Homework of 11.16
Differential geometry

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Problem 72.

Lemma 1. *Let $\gamma : [0;1] \to \mathbb{R}^n$ be regular a curve from $\vec{0}$ to $\vec{v}$ of length l, whose curvature is everywhere not less than $k > 0$ and which is concatenable to itself (that means concatenation of $\gamma : [0;1] \to \mathbb{R}^n$ and $\gamma + \vec{v} : [0;1] \to \mathbb{R}^n$ is regular too). Then for every $m \in \mathbb{N} \setminus \{0\}$ curve*

$$\tau_m : [0;1] \to \mathbb{R}^n, t \mapsto \begin{cases} \frac{1}{m}\gamma(tm) & \text{if } \frac{0}{m} \leqslant t \leqslant \frac{1}{m} \\ \dots & \\ \frac{1}{m}(\gamma(tm - p) + p\vec{v}) & \text{if } \frac{p}{m} \leqslant t \leqslant \frac{p+1}{m} \qquad (p \in \{0; \dots; m-1\}) \\ \dots & \\ \frac{1}{m}(\gamma(tm - m + 1) + (m-1)\vec{v}) & \text{if } \frac{m-1}{m} \leqslant t \leqslant \frac{m}{m} \end{cases}$$

is regular, has length l, goes from $\vec{0}$ to $\vec{v}$, and whose curvature is everywhere not less than mk.

Proof.

- τ_m is correctly defined, because for every $p \in \{0; \dots; m-1\}$

$$\begin{aligned} \frac{1}{m}\left(\gamma\left(\frac{p+1}{m}m - p\right) + p\vec{v}\right) &= \frac{1}{m}(\gamma(1) + p\vec{v}) \\ &= \frac{1}{m}((p+1)\vec{v}) \\ &= \frac{1}{m}(\gamma(0) + (p+1)\vec{v}) \\ &= \frac{1}{m}\left(\gamma\left(\frac{p+1}{m}m - (p+1)\right) + (p+1)\vec{v}\right), \end{aligned}$$

  i.e. $\tau_m(\frac{p+1}{m})$ is correctly defined.

- For any $p \in \{0; \dots; m-2\}$ curve $(m\tau_m - p\vec{v})|_{[\frac{p}{m}; \frac{p+2}{m}]}$ is just concatenation of γ and $\gamma + \vec{v}$, which is regular. Regularity of τ_m on $[0;1] \setminus \{\frac{p}{m}\}_{p=1}^{m-1}$ is obvious. So now regularity of τ_m in any point $\frac{p}{m}$ is obvious too. Hence τ_m is regular everywhere.

- Length of τ_m is

$$\begin{aligned}\int_0^1 \|\tau_m'(t)\|dt &= \sum_{p=0}^{m-1}\int_{p/m}^{(p+1)/m}\left\|\left(\frac{1}{m}(\gamma(tm-p)+p\vec{v})\right)'\right\|dt\\ &= \sum_{p=0}^{m-1}\int_{p/m}^{(p+1)/m}\left\|\left(\frac{1}{m}\gamma(tm-p)\right)'\right\|dt\\ &= \sum_{p=0}^{m-1}\int_{p/m}^{(p+1)/m}\|\gamma'(tm-p)\|dt\\ &= \sum_{p=0}^{m-1}\frac{1}{m}\int_{p/m}^{(p+1)/m}\|\gamma'(tm-p)\|d(tm-p)\\ &= \sum_{p=0}^{m-1}\frac{1}{m}\int_0^1\|\gamma'(s)\|ds\\ &= \sum_{p=0}^{m-1}\frac{l}{m}\\ &= l.\end{aligned}$$

