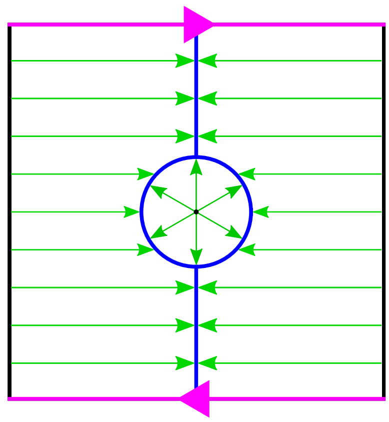
Homework of 09.07
Differential geometry

Gleb Minaev @ 204 (20.Б04-мкн)

12 сентября 2021 г.

Задача 7. At first let's show that blue set on the picture (that is embedding of graph with 2 vertices and 3 multiple edges between them) is a retract of a handle.



Let's take function f from the handle to the blue set such that it's identical on the blue set, it projects points outside the blue set horizontally to the nearest point in the direction and it projects points inside the blue set to the nearest point of the circle. All ways of projections are drawn by green arrows. Obviously f is continuous. (Also, using linear projection by the arrows we can get homotopy from the handle to the blue set. So the set is deformation retract.)

Then it's obvious that we can contract on the handle any of two edges of the embedded graph that is a part of circle's boundary to a point. So composition of the contraction and the retraction f is continuous map from the handle to graph with single vertex and two loops, i.e. wedge sum of two circles. (Also, the contraction is homotopy from the first graph to the second. So concatenation of the homotopies shows that the wedge sum is deformation retract).

Задача 8. Let X and Y be homotopy equivalent topological spaces such that their homotopy equivalence is raised by functions $f : X \to Y$ and $g : Y \to X$. Let also Σ_X and Σ_Y be the sets of connected components of X and Y respectively and Θ_X and Θ_Y be the sets of path-connected ones respectively.

Лемма 1. *Image under f of connected component is connected.*

Доказательство. If image of connected component A is not connected, then there non-empty open (in $f(A)$) disjoint S and T such that $f(A) = S \cup T$. Thus $f^{-1}(S)$ and $f^{-1}(T)$ are also open (in A) and disjoint. So A is not connected — contradiction. $\square$

Следствие 1.1. *f induces map $\Sigma_X \to \Sigma_Y$.*

Лемма 2. *Image under f of path-connected component is path-connected.*

Доказательство. For any two points x_1 and x_2 in path-connected component A there is a path α from x_1 to x_2. α is a continuous map $[0;1] \to X$. Thus $f \circ \alpha : [0;1] \to Y$ is a path from $f(x_1)$ to $f(x_2)$. So $f(x_1)$ and $f(x_2)$ are path-connected. So image of A is path-connected. □

Следствие 2.1. *f induces map $\Theta_X \to \Theta_Y$.*

Лемма 3. *If images of $x_1, x_2 \in X$ are connected (path-connected), then x_1 and x_2 are also connected (path-connected).*

Доказательство. Let $y_1 := f(x_1)$, $y_2 := f(x_2)$, $z_1 := g(y_1)$, $z_2 := g(y_2)$, . As we know $g \circ f \sim \mathrm{Id}_X$. It means that there is homotopy H between $g \circ f$ and Id_X ($H(x,0) = g(f(x))$, $H(x,1) = x$). Thus a map

$$\alpha_x : [0;1] \to X, t \mapsto H(x,t)$$

is a path from $(g \circ f)(x)$ to $\mathrm{Id}_X(x) = x$. By substituting x with x_1 and x_2 we've got that x_1 and z_1 are path-connected and x_2 and z_2 are path-connected. Also y_1 and y_2 are connected (path-connected), so (by lemmas 1 and 2) z_1 and z_2 are too. Thus x_1 and x_2 are connected (path-connected). □

Следствие 3.1. *Images under f of connected (path-connected) components do not share connected (path-connected) components.*

Следствие 3.2. *f induces injection $\Sigma_X \to \Sigma_Y$ ($\Theta_X \to \Theta_Y$).*

So f and g induce injections from set of connected (path-connected) components of X to set of connected (path-connected) components of Y and vice versa. Thus we have got by Schröder–Bernstein theorem that the sets of connected (path-connected) components of X and Y are equinumerous. But there is another way of proofing the equinumerousity.

Лемма 4. *$g \circ f$ induces identical map $\Sigma_X \to \Sigma_Y$ ($\Theta_X \to \Theta_Y$).*

Доказательство. Let A be some set from Σ_X (Θ_X) and $A' := g(f(A))$. Let x be some point of A and $x' := g(f(x)) \in A'$. So there is a path from x' to x, thus they are path-connected. It means $A = A'$. □

Следствие 4.1. *f induces bijection $\Sigma_X \to \Sigma_Y$ ($\Theta_X \to \Theta_Y$).*

Следствие 4.2. *$|\Sigma_X| = |\Sigma_Y|$ ($|\Theta_X| = |\Theta_Y|$).*

Задача 9. Let X be a topological space with antidescrete topology and x_1, x_2 — some points of X. Define function

$$\alpha : [0;1] \to X, t \mapsto \begin{cases} x_1 & \text{if } t \in [0;1), \\ x_2 & \text{if } t = 1. \end{cases}$$

Thus α is continuous, because preimages of the only open sets $\varnothing$ and X are $\varnothing$ and $[0;1]$ respectively, that are open sets. So α is a path from x_1 to x_2. So X is path-connected and $\pi_1(X,x)$ does not depends on x.

Thus let α_1 and α_2 be paths from x_1 to x_2. Then define function

$$H : [0;1] \times [0;1] \to X, (s,t) \mapsto \begin{cases} \alpha_1(s) & \text{if } t = 0, \\ \alpha_2(s) & \text{if } t = 1, \\ x_1 & \text{if } s = 0, \\ x_2 & \text{if } s = 1, \\ \text{any point from } X & \text{otherwise.} \end{cases}$$

Because of the same reason, H is continuous, thus homotopy between α_1 and α_2. Thus $\pi_1(X)$ is trivial.
