Homework of 09.27
Differential geometry

Gleb Minaev @ 204 (20.Б04-мкн)

Problem 37. Notice that a sphere with p handles is a sphere with $2p$ holes glued with p cylinders (lateral surfaces of cylinders). Let's name holes as $h_{1,a}$, $h_{1,b}$, …, $h_{p,a}$, and $h_{p,b}$ and cylinders as c_1, …, c_p. So the sphere with p handles can be obtained by glueing upper border of each c_i to hole $h_{i,a}$ and its lower border to hole $h_{i,b}$.

Then consider d spheres with $2p$ holes each and pd cylinders. Let's name each sphere with holes as s_1, …, s_d, holes of sphere s_i as $h_{i,1,a}$, $h_{i,1,b}$, …, $h_{i,p,a}$, $h_{i,p,b}$ and cylinders as $c_{1,1}$, …, $c_{1,p}$, $c_{2,1}$, …, $c_{d,p}$. Then consider glueing upper border of each cylinder $c_{i,j}$ to hole $h_{i,j,a}$ and its lower border to hole $h_{i+1,j,b}$ (where $i+1$ is considered with respect to congruence modulo d). Hence we've got some (path-connected without boundary) surfaces C.

Let B be the sphere with p handles, B' be a disjoint union of sphere with $2p$ holes and p cylinders and C' be a disjoint union of d spheres with $2p$ holes each and pd cylinders. Then there are continuous maps $p_B : B' \to B$, $p_C : C' \to C$ that are results of glueing and

$$f' : C' \to B', s_i \mapsto s, h_{i,j,l} \mapsto h_{j,l}, c_{i,j} \mapsto c_j$$

that is obvious covering space. It's also obvious there is a continuous map $f : C \to B$ such that diagram

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d, "p_C"'] & B' \arrow[d, "p_B"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative. Hence obviously f is d-sheeted covering space.

Also $\chi(s_i) = 2 - 2p$, $\chi(c_{i,j}) = 0$. So

$$\chi(C) = \sum_{i=1}^{d} \chi(s_i) + \sum_{i=1}^{d} \sum_{j=1}^{p} \chi(c_{i,j}) = d \cdot (2 - 2p) + d \cdot p \cdot 0 = 2d(1-p).$$

So as far as C is oriented surface, it is a sphere with $d(p-1)+1$ handles.

Problem 38. We will repeat the trick. Notice that there is obvious covering space $S^2 \to \mathbb{R}P^2$. Hence there is obvious covering space of cross-cap by a sphere with 2 holes.

Let B' be a disjoint union of sphere with p holes and p cross-caps and C' be a disjoint union of two spheres with p holes each and p spheres with two holes each (cylinders). Let's name sphere in B' as s, holes in s as h_1, …, h_p, cross-caps in B' as m_1, …, m_p, spheres in C' as s_1 and s_2, holes in s_i as $h_{i,1}$, …, $h_{i,p}$, cylinders in C' as c_1, …, c_p.

There is obvious covering space $f' : C' \to B'$ where s_1 and s_2 identically cover s and c_i 2-sheetedly covers m_i (as we noticed at the beginning of the proof) for each i. Glueing all m_i to h_i respectively we obtain a surface B and a projection $p_B : B' \to B$. Glueing all upper borders of c_i to $h_{1,i}$ and lower borders of c_i to $h_{2,i}$ with respect to f' and p_B (it will be glued in not usually but turned out because of nature of the covering space of cross-cap; if previously we turn out s_2 then glueing will be usual). Hence there is a continuous map $f : C \to B$ such that diagram

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d, "p_C"'] & B' \arrow[d, "p_B"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative. Hence obviously f is 2-sheeted covering space.

Also C is oriented (because s_1 and all c_i will get usual orientation and s_2 will get opposite orientation). Hence C is a sphere with handles (more precisely, with $p-1$ handles).
