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Structures for Grothendieck Fibrations

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Introduction

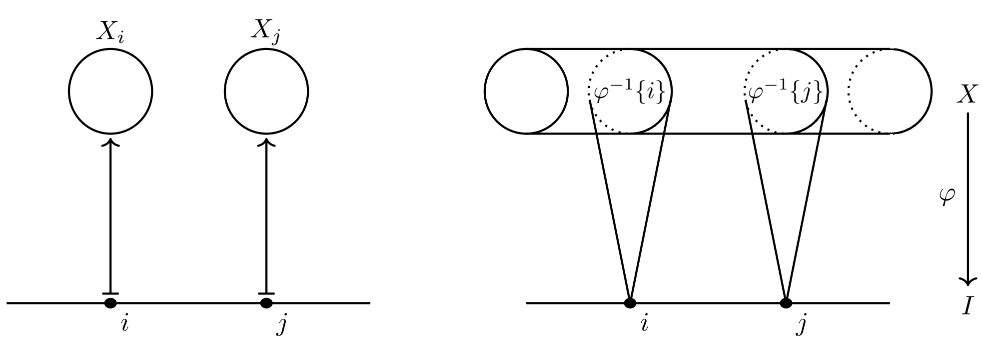
The aim of this thesis is to prove a kind of monadicity and comonadicity of the *Grothendieck fibrations*. To reach this aim, we first have to restrict to a particular kind of Grothendieck fibrations, namely the *split fibrations* (and to the *split fibrations over a fixed base category* for the comonadicity).

The non-specialist reader should consider that proving the monadicity, or the comonadicity, of the Grothendieck fibrations allows us to see the Grothendieck fibrations as algebraic structures with respect to more elementary mathematical objects, hence understanding how properties of fibrations derive from properties of the more elementary objects.

More precisely, we prove that the split fibrations are monadic on $\mathcal{CAT}^{\rightarrow}$, the arrow category of the 2-category $\mathcal{CAT}$ of small categories, functors and natural transformations. This means that a split fibration can be seen as a functor endowed with a certain algebraic structure.

We prove also that split fibrations over a fixed base category $\mathcal{C}$ are comonadic on $\mathcal{CAT} \downarrow \mathcal{C}$, the slice of $\mathcal{CAT}$ over $\mathcal{C}$. This will yield that a split fibration over $\mathcal{C}$ can also be seen as a functor over $\mathcal{C}$ endowed with a certain "coalgebraic" structure.

We begin introducing the categorical concept of *(Grothendieck) fibration*. This is a structure that captures both concepts of indexing and change of base. The introduction of such a concept starts from a change of perspective about indexing. Indeed, rather than viewing an indexing as a function from an index set to a universe of mathematical objects (for example of sets or of categories), we will view it as the collection of fibres of a map which lands in the index set, as the following figure might suggest:



This is exactly the idea which gave rise to *fibre bundles* and all the subsequent developments in differential geometry, algebraic topology and algebraic geometry.

It is easy to show that these two points of view are equivalent for sets. But when we generalize it and consider an indexed category, in order to view it as a *fibration* we need a concept of change of base. Indeed we then need to start from a functor which lands in $\mathcal{C}at$, the category of small categories and functors, and the assignment on objects will correspond to the indexing, whereas the assignment on morphisms will correspond to the change of base. The simplest instance of change of base is given by pulling-back. And the universal property of the pullback is what we will generalize to get the fundamental notion of *cartesian morphisms*.

We will then need the notion of *cloven fibration*, which under the axiom of choice for classes, is equivalent to the notion of fibration. And we will also need the more restrictive notion of *split fibration*. This is more restrictive in the sense that not every *fibration* can be made *split*. For this reason, we will try to get rid of the hypothesis on a splitting for a fibration in order to view a fibration simply as a functor with certain properties. Indeed we manage to prove that the pseudo-algebras for the monad which gives the monadicity of the split fibrations over a fixed base category are the cloven fibrations over such base category.

In the first chapter we present some preliminary notions which we will need in the following chapters. We assume throughout that the reader is familiar with the basic definitions and tools of category theory, and we just reduce ourselves to list the notations we shall use for such basic notions just after this introduction.

In the second chapter we introduce the notions of *Grothendieck fibrations*, *cloven fibrations* and *split fibrations*. And we show two fundamental constructions of fibrations. We will then describe the 2-categories of fibrations.

In the third chapter we prove the monadicity result for split fibrations, explaining why we have to restrict ourselves first to the cloven fibrations and then to the split fibrations. Then, in the last section of the third chapter, we prove that the pseudo-algebras for the monad which gives the monadicity result for split fibrations over a fixed base category are the cloven fibrations over such base category.

In the fourth chapter we prove the comonadicity of split fibrations over a fixed base category, arguing also why we have to restrict ourselves to such fibrations. The proof of the comonadicity is more technical, and use arguments inspired by the ideas of Yoneda Lemma.

Notations

We will use the following notations for the basic notions we will need.

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathcal{Set}$	the category of sets and functions between sets
$\mathcal{Grp}$	the category of groups and group homomorphisms between them
$\mathcal{Ab}$	the category of abelian groups and groups homomorphisms between them
$\mathcal{Mod}(R)$	the category of R-modules and R-modules homomorphisms between them
$\mathcal{Alg}(R)$	the category of R-algebras and R-algebras homomorphisms
$\mathcal{CRng}$	the category of commutative rings with unit and unitary ring homomorphisms
$\mathcal{Top}$	the category of topological spaces and continuous functions between them
$\mathcal{Cat}$	the category of small categories and functors between them
$\mathcal{CAT}$	the 2-category of small categories, functors and natural transformations
$s\mathcal{Set}$	the category of simplicial sets
$c \in \mathcal{C}$	c is an object of the category $\mathcal{C}$
$c \xrightarrow{\sim} d$	an isomorphism from c to d
$\mathcal{C}(a, b)$	the set of morphisms $a \to b$ in the category $\mathcal{C}$
$\mathrm{Hom}(a, b)$	the set of morphisms $a \to b$ in the evident category we are considering
$\langle L, R, \varphi \rangle$	an adjunction $L \dashv R$ with hom-sets natural bijection $\varphi\colon \mathrm{Hom}(L-, +) \xrightarrow{\sim} \mathrm{Hom}(-, R+)$
id_a	the identity morphism $a \to a$
$\mathrm{Id}_{\mathcal{C}}$	the idenitity functor from a category $\mathcal{C}$ to itself

Chapter 1

Preliminaries

We shall assume that the reader is familiar with the basic definitions and tools of category theory. In this first chapter we will present some preliminary notions which we will need in the following chapters. For the basic notions, we will use the notations written above.

1.1 Horizontal Composition

In this section we will describe the *horizontal composition* of natural transformations and prove the *interchange law*. The main reference for this section will be [ML71].

Firstly, we recall the *vertical composition* between natural transformations.

Construction 1.1.1. Let $\mathcal{C}, \mathcal{D}$ be two categories and let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be three functors. Let then $\lambda \colon F \Rightarrow G$ and $\mu \colon G \Rightarrow H$ be two natural transformations:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "\lambda"] \arrow[Rightarrow, from=M, to=D, "\mu"] \end{tikzcd}$$

We shall then construct a natural transformation (as we shall promplty prove it is such) from F to H, which we denote $\mu \circ \lambda$, defining it on components as

$$(\mu \circ \lambda)_c := \mu_c \circ \lambda_c \colon Fc \to Hc$$

for every $c \in \mathcal{C}$.

Proposition 1.1.2. *Let $\mathcal{C}, \mathcal{D}$ be two categories and let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be three functors. Let then $\lambda \colon F \Rightarrow G$ and $\mu \colon G \Rightarrow H$ be two natural transformations.*
Then the morphisms $(\mu \circ \lambda)_c$ with $c \in \mathcal{C}$ form a natural transformation $\mu \circ \lambda \colon F \Rightarrow H$, called the ***vertical composition of** λ **and** μ.*

Proof. The proof is trivial. □

Remark 1.1.3. We see that the vertical composition of natural transformations is associative and has units id_F for every F functor (where id_F is defined to be the identity on each component).

Now we want to construct an *horizontal composition* of natural transformations.

We first show how to compose "horizontally" a natural transformation and a functor. This operation is called *whiskering*.

Construction 1.1.4. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two parallel functors and let $\lambda\colon F \Rightarrow G$ be a natural transformation. Let then $H\colon \mathcal{B} \to \mathcal{C}$ and $K\colon \mathcal{D} \to \mathcal{E}$ be two functors.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "H"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{D} \arrow[r, "K"] & \mathcal{E} \end{tikzcd}$$

We shall construct a natural transformation (as we shall promptly prove it is such)

$$\lambda H\colon FH \Rightarrow GH$$

called the ***whiskering of*** λ ***with*** H, defining it on components as

$$(\lambda H)_b := \lambda_{Hb}\colon FHb \to GHb$$

for every $b \in \mathcal{B}$.

And we shall also construct a natural transformation (as we shall promptly prove it is such)

$$K\lambda\colon KF \Rightarrow KG$$

called the ***whiskering of*** λ ***with*** K, defining it on components as

$$(K\lambda)_c := K(\lambda_c)\colon KFc \to KGc$$

for every $c \in \mathcal{C}$.

Proposition 1.1.5. *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two parallel functors and let $\lambda\colon F \Rightarrow G$ be a natural transformation. Let then $H\colon \mathcal{B} \to \mathcal{C}$ and $K\colon \mathcal{D} \to \mathcal{E}$ be two functors. Then the morphisms $(\lambda H)_b$ with $b \in \mathcal{B}$ form a natural transformation from FH to GH, and the morphisms $(K\lambda)_c$ with $c \in \mathcal{C}$ form a natural transformation from KF to KG.*

Proof. The proof is trivial. □

Remark 1.1.6. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Let then $F, G\colon \mathcal{C} \to \mathcal{D}$ be two parallel functors and let $H, K\colon \mathcal{D} \to \mathcal{E}$ be two other parallel functors:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{E} \end{tikzcd}$$

We want to construct a natural tranformation $HF \Rightarrow KG$. We see that there are two ways of defining such a natural transformation, which, as we shall promptly prove, are equivalent.

The first possibility is defining it as $\mu G \circ H\lambda$; the second one is defining it as $K\lambda \circ \mu F$.

Proposition 1.1.7. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Let then $F, G\colon \mathcal{C} \to \mathcal{D}$ be two parallel functors and let $H, K\colon \mathcal{D} \to \mathcal{E}$ be two other parallel functors:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{E} \end{tikzcd}$$

Then both $\mu G \circ H\lambda$ and $K\lambda \circ \mu F$ are natural transformation from HF to KG. Furthermore,

$$\mu G \circ H\lambda = K\lambda \circ \mu F.$$

This equality is called ***interchange law****. We shall write the interchange law in general as*

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \\ \cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \\ \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

Proof. The first assertion is clear. Let now $c \in \mathcal{C}$. We need to prove that

$$(\mu G)_c \circ (H\lambda)_c = (K\lambda)_c \circ (\mu F)_c.$$

But by Construction 1.1.4 this means

$$\mu_{Gc} \circ H(\lambda_c) = K(\lambda_c) \circ \mu_{Fc}.$$

And this holds by naturality of μ:

$$\begin{tikzcd} HFc \arrow[r, "\mu_{Fc}"] \arrow[d, "H\lambda_c"'] & KFc \arrow[d, "K\lambda_c"] \\ HGc \arrow[r, "\mu_{Gc}"'] & KGc \end{tikzcd}$$

□

Definition 1.1.8. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Let then $F, G\colon \mathcal{C} \to \mathcal{D}$ be two parallel functors and let $H, K\colon \mathcal{D} \to \mathcal{E}$ be two other parallel functors. We call any of the two equal natural transformation $\mu G \circ H\lambda$ and $K\lambda \circ \mu F$ from HF to KG the ***horizontal composition of*** λ ***and*** μ. We shall denote such a composition of λ and μ as $\mu * \lambda$.

Remark 1.1.9. The horizontal composition of natural transformations is clearly associative and has units given by $\mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}$ with $\mathcal{C}$ a category and $\mathrm{Id}_{\mathcal{C}}$ the identity functor from $\mathcal{C}$ to $\mathcal{C}$.

Note also that with the notion of horizontal composition we recover the notion of whiskering by considering one of the two natural transformations as an identity. Namely, in the notations of Definition 1.1.8, we have that $\mu * \mathrm{id}_F = \mu F$ and that $\mathrm{id}_H * \lambda = H\lambda$.

1.2 Adjunctions

We want to show some useful characterizations of adjunction, given by universal morphisms. We will then be interested in generalizing such characterizations to 2-categories, in Theorem 1.9.5.

Firstly, we recall the basic theorem which provides us with the concepts of unit and counit of an adjunction.

Theorem 1.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and consider* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$ *an adjunction with hom-sets natural bijection φ given by*

$$\varphi_{c,d}\colon \mathcal{D}(Lc,d) \xrightarrow{\sim} \mathcal{C}(c,Rd)$$

for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
Then this adjunction $\langle L, R, \varphi\rangle$ determines:

(i) *a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$, called the* ***unit of the adjunction****, such that for every $c \in \mathcal{C}$ the arrow η_c in $\mathcal{C}$ is universal from c to R, and for every morphism $f\colon Fc \to d$ in $\mathcal{D}$ the right adjunct of f is*

$$\varphi f = Rf \circ \eta_c\colon c \to Rd; \tag{1.1}$$

(ii) *a natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$, called the* ***counit of the adjunction****, such that for every $d \in \mathcal{D}$ the arrow ε_a in $\mathcal{D}$ is universal from L to d, and for every morphism $g\colon c \to Rd$ in $\mathcal{C}$ the left adjunct of g is*

$$\varphi^{-1} g = \varepsilon_d \circ Lg\colon Lc \to d. \tag{1.2}$$

Moreover both the following equalities, called the ***triangular identities****, hold:*

$$\left(L \overset{L\eta}{\Rightarrow} LRL \overset{\varepsilon L}{\Rightarrow} L\right) = \mathrm{id}_L \qquad \left(R \overset{\eta R}{\Rightarrow} RLR \overset{R\varepsilon}{\Rightarrow} R\right) = \mathrm{id}_R \tag{1.3}$$

Proof. The less expert reader shall find a proof of this theorem in [ML71] [IV.1 Theorem 1]. □

It is interesting to note that various portions of these data determined by an adjunction actually determine an adjunction, as we show in the following theorem.

Theorem 1.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then each adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$ *with hom-sets natural bijection φ given by*

$$\varphi_{c,d}\colon \mathcal{D}(Lc,d) \xrightarrow{\sim} \mathcal{C}(c,Rd)$$

for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ is completely determined by the data in any one of the following lists:

(*i*) *functors L and R, and a natual transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ such that for every $c \in \mathcal{C}$ the arrow $\eta_c\colon c \to RLc$ is universal from c to R; then φ is determined by equation (1.1);*

(*ii*) *the functor $R\colon \mathcal{D} \to \mathcal{C}$ and for every $c \in \mathcal{C}$ an object $L_0c \in \mathcal{D}$ and a universal arrow $\eta_c\colon c \to RL_0c$ from c to R; then the functor L has object function L_0 and is defined on arrows $h\colon c \to c'$ by*

$$RLh \circ \eta_c = \eta_{c'} \circ h;$$

(*iii*) *functors L and R, and a natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ such that for every $d \in \mathcal{D}$ the arrow $\varepsilon_d\colon LRd \to d$ is universal from L to d; then φ^{-1} is determined by equation (1.2);*

(*iv*) *the functor $L\colon \mathcal{C} \to \mathcal{D}$ and for every $d \in \mathcal{D}$ an object $R_0d \in \mathcal{C}$ and a universal arrow $\varepsilon_d\colon LR_0d \to d$ from L to d; then the functor R is determined in a way dual to the one described in point (ii);*

(*v*) *functors L and R, and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to RL$ and $\varepsilon\colon LR \to \mathrm{Id}_{\mathcal{D}}$ such that the triangular identities (described in equation (1.3)) hold; then φ is determined by equation (1.1) and φ^{-1} is determined by equation (1.2).*

Proof. (*i*) Let $c \in \mathcal{C}$. The statement that η_c is universal means that for every $f\colon c \to Rd$ in $\mathcal{C}$ there is exactly one morphism $g\colon Lc \to d$ such that the diagram

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "f"'] & RLc \arrow[d, dashed, "Rg"] \\ & Rd \end{tikzcd}$$

is commutative. This states precisely that the assignment

$$g \longmapsto \theta(g) = Rg \circ \eta_c$$

defines a bijection

$$\theta\colon \mathcal{C}(Lc, d) \to \mathcal{D}(c, Rd)$$

which is natural in c because η is natural, and is natural in d because R is a functor. Thus, θ gives an adjunction $\langle L, R, \theta\rangle$. In case η was the unit obtained from an adjunction $\langle L, R, \varphi\rangle$, then $\theta = \varphi$.

(*ii*) We show that the data (*ii*) can be expanded to (*i*), and hence determine the adjunction. We show that there is exactly one way to construct a functor L which has object function L_0 and such that $\eta\colon \mathrm{Id}_{\mathcal{C}} \to RL$ is a natural transformation. Indeed for every morphism $h\colon c \to c'$ in $\mathcal{C}$ the fact that η_c is a universal arrow states that there is exactly one arrow

$Lh\colon F_0c \to F_0c'$ such that the diagram

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "h"'] & RL_0c \arrow[d, dashed, "RLh"] \\ c' \arrow[r, "\eta_{c'}"'] & LR_0c' \end{tikzcd}$$

is commutative, and we define in this way the image of h under L. The commutativity of the diagram above then states that η is a natural transformation. Finally, it is straightforward to prove that these assignments Lh with $h\colon c \to c'$ in $\mathcal{C}$ make L into a functor from $\mathcal{C}$ to $\mathcal{D}$.

(iii) It is dual to (i).

(iv) It is dual to (ii).

(v) We use the natural transformations η and ε to define for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ functions $\varphi_{c,d}$ and $\psi_{c,d}$

$$\mathcal{C}(Lc, d) \underset{\psi_{c,d}}{\overset{\varphi_{c,d}}{\rightleftarrows}} \mathcal{D}(c, Rd)$$

by setting

$$\varphi_{c,d} f = Rf \circ \eta_c$$

for every $f\colon Fc \to d$ in $\mathcal{D}$ and

$$\psi_{c,d}\, g = \varepsilon_d \circ Lg$$

for every $g\colon c \to Rd$ in $\mathcal{C}$.

Then since R is a functor and η is natural we get that

$$(\varphi_{c,d} \circ \psi_{c,d})\, g = R\varepsilon_d \circ RLg \circ \eta_c = R\varepsilon_d \circ \eta_{Rd} \circ g.$$

But since the triangular identities (described in equation (1.3)) hold by assumption, we get that $G\varepsilon_d \circ \eta_{Gd} = \mathrm{id}$, whence we get that $\varphi_{c,d} \circ \psi_{c,d} = \mathrm{id}$. Dually, we also get that $\psi_{c,d} \circ \varphi_{c,d} = \mathrm{id}$.

Therefore $\varphi_{c,d}$ is a bijection for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Moreover these morphisms $\varphi_{c,d}$ clearly form a natural transformation. We then get an adjunction, which, in case we started from an adjunction $\langle L, R, \theta \rangle$, coincides with that original one. □

1.3 Pasting Diagrams

In this section we present some ideas about the *pasting diagrams*, and we give an explicit definition for simple cases, which are the only ones we will use.

Firstly, we show the most basic cases of pasting diagrams.

Definition 1.3.1. Let $\mathcal{C}$ be a category. A ***pasting diagram in*** $\mathcal{C}$ is a sequence of composable morphisms in $\mathcal{C}$

$$a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_{n+1}.$$

We think of these arrows as not yet composed, but "pasted together" at their objects, forming a composable sequence. Note that there is a unique way to assign a "value" $f\colon a_1 \longrightarrow a_{n+1}$ to the sequence above, and in this case this is just the composite of the sequence.
We call the ***value of the pasting diagram*** the composite of the sequence.

Now, we consider $\mathcal{C}at$, and try to generalize the intuitive idea we have just seen to encode in a diagram vertical composites of horizontal composites of natural transformations. In general, we shall apply the same ideas to an arbitrary 2-category (which we will define in Section 1.7).

The idea is to call *pasting diagram* a juxtaposition of diagrams showing the domain and the codomain of some natural transformations which has a unique value, in terms of vertical composition of horizontal compositions of the natural transformations of the diagram.

A formal definition of *pasting diagrams* is not easy. We will then show the explicit definition for an easy case in $\mathcal{C}at$. However, we will not use bigger diagrams than the one we describe below.

Construction 1.3.2. Consider the following diagram in $\mathcal{C}at$

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Nearrow\lambda"] & b \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\Nearrow\mu"] & c \arrow[d, "c"] \\
x \arrow[r, "h"'] & y \arrow[r, "k"'] & z
\end{tikzcd} \tag{1.4}$$

with λ and μ two natural transformations (in general, for a 2-category $\mathcal{K}$, these would be 2-cells in $\mathcal{K}$). This is the prototype of a ***pasting diagram in*** $\mathcal{C}at$ (in general, ***in the*** 2***-category*** $\mathcal{K}$).

We want to assign a *value* to this diagram, that is, a natural transformation

$$\theta\colon cgf \Rightarrow kha.$$

And we want to consider a sort of "directionality". Then we see that we need some parts of the codomain of λ in order to apply μ, and we decide to assign a value which starts from λ. At this point, also looking at Proposition 1.1.7 we define the ***value of the pasting diagram*** 1.4 to be the composite

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\Nearrow\mu"] & c \arrow[d, "c"] \\
 & y \arrow[r, "k"'] & z
\end{tikzcd}
\circ
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Nearrow\lambda"] & b \arrow[d, "b"] & \\
x \arrow[r, "h"'] & y \arrow[r, "k"'] & z
\end{tikzcd}$$

that is, $\mu f \circ k\lambda$. It is easy to show that this is an acceptable solution for θ.

We will thus write (pasting) diagrams like

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "a"] & b \arrow[r, "g"] \arrow[d, "b"] & c \arrow[d, "c"] \\ x \arrow[r, "h"'] \arrow[ur, phantom, "\Nearrow\lambda"] & y \arrow[r, "k"'] \arrow[ur, phantom, "\Nearrow\mu"] & z \end{tikzcd}$$

with the meaning of the composition $\mu f \circ k\lambda$. Note that we shall now paste also a triangle and a square or two triangles (and not only two squares as above), since we can always view a triangle as a square with the identity on one edge.

Analogously, if we have the same kind of diagram but with reversed natural transformations λ and μ

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "a"] \arrow[dr, phantom, "\Swarrow\lambda"] & b \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\Swarrow\mu"] & c \arrow[d, "c"] \\ x \arrow[r, "h"'] & y \arrow[r, "k"'] & z \end{tikzcd} \tag{1.5}$$

we have that this is another ***pasting diagram in*** $\mathcal{C}at$ (in general, ***in the*** 2***-category*** $\mathcal{K}$). Thinking again to a sort of "directionality" we define the ***value of the pasting diagram*** 1.5 to be the composite

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "a"] \arrow[dr, phantom, "\Swarrow\lambda"] & b \arrow[d, "b"] & \\ x \arrow[r, "h"'] & y \arrow[r, "k"'] & z \end{tikzcd} \quad \circ \quad \begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\Swarrow\mu"] & c \arrow[d, "c"] \\ & y \arrow[r, "k"'] & z \end{tikzcd}$$

that is, $k\lambda \circ \mu f$.

We will need just these two kinds of pasting diagrams. Anyway, these definitions shall be extended to bigger diagrams with various shapes. The idea is viewing diagram (1.4) and diagram (1.5) as building blocks for bigger diagrams, and define a ***pasting diagram*** to be some diagram which can be assigned a unique value decomposing it into pieces like diagram (1.4) and diagram (1.5), that is, when every possible path to break it into such pieces gives the same value (the same natural transformation, or, in general, the same 2-cell in the 2-category $\mathcal{K}$). However a formal definition for general diagrams is not easy, since it requires to impose a "consistent directionality". In fact if, for example, we have a diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "a"] & b \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\Swarrow\mu"] & c \arrow[d, "c"] \\ x \arrow[r, "h"'] \arrow[ur, phantom, "\Nearrow\lambda"] & y \arrow[r, "k"'] & z \end{tikzcd}$$

we cannot decide a value of it, since we cannot decide whether starting with λ or with μ. Then we say that this diagram is not a pasting diagram.

Remark 1.3.3. Note that the pasting diagrams show in a clearer way more information than the composite written as equation, for example the domains and the codomains of the natural transformations and of the functors involved. And the squares which form the pasting diagram will also be "more basic" than the whole composite, such as in Definition 1.3.1 one might expect the factors f_i of the composite to be "more basic" than the whole composition.

Example 1.3.4. In order to understand better the language of pasting diagrams, we want to show how we can translate into this language the triangular identities of an adjunction $\langle L, R, \varphi \rangle$ with unit η and counit ε.

Remember that these are

$$\left(L \overset{L\eta}{\Rightarrow} LRL \overset{\varepsilon L}{\Rightarrow} L\right) = \mathrm{id}_L \qquad\qquad \left(R \overset{\eta R}{\Rightarrow} RLR \overset{R\varepsilon}{\Rightarrow} R\right) = \mathrm{id}_R$$

We then easily see that the triangular identity on the left can be rephrased as the equality

$$\begin{tikzcd}
\cdot \arrow[r, "L"] \arrow[d, equal] \arrow[dr, phantom, pos=0.3, "\overset{\eta}{\Rightarrow}"] \arrow[dr, phantom, pos=0.75, "\overset{\varepsilon}{\Rightarrow}"] & \cdot \arrow[d, equal] \arrow[dl, "R"] \\
\cdot \arrow[r, "L"'] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r, bend left, "L"{name=U}] \arrow[r, bend right, "L"'{name=D}] & \cdot
\arrow[from=U, to=D, equal, "\mathrm{id}_L"]
\end{tikzcd}$$

and that the triangular identity on the right can be rephrased as the equality

$$\begin{tikzcd}
\cdot \arrow[r, "R"] \arrow[d, equal] \arrow[dr, phantom, pos=0.3, "\overset{\varepsilon}{\Leftarrow}"] \arrow[dr, phantom, pos=0.75, "\overset{\eta}{\Leftarrow}"] & \cdot \arrow[d, equal] \arrow[dl, "L"] \\
\cdot \arrow[r, "R"'] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r, bend left, "R"{name=U}] \arrow[r, bend right, "R"'{name=D}] & \cdot
\arrow[from=U, to=D, equal, "\mathrm{id}_R"]
\end{tikzcd}$$

Note that in the language of pasting diagrams the triangular identities appear to be much more natural.

1.4 Monads and Algebras

In this section, we introduce the concepts of monads and of algebras for a monad. The main references will be [ML71] and [Sch19].

Throughout mathematics we encounter structures defined by some action morphisms. Here we give some examples.

Example 1.4.1. 1. Given a group G, we may consider a G-set X described by an action map $G \times X \to X$.

2. Given an abelian group M and a ring R, we can get an R-module M by fixing a group homomorphism $R \otimes_{\mathbb{Z}} M \to M$.

3. Given a monoid M in $\mathcal{S}et$, we get a map

$$\Pi_{k=1}^n M \to M, \quad (m_1, \dots, m_n) \mapsto ((\dots((m_1 m_2)m_3)\dots)m_{n-1})m_n.$$

This induces an action map from $W(M) = \amalg_{n\in\mathbb{N}} \Pi_{k=1}^n M$, the set of words on M, to M.

4. Given a directed graph $D = (V, E, E \overset{s}{\underset{t}{\rightrightarrows}} V)$, we can create its free category FD, where the objects are the vertices and for every $v, w \in V$ the set of morphisms from v to w is $FD(v, w) = \{\text{finite paths } v \to \cdots \to w\}$. For every $v \in V$ we set id_v to be the path of length 0, while composition is defined to be just the concatenation of paths.

   In particular, if D is the directed graph with $V = \{0, \dots, n\}$ and an edge $j \to k$ if and only if $k = j + 1$, we have $FD \cong [n]$.

   If $D = \{*\}$ and $E = \{* \to *\}$, then $FD(*, *) \cong \mathbb{N}$.

   Given a small category $\mathcal{C}$, we may consider the underlying directed graph $U\mathcal{C} = D$ with $V = \mathrm{Ob}(\mathcal{C})$, $E = \mathcal{C}^{\to}$, $s = \mathrm{dom}$ and $t = \mathrm{cod}$. We then get an action map $UFU\mathcal{C} \longrightarrow U\mathcal{C}$ sending a finite path to its composite. This map is a morphism of directed graphs.

Note that in all these examples we have a category $\mathcal{C}$ and some functor $T\colon \mathcal{C} \to \mathcal{C}$, which can be used to give structure to an object C of $\mathcal{C}$, by giving an action map $T(C) \to C$ in $\mathcal{C}$.

We can see these examples as specific instances of a general phenomenon, introducing the concept of *monad*.

Definition 1.4.2. A ***monad*** on a category $\mathcal{C}$ is a triple (T, μ, η) where $T\colon \mathcal{C} \to \mathcal{C}$ is a functor, and $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow T$ are natural transformations such that the diagrams

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \tag{1.6}$$

commute. Sometimes, we will refer to a monad (T, μ, η) writing just T.

The natural transformation μ is called the ***multiplication of*** T, while η is called the ***unit of*** T. These names come from the fact that the definition of a monad is, formally, like the definition of a monoid in $\mathcal{S}et$.

We can then view the two commutative diagrams above as an axiom of associativity and two axioms of unit. To appreciate this point of view, it might be useful to see the commutativity of the two diagrams also in the language of pasting diagrams.

The commutativity of the diagram on the left of equation (1.6) is translated in the language of pasting diagrams as the equality of the following two diagrams:

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
$$

Whereas the commutativity of the diagram on the right of equation (1.6) can be rephrased as the following two equalities:

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\eta \quad \Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, bend left=60, equal] \arrow[ur, bend right, "T"'] \arrow[rr, "T"'] & {} & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "T"] \arrow[d, phantom, "="] \\
\mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, bend right, "T"'] \arrow[dr, bend left=60, equal] \arrow[d, phantom, "\Downarrow\mu \quad \Downarrow\eta"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathcal{C}
\end{tikzcd}
$$

A monad naturally defines other algebraic structures, as we now describe. This will be the desired generalization of the phenomenon we have seen in Example 1.4.1.

Definition 1.4.3. Given a monad (T, μ, η) on $\mathcal{C}$, a ***T-algebra*** (or ***T-module***) is a pair (a, α) with $a \in \mathcal{C}$ and $\alpha \colon Ta \to a$ a morphism in $\mathcal{C}$ such that the following diagrams commute:

$$
\begin{tikzcd}
T^2 a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\
Ta \arrow[r, "\alpha"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\
 & a
\end{tikzcd}
$$

We will sometimes refer to the commutativity of the diagram on the left and of the diagram on the right as the two axioms of T-algebra for (a, α).

A ***morphism of T-algebras*** $(a, \alpha) \to (b, \beta)$ is a morphism $f \colon a \to b$ in $\mathcal{C}$ such that the following diagram commutes in $\mathcal{C}$:

$$
\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
$$

We will sometimes refer to the commutativity of this diagram as the axiom of morphism of T-algebras for $f \colon (a, \alpha) \to (b, \beta)$.

Construction 1.4.4. Let (T, μ, η) be a monad on $\mathcal{C}$. Consider the following data, which we shall denote $T\!-\!\mathcal{A}\!\ell g$ and call the ***category of T-algebras*** (as we shall promptly prove it is such):

an object of $T\!-\!\mathcal{A}\!\ell g$ is a T-algebra (a, α);

a morphism $(a, \alpha) \to (b, \beta)$ **in** $T\!-\!\mathcal{A}\!\ell g$ is a morphism of T-algebras $(a, \alpha) \xrightarrow{f} (b, \beta)$.

Proposition 1.4.5. *Let (T, μ, η) be a monad on $\mathcal{C}$. Then the data $T\!-\!\mathcal{Alg}$, with composition inherited by that of $\mathcal{C}$, form a category. Moreover there is a natural forgetful functor*

$$U^T : T\!-\!\mathcal{Alg} \longrightarrow \mathcal{C}$$
$$(a, \alpha) \longmapsto a$$
$$((a, \alpha) \xrightarrow{f} (b, \beta)) \longmapsto (a \xrightarrow{f} b)$$

Proof. The proof is trivial. □

We now show how to recover the examples we have seen in Example 1.4.1 with the language we have introduced.

Example 1.4.6. 1. Let G be a group. Then the functor

$$T = G \times - : \mathcal{Set} \longrightarrow \mathcal{Set}$$

equipped with the natural transformations $\mu \colon T^2 \Rightarrow T$ and $\eta \colon \mathrm{Id}_{\mathcal{Set}} \Rightarrow T$ defined by

$$\begin{aligned} \mu_A \colon \ & G \times (G \times A) \to G \times A \\ & (g, (h, a)) \mapsto (gh, a) \end{aligned}$$

$$\begin{aligned} \eta_A \colon \ & A \to G \times A \\ & a \mapsto (e, a) \end{aligned}$$

for every $A \in \mathcal{Set}$ is a monad. Moreover it is easy to see that (A, α) is a T-algebra if and only if A is a G-set, with structure of G-set given by the action map α.

It is straightforward to show that the category $T\!-\!\mathcal{Alg}$ is isomorphic to the category of G-sets (with G-equivariant morphisms).

2. Given a ring R, the functor $T = R \otimes_{\mathbb{Z}} - \colon \mathcal{Ab} \to \mathcal{Ab}$ equipped with the natural transformations $\mu \colon T^2 \Rightarrow T$ and $\eta \colon \mathrm{Id}_{\mathcal{Ab}} \Rightarrow T$ defined by

$$\begin{aligned} \mu_- \colon \ & R \otimes_{\mathbb{Z}} (R \otimes_{\mathbb{Z}} -) \cong (R \otimes_{\mathbb{Z}} R) \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \\ \eta_- \colon \ & - \cong \mathbb{Z} \otimes_{\mathbb{Z}} - \Rightarrow R \otimes_{\mathbb{Z}} - \end{aligned}$$

is a monad. Moreover $(R \otimes_{\mathbb{Z}} -)\!-\!\mathcal{Alg} \cong \mathcal{Mod}(R)$.

3. The assignment $WX = \amalg_{n \in \mathbb{N}} \Pi_{k=1}^{n} X$ for every $X \in \mathcal{Set}$ extends to a functor

$$W \colon \mathcal{Set} \longrightarrow \mathcal{Set}.$$

Equipping the functor W with the natural transformations $\mu = (\mu_X \colon WWX \to WX)_{X \in \mathcal{Set}}$ given by concatenation of words and and $\eta \colon \mathrm{Id}_{\mathcal{Ab}} \Rightarrow T$ defined by

$$\begin{aligned} \eta_X \colon X & \longrightarrow WX \\ x & \longmapsto (x) \end{aligned}$$

we obtain a monad. Moreover $W\!-\!\mathcal{Alg} \cong \mathcal{Mon}(\mathcal{Set})$.

4. The free-forgetful adjunction $F \dashv U$ between the category of directed graphs and the category $\mathcal{C}at$ of small categories induces a monad $T = (UF, \mu, \eta)$ on the former, in a general way which we shall describe below, in Proposition 1.4.9. Moreover this monad T is such that $T\!-\!\mathcal{A}lg \cong \mathcal{C}at$.

Now that we have introduced these structures, we aim at defining a *monadic functor.* This concept is related with the strong connection between monads and adjunctions.
In the following proposition we show that from a monad we always get an associated adjunction. Below, we will also see that from an adjunction we can always construct a monad.

Proposition 1.4.7. *Let (T, μ, η) be a monad on $\mathcal{C}$. Then the functor $U^T\colon T\!-\!\mathcal{A}lg \longrightarrow \mathcal{C}$ has a left adjoint*

$$F^T\colon \mathcal{C} \longrightarrow T\!-\!\mathcal{A}lg$$

$$c \longmapsto (Tc, \mu_c)$$

$$(f\colon c \to d) \longmapsto (Tf\colon (Tc, \mu_c) \to (Td, \mu_d))$$

such that $U^T F^T = T$. Furthermore, the unit η^T of this adjunction is given by η and the counit ε^T has components $\varepsilon^T_{(a,\alpha)} = \alpha\colon (Ta, \mu_a) \to (a, \alpha)$ for every $(a, \alpha) \in T\!-\!\mathcal{A}lg$.

Proof. (i) Let $c \in \mathcal{C}$. To prove that (Tc, μ_c) is a T-algebra we need to show that the following two diagrams are commutative:

$$\begin{tikzcd} T^3c \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] & T^2c \arrow[d, "\mu_c"] \\ T^2c \arrow[r, "\mu_c"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} Tc \arrow[r, "\eta_{Tc}"] \arrow[dr, equal] & T^2c \arrow[d, "\mu_c"] \\ & Tc \end{tikzcd}$$

But these are exactly the associativity axiom and one of the two unit axioms for the monad (T, μ, η).

(ii) Let now $f\colon c \to c'$ be a morphism in $\mathcal{C}$. Then Tf is a morphism of algebras from (Tc, μ_c) to $(Tc', \mu_{c'})$ because the diagram

$$\begin{tikzcd} T^2c \arrow[r, "T^2f"] \arrow[d, "\mu_c"'] & T^2c' \arrow[d, "\mu_{c'}"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

is commutative by naturality of μ. Thus F^T is defined on morphisms. It is a functor by functoriality of T, and it is clear that $U^{\mathrm{T}}F^{\mathrm{T}} = T$, by construction of F^{T}.

(iii) The unit η is a natural transformation by assumption.

Now consider $(a,\alpha) \in T\!-\!\mathcal{Alg}$. We see that

$$F^T U^T(a,\alpha) = F^T a = (Ta, \mu_a).$$

Therefore, in order to show that $\varepsilon^T_{(a,\alpha)} = \alpha$ is a morphism of T-algebras from $F^T U^T(a,\alpha)$ to $\mathrm{Id}_{T\text{-}\mathcal{Alg}}(a,\alpha)$, it remains to show that the square

$$\begin{tikzcd} T^2 a \arrow[r, "T\alpha"] \arrow[d, "\mu_a"'] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"'] & a \end{tikzcd}$$

is commutative. But this is one of the two axioms of T-algebra for (a,α).

To prove that $\varepsilon^T \colon F^T U^T \Rightarrow \mathrm{Id}_{T\text{-}\mathcal{Alg}}$ is a natural transformation we need to show that the square

$$\begin{tikzcd} (Ta,\mu_a) \arrow[r, "\alpha=\varepsilon^T_{(a,\alpha)}"] \arrow[d, "Tf"'] & (a,\alpha) \arrow[d, "f"] \\ (Tb,\mu_b) \arrow[r, "\beta=\varepsilon^T_{(b,\beta)}"'] & (b,\beta) \end{tikzcd}$$

is commutative, but this is the axiom of morphism of T-algebras for f.

(iv) It only remains to check the two triangular identities, which are

$$\varepsilon^T F^T \circ F^T \eta = \mathrm{id}_{F^T} \qquad \text{and} \qquad U^T \varepsilon^T \circ \eta U^T = \mathrm{id}_{U^T}\,.$$

We check these two equalities on components. Let $c \in \mathcal{C}$ and let $(a,\alpha) \in T\!-\!\mathcal{Alg}$. On these components the two equalities above can be rephrased as the commutativity of the diagrams

$$\begin{tikzcd} (Tc,\mu_c) \arrow[r, "T\eta_c"] \arrow[dr, equal] & (T^2c,\mu_{Tc}) \arrow[d, "\mu_{Tc}"] \\ & (Tc,\mu_c) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, equal] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

And the commutativity of diagram on the left is given by the second unit axiom for the monad T, while that of the diagram on the right is one of the two axioms of T-algebra for (a,α). ☐

Definition 1.4.8. Given a monad (T,μ,η) on $\mathcal{C}$, the T-algebras of the form (Tc,μ_c) with $c \in \mathcal{C}$ are called ***free*** T***-algebras***.

Proposition 1.4.9. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ *be an adjunction, with unit* η *and counit* ε*. Then*

$$(UF, U\varepsilon F, \eta)$$

is a monad on $\mathcal{C}$. Furthermore, if (T, μ, η^T) is a monad on $\mathcal{C}$, and we consider the associated adjunction $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ with unit η^T and counit ε^T described in Proposition 1.4.7, we have that

$$(U^T F^T, U^T \varepsilon^T F^T, \eta^T) = (T, \mu, \eta^T).$$

Proof. To prove the first assertion, we check the axioms of monad for $(UF, U\varepsilon F, \eta)$. The axiom of associativity

$$\begin{tikzcd} UFUFUF \arrow[r, Rightarrow, "UFU\varepsilon F"] \arrow[d, Rightarrow, "U\varepsilon FUF"'] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\ UFUF \arrow[r, Rightarrow, "U\varepsilon F"'] & UF \end{tikzcd}$$

holds by naturality of the natural transformation $U\varepsilon \colon UFU \Rightarrow U$, whereas the unit axioms of monad

$$\begin{tikzcd} UF \arrow[r, Rightarrow, "\eta UF"] \arrow[dr, equal] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] & UF \arrow[l, Rightarrow, "UF\eta"'] \arrow[dl, equal] \\ & UF & \end{tikzcd}$$

hold by the triangular identities of the adjunction $F \dashv U$:

$$U\varepsilon \circ \eta U = \mathrm{id}_U \qquad \text{and} \qquad \varepsilon F \circ F\eta = \mathrm{id}_F\,.$$

Finally, the fact that

$$(U^T F^T, U^T \varepsilon^T F^T, \eta^T) = (T, \mu, \eta^T)$$

immediatly follows by Proposition 1.4.7. Notice in particular that for every $c \in \mathcal{C}$ we have that

$$(U^{\mathrm{T}} \varepsilon^T F^{\mathrm{T}})_c = U^{\mathrm{T}}(\varepsilon^T_{Tc,\mu_c}) = U^{\mathrm{T}}(\mu_c) = \mu_c. \qquad \square$$

Remark 1.4.10. Proposition 1.4.9 shows that an adjunction always produces a monad. This also justifies what we have said in Example 1.4.6 (4).

Furthermore, the last assertion of Proposition 1.4.9 ensures that if we start from a monad T, consider the associated adjunction $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ and then consider the monad associated to such adjunction we obtain the monad we started from.

However if we start from an adjunction, consider the associated monad and then consider the adjunction associated to this monad we do not find in general the adjunction we started from. However we have a canonical *comparison functor*, which we describe in the following theorem. This fundamental difference we have in general between the two paths we have just described gives rise to the concept of *monadic functor*.

Theorem 1.4.11. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ *be an adjunction with unit* η *and counit* ε*. Consider the associated monad on* $\mathcal{C}$

$$T = (UF, U\varepsilon F, \eta)$$

(*see Proposition 1.4.9*) *and then consider the adjunction* $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ *associated to the monad* T (*see Proposition 1.4.7*)*. Then there exists a unique functor*

$$\overline{U}\colon \mathcal{D} \longrightarrow T\text{–}\mathcal{Alg}$$

such that $U^{\mathrm{T}}\overline{U} = U$ *and* $\overline{U}F = F^{\mathrm{T}}$*:*

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed, "\overline{U}"] \arrow[dr, shift right, "U"'] & T\text{–}\mathcal{Alg} \arrow[d, shift left, "U^{\mathrm{T}}"] \\ & \mathcal{C} \arrow[ul, shift right, "F"'] \arrow[u, shift left, "F^{\mathrm{T}}"] \end{tikzcd}$$

Moreover $\overline{U}$ *is given by*

$$\begin{aligned} \overline{U}\colon \mathcal{D} &\longrightarrow T\text{–}\mathcal{Alg} \\ d &\longmapsto (Ud, U\varepsilon_d) \\ (d \xrightarrow{h} d') &\longmapsto (Uh\colon Ud \to Ud') \end{aligned}$$

Proof. Let $d \in \mathcal{D}$. Note that

$$U\varepsilon_d\colon UFUd \longrightarrow Ud$$

and thus, in order to prove that $(Ud, U\varepsilon_d)$ is a T-algebra, it only remains to show that the following two diagrams are commutative:

$$\begin{tikzcd} UFUFUd \arrow[r, "UFU\varepsilon_d"] \arrow[d, "(U\varepsilon F)_{Ud}"'] & UFUd \arrow[d, "U\varepsilon_d"] \\ UFUd \arrow[r, "U\varepsilon_d"'] & Ud \end{tikzcd} \qquad \begin{tikzcd} Ud \arrow[r, "\eta_{Ud}"] \arrow[dr, equal] & UFUd \arrow[d, "U\varepsilon_d"] \\ & Ud \end{tikzcd}$$

The diagram on the left is commutative because it is the image under the functor U of a diagram which is commutative by the interchange law (see Proposition 1.1.7) for

$$\begin{tikzcd} \cdot \arrow[r, bend left, "FU"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \cdot \arrow[r, bend left, "FU"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \cdot \end{tikzcd}$$

Whereas the diagram on the right is commutative by one of the two triangular identities for the adjunction $F \dashv U$.

Let now $h\colon d \to d'$ be a morphism in $\mathcal{D}$. In order to prove that Uh is a morphism of T-algebras, it suffices to show that the square

$$\begin{tikzcd} UFUd \arrow[r, "UFUh"] \arrow[d, "U\varepsilon_d"'] & UFUd' \arrow[d, "U\varepsilon_{d'}"] \\ UFd \arrow[r, "Uh"'] & UFd' \end{tikzcd}$$

is commutative. But this immediatly follows by the naturality of ε.

At this point, it is straightforward to verify that $\overline{U}$ is a functor and that $U^{\mathrm{T}}\overline{U} = U$ and $\overline{U}F = F^{\mathrm{T}}$.

Then it remains to prove the uniqueness of $\overline{U}$. Since it must hold that $U^{\mathrm{T}}\overline{U} = U$, we see that for every $d \in \mathcal{D}$ we must have that $\overline{U}d$ is a T-algebra with underlying object Ud. Say that $\overline{U}d = (Ud, w)$, with $w\colon UFUd \longrightarrow Ud$. For the proof that w has to coincide with $U\varepsilon_d$ see [ML71] [VI.3.Theorem 1]. (However we will not use the uniqueness of $\overline{U}$.)

Finally, on morphisms h in $\mathcal{D}$ we have to define $\overline{U}(h)$ to be Uh, because it must hold that $U^{\mathrm{T}}\overline{U} = U$. $\square$

Definition 1.4.12. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ be an adjunction. The functor $\overline{U}$ described in Theorem 1.4.11 is called the ***comparison functor associated to the adjunction*** $F \dashv U$ (sometimes we say ***to the functor*** U, remembering that U is a right adjoint) ***and to the monad*** T.

Definition 1.4.13. Let $U\colon \mathcal{D} \longrightarrow \mathcal{C}$ be a right adjoint. Say that $F \dashv U$ with unit η and counit ε and consider the monad $(UF, U\varepsilon F, \eta)$ associated to that adjunction (see Proposition 1.4.9). We say that U is a ***monadic functor*** if the comparison functor $\overline{U}\colon \mathcal{D} \longrightarrow T\!-\!\mathcal{A}lg$ is an equivalence of categories.

Remark 1.4.14. Informally, we call a right adjoint U a monadic functor if also the second path we have described in Remark 1.4.10 gives essentially what we started from. That is, if considering the monad associated to the adjunction of which U is part and then considering the adjunction associated to such monad we get essentially the same adjunction we started from.

We conclude this section showing a simple tool we can use to establish that a functor is not monadic. We will not need this result, but it might be useful to see an example of a non-monadic functor.

Lemma 1.4.15. *Let (T, μ, η) be a monad on $\mathcal{C}$. Then functor U^T we have described in Proposition 1.4.5 is conservative, that is, for every morphism f in $T\!-\!\mathcal{A}lg$, if $U^T f$ is an isomorphism in $\mathcal{C}$ then f is an isomorphism of T-algebras.*

Proof. Let $f\colon (a,\alpha) \to (b,\beta)$ be a morphism in $T-\mathcal{Alg}$. Suppose that $g\colon b \to a$ is the inverse of $U^T f = f\colon a \to b$. We only need to prove that in the diagram

$$\begin{tikzcd} Tb \arrow[r, "Tg"] \arrow[d, "\beta"'] & Ta \arrow[r, "Tf"] \arrow[d, "\alpha"'] & Tb \arrow[d, "\beta"] \\ b \arrow[r, "g"'] & a \arrow[r, "f"'] & b \end{tikzcd}$$

the square on the left commutes, since then it follows that g is a right inverse of f in $T-\mathcal{Alg}$, and with an analogous argument we can see that g is a left inverse of f as well. We see that $f \circ g \circ \beta = \beta$ and $f \circ \alpha \circ Tg = \beta \circ Tf \circ Tg = \beta \circ T(f \cdot g) = \beta \circ T\,\mathrm{id}_b = \beta$. Then we conclude by injectivity of f. $\square$

Example 1.4.16. Thanks to Lemma 1.4.15, we have an easy tool to say that some functor cannot be monadic.

For example the forgetful functor $U\colon \mathcal{Top} \to \mathcal{Set}$ cannot be monadic since it does not reflect isomorphisms. However, if we restrict ourselves to the full subcategory of $\mathcal{Top}$ spanned by compact Hausdorff spaces we indeed obtain a monadic functor.

In this case, we can also see more explicitly why the functor U is not monadic. We calculate the category of algebras associated to U, which is a right adjoint, and see that it is not equivalent to $\mathcal{Top}$.

We know that U is part of the adjunction $\begin{tikzcd} \mathcal{Set} \arrow[r, bend left, "\mathrm{Disc}"] \arrow[r, phantom, "\perp"] & \mathcal{Top} \arrow[l, bend left, "U"] \end{tikzcd}$, with $\mathrm{Disc} =: F$ the functor which endows every set with the discrete topology. We immediatly see that $UFX = X$ for every $X \in \mathcal{Set}$, whence $UF = \mathrm{Id}_{\mathcal{Set}}$. But there is a unique natural transformation $\alpha\colon \mathrm{Id}_{\mathcal{Set}} \Rightarrow \mathrm{Id}_{\mathcal{Set}}$, which is the identity $\mathrm{id}_{\mathrm{Id}_{\mathcal{Set}}}$. In fact $\mathrm{Id}_{\mathcal{Set}} \cong \mathrm{Hom}(*,-)$, with $*$ the terminal object of $\mathcal{Set}$, and we then see that

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{Set}}, \mathrm{Id}_{\mathcal{Set}}) \cong \mathrm{Nat}(\mathrm{Hom}(*,-), \mathrm{Hom}(*,-)) \cong \mathrm{Hom}(*,*) = \{\mathrm{id}_*\}$$

by Yoneda Lemma, whence $\alpha = \mathrm{Id}_{\mathcal{Set}}$.

It follows that the monad T associated to the adjunction $F \dashv U$ is

$$\mathrm{T} := (UF, U\varepsilon F, \eta) = (\mathrm{Id}_{\mathcal{Set}}, \mathrm{id}, \mathrm{id}).$$

Therefore $T-\mathcal{Alg} \cong \mathcal{Set}$, and $\mathcal{Set}$ is not equivalent to $\mathcal{Top}$, whence U is not monadic.

1.5 Comonads and Coalgebras

All the definitions of Section 1.4 can be dualized. A dualized version of all the results we have seen in Section 1.4 will continue to hold.

We state here for clarity the main definitions and results in dualized version.

Definition 1.5.1. A ***comonad*** on a category $\mathcal{C}$ is a triple $(\Omega, \delta, \varepsilon)$ where $\Omega\colon \mathcal{C} \to \mathcal{C}$ is a functor, and $\delta\colon \Omega \Rightarrow \Omega^2$ and $\varepsilon\colon \Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ are natural transformations such that the diagrams

$$
\begin{tikzcd}
\Omega \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & \Omega^2 \arrow[d, Rightarrow, "\Omega\delta"] \\
\Omega^2 \arrow[r, Rightarrow, "\delta\Omega"'] & \Omega^3
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega & \Omega^2 \arrow[l, Rightarrow, "\varepsilon\Omega"'] & \Omega \arrow[l, Rightarrow, "\Omega\varepsilon"'] \\
& \Omega \arrow[ul, equal] \arrow[u, Rightarrow, "\delta"'] \arrow[ur, equal] &
\end{tikzcd}
\tag{1.7}
$$

commute. Sometimes we will refer to a comonad $(\Omega, \delta, \varepsilon)$ writing just Ω.

The natural transformation δ is called the ***comultiplication of*** Ω, while ε is called the ***counit of*** Ω.

We can then view the two commutative diagrams above as an axiom of coassociativity and two axioms of counit. To appreciate this point of view, it might be useful to see the commutativity of the two diagrams also in the language of pasting diagrams.

The commutativity of the diagram on the left of equation (1.7) is translated in the language of pasting diagrams as the equality of the following two diagrams:

$$
\begin{tikzcd}
& \mathcal{C} \arrow[r, "\Omega"] \arrow[drr, bend right, "\Omega"] \arrow[d, phantom, "\Uparrow\delta"] & \mathcal{C} \arrow[dr, "\Omega"] \arrow[d, phantom, "\Uparrow\delta"] & \\
\mathcal{C} \arrow[ur, "\Omega"] \arrow[rrr, "\Omega"'] & {} & {} & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[r, "\Omega"] \arrow[d, phantom, "\Uparrow\delta"] & \mathcal{C} \arrow[dr, "\Omega"] \arrow[d, phantom, "\Uparrow\delta"] & \\
\mathcal{C} \arrow[ur, "\Omega"] \arrow[urr, bend right, "\Omega"] \arrow[rrr, "\Omega"'] & {} & {} & \mathcal{C}
\end{tikzcd}
$$

Whereas the commutativity of the diagram on the right of equation (1.7) can be rephrased as the following two equalities:

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dr, "\Omega"] \arrow[d, phantom, "\Uparrow\varepsilon\ \ \Uparrow\delta"] & \\
\mathcal{C} \arrow[ur, equal, bend left] \arrow[ur, bend right, "\Omega"'] \arrow[rr, "\Omega"'] & {} & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[d, bend right, "\Omega"'] \arrow[d, bend left, "\Omega"] \arrow[d, phantom, "="] \\
\mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[dr, equal, bend left] \arrow[dr, bend right, "\Omega"'] \arrow[d, phantom, "\Uparrow\delta\ \ \Uparrow\varepsilon"] & \\
\mathcal{C} \arrow[ur, "\Omega"] \arrow[rr, "\Omega"'] & {} & \mathcal{C}
\end{tikzcd}
$$

Definition 1.5.2. Given a comonad $(\Omega, \delta, \varepsilon)$ on $\mathcal{C}$, an Ω***-coalgebra*** is a pair (b, β) with $b \in \mathcal{C}$ and $\beta\colon b \to \Omega b$ a morphism in $\mathcal{C}$ such that the following diagrams commute:

$$
\begin{tikzcd}
b \arrow[r, "\beta"] \arrow[d, "\beta"'] & Tb \arrow[d, "\Omega\beta"] \\
Tb \arrow[r, "\delta_b"'] & T^2 b
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "\beta"] \arrow[dr, equal] & \Omega b \arrow[d, "\varepsilon_b"] \\
& b
\end{tikzcd}
$$

We will sometimes refer to the commutativity of the diagram on the left and of the diagram on the right as the two axioms of Ω-coalgebra for (b, β).

A ***morphism of* Ω*-coalgebras*** $(b,\beta) \to (c,\gamma)$ is a morphism $f\colon b \to c$ such that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "\beta"'] & c \arrow[d, "\gamma"] \\ \Omega b \arrow[r, "\Omega f"'] & \Omega c \end{tikzcd}$$

We will sometimes refer to the commutativity of this diagram as the axiom of morphism of Ω-coalgebras for $f\colon (b,\beta) \to (c,\gamma)$.

Construction 1.5.3. Let $(\Omega,\delta,\varepsilon)$ be a comonad on $\mathcal{C}$. Consider the following data, which we shall denote $\Omega-\mathcal{CoAlg}$ and call the category of Ω-coalgebras (as we shall promptly prove it is such):

an object of $\Omega-\mathcal{CoAlg}$ is an Ω-coalgebra (b,β);

a morphism $(b,\beta) \to (c,\gamma)$ **in** $\Omega-\mathcal{CoAlg}$ is a morphism of Ω-coalgebras $(b,\beta) \xrightarrow{f} (c,\gamma)$.

Proposition 1.5.4. *Let $(\Omega,\delta,\varepsilon)$ be a comonad on $\mathcal{C}$. Then the data Ω–$\mathcal{CoAlg}$, with composition inherited by that of $\mathcal{C}$, form a category. Moreover there is a natural forgetful functor*

$$U_\Omega\colon \Omega-\mathcal{CoAlg} \longrightarrow \mathcal{C}$$
$$(b,\beta) \longmapsto b$$
$$((b,\beta) \xrightarrow{f} (c,\gamma)) \longmapsto (b \xrightarrow{f} c)$$

Proof. The proof is trivial. □

We show that a comonad always have an associated adjunction.

Proposition 1.5.5. *Let $(\Omega,\delta,\varepsilon)$ be a comonad on $\mathcal{C}$. Then the functor $U_\Omega\colon \Omega-\mathcal{CoAlg} \longrightarrow \mathcal{C}$ has a right adjoint*

$$R_\Omega\colon \mathcal{C} \longrightarrow \Omega-\mathcal{CoAlg}$$
$$c \longmapsto (\Omega c, \delta_c)$$
$$(f\colon c \to d) \longmapsto (\Omega f\colon (\Omega c,\delta_c) \to (\Omega d,\delta_d))$$

such that $U_\Omega R_\Omega = \Omega$. Furthermore, the counit ε_Ω of this adjunction is given by ε and the unit η_Ω has components $\eta_{\Omega,(b,\beta)} = \beta\colon (b,\beta) \to (\Omega b,\delta_b)$ for every $(b,\beta) \in \Omega-\mathcal{CoAlg}$.

Proof. Immediate, since it is just a dualization of Proposition 1.4.7. □

Definition 1.5.6. Given a comonad $(\Omega,\delta,\varepsilon)$ on $\mathcal{C}$, the Ω-coalgebras of the form $(\Omega c,\delta_c)$ with $c \in \mathcal{C}$ are called ***free*** Ω***-coalgebras***.

Now we show that an adjunction always produces a comonad.

Proposition 1.5.7. *Let* $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$ *be an adjunction, with unit* η *and counit* ε*. Then*

$$(LR, L\eta R, \varepsilon)$$

is a comonad on $\mathcal{C}$*. Furthermore, if* $(\Omega, \delta, \varepsilon_\Omega)$ *is a comonad on* $\mathcal{C}$*, and we consider the associated adjunction* $U_\Omega \dashv R_\Omega$ *with unit* η_Ω *and counit* ε_Ω *described in Proposition 1.5.5, we have that*

$$(U_\Omega R_\Omega, U_\Omega \eta_\Omega R_\Omega, \varepsilon_\Omega) = (\Omega, \delta, \varepsilon_\Omega).$$

Proof. Immediate, since it is just a dualization of Proposition 1.4.9. □

Theorem 1.5.8. *Let* $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$ *be an adjunction with unit* η *and counit* ε*. Consider the associated comonad on* $\mathcal{C}$

$$\Omega = (LR, L\eta R, \varepsilon)$$

(*see Proposition 1.5.7*) *and then consider the adjunction* $U_\Omega \dashv R_\Omega$ *associated to the comonad* Ω (*see Proposition 1.5.5*)*. Then there exists a unique functor*

$$\overline{L}\colon \mathcal{D} \longrightarrow T\!-\!\mathcal{Alg}$$

such that $U_\Omega \overline{L} = L$ *and* $\overline{L}R = R_\Omega$*:*

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed, "\overline{L}"] \arrow[dr, shift left, "L"] & \Omega\!-\!\mathcal{CoAlg} \arrow[d, shift right, "U_\Omega"'] \\ & \mathcal{C} \arrow[ul, shift left, "R"] \arrow[u, shift right, "R_\Omega"'] \end{tikzcd}$$

Moreover $\overline{L}$ *is given by*

$$\overline{L}\colon \mathcal{D} \longrightarrow \Omega\!-\!\mathcal{CoAlg}$$
$$d \longmapsto (Ld, L\eta_d)$$
$$(d \xrightarrow{h} d') \longmapsto (Lh\colon Ld \to Ld')$$

Proof. Immediate, since it is just a dualization of Theorem 1.4.11 □

Definition 1.5.9. Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "U"] \end{tikzcd}$ be an adjunction. The functor $\overline{L}$ described in Theorem 1.5.8 is called the ***comparison functor associated to the adjunction*** $L \dashv R$ (sometimes we say ***to the functor*** L, remembering that L is a left adjoint) ***and to the comonad*** Ω.

Definition 1.5.10. Let $L\colon \mathcal{D} \longrightarrow \mathcal{C}$ be a left adjoint. Say that $L \dashv R$ with unit η and counit ε and consider the comonad $(LR, L\eta R, \varepsilon)$ associated to that adjunction (see Proposition 1.5.7). We say that L is a ***comonadic functor*** if the comparison functor $\overline{L}\colon \mathcal{D} \longrightarrow \Omega-\mathcal{CoAlg}$ is an equivalence of categories.

1.6 Monoidal Categories

We aim at defining an *enriched category*. Firstly, we need to introduce the *monoidal categories*. Then we will see that we can consider mathematical objects *in a monoidal category* $\mathcal{V}$, such as *monoids in* $\mathcal{V}$. Finally we will give the definition of a *category enriched over a monoidal category* $\mathcal{V}$ (or $\mathcal{V}$*-category*). The main references for this section are [Kel82] and [Sch19].

Definition 1.6.1. A ***monoidal category*** is a sextuple $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$, where $\mathcal{V}$ is a category, $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a functor, $I \in \mathcal{V}$ is an object, and $\alpha\colon (-\otimes -)\otimes - \Rightarrow - \otimes (-\otimes -)$, $\lambda\colon I \otimes - \Rightarrow \mathrm{id}$ and $\rho\colon -\otimes I \Rightarrow \mathrm{id}$ are natural isomorphisms such that for every $W, X, Y, Z \in \mathcal{V}$ the diagrams

$$
\begin{tikzcd}
 & (W\otimes X)\otimes(Y\otimes Z) \arrow[rrd, "\alpha_{W,X,Y\otimes Z}"] & & \\
((W\otimes X)\otimes Y)\otimes Z \arrow[ru, "\alpha_{W\otimes X,Y,Z}"] \arrow[rd, "\alpha_{W,X,Y}\otimes \mathrm{id}_Z"'] & & & W\otimes(X\otimes(Y\otimes Z)) \\
 & (W\otimes(X\otimes Y))\otimes Z \arrow[r, "\alpha_{W,X\otimes Y,Z}"'] & W\otimes((X\otimes Y)\otimes Z) \arrow[ru, "\mathrm{id}_W\otimes\alpha_{X,Y,Z}"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(X\otimes I)\otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[rd, "\rho_X\otimes \mathrm{id}_Y"'] & & X\otimes(I\otimes Y) \arrow[ld, "\mathrm{id}_X\otimes\lambda_Y"] \\
 & X\otimes Y &
\end{tikzcd}
$$

are commutative. Sometimes we will refer to a monoidal category $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ writing just $(\mathcal{V}, \otimes, I)$ or even just $\mathcal{V}$.

We call $\otimes$ the ***tensor product of*** $\mathcal{V}$, I the ***unit object*** or ***tensor unit of*** $\mathcal{V}$, α the ***associator***, λ the ***left unitor*** and ρ the ***right unitor***.

Notation 1.6.2. For simplicity, for the rest of this section we will denote $-\otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes -$ as $-\otimes X$ and $X \otimes -$ respectively (for every coherent object X).

We now show some basic examples of monoidal categories.

Example 1.6.3. 1. If $\mathcal{E}$ is a category with finite products, then $(\mathcal{E}, \times, *)$ (where $*$ denotes the terminal object of $\mathcal{E}$) is a monoidal category, with α, λ and ρ induced by the universal property of product. Instances of this example are $\mathcal{Set}$, $\mathcal{Cat}$, $\mathcal{Grp}$, $s\mathcal{Set}$, $\mathcal{Top}$.

2. $(\mathcal{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and in general, given a commutative ring R, $(\mathcal{Mod}(R), \otimes_R, R)$ are monoidal categories, with α, λ and ρ induced by the universal property of tensor product.

3. The order $\overline{\mathbb{R}}_+ = [0, \infty]$ with $\otimes = +$, $I = 0$ and α, λ and ρ defined to be the identity is a monoidal category.