- $$\tau_m(0)=\frac{1}{m}\gamma(0\cdot m)=\frac{1}{m}\gamma(0)=\frac{1}{m}\vec{0}=\vec{0}$$

  and

  $$\tau_m(1)=\frac{1}{m}(\gamma(1\cdot m-m+1)+(m-1)\vec{v})=\frac{1}{m}(\gamma(1)+(m-1)\vec{v})=\frac{1}{m}(\vec{v}+(m-1)\vec{v})=\frac{m}{m}\vec{v}=\vec{v}.$$

- For every $p\in\{0;\dots;m-1\}$ and every $t\in[\frac{p}{m};\frac{p+1}{m}]$

  $$\tau_m'(t)=\left(\frac{1}{m}(\gamma(tm-p)-p\vec{v})\right)'=\gamma'(tm-p)$$

  and $\tau_m''(t)=m\gamma''(tm-p)$. Hence

  $$\kappa_{\tau_m}(t)=\frac{|\tau_m''(t)\times\tau_m'(t)|}{\|\tau_m'(t)\|^3}=\frac{|m\gamma''(tm-p)\times\gamma'(tm-p)|}{\|\gamma'(tm-p)\|^3}=m\kappa_\gamma(tm-p)\geqslant mk.$$

□

Corollary 1.1. *Let $\gamma:[0;1]\to\mathbb{R}^n$ be regular a curve from $\vec{0}$ to $\vec{v}$ of length l, whose curvature is everywhere positive and which is concatenable to itself. Then for every $k>0$ there is $m\in\mathbb{N}\setminus\{0\}$. Such that curvature τ_m is everywhere not less than k.*

Proof. Curvature of τ_m in t is a continuous function on compact $[0;1]$. Hence there is $\varepsilon>0$ such that the curvature is everywhere not less ε. Then there is $m\in\mathbb{N}$ such that $m\geqslant k/\varepsilon$. So curvature of τ_m is everywhere not less than $m\varepsilon\geqslant k$. □

a) Let's consider almost trochoid ($h>1$, $\alpha>0$)

$$\gamma:[0;1]\to\mathbb{R}^2, t\mapsto\left(t-\frac{h}{2\pi}\sin(2\pi t);\frac{\alpha h}{2\pi}(1-\cos(2\pi t))\right).$$

Then

$$\gamma'(t) = (1 - h\cos(2\pi t); \alpha h \sin(2\pi t)), \qquad \gamma''(t) = 2\pi h(\sin(2\pi t); \alpha\cos(2\pi t)).$$

So

$$\begin{aligned} |\gamma'(t) \times \gamma''(t)| &= |(1 - h\cos(2\pi t))2\pi h\alpha\cos(2\pi t) - \alpha h \sin(2\pi t)2\pi h \sin(2\pi t)| \\ &= 2\pi h\alpha|\cos(2\pi t) - h| \\ &\geqslant 2\pi h\alpha(h-1) > 0. \end{aligned}$$

Also when $\|\gamma'(t)\| = 0$, $1 = h\cos(2\pi t)$ and $\alpha h\sin(2\pi t)$. Then $\sin(2\pi t) = 0$, so $t \in \mathbb{Z}$, $h\cos(2\pi t) = h > 1$. Hence $\|\gamma'(t)\|$ is always positive. Hence γ is regular, self-concatenable, goes from $\vec{0}$ to $(1; 0)$, and has positive curvature.

Let l be a length of γ; obviously $l > 1$. We know that for some $m \in \mathbb{N} \setminus \{0\}$ curve τ_m is also regular, goes from $\vec{0}$ to $(1; 0)$, and has length l, but also has curvature that is not less $1.1 > 1$. Then let's consider curve $\delta := \gamma/l$. Then δ is regular, goes from $\vec{0}$ to $(1/l; 0)$, has length 1, and curvature > 1 (it increased, because of the homotety with coefficient from $(0; 1)$). Then let's prove that fro every $\varepsilon > 0$ l may be $< 1 + \varepsilon$; in that case

