Listochek 9.
Calculus.
Solutions.

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Contents

Basic problems

Problem 1. TBA

Problem 2. Let
$$\rho : \mathbb{R}^2 \to \mathbb{R}, p \mapsto \frac{|p|^2}{\pi}.$$
Then for any point $p \in \mathbb{R}^2$ and radius $R \geqslant 0$
$$\begin{aligned}
\iint\limits_{B_R(p)} \rho(p + (x; y)) dx dy &= \int\limits_0^R \int\limits_0^{2\pi} \rho(p + r(\cos(\varphi); \sin(\varphi)))r \, d\varphi \, dr \\
&= \int\limits_0^R \int\limits_0^{2\pi} \frac{(p_x + r\cos(\varphi))^2 + (p_y + r\sin(\varphi))^2}{\pi} r \, d\varphi \, dr \\
&= \int\limits_0^R \int\limits_0^{2\pi} r \frac{p_x^2 + p_y^2 + r^2 + 2r(p_x \cos(\varphi) + p_y \sin(\varphi))}{\pi} \, d\varphi \, dr \\
&= \int\limits_0^R 2r(p_x^2 + p_y^2 + r^2) \, dr
\end{aligned}$$

$$
\begin{aligned}
&= \left(r^2|p|^2 + \frac{r^4}{2} \right)\Bigg|_0^R \\
&= R^2|p|^2 + \frac{R^4}{2} \\
&= \mu(B_R(p))
\end{aligned}
$$

Thus measure $A \mapsto \int_A \rho(p)\lambda(dp)$ is defined in the same way as μ, so $\mu(A) = \int_A \rho(p)\lambda(dp)$. It already means that μ is absolutely continuous with respect to the Lebesgue measure λ: if $\lambda(A) = 0$, then $\mu(A) = \int_A \rho(p)dp = 0$. But we can show that explicitly.

Lemma 1. *Let $A \subseteq B_R(0)$ be a set with zero Lebesgue measure. Then $\mu(A) = 0$.*

Proof. Let $\varepsilon > 0$ and $\{B_i\}$ be a cover of A with balls such that $\sum_i \lambda(B_i) < \varepsilon$. If there are balls in the cover with center not in $B_R(0)$ we can change them in such way that their centers lie in $B_R(0)$, $\{B_i\}$ still is a cover of A, and $\sum_i \lambda(B_i)$ does not increase. Hence (if r_i and c_i are radius and center of B_i)

$$
\begin{aligned}
\sum_i \mu(B_i) &= \sum_i (|c_i|^2|r_i|^2 + |r_i|^4/2) \\
&\leqslant R^2 \left(\sum_i r_i^2 \right) + \left(\sum_i r_i^2 \right)^2 /2 \\
&\leqslant \left(\sum_i \lambda(B_i) \right) \left(\frac{R^2}{\pi} + \frac{1}{\pi^2} \left(\sum_i \lambda(B_i) \right) /2 \right) \\
&\leqslant \frac{\varepsilon}{\pi}(R^2 + \frac{\varepsilon}{2\pi}).
\end{aligned}
$$

Thus $\lambda(A) = 0$. $\square$

Corollary 1.1. *If A is a set with zero Lebesgue measure, then $\mu(A) = 0$.*

Proof.

$$\mu(A) = \lim_{R\to\infty} \mu(A \cap B_R(0)) = \lim_{R\to\infty} 0 = 0.$$

$\square$

Problem 3. We need to show that if $F \subseteq E$, then $|F|O_F(f) \leqslant |E|O_E(f)$, which is the same as

$$\int_F \left| f - \fint_F f \right| \leqslant \int_E \left| f - \fint_E f \right|.$$

WLOG $\fint_F f \leqslant \fint_E f$. Let

$$a := \fint_F f \qquad b := \fint_E f$$

$$F_1 := \{x \in F \mid x < a\} \qquad F_2 := \{x \in F \mid x \in [a;b]\} \qquad F_3 := \{x \in F \mid x > b\}$$

$$G_1 := \{x \in E \setminus F \mid x < b\} \qquad G_2 := \{x \in E \setminus F \mid x \geqslant b\}$$

$$m_1 := |F_1| \qquad m_2 := |F_2| \qquad m_3 := |F_3| \qquad n_1 := |G_1| \qquad n_1 := |G_2|$$

$$f_1 := \fint_{F_1} f \qquad f_2 := \fint_{F_2} f \qquad f_3 := \fint_{F_3} f \qquad g_1 := \fint_{G_1} f \qquad g_2 := \fint_{G_2} f.$$

