Homework of 10.12
Differential equations and dynamic systems.
Solutions.

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Problem 214.

$$(x^2 - \sin(y)^2)dx + x\sin(2y)dy = 0$$

Let $f = x^2 - \sin(y)^2$, $g = x\sin(2y)$. Try to find function $\mu(x)$ such that

$$\frac{\partial(f\mu)}{\partial y} = \frac{\partial(g\mu)}{\partial x}$$
$$\frac{\partial f}{\partial y}\mu = \frac{\partial g}{\partial x}\mu + \frac{\partial \mu}{\partial x}g$$
$$\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\mu = g\mu'$$
$$\mu = Ce^{\int_{x_0}^{x} \frac{\frac{\partial f}{\partial y}(t) - \frac{\partial g}{\partial x}(t)}{g(t)}dt} = Ce^{\int_{x_0}^{x} \frac{-\sin(2y)-\sin(2y)}{t\sin(2y)}dt} = Ce^{\int_1^x \frac{-2}{t}dt} = Cx^{-2} = x^{-2}.$$

Hence

$$\begin{aligned}
U(x,y) &:= \int_{x_0}^{x}(f\mu)(t,y_0)dt + \int_{y_0}^{y}(g\mu)(x,t)dt\\
&= \int_{x_0}^{x}\left(1 - \frac{\sin(y_0)^2}{t^2}\right)dt + \int_{y_0}^{y}\frac{\sin(2t)}{x}dt\\
&= \left(t + \frac{\sin(y_0)^2}{t}\right)\Bigg|_{x_0}^{x} + \frac{-\cos(2t)}{2x}\Bigg|_{y_0}^{y}\\
&= x - x_0 - \frac{1-\cos(2y_0)}{2x_0} + \frac{1-\cos(2y_0)}{2x} + \frac{-\cos(2y)}{2x} - \frac{-\cos(2y_0)}{2x}\\
&= x - x_0 - \frac{1-\cos(2y_0)}{2x_0} + \frac{1}{2x} + \frac{-\cos(2y)}{2x}\\
&= x - x_0 - \frac{\sin(y_0)^2}{x_0} + \frac{\sin(y)^2}{x}\\
&= x + \frac{\sin(y)^2}{x}
\end{aligned}$$

is integral of the equation. Hence

$$x + \frac{\sin(y)^2}{x} = C$$
$$\sin(y)^2 = Cx - x^2$$
$$y = \sin^{-1}(\sqrt{Cx - x^2}).$$

Problem 215.

$$x(\ln(y)+2\ln(x)-1)dy = 2ydx$$
$$2ydx + x(1-\ln(y)-2\ln(x))dy = 0$$

Let $f = 2y$, $g = x(1-\ln(y)-2\ln(x))$. Try to find function $\mu(x,y)$ such that

$$\frac{\partial(f\mu)}{\partial y} = \frac{\partial(g\mu)}{\partial x}$$
$$\frac{\partial f}{\partial y}\mu + \frac{\partial \mu}{\partial y} f = \frac{\partial g}{\partial x}\mu + \frac{\partial \mu}{\partial x} g$$
$$\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\mu + \frac{\partial \mu}{\partial y} f - \frac{\partial \mu}{\partial x} g = 0$$

Let's consider μ as composition $\sigma \circ \eta$ for some $\sigma(t)$, $\eta(x,y)$. So

$$\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\sigma(\eta) + \left(\frac{\partial \eta}{\partial y} f - \frac{\partial \eta}{\partial x} g\right)\sigma'(\eta) = 0$$
$$(\ln(\sigma))'(\eta) = \frac{\sigma'(\eta)}{\sigma(\eta)} = \frac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}}{\frac{\partial \eta}{\partial x} g - \frac{\partial \eta}{\partial y} f}$$

In terms of current f and g

$$(\ln(\sigma))'(\eta) = \frac{3+2\ln(x)+\ln(y)}{\frac{\partial \eta}{\partial x} x(1-2\ln(x)-\ln(y)) - \frac{\partial \eta}{\partial y} 2y}$$