Definition 1.6.4. A ***lax monoidal functor*** from a monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}, \alpha^{\mathcal{V}}, \lambda^{\mathcal{V}}, \rho^{\mathcal{V}})$ to a monoidal category $(\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}}, \alpha^{\mathcal{W}}, \lambda^{\mathcal{W}}, \rho^{\mathcal{W}})$ is a triple (F, φ_0, φ), where $F\colon \mathcal{V} \to \mathcal{W}$ is a functor, $\varphi_0\colon I_{\mathcal{W}} \to FI_{\mathcal{V}}$ a morphism in $\mathcal{W}$ and $\varphi\colon \otimes_{\mathcal{W}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{W}}$ is a natural transformation such that for every $X, Y, Z \in \mathcal{V}$ the diagrams

$$\begin{tikzcd}
(FX \otimes_{\mathcal{W}} FY) \otimes_{\mathcal{W}} FZ \arrow[r, "\alpha^{\mathcal{W}}_{FX,FY,FZ}"] \arrow[d, "\varphi_{X,Y} \otimes_{\mathcal{W}} FZ"'] & FX \otimes_{\mathcal{W}} (FY \otimes_{\mathcal{W}} FZ) \arrow[d, "FX \otimes_{\mathcal{W}} \varphi_{Y,Z}"] \\
F(X \otimes_{\mathcal{V}} Y) \otimes_{\mathcal{W}} FZ \arrow[d, "\varphi_{X \otimes_{\mathcal{V}} Y, Z}"'] & FX \otimes_{\mathcal{W}} F(Y \otimes_{\mathcal{V}} Z) \arrow[d, "\varphi_{X, Y \otimes_{\mathcal{V}} Z}"] \\
F((X \otimes_{\mathcal{V}} Y) \otimes_{\mathcal{V}} Z) \arrow[r, "F\alpha^{\mathcal{V}}_{X,Y,Z}"'] & F(X \otimes_{\mathcal{V}} (Y \otimes_{\mathcal{V}} Z))
\end{tikzcd}$$

and

$$\begin{tikzcd}
I_{\mathcal{W}} \otimes_{\mathcal{W}} FX \arrow[r, "\varphi_0 \otimes_{\mathcal{W}} FX"] \arrow[d, "\lambda^{\mathcal{W}}_{FX}"'] & FI_{\mathcal{V}} \otimes_{\mathcal{W}} FX \arrow[d, "\varphi_{I_{\mathcal{V}}, X}"] \\
FX & F(I_{\mathcal{V}} \otimes_{\mathcal{V}} X) \arrow[l, "F\lambda^{\mathcal{V}}_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes_{\mathcal{W}} I_{\mathcal{W}} \arrow[r, "FX \otimes_{\mathcal{W}} \varphi_0"] \arrow[d, "\rho^{\mathcal{W}}_{FX}"'] & FX \otimes_{\mathcal{W}} FI_{\mathcal{V}} \arrow[d, "\varphi_{X, I_{\mathcal{V}}}"] \\
FX & F(X \otimes_{\mathcal{V}} I_{\mathcal{V}}) \arrow[l, "F\rho^{\mathcal{V}}_X"]
\end{tikzcd}$$

are commutative.

Reversing the direction of φ_0 and φ we get the definition of an ***oplax monoidal functor***.

A ***strong monoidal functor*** is a lax monoidal functor (F, φ_0, φ) such that φ_0 and φ are isomorphisms.

A ***strict monoidal functor*** is a lax monoidal functor (F, φ_0, φ) such that φ_0 and φ are identities.

Definition 1.6.5. Let $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}, \alpha^{\mathcal{V}}, \lambda^{\mathcal{V}}, \rho^{\mathcal{V}})$ and $(\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}}, \alpha^{\mathcal{W}}, \lambda^{\mathcal{W}}, \rho^{\mathcal{W}})$ two monoidal categories and let (F, φ_0, φ) and (G, ψ_0, ψ) two lax monoidal functors.
A ***monoidal natural transformation*** from (F, φ_0, φ) to (G, ψ_0, ψ) is a natural transformation $\gamma\colon F \Rightarrow G$ such that the diagrams

$$\begin{tikzcd} FX \otimes_{\mathcal{W}} FY \arrow[r, "\gamma_X \otimes_{\mathcal{W}} \gamma_Y"] \arrow[d, "\varphi_{X,Y}"'] & GX \otimes_{\mathcal{W}} GY \arrow[d, "\psi_{X,Y}"] \\ F(X \otimes_{\mathcal{V}} Y) \arrow[r, "\gamma_{X \otimes_{\mathcal{V}} Y}"'] & G(X \otimes_{\mathcal{V}} Y) \end{tikzcd} \qquad \begin{tikzcd} & I_{\mathcal{W}} \arrow[dl, "\varphi_0"'] \arrow[dr, "\psi_0"] & \\ FI_{\mathcal{V}} \arrow[rr, "\gamma_{I_{\mathcal{V}}}"'] & & GI_{\mathcal{V}} \end{tikzcd}$$

are commutative.

Proposition 1.6.6. *Lax monoidal functors compose and monoidal natural transformations whisker with lax monoidal functors.*

Proof. The proof is straightforward. □

Example 1.6.7. Given a locally small monoidal category $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$, the hom-functor

$$\mathcal{V}(I, -)\colon \mathcal{V} \to \mathcal{S}et$$

is a lax monoidal functor, with

$$\varphi_0\colon \{*\} \to \mathcal{V}(I, I), \quad * \mapsto \mathrm{id}_I$$

and

$$\varphi_{X,Y}\colon \mathcal{V}(I,X) \times \mathcal{V}(I,Y) \longrightarrow \mathcal{V}(I, X \otimes Y)$$
$$(f,g) \longmapsto (f \otimes g) \circ \lambda_I^{-1} = (f \otimes g) \circ \rho_I^{-1}\colon I \xrightarrow{\sim} I \otimes I \to X \otimes Y$$

for every $X, Y \in \mathcal{V}$.
This lax monoidal functor is universally denoted by $V\colon \mathcal{V} \to \mathcal{S}et$.

If $\mathcal{V}$ has coproducts, then V has a left adjoint given by $F\colon \mathcal{S}et \to \mathcal{V}$, $S \mapsto \amalg_S I$. Assuming for simplicity that $\mathcal{V}$ is cocomplete, it is easy to show that F is strong monoidal if $\otimes$ preserves colimits in each variable by using that $\mathcal{S}et$ is the free cocomplete category on $\{*\}$.

1.7 Categories Enriched over a Monoidal Category

We now want to show that we can consider mathematical objects *in a monoidal category*. We see the example of a *monoid in a monoidal category*. This will also provide us with an intuitive way to introduce *categories enriched over a monoidal category*, requiring though good enough monoidal categories to *enrich over*. Finally, we will extend this definition to that of a category enriched over an arbitrary monoidal category. The main references for this section are still [Kel82] and [Sch19].

Definition 1.7.1. A ***monoid in a monoidal category*** $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ is a triple (M, m, u) with $M \in \mathcal{V}$, and $m\colon M \otimes M \to M$ and $u\colon I \to M$ are morphisms in $\mathcal{V}$, called respectively the ***multiplication of*** M and the ***unit of*** M, such that the diagrams

$$
\begin{tikzcd}
 & (M \otimes M) \otimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[dl, "m\otimes M"'] & M \otimes (M \otimes M) \arrow[dr, "M\otimes m"] & \\
M \otimes M \arrow[drr, "m"'] & & & M \otimes M \arrow[dll, "m"] \\
 & & M &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
I \otimes M \arrow[r, "u\otimes M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "m"] & M \otimes I \arrow[l, "M\otimes u"'] \arrow[dl, "\rho_M"] \\
 & M &
\end{tikzcd}
$$

are commutative.

A ***morphism of monoids in*** $\mathcal{V}$ from (M, m, u) to (M', m', u') is a morphism $f\colon M \to M'$ in $\mathcal{V}$ such that $m' \circ (f \otimes f) = f \circ m$ and $f \circ u = u'$.

Construction 1.7.2. Consider the following data, which we shall denote $\mathcal{Mon}(\mathcal{V})$ and call the ***category of monoids in*** $\mathcal{V}$ (as we shall promptly prove it is such):

an object of $\mathcal{Mon}(\mathcal{V})$ is a monoid (M, m, u) in $\mathcal{V}$;

a morphism $(M, m, u) \to (M', m', u')$ **in** $\mathcal{Mon}(\mathcal{V})$ is a morphism f of monoids in $\mathcal{V}$.

Proposition 1.7.3. *The data* $\mathcal{Mon}(\mathcal{V})$ *form a category.*

Proof. The proof is trivial. □

Example 1.7.4. Consider $\mathcal{V} = \mathcal{Mod}(R)$ with R a commutative ring. We have said in Example 1.6.3 that $\mathcal{V}$ is a monoidal category. It is easy to see that $\mathcal{Mon}(\mathcal{Mod}(R)) = \mathcal{Alg}(R)$.

Because of this example, if $\mathcal{V}$ is additive, monoids in $\mathcal{V}$ are often called algebras in $\mathcal{V}$ as well.

Now we show an intuitive way to introduce *enriched categories*, which requires, though, to assume that the monoidal category we start from has all the coproducts and is such that for every $X \in \mathcal{V}$ both the functors $X \otimes -$ and $- \otimes X$ preserve coproducts. Finally, we will see how to define *enriched categories* over arbitrary monoidal categories.

Construction 1.7.5. Let $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category with all the coproducts and such that for every $X \in \mathcal{V}$ both the functors $X \otimes -$ and $- \otimes X$ preserve coproducts. Let then S be a set.

Consider the functor category $[S \times S, \mathcal{V}] = \prod_{S\times S} \mathcal{V}$, which we shall denote $\mathcal{Mat}(\mathcal{V}, S)$ and call the ***category of*** $\mathcal{V}$***-matrices with index set*** S, viewing each functor $M\colon S \times S \to \mathcal{V}$ as a sort of matrix of size $|S| \times |S|$ and entries in $\mathcal{V}$.

We shall then endow $\mathcal{M}at(\mathcal{V}, S)$ with the matrix multiplication

$$(M(x,y))_{(x,y)\in S^2} \otimes (N(x,y))_{(x,y)\in S^2} = \left(\sum_{z\in S} (M(z,y)\otimes N(x,z))\right)_{(x,y)\in S^2}$$

and the unit $(I_{x,y})_{(x,y)\in S^2}$ defined by

$$\begin{cases} I_{x,y} = I & \text{if } x = y \\ I_{x,y} = \emptyset & \text{otherwise.} \end{cases}$$

(where $\emptyset$ denotes the initial object of $\mathcal{V}$).

Proposition 1.7.6. *Let $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category with all the coproducts and such that for every $X \in \mathcal{V}$ both the functors $X \otimes -$ and $- \otimes X$ preserve coproducts. Let then S be a set.*
Then the category $\mathcal{M}at(\mathcal{V}, S)$ endowed with the matrix multiplication and the unit described in Construction 1.7.5, and with α', λ' and ρ' induced by the ones of $\mathcal{V}$ via the universal property of coproducts is a monoidal category.

Proof. The proof is straightforward. $\square$

Definition 1.7.7. Let $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category with all the coproducts and such that for every $X \in \mathcal{V}$ both the functors $X \otimes -$ and $- \otimes X$ preserve coproducts. Let then S be a set.
A (***small***) ***category enriched over*** $\mathcal{V}$, or (***small***) $\mathcal{V}$***-category***, ***with object set S*** is a monoid in $\mathcal{M}at(\mathcal{V}, S)$.
That is, $\mathcal{A}$ is a category enriched over $\mathcal{V}$ with object set S if for every pair $(a,b) \in S^2$ there is an object $\mathcal{A}(a,b) \in \mathcal{V}$, called the $\mathcal{V}$***-object of homomorphisms***, for every $a \in S$ there is a morphism $\mathrm{id}_a \colon I \to \mathcal{A}(a,a)$ in $\mathcal{V}$ (given by the unit of the monoid), called the ***identity of*** a, and for every $a, b, c \in S$ there is an homomorphism

$$\mathcal{A}(b,c) \otimes \mathcal{A}(a,b) \xrightarrow{C_{a,b,c}} \mathcal{A}(a,c)$$

in $\mathcal{V}$ (given by the matrix multiplication and the injections in the coproduct), called the ***composition homomorphism***, such that for every $a, b, c, d \in S$ the diagrams

$$\begin{tikzcd}
(\mathcal{A}(c,d)\otimes\mathcal{A}(b,c))\otimes\mathcal{A}(a,b) \arrow[rr, "\cong", "\alpha_{\mathcal{A}(c,d),\mathcal{A}(b,c),\mathcal{A}(a,b)}"'] \arrow[d, "C_{b,c,d}\otimes\mathcal{A}(a,b)"'] & & \mathcal{A}(c,d)\otimes(\mathcal{A}(b,c)\otimes\mathcal{A}(a,b)) \arrow[d, "\mathcal{A}(c,d)\otimes C_{a,b,c}"] \\
\mathcal{A}(b,d)\otimes\mathcal{A}(a,b) \arrow[dr, "C_{a,b,d}"'] & & \mathcal{A}(c,d)\otimes\mathcal{A}(a,c) \arrow[dl, "C_{a,c,d}"] \\
& \mathcal{A}(a,d) &
\end{tikzcd}$$

and

$$
\begin{tikzcd}
I \otimes \mathcal{A}(a,b) \arrow[r, "\mathrm{id}_b \otimes \mathcal{A}(a,b)"] \arrow[dr, "\lambda_{\mathcal{A}(a,b)}"'] & \mathcal{A}(b,b) \otimes \mathcal{A}(a,b) \arrow[d, "C_{a,b,b}"] \\
& \mathcal{A}(a,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(a,b) \otimes \mathcal{A}(a,a) \arrow[d, "C_{a,a,b}"'] & \mathcal{A}(a,b) \otimes I \arrow[l, "\mathcal{A}(a,b)\otimes \mathrm{id}_a"'] \arrow[dl, "\rho_{\mathcal{A}(a,b)}"] \\
\mathcal{A}(a,b) &
\end{tikzcd}
$$

are commutative.

Note now that this unravelled definition does not require anymore that $\mathcal{V}$ has all the coproducts or that for every $X \in \mathcal{X}$ the functors $X \otimes -$ and $- \otimes X$ preserve coproducts.
Then we shall take this as the definition for (***small***) ***category enriched over*** $\mathcal{V}$, or (***small***) $\mathcal{V}$***-category***, ***with object set*** S when $\mathcal{V}$ is an arbitrary monoidal category.

A (***small***) ***category enriched over*** $\mathcal{V}$, or (***small***) $\mathcal{V}$***-category***, is a pair $(S, \mathcal{A})$ with S a set and $\mathcal{A}$ a category enriched over $\mathcal{V}$ with object set S.
Sometimes we will refer to a $\mathcal{V}$-category $(S, \mathcal{A})$ writing just $\mathcal{A}$; we will then refer to the object set S as $\mathrm{Ob}(\mathcal{A})$. Sometimes we will write $A \in \mathcal{A}$ with the meaning that $A \in \mathrm{Ob}(\mathcal{A})$.

Example 1.7.8. We can see many known structured categories as examples of enriched categories:

- $\mathcal{V} = \mathcal{S}et \rightsquigarrow$ categories;
- $\mathcal{V} = \mathcal{A}b \rightsquigarrow$ additive categories;
- $\mathcal{V} = \mathcal{M}od(R) \rightsquigarrow$ linear categories;
- $\mathcal{V} = \mathcal{T}op \rightsquigarrow$ topological categories;
- $\mathcal{V} = \mathcal{C}at \rightsquigarrow$ 2-categories;
- $\mathcal{V} = n\text{-}\mathcal{C}at \rightsquigarrow$ strict $(n+1)$-categories.

All of these are straightforward to prove, since unravelling the definition of $\mathcal{V}$-category for each of these particular monoidal categories $\mathcal{V}$ (see also Example 1.6.3) we just get a rewriting of the original definitions. Here we are assuming that the reader already knows the definitions of these structured categories in order to understand the example. Anyway, we will unravel the definition of a $\mathcal{C}at$-category in Remark 1.7.14, and the less expert reader shall take that as the definition of a 2-category.

Construction 1.7.9. Let $\mathcal{V}$ be a monoidal category with all the coproducts, and such that for every $X \in \mathcal{V}$ both the functors $X \otimes -$ and $- \otimes X$ preserve coproducts. Let then $f : S \to T$ be a function between sets. We denote

$$
\begin{aligned}
f^* \colon \mathcal{M}at(\mathcal{V}, T) &\to \mathcal{M}at(\mathcal{V}, S) \\
(T \times T \xrightarrow{M} \mathcal{V}) &\mapsto (S \times S \xrightarrow{f \times f} T \times T \xrightarrow{M} \mathcal{V})
\end{aligned}
$$

and $f_*\colon \mathcal{M}at(\mathcal{V}, S) \to \mathcal{M}at(\mathcal{V}, T)$ its left adjoint, given by

$$(f_* M)(a, b) = \sum_{\{(x,y)\colon\, fx=a, fy=b\}} M(x, y).$$

Definition 1.7.10. A ***$\mathcal{V}$-functor*** $(S, \mathcal{A}) \to (T, \mathcal{B})$ from a $\mathcal{V}$-category (S, A) to a $\mathcal{V}$-category $(T, \mathcal{B})$ is a pair $(F, (F_{a,b})_{(a,b)\in S^2})$ with $F\colon S \to T$ a function and $(F_{a,b})_{(a,b)\in S^2}$ a morphism $\mathcal{A} \to F^*\mathcal{B}$ of monoids in $\mathcal{M}at(\mathcal{V}, S)$.
That is, a $\mathcal{V}$-functor from a $\mathcal{V}$-category $(S, \mathcal{A})$ to a $\mathcal{V}$-category $(T, \mathcal{B})$ is a function $F\colon S \to T$ and for every pair $(a, b) \in S^2$ a morphism

$$F_{a,b}\colon \mathcal{A}(a, b) \to \mathcal{B}(Fa, Fb)$$

in $\mathcal{V}$ such that for every $a, b, c \in S$

$$\begin{tikzcd}
\mathcal{A}(b,c) \otimes \mathcal{A}(a,b) \arrow[r, "F_{b,c}\otimes F_{a,b}"] \arrow[d, "C^{\mathcal{A}}_{a,b,c}"'] & \mathcal{B}(Fb,Fc) \otimes \mathcal{B}(Fa,Fb) \arrow[d, "C^{\mathcal{B}}_{Fa,Fb,Fc}"] \\
\mathcal{A}(a,c) \arrow[r, "F_{a,c}"'] & \mathcal{B}(Fa,Fc)
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \arrow[r, "\mathrm{id}^{\mathcal{A}}_a"] \arrow[dr, "\mathrm{id}^{\mathcal{B}}_{Fa}"'] & \mathcal{A}(a,a) \arrow[d, "F_{a,a}"] \\
& \mathcal{B}(Fa,Fa)
\end{tikzcd}$$

where $C^{\mathcal{A}}_{a,b,c}$ and $C^{\mathcal{B}}_{Fa,Fb,Fc}$ are the composition morphisms of, respectively, $\mathcal{A}$ and $\mathcal{B}$, and $\mathrm{id}^{\mathcal{A}}_a$ and $\mathrm{id}^{\mathcal{B}}_{Fa}$ are, respectively, the identity of a in $\mathcal{A}$ and the identity of Fa in $\mathcal{B}$.

Exactly as happened in Definition 1.7.7, this unravelled definition no longer requires that $\mathcal{V}$ has all the coproducts or that for every $X \in \mathcal{X}$ the functors $X \otimes -$ and $- \otimes X$ preserve coproducts.
Then we shall take this as the definition for $\mathcal{V}$ ***functor*** from a $\mathcal{V}$-category $(S, \mathcal{A})$ to a $\mathcal{V}$-category $(T, \mathcal{B})$ when $\mathcal{V}$ is an arbitrary monoidal category.

Construction 1.7.11. Consider the following data, which we shall denote $\mathcal{V}{-}\mathcal{C}at$ and call the ***category of*** (***small***) ***$\mathcal{V}$-categories*** (as we shall promptly prove it is such):

an object of $\mathcal{V}{-}\mathcal{C}at$ is a $\mathcal{V}$-category $(S, \mathcal{A})$;

a morphism $(S, \mathcal{A}) \to (T, \mathcal{B})$ **in $\mathcal{V}{-}\mathcal{C}at$** is a $\mathcal{V}$-functor from $(S, \mathcal{A})$ to $(T, \mathcal{B})$.

Proposition 1.7.12. *The data $\mathcal{V}{-}\mathcal{C}at$ form a category.*

Proof. The proof is trivial. □

Remark 1.7.13. We may define the category of $\mathcal{V}$-*graphs* analogously. We have a natural forgetful functor from $\mathcal{V}-\mathcal{C}at$ to the category of $\mathcal{V}$-graphs.

Remark 1.7.14. We now want to unravel the definition of a $\mathcal{C}at$-category. The less expert reader shall then take this as the definition of a ***2-category***.

A (small) $\mathcal{C}at$-category $\mathcal{K}$ is a pair $(\mathrm{Ob}(\mathcal{K}), \mathcal{K}'')$ with $\mathrm{Ob}(\mathcal{K})$ a set, which we shall call the ***set of objects of*** $\mathcal{K}$ or ***0-cells of*** $\mathcal{K}$ and $\mathcal{K}''$ a $\mathcal{C}at$-category with object set $\mathrm{Ob}(\mathcal{K})$. We shall denote $\mathcal{K}''$ as $\mathcal{K}$.
Then a (small) $\mathcal{C}at$-category is a set $\mathrm{Ob}(\mathcal{K})$ of 0-cells of $\mathcal{K}$ and for every $A, B \in \mathrm{Ob}(\mathcal{K})$ a category $\mathcal{K}(A, B)$ such that all the properties we promptly describe hold.

For every $A, B \in \mathrm{Ob}(\mathcal{K})$, we shall call an object f of the category $\mathcal{K}(A, B)$ a ***1-cell from*** A ***to*** B ***in*** $\mathcal{K}$ and denote it $f\colon A \to B$, and we shall call a morphism λ in $\mathcal{K}(A, B)$ from a 1-cell $f\colon A \to B$ in $\mathcal{K}$ to a 1-cell $g\colon A \to B$ in $\mathcal{K}$ a ***2-cell from*** f ***to*** g, and denote it as

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\lambda"] & B \end{tikzcd}$$

The structure of category of $\mathcal{K}(A, B)$ now tells us that we have an operation of vertical composition on the 2-cells in $\mathcal{K}$

$$\begin{tikzcd} A \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & B \end{tikzcd}$$

(where α and β are two (vertically) composable 2-cells in $\mathcal{K}$) which we shall denote as $\beta \circ \alpha$ and which is associative and has units $\mathrm{id}_f\colon f \Rightarrow f$ for every 1-cell $f\colon A \to B$.
We also have for every $A, B, C \in \mathrm{Ob}(\mathcal{K})$ composition functors

$$\mathcal{K}(B, C) \times \mathcal{K}(A, B) \xrightarrow{C_{A,B,C}} \mathcal{K}(A, C)$$

and identities $\mathrm{id}_A\colon * \to \mathcal{K}(A, A)$, which we shall write as $\mathrm{id}_A\colon A \to A$, such that for every $A, B, C, D \in \mathrm{Ob}(\mathcal{K})$

$$\begin{tikzcd}[column sep=small] (\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) \arrow[rr, equal] \arrow[d, "C_{B,C,D} \otimes \mathcal{A}(A,B)"'] & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \arrow[d, "\mathcal{A}(C,D) \otimes C_{A,B,C}"] \\ \mathcal{A}(B,D) \otimes \mathcal{A}(A,B) \arrow[dr, "C_{A,B,D}"'] & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[dl, "C_{A,C,D}"] \\ & \mathcal{A}(A,D) & \end{tikzcd}$$

and

$$
\begin{tikzcd}
I \otimes \mathcal{A}(A,B) \arrow[r, "\mathrm{id}_B \otimes \mathcal{A}(A,B)"] \arrow[dr, equal] & \mathcal{A}(B,B) \otimes \mathcal{A}(A,B) \arrow[d, "C_{A,B,B}"] \\
& \mathcal{A}(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \arrow[d, "C_{A,A,B}"'] & \mathcal{A}(A,B) \otimes I \arrow[l, "\mathcal{A}(A,B) \otimes \mathrm{id}_A"'] \arrow[dl, equal] \\
\mathcal{A}(A,B)
\end{tikzcd}
$$

are commutative (remember that for the $\mathcal{V}$-category $\mathcal{C}\mathbf{at}$ the natural isomorphisms α, λ and ρ are identities).

These composition functors give, in particular, an operation of horizontal composition of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, which is associative and unital by the diagrams above, and also whiskering operations

$$
\left(
\begin{tikzcd}
B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & C,
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "f"] & B
\end{tikzcd}
\right)
\longmapsto
\begin{tikzcd}
A \arrow[r, bend left, "gf"] \arrow[r, bend right, "hf"'] \arrow[r, phantom, "\Downarrow\alpha f"] & B
\end{tikzcd}
$$

and similarly on the other side. Saying that $C_{A,B,C}$ is a functor means that these operations satisfy the interchange law, that is, that given a diagram of the form

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}
$$

we have $\beta g \circ h\alpha = k\alpha \circ \beta f$. In pictures

$$
\begin{array}{c}
\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\\
\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot
\end{tikzcd}
\end{array}
=
\begin{array}{c}
\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot
\end{tikzcd}
\\
\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\end{array}
$$

Ideed this follows from the fact that giving a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ amounts to giving a compatible collection of functors $F(-,d)$ and $F(c,-)$ for all $(c,d) \in \mathcal{C} \times \mathcal{D}$.

Sometimes we will call a 1-cell in a 2-category $\mathcal{K}$ also ***1-morphism*** or even just ***morphism***, and a 2-cell in $\mathcal{K}$ a ***2-morphism***.

Construction 1.7.15. Consider the following data, which we shall denote $\mathcal{CAT}$ and call the 2-category of (small) categories (as we shall promptly prove it is such):

a 0-cell of $\mathcal{CAT}$ is a (small) category $\mathcal{C}$;

a 1-cell $\mathcal{C} \to \mathcal{D}$ in $\mathcal{CAT}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$;

a 1-cell $F \Rightarrow G$ in $\mathcal{CAT}$ is a natural transformation $\lambda\colon F \Rightarrow G$.

Proposition 1.7.16. *The data $\mathcal{CAT}$ form a 2-category.*

Proof. The proof is trivial. □

Example 1.7.17. 1. Analogously to Construction 1.7.15, one could see that monoidal categories, lax monoidal functors and monoidal natural transformations form a 2-category.

2. Again analogously to Construction 1.7.15, one could see that for $\mathcal{V}$ a monoidal category, (small) $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$ natural transformations (which we shall promptly define) form a 2-category.

Definition 1.7.18. Let $\mathcal{V}$ be a monoidal category. Let $\mathcal{A}$ and $\mathcal{B}$ be two (small) $\mathcal{V}$-categories and $F, G\colon \mathcal{A} \to \mathcal{B}$ be two $\mathcal{V}$-functors. Then a ***$\mathcal{V}$-natural transformation*** $F \Rightarrow G$ is a collection of morphisms in $\mathcal{V}$

$$(\alpha_A\colon I \to \mathcal{B}(FA, GA))_{A\in\mathcal{A}}$$

such that for every $A, B \in \mathrm{Ob}(\mathcal{A})$ the diagram

$$\begin{tikzcd}
 & I \otimes \mathcal{A}(A,B) \arrow[r, "\alpha_B \otimes F"] & \mathcal{B}(FB,GB) \otimes \mathcal{B}(FA,FB) \arrow[dr, "\circ"] & \\
\mathcal{A}(A,B) \arrow[ur, "\lambda^{-1}_{\mathcal{A}(A,B)}"] \arrow[dr, "\rho^{-1}_{\mathcal{A}(A,B)}"'] & & & \mathcal{B}(FA,GB) \\
 & \mathcal{A}(A,B) \otimes I \arrow[r, "G \otimes \alpha_A"'] & \mathcal{B}(GA,GB) \otimes \mathcal{B}(FA,GA) \arrow[ur, "\circ"'] &
\end{tikzcd} \tag{1.8}$$

is commutative.

Construction 1.7.19. We now want to define the whiskering operations. Consider the diagram

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "K"] & \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, "L"] & \mathcal{B}'
\end{tikzcd}$$

where F, G, K, L are $\mathcal{V}$-functors and α is a $\mathcal{V}$-natural transformation (with $\mathcal{V}$ a monoidal category).
We define a $\mathcal{V}$-natural transformation (as we shall promptly prove it is such)

$$L\alpha = ((L\alpha)_A)_{A\in\mathcal{A}} : LF \Longrightarrow LG$$

defining $(L\alpha)_A$ as the composite

$$I \xrightarrow{\alpha_A} \mathcal{B}(FA, GA) \xrightarrow{L} \mathcal{B}'(LFB, LGB)$$

for every $A \in \mathcal{A}$, and a $\mathcal{V}$-natural transformation (as we shall promptly prove it is such)

$$\alpha K = ((\alpha K)_{A'})_{A'\in\mathcal{A}'} : FK \Longrightarrow GK$$

defining $(\alpha K)_{A'}$ as $\alpha_{KA'}\colon I \to \mathcal{B}(FKA', GKA')$.

Proposition 1.7.20. *Consider the diagram*

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "K"] & \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, "L"] & \mathcal{B}'
\end{tikzcd}$$

where F, G, K, L are $\mathcal{V}$-functors and α is a $\mathcal{V}$-natural transformation (with $\mathcal{V}$ a monoidal category).
Then $L\alpha$ and αK, which we have produced in Construction 1.7.19, are $\mathcal{V}$-natural transformations.

Proof. Clearly αK is a $\mathcal{V}$-natural transformation $FK \Rightarrow GK$. To see that also $L\alpha$ is a $\mathcal{V}$-natural transformation, it suffices to glue diagram (1.8) with the following commutative diagram (given by the fact that L is a $\mathcal{V}$-functor):

$$\begin{tikzcd}
\mathcal{B}(FB,GB)\otimes\mathcal{B}(FA,FB) \arrow[r, "L_{FB,GB}\otimes L_{FA,FB}"] \arrow[dr, "\circ"] & \mathcal{B}'(LFB,LGB)\otimes\mathcal{B}'(LFA,LFB) \arrow[dr, "\circ"] & \\
& \mathcal{B}(FA,GB) \arrow[r, "L_{FA,GB}"] & \mathcal{B}'(LFA,LGB) \\
\mathcal{B}(GA,GB)\otimes\mathcal{B}(FA,GA) \arrow[r, "L_{GA,GB}\otimes L_{FA,GA}"'] \arrow[ur, "\circ"'] & \mathcal{B}'(LGA,LGB)\otimes\mathcal{B}'(LFA,LGA) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

□

Construction 1.7.21. Now given a diagram of the form

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{B}
\end{tikzcd}$$

where F, G, H are $\mathcal{V}$-functors and α, β are $\mathcal{V}$-natural transformations (with $\mathcal{V}$ a monoidal category), we define the vertical composition

$$\beta \circ \alpha = ((\beta \circ \alpha)_A)_{A \in \mathcal{A}} : F \Longrightarrow H$$

defining $(\beta \circ \alpha)_A$ as the composite

$$I \xrightarrow{\sim} I \otimes I \xrightarrow{\beta_A \otimes \alpha_A} \mathcal{B}(GA, HA) \otimes \mathcal{B}(FA, GA) \xrightarrow{\circ} \mathcal{B}(FA, HA).$$

The identities for this vertical composition are given, for every $\mathcal{V}$ functor F, by

$$(\mathrm{id}_F)_A := \mathrm{id}_{FA} \colon I \to \mathcal{B}(FA, FA)$$

for every $A \in \mathcal{A}$. We leave it to the reader to check associativity and the interchange law.

Proposition 1.7.22. *Let $\mathcal{V}$ be a monoidal category. The vertical composition of $\mathcal{V}$-natural transformations described in Construction 1.7.21 is associative and satisfies the interchange law.*

Proof. The proof is straightforward. □

Remark 1.7.23. Let $\mathcal{V}$ be a monoidal category. Having defined vertical and horizontal composition of $\mathcal{V}$-natural transformations which satisfy the interchange law, we can extend the notion of pasting diagrams to $\mathcal{V}$-categories.

Remark 1.7.24. We have already said in Example 1.7.8 that if we consider $\mathcal{V} = \mathcal{S}et$ we obtain that the $\mathcal{V}$-categories are just the categories. It is easy to see that the $\mathcal{S}et$-functors are just the functors and that the $\mathcal{S}et$-natural transformations are just the natural transformations.

We have already seen that the case $\mathcal{V} = \mathcal{C}at$ is much more interesting, since the $\mathcal{C}at$-categories are the 2-categories (see Remark 1.7.14). We now want to unravel the definitions of a $\mathcal{C}at$-functor and of a $\mathcal{C}at$-natural transformation. We will see that we get precisely the definitions of a 2-functor and of a 2-natural transformation. The less expert reader shall instead take these as the definitions respectively of a ***2-functor*** and of a ***2-natural transformation***.

Remember that for the monoidal category $\mathcal{C}at$ the natural isomorphisms α, λ and ρ are identities.

Given two 2-categories $\mathcal{K}$ and $\mathcal{K}'$, a $\mathcal{C}at$-functor is a function

$$\begin{aligned} F\colon \operatorname{Ob}(\mathcal{K}) &\longrightarrow \operatorname{Ob}(\mathcal{K}') \\ A &\longmapsto FA \end{aligned}$$

and for every pair $(A, B) \in \operatorname{Ob}(\mathcal{K}) \times \operatorname{Ob}(\mathcal{K})$ a functor

$$F_{A,B}\colon \mathcal{K}(A, B) \longrightarrow \mathcal{K}'(FA, FB)$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B & \longmapsto & FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \end{tikzcd}$$

which satisfies the properties we promptly describe.
Saying that $F_{A,B}$ defines a functor means exactly that the assignment $F_{A,B}$ respects vertical composition. The $\mathcal{C}at$-functor axiom related to units says that for every $A \in \operatorname{Ob}(\mathcal{K})$

$$F \operatorname{id}_A = \operatorname{id}_{FA}\,.$$

The other $\mathcal{C}at$-functor axiom says that the diagram in $\mathcal{C}at$

$$\begin{tikzcd} \mathcal{K}(B, C) \times \mathcal{K}(A, B) \arrow[r, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ"'] & \mathcal{K}'(FB, FC) \times \mathcal{K}'(FA, FB) \arrow[d, "\circ"] \\ \mathcal{K}(A, C) \arrow[r, "F_{A,C}"'] & \mathcal{K}'(FA, FC) \end{tikzcd}$$

is commutative. This means that the collection $(F_{A,B})_{(A,B)\in \mathrm{Ob}(\mathcal{K})\times \mathrm{Ob}(\mathcal{K})}$ preserves the whiskering operation.

Now, we want to unravel the definition of a $\mathcal{C}at$-natural transformation $\alpha\colon F \Longrightarrow G$

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{K}' \end{tikzcd}$$

where F and G are $\mathcal{C}at$-functors.
We have that α is a collection of functors

$$(\alpha_A\colon * \longrightarrow \mathcal{K}'(FA, GA))_{A\in\mathcal{A}},$$

that is, a collection of 1-cells in $\mathcal{K}'$

$$(\alpha_A\colon FA \longrightarrow GA)_{A\in\mathcal{A}}$$

such that the diagram in $\mathcal{C}at$

$$\begin{tikzcd} & \mathcal{K}'(FB,GB)\times\mathcal{K}'(FA,FB) \arrow[dr, "\circ"] & \\ \mathcal{K}(A,B) \arrow[ur, "\alpha_B\times F\circ\lambda^{-1}_{\mathcal{K}(A,B)}"] \arrow[dr, "G\times\alpha_A\circ\rho^{1}_{\mathcal{K}(A,B)}"'] & & \mathcal{K}'(FA,GB) \\ & \mathcal{K}'(GA,GB)\times\mathcal{K}'(FA,GA) \arrow[ur, "\circ"'] & \end{tikzcd}$$

is commutative. On objects it means that for every 1-cell $f\colon A \to B$ in $\mathcal{K}$ the diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

is commutative, whereas on morphisms it means that for every 2-cell $\varphi\colon f \Rightarrow g$ in $\mathcal{K}$ the following equality, written in the language of pasting diagrams, holds:

$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\varphi"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, bend right, "Gg"'] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\varphi"] & GB \end{tikzcd}$$

By Example 1.7.17, we know that taking the 2-categories as 0-cells, the 2-functors as 1-cells and the 2-natural transformations as 2-cells we obtain a 2-category (since it is precisely $\mathcal{V}$-$\mathcal{C}at$ with $\mathcal{V} = \mathcal{C}at$). We will denote such 2-category as 2-$\mathcal{C}at$.

1.8 Slices and Arrow Categories

In this section we show the construction of the *slice of a category* $\mathcal{C}$ *over an object* $C \in \mathcal{C}$ and of the *arrow category* of a category, and then of a 2-category. We will then see that these two constructions have a common generalization, given by the *comma category.*

Construction 1.8.1. Let $\mathcal{C}$ be a category and let C be an object of $\mathcal{C}$. Consider the following data, called the ***slice over*** C ***of*** $\mathcal{C}$ and denoted $\mathcal{C}{\downarrow}C$ or $\mathcal{C}/C$:

an object of $\mathcal{C}{\downarrow}C$ is a morphism $\begin{array}{c} X \\ \downarrow\varphi \\ C \end{array}$ in $\mathcal{C}$;

a morphism $\begin{array}{c} X \\ \downarrow\varphi \\ C \end{array} \xrightarrow{f} \begin{array}{c} Y \\ \downarrow\psi \\ C \end{array}$ **in** $\mathcal{C}{\downarrow}C$ is a morphism $f\colon X \to Y$ in $\mathcal{C}$ and a commutative triangle in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & C & \end{tikzcd}$$

Proposition 1.8.2. *With composition given by that in* $\mathcal{C}$ *via diagram pasting, the data* $\mathcal{C}{\downarrow}C$ *form a category.*

Proof. The proof is trivial. □

Construction 1.8.3. Let $\mathcal{C}$ be a category. Consider the following data which we denote $\mathcal{C}^{\to}$ and call the ***arrow category of*** $\mathcal{C}$ (as we shall promptly prove it is such):

an object of $\mathcal{C}^{\to}$ is a morphism $\begin{array}{c} X \\ \downarrow\varphi \\ C \end{array}$ in $\mathcal{C}$, for arbitrary objects X and C of $\mathcal{C}$;

a morphism $\begin{array}{c} X \\ \downarrow\varphi \\ C \end{array} \xrightarrow{f} \begin{array}{c} Y \\ \downarrow\psi \\ D \end{array}$ **in** $\mathcal{C}^{\to}$ is a pair of morphisms $(f_0\colon C \to D,\ f_1\colon X \to Y)$ in $\mathcal{C}$ and a commutative square

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ C \arrow[r, "f_0"'] & D \end{tikzcd}$$

Proposition 1.8.4. *With composition given componentwise from $\mathcal{C}$ via diagram pasting, the data $\mathcal{C}^{\rightarrow}$ form a category.*

Proof. The proof is trivial. □

Remark 1.8.5. The arrow category of a category $\mathcal{C}$ is denoted $\mathcal{C}^{\rightarrow}$ because it coincides with the functor category $[\mathit{2}, \mathcal{C}]$. That is, the category of functors from the category $\mathit{2}$ with two objects and one non-trivial arrow to the category $\mathcal{C}$.

Now we would like to construct the *arrow category of a* 2*-category*. This would be possible in a very natural way using the notion of *cotensor* from enriched category theory, and this way would also ensures that we find a 2-category.

We will instead construct explicitly the *arrow category of a* 2*-category* $\mathcal{C}$, taking as 0-cells the 2-functors from $\mathit{2}$ to $\mathcal{C}$, as 1-cells the 2-natural transformations, and as 2-cells the so-called *modifications*, described in the following definition.

Definition 1.8.6. Let $F, G\colon \mathcal{A} \to \mathcal{K}$ be 2-functors, and let $\alpha, \beta\colon F \to G$ be 2-natural transformations. A ***modification*** $\lambda\colon \alpha \Rrightarrow \beta$ is a collection of 2-cells $(\lambda_A\colon \alpha_A \to \beta_A)_{A\in\mathcal{A}}$ in $\mathcal{K}$ such that

$$\begin{tikzcd} FA \arrow[r, bend left, "\alpha_A"] \arrow[r, bend right, "\beta_A"'] \arrow[r, phantom, "\Downarrow\lambda_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, bend right, "\beta_B"'] & GB \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, bend left, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, bend left, "\alpha_B"] \arrow[r, bend right, "\beta_B"'] \arrow[r, phantom, "\Downarrow\lambda_B"] & GB \end{tikzcd}$$

holds for every 1-cell $f\colon A \to B$ in $\mathcal{A}$.

Remark 1.8.7. In Definition 1.8.6 one should imagine that the identity between the two diagrams glues the first one to the second along the only two paths which appear with identical denotations in both diagrams: $FA \xrightarrow{\alpha_A} GA \xrightarrow{Gf} GB$ and $FA \xrightarrow{Ff} FB \xrightarrow{\beta_B} GB$. This is to highlight the fact that the functors which are the domain and the codomain of the two 2-cells can be taken to be $Gf\alpha_A$ and $\beta_B Ff$ respectively. Note that, while the domain of the 2-cell $Gf\lambda_A$ appears clearly to be $Gf\alpha_A$, its "natural" codomain is $Gf\beta_A$. But $Gf\beta_A = \beta_B Ff$ by naturality of β. Similarly for the codomain of the 2-cell $\lambda_B Ff$.

We used the language of pasting diagrams; recall from Section 1.3 that in the language of horizontal compositions the 2-cell identity is written as

$$
\begin{tikzcd}
FA \arrow[r, bend left, "\alpha_A"] \arrow[r, bend right, "\beta_A"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & GA \arrow[r, "Gf"] & GB
\end{tikzcd}
$$
$$\|$$
$$
\begin{tikzcd}
FA \arrow[r, "Ff"'] & FB \arrow[r, bend left, "\alpha_B"] \arrow[r, bend right, "\beta_B"'] \arrow[r, phantom, "\Downarrow\lambda_B"] & GB
\end{tikzcd}
$$

By enrichment, we repeat Construction 1.8.3 for 2-categories thanks to Remark 1.8.5 and Definition 1.8.6.

Construction 1.8.8. Let $\mathcal{K}$ be a 2-category. Consider the following data which we shall denote $\mathcal{K}^{\to}$ and call the ***arrow category of the* 2*-category* $\mathcal{K}$**:

a 0-cell of $\mathcal{K}^{\to}$ is a 1-cell $\begin{smallmatrix} X \\ \downarrow\varphi \\ C \end{smallmatrix}$ in $\mathcal{K}$, for arbitrary objects X and C of $\mathcal{K}$;

a 1-cell $\begin{smallmatrix} X \\ \downarrow\varphi \\ C \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} Y \\ \downarrow\psi \\ D \end{smallmatrix}$ in $\mathcal{K}^{\to}$ is a pair of 1-cells $(f_0\colon C \to D,\ f_1\colon X \to Y)$ in $\mathcal{K}$ and a commutative square

$$
\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
C \arrow[r, "f_0"'] & D;
\end{tikzcd}
$$

a 2-cell $\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\lambda"] & Y \arrow[d, "\psi"] \\ C & D \end{tikzcd}$ in $\mathcal{K}^{\to}$ is a pair of 2-cells $(\lambda_0\colon f_0 \to g_0,\ \lambda_1\colon f_1 \to g_1)$ in $\mathcal{K}$ such that

$$
\begin{tikzcd}
X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
C \arrow[r, bend right, "g_0"'] & D
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "f_1"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
C \arrow[r, bend left, "f_0"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & D
\end{tikzcd}
$$

holds (notice that there is only one non-trivial arrow in $\mathit{2}$).

Proposition 1.8.9. *With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells component-wise inherited from $\mathcal{K}$, the data $\mathcal{K}^{\to}$ form a 2-category.*

Proof. The proof is trivial. □

There is a common generalization of Construction 1.8.1 and Construction 1.8.3 which we shall now describe. It is obviously available via enrichment also for 2-categories, but we shall not need it.

Construction 1.8.10. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, and let S, T (for source and target) be functors as in the diagram:

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B}$$

Consider the following data which we denote $S{\downarrow}T$ and call the ***comma*** of S and T:

an object of $S{\downarrow}T$ is a triple (A, B, h) with A an object of $\mathcal{A}$, B an object of $\mathcal{B}$ and $h\colon S(A) \to T(B)$ a morphism in $\mathcal{C}$;

a morphism $(A, B, h) \to (A', B', h')$ **in** $S{\downarrow}T$ is a pair of morphisms $(f\colon A \to A',\ g\colon B \to B')$ in $\mathcal{C}$ and a commutative square

$$\begin{tikzcd} S(A) \arrow[r, "h"] \arrow[d, "S(f)"'] & T(B) \arrow[d, "T(g)"] \\ S(A') \arrow[r, "h'"'] & T(B') \end{tikzcd}$$

When $S\colon \mathcal{A} \to \mathcal{C}$ is the identity, we may write $S{\downarrow}T$ as $\mathcal{C}{\downarrow}T$.

Proposition 1.8.11. *With composition componentwise inherited from $\mathcal{A}$ (for the first component) and from $\mathcal{B}$ (for the second component), the data $S{\downarrow}T$ form a category.*

Proof. The proof is trivial. □

Remark 1.8.12. From Construction 1.8.10, we recover the notion of slice over C of a category $\mathcal{C}$ (Construction 1.8.1) taking $\mathcal{A} = \mathcal{C}$, S the identity functor on $\mathcal{A}$, $\mathcal{B} = \mathit{1}$ (where $\mathit{1}$ is the category with one object $*$ and one morphism) and T the constant functor at C (that is, $T(*) = C$); an object $(a, *, h)$ can be simplified to a pair (A, h) and a square which represents a morphism can be simplified to a triangle.

We also recover the notion of arrow category (Construction 1.8.3) taking $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $S = T = \mathrm{Id}$.

1.9 2-adjunctions

In this section we generalize the concept of adjunction to $\mathit{2}$-categories. We then describe some characterizations of 2-adjunction, which will be extremely useful for us.

Definition 1.9.1. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A ***2-adjunction between $\mathcal{C}$ and $\mathcal{D}$*** is a triple $\langle L, R, \varphi\rangle$ with $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ two 2-functors, and φ a collection of isomorphisms of categories

$$\varphi_{c,d}\colon \mathcal{D}(Lc, d) \xrightarrow{\sim} \mathcal{C}(c, Rd)$$

2-natural in $c \in \mathcal{C}$ and in $d \in \mathcal{D}$.

The 2-functor L is called the ***left 2-adjoint to*** R, whereas the 2-functor R is called the ***right 2-adjoint to*** L.

Remark 1.9.2. Note that this is a natural definition to give, since for categories we ask for a natural bijection, that is, a natural isomorphism in $\mathcal{S}\mathit{et}$, whereas here for $\mathcal{C}\mathit{at}$-categories we ask for a $\mathcal{C}\mathit{at}$-natural isomorphism in $\mathcal{C}\mathit{at}$.

Aiming at generalizing the characterizations of adjunction provided by Theorem 1.2.2, we now need to generalize the concept of universal arrow to 2-categories.

Definition 1.9.3. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Let then $c \in \mathcal{C}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be a 2-functor. A ***2-universal arrow from c to the 2-functor*** R is a pair $\langle d, \eta^d_c\colon c \to Rd\rangle$ with $d \in \mathcal{D}$ and $\eta^d_c\colon c \to Rd$ a 1-cell in $\mathcal{C}$ such that for every 1-cell $h\colon c \to Rd'$ in $\mathcal{C}$ with $d' \in \mathcal{D}$, there exists a unique 1-cell $g_h\colon d \to d'$ in $\mathcal{D}$ which makes the diagram of 1-cells in $\mathcal{C}$

$$\begin{tikzcd} c \arrow[r, "\eta^d_c"] \arrow[dr, "h"'] & Rd \arrow[d, dashed, "Rg_h"] \\ & Rd' \end{tikzcd} \tag{1.9}$$

commute, and such that for every 2-cell $\lambda\colon h \Rightarrow k\colon c \to Rd'$ in $\mathcal{D}$ with $d' \in \mathcal{D}$ and h, k two 1-cells from c to Rd' in $\mathcal{K}$, there exists a unique 2-cell $\delta\colon g_h \Rightarrow g_k\colon d \to d'$ in $\mathcal{D}$ (with g_h and g_k the unique 1-cells which make, respectively, diagram (1.9) and the analogous diagram with k instead of h commute) which makes the diagram of 2-cells in $\mathcal{C}$

$$\begin{tikzcd} c \arrow[r, "\eta^d_c"] \arrow[dr, bend left=20, "k"] \arrow[dr, bend right=30, "h"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & Rd \arrow[d, dashed, bend right=40, "Rg_h"'] \arrow[d, dashed, bend left=40, "Rg_k"] \arrow[d, phantom, "\overset{\delta}{\dashrightarrow}"] \\ & Rd' \end{tikzcd}$$

commute (that is, such that $\delta\eta^d_c = \lambda$).

Sometimes we will say that a 1-cell $\eta_c^d\colon c \to Rd$ in $\mathcal{C}$ with $d \in \mathcal{D}$ is a 2-universal arrow from c to the 2-functor R with the meaning that $\langle d, \eta_c^d\rangle$ is a 2-universal arrow from c to the 2-functor R.

Let now $d \in \mathcal{D}$ and $L\colon \mathcal{C} \to \mathcal{D}$ be a 2-functor. We define a ***2-universal arrow from the 2-functor L to d*** dualizing the definition above of 2-universal arrow from c to the 2-functor R.

Now we can trivially generalize to 2-categories Theorem 1.2.1 (which provided us with the concepts of unit and counit of an adjunction).

Theorem 1.9.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and consider* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$ *a 2-adjunction with hom-categories natural isomorphism φ given by*

$$\varphi_{c,d}\colon \mathcal{D}(Lc, d) \xrightarrow{\sim} \mathcal{C}(c, Rd)$$

for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
Then this 2-adjunction $\langle L, R, \varphi\rangle$ determines:

*(i) a 2-natural transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$, called the **unit of the 2-adjunction**, such that for every $c \in \mathcal{C}$ the arrow η_c in $\mathcal{C}$ is 2-universal from c to the 2-functor R, and for every morphism $f\colon Fc \to d$ in $\mathcal{D}$ the right adjunct of f is*

$$\varphi f = Rf \circ \eta_c\colon c \to Rd; \tag{1.10}$$

*(ii) a 2-natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$, called the **counit of the 2-adjunction**, such that for every $d \in \mathcal{D}$ the arrow ε_a in $\mathcal{D}$ is 2-universal from the 2-functor L to d, and for every morphism $g\colon c \to Rd$ in $\mathcal{C}$ the left adjunct of g is*

$$\varphi^{-1} g = \varepsilon_d \circ Lg\colon Fc \to d. \tag{1.11}$$

*Moreover both the following equalities of 2-natural transformations, called the **triangular identities**, hold:*

$$F \overset{F\eta}{\Rightarrow} FGF \overset{\varepsilon F}{\Rightarrow} F = \mathrm{id}_F \qquad\qquad G \overset{\eta G}{\Rightarrow} GFG \overset{G\varepsilon}{\Rightarrow} G = \mathrm{id}_G \tag{1.12}$$

Proof. The proof is straightforward. It suffices to generalize the ideas of the proof of the unenriched version of the theorem (which is Theorem 1.2.1) to $\mathcal{C}at$-enriched categories. □

We now conclude this section generalizing Theorem 1.2.2 (on the characterizations of adjunction) to 2-categories.

Theorem 1.9.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then each 2-adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$ *with hom-categories natural isomorphisms φ given by*

$$\varphi_{c,d} \colon \mathcal{D}(Lc, d) \xrightarrow{\sim} \mathcal{C}(c, Rd)$$

for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ is completely determined by the data in any one of the following lists:

(*i*) *2-functors L and R, and a 2-natual transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ such that for every $c \in \mathcal{C}$ the arrow $\eta_c \colon c \to RLc$ is 2-universal from c to the 2-functor R; then φ is determined by equation* (*1.10*)*;*

(*ii*) *the 2-functor $R\colon \mathcal{D} \to \mathcal{C}$ and for every $c \in \mathcal{C}$ a 0-cell $L_0c \in \mathcal{D}$ and a 2-universal arrow $\eta_c \colon c \to RL_0c$ from c to the 2-functor R; then the 2-functor L has object function L_0 and is defined on 1-cells $h\colon c \to c'$ by*

$$RLh \circ \eta_c = \eta_{c'} \circ h$$

and on 2-cells $\lambda\colon h \Rightarrow k \colon c \to c'$ by

$$RL\lambda\, \eta_c = \eta_{c'} \lambda;$$

(*iii*) *2-functors L and R, and a 2-natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ such that for every $d \in \mathcal{D}$ the arrow $\varepsilon_d \colon LRd \to d$ is 2-universal from the 2-functor L to d; then φ^{-1} is determined by equation* (*1.11*)*;*

(*iv*) *the 2-functor $L\colon \mathcal{C} \to \mathcal{D}$ and for every $d \in \mathcal{D}$ a 0-cell $R_0d \in \mathcal{C}$ and a 2-universal arrow $\varepsilon_d \colon LR_0d \to d$ from the 2-functor L to d; then the 2-functor R is determined in a way dual to the one described in point* (*ii*)*;*

(*v*) *2-functors L and R, and 2-natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to RL$ and $\varepsilon\colon LR \to \mathrm{Id}_{\mathcal{D}}$ such that the triangular identities* (*described in equation* (*1.12*)) *hold; then φ is determined by equation* (*1.10*) *and φ^{-1} is determined by equation* (*1.11*)*.*

Proof. The proof is straightforward. It suffices to generalize the ideas of the proof of the unenriched version of the theorem (which is Theorem 1.2.2) to $\mathcal{C}at$-enriched categories. □

1.10 Enriched Monads and Enriched Comonads

In this section we will generalize the definitions of Section 1.4 and Section 1.5 to enriched categories. In particular we will introduce the enriched monads and comonads and the enriched categories of algebras and coalgebras. The main references for this section are still [Kel82] and [Sch19].

Definition 1.10.1. Let $\mathcal{V}$ be a monoidal category. A ***$\mathcal{V}$-monad*** is a monad in $\mathcal{V}\text{−}\mathcal{C}at$. In other words, a $\mathcal{V}$-monad on a $\mathcal{V}$-category $\mathcal{C}$ (which is a 0-cell in $\mathcal{V}\text{-}\mathcal{CAT}$) is a triple (T, μ, η) with $T\colon \mathcal{C} \to \mathcal{C}$ a $\mathcal{V}$-functor and $\mu\colon T^2 \Rightarrow T$ and $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow T$ two $\mathcal{V}$-natural transformations such that the following equalities of pasting diagrams hold:

$$
\begin{tikzcd}
& \mathcal{C} \arrow[r, "T"] \arrow[drr, bend right, "T"'] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[urr, bend right, "T"'] \arrow[rrr, "T"'] & {} & {} & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, equal, bend left] \arrow[ur, bend right, "T"'] \arrow[ur, phantom, bend left, "\Downarrow\eta"] \arrow[rr, "T"'] & {} & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[d, bend right, "T"'] \arrow[d, bend left, "T"] \arrow[d, phantom, "="] \\
\mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C} \arrow[dr, equal, bend left] \arrow[dr, bend right, "T"'] \arrow[dr, phantom, bend left, "\Downarrow\eta"] \arrow[d, phantom, "\Downarrow\mu"] & \\
\mathcal{C} \arrow[ur, "T"] \arrow[rr, "T"'] & {} & \mathcal{C}
\end{tikzcd}
$$

Sometimes, we will refer to a $\mathcal{V}$-monad (T, μ, η) writing just T.

The natural transformation μ is called the ***multiplication of*** T, while η is called the ***unit of*** T.

A $\mathcal{V}$-monad naturally defines other algebraic structures, as we now describe. This will be the enriched version of the construction of the category of algebras for a monad, which was Construction 1.4.4.

Construction 1.10.2. We want to construct a new $\mathcal{V}$-category of T-algebras out of a $\mathcal{V}$-monad T. For this, we need to define the underlying *ordinary* or *unenriched category* of a $\mathcal{V}$-category $\mathcal{C}$.

By Example 1.6.7, we have a lax monoidal functor $V\colon \mathcal{V} \to \mathcal{S}et$. This induces a functor

$$
\begin{aligned}
V_*\colon \mathcal{V}\text{−}\mathcal{C}at &\longrightarrow \mathcal{S}et\text{−}\mathcal{C}at = \mathcal{C}at \\
\mathcal{C} &\longmapsto V_*\mathcal{C} \\
(F\colon \mathcal{C} \to \mathcal{D}) &\longmapsto (V_*F\colon V_*\mathcal{C} \to V_*\mathcal{D})
\end{aligned}
$$

defining the category $V_*\mathcal{C}$ as the category (as we shall promptly prove it is such) which has the same object class as $\mathcal{C}$ and such that for every $a, b \in \mathcal{C}$ we have that

$$V_*\mathcal{C}(a, b) = V(\mathcal{C}(a, b)).$$

The composition is induced by the monoidal structure of $\mathcal{V}$. That is, the morphisms in $V_*\mathcal{C}$ from a to b, with $a, b \in \mathcal{C}$, are given by morphisms $I \xrightarrow{f} \mathcal{C}(a, b)$, and the composite $g \circ f$ of two morphisms $f\colon a \to b$ and $g\colon b \to c$ in $V_*\mathcal{C}$ is defined to be the composite

$$I \xrightarrow{\sim} I \otimes I \xrightarrow{g \otimes f} \mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \xrightarrow{\circ} \mathcal{C}(a, c).$$

The functor V_*F is defined as F on objects, whereas on morphisms $f\colon a \to b$ in $V_*\mathcal{C}$ it produces a morphism $F(f)$ in $V_*\mathcal{D}(Fa, Fb)$ defined as

$$I \xrightarrow{f} \mathcal{C}(a,b) \xrightarrow{F} \mathcal{D}(Fa, Fb)$$

Proposition 1.10.3. *Let $\mathcal{V}$ be a monoidal category and let $V\colon \mathcal{V} \longrightarrow \mathcal{S}et$ be the lax monoidal functor we have produced in Example 1.6.7. Let then $\mathcal{C}$ be a $\mathcal{V}$-category.*
Then the data $V_*\mathcal{C}$ we have described in Construction 1.10.2 form a category, called the ***ordinary*** *or* ***unenriched category of*** *$\mathcal{C}$ and denoted $\mathcal{C}_0$. Moreover if $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-functor, the data V_*F define a functor from $V_*\mathcal{C}$ to $V_*\mathcal{D}$, called the* ***ordinary*** *or* ***unenriched functor of*** *F and denoted F_0. Finally, the assignment V_* described in Construction 1.10.2 is a functor from $\mathcal{V}{-}\mathcal{C}at$ to $\mathcal{C}at$.*

Proof. The proof is trivial. □

Example 1.10.4. We now show some examples of the action of the functor $V_*\colon \mathcal{V}{-}\mathcal{C}at \longrightarrow \mathcal{C}at$.

1. If $\mathcal{V} = \mathcal{A}b$ the functor V_* forgets the additive structure of the hom-sets;
2. If $\mathcal{V} = \mathcal{M}od(R)$ the functor V_* forgets the R-module structure of the hom-sets;
3. if $\mathcal{V} = \mathcal{T}op$ the functor V_* forgets the topology;
4. if $\mathcal{V} = (s\mathcal{S}et, \times)$ and $\mathcal{C}$ is a $\mathcal{V}$-category, the action of V_* on $\mathcal{C}$ forgets all the simplices in the hom-sets $\mathcal{C}(a,b)$ except the 0-simplices, i.e. the vertices.

Now we want to construct a $\mathcal{V}$-category of T-algebras for a $\mathcal{V}$-monad T.

Construction 1.10.5. Let $\mathcal{V}$ be a category with equalizers, $\mathcal{C}$ be a $\mathcal{V}$-category and (T, μ, η) be a $\mathcal{V}$-monad on $\mathcal{C}$.
Consider the following data, which we shall denote $T{-}\mathcal{A}lg$ and call the $\mathcal{V}$-category of T-algebras (since we shall promptly prove that these data extend to a $\mathcal{V}$-category):

the object set of $T{-}\mathcal{A}lg$ is $T_0{-}\mathcal{A}lg$, that is, an object of $T{-}\mathcal{A}lg$ is a pair (A, α) with $A \in \mathcal{C}$ and $\alpha\colon TA \to A$ a morphism $I \xrightarrow{\alpha} \mathcal{C}(TA, A)$ in $\mathcal{V}$ such that

$$\begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "\alpha"] \\ & A; \end{tikzcd}$$

for every $(A,\alpha), (B,\beta) \in \mathrm{Ob}(T{-}\mathcal{A}lg)$ **the** $\mathcal{V}$ **object of morphisms** $T{-}\mathcal{A}lg((A,\alpha),(B,\beta))$

is the equalizer in $\mathcal{V}$

$$\begin{tikzcd} & & \mathcal{C}(TA,TB) \arrow[dr, "\beta_*"] & \\ T-\mathcal{Alg}((A,\alpha),(B,\beta)) \arrow[r, tail, "U_{(A,\alpha),(B,\beta)}"] & \mathcal{C}(A,B) \arrow[ur, "T_{A,B}"] \arrow[rr, "\alpha^*"'] & & \mathcal{C}(TA,B) \end{tikzcd} \tag{1.13}$$

where β_* is the composite

$$\mathcal{C}(TA,TB) \xrightarrow{\simeq} I \otimes \mathcal{C}(TA,TB) \xrightarrow{\beta\otimes\mathrm{id}} \mathcal{C}(TB,B) \otimes \mathcal{C}(TA,TB) \xrightarrow{\circ} \mathcal{C}(TA,B)$$

and similarly α^* is the composite

$$\mathcal{C}(A,B) \xrightarrow{\simeq} \mathcal{C}(A,B) \otimes I \xrightarrow{\mathrm{id}\otimes\alpha} \mathcal{C}(A,B) \otimes \mathcal{C}(TA,A) \xrightarrow{\circ} \mathcal{C}(TA,B).$$

Proposition 1.10.6. *Let $\mathcal{V}$ be a monoidal category, $\mathcal{C}$ be a $\mathcal{V}$-category and (T,μ,η) be a $\mathcal{V}$-monad on $\mathcal{C}$. Then the data $T-\mathcal{Alg}$ we have described in Construction 1.10.5 extend in a unique way to a structure of $\mathcal{V}$-category such that the morphisms $U_{(A,\alpha),(B,\beta)}$ in $\mathcal{V}$ extend to a $\mathcal{V}$-functor*

$$U\colon T-\mathcal{Alg} \longrightarrow \mathcal{C}.$$

Proof. To construct the identities, note that for every $(A,\alpha) \in T_0-\mathcal{Alg}$ we have that $\mathrm{id}_A\colon I \to \mathcal{C}(A,A)$ equalizes the two morphisms α^* and $\alpha_* \circ T_{A,A}$ (of diagram (1.13) with $(A,\alpha)=(B,\beta)$), that is, $\mathrm{id}_A \circ \alpha = \alpha \circ T\,\mathrm{id}_A$.
Thus there exists a unique morphism

$$\mathrm{id}_{(a,\alpha)}\colon I \longrightarrow T-\mathcal{Alg}((A,\alpha),(A,\alpha))$$

such that

$$\begin{tikzcd} T-\mathcal{Alg}((A,\alpha),(A,\alpha)) \arrow[rr, tail, "U_{(A,\alpha),(A,\alpha)}"] & & \mathcal{C}(A,A) \\ & I \arrow[ul, dashed, "\mathrm{id}_{(A,\alpha)}"] \arrow[ur, "\mathrm{id}_A"'] & \end{tikzcd}$$

by the universal property of the equalizer. Since $U_{(A,\alpha),(A,\alpha)}$ is monic we have to define $\mathrm{id}_{(A,\alpha)}$ as this dashed arrow if we want U to be a $\mathcal{V}$-functor.

Similarly, if we want U to be $\mathcal{V}$-functor, for every $(A,\alpha),(B,\beta),(C,\gamma) \in T_0-\mathcal{Alg}$ we need to define the composition morphism in $\mathcal{V}$

$$C_{(A,\alpha),(B,\beta),(C,\gamma)}\colon T-\mathcal{Alg}((B,\beta),(C,\gamma)) \otimes T-\mathcal{Alg}((A,\alpha),(B,\beta)) \longrightarrow T-\mathcal{Alg}((A,\alpha),(C,\gamma))$$

in a way such that the diagram

$$\begin{tikzcd} T-\mathcal{Alg}((B,\beta),(C,\gamma)) \otimes T-\mathcal{Alg}((A,\alpha),(B,\beta)) \arrow[r, "U_{(B,\beta),(C,\gamma)}\otimes U_{(A,\alpha),(B,\beta)}"] \arrow[d, dashed, "C_{(A,\alpha),(B,\beta),(C,\gamma)}"'] & \mathcal{C}(B,C)\otimes\mathcal{C}(A,B) \arrow[d, "\circ"] \\ T-\mathcal{Alg}((A,\alpha),(C,\gamma)) \arrow[r, "U_{(A,\alpha),(C,\gamma)}"'] & \mathcal{C}(A,C) \end{tikzcd}$$

is commutative. And then we conclude that there is at most one solution for such composition morphism by the fact that $U_{(A,\alpha),(C,\gamma)}$ is monic. Therefore we have already proved the desired uniqueness of the $\mathcal{V}$-category structure for $T-\mathcal{Alg}$.
Moreover, we find a composition morphism $C_{(A,\alpha),(B,\beta),(C,\gamma)}$ which makes the diagram above commute by the universal property of the equalizer, since it is straightforward to show that the composite on the right of the diagram above equalizes the morphisms α^* and $\gamma_* \circ T_{A,C}$ (of diagram (1.13) with (C,γ) instead of (B,β)).

