Homework of 11.27
Differential equations and dynamic systems.
Solutions.

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Problem 513.

$$y'' - 2y' = 0$$

Characteristic poylomial $\lambda^2 - 2\lambda = 0$ has roots 0 and 2. Hence

$$y = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}.$$

Problem 519.

$$y^{(4)} - y = 0.$$

Characteristic polynomial $\lambda^4 - 1 = 0$ has roots 1, -1, i and $-i$. Hence

$$y = c_1 e^x + c_2 e^{-x} + c_3 \sin(x) + c_4 \cos(x).$$

Problem 524.

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} = 0.$$

Characteristic polynomial $\lambda^5 - 6\lambda^4 + 9\lambda^3 = 0$ has roots 0, 0, 0, 3 and 3. Hence

$$y = (c_1 + c_2 x + c_3 x^2)e^{0x} + (c_3 + c_4 x)e^{3x} = (c_1 + c_2 x + c_3 x^2) + (c_3 + c_4 x)e^{3x}.$$

Problem 575.

$$y'' - 2y' + y = \frac{e^x}{x}.$$

Characteristic polynomial $\lambda^2 - 2\lambda + 1 = 0$ has roots 1 and 1. Hence

$$y = C_1(x)e^x + C_2(x)xe^x,$$

such that

$$\begin{cases} C_1'e^x + C_2'xe^x = 0 \\ C_1'(e^x)' + C_2'(xe^x)' = \dfrac{e^x}{x} \end{cases} \qquad \begin{cases} C_1'e^x + C_2'xe^x = 0 \\ C_1'e^x + C_2'(x+1)e^x = \dfrac{e^x}{x} \end{cases} \qquad \begin{cases} C_1' + C_2'x = 0 \\ C_1' + C_2'(x+1) = \dfrac{1}{x} \end{cases}$$

$$\begin{cases} C_1' + C_2'x = 0 \\ C_2' = \dfrac{1}{x} \end{cases} \qquad \begin{cases} C_1' = -1 \\ C_2' = \dfrac{1}{x} \end{cases} \qquad \begin{cases} C_1 = -x + c_1 \\ C_2 = \ln(x) + c_2 \end{cases}$$

Hence

$$y = (-x + c_1)e^x + (\ln(x) + c_2)xe^x = (x\ln(x) + (c_2 - 1)x + c_1)e^x.$$

Problem 620.

$$y'' + ay' + by = 0.$$

Let α and β be roots of characteristic polynomial $\lambda^2 + a\lambda + b$. We know that $e^{\alpha x} + e^{\overline{\alpha} x}$ are solutions of the equations. Then α must have negative real part. Because

$$0 = \lim_{x\to+\infty} e^{\alpha x} + e^{\overline{\alpha} x} = \lim_{x\to+\infty} e^{\mathrm{Re}(\alpha)x} \cdot \cos(\mathrm{Im}(\alpha)x),$$

hence either $\mathrm{Im}(\alpha) = 0$ and $\lim_{x\to+\infty} e^{\mathrm{Re}(\alpha)x} = 0$ means $\mathrm{Re}(\alpha) < 0$ or $\mathrm{Im}(\alpha) \neq 0$ and

$$0 = \lim_{n\to+\infty} e^{2\pi n \mathrm{Re}(\alpha)/|\mathrm{Im}(\alpha)|} \cos(\mathrm{sign}(\mathrm{Im}(\alpha))2\pi n) = \lim_{n\to+\infty} e^{2\pi n \mathrm{Re}(\alpha)/|\mathrm{Im}(\alpha)|}$$

means $\mathrm{Re}(\alpha) < 0$. And if real parts of α and β are negative then any solution $y = c_1 e^{\alpha x} + c_2 e^{\beta x} \to 0$ when $x \to +\infty$ (in case $\alpha = \beta$ $y = (c_1 + c_2 x)e^{\alpha x} \to 0$ when $x \to 0$).

But

$$\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad \beta = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

So if $b^2 - 4c < 0$, then $b = -2\mathrm{Re}(\alpha) = -2\mathrm{Re}(\beta) > 0$. And if $b^2 - 4c \geqslant 0$, then

$$0 > \mathrm{Re}(\alpha) = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

that is equivalent to

$$\begin{gathered} b > \sqrt{b^2 - 4c} \\ b^2 > b^2 - 4c \\ c > 0 \end{gathered}$$

(in that case if $\mathrm{Re}(a) < 0$, then $\mathrm{Re}(b) = \frac{-b-\sqrt{b^2-4c}}{2} < 0$).

Hence the sought condition is

$$\begin{cases} b > 0 \\ \left[\begin{array}{l} c > \dfrac{b^2}{4} \\ 0 < c \leqslant \dfrac{b^2}{4} \end{array}\right. \end{cases} \iff \begin{cases} b > 0 \\ c > 0 \end{cases}$$

Problem 590.

$$x^2 y'' - xy' - 3y = 0.$$

Characteristic polynomial $\lambda(\lambda - 1) - \lambda - 3 = \lambda^2 - 2\lambda - 3$ has roots -1 and 3. Hence

$$y(e^t) = c_1 e^{-t} + c^2 e^{3t}.$$