So we have that

$$
\begin{gathered}
a(m_1+m_2+m_3) = f_1m_1+f_2m_2+f_3m_3 \\
b(m_1+m_2+m_3+n_1+n_2) = f_1m_1+f_2m_2+f_3m_3+g_1n_1+g_2n_2 \\
f_1 \leqslant a \leqslant f_2 \leqslant b \leqslant f_3 \\
g_1 \leqslant b \leqslant g_3 \\
m_1, m_2, m_3, n_1, n_2 \geqslant 0.
\end{gathered}
$$

And we need to show

$$m_1(a-f_1)+m_2(f_2-a)+m_3(f_3-a) \leqslant m_1(b-f_1)+m_2(b-f_2)+m_3(f_3-b)+n_1(b-g_1)+n_2(g_2-b).$$

The last is equivivalent to

$$m_1(a-b)+m_2(2f_2-a-b)+m_3(b-a) \leqslant n_1(b-g_1)+n_2(g_2-b).$$

Let's fix a, b, m_1, m_2, m_3, f_1, f_2, f_3, g_1 and g_2. Let $m = m_1+m_2+m_3$. Then the only condition that restricts g_1 and/or g_2 is

$$
\begin{gathered}
b(m+n_1+n_2) = am+g_1n_1+g_2n_2 \\
(b-a)m+(b-g_1)n_1 = (g_2-b)n_2.
\end{gathered}
$$

Note that $b-a \geqslant 0$, $(b-g_1) \geqslant 0$ and $g_2-b \geqslant 0$. Then let's decrease n_1 and n_2 in such way that $n_1 = 0$. n_2 will be $\geqslant 0$, because $(b-a)m \geqslant 0$. But right-hand side of

$$m_1(a-b)+m_2(2f_2-a-b)+m_3(b-a) \leqslant n_1(b-g_1)+n_2(g_2-b)$$

will decrease, so we need to prove stricter condition. Then we have that

$$(b-a)m = (g_2-b)n_2,$$

and need to prove that

$$m_1(a-b)+m_2(2f_2-a-b)+m_3(b-a) \leqslant n_2(g_2-b).$$

Obviously it means we need to show that

$$
\begin{gathered}
m_1(a-b)+m_2(2f_2-a-b)+m_3(b-a) \leqslant (b-a)(m_1+m_2+m_3) \\
m_2(2f_2-2b) \leqslant (b-a)2m_1 \\
(f_2-b)m_2 \leqslant (b-a)m_1,
\end{gathered}
$$

that is now obvious, because $f_2-b \leqslant 0$ and $b-a \geqslant 0$.

Problem 4. Set E of points x for which $\lim_{n\to\infty} f_n(x)$ is not defined or is not $f(x)$ has zero measure. Hence if we change all f_n and f such that $f_n|_E = f_E = 0$, then $f_n \to f$ on all X and all f_n and f won't change their class in L^2. So now we may assume that $f_n \to f$ on all X.

Lemma 2. *For any* $t > 0$

$$\mu(\{x \in X \mid |f| \geqslant t\}) \leqslant \frac{2}{1+t^3}.$$

Proof. Let for some $t > 0$ $E := \{x \in X \mid |f| \geqslant t\}$ and

$$\mu(E) > \frac{2}{1+t^3}.$$

Then there is such $\delta > 0$ that

$$\mu(E) \geqslant \frac{2}{1+t^3} + 2\delta.$$

By continuity of $1/(1+t^3)$ there is such $\varepsilon > 0$ that

$$\frac{2}{1+t^3} + \delta > \frac{2}{1+(t-\varepsilon)^3}.$$

By definition of convergence for any $x \in X$ there is such $n_x \in \mathbb{N}$ that for any $n \geqslant n_x$

$$|f_n(x) - f(x)| \leqslant \varepsilon.$$

Hence there are sets

$$E_n := \{x \in E \mid n_x \leqslant n\},$$

$E_{n+1} \supseteq E_n$ and $E = \bigcup_{n=0}^{\infty} E_n$. Hence there is such m that

$$\mu(E_m) \geqslant \frac{2}{1+t^3} + \delta.$$

Then for any $x \in E_m$ and any $n \geqslant m$ we have that $n \geqslant m \geqslant n_x$, so

$$|f_n(x) - f(x)| \leqslant \varepsilon.$$

But $f(x) \geqslant t$, so $f_n(x) \geqslant t - \varepsilon$. Hence

$$\{x \in X \mid |f_m(x)| \geqslant t - \varepsilon\} \supseteq E_m.$$

But then

$$\mu(\{x \in X \mid |f_m(x)| \geqslant t - \varepsilon\}) \geqslant \mu(E_m) \geqslant \frac{2}{1+t^3} + \delta > \frac{2}{1+(t-\varepsilon)^3}.$$