Now, the fact that these data make $T-\mathcal{Alg}$ into a $\mathcal{V}$-category is easy to prove using that $\mathcal{C}$ is a $\mathcal{V}$-category and that for every $(A,\alpha),(B,\beta) \in T_0-\mathcal{Alg}$ the morphism $U_{(A,\alpha),(B,\beta)}$ in $\mathcal{V}$ is monic.

Furthermore, the diagrams above show that the morphisms $U_{(A,\alpha),(B,\beta)}$ (with $(A,\alpha),(B,\beta) \in T_0-\mathcal{Alg}$) extend to a $\mathcal{V}$-functor from $T-\mathcal{Alg}$ to $\mathcal{C}$. Note indeed that those two diagrams were exactly the diagrams of the two axioms of $\mathcal{V}$-functor. □

Example 1.10.7. Let $\mathcal{V} = \mathcal{Ab}$ and let R be a commutative ring. Then $\mathcal{Ab}$ is an $\mathcal{Ab}$-category since we can sum morphisms of abelian groups and the composition is $\mathbb{Z}$-bilinear. Moreover

$$T = - \otimes_{\mathbb{Z}} R\colon \mathcal{Ab} \to \mathcal{Ab}$$

is an $\mathcal{Ab}$-monad. The $\mathcal{Ab}$ category $T-\mathcal{Alg}$ of T-algebras coincides with $\mathcal{Mod}(R)$ equipped with the addition of R-module homomorphisms.

Let $\mathcal{V}$ be a monoidal category. At this point, we shall dualize the definitions we have seen in this section so far and obtain the definitions of a ***$\mathcal{V}$-comonad*** Ω and of the ***$\mathcal{V}$-category*** $\Omega-\mathcal{CoAlg}$ ***of coalgebras for a $\mathcal{V}$-comonad*** Ω.
This operation of dualization is exactly the same of the one we have shown in Section 1.5.

1.11 The 2-category of Algebras for a 2-monad

We are mostly interested in the enrichment over $\mathcal{V} = \mathcal{Cat}$. We describe explicitly the 2-category $T-\mathcal{Alg}$ for a 2-monad T. We will then show the $\mathcal{Cat}$-enriched version of Theorem 1.4.11 and Theorem 1.5.8 (on comparison functors).

Remark 1.11.1. Let then $\mathcal{K}$ be a 2-category and consider (T,μ,η) a $\mathcal{Cat}$-monad on $\mathcal{K}$. A $\mathcal{Cat}$-monad is usually called a ***2-monad***.

Now, we want to describe explicitly the 2-category $T-\mathcal{Alg}$.

A 0-cell of $T-\mathcal{Alg}$ is an object of $T_0-\mathcal{Alg}$, that is, a pair (A,α) with $A \in \mathcal{K}$ and $\alpha\colon TA \to A$ a functor $I \to \mathcal{K}(TA,A)$ (which means $\alpha\colon TA \to A$ a 1-cell in $\mathcal{K}$) such that the diagrams

$$\begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "\alpha"] \\ & A; \end{tikzcd}$$

are commutative.

A 1-cell $f\colon (A,\alpha) \to (B,\beta)$ in $T-\mathcal{Alg}$ is simply a morphism f in $T_0-\mathcal{Alg}$, that is, a 1-cell $f\colon A \to B$ in $\mathcal{K}$ such that the diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative.

A 2-cell $\varphi\colon f \Rightarrow g\colon (A,\alpha) \to (B,\beta)$ in $T-\mathcal{Alg}$ is a morphism in the equalizer

$$\begin{tikzcd} & & \mathcal{K}(TA,TB) \arrow[dr, "\beta_*"] & \\ T-\mathcal{Alg}((A,\alpha),(B,\beta)) \arrow[r, tail, "U_{(A,\alpha),(B,\beta)}"] & \mathcal{K}(A,B) \arrow[ur, "T_{A,B}"] \arrow[rr, "\alpha^*"'] & & \mathcal{K}(TA,B) \end{tikzcd}$$

(which moreover is from $f \in \mathcal{K}(A,B)$ to $g \in \mathcal{K}(A,B)$), that is, a 2-cell $\varphi\colon f \Rightarrow g : A \to B$ in $\mathcal{K}$ such that

$$\begin{tikzcd} TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T\varphi"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, bend right, "g"'] & B \end{tikzcd} = \begin{tikzcd} TA \arrow[r, bend left, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}$$

Proposition 1.11.2. *Let (T,μ,η) be a 2-monad on a 2-category $\mathcal{C}$. Then the 2-functor $U^T := U\colon T-\mathcal{Alg} \longrightarrow \mathcal{C}$ produced in Proposition 1.10.6 has a left 2-adjoint*

$$F^T\colon \mathcal{C} \longrightarrow T-\mathcal{Alg}$$

$$c \longmapsto (Tc, \mu_c)$$

$$(f\colon c \to d) \longmapsto (Tf\colon (Tc,\mu_c) \to (Td,\mu_d))$$

$$(\lambda\colon f \Rightarrow g\colon c \to d) \longmapsto (T\lambda\colon Tf \Rightarrow Tg\colon (Tc,\mu_c) \to (Td,\mu_d))$$

such that $U^T F^T = T$. Furthermore, the unit η^T of this 2-adjunction is given by η and the counit ε^T has components $\varepsilon^T_{(a,\alpha)} = \alpha\colon (Ta,\mu_a) \to (a,\alpha)$ for every $(a,\alpha) \in T-\mathcal{Alg}$.

Proof. The proof is straightforward. It suffices to generalize the ideas of the proof of the unenriched version of the proposition (which is Proposition 1.4.7) to $\mathcal{Cat}$-enriched categories. □

Definition 1.11.3. Given a 2-monad (T, μ, η) on $\mathcal{C}$, the T-algebras of the form (Tc, μ_c) with $c \in \mathcal{C}$ are called ***free T-algebras***.

Proposition 1.11.4. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ *be a 2-adjunction, with unit η and counit ε. Then*

$$(UF, U\varepsilon F, \eta)$$

is a 2-monad on $\mathcal{C}$. Furthermore, if (T, μ, η^T) is a 2-monad on $\mathcal{C}$, and we consider the associated 2-adjunction $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ with unit η^T and counit ε^T described in Proposition 1.11.2, we have that

$$(U^T F^T, U^T \varepsilon^T F^T, \eta^T) = (T, \mu, \eta^T).$$

Proof. The proof is straightforward. It suffices to generalize the ideas of the proof of the unenriched version of the proposition (which is Proposition 1.4.9) to $\mathcal{C}at$-enriched categories. □

Remark 1.11.5. Proposition 1.11.4 shows that a 2-adjunction always produces a 2-monad. Furthermore, the last assertion of Proposition 1.11.4 ensures that if we start from a 2-monad T, consider the associated 2-adjunction $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ and then consider the 2-monad associated to such 2-adjunction we obtain the 2-monad we started from.

However if we start from a 2-adjunction, consider the associated 2-monad and then consider the 2-adjunction associated to this 2-monad we do not find in general the 2-adjunction we started from. However we have a canonical *comparison 2-functor*, which we describe in the following theorem.

Theorem 1.11.6. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ *be a 2-adjunction with unit η and counit ε. Consider the associated 2-monad on $\mathcal{C}$*

$$T = (UF, U\varepsilon F, \eta)$$

(see Proposition 1.11.4) and then consider the 2-adjunction $F^{\mathrm{T}} \dashv U^{\mathrm{T}}$ associated to the 2-monad T (see Proposition 1.11.2). Then there exists a unique 2-functor

$$\overline{U} \colon \mathcal{D} \longrightarrow T\text{-}\mathcal{A}lg$$

such that $U^{\mathrm{T}}\overline{U} = U$ and $\overline{U}F = F^{\mathrm{T}}$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, dashed, "\overline{U}"] \arrow[dr, shift right, "U"'] & T\text{-}\mathcal{A}lg \arrow[d, shift left, "U^{\mathrm{T}}"] \\ & \mathcal{C} \arrow[ul, shift right, "F"'] \arrow[u, shift left, "F^{\mathrm{T}}"] \end{tikzcd}$$

Moreover $\overline{U}$ is given by

$$\overline{U}\colon \mathcal{D} \longrightarrow T\text{-}\mathcal{A}lg$$

$$d \longmapsto (Ud, U\varepsilon_d)$$

$$(d \xrightarrow{h} d') \longmapsto (Uh\colon Ud \to Ud')$$

$$(h \overset{\lambda}{\Rightarrow} k) \longmapsto (U\lambda\colon Uh \Rightarrow Uk)$$

Proof. The proof is straightforward. It suffices to generalize the ideas of the proof of the unenriched version of the theorem (which is Theorem 1.4.11) to $\mathcal{C}at$-enriched categories. □

Definition 1.11.7. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$ be a 2-adjunction. The functor $\overline{U}$ described in Theorem 1.11.6 is called the ***comparison* 2*-functor associated to the* 2*-adjunction*** $F \dashv U$ (sometimes we say ***to the* 2*-functor*** U, remembering that U is a right 2-adjoint) ***and to the* 2*-monad*** T.

Definition 1.11.8. Let $U\colon \mathcal{D} \longrightarrow \mathcal{C}$ be a right 2-adjoint. Say that $F \dashv U$ with unit η and counit ε and consider the 2-monad $(UF, U\varepsilon F, \eta)$ associated to that 2-adjunction (see Proposition 1.4.9). We say that U is a ***monadic* 2*-functor*** if the comparison 2-functor $\overline{U}\colon \mathcal{D} \longrightarrow T\text{-}\mathcal{A}lg$ is an equivalence of 2-categories (defined in the evident way using 2-natural transformations).

At this point, we shall dualize what we have seen in this section so far and obtain the analogous results for a 2-comonad (that is, a $\mathcal{C}at$-comonad) Ω. In particual we will use the dualized version of Theorem 1.11.6 (on the comparison 2-functor), which becomes the $\mathcal{C}at$-enriched version of Theorem 1.5.8.
This operation of dualization is exactly the same of the one we have shown in Section 1.5.

1.12 The 2-category of Pseudo-algebras for a 2-monad

In this section we want to weaken the data which form the 2-category $T\text{-}\mathcal{A}lg$ of algebras for a 2-monad T. Remember that we have described explicitly such 2-category in Remark 1.11.1. One shall compare such remark with the definitions we give in this section. The main references for this section are [CHP03] and [Tro19].

Definition 1.12.1. Let $\mathcal{K}$ be a 2-category and let (T, μ, η) be a 2-monad on $\mathcal{K}$. A ***pseudo-algebra for the* 2*-monad*** T or ***pseudo-T-algebra*** is a quadruple $(A, \alpha, \alpha_\mu, \alpha_\eta)$ with $A \in \mathcal{K}$

a 0-cell, $\alpha\colon TA \to A$ a 1-cell in $\mathcal{K}$, and α_μ and α_η invertible 2-cells in $\mathcal{K}$ as in the figures

$$
\begin{tikzcd}
T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\swArrow\alpha_\mu"] & TA \arrow[d, "\alpha"] \\
TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\swArrow\alpha_\eta"] & TA \arrow[d, "\alpha"] \\
& A;
\end{tikzcd}
$$

subject to the following two coherence axioms, written in the language of pasting diagrams:

$$
\begin{tikzcd}
T^3A \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TA}"'] \arrow[dr, "T\mu_A"] & T^2A \arrow[dr, "T\alpha"] \arrow[d, phantom, "\Downarrow T\alpha_\mu"] & \\
T^2A \arrow[r, phantom, "="] \arrow[dr, "\mu_A"'] & T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\swArrow\alpha_\mu"] & TA \arrow[d, "\alpha"] \\
& TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
=
\begin{tikzcd}
T^3A \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TA}"'] \arrow[dr, phantom, "="] & T^2A \arrow[d, "\mu_A"] \arrow[dr, "T\alpha"] & \\
T^2A \arrow[r, "T\alpha"] \arrow[dr, "\mu_A"'] & TA \arrow[d, phantom, "\Downarrow\alpha_\mu"] \arrow[dr, "\alpha"] & TA \arrow[l, phantom, "\Leftarrow\alpha_\mu"] \arrow[d, "\alpha"] \\
& TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
& T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\swArrow\alpha_\mu"] & TA \arrow[d, "\alpha"] \\
TA \arrow[ur, "T\eta_A"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "="] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
=
\begin{tikzcd}
& T^2A \arrow[r, "T\alpha"] \arrow[d, phantom, "\Downarrow T\alpha_\eta"] & TA \arrow[d, "\alpha"] \\
TA \arrow[ur, "T\eta_A"] \arrow[r, equal] & TA \arrow[r, "\alpha"'] \arrow[ur, equal] \arrow[ur, phantom, shift right=3, "="] & A
\end{tikzcd}
$$

(where the equalities we have written inside the squares and inside the triangles represent identity 2-cells in $\mathcal{K}$, and these are given by the axioms of 2-monad and by naturality of μ (for the one above on the right)).

A second identity axiom, related with η_{TA}, follows from these two axioms:

$$\alpha_\mu \eta_{TA} = \alpha_\eta \alpha.$$

Definition 1.12.2. Let $\mathcal{K}$ be a 2-category and let (T, μ, η) be a 2-monad on $\mathcal{K}$. A ***pseudo-morphism of pseudo-T-algebras from*** $(A, \alpha, \alpha_\mu, \alpha_\eta)$ ***to*** $(B, \beta, \beta_\mu, \beta_\eta)$, with $(A, \alpha, \alpha_\mu, \alpha_\eta)$ and $(B, \beta, \beta_\mu, \beta_\eta)$ pseudo-T-algebras is a pair $(f, \overline{f})$ with $f\colon A \to B$ a 1-cell in $\mathcal{K}$ and $\overline{f}$ an invertible 2-cell in $\mathcal{K}$

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\swArrow\overline{f}"] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

subject to the following coherence axioms:

$$
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] \arrow[dr, "T\alpha"] & T^2B \arrow[dr, "T\beta"] \arrow[d, phantom, "\Downarrow T\overline{f}"] & \\
TA \arrow[dr, "\alpha"'] & TA \arrow[l, phantom, "\overset{\Longleftarrow}{\alpha_\mu}"] \arrow[r, "Tf"] \arrow[d, "\alpha"] & TB \arrow[d, "\beta"] \arrow[dl, phantom, "\swarrow \overline{f}"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "="] & T^2B \arrow[d, "\mu_B"'] \arrow[dr, "T\beta"] & \\
TA \arrow[r, "Tf"] \arrow[dr, "\alpha"'] & TB \arrow[d, phantom, "\Downarrow \overline{f}"] \arrow[dr, "\beta"] & TB \arrow[l, phantom, "\overset{\Longleftarrow}{\beta_\mu}"] \arrow[d, "\beta"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, "\eta_A"] & B \arrow[d, phantom, "\|"] \arrow[dr, "\eta_B"] & \\
A \arrow[dr, equal] & TA \arrow[l, phantom, "\overset{\alpha_\eta}{\Longleftarrow}"] \arrow[r, "Tf"] \arrow[d, "\alpha"] & TB \arrow[d, "\beta"] \arrow[dl, phantom, "\swarrow \overline{f}"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \arrow[dr, "\eta_B"] & \\
A \arrow[r, "f"] \arrow[dr, equal] & B \arrow[d, phantom, "\|"] \arrow[dr, equal] & TB \arrow[l, phantom, "\overset{\Longleftarrow}{\beta_\eta}"] \arrow[d, "\beta"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Definition 1.12.3. Let $\mathcal{K}$ be a 2-category and let (T, μ, η) be a 2-monad on $\mathcal{K}$. A ***pseudo-T-transformation from*** $(f, \overline{f})$ ***to*** $(g, \overline{g})$ with $(f, \overline{f}), (g, \overline{g}) \colon (A, \alpha, \alpha_\mu, \alpha_\eta) \to (B, \beta, \beta_\mu, \beta_\eta)$ pseudo-morphisms of pseudo-T-algebras is a 2-cell $\varphi \colon f \Rightarrow g \colon A \to B$ in $\mathcal{K}$ such that

$$
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T\varphi"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \arrow[dl, phantom, "\swarrow \overline{g}"] \\
A \arrow[r, bend right, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \arrow[dl, phantom, "\swarrow \overline{f}"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \varphi"] & B
\end{tikzcd}
$$

Construction 1.12.4. Let $\mathcal{K}$ be a 2-category and let (T, μ, η) be a 2-monad on $\mathcal{K}$. Consider the following data, which we shall denote $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ and call the 2-category of pseudo-T-algebras (as we shall promptly prove it is such):

a 0-cell of $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ is a pseudo-T-algebra $(A, \alpha, \alpha_\mu, \alpha_\eta)$;

a 1-cell $(A, \alpha, \alpha_\mu, \alpha_\eta) \to (B, \beta, \beta_\mu, \beta_\eta)$ in $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ is a pseudo-morphism of pseudo-T-algebras $(f, \overline{f})$ from $(A, \alpha, \alpha_\mu, \alpha_\eta)$ to $(B, \beta, \beta_\mu, \beta_\eta)$;

a 2-cell $(f, \overline{f}) \Rightarrow (g, \overline{g})$ in $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ is a pseudo-T-transformation from $(f, \overline{f})$ to $(g, \overline{g})$;

the composition functors are for every $(A, \alpha, \alpha_\mu, \alpha_\eta)$, $(B, \beta, \beta_\mu, \beta_\eta)$ and $(C, \gamma, \gamma_\mu, \gamma_\eta)$ pseudo-T-algebras

$$C_{(A,\alpha),(B,\beta),(C,\gamma)} \colon \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((B,\beta),(C,\gamma)) \times \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((A,\alpha),(B,\beta)) \longrightarrow \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((A,\alpha),(C,\gamma))$$

defined by sending a pair

$$((f,\overline{f}),(g,\overline{g})) \in \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((B,\beta),(C,\gamma)) \times \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((A,\alpha),(B,\beta))$$

to the 1-cell

$$(gf, \overline{gf}) \in \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}((A,\alpha),(C,\gamma))$$

where gf is the composite of 1-cells in $\mathcal{K}$ and $\overline{gf}$ is the composite given by the following pasting diagram:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\swarrow \overline{f}"] & TB \arrow[r, "Tg"] \arrow[d, "\beta"] \arrow[dr, phantom, "\swarrow \overline{g}"] & TC \arrow[d, "\gamma"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

(it is straightforward to see that $(gf, \overline{gf})$ is a pseudo-morphism of pseudo-T-algebras).

Proposition 1.12.5. *The data $\mathcal{Ps}$-T-$\mathcal{Alg}$ form a 2-category.*

Proof. The proof is straightforward. □

Remark 1.12.6. Let T be a 2-monad on a 2-category $\mathcal{K}$. We easily note that any T-algebra is a pseudo-T-algebra, that any morphism of T-algebras is a pseudo-morphism of pseudo T-algebras and that any 2-morphism of T-algebras is pseudo-T-transformation. In fact it suffices to take the identities as invertible 2-cells in $\mathcal{K}$ and we get that all the coherence diagrams are trivially satisfied.

Dualizing all the definitions we shall then get the description of the 2-category $\mathcal{Ps}$–Ω–$\mathcal{CoAlg}$ of pseudo-Ω-coalgebras for a 2-comonad Ω.

Chapter 2

Grothendieck Fibrations

In this second chapter we introduce the notion of *Grothendieck fibration*. In order to motivate that notion, we start reviewing some examples. We will then introduce the notions of *cloven* and *split fibrations* and describe the 2-categories of fibrations.

2.1 Motivation and Examples

A *(Grothendieck) fibration* is a categorical structure which captures the concepts of indexing and change of base. We are going to see a few examples which show these two concepts and the historical reason that brings to the name "fibration". The main reference for this section is [Jac99].

The first elementary example we review is that of family of sets whose indices, or parameters, range over a set I. We can view this in two ways.

The first, plain way to look at it is what one may call ***pointwise indexing***: as a collection $(X_i)_{i\in I}$, where X_i is a set as i varies in the set I. This is probably the most elementary way; this collection is actually a function $[i \mapsto X_i] : I \to \mathcal{U}$ where $\mathcal{U}$ is the universe of all sets. Although the intution is very simple, the explicit determination of the mathematical data becomes clumsy: one has to introduce a universe of sets in order to fit the "direct" intution within the standard framework for everyday mathematics.

The second way is to "turn things around" and consider a ***display indexing***: a function $\varphi\colon X \to I$ determines a pointwise indexed family of sets taking the fibres over each singleton $\{i\}$

$$\varphi^{-1}\{i\} = \{x \in X \mid \varphi(x) = i\}$$

for each $i \in I$. This second point of view may appear less natural at first sight, but it is the one that has been proved extremely useful for many branches of mathematics. And it will be the useful point of view to consider for our generalization, as it enjoys better properties when applied to the situation of categories.

We represent the two points of view in Figure 2.1.

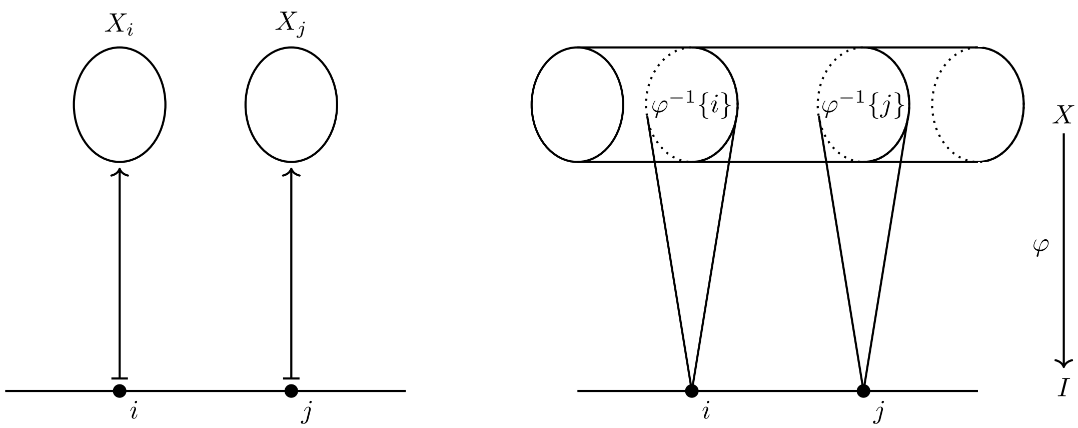


Figure 2.1: Pointwise indexing and display indexing

It is easy to see how the two ways are essentially the same for families of sets, but such an equivalence between *indexed categories* (that is, families of categories) and *fibrations* will be given a satisfactory mathematical status only in 2.4.14.

Given a family $(X_i)_{i\in I}$ of sets we can set X to be the disjoint union

$$\coprod_{i\in I} X_i := \{(i,x) \mid i \in I, x \in X_i\}$$

and consider the "first projection" function

$$\begin{aligned} \pi\colon \textstyle\coprod_{i\in I} X_i &\to I \\ (i,x) &\mapsto i \end{aligned}$$

Then, for all $i \in I$, the fibre $\pi^{-1}\{i\}$ is, up to a canonical isomorphism, the original X_i.

Conversely, given a function $\varphi\colon X \to I$, for each $i \in I$ set $X_i := \varphi^{-1}\{i\}$. This yields a family $(X_i)_{i\in I}$; and one finds a canonical isomorphism $\coprod_{i\in I} X_i \xrightarrow{\sim} X$.

The idea was put at use in topology in the early 1930's with the notion of family of topological spaces which vary "continuously" with respect to a topological parameter. In such a situation the first approach is doomed to fail as noone has yet come up with a topology on the universe of sets. While the second way proved extremely useful as it gave rise to fibre bundles—requiring in addition a "local" condition on the "family" of topological spaces—and all the subsequent developments in differential geometry, algebraic topology, and algebraic geometry.

In the following, we will often describe families of sets or of other mathematical objects as functions $\varphi\colon X \to I$ as in the display indexing way. In order to emphasise the fact that we think of such maps φ as families, or fibre bundles, or projections, we will often write them vertically as $\begin{array}{c} X \\ \downarrow\varphi \\ I \end{array}$.

Notice that if we realize in this way a constant family of value X over a set I we obtain a morphism $\begin{array}{c} I \times X \\ \downarrow \pi \\ I \end{array}$, where π is precisely the projection from the Cartesian product; all the fibres of this constant family are isomorphic to X. Note that this determines the family uniquely up to isomorphism in the case of families of sets, but fails interestingly in the case of fibre bundles.

Families $\begin{array}{c} X \\ \downarrow \varphi \\ I \end{array}$ of sets appear as objects in two prominent categories: the *slice category* $\mathcal{S}et \downarrow I$ and the *arrow category* $\mathcal{S}et^{\to}$. The former describes the I-indexed families for a fixed set I, while the latter describes all the families for all possible sets of indices. The less expert reader shall find the construction of such categories in Section 1.8.

Construction 2.1.1. Now that we have spoken about indexing, we want to describe change of base.

Let $\begin{array}{c} Y \\ \downarrow \psi \\ J \end{array}$ be a family of sets over a set J, that is $\varphi \in \mathcal{S}et \downarrow J$. *Change of base* (which appears in other situations also with the name *substitution* or *pullback*) involves changing the index set J. More specifically, change of base along a function $u \colon I \to J$ involves creating a family of sets with the domain I of u as new index set and with fibres $Y_{u(i)}$ for $i \in I$. Thus the family $(Y_j)_{j \in J}$ is turned into a family $(X_i)_{i \in I}$ with $X_i = Y_{u(i)}$. What we have actually done is taking the pullback of ψ against u:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\psi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

That is, we form the set $X = \{(i, y) \in I \times Y \mid u(i) = \psi(y)\}$ and use the restrictions of the two projection functions from $I \times Y$ to get $I \overset{\varphi}{\leftarrow} X \overset{f}{\to} Y$. We obtain a new family $\begin{array}{c} X \\ \downarrow \varphi \\ I \end{array}$ over I with fibres

$$X_i = \varphi^{-1}\{i\} \xrightarrow{\sim} \{y \in Y \mid \psi(y) = u(i)\} = \psi^{-1}\{u(i)\} = Y_{u(i)}.$$

We usually write $u^*(\psi)$ for the result φ of substituting ψ along u.

Remark 2.1.2. Notice that the pair (u, f) in the pullback diagram of Construction 2.1.1 is a morphism $u^*(\psi) \to \psi$ in the arrow category $\mathcal{S}et^{\to}$. Later, we will call (u, f) a *cartesian morphism.* This morphism has an important universal property, which we now describe.

Suppose we have another morphism

$$\begin{array}{c} Z \\ \downarrow \chi \\ K \end{array} \xrightarrow{(v,g)} \begin{array}{c} Y \\ \downarrow \psi \\ J \end{array}$$

in $\mathcal{S}et^{\rightarrow}$ such that $v\colon K \to J$ factors through $u\colon I \to J$, say via $w\colon K \to I$ (with $v = u \circ w$), as in the diagram

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[dd, "\chi"'] & & \\
& X \arrow[r, "f"] \arrow[dd, "\varphi"'] \arrow[ddr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "\psi"] \\
K \arrow[rd, "w"'] \arrow[rrd, bend left, "v"] & & \\
& I \arrow[r, "u"'] & J
\end{tikzcd}$$

Then there exists a unique morphism

$$\begin{array}{ccc} Z & & X \\ \downarrow\chi & \xrightarrow{(w,h)} & \downarrow\psi \\ K & & I \end{array}$$

in $\mathcal{S}et^{\rightarrow}$ which is sent to w by the codomain functor $\mathrm{cod}\colon \mathcal{S}et^{\rightarrow} \to \mathcal{S}et$, and such that

$$\begin{array}{ccccc} Z & & X & & Y \\ \downarrow\chi & \overset{(w,h)}{\dashrightarrow} & \downarrow\psi & \xrightarrow{(u,f)} & \downarrow\varphi \\ K & & I & & J \end{array} \quad = \quad \begin{array}{ccc} Z & & Y \\ \downarrow\chi & \xrightarrow{(v,g)} & \downarrow\psi \\ K & & J \end{array}$$

This holds because we have constructed X as a pullback:

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "h"] \arrow[dd, "\chi"'] & & \\
& X \arrow[r, "f"] \arrow[dd, "\varphi"'] \arrow[ddr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "\psi"] \\
K \arrow[rd, "w"'] \arrow[rrd, bend left, "v"] & & \\
& I \arrow[r, "u"'] & J
\end{tikzcd}$$

The universal property of what we will call a *cartesian morphism* indeed is, in this case, the universal property of the pullback.

We will see that the presence of such morphisms with this universal property is what makes the codomain functor $\mathrm{cod}\colon \mathcal{S}et^{\rightarrow} \to \mathcal{S}et$ a Grothendieck fibration.

Now, we want to abstractly capture this property in purely categorical terms.

2.2 Grothendieck Fibrations

We would like to generalize to categories what we have seen in Section 2.1 and eventually get to the definition of a Grothendieck fibration. The main reference for this section is still [Jac99]. Firstly, we need some terminology.

Construction 2.2.1. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor. Then p can be seen as a display family $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ of categories. For every object $I \in \mathcal{B}$, consider the following data, which we denote $\mathcal{E}_I$ and call the ***fibre of*** $\begin{array}{c}\mathcal{E}\\ \downarrow \varphi\\ \mathcal{B}\end{array}$ ***over*** I:

an object of $\mathcal{E}_I$ is an object $X \in \mathcal{E}$ with $p(X) = I$;

a morphism $X \to Y$ **in** $\mathcal{E}_I$ is a morphism $f\colon X \to Y$ in $\mathcal{E}$ such that $p(f) = \mathrm{id}_I$ in $\mathcal{B}$.

Proposition 2.2.2. *The fibre $\mathcal{E}_I$ of $\begin{array}{c}\mathcal{E}\\ \downarrow \varphi\\ \mathcal{B}\end{array}$ over I is a subcategory of $\mathcal{E}$.*

Proof. The proof is trivial. □

Definition 2.2.3. An object $X \in \mathcal{E}$ such that $p(X) = I$ (that is, $X \in \mathcal{E}_I$)) is said to be ***above*** I; similarly, a morphism $f \in \mathcal{E}$ with $p(f) = u$ in $\mathcal{B}$ is said to be ***above*** u. This terminology is in accordance with our "vertical" notation $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$.

A morphism in $\mathcal{E}$ which is above some identity morphism in $\mathcal{B}$ (that is, when it is in a fibre) is called ***vertical***.

When considering such a family of categories $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$, we call $\mathcal{B}$ the ***base category*** and we call $\mathcal{E}$ the ***total category***.

Example 2.2.4. Consider the codomain functor $\begin{array}{c}\mathcal{S}et^{\to}\\ \downarrow \mathrm{cod}\\ \mathcal{S}et\end{array}$. An object above $I \in \mathcal{S}et$ is a family $\begin{array}{c}X\\ \downarrow \varphi\\ I\end{array}$ over I. A vertical morphism has the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

Thus the fibre category above $I \in \mathcal{S}et$ can be identified with the slice category $\mathcal{S}et \downarrow I$ of families over I and commuting triangles.

Notice, moreover, that the fibre $\mathcal{S}et \downarrow 1$ over the terminal object of $\mathcal{S}et$ (that is, the singleton) can be identified with the base category $\mathcal{S}et$ itself.

Looking at Construction 2.1.1, we also see that we can construct the fibres X_i with $i \in I$ of a family $\begin{smallmatrix} X \\ \downarrow\varphi \\ I \end{smallmatrix}$ via the pullback of sets

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\ 1 \arrow[r] & I \end{tikzcd}$$

Example 2.2.5. Let $\begin{smallmatrix} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{smallmatrix}$ be a functor. The fibre category over $I \in \mathcal{B}$ can be constructed via the pullback of categories

$$\begin{tikzcd} \mathcal{E}_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \mathbb{1} \arrow[r] & \mathcal{B} \end{tikzcd}$$

as we might expect after seeing Example 2.2.4.

We now come to the definition of cartesian morphisms, which will abstractly capture the universal property we have seen in Remark 2.1.2.

Definition 2.2.6. Let $\begin{smallmatrix} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{smallmatrix}$ be a functor. A morphism $f\colon X \to Y$ in $\mathcal{E}$ above $u := p(f)\colon B \to C$ is called ***cartesian (with respect to*** p***)*** if for every $Z \in \mathcal{E}$, for every $w\colon p(Z) \to B$ and for every $g\colon Z \to Y$ such that $p(g) = u \circ w$, there exists a unique $h\colon Z \to X$ in $\mathcal{E}$ above w such that $f \circ h = g$. We can represent the whole situation in a diagram as

$$\begin{tikzcd} Z \arrow[r, dashed, "h"] \arrow[rr, bend left, "g"] \arrow[d, mapsto, "p"] & X \arrow[r, "f"] \arrow[d, mapsto, "p"] & Y \arrow[d, mapsto, "p"] & \mathcal{E} \arrow[d, "p"] \\ p(Z) \arrow[r, "w"'] \arrow[rr, bend left, "u \circ w"] & B \arrow[r, "u"'] & C & \mathcal{B} \end{tikzcd}$$

From now on, we will often write such diagrams; we will intend that what is represented in the upper part of the diagram is above what is represented in the lower part, and also the both the upper and the lower parts of the diagram are commutative.

Sometimes we will just say that f is cartesian without specifying the morphism u (and we will intend that $u = p(f)$) or even without specifying the functor p (when it is already clear).

Definition 2.2.7. A functor $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ is called a ***Grothendieck fibration*** (or just ***fibration***) if for every $E \in \mathcal{E}$ and for every $u\colon B \to p(E)$ in $\mathcal{B}$, there is a cartesian morphism $f\colon X \to E$ above u, which is often called a cartesian lifting of u to E.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "h"] \arrow[dd, mapsto, "p"] & & \\
& X \arrow[r, "f"] \arrow[dd, mapsto, "p"] & E \arrow[dd, mapsto, "p"] \\
p(Z) \arrow[rrd, bend left, "u\circ w"] \arrow[rd, "w"'] & & \\
& B \arrow[r, "u"'] & p(E)
\end{tikzcd}$$

Here we are actually talking about the morphism in violet, but it may be useful to consider the entire diagram the morphism in violet "*fits well into*" (in the sense that it is above what is represented below it and that it makes the upper part of the diagram commute).

From now on, in an analogous way, we will often represent in violet the morphisms we shall focus on. We will also say that a morphism *fits well into a diagram* in the sense we have just described above.

Sometimes a Grothendieck fibration $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ is also called a fibred category or a category (fibred) over $\mathcal{B}$. This last name comes from the fact Grothendieck fibrations have been developed in order to describe category theory over a base category.

Proposition 2.2.8. *Cartesian liftings are unique up to isomorphism.*

Proof. Let $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ be a fibration. Let $E \in \mathcal{E}$ and let $u\colon B \to p(E)$ be a morphism in $\mathcal{B}$. Take also $f\colon X \to E$ and $f'\colon X' \to E$ two cartesian liftings of u to E. Then these morphisms fit well into the diagram

$$\begin{tikzcd}
X' \arrow[rrd, bend left, "f'"] \arrow[dd, mapsto, "p"] & & \\
& X \arrow[r, "f"] \arrow[dd, mapsto, "p"] & E \arrow[dd, mapsto, "p"] \\
B \arrow[rrd, bend left, "u"] \arrow[rd, equal] & & \\
& B \arrow[r, "u"'] & p(E)
\end{tikzcd}$$

Therefore, since f is cartesian, there exists a unique $h\colon X' \to X$ over the identity such that $f \circ h = f'$. But also f' is cartesian, and then there exists a unique $k\colon X \to X'$ over the identity such that $f' \circ k = f$.

$$
\begin{tikzcd}
X' \arrow[drr, bend left, "f'"] \arrow[dr, dashed, bend right, "h"'] \arrow[dd, "p"] & & \\
& X \arrow[ul, dashed, bend right, "k"'] \arrow[r, "f"] \arrow[dd, "p"] & E \arrow[dd, "p"] \\
B \arrow[drr, bend left, "u"] \arrow[dr, equal] & & \\
& B \arrow[r, "u"'] & p(E)
\end{tikzcd}
$$

Now we shall use the uniqueness given by a cartesian morphism to conclude that $h \circ k$ and $k \circ h$ are the identity. We will prove that $h \circ k = \mathrm{id}_X$; for the other composite the proof is analogous. The composite $h \circ k$ fits well into the diagram

$$
\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, bend left, "h\circ k"] \arrow[dr, bend right, "\mathrm{id}"'] \arrow[dd, "p"] & & \\
& X \arrow[r, "f"] \arrow[dd, "p"] & E \arrow[dd, "p"] \\
B \arrow[drr, bend left, "u"] \arrow[dr, equal] & & \\
& B \arrow[r, "u"'] & p(E)
\end{tikzcd}
$$

since $p(h \circ k) = p(h) \circ p(k) = \mathrm{id}_B \circ \mathrm{id}_B = \mathrm{id}_B$. But f is cartesian and both id_X and $h \circ k$ are over id_B and such that the upper triangle commutes, so we conclude that $h \circ k = \mathrm{id}_X$. Analogously, $k \circ h = \mathrm{id}_{X'}$. Therefore h is an isomorphism and so f and f' differ by an isomorphism. □

Remark 2.2.9. Note that the proof is based on the general argument which ensures that the mathematical structures defined via a universal property are unique up to isomorphism.

Now we would like to see that Grothendieck fibrations, which are also called fibred categories, are a generalization of the concept of category. That is, we want to show that categories are fibred categories.

Proposition 2.2.10. *Let $\mathcal{C}$ be a category. Then the constant functor $\mathcal{C} \to \mathit{1}$ is a fibration.*

Proof. Say that $*$ is the only object of the category $\mathit{1}$. Then the only morphism in $\mathit{1}$ is id_*. Let $C \in \mathcal{C}$. It suffices to show that there is a cartesian lifting of id_* to C. But we can take the identity id_C:

$$
\begin{tikzcd}
D \arrow[drr, bend left, "g"] \arrow[dr, dashed, bend left=10, "h"] \arrow[dr, bend right=10, "g"'] \arrow[dr, phantom, "\shortparallel"] \arrow[dd] & & \\
& C \arrow[r, "\mathrm{id}_C"] \arrow[dd] & C \arrow[dd] \\
* \arrow[drr, equal, bend left] \arrow[dr, equal] & & \\
& * \arrow[r, equal] & *
\end{tikzcd}
$$

□

Remark 2.2.11. Notice that $\mathcal{C} \to \mathit{1}$ gives exactly the data of a category. For us, this is even more meaningful, since there is only one fibre and this is the category $\mathcal{C}$.

From the proof of Proposition 2.2.10 we clearly see that the identity is always a cartesian morphism. Actually, more is true: every isomorphism is cartesian (with respect to any functor).

Proposition 2.2.12. *Let* $\begin{array}{c}\mathcal{X}\\ \downarrow_F\\ \mathcal{C}\end{array}$ *be a functor and let f be an isomorphism in $\mathcal{X}$. Then f is cartesian with respect to F.*

Proof. Consider the diagram

$$\begin{tikzcd}
Z \arrow[r, dashed, "h"] \arrow[rr, bend left, "g"] \arrow[d, mapsto, "F"] & X \arrow[r, "f"] \arrow[d, mapsto, "F"] & Y \arrow[d, mapsto, "F"] \\
p(Z) \arrow[r, "w"'] \arrow[rr, bend left, "F(f)\circ w"] & F(X) \arrow[r, "F(f)"'] & F(Y)
\end{tikzcd}$$

for arbitrary g and w. Then every morphism h which makes the upper triangle commute must satisfy $f \circ h = g$ and then $h = f^{-1} \circ g$. Moreover, $h = f^{-1} \circ g$ is a morphism over $F(f)^{-1} \circ F(f) \circ w = w$ and makes the upper triangle commute. $\square$

We now might wonder if the composite of cartesian morphisms is cartesian, and also if the converse is true. The answer is given by the following proposition.

Proposition 2.2.13. *Let* $\begin{array}{c}\mathcal{X}\\ \downarrow_F\\ \mathcal{C}\end{array}$ *be a functor and let f and g be two composable morphisms in $\mathcal{X}$. Then*

(i) If f and g are cartesian, then also $g \circ f$ is cartesian.

(ii) If g and $g \circ f$ are cartesian, then also f is cartesian.

Proof. For (i) it suffices to consider the diagram

$$\begin{tikzcd}
Z \arrow[r, dashed, "h"] \arrow[rr, dashed, bend left, "\tilde{h}"] \arrow[rrr, bend left=40, "k"] \arrow[d, mapsto, "F"] & X \arrow[r, "f"] \arrow[d, mapsto, "F"] & Y \arrow[r, "g"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
p(Z) \arrow[r, "w"'] \arrow[rr, bend left, "F(f)\circ w"] \arrow[rrr, bend left=40, "F(g)\circ F(f)\circ w"] & F(X) \arrow[r, "F(f)"'] & F(Y) \arrow[r, "F(g)"'] & F(E)
\end{tikzcd}$$

and use the fact that g is cartesian to find a unique $\tilde{h}$ and then the fact that f is cartesian to find a unique h such that the whole diagram commute and the upper part of the diagram is above the lower part of the diagram. Notice that if we take another morphism $h'\colon Z \to X$ over w such that $g \circ f \circ h' = k$, we get that $f \circ h' = \tilde{h} = f \circ h$ by uniqueness of $\tilde{h}$ (given by the fact that g is cartesian) and then $h' = h$ by uniqueness of h (given by the fact that f is cartesian).

To prove (ii), consider the diagram

$$\begin{tikzcd}
Z \arrow[rrrd, bend left, "g\circ q=:k"] \arrow[rrd, bend left, "q"] \arrow[rd, dashed, "h"] \arrow[dd, "F"] & & & \\
& X \arrow[r, "f"] \arrow[dd, "F"] & Y \arrow[r, "g"] \arrow[dd, "F"] & E \arrow[dd, "F"] \\
p(Z) \arrow[rd, "w"'] \arrow[rrd, "F(f)\circ w"] \arrow[rrrd, "F(g)\circ F(f)\circ w"] & & & \\
& F(X) \arrow[r, "F(f)"'] & F(Y) \arrow[r, "F(g)"'] & F(E)
\end{tikzcd}$$

Let $k := g \circ q$. Since g is cartesian, q is the only morphism above $F(f) \circ w$ such that $g \circ q = k$. Since $g \circ f$ is cartesian, we find a unique h over w such that $g \circ f \circ h = k = g \circ q$. It remains to prove that h is such that $f \circ h = q$ and that it is the only one above w which satisfies it. Since g followed both $f \circ h$ or q gives k, we get that $f \circ h = q$ by uniqueness of q. Finally, if we take another morphism $h'\colon Z \to X$ over w such that $f \circ h' = q$ we get that $g \circ f \circ h' = g \circ q = k$, whence $h' = h$ by uniqueness of h. $\square$

2.3 Two Fundamental Constructions

We now want to see two fundamental constructions of Grothendieck fibrations. This section will also give us some examples of fibrations. In particular we will prove that the examples which we have seen in Section 2.1 are examples of fibrations.

The main reference for this section will be [Str18].

Remark 2.3.1. We have seen in Remark 2.1.2 that the morphism we have produced in Construction 2.1.1 satisfies the universal property of a cartesian morphism. Now that we know the definition, we notice that the morphism (u, f) in $\mathcal{S}et^{\to}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\psi"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

was a cartesian morphism above u with respect to the codomain functor $\begin{array}{c}\mathcal{S}et^{\to}\\ \downarrow\mathrm{cod}\\ \mathcal{S}et\end{array}$.

We notice that we only used that $\mathcal{S}et$ has pullbacks, so we might try to generalize that result to any category $\mathcal{C}$ with pullbacks instead of $\mathcal{S}et$.

Theorem 2.3.2. *Let $\mathcal{C}$ be a category. Then the codomain functor* $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ *is a Grothendieck fibration if and only if $\mathcal{C}$ has pullbacks.*

Proof. Assume first that $\mathcal{C}$ has pullbacks. For every diagram

$$\begin{tikzcd} & Y \arrow[d, "\psi"] & \mathcal{C}^{\to} \arrow[dd, "\mathrm{cod}"] \\ & J & \\ I \arrow[r, "u"] & J & \mathcal{C} \end{tikzcd}$$

we exhibit a cartesian lifting of u to $\begin{array}{c}Y\\ \downarrow{\scriptstyle \psi}\\ J\end{array}$. After Remark 2.3.1 we define it looking at Construction 2.1.1. We consider the object $\varphi \in \mathcal{C}^{\to}$ and the morphism (u, f) in $\mathcal{C}^{\to}$ given by the pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\psi"] & \mathcal{C}^{\to} \arrow[dd, "\mathrm{cod}"] \\ I \arrow[r, "u"] & J & \\ I \arrow[r, "u"] & J & \mathcal{C} \end{tikzcd}$$

To prove that (u, f) is a cartesian morphism with respect to cod, it suffices to follow the construction we have seen in Remark 2.1.2, just changing $\mathcal{S}et$ with $\mathcal{C}$ and $\mathcal{S}et^{\to}$ with $\mathcal{C}^{\to}$. We write here again the diagram we had obtained by the universal property of pullback

$$\begin{tikzcd} Z \arrow[drr, bend left, "g"] \arrow[dr, dashed, "h"] \arrow[dd, "\chi"'] & & \\ & X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\psi"] \\ K \arrow[r, "w"'] \arrow[rr, bend left, "u \circ w"] & I \arrow[r, "u"'] & J \end{tikzcd}$$

We see that the morphism (w, h) in $\mathcal{C}^{\to}$ is the unique morphism above w (it just means that the first component must be w) which satisfies $(u, f) \circ (w, h) = (u \circ w, g)$, by the universal property of pullback.

Therefore $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ is a fibration.

Assume now that $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ is a fibration. Then for every morphism $u\colon I \to J$ and every object $\begin{array}{c}Y\\ \downarrow{\scriptstyle \psi}\\ J\end{array} \in \mathcal{C}^{\to}$, there exists a cartesian lifting of u to $\begin{array}{c}Y\\ \downarrow{\scriptstyle \psi}\\ J\end{array}$, say

$$\begin{array}{c}X\\ \downarrow{\scriptstyle \varphi}\\ I\end{array} \xrightarrow{(u,f)} \begin{array}{c}Y\\ \downarrow{\scriptstyle \psi}\\ J\end{array}$$

Now we show that X with the morphisms f and φ in $\mathcal{C}$ is a pullback of u and ψ. Since (u, f) is a morphism in $\mathcal{C}^{\to}$, we have that the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

commutes. Now consider another object Z of $\mathcal{C}$ and two morphisms $g\colon Z \to Y$ and $\chi\colon Z \to I$ which form another commutative square with u and ψ

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\chi"'] & & \\
& X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
& I \arrow[r, "u"'] & J
\end{tikzcd}$$

But since (u, f) is a cartesian lifting of u to $\begin{array}{c}Y\\ \downarrow{\scriptstyle \psi}\\ J\end{array}$, we can find a morphism $h\colon Z \to X$ above id_I and such that $f \circ h = g$

$$\begin{tikzcd}
Z \arrow[rr, bend left, "g"] \arrow[r, dashed, "h"] \arrow[d, "\chi"'] & X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\
I \arrow[r, "\mathrm{id}"'] \arrow[rr, bend left, "u"] & I \arrow[r, "u"'] & J
\end{tikzcd}$$

Since (id_I, h) needs to be a morphism in $\mathcal{C}^{\to}$, we also have that $\varphi \circ h = \mathrm{id} \circ \chi = \chi$. Then we have found a pullback of u and ψ. By arbitrariness of u and ψ it follows that $\mathcal{C}$ has pullbacks. $\square$

Remark 2.3.3. The proof Theorem 2.3.2 also justifies what we said in Remark 2.3.1, that is, that $\begin{array}{c}\mathcal{S}et^{\rightarrow}\\ \downarrow \mathrm{cod}\\ \mathcal{S}et\end{array}$ is a fibration.

Definition 2.3.4. If $\mathcal{C}$ has pullbacks, the Grothendieck fibration $\begin{array}{c}\mathcal{C}^{\rightarrow}\\ \downarrow \mathrm{cod}\\ \mathcal{C}\end{array}$ is called the ***fundamental fibration*** on $\mathcal{C}$.

We have just seen the first fundamental construction of a fibration. Now, we would like to see that also display indexing, which we have seen in Section 2.1 gives rise to a fibration.

To do this, a nice way goes through presheaves (here, of sets), that is, functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.

Construction 2.3.5. Given a function $\varphi \colon X \to I$, which we think as a collection of sets through dispaly indexing, as we saw in Section 2.1, we can construct the presheaf

$$\begin{aligned} \mathrm{Fam} \colon I^{\mathrm{op}} &\longrightarrow \mathcal{S}et \\ i &\longmapsto X_i := \varphi^{-1}\{i\} \\ \mathrm{id}_i &\longmapsto \mathrm{id}_{X_i} \end{aligned}$$

where we consider I as a discrete category (that is, we take as morphisms just the identities) and we send each $i \in I$ to its fibre, thus representing the collection of sets with this presheaf.

Remark 2.3.6. We have seen in Construction 2.2.1 that display indexing can be thought with arbitrary categories instead of sets (which can be considered as discrete categories). So we would like to generalize Construction 2.3.5 to have a presheaf of categories $\mathrm{Fam} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ starting from an arbitrary functor $F \colon \mathcal{X} \to \mathcal{C}$.

The problem is that, with this generality, this cannot be done. Construction 2.3.5 works well because there is a canonical way to define the functor Fam on morphisms (that is, the only way that makes it a functor, considering I as a discrete category).

In general, this is related with what we have called change of base or substitution in Construction 2.1.1, and there is not a canonical change of base starting from any functor instead of φ. We then might think, also remembering that Grothendieck fibrations capture both indexing and change of base, as we have said in Section 2.1, that it is actually better to start from a presheaf of categories rather than from a functor thought as display indexing, and to try to construct a fibration from a presheaf of categories.
In this way we will start having information on both indexing and change of base, and we will just try to encode this information in a fibration.

Example 2.3.7. Consider the category $\mathcal{CR}ng$ of commutative rings with unit. After Remark 2.3.6, we would like to construct a presheaf of categories $\mathcal{H} \colon \mathcal{CR}ng^{\mathrm{op}} \longrightarrow \mathcal{C}at$. As we said in Remark 2.3.6, this is related to giving an indexing over $\mathcal{CR}ng$ and a change of base along each morphism h in $\mathcal{CR}ng$.

As indexing, we consider

$$(\mathcal{M}od(R))_{R\in \mathcal{CRng}},$$

where $\mathcal{M}od(R)$ is the category of R-modules.
As change of base along a morphism $h\colon R\to R'$ in $\mathcal{CRng}$, we consider

$$(\rho_{\mathrm{h}}\colon \mathcal{M}od(R')\to \mathcal{M}od(R))$$

the restriction of scalars along h, that is, if M is an R-module, $\rho_h(M)$ is a module with the same addition of M and scalar multiplication given by $r\cdot x = h(r)\cdot_M x$.

Then we shall consider the assignment

$$\begin{aligned} \mathcal{H}\colon \mathcal{CRng}^{\mathrm{op}} &\longrightarrow \mathcal{Cat} \\ R &\longmapsto \mathcal{M}od(R) \\ (h\colon R\to R') &\longmapsto (\rho_{\mathrm{h}}\colon \mathcal{M}od(R')\to \mathcal{M}od(R)) \end{aligned}$$

And it is easy to prove that $\mathcal{H}$ is a functor, that is, that $\mathcal{H}$ is a presheaf of categories.

Now, as we said in Remark 2.3.6, we would like to construct from this presheaf of categories a fibration, encoding in some sense both the indexing and the change of base.
We might have the idea to consider the category $\mathcal{Mod}$ of "all" modules, given by the following data:

an object of $\mathcal{Mod}$ is a pair (R,M) with $R\in\mathcal{CRng}$ and $M\in\mathcal{M}od(R)$;

a morphism $(R,M)\longrightarrow(R',M')$ **in $\mathcal{Mod}$** is a pair (h,κ) with $h\colon R\to R'$ a morphism in $\mathcal{CRng}$ and $\kappa\colon M\to\rho_h(M')$ a morphism in $\mathcal{M}od(R)$.

It is easy to prove that, with composition and identities componentwise inherited from $\mathcal{CRng}$ (for the first component) and from $\mathcal{M}od(R)$ (for the second component), $\mathcal{Mod}$ is a category.

Now it is natural to consider as fibration (as we will prove later it is such) the projection on the first component

$$\begin{aligned} \mathrm{pr}_1\colon \mathcal{Mod} &\longrightarrow \mathcal{CRng} \\ (R,M) &\longmapsto R \\ (h,\kappa) &\longmapsto h \end{aligned}$$

The fact that this projection functor is a fibration will follow as a particular case by Construction 2.3.9, which gives the general construction of a fibration starting from a presheaf of categories.

Example 2.3.8. Let $\mathcal{C}$ be a category with pullbacks. We can consider

$$\begin{aligned} \mathcal{H}\colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{Cat} \\ C &\longmapsto \mathcal{C}\!\downarrow\! C \\ (u\colon C'\to C) &\longmapsto (u^*\colon \mathcal{C}\!\downarrow\! C\to \mathcal{C}\!\downarrow\! C') \end{aligned}$$

where u^* is a pullback functor (which one might see as a right adjoint to $u \circ -$, the functor of postcomposition with u).

Generalizing Example 2.2.4 to a functor $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow \text{cod}\\ \mathcal{C}\end{array}$, we can think of the slice category $\mathcal{C} \downarrow C$ as the fibre of cod over C. Then one might recall Construction 2.3.5 and see that we are trying to generalize that construction to the functor $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow \text{cod}\\ \mathcal{C}\end{array}$. As we have discussed in Remark 2.3.6, to do this we actually need a change of base, which we indeed have, as we have seen in Theorem 2.3.2, and is given by pullbacks. The problem here is that we actually have to choose the pullbacks (or equivalently the right adjoints to postcomposition functors), which are unique only up to isomorphism. This has the consequence that $\mathcal{H}(u \circ v)$ is not equal to $\mathcal{H}(v) \circ \mathcal{H}(u)$ in general, but only isomorphic. Then, in general, this $\mathcal{H}$ is not a functor, and thus not a presheaf of categories.

Nonetheless, we are just trying to write into a fibration what we have already seen that actually is a fibration, in Theorem 2.3.2. We do not want to exclude this situation, and so we will accept such "functors" which preserve composition and (maybe also) identities only up to natural isomorphisms (and such that these isomorphisms satisfy a diagram of associativity and two diagrams of unit). These are called ***pseudofunctors***.

We now see the second and most important fundamental construction of this section. We will start from a pseudofunctor $\mathcal{C}^{\text{op}} \to \mathcal{C}at$ and we will produce a fibration. This construction takes the name of ***Grothendieck construction***, and is actually a generalization of a more basic and famous construction, the ***category of elements*** of a functor which lands in $\mathcal{S}et$.

Construction 2.3.9 (Grothendieck construction)**.** Let $\mathcal{H}\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ be a pseudofunctor. Starting from $\mathcal{H}$, we construct a new category $\mathcal{G}_{\mathcal{H}}$. Since there is a standard notation only for the whole Grothendieck construction, the author has decided to use this non-standard notation $\mathcal{G}_{\mathcal{H}}$.

Consider the following data:

an object of $\mathcal{G}_{\mathcal{H}}$ is a pair (B, X) with $B \in \mathcal{C}$ and $X \in \mathcal{H}(B)$;

a morphism $(B, X) \to (C, Y)$ **in** $\mathcal{G}_{\mathcal{H}}$ is a pair of morphisms (b, x) with $b\colon B \to C$ in $\mathcal{C}$ and $x\colon X \to \mathcal{H}(b)(Y)$ in $\mathcal{H}(B)$ (notice that $H(b)\colon \mathcal{H}(C) \to \mathcal{H}(B)$);

a composition $(B, X) \xrightarrow{(b,x)} (C, Y) \xrightarrow{(c,y)} (D, Z)$ **in** $\mathcal{G}_{\mathcal{H}}$ is given by

$$(c, y) \circ (b, x) := (c \circ b,\, \alpha_Z^{b,c} \circ \mathcal{H}(b)(y) \circ x)$$

where $\alpha^{b,c}\colon \mathcal{H}(b) \circ \mathcal{H}(c) \overset{\cong}{\Rrightarrow} \mathcal{H}(c \circ b)$ is the natural isomorphism given by the fact that $\mathcal{H}$ is a pseudofunctor (notice that $x\colon X \to \mathcal{H}(b)(y)$, that $\mathcal{H}(b)\colon \mathcal{H}(c) \to \mathcal{H}(b)$ is a functor and that $\mathcal{H}(b)(y)\colon \mathcal{H}(b)(y) \to \mathcal{H}(b)(\mathcal{H}(c)(Z)))$;

an identity $\mathrm{id}_{(B,X)}$ **in** $\mathcal{G}_{\mathcal{H}}$ is $(\mathrm{id}_B, (\lambda_X^B)^{-1} \circ \mathrm{id}_X)$, where $\lambda^B \colon \mathcal{H}(\mathrm{id}_B) \overset{\cong}{\Rightarrow} \mathrm{Id}_{\mathcal{H}(B)}$ (notice that $(\lambda_X^B)^{-1} \circ \mathrm{id}_X \colon X \to \mathcal{H}(\mathrm{id}_B(X)))$.

One can easily prove that the data $\mathcal{G}_{\mathcal{H}}$ form a category, although this requires to use the axioms of associativity and unit that the natural isomorphisms $\alpha^{b,c}$ and λ^B satisfy.

However the definition of $\mathcal{G}_{\mathcal{H}}$ simplifies when $\mathcal{H}$ is a functor, that is, when $\alpha^{b,c} = \mathrm{Id}$ for every b, c composable morphisms in $\mathcal{C}$ and $\lambda^B = \mathrm{Id}$ for every $B \in \mathcal{C}$, and in this case it is much easier to verify that $\mathcal{G}_{\mathcal{H}}$ is a category.

We can view the objects of the category $\mathcal{G}_{\mathcal{H}}$ as pairs (B, X) with $B \in \mathcal{C}$ and $\mathit{1} \xrightarrow{X} \mathcal{H}(B)$ a functor from the category $\mathit{1}$ with one object and the identity to the category $\mathcal{H}(B)$.

Then we can also visualize the morphisms of the category $\mathcal{G}_{\mathcal{H}}$ as pairs (b, x) with x a natural transformation

$$\begin{tikzcd}
& \mathit{1} \arrow[dl, "X"'] \arrow[dr, "Y"] \arrow[d, phantom, "\overset{x}{\Longrightarrow}"] & \\
\mathcal{H}(B) & & \mathcal{H}(C) \arrow[ll, "\mathcal{H}(b)"] \\
B \arrow[rr, "b"] & & C
\end{tikzcd}$$

And composition seems now much more natural, since it just corresponds to the composition of the natural transformations:

$$\begin{tikzcd}
& & \mathit{1} \arrow[dll, "X"'] \arrow[d, "Y"] \arrow[drr, "Z"] \arrow[dl, phantom, "\overset{x}{\Rightarrow}"] \arrow[dr, phantom, "\overset{y}{\Rightarrow}"] & & \\
\mathcal{H}(B) & & \mathcal{H}(C) \arrow[ll, "\mathcal{H}(b)"] \arrow[d, phantom, "\Downarrow \alpha^{b,c}"] & & \mathcal{H}(D) \arrow[ll, "\mathcal{H}(c)"] \arrow[llll, bend left, "\mathcal{H}(cb)"] \\
& & & & \\
B \arrow[rr, "b"] & & C \arrow[rr, "c"] & & D
\end{tikzcd}$$

The identities seem now more natural as well:

$$\begin{tikzcd}
 & 1 \arrow[dl, "X"'] \arrow[dr, "X"] \arrow[d, phantom, "\Rrightarrow \mathrm{id}_X"] & \\
\mathcal{H}(B) & & \mathcal{H}(B) \arrow[ll, "\mathrm{Id}_{\mathcal{H}(B)}"'] \arrow[ll, bend left=60, "\mathcal{H}(\mathrm{id}_B)"] \arrow[ll, phantom, bend left=30, "\Rrightarrow (\lambda^B)^{-1}"]
\end{tikzcd}$$

$$B \xrightarrow{\mathrm{id}_B} B$$

Finally, the fact that $\mathcal{G}_{\mathcal{H}}$ is a category might appear more natural as well, with pasting diagrams arguments.

We then produce a fibration (as we shall promptly prove it is such):

$$\begin{aligned} \textstyle\int H \colon \mathcal{G}_{\mathcal{H}} &\longrightarrow \mathcal{C} \\ (B, X) &\longmapsto B \\ (b, x) &\longmapsto b \end{aligned}$$

To prove that this really is a fibration, we start from a diagram

$$\begin{tikzcd}
 & (C, Y) \arrow[d, mapsto, "\int H"] \\
B \arrow[r, "b"] & C
\end{tikzcd}$$

(with arbitrary b and Y) and we find a cartesian lifting of b to (C, Y).
The first component has to be b. The object above B will then be of the form (B, X) for some $X \in \mathcal{H}(B)$. We now need a morphism $X \to \mathcal{H}(b)(Y)$, and, also thinking about the arbitrariness of $\mathcal{H}$, one might think that it is clever to define $X := H(b)(Y)$ and the cartesian lifting of b to (C, Y) as $(b, \mathrm{id}_{\mathcal{H}(b)(Y)})$. To see that it really works, we consider our usual diagram

$$\begin{tikzcd}
(A, K) \arrow[d, mapsto, "\int H"] \arrow[dr, dashed, "\exists! v = (a, (\alpha_Y^{a,b})^{-1} \circ k)"] \arrow[drr, bend left, "(ba, k)"] & & \\
 & (B, \mathcal{H}(b)(Y)) \arrow[r, "(b, \mathrm{id})"] \arrow[d, mapsto, "\int H"] & (C, Y) \arrow[d, mapsto, "\int H"] \\
A \arrow[r, "a"'] \arrow[rr, bend left, "ba"] & B \arrow[r, "b"'] & C
\end{tikzcd}$$

with arbitrary a and k (note that the first component of (ba, k) must be like that because it is a morphism above ba. Clearly the first component of v must be a. Then we must search for a

v of the form (a, w) with $w\colon K \to \mathcal{H}(a)(\mathcal{H}(b)(Y))$. We must have that $(b, \mathrm{id}) \circ (a, w) = (ba, k)$, that is $(ba, \alpha_Y^{a,b} \circ \mathcal{H}(a)(\mathrm{id}_{H(b)(Y)}) \circ w) = (ba, k)$.
Since $\mathcal{H}(a)$ is a functor, it preserves identities. Then we obtain that $\alpha_Y^{a,b} \circ w = k$. But $\alpha_Y^{a,b}$ is an isomorphism, so we get that w must be equal to $(\alpha_Y^{a,b})^{-1} \circ k$ and we have proved the uniqueness of v. Now, we just need to check that $v := (a, (\alpha_Y^{a,b})^{-1} \circ k)$ fits well in the diagram making it commute. But it is clear that $\int H(v) = a$ and that

$$(b, \mathrm{id}) \circ (a, (\alpha_Y^{a,b})^{-1} \circ k) = (ba, \alpha_Y^{a,b} \circ \mathcal{H}(a)(\mathrm{id}) \circ (\alpha_Y^{a,b})^{-1} \circ k) = (ba, k).$$

Therefore $\int H$ is a fibration.

Example 2.3.10. Let $\mathcal{C}$ be a category and let $\mathcal{H}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ a constant functor, say of value (a category) $\mathcal{D}$. Following Construction 2.3.9, which simplifies a bit since $\mathcal{H}$ is a proper functor, we get a category $\mathcal{G}_{\mathcal{H}}$ which consists of the following data:

an object of $\mathcal{G}_{\mathcal{H}}$ is a pair (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{H}(C) = \mathcal{D}$;

a morphism $(C, D) \to (C', D')$ in $\mathcal{G}_{\mathcal{H}}$ is a pair of morphisms (c, d) with $c\colon C \to C'$ in $\mathcal{C}$ and $d\colon D \to \mathcal{H}(c)(D') = \mathrm{id}_{\mathcal{D}}(D') = D'$ in $\mathcal{H}(C) = \mathcal{D}$;

a composition $(C, D) \overset{(c,d)}{\to} (C', D') \overset{(c',d')}{\to} (C'', D'')$ in $\mathcal{G}_{\mathcal{H}}$ is given by
$$(c', d') \circ (c, d) := (c'c, \mathrm{id}_{\mathcal{D}}(d') \circ d);$$

an identity $\mathrm{id}_{(C,D)}$ in $\mathcal{G}_{\mathcal{H}}$ is $(\mathrm{id}_C, \mathrm{id}_D)$.

Therefore $\mathcal{G}_{\mathcal{H}}$ is just the product category $\mathcal{C} \times \mathcal{D}$ and we deduce from Construction 2.3.9 that the projection to the first component $\begin{array}{c}\mathcal{C} \times \mathcal{D} \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C}\end{array}$ is a Grothendieck fibration.

