Homework of 09.27
Differential geometry

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Problem 50. Let α and β be some two loops starting at $e \in G$ (identity element of the group G). Consider map $S : I \times I \to G, (s, t) \mapsto \alpha(s) \cdot \beta(t)$. It is composition of maps $I \times I \to G \times G, (s, t) \mapsto (\alpha(s), \beta(t))$ (that has continuous coordinates, hence is continuous itself) and $G \times G \to G, (x, y) \mapsto x \cdot y$ (that is continuous by definition). Note that $S|_{I \times \{0\}} = S|_{I \times \{1\}} = \alpha$ and $S|_{\{0\} \times I} = S|_{\{1\} \times I} = \beta$.

Consider on $I \times I$ paths $\varphi : (0; 0) \rightsquigarrow (1; 0) \rightsquigarrow (1; 1)$ and $\psi : (0; 0) \rightsquigarrow (0; 1) \rightsquigarrow (1; 1)$. So $S \circ \varphi = \alpha\beta$, $S \circ \psi = \beta\alpha$. Hence S is almost homotopy between paths $\alpha\beta$ and $\beta\alpha$. To make it clear (and show explicit homotopy) construct homotopy H between φ and ψ (it exists because $I \times I \cong D^2$ is simply connected), so $H \circ S$ is homotopy between $\alpha\beta$ and $\beta\alpha$.

Problem 51. Let ST be a solid torus in neighbourhood of S^1. Then obviously $D^3 \setminus ST$ is a deformation retract of $\mathbb{R}^3 \setminus S^1$ (because we know that $\mathbb{R}^3 \setminus \mathrm{Int}(D^3)$ (where interior $\mathrm{Int}(D^3)$ of D^3 means $D^3 \setminus S^2$) deformation retracts to its border S^2 and $ST \setminus S^1$ deformation retracts to its border). Hence $D^3 \setminus ST$ is a homotopy equivalent to $\mathbb{R}^3 \setminus S^1$.

Consider inversion (in geometrical meaning) with center inside ST and any positive radius. Then result will be some ~~crumpled solid torus without Int(D^3)~~ space that is homeomorphic to solid torus without $\mathrm{Int}(D^3)$. And similarly to strip with a hole (which defromation retract is a circle with a path from one its point to an opposit one) solid torus with a (3-dimensional) hole deformation retracts to S^2 with a path from one its point to another one. Obviously obtained topological space is homotopy equivalent to $S^2 \vee S^1$. Hence $S^2 \vee S^1$ is homotopy equivalent to $\mathbb{R}^3 \setminus S^1$.
