Homework of 09.28
Differential equations and dynamic systems.
Solutions.

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Problem 156.

$$\begin{aligned} xy' - 2x^2\sqrt{y} &= 4y \\ xy' &= 4y + 2x^2y^{1/2} \\ y' &= \frac{4}{x}y + 2xy^{1/2} \end{aligned}$$

Obtained Bernoulli equation for degree $1/2$. Substitute $v = y^{1-1/2} = y^{1/2}$. Hence

$$\begin{aligned} 2v' &= \frac{4}{x}v + 2x \\ v' - \frac{2}{x}v &= x \end{aligned}$$

Consider map $\varphi(x) := e^{\int_1^x \frac{-2}{t}dt} = e^{-2\ln(x)} = x^{-2}$. Then $\varphi' = \frac{-2}{x}\varphi$. So

$$\begin{aligned} \varphi v' - \frac{2}{x}\varphi v &= x\varphi \\ \varphi v' + \varphi' v &= x\varphi \\ (\varphi v)' &= x\varphi \\ \varphi v = \int t\varphi(t)dt &= \int t^{-1}dt = \ln(x) + C \\ v &= x^2\ln(x) + Cx^2 \\ y = v^2 &= (x^2\ln(x) + Cx^2)^2. \end{aligned}$$

Problem 166.

$$\int_0^x (x-t)y(t)dt = 2x + \int_0^x y(t)dt.$$

Consider map $Y(x) := \int_0^x y(t)dt$. Then

$$\int_0^x (x-t)dY(t) = 2x + Y(x)$$
$$(x-t)Y(t)|_0^x - \int_0^x Y(t)d(x-t) = 2x + Y(x)$$
$$(x-x)Y(x) - (x-0)Y(0) + \int_0^x Y(t)dt = 2x + Y(x)$$
$$0 - 0 + \int_0^x Y(t)dt = 2x + Y(x)$$
$$\int_0^x Y(t)dt = 2x + Y(x)$$

Consider map $z(x) := \int_0^x Y(t)dt$. Then

$$z = 2x + z'$$
$$z' - z = -2x$$

Consider map $\varphi(x) := e^{\int_0^x -1dt} = e^{-x}$. Then $\varphi' = -\varphi$. So

$$\varphi z' - \varphi z = -2x\varphi$$
$$\varphi z' + \varphi' z = -2x\varphi$$
$$(\varphi z)' = -2x\varphi$$
$$\varphi z = \int -2t\varphi(t)dt = \int -2te^{-t}dt = \int 2tde^{-t} = 2xe^{-x} - \int 2e^{-t} = 2xe^{-x} + 2e^{-x} + C$$
$$z = 2x + 2 + Ce^x$$
$$y = z'' = Ce^x$$

The only remaining conditions are $z(0) = 0$ and $z'(0) = 0$. From the first condition we have that

$$0 = z(0) = 2 \cdot 0 + 2 + Ce^0 = 2 + C.$$

So $C = -2$. Considering the second condition we have that

$$z'(0) = 2 - 2e^0 = 2 - 2 = 0.$$

So $2 + 2x - 2e^x$ is the only possible z. Hence $y(x) = z''(x) = -2e^x$ is the only solution of the equation.

Problem 168.

$$3y' + y^2 + \frac{2}{x^2} = 0$$
$$3y' = -y^2 - \frac{2}{x^2}$$
$$y' = -\frac{1}{3}y^2 - \frac{2}{3x^2}$$

Obtained Riccati equation. Consider $\alpha(x) := \frac{1}{x}$. Then

$$\alpha' = -\frac{1}{x^2} = -\frac{1}{3x^2} - \frac{2}{3x^2} = -\frac{1}{3}y^2 - \frac{2}{3x^2}.$$

Hence α is one of solutions. Substitute $y = z + \alpha$. Then

$$z' = (y-\alpha)' = -\frac{1}{3}(y^2 - \alpha^2) = -\frac{1}{3}(y-\alpha)(y+\alpha) = -\frac{1}{3}z(z+2\alpha) = -\frac{2\alpha}{3}z - \frac{1}{3}z^2.$$

Obtained Bernoulli equation for degree 2. Substitute $z = v^{1-2} = v^{-1}$. Then

$$\begin{gathered}
-v' = -\frac{2\alpha}{3}v - \frac{1}{3} \\
v' - \frac{2\alpha}{3}v = \frac{1}{3}.
\end{gathered}$$

Consider map $\varphi(x) := e^{\int_1^x \frac{-2\alpha(t)}{3}dt} = e^{\int_1^x \frac{-2dt}{3t}} = e^{-\frac{2}{3}\ln(x)} = x^{-2/3}$. Then $\varphi' = -\frac{2\alpha}{3}\varphi$. So

$$\begin{gathered}
\varphi v' - \frac{2\alpha}{3}\varphi v = \frac{\varphi}{3} \\
\varphi v' + \varphi' v = \frac{\varphi}{3} \\
(\varphi v)' = \frac{\varphi}{3} \\
\varphi v = \int \frac{\varphi(t)}{3}dt = \int \frac{t^{-2/3}}{3}dt = x^{1/3} + C \\
v = x + Cx^{2/3} \\
z = \frac{1}{v} = \frac{1}{x + Cx^{2/3}} \\
y = z + \alpha = \frac{1}{x} \cdot \frac{2x^{1/3} + C}{x^{1/3} + C}
\end{gathered}$$

Problem 104.

$$\begin{gathered}
2x^3y' = y(2x^2 - y^2) \\
2x^3y' = 2x^2y - y^3 \\
y' = \frac{1}{x}y - \frac{1}{2x^3}y^3
\end{gathered}$$

Obtained Bernoulli equation for degree 3. Substitute $v = y^{1-3} = y^{-2}$. Then

$$\begin{gathered}
\frac{v'}{-2} = \frac{1}{x}v - \frac{1}{2x^3} \\
v' = -\frac{2}{x}v + \frac{1}{x^3} \\
v' + \frac{2}{x}v = \frac{1}{x^3}
\end{gathered}$$

Consider map $\varphi(x) := e^{\int_1^x \frac{2dt}{t}} = e^{2\ln(x)} = x^2$. Then $\varphi' = \frac{2}{x}\varphi$. So

$$\varphi v' + \frac{2}{x}\varphi v = \frac{\varphi}{x^3}$$
$$\varphi v' + \varphi' v = \frac{\varphi}{x^3}$$
$$(\varphi v)' = \frac{\varphi}{x^3}$$
$$\varphi v = \int \frac{\varphi(t)}{t^3} dt = \int \frac{dt}{t} = \ln(x) + C$$
$$v = \frac{\ln(x) + C}{x^2}$$
$$y = \frac{1}{\sqrt{v}} = \frac{x}{\sqrt{\ln(x) + C}}.$$