Example 2.3.11. Consider the category $\mathcal{CR}ng$ of commutative rings with unit, and consider the presheaf of categories

$$\begin{aligned} \mathcal{H}\colon \mathcal{CR}ng^{\mathrm{op}} &\longrightarrow \mathcal{C}at \\ R &\longmapsto \mathcal{M}od(R) \\ (h\colon R \to R') &\longmapsto (\rho_{\mathrm{h}}\colon \mathcal{M}od(R') \to \mathcal{M}od(R)) \end{aligned}$$

described in Example 2.3.7. Then we can easily see that the Grothendieck construction applied to $\mathcal{H}$ gives exactly the projection functor

$$\begin{aligned} \mathrm{pr}_1\colon \mathcal{M}od &\longrightarrow \mathcal{CR}ng \\ (R, M) &\longmapsto R \\ (h, \kappa) &\longmapsto h \end{aligned}$$

we have described in Example 2.3.7. We might understand better from this the definition of the category $\mathcal{G}_{\mathcal{H}}$, seeing how it encodes the information on both indexing and change of base.

This also justifies that this projection functor pr_1 is a fibration, as we had anticipated in Example 2.3.7.

Example 2.3.12. Let $\mathcal{C}$ be a category with pullbacks and let $\mathcal{H}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\!at$ be the pseudo-functor described in Example 2.3.8. Following Construction 2.3.9, which simplifies a bit since $\mathcal{H}$ is a proper functor, we get a category $\mathcal{G}_{\mathcal{H}}$ which consists of the following data:

an object of $\mathcal{G}_{\mathcal{H}}$ is a pair $(C, D \xrightarrow{f} C)$ with $C \in \mathcal{C}$ and $f\colon D \to C$ a morphism in $\mathcal{C}$ (indeeed $f \in \mathcal{C}{\downarrow}C$);

a morphism $(C, D \xrightarrow{f} C) \to (C', D' \xrightarrow{f'} C')$ **in** $\mathcal{G}_{\mathcal{H}}$ is a pair of morphisms (c, d) with $c\colon C \to C'$ in $\mathcal{C}$ and $d\colon (D \xrightarrow{f} C) \to c^*(D' \xrightarrow{f'} C')$ in $\mathcal{C}{\downarrow}C$, but by adjunction d corresponds to a unique morphism $(D \xrightarrow{f} C \xrightarrow{c} C') \to (D' \xrightarrow{f'} C')$ and so to a unique morphism $D \xrightarrow{d} D'$ such that $cf = f'd$;

composition and identities are, thinking of morphisms with the correspondence described right above, just the composition and the identities component-wise inherited by $\mathcal{C}$.

Therefore $\mathcal{G}_{\mathcal{H}}$ is precisely the arrow category $\mathcal{C}^{\to}$, and we see that $\int H$ is just the codomain functor $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow\mathrm{cod}\\ \mathcal{C}\end{array}$. We have then proved in another way that this codomain functor is a fibration. Thus this second fundamental construction generalizes the first one.

2.4 Cloven and Split Fibrations

The main reference for this section is still [Str18].

Let $\mathcal{C}$ be a category with pullbacks. We have seen in the proof of Theorem 2.3.2 and more explicitly in Example 2.3.8 that to make $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow\mathrm{cod}\\ \mathcal{C}\end{array}$ into a fibration we have to choose the pullbacks, actually utilizing the axiom of choice for classes. This says to us that in general there might be more ways to choose the cartesian liftings, even if they are always unique up to isomorphism, by Proposition 2.2.8.

Definition 2.4.1. A ***cleavage*** for a functor $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ is a choice, for every $E \in \mathcal{E}$ and for every $u\colon B \to p(E)$ in $\mathcal{B}$, of a cartesian lifting of u to E. When we have a cleavage, we denote the chosen cartesian lifting of u to E

$$\mathrm{Cart}_p(u, E) = \mathrm{Cart}\,(u, E)\colon u^*E \to E$$

for every $E \in \mathcal{E}$ and every $u\colon B \to p(E)$. In a diagram

$$\begin{tikzcd}
u^*E \arrow[r, "\mathrm{Cart}(u,E)"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\
B \arrow[r, "b"] & C
\end{tikzcd}$$

Remark 2.4.2. A functor with a cleavage obviously is a Grothendieck fibration. If, instead, we start from a fibration, we can get a cleavage using the axiom of choice. Therefore, if one is in the situation in which he can use the axiom of choice, a functor with a cleavage is exactly a Grothendieck fibration.

Definition 2.4.3. A functor with a cleavage is called a ***cloven fibration***.

Remark 2.4.4. By Remark 2.4.2, a cloven fibration really is a fibration. Furthermore, there is the advantage that one can talk about cloven fibrations even when the axiom of choice is not assumed or cannot be assumed. For us, the axiom of choice will always be assumed, but it is interesting to work with fixed cleavages and to ask ourselves when they are preserved.

Example 2.4.5. When we described the Grothendieck construction (Construction 2.3.9 we actually constructed a cleavage for the functor $\int H$.

To see this more explicitly, recall Example 2.3.10, in which we proved using the Grothendieck construction that the projection from the product category $\begin{array}{c}\mathcal{C}\times\mathcal{D}\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is a Grothendieck fibration.

Following the entire Construction 2.3.9, we see that we construct a cleavage for $\begin{array}{c}\mathcal{C}\times\mathcal{D}\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$

$$\begin{tikzcd}
(C', D) \arrow[r, "(c,\mathrm{id})"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (C, D) \arrow[d, mapsto, "\mathrm{pr}_1"] \\
C' \arrow[r, "c"'] & C
\end{tikzcd}$$

setting $\mathrm{Cart}\,(c,(C,D)) = ((C',D) \overset{(c,\mathrm{id})}{\to} (C,D))$ for every $(C,D) \in \mathcal{C}\times\mathcal{D}$ and every $c\colon C' \to C$.

In particular, if $c = \mathrm{id}_C$, $\mathrm{Cart}\,(\mathrm{id},(C,D)) = \mathrm{id}_{(C,D)}$.

However, even if this particular cleavage satisfies this property, we can find (in general) other cleavages in which the chosen cartesian lifting of an identity is not an identity.

For example, fixed an isomorphism $D' \overset{\varphi}{\to} D$ different from the identity (assumed that it exists in $\mathcal{D}$), we could choose as a cartesian lifting for an identity id_C to (C,D) the morphism (id_C,φ), which is cartesian with respect to pr_1 since it is an isomorphism in $\mathcal{C}\times\mathcal{D}$, by Proposition 2.2.12.

Definition 2.4.6. A cloven fibration $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ such that $\mathrm{Cart}\left(\mathrm{id}_{p(E)}, E\right) = \mathrm{id}_E$ for every $E \in \mathcal{E}$ is called a ***normal*** fibration.

Remark 2.4.7. One might easily see that Example 2.4.5 can be generalized to say that every time we start from a (proper) functor $\mathcal{H}$ rather than just a pseudofunctor, the cleavage produced in Construction 2.3.9 makes $\int H$ into a normal fibration. (Remember that we saw that $\begin{array}{c}\mathcal{C}\times\mathcal{D}\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is the Grothendieck construction of a proper functor, in Example 2.3.10.)

Another question which might arise is about functoriality of the assignment Cart we have when we consider a cloven fibration.

Proposition 2.4.8. *Let* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ *be a cloven fibration. Fixed a certain* $E \in \mathcal{E}$*, the assignment* $[u \mapsto u^*E]$ *extends to a functor* $\mathcal{B}{\downarrow}p(E) \to \mathcal{E}$*, which we shall denote* $\mathrm{Cart}\,(-, E)$*.*

Proof. First note that we have a function on objects and a function on morphisms.
As function on objects just take $(B \xrightarrow{b} p(E)) \mapsto b^*E$.

Take now a morphism $\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "b"] & \\ C \arrow[r, "c"'] & p(E) \end{tikzcd}$ in $\mathcal{B}{\downarrow}p(E)$. We need a morphism $b^*E \to c^*E$. But we can produce such a morphism by the definition of cartesian morphism:

$$\begin{tikzcd}
b^*E \arrow[r, dashed, "\exists! v"] \arrow[rr, bend left, "\mathrm{Cart}(b,E)"] \arrow[d, mapsto, "p"] & c^*E \arrow[r, "\mathrm{Cart}(c,E)"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\
B \arrow[r, "h"'] \arrow[rr, bend left, "ch=b"] & C \arrow[r, "c"'] & p(E)
\end{tikzcd}$$

We can then define $\mathrm{Cart}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "b"] & \\ C \arrow[r, "c"'] & p(E) \end{tikzcd}, E\right)$ to be this v.

Now that we have a function on morphisms, it remains to verify the functoriality.
The preservation of identities is trivial, since for every $b \in \mathcal{B}{\downarrow}p(E)$ when $h = \mathrm{id}_B$ we have the diagram

$$\begin{tikzcd}
b^*E \arrow[rrr, bend left, "\mathrm{Cart}(b,E)"] \arrow[r, dashed, "v"] \arrow[r, shift right=2, "\mathrm{id}"'] \arrow[d, mapsto, "p"] & b^*E \arrow[rr, "\mathrm{Cart}(b,E)"] \arrow[d, mapsto, "p"] & & E \arrow[d, mapsto, "p"] \\
B \arrow[r, "\mathrm{id}"'] \arrow[rrr, bend left, "b"] & B \arrow[rr, "b"'] & & p(E)
\end{tikzcd}$$

where v is the only morphism above id_B which fits well in the diagram. Since id_{b^*E} fits well in it too, we conclude by uniqueness of v.

To verify the preservation of composites, we consider the diagram

$$\begin{tikzcd}
a^*E \arrow[rrr, bend left, "\mathrm{Cart}(a,E)"] \arrow[r, dashed, "w"] \arrow[rr, dashed, bend right, "z"'] \arrow[d, mapsto, "p"] & b^*E \arrow[rr, bend left, "\mathrm{Cart}(b,E)"] \arrow[r, dashed, "v"] \arrow[d, mapsto, "p"] & c^*E \arrow[r, "\mathrm{Cart}(c,E)"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\
A \arrow[r, "k"] \arrow[rrr, bend left, "bk=a"] \arrow[rr, bend right, "hk"'] & B \arrow[r, "h"] \arrow[rr, bend left, "ch=b"] & C \arrow[r, "c"'] & p(E)
\end{tikzcd}$$

corresponding to the composite

$$\begin{tikzcd}
A \arrow[d, "k"'] \arrow[ddr, "a"] & \\
B \arrow[d, "h"'] \arrow[dr, "b"] & \\
C \arrow[r, "c"'] & p(E)
\end{tikzcd}$$

in $\mathcal{B}\downarrow p(E)$, where v is produced by cartesianity of $\mathrm{Cart}\,(c, E)$, w is produced by cartesianity of $\mathrm{Cart}\,(b, E)$ and z is produced by cartesianity of $\mathrm{Cart}\,(c, E)$ considering the composite hk.

We would like to have that $vw = z$, but this is granted by cartesianity of $\mathrm{Cart}\,(c, E)$ since both the morphisms fit well in the diagram and z needs to be unique. □

Remark 2.4.9. We will sometimes use the notation introduced in the proof of Proposition 2.4.8 for the function on morphisms produced by the cleavage of a cloven fibration.

Even if every cloven fibration produces a functor $\mathrm{Cart}\,(-, E)$, we have a precious property in addition when the fibration we start from is normal.

In the notation introduced in the proof of Proposition 2.4.8, if we take $c = \mathrm{id}_{p(E)}$ then $\mathrm{Cart}\,(c, E) = \mathrm{id}_E$ and therefore v must be equal to $\mathrm{Cart}\,(b, E)$ (because this works well and there can be only one morphism which fits well in the diagram). Thus we have interpreted

$\mathrm{Cart}\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "b"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}, E\right)$ as $\mathrm{Cart}\,(b, E)$. This interpretation will be useful to us many times.

Remark 2.4.10. Looking at the proof of Proposition 2.4.8, and also after Remark 2.4.9, one might wonder whether $\mathrm{Cart}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "b"] & \\ C \arrow[r, "c"'] & p(E) \end{tikzcd}, E\right)$ corresponds to $\mathrm{Cart}\,(h, c^*E)$ or not. After all, $\mathrm{Cart}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "b"] & \\ C \arrow[r, "c"'] & p(E) \end{tikzcd}, E\right)$ is a cartesian lifting of h to c^*E, since it is above h, it lands in c^*E and it is cartesian by Proposition 2.2.13, because both $\mathrm{Cart}\,(c, E)$ and $\mathrm{Cart}\,(c, E) \circ \mathrm{Cart}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "b"] & \\ C \arrow[r, "c"'] & p(E) \end{tikzcd}, E\right) = \mathrm{Cart}\,(b, E)$ are cartesian. And we obviously have that $\mathrm{Cart}\,(h, c^*E)$ is a cartesian lifting of h to c^*E.
Moreover Remark 2.4.9 says to us that, at least when we have a normal fibration, a particular case of this property is true.

However, even if we always have the equality stated above up to isomorphism, since cartesian liftings are unique up to isomorphisms by Proposition 2.2.8, in general we do not have the equality.

An example could be given for $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$, when $\mathcal{C}$ has pullbacks (which we need to have that $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ is a fibration). We said in Example 2.3.8 that the assignment $\mathcal{H}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ associated to $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ is in general just a pseudofunctor, and this problem is strongly related to that one, as we will prove below in Theorem 2.4.14.

We can give an explicit idea of the reason why for the functor $\begin{array}{c}\mathcal{C}^{\to}\\ \downarrow{\scriptstyle \mathrm{cod}}\\ \mathcal{C}\end{array}$ we do not have in general that $\mathrm{Cart}\,(c, E) \circ \mathrm{Cart}\,(h, c^*E) = \mathrm{Cart}\,(b, E)$ (for b, c, h taken as above) recalling what we have said in Example 2.3.8.
When we consider the diagram of cartesian liftings below with $(u, c_u) = \mathrm{Cart}\,(u, \psi)$, $(w, c_w) = \mathrm{Cart}\,(w, \varphi)$ and $(uw, c_{uw}) = \mathrm{Cart}\,(uw, \psi)$, which are the unique liftings given by the cleavage produced in Grothendieck construction (Construction 2.3.9) (see also Example 2.4.5) and thus

are given by the pullbacks of the three squares in the diagram,

$$
\begin{tikzcd}
Z \arrow[dr, dotted, "c_w"] \arrow[drr, dotted, bend left, "c_{uw}"] \arrow[dd, "\chi"'] & \neq & \\
& X \arrow[r, dotted, "c_u"] \arrow[dd, "\varphi"] & Y \arrow[dd, "\psi"] \\
K \arrow[dr, "w"'] \arrow[drr, bend left, "uw"] & & \\
& I \arrow[r, "u"'] & J \\
K \arrow[dr, "w"'] \arrow[drr, bend left, "uw"] & & \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathcal{C}
\end{tikzcd}
$$

we cannot conclude that $c_u \circ c_w$ is equal to c_{uw} (here we cannot use the cartesianity of any morphism to conclude, since c_w has not been built with cartesianity but with the cleavage). In general when we compose the two chosen pullbacks in the foreground (that is, when we compose the two commutative squares), the result will not be the chosen pullback in the background.

Definition 2.4.11. Let $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{smallmatrix}$ be a cloven fibration. If $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{smallmatrix}$ is a normal fibration and for every $E \in \mathcal{E}$ and every $c\colon C \to p(E)$ and $h\colon B \to C$ in $\mathcal{B}$

$$\mathrm{Cart}\,(c, E) \circ \mathrm{Cart}\,(h, c^*E) = \mathrm{Cart}\,(ch, E),$$

p is called a ***split*** fibration.

A cleavage which makes a functor into a split fibration is called a ***splitting cleavage***.

Example 2.4.12. In general a fibration $\begin{smallmatrix}\mathcal{C}^{\to}\\ \downarrow \mathrm{cod}\\ \mathcal{C}\end{smallmatrix}$ need not be split, see Remark 2.4.10.

We saw in Construction 2.3.9 that all the fibrations produced with the Grothendieck construction starting from a (proper) functor $\mathcal{H}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ have a (canonical) cleavage which makes them normal (see also Remark 2.4.7). It is actually true that every fibration has a cleavage which makes it a normal fibration, since one can just modify the cartesian liftings of the identities into identities, which are always cartesian with respect to $\int H$ by Proposition 2.2.12.

However, not all the fibrations can be made split, as the following example, which is taken from [Str18], shows.

Example 2.4.13. Consider the groups $\mathcal{B} := (\mathbb{Z}_2, +_2)$ and $\mathcal{E} := (\mathbb{Z}, +)$ as categories (that is, as categories with one objects and morphisms corresponding to the object of the group), and take the fibration $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{smallmatrix}$ defined by $a \mapsto p(a) := a \bmod 2$, for every morphism a in $\mathcal{E}$. It is an easy exercise to verify that this really is a fibration, since every morphism in $\mathcal{B}$ has a lifting

to $\mathcal{E}$ and every morphism in $\mathcal{E}$ is an isomorphism and thus cartesian with respect to p, by Proposition 2.2.12.
A splitting cleavage for p would give rise to a functor $F\colon \mathcal{B} \to \mathcal{E}$, defined as $j \mapsto \operatorname{Cart}(j, *)$ for every morphism j in $\mathcal{B}$, which is such that $p \circ F = \mathrm{Id}_{\mathcal{B}}$. The property of splitting cleavage is precisely the functoriality of such F. But this cannot be since there is no group homomorphism $h\colon (\mathbb{Z}_2, +_2) \to (\mathbb{Z}, +)$ with $h(1)$ an odd number of $\mathbb{Z}$.

We now show the result we had anticipated in Remark 2.4.10.

Theorem 2.4.14. *Let $\mathcal{H}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ be a pseudofunctor. Then the following facts are equivalent:*

1. *$\mathcal{H}$ is a functor;*
2. *$\int \mathcal{H}$ with the cleavage produced in Construction 2.3.9 is a split fibration.*

Proof. "*1 $\Rightarrow$ 2*" : Assume that $\mathcal{H}$ is a functor. Then the definition of the category $\mathcal{G}_{\mathcal{H}}$ simplifies. We have already seen that when $\mathcal{H}$ is a functor, $\int H$ is a normal fibration, in Remark 2.4.7. Take now $(C, Y) \in \mathcal{G}_{\mathcal{H}}$, and $b\colon B \to C$, $a\colon A \to B$ morphisms in $\mathcal{C}$. Then the composite $\operatorname{Cart}(b, (C, Y)) \circ \operatorname{Cart}(a, b^*(C, Y)))$ is given by

$$(A, \mathcal{H}(a)(\mathcal{H}(b)(Y))) \xrightarrow{(a,\mathrm{id})} (B, \mathcal{H}(b)(Y)) \xrightarrow{(b,\mathrm{id})} (C, Y)$$

while $\operatorname{Cart}(ba, (C, Y))$ is given by

$$(A, \mathcal{H}(ba)(Y)) \xrightarrow{(ba,\mathrm{id})} (C, Y).$$

But since the natural isomorphism $\alpha^{a,b}\colon \mathcal{H}(a) \circ \mathcal{H}(b) \overset{\cong}{\Rightarrow} \mathcal{H}(b \circ a)$ given by the pseudofunctor $\mathcal{H}$ is the identity (because we have assumed that $\mathcal{H}$ is a proper functor), we have that $(A, \mathcal{H}(a)(\mathcal{H}(b)(Y))) = (A, \mathcal{H}(ba)(Y))$, and also that

$$(b, \mathrm{id}) \circ (a, \mathrm{id}) = (ba, \mathrm{id} \circ \mathcal{H}(b)(\mathrm{id}) \circ \mathrm{id}) = (ba, \mathrm{id}).$$

"*2 $\Rightarrow$ 1*" : Assume now that $\int H$ with the cleavage produced in Construction 2.3.9 is a split fibration. In particular $\int H$ is a normal fibration and thus for every $(C, Y) \in \mathcal{G}_{\mathcal{H}}$ we have that $\operatorname{Cart}(\mathrm{id}_C, (C, Y)) = \mathrm{id}_{(C,Y)}$. But $\operatorname{Cart}(\mathrm{id}_C, (C, Y))$ is given by $(C, \mathcal{H}(\mathrm{id}_C)(Y)) \xrightarrow{(\mathrm{id},\mathrm{id})} (C, Y)$, whence $\mathcal{H}(\mathrm{id}_C)(Y) = Y = \mathrm{id}_{\mathcal{H}(C)}(Y)$. Thus by arbitrariness of Y, the natural isomorphism $\lambda^C\colon \mathcal{H}(\mathrm{id}_C) \overset{\cong}{\Rightarrow} \mathrm{Id}_{\mathcal{H}(C)}$ is the identity.
Now take $(C, Y) \in \mathcal{G}_{\mathcal{H}}$, and $b\colon B \to C$, $a\colon A \to B$ morphisms in $\mathcal{C}$. Then the composite $\operatorname{Cart}(b, (C, Y)) \circ \operatorname{Cart}(a, b^*(C, Y)))$ is given by

$$(A, \mathcal{H}(a)(\mathcal{H}(b)(Y))) \xrightarrow{(a,\mathrm{id})} (B, \mathcal{H}(b)(Y)) \xrightarrow{(b,\mathrm{id})} (C, Y)$$

and $\operatorname{Cart}(ba, (C, Y))$ is given by

$$(A, \mathcal{H}(ba)(Y)) \xrightarrow{(ba,\mathrm{id})} (C, Y)$$

(we have written it also in the first part of the proof). But $\int H$ is a split fibration and thus these two morphisms must coincide, whence

$$(A, \mathcal{H}(a)(\mathcal{H}(b)(Y))) = (A, \mathcal{H}(ba)(Y))$$

and then $\mathcal{H}(a)(\mathcal{H}(b)(Y)) = \mathcal{H}(ba)(Y)$. By arbitratriness of Y it follows that the natural isomorphism $\alpha^{a,b} \colon \mathcal{H}(a) \circ \mathcal{H}(b) \overset{\cong}{\Rightarrow} \mathcal{H}(b \circ a)$ is the identity.
By arbitrariness of C, a and b this precisely means that the pseudofunctor $\mathcal{H}$ is a functor, as we said in Construction 2.3.9. □

2.5 The 2-categories of Fibrations

The main reference for this section is [Str18], even if we here generalize the definitions to fibrations with different base category. The aim of this section is to say that the Grothendieck fibrations as well as the cloven fibrations form a 2-category.

We need to define 1-morphisms and 2-morphisms of Grothendieck fibrations, and since they are, first of all, functors, we get inspiration from the definition of the arrow category of the 2-category $\mathcal{CAT}$, which is a particular case of Construction 1.8.8. (Recall also the definition of the 2-category $\mathcal{CAT}$, which was given in Construction 1.7.15).

For clarity, we write explicity the definition of the arrow category of $\mathcal{CAT}$.

Definition 2.5.1. The ***arrow category*** $\mathcal{CAT}^{\rightarrow}$ of $\mathcal{CAT}$ is the 2-category which consists of the following data:

a 0-cell of $\mathcal{CAT}^{\rightarrow}$ is a functor $\begin{smallmatrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{smallmatrix}$, for arbitrary categories $\mathcal{X}$ and $\mathcal{C}$;

a 1-cell $\begin{smallmatrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{smallmatrix} \xrightarrow{H} \begin{smallmatrix}\mathcal{Y}\\ \downarrow G\\ \mathcal{D}\end{smallmatrix}$ **in** $\mathcal{CAT}^{\rightarrow}$ is a pair of functors $(H_0 \colon \mathcal{C} \to \mathcal{D},\ H_1 \colon \mathcal{X} \to \mathcal{Y})$ and a commutative square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D};
\end{tikzcd}$$

a 2-cell $\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] \arrow[r, bend left, "H"] \arrow[r, phantom, "\Downarrow\lambda"] \arrow[r, bend right, "K"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} & \mathcal{D} \end{tikzcd}$ **in** $\mathcal{CAT}^{\rightarrow}$ is a pair of natural transformations $(\lambda_0 \colon H_0 \Rightarrow K_0,\ \lambda_1 \colon H_1 \Rightarrow K_1)$ such that

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D}
\end{tikzcd}
$$

With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells component-wise inherited from $\mathcal{CAT}$, $\mathcal{CAT}^{\to}$ is a 2-category, as we had said in general in Construction 1.8.8.

To define the 2-category of Grothendieck fibrations we need to ask in addition a preservation of cartesian morphisms, as one might expect.

Definition 2.5.2. We define the 2-category $\mathcal{FIB}$ as the 2-category (as we shall promptly prove it is such) which consists of the following data:

a 0-cell of $\mathcal{FIB}$ is a Grothendieck fibration $\begin{smallmatrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{smallmatrix}$, for arbitrary categories $\mathcal{X}$ and $\mathcal{C}$;

a 1-cell $\begin{smallmatrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{smallmatrix} \xrightarrow{H} \begin{smallmatrix}\mathcal{Y}\\ \downarrow G\\ \mathcal{D}\end{smallmatrix}$ **in $\mathcal{FIB}$** is a pair of functors $(H_0\colon \mathcal{C} \to \mathcal{D},\ H_1\colon \mathcal{X} \to \mathcal{Y})$ and a commutative square

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
$$

such that, for every $X \in \mathcal{X}$ and every $f\colon C \to F(X)$ in $\mathcal{C}$, H_1 sends a cartesian lifting of f to X with respect to F to a cartesian lifting of $H_0(f)$ to $H_1(X)$ with respect to G:

$$
\begin{tikzcd}
E \arrow[r, "c_f"] \arrow[d, mapsto, "F"] & X \arrow[d, mapsto, "F"] & & H_1(E) \arrow[r, "H_1(c_f)"] \arrow[d, mapsto, "G"] & H_1(X) \arrow[d, mapsto, "G"] \\
C \arrow[r, "f"'] & F(X) \arrow[r, phantom, "\overset{H_1}{\underset{H_0}{\longmapsto}}"] & & H_0(C) \arrow[r, "H_0(f)"'] & H_0(F(X))
\end{tikzcd}
$$

(Notice that if H_0 is the identity, this last request of preservations of cartesian liftings simplifies to say that H_1 sends cartesian morphisms with respect to F to cartesian morphisms with respect to G);

a 2-cell $\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] \arrow[r, bend left, "H"] \arrow[r, phantom, "\Downarrow\lambda"] \arrow[r, bend right, "K"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} & \mathcal{D} \end{tikzcd}$ **in** $\mathcal{FIB}$ is a pair of natural transformations ($\lambda_0\colon H_0 \Rightarrow K_0$, $\lambda_1\colon H_1 \Rightarrow K_1$) such that

$$\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] \arrow[r, bend left, "H_1"] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[r, bend right, "K_1"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[d, "F"'] \arrow[r, bend left, "H_1"] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, phantom, "\Downarrow\lambda_0"] \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd}$$

A 1-cell of $\mathcal{FIB}$ is sometimes called a ***cartesian functor***.

Proposition 2.5.3. *With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells component-wise inherited from $\mathcal{CAT}$, the data$\mathcal{FIB}$ form a 2-category (which will thus be a (in general not full and not wide) subcategory of $\mathcal{CAT}^{\to}$).*

Proof. The proof is trivial. □

It is often useful to restrict our attention to Grothendieck fibrations over the same base category $\mathcal{C}$. So we are interested in defining the 2-category of such fibrations. We then simplify Definition 2.5.2 forcing H_0 and λ_0 to be the identity.

Definition 2.5.4. We define the 2-category $\mathcal{FIB}_{\mathcal{C}}$ of Grothendieck fibrations over the same base category $\mathcal{C}$ as the 2-category (as we shall promptly prove it is such) which consists of the following data:

a 0-cell of $\mathcal{FIB}_{\mathcal{C}}$ is a Grothendieck fibration $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$, for an arbitrary $\mathcal{X}$ but fixed base category $\mathcal{C}$;

a 1-cell $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array} \xrightarrow{H_1} \begin{array}{c}\mathcal{Y}\\ \downarrow G\\ \mathcal{C}\end{array}$ **in** $\mathcal{FIB}_{\mathcal{C}}$ is a functor $H_1\colon \mathcal{X} \to \mathcal{Y}$ and a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[dr, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ & \mathcal{C} \end{tikzcd}$$

such that H_1 sends cartesian morphisms with respect to F to cartesian morphisms with respect to G;

a 2-cell $\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} & \mathcal{C} \end{tikzcd}$ **in** $\mathcal{FIB}_{\mathcal{C}}$ is a natural transformation $\lambda_1 \colon H_1 \Rightarrow K_1$ such that

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd}$$

that is, a natural transformation $\lambda_1 \colon H_1 \Rightarrow K_1$ such that for every $X \in \mathcal{X}$, $(\lambda_1)_X$ is vertical with respect to G (indeed $G((\lambda_1)_X) = \mathrm{id}_{F(X)}$).

A 1-cell of $\mathcal{FIB}_{\mathcal{C}}$ is sometimes called a ***cartesian functor over*** $\mathcal{C}$.

Proposition 2.5.5. *With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells inherited from* $\mathcal{CAT}$*, the data* $\mathcal{FIB}_{\mathcal{C}}$ *form a 2-category (which will thus be a (in general not full and not wide) subcategory of* $\mathcal{CAT} \downarrow \mathcal{C}$*).*

Proof. The proof is trivial. □

We have also defined cloven fibrations and told why they are important. Then we would like to consider also the 2-category of cloven fibrations. In this case, we do want that cartesian liftings of the cleavage of the first functor get sent to cartesian liftings of the cleavage of the second functor, rather than asking that they get sent to an arbitrary cartesian lifting.

Definition 2.5.6. We define the 2-category $\mathcal{ClFIB}$ as the 2-category (as we shall promptly prove it is such) which consists of the following data:

a 0-cell of $\mathcal{ClFIB}$ is a cloven fibration $\begin{smallmatrix} \mathcal{X} \\ \downarrow p \\ \mathcal{C} \end{smallmatrix}$, for arbitrary categories $\mathcal{X}$ and $\mathcal{C}$;

a 1-cell $\begin{smallmatrix} \mathcal{X} \\ \downarrow p \\ \mathcal{C} \end{smallmatrix} \xrightarrow{H} \begin{smallmatrix} \mathcal{Y} \\ \downarrow q \\ \mathcal{D} \end{smallmatrix}$ **in** $\mathcal{ClFIB}$ is a pair of functors ($H_0 \colon \mathcal{C} \to \mathcal{D}$, $H_1 \colon \mathcal{X} \to \mathcal{Y}$) and a commutative square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd}$$

such that for every $X \in \mathcal{X}$ and every $u \colon C \to p(X)$ in $\mathcal{C}$

$$H_1(\mathrm{Cart}_p(u, X)) = \mathrm{Cart}_q(H_0(u), H_1(X))$$

$$\begin{tikzcd}
u^*X \arrow[r, "\mathrm{Cart}_p(u,X)"] \arrow[d, mapsto, "p"] & X \arrow[d, mapsto, "p"] & & H_0(u)^*H_1(X) \arrow[r, "\mathrm{Cart}_q(H_0(u),H_1(X))"] \arrow[d, mapsto, "q"] & H_1(X) \arrow[d, mapsto, "q"] \\
C \arrow[r, "u"'] & p(X) \arrow[r, phantom, "\overset{H_1}{\underset{H_0}{\longmapsto}}"] & & H_0(C) \arrow[r, "H_0(u)"'] & H_0(p(X));
\end{tikzcd}$$

a 2-cell $\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{Y} \arrow[d, "q"] \\ \mathcal{C} & \mathcal{D} \end{tikzcd}$ **in $\mathcal{ClFIB}$** is a pair of natural transformations ($\lambda_0 \colon H_0 \Rightarrow K_0$, $\lambda_1 \colon H_1 \Rightarrow K_1$) such that

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D}
\end{tikzcd}$$

A 1-cell of $\mathcal{ClFIB}$ is sometimes called a ***cartesian functor between cloven fibrations*** or ***cloven cartesian functor***.

Proposition 2.5.7. *With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells component-wise inherited from $\mathcal{CAT}$, the data $\mathcal{ClFIB}$ form a 2-category.*

Proof. The proof is trivial. □

And also in this case it is useful to define the 2-category of cloven fibrations over the same base category $\mathcal{C}$.

Definition 2.5.8. We define the 2-category $\mathcal{ClFIB}_{\mathcal{C}}$ of cloven fibrations over the same base category $\mathcal{C}$ as the 2-category (as we shall promptly prove it is such) which consists of the following data:

a 0-cell of $\mathcal{ClFIB}_{\mathcal{C}}$ is a cloven fibration $\begin{smallmatrix}\mathcal{X}\\ \downarrow p\\ \mathcal{C}\end{smallmatrix}$, for an arbitrary $\mathcal{X}$ but fixed base category $\mathcal{C}$;

a 1-cell $\begin{smallmatrix}\mathcal{X}\\ \downarrow p\\ \mathcal{C}\end{smallmatrix} \xrightarrow{H_1} \begin{smallmatrix}\mathcal{Y}\\ \downarrow q\\ \mathcal{C}\end{smallmatrix}$ **in $\mathcal{ClFIB}_{\mathcal{C}}$** is a functor $H_1 \colon \mathcal{X} \to \mathcal{Y}$ and a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[dr, "p"'] & \mathcal{Y} \arrow[d, "q"] \\ & \mathcal{C} \end{tikzcd}$$

such that for every $X \in \mathcal{X}$ and every $u\colon C \to p(X)$ in $\mathcal{C}$

$$H_1(\mathrm{Cart}_p(u,X)) = \mathrm{Cart}_q(u, H_1(X));$$

a 2-cell $\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] & \mathcal{Y} \arrow[d, "q"] \\ \mathcal{C} & \mathcal{C} \end{tikzcd}$ **in** $\mathcal{C}\ell\mathcal{FIB}_{\mathcal{C}}$ is a natural transformation $\lambda_1\colon H_1 \Rightarrow K_1$ such that

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] & \mathcal{Y} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[r, bend left, "H_1"] & \mathcal{Y} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd}$$

that is, a natural transformation $\lambda_1\colon H_1 \Rightarrow K_1$ such that for every $X \in \mathcal{X}$, $(\lambda_1)_X$ is vertical (with respect to G) (indeed $G((\lambda_1)_X) = \mathrm{id}_{F(X)}$).

A 1-cell of $\mathcal{C}\ell\mathcal{FIB}_{\mathcal{C}}$ is sometimes called a ***cloven cartesian functor over*** $\mathcal{C}$.

Proposition 2.5.9. *With composition and identities of 1-cells, and vertical and horizontal composition and identities of 2-cells inherited from $\mathcal{CAT}$, the data $\mathcal{C}\ell\mathcal{FIB}_{\mathcal{C}}$ form a 2-category.*

Proof. The proof is trivial. □

Remark 2.5.10. Definition 2.5.6 is the natural definition for the 2-category of functors with cleavage, as we ask that the 1-cells preserve the cleavage.

There is however another useful 2-category of cloven fibrations to consider, which has as 0-cells the cloven fibrations, as 1-cells the cartesian functors and as 2-cells the modifications. That is, the full sub-2-category of $\mathcal{FIB}$ given by the cloven fibrations. In certain situations, it is more natural to consider this 2-category of cloven fibrations rather than the one described in Definition 2.5.6. For example in Section 3.4, we will prove that the pseudo-algebras for the monad which gives the monadicity of split fibrations over a fixed base category is exactly the 2-category of cloven fibrations and cartesian functors over such fixed base category. We then now define such a 2-category.

However, when we will talk about the 2-category of cloven fibrations without specifying the 1-cells we will always refer to the 2-category $\mathcal{C}\ell\mathcal{FIB}$ described in Definition 2.5.6.

Definition 2.5.11. We define the 2-category $\mathcal{CloFIB}$ of cloven fibrations and cartesian functors as the full sub-2-category of $\mathcal{FIB}$ of cloven fibrations. That is, the 0-cells of $\mathcal{CloFIB}$ are the cloven fibrations, whereas the 1-cells and the 2-cells are the same of $\mathcal{FIB}$.

Let $\mathcal{C}$ be a category. We define the 2-category $\mathcal{CloFIB}_{\mathcal{C}}$ of cloven fibrations over $\mathcal{C}$ and cartesian functors over $\mathcal{C}$ as the full sub-2-category of $\mathcal{FIB}_{\mathcal{C}}$ of cloven fibrations over $\mathcal{C}$.

Definition 2.5.12. We define the 2-category $\mathcal{SpFIB}$ of split fibrations as the full sub-2-category of $\mathcal{ClFIB}$ of split fibrations.

Let $\mathcal{C}$ be a category. We define the 2-category $\mathcal{SpFIB}_{\mathcal{C}}$ of split fibrations over the same base category $\mathcal{C}$ as the full sub-2-category of $\mathcal{ClFIB}_{\mathcal{C}}$ of split fibrations over $\mathcal{C}$.

Chapter 3

Monadicity of Fibrations

In this third chapter we ask ourselves whether or not there is a 2-monad whose algebras are the Grothendieck fibrations. We will see that we need to restrict not only to the cloven fibrations but actually to the split fibrations to have monadicity, and we will try to get rid of this quite restrictive hypothesis in Section 3.4. In such section, we will manage to prove that the 2-category of pseudo-algebras for the monad which gives the monadicity of split fibrations over a fixed base category $\mathcal{C}$ is the 2-category $\mathcal{C}lo\mathcal{FIB}_{\mathcal{C}}$ of cloven fibrations over such base category and cartesian functors over $\mathcal{C}$.

3.1 A First Attempt

Since every monad comes from an adjunction, we first search for an adjunction. Since we have defined the 2-categories of fibrations by adding structure to the 2-category $\mathcal{CAT}^{\to}$, the arrow category of $\mathcal{CAT}$, it is natural to search for a 2-adjunction with $\mathcal{CAT}^{\to}$.

As right 2-adjoint $U\colon \mathcal{FIB} \to \mathcal{CAT}^{\to}$ it is natural to take the forgetful functor:

$$U\colon \mathcal{FIB} \to \mathcal{CAT}^{\to}$$

$$p\colon \mathcal{E} \to \mathcal{B} \longmapsto p\colon \mathcal{E} \to \mathcal{B}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \arrow[r, phantom, "\longmapsto" near start] & \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D}
\end{tikzcd}$$

Now, starting from an arbitrary functor $F\colon \mathcal{X} \to \mathcal{C}$, we would like to construct a free Grothendieck fibration associated to F.

Since the structure of a Grothendieck fibration is actually related only to 1-morphisms, it is natural to expect that all the issues arise in the attempt to find a 1-adjunction, and that a 1-adjunction would easily extend to a 2-adjunction. Then, in this section, we will make an attempt to find a left adjoint for the functor U (treated for the rest of this section as a 1-functor, forgetting its action on 2-morphisms in $\mathcal{FIB}$).

By Theorem 1.2.2, to find a left adjoint for the functor U it suffices to find for each functor $F\colon \mathcal{X} \to \mathcal{C}$ a universal morphism from F to U, say

$$\eta_F = ((\eta_F)_0, (\eta_F)_1)\colon \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array} \to U\left(\begin{array}{c}\mathcal{A}_F\\ \downarrow L(F)\\ \mathcal{D}_F\end{array}\right).$$

(This $L(F)$ will be the free Grothendieck fibration associated to F.)

But this means that for every $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array} \in \mathcal{FIB}$ and for every morphism $H = (H_0\colon \mathcal{C} \to \mathcal{B}, H_1\colon \mathcal{X} \to \mathcal{E})$ from $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$ to $U\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}\right)$ in $\mathcal{CAT}^{\to}$, there exists a unique morphism

$$Q_H = ((Q_H)_0, (Q_H)_1)\colon \begin{array}{c}\mathcal{A}_F\\ \downarrow L(F)\\ \mathcal{D}_F\end{array} \to \begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$$

in $\mathcal{FIB}$ such that the following triangle in $\mathcal{CAT}^{\to}$ is commutative:

$$\begin{tikzcd}
 & \mathcal{A}_F \arrow[dd, "L(F)"] \arrow[dr, bend left, "(Q_H)_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"'] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{D}_F \arrow[dr, bend left, "(Q_H)_0"] & \\
\mathcal{C} \arrow[ur, bend left, "(\eta_F)_0"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}$$

Remark 3.1.1. We might consider the important particular case of the diagram above with $F = p\colon \mathcal{E} \to \mathcal{B}$ a Grothendieck fibration and $H = (\mathrm{Id}, \mathrm{Id})$. We see that $(\eta_F)_0$ needs then to factorize the identity. One might then think that it is natural to take $\mathcal{D}_F = \mathcal{B}$ and $(\eta_F)_0 = \mathrm{Id}$,

also because in this way the free fibration associated to a fibration over $\mathcal{B}$ would still be a fibration over $\mathcal{B}$.
Then $(Q_H)_0$ needs to be equal to H_0.

At this point, fixed the base category of $L(F)$, we want to set $\mathcal{A}_F$. To make $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$ into a fibration, we need to find, for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$, a cartesian lifting of f to E. The idea is to define the free Grothendieck fibration associated to F by formally adding the data $(E, f\colon A \to F(E))$ and pretending that these are the cartesian lifting we want. But notice that these data are precisely the objects of the comma category $\mathcal{C}\downarrow F$ (see Construction 1.8.10.

We then set $L(F)$ to be $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$, which clearly is a functor.

Proposition 3.1.2. *Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Then* $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ *is a Grothendieck fibration.*

Proof. Let $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ and $A' \xrightarrow{a} A$ in $\mathcal{C}$. We find a cartesian lifting of a to $(A, E, A \xrightarrow{f} F(E))$. Also looking at Example 2.4.5, we define this cartesian lifting to be (a, id):

$$\begin{tikzcd}
(A', E, A' \xrightarrow{fa} F(E)) \arrow[r, "{(a,\mathrm{id})}"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (A, E, A \xrightarrow{f} F(E)) \arrow[d, mapsto, "\mathrm{pr}_1"] \\
A' \arrow[r, "a"'] & A
\end{tikzcd}$$

The morphism (a, id) obvioulsy is above a, and it is cartesian since, for every $(A'', X, A'' \xrightarrow{h} F(X)) \in \mathcal{C}\downarrow F$, every $w\colon A'' \to A'$ in $\mathcal{C}$ and every $(aw, k)\colon (A'', X, A'' \xrightarrow{h} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$ in $\mathcal{C}\downarrow F$ above aw, there exists a unique morphism

$$v\colon (A'', X, A'' \xrightarrow{h} F(X)) \longrightarrow (A', E, A' \xrightarrow{fa} F(E))$$

above w which makes the above triangle commute in the following diagram

$$\begin{tikzcd}
(A'', X, A'' \xrightarrow{h} F(X)) \arrow[rr, bend left, "{(aw,k)}"] \arrow[r, dashed, "v"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (A', E, A' \xrightarrow{fa} F(E)) \arrow[r, "{(a,\mathrm{id})}"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (A, E, A \xrightarrow{f} F(E)) \arrow[d, mapsto, "\mathrm{pr}_1"] \\
A'' \arrow[r, "w"'] \arrow[rr, bend left, "u\circ w"] & A' \arrow[r, "a"'] & A
\end{tikzcd}$$

The first component of v must be w and the second component of v must be k since the composite with id must be k, and $v = (w, k)$ fits in the diagram.

Therefore (a, id) is a cartesian lifting of a and $\begin{array}{c}\mathcal{C}{\downarrow}F\\ \downarrow{\scriptstyle \mathrm{pr}_1}\\ \mathcal{C}\end{array}$ is a fibration. $\square$

Construction 3.1.3. At this point, it is natural to define $(\eta_F)_1$ to be

$$(\eta_F)_1 \colon \mathcal{X} \to \mathcal{C}{\downarrow}F$$

$$\begin{tikzcd}
E \arrow[d, "e"'] \arrow[r, mapsto] & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[d, "F(e)"', "e", "F(e)", "F(e)"] \\
X \arrow[r, mapsto] & (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X))
\end{tikzcd}$$

It is clear that this is a functor and that the square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(\eta_F)_1"] \arrow[d, "F"'] & \mathcal{C}{\downarrow}F \arrow[d, "\mathrm{pr}_1"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}_{\mathcal{C}}"'] & \mathcal{C}
\end{tikzcd}$$

is commutative.

It now remains to find a functor $(Q_H)_1 \colon \mathcal{C}{\downarrow}F \to \mathcal{E}$ such that Q_H is a cartesian functor. It might be useful to see again the triangle in $\mathcal{CAT}^{\to}$ which we want to be commutative for a unique cartesian functor $Q_H = (H_0, (Q_H)_1)$:

$$\begin{tikzcd}
 & \mathcal{C}{\downarrow}F \arrow[dd, "\mathrm{pr}_1"] \arrow[dr, bend left, "(Q_H)_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{C} \arrow[dr, bend left, "H_0"] & \\
\mathcal{C} \arrow[ur, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}$$

To define $(Q_H)_1$ we start from an object $(A, E, A \xrightarrow{f} F(E))$ of $\mathcal{C}{\downarrow}F$ and we see that we have to construct an object in $\mathcal{E}$ which is above $H_0(A)$. But if we start from the diagram

$$\begin{tikzcd}
 & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and we apply (H_0, H_1) to it, we get the diagram

$$\begin{tikzcd}
{\color{violet}H_0(f)^{\sim}H_1(E)} \arrow[r, violet, "c_{H_0(f)}"] \arrow[d, mapsto, "p"] & H_1(E) \arrow[d, mapsto, "p"] \\
H_0(A) \arrow[r, "H_0(f)"'] & H_0(F(E))
\end{tikzcd}$$

from which we can construct an object $H_0(f)^{\sim}H_1(E)$ above $H_0(A)$ and a cartesian morphism $c_{H_0(f)}$ above $H_0(f)$, written in violet, since p is a fibration.
Here we know that a cartesian lifting of $H_0(f)$ to $H_1(E)$ exists and we use this non-standard notation $c_{H_0(f)}\colon H_0(f)^{\sim}H_1(E) \to H_1(E)$ to indicate one chosen such lifitng.

Therefore, using the axiom of choice, we construct a function

$$\begin{aligned}
q\colon\ \mathrm{Ob}(\mathcal{C}{\downarrow}F) &\longrightarrow \mathrm{Ob}(\mathcal{E}) \times \mathrm{Ob}(\mathcal{E}^{\to}) \\
(A, E, A \xrightarrow{f} F(E)) &\overset{q}{\longmapsto} (H_0(f)^{\sim}H_1(E), c_{H_0(f)})
\end{aligned}$$

and we can ask in addition that $q(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) = (H_1(E), \mathrm{id})$ for every $E \in \mathcal{X}$, since this is an acceptable choice. We will see why we need this further property in Remark 3.1.5.

We then define $(Q_H)_1$ in the following way:

$$\begin{aligned}
(Q_H)_1\colon \mathcal{C}{\downarrow}F &\longrightarrow \mathcal{E} \\
(A, E, A \xrightarrow{f} F(E)) &\longmapsto \mathrm{pr}_1(q(A, E, A \xrightarrow{f} F(E)))
\end{aligned}$$

$$\begin{tikzcd}
 & & E' \arrow[d, mapsto, "F"] \arrow[dr, violet, bend left, "e"] & \\
 & & & E \arrow[dd, mapsto, "F"] \\
A' \arrow[rr, "f'"'] \arrow[dr, violet, bend right, "a"'] & & F(E') \arrow[dr, violet, "F(e)"] & \\
 & A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}
\longmapsto
\begin{tikzcd}
H_0(f')^{\sim}H_1(E') \arrow[rr, "c_{H_0(f')}"] \arrow[dd, mapsto, "p"] \arrow[dr, violet, dashed, "(Q_H)_1(a{,}e)"] & & H_1(E') \arrow[dr, bend left, "H_1(e)"] \arrow[dd, mapsto, "p"] & \\
 & H_0(f)^{\sim}H_1(E) \arrow[rr, "c_{H_0(f)}"] \arrow[dd, mapsto, "p"] & & H_1(E) \arrow[dd, mapsto, "p"] \\
H_0(A') \arrow[rr, "H_0(f')"'] \arrow[dr, bend right, "H_0(a)"'] & & H_0(F(E')) \arrow[dr, "H_0(F(e))"] & \\
 & H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd}$$

Proposition 3.1.4. *The assignment $(Q_H)_1$ produced in Construction 3.1.3 is a functor from $\mathcal{C}{\downarrow}F$ to $\mathcal{E}$.*

Proof. We first show that $(Q_H)_1$ preserves identities. We thus consider an arbitrary diagram of the form

$$\begin{tikzcd}
H_0(f)^{\sim}H_1(E) \arrow[rr, "c_{H_0(f)}"] \arrow[dd, mapsto, "p"] \arrow[dr, dashed, "(Q_H)_1(\mathrm{id},\mathrm{id})"] & & H_1(E) \arrow[dr, equal, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & \\
& H_0(f)^{\sim}H_1(E) \arrow[rr, "c_{H_0(f)}"] \arrow[dd, mapsto, "p"] & & H_1(E) \arrow[dd, mapsto, "p"] \\
H_0(A) \arrow[rr, "H_0(f)"] \arrow[dr, equal, "\mathrm{id}"'] & & H_0(F(E)) \arrow[dr, equal] & \\
& H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd}$$

Since $(Q_H)_1(\mathrm{id},\mathrm{id})$ is the unique morphism which fits well into the diagram, in the place of the morphism in violet, and also the identity $\mathrm{id}_{H_0(f)^{\sim}H_1(E)}$ fits well into it, it follows that

$$(Q_H)_1(\mathrm{id},\mathrm{id}) = \mathrm{id}_{H_0(f)^{\sim}H_1(E)}.$$

Now, consider an arbitrary composite

$$\begin{tikzcd}
& & E'' \arrow[dr, bend left, "e'"] \arrow[dd, mapsto, "F"] & & \\
& & & E' \arrow[dr, bend left, "e"] \arrow[dd, mapsto, "F"] & \\
A'' \arrow[rr, "f''"'] \arrow[dr, bend right, "a'"'] & & F(E'') \arrow[dr, "F(e')"] & & E \arrow[dd, mapsto, "F"] \\
& A' \arrow[rr, "f'"'] \arrow[dr, bend right, "a"'] & & F(E') \arrow[dr, "F(e)"] & \\
& & A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}$$

in $\mathcal{C}{\downarrow}F$. In order to compute the assignment $(Q_H)_1$ on this composite we consider the diagram

$$\begin{tikzcd}
H_0(f'')^{\sim}H_1(E'') \arrow[rrr, "c_{H_0(f'')}"] \arrow[dd, mapsto, "p"'] \arrow[dr, "(Q_H)_1(a',e')"] \arrow[ddrr, bend right, "(Q_H)_1(aa',ee')"'] & & & H_1(E'') \arrow[dr, bend left, "H_1(e')"] \arrow[dd, mapsto, "p"] & & \\
& H_0(f')^{\sim}H_1(E') \arrow[rrr, "c_{H_0(f')}"] \arrow[ddd, mapsto, "p"] \arrow[ddr, "(Q_H)_1(a,e)"] & & & H_1(E') \arrow[ddr, bend left, "H_1(e)"] \arrow[ddd, mapsto, "p"] & \\
H_0(A'') \arrow[rrr, "H_0(f'')"] \arrow[ddr, bend right, "H_0(a')"'] & & & H_0(F(E'')) \arrow[ddr, "H_0(F(e'))"'] & & \\
& & H_0(f)^{\sim}H_1(E) \arrow[rrr, "c_{H_0(f)}"] \arrow[dd, mapsto, "p"] & & & H_1(E) \arrow[dd, mapsto, "p"] \\
& H_0(A') \arrow[rrr, "H_0(f')"] \arrow[dr, bend right, "H_0(a)"'] & & & H_0(F(E')) \arrow[dr, "H_0(F(e))"'] & \\
& & H_0(A) \arrow[rrr, "H_0(f)"'] & & & H_0(F(E))
\end{tikzcd}$$

Since $c_{H_0(f)}$ is a cartesian morphism, we obtain that

$$(Q_H)_1((a,e)\circ(a',e')) = (Q_H)_1(aa',ee') = (Q_H)_1(a,e)\circ(Q_H)_1(a',e').$$

Therefore $(Q_H)_1$ is a functor. $\square$

Remark 3.1.5. By construction, the functor $(Q_H)_1$ defined in Construction 3.1.3 makes the diagram

$$\begin{tikzcd}
& \mathcal{C}\downarrow F \arrow[dr, bend left, "(Q_H)_1"] \arrow[dd, "\mathrm{pr}_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"'] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
& \mathcal{C} \arrow[dr, bend left, "H_0"] & \\
\mathcal{C} \arrow[ur, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}$$

commute. Indeed we started from the commutative square in the foreground, we see that the lower triangle is trivially commutative and we have already shown the commutativity of the left square in the background in Construction 3.1.3.

Whereas the right square in the background commutes because for every (a,e) morphism in $\mathcal{C}\downarrow F$, we have that $(Q_H)_1(a,e)$ is a morphism in $\mathcal{E}$ above $H_0(a)$, by Construction 3.1.3.

It now remains to check the upper triangle, but it commutes because we have constructed the function $q\colon \operatorname{Ob}(\mathcal{C}\downarrow F) \to \operatorname{Ob}(\mathcal{E}) \times \operatorname{Ob}(\mathcal{E}^{\to})$ choosing

$$q((\eta_F)_1(E)) = q(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) = (H_1(E), \mathrm{id}).$$

for every $E \in \mathcal{X}$, in Construction 3.1.3.

Lemma 3.1.6. *Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor and let*

$$\begin{tikzcd}
& & E' \arrow[dd, mapsto, "F"] \arrow[dr, bend left, "e"] & \\
& & & E \arrow[dd, mapsto, "F"] \\
A' \arrow[rr, "f'"'] \arrow[dr, bend right, "a"'] & & F(E') \arrow[dr, "F(e)"] & \\
& A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}$$

be a cartesian morphism in $\mathcal{C}\downarrow F$. Then e is an isomorphism.

Proof. Since (a,e) is a cartesian morphism, considering the diagram

$$
\begin{tikzcd}
(A', E, A' \xrightarrow{F(e)f'} F(E)) \arrow[dr, dashed, "v"] \arrow[drr, bend left, "(a,\mathrm{id})"] \arrow[dd, mapsto, "\mathrm{pr}_1"'] & & \\
& (A', E', A' \xrightarrow{f'} F(E')) \arrow[r, "(a,e)"] \arrow[dd, mapsto, "\mathrm{pr}_1"'] & (A, E, A \xrightarrow{f} F(E)) \arrow[dd, mapsto, "\mathrm{pr}_1"] \\
A' \arrow[dr, "\mathrm{id}_{A'}"'] \arrow[drr, bend left, "a"] & & \\
& A' \arrow[r, "a"'] & A
\end{tikzcd}
$$

we get a unique morphism v above $\mathrm{id}_{A'}$ such that $(a, e) \circ v = (a, \mathrm{id})$. Then v has to be of the form $(\mathrm{id}_{A'}, w)$ with $w \colon E \to E'$ such that $e \circ w = \mathrm{id}$. Thus, we have found a right inverse of e. We now want to consider $w \circ e$, and we see that it fits well in the diagram

$$
\begin{tikzcd}
(A', E', A' \xrightarrow{f'} F(E')) \arrow[dr, shift left, "(\mathrm{id},w\circ e)"] \arrow[dr, shift right, "(\mathrm{id},\mathrm{id})"'] \arrow[drr, bend left, "(a,\mathrm{id})"] \arrow[dd, mapsto, "\mathrm{pr}_1"'] & & \\
& (A', E', A' \xrightarrow{f'} F(E')) \arrow[r, "(a,\mathrm{id})"] \arrow[dd, mapsto, "\mathrm{pr}_1"'] & (A, E, A \xrightarrow{f} F(E)) \arrow[dd, mapsto, "\mathrm{pr}_1"] \\
A' \arrow[dr, "\mathrm{id}_{A'}"'] \arrow[drr, bend left, "a"] & & \\
& A' \arrow[r, "a"'] & A
\end{tikzcd}
$$

But also $\mathrm{id}_{(A', E', A' \xrightarrow{f'} F(E'))}$ fits well in the same diagram, and (a, id) is cartesian by the proof of Proposition 3.1.2, whence it follows that $w \circ e$ needs to be the identity.

Therefore e is an isomorphism. □

Remark 3.1.7. There is another instructive way to prove Lemma 3.1.6. In the notation introduced in the proof of Lemma 3.1.6, one could notice that (a, e) and (a, id) are two cartesian liftings of a to $(A, E, A \xrightarrow{f} F(E))$. Therefore they differ by an isomorphism, by Proposition 2.2.8. In particular, e and id differ by an isomorphism. This means that there exists an isomorphism w such that $e \circ w = \mathrm{id}$, whence e is an isomorphism.

Proposition 3.1.8. *The morphism Q_H in $\mathcal{CAT}^{\to}$ defined in Construction 3.1.3 is a cartesian functor from* $\begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C} \end{array}$ *to* $\begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$.

Proof. Since Q_H is a morphism in $\mathcal{CAT}^{\to}$ (by Remark 3.1.5), it only remains to show that $(Q_H)_1$ sends cartesian morphisms with respect to pr_1 to cartesian morphisms with respect to p. Then consider a cartesian morphism $(A', E', A' \xrightarrow{f'} F(E')) \xrightarrow{(a,e)} (A, E, A \xrightarrow{f} F(E))$ with respect to pr_1. By Construction 3.1.3, in order to compute $(Q_H)_1(a,e)$, we consider the diagram

$$\begin{tikzcd}
H_0(f')^{\sim}H_1(E') \arrow[rr, "c_{H_0(f')}"] \arrow[dr, dashed, "(Q_H)_1(a{,}e)"] \arrow[dd, "p"] & & H_1(E') \arrow[dr, bend left, "H_1(e)"] \arrow[dd, "p"] & \\
& H_0(f)^{\sim}H_1(E) \arrow[rr, "c_{H_0(f)}"] \arrow[dd, "p"] & & H_1(E) \arrow[dd, "p"] \\
H_0(A') \arrow[rr, "H_0(f')"'] \arrow[dr, bend right, "H_0(a)"'] & & H_0(F(E')) \arrow[dr, "H_0(F(e))"] & \\
& H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd}$$

where $c_{H_0(f)}$ and $c_{H_0(f')}$ are the cartesian morphisms we have chosen in Construction 3.1.3. But, by Lemma 3.1.6, e is an isomorphism. Thus also $H_1(e)$ is an isomorphism and hence a cartesian morphism with respect to p by Proposition 2.2.12. Then $H_1(e) \circ c_{H_0(f)}$ is cartesian because composite of cartesian morphisms, by Proposition 2.2.13 (i). It follows that $(Q_H)_1(a,e)$ is a cartesian morphism with respect to p, by Proposition 2.2.13 (ii), since $c_{H_0(f)}$ and $c_{H_0(f)} \circ (Q_H)_1(a,e) = H_1(e) \circ c_{H_0(f)}$ are cartesian. $\square$

Remark 3.1.9. To have an adjunction $\begin{tikzcd}\mathcal{CAT}^{\to} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{FIB} \arrow[l, bend left, "U"]\end{tikzcd}$, it remains to show the uniqueness of (Q_H), that is, that if the diagram

$$\begin{tikzcd}
& \mathcal{C}\downarrow F \arrow[dr, bend left, "Q'"] \arrow[dd, "\mathrm{pr}_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
& \mathcal{C} \arrow[dr, bend left, "H_0"] & \\
\mathcal{C} \arrow[ur, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}$$

commutes also with another functor Q' such that (H_0, Q') is a cartesian functor, then Q' needs to be equal to $(Q_H)_1$.

But one shall recall that in Construction 3.1.3 we have produced the functor $(Q_H)_1$ constructing first a function $q\colon \mathrm{Ob}(\mathcal{C}\downarrow F) \to \mathrm{Ob}(\mathcal{E}) \times \mathrm{Ob}(\mathcal{E}^{\to})$ using the axiom of choice. Any different choice of cartesian liftings $c_{H_0(f)}$ of $H_0(f)$ would give rise to a different function $\mathrm{Ob}(\mathcal{C}\downarrow F) \to \mathrm{Ob}(\mathcal{E}) \times \mathrm{Ob}(\mathcal{E}^{\to})$ and hence to a functor $Q'\colon \mathcal{C}\downarrow F \to \mathcal{E}$ different from $(Q_H)_1$ but such that (H_0, Q') is a cartesian functor, by the same argument of Proposition 3.1.8, and such that the diagram above commutes, by the same argument of Remark 3.1.5, as long as we choose again the identity as cartesian lifting of each identity in the construction of the new function $\mathrm{Ob}(\mathcal{C}\!\downarrow\! F) \to \mathrm{Ob}(\mathcal{E}) \times \mathrm{Ob}(\mathcal{E}^{\to})$ (as we have done in Construction 3.1.3).

Therefore the natural attempt we have described in this section does not bring to a left adjoint of the forgetful functor $U\colon \mathcal{FIB} \to \mathcal{CAT}^{\to}$.

Nonetheless, one might think that the problem of choices we have just described could be fixed considering cloven fibrations. A new problem arises though, since being a cloven cartesian functor is a more restrictive notion than being a cartesian functor.

In order to have an analogue of Proposition 3.1.8 with "cloven cartesian functor" instead of "cartesian functor", we have to restrict ourselves to split fibrations, as we will see in Remark 3.2.6. In the next section we will then work with split fibrations and we will manage to show a 2-adjunction between $\mathcal{S}p\mathcal{FIB}$ and $\mathcal{CAT}^{\to}$.

3.2 A 2-adjunction between $\mathcal{S}p\mathcal{FIB}$ and $\mathcal{CAT}^{\to}$

In this section we show a 2-adjunction between $\mathcal{S}p\mathcal{FIB}$ and $\mathcal{CAT}^{\to}$, in order to prove, in Section 3.3, that the forgetful 2-functor $U\colon \mathcal{S}p\mathcal{FIB} \to \mathcal{CAT}^{\to}$ is monadic. To show such a 2-adjunction, we go over Section 3.1 again considering split fibrations.

As right 2-adjoint $U\colon \mathcal{S}p\mathcal{FIB} \to CAT^{\to}$ it is natural to take the forgetful functor:

$$U\colon \mathcal{S}p\mathcal{FIB} \to \mathcal{CAT}^{\to}$$

$$p\colon \mathcal{E} \to \mathcal{B} \longmapsto p\colon \mathcal{E} \to \mathcal{B}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd}$$

By Theorem 1.9.5, to have a left 2-adjoint to U it suffices to find for each functor $F\colon \mathcal{X} \to \mathcal{C}$ a 2-universal morphism from F to U, say $\eta_F = ((\eta_F)_0, (\eta_F)_1)\colon \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array} \to U\left(\begin{array}{c}\mathcal{A}_F\\ \downarrow L(F)\\ \mathcal{D}_F\end{array}\right)$. (This $L(F)$ will be the free split fibration associated to F.)

But this means that for every $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix} \in \mathcal{SpFIB}$ and for every 1-morphism $H = (H_0\colon \mathcal{C} \to \mathcal{B}, H_1\colon \mathcal{X} \to \mathcal{E})$ from $\begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$ to $U\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}\right)$ in $\mathcal{CAT}^{\to}$, there exists a unique 1-morphism

$$Q_H = ((Q_H)_0, (Q_H)_1)\colon \begin{matrix}\mathcal{A}_F\\ \downarrow L(F)\\ \mathcal{D}_F\end{matrix} \to \begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}$$

in $\mathcal{SpFIB}$ such that the triangle in $\mathcal{CAT}^{\to}$

$$\begin{tikzcd}
 & \mathcal{A}_F \arrow[dr, bend left, "(Q_H)_1"] \arrow[dd, "L(F)"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"'] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{D}_F \arrow[dr, bend left, "(Q_H)_0"] & \\
\mathcal{C} \arrow[ur, bend left, "(\eta_F)_0"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}$$

is commutative, and that for every 2-morphism

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{B}
\end{tikzcd}$$

in $\mathcal{CAT}^{\to}$, there exists a unique 2-morphism

$$\begin{tikzcd}
\mathcal{A}_F \arrow[r, bend left, "(Q_H)_1"] \arrow[r, bend right, "(Q_K)_1"'] \arrow[r, phantom, "\Downarrow\delta_1"] \arrow[d, "L(F)"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{D}_F \arrow[r, bend right, "(Q_K)_0"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A}_F \arrow[r, bend left, "(Q_H)_1"] \arrow[d, "L(F)"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{D}_F \arrow[r, bend left, "(Q_H)_0"] \arrow[r, bend right, "(Q_K)_0"'] \arrow[r, phantom, "\Downarrow\delta_0"] & \mathcal{B}
\end{tikzcd}$$

in $\mathcal{SpFIB}$ which makes the following diagram of 2-morphisms in $\mathcal{CAT}^{\rightarrow}$ commute:

$$
\begin{tikzcd}
& \mathcal{A}_F \arrow[dd, "L(F)"'] \arrow[dr, bend left, "Q_H"] \arrow[dr, "Q_K"'] \arrow[dr, phantom, "\Downarrow\delta"] & \\
\mathcal{X} \arrow[ur, "\eta_F"] \arrow[ddd, "F"'] \arrow[rr, bend left, "H"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \mathcal{E} \arrow[ddd, "p"] \\
& \mathcal{D}_F & \\
& & \\
\mathcal{C} & & \mathcal{B}
\end{tikzcd}
\tag{3.1}
$$

Remark 3.2.1. Looking at Remark 3.1.1, we set $L(F) := \begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ and $(\eta_F)_0 := \mathrm{Id}_{\mathcal{C}}$. Then $(Q_H)_0$ needs to be the identity.