Contradiction. □

Lemma 3. *Let $g : X \to \mathbb{R}$ be such measurable map that for any $t > 0$*

$$\mu(\{x \in X \mid |g(x)| \geqslant t\}) \leqslant \frac{2}{1+t^3},$$

and E be a set of measure $\leqslant \frac{2}{a^{3/2}}$. Then

$$\int_E |g|^2 \leqslant \frac{6}{\sqrt{a}}$$

Proof. Let's quickly note that

$$\mu(\{x \in X \mid |g(x)| \geqslant t\}) \leqslant \frac{2}{t^3}.$$

Then

$$\mu(\{x \in X \mid |g(x)|^2 \geqslant t\}) = \mu(\{x \in X \mid |g(x)| \geqslant \sqrt{t}\}) \leqslant \frac{2}{1+t^{3/2}}.$$

Then

$$
\begin{aligned}
\int_E |g|^2 &= \int_0^{+\infty} \mu(\{x \in X \mid |g(x)|^2 \geqslant t\} \cap E) \\
&\leqslant \int_0^{+\infty} \min(\mu(E), \mu(\{x \in X \mid |g(x)|^2 \geqslant t\})) \\
&\leqslant \int_0^{+\infty} \min\left(\frac{2}{a^{3/2}}, \frac{2}{t^{3/2}}\right) \\
&= \int_0^{a} \frac{2}{a^{3/2}} + \int_a^{+\infty} \frac{2}{t^{3/2}} \\
&= \frac{2}{\sqrt{a}} + \frac{4}{\sqrt{a}} \\
&= \frac{6}{\sqrt{a}}
\end{aligned}
$$

□

Let $\varepsilon > 0$. Then $a := \frac{(4 \cdot 6 \cdot 2)^2}{\varepsilon^2}$. By definition of convergence for any $x \in X$ there is such $n_x \in \mathbb{N}$ that for any $n \geqslant n_x$

$$
|f_n(x) - f(x)| \leqslant \delta := \frac{\varepsilon^2}{2} / \left(1 - \frac{2}{a^{3/2}}\right).
$$

Then there are sets

$$
X_n := \{x \in X \mid n_x \leqslant n\},
$$

$X_{n+1} \supseteq X_n$ and $X = \bigcup_{n=0}^{\infty} X_n$. Hence there is such m that

$$
\mu(X_m) \geqslant 1 - \frac{2}{a^{3/2}}.
$$

Then for any $x \in X_m$ and any $n \geqslant m$ we have that $n \geqslant m \geqslant n_x$, so

$$
|f_n(x) - f(x)| \leqslant \delta.
$$

Thus for any $n \geqslant m$

$$
\begin{aligned}
\int_X |f_n - f|^2 &= \int_{X_m} |f_n - f|^2 + \int_{X \setminus X_m} |f_n - f|^2 \\
&\leqslant \int_{X_m} \delta + \int_{X \setminus X_m} 2(|f_n|^2 + |f|^2) \\
&\leqslant \delta\mu(X_m) + 4\frac{6}{\sqrt{a}} \\
&\leqslant \frac{\varepsilon^2}{2} + \frac{4 \cdot 6}{(4 \cdot 6 \cdot 2)/\varepsilon} \\
&\leqslant \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} \\
&\leqslant \varepsilon^2.
\end{aligned}
$$

Thus for any $n \geqslant f_n$.

$$
\|f_n - f\|_{L^2} = \left(\int_X |f_n - f|^2\right)^{1/2} \leqslant \varepsilon,
$$

which means that $f_n \to f$ in L^2.

Problem 5. We need to show that
$$\int_X (\|f+v\| - \|f\|) \geqslant \frac{1}{8}\|v\|,$$
with conditions only on $\|f\|$. Then let's minimize the left-hand side by changing f without changing $\|f\|$. It means we should minimize $\|f+v\|$. In that case f should be contradirectional to v, so
$$\int_X (\|f+v\| - \|f\|) \geqslant \int_X \left(\Big|\|f\| - \|v\|\Big| - \|f\|\right).$$
Let
$$X_1 := \{x \in X \mid \|f\| \leqslant \tfrac{1}{4}\|v\|\}, \qquad X_2 := \{x \in X \mid \tfrac{1}{4}\|v\| < \|f\| \leqslant \|v\|\}, \qquad X_3 := \{x \in X \mid \|v\| < \|f\|\}.$$
Then
$$\int_{X_1} \left(\Big|\|f\| - \|v\|\Big| - \|f\|\right) = \int_{X_1} (\|v\| - 2\|f\|) \geqslant \int_{X_1} (\|v\| - 2\cdot\tfrac{1}{4}\|v\|) = \frac{3}{4}\left(1 - \frac{1}{2}\right)\|v\| = \frac{3}{8}\|v\|,$$
$$\int_{X_2} \left(\Big|\|f\| - \|v\|\Big| - \|f\|\right) = \int_{X_2} (\|v\| - 2\|f\|) \geqslant \int_{X_1} (\|v\| - 2\|v\|) = -\mu(X_1)\|v\|,$$
$$\int_{X_3} \left(\Big|\|f\| - \|v\|\Big| - \|f\|\right) = \int_{X_3} -\|v\| = -\mu(X_3)\|v\|.$$
Hence
$$\int_X \left(\Big|\|f\| - \|v\|\Big| - \|f\|\right) \geqslant \frac{3}{8}\|v\| - (\mu(X_1) + \mu(X_2))\|v\| = \frac{3}{8}\|v\| - \frac{1}{4}\|v\| = \frac{1}{8}\|v\|.$$

