Homework for 11.11
Algebra

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Problem 1.

1. The graph is irrealisable, because then there must be an arrow from the left to the right vertex.

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[rr, bend left, dashed] & \bullet \arrow[l, shift left] \arrow[r] & \bullet \end{tikzcd}$$

2. The graph is realisable.

$$\begin{tikzcd} \bullet & \bullet \arrow[l, shift left] \arrow[l, shift right] \arrow[r] & \bullet \end{tikzcd}$$

   Let the category be the subcategory of Sets formed with some sets A, B, and C and some (different) maps $f : B \to A$, $g : B \to A$, and $h : B \to C$ (and also bijections of A, B, and C on themselves).

$$\begin{tikzcd} A & B \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \arrow[r, "h"] & C \end{tikzcd}$$

   Obviously the considered morphisms are closed on themselves: there is no need in other morphisms, because any possible compositions uses at least one id-argument. Hence it's subcategory, so category.

3. The graph is realisable.

$$\begin{tikzcd} \bullet \arrow[rr, shift left] \arrow[dr] & & \bullet \arrow[ll, shift left] \arrow[dl] \\ & \bullet & \end{tikzcd}$$

   Let the category be the subcategory of Sets formed with some sets A, B, and C and some bijections $f : A \to B$, $g : A \to C$, $h : B \to A$, and $i : B \to C$ (and also bijections of A, B, and C on themselves), such that the diagram is commutative. (The silliest example is three sets, each is of its unique element, and the only possible bijections between them. But we may also consider any three "equal spaces with different points of view"(homeomorphic topological spaces or isomorphic vector spaces with different considered basises or isomorphic graphs with different disposition of their vertices) and isomorphisms between them that turn their "points of view"(basises in vector spaces or dispositions of vertices in graphs) into each other).

$$\begin{tikzcd} A \arrow[rr, shift left, "f"] \arrow[dr, "g"'] & & B \arrow[ll, shift left, "h"] \arrow[dl, "i"] \\ & C & \end{tikzcd}$$

   Obviously the considered morphisms are closed on themselves. Hence it's subcategory, so category.

4. The graph is irrealisable.

$$\bullet \rightleftarrows\!\!\leftarrow \bullet$$

Let name the arrows like in next diagram.

$$\begin{tikzcd} A \arrow[r, "g"] & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "h"] \end{tikzcd}$$

Then $g \circ f$ must be id_B, and $h \circ g$ must be id_A (because there are no other arrows from A to itself and from B to itself). Hence

$$f = \mathrm{id}_A \circ f = h \circ g \circ f = h \circ \mathrm{id}_B = h.$$

It's contradiction.

Problem 16.

(a) Let $\tau : \mathrm{Id}_{\mathrm{Sets}} \to \mathrm{Id}_{\mathrm{Sets}}$ be a natural transformation. It means that for every $X \in \mathrm{Ob(Sets)}$ there is a morphism $\tau_X : X \to X$ that is a map $X \to X$, such that for any morphism (map) $f : X \to Y$ diagram below is commutative.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau_X"'] & Y \arrow[d, "\tau_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then for any X and any $x \in X$ let's consider $f : \{x\} \to X, x \mapsto x$.

$$\begin{tikzcd} \{x\} \arrow[r, "x \mapsto x"] \arrow[d, "x \mapsto x"', "\tau_{\{x\}}"] & X \arrow[d, "\tau_X"] \\ \{x\} \arrow[r, "x \mapsto x"'] & X \end{tikzcd}$$

Then there is only one possible $\tau_{\{x\}}$ — a map $x \mapsto x$ (because there are no other morphisms from $\{x\}$ to $\{x\}$). Then

$$\tau_X(x) = \tau_X(f(x)) = f(\tau_{\{x\}}(x)) = f(x) = x.$$

Hence $\tau_X = \mathrm{Id}_X$. So there any natural transformation $\mathrm{Id}_{\mathrm{Sets}} \to \mathrm{Id}_{\mathrm{Sets}}$ is identity.

(б) Let $\tau : \mathrm{Id}_M \to \mathrm{Id}_M$ be a natural transformation, where M is a category of a single object O (hence monoid). It means there is morphism $\varphi : O \to O$ such that for any morphism $f : O \to O$ diagram below is commutative.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

That means φ is commutative with every element in M. Obviously any commutative element $\varphi \in M$ also forms a natural transformation. Hence a natural transformations $\mathrm{Id}_M \to \mathrm{Id}_M$ are the same as commutative elements of M.

(в) Let there be functors

$$\begin{aligned} F &: \text{Sets} \to \text{Sets}, \\ &X \mapsto X \times X, \\ &(f : X \to Y) \mapsto (f : X^2 \mapsto Y^2, (x_1; x_2) \mapsto (f(x_1); f(x_2))) \end{aligned}$$

and

$$\begin{aligned} G &: \text{Sets} \to \text{Sets}, \\ &X \mapsto 2^X, \\ &(f : X \to Y) \mapsto (f : 2^X \mapsto 2^Y, U \mapsto f[U]), \end{aligned}$$

and natural transformation $\tau : F \to G$. It means for any $X \in$ Sets there is a morphism $\tau_X : X^2 \to 2^X$ that is a map $X^2 \to 2^X$, such that for any morphism (map) $f : X \to Y$ diagram below is commutative.

$$\begin{tikzcd} X^2 \arrow[r, "(f;f)"] \arrow[d, "\tau_X"'] & Y^2 \arrow[d, "\tau_Y"] \\ 2^X \arrow[r, "f[\cdot]"'] & 2^Y \end{tikzcd}$$

Lemma 1. $\tau_X((x_1; x_2)) \subseteq \{x_1; x_2\}$. *(Also $\tau((x; x)) \subseteq \{x\}$.)*

Доказательство. Let $Y = \{x_1; x_2\}$, $f : Y \to X, x_1 \mapsto x_1, x_2 \mapsto x_2$. Then

$$\begin{tikzcd} \{x_1; x_2\}^2 \arrow[r, "(f;f)"] \arrow[d, "\tau_{\{x_1;x_2\}}"'] & X^2 \arrow[d, "\tau_X"] \\ 2^{\{x_1;x_2\}} \arrow[r, "f[\cdot]"'] & 2^X \end{tikzcd}$$

So

$$\tau_X((x_1; x_2)) = [\tau_X((f(x_1); f(x_2))) = f[\tau_Y((x_1; x_2))] \subseteq \{x_1; x_2\},$$

because $f[Y] = \{x_1; x_2\}$.