$$\frac{1}{l} > \frac{1}{1+\varepsilon} > \frac{1-\varepsilon^2}{1+\varepsilon} = 1 - \varepsilon.$$

$$\begin{aligned} l &= \int_0^1 \|\gamma'(t)\| dt \\ &= \int_0^1 \sqrt{(1 - h\cos(2\pi t))^2 + (\alpha h\sin(2\pi t))^2} dt \\ &\leqslant \int_0^1 |1 - h\cos(2\pi t)| + |\alpha h \sin(2\pi t)| dt \\ &\leqslant \int_0^1 |1 - h\cos(2\pi t)| + \alpha h dt \\ &\leqslant \int_0^1 |1 - h\cos(2\pi t)| dt + \alpha h. \end{aligned}$$

We already have that

$$\lim_{\substack{h \to 1^+ \\ \alpha \to 0^+}} \alpha h = 0.$$

Let $\varphi := \cos^{-1}(1/h)$. Then $\lim_{h \to 1^+} \varphi = 0^+$ and

$$\begin{aligned} \int_0^1 |1 - h\cos(2\pi t)| dt &= \int_0^1 (1 - h\cos(2\pi t)) dt + 2\int_{-\varphi}^{\varphi} (h\cos(2\pi t) - 1) dt \\ &= 1 + 2\int_{-\varphi}^{\varphi} (h\cos(2\pi t) - 1) dt \\ &\leqslant 1 + 2\int_{-\varphi}^{\varphi} (h - 1) dt \\ &\leqslant 1 + 4\varphi(h-1) dt. \end{aligned}$$

Hence

$$\lim_{\substack{h\to 1^+\\ \alpha\to 0^+}} l \leqslant \lim_{\substack{h\to 1^+\\ \alpha\to 0^+}} \int_0^1 |1-h\cos(2\pi t)|dt + \alpha h \leqslant \lim_{\substack{h\to 1^+\\ \alpha\to 0^+}} 1+4\varphi(h-1)+\alpha h = 1+0+0 = 1.$$

Then there are $h>1$ and $\alpha>0$ such that $l<1+\varepsilon$.

b) Let's consider spiral $(r>0)$

$$\gamma : [0;1] \to \mathbb{R}^3, t\mapsto (t; r(\cos(2\pi t)-1); r\sin(2\pi t)).$$

Then

$$\gamma'(t) = (1; -2\pi r\sin(2\pi t); 2\pi r\cos(2\pi t)), \qquad \gamma''(t) = (0; -4\pi^2 r\cos(2\pi t); -4\pi^2 r\sin(2\pi t)).$$

So

$$\begin{aligned}|\gamma'(t)\times\gamma''(t)| &= \sqrt{(-4\pi^2 r\cos(2\pi t))^2 + (-4\pi^2 r\sin(2\pi t))^2 + (8\pi^3 r^2\sin(2\pi t)^2 + 8\pi^3 r^2\cos(2\pi t)^2)^2}\\ &= \sqrt{16\pi^4 r^2 + 64\pi^6 r^4} > 0\end{aligned}$$

Also $\|\gamma'(t)\| \geqslant 1$. Hence γ is regular, self-concatenable, goes from $\vec{0}$ to $(1;0;0)$, and has positive curvature. Also length of γ is

$$\begin{aligned}\int_0^1 \|\gamma'(t)\|dt &= \int_0^1 \sqrt{1+4\pi^2r^2\sin(2\pi t)^2 + 4\pi^2 r^2\cos(2\pi t)^2}\\ &= \int_0^1 \sqrt{1+4\pi^2 r^2}\\ &= \sqrt{1+4\pi^2r^2}\end{aligned}$$

Hence for every $\varepsilon>0$ if $r<\frac{\varepsilon}{2\pi}$ then the length is $<1+\varepsilon$.

So by the same reasoning we construct τ_m with the same characteristics but curvature, which increased in needed way, and homotate it with coefficient $1/l$ to get the asked curve.