We already know that $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is a Grothendieck fibration, by Proposition 3.1.2, but now we need it to be a split fibration.

Proposition 3.2.2. *Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Then* $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ *is a split fibration.*

Proof. In the proof of Proposition 3.1.2 we have actually constructed a cleavage for $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$.

More precisely, we proved that $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is a cloven fibration setting for every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ and every $a\colon A' \to A$

$$\mathrm{Cart}\left(a, (A, E, A \xrightarrow{f} F(E))\right) := ((A', E, A' \xrightarrow{fa} F(E)) \xrightarrow{(a,\mathrm{id})} (A, E, A \xrightarrow{f} F(E))).$$

Now we immediatly see that this cleavage makes $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ into a normal fibration, since for every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ we have that

$$\mathrm{Cart}\left(\mathrm{id}_A, (A, E, A \xrightarrow{f} F(E))\right) = ((A, E, A \xrightarrow{f\circ\mathrm{id}} F(E)) \xrightarrow{(\mathrm{id},\mathrm{id})} (A, E, A \xrightarrow{f} F(E))).$$

Furthermore, $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is a split fibration, since for every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ and every $a\colon A' \to A$ and $h\colon A'' \to A'$ in $\mathcal{C}$ we have that

$$\begin{aligned}
&\mathrm{Cart}\Big(a, (A, E, A \xrightarrow{f} F(E))\Big) \circ \mathrm{Cart}\Big(h, a^*(A, E, A \xrightarrow{f} F(E))\Big) = \\
&= ((A'', E, A'' \xrightarrow{fah} F(E)) \xrightarrow{(h,\mathrm{id})} (A', E, A' \xrightarrow{fa} F(E)) \xrightarrow{(a,\mathrm{id})} (A, E, A \xrightarrow{f} F(E))) = \\
&= ((A'', E, A'' \xrightarrow{fah} F(E)) \xrightarrow{(ah,\mathrm{id})} (A, E, A \xrightarrow{f} F(E))) = \\
&= \mathrm{Cart}\Big(ah, (A, E, A \xrightarrow{f} F(E))\Big).
\end{aligned}$$

□

Now we want to go over Construction 3.1.3 again considering split fibrations.

Construction 3.2.3. At this point, as we have done in Construction 3.1.3, it is natural to define $(\eta_F)_1$ to be

$$(\eta_F)_1\colon \mathcal{X} \to \mathcal{C}\downarrow F$$

$$\begin{array}{ccc}
E & \longmapsto & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \\
{\scriptstyle e}\big\downarrow & \longmapsto & {\scriptstyle F(e)}\big\downarrow \quad {\scriptstyle e}\big\downarrow {\scriptstyle F(e)}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle F(e)} \\
X & \longmapsto & (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X))
\end{array}$$

It is clear that this is a functor and that the square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(\eta_F)_1"] \arrow[d, "F"'] & \mathcal{C}\downarrow F \arrow[d, "pr1"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}_{\mathcal{C}}"'] & \mathcal{C}
\end{tikzcd}$$

is commutative.

It now remains to find a functor $(Q_H)_1\colon \mathcal{C}\downarrow F \to \mathcal{E}$ such that Q_H is a cloven cartesian functor.

It might be useful to see again the triangle in $\mathcal{CAT}^{\to}$ which we want to be commutative for a unique cloven cartesian functor $Q_H = (H_0, (Q_H)_1)$:

$$
\begin{tikzcd}
& \mathcal{C}\downarrow F \arrow[dr, bend left, "(Q_H)_1"] \arrow[dd, "\mathrm{pr}_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"'] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
& \mathcal{C} \arrow[dr, bend left, "H_0"] & \\
\mathcal{C} \arrow[ur, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}
$$

To define $(Q_H)_1$ we start from an object $(A, E, A \xrightarrow{f} F(E))$ of $\mathcal{C}\downarrow F$ and we see that we have to construct an object in $\mathcal{E}$ which is above $H_0(A)$. But if we start from the diagram

$$
\begin{tikzcd}
& E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}
$$

and we apply (H_0, H_1) to it, we get the diagram

$$
\begin{tikzcd}
H_0(f)^* H_1(E) \arrow[r, "\mathrm{Cart}(H_0(f), H_1(E))"] \arrow[d, mapsto, "p"] & H_1(E) \arrow[d, mapsto, "p"] \\
H_0(A) \arrow[r, "H_0(f)"'] & H_0(F(E))
\end{tikzcd}
$$

from which we can construct an object $H_0(f)^* H_1(E)$ above $H_0(A)$ and a cartesian morphism $\mathrm{Cart}\,(H_0(f), H_1(E))$ above $H_0(f)$, written in violet, since p is a cloven (indeed split) fibration. This is similar to what we have done in Construction 3.1.3, but there is an important difference: here we are not arbitrarily choosing a cartesian lifting of $H_0(f)$ to $H_1(E)$, but we are taking the unique lifting of $H_0(f)$ to $H_1(E)$ which is in the (fixed) cleavage of p. This will solve the problem we have seen in Remark 3.1.9.

We then define $(Q_H)_1$ in the following way:

$$
\begin{aligned}
(Q_H)_1 \colon \mathcal{C}\downarrow F &\longrightarrow \mathcal{E} \\
(A, E, A \xrightarrow{f} F(E)) &\longmapsto H_0(f)^* H_1(E)
\end{aligned}
$$

$$
\begin{tikzcd}
 & & E' \arrow[d, mapsto, "F"] \arrow[dr, bend left, "e"] & \\
 & & & E \arrow[dd, mapsto, "F"] \\
A' \arrow[r, "f'"'] \arrow[dr, bend right, "a"'] & & F(E') \arrow[dr, "F(e)"] & \\
 & A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}
\quad\longmapsto\quad
\begin{tikzcd}
H_0(f')^*H_1(E') \arrow[rr, "\mathrm{Cart}(H_0(f'),H_1(E'))"] \arrow[dd, mapsto, "p"] \arrow[dr, dashed, "(Q_H)_1(a,e)"] & & H_1(E') \arrow[dd, mapsto, "p"] \arrow[dr, bend left, "H_1(e)"] & \\
 & H_0(f)^*H_1(E) \arrow[rr, "\mathrm{Cart}(H_0(f),H_1(E))"] \arrow[dd, mapsto, "p"] & & H_1(E) \arrow[dd, mapsto, "p"] \\
H_0(A') \arrow[rr, "H_0(f')"'] \arrow[dr, bend right, "H_0(a)"'] & & H_0(F(E')) \arrow[dr, "H_0(F(e))"] & \\
 & H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd}
$$

Remark 3.2.4. Notice that we have actually defined $(Q_H)_1$ exactly as we had done in Construction 3.1.3, just fixing one of all the possible functions $q\colon \operatorname{Ob}(\mathcal{C}\downarrow F) \longrightarrow \operatorname{Ob}(\mathcal{E})\times\operatorname{Ob}(\mathcal{E}^{\to})$, setting

$$q(A, E, A \xrightarrow{f} F(E)) := (H_0(f)^*H_1(E), \mathrm{Cart}\,(H_0(f), H_1(E)))$$

We then already know that $(Q_H)_1$ is a functor $\mathcal{C}\downarrow F \to \mathcal{E}$, by Proposition 3.1.4.

Furthermore, since we have defined both $(\eta_F)_1$ and $(Q_H)_1$ in the same way we had defined them in Section 3.1, we already know that the diagram

$$
\begin{tikzcd}
 & \mathcal{C}\downarrow F \arrow[dd, "\mathrm{pr}_1"] \arrow[dr, bend left, "(Q_H)_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{C} \arrow[dr, bend left, "H_0"] & \\
\mathcal{C} \arrow[ur, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd}
$$

commutes, by Remark 3.1.5. Notice that to ensure the commutativity of the upper triangle we are using that p is a normal fibration (as in Remark 3.1.5 we used that $q(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) = (H_1(E), \mathrm{id})$ for every $E \in \mathcal{X}$).

So far, we have only used that p is a normal fibration, and not that p is a split fibration.

We now want to prove that Q_H is a cloven cartesian functor. We have already seen that Q_H is a cartesian functor, in Proposition 3.1.8, but being a cloven cartesian functor is more restrictive, and we now see that this is equivalent to starting from a cloven fibration $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{smallmatrix}$ which is split.

Proposition 3.2.5. *The morphism Q_H in $CAT^{\to}$ defined in Construction 3.2.3 is a cloven cartesian functor from* $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ *to* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ *if and only if the cloven fibration* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ *is split.*

Proof. Recall that the splitting cleavage of $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$ is given by

$$\mathrm{Cart}_{\mathrm{pr}_1}\left(a,(A,E,A\xrightarrow{f}F(E))\right) := ((A',E,A'\xrightarrow{fa}F(E))\xrightarrow{(a,\mathrm{id})}(A,E,A\xrightarrow{f}F(E))).$$

Starting from a diagram

$$\begin{tikzcd}
(A',E,A'\xrightarrow{fa}F(E)) \arrow[r, "{(a,\mathrm{id})}"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (A,E,A\xrightarrow{f}F(E)) \arrow[d, mapsto, "\mathrm{pr}_1"] \\
A' \arrow[r, "a"'] & A
\end{tikzcd}$$

(representing the cleavage of $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$) we already know that $(Q_H)_1(a,\mathrm{id})$ fits into the diagram

$$\begin{tikzcd}
(Q_H)_1(A',E,A'\xrightarrow{fa}F(E)) \arrow[r, "{(Q_H)_1(a,\mathrm{id})}"] \arrow[d, mapsto, "p"] & (Q_H)_1(A,E,A\xrightarrow{f}F(E)) \arrow[d, mapsto, "p"] \\
H_0(A') \arrow[r, "H_0(a)"'] & H_0(A)
\end{tikzcd}$$

that is, that $(Q_H)_1(a,\mathrm{id})$ is a (cartesian) lifting of $H_0(a)$ to $(Q_H)_1(A,E,A\xrightarrow{f}F(E))$, by Proposition 3.1.8. We ask ourselves when $(Q_H)_1(a,\mathrm{id})$ is the one cartesian lifting of $H_0(a)$ to $(Q_H)_1(A,E,A\xrightarrow{f}F(E))$ which is in the cleavage of p. By construction, $(Q_H)_1(a,\mathrm{id})$ fits well also into the diagram

$$\begin{tikzcd}
H_0(fa)^*H_1(E) \arrow[rr, "{\mathrm{Cart}_p(H_0(fa),H_1(E))}"] \arrow[dd, mapsto, "p"] \arrow[dr, dashed, "{(Q_H)_1(a,\mathrm{id})}"] & & H_1(E) \arrow[dr, equal] \arrow[dd, mapsto, "p"] & \\
& H_0(f)^*H_1(E) \arrow[rr, "{\mathrm{Cart}_p(H_0(f),H_1(E))}"] \arrow[dd, mapsto, "p"] & & H_1(E) \arrow[dd, mapsto, "p"] \\
H_0(A') \arrow[rr, "H_0(fa)"'] \arrow[dr, bend right, "H_0(a)"'] & & H_0(F(E)) \arrow[dr, equal] & \\
& H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd}$$

Therefore we see that (Q_H) is a cloven cartesian functor if and only if p is a split fibration. In fact, if p is a split fibration, we know that also $\mathrm{Cart}_p(H_0(a), H_0(f)^* H_1(E))$ is an acceptable choice for the morphism in violet in the diagram right above, and by uniqueness it follows that

$$(Q_H)_1(a, \mathrm{id}) = \mathrm{Cart}_p(H_0(a), H_0(f)^* H_1(E)).$$

Whereas if we assume that $(Q_H)_1$ preserves the cleavage of $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$, that is, that for every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ and for every $a\colon A' \to A$ in $\mathcal{C}$ we have that

$$(Q_H)_1(a, \mathrm{id}) = \mathrm{Cart}_p(H_0(a), H_0(f)^* H_1(E))$$

then by the commutative diagram above we see that for every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ and for every $a\colon A' \to A$ in $\mathcal{C}$

$$\mathrm{Cart}_p(H_0(f), H_1(E)) \circ \mathrm{Cart}_p(H_0(a), H_0(f)^* H_1(E)) = \mathrm{Cart}_p(H_0(fa), H_1(E))$$

which exactly means that p is a split fibration. $\square$

Remark 3.2.6. Proposition 3.2.5 explains why we had to restrict ourselves to split fibrations in order to prove that the function on objects L we have constructed extends to a left 2-adjoint of the forgetful 2-functor from cloven fibrations to $\mathcal{CAT}^{\to}$.

Theorem 3.2.7. *The forgetful 2-functor $U\colon \mathcal{SpFIB} \to \mathcal{CAT}^{\to}$ has a left 2-adjoint $L\colon \mathcal{CAT}^{\to} \to \mathcal{SpFIB}$.*

Proof. We prove that U has a left 2-adjoint by using the characterization with 2-universal morphisms, given in Theorem 1.9.5. For every $F\colon \mathcal{X} \to \mathcal{C}$ in $\mathcal{CAT}^{\to}$, set $L(F) := \begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$, as in Remark 3.2.1. Then $L(F)$ is a split fibration, by Proposition 3.2.2.

We thus search for a 2-universal morphism from F to the 2-functor U of the form

$$\eta_F\colon \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array} \longrightarrow U\left(\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}\right).$$

We construct η_F as in Construction 3.2.3. To prove that η_F is a (1-)universal morphism from F to U (treated as a 1-functor), we have to prove that for every $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array} \in \mathcal{SpFIB}$ and for every 1-morphism $H = (H_0\colon \mathcal{C} \to \mathcal{B}, H_1\colon \mathcal{X} \to \mathcal{E})$ from $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$ to $U\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}\right)$ in $\mathcal{CAT}^{\to}$, there exists a unique 1-morphism

$$Q_H = ((Q_H)_0, (Q_H)_1) \colon \begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C} \end{array} \to \begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$$

in $\mathcal{S}p\mathcal{FIB}$ such that the following triangle in $\mathcal{CAT}^{\to}$ is commutative:

$$\begin{tikzcd}
 & \mathcal{C}\downarrow F \arrow[dd, "\mathrm{pr}_1"] \arrow[dr, bend left, "(Q_H)_1"] & \\
\mathcal{X} \arrow[ur, bend left, "(\eta_F)_1"] \arrow[rr, bend right, "H_1"'] \arrow[dd, "F"'] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{C} \arrow[dr, bend left, "(Q_H)_0"] & \\
\mathcal{C} \arrow[ur, bend left, "(\eta_F)_0"] \arrow[rr, bend right, "H_0"'] & & \mathcal{B}
\end{tikzcd} \tag{3.2}$$

Fixed a split fibration $\begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$ and a morphism H from $\begin{array}{c} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{array}$ to $U\left(\begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}\right)$ in $\mathcal{CAT}^{\to}$, we construct the morphism $Q_H \colon \begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C} \end{array} \to \begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$ as in Construction 3.2.3.

We then know that Q_H is a morphism in $\mathcal{S}p\mathcal{FIB}$, that is, a cloven cartesian functor, by Proposition 3.2.5, and also that the diagram (3.2) commutes, by Remark 3.2.4.

In order to prove that η_F is a (1-)universal morphism, it now remains to prove that this Q_H is the unique cloven cartesian functor which makes diagram (3.2) commute. Suppose then that there is another cloven cartesian functor $Q' = (Q'_0, Q'_1) \colon \mathcal{C}\downarrow F \to \mathcal{E}$ which makes diagram (3.2) commute.

We immediatly see that Q'_0 must be equal to H_0, by commutativity of diagram (3.2), and then it must coincide with $(Q_H)_0$. We show that Q'_1 needs to coincide with $(Q_H)_1$.
For every $\mathcal{E} \in \mathcal{X}$ we have that

$$Q'_1(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) = Q'_1((\eta_F)_1(E)) = H_1(E) = (Q_H)_1(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E))$$

Thus Q'_1 and $(Q_H)_1$ need to coincide at least on all the objects of the form $(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E))$. Take now an arbitrary object $(A, E, A \xrightarrow{f} F(E))$ of $\mathcal{C}\downarrow F$. Since the morphism (f, id)

of the diagram

$$\begin{tikzcd}
(A, E, A \xrightarrow{f} F(E)) \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, maps to, "\mathrm{pr}_1"] & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[d, maps to, "\mathrm{pr}_1"] \\
A' \arrow[r, "a"'] & A
\end{tikzcd}$$

is in the cleavage of $\begin{array}{c}\mathcal{C}\downarrow F\\ \downarrow \mathrm{pr}_1\\ \mathcal{C}\end{array}$, and since Q' is a cloven cartesian functor, both $Q'_1(f, \mathrm{id})$ and $(Q_H)_1(f, \mathrm{id})$ need to be the one cartesian lifting of $H_0(f)$ to $H_1(E)$ which is in the cleavage of the split fibration p:

$$\begin{tikzcd}
\begin{array}{c} Q'_1(A, E, A \xrightarrow{f} F(E)) \\ (Q_H)_1(A, E, A \xrightarrow{f} F(E)) \end{array} \arrow[r, shift left, "{Q'_1(f,\mathrm{id})}"] \arrow[r, shift right, "{(Q_H)_1(f,\mathrm{id})}"'] \arrow[d, maps to, "p"] & H_1(E) \arrow[d, maps to, "p"] \\
H_0(A) \arrow[r, "H_0(f)"'] & H_0(F(E))
\end{tikzcd}$$

This holds because we have already proved that Q'_1 and $(Q_H)_1$ need to coincide at least on all the objects of the form $(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E))$, with $E \in \mathcal{X}$.

But then we have proved that Q'_1 and $(Q_H)_1$ must coincide on all the objects of $\mathcal{C}\downarrow F$ and, moreover, that

$$Q'_1(f, \mathrm{id}) = (Q_H)_1(f, \mathrm{id})$$

for every morphism $(A, E, A \xrightarrow{f} F(E)) \xrightarrow{(f,\mathrm{id})} (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E))$ in $\mathcal{C}\downarrow F$.

Take now an arbitrary morphism

$$(A', E', A' \xrightarrow{f'} F(E')) \xrightarrow{(a,e)} (A, E, A \xrightarrow{f} F(E)).$$

Recall that by construction of $(Q_H)_1$ the morphism $(Q_H)_1(a, e)$ fits well into the diagram

$$\begin{tikzcd}
H_0(f')^*H_1(E') \arrow[rr, "{\mathrm{Cart}_p(H_0(f'),H_1(E'))}"] \arrow[dr, dashed, "{(Q_H)_1(a,e)}"] \arrow[dd, maps to, "p"] & & H_1(E') \arrow[dr, bend left, "H_1(e)"] \arrow[dd, maps to, "p"] & \\
& H_0(f)^*H_1(E) \arrow[rr, "{\mathrm{Cart}_p(H_0(f),H_1(E))}"] \arrow[dd, maps to, "p"] & & H_1(E) \arrow[dd, maps to, "p"] \\
H_0(A') \arrow[rr, "H_0(f')"'] \arrow[dr, bend right, "H_0(a)"'] & & H_0(F(E')) \arrow[dr, "H_0(F(e))"] & \\
& H_0(A) \arrow[rr, "H_0(f)"'] & & H_0(F(E))
\end{tikzcd} \tag{3.3}$$

We show that also $Q'_1(a,e)$ is an acceptable choice for the morphism in violet in diagram (3.3), whence we will conclude that $Q'_1(a,e)$ and $(Q_H)_1(a,e)$ must coincide by uniqueness.
We already know that

$$H_0(f')^* H_1(E') = Q'_1(A', E', A' \xrightarrow{f'} F(E')) \text{ and } H_0(f)^* H_1(E) = Q'_1(A, E, A \xrightarrow{f} F(E))$$

and also that

$$\mathrm{Cart}_p(H_0(f), H_1(E)) = (Q_H)_1(f, \mathrm{id}) = Q'_1(f, \mathrm{id}).$$

Furthermore, since $Q'_1 \circ (\eta_F)_1 = H_1$, by commutativity of diagram (3.2), we have that

$$\begin{aligned} & H_1(e) \circ \mathrm{Cart}_p(H_0(f'), H_1(E')) = Q'_1(F(e), e) \circ Q'_1(f', \mathrm{id}) = Q'_1(F(e) \circ f', e) = \\ & = Q'_1(fa, e) = Q'_1(f, \mathrm{id}) \circ Q'_1(a, e) \end{aligned}$$

(where we have also used that (a, e) is a morphism in $\mathcal{C} \downarrow F$, for the last equality of the first row). Then we have shown that $Q'_1(a,e)$ (in the place of the violet morphism in diagram (3.3)) makes the top square of diagram (3.3) commute.
Finally, by commutativity of diagram (3.2), we have that

$$p(Q'_1(a,e)) = (Q_H)_0(a) = H_0(a).$$

Therefore we have proved that also $Q'_1(a,e)$ is an acceptable choice for the morphism in violet in diagram (3.3), and we conclude that $Q'_1(a,e)$ and $(Q_H)_1(a,e)$ must coincide by uniqueness. By arbitrariness of (a,e) we conclude that Q'_1 and $(Q_H)_1$ must coincide as functors, whence Q' and Q_H must coincide as morphism in $\mathcal{SpFIB}$, that is, as cloven cartesian functors.

Finally, we prove that the (1-)universal morphism η_F we have constructed is a 2-universal morphism from F to the 2-functor U. Given a 2-morphism

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{B} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{B} \end{tikzcd} \tag{3.4}$$

in $\mathcal{CAT}^{\to}$, we consider the cloven cartesian functors Q_H and Q_K respectively associated to H

and to K. We need to prove that there is a unique 2-morphism

$$
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"{name=A}] \arrow[r, bend right, "(Q_K)_1"'{name=B}] \arrow[d, "\mathrm{pr}_1"'] \arrow[from=A, to=B, Rightarrow, "\delta_1"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend right, "(Q_K)_0"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend left, "(Q_H)_0"{name=C}] \arrow[r, bend right, "(Q_K)_0"'{name=D}] \arrow[from=C, to=D, Rightarrow, "\delta_0"] & \mathcal{B}
\end{tikzcd}
$$

in $\mathcal{SpFIB}$ which makes the following diagram of 2-morphisms in $\mathcal{CAT}^{\to}$ commute:

$$
\begin{tikzcd}
& \mathcal{C}\downarrow F \arrow[dd, "\mathrm{pr}_1"'{name=P}] & \\
\mathcal{X} \arrow[dd, "F"'{name=F}] & & \mathcal{E} \arrow[dd, "p"{name=Q}] \\
& \mathcal{C} & \\
\mathcal{C} & & \mathcal{B}
\arrow[from=F, to=P, "\eta_F"]
\arrow[from=P, to=Q, bend left, "Q_H"{name=QH}]
\arrow[from=P, to=Q, bend right, "Q_K"'{name=QK}]
\arrow[from=QH, to=QK, Rightarrow, dashed, "\delta"]
\arrow[from=F, to=Q, bend left=10, "H"{name=HH}]
\arrow[from=F, to=Q, bend right, "K"'{name=KK}]
\arrow[from=HH, to=KK, Rightarrow, "\lambda"']
\end{tikzcd}
\tag{3.5}
$$

(which is diagram (3.1) after the choice of $L(F)$).

diagram (3.5) can be written in equations as

$$
\begin{cases}
\lambda_0 = \delta_0(\eta_F)_0 \\
\lambda_1 = \delta_1(\eta_F)_1
\end{cases}
$$

which is equivalent to ask that for every $A \in \mathcal{C}$ and for every $E \in \mathcal{X}$

$$
\begin{cases}
\lambda_{0,A} = \delta_{0,A} \\
\lambda_{1,E} = \delta_{1,(F(E),E,F(E)\overset{\mathrm{id}}{\to}F(E))}
\end{cases}
$$

Therefore the unique possible choice for δ_0 is

$$\delta_0 := \lambda_0, \tag{3.6}$$

and δ_1 has to be defined on objects of the form $(F(E), E, F(E) \overset{\mathrm{id}}{\to} F(E))$ as

$$\delta_{1,(F(E),E,F(E)\overset{\mathrm{id}}{\to}F(E))} := \lambda_{1,E}. \tag{3.7}$$

Notice that if $(A, E, A \overset{f}{\to} F(E))$ is an arbitrary object of $\mathcal{C}\downarrow F$,

$$\delta_{1,(A,E,A\overset{f}{\to}F(E))} \colon (Q_H)_1(A, E, A \overset{f}{\to} F(E)) \longrightarrow (Q_K)_1(A, E, A \overset{f}{\to} F(E)),$$

that is, by construction of $(Q_H)_1$ and $(Q_K)_1$,

$$\delta_{1,(A,E,A\xrightarrow{f}F(E))} \colon H_0(f)^*H_1(E) \longrightarrow K_0(f)^*K_1(E).$$

Then it is natural to consider the diagrams $H_0(f)^*H_1(E)$ and $K_0(f)^*K_1(E)$ fit well into and connect them using the natural transformations λ_0 and λ_1:

$$\begin{tikzcd}
H_0(f)^*H_1(E) \arrow[rr, "\mathrm{Cart}(H_0(f),H_1(E))"] \arrow[dd, "p"] & & H_1(E) \arrow[dr, bend left, "\lambda_{1,E}"] \arrow[dd, "p"] & \\
& K_0(f)^*K_1(E) \arrow[rr, "\mathrm{Cart}(K_0(f),K_1(E))"] \arrow[dd, "p"] & & K_1(E) \arrow[dd, "p"] \\
H_0(A) \arrow[rr, "H_0(f)"'] \arrow[dr, bend right, "\lambda_{0,A}"'] & & H_0(F(E)) \arrow[dr, "\lambda_{0,F(E)}"] & \\
& K_0(A) \arrow[rr, "K_0(f)"'] & & K_0(F(E))
\end{tikzcd} \tag{3.8}$$

Notice that the lower square of diagram (3.8) commutes by naturality of λ_0 and that

$$p\,(\lambda_{1,E}) = (p\lambda_1)_E = (\lambda_0 F)_E = \lambda_{0,F(E)}$$

by equation (3.4).

Thus, to prove that there is at most one 2-morphism

$$\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"{name=U}] \arrow[r, bend right, "(Q_K)_1"'{name=D}] \arrow[d, "\mathrm{pr}_1"'] \arrow[Rightarrow, from=U, to=D, "\delta_1"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend right, "(Q_K)_0"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend left, "(Q_H)_0"{name=U}] \arrow[r, bend right, "(Q_K)_0"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\delta_0"] & \mathcal{B}
\end{tikzcd} \tag{3.9}$$

in $\mathcal{SpFIB}$ which makes diagram (3.5) commute, it suffices to show that any such 2-morphism δ in $\mathcal{SpFIB}$ has to be such that $\delta_{1,(A,E,A\xrightarrow{f}F(E))}$ fits well into diagram (3.8):

$$\begin{tikzcd}
H_0(f)^*H_1(E) \arrow[rr, "\mathrm{Cart}(H_0(f),H_1(E))"] \arrow[dd, "p"] \arrow[dr, dashed, "\delta_{1,(A,E,A\xrightarrow{f}F(E))}"] & & H_1(E) \arrow[dr, bend left, "\lambda_{1,E}"] \arrow[dd, "p"] & \\
& K_0(f)^*K_1(E) \arrow[rr, "\mathrm{Cart}(K_0(f),K_1(E))"] \arrow[dd, "p"] & & K_1(E) \arrow[dd, "p"] \\
H_0(A) \arrow[rr, "H_0(f)"'] \arrow[dr, bend right, "\lambda_{0,A}"'] & & H_0(F(E)) \arrow[dr, "\lambda_{0,F(E)}"] & \\
& K_0(A) \arrow[rr, "K_0(f)"'] & & K_0(F(E))
\end{tikzcd} \tag{3.10}$$

In fact we will then conclude the uniqueness of δ by the uniqueness of the morphism in violet in diagram (3.10), given by the fact that $\mathrm{Cart}\,(K_0(f), K_1(E))$ is a cartesian morphism.

We have that

$$p\left(\delta_{1,(A,E,A\xrightarrow{f}F(E))}\right) = (p\delta_1)_{(A,E,A\xrightarrow{f}F(E))} = (\delta_0\,\mathrm{pr}_1)_{(A,E,A\xrightarrow{f}F(E))} = \delta_{0,A} = \lambda_{0,A}$$

by equations (3.9) and (3.6). To see that the upper square of diagram (3.10) commutes, it is useful to notice that such square is equal to the square

$$\begin{tikzcd}
H_0(f)^*H_1(E) \arrow[r, "(Q_H)_1(f{,}\mathrm{id})"] \arrow[d, "\delta_{1,(A,E,A\xrightarrow{f}F(E))}"] & H_1(E) \arrow[d, "\delta_{1,(F(E),E,F(E)\xrightarrow{\mathrm{id}}F(E))}"] \\
K_0(f)^*K_1(E) \arrow[r, "(Q_K)_1(f{,}\mathrm{id})"] & K_1(E)
\end{tikzcd}$$

by equation (3.7) and by construction of $(Q_H)_1$ and $(Q_K)_1$. Then we conclude the commutativity of the upper square of diagram (3.10) by using the naturality of δ_1.

It only remains to show that, setting $\delta_0 = \lambda_0$ (as in equation (3.6)), $\delta_{1,(F(E),E,F(E)\xrightarrow{\mathrm{id}}F(E))} := \lambda_{1,E}$ (as in equation (3.7)) and $\delta_{1,(A,E,A\xrightarrow{f}F(E))}$ to be the unique morphism which fits well into diagram (3.10) in the place of the morphism in violet, we have that $\delta = (\delta_0, \delta_1)$ is a 2-morphism in $\mathcal{SpFIB}$ which makes diagram (3.5) commute. The fact that δ makes diagram (3.5) commute is trivial by construction. Since we have defined a 2-morphism in $\mathcal{SpFIB}$ as a mere modification, it suffices to show that δ_0 and δ_1 are natural transformations such that

$$\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"] \arrow[r, bend right, "(Q_K)_1"'] \arrow[r, phantom, "\Downarrow\delta_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend right, "(Q_K)_0"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, bend left, "(Q_H)_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, bend left, "(Q_H)_0"] \arrow[r, bend right, "(Q_K)_0"'] \arrow[r, phantom, "\Downarrow\delta_0"] & \mathcal{B}
\end{tikzcd}$$

But the naturality of δ_0 is trivial and the naturality of δ_1 easily follows by composing diagram (3.10) with the square of naturality (whose we need to show the commutativity) and by the fact that $\mathrm{Cart}\,(K_0(f), K_1(E))$ is a cartesian morphism. Moreover, given $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$, the fact that $\delta_{1,(A,E,A\xrightarrow{f}F(E))}$ fits well into diagram (3.10) implies that

$$(p\delta_1)_{(A,E,A\xrightarrow{f}F(E))} = p\left(\delta_{1,(A,E,A\xrightarrow{f}F(E))}\right) = \lambda_{0,A} = \delta_{0,A} = \delta_{0,\mathrm{pr}_1(A,E,A\xrightarrow{f}F(E))} = (\delta_0\,\mathrm{pr}_1)_{(A,E,A\xrightarrow{f}F(E))}$$

(also using that $\delta_0 = \lambda_0$ by construction).

Therefore we have proved that η_F is a 2-universal morphism from F to the 2-functor U, and we conclude by Theorem 1.9.5 that the function on objects L extends to a left 2-adjoint to the forgetful 2-functor U. □

Remark 3.2.8. By Theorem 1.9.5, the morphisms η_F, defined in Construction 3.2.3, combine to a natural transformation $\eta\colon \operatorname{Id}_{\mathcal{CAT}^{\to}} \longrightarrow UL$, which corresponds to the unit of the 2-adjunction $L \dashv U$ constructed in the proof of Theorem 3.2.7.

By Theorem 1.9.5, we see that the functor $L\colon \mathcal{CAT}^{\to} \to \mathcal{S}p\mathcal{FIB}$ constructed in the proof of Theorem 3.2.7 acts on 1-cells as

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd} \quad \overset{L}{\longmapsto} \quad \begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "(H_0,H_1)"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{D}\downarrow G \arrow[d, "\mathrm{pr}_1"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd}$$

Analogously, for every 2-cell λ in $\mathcal{CAT}^{\to}$, we have that

$$L(\lambda) = (\lambda_0, (\lambda_0, \lambda_1)).$$

Furthermore, by Theorem 1.9.5, we see that the counit ε of the 2-adjunction $L \dashv U$ is given by

$$\varepsilon_p\colon \quad \begin{tikzcd} \mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B} \end{tikzcd}$$

for every $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{smallmatrix} \in \mathcal{S}p\mathcal{FIB}$, where we denote Q_p the functor $(Q_{\mathrm{Id}_p})_1$ produced in Construction 3.2.3 (with $H = \mathrm{Id}_p$).

3.3 Monadicity of Split Fibrations

In this section we prove that the forgetful 2-functor $U\colon \mathcal{S}p\mathcal{FIB} \longrightarrow \mathcal{CAT}^{\to}$ is monadic. We will use the 2-adjunction we have shown in the last section.

Notation 3.3.1. Till the end of this chapter, we will denote L and U respectively the left 2-adjoint and the right 2-adjoint of the 2-adjunction produced in the proof of Theorem 3.2.7. We will also denote η and ε respectively the unit and the counit of the 2-adjuction $L \dashv U$ (see Remark 3.2.8 for the description of such natural transformations). Finally, we will sometimes use also the notation Q_p introduced in Remark 3.2.8.

Now that we have a 2-adjunction $\begin{tikzcd} \mathcal{CAT}^{\to} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{FIB} \arrow[l, bend left, "U"] \end{tikzcd}$, by Theorem 3.2.7, we can consider the 2-monad generated by this adjunction.

Remark 3.3.2. By Proposition 1.11.4, the 2-monad T generated by the 2-adjunction $L \dashv U$ is given by

$$T = (UL, U\varepsilon L, \eta).$$

By Remark 1.11.1, a T-algebra is a pair

$$\left(\begin{array}{c} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{array}, \quad \begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C} \end{array} \xrightarrow{\alpha} \begin{array}{c} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{array} \right)$$

with $F \in \mathcal{CAT}^{\to}$ and $\alpha = (\alpha_0, \alpha_1)$ a morphism in $\mathcal{CAT}^{\to}$ (sometimes we will say that a T-algebra is just a commutative square

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C} \end{tikzcd}$$

in $\mathcal{C}at$) subject to the following axioms:

(ALG1)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "(\eta_F)_1"] \arrow[d, "F"] & \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C} \end{tikzcd} \quad = \quad \mathrm{id}_F$$

(ALG2)
$$\begin{tikzcd} \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "((U\varepsilon L)_F)_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "(UL\alpha)_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C} \end{tikzcd}$$

We immediatly see that (ALG1) implies that α_0 needs to be the identity $\mathrm{Id}_{\mathcal{C}}$. Then (ALG2) simplifies a bit. Using Remark 3.2.8 to describe the functors $(U\varepsilon L)_F$ and $UL\alpha$, we get a more explicit form of (ALG2) (assumed that (ALG1) holds):

(ALG2*)
$$\mathcal{C}\downarrow \mathrm{pr}_1 \xrightarrow{Q_{\mathrm{pr}_1}} \mathcal{C}\downarrow F \xrightarrow{\alpha_1} \mathcal{X} \quad = \quad \mathcal{C}\downarrow \mathrm{pr}_1 \xrightarrow{(\mathrm{Id},\alpha_1)} \mathcal{C}\downarrow F \xrightarrow{\alpha_1} \mathcal{X}$$

If we take an object $(A', (A, E, A \xrightarrow{f} F(E)), A' \xrightarrow{a} A) \in \mathcal{C}\downarrow \mathrm{pr}_1$, looking at the cleavage of $\begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow \mathrm{pr}_1 \\ \mathcal{C} \end{array}$ produced in the proof of Proposition 3.1.2, we get that

$$Q_{\mathrm{pr}_1}(A',(A,E,A \xrightarrow{f} F(E)),A' \xrightarrow{a} A) = (A',E,A' \xrightarrow{fa} F(E)). \tag{3.11}$$

Thus we see that, on objects, (ALG2*) means that for every $(A',(A,E,A \xrightarrow{f} F(E)),A' \xrightarrow{a} A) \in \mathcal{C}{\downarrow}\mathrm{pr}_1$ we have

$$\alpha_1(A',E,A' \xrightarrow{fa} F(E)) = \alpha_1(A',\alpha_1(A,E,A \xrightarrow{f} F(E)),A' \xrightarrow{a} A)$$

Whereas if we take a morphism $(A',(A,E,A \xrightarrow{f} F(E)),A' \xrightarrow{a} A) \xrightarrow{(j,(i,e))} (C',(C,X,C \xrightarrow{g} F(X)),C' \xrightarrow{c} C)$ in $\mathcal{C}{\downarrow}\mathrm{pr}_1$, in a diagram

$$\begin{tikzcd}
 & & & E \arrow[dd, mapsto, "F"] \arrow[dr, bend left, "e"] & \\
 & & & & X \arrow[dd, mapsto, "F"] \\
A' \arrow[r, "a"'] \arrow[dr, bend right, "j"'] & A \arrow[rr, "f"'] \arrow[dr, bend right, "i"'] & & F(E) \arrow[dr, "F(e)"] & \\
 & C' \arrow[r, "c"'] & C \arrow[rr, "g"'] & & F(X),
\end{tikzcd}$$

by Construction 3.2.3 and equation (3.11) we know that $Q_{\mathrm{pr}_1}(j,(i,e))$ is the unique morphism which fits well in the diagram

$$\begin{tikzcd}
(A',E,A' \xrightarrow{fa} F(E)) \arrow[rr, "{(a,\mathrm{id})}"] \arrow[dd, mapsto, "p"] \arrow[dr, dashed, "{Q_{\mathrm{pr}_1}(j,(i,e))}"] & & (A,E,A \xrightarrow{f} F(E)) \arrow[dr, bend left, "{(i,e)}"] \arrow[dd, mapsto, "p"] & \\
 & (C',X,C' \xrightarrow{gc} F(X)) \arrow[rr, "{(c,\mathrm{id})}"] \arrow[dd, mapsto, "p"] & & (C,X,C \xrightarrow{g} F(X)) \arrow[dd, mapsto, "p"] \\
A' \arrow[rr, "a"'] \arrow[dr, bend right, "j"'] & & A \arrow[dr, "H_0(F(e))"] & \\
 & C' \arrow[rr, "c"'] & & C
\end{tikzcd}$$

in the place of the morphism in violet. Therefore

$$Q_{\mathrm{pr}_1}(j,(i,e)) = (j,e)$$

by uniqueness, since this is an acceptable choice.
We then get that, on morphisms, (ALG2*) means that for every $(j,(i,e))$ morphism in $\mathcal{C}{\downarrow}\mathrm{pr}_1$ as above

$$\alpha_1(j,e) = \alpha_1(j,\alpha_1(i,e)).$$

We can now describe the 1-morphisms of T-algebras.
By Remark 1.11.1, given two T-algebras (F,α) and (G,β), that is,

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D}\downarrow G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D}, \end{tikzcd}$$

a morphism of T-algebras from (F,α) to (G,β) is a morphism

$$H = (H_0, H_1)\colon \begin{matrix} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{matrix} \to \begin{matrix} \mathcal{Y} \\ \downarrow G \\ \mathcal{D} \end{matrix}$$

in $\mathcal{CAT}^{\to}$, and we will sometimes write just a square $\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd}$, such that

(MorAlg)
$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "(H_0,H_1)"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{D}\downarrow G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D} \end{tikzcd}$$

(we are also using Remark 3.2.8 to say that $T(H) = (H_0, H_1)$). The axiom (MorAlg) is obviously equivalent to

(MorAlg*)
$$\mathcal{C}\downarrow F \xrightarrow{\alpha_1} \mathcal{X} \xrightarrow{H_1} \mathcal{Y} \quad = \quad \mathcal{C}\downarrow F \xrightarrow{(H_0,H_1)} \mathcal{D}\downarrow G \xrightarrow{\beta_1} \mathcal{Y}$$

Finally, given $H, K\colon (F,\alpha) \to (G,\beta)$ two 1-morphisms of algebras, by Remark 1.11.1 a 2-morphism of algebras is a 2-cell $\lambda\colon H \Rightarrow K : F \to G$ in $\mathcal{CAT}^{\to}$, which means that

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd}$$

holds, such that the following additional property holds as well:

$$\textbf{(2MorAlg)}\quad \begin{tikzcd} \mathcal{C}{\downarrow}F \arrow[r, bend left, "{(H_0,H_1)}"] \arrow[r, bend right, "{(K_0,K_1)}"'] \arrow[r, phantom, "\Downarrow(\lambda_0,\lambda_1)"] & \mathcal{D}{\downarrow}G \arrow[r, "\beta_1"] & \mathcal{Y} \end{tikzcd} = \begin{tikzcd} \mathcal{C}{\downarrow}F \arrow[r, "\alpha_1"] & \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] & \mathcal{Y} \end{tikzcd}$$

Here we have used the equivalent characterization (MorAlg*) of the axiom (MorAlg) and Remark 3.2.8 to say that $(T(\lambda))_1 = (\lambda_0, \lambda_1)$.

Notation 3.3.3. From now on, we will denote T the 2-monad generated by the 2-adjunction $L \dashv U$. We will then denote $T-\mathcal{Alg}$ the 2-category of T-algebras.

Now, aiming at proving that the forgetful functor $U\colon \mathcal{SpFIB} \to \mathcal{CAT}^{\to}$ is monadic, we consider the canonical comparison 2-functor $\overline{U}\colon \mathcal{SpFIB} \to T-\mathcal{Alg}$. We will then show that it is an isomorphism of categories.

Remark 3.3.4. By Theorem 1.11.6 (and Definition 1.11.7), the comparison 2-functor associated to $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}$ and to the monad T

$$\begin{tikzcd} \mathcal{SpFIB} \arrow[r, dashed, "\overline{U}"] \arrow[dr, "U"'] & T-\mathcal{Alg} \arrow[d, "U^{\mathrm{T}}"] \\ & \mathcal{CAT}^{\to} \end{tikzcd}$$

is given by

$$\overline{U}\colon \mathcal{SpFIB} \to T-\mathcal{Alg}$$

$$\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix} \longmapsto \left(\begin{matrix}\mathcal{E}\\ \downarrow U(p)\\ \mathcal{B}\end{matrix}\,,\quad \begin{matrix}\mathcal{B}\downarrow p\\ \downarrow \mathrm{pr}_1\\ \mathcal{B}\end{matrix} \xrightarrow{(U\varepsilon)_p} \begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}\right)$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd}$$

We also know, by Remark 3.2.8, that $(U\varepsilon)_p$ is given by the square

$$\begin{tikzcd} \mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B} \end{tikzcd}$$

Thus we have already found a canonical way to associate to every object, 1-morphism and 2-morphism in $\mathcal{S}p\mathcal{FIB}$ respectively an algebra for the monad T, a 1-morphism of algebras and a 2-morphism of algebras.
Note that Theorem 1.11.6 ensures that a 2-morphism of split fibrations, that is, a mere modification, is a 2-morphism of T-algebras. We will see an explicit justification of this fact in Proposition 3.3.9, but first we will need to show that a T-algebra is a split fibration.

We now want to show that if we start from an algebra for the monad T we can construct a split fibration from it. After this, we will consider 1-morphisms and 2-morphisms of algebras.

Construction 3.3.5. Let

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}$$

be a T-algebra. We show that the structure of algebra on the functor $\begin{smallmatrix} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{smallmatrix}$ makes F into a split fibration. Firstly, we have to find a cleavage for F.

We then start from a diagram

$$\begin{tikzcd} & E \arrow[d, mapsto, "F"] \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

and we search for a cartesian lifting of f to E. To do this, one might think to apply the unit η_F of the 2-adjunction $L \dashv U$ to the diagram above

$$\begin{tikzcd} & E \arrow[d, mapsto, "F"] & & (A, E, A \xrightarrow{f} F(E)) \arrow[r, "(f,\mathrm{id})"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[d, mapsto, "\mathrm{pr}_1"] \\ A \arrow[r, "f"'] & F(E) & {} \arrow[r, mapsto, "(\eta_F)_1", "\mathrm{Id}"'] & A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

and find the data in violet using the fact that pr_1 is a cloven fibration. We now recall that axiom (ALG1) of algebra for the monad T (see Remark 3.3.2) implies that $\alpha_1 \circ (\eta_F)_1 = \mathrm{id}_F$. Therefore applying (Id, α_1) to the diagram above on the right we restore our starting data

$$\begin{tikzcd}
\alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and we see that we have produced, in addition, a morphism above f (in violet). Thus, it suffices to prove that this morphism $\alpha_1(f, \mathrm{id})$ is cartesian.

Consider then a diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "k"] \arrow[dd, mapsto, "F"] & & \\
 & \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd} \tag{3.12}$$

with arbitrary $X \in \mathcal{X}$, $a\colon F(X) \to A$ in $\mathcal{C}$ and $k\colon X \to E$ above fa. We would like to show that there is a unique morphism $v\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ which fits well into diagram (3.12). To find a suitable such morphism v we repeat the same argument of above: we apply $(\mathrm{Id}, (\eta_F)_1)$ and then (Id, α_1) to diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "k"] \arrow[dd, mapsto, "F"] & & \\
 & & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd} \tag{3.13}$$

in order to restore the data of diagram (3.12) but trying to find some further data in the middle step, using the fact that $\begin{array}{c} \mathcal{C}\downarrow F \\ \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathcal{C} \end{array}$ is a cloven fibration.

If we apply $(\mathrm{Id},(\eta_F)_1)$ to diagram (3.13) we get the diagram in black

$$
\begin{tikzcd}
(F(X),X,F(X)\overset{\mathrm{id}}{\to}F(X)) \arrow[dr, dashed, "(a,k)"] \arrow[drr, bend left, "(fa,k)"] \arrow[dd, "\mathrm{pr}_1"] & & \\
& (A,E,A\overset{f}{\to}F(E)) \arrow[r, "(f,\mathrm{id})"] \arrow[dd, "\mathrm{pr}_1"] & (F(E),E,F(E)\overset{\mathrm{id}}{\to}F(E)) \arrow[dd, "\mathrm{pr}_1"] \\
F(X) \arrow[dr, "a"'] \arrow[drr, bend left, "fa"] & & \\
& A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{3.14}
$$

from which we can produce the morphism (f,id) of the cleavage of $\begin{array}{c}\mathcal{C}\!\downarrow\! F\\ \downarrow{\scriptstyle \mathrm{pr}_1}\\ \mathcal{C}\end{array}$ as above, and we also know that (a,k) is the unique morphism from $(F(X),X,F(X)\overset{\mathrm{id}}{\to}F(X))$ to $(A,E,A\overset{f}{\to}F(E))$ which fits well into diagram (3.14) in the place of the morphism in violet on the left, since it is an acceptable choice and (f,id) is cartesian with respect to pr_1. Applying (Id,α_1) to diagram (3.14) we then get

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\alpha_1(a,k)"] \arrow[drr, bend left, "k"] \arrow[dd, "F"] & & \\
& \alpha_1(A,E,A\overset{f}{\to}F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[dd, "F"] & E \arrow[dd, "F"] \\
F(X) \arrow[dr, "a"'] \arrow[drr, bend left, "fa"] & & \\
& A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{3.15}
$$

using again the axiom (ALG1) of algebra for the monad T (from Remark 3.3.2), as above, to get that $\alpha_1\circ(\eta_F)_1=\mathrm{id}_F$.

Therefore we have found a morphism $v\colon X\to\alpha_1(A,E,A\overset{f}{\to}F(E))$ which fits well into diagram (3.12). It remains to prove that $\alpha_1(a,k)$ is the unique such morphism v. In order to prove this, we might see existence and uniqueness of this map v connected to a bijection of hom-sets, which reveals to be an adjunction, as we now prove.

Proposition 3.3.6. *Given*

$$
\begin{tikzcd}
\mathcal{C}\!\downarrow\! F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C}
\end{tikzcd}
$$

a T-algebra, we have that $\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "(\eta_F)_1"] \arrow[r, phantom, "\perp"] & \mathcal{C}\downarrow F \arrow[l, bend left, "\alpha_1"] \end{tikzcd}$ *is an adjunction.*

Proof. We first prove that for every $X \in \mathcal{X}$ and every $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$ there is a bijection

$$\mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E))) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E))) \tag{3.16}$$

by constructing the functions φ and ψ and proving that they are inverses of each other.

We can construct the map φ as follows. Given a morphism

$$(a, e) \colon (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$$

in $C\downarrow F$, we define

$$\varphi(a, e) := \alpha_1(a, e).$$

We have that $\alpha_1(a, e)$ is a morphism from X to $\alpha_1(A, E, A \xrightarrow{f} F(E))$ by axiom (ALG1) of Remark 3.3.2.
We then see that $\varphi(a, e)$ fits well in the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "e"] \arrow[dr, dashed, "\varphi(a,e)"] \arrow[dd, mapsto, "F"] & & \\
& \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd} \tag{3.17}$$

that is, $F(\varphi(a, e)) = a$ and $\alpha_1(f, \mathrm{id}) \circ \varphi(a, e) = e$.

Now, we construct the map ψ. Given a morphism $q \colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ in $\mathcal{X}$, we see that q fits well in the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "\alpha_1(f,\mathrm{id})\circ q"] \arrow[dr, "q"] \arrow[dd, mapsto, "F"] & & \\
& \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "F(q)"'] \arrow[rr, bend left, "f\circ F(q)"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

We then define

$$\psi(q) := (F(q), \alpha_1(f, \mathrm{id}) \circ q),$$

so that it surely holds that $\psi \circ \varphi = \mathrm{id}$, by figure 3.17:

$$\psi(\varphi(a, e)) = (F(\varphi(a, e)), \alpha_1(f, \mathrm{id}) \circ \varphi(a, e)) = (a, e)$$

for every morphism (a, e) in $\mathcal{C}{\downarrow}F$.

To prove that we have the bijection of equation (3.16), it then remains to show that $\varphi \circ \psi = \mathrm{id}$. To do this, we need to use the fact that α_1 is a full functor, by axiom (ALG1) of Remark 3.3.2. Starting from a morphism $q\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ in $\mathcal{X}$, we can see it as

$$q\colon \alpha_1(F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow \alpha_1(A, E, A \xrightarrow{f} F(E)),$$

by axiom (ALG1) of Remark 3.3.2.
Then, by fullness of α_1, there exist a morphism

$$(k, l)\colon (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$$

in $\mathcal{C}{\downarrow}F$ such that

$$q = \alpha_1(k, l).$$

And we notice that $k = \mathrm{pr}_1(k, l) = F(\alpha_1(k, l)) = F(q)$, by using the fact that (Id, α_1) is the structure morphism of an algebra.

Therefore, by using axiom (ALG2*) on morphisms, we see that

$$\varphi(\psi(q)) = \varphi(F(q), \alpha_1(f, \mathrm{id}) \circ q) = \alpha_1(F(q), \alpha_1(f, \mathrm{id}) \circ q) = \alpha_1(F(q), \alpha_1(f, \mathrm{id}) \circ \alpha_1(k, l)) =$$
$$= \alpha_1(F(q), \alpha_1(f, \mathrm{id}) \circ \alpha_1(F(q), l)) = \alpha_1(F(q), \alpha_1(f \circ F(q), l)) = \alpha_1(F(q), l) = \alpha_1(k, l) = q.$$

By arbitrariness of q, it follows that $\varphi \circ \psi = \mathrm{id}$.

Finally, we prove naturality. We write $\varphi_{X,(A,E,f)}$ to specify that it is the morphism φ constructed starting from $X \in \mathcal{X}$ and $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}{\downarrow}F$. Given a morphism $g\colon Y \to X$ in $\mathcal{X}$ and a morphism $(w, k)\colon (A, E, A \xrightarrow{f} F(E)) \longrightarrow (A', E', A' \xrightarrow{f'} F(E'))$, we need to show the commutativity of the square

$$\begin{tikzcd}
\mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E))) \arrow[r, "\varphi_{X,(A,E,f)}"] \arrow[d, "\mathrm{Hom}((\eta_F)_1(g),(w,k))"'] & \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E))) \arrow[d, "\mathrm{Hom}(g,\alpha_1(w,k))"] \\
\mathrm{Hom}((\eta_F)_1(Y), (A', E', A' \xrightarrow{f'} F(E'))) \arrow[r, "\varphi_{Y,(A',E',f')}"] & \mathrm{Hom}(Y, \alpha_1(A', E', A' \xrightarrow{f'} F(E')))
\end{tikzcd} \tag{3.18}$$

Given $(a, e)\colon (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$ in $\mathcal{C}{\downarrow}F$, the path on the right of diagram (3.18) corresponds to

$$(a, e) \longmapsto \alpha_1(a, e) \longmapsto \alpha_1(w, k) \circ \alpha_1(a, e) \circ g$$

whereas the path on the left corresponds to

$$(a, e) \longmapsto (w, k) \circ (a, e) \circ (\eta_F)_1(g) \longmapsto \alpha_1(w, k) \circ \alpha_1(a, e) \circ \alpha_1((\eta_F)_1(g)).$$

Therefore the two paths of diagram (3.18) gives the same result, by axiom (ALG1) of Remark 3.3.2, which means that diagram (3.18) is commutative. $\square$

Corollary 3.3.7. *Each functor $F\colon X \to \mathcal{C}$ can have at most one structure of algebra, up to (a unique) isomorphism.*

Proof. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Recall from Construction 3.3.2 that a structure of T-algebra for F is a square

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}$$

Since both Id and α_1 are adjoints, respectively to Id and $(\eta_F)_1$, it follows from uniqueness of adjoints (up to a unique isomorphism) that F can have at most one structure of algebra, up to (a unique) isomorphism. $\square$

We are now ready to prove that the structure of a T-algebra (F, α) defines a structure of a split fibration for F. We will say that a T-algebra is a split fibration.

Theorem 3.3.8. *Let*

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}$$

be a T-algebra. Then the assignment

$$\mathrm{Cart}\,(f, E) := \alpha_1(f, \mathrm{id})\colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \longrightarrow E \tag{3.19}$$

for every diagram

$$\begin{tikzcd} & E \arrow[d, mapsto, "F"] \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

with $E \in \mathcal{X}$ and f a morphism in $\mathcal{C}$, which we have produced in Construction 3.3.5, defines a splitting cleavage for F.

Proof. We prove that the assignment in equation (3.19) defines a cleavage for F. After what we have shown in Construction 3.3.5, it remains to prove that $\alpha_1(a, k)$ is the only morphism which fits well into diagram (3.15), which we write again below, in the place of the morphism in violet:

$$
\begin{tikzcd}
X \arrow[d, "F"] \arrow[rr, bend left, "k"] \arrow[r, dashed, "\alpha_1(a,k)"] & \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, "F"] & E \arrow[d, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{3.20}
$$

In fact we will then conlcude by arbitrariness of k and a that $\mathrm{Cart}\,(f, E) = \alpha_1(f, \mathrm{id})$ is a cartesian morphism.

Let then $q\colon X \longrightarrow (A, E, A \xrightarrow{f} F(E))$ be another morphism which fits well into diagram (3.20) in the place of the morphism in violet.
We have seen in the proof of Proposition 3.3.6 that the maps φ and ψ constructed in that proof are inverses of each other and yield a natural bijection

$$\mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E))) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E)))$$

(corresponding to equation (3.16)).

We then appy ψ to q and get

$$\psi(q) = (F(q), \alpha_1(f, \mathrm{id}) \circ q) = (a, k)$$

by hypothesis, whence we obtain

$$q = \varphi(\psi(q)) = \varphi(a, k) = \alpha_1(a, k).$$

Therefore we have proved that the assignment in equation (3.19) defines a cleavage for F. We show that it makes F into a split fibration.

We first see that F, with the constructed cleavage, is a normal fibration. We need to show that for every $E \in \mathcal{X}$ we have that

$$\mathrm{Cart}\left(\mathrm{id}_{F(E)}, E\right) = \mathrm{id}_E\colon E \to E.$$

But this holds because if $E \in \mathcal{X}$ then

$$
\begin{aligned}
&\mathrm{Cart}\left(\mathrm{id}_{F(E)}, E\right) = \left(\alpha_1(\mathrm{id}, \mathrm{id})\colon \alpha_1(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \longrightarrow E\right) = \\
&= (\alpha_1(\mathrm{id}, \mathrm{id})\colon \alpha_1((\eta_F)_1(E)) \longrightarrow E) = (\mathrm{id}\colon E \longrightarrow E),
\end{aligned}
$$

by the axiom (ALG1) of T-algebra, from Remark 3.3.2.

Now, in order to prove that F is split, given $E \in \mathcal{X}$ and given $f\colon A \to F(E)$ and $a\colon A' \to A$ in $\mathcal{C}$, we prove that

$$\mathrm{Cart}\,(f, E) \circ \mathrm{Cart}\,(a, f^*E) = \mathrm{Cart}\,(fa, E).$$

Then we need to prove the equality in the diagram

$$\begin{tikzcd}
\alpha_1(A', E, A' \xrightarrow{fa} F(E)) \arrow[d, equal] \arrow[rrdd, bend left, "\alpha_1(fa,\mathrm{id})"] & & \\
\alpha_1(A', \alpha_1(A, E, A \xrightarrow{f} F(E)), A' \xrightarrow{a} A) \arrow[rd, "\alpha_1(a,\mathrm{id})"] \arrow[dd, mapsto, "F"] & & \\
& \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and verify that it is a commutative diagram. But the equality is given by axiom (ALG2*) of T-algebra (from Remark 3.3.2) on objects, and the commutativity follows by the fact that α_1 is a functor from $\mathcal{C}{\downarrow}F$ to $\mathcal{X}$. $\square$

Now we have all the instruments we need to give an explicit justification to the fact that a mere modification is a 2-morphism of algebras.

Proposition 3.3.9. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{X} \arrow[r, "K_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "K_0"'] & \mathcal{D}
\end{tikzcd}$$

be two 1*-morphisms of T-algebras, where F and G have structures of T-algebra given by*

$$\begin{tikzcd}
\mathcal{C}{\downarrow}F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{D}{\downarrow}G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D}
\end{tikzcd}$$

Let

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd}$$

be a modification, that is, a 2-cell in $\mathcal{CAT}^{\to}$. *Then* (λ_0, λ_1) *automatically satisfies axiom* (2MorAlg) (*from Remark 3.3.2*), *that is,* (λ_0, λ_1) *is a 2-morphism of* T*-algebras.*

Proof. Given $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C} \downarrow F$, we prove that

$$(\beta_1(\lambda_0, \lambda_1))_{(A,E,A \xrightarrow{f} F(E))} = (\lambda_1 \alpha_1)_{(A,E,A \xrightarrow{f} F(E))}.$$

We have that

$$(\beta_1(\lambda_0, \lambda_1))_{(A,E,A \xrightarrow{f} F(E))} = \beta_1(\lambda_{0,A}, \lambda_{1,E}) \quad \text{and} \quad (\lambda_1 \alpha_1)_{(A,E,A \xrightarrow{f} F(E))} = \lambda_{1,\alpha_1(A,E,A \xrightarrow{f} F(E))}.$$

To prove that they are equal, it suffices to show that both these morphisms fits well in the diagram

$$\begin{tikzcd} H_1(\alpha_1(A, E, A \xrightarrow{f} F(E))) \arrow[rr, "H_1(\alpha_1(f{,}\mathrm{id}))", "\beta_1(H_0(f){,}\mathrm{id})"'] \arrow[dd, "G"] \arrow[dr, dashed, "\lambda_{1,\alpha_1(A,E,A \xrightarrow{f} F(E))}", "\beta_1(\lambda_{0,A}{,}\lambda_{1,E})"'] & & H_1(E) \arrow[dr, bend left, "\lambda_{1,E}"] \arrow[dd, "G"] & \\ & K_1(\alpha_1(A, E, A \xrightarrow{f} F(E))) \arrow[rr, "K_1(\alpha_1(f{,}\mathrm{id}))", "\beta_1(K_0(f){,}\mathrm{id})"'] \arrow[dd, "G"] & & K_1(E) \arrow[dd, "G"] \\ H_0(A) \arrow[rr, "H_0(f)"'] \arrow[dr, bend right, "\lambda_{0,A}"'] & & H_0(F(E)) \arrow[dr, "G(\lambda_{1,E})", "\lambda_{0,F(E)}"'] & \\ & K_0(A) \arrow[rr, "K_0(f)"'] & & K_0(F(E)) \end{tikzcd} \tag{3.21}$$

in the place of the morphism in violet, where we have written two labels for the same arrow in black whenever they are equal morphisms, since

$$\beta_1(K_0(f), \mathrm{id}) = \mathrm{Cart}_G(K_0(f), K_1(E))$$

is a cartesian morphism, by Theorem 3.3.8. For clarity, all the equalities of diagram (3.21) are easily justified by using axiom (MorAlg*) (from Remark 3.3.2) and the fact that (λ_0, λ_1) is a modification.

The morphism $\lambda_{1,\alpha_1(A,E,A\xrightarrow{f}F(E))}$ fits well in diagram (3.21) because

$$G\left(\lambda_{1,\alpha_1(A,E,A\xrightarrow{f}F(E))}\right) = \lambda_{0,F(\alpha_1(A,E,A\xrightarrow{f}F(E)))} = \lambda_{0,A}$$

(by the fact that (λ_0, λ_1) is a modification) and because it makes the upper square of diagram (3.21) commute by naturality of λ_1.

The morphism $\beta_1(\lambda_{0,A}, \lambda_{1,E})$ satisfies

$$G\left(\beta_1(\lambda_{0,A}, \lambda_{1,E})\right) = \mathrm{pr}_1(\lambda_{0,A}, \lambda_{1,E}) = \lambda_{0,A},$$

by the fact that $(G, (\mathrm{Id}, \beta_1))$ is a T-algebra. Then it only remains to prove that $\beta_1(\lambda_{0,A}, \lambda_{1,E})$ makes the upper square of diagram (3.21) commute. To show this, it is useful to see that

$$\lambda_{1,E} = \beta_1(\lambda_{0,F(E)}, \lambda_{1,E}),$$

by axiom (ALG1) (from Remark 3.3.2) for $(G, (\mathrm{Id}, \beta_1))$. From this, we obtain that

$$\lambda_{1,E} \circ \beta_1(H_0(f), \mathrm{id}) = \beta_1(\lambda_{0,F(E)} \circ H_0(f), \lambda_{1,E}),$$

whereas

$$\beta_1(K_0(f), \mathrm{id}) \circ \beta_1(\lambda_{0,A}, \lambda_{1,E}) = \beta_1(K_0(f) \circ \lambda_{0,A}, \lambda_{1,E}).$$

We then conclude by noting that

$$\begin{tikzcd}
H_0(A) \arrow[r, "\lambda_{0,A}"] \arrow[d, "H_0(f)"'] & K_0(A) \arrow[d, "K_0(f)"] \\
H_0(F(E)) \arrow[r, "\lambda_{0,F(E)}"'] & K_0(F(E))
\end{tikzcd}$$

by naturality of λ_0. □

Recall from Remark 3.3.4 that aiming at proving the monadicity of split fibrations, now that we have shown that a T-algebra is a split fibration, it remains to show that 1-morphisms and 2-morphisms of algebras are respectively 1-morphisms and 2-morphisms of split fibrations (that is, of cloven fibrations).

Proposition 3.3.10. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}$$

be a 1*-morphism of algebras between two T-algebras*

$$
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{D}\downarrow G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D}
\end{tikzcd}
$$

Consider F and G as split fibrations by using Theorem 3.3.8. Then (H_0, H_1) is a cloven cartesian functor.

Proof. We need to prove that for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$

$$H_1(\mathrm{Cart}_F(f, E)) = \mathrm{Cart}_G(H_0(f), H_1(E)). \tag{3.22}$$

Given $E \in \mathcal{X}$ and $f\colon A \to F(E)$ in $\mathcal{C}$, we have that

$$\mathrm{Cart}_F(f, E) = \alpha_1(f, \mathrm{id}_E)\colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \to E$$

by construction (see Theorem 3.3.8 and Construction 3.3.5). Then

$$H_1(\mathrm{Cart}_F(f, E)) = H_1(\alpha_1(f, \mathrm{id}_E)).$$

But (H_0, H_1) is a 1-morphism of T-algebras, which means that it satisfies axiom (MorAlg) (from Remark 3.3.2), which we write again here:

(MorAlg)
$$
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, "(H_0,H_1)"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{D}\downarrow G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D}
\end{tikzcd}
$$

It follows that

$$
\begin{aligned}
&H_1(\mathrm{Cart}_F(f, E)) = H_1(\alpha_1(f, \mathrm{id}_E)) = \beta_1((H_0, H_1)(f, \mathrm{id}_E)) = \beta_1(H_0(f), H_1(\mathrm{id})) = \\
&= \beta_1(H_0(f), \mathrm{id}_{H_1(E)}) = \mathrm{Cart}_G(H_0(f), H_1(E)),
\end{aligned}
$$

which shows that equation (3.34) holds. □

It now remains to consider 2-morphisms of T-algebras and prove that they are 2-morphisms of split fibrations.

Proposition 3.3.11. *Let*

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D}
\end{tikzcd}
$$

be a 2-morphism of T-algebras, where F and G have structures of algebra given by

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{D}\downarrow G \arrow[r, "\beta_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{D} \arrow[r, "\mathrm{Id}"'] & \mathcal{D} \end{tikzcd}$$

Consider F and G as split fibrations by using Theorem 3.3.8, and consider (H_0, H_1) and (K_0, K_1) as cloven cartesian functors by using Proposition 3.3.10. Then (λ_0, λ_1) is a 2-morphisms of split fibrations.

