Homework of 09.13
Differential geometry

Gleb Minaev @ 204 (20.Б04-мкн)

Problem 16. The problem is a particular case of the next problem.

Problem 17. Let $f : D^n \to Y$ be a continuous map and $F : S^{n-1} \times [0; 1] \to Y$ be a homotopy such that $F_0 = f|_{S^{n-1}}$. Than let's construct function

$$G : D^n \times [0; 1] \to Y, (x, t) \mapsto \begin{cases} f((1+t)x) & \text{if } |x| \leqslant \frac{1}{1+t}, \\ F(x/|x|, \frac{(1+t)|x|-1}{t}) & \text{if } |x| \geqslant \frac{1}{1+t}. \end{cases}$$

It's defined correctly, i.e. for every point x that $|x| = \frac{1}{1+t}$

$$f((1+t)x) = F(x/|x|, \frac{(1+t)|x|-1}{t}),$$

because $\frac{(1+t)|x|-1}{t} = 0$ and $(1+t)x = x/|x| \in S^{n-1}$, so

$$F(x/|x|, \frac{(1+t)|x|-1}{t}) = F_0(x/|x|) = F_0((1+t)|x|) = f|_{S^{n-1}}((1+t)|x|).$$

Also it's obvious that G is continuous (it's linear expansion of embedding of D^n under f on embedding of $S^{n-1} \times [0; 1]$ under F).

Hence G is homotopy such that $G_t|_{S^{n-1}} = F_t$ and $G_0 = f$. It means that (D^n, S^{n-1}) is a Borsuk pair.

Problem 18. Let's consider pair $(X, A) := ([0; 2], [0; 1) \cup (1; 2])$. Let's also consider a continuous map $f := \mathrm{Id}_{[0;2]}$ and a map

$$H : A \times [0; 1] \to X, (x, t) \mapsto \begin{cases} (1-t)x + t(1-x) & \text{if } x \in [0; 1), \\ (1-t)x + t(3-x) & \text{if } x \in (1; 2]. \end{cases}$$

Saying simply, H (linearly) turns round intervals $[0; 1)$ and $(1; 2]$. So obviously H is continuous (hence homotopy) and $H_0 = f|_A$.

Let's show that H cannot be raised to X. Assume that G is a raising of H to X. Then the only difference between G and F is determination of path of point 1. Also it means that G_1 is continuous map $[0; 2] \to [0; 2]$. Let p be $G_1(1)$.

If $p = 0$ then let's take 1/2-neighbourhood U_p of p. It is clear that $U_p = [0; 1/2)$. Preimage of U_p under G_1 is interval $(1/2; 1]$ which is not open. Hence G_1 is not continuous. The same goes for the case $p = 2$. Then $p \in (0; 2)$. So there is neighbourhood U_p of p that does not intersect with some neighbourhoods of 0 and 2. Hence preimage of U_p contains 1 and does not intersect some deleted neighbourhood of 1. That means $G_1^{-1}(U_p)$ is not open. Hence G_1 is not continuous after all.

Problem 19. Consider maps (inclusions)

$$f : X \to X \cup A \times I, x \mapsto x, \qquad F : A \times I \to X \cup A \times I, (a,t) \mapsto (a,t).$$

Obviously $f|_A = F|_A$. So there exists a homotopy $G : X \times I \to X \cup A \times I$ that is a raising of F and f. That means that G is continuous map from $X \times I$ to its subset $X \cup A \times I$ that is identity map on the subset. Hence the subset $X \cup A \times I$ is a retract.

Problem 20. Let $f : X \to Y$ and $H : A \times I \to Y$ be continuous maps that $f|_A = H|_{A\times\{0\}}$. Then consider

$$\widetilde{H} : X \cup A \times I \to Y, x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ H(x) & \text{if } x \in A \times I. \end{cases}$$

Because of the condition $f|_A = H|_{A\times\{0\}}$, $\widetilde{H}$ is defined correctly.

Let's prove that X and $A \times I$ are closed sets in $X \cup A \times I$.

$$(X \cup A \times I) \setminus X = A \times (I \setminus \{0\})$$

which is open in $A \times I$ (and does not intersect X), hence in $X \cup A \times I$ too. So X is closed.

$$A \times \{1\} = (X \cup A \times I) \cap (X \times \{1\})$$

is closed in $X \times \{1\}$, because $X \cup A \times I$ is closed in $X \times I$, because $X \cup A \times I$ is a retract of $X \times I$ (and $X \times I$ is Hausdorff). Hence A is closed in X. Then

$$(X \cup A \times I) \setminus A \times I = X \setminus A$$

which is open in X, because A is closed in X. Hence $(X \cup A \times I) \setminus A \times I$ is open, so $A \times I$ is closed.

So $\{X; A \times I\}$ is finite closed cover of $X \cup A \times I$ and $\widetilde{H}$ is continuous on every set of the cover. Hence $\widetilde{H}$ is continuous on whole space. Let $F : X \times I \to X \cup A \times I$ be a retraction. So $\widetilde{H} \circ F$ is a homotopy from X to Y that is a raising of $\widetilde{H}$ (hence of f and H too). It means (X, A) is a Borsuk pair.