Let $\eta = xy^2$. So $\frac{\partial \eta}{\partial x} = \eta/x$, $\frac{\partial \eta}{\partial y} = 2\eta/y$,

$$(\ln(\sigma))'(\eta) = \frac{3+2\ln(x)+\ln(y)}{\eta(1-2\ln(x)-\ln(y)) - 4\eta}$$
$$= \frac{3+2\ln(x)+\ln(y)}{\eta(-3-2\ln(x)-\ln(y))}$$
$$= \frac{-1}{\eta}$$

$$\ln(\mu) = \ln(\sigma(\eta)) = -\ln(\eta)$$
$$\mu = \frac{1}{xy^2}$$

Hence

$$
\begin{aligned}
U(x,y) &:= \int_{x_0}^{x} (f\mu)(t,y_0)dt + \int_{y_0}^{y} (g\mu)(x,t)dt \\
&= \int_{x_0}^{x} \frac{2}{ty_0}dt + \int_{y_0}^{y} \frac{1-\ln(t)-2\ln(x)}{t^2}dt \\
&= \frac{2}{y_0}\ln(t)\Big|_{x_0}^{x} + \frac{2\ln(x)+\ln(y)}{y}\Big|_{y_0}^{y} \\
&= -\frac{2\ln(x_0)}{y_0} + \frac{2\ln(x)+\ln(y)-\ln(y_0)}{y} \\
&= \frac{2\ln(x)+\ln(y)}{y}
\end{aligned}
$$

is integral of the equation. Hence

$$
\begin{gathered}
\frac{2\ln(x)+\ln(y)}{y} = C \\
Cy = \ln(x^2 y)
\end{gathered}
$$

which solutions is not elemantary functions.

Problem 242.

$$
\begin{gathered}
8(y')^3 = 27y \\
2y' = 3\sqrt[3]{y}
\end{gathered}
$$

(dividing by $\sqrt[3]{y}$ lose solution $y \equiv 0$)

$$
\begin{gathered}
\frac{2}{3}\frac{y'}{y^{1/3}} = 1 \\
(y^{2/3})' = 1 \\
y^{2/3} = C + x \\
y = \pm(C+x)^{3/2}.
\end{gathered}
$$

It's easy to see that any point $(x_0; y_0)$ where $y_0 \neq 0$ is a point of uniqness, because it's not solution of $y \equiv 0$, so it can be a solution of only $y = \pm(C+x)^{3/2}$, then $C = y_0^{2/3} - x_0$ and sign in the formula is determined by sign of y_0, so the solution is realy unique. But any point $(x_0; 0)$ is not a point of uniqness, because there are three different solutions $y \equiv 0$, $y = (x-x_0)^{3/2}$ and $y = -(x-x_0)^{3/2}$. Hence $y \equiv 0$ is only singular solution.

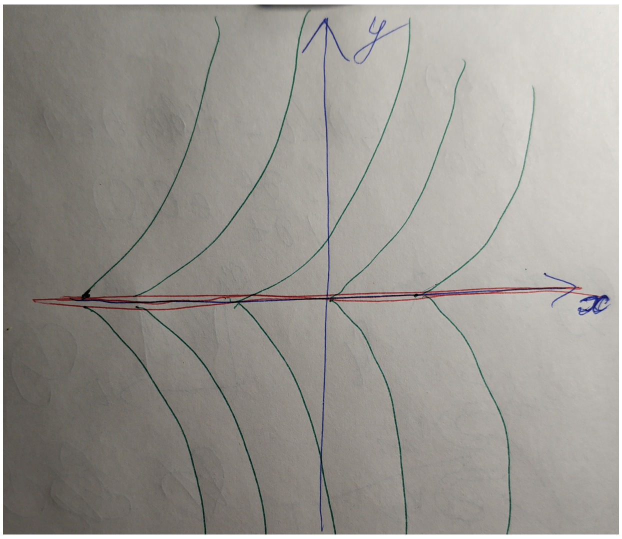

Figure 1: Red line — singular solution, green lines — samples of other solution.