Proof. The proof is trivial since we have defined a 2-morphism of split fibrations to be just a 2-cell in $\mathcal{CAT}^{\rightarrow}$. □

Remark 3.3.12. Notice that the proof of Proposition 3.3.11 is trivial because the structure of a split fibration is related to 1-morphisms only. Notice also that a 2-morphism of T-algebras does not actually have more structure than a modification, by Proposition 3.3.9.

Proposition 3.3.13. *Theorem 3.3.8, Proposition 3.3.10 and Proposition 3.3.11 jointly yield a 2-functor*

$$W\colon T-\mathcal{Alg} \longrightarrow \mathcal{SpFIB}$$

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd}$$

where $\begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$ *has the splitting cleavage defined by*

$$\mathrm{Cart}\,(f,E) := \alpha_1(f,\mathrm{id})\colon \alpha_1(A,E,A \xrightarrow{f} F(E)) \to E$$

for every $E \in \mathcal{X}$ *and every* $f\colon A \to F(E)$ *in* $\mathcal{C}$.

Proof. The proof is trivial. $\square$

Theorem 3.3.14. *The comparison 2-functor* $\overline{U}\colon \mathcal{SpFIB} \longrightarrow T\text{–}\mathcal{Alg}$ *associated to the* 2*-functor* $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}$ *and to the 2-monad* T

$$\begin{tikzcd} \mathcal{SpFIB} \arrow[r, dashed, "\overline{U}"] \arrow[dr, "U"'] & T-\mathcal{Alg} \arrow[d, "U^{\mathrm{T}}"] \\ & \mathcal{CAT}^{\to} \end{tikzcd}$$

is an isomorphism of categories. In particular, $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}$ *is monadic.*

Proof. We prove that the 2-functor

$$W\colon T-\mathcal{Alg} \longrightarrow \mathcal{SpFIB}.$$

produced in Proposition 3.3.13 is an inverse 2-functor of

$$\overline{U}\colon \mathcal{SpFIB} \longrightarrow T-\mathcal{Alg}.$$

We first prove that $\overline{U}$ and W are inverses of each other on objects.

Given a split fibration $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}$, with cleavage denoted $\mathrm{Cart}_p(-,\cdot)$, we prove that

$$W\left(\overline{U}\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}\right)\right) = \begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}$$

We have that

$$W\left(\overline{U}\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}\right)\right) = W\left(\begin{tikzcd} \mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B} \end{tikzcd}\right)$$

by Remark 3.3.4, and

$$W\left(\begin{tikzcd} \mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B} \end{tikzcd}\right)$$

coincides with $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ equipped with the splitting clevage defined by

$$\mathrm{Cart}\,(u,E) := Q_p(u,\mathrm{id})\colon Q_p(B,E,B \xrightarrow{u} p(E)) \longrightarrow E$$

for every $E \in \mathcal{E}$ and every $u\colon B \to p(E)$ in $\mathcal{B}$, by Proposition 3.3.13.
But this cleavage coincides with the original one $\mathrm{Cart}_p(-,\cdot)$, since

$$(B,E,B \xrightarrow{u} p(E)) \overset{Q_p}{\longmapsto} u^*E, \qquad (p(E),E,p(E) \xrightarrow{\mathrm{id}} p(E)) \overset{Q_p}{\longmapsto} E$$

and

$$\begin{tikzcd}
 & E \arrow[dd, mapsto, "p"] \arrow[dr, bend left, "\mathrm{id}"] & & & & u^*E \arrow[rr, "\mathrm{Cart}_p(u,E)"] \arrow[dr, dashed, "Q_p(u,\mathrm{id})"'] \arrow[dd, mapsto, "p"] & & E \arrow[dr, bend left, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & \\
 & & E \arrow[dd, mapsto, "p"] & & & & E \arrow[rr, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & & E \arrow[dd, mapsto, "p"] \\
B \arrow[r, "u"'] \arrow[dr, bend right, "u"'] & p(E) \arrow[dr, "\mathrm{id}"] & & \overset{Q_p}{\longmapsto} & & B \arrow[rr, "u"'] \arrow[dr, bend right, "u"'] & & p(E) \arrow[dr, "\mathrm{id}"] & \\
 & p(E) \arrow[r, "\mathrm{id}"'] & p(E) & & & & p(E) \arrow[rr, "\mathrm{id}"'] & & p(E)
\end{tikzcd}$$

by Construction 3.2.3 (recall that Q_p denotes the functor $(Q_{\mathrm{Id}_p})_1$ produced in Construction 3.2.3, with $H = \mathrm{Id}_p$, after Remark 3.2.8).

Now, given

$$\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C}
\end{tikzcd}$$

a T-algebra, we prove that

$$\overline{U}\left(W\left(\begin{array}{ccc} \mathcal{C}\downarrow F & \xrightarrow{\alpha_1} & \mathcal{X} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow[\mathrm{Id}]{} & \mathcal{C} \end{array}\right)\right) = \begin{array}{ccc} \mathcal{C}\downarrow F & \xrightarrow{\alpha_1} & \mathcal{X} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow[\mathrm{Id}]{} & \mathcal{C} \end{array}$$

By Proposition 3.3.13,

$$W\left(\begin{array}{ccc} \mathcal{C}\downarrow F & \xrightarrow{\alpha_1} & \mathcal{X} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow[\mathrm{Id}]{} & \mathcal{C} \end{array}\right)$$

coincides with $\begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$ equipped with the splitting cleavage defined by

$$\operatorname{Cart}(f,E) := \alpha_1(f,\mathrm{id})\colon \alpha_1(A,E,A \xrightarrow{f} F(E)) \to E \tag{3.23}$$

for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$. Then

$$\overline{U}\left(W\left(\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}\right)\right) = \begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "Q_F"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}$$

and it remains to prove that $Q_F = \alpha_1$.

On objects $(A,E,A \xrightarrow{f} F(E)) \in \mathcal{C}\downarrow F$, we have that

$$Q_F(A,E,A \xrightarrow{f} F(E)) = f^*E = \alpha_1(A,E,A \xrightarrow{f} F(E)),$$

by equation (3.23).

On morphisms $(A',E',A' \xrightarrow{f'} F(E')) \xrightarrow{(a,e)} (A,E,A \xrightarrow{f} F(E))$ in $\mathcal{C}\downarrow F$, we have that Q_F acts as

$$\begin{tikzcd} & E' \arrow[d, "F"] \arrow[r, bend left, "e"] & E \arrow[d, "F"] \\ A' \arrow[r, "f'"'] \arrow[d, bend right, "a"'] & F(E') \arrow[dr, "F(e)"] & F(E) \\ A \arrow[rr, "f"'] & & F(E) \end{tikzcd} \overset{Q_F}{\longmapsto} \begin{tikzcd} \alpha_1(A',E',A' \xrightarrow{f'} F(E')) \arrow[rr, "\alpha_1(f',\mathrm{id})"] \arrow[dd, "F"] \arrow[dr, dashed, "Q_F(a,e)"] & & E' \arrow[dr, bend left, "e"] \arrow[dd, "F"] & \\ & \alpha_1(A,E,A \xrightarrow{f} F(E)) \arrow[rr, "\alpha_1(f,\mathrm{id})"] \arrow[dd, "F"] & & E \arrow[dd, "F"] \\ A' \arrow[rr, "f'"'] \arrow[dr, bend right, "a"'] & & F(E') \arrow[dr, "F(e)"] & \\ & A \arrow[rr, "f"'] & & F(E) \end{tikzcd}$$

But $\alpha_1(f,\mathrm{id})$ is a cartesian morphism and also $\alpha_1(a,e)$ fits well in the place of the morphism in violet in the diagram above on the right. In fact

$$F(\alpha_1(a,e)) = \mathrm{pr}_1(a,e) = a,$$

by the fact that $(F,(\mathrm{Id},\alpha_1))$ is a T-algebra, and

$$\alpha_1(f,\mathrm{id}) \circ \alpha_1(a,e) = \alpha_1(fa,e) = \alpha_1(F(e)\circ f',e) = \alpha_1(F(e),e)\circ\alpha_1(f',\mathrm{id}) = e \circ \alpha_1(f',\mathrm{id})$$

by using axiom (ALG1) of T-algebra (from Remark 3.3.2).
Therefore Q_F and α_1 need to coincide on morphisms and then they are equal as functors, whence $\overline{U}\circ W$ coincides with the identity functor on objects.

We have thus proved that $\overline{U}$ and W are inverses of each other on objects. But on 1-morphisms and on 2-morphisms both $\overline{U}$ and W are essentially defined as the identity, by Remark 3.3.4 and Proposition 3.3.13 (see also Proposition 3.3.9 to have an explicit proof that a 2-morphism of T-algebras is the same thing of a 2-morphism of fibrations, that is, a mere modification). Therefore we immediatly get that

$$W\circ\overline{U} = \mathrm{Id}_{\mathcal{SpFIB}} \quad \text{and} \quad \overline{U}\circ W = \mathrm{Id}_{T\text{-}\mathcal{Alg}},$$

which means that the 2-functor $\overline{U}$ is an isomorphism of 2-categories with inverse 2-functor W.

In particular, it trivially follows that the comparison 2-functor $\overline{U}$ is an equivalence of 2-categories, that is, that the 2-functor $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\rightarrow}$ is monadic. $\square$

3.4 Pseudo-T-algebras

In this section we get rid of the quite restrictive hypothesis to start from split fibrations to have an algebraic structure on the fibrations.

We will need to restrict the 2-monad T (which we have described in Remark 3.3.2) to a 2-monad T' on $\mathcal{CAT} \downarrow \mathcal{C}$ (with $\mathcal{C}$ a fixed category), and we will manage to prove that the 2-category $\mathcal{Ps}\text{-}T'\text{-}\mathcal{Alg}$ is exactly the 2-category $\mathcal{CloFIB}_{\mathcal{C}}$ of cloven fibrations over $\mathcal{C}$ and cartesian functors over $\mathcal{C}$.

For this, we will remember what we have shown in the last section to prove that the forgetful 2-functor $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\rightarrow}$ is monadic.

Recall the definition of the 2-category $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ of pseudo-T-algebras from Section 1.12. Now we shall write explicitly such definition for our 2-monad T, also using Remark 3.3.2. Recall also the notation Q_p for $(Q_{\mathrm{Id}_p})_1$, from Remark 3.2.8.

Remark 3.4.1. A **pseudo-T-algebra** is a quadruple $(F, \alpha, \alpha_\mu, \alpha_\eta)$ with $F\colon \mathcal{X} \to \mathcal{C}$ a functor, $\alpha = (\alpha_0, \alpha_1)$

$$\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C}
\end{tikzcd}$$

a 1-cell in $\mathcal{CAT}^{\rightarrow}$, and $\alpha_\mu = ((\alpha_\mu)_0, (\alpha_\mu)_1)$ and $\alpha_\eta = ((\alpha_\eta)_0, (\alpha_\eta)_1)$ invertible 2-cells in $\mathcal{CAT}^{\rightarrow}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swarrow (\alpha_\mu)_0"] & \mathcal{C} \arrow[d, "\alpha_0"] \\
\mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "(\alpha_0{,}\alpha_1)"] \arrow[d, "Q_{\mathrm{pr}_1}"'] \arrow[dr, phantom, "\swarrow (\alpha_\mu)_1"] & \mathcal{C}\downarrow F \arrow[d, "\alpha_1"] \\
\mathcal{C}\downarrow F \arrow[r, "\alpha_1"'] & \mathcal{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{Id}"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\swarrow (\alpha_\eta)_0"] & \mathcal{C} \arrow[d, "\alpha_0"] \\
& \mathcal{C};
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X} \arrow[r, "(\eta_F)_1"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\swarrow (\alpha_\eta)_1"] & \mathcal{C}\downarrow F \arrow[d, "\alpha_1"] \\
& \mathcal{X};
\end{tikzcd}$$

subject to the following two coherence axioms (with two diagrams for each axiom representing the two levels), written in the language of pasting diagrams:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \mathcal{C} \arrow[d, phantom, "\Downarrow(\alpha_\mu)_0"] \arrow[dr, "\alpha_0"] & \\
\mathcal{C} \arrow[r, phantom, "="] \arrow[dr, "\mathrm{Id}"'] & \mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow(\alpha_\mu)_0"] & \mathcal{C} \arrow[d, "\alpha_0"] \\
& \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "="] & \mathcal{C} \arrow[d, "\mathrm{Id}"] \arrow[dr, "\alpha_0"] & \\
\mathcal{C} \arrow[r, "\alpha_0"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{C} \arrow[d, phantom, "\Downarrow(\alpha_\mu)_0"] \arrow[dr, "\alpha_0"] & \mathcal{C} \arrow[l, phantom, "\Leftarrow(\alpha_\mu)_0"] \arrow[d, "\alpha_0"] \\
& \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C}\downarrow \mathrm{pr}_1' \arrow[r, "{(\alpha_0,(\alpha_0,\alpha_1))}"] \arrow[d, "Q_{\mathrm{pr}_1'}"'] \arrow[dr, "{(\mathrm{Id},Q_{\mathrm{pr}_1})}"] & \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[d, phantom, "\Downarrow((\alpha_\mu)_0,(\alpha_\mu)_1)"] \arrow[dr, "{(\alpha_0,\alpha_1)}"] & \\
\mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, phantom, "="] \arrow[dr, "Q_{\mathrm{pr}_1}"'] & \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "{(\alpha_0,\alpha_1)}"] \arrow[d, "Q_{\mathrm{pr}_1}"] \arrow[dr, phantom, "\swarrow(\alpha_\mu)_1"] & \mathcal{C}\downarrow F \arrow[d, "\alpha_1"] \\
& \mathcal{C}\downarrow F \arrow[r, "\alpha_1"'] & \mathcal{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}\downarrow \mathrm{pr}_1' \arrow[r, "{(\alpha_0,(\alpha_0,\alpha_1))}"] \arrow[d, "Q_{\mathrm{pr}_1'}"'] \arrow[dr, phantom, "="] & \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[d, "Q_{\mathrm{pr}_1}"] \arrow[dr, "{(\alpha_0,\alpha_1)}"] & \\
\mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "{(\alpha_0,\alpha_1)}"] \arrow[dr, "Q_{\mathrm{pr}_1}"'] & \mathcal{C}\downarrow F \arrow[d, phantom, "\Downarrow(\alpha_\mu)_1"] \arrow[dr, "\alpha_1"] & \mathcal{C}\downarrow F \arrow[l, phantom, "\Leftarrow(\alpha_\mu)_1"] \arrow[d, "\alpha_1"] \\
& \mathcal{C}\downarrow F \arrow[r, "\alpha_1"'] & \mathcal{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow(\alpha_\mu)_0"] & \mathcal{C} \arrow[d, "\alpha_0"] \\
\mathcal{C} \arrow[ur, "\mathrm{Id}"] \arrow[r, equal] \arrow[r, phantom, bend left, "="] & \mathcal{C} \arrow[r, "\alpha_0"'] & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C} \arrow[r, "\alpha_0"] \arrow[d, phantom, "\Downarrow(\alpha_\eta)_0"] & \mathcal{C} \arrow[d, "\alpha_0"] \\
\mathcal{C} \arrow[ur, "\mathrm{Id}"] \arrow[r, equal] & \mathcal{C} \arrow[ur, equal] \arrow[r, "\alpha_0"'] \arrow[ur, phantom, shift right=3, "="] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "{(\alpha_0,\alpha_1)}"] \arrow[d, "Q_{\mathrm{pr}_1}"] \arrow[dr, phantom, "\swarrow(\alpha_\mu)_1"] & \mathcal{C}\downarrow F \arrow[d, "\alpha_1"] \\
\mathcal{C}\downarrow F \arrow[ur, "{(\mathrm{Id},(\eta_F)_1)}"] \arrow[r, equal] \arrow[r, phantom, bend left, "="] & \mathcal{C}\downarrow F \arrow[r, "\alpha_1"'] & \mathcal{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C}\downarrow \mathrm{pr}_1 \arrow[r, "{(\alpha_0,\alpha_1)}"] \arrow[d, phantom, "\Downarrow((\alpha_\eta)_0,(\alpha_\eta)_1)"] & \mathcal{C}\downarrow F \arrow[d, "\alpha_1"] \\
\mathcal{C}\downarrow F \arrow[ur, "{(\mathrm{Id},(\eta_F)_1)}"] \arrow[r, equal] & \mathcal{C}\downarrow F \arrow[ur, equal] \arrow[r, "\alpha_1"'] \arrow[ur, phantom, shift right=3, "="] & \mathcal{X}
\end{tikzcd}
$$

where the equalities we have written inside the squares and inside the triangles represent identity natural transformations, and we have called pr_1' the functor $\mathrm{pr}_1 \colon \mathcal{C}\downarrow \mathrm{pr}_1 \to \mathcal{C}$.

A second identity axiom follows from these two axioms:

$$\alpha_\mu(\eta_{\mathrm{pr}_1}) = \alpha_\eta \alpha.$$

A **pseudo-morphism of pseudo-T-algebras from** $(F, \alpha, \alpha_\mu, \alpha_\eta)$ **to** $(G, \beta, \beta_\mu, \beta_\eta)$, with

$(F,\alpha,\alpha_\mu,\alpha_\eta)$ and $(G,\beta,\beta_\mu,\beta_\eta)$ pseudo-T-algebras is a pair $(H,\overline{H})$ with $H=(H_0,H_1)$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}$$

a 1-cell in $\mathcal{CAT}^{\to}$ and $\overline{H}=(\overline{H}_0,\overline{H}_1)$ an invertible 2-cell in $\mathcal{CAT}^{\to}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H_0"] \arrow[d, "\alpha_0"'] & \mathcal{D} \arrow[d, "\beta_0"] \arrow[dl, phantom, "\swarrow \overline{H}_0"] \\
\mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}{\downarrow}F \arrow[r, "{(H_0,H_1)}"] \arrow[d, "\alpha_1"'] & \mathcal{D}{\downarrow}G \arrow[d, "\beta_1"] \arrow[dl, phantom, "\swarrow \overline{H}_1"] \\
\mathcal{X} \arrow[r, "H_1"'] & \mathcal{Y}
\end{tikzcd}$$

subject to the following coherence axioms:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H_0"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, "\alpha_0"] & \mathcal{D} \arrow[dr, "\beta_0"] \arrow[d, phantom, "\Downarrow \overline{H}_0"] & \\
\mathcal{C} \arrow[dr, "\alpha_0"'] & \mathcal{C} \arrow[l, phantom, "\Leftarrow (\alpha_\mu)_0"] \arrow[r, "H_0"] \arrow[d, "\alpha_0"] & \mathcal{D} \arrow[d, "\beta_0"] \arrow[dl, phantom, "\swarrow \overline{H}_0"] \\
& \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[r, "H_0"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "="] & \mathcal{D} \arrow[d, "\mathrm{Id}"] \arrow[dr, "\beta_0"] & \\
\mathcal{C} \arrow[r, "H_0"] \arrow[dr, "\alpha_0"'] & \mathcal{D} \arrow[d, phantom, "\Downarrow \overline{H}_0"] \arrow[dr, "\beta_0"] & \mathcal{D} \arrow[l, phantom, "\Leftarrow (\beta_\mu)_0"] \arrow[d, "\beta_0"] \\
& \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}{\downarrow}\mathrm{pr}_1 \arrow[r, "{(H_0,(H_0,H_1))}"] \arrow[d, "Q_{\mathrm{pr}_1}"'] \arrow[dr, "{(\alpha_0,\alpha_1)}"] & \mathcal{D}{\downarrow}\mathrm{pr}_1 \arrow[dr, "{(\beta_0,\beta_1)}"] \arrow[d, phantom, "\Downarrow (\overline{H}_0,\overline{H}_1)"] & \\
\mathcal{C}{\downarrow}F \arrow[dr, "\alpha_1"'] & \mathcal{C}{\downarrow}F \arrow[l, phantom, "\Leftarrow (\alpha_\mu)_1"] \arrow[r, "{(H_0,H_1)}"] \arrow[d, "\alpha_1"] & \mathcal{D}{\downarrow}G \arrow[d, "\beta_1"] \arrow[dl, phantom, "\swarrow \overline{H}_1"] \\
& \mathcal{X} \arrow[r, "H_1"'] & \mathcal{Y}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}{\downarrow}\mathrm{pr}_1 \arrow[r, "{(H_0,(H_0,H_1))}"] \arrow[d, "Q_{\mathrm{pr}_1}"'] \arrow[dr, phantom, "="] & \mathcal{D}{\downarrow}\mathrm{pr}_1 \arrow[d, "Q_{\mathrm{pr}_1}"] \arrow[dr, "{(\beta_0,\beta_1)}"] & \\
\mathcal{C}{\downarrow}F \arrow[r, "{(H_0,H_1)}"] \arrow[dr, "\alpha_1"'] & \mathcal{D}{\downarrow}G \arrow[d, phantom, "\Downarrow \overline{H}_1"] \arrow[dr, "\beta_1"] & \mathcal{D}{\downarrow}G \arrow[l, phantom, "\Leftarrow (\beta_\mu)_1"] \arrow[d, "\beta_1"] \\
& \mathcal{X} \arrow[r, "H_1"'] & \mathcal{Y}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "H_0"] \arrow[d, equal] \arrow[dr, "\mathrm{Id}"] & \mathcal{D} \arrow[d, equal] \arrow[dr, "\mathrm{Id}"] & \\
A \arrow[dr, equal] & \mathcal{C} \arrow[l, Rightarrow, "(\alpha_\eta)_0"'] \arrow[r, "H_0"] \arrow[d, "\alpha_0"] \arrow[dr, phantom, "\Swarrow \overline{H}_0"] & \mathcal{D} \arrow[d, "\beta_0"] \\
 & \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[r, "H_0"] \arrow[d, equal] \arrow[dr, phantom, "="] & \mathcal{D} \arrow[d, equal] \arrow[dr, "\mathrm{Id}"] & \\
\mathcal{C} \arrow[r, "H_0"] \arrow[dr, equal] & \mathcal{D} \arrow[d, equal] \arrow[dr, equal] & \mathcal{D} \arrow[l, Rightarrow, "(\beta_\eta)_0"] \arrow[d, "\beta_0"] \\
 & \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, equal] \arrow[dr, "(\eta_F)_1"] & \mathcal{Y} \arrow[d, equal] \arrow[dr, "(\eta_G)_1"] & \\
\mathcal{X} \arrow[dr, equal] & \mathcal{C}{\downarrow}F \arrow[l, Rightarrow, "(\alpha_\eta)_1"'] \arrow[r, "{(H_0,H_1)}"] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\Swarrow \overline{H}_1"] & \mathcal{D}{\downarrow}G \arrow[d, "\beta_1"] \\
 & \mathcal{X} \arrow[r, "H_1"'] & \mathcal{Y}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, equal] \arrow[dr, phantom, "="] & \mathcal{Y} \arrow[d, equal] \arrow[dr, "(\eta_G)_1"] & \\
\mathcal{X} \arrow[r, "H_1"] \arrow[dr, equal] & \mathcal{Y} \arrow[d, equal] \arrow[dr, equal] & \mathcal{D}{\downarrow}G \arrow[l, Rightarrow, "(\beta_\eta)_1"] \arrow[d, "\beta_1"] \\
 & \mathcal{X} \arrow[r, "H_1"'] & \mathcal{Y}
\end{tikzcd}
$$

A **pseudo-T-transformation from** $(H, \overline{H})$ **to** $(K, \overline{K})$ with $(H, \overline{H}), (K, \overline{K})\colon (F, \alpha, \alpha_\mu, \alpha_\eta) \to (B, \beta, \beta_\mu, \beta_\eta)$ pseudo-morphisms of pseudo-T-algebras is a 2-cell $\varphi\colon H \Rightarrow K\colon F \to G$ in $\mathcal{CAT}^{\to}$

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\varphi_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\varphi_0"] & \mathcal{D}
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\varphi_0"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Swarrow \overline{K}_0"] & \mathcal{D} \arrow[d, "\beta_0"] \\
\mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Swarrow \overline{H}_0"] & \mathcal{D} \arrow[d, "\beta_0"] \\
\mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\varphi_0"] & \mathcal{D}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[rr, bend left, "{(H_0,H_1)}"] \arrow[rr, bend right, "{(K_0,K_1)}"'] \arrow[rr, phantom, "\Downarrow(\varphi_0,\varphi_1)"] \arrow[d, "\alpha_1"'] & & \mathcal{D}\downarrow G \arrow[d, "\beta_1"] \arrow[dll, Rightarrow, "\overline{K}_1"] \\
\mathcal{X} \arrow[rr, bend right, "K_1"'] & & \mathcal{Y}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow F \arrow[rr, bend left, "{(H_0,H_1)}"] \arrow[d, "\alpha_1"'] & & \mathcal{D}\downarrow G \arrow[d, "\beta_1"] \arrow[dll, Rightarrow, "\overline{H}_1"'] \\
\mathcal{X} \arrow[rr, bend left, "H_1"] \arrow[rr, bend right, "K_1"'] \arrow[rr, phantom, "\Downarrow\varphi_1"] & & \mathcal{Y}
\end{tikzcd}
$$

Now, we try to get rid of the hypothesy to start from split cartesian functor to see an algebraic structure on the fibrations. The first attempt is to show that the 2-category of pseudo-T-algebras is the 2-category $\mathcal{ClFIB}$ of cloven fibrations. However, we will see that we need to restrict ourselves over a fixed base category $\mathcal{C}$, and that the pseudo-morphisms of pseudo-algebras for this "restricted" monad preserve the cleavage only up to isomorphism, which means that they are not 1-cells in $\mathcal{ClFIB}$ but 1-cells in $\mathcal{CloFIB}$.

Construction 3.4.2. Firstly, we construct a 2-functor $J\colon \mathcal{ClFIB} \longrightarrow T\!-\!\mathcal{Alg}$.

Let $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}$ be a cloven fibration. Looking at Remark 3.3.4, we would like to define the 2-functor J as

$$J\colon \mathcal{ClFIB} \to \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, "p"] \\
\mathcal{B}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "H_0"'] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "H_0"'] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C}
\end{tikzcd}
$$

where $Q_p = (Q_{\mathrm{Id}_p})_1$ (from Remark 3.2.8).

This is surely well defined on 0-cells because we did not use the hypothesis that p was split to define the functor $(Q_H)_1$ in Construction 3.2.3 (see also Remark 3.2.4). Furthermore we know by Remark 3.2.4 that Q_p is a functor, whence we get that (Id, Q_p) is a 1-cell in $\mathcal{CAT}^{\to}$.

Now, we show that

$$\begin{tikzcd}
\mathcal{B}{\downarrow}p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B}
\end{tikzcd}$$

is a pseudo-T-algebra.

Note that the only interesting level is the one of the total categories. Indeed below we have always just the identity functor. We thus focus on the above level only and construct two invertible 2-cells $(\mathrm{Id}, \alpha_\mu)$ (for the multiplication) and $(\mathrm{Id}, \alpha_\eta)$ (for the unit) in $\mathcal{CAT}^{\to}$.

We now construct a natural isomorphism α_η

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "(\eta_p)_1"] \arrow[dr, equal] & \mathcal{B}{\downarrow}p \arrow[d, "Q_p"] \arrow[dl, phantom, "\Swarrow \alpha_\eta"] \\
 & \mathcal{E};
\end{tikzcd}$$

On components we need to construct isomorphisms in $\mathcal{E}$

$$(\alpha_\eta)_E : Q_p((\eta_p)_1(E)) \xrightarrow{\sim} E$$

that is,

$$(\alpha_\eta)_E \colon \ \mathrm{id}^* E \xrightarrow{\sim} E.$$

And we know how to find such isomorphisms, since by Proposition 2.2.8 we know that cartesian liftings are unique up to isomorphisms. More precisely we know that there exists an isomorphism χ which fits well into the diagram

$$\begin{tikzcd}
\mathrm{id}^* E \arrow[r, dashed, bend right, "\chi"'] \arrow[rr, bend left, "\mathrm{Cart}_p(\mathrm{id}, E)"] \arrow[d, mapsto, "p"] & E \arrow[l, dashed, bend right, "\chi^{-1}"'] \arrow[r, "\mathrm{id}"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\
p(E) \arrow[r, equal] \arrow[rr, bend left, "\mathrm{id}"] & p(E) \arrow[r, "\mathrm{id}"'] & p(E)
\end{tikzcd}$$

and we define $(\alpha_\eta)_E := \chi \colon \ \mathrm{id}^* E \xrightarrow{\sim} E$.

Note that we have thus defined

$$(\alpha_\eta)_E := \mathrm{Cart}_p(\mathrm{id}, E)\colon\ \mathrm{id}^* E \xrightarrow{\sim} E.$$

Now, we want to define a natural isomorphism α_μ

$$\begin{tikzcd} \mathcal{B}{\downarrow}\mathrm{pr}_1 \arrow[r, "(\mathrm{Id}, Q_p)"] \arrow[d, "Q_{\mathrm{pr}_1}"'] \arrow[dr, phantom, "\swarrow\alpha_\mu"] & \mathcal{B}{\downarrow}p \arrow[d, "Q_p"] \\ \mathcal{B}{\downarrow}p \arrow[r, "Q_p"'] & \mathcal{E} \end{tikzcd}$$

By Remark 3.4.1 we see that on components we need to construct isomorphisms in $\mathcal{E}$

$$(\alpha_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)}\colon Q_p(A', Q_p(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A) \xrightarrow{\sim} Q_p(A',E,A'\xrightarrow{fa}p(E))$$

that is, isomorphisms

$$(\alpha_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)}\colon a^*f^*E \xrightarrow{\sim} (fa)^*E.$$

But we know how to find such isomorphisms χ, since by Proposition 2.2.8 we know that cartesian liftings are unique up to isomorphisms. More precisely, we know that there exists an isomorphism γ which fits well into the diagram

$$\begin{tikzcd} a^*f^*E \arrow[rr, bend left, "\mathrm{Cart}_p(f,E)\circ\mathrm{Cart}_p(a,f^*E)"] \arrow[r, dashed, bend right, "\gamma"'] \arrow[d, mapsto, "p"] & (fa)^*E \arrow[l, dashed, bend right, "\gamma^{-1}"'] \arrow[r, "\mathrm{Cart}_p(fa,E)"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\ A' \arrow[rr, bend left, "fa"] \arrow[r, equal] & A' \arrow[r, "fa"'] & p(E) \end{tikzcd}$$

and we define

$$(\alpha_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)} := \gamma\colon a^*f^*E \xrightarrow{\sim} (fa)^*E.$$

Proposition 3.4.3. *Let* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ *be a cloven cartesian functor and consider the assignment* J *produced in Construction 3.4.2. Then the data* $J(p)$ *equipped with the data* α_μ *and* α_η *produced in Construction 3.4.2 form a pseudo-*T*-algebra.*

Proof. We prove the naturality of α_η. Given a morphism $e\colon E \to E'$ in $\mathcal{E}$, we need to show that the square

$$
\begin{tikzcd}
\mathrm{id}^* E \arrow[r, "(\alpha_\eta)_E"] \arrow[d, "Q_p(p(e),e)"'] & E \arrow[d, "e"] \\
\mathrm{id}^* E' \arrow[r, "(\alpha_\eta)_{E'}"'] & E'
\end{tikzcd}
\tag{3.24}
$$

is commutative.
And by construction $Q_p(p(e), e)$ fits well into the diagram

$$
\begin{tikzcd}
\mathrm{id}^* E \arrow[rr, "\mathrm{Cart}(\mathrm{id},E)"] \arrow[dr, dashed, "Q_p(p(e),e)"] \arrow[dd, mapsto, "p"] & & E \arrow[dr, bend left, "e"] \arrow[dd, mapsto, "p"' near end] & \\
& \mathrm{id}^* E' \arrow[rr, "\mathrm{Cart}(\mathrm{id},E')"] \arrow[dd, mapsto, "p" near start] & & E' \arrow[dd, mapsto, "p"] \\
p(E) \arrow[rr, "\mathrm{id}"' near start] \arrow[dr, bend right, "p(e)"'] & & p(E) \arrow[dr, "p(e)"] & \\
& p(E') \arrow[rr, "\mathrm{id}"'] & & p(E')
\end{tikzcd}
$$

But the upper square of this diagram coincide with the square of naturality (diagram (3.24)), and then we conclude the naturality of α_η.

The naturality of α_μ follows by cartesianity diagrams (that is, by the universal property of cartesian morphisms seen in an appropriate diagram.

Moreover we have that $(\mathrm{Id}, \alpha_\eta)$ and $(\mathrm{Id}, \alpha_\mu)$ are modifications since by construction each component of both α_η and α_μ is above the identity.

We now show that the coherence diagrams of pseudo-T-algebras (from Remark 3.4.1, see also Definition 1.12.1) hold.

The first coherence diagram asks for a uniqueness of the construction of a morphism from

$$Q_p((\mathrm{Id}, Q_p)((\mathrm{Id}, (\mathrm{Id}, Q_p))(C, (A', (A, E, A \xrightarrow{f} p(E)), A' \xrightarrow{a} A), C \xrightarrow{w} A'))) = w^* a^* f^* E$$

to

$$Q_p(Q_{\mathrm{pr}_1}(Q_{\mathrm{pr}'_1}(C, (A', (A, E, A \xrightarrow{f} p(E)), A' \xrightarrow{a} A), C \xrightarrow{w} A'))) = (faw)^* E$$

(where we called pr'_1 the functor $\mathrm{pr}_1 \colon \mathcal{B} \downarrow \mathrm{pr}_1 \longrightarrow \mathcal{B}$). Such a uniqueness is easily proved by cartesianity diagrams.

The second coherence diagram can be written as the equality

$$Q_p \,\mathrm{id} \circ \alpha_\mu T(\eta_p)_1 = \mathrm{id} \circ Q_p T \alpha_\eta.$$

But this means

$$\alpha_\mu T(\eta_p)_1 = Q_p T \alpha_\eta.$$

Let then $(A, E, A \xrightarrow{f} p(E)) \in \mathcal{B}{\downarrow}p$. We have that

$$(\alpha_\mu T(\eta_p)_1)_{(A,E,A\xrightarrow{f}p(E))} = (\alpha_\mu)_{(A,(p(E),E,p(E)\xrightarrow{\mathrm{id}}p(E)),A\xrightarrow{f}p(E))}$$

whereas

$$(Q_p T\alpha_\eta)_{(A,E,A\xrightarrow{f}p(E))} = Q_p(\mathrm{id}_A, (\alpha_\eta)_E)$$

and $Q_p(\mathrm{id}_A, (\alpha_\eta)_E)$ is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
f^*\,\mathrm{id}^*\,E \arrow[rr, "\mathrm{Cart}(f{,}\mathrm{id}^*\,E)"] \arrow[dd, mapsto, "p"'] \arrow[dr, dashed, "Q_p(\mathrm{id}_A{,}(\alpha_\eta)_E)"] & & \mathrm{id}^*\,E \arrow[dr, bend left, "(\alpha_\eta)_E"] \arrow[dd, mapsto, "p"] & \\
& f^*E \arrow[rr, "\mathrm{Cart}(f{,}E)"] \arrow[dd, mapsto, "p"] & & E \arrow[dd, mapsto, "p"] \\
A \arrow[rr, "f"'] \arrow[dr, bend right, "\mathrm{id}_A"'] & & p(E) \arrow[dr, "\mathrm{id}"] & \\
& A \arrow[rr, "f"'] & & p(E)
\end{tikzcd}$$

in the place of the morphism in violet.
Therefore we conlcude that the second coherence diagram is satisfied since by construction of α_μ we know that $(\alpha_\mu)_{(A,(p(E),E,p(E)\xrightarrow{\mathrm{id}}p(E)),A\xrightarrow{f}p(E))}$ fits well into the diagram above in the place of the morphism in violet, since $f \circ \mathrm{id} = f$ and $(\alpha_\eta)_E = \mathrm{Cart}\,(\mathrm{id}, E)$. $\square$

Now that we have proved that $J\,(p)$ is a pseudo-T-algebra for every cloven fibration p, we consider the 1-cells and the 2-cells in $\mathcal{ClFIB}$.

Construction 3.4.4. Let $H = (H_0, H_1)$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "H_0"'] & \mathcal{C}
\end{tikzcd}$$

be a cloven cartesian functor from a cloven fibration p to a cloven fibration q. We want to prove that H is a pseudo-morphism of pseudo-T-algebras.

Also this time, the only interesting level is the one of the total categories. Then we construct a modification of the form $(\mathrm{Id}, \overline{H})$ with $\overline{H}$ a natural isomorphism

$$\begin{tikzcd}
\mathcal{B}{\downarrow}p \arrow[r, "(H_0{,}H_1)"] \arrow[d, "Q_p"'] & \mathcal{C}{\downarrow}q \arrow[d, "Q_q"] \arrow[dl, phantom, "\Swarrow \overline{H}"] \\
\mathcal{E} \arrow[r, "H_1"'] & \mathcal{X}
\end{tikzcd}$$

On components $(A, E, A \xrightarrow{f} p(E)) \in \mathcal{B}\downarrow p$, we need to find an isomorphism in $\mathcal{X}$

$$\overline{H}_{(A,E,A\xrightarrow{f}p(E))} \colon Q_q((H_0, H_1)(A, E, A \xrightarrow{f} p(E))) \xrightarrow{\sim} H_1(Q_p(A, E, A \xrightarrow{f} p(E)))$$

that is,

$$\overline{H}_{(A,E,A\xrightarrow{f}p(E))} \colon H_0(f)^* H_1(E) \xrightarrow{\sim} H_1(f^* E).$$

But we started from a cloven cartesian functor H and then we know that

$$H_1(\mathrm{Cart}_p(f, E)) = \mathrm{Cart}_q(H_0(f), H_1(E)).$$

Thus we can take $\overline{H}$ to be just the identity.

Proposition 3.4.5. *Let* $H = (H_0, H_1)$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "H_0"'] & \mathcal{C}
\end{tikzcd}$$

be a cloven cartesian functor from a cloven fibration p *to a cloven fibration* q*. Then* H*, equipped with the identity modification* $\overline{H} = \mathrm{id}$ *is a pseudo-morphism of pseudo-*T*-algebras.*

Proof. We have seen in Proposition 3.4.3 how we can view p and q as pseudo-T-algebras $(p, Q_p, \alpha_m u^p, \alpha^p_\eta)$ and $(q, Q_q, \alpha_m u^q, \alpha^q_\eta)$.

It suffices to show that the coherence axioms of pseudo-morphism of pseudo-T-algebras are satisfied, and these simplify a lot since we have defined $\overline{H}$ to be the identity. Moreover it suffices to check the level of the total categories.

The first coherence axioms can then be written as the equality

$$H_1 \alpha^p_\mu = \alpha^q_\mu (T^2 H)_1.$$

On components $(A', (A, E, A \xrightarrow{f} p(E)), A' \xrightarrow{a} A)$ this means

$$H_1\left((\alpha^p_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)}\right) = (\alpha^q_\mu)_{(H_0(A'),(H_0(A),H_1(E),H_0(A)\xrightarrow{H_0(f)}H_0(p(E))),H_0(A')\xrightarrow{H_0(a)}H_0(A))}$$

And this follows by construction of α^p_μ and α^q_μ, since $(\alpha^p_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)}$ is defined to be the unique morphism which fits well into the diagram

$$
\begin{tikzcd}
a^*f^*E \arrow[rr, bend left, "\mathrm{Cart}_p(f,E)\circ\mathrm{Cart}_p(a,f^*E)"] \arrow[r, dashed, "(\alpha^p_\mu)_{(A',(A,E,A\xrightarrow{f}p(E)),A'\xrightarrow{a}A)}"'] \arrow[d, "p"] & (fa)^*E \arrow[r, "\mathrm{Cart}_p(fa,E)"] \arrow[d, "p"] & E \arrow[d, "p"] \\
A' \arrow[r, equal] \arrow[rr, bend left, "fa"] & A' \arrow[r, "fa"'] & p(E)
\end{tikzcd}
$$

in the place of the morphism in violet, and taking the image of this diagram under (H_0, H_1) we get precisely the diagram which defines

$$(\alpha^q_\mu)_{(H_0(A'),(H_0(A),H_1(E),H_0(A)\overset{H_0(f)}{\longrightarrow}H_0(p(E))),H_0(A')\overset{H_0(a)}{\longrightarrow}H_0(A))}$$

via cartesianity of $\mathrm{Cart}_q(H_0(fa), H_1(E))$, since H is a cloven cartesian functor.

The second coherence diagram can be written as the equality

$$H_1\alpha^p_\eta = \alpha^q_\eta H_1.$$

On components $E \in \mathcal{E}$ this means

$$H_1(\mathrm{Cart}_p(\mathrm{id}, E)) = \mathrm{Cart}_q(\mathrm{id}, H_1(E)).$$

But this is true since H is a cloven cartesian functor. $\square$

Now we consider 2-cells in $\mathcal{ClFIB}$ and we prove that they are pseudo-T-transformations.

Proposition 3.4.6. *Let*

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C}
\end{tikzcd}
$$

be a 2-morphism in $\mathcal{ClFIB}$, that is, a mere modification. Then (λ_0, λ_1) is a pseudo-T-transformation.

Proof. We have seen in Proposition 3.4.5 that (H_0, H_1) and (K_0, K_1) are pseudo-morphisms of pseudo-T-algebras taking as invertible 2-cells in $\mathcal{CAT}^{\to}$ the identities. Then the coherence axiom of pseudo-T-transformation become trivial, since it coincides with the request that (λ_0, λ_1) is a modification. $\square$

Remark 3.4.7. Note that the proofs of Proposition 3.4.5 and Proposition 3.4.6 are almost trivial, since the 1-cells of $\mathcal{ClFIB}$ are pseudo-morphisms of pseudo-T-transformations taking as invertible 2-cells in $\mathcal{CAT}^{\to}$ just the identities. After all the 1-cells and the 2-cells in $\mathcal{ClFIB}$ are exactly the 1-cells and the 2-cells in $\mathcal{SpFIB}$, and we already know that these are respectively 1-morphisms of T-algebras and 2-morphisms of T-algebras, and thus, respectively, pseudo-morphisms of pseudo-T-algebras and pseudo-T-transformations, by Remark 1.12.6.

It is then natural to think that the assignment J forms a 2-functor, since on 1-cells and on 2-cells we have defined J exactly like the comparison 2-functor $\overline{U}$ associated to the forgetful 2-functor $U\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}$ and to the 2-monad T. One could anyway check explicitly the 2-functoriality of J.

Proposition 3.4.8. *Proposition 3.4.3, Proposition 3.4.5 and Proposition 3.4.6 jointly yield a 2-functor*

$$J\colon \mathcal{ClFIB} \to \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{B}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "H_0"'] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend right, "K_0"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{C} \end{tikzcd}$$

Proof. The proof is trivial. □

Now, we would like to construct a 2-functor

$$W\colon \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg} \longrightarrow \mathcal{ClFIB}.$$

Construction 3.4.9. Let $(F, \alpha, \alpha_\mu, \alpha_\eta)$ be a pseudo-T-algebra with $F\colon \mathcal{X} \to \mathcal{C}$ a functor.

We would like to show that the structure of pseudo-T-algebra on F makes F into a cloven fibration. We thus want to find a cleavage for F.

We start from a diagram

$$\begin{tikzcd} & E \arrow[d, "F"] \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

and we search for a cartesian lifting of f to E. To do this, one might think to apply the unit η_F of the 2-adjunction $L \dashv U$ to the diagram above

$$\begin{tikzcd} & E \arrow[d, "F"] & & (A, E, A \xrightarrow{f} F(E)) \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "\mathrm{pr}_1"] & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[d, "\mathrm{pr}_1"] \\ A \arrow[r, "f"'] & F(E) \arrow[r, phantom, "\xmapsto[\mathrm{Id}]{(\eta_F)_1}"] & & A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

and find the data in violet using the fact that pr_1 is a cloven fibration. Remember that at this point in Construction 3.3.5 we applied (α_0, α_1) to such diagram restoring the starting data and noting that we had produced in this way a new morphism above f, which then turned out to be the desiderd cartesian lifting.

The problem here is that applying (α_0, α_1) we do not restore our starting data, because $\alpha \circ \eta_F$ is now only isomorphic to the identity, via α_η. Applying (α_0, α_1) to the diagram above we get

$$\begin{tikzcd} \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "{\alpha_1(f,\mathrm{id})}"] \arrow[d, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[d, "F"] \\ \alpha_0(A) \arrow[r, "\alpha_0(f)"'] & \alpha_0(F(E)) \end{tikzcd}$$

Now, in order to restore our starting data, we could try to compose with the isomorphisms given by α_η:

$$\begin{tikzcd} ? \arrow[r, "(\alpha_\eta)^{-1}_{1,?}"] \arrow[d, "F"] & \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "{\alpha_1(f,\mathrm{id})}"] \arrow[d, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[r, "(\alpha_\eta)_{1,E}"] \arrow[d, "F"] & E \arrow[d, "F"] \\ A \arrow[r, "(\alpha_\eta)^{-1}_{0,A}"'] & \alpha_0(A) \arrow[r, "\alpha_0(f)"'] & \alpha_0(F(E)) \arrow[r, "(\alpha_\eta)_{0,F(E)}"'] & F(E) \end{tikzcd} \tag{3.25}$$

In fact note that the morphism below from A to $F(E)$ is $f \colon A \to F(E)$, since by naturality of $(\alpha_\eta)_0$ we have the the diagram

$$\begin{tikzcd} \alpha_0(A) \arrow[r, "\alpha_0(f)"] \arrow[d, "(\alpha_\eta)_{0,A}"'] & \alpha_0(F(E)) \arrow[d, "(\alpha_\eta)_{0,F(E)}"] \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

is commutative, and $(\alpha_\eta)_{0,A}$ is an isomorphism. Note also that the morphisms in upper part of the diagram are above the morphisms in the lower part of the diagram since α_η is a modification.

Remark 3.4.10. The problem is that we do not know how to construct the object ? of equation (3.25). We would need to find a lifting of $(\alpha_\eta)_{0,A}^{-1}$ and the most natural one would be given by $(\alpha_\eta)_1^{-1}$, but $\alpha_1(A, E, A \xrightarrow{f} F(E))$ is not of the form $\alpha_1((\eta_F)_1(X))$ for some $X \in \mathcal{X}$.

Then we restrict ourselves to fibrations over a fixed base category $\mathcal{C}$. We can trivially adapt what we have seen in Section 3.2 to find a left 2-adjoint to the forgetful 2-functor (which we call U as the one we were considering before)

$$U\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"] & & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} & \longmapsto & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] & & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} & \longmapsto & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend right, equal] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend right, equal] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd}$$

Till the end of this chapter, we will call such left 2-adjoint L (as we were calling the left 2-adjoint of $U\colon \mathcal{SpFIB} \to \mathcal{CAT}^{\to}$) and we will call T the 2-monad generated by this "restricted adjunction".

We want to prove that the pseudo-T-algebras (for this "restricted" 2-monad T) are the cloven fibrations over the fixed base category $\mathcal{C}$.

We can trivially find an explicit description of the 2-category $\mathcal{Ps}$-T-$\mathcal{Alg}$ adapting the description given in Remark 3.4.1 for the pseudo-algebras for the "non-restricted" monad (just trivializing the lower level, the one of the base categories).

We can then obviously adapt what we have said in Proposition 3.4.8 to get a 2-functor

$$J\colon \mathcal{ClFIB}_{\mathcal{C}} \to \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"] & & \mathcal{C}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} & \longmapsto & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
$$

At this point, following Construction 3.4.9, starting from a pseudo-T-algebra $(F, \alpha, \alpha_\mu, \alpha_\eta)$ with $F\colon \mathcal{X} \to \mathcal{C}$ a functor, and from a diagram

$$
\begin{tikzcd}
& E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}
$$

we construct a lifting of f to E as in the following diagram:

$$
\begin{tikzcd}
\alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[r, "(\alpha_\eta)_{1,E}"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E) \arrow[r, "\mathrm{Id}"'] & F(E)
\end{tikzcd}
\tag{3.26}
$$

Now, in order to prove that a pseudo-T-algebra is a cloven fibration, it only remains to prove that

$$
(\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})
$$

is cartesian with respect to F.

Construction 3.4.11. Let $(F, \alpha, \alpha_\mu, \alpha_\eta)$ be a pseudo-T-algebra with $F\colon \mathcal{X} \to \mathcal{C}$ a functor and consider the diagram

$$\begin{tikzcd}
X \arrow[drrr, bend left, "k"] \arrow[dd, mapsto, "F"] & & & \\
& \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[dd, mapsto, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[r, "(\alpha_\eta)_{1,E}"] \arrow[dd, mapsto, "F"] & E \arrow[dd, mapsto, "F"] \\
F(X) \arrow[dr, "a"'] \arrow[drr, bend left, "fa"] & & & \\
& A \arrow[r, "f"'] & F(E) \arrow[r, equal] & F(E)
\end{tikzcd} \tag{3.27}$$

with arbitrary $X \in \mathcal{X}$, $a\colon F(X) \to A$ in $\mathcal{C}$ and $k\colon X \to E$ above fa. We would like to show that there is a unique morphism $v\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ which fits well into diagram (3.27).

To find a suitable such morphism v we repeat the same argument of Construction 3.4.9. Applying $(\mathrm{Id}, (\eta_F)_1)$ to the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "k"] \arrow[dd, mapsto, "F"] & & \\
& & E \arrow[dd, mapsto, "F"] \\
F(X) \arrow[dr, "a"'] \arrow[drr, bend left, "fa"] & & \\
& A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and then using the fact that pr_1 is a cloven fibration, we get the diagram in black

$$\begin{tikzcd}
(F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \arrow[drr, bend left, "(fa,k)"] \arrow[dr, dashed, "(a,k)"] \arrow[dd, mapsto, "\mathrm{pr}_1"] & & \\
& (A, E, A \xrightarrow{f} F(E)) \arrow[r, "(f,\mathrm{id})"] \arrow[dd, mapsto, "\mathrm{pr}_1"] & (F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[dd, mapsto, "\mathrm{pr}_1"] \\
F(X) \arrow[dr, "a"'] \arrow[drr, bend left, "fa"] & & \\
& A \arrow[r, "f"'] & F(E)
\end{tikzcd} \tag{3.28}$$

and we know that (a, k) is the unique morphism from $(F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X))$ to $(A, E, A \xrightarrow{f} F(E))$ which fits well into diagram (3.28) in the place of the morphism in violet, since it is an acceptable choice and (f, id) is cartesian with respect to pr_1. Applying (Id, α_1) to diagram (3.28) we then get

$$\begin{tikzcd}
\alpha_1((\eta_F)_1(X)) \arrow[r, dashed, "\alpha_1(a,k)"] \arrow[rr, bend left, "\alpha_1((\eta_F)_1(k))"] \arrow[d, mapsto, "F"] & \alpha_1(A,E,A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

At this point, we find an acceptable morphism $v\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ which fits well into diagram (3.27) by composition with the isomorphisms given by α_η:

$$\begin{tikzcd}
X \arrow[r, "(\alpha_\eta)^{-1}_{1,X}"] \arrow[d, mapsto, "F"] & \alpha_1((\eta_F)_1(X)) \arrow[r, dashed, "\alpha_1(a,k)"] \arrow[rr, bend left, "\alpha_1((\eta_F)_1(k))"] \arrow[d, mapsto, "F"] & \alpha_1(A,E,A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[r, "(\alpha_\eta)_{1,E}"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, equal] & F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E) \arrow[r, equal] & F(E)
\end{tikzcd} \tag{3.29}$$

Note also that the following diagram is commutative by naturality of $(\alpha_\eta)_1$:

$$\begin{tikzcd}
\alpha_1((\eta_F)_1(X)) \arrow[r, "\alpha_1((\eta_F)_1(k))"] \arrow[d, "(\alpha_\eta)_{1,X}"'] & \alpha_1((\eta_F)_1(E)) \arrow[d, "(\alpha_\eta)_{1,E}"] \\
X \arrow[r, "k"'] & E
\end{tikzcd}$$

It remains to prove that $\alpha_1(a,k) \circ (\alpha_\eta)^{-1}_{1,X}$ is the unique such morphism v. In order to prove this, we start from another such morphism v and we try to go back to diagram (3.28), in which we have a uniqueness of the morphism in violet given by the fact that (f, id) is a cartesian morphism with respect to pr_1.

Construction 3.4.12. Let $(F, \alpha, \alpha_\mu, \alpha_\eta)$ be a pseudo-T-algebra. We define a function

$$\varphi\colon\ \mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E))) \longrightarrow \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E)))$$

which represents the construction of the acceptable morphism $v\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ which fits well into diagram (3.27), produced in Construction 3.4.9.

For every

$$(a, k)\colon (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$$

in $C \downarrow F$, we note that $\alpha_1(a,k)$ fits well into diagram (3.29) and we define

$$\varphi(a,k) := \alpha_1(a,k) \circ (\alpha_\eta)^{-1}_{1,X}.$$

Now, given a morphism $q\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ in $\mathcal{X}$ which fits well into diagram (3.27), we want to go back with it to diagram (3.28). We then search for a function

$$\psi\colon \ \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E))) \longrightarrow \mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E)))$$

which is an inverse of φ.

In particular, we want that $\psi \circ \varphi = \mathrm{id}$. In order to define the function ψ, we note that we have the following commutative diagram by naturality of $(\alpha_\eta)_1$:

$$\begin{tikzcd} \alpha_1((\eta_F)_1(X)) \arrow[r, "\alpha_1((\eta_F)_1(k))"] \arrow[d, "(\alpha_\eta)_{1,X}"'] & \alpha_1((\eta_F)_1(E)) \arrow[d, "(\alpha_\eta)_{1,E}"] \\ X \arrow[r, "k"'] & E \end{tikzcd} \tag{3.30}$$

Then we define the map ψ as

$$\psi(q) := (F(q), (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \circ q)$$

for every morphism $q\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ in $\mathcal{X}$.
And we immediatly see, by diagram (3.29) and diagram (3.30), that

$$\begin{aligned} &\psi(\varphi(a,k)) = \psi(\alpha_1(a,k) \circ (\alpha_\eta)^{-1}_{1,X}) = \\ &= \big(a, (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \circ \alpha_1(a,k) \circ (\alpha_\eta)^{-1}_{1,X}\big) = \\ &= (a,k) \end{aligned}$$

for every $(a,k)\colon (F(X), X, F(X) \xrightarrow{\mathrm{id}} F(X)) \longrightarrow (A, E, A \xrightarrow{f} F(E))$ in $C \downarrow F$.

Now, we want to prove that ψ is an inverse of φ.

Proposition 3.4.13. *Let $(F, \alpha, \alpha_\mu, \alpha_\eta)$ be a pseudo-T-algebra, with $F\colon \mathcal{X} \to \mathcal{C}$. Then the functions φ and ψ produced in Construction 3.4.12*

$$\mathrm{Hom}((\eta_F)_1(X), (A, E, A \xrightarrow{f} F(E))) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(X, \alpha_1(A, E, A \xrightarrow{f} F(E)))$$

are inverses of each other.

Proof. We already know by Construction 3.4.12 that $\psi \circ \varphi = \mathrm{id}$.

It remains to prove that $\varphi \circ \psi = \mathrm{id}$.
Let then $q\colon X \to \alpha_1(A, E, A \xrightarrow{f} F(E))$ be a morphism in $\mathcal{X}$. We have that

$$\begin{aligned}\varphi(\psi(q)) &= \alpha_1\left(F(q), (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \circ q\right) \circ (\alpha_\eta)^{-1}_{1,X} = \\ &= \alpha_1\left(\mathrm{id}_A, (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})\right) \circ \alpha_1(F(q), q) \circ (\alpha_\eta)^{-1}_{1,X}\end{aligned}$$

Now note that

$$\alpha_1(F(q), q) = \alpha_1((\eta_F)_1(q))$$

and that by naturality of $(\alpha_\eta)_1$ we have the following commutative square:

$$\begin{tikzcd}
\alpha_1((\eta_F)_1(X)) \arrow[r, "\alpha_1((\eta_F)_1(q))"] \arrow[d, "(\alpha_\eta)_{1,X}"'] & \alpha_1((\eta_F)_1(\alpha_1(A, E, A \xrightarrow{f} F(E)))) \arrow[d, "(\alpha_\eta)_{1,\alpha_1(A,E,A\xrightarrow{f}F(E))}"] \\
X \arrow[r, "q"'] & \alpha_1(A, E, A \xrightarrow{f} F(E))
\end{tikzcd}$$

Then it suffices to prove that

$$\alpha_1\left(\mathrm{id}_A, (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})\right) = (\alpha_\eta)_{1,\alpha_1(A,E,A\xrightarrow{f}F(E))}$$

But by the axioms of pseudo-T-algebra we have that

$$\alpha_1\left(\mathrm{id}_A, (\alpha_\eta)_{1,E}\right) = \alpha_1\left(\left((\alpha_\eta)_0, (\alpha_\eta)_1\right)(A, E, A \xrightarrow{f} F(E))\right) = (\alpha_\mu)_{1,\left(A,(\eta_F)_1(E),A\xrightarrow{f}F(E)\right)}$$

and that

$$(\alpha_\eta)_{1,\alpha_1(A,E,A\xrightarrow{f}F(E))} = (\alpha_\mu)_{1,(\eta_{\mathrm{pr}_1})_1(A,E,A\xrightarrow{f}F(E))} = (\alpha_\mu)_{1,\left(A,(A,E,A\xrightarrow{f}F(E)),A\xrightarrow{\mathrm{id}}A\right)}.$$

And we conclude noting that the following square is commutative by naturality of $(\alpha_\mu)_1$:

$$\begin{tikzcd}
\alpha_1\left(A, \alpha_1(A, E, A \xrightarrow{f} F(E)), A \xrightarrow{\mathrm{id}} A\right) \arrow[r, "\alpha_1(\mathrm{id},\alpha_1(f,\mathrm{id}))"] \arrow[d, "(\alpha_\mu)_{1,\left(A,(A,E,A\xrightarrow{f}F(E)),A\xrightarrow{\mathrm{id}}A\right)}"'] & \alpha_1\left(A, \alpha_1(F(E), E, F(E) \xrightarrow{\mathrm{id}} F(E))\right) \arrow[d, "(\alpha_\mu)_{1,\left(A,(F(E),E,F(E)\xrightarrow{\mathrm{id}}F(E))\right)}"] \\
\alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[r, equal, "\alpha_1(\mathrm{id},\mathrm{id})"'] & \alpha_1(A, E, A \xrightarrow{f} F(E))
\end{tikzcd}$$

□

We are now ready to prove that the structure of a pseudo-T-algebra $(F, \alpha, \alpha_\mu, \alpha_\eta)$ defines a structure of a cloven fibration for F. We will say that a pseudo-T-algebra is a cloven fibration.

Theorem 3.4.14. *Let $(F, \alpha, \alpha_\mu, \alpha_\eta)$ be a pseudo-T-algebra, with $F\colon \mathcal{X} \to \mathcal{C}$. Then the assignment*

$$\mathrm{Cart}\,(f, E) := (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})\colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \longrightarrow E \tag{3.31}$$

for every diagram

$$
\begin{tikzcd}
 & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}
$$

with $E \in \mathcal{X}$ and f a morphism in $\mathcal{C}$, which we have produced in Construction 3.4.9, defines a cleavage for F.

Proof. After what we have shown in Construction 3.4.9, it remains to prove that $\alpha_1(a,k)\circ(\alpha_\eta)_{1,X}^{-1}$ is the only morphism which fits well into diagram (3.27), which we write again below, in the place of the morphism in violet:

$$
\begin{tikzcd}
X \arrow[rrr, bend left, "k"] \arrow[r, dashed, "\alpha_1(a,k)\circ(\alpha_\eta)_{1,X}^{-1}"] \arrow[d, mapsto, "F"] & \alpha_1(A,E,A \xrightarrow{f} F(E)) \arrow[r, "\alpha_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & \alpha_1((\eta_F)_1(E)) \arrow[r, "(\alpha_\eta)_{1,E}"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E) \arrow[r, equal] & F(E)
\end{tikzcd}
\tag{3.32}
$$

In fact we will then conlcude by arbitrariness of k and a that $\mathrm{Cart}\,(f,E)$ is a cartesian morphism.

Let then $q\colon X \longrightarrow \alpha_1(A,E,A \xrightarrow{f} F(E))$ be another morphism which fits well into diagram (3.32) in the place of the morphism in violet.

Recall the bijection

$$
\mathrm{Hom}((\eta_F)_1(X),(A,E,A \xrightarrow{f} F(E))) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(X,\alpha_1(A,E,A \xrightarrow{f} F(E)))
$$

described in Proposition 3.4.13.

We prove that $\psi(q)$ fits well into diagram (3.28), which we write again below, in the place of the morphism in violet:

$$
\begin{tikzcd}
(F(X),X,F(X) \xrightarrow{\mathrm{id}} F(X)) \arrow[rr, bend left, "(fa,k)"] \arrow[r, dashed, "(a,k)", "\psi(q)"'] \arrow[d, mapsto, "\mathrm{pr}_1"] & (A,E,A \xrightarrow{f} F(E)) \arrow[r, "(f,\mathrm{id})"] \arrow[d, mapsto, "\mathrm{pr}_1"] & (F(E),E,F(E) \xrightarrow{\mathrm{id}} F(E)) \arrow[d, mapsto, "\mathrm{pr}_1"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{3.33}
$$

Firstly, we note that

$$\mathrm{pr}_1(\psi(q)) = F(q) = a.$$

Then note that

$$(f, \mathrm{id}) \circ \psi(q) = (f, \mathrm{id}) \circ (F(q), (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \circ q) = (fa, k).$$

Therefore we have proved that $\psi(q)$ fits well into diagram (3.33), whence we get that

$$\psi(q) = (a, k)$$

by cartesianity of (f, id) with respect to pr_1. And now we conclude applying φ (which is an inverse of ψ):

$$q = \varphi(\psi(q)) = \varphi(a, k) = \alpha_1(a, k) \circ (\alpha_\eta)^{-1}_{1,X}.$$

□

Remark 3.4.15. Let $(H, \overline{H})$ be a pseudo-morphism of pseudo -T-algebras, with $H = (\mathrm{Id}, H_1)$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

between two pseudo-T-algebras $(F, \alpha, \alpha_\mu, \alpha_\eta)$ and $(G, \beta, \beta_\mu, \beta_\eta)$.
Consider F and G as split fibrations by using Theorem 3.4.14. We would like to prove that (Id, H_1) is a cloven cartesian functor. However, this will not be true, and we will not get the isomorphism we were searching for.

We will instead get an isomorphism between the 2-category $\mathcal{Ps}$-T-$\mathcal{Alg}$ and the 2-category $\mathcal{CloFIB}_{\mathcal{C}}$ of cloven fibrations over $\mathcal{C}$ and cartesian functors over $\mathcal{C}$.

In order to prove that the pseudo-morphisms of pseudo-T-algebras are the cloven cartesian functors over $\mathcal{C}$, we would need to prove that for every $E \in \mathcal{X}$ and every $f \colon A \to F(E)$ in $\mathcal{C}$

$$H_1(\mathrm{Cart}_F(f, E)) = \mathrm{Cart}_G(f, H_1(E)). \tag{3.34}$$

Given $E \in \mathcal{X}$ and $f \colon A \to F(E)$ in $\mathcal{C}$, we have that

$$\mathrm{Cart}_F(f, E) = (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \longrightarrow E$$

by construction (see Theorem 3.4.14 and Remark 3.4.10). Then

$$H_1(\mathrm{Cart}_F(f, E)) = H_1((\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})) = H_1((\alpha_\eta)_{1,E}) \circ H_1(\alpha_1(f, \mathrm{id})).$$

And by the axioms of pseudo-morphism of pseudo-T-algebra we get that

$$H_1((\alpha_\eta)_{1,E}) \circ \overline{H}_{1,(\eta_F)_1(E)} = (\beta_\eta)_{1,H_1(E)}$$

Moreover we have the following commutative diagram by naturality of $\overline{H}_1^{-1}$:

$$\begin{tikzcd}
H_1(\alpha_1((A,E,A \xrightarrow{f} F(E)))) \arrow[r, "H_1(\alpha_1(f,\mathrm{id}))"] \arrow[d, "\overline{H}^{-1}_{1,(A,E,A\xrightarrow{f}F(E))}"'] & H_1(\alpha_1((\eta_F)_1(E))) \arrow[d, "\overline{H}^{-1}_{1,(\eta_F)_1(E)}"] \\
\beta_1(A, H_1(E), A \xrightarrow{f} F(E)) \arrow[r, "\beta_1(f,\mathrm{id})"'] & \beta_1(F(E), H_1(E), F(E) \xrightarrow{\mathrm{id}} F(E))
\end{tikzcd}$$

Therefore we get that

$$H_1(\mathrm{Cart}_F(f,E)) = (\beta_\eta)_{1,H_1(E)} \circ \overline{H}^{-1}_{1,(\eta_F)_1(E)} \circ H_1(\alpha_1(f,\mathrm{id})) = (\beta_\eta)_{1,H_1(E)} \circ \beta_1(f,\mathrm{id}) \circ \overline{H}^{-1}_{1,(A,E,A\xrightarrow{f}F(E))}$$

whence

$$H_1(\mathrm{Cart}_F(f,E)) \circ \overline{H}_{1,(A,E,A\xrightarrow{f}F(E))} = \mathrm{Cart}_G(f, H_1(E)).$$

We thus see that we will not have that the 2-functor $J\colon \mathcal{ClFIB}_{\mathcal{C}} \to \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ is an isomorphism of categories. Indeed the pseudo-morphisms of pseudo-T-algebras do not preserve the cleavage on the nose, but only up to isomorphism. Therefore we have just proved that the pseudo-morphisms of pseudo-T-algebras are the 1-cells of the 2-category $\mathcal{CloFIB}_{\mathcal{C}}$, and we expect to find an isomorphism between $\mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$ and $\mathcal{CloFIB}_{\mathcal{C}}$.

