Homework for 12.16
Algebra

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Contents

Lemma 1. *If $A^p = I$ for some linear automorphism A and positive integer p, then A is diagonalizable.*

Proof. Let SJS^{-1} be a diagonalization, i.e. J is a Jordan matrix, and S is a change of basis. Then

$$I = S^{-1}IS = S^{-1}A^pS = S^{-1}(SJS^{-1})^pS = S^{-1}(SJ^pS^{-1})S = J^p.$$

Let J contain Jordan blocks of size > 1. Any Jordan block can be represented as $\lambda I + T$ where λ is any scalar and

$$T = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Hence

$$(\lambda I + T)^p = \sum_{t=0}^{p} \binom{p}{t} \lambda^{p-t} T^t = \begin{pmatrix} \lambda^p & \lambda^{p-1} & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \lambda^{p-1} \\ & & & \lambda^p \end{pmatrix}$$

that means J^p contains this matrix (which size is > 1) on its diagonal, thus $J^p \neq I$. Hence J consist only of Jordan blocks 1×1, that means A is diagonalizable. □

Corollary 1.1. *If ρ is representation of finite group G, then for any $g \in G$ operator $\rho(g)$ is representable.*

Proof. $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e) = I$. □

Definition 1. Let's denote for any variables $t_1, \dots, t_n$

- $$p_k(\bar{t}) := p_k(t_1, \dots, t_n) := \sum_{i=1}^{n} t_i^k,$$

- $$h_k(\bar{t}) := h_k(t_1, \ldots, t_n) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n} t_{i_1} \cdots t_{i_k},$$
- $$e_k(\bar{t}) := h_k(t_1, \ldots, t_n) := \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} t_{i_1} \cdots t_{i_k}.$$

Lemma 2. *[Newton's identity]*

1.
$$kh_k = \sum_{i=1}^{k} h_{k-i} p_i$$

2.
$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

Proof.

1. Obviously
$$h_m = \sum_{d_1 + \cdots + d_n = m} \lambda_1^{d_1} \cdots \lambda_n^{d_n}.$$
Thus let's consider any monomial $\lambda_1^{d_1} \cdots \lambda_n^{d_n}$, where $d_1 + \cdots + d_n = k$. It's counted on left side of the equation with coefficient k. Then on the right side it can be contained by some addendum $h_{k-t}p_t$ iff is represented as
$$\lambda_1^{d_1} \cdots \lambda_{m-1}^{d_{m-1}} \lambda_m^{d_m - t} \lambda_{m+1}^{d_{m+1}} \lambda_n^{d_n} \ \cdot \ \lambda_m^t.$$
Obviously there are exactly $d_1 + \cdots + d_n = k$ such representations of the monomial. Hence it's counted on both sides with coefficient k.

2. Let's consider polynomial $\sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$ (of $\lambda_1, \ldots, \lambda_n$). Every addendum $e_{k-t}p_t$ consists only of monomials of kind $\lambda_{j_1} \ldots \lambda_{j_{k-t}} \lambda_i^t$. So let's consider some monomial $\lambda_{j_1} \ldots \lambda_{j_{k-t}} \lambda_i^t$, and WLOG let's assume that $i \notin \{j_1; \ldots; j_{k-t}\}$. Then it may be contained (with nonzero coefficient) in only two addendums: as $(-1)^{t-1} \cdot \lambda_{j_1} \ldots \lambda_{j_{k-t}} \cdot \lambda_i^t$ in $(-1)^{t-1} e_{k-t} p_t$ and as $(-1)^{t-2} \cdot \lambda_{j_1} \ldots \lambda_{j_{k-t}} \lambda_i \cdot \lambda_i^{t-1}$ in $(-1)^{t-2} e_{k-t+1} p_{t-1}$. But the second expression is one of the addendums iff $t \geqslant 2$, and in that case total coefficient before it is $0 = (-1)^{t-1} + (-1)^{t-2}$. Otherwise $t = 1$, and it's counted with coefficient $(-1)^{t-1} = 1$. Hence
$$\sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i = \sum_{1 \leqslant j_1 < \cdots < j_{k-1} \leqslant n} \ \sum_{i \notin \{j_1; \ldots; j_{k-1}\}} \lambda_i \lambda_{j_1} \cdots \lambda_{j_{k-1}} = ke_k.$$

$\square$

Lemma 3. *In terms of generation functions*
$$\sum_{k=0}^{\infty} h_k t^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k} t^k \right), \qquad \sum_{k=0}^{\infty} e_k t^k = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k}{k} t^k \right).$$

Proof. Let

$$H(t) := \sum_{k=0}^{\infty} h_k t^k, \qquad E(t) := \sum_{k=0}^{\infty} e_k t^k, \qquad P(t) := \sum_{k=0}^{\infty} p_k t^k, \qquad Q(t) := \sum_{k=1}^{\infty} \frac{p_k}{k} t^k,$$

