2\sin(2\varphi)$. Hence the curve $(p^2+q^2)^2 = 2pq$ is equivalent to curve $l^4 = l^2\sin(2\varphi) \Leftrightarrow l^2 = \sin(2\varphi)$. Hence we are looking for area of the surface in region $l^2 \leqslant \sin(2\varphi)$ (region $l^2 \geqslant \sin(2\varphi)$ is not bounded, because contatin region $l \geqslant 1$).

Jacobian of substitution $(p;q) \mapsto (x;y)$ is obviously a^2, and Jacobian of $(r;\varphi) \mapsto (p;q)$ is

$$\begin{vmatrix} \frac{\partial p}{\partial l} & \frac{\partial p}{\partial \varphi} \\ \frac{\partial q}{\partial l} & \frac{\partial q}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos(\varphi) & -l\sin(\varphi) \\ \sin(\varphi) & l\cos(\varphi) \end{vmatrix} = l.$$

So the area is

$$\begin{aligned}
A &= \int_\Omega \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \\
&= \int_\Omega \sqrt{1+\left(\frac{\partial (ar)}{\partial (ap)}\right)^2+\left(\frac{\partial (ar)}{\partial (aq)}\right)^2}a^2dpdq \\
&= a^2\int_\Omega \sqrt{1+\left(\frac{\partial r}{\partial p}\right)^2+\left(\frac{\partial r}{\partial q}\right)^2}dpdq \\
&= a^2\int_\Omega \sqrt{1+p^2+q^2}dpdq \\
&= a^2\int_\Omega \sqrt{1+l^2}ldld\varphi \\
&= 2a^2\int_0^{\pi/2}\int_0^{\sqrt{\sin(2\varphi)}} \sqrt{1+l^2}ldld\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(1+l^2)^{3/2}\Big|_0^{\sqrt{\sin(2\varphi)}} d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}((1+\sin(2\varphi))^{3/2}-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}((2\sin(\varphi+\pi/4)^2)^{3/2}-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(2\sqrt{2}\sin(\varphi+\pi/4)^3-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(\sqrt{2}(2\sin(\varphi+\pi/4)^2)\sin(\varphi+\pi/4)-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(\sqrt{2}(\cos(0)-\cos(2\varphi+\pi/2))\sin(\varphi+\pi/4)-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(\sqrt{2}\sin(\varphi+\pi/4)-\sqrt{2}\cos(2\varphi+\pi/2)\sin(\varphi+\pi/4)-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}(\sqrt{2}\sin(\varphi+\pi/4)-\frac{1}{\sqrt{2}}(\sin(3\varphi+3\pi/4)+\sin(-\varphi-\pi/4))-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3}((\sqrt{2}+\frac{1}{\sqrt{2}})\sin(\varphi+\pi/4)-\frac{1}{\sqrt{2}}\sin(3\varphi+3\pi/4)-1)d\varphi \\
&= 2a^2\int_0^{\pi/2} \frac{1}{3\sqrt{2}}(3\sin(\varphi+\pi/4)-\sin(3\varphi+3\pi/4)-\sqrt{2})d\varphi
\end{aligned}$$

$$
\begin{aligned}
&= 2a^2 \frac{1}{3\sqrt{2}}(-3\cos(\varphi+\pi/4) + \frac{1}{3}\cos(3\varphi+3\pi/4) - \sqrt{2}\varphi)\Big|_0^{\pi/2} \\
&= 2a^2 \frac{1}{3\sqrt{2}}\Big((-3\cos(3\pi/4) + \frac{1}{3}\cos(9\pi/4) - \pi/\sqrt{2}) \\
&\quad - (-3\cos(\pi/4) + \frac{1}{3}\cos(3\pi/4))\Big) \\
&= 2a^2 \frac{1}{3\sqrt{2}}\Big(-3\frac{-1}{\sqrt{2}} + \frac{1}{3}\frac{1}{\sqrt{2}} - \pi/\sqrt{2} + 3\frac{1}{\sqrt{2}} - \frac{1}{3}\frac{-1}{\sqrt{2}}\Big) \\
&= 2a^2 \frac{1}{3\sqrt{2}}\Big(\frac{3}{\sqrt{2}} + \frac{1}{3\sqrt{2}} + \frac{3}{\sqrt{2}} + \frac{1}{3\sqrt{2}} - \pi/\sqrt{2}\Big) \\
&= 2a^2 \frac{1}{3\sqrt{2}}\Big(\sqrt{2}\frac{10}{3} - \pi/\sqrt{2}\Big) \\
&= 2a^2 \frac{1}{3}\Big(\frac{10}{3} - \pi/2\Big) \\
&= 2a^2 \Big(\frac{10}{9} - \pi/6\Big) \\
&= a^2 \Big(\frac{20}{9} - \pi/3\Big)
\end{aligned}
$$