Now, we want to change a bit the functor J on the 1-cells to obtain a functor $\tilde{J}\colon \mathcal{CloFIB}_{\mathcal{C}} \longrightarrow \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$.

Proposition 3.4.16. *Let $\mathcal{C}$ be a category. The assignment*

$$\tilde{J}\colon \mathcal{CloFIB}_{\mathcal{C}} \longrightarrow \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"] & & \mathcal{C}\downarrow p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, phantom, "\longmapsto"] & {} & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, phantom, "\longmapsto"] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] & \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] & \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}$$

is a 2-functor.

Proof. We have already proved that a cloven fibration over $\mathcal{C}$ is a pseudo-T-algebra.

Now, consider

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

a cartesian functor over $\mathcal{C}$.

We can see p as a pseudo-T-algebra

$$\begin{tikzcd}
\mathcal{C}{\downarrow}p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

with natural isomorphisms α_μ and α_η and we can see q as a pseudo-T-algebra

$$\begin{tikzcd}
\mathcal{C}{\downarrow}q \arrow[r, "Q_q"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

with natural isomorphisms β_μ and β_η.

We want to prove that $H = (\mathrm{Id}, H_1)$ is a pseudo-morphism of pseudo-T-algebras. We need to construct a natural isomorphism

$$\overline{H}_1 \colon Q_q(\mathrm{Id}, H_1) \Rightarrow H_1 Q_p.$$

On components $(A, E, A \xrightarrow{f} p(E)) \in \mathcal{C}{\downarrow}p$ we need to produce

$$\overline{H}_{1,(A,E,A\xrightarrow{f}p(E))} \colon Q_q(A, H_1(E), A \xrightarrow{f} p(E)) \longrightarrow H_1(Q_p(A, E, A \xrightarrow{f} p(E))).$$

But we can consider the diagram in black

$$\begin{tikzcd}
Q_q(A, H_1(E), A \xrightarrow{f} p(E)) \arrow[d, "q"] \arrow[r, dashed, "\overline{H}_{1,(A,E,A\xrightarrow{f}p(E))}"'] \arrow[rr, bend left, "\mathrm{Cart}_q(f, H_1(E))"] & H_1(Q_p(A, E, A \xrightarrow{f} p(E))) \arrow[r, "H_1(\mathrm{Cart}_p(f,E))"] \arrow[d, "q"] & H_1(E) \arrow[d, "q"] \\
A \arrow[r, equal] \arrow[rr, bend left, "f"] & A \arrow[r, "f"'] & p(E)
\end{tikzcd} \tag{3.35}$$

and produce the morphism in violet $\overline{H}_{1,(A,E,A\xrightarrow{f}p(E))}$ by cartesianity of $H_1(\mathrm{Cart}_p(f,E)$ (which is cartesian since H_1 is a cartesian functor over $\mathcal{C}$).

We also know that such a morphism $\overline{H}_{1,(A,E,A\xrightarrow{f}p(E))}$ is an isomorphism because $H_1(\mathrm{Cart}_p(f,E)$ and $\mathrm{Cart}_q(f,H_1(E))$ are two cartesian lifting of the same morphism f to the same object $H_1(E)$.

Therefore we have defined a natural isomorphism $\overline{H}$. Now, we prove that the axioms of pseudo-morphism of pseudo-T-algebras are satisfied.

The first such axiom is

$$H_1(\alpha_\mu)_1 \circ \overline{H_1}(\mathrm{Id},Q_p) \circ Q_q(\mathrm{Id},\overline{H}_1) = \overline{H}_1 Q_{\mathrm{pr}_1} \circ (\beta_\mu)_1(\mathrm{Id},(\mathrm{Id},H_1))$$

And this can be easily proved on components via cartesianity diagrams.

The second axiom of pseudo-morphism of pseudo-T-algebras is

$$H_1(\alpha_\eta)_1 \circ \overline{H}_1(\eta_F)_1 = (\beta_\eta)_1 H_1.$$

On components $E \in \mathcal{E}$ this means

$$H_1((\alpha_\eta)_{1,E}) \circ \overline{H}_{1,(F(E),E,F(E)\xrightarrow{\mathrm{id}}F(E))} = (\beta_\eta)_{1,H_1(E)}$$

and this means

$$H_1(\mathrm{Cart}_p(\mathrm{id},E)) \circ \overline{H}_{1,(F(E),E,F(E)\xrightarrow{\mathrm{id}}F(E))} = \mathrm{Cart}_q(\mathrm{id},H_1(E)),$$

which is true by construction of $\overline{H}_1$.

Now, consider $(\mathrm{Id},\lambda_1)\colon H \Rightarrow K : p \to q$ a 2-cell in $\mathcal{C}\ell o\mathcal{FIB}_{\mathcal{C}}$. Construct $\overline{H}_1$ and $\overline{K}_1$ as above, such that $(H,\overline{H})$ and $(K,\overline{K})$ are pseudo-morphisms of pseudo-T-algebras.

We want to prove that (id,λ_1) is a pseudo-T-transformation. Then it suffices to verify that

$$\overline{K}_1 \circ Q_q(\mathrm{id},\lambda_1) = \lambda_1 Q_p \circ \overline{H}_1.$$

On components $(A,E,A\xrightarrow{f}p(E)) \in \mathcal{C}\downarrow p$, this means

$$\overline{K}_{1,(A,E,A\xrightarrow{f}p(E))} \circ Q_q(\mathrm{id},\lambda_{1,E}) = \lambda_{1,Q_p(A,E,A\xrightarrow{f}p(E))} \circ \overline{H}_{1,(A,E,A\xrightarrow{f}p(E))}$$

Consider the diagram

$$\begin{tikzcd}
Q_q(A,H_1(E),A\xrightarrow{f}H_1(E)) \arrow[rr, "\mathrm{Cart}_q(f,H_1(E))"] \arrow[dd, "q"] \arrow[dr, dashed, "Q_p(\mathrm{id}_A,\lambda_{1,E})"] & & H_1(E) \arrow[dd, "q"] \arrow[dr, bend left, "\lambda_{1,E}"] & \\
& Q_q(A,K_1(E),A\xrightarrow{f}p(E)) \arrow[rr, "\mathrm{Cart}_q(f,K_1(E))"] \arrow[dd, "q"] & & K_1(E) \arrow[dd, "q"] \\
A \arrow[rr, "f"'] \arrow[dr, bend right, "\mathrm{id}_A"'] & & p(E) \arrow[dr, equal] & \\
& A \arrow[rr, "f"'] & & p(E)
\end{tikzcd}$$

and also diagram (3.35) for H_1 and for K_1 instead of H_1.

Then we see that we conclude by cartesianity of the morphism $K_1(\mathrm{Cart}_p(f,E))$.

Finally, the 2-functoriality is now trivial. $\square$

Now, we want to prove that the 2-functor $\tilde{J}$ is an isomorphism of categories.

We have already described the pseudo-T-algebras and the pseudo-morphisms of pseudo-T-algebras. Now we consider pseudo-T-transformations.

Proposition 3.4.17. *Let $\varphi\colon H \Rightarrow K\colon F \to G$ be a pseudo-T-transformation with*

$$(H, \overline{H}), (K, \overline{K})\colon (F, \alpha, \alpha_\mu, \alpha_\eta) \to (B, \beta, \beta_\mu, \beta_\eta)$$

pseudo-morphisms of pseudo-T-algebras with $F\colon \mathcal{X} \to \mathcal{C}$ and $G\colon \mathcal{Y} \to \mathcal{D}$.
Consider F and G as cloven fibrations over $\mathcal{C}$ by using Theorem 3.4.14, and consider H and K as cartesian functors over $\mathcal{C}$ by using Remark 3.4.15. Then φ is a 2-cell in $\mathcal{CloFIB}_{\mathcal{C}}$.

Proof. The proof is trivial, since we have defined the 2-cells in $\mathcal{CloFIB}_{\mathcal{C}}$ to be mere modifications. $\square$

Proposition 3.4.18. *Theorem 3.4.14, Remark 3.4.15 and Proposition 3.4.17 jointly yield a 2-functor*

$$W\colon \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg} \longrightarrow \mathcal{CloFIB}_{\mathcal{C}}$$

$$\left(\begin{tikzcd} \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \end{tikzcd}, \begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}, \alpha_\mu, \alpha_\eta\right) \longmapsto \begin{tikzcd} \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \end{tikzcd}$$

$$\left(\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}, \overline{H}\right) \longmapsto \begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend right, "K_0"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, bend left, "H_0"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathcal{D} \end{tikzcd}$$

where $\begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix} = W(F, \alpha, \alpha_\mu, \alpha_\eta)$ *has the cleavage defined by*

$$\mathrm{Cart}\,(f, E) := (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})\colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \longrightarrow E$$

for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$.

Proof. The proof is trivial. □

Theorem 3.4.19. *The 2-functor*

$$\tilde{J} \colon \mathcal{CloFIB}_{\mathcal{C}} \longrightarrow \mathcal{Ps}\text{-}T\text{-}\mathcal{Alg}$$

described in Proposition 3.4.16 is an isomorphism of categories. That is, the 2*-category of pseudo-*T*-algebras is precisely the* 2*-category of cloven fibrations over* $\mathcal{C}$ *and cartesian functors over* $\mathcal{C}$*.*

Proof. We prove that the 2-functor $\tilde{J}$ and the 2-functor W produced in Proposition 3.4.18 are inverses of each other.

Firstly, we prove that $\tilde{J}$ and W are inverses of each other on 0-cells.
Let then

$$\left(\begin{array}{c} \mathcal{X} \\ \big\downarrow{\scriptstyle F} \\ \mathcal{C} \end{array}, \begin{tikzcd} \mathcal{C}\!\downarrow\! F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}, \alpha_\mu, \alpha_\eta\right)$$

be a pseudo-T-algebra. We want to prove that

$$\tilde{J}(W(F, \alpha, \alpha_\mu, \alpha_\eta)) = (F, \alpha, \alpha_\mu, \alpha_\eta)$$

We have that

$$W\left(\begin{array}{c} \mathcal{X} \\ \big\downarrow{\scriptstyle F} \\ \mathcal{C} \end{array}, \begin{tikzcd} \mathcal{C}\!\downarrow\! F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}, \alpha_\mu, \alpha_\eta\right)$$

is the functor $\begin{array}{c} \mathcal{X} \\ \downarrow{\scriptstyle F} \\ \mathcal{C} \end{array}$ endowed with the cleavage defined by

$$\mathrm{Cart}\,(f, E) := (\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id}) \colon \alpha_1(A, E, A \xrightarrow{f} F(E)) \longrightarrow E \tag{3.36}$$

for every $E \in \mathcal{X}$ and every $f \colon A \to F(E)$ in $\mathcal{C}$.
And then by construction $\tilde{J}(W(F, \alpha, \alpha_\mu, \alpha_\eta))$ is

$$\begin{tikzcd} \mathcal{C}\!\downarrow\! F \arrow[r, "Q_F"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

endowed with the natural transformations α_μ^F and α_η^F described in Construction 3.4.2.

It suffices to prove, then, that $Q_F = \alpha_1$, $\alpha_\mu^F = \alpha_\mu$ and $\alpha_\eta^F = \alpha_\eta$.

On objects $(A, E, A \xrightarrow{f} F(E)) \in \mathcal{C}{\downarrow}F$, we have that

$$Q_F(A, E, A \xrightarrow{f} F(E)) = \alpha_1(A, E, A \xrightarrow{f} F(E))$$

by equation (3.36).

On morphisms $(A', E', A' \xrightarrow{f'} F(E')) \xrightarrow{(a,e)} (A, E, A \xrightarrow{f} F(E))$ in $\mathcal{C}{\downarrow}F$, we have that Q_F acts as

$$\begin{tikzcd}
& E' \arrow[d, mapsto, "F"] \arrow[dr, bend left, "e"] & \\
& & E \arrow[dd, mapsto, "F"] \\
A' \arrow[r, "f'"'] \arrow[d, bend right, "a"'] & F(E') \arrow[dr, "F(e)"] & \\
A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}
\quad \overset{Q_F}{\longmapsto} \quad
\begin{tikzcd}
\alpha_1(A', E', A' \xrightarrow{f'} F(E')) \arrow[rr, "(\alpha_\eta)_{1,E'} \circ \alpha_1(f',\mathrm{id})"] \arrow[dd, mapsto, "F"] \arrow[dr, dashed, "Q_F(a,e)"] & & E' \arrow[dd, mapsto, "F"] \arrow[dr, bend left, "e"] & \\
& \alpha_1(A, E, A \xrightarrow{f} F(E)) \arrow[rr, "(\alpha_\eta)_{1,E} \circ \alpha_1(f,\mathrm{id})"] & & E \arrow[dd, mapsto, "F"] \\
A' \arrow[rr, "f'"'] \arrow[dr, bend right, "a"'] & & F(E') \arrow[dr, "F(e)"] & \\
& A \arrow[rr, "f"'] \arrow[from=uu, mapsto, crossing over, "F"] & & F(E)
\end{tikzcd}$$

But $(\alpha_\eta)_{1,E} \circ \alpha_1(f, \mathrm{id})$ is a cartesian morphism and also $\alpha_1(a, e)$ fits well into the diagram above on the right in the place of the morphism in violet. In fact

$$F(\alpha_1(a, e)) = \mathrm{pr}_1(a, e) = a,$$

by the fact that $(F, (\mathrm{Id}, \alpha_1), \alpha_\mu, \alpha_\eta)$ is a pseudo-T-algebra, and by naturality of $(\alpha_\eta)_1$.

Therefore Q_F and α_1 need to coincide on morphisms and then they are equal as functors. At this point, the fact that $\alpha_\mu^F = \alpha_\mu$ and $\alpha_\eta^F = \alpha_\eta$ easily follows with cartesianity diagrams, whence $\tilde{J} \circ W$ coincides with the identity functor on 0-cells.

Now let $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}$ be a cloven fibration over $\mathcal{C}$, with cleavage denoted $\mathrm{Cart}_p(-, \cdot)$. We prove that

$$W\left(\tilde{J}\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\right)\right) = \begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}$$

We have that

$$\tilde{J}\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\right) = \begin{tikzcd}
\mathcal{C}{\downarrow}p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

equipped with the natural isomorphisms α_μ and α_η produced in Construction 3.4.2, and

$$W\left(\begin{tikzcd}
\mathcal{C}{\downarrow}p \arrow[r, "Q_p"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}\right)$$

coincides with $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}$ equipped with the splitting clevage defined by

$$\operatorname{Cart}(f,E) := (\alpha_\eta)_{1,E} \circ Q_p(f,\mathrm{id})\colon \operatorname{Cart}_p(f,E) \longrightarrow E$$

for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$.
But this cleavage coincides with the original one $\operatorname{Cart}_p(-,\cdot)$, since

$$(A, E, A \xrightarrow{f} p(E)) \overset{Q_p}{\longmapsto} f^*E, \qquad (p(E), E, p(E) \xrightarrow{\mathrm{id}} p(E)) \overset{Q_p}{\longmapsto} \mathrm{id}^* E$$

and

$$\begin{tikzcd}
 & & E \arrow[dr, bend left, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & \\
 & & & E \arrow[dd, mapsto, "p"] \\
A \arrow[rr, "u"'] \arrow[dr, bend right, "f"'] & & p(E) \arrow[dr, "\mathrm{id}"] & \\
 & p(E) \arrow[rr, "\mathrm{id}"'] & & p(E)
\end{tikzcd}
\quad \overset{Q_p}{\longmapsto} \quad
\begin{tikzcd}
f^*E \arrow[rr, "\operatorname{Cart}_p(f,E)"] \arrow[dr, dashed, "Q_p(f,\mathrm{id})"] \arrow[dd, mapsto, "p"] & & E \arrow[dr, bend left, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & \\
 & \mathrm{id}^* E \arrow[rr, "\operatorname{Cart}_p(\mathrm{id},E)"] \arrow[dd, mapsto, "p"] & & E \arrow[dd, mapsto, "p"] \\
A \arrow[rr, "f"'] \arrow[dr, bend right, "f"'] & & p(E) \arrow[dr, "\mathrm{id}"] & \\
 & p(E) \arrow[rr, "\mathrm{id}"'] & & p(E)
\end{tikzcd}$$

whence we get that

$$\operatorname{Cart}_p(f,E) = \operatorname{Cart}_p(\mathrm{id},E) \circ Q_p(f,\mathrm{id}) = (\alpha_\eta)_{1,E} \circ Q_p(f,\mathrm{id}) = \operatorname{Cart}(f,E).$$

We have thus proved that $\tilde{J}$ and W are inverses of each other on 0-cells. But on 1-cells and on 2-cells both $\tilde{J}$ and W are essentially defined as the identity.
Therefore we immediatly get that

$$W \circ \tilde{J} = \mathrm{Id}_{\mathcal{Clo}\mathcal{FIB}_{\mathcal{C}}} \quad \text{and} \quad \tilde{J} \circ W = \mathrm{Id}_{\mathcal{Ps}\text{-}\mathcal{T}\text{-}\mathcal{Alg}},$$

which means that the 2-functor $\tilde{J}$ is an isomorphism of 2-categories with inverse 2-functor W. $\square$

Chapter 4

Comonadicity of Fibrations

In the last chapter we have proved that the 2-category $\mathcal{SpFIB}$ is 2-monadic over $\mathcal{CAT}^{\to}$. In this chapter we investigate whether $\mathcal{SpFIB}$ is also 2-comonadic over the same 2-category $\mathcal{CAT}^{\to}$.

We will see that we have to restrict ourselves to $\mathcal{SpFIB}_{\mathcal{C}}$, that is, to split fibrations over a fixed base category $\mathcal{C}$. Then we will prove that the forgetful 2-functor $L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$ is comonadic.

4.1 A First Attempt

We aim at finding a comonad whose category of coalgebras is equivalent to a category of Grothendieck fibrations. For the same reasons we have discussed in Remark 3.1.9, we have to restrict ourselves to split fibrations. We will see a more explicit justification of this in Remark 4.2.7.

Moreover, since we have defined (split) Grothendieck fibrations by adding structure to the 2-category $\mathcal{CAT}^{\to}$, it is natural to search for a comonad on $\mathcal{CAT}^{\to}$.

Since every 2-comonad comes from a 2-adjunction, we first search for a 2-adjunction between $\mathcal{SpFIB}$ and $\mathcal{CAT}^{\to}$.
As left adjoint, it is natural to consider the forgetful 2-functor

$$L\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}.$$

Now, we would like to construct a split fibration starting from a functor $F\colon \mathcal{X} \to \mathcal{C}$.

We might recall Proposition 2.4.8, which showed that the structure of a cloven fibration $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{matrix}$ produces a functor

$$\mathrm{Cart}_p(-,E)\colon \mathcal{B}\downarrow p(E) \to \mathcal{E}$$

for every $E \in \mathcal{E}$ fixed. Actually, we could prove something stronger.

Proposition 4.1.1. *Let* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ *be a cloven fibration. Fixed a certain* $E \in \mathcal{E}$*, the assignment* $\mathrm{Cart}_p(-,E)$ *extends to a functor* $\mathcal{B}\downarrow p(E) \to \mathcal{E}$ *such that the triangle*

$$\begin{tikzcd}
\mathcal{B}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-{,}E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[d, "p"] \\
& \mathcal{B}
\end{tikzcd}$$

in $\mathcal{C}at$ *is commutative, where* dom *is the domain functor, defined by:*

$$\mathrm{dom}\colon \mathcal{B}\downarrow p(E) \longrightarrow \mathcal{B}$$

$$(B \xrightarrow{u} p(E)) \longmapsto B$$

$$\begin{array}{c}\begin{tikzcd}
B' \arrow[d, "h"'] \arrow[dr, "u'"] & \\
B \arrow[r, "u"'] & p(E)
\end{tikzcd}\end{array} \longmapsto h$$

Proof. We have proved in Proposition 2.4.8 that the assignment $\mathrm{Cart}_p(-,E)$ extends to a functor $\mathcal{B}\downarrow p(E) \to \mathcal{E}$, by defining $\mathrm{Cart}_p(-,E)(B \xrightarrow{b} p(E)) := \mathrm{Cart}_p\left(B \xrightarrow{b} p(E), E\right) = b^*E$ and

$$\mathrm{Cart}_p\left(\begin{array}{c}\begin{tikzcd}
B \arrow[d, "h"'] \arrow[dr, "b"] & \\
C \arrow[r, "c"'] & p(E)
\end{tikzcd}\end{array}, E\right)$$

to be the unique morphism $v\colon b^*E \to c^*E$ which fits well into the diagram

$$\begin{tikzcd}
b^*E \arrow[dr, dashed, "v"] \arrow[drr, bend left, "\mathrm{Cart}(b{,}E)"] \arrow[dd, "p"] & & \\
& c^*E \arrow[r, "\mathrm{Cart}(c{,}E)"] \arrow[dd, "p"] & E \arrow[dd, "p"] \\
B \arrow[dr, "h"'] \arrow[drr, bend left, "ch=b"] & & \\
& C \arrow[r, "c"'] & p(E)
\end{tikzcd}$$

We then see that

$$p\left(\mathrm{Cart}_p(-,E)(B \xrightarrow{b} p(E))\right) = p\left(b^*E\right) = B$$

and

$$p\left(\mathrm{Cart}_p\left(\begin{array}{c}\begin{tikzcd}
B \arrow[d, "h"'] \arrow[dr, "b"] & \\
C \arrow[r, "c"'] & p(E)
\end{tikzcd}\end{array}, E\right)\right) = h.$$

It is thus clear that the triangle

$$\begin{tikzcd} \mathcal{B}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{B} \end{tikzcd}$$

is commutative. □

We might wonder if we can use a sort of converse of Proposition 4.1.1 to construct a cloven fibration starting from a functor $F\colon \mathcal{X} \to \mathcal{C}$.

We ask ourselves if starting from a fixed $E \in \mathcal{X}$ and from a commutative triangle

$$\begin{tikzcd} \mathcal{C}\downarrow F(E) \arrow[r, "\alpha_1"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[d, "F"] \\ & \mathcal{B} \end{tikzcd}$$

in $\mathcal{C}at$ we could view this triangle as an assignment of cartesian liftings to E of each morphism $A \xrightarrow{f} F(E)$ in $\mathcal{C}$.

Given a morphism $A \xrightarrow{f} F(E)$ in $\mathcal{C}$, we see that $\alpha_1(A \xrightarrow{f} F(E))$ is an object of $\mathcal{E}$ which is above B, by commutativity of the triangle, and thus it is a good candidate for f^*E. In order to produce a lifting of f, it might be useful to consider the morphism in $\mathcal{C}\downarrow F(E)$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}$$

We have that

$$\alpha_1\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) : \alpha_1(A \xrightarrow{f} F(E)) \longrightarrow \alpha_1(\mathrm{id}_{F(E)}). \tag{4.1}$$

We have said above that $\alpha_1(A \xrightarrow{f} F(E))$ might be a good candidate for f^*E, then, in order to have a cartesian lifting of f to E, it remains to see if $\alpha_1(\mathrm{id}_{F(E)})$ is equal to E and if the morphism in equation (4.1) is cartesian.

The problem is that nothing can ensure that $\alpha_1(\mathrm{id}_{F(E)})$ is E; we just know that it is an object above $F(E)$. Remember that we have already faced a similar situation, when we proved that a T-algebra

$$\begin{tikzcd} \mathcal{C}\downarrow F \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd}$$

is a split fibration, in Theorem 3.3.8. To prove that theorem, we strongly needed axiom (ALG1) of T-algebra (from Remark 3.3.2), and here we do not have such an axiom or a similar one. For this reason, an easy converse of Proposition 4.1.1 cannot hold.

However, there is a more subtle way to construct a cloven fibration (which will be also split) starting from commutative triangles in $\mathcal{C}at$ of the form

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

with $C \in \mathcal{C}$.

Construction 4.1.2. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor and let $C \in \mathcal{C}$. Consider the following data, which we may denote $\mathrm{Tr}(C, F)$:

an object of $\mathrm{Tr}(C, F)$ is a commutative triangle in $\mathcal{C}at$ from the functor $\mathrm{dom}\colon \mathcal{C}\downarrow C \to \mathcal{C}$ to the functor $F\colon \mathcal{X} \to \mathcal{C}$:

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd};$$

a morphism $\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \longrightarrow \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha'"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$ **in** $\mathrm{Tr}(C, F)$ is a natural transformation $\lambda\colon \alpha \Rightarrow \alpha'$ such that

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\Downarrow\lambda"] \arrow[d, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, bend right, "\mathrm{Id}"'] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, bend left, "\alpha"] \arrow[d, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & \mathcal{C} \end{tikzcd}$$

(that is, such that (Id, λ) is a modification from (Id, α) to (Id, α')).

Proposition 4.1.3. *The data* $\mathrm{Tr}(C, F)$ *we have described in Construction 4.1.2 form a category.*

Proof. The proof is trivial, since $\mathrm{Tr}(C, F)$ coincides with the hom-category

$$\mathrm{Hom}_{\mathcal{CAT}\downarrow\mathcal{C}}\left(\begin{array}{c} \mathcal{C}\downarrow C \\ \downarrow \mathrm{dom} \\ \mathcal{C} \end{array}, \begin{array}{c} \mathcal{X} \\ \downarrow F \\ \mathcal{C} \end{array}\right)$$

of the 2-category $\mathcal{CAT}\downarrow\mathcal{C}$. $\square$

Remark 4.1.4. Notice that a natural transformation $\lambda\colon \alpha \Rightarrow \alpha'$ is a morphism

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \xrightarrow{\lambda} \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha'"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

in $\mathrm{Tr}(C, F)$ if and only if for every object $C' \xrightarrow{c} C$ of $\mathcal{C}\downarrow C$

$$F\left(\lambda_{C' \xrightarrow{c} C}\right) = \mathrm{id}_{C'}\,.$$

We will often write a morphism in $\mathrm{Tr}(C, F)$ as

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, bend left, "\alpha"{name=A}] \arrow[r, bend right, "\alpha'"'{name=B}] \arrow[dr, bend right, "\mathrm{dom}"'] \arrow[from=A, to=B, Rightarrow, "\lambda"] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

We now might wonder what happens as C varies in $\mathcal{C}$. As we might expect, this will produce a functor, which needs to be from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}at$.

Construction 4.1.5. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor and let $C' \xrightarrow{f} C$ a morphism in $\mathcal{C}$. Starting from an object of $\mathrm{Tr}(C, F)$, that is, a commutative triangle

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd},$$

in $\mathcal{C}at$, we need to construct an object of $\mathrm{Tr}(C', F)$. It is natural to consider the pre-composition with the post-composition functor with f:

$$\begin{tikzcd} & \mathcal{C}\downarrow C \arrow[dr, "\alpha"] & \\ \mathcal{C}\downarrow C' \arrow[ur, "f\circ -"] \arrow[rr, "\alpha(f\circ -)"] \arrow[drr, "\mathrm{dom}"'] & & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

Analogously, if we start from a morphism

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, bend left, "\alpha"{name=A}] \arrow[r, bend right, "\alpha'"'{name=B}] \arrow[dr, bend right, "\mathrm{dom}"'] \arrow[from=A, to=B, Rightarrow, "\lambda"] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

in $\mathrm{Tr}(C,F)$, we construct a morphism in $\mathrm{Tr}(C',F)$ as

$$\begin{tikzcd}
\mathcal{C}\downarrow C' \arrow[r, "f\circ -"] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow C \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{X} \arrow[d, "F"] \\
 & & \mathcal{C}
\end{tikzcd}$$

We denote

$$\cdot(f\circ -)$$

the assignment from $\mathrm{Tr}(C,F)$ to $\mathrm{Tr}(C',F)$ (on objects and morphisms) we have just described.

Proposition 4.1.6. *Let $F\colon \mathcal{X}\to\mathcal{C}$ be a functor. If $C' \xrightarrow{f} C$ is a morphism in $\mathcal{C}$, then*

$$\cdot(f\circ -)\colon \mathrm{Tr}(C,F) \longrightarrow \mathrm{Tr}(C',F)$$

is a functor. Moreover the assignment

$$\begin{aligned}
\mathrm{Tr}(-,F) : \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{C}at \\
C &\longmapsto \mathrm{Tr}(C,F)
\end{aligned}$$

$$\begin{tikzcd}
C \arrow[r, mapsto] & \mathrm{Tr}(C,F) \arrow[d, "\cdot(f\circ -)"] \\
C' \arrow[u, "f"] \arrow[r, mapsto] & \mathrm{Tr}(C',F)
\end{tikzcd}$$

is a functor, that is, $\mathrm{Tr}(-,F)$ *is a presheaf of categories on* $\mathcal{C}$.

Proof. We prove that the assignment

$$\cdot(f\circ -)\colon \mathrm{Tr}(C,F) \longrightarrow \mathrm{Tr}(C',F)$$

satisfies functoriality.
The preservation of identities is trivial. Then it remains to show that, given two composable morphisms λ and μ in $\mathrm{Tr}(C,F)$, we have that

$$(\mu\circ\lambda)(f\circ -) = \mu(f\circ -)\circ\lambda(f\circ -).$$

But this is true by the properties of horizontal composition of natural morphisms. In fact by Proposition 1.1.7 we have that

$$\begin{tikzcd} \cdot \arrow[r, "f\circ -"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\lambda"] & \cdot \end{tikzcd} \circ \begin{tikzcd} \cdot \arrow[r, "f\circ -"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\mu"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left, "f\circ -"] \arrow[r, bend right, "f\circ -"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\mu\circ\lambda"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "f\circ -"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\mu\circ\lambda"] & \cdot \end{tikzcd}$$

□

Corollary 4.1.7. *Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Then the assignment*

$$\mathrm{Tr}(-,F) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}at$$

yields a split fibration over $\mathcal{C}$

$$\begin{tikzcd} \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \end{tikzcd}$$

Proof. By Proposition 4.1.6, the assignment

$$\mathrm{Tr}(-,F) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}at$$

is a presheaf of categories on $\mathcal{C}$. We then conclude by Theorem 2.4.14 that applying the Grothendieck construction (Construction 2.3.9) to $\mathrm{Tr}(-,F)$ we obtain a fibration $\int \mathrm{Tr}(-,F)$ over $\mathcal{C}$ which is also split. □

Remark 4.1.8. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Looking at the explicit construction of the category $\mathcal{G}_{\mathrm{Tr}(-,F)}$, from Construction 2.3.9, we see that:

an object of $\mathcal{G}_{\mathrm{Tr}(-,F)}$ is a pair

$$\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$$

with $C \in \mathcal{C}$ and $\begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$ a commutative triangle in $\mathcal{C}at$;

a morphism $\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right) \longrightarrow \left(C', \begin{tikzcd} \mathcal{C}\downarrow C' \arrow[r, "\alpha'"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$ **in** $\mathcal{G}_{\mathrm{Tr}(-,F)}$ is a pair (f,λ) with $f\colon C \to C'$ a morphism in $\mathcal{C}$ and

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[rr, bend left, "\alpha"] \arrow[r, "f\circ -"'] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow C' \arrow[r, "\alpha'"'] \arrow[u, phantom, "\Downarrow\lambda"] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

a morphism in $\mathrm{Tr}(C,F)$.

Remark 4.1.9. Now that we have created a split fibration $\int \mathrm{Tr}\,(-,F)$ starting from an arbitrary functor $F\colon \mathcal{X} \to \mathcal{C}$, we can produce a function on objects

$$U\colon\ \mathrm{Ob}(\mathcal{CAT}^{\to}) \longrightarrow \mathrm{Ob}(\mathcal{SpFIB})$$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "F"] & & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, phantom, "\longmapsto"] & & \mathcal{C} \end{tikzcd}$$

The problem of such a function on objects is that it cannot be extended, at least in an easy and natural way, to a 2-functor (not even to a 1-functor) which is right adjoint to the forgetful functor $L\colon \mathcal{SpFIB} \longrightarrow \mathcal{CAT}^{\to}$. In fact if we start from a morphism

$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "H_0"'] & \mathcal{D} \end{tikzcd}$$

in $\mathcal{CAT}^{\to}$, we would need to define a morphism

$$\begin{tikzcd} \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "?_1"] \arrow[d, "\int \mathrm{Tr}(-,F)"'] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\ \mathcal{C} \arrow[r, "?_0"'] & \mathcal{D} \end{tikzcd} \tag{4.2}$$

in $\mathcal{SpFIB}$. If we want that the function on objects U extends to a right adjoint of L, we need to set $?_0 = H_0$. Then to define $?_1$ we need to construct, starting from a pair

$$\left(C,\ \begin{tikzcd} \mathcal{C}{\downarrow}C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$$

with $C \in \mathcal{C}$ and $\begin{tikzcd} \mathcal{C}{\downarrow}C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$ a commutative triangle in $\mathcal{C}at$, an object of $\mathcal{G}_{\mathrm{Tr}(-,G)}$ of the form

$$\left(H_0(C),\ \begin{tikzcd} \mathcal{D}{\downarrow}H_0(C) \arrow[r, "\beta"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{Y} \arrow[d, "G"] \\ & \mathcal{D} \end{tikzcd} \right)$$

(where we have used the explicit description of the category $\mathcal{G}_{\mathrm{Tr}(-,F)}$, from Remark 4.1.8, and also the commutativity of diagram (4.2)). And the problem is that we should define such a functor $\beta\colon \mathcal{D}\downarrow H_0(C) \longrightarrow \mathcal{Y}$ from the situation

$$\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[d, "H_0"'] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"] & \mathcal{Y} \arrow[d, "G"] \\
 & \mathcal{C} \arrow[r, "H_0"] & \mathcal{D} \\
\mathcal{D}\downarrow H_0(C) & &
\end{tikzcd} \tag{4.3}$$

A solution to this problem is to restrict ourselves to split fibrations over a fixed base category $\mathcal{C}$, and to search for an adjunction

$$\begin{tikzcd}
\mathcal{SpFIB}_{\mathcal{C}} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{CAT}\downarrow\mathcal{C} \arrow[l, bend left, "U"]
\end{tikzcd}$$

In fact, in this way, the situation of diagram (4.3) will strongly improve, as, with informal words, $H_0\colon \mathcal{C}\to\mathcal{D}$ becomes the identity functor and $\mathcal{D}\downarrow H_0(C)$ becomes $\mathcal{C}\downarrow C$. From this, it is clear that it suffices to define the functor β as

$$\beta := H_1\circ\alpha$$

since we have the commutativity of diagram

$$\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[rr, bend left, "\beta"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"] & \mathcal{Y} \arrow[d, "G"] \\
 & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

4.2 A 2-adjunction between $\mathcal{SpFIB}_{\mathcal{C}}$ and $\mathcal{CAT}\downarrow\mathcal{C}$

In this section we will apply what we have done in Section 4.1 to functors and split fibrations over a fixed base category $\mathcal{C}$. We will then formalise what we have informally said at the end of Remark 4.1.9 and prove that there is an adjunction between $\mathcal{SpFIB}_{\mathcal{C}}$ and $\mathcal{CAT}\downarrow\mathcal{C}$, in order to prove, in Section 4.3, that the category $\mathcal{SpFIB}_{\mathcal{C}}$ is comonadic over $\mathcal{CAT}\downarrow\mathcal{C}$.

Let

$$L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$$

$$\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \longmapsto \begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend right, equal] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend right, equal] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd}$$

be the forgetful 2-functor.

We aim at proving that the 2-functor L is (2-)comonadic. Firstly, we search for a right 2-adjoint to L.

By Theorem 1.9.5, to have a right 2-adjoint to L it suffices to find for each functor $F\colon \mathcal{X} \to \mathcal{C}$ a 2-universal morphism from the 2-functor L to F, say

$$(\varepsilon_F)_1\colon\ L\left(\begin{array}{c}\mathcal{A}_F\\ \downarrow R(F)\\ \mathcal{C}\end{array}\right) \longrightarrow \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}.$$

(This $R(F)$ will be the cofree split fibration over $\mathcal{C}$ associated to F.)

But this means that for every $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \in \mathcal{S}p\mathcal{FIB}_{\mathcal{C}}$ and for every 1-morphism

$$H_1\colon\ L\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\right) \longrightarrow \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$$

in $\mathcal{CAT}\downarrow\mathcal{C}$, there exists a unique 1-morphism

$$V_{H_1}\colon\ \begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \longrightarrow \begin{array}{c}\mathcal{A}_F\\ \downarrow R(F)\\ \mathcal{C}\end{array}$$

in $\mathcal{S}p\mathcal{FIB}_{\mathcal{C}}$ such that the triangle in $\mathcal{CAT}\downarrow\mathcal{C}$

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[dr, "V_{H_1}"'] \arrow[dd, "p"'] & & \mathcal{X} \arrow[dd, "F"] \\
& \mathcal{A}_F \arrow[ur, "(\varepsilon_F)_1"'] \arrow[dd, "R(F)"] & \\
\mathcal{C} \arrow[rr, bend left, equal] \arrow[dr, equal] & & \mathcal{C} \\
& \mathcal{C} \arrow[ur, equal] &
\end{tikzcd}
\tag{4.4}
$$

is commutative, and that for every 2-morphism

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, bend right, equal] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
$$

in $\mathcal{CAT}\downarrow\mathcal{C}$, there exists a unique 2-morphism

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[r, bend right, "V_{K_1}"'] \arrow[r, phantom, "\Downarrow\delta_1"] \arrow[d, "p"'] & \mathcal{A}_F \arrow[d, "R(F)"] \\
\mathcal{C} \arrow[r, bend right, equal] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[d, "p"'] & \mathcal{A}_F \arrow[d, "R(F)"] \\
\mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & \mathcal{C}
\end{tikzcd}
$$

in $\mathcal{SpFIB}_{\mathcal{C}}$ which makes the following diagram of 2-morphisms in $\mathcal{CAT}\downarrow\mathcal{C}$ commute:

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[rr, "K_1"'] \arrow[rr, phantom, "\Downarrow\lambda_1"] \arrow[dr, bend left, "V_{H_1}"] \arrow[dr, bend right, "V_{K_1}"'] \arrow[dr, phantom, "\Downarrow\delta_1"] \arrow[d, "p"'] & & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} & \mathcal{A}_F \arrow[ur, "(\varepsilon_F)_1"'] \arrow[d, "R(F)"] & \mathcal{C} \\
& \mathcal{C} &
\end{tikzcd}
\tag{4.5}
$$

Remark 4.2.1. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. Looking at Remark 4.1.9, we set

$$R(F) := \begin{tikzcd} \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \end{tikzcd}$$

By Corollary 4.1.7, we know that $R(F)$ is a split fibration over $\mathcal{C}$.

We then search for a 2-universal morphism from the 2-functor L to F of the form

$$(\varepsilon_F)_1\colon L\left(\begin{tikzcd} \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \end{tikzcd}\right) \longrightarrow \begin{tikzcd} \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \end{tikzcd}.$$

Construction 4.2.2. Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor. We need to construct a functor

$$(\varepsilon_F)_1\colon \mathcal{G}_{\mathrm{Tr}(-,F)} \longrightarrow \mathcal{X}$$

which makes the square in $\mathcal{C}at$

$$\begin{tikzcd} \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "(\varepsilon_F)_1"] \arrow[d, "\int \mathrm{Tr}(-,F)"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \tag{4.6}$$

(which actually is a triangle) commute.

Given an object of $\mathcal{G}_{\mathrm{Tr}(-,F)}$

$$\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right),$$

we have that

$$\int \mathrm{Tr}\,(-,F)\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right) = C,$$

whence it must hold that

$$F\left((\varepsilon_F)_1\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right)\right) = C.$$

We then define

$$(\varepsilon_F)_1\left(C,\ \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right) = \alpha\left(C \overset{\mathrm{id}}{\to} C\right),$$

so that diagram (4.6) trivially commutes on objects:

$$F(\alpha(C \overset{\mathrm{id}}{\to} C)) = \mathrm{dom}(C \overset{\mathrm{id}}{\to} C) = C.$$

Now, given a morphism (f,λ) in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, that is, a pair (f,λ) with $f\colon C \to C'$ a morphism in $\mathcal{C}$ and

$$\begin{tikzcd} \mathcal{C}\downarrow C \arrow[rr, bend left, "\alpha"] \arrow[r, "f\circ -"'] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow C' \arrow[u, phantom, "\Downarrow\lambda"] \arrow[r, "\alpha'"'] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

a morphism in $\mathrm{Tr}(C,F)$, we need to define a morphism

$$(\varepsilon_F)_1(f,\lambda)\colon \alpha(C \overset{\mathrm{id}}{\to} C) \longrightarrow \alpha'(C' \overset{\mathrm{id}}{\to} C').$$

The natural transformation λ gives us a morphism

$$\alpha(C \overset{\mathrm{id}}{\to} C) \xrightarrow{\lambda_{(C\overset{\mathrm{id}}{\to}C)}} \alpha'(f\circ -)(C \overset{\mathrm{id}}{\to} C) = \alpha'(C \overset{f}{\to} C')$$

We then define

$$(\varepsilon_F)_1(f,\lambda) := \alpha'\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ C' \arrow[r, "\mathrm{id}"'] & C' \end{tikzcd}\right)\circ\lambda_{(C\overset{\mathrm{id}}{\to}C)}.$$

And we see that $(\varepsilon_F)_1$ (assumed that it is a functor, as we shall promptly prove) makes diagram (4.6) commute:

$$F\left(\alpha'\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ C' \arrow[r, "\mathrm{id}"'] & C' \end{tikzcd}\right)\circ\lambda_{(C\overset{\mathrm{id}}{\to}C)}\right) = F\left(\alpha'\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ C' \arrow[r, "\mathrm{id}"'] & C' \end{tikzcd}\right)\right)\circ F(\lambda_{(C\overset{\mathrm{id}}{\to}C)}) = f\circ\mathrm{id}_{C'} = f$$

Proposition 4.2.3. *Let $F\colon \mathcal{X}\to\mathcal{C}$ be a functor. Then the assignment*

$$(\varepsilon_F)_1\colon \mathcal{G}_{\mathrm{Tr}(-,F)} \longrightarrow \mathcal{X}$$

produced in Construction 4.2.2 is a functor.

Proof. We immediatly see that for every $\left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right) \in \mathcal{G}_{\mathrm{Tr}(-,F)}$

$$(\varepsilon_F)_1(\mathrm{id}_C, \mathrm{id}_\alpha) = \mathrm{id}_C,$$

which means that $(\varepsilon_F)_1$ preserves the identities.

Consider now a composite

$$\left(B, \begin{tikzcd} \mathcal{C}\downarrow B \arrow[r, "\beta"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right) \xrightarrow{(b,\mathtt{x})} \left(C, \begin{tikzcd} \mathcal{C}\downarrow C \arrow[r, "\gamma"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right) \xrightarrow{(c,\mathtt{y})} \left(D, \begin{tikzcd} \mathcal{C}\downarrow D \arrow[r, "\delta"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}\right)$$

in $\mathcal{G}_{\mathrm{Tr}(-,F)}$.
We have that

$$(\varepsilon_F)_1(c,\mathtt{y}) \circ (\varepsilon_F)_1(b,\mathtt{x}) = \delta\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "c"] & \\ D \arrow[r, "\mathrm{id}"'] & D \end{tikzcd}\right) \circ \mathtt{y}_{(C \xrightarrow{\mathrm{id}} C)} \circ \gamma\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "b"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right) \circ \mathtt{x}_{(B \xrightarrow{\mathrm{id}} B)},$$

whereas

$$(\varepsilon_F)_1((c,\mathtt{y}) \circ (b,\mathtt{x})) = (\varepsilon_F)_1(c \circ b, \mathtt{y}(b \circ -) \circ \mathtt{x}) = \delta\left(\begin{tikzcd} B \arrow[d, "cb"'] \arrow[dr, "cb"] & \\ D \arrow[r, "\mathrm{id}"'] & D \end{tikzcd}\right) \circ (\mathtt{y}(b \circ -))_{(B \xrightarrow{\mathrm{id}} B)} \circ \mathtt{x}_{(B \xrightarrow{\mathrm{id}} B)} =$$

$$= \delta\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "c"] & \\ D \arrow[r, "\mathrm{id}"'] & D \end{tikzcd}\right) \circ \delta\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "cb"] & \\ C \arrow[r, "c"'] & D \end{tikzcd}\right) \circ \mathtt{y}_{(B \xrightarrow{b} C)} \circ \mathtt{x}_{(B \xrightarrow{\mathrm{id}} B)}.$$

It then suffices to prove that

$$\mathtt{y}_{(C \xrightarrow{\mathrm{id}} C)} \circ \gamma\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "b"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right) = \delta\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "cb"] & \\ C \arrow[r, "c"'] & D \end{tikzcd}\right) \circ \mathtt{y}_{(B \xrightarrow{b} C)}. \tag{4.7}$$

But

$$\delta\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "cb"] & \\ C \arrow[r, "c"'] & D \end{tikzcd}\right) = \delta\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "b"] & & \\ C \arrow[r, "\mathrm{id}"'] & C \arrow[r, "c"] & D \end{tikzcd}\right) = \delta(c \circ -)\left(\begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, "b"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right)$$

and then we conclude that equation (4.7) holds by naturality of

$$\mathrm{y}\colon \gamma \Longrightarrow \delta(c\circ -).$$

□

We now aim at proving that the functor $(\varepsilon_F)_1$ is a 2-universal morphism.

Construction 4.2.4. Let $F\colon \mathcal{X}\to\mathcal{C}$ be a functor, $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \in \mathcal{SpFIB}_{\mathcal{C}}$ and

$$H_1\colon L\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\right)\longrightarrow \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$$

be a 1-morphism in $\mathcal{CAT}\downarrow\mathcal{C}$.
We need to construct (and then prove its uniqueness) a 1-morphism

$$V_{H_1}\colon \begin{array}{c}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{array}\longrightarrow \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$$

in $\mathcal{SpFIB}_{\mathcal{C}}$ which makes the triangle in $\mathcal{CAT}\downarrow\mathcal{C}$

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[r, bend right, "V_{H_1}"'] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, bend right, "(\varepsilon_F)_1"'] \arrow[dd, "\int\mathrm{Tr}(-,F)"] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[rr, equal, bend left] \arrow[dr, equal, bend right] & & \mathcal{C} \\
& \mathcal{C} \arrow[ur, equal, bend right] &
\end{tikzcd}$$

(which is diagram (4.4) after the choice of $R(F)$) commute.

We try to start from $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \in \mathcal{ClFIB}_{\mathcal{C}}$ a normal fibration and define V_{H_1} from it. We will see later, in Remark 4.2.7, where we really need to start from a split fibration, justifying why we need to restrict to split fibrations over $\mathcal{C}$ to have a right 2-adjoint to L.

Since in particular the square in $\mathcal{Cat}$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "V_{H_1}"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int\mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

needs to commute, we have to define V_{H_1} such that for every $E \in \mathcal{E}$ the first component of $V_{H_1}(E)$ is $p(E)$ and for every morphism $E \xrightarrow{e} E'$ in $\mathcal{E}$ the first component of $V_{H_1}(e)$ is $p(e)$.

Given $E \in \mathcal{E}$, we then have to search for an object $V_{H_1}(E)$ of $\mathcal{G}_{\mathrm{Tr}(-,F)}$ of the form

$$\begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\beta"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}.$$

Also looking at Remark 4.1.9, we might have the idea to define such a functor β as $H_1 \circ \alpha$

$$\begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] \arrow[rr, bend left, "\beta"] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

for some functor $\alpha\colon \mathcal{C}\downarrow p(E) \to \mathcal{E}$ such that the triangle in $\mathcal{C}at$

$$\begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{C} \end{tikzcd}$$

commutes. But we have already constructed a such a functor α in Proposition 4.1.1, that is, the functor

$$\mathrm{Cart}_p(-,E)\colon \mathcal{C}\downarrow p(E) \longrightarrow \mathcal{E}.$$

We thus define

$$V_{H_1}(E) := \left(p(E), \begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \right)$$

Let now $E \xrightarrow{e} E'$ be a morphism in $\mathcal{E}$. We already know that the first component of $V_{H_1}(e)$ has to be $p(e)$. Then $V_{H_1}(e)$ will be of the form $(p(e), \mu^e)$ with

$$\begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[rr, bend left, "H_1\mathrm{Cart}_p(-,E)"] \arrow[r, "p(e)\circ -"'] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow p(E') \arrow[u, phantom, "\Downarrow \mu^e"] \arrow[r, "H_1\mathrm{Cart}_p(-,E')"'] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

a morphism in $\mathrm{Tr}\,(p(E),F)$. It is natural to take $\mu^e := H_1\lambda^e$ with λ^e a morphism in $\mathrm{Tr}\,(p(E),p)$ of the form

$$\begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[rr, bend left, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "p(e)\circ -"'] \arrow[ddrr, bend right, "\mathrm{dom}"'] & \arrow[d, phantom, "\Downarrow \lambda^e"] & \mathcal{E} \arrow[r, "H_1"] \arrow[dd, "p"] & \mathcal{X} \arrow[dd, "F"] \\
& \mathcal{C}\downarrow p(E') \arrow[ur, "\mathrm{Cart}_p(-,E')"'] & & \\
& & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

It remains to construct, given $(C \xrightarrow{f} p(E)) \in \mathcal{C}\downarrow p(E)$, a morphism

$$\lambda^e_f \colon f^*E \longrightarrow (p(e)f)^*E'.$$

Then we shall consider the diagrams f^*E and $(p(e)f)^*E$ fits well into, and define λ^e_f to be the unique morphism which fits well into the diagram

$$\begin{tikzcd}
f^*E \arrow[rr, "\mathrm{Cart}_p(f,E)"] \arrow[dr, dashed, violet, "\lambda^e_f"] \arrow[dd, "p"] & & E \arrow[dr, bend left, "e"] \arrow[dd, "p"] & \\
& (p(e)f)^*E' \arrow[rr, "\mathrm{Cart}_p(p(e)f,E')"] \arrow[dd, "p"] & & E' \arrow[dd, "p"] \\
C \arrow[rr, "f"'] \arrow[dr, equal, bend right] & & p(E) \arrow[dr, "p(e)"] & \\
& C \arrow[rr, "p(e)f"'] & & p(E')
\end{tikzcd}$$

in the place of the morphism in violet.

And now we define

$$V_{H_1}(e) := (p(e), H_1\lambda^e).$$

Then, by construction, it is clear that the square in $\mathcal{Cat}$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "V_{H_1}"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is commutative.

Theorem 4.2.5. *Let $F\colon \mathcal{X} \to \mathcal{C}$ be a functor, $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array} \in \mathcal{C\ell FIB}_{\mathcal{C}}$ be a normal fibration and*

$$H_1\colon L\begin{pmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{pmatrix}\longrightarrow \begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$$

be a 1*-morphism in* $\mathcal{CAT}\downarrow\mathcal{C}$. *Then the assignments* λ^e_f *produced in Construction 4.2.4 for every* $(C\xrightarrow{f}p(E))\in\mathcal{C}\downarrow p(E)$ *form a morphism* λ^e *in* $\mathrm{Tr}(p(E),p)$.
Moreover the assignment V_{H_1} *(on objects and morphisms) produced in Construction 4.2.4 is a functor from* $\begin{matrix}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{matrix}$ *to* $\begin{matrix}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{matrix}$.

Proof. Let $\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "f"] & \\ C' \arrow[r, "f'"'] & p(E)\end{tikzcd}$ be a morphism in $\mathcal{C}\downarrow p(E)$. We need to prove that

$$\mathrm{Cart}_p(-,E')\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "f"] & & \\ C' \arrow[r, "f'"'] & p(E) \arrow[r, "p(e)"] & p(E')\end{tikzcd}\right)\circ\lambda^e_f=\lambda^e_{f'}\circ\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "f"] & \\ C' \arrow[r, "f'"'] & p(E)\end{tikzcd}\right). \quad (4.8)$$

But looking at the proof of Proposition 2.4.8, we see that

$$\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "f"] & \\ C' \arrow[r, "f'"'] & p(E)\end{tikzcd}\right)$$

is defined to be the unique morphism v which fits well into the diagram

$$\begin{tikzcd} f^*E \arrow[d, mapsto, "p"] \arrow[r, dashed, "\exists! v"] \arrow[rr, bend left, "\mathrm{Cart}_p(f,E)"] & (f')^*E \arrow[d, mapsto, "p"] \arrow[r, "\mathrm{Cart}_p(f',E)"] & E \arrow[d, mapsto, "p"] \\ C \arrow[r, "c"'] \arrow[rr, bend left, "f"] & C' \arrow[r, "f'"'] & p(E)\end{tikzcd}$$

in the place of the morphism in violet, and, analogously, that

$$\mathrm{Cart}_p(-,E')\left(\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, "f"] & & \\ C' \arrow[r, "f'"'] & p(E) \arrow[r, "p(e)"] & p(E')\end{tikzcd}\right)$$

is the unique morphism w which fits well into the diagram

$$
\begin{tikzcd}
(p(e)f)^*E' \arrow[rrd, bend left, "\mathrm{Cart}_p(p(e)f,E')"] \arrow[rd, dashed, "\exists! w"] \arrow[dd, mapsto, "p"] & & \\
& (p(e)f')^*E' \arrow[r, "\mathrm{Cart}_p(p(e)f',E')"] \arrow[dd, mapsto, "p"] & E' \arrow[dd, mapsto, "p"] \\
C \arrow[rd, "c"'] \arrow[rrd, bend left, "p(e)f"] & & \\
& C' \arrow[r, "p(e)f'"'] & p(E')
\end{tikzcd}
$$

in the place of the morphism in violet.

We then have that both $\lambda^e_{f'} \circ v$ and $w \circ \lambda^e_f$ fit well into the diagram

$$
\begin{tikzcd}
& f^*E \arrow[rrrd, bend left, "\mathrm{Cart}_p(f,E)"] \arrow[rd, "v"] \arrow[dddl, bend right, "\lambda^e_f"'] \arrow[dd, mapsto, "p"] & & & & \\
& & (f')^*E \arrow[rr, "\mathrm{Cart}_p(f',E)"] \arrow[ddr, "\lambda^e_{f'}"] \arrow[ddd, mapsto, "p"] & & E \arrow[ddr, "e"] \arrow[ddd, mapsto, "p"] & \\
& C \arrow[ddrrr, "f"] \arrow[ddr, "c"] \arrow[dddl, equals] & & & & \\
(p(e)f)^*E' \arrow[rrr, "w"] \arrow[rrrrr, bend right, "\mathrm{Cart}_p(p(e)f,E')"'] \arrow[dd, mapsto, "p"] & & & (p(e)f')^*E' \arrow[rr, "\mathrm{Cart}_p(p(e)f',E')"] \arrow[dd, mapsto, "p"] & & E' \arrow[dd, mapsto, "p"] \\
& & C' \arrow[rr, "f'"'] \arrow[dr, equals] & & p(E) \arrow[dr, "p(e)"] & \\
C \arrow[rrr, bend right, "c"] \arrow[rrrrr, bend right=40, "p(e)f"'] & & & C' \arrow[rr, "p(e)f'"'] & & p(E')
\end{tikzcd}
$$

whence we conclude that equation (4.8) holds by the fact that $\mathrm{Cart}_p(p(e)f', E')$ is a cartesian morphism. Therefore we have proved that the assignments λ^e_f form a natural transformation

$$\lambda^e \colon \mathrm{Cart}_p(-,E) \Longrightarrow \mathrm{Cart}_p(-,E')(p(e)\circ -).$$

To prove that λ^e is a morphism in $\mathrm{Tr}(p(E),p)$, it only remains to show that for every object $C \xrightarrow{f} p(E)$ of $\mathcal{C}\downarrow p(E)$

$$p\left(\lambda^e_{C\xrightarrow{f}p(E)}\right) = \mathrm{id}_C,$$

by Remark 4.1.4.
But this is true by construction of $\lambda^e_{C\xrightarrow{f}p(E)}$ (see Construction 4.2.4).

Now, we prove that the assignment V_{H_1} is a functor from $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}$ to $\begin{matrix}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \downarrow \int \mathrm{Tr}(-,F)\\ \mathcal{C}\end{matrix}$.

Firstly, we show that V_{H_1} preserves the identities. Given $E \in \mathcal{E}$, we have, by Construction 4.2.4, that

$$V_{H_1}(\mathrm{id}_E) = (p(\mathrm{id}_E), H_1\lambda^{\mathrm{id}_E})$$

with

$$\begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{id}\circ -"'] \arrow[rr, bend left, "\mathrm{Cart}_p(-,E)"] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"'] \arrow[u, phantom, "\Downarrow \lambda^{\mathrm{id}_E}"] & \mathcal{E} \arrow[d, "p"] \\
 & & \mathcal{C}
\end{tikzcd}$$

defined to be such that for every object $C \xrightarrow{f} p(E)$ of $\mathcal{C}\downarrow p(E)$ the morphism $\lambda_f^{\mathrm{id}_E}$ is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
f^*E \arrow[rr, "\mathrm{Cart}_p(f,E)"] \arrow[dd, mapsto, "p"] \arrow[dr, dashed, "\lambda_f^{\mathrm{id}_E}"] & & E \arrow[dd, mapsto, "p"] \arrow[dr, bend left, "\mathrm{id}_E"] & \\
 & f^*E \arrow[rr, "\mathrm{Cart}_p(f,E)"] \arrow[dd, mapsto, "p"] & & E \arrow[dd, mapsto, "p"] \\
C \arrow[rr, "f"] \arrow[dr, equal, bend right] & & p(E) \arrow[dr, "\mathrm{id}_{p(E)}"] & \\
 & C \arrow[rr, "f"'] & & p(E)
\end{tikzcd}$$

in the place of the morphism in violet. Given an object $C \xrightarrow{f} p(E)$ of $\mathcal{C}\downarrow p(E)$, we see that id_{f^*E} is an acceptable choice for the morphism in violet, whence we obtain that

$$\lambda_f^{\mathrm{id}_E} = \mathrm{id}_{f^*E}\,.$$

And it follows that

$$(H_1\lambda^{\mathrm{id}_E})_f = \mathrm{id}_{H_1(f^*E)}\,.$$

Therefore V_{H_1} preserves the identities.

Consider now a composite in $\mathcal{E}$

$$E \xrightarrow{e} E' \xrightarrow{e'} E''.$$

We have that

$$V_{H_1}(e'e) = (p(e'e), H_1\lambda^{e'e})$$

whereas

$$V_{H_1}(e') \circ V_{H_1}(e) = (p(e'), H_1\lambda^{e'}) \circ (p(e), H_1\lambda^{e}) = (p(e')\circ p(e), H_1\lambda^{e'}(p(e)\circ -) \circ H_1\lambda^{e}).$$

It then suffices to show that

$$\lambda^{e'e} = \lambda^{e'}(p(e)\circ -)\circ\lambda^{e}. \tag{4.9}$$

Let $C \xrightarrow{f} p(E)$ be an object of $\mathcal{C}\downarrow p(E)$. Then both $\lambda_f^{e'e}$ and $\lambda^{e'}_{p(e)f}\circ\lambda^{e}_f$ fit well into the diagram

$$\begin{tikzcd}
f^*E \arrow[rrr, "\mathrm{Cart}_p(f,E)"] \arrow[dd, "p"'] \arrow[dr, "\lambda_f^e"] \arrow[dddrr, bend right, "\lambda_f^{e'e}"'] & & & E \arrow[dr, bend left, "e"] \arrow[dd, "p"] & & \\
& (p(e)f)^*E' \arrow[rrr, "\mathrm{Cart}_p(p(e)f,E')"] \arrow[ddd, "p"'] \arrow[ddr, "\lambda^{e'}_{p(e)f}"] & & & E' \arrow[ddr, bend left, "e'"] \arrow[ddd, "p"] & \\
C \arrow[rrr, "f"] \arrow[ddr, equal] & & & p(E) \arrow[ddr, "p(e)"] & & \\
& & (p(e')p(e)f)^*E'' \arrow[rrr, "\mathrm{Cart}_p(p(e')p(e)f,E'')"] \arrow[dd, "p"'] & & & E'' \arrow[dd, "p"] \\
& C \arrow[rrr, "p(e)f"] \arrow[dr, equal] & & & p(E') \arrow[dr, "p(e')"] & \\
& & C \arrow[rrr, "p(e')p(e)f"'] & & & p(E'')
\end{tikzcd}$$

whence we conclude that equation (4.9) holds by the fact that $\mathrm{Cart}_p(p(e')p(e)f, E'')$ is a cartesian morphism. □

Proposition 4.2.6. *Let $F\colon \mathcal{X}\to\mathcal{C}$ be a functor, $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix} \in \mathcal{ClFIB}_{\mathcal{C}}$ be a normal fibration and*

$$H_1\colon L\begin{pmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{pmatrix} \longrightarrow \begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$$

be a 1-morphism in $\mathcal{CAT}\downarrow\mathcal{C}$. Consider

$$V_{H_1}\colon \begin{matrix}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{matrix} \longrightarrow \begin{matrix}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \textstyle\int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{matrix}$$

the functor produced in Construction 4.2.4 (see also Theorem 4.2.5).

Then V_{H_1} makes the triangle in $\mathcal{CAT}\downarrow\mathcal{C}$

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[dr, bend right, "V_{H_1}"'] \arrow[dd, "p"'] & & \mathcal{X} \arrow[dd, "F"] \\
& \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[ur, bend right, "(\varepsilon_F)_1"'] \arrow[dd, "\int\mathrm{Tr}(-,F)"] & \\
\mathcal{C} \arrow[rr, equal, bend left] \arrow[dr, equal, bend right] & & \mathcal{C} \\
& \mathcal{C} \arrow[ur, equal, bend right] &
\end{tikzcd}
\tag{4.10}
$$

commute.

Proof. The square in the background of diagram (4.10) is commutative because H_1 is a 1-morphism in $\mathcal{CAT}\downarrow\mathcal{C}$, and we have already proved that the square on the right in the foreground is commutative, in Construction 4.2.2. Moreover, we have aleady proved also that the square on the left in the foreground commutes, in Construction 4.2.4.

It then remains to show that the upper triangle of diagram (4.10) commutes. Let $E \in \mathcal{E}$. Then

$$
V_{H_1}(E) = \left(p(E),\quad
\begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "F"] \\
& \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\right)
$$

and therefore, by Construction 4.2.2, we have that

$$
(\varepsilon_F)_1(V_{H_1}(E)) = H_1\mathrm{Cart}_p(-,E)(p(E) \xrightarrow{\mathrm{id}} p(E)) = H_1(E),
$$

since p is a normal fibration.

Now take $E \xrightarrow{e} E'$ a morphism in $\mathcal{E}$. Then

$$
V_{H_1}(e) = (p(e), H_1\lambda^e)
$$

and therefore, by Construction 4.2.2,

$$
\begin{aligned}
&(\varepsilon_F)_1(V_{H_1}(e)) = (\varepsilon_F)_1(p(e), H_1\lambda^e) = \\
&= H_1\mathrm{Cart}_p(-,E') \left(
\begin{tikzcd}
p(E) \arrow[d, "p(e)"'] \arrow[dr, "p(e)"] & \\
p(E') \arrow[r, "\mathrm{id}"'] & p(E')
\end{tikzcd}
\right) \circ (H_1\lambda^e)_{(p(E)\xrightarrow{\mathrm{id}} p(E))},
\end{aligned}
$$

whence, in order to prove that the upper triangle of diagram (4.10) commutes, it suffices to prove that

$$\mathrm{Cart}_p(-,E')\left(\begin{tikzcd} p(E) \arrow[d, "p(e)"'] \arrow[dr, "p(e)"] & \\ p(E') \arrow[r, "\mathrm{id}"'] & p(E') \end{tikzcd}\right)\circ \lambda^e_{(p(E)\xrightarrow{\mathrm{id}}p(E))} = e.$$

Call

$$v := \mathrm{Cart}_p(-,E')\left(\begin{tikzcd} p(E) \arrow[d, "p(e)"'] \arrow[dr, "p(e)"] & \\ p(E') \arrow[r, "\mathrm{id}"'] & p(E') \end{tikzcd}\right).$$

By construction (see the proof of Proposition 2.4.8 and Construction 4.2.4) and by the fact that p is a normal fibration, both $v\circ\lambda^e_{(p(E)\xrightarrow{\mathrm{id}}p(E))}$ and e fit well into the diagram

$$\begin{tikzcd}
E \arrow[rrr, "\mathrm{id}"] \arrow[dd, mapsto, "p"'] \arrow[dr, "\lambda^e_{(p(E)p(E))}"] \arrow[drrdd, bend right, "e"'] & & & E \arrow[dr, bend left, "e"] \arrow[dd, mapsto, "p"] & & \\
& p(e)^*E' \arrow[rrr, "\mathrm{Cart}_p(p(e),E')"] \arrow[ddd, mapsto, "p"] \arrow[drr, "v"] & & & E' \arrow[ddr, bend left, "\mathrm{id}"] \arrow[ddd, mapsto, "p"] & \\
p(E) \arrow[rrr, "\mathrm{id}"] \arrow[ddr, equal] & & & p(E) \arrow[ddr, "p(e)"] & & \\
& & E' \arrow[rrr, "\mathrm{id}"] \arrow[dd, mapsto, "p"] & & & E' \arrow[dd, mapsto, "p"] \\
& p(E) \arrow[rrr, "p(e)"] \arrow[dr, bend right, "p(e)"'] & & & p(E') \arrow[dr, "\mathrm{id}"] & \\
& & p(E') \arrow[rrr, "\mathrm{id}"'] & & & p(E')
\end{tikzcd}$$

And we then conclude by the fact that $\mathrm{id}_{E'}$ is a cartesian morphism. $\square$

Remark 4.2.7. So far, we have only used that the fibration p we start from is a normal fibration, and not that p is a split fibration.