We need to show that

$$H = \exp(Q) \qquad \text{and} \qquad E = \exp(-Q(-t)).$$

By previous lemma we have that

$$H' t = \sum_{k=0}^{\infty} k h_k t^k = \sum_{k=0}^{\infty} t_k \sum_{l=1}^{k} e_{k-l} p_l = H(P - p_0),$$
$$E' t = \sum_{k=0}^{\infty} k e_k t^k = \sum_{k=0}^{\infty} t_k \sum_{l=1}^{k} (-1)^{l+1} e_{k-l} p_l = E(p_0 - P(-t)).$$

Hence

$$(\ln(H))' = \frac{H'}{H} = \frac{P - p_0}{t}, \qquad (\ln(E))' = \frac{E'}{E} = \frac{p_0 - P(-t)}{t} = \frac{P(-t) - p_0}{-t}.$$

But

$$Q' = \sum_{k=1}^{\infty} p_k t^{k-1} = \frac{\sum_{k=1}^{\infty} p_k t^k}{t} = \frac{P - p_0}{t}.$$

Hence

$$\ln(H) = Q + C_2, \qquad \ln(E) = -Q(-t) + C_1.$$

Checking the equations for $t = 0$, get $C_1 = C_2 = 0$. Then the needed equations are obvious. $\square$

Corollary 3.1.

$$h_k = \sum_{m_1 + 2m_2 + \cdots + km_k = k} \prod_{i=1}^{k} \frac{(p_i)^{m_i}}{m_i! i^{m_i}}, \qquad e_k = (-1)^k \sum_{m_1 + 2m_2 + \cdots + km_k = k} \prod_{i=1}^{k} \frac{(-p_i)^{m_i}}{m_i! i^{m_i}}.$$

Problem 1. Let n be dimension of representing vector space, $\{v_i\}$ be eigenbasis of $V(g)$, and $\{\lambda_i\}$ be its set of associated eigenvalues. Then $\{v_i v_j\}_{i \leqslant j}$ is a basis of algebra $\mathrm{Sym}^2(V)$ and eigenbasis of operator $\mathrm{Sym}^2(V)(g)$ (and $\{\lambda_i \lambda_j\}_{i \leqslant j}$ is its associated set of eigenvalues). Hence

$$\begin{aligned}
\chi_{\mathrm{Sym}^2(V)}(g) = \mathrm{tr}(\mathrm{Sym}^2(V)(g)) && = \sum_{i \leqslant j} \lambda_i \lambda_j \\
= \frac{1}{2}\left(\left(2\sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2\right) + \sum_i \lambda_i^2\right) && = \frac{1}{2}\left(\sum_{i,j} \lambda_i \lambda_j + \sum_i \lambda_i^2\right) \\
= \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2\right) && = \frac{1}{2}(\mathrm{tr}(V(g))^2 + \mathrm{tr}(V(g)^2)) \\
= \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)).
\end{aligned}$$

Problem 2. Let $\{v_i\}$ be eigenbasis of $V(g)$, and $\{\lambda_i\}$ be its set of associated eigenvalues. Then set of pairs

$$(\lambda_{i_1}\cdots\lambda_{i_k}; v_{i_1}\cdots v_{i_k})$$

for any nondecreasing sequence $(i_1;\ldots;i_k)$ of indices (i.e. integers from $\{1;\ldots;n\}$) is set of eigenpairs of $\mathrm{Sym}^k(V)$, and set of pairs

$$(\lambda_{i_1}\cdots\lambda_{i_k}; v_{i_1}\wedge\cdots\wedge v_{i_k})$$

for any increasing sequence $(i_1;\ldots;i_k)$ of indices is set of eigenpairs of $\wedge^k(V)$. Thus

$$\chi_{\mathrm{Sym}^k}(g) = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n}\lambda_{i_1}\cdots\lambda_{i_k} = h_k(\overline{\lambda}), \qquad \chi_{\wedge^k}(g) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\lambda_{i_1}\cdots\lambda_{i_k} = e_k(\overline{\lambda}).$$

But we know that $\{(\lambda_i^t; v_i)\}$ is set of eigenpairs of $V(g)^t = V(g^t)$, thus

$$\chi_V(g^t) = \sum_{i=1}^n \lambda_i^t = p_t(\overline{\lambda}).$$

So the problem now is to express h_k and e_k with p_t's. Then by corollary 3.1 we have that

$$\chi_{\mathrm{Sym}^k}(g) = \sum_{m_1+2m_2+\cdots+km_k=k}\prod_{i=1}^k\frac{(\chi_V(g^i))^{m_i}}{m_i!i^{m_i}}, \quad \chi_{\wedge^k}(g) = (-1)^k\sum_{m_1+2m_2+\cdots+km_k=k}\prod_{i=1}^k\frac{(-\chi_V(g^i))^{m_i}}{m_i!i^{m_i}}.$$