Problem 6. Let E do not contain any two points with rational distance between them. Then there is set E' such that $\lambda(E') > 0$, $E' + s \subseteq E$ for some $s \in \mathbb{R}$, and $E' \subseteq [0;1]$. Then
$$F := \bigsqcup_{q \in [0;1] \cap \mathbb{Q}} E' + q$$
(for any different $q_1, q_2 \in Q$ sets $E' + q_1$ and $E' + q_2$ are disjoint) has infinite measure (because $\lambda(F) = \sum_{q \in [0;1]\cap\mathbb{Q}} \lambda(E' + q) = \sum_{q \in [0;1]\cap\mathbb{Q}} \lambda(E') = |[0;1] \cap \mathbb{Q}| \cdot \lambda(E') = \infty$), and is contained in $[0;3]$, which is contradiction. Thus E contains rationaly distanced couple of points.

Problem 7. If $j = k$, then
$$\int_0^1 \varphi_j \varphi_k d\lambda = \int_0^1 \varphi_j^2 d\lambda = \int_0^1 d\lambda = 1.$$
Then WLOG $j < k$. So
$$\begin{aligned}\int\limits_{[0;1]} \varphi_j \varphi_k d\lambda &= \int\limits_{[0;1)} \varphi_j \varphi_k d\lambda \\ &= \sum_{n=1}^{2^j} \sum_{m=1}^{2^{k-j-1}} \int\limits_{\left[\frac{2^{k-j}(n-1)+2(m-1)}{2^k};\frac{2^{k-j}(n-1)+2(m-1)+1}{2^k}\right)} \varphi_j \varphi_k d\lambda \\ &\quad + \int\limits_{\left[\frac{2^{k-j}(n-1)+2(m-1)+1}{2^k};\frac{2^{k-j}(n-1)+2(m-1)+2}{2^k}\right)} \varphi_j \varphi_k d\lambda\end{aligned}$$

$$= \sum_{n=1}^{2^j} \sum_{m=1}^{2^{k-j-1}} \int\limits_{\left[\frac{2^{k-j}(n-1)+2(m-1)}{2^k}; \frac{2^{k-j}(n-1)+2(m-1)+1}{2^k}\right)} (-1)^{n-1} \cdot 1 \, d\lambda$$

$$+ \int\limits_{\left[\frac{2^{k-j}(n-1)+2(m-1)+1}{2^k}; \frac{2^{k-j}(n-1)+2(m-1)+2}{2^k}\right)} (-1)^{n-1} \cdot (-1) \, d\lambda$$

$$= \sum_{n=1}^{2^j} \sum_{m=1}^{2^{k-j-1}} (-1)^{n-1} \left(\frac{1}{2^k} - \frac{1}{2^k}\right)$$

$$= 0$$

Problem 8. f is summable iff $\int_E f < +\infty$ (because $f \geqslant 0$).

Let f be summable. Then

$$\begin{aligned}
\sum_{k=1}^{+\infty} \mu(\widetilde{E}_k) &= \sum_{k=1}^{+\infty} \sum_{l=k}^{+\infty} \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=1}^{+\infty} \sum_{k=1}^{l} \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=1}^{+\infty} l \cdot \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&\leqslant \int_E f \\
&< +\infty.
\end{aligned}$$

Let f not be summable. Then

$$\begin{aligned}
\sum_{k=1}^{+\infty} \mu(\widetilde{E}_k) &= \sum_{k=1}^{+\infty} \sum_{l=k}^{+\infty} \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=1}^{+\infty} \sum_{k=1}^{l} \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=1}^{+\infty} l \cdot \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=0}^{+\infty} l \cdot \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&= \sum_{l=0}^{+\infty} (l+1) \cdot \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&\quad - \sum_{l=0}^{+\infty} \mu(\{x \in E \mid f(x) \in [l; l+1)\}) \\
&\geqslant \int_E f - \mu(E) \\
&= +\infty - \mu(E) \\
&= +\infty.
\end{aligned}$$
