Homework of 11.02
Differential equations and dynamic systems.
Solutions.

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Problem.

$$(1 + e^y)dx - e^{2y}\sin(x)^3 dy = 0$$

(lose case $\sin(x) = 0$, that is $x \equiv \pi n$ for some $n \in \mathbb{Z}$)

$$y' = \frac{dy}{dx} = \frac{1 + e^y}{e^{2y}} \frac{1}{\sin(x)^3}.$$

So there is integral

$$\begin{aligned} G &:= \int \frac{e^{2y}}{1 + e^y} dy - \int \frac{dx}{\sin(x)^3} \\ &= e^y - \ln(1 + e^y) - \frac{\cos(x)}{2(\cos(x)^2 - 1)} - \frac{1}{4}\ln\left(\frac{1 - \cos(x)}{1 + \cos(x)}\right) + C = 0 \end{aligned}$$

So because function $f(t) = e^t - t$ is hard to inverse, y cannot be expressed elementary of x.

Problem.

$$y' = \left(\frac{y + 1}{x + y - 2}\right)^2$$

Let $x = u + 3$, $y = v - 1$. Then

$$v' = \left(\frac{v}{u + v}\right)^2$$

Let $v = pu$. Then

$$\begin{aligned} up' + p = v' = \left(\frac{1}{1 + v/u}\right)^2 &= \frac{1}{(1 + p)^2} \\ p' &= \frac{1 - p(1 + p)^2}{(1 + p)^2} \frac{1}{u}. \end{aligned}$$

So there is integral

$$\begin{aligned} G &:= \int \frac{(1 + p)^2 dp}{1 - p(1 + p)^2} - \int \frac{du}{u} \\ &= \sum_{\alpha^3 - \alpha^2 - 1 = 0} \frac{\alpha}{3\alpha - 2} \ln(p - \alpha + 1) - \ln(u) - C = 0. \end{aligned}$$

Problem 426.

$$yy'' + 1 = y'^2$$

Let $p(y) = y'$. Then $y'' = pp'$, so

$$\begin{aligned} ypp' + 1 &= p^2 \\ p' &= \frac{p^2 - 1}{p}\frac{1}{y} \end{aligned}$$

If $\frac{p^2-1}{p} = 0$, then $p^2 = 1$, so $p = \pm 1$, $y = C \pm x$. Otherwise there is integral

$$\begin{aligned} G &:= \int \frac{pdp}{p^2 - 1} - \int \frac{dy}{y} \\ &= \frac{1}{2}\ln(p^2 - 1) - \ln(y) + C = 0 \end{aligned}$$

Hence

$$\begin{aligned} \ln(p^2 - 1) &= C + 2\ln(y) \\ p^2 - 1 &= Cy^2 \\ y' = p &= \pm\sqrt{1 + Cy^2}. \end{aligned}$$

If $\pm\sqrt{1 + Cy^2} = 0$, then $y = \pm 1/\sqrt{-C}$. Otherwise there is integral

$$\begin{aligned} H &:= \int \frac{dy}{\pm\sqrt{1 + Cy^2}} - \int dx \\ &= \pm\frac{\sinh^{-1}(\sqrt{C}y)}{\sqrt{C}} - x + D = 0. \end{aligned}$$

Hence

$$y = \pm\frac{\sinh(\sqrt{C}x + \sqrt{C}D)}{\sqrt{C}} = \pm\frac{\sinh(ax + b)}{a}.$$

Problem 446.

$$(y' + 2y)y'' = y'^2$$

Let $p(y) = y'$. Then $y'' = pp'$, so

$$(p + 2y)pp' = p^2$$

(lose case $p \equiv 0$, where $y \equiv \text{const}$)

$$p' = \frac{p}{p + 2y}$$

Let $p = qy$. Then

$$\begin{aligned} yq' + q &= \frac{q}{q + 2} \\ q' &= \frac{-q(q + 1)}{q + 2}\frac{1}{y} \end{aligned}$$

If $\frac{-q(q+1)}{q+2} = 0$, then $q \in \{0; -1\}$, $p = 0$ or $p = -y$, $y \equiv \text{const}$ or $y = Ce^{-x}$. Otherwise there is integral

$$\begin{aligned} G &:= \int -\frac{q+2}{q(q+1)} dq - \int \frac{dy}{y} \\ &= \int \left(\frac{1}{q+1} - \frac{2}{q} \right) dq - \ln(y) \\ &= \ln(q+1) - 2\ln(q) - \ln(y) + C = 0 \end{aligned}$$

Hence

$$\begin{gathered} \ln\left(\frac{q+1}{q^2}\right) = \ln(Cy) \\ \frac{q+1}{q^2} = Cy \\ q = \frac{1 \pm \sqrt{1+4Cy}}{2Cy} \\ y' = p = qy = \frac{1 \pm \sqrt{1+4Cy}}{2C} \end{gathered}$$

So there is integral

$$\begin{aligned} H &:= \int \frac{2Cy}{1 \pm \sqrt{1+4Cy}} dy - \int dx \\ &= \pm\sqrt{1+4Cy} - \ln(1 \pm \sqrt{1+4Cy}) - x + D = 0 \end{aligned}$$