Problem 3. Let $\{v_i\}$ be eigenbasis of $V(g)$, and $\{\lambda_i\}$ be its set of associated eigenvalues. Then characteristic polynomial of $V(g)$ is

$$\prod_{i=1}^n(\lambda_i - x) = (-1)^n\sum_{k=0}^n x^{n-k}(-1)^k e_k(\overline{\lambda}) = (-1)^n\sum_{k=0}^n x^{n-k}\sum_{m_1+2m_2+\cdots+km_k=k}\prod_{i=1}^k\frac{(-\chi_V(g^i))^{m_i}}{m_i!i^{m_i}}.$$

So

$$\begin{aligned}
e_0 &= 1\\
e_1 &= \sum_i \lambda_i = p_1 = \chi_V(g)\\
e_2 &= \sum_{i<j}\lambda_i\lambda_j = \frac{1}{2}\left(\sum_{i,j}\lambda_i\lambda_j - \sum_i\lambda_i^2\right) = \frac{1}{2}(p_1^2 - p_2) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))\\
e_3 &= \sum_{i<j<k}\lambda_i\lambda_j\lambda_k\\
&= \frac{1}{6}p_1^3 - \frac{1}{2}\sum_{i,j}x_i^2x_j - \frac{1}{6}\sum_i\lambda_i^3 = \frac{1}{6}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{3}\sum_i\lambda_i^3\\
&= \frac{1}{6}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{3}p_3\\
e_4 &= \sum_{i<j<k<l}\lambda_i\lambda_j\lambda_k\lambda_l\\
&= \frac{1}{24}p_1^4 - \frac{1}{2}\sum_{i,j<k}\lambda_i^2\lambda_j\lambda_k - \frac{1}{4}\sum_{i<j}\lambda_i^2\lambda_j^2 - \frac{1}{6}\sum_{i,j}\lambda_i^3\lambda_j - \frac{1}{24}\sum_i\lambda_i^4\\
&= \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{4}\sum_{i<j}\lambda_i^2\lambda_j^2 + \frac{1}{3}\sum_{i,j}\lambda_i^3\lambda_j + \frac{5}{24}\sum_i\lambda_i^4
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{8}p_2^2 + \frac{1}{3}\sum_{i,j}\lambda_i^3\lambda_j + \frac{1}{12}\sum_i \lambda_i^4 \\
&= \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{8}p_2^2 + \frac{1}{3}p_1p_3 - \frac{1}{4}\sum_i \lambda_i^4 \\
&= \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{8}p_2^2 + \frac{1}{3}p_1p_3 - \frac{1}{4}p_4
\end{aligned}
$$

Thus the characteristic polynomial is

ord(g)\n	2
2\2	$\frac{1}{2}\chi_V(g) - 1 - \chi_V(g)x + x^2$
2\3	$\frac{1}{6}\chi_V(g)^3 - \frac{7}{6}\chi_V(g) - \left(\frac{1}{2}\chi_V(g) - \frac{3}{2}\right)x + \chi_V(g)x^2 - x^3$
2\4	$\frac{1}{24}p_1^4 - \frac{2}{3}p_1^2 - \frac{1}{2} - \left(\frac{1}{6}\chi_V(g)^3 - \frac{5}{3}\chi_V(g)\right)x - \left(\frac{1}{2}\chi_V(g) - 2\right)x^2 - \chi_V(g)x^3 + x^4$
3\2	$\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2) - \chi_V(g)x + x^2$
3\3	$\frac{1}{6}\chi_V(g)^3 - \frac{1}{2}\chi_V(g)\chi_V(g^2) + 1 - \left(\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2)\right)x + \chi_V(g)x^2 - x^3$
3\4	$\frac{1}{24}\chi_V(g)^4 - \frac{1}{4}\chi_V(g)^2\chi_V(g^2) + \frac{1}{8}\chi_V(g^2)^2 + \frac{4}{3}\chi_V(g) - \frac{1}{4}\chi_V(g)$ $- \left(\frac{1}{6}\chi_V(g)^3 - \frac{1}{2}\chi_V(g)\chi_V(g^2) + \frac{4}{3}\right)x + \left(\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2)\right)x^2 - \chi_V(g)x^3 + x^4$
4\2	$\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2) - \chi_V(g)x + x^2$
3\3	$\frac{1}{6}\chi_V(g)^3 - \frac{1}{2}\chi_V(g)\chi_V(g^2) + \frac{1}{3}\chi_V(g^3) - \left(\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2)\right)x + \chi_V(g)x^2 - x^3$
3\4	$\frac{1}{24}\chi_V(g)^4 - \frac{1}{4}\chi_V(g)^2\chi_V(g^2) + \frac{1}{8}\chi_V(g^2)^2 + \frac{1}{3}\chi_V(g)\chi_V(g^3) - 1$ $- \left(\frac{1}{6}\chi_V(g)^3 - \frac{1}{2}\chi_V(g)\chi_V(g^2) + \frac{1}{3}\chi_V(g^3)\right)x + \left(\frac{1}{2}\chi_V(g) - \frac{1}{2}\chi_V(g^2)\right)x^2 - \chi_V(g)x^3 + x^4$