P.S. The reasoning also works in case $x_1 = x_2$. □

Lemma 2. *If for some $(x_1; x_2)$ in some X^2 where $x_1 \neq x_2$*

(a) $\tau_X((x_1; x_2)) = \{x_1; x_2\}$,

(b) $\tau_X((x_1; x_2)) = \{x_1\}$,

(c) $\tau_X((x_1; x_2)) = \{x_2\}$,

(d) $\tau_X((x_1; x_2)) = \varnothing$,

then it's true for any pair $(y_1; y_2)$ in any Y^2 (and y_1 and y_2 may be equal).

Доказательство. For any pair $(y_1; y_2) \in Y^2$ we may consider any map $f : X \to Y$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then there is commutative diagram

$$\begin{tikzcd} (x_1; x_2) \arrow[r, mapsto, "(f;f)"] \arrow[d, mapsto, "\tau_X"'] & (y_1; y_2) \arrow[d, mapsto, "\tau_Y"] \\ U \arrow[r, mapsto, "f[\cdot]"'] & f(U) \end{tikzcd}$$

Obviously

(a) $f(\{x_1; x_2\}) = \{y_1; y_2\}$,
(b) $f(\{x_1\}) = \{y_1\}$,
(c) $f(\{x_2\}) = \{y_2\}$,
(d) $f(\varnothing) = \varnothing$.

Then

(a) $\tau_Y((y_1; y_2)) = \{y_1; y_2\}$,
(b) $\tau_Y((y_1; y_2)) = \{y_1\}$,
(c) $\tau_Y((y_1; y_2)) = \{y_2\}$,
(d) $\tau_Y((y_1; y_2)) = \varnothing$.

$\square$

But there is at least one pair $(x_1; x_2)$ in some X^2 where $x_1 \neq x_2$. Hence independently on X τ_X maps any pair $(x_1; x_2)$ to set of either both, left, right or none of x_1 and x_2 (and kind of choice is always the same). And obviously any of the kinds of choises forms a natural transformation (and only one).

Problem 18. For any morphism $\varphi : X \to Y$ the diagram of τ is

$$\begin{tikzcd}
\mathrm{Hom}(A,X) \arrow[r, "f\mapsto\varphi\circ f"] \arrow[d, shift right, "\tau_X"'] & \mathrm{Hom}(A,Y) \arrow[d, shift left, "\tau_Y"] \\
\mathrm{Hom}(B,X) \arrow[r, "g\mapsto\varphi\circ g"'] \arrow[u, shift right, "\tau_X^{-1}"'] & \mathrm{Hom}(B,Y) \arrow[u, shift left, "\tau_Y^{-1}"]
\end{tikzcd}$$

Let $p := \tau_B^{-1}(\mathrm{id}_B) \in \mathrm{Hom}(A,B)$ and $q := \tau_A(\mathrm{id}_A) \in \mathrm{Hom}(B,A)$. Then let's consider cases $\varphi = p$ and $\varphi = q$:

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "f\mapsto p\circ f"] \arrow[d, shift right, "\tau_A"'] & \mathrm{Hom}(A,B) \arrow[d, shift left, "\tau_B"] \\
\mathrm{Hom}(B,A) \arrow[r, "g\mapsto p\circ g"'] \arrow[u, shift right, "\tau_A^{-1}"'] & \mathrm{Hom}(B,B) \arrow[u, shift left, "\tau_B^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto, "f\mapsto p\circ f"] \arrow[d, mapsto, shift right, "\tau_A"'] & p \arrow[d, mapsto, shift left, "\tau_B"] \\
q \arrow[r, mapsto, dashed, "g\mapsto p\circ g"'] \arrow[u, mapsto, shift right, "\tau_A^{-1}"'] & \mathrm{id}_B \arrow[u, mapsto, shift left, "\tau_B^{-1}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(A,B) \arrow[r, "f\mapsto q\circ f"] \arrow[d, shift right, "\tau_B"'] & \mathrm{Hom}(A,A) \arrow[d, shift left, "\tau_A"] \\
\mathrm{Hom}(B,B) \arrow[r, "g\mapsto q\circ g"'] \arrow[u, shift right, "\tau_B^{-1}"'] & \mathrm{Hom}(B,A) \arrow[u, shift left, "\tau_A^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
p \arrow[r, mapsto, dashed, "f\mapsto q\circ f"] \arrow[d, mapsto, shift right, "\tau_B"'] & \mathrm{id}_A \arrow[d, mapsto, shift left, "\tau_A"] \\
\mathrm{id}_B \arrow[r, mapsto, "g\mapsto q\circ g"'] \arrow[u, mapsto, shift right, "\tau_B^{-1}"'] & q \arrow[u, mapsto, shift left, "\tau_A^{-1}"]
\end{tikzcd}$$

So $\mathrm{id}_B = p \circ q$ and $\mathrm{id}_A = q \circ p$. Hence p and q are isomorphisms.
