Homework of 10.12
Differential geometry

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Problem 60. We know that $|\gamma(t+\delta)-\gamma(t)|$ depends on only δ; then let $l(\delta) := |\gamma(t+\delta)-\gamma(t)|$. So for any $t \in (a;b)$

$$|\gamma'(t)| = \left|\lim_{\delta\to 0} \frac{\gamma(t+\delta)+\gamma(t)}{\delta}\right| = \lim_{\delta\to 0} \frac{|\gamma(t+\delta)+\gamma(t)|}{|\delta|} = \lim_{\delta\to 0} \frac{l(\delta)}{|\delta|}$$

does not depend on t, i.e. $|\gamma'| = \text{const}$ on $[a;b]$. Also $|\gamma(t+\delta)-\gamma(t)|$ being a constant function of t for any fixed δ means that

$$(\gamma(t+\delta)-\gamma(t)) \cdot (\gamma'(t+\delta)-\gamma'(t)) = 0,$$

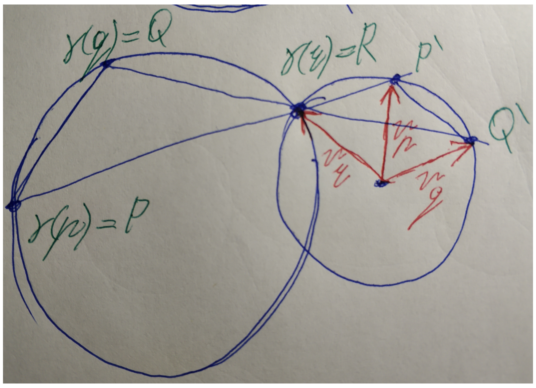
i.e. $\gamma(t+\delta)-\gamma(t)$ and $\gamma'(t+\delta)-\gamma'(t)$ are perpendicular (or one of them is a zero vector).

If $|\gamma'| = 0$, then $\gamma' = 0$ and $\gamma = \text{const}$. Hence in that case problem is obvious. Then let's consider case $|\gamma'| > 0$.

Lemma 1. *Let $p, q, r \in [a;b]$ be three different points such that $\gamma(p)$, $\gamma(q)$ and $\gamma(r)$ are different in pairs and lie on some circle ω (so are not collinear). Let O be a center of ω and R_φ is operator of rotation (of vector) by φ counterclockwise. Then $\gamma'(p) = \gamma'(q) = \gamma'(r)$ or there is $\lambda \in \mathbb{R}$ such that*

$$\gamma'(p) = \lambda \mathrm{R}_{\pi/2}(\gamma(p)-O), \qquad \gamma'(q) = \lambda \mathrm{R}_{\pi/2}(\gamma(q)-O), \qquad \gamma'(r) = \lambda \mathrm{R}_{\pi/2}(\gamma(r)-O).$$

Proof. Let's name $P := \gamma(p)$, $Q := \gamma(q)$ and $R := \gamma(r)$, $v_p := \mathrm{R}_{-\pi/2}(\gamma'(p))$, $v_q := \mathrm{R}_{-\pi/2}(\gamma'(q))$ and $v_r := \mathrm{R}_{-\pi/2}(\gamma'(r))$. Then $v_p - v_q$ is (zero vector or) parallel to $\overline{PQ}$; and the same goes for p and r, q and r. Let σ be a circle with center $R - v_r$ and goes through R, and P', Q' be secondary intersection of σ with $\overline{RP}$ and $\overline{RQ}$.



Let O' be center of σ. $v_p - v_r \parallel \overline{PR}$, so $R - v_r + v_p \in \overline{PR}$, i.e. $O' + v_p$ is R or P'; similarly $O' + v_q \in \{R; Q'\}$.

If $v_p = v_r$ then $v_p - v_q = v_r - v_q$ is parallel to both $\overline{PQ}$ and $\overline{RQ}$ (that are not parallel, so $v_q - v_r = 0$, $v_q = v_r$. Similarly if $v_q = v_r$ then $v_p = v_r$. Then let's consider case $v_p = P' - O'$ and $v_q = Q' - O'$.

Then $v_p - v_q = P' - Q'$ is parallel to $\overline{PQ}$. So homotety with center R and coefficient $\lambda := \overrightarrow{RP}/\overrightarrow{RP'}$ maps R to R, P to P', line $\overline{RQ'}$ to itself, $\overline{P'Q'}$ to parallel line through P that is $\overline{PQ}$, so Q' to Q. Hence it maps v_p, v_q and v_r to $P - O$, $Q - O$ and $R - O$. It means

$$\gamma'(p) = \mathrm{R}_{\pi/2}(v_p) = \mathrm{R}_{\pi/2}(\lambda(\gamma(p) - O)) = \lambda\mathrm{R}_{\pi/2}(\gamma(p) - O);$$

the same goes to q and r. $\square$

Corollary 1.1. *If $\gamma(p)$, $\gamma(q)$ and $\gamma(r)$ are not collinear and $\gamma'(p)$, $\gamma'(q)$ and $\gamma'(r)$ are different, then $\gamma'(p)$, $\gamma'(q)$ and $\gamma'(r)$ constructed from $\gamma(p)$, $\gamma(q)$ and $\gamma(r)$ respectievly are tangent to circumcircle of $\gamma(p)$, $\gamma(q)$ and $\gamma(r)$, have the same length and are oriented in the same direction along the circumcircle.*

Corollary 1.2. *If $\gamma(p) \neq \gamma(q)$ and $\gamma'(p) \neq \gamma'(q)$ then for every r if $\gamma(r) \notin \overline{\gamma(p)\gamma(q)}$ then $\gamma(r)$ is lying on circle through $\gamma(p)$ and $\gamma(q)$ that is tangent to $\gamma'(p)$ constructed from $\gamma(p)$.*

So if there are two arguments $p, q \in [a; b]$ such that $\gamma(p) \neq \gamma(q)$ and $\gamma'(p) \neq \gamma'(q)$, then all points of γ are lying in union of a circle and a line that have intersection $\{\gamma(p); \gamma(q)\}$. But obviously because of smoothness of γ it can not go from the circle to the line (there will be a breaking in point where γ will change its trajectory from the circle to the line or vice versa). Then γ is lying on some line or some circle.

Then the remaining case is where for any $p, q \in [a; b]$ $\gamma(p) = \gamma(q)$ or $\gamma'(p) = \gamma'(q)$. Then if $\gamma = \text{const}$ the problem is obvious; otherwise there are $p, q \in [a; b]$ such that $\gamma(p) \neq \gamma(q)$. Then $\gamma'(p) = \gamma'(q)$. So for any $r \in [a; b]$ $\gamma(r) \neq \gamma(p)$ or $\gamma(r) \neq \gamma(q)$, so then $\gamma'(r) = \gamma'(p) = \gamma'(q)$. Hence $\gamma' = \text{const}$, so $\gamma = A + tv$ that is lying on a line through A and is parallel to v.