Problem 4017. Similarly to previous problem we are looking for volume of solid figure

$$\Omega := \{(x;y;z) \mid x^2+y^2 \geqslant az \wedge z \geqslant 0 \wedge (x^2+y^2)^2 \leqslant a^2(x^2-y^2)\}.$$

Let $x = ar\cos(\varphi)$, $y = ar\sin(\varphi)$, $z = aq$. Then $(x;y;z) \in \Omega$ iff

$$(r;\varphi;q) \in \Gamma := \{(r;\varphi;q) \mid r^2 \geqslant q \geqslant 0 \wedge r^2 \leqslant \cos(2\varphi) \wedge 0 \leqslant r \wedge \varphi \in [0;2\pi]\}.$$

Also Jacobian of the substitution $(r;\varphi;q) \to (x;y;z)$ is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial q} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial q} \end{vmatrix} = \begin{vmatrix} a\cos(\varphi) & -ar\sin(\varphi) & \\ a\sin(\varphi) & ar\cos(\varphi) & \\ & & a \end{vmatrix} = a^3 r.$$

Hence volume of solid figure Ω is

$$
\begin{aligned}
V &= \int_\Omega 1 dxdydz \\
&= \int_\Gamma a^3 r dqdrd\varphi \\
&= 4\int_0^{\pi/4}\int_0^{\sqrt{\cos(2\varphi)}}\int_0^{r^2} a^3 r dqdrd\varphi \\
&= 4a^3\int_0^{\pi/4}\int_0^{\sqrt{\cos(2\varphi)}} r\int_0^{r^2} dqdrd\varphi \\
&= 4a^3\int_0^{\pi/4}\int_0^{\sqrt{\cos(2\varphi)}} r^3 drd\varphi \\
&= 4a^3\int_0^{\pi/4} \frac{r^4}{4}\Big|_0^{\sqrt{\cos(2\varphi)}} d\varphi
\end{aligned}
$$

$$
\begin{aligned}
&= a^3 \int_0^{\pi/4} \cos(2\varphi)^2 d\varphi \\
&= \frac{1}{2}a^3 \int_0^{\pi/4} 1 + \cos(4\varphi) d\varphi \\
&= \frac{1}{2}a^3 \, \varphi + \frac{\sin(4\varphi)}{4}\Big|_0^{\pi/4} \\
&= \frac{1}{2}a^3 \left(\varphi + \frac{\sin(4\varphi)}{4}\right)\Big|_0^{\pi/4} \\
&= \frac{1}{2}a^3 \frac{\pi}{4} \\
&= \frac{\pi a^3}{8}
\end{aligned}
$$

Problem 4104. Similarly we are looking for volume of solide figure

$$\Omega := \{(x;y;z) \mid \sqrt{x^2+y^2} \geqslant z \wedge az \geqslant x^2+y^2\}.$$

Let $x = ar\cos(\varphi)$, $y = ar\sin(\varphi)$, $z = aq$. Then $(x;y;z) \in \Omega$ iff

$$(r;\varphi;q) \in \Gamma := \{(r;\varphi;q) \mid r \geqslant q \geqslant r^2 \wedge r \geqslant 0 \wedge \varphi \in [0;2\pi]\}.$$

Also Jacobian of the substitution $(r;\varphi;q) \to (x;y;z)$ is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial q} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial q} \end{vmatrix} = \begin{vmatrix} a\cos(\varphi) & -ar\sin(\varphi) & \\ a\sin(\varphi) & ar\cos(\varphi) & \\ & & a \end{vmatrix} = a^3 r.$$

Hence volume of solid figure Ω is

$$
\begin{aligned}
V &= \int_\Omega 1 dxdydz \\
&= \int_\Gamma a^3 r dq dr d\varphi \\
&= \int_0^{2\pi} \int_0^1 \int_{r^2}^r a^3 r dq dr d\varphi \\
&= 2\pi a^3 \int_0^1 r \int_{r^2}^r dq dr \\
&= 2\pi a^3 \int_0^1 r(r - r^2) dr \\
&= 2\pi a^3 \int_0^1 r^2 - r^3 dr \\
&= 2\pi a^3 \left(\frac{r^3}{3} - \frac{r^4}{4}\right)\Big|_0^1 \\
&= \frac{\pi a^3}{6}
\end{aligned}
$$
