Homework of 12.07
Differential equations and dynamic systems.
Solutions.

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Problem 788.

$$\begin{cases} \dot{x} + x - 8y = 0 \\ \dot{y} - x - y = 0 \end{cases}$$

$$\begin{cases} \dot{x} = -x + 8y \\ \dot{y} = x + y \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Whereas

$$\begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -3 & \\ & 3 \end{pmatrix}\begin{pmatrix} -4 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

that means $\left(\begin{smallmatrix} -1 & 8 \\ 1 & 1 \end{smallmatrix}\right)$ has eigenvectors $\left(\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ with eigenvalues -3 and 3 respectievely. Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} -4 \\ 1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{3t} = \begin{pmatrix} -4c_1e^{-3t} + 2c_2e^{3t} \\ c_1e^{-3t} + c_2e^{3t} \end{pmatrix}$$

Problem 791.

$$\begin{cases} \dot{x} + x + 5y = 0 \\ \dot{y} - x - y = 0 \end{cases}$$

$$\begin{cases} \dot{x} = -x - 5y \\ \dot{y} = x + y \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Whereas

$$\begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1-2i & -1+2i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2i & \\ & 2i \end{pmatrix}\begin{pmatrix} -1-2i & -1+2i \\ 1 & 1 \end{pmatrix}^{-1}$$

that means $\left(\begin{smallmatrix} -1 & -5 \\ 1 & 1 \end{smallmatrix}\right)$ has eigenvectors $\left(\begin{smallmatrix} -1-2i \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1+2i \\ 1 \end{smallmatrix}\right)$ with eigenvalues $-2i$ and $2i$ respectievely. Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} -1-2i \\ 1 \end{pmatrix}e^{-2it} + c_2\begin{pmatrix} -1+2i \\ 1 \end{pmatrix}e^{2it} = \begin{pmatrix} (-1-2i)c_1e^{-2it} + (-1+2i)c_2e^{2it} \\ c_1e^{-2it} + c_2e^{2it} \end{pmatrix}$$

If we are looking for real valued solutions, then we must assume that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-(c_1+c_2) - (c_1-c_2)2i)\cos(2t) + ((c_1-c_2)i - (c_1+c_2)2)\sin(2t) \\ (c_1+c_2)\cos(2t) + (c_1-c_2)i\sin(2t) \end{pmatrix}$$

has real coefficients before sines and cosines. So c_1+c_2 is real and c_1-c_2 is imaginary, which means $c_1 = a+bi$ and $c_2 = a-bi$ for some real a and b. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-2a+4b)\cos(2t) + (-2b-4a)\sin(2t) \\ 2a\cos(2t) - 2b\sin(2t) \end{pmatrix}$$

Problem 795.

$$\begin{cases} \dot{x} - 5x - 3y = 0 \\ \dot{y} + 3x + y = 0 \end{cases}$$

$$\begin{cases} \dot{x} = 5x + 3y \\ \dot{y} = -3x - y \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Whereas

$$\begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & \end{pmatrix}\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -3 & \end{pmatrix}^{-1}$$

that means $A = \left(\begin{smallmatrix} 5 & 3 \\ -3 & -1 \end{smallmatrix}\right)$ has generalized eigenvectors $v_1 = \left(\begin{smallmatrix} 3 \\ -3 \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with eigenvalue 2: $v_1 A = 2v_1$ and $v_2 A = 2v_2 + v_1$. Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = (c_1(v_1 t + v_2) + c_2 v_1)e^{2t} = \begin{pmatrix} (c_1(3t+1) + 3c_2)e^{2t} \\ -3c_1 t e^{2t} \end{pmatrix} = \begin{pmatrix} (3c_1 t + (c_1 + 3c_2))e^{2t} \\ -3c_1 t e^{2t} \end{pmatrix}$$