Now, we want to prove that V_{H_1} is a cloven cartesian functor over $\mathcal{C}$. Analogously to what happened to prove that $U\colon \mathcal{SpFIB}\longrightarrow\mathcal{CAT}^{\to}$ is monadic (in Chapter 3), this fact turns out to be equivalent to the fact that the cloven fibration p we start from is split, as we now show, in Proposition 4.2.8.
This explains why we had to restrict to split fibrations also this time, to prove comonadicity of a 2-category of fibrations.

Proposition 4.2.8. *Let $F\colon\mathcal{X}\to\mathcal{C}$ be a functor,* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\in\mathcal{ClFIB}_{\mathcal{C}}$ *be a normal fibration and*

$$H_1 \colon L\left(\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}\right) \longrightarrow \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array}$$

be a 1*-morphism in* $\mathcal{CAT}\downarrow\mathcal{C}$*. Consider*

$$V_{H_1}\colon \begin{array}{c}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{array} \longrightarrow \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$$

the functor produced in Construction 4.2.4 (see also Theorem 4.2.5).

Then V_{H_1} *is a cloven cartesian functor over* $\mathcal{C}$ *if and only if the normal fibration* $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}$ *is split.*

Proof. We have already proved in Proposition 4.2.6 that the triangle in $\mathcal{C}at$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "V_{H_1}"] \arrow[dr, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int\mathrm{Tr}(-,F)"] \\
& \mathcal{C}
\end{tikzcd}$$

is commutative, in Construction 4.2.4. Then V_{H_1} is a cloven cartesian functor over $\mathcal{C}$ if and only if for every $E \in \mathcal{E}$ and every $f\colon C \to p(E)$ in $\mathcal{C}$

$$V_{H_1}(\mathrm{Cart}_p(f,E)) = \mathrm{Cart}_{\int\mathrm{Tr}(-,F)}(f, V_{H_1}(E)). \tag{4.11}$$

Let then $E \in \mathcal{E}$ and $f\colon C \to p(E)$ in $\mathcal{C}$. Clearly

$$\begin{tikzcd}
f^*E \arrow[r, "\mathrm{Cart}_p(f,E)"] \arrow[d, mapsto, "p"] & E \arrow[d, mapsto, "p"] \\
C \arrow[r, "f"] & p(E)
\end{tikzcd}$$

We have that

$$V_{H_1}(\mathrm{Cart}_p(f,E)) = \left(p(\mathrm{Cart}_p(f,E)), H_1\lambda^{\mathrm{Cart}_p(f,E)}\right) = \left(f, H_1\lambda^{\mathrm{Cart}_p(f,E)}\right)$$

with

$$\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[rr, bend left, "\mathrm{Cart}_p(-,f^*E)"] \arrow[r, "f\circ-"'] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"'] \arrow[u, phantom, "\Downarrow\lambda^{\mathrm{Cart}_p(f,E)}"] & \mathcal{E} \arrow[d, "p"] \\
& & \mathcal{C}
\end{tikzcd}$$

defined to be such that for every object $B \xrightarrow{h} C$ of $\mathcal{C}\downarrow C$ the morphism $\lambda_h^{\mathrm{Cart}_p(f,E)}$ is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
h^*f^*E \arrow[rr, "\mathrm{Cart}_p(h,f^*E)"] \arrow[dr, dashed, "\lambda_h^{\mathrm{Cart}_p(f,E)}"] \arrow[dd, mapsto, "p"] & & f^*E \arrow[dr, bend left, "\mathrm{Cart}_p(f,E)"] \arrow[dd, mapsto, "p"] & \\
& (fh)^*E \arrow[rr, "\mathrm{Cart}_p(fh,E)"] \arrow[dd, mapsto, "p"] & & E \arrow[dd, mapsto, "p"] \\
B \arrow[rr, "h"] \arrow[dr, equal] & & C \arrow[dr, "f"] & \\
& B \arrow[rr, "fh"'] & & p(E)
\end{tikzcd}$$

in the place of the morphism in violet.

Whereas, by Construction 2.3.9 (see also Example 2.4.5), we have that

$$\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(f, V_{H_1}(E)) = (f, \mathrm{id}_{H_1 \mathrm{Cart}_p(-,E)(f\circ -)}).$$

Therefore we have that V_{H_1} is a cloven cartesian functor over $\mathcal{C}$ if and only if for every $E \in \mathcal{E}$ and for every $f\colon C \to p(E)$ in $\mathcal{C}$ equation (4.11) holds if and only if for every $E \in \mathcal{E}$, for every $f\colon C \to p(E)$ in $\mathcal{C}$ and for every $h\colon B \to C$

$$\lambda_h^{\mathrm{Cart}_p(f,E)} = \mathrm{id}$$

if and only if for every $E \in \mathcal{E}$, for every $f\colon C \to p(E)$ in $\mathcal{C}$ and for every $h\colon B \to C$

$$\mathrm{Cart}_p(fh, E) = \mathrm{Cart}_p(f, E) \circ \mathrm{Cart}_p(h, f^*E)$$

if and only if p is a split fibration. □

Remark 4.2.9. Notice that the proof of Proposition 4.2.8 shows that if p is a split fibration then for every $f\colon C \to p(E)$ in $\mathcal{C}$

$$\mathrm{Cart}_p(-, f^*E) = \mathrm{Cart}_p(-, E)(f \circ -), \tag{4.12}$$

which is actually provable with a much easier argument.

The interesting fact is that the proof of Proposition 4.2.8 shows that any cloven cartesian functor over $\mathcal{C}$

$$V_1'\colon \begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array} \longrightarrow \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \smallint\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$$

satisfies a property similar to that in equation (4.12). Say that

$$V_1'(E) = \left(p(E),\ \begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\alpha_E^{V'}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$$

for every $E \in \mathcal{E}$ (we know that $V_1'(E)$ is of this form because V_1' is a cloven cartesian functor over $\mathcal{C}$ from $\begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array}$ to $\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \smallint\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$). Then, given $f\colon C \to p(E)$ in $\mathcal{C}$, from the fact that

$$V_1'\left(\mathrm{Cart}_p(f, E)\right) = \mathrm{Cart}_{\smallint\mathrm{Tr}(-,F)}(f, V_1'(E)) = (f, \mathrm{id}_{\alpha_E^{V'}(f\circ -)})$$

we obtain that

$$\alpha_{f^*E}^{V'} = \alpha_E^{V'}(f \circ -),$$

as the proof of Proposition 4.2.8 showed for V_{H_1}.

Theorem 4.2.10. *The forgetful* 2*-functor* $L\colon \mathcal{S}p\mathcal{FIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$ *has a right* 2*-adjoint* $R\colon \mathcal{CAT}\downarrow\mathcal{C} \longrightarrow \mathcal{S}p\mathcal{FIB}_{\mathcal{C}}$.

Proof. We prove that L has a right 2-adjoint by using the characterization with 2-universal morphisms, given in Theorem 1.9.5. For every functor $F\colon \mathcal{X} \to \mathcal{C}$ (that is, $F \in \mathcal{CAT}\downarrow\mathcal{C}$), we set

$$R(F) := \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \smallint\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$$

as in Remark 4.2.1. Then $R(F)$ is a split fibration, by Corollary 4.1.7.

We then search for a 2-universal morphism from the 2-functor L to F of the form

$$(\varepsilon_F)_1\colon L\left(\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \smallint\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}\right) \longrightarrow \begin{array}{c}\mathcal{X}\\ \big\downarrow F\\ \mathcal{C}\end{array}.$$

We construct $(\varepsilon_F)_1$ as in Construction 4.2.2, and we know it is a functor by Proposition 4.2.3. To prove that $(\varepsilon_F)_1$ is a (1-)universal morphism from the functor L (treated as a 1-functor) to F, we have to prove that for every $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix} \in \mathcal{SpFIB}_{\mathcal{C}}$ and for every (1-)morphism

$$H_1\colon L\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}\right) \longrightarrow \begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$$

in $\mathcal{CAT}\downarrow\mathcal{C}$, there exists a unique (1-)morphism

$$V_{H_1}\colon \begin{matrix}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{matrix} \longrightarrow \begin{matrix}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{matrix}$$

in $\mathcal{SpFIB}_{\mathcal{C}}$ such that the triangle in $\mathcal{CAT}\downarrow\mathcal{C}$

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[r, bend right, "V_{H_1}"'] \arrow[dd, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, bend right, "(\varepsilon_F)_1"'] \arrow[ddd, "\int\mathrm{Tr}(-,F)"] & \mathcal{X} \arrow[dd, "F"] \\
& & \\
\mathcal{C} \arrow[rr, bend left, equal] \arrow[dr, bend right, equal] & & \mathcal{C} \\
& \mathcal{C} \arrow[ur, bend right, equal] &
\end{tikzcd} \tag{4.13}$$

is commutative.

Fixed $\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}$ a split fibration over $\mathcal{C}$ and

$$H_1\colon L\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}\right) \longrightarrow \begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$$

a 1-morphism in $\mathcal{CAT}\downarrow\mathcal{C}$, we construct

$$V_{H_1}\colon \begin{matrix}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{matrix} \longrightarrow \begin{matrix}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{matrix}$$

as in Construction 4.2.4. And we know that it is a cloven cartesian functor over $\mathcal{C}$ by Theorem 4.2.5 and Proposition 4.2.8.

Moreover, we know that V_{H_1} makes diagram (4.13) commute by Proposition 4.2.6.

In order to prove that $(\varepsilon_F)_1$ is a (1-)universal morphism, it now remains to prove that this V_{H_1} is the unique cloven cartesian functor over $\mathcal{C}$ which makes diagram (4.13) commute. Suppose then that there is another cloven cartesian functor over $\mathcal{C}$, say

$$V_1'\colon \begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array} \longrightarrow \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$$

which makes diagram (4.13) commute, that is, such that

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "H_1"] \arrow[r, bend right, "V_1'"'] \arrow[dd, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, bend right, "(\varepsilon_F)_1"'] \arrow[ddd, "\int\mathrm{Tr}(-,F)"] & \mathcal{X} \arrow[dd, "F"] \\
 & & \\
\mathcal{C} \arrow[rr, bend left, equal] \arrow[dr, bend right, equal] & & \mathcal{C} \\
 & \mathcal{C} \arrow[ur, bend right, equal] &
\end{tikzcd} \tag{4.14}$$

We show that V_1' needs to coincide with V_{H_1}.
It clearly suffices to show that V_1' and V_{H_1} coincide as functors from $\mathcal{E}$ to $\mathcal{G}_{\mathrm{Tr}(-,F)}$.

Let $E \in \mathcal{E}$. We have that

$$V_{H_1}(E) = \left(p(E), \begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "F"] \\
 & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd} \right),$$

whereas $V_1'(E)$ will be of the form

$$V_1'(E) = \left(p(E), \begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[r, "\alpha_E^{V'}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\
 & \mathcal{C}
\end{tikzcd} \right)$$

for some functor $\alpha_E^{V'}\colon \mathcal{C}\downarrow p(E) \to \mathcal{X}$, by the fact that V_1' is a cloven cartesian functor over $\mathcal{C}$ from $\begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array}$ to $\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow \int\mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}$.

Since $\mathrm{id}_E = \mathrm{Cart}_p\left(\mathrm{id}_{p(E)}, E\right)$ and V_1' is a cloven cartesian functor over $\mathcal{C}$, we get that

$$V_1'(E \overset{\mathrm{id}_E}{\to} E) = V_1'\left(\mathrm{Cart}_p\left(\mathrm{id}_{p(E)}, E\right)\right) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}\left(\mathrm{id}_{p(E)}, V_1'(E)\right) = \left(\mathrm{id}_{p(E)}, \mathrm{id}_{\alpha_E^{V'}(\mathrm{id}_{p(E)}\,\circ -)}\right)$$

(using Construction 2.3.9 to see the explicit cleavage of the Grothendieck construction).
But then

$$(\varepsilon_F)_1(V_1'(E \overset{\mathrm{id}}{\to} E)) = \alpha_E^{V'}\left(\begin{tikzcd} p(E) \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right) \circ \mathrm{id}_{(p(E)\overset{\mathrm{id}}{\to}p(E))} = \alpha_E^{V'}\left(\begin{tikzcd} p(E) \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right)$$

and we also have that

$$(\varepsilon_F)_1(V_1'(E \overset{\mathrm{id}}{\to} E)) = H_1(E \overset{\mathrm{id}}{\to} E) = H_1\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} p(E) \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right)$$

by commutativity of diagram (4.14).
Therefore

$$\alpha_E^{V'}\left(\begin{tikzcd} p(E) \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right) = H_1\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} p(E) \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right)$$

In particular $\alpha_E^{V'}(p(E) \overset{\mathrm{id}}{\to} p(E)) = H_1\mathrm{Cart}_p(-,E)(p(E) \overset{\mathrm{id}}{\to} p(E))$.

Aiming at proving that $\alpha_E^{V'}$ and $H_1\mathrm{Cart}_p(-,E)$ coincide on all the objects of $\mathcal{C}\downarrow p(E)$, we now consider an arbitrary morphism $f\colon C \longrightarrow p(E)$ in $\mathcal{C}$. Since V_1' is a cloven cartesian functor over $\mathcal{C}$, we get that

$$V_1'\left(\mathrm{Cart}_p(f,E)\right) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(f, V_1'(E)) = \left(f, \mathrm{id}_{\alpha_E^{V'}(f\circ -)}\right) \tag{4.15}$$

(using Construction 2.3.9 to see the explicit cleavage of the Grothendieck construction).
But then

$$(\varepsilon_F)_1\left(V_1'\left(\mathrm{Cart}_p(f,E)\right)\right) = \alpha_E^{V'}\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right) \circ \mathrm{id}_{(p(E)\overset{\mathrm{id}}{\to}p(E))} = \alpha_E^{V'}\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right)$$

and we also have that

$$(\varepsilon_F)_1(V_1'(\mathrm{Cart}_p(f,E))) = H_1(\mathrm{Cart}_p(f,E)) = H_1\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right)$$

by commutativity of diagram (4.14) and by the fact that p is a normal fibration (see Remark 2.4.9).
Therefore

$$\alpha_E^{V'}\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right) = H_1(\mathrm{Cart}_p(f,E)) = H_1\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, "f"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd}\right). \tag{4.16}$$

In particular we get that $\alpha_E^{V'}$ and $H_1\mathrm{Cart}_p(-,E)$ coincide on objects.

Consider now an arbitrary morphism

$$\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "u"] & \\ C \arrow[r, "f"'] & p(E) \end{tikzcd}$$

in $\mathcal{C}\downarrow p(E)$. We need to prove that

$$\alpha_E^{V'}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "u"] & \\ C \arrow[r, "f"'] & p(E) \end{tikzcd}\right) = H_1\mathrm{Cart}_p(-,E)\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "u"] & \\ C \arrow[r, "f"'] & p(E) \end{tikzcd}\right). \tag{4.17}$$

But we see that

$$\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "u"] & \\ C \arrow[r, "f"'] & p(E) \end{tikzcd} \quad = \quad \begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "h"] & & \\ C \arrow[r, "\mathrm{id}"'] & C \arrow[r, "f"'] & p(E) \end{tikzcd}$$

and thus we can rewrite equation (4.17) as

$$\alpha_E^{V'}(f\circ -)\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "h"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right) = H_1\mathrm{Cart}_p(-,E)(f\circ -)\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "h"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right)$$

and thus, by Remark 4.2.9, as

$$\alpha_{f^*E}^{V'}\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "h"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right) = H_1\mathrm{Cart}_p(-,f^*E)\left(\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "h"] & \\ C \arrow[r, "\mathrm{id}"'] & C \end{tikzcd}\right) \tag{4.18}$$

Then we conlcude that equation (4.18) holds by applying equation (4.16) to f^*E instead of E (notice that we have proved equation (4.16) for an arbitrary object E of $\mathcal{E}$).

Therefore $\alpha_E^{V'} = H_1\mathrm{Cart}_p(-,E)$ (as functors) and thus V_1' and V_{H_1} coincide on objects.

Notice that we have already proved also that V_1' and V_{H_1} coincide on all the morphisms of the cleavage of p: for every $f\colon C \to p(E)$ in $\mathcal{C}$

$$
\begin{aligned}
&V_1'\left(\mathrm{Cart}_p(f,E)\right) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(f, V_1'(E)) = \left(f, \mathrm{id}_{\alpha_E^{V'}(f\circ -)}\right) = \\
&= \left(f, \mathrm{id}_{H_1\mathrm{Cart}_p(-,E)(f\circ -)}\right) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(f, V_{H_1}(E)) = V_{H_1}\left(\mathrm{Cart}_p(f,E)\right)
\end{aligned}
\tag{4.19}
$$

(where we have used equation (4.15)).

Let now $e\colon E \to E'$ be an arbitrary morphism in $\mathcal{E}$. We can write e as

$$e = \mathrm{Cart}_p(p(e),E) \circ w$$

where w is the unique morphism which fits well into the diagram

$$
\begin{tikzcd}
E \arrow[rr, bend left, "e"] \arrow[r, dashed, "\exists! w"] \arrow[d, mapsto, "p"] & p(e)^*E' \arrow[r, "\mathrm{Cart}_p(p(e),E')"] \arrow[d, mapsto, "p"] & E' \arrow[d, mapsto, "p"] \\
p(E) \arrow[r, equal] \arrow[rr, bend left, "p(e)"] & p(E) \arrow[r, "p(e)"'] & p(E')
\end{tikzcd}
$$

We then have that

$$V_1'\left(E \xrightarrow{e} E'\right) = V_1'\left(\mathrm{Cart}_p(p(e),E)\right) \circ V_1'\left(E \xrightarrow{w} p(e)^*E'\right)$$

and thus by equation (4.19) it suffices to show that

$$V_1'\left(E \xrightarrow{w} p(e)^*E'\right) = V_{H_1}\left(E \xrightarrow{w} p(e)^*E'\right).$$

We have that

$$V_{H_1}\left(E \xrightarrow{w} p(e)^*E'\right) = \left(\mathrm{id}_{p(E)}, H_1\lambda^w\right)$$

with

$$
\begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[rr, bend left, "\mathrm{Cart}_p(-,E)"] \arrow[r, "\mathrm{id}\circ -"'] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,p(e)^*E')"'] \arrow[u, phantom, "\Downarrow \lambda^w"] & \mathcal{E} \arrow[d, "p"] \\
& & \mathcal{C}
\end{tikzcd}
$$

defined to be such that for every object $C \xrightarrow{f} p(E)$ of $\mathcal{C}{\downarrow}p(E)$ the morphism λ^w_f is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
f^*E \arrow[rr, "\mathrm{Cart}_p(f,E)"] \arrow[dr, dashed, "\lambda^w_f"] \arrow[dd, "p"] & & E \arrow[dr, bend left, "w"] \arrow[dd, "p"] & \\
& f^*p(e)^*E' \arrow[rr, "\mathrm{Cart}_p(f,p(e)^*E')"] \arrow[dd, "p"] & & p(e)^*E' \arrow[dd, "p"] \\
C \arrow[rr, "f"'] \arrow[dr, equal, bend right] & & p(E) \arrow[dr, equal] & \\
& C \arrow[rr, "f"'] & & p(E)
\end{tikzcd} \tag{4.20}$$

in the place of the morphism in violet.
Whereas $V_1'\left(E \xrightarrow{w} p(e)^*E'\right)$ will be of the form

$$V_1'\left(E \xrightarrow{w} p(e)^*E'\right) = \left(\mathrm{id}_{p(E)}, \lambda^{V',w}\right)$$

with

$$\begin{tikzcd}
\mathcal{C}{\downarrow}p(E) \arrow[rr, bend left=40, "\alpha^{V'}_E"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[drr, bend right=50, "\mathrm{dom}"'] & \arrow[d, phantom, "\Downarrow \lambda^{V',w}"] & \mathcal{X} \arrow[d, "F"] \\
& \mathcal{C}{\downarrow}p(E) \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] & \mathcal{C}
\end{tikzcd}$$

because V_1' is a cloven cartesian functor over $\mathcal{C}$ from $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{array}$ to $\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \downarrow \int \mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}\big)$.

To prove that V_1' and V_{H_1} coincide, it only remains to prove that $\lambda^{V',w} = H_1\lambda^w$.
We have that

$$(\varepsilon_F)_1\left(V_1'\left(E \xrightarrow{w} p(e)^*E'\right)\right) = \mathrm{id} \circ \lambda^{V',w}_{(p(E) \xrightarrow{\mathrm{id}} p(E))}$$

and that

$$(\varepsilon_F)_1\left(V_1'\left(E \xrightarrow{w} p(e)^*E'\right)\right) = \mathrm{id} \circ H_1\lambda^{w}_{(p(E) \xrightarrow{\mathrm{id}} p(E))}$$

by commutativity of diagram (4.14), whence we obtain that

$$\lambda^{V',w}_{(p(E) \xrightarrow{\mathrm{id}} p(E))} = H_1\lambda^{w}_{(p(E) \xrightarrow{\mathrm{id}} p(E))}. \tag{4.21}$$

Consider now an arbitrary object $f \colon C \to p(E)$ in $\mathcal{C}$. We use the same strategy as before (to prove that $\alpha^{V'}_E$ and $H_1\mathrm{Cart}_p(-,E)$ coincide on morphisms), writing

$$\lambda^{V',w}_{(C \xrightarrow{f} p(E))} = \left(\lambda^{V',w}(f \circ -)\right)_{(C \xrightarrow{\mathrm{id}} C)}.$$

So we need to prove that $\lambda^{V',w}(f\circ -)$ coincide with $\lambda^{V',s}$ for some vertical morphism s in $\mathcal{E}$. The best candidate is $s := \lambda^w_f\colon f^*E \longrightarrow f^*p(e)^*E'$ (see diagram (4.20)). We prove that

$$
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[r, "f\circ -"] & \mathcal{C}\downarrow p(E) \arrow[rr, bend left, "\alpha^{V'}_{E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rr, phantom, "\Downarrow\lambda^{V',w}"] & & \mathcal{X} \\
& & \mathcal{C}\downarrow p(E) \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[rr, bend left, "\alpha^{V'}_{f^*E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rr, phantom, "\Downarrow\lambda^{V',\lambda^w_f}"] & & \mathcal{X} \\
& \mathcal{C}\downarrow C \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
$$

It is easy to see that

$$
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[r, "f\circ -"] & \mathcal{C}\downarrow p(E) \arrow[rr, bend left, "\alpha^{V'}_{E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rr, phantom, "\Downarrow\lambda^{V',w}"] & & \mathcal{X} \\
& & \mathcal{C}\downarrow p(E) \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[rrr, bend left, "\alpha^{V'}_{f^*E}"] \arrow[dr, "f\circ -"'] \arrow[rrr, phantom, "\Downarrow\mathrm{id}"] & & & \\
& \mathcal{C}\downarrow p(E) \arrow[rr, bend left, "\alpha^{V'}_{E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rr, phantom, "\Downarrow\lambda^{V',w}"] & & \mathcal{X} \\
& & \mathcal{C}\downarrow p(E) \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
=
$$

$$
\begin{aligned}
&= \mathrm{pr}_2\left(V'_1\left(E \xrightarrow{\ w\ } p(e)^*E'\right)\circ V'_1\left(f^*E \xrightarrow{\mathrm{Cart}_p(f,E)} E\right)\right) = \\
&= \mathrm{pr}_2\left(V'_1\left(f^*E \xrightarrow{w\circ\mathrm{Cart}_p(f,E)} p(e)^*E'\right)\right) = \\
&= \mathrm{pr}_2\left(V'_1\left(f^*p(e)^*E' \xrightarrow{\mathrm{Cart}_p(f,p(e)^*E')} p(e)^*E'\right)\circ V'_1\left(f^*E \xrightarrow{\lambda^w_f} f^*p(e)^*E'\right)\right) =
\end{aligned}
$$

$$
=
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[rrr, bend left, "\alpha^{V'}_{f^*E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rrr, phantom, "\Downarrow\lambda^{V',\lambda^w_f}"] & & & \\
& \mathcal{C}\downarrow C \arrow[rr, bend left, "\alpha^{V'}_{u^*p(e)^*E'}"] \arrow[dr, "f\circ -"'] \arrow[rr, phantom, "\Downarrow\mathrm{id}"] & & \mathcal{X} \\
& & \mathcal{C}\downarrow p(E) \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}\downarrow C \arrow[rr, bend left, "\alpha^{V'}_{f^*E}"] \arrow[dr, "\mathrm{id}\circ -"'] \arrow[rr, phantom, "\Downarrow\lambda^{V',\lambda^w_f}"] & & \mathcal{X} \\
& \mathcal{C}\downarrow C \arrow[ur, "\alpha^{V'}_{p(e)^*E'}"'] &
\end{tikzcd}
$$

by commutativity of the diagram

$$
\begin{tikzcd}
f^*E \arrow[rr, "\mathrm{Cart}_p(f,E)"] \arrow[dr, dashed, "\lambda^w_f"] \arrow[dd, "p"] & & E \arrow[dr, bend left, "w"] \arrow[dd, "p"] & \\
& f^*p(e)^*E' \arrow[rr, "\mathrm{Cart}_p(f,p(e)^*E')"] \arrow[dd, "p"] & & p(e)^*E' \arrow[dd, "p"] \\
C \arrow[rr, "f"'] \arrow[dr, equal, bend right] & & p(E) \arrow[dr, equal] & \\
& C \arrow[rr, "f"'] & & p(E)
\end{tikzcd}
$$

Notice that this argument needs to hold also for V_{H_1}, and then we have proved also that for every $f\colon C \to p(E)$ in $\mathcal{C}$

$$H_1\lambda^w(f \circ -) = H_1\lambda^{\lambda^w_f}.$$

Therefore

$$\lambda^{V',w}_{(C \xrightarrow{f} p(E))} = \left(\lambda^{V',w}(f \circ -)\right)_{(C \xrightarrow{\mathrm{id}} C)} = \left(\lambda^{V',\lambda^w_f}\right)_{(C \xrightarrow{\mathrm{id}} C)} = \left(H_1\lambda^{\lambda^w_f}\right)_{(C \xrightarrow{\mathrm{id}} C)} =$$
$$= \left(H_1\lambda^w(f \circ -)\right)_{(C \xrightarrow{\mathrm{id}} C)} = \left(H_1\lambda^w\right)_{(C \xrightarrow{f} p(E))}$$

by equation (4.21) (applied to λ^w_f, which is a vertical morphism in $\mathcal{E}$).
We have thus proved that V_1' has to coincide with V_{H_1}. Then $(\varepsilon_F)_1$ is a 1-universal morphism.

Finally, we prove that $(\varepsilon_F)_1$ is a 2-universal morphism from the 2-functor L to F.
Given a 2-morphism

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C} \end{tikzcd}$$

in $\mathcal{CAT}\downarrow\mathcal{C}$, we consider V_{H_1} and V_{K_1} the cloven cartesian functors over $\mathcal{C}$ respectively associated to H_1 and to K_1. We prove there exists a unique 2-morphism

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[r, bend right, "V_{K_1}"'] \arrow[r, phantom, "\Downarrow\delta_1"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal, bend right] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & \mathcal{C} \end{tikzcd}$$

in $\mathcal{S}p\mathcal{FIB}_{\mathcal{C}}$ which makes the following diagram of 2-morphisms in $\mathcal{CAT}\downarrow\mathcal{C}$ commute:

$$\begin{tikzcd} \mathcal{E} \arrow[dd, "p"'] \arrow[rr, bend left, "H_1"] \arrow[rr, bend right, "K_1"'] \arrow[rr, phantom, "\Downarrow\lambda_1"] \arrow[dr, bend left, "V_{H_1}"] \arrow[dr, bend right, "V_{K_1}"'] \arrow[dr, phantom, "\Downarrow\delta_1"] & & \mathcal{X} \arrow[dd, "F"] \\ & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[ur, "(\varepsilon_F)_1"'] \arrow[dd, "\int \mathrm{Tr}(-,F)"] & \\ \mathcal{C} & & \mathcal{C} \\ & \mathcal{C} & \end{tikzcd} \tag{4.22}$$

(which is diagram (4.5) after the choice of $R(F)$).

Let

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[r, bend right, "V_{K_1}"'] \arrow[r, phantom, "\Downarrow\delta_1"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "V_{H_1}"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow \mathrm{id}"] & \mathcal{C}
\end{tikzcd}
\tag{4.23}
$$

be a 2-moprhism in $\mathcal{SpFIB}_{\mathcal{C}}$ which makes diagram 4.22 commute, that is, such that for every $E \in \mathcal{E}$

$$(\varepsilon_F)_1(\delta_{1,E}) = \lambda_{1,E}. \tag{4.24}$$

Let $E \in \mathcal{E}$. Recall that $V_{H_1}(E)$ is defined as

$$
V_{H_1}(E) = \left(p(E), \quad
\begin{tikzcd}
\mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "F"] \\
& \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\right),
$$

and $V_{K_1}(E)$ is defined analogously.

Then $\delta_{1,E}$ has to be a morphism in $\mathcal{G}_{\mathrm{Tr}(-,F)}$ of the form

$$\left(p(E) \xrightarrow{h} p(E), \mu^E \right)$$

with

$$
\begin{tikzcd}
& & & \mathcal{E} \arrow[dr, "H_1"] & \\
\mathcal{C}\downarrow p(E) \arrow[urrr, bend left, "\mathrm{Cart}_p(-,E)"] \arrow[drr, "h\circ -"'] \arrow[ddrrrr, bend right, "\mathrm{dom}"'] & & & & \mathcal{X} \arrow[dd, "F"] \\
& & \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"'] \arrow[from=1-3, to=3-3, phantom, "\Downarrow \mu^E"] & \mathcal{E} \arrow[ur, "K_1"'] & \\
& & & & \mathcal{C}
\end{tikzcd}
$$

By diagram (4.23), we see that $h\colon p(E) \to p(E)$ has to be the identity $\mathrm{id}_{p(E)}$. And then we see that

$$(\varepsilon_F)_1(\delta_{1,E}) = \mathrm{id} \circ \mu^E_{(p(E) \xrightarrow{\mathrm{id}} p(E))} = \mu^E_{(p(E) \xrightarrow{\mathrm{id}} p(E))}.$$

But by equation (4.24) we have that

$$(\varepsilon_F)_1(\delta_{1,E}) = \lambda_{1,E} = \left(\lambda_1 \mathrm{Cart}_p(-,E)\right)_{(p(E) \xrightarrow{\mathrm{id}} p(E))},$$

whence we get that

$$\mu^E_{(p(E)\xrightarrow{\mathrm{id}}p(E))} = (\lambda_1 \operatorname{Cart}_p(-,E))_{(p(E)\xrightarrow{\mathrm{id}}p(E))}. \tag{4.25}$$

Now, we want to show that $\delta_{1,E}$ has to coincide with $\left(\mathrm{id}_{p(E)}, \lambda_1 \operatorname{Cart}_p(-,E)\right)$.
To show this, we take $f\colon C \to p(E)$ an arbitrary object of $\mathcal{C}{\downarrow}p(E)$ and we prove that $\mu^E_{(C\xrightarrow{f}p(E))}$ has to coincide with $(\lambda_1 \operatorname{Cart}_p(-,E))_{(C\xrightarrow{f}p(E))}$.

We use the same strategy as above (to prove that $\lambda^{V',w}_{(C\xrightarrow{f}p(E))} = (H_1\lambda^w)_{(C\xrightarrow{f}p(E))}$).
We have that

$$\begin{tikzcd}
\mathcal{C}{\downarrow}C \arrow[r, "f\circ-"] & \mathcal{C}{\downarrow}p(E) \arrow[rr, bend left, "H_1\operatorname{Cart}_p(-,E)"] \arrow[dr, "\mathrm{id}\circ-"'] & \arrow[d, phantom, "\Downarrow\mu^E"] & \mathcal{X} \\
& & \mathcal{C}{\downarrow}p(E) \arrow[ur, "K_1\operatorname{Cart}_p(-,E)"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}{\downarrow}C \arrow[drrr, bend left, "H_1\operatorname{Cart}_p(-,f^*E)"] \arrow[dr, "f\circ-"'] & \arrow[d, phantom, "\Downarrow\mathrm{id}"] & & \\
& \mathcal{C}{\downarrow}p(E) \arrow[rr, bend left, "H_1\operatorname{Cart}_p(-,E)"] \arrow[dr, "\mathrm{id}\circ-"'] & \arrow[d, phantom, "\Downarrow\mu^E"] & \mathcal{X} \\
& & \mathcal{C}{\downarrow}p(E) \arrow[ur, "K_1\operatorname{Cart}_p(-,E)"'] &
\end{tikzcd}
=$$

$$= \mathrm{pr}_2\left(\delta_{1,E}\circ V_{H_1}\left(f^*E \xrightarrow{\operatorname{Cart}_p(f,E)} E\right)\right) =$$
$$= \mathrm{pr}_2\left(V_{K_1}\left(f^*E \xrightarrow{\operatorname{Cart}_p(f,E)} E\right)\circ\delta_{1,f^*E}\right) =$$

$$=
\begin{tikzcd}
\mathcal{C}{\downarrow}C \arrow[drr, bend left, "H_1\operatorname{Cart}_p(-,f^*E)"] \arrow[d, "\mathrm{id}\circ-"'] & \arrow[d, phantom, "\Downarrow\mu^{f^*E}"] & \\
\mathcal{C}{\downarrow}C \arrow[rr, bend left, "K_1\operatorname{Cart}_p(-,f^*E)"] \arrow[dr, "f\circ-"'] & \arrow[d, phantom, "\Downarrow\mathrm{id}"] & \mathcal{X} \\
& \mathcal{C}{\downarrow}p(E) \arrow[ur, "K_1\operatorname{Cart}_p(-,E)"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}{\downarrow}C \arrow[rr, bend left, "H_1\operatorname{Cart}_p(-,f^*E)"] \arrow[dr, "\mathrm{id}\circ-"'] & \arrow[d, phantom, "\Downarrow\mu^{f^*E}"] & \mathcal{X} \\
& \mathcal{C}{\downarrow}C \arrow[ur, "K_1\operatorname{Cart}_p(-,f^*E)"'] &
\end{tikzcd}$$

Therefore

$$\begin{aligned}
\mu^E_{(C\xrightarrow{f}p(E))} &= \left(\mu^E(f\circ -)\right)_{(C\xrightarrow{\mathrm{id}}C)} = \left(\mu^{f^*E}\right)_{(C\xrightarrow{\mathrm{id}}C)} = (\lambda_1 \operatorname{Cart}_p(-,f^*E))_{(C\xrightarrow{\mathrm{id}}C)} = \\
&= (\lambda_1 \operatorname{Cart}_p(-,E)(f\circ -))_{(C\xrightarrow{\mathrm{id}}C)} = (\lambda_1 \operatorname{Cart}_p(-,E))_{(C\xrightarrow{F}P(E))}.
\end{aligned}$$

by equation (4.25) applied to $f^*E \in \mathcal{E}$ instead of E.

We have thus proved that there can be at most one 2-morphism in $\mathcal{SpFIB}_{\mathcal{C}}$ which makes diagram (4.22) commute.

Now, it only remains to show that if we define $\delta_1\colon V_{H_1} \Rightarrow V_{K_1}$ as

$$\delta_{1,E} := \left(\mathrm{id}_{p(E)}, \lambda_1 \operatorname{Cart}_p(-,E)\right)$$

then δ_1 is a 2-morphism in $\mathcal{SpFIB}_{\mathcal{C}}$ which makes diagram (4.22) commute.

The naturality of δ_1 easily follows by the naturality of λ_1, and the fact that δ_1 makes diagram (4.22) commute is trivial by construction.
Then it only remains to show that

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "V_{H_1}"] \arrow[rr, bend right, "V_{K_1}"'] \arrow[rr, phantom, "\Downarrow \delta_1"] \arrow[d, "p"'] & & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[rr, equal, bend right] & & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "V_{H_1}"] \arrow[d, "p"'] & & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[rr, equal, bend left] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow \mathrm{id}"] & & \mathcal{C}
\end{tikzcd}
$$

But, given $E \in \mathcal{E}$, we have that

$$
\int \mathrm{Tr}\,(-, F)\,(\delta_{1,E}) = \mathrm{id}_{p(E)}
$$

by construction.

Therefore we have proved that $(\varepsilon_F)_1$ is a 2-universal morphism from the 2-functor L to F, and we conclude by Theorem 1.9.5 that the function on objects R extends to a right 2-adjoint to the forgetful 2-functor L. $\square$

Remark 4.2.11. By 1.9.5, the morphisms $(\varepsilon_F)_1$, defined in Construction 4.2.2, combine to a natural transformation $\varepsilon_1\colon LR \longrightarrow \mathrm{Id}_{\mathcal{CAT}\downarrow\mathcal{C}}$, which corresponds to the counit of the 2-adjunction $L \dashv R$ constructed in the proof of Theorem 4.2.10.

By 1.9.5, we see that the 2-functor $R\colon \mathcal{CAT}\downarrow\mathcal{C} \longrightarrow \mathcal{SpFIB}_{\mathcal{C}}$ constructed in the proof of Theorem 4.2.10 acts on 1-cells as

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\overset{R}{\longmapsto}
\begin{tikzcd}
\mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "(\mathrm{Id}, H_1\cdot)"] \arrow[d, "\int \mathrm{Tr}(-,F)"'] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
$$

that is, if

$$
\left(C, \begin{tikzcd}
C\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\
 & \mathcal{C}
\end{tikzcd} \right)
$$

is an object of $\mathcal{G}_{\mathrm{Tr}(-,F)}$, then

$$
R\,(H_1) \left(C, \begin{tikzcd}
C\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\
 & \mathcal{C}
\end{tikzcd} \right)
=
\left(C, \begin{tikzcd}
C\downarrow C \arrow[r, "\alpha"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \arrow[r, "H_1"] & \mathcal{Y} \arrow[d, "G"] \\
 & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd} \right),
$$

and if (h, μ) is a morphism in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, then

$$R(H_1)(h, \mu) = (h, H_1\mu).$$

Analogously, for every 2-cell λ_1 in $\mathcal{CAT} \downarrow \mathcal{C}$, we have that

$$R(\lambda_1) = (\mathrm{id}, \lambda_1 \cdot).$$

Furthermore, by 1.9.5, we see that the unit η_1 of the 2-adjunction $L \dashv R$ is given by

$$\eta_{1,p}\colon \quad \begin{tikzcd} \mathcal{E} \arrow[r, "V_p"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,p)} \arrow[d, "\int \mathrm{Tr}(-,p)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

for every $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{smallmatrix} \in \mathcal{SpFIB}_{\mathcal{C}}$, where we denote V_p the functor V_{Id_p} produced in Construction 4.2.4 (with $H_1 = \mathrm{Id}_p$).

4.3 Comonadicity of Split Fibrations over a Fixed Base Category

In this section we prove that the forgetful 2-functor $L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT} \downarrow \mathcal{C}$ is comonadic. We will use the 2-adjunction we have shown in the last section.

Notation 4.3.1. Throughout all this section, we will denote L and R respectively the left 2-adjoint and the right 2-adjoint of the 2-adjunction produced in the proof of Theorem 4.2.10. We will also denote η_1 and ε_1 respectively the unit and the counit of the 2-adjuction $L \dashv U$ (see Remark 4.2.11 for the description of such natural transformations). Finally, we will sometimes use also the notation V_p introduced in Remark 4.2.11.

Now that we have an adjunction

$$\begin{tikzcd} \mathcal{SpFIB}_{\mathcal{C}} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{CAT} \downarrow \mathcal{C} \arrow[l, bend left, "U"] \end{tikzcd}$$

by Theorem 4.2.10, we can consider the 2-comonad generated by this adjunction.

Remark 4.3.2. By the dualized version of Proposition 1.11.4 (which is the enriched version of Proposition 1.5.7), the 2-comonad Ω generated by the 2-adjunction $L \dashv R$ is given by

$$\Omega = (LR, L\eta_1 R, \varepsilon_1).$$

By the dualized version of Remark 1.11.1, a Ω-coalgebra is a pair

$$\left(\begin{array}{c}\mathcal{X}\\ \Big\downarrow F\\ \mathcal{C}\end{array},\ \begin{array}{c}\mathcal{X}\\ \Big\downarrow F\\ \mathcal{C}\end{array} \xrightarrow{\beta_1} \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \Big\downarrow \int \mathrm{Tr}(-,F)\\ \mathcal{C}\end{array}\right)$$

with $F \in \mathcal{CAT}\downarrow\mathcal{C}$ and β_1 a 1-morphism in $\mathcal{CAT}\downarrow\mathcal{C}$ (sometimes we will say that a Ω-coalgebra is just a commutative square (actually, a triangle)

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \tag{4.26}$$

in $\mathcal{C}at$) subject to the following axioms:

(CoAlg1)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "(\varepsilon_F)_1"] \arrow[d, "\int \mathrm{Tr}(-,F)"] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \quad = \quad \mathrm{id}_F$$

(CoAlg2)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "(L\eta_1 R)_F"] \arrow[d, "\int \mathrm{Tr}(-,F)"'] & \mathcal{G}_{\mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)} \arrow[d, "\int \mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)"'] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "LR\beta_1"] \arrow[d, "\int \mathrm{Tr}(-,F)"'] & \mathcal{G}_{\mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)} \arrow[d, "\int \mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)"'] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Using Remark 4.2.11 to describe the functors $(L\eta_1 R)_F$ and $LR\beta$, we get a more explicit form of (CoAlg2):

(CoAlg2*)
$$\mathcal{X} \xrightarrow{\beta_1} \mathcal{G}_{\mathrm{Tr}(-,F)} \xrightarrow{V_{\int \mathrm{Tr}(-,F)}} \mathcal{G}_{\mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)} \quad = \quad \mathcal{X} \xrightarrow{\beta_1} \mathcal{G}_{\mathrm{Tr}(-,F)} \xrightarrow{(\mathrm{Id},\beta_1\cdot)} \mathcal{G}_{\mathrm{Tr}\left(-,\int \mathrm{Tr}(-,F)\right)}$$

Say that for every object $X \in \mathcal{X}$,

$$\beta_1(X) = \left(F(X),\ \begin{tikzcd} \mathcal{C}\downarrow F(X) \arrow[r, "\alpha_X^{\beta_1}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$$

for some functor $\alpha_X^{\beta_1} \colon \mathcal{C}\downarrow F(X) \longrightarrow \mathcal{X}$. Notice that we know that for every $X \in \mathcal{X}$ we know that $\beta_1(X)$ is of this form by commutativity of diagram (4.26).

Let $X \in \mathcal{X}$. Then

$$V_{\int \mathrm{Tr}(-,F)}(\beta_1(X)) = \left(F(X), \begin{tikzcd} \mathcal{C}{\downarrow}F(X) \arrow[r, "\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(X))"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "\mathrm{Id}_{\mathcal{G}_{\mathrm{Tr}(-,F)}}"] \arrow[d, "\int \mathrm{Tr}(-,F)"] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \right),$$

whereas

$$(\mathrm{Id}, \beta_1 \cdot)(\beta_1(X)) = \left(F(X), \begin{tikzcd} \mathcal{C}{\downarrow}F(X) \arrow[r, "\alpha_X^{\beta_1}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \right).$$

Therefore, on objects, (CoAlg2*) means that for every $X \in \mathcal{X}$

$$\beta_1 \circ \alpha_X^{\beta_1} = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-, \beta_1(X)).$$

Let now $X \xrightarrow{f} Y$ be a morphism in $\mathcal{X}$. Then by commutativity of diagram (4.26) we know that $\beta_1(f)$ will be of the form

$$\beta_1(X \xrightarrow{f} Y) = \left(F(X) \xrightarrow{F(f)} F(Y), \lambda^{\beta_1, f} \right)$$

with

$$\begin{tikzcd} \mathcal{C}{\downarrow}F(X) \arrow[rr, bend left, "\alpha_X^{\beta_1}"] \arrow[r, "F(f)\circ -"'] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}{\downarrow}F(Y) \arrow[r, "\alpha_Y^{\beta_1}"'] \arrow[u, phantom, "\Downarrow \lambda^{\beta_1, f}"] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

a morphism in $\mathrm{Tr}(F(X), F)$.
We have that

$$(\mathrm{Id}, \beta_1 \cdot)(\beta_1(X \xrightarrow{f} Y)) = \left(F(f), \beta_1 \lambda^{\beta_1, f} \right)$$

whereas

$$V_{\int \mathrm{Tr}(-,F)}(\beta_1(X \xrightarrow{f} Y)) = \left(F(f), \lambda^{\beta_1(f)} \right)$$

where $\lambda^{\beta_1(f)}$ is defined to be such that for every object $g\colon C \to F(X)$ in $\mathcal{C}{\downarrow}F(X)$ the morphism $\lambda_g^{\beta_1(f)}$ is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
\left(C, \mathcal{C}{\downarrow}C \xrightarrow{g\circ -} \mathcal{C}{\downarrow}F(X) \xrightarrow{\alpha_X^{\beta_1}} \mathcal{X}\right) \arrow[rr, "{(g,\mathrm{id})}"] \arrow[dd, "\int \mathrm{Tr}(-,F)"'] \arrow[dr, dashed, "\lambda_g^{\beta_1(f)}"] & & \left(F(X), \mathcal{C}{\downarrow}F(X) \xrightarrow{\alpha_X^{\beta_1}} \mathcal{X}\right) \arrow[dr, bend left, "{\left(F(f),\lambda^{\beta_1,f}\right)}"] \arrow[dd, "\int \mathrm{Tr}(-,F)"] & \\
& \left(C, \mathcal{C}{\downarrow}C \xrightarrow{g\circ -} \mathcal{C}{\downarrow}F(X) \xrightarrow{F(f)\circ -} \mathcal{C}{\downarrow}F(Y) \xrightarrow{\alpha_Y^{\beta_1}} \mathcal{X}\right) \arrow[rr, "{(F(f)\circ g,\mathrm{id})}"] \arrow[dd, "\int \mathrm{Tr}(-,F)"] & & \left(F(Y), \mathcal{C}{\downarrow}F(Y) \xrightarrow{\alpha_Y^{\beta_1}} \mathcal{X}\right) \arrow[dd, "\int \mathrm{Tr}(-,F)"] \\
C \arrow[rr, "g"] \arrow[dr, equal, bend right] & & F(X) \arrow[dr, "F(f)"] & \\
& C \arrow[rr, "F(f)g"'] & & F(Y)
\end{tikzcd}$$

in the place of the morphism in violet, that is

$$\lambda_g^{\beta_1(f)} = \left(\mathrm{id}_C, \lambda^{\beta_1,f}(g\circ -)\right)$$

Then, on morphisms, (CoAlg2*) means that for every $f\colon X \to Y$ morphism in $\mathcal{X}$

$$\beta_1\lambda^{\beta_1,f} = \lambda^{\beta_1(f)}.$$

In particular it means that for every $f\colon X \to Y$ in $\mathcal{X}$ and for every $g\colon C \to F(X)$ in $\mathcal{C}$

$$\left(\beta_1\lambda^{\beta_1,f}\right)_g = \left(\mathrm{id}_C, \lambda^{\beta_1,f}(g\circ -)\right).$$

We can now describe the 1-morphisms of Ω-coalgebras.

By the dualized version of Remark 1.11.1, given two Ω-coalgebras (F,β_1) and (G,γ_1), that is

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{X} \arrow[r, "\gamma_1"] \arrow[d, "G"'] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

a morphism of T-algebras from (F,α) to (G,β) is a (1-)morphism

$$H_1\colon \begin{array}{c}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{array} \longrightarrow \begin{array}{c}\mathcal{Y}\\ \downarrow G\\ \mathcal{C}\end{array}$$

in $\mathcal{CAT}{\downarrow}\mathcal{C}$ such that

(MCoAlg)
$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, "{(\mathrm{Id},H_1\cdot)}"] \arrow[d, "\int \mathrm{Tr}(-,F)"] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[r, "\gamma_1"] \arrow[d, "G"] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

(here we have used Remark 4.2.11 to say that $\Omega(H) = (\mathrm{Id}, H_1\cdot))$.

We will sometimes write a morphism of Ω-coalgebras just as a square
$$\begin{tikzcd} \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

Say that for every $X \in \mathcal{X}$

$$\beta_1(X) = \left(F(X), \begin{tikzcd} \mathcal{C}{\downarrow}F(X) \arrow[r, "\alpha_X^{\beta_1}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{X} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd} \right)$$

and for every $Y \in \mathcal{Y}$

$$\gamma_1(Y) = \left(G(Y), \begin{tikzcd} \mathcal{C}{\downarrow}G(Y) \arrow[r, "\alpha_Y^{\gamma_1}"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{Y} \arrow[d, "G"] \\ & \mathcal{C} \end{tikzcd} \right).$$

On objects, axiom (MCoAlg) means that for every $X \in \mathcal{X}$

$$H_1\alpha_X^{\beta_1} = \alpha_{H_1(X)}^{\gamma_1}$$

(notice that we have used that $G \circ H_1 = F$).

Now, say that for every $f\colon X \to E$ morphism in $\mathcal{X}$

$$\beta_1(f) = \big(F(f), \lambda^{\beta_1,f}\big)$$

with

$$\begin{tikzcd} \mathcal{C}{\downarrow}F(X) \arrow[rr, bend left, "\alpha_X^{\beta_1}"] \arrow[r, "F(f)\circ-"'] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}{\downarrow}F(E) \arrow[u, phantom, "\Downarrow\lambda^{\beta_1,f}"] \arrow[r, "\alpha_E^{\beta_1}"'] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}$$

a morphism in $\mathrm{Tr}(F(X), F)$, and that for every $g\colon Y \to M$ morphism in $\mathcal{Y}$

$$\gamma_1(g) = (G(g), \lambda^{\gamma_1,g})$$

with

$$\begin{tikzcd}
\mathcal{C}\downarrow G(Y) \arrow[r, "G(g)\circ-"'] \arrow[rr, bend left, "\alpha_Y^{\gamma_1}"] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow G(M) \arrow[r, "\alpha_M^{\gamma_1}"'] \arrow[u, phantom, "\Downarrow\lambda^{\gamma_1,g}"] & \mathcal{Y} \arrow[d, "G"] \\
& & \mathcal{C}
\end{tikzcd}$$

On morphisms, axiom (MCoAlg) means that for every $f\colon X \to E$ in $\mathcal{X}$

$$H_1\lambda^{\beta_1,f} = \lambda^{\gamma_1,H_1(f)}.$$

Finally, given $H_1, K_1\colon (F,\beta_1) \to (G,\gamma_1)$ two 1-morphisms of algebras, by the dualized version of Remark 1.11.1 a 2-morphism of algebras is a 2-cell $\lambda_1\colon H_1 \Rightarrow K_1 : F \to G$ in $\mathcal{CAT}\downarrow\mathcal{C}$, which means that

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}$$

holds, such that the following additional property holds as well:

$$\textbf{(2MCoAlg)}\qquad \begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[r, bend left, "(\mathrm{Id},H_1\cdot)"] \arrow[r, bend right, "(\mathrm{Id},K_1\cdot)"'] \arrow[r, phantom, "\Downarrow(\mathrm{id},\lambda_1\cdot)"] & \mathcal{G}_{\mathrm{Tr}(-,G)}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] & \mathcal{Y} \arrow[r, "\gamma_1"] & \mathcal{G}_{\mathrm{Tr}(-,G)}
\end{tikzcd}$$

(here we have used Remark 4.2.11 to say that $\Omega(\lambda_1) = (\mathrm{id}, \lambda_1\cdot)$).

Notation 4.3.3. From now on, we will denote Ω the 2-comonad generated by the 2-adjunction $L \dashv R$. We will then denote $\Omega-\mathcal{CoAlg}$ the 2-category of Ω-coalgebras, described explicitly in Remark 4.3.2.

Now, aiming at proving that the forgetful 2-functor $L\colon \mathcal{SpFIB}_{\mathcal{C}} \to \mathcal{CAT}\downarrow\mathcal{C}$ is comonadic, we consider the canonical comparison 2-functor $\overline{L}\colon \mathcal{SpFIB}_{\mathcal{C}} \to T-\mathcal{CoAlg}$. We will then show that it is an isomorphism of categories.

Remark 4.3.4. By Theorem 1.11.6 (which is the enriched version of Theorem 1.5.8), the comparison 2-functor associated to $L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$ and to the comonad Ω

$$\begin{tikzcd}
\mathcal{SpFIB}_{\mathcal{C}} \arrow[r, dashed, "\overline{L}"] \arrow[dr, "L"'] & \Omega\text{–}\mathcal{CoAlg} \arrow[d, "U_\Omega"] \\
& \mathcal{CAT} \downarrow \mathcal{C}
\end{tikzcd}$$

is given by

$$\overline{L} \colon \mathcal{SpFIB}_{\mathcal{C}} \to \Omega\text{–}\mathcal{CoAlg}$$

$$\begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array} \longmapsto \left(\begin{array}{c}\mathcal{E}\\ \big\downarrow L(p)\\ \mathcal{C}\end{array}, \quad \begin{array}{c}\mathcal{E}\\ \big\downarrow p\\ \mathcal{C}\end{array} \xrightarrow{(L\eta_1)_p} \begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,p)}\\ \big\downarrow \int \mathrm{Tr}(-,p)\\ \mathcal{C}\end{array} \right)$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend right] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "H_1"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, equal, bend left] \arrow[r, equal, bend right] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}$$

We also know, by Remark 4.2.11, that $(L\eta_1)_p$ is given by the square

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "V_p"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,p)} \arrow[d, "\int \mathrm{Tr}(-,p)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Thus we have already found a canonical way to associate to every object, 1-morphism and 2-morphism in $\mathcal{SpFIB}_{\mathcal{C}}$ respectively a coalgebra for the comonad Ω, a 1-morphism of coalgebras and a 2-morphism of coalgebras.

We now want to show that if we start from a coalgebra for the comonad Ω we can construct a split fibration over $\mathcal{C}$ from it. After this, we will consider 1-morphisms and 2-morphisms of coalgebras.

Construction 4.3.5. Let

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

be an Ω-coalgebra. We show that the structure of coalgebra on the functor $\begin{smallmatrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{smallmatrix}$ makes F into a split fibration over $\mathcal{C}$. Firstly, we have to find a cleavage for F.

We then start from a diagram

$$\begin{tikzcd}
 & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and we search for a cartesian lifting of f to E. To do this, we apply the structure morphism β_1 to the diagram above

$$\begin{tikzcd}
 & E \arrow[d, mapsto, "F"] & & \left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, mapsto, "\int \mathrm{Tr}(-,F)"] & \left(F(E), \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[d, mapsto, "\int \mathrm{Tr}(-,F)"] \\
A \arrow[r, "f"'] & F(E) & \arrow[r, mapsto, "\beta_1", "\mathrm{Id}"'] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and find the data in violet using the fact that $\int \mathrm{Tr}\,(-,F)$ is a cloven fibration. We now recall that axiom (CoAlg1) of coalgebra for the comonad Ω (see Remark 4.3.2) implies that $(\varepsilon_F)_1 \circ \beta_1 = \mathrm{id}_F$. Therefore applying $(\mathrm{Id}, (\varepsilon_F)_1)$ to the diagram above on the right we restore our starting data

$$\begin{tikzcd}
(\varepsilon_F)_1 \left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "{(\varepsilon_F)_1(f,\mathrm{id})}"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and we see that we have produced, in addition, a morphism above f (in violet). Thus, it suffices to prove that this morphism $(\varepsilon_F)_1(f, \mathrm{id})$ is cartesian.

Consider then a diagram

$$
\begin{tikzcd}
X \arrow[rrd, bend left, "k"] \arrow[dd, "F"] & & \\
& (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "(\varepsilon_F)_1(f,\mathrm{id})"] \arrow[d, "F"] & E \arrow[d, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{4.27}
$$

with arbitrary $X \in \mathcal{X}$, $a\colon F(X) \to A$ in $\mathcal{C}$ and $k\colon X \to E$ above fa. We would like to show that there is a unique morphism

$$
v\colon X \longrightarrow (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)
$$

which fits well into diagram (4.27).
To find a suitable such morphism v we repeat the same argument of above: we apply (Id, β_1) and then $(\mathrm{Id}, (\varepsilon_F)_1)$ to diagram

$$
\begin{tikzcd}
X \arrow[rrd, bend left, "k"] \arrow[dd, "F"] & & \\
& & E \arrow[d, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{4.28}
$$

in order to restore the data of diagram (4.27) but trying to find some further data in the middle step, using the fact that $\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)} \\ \big\downarrow {\textstyle\int}\mathrm{Tr}(-,F) \\ \mathcal{C}\end{array}$ is a cloven fibration.

If we apply (Id, β_1) to diagram (4.28) we get the diagram in black

$$
\begin{tikzcd}
\left(F(X), \mathcal{C}\downarrow F(X) \xrightarrow{\alpha_X^{\beta_1}} \mathcal{X}\right) \arrow[rrd, bend left, "(fa,\lambda^{\beta_1,k})"] \arrow[rd, dashed, "(a,\lambda^{\beta_1,k})"] \arrow[dd, "\int\mathrm{Tr}(-,F)"] & & \\
& \left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "(f,\mathrm{id})"] \arrow[d, "\int\mathrm{Tr}(-,F)"] & \left(F(E), \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[d, "\int\mathrm{Tr}(-,F)"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{4.29}
$$

from which we can produce the morphism (f, id) of the cleavage of $\begin{array}{c}\mathcal{G}_{\mathrm{Tr}(-,F)}\\ \big\downarrow{\scriptstyle \int \mathrm{Tr}(-,F)}\\ \mathcal{C}\end{array}$ as above, and we also know that $\left(a, \lambda^{\beta_1,k}\right)$ is the unique morphism from $\left(F(X), \mathcal{C}\!\downarrow\! F(X) \xrightarrow{\alpha_X^{\beta_1}} \mathcal{X}\right)$ to $\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{f\circ -} \mathcal{C}\!\downarrow\! F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$ which fits well into diagram (4.29) in the place of the morphism in violet, since it is an acceptable choice and (f, id) is cartesian with respect to $\int \mathrm{Tr}\,(-, F)$. Applying $(\mathrm{Id}, (\varepsilon_F)_1)$ to diagram (4.29) we then get

$$\begin{tikzcd}
X \arrow[dd, mapsto, "F"'] \arrow[dr, dashed, "(\varepsilon_F)_1\left(a,\lambda^{\beta_1,k}\right)"] \arrow[drr, bend left, "k"] & & \\
& (\varepsilon_F)_1\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{f\circ -} \mathcal{C}\!\downarrow\! F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "(\varepsilon_F)_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd} \tag{4.30}$$

using again the axiom (CoAlg1) of coalgebra for the comonad Ω (from Remark 4.3.2), as above, to get that $(\varepsilon_F)_1 \circ \beta_1 = \mathrm{id}_F$.

Therefore we have found a morphism $v\colon X \to (\varepsilon_F)_1\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{f\circ -} \mathcal{C}\!\downarrow\! F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$ which fits well into diagram (4.27). It remains to prove that $(\varepsilon_F)_1\left(a, \lambda^{\beta_1,k}\right)$ is the unique such morphism v. In order to prove this, we recall what we did in the same situation for the monad case (in Section 3.3). Then we investigate whether we have a useful bijection of hom-sets also this time or not. We won't be able to find such a bijection, but the attempt will lead us to the proof of the uniqueness of v we are searching for.

Construction 4.3.6. Given

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

an Ω-coalgebra, $X \in \mathcal{X}$ and $\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{\alpha} \mathcal{X}\right)$ an object of $\mathcal{G}_{\mathrm{Tr}(-,F)}$, we try to construct two functions φ and ψ (which will depend on X and (A, α)) which are inverses of each other and yield a bijection of hom-sets

$$\mathrm{Hom}\left(\beta_1(X), \left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{\alpha} \mathcal{X}\right)\right) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}\left(X, (\varepsilon_F)_1\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{\alpha} \mathcal{X}\right)\right) \tag{4.31}$$

which would sum up to an adjunction between β_1 and $(\varepsilon_F)_1$. Our attempt will be doomed to fail, but it will be useful anyway to see how the functions φ and ψ work, even if they won't be inverses of each other. The idea will be, rather than applying ψ and then φ obtaining the identity, to use the universal property of cartesian morphisms in the middle step.

Start from a morphism $\beta_1(X) \longrightarrow \left(A, \mathcal{C}{\downarrow}A \xrightarrow{\alpha} \mathcal{X}\right)$ in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, which will be of the form (a, λ) with $a\colon F(X) \longrightarrow A$ and

$$\begin{tikzcd}
\mathcal{C}{\downarrow}F(X) \arrow[r, "a\circ -"'] \arrow[rr, bend left=40, "\alpha_X^{\beta_1}"] \arrow[rr, phantom, bend left=20, "\Downarrow\lambda"] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}{\downarrow}A \arrow[r, "\alpha"'] & \mathcal{X} \arrow[d, "F"] \\
& & \mathcal{C}
\end{tikzcd}$$

Recall that axiom (CoAlg1) of coalgebras for the comonad Ω implies that $(\varepsilon_F)_1(\beta_1(X)) = X$. Then it is natural to choose φ as the function on morphisms given by the functor $(\varepsilon_F)_1$:

$$\varphi(a, \lambda) := (\varepsilon_F)_1(a, \lambda)\colon X \longrightarrow (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{\alpha} \mathcal{X}\right).$$

Remember that this is the way in which we have produced, in Construction 4.3.5, the morphism $v\colon X \to (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$ whose uniqueness we aim at proving.
We also notice that

$$F(\varphi(a, \lambda) = F((\varepsilon_F)_1(a, \lambda)) = \int \mathrm{Tr}\,(-, F)(a, \lambda) = a.$$

Now, we would like to construct an inverse ψ of the function φ. Start then from a morphism

$$q\colon X \longrightarrow (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{\alpha} \mathcal{X}\right),$$

that is $q\colon X \longrightarrow \alpha(A \xrightarrow{\mathrm{id}} A)$. We need to construct a morphism $\psi(q) = (a, \mu)$ with $a\colon F(X) \to A$ and

$$\begin{tikzcd}
\mathcal{C}{\downarrow}F(X) \arrow[r, "a\circ -"'] \arrow[rr, bend left=40, "\alpha_X^{\beta_1}"] \arrow[rr, phantom, bend left=20, "\Downarrow\mu"] \arrow[drr, bend right, "\mathrm{dom}"'] & \mathcal{C}{\downarrow}A \arrow[r, "\alpha"'] & \mathcal{X} \arrow[d, "F"] \\
& & \mathcal{C}
\end{tikzcd}.$$

It is natural to consider $a := F(q)\colon F(X) \to F\left(\alpha(A \xrightarrow{\mathrm{id}} A)\right) = A$. Then μ needs to be of the form

$$\begin{tikzcd}
\mathcal{C}\downarrow F(X) \arrow[r, "F(q)\circ -"'] \arrow[rr, bend left, "\alpha_X^{\beta_1}"] \arrow[rrd, bend right, "\mathrm{dom}"'] & \mathcal{C}\downarrow A \arrow[r, "\alpha"'] \arrow[u, phantom, "\Downarrow\mu"] & \mathcal{X} \arrow[d, "F"] \\
& & \mathcal{C}
\end{tikzcd}.$$

We also want that $\varphi(\psi(q)) = q$, that is

$$\alpha\left(\begin{tikzcd} F(X) \arrow[d, "F(q)"'] \arrow[dr, "F(q)"] & \\ A \arrow[r, "\mathrm{id}"'] & A \end{tikzcd}\right) \circ \mu_{(F(X)\xrightarrow{\mathrm{id}}F(X))} = q. \tag{4.32}$$

But now we don't know how to find $\mu_{(F(X)\xrightarrow{\mathrm{id}}F(X))}$ solving equation (4.32), since there is a lack of information for such a general $\left(A, \mathcal{C}\downarrow A \xrightarrow{\alpha} \mathcal{X}\right) \in \mathcal{G}_{\mathrm{Tr}(-,F)}$.

However, we just aim at proving the uniqueness of the morphism

$$v\colon X \to (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$$

we have produced in Construction 4.3.5. Then, for us, it would suffice to find a bijection of the form of equation (4.31) just for $X \in \mathcal{X}$ and $\left(A, \mathcal{C}\downarrow A \xrightarrow{\alpha} \mathcal{X}\right) \in \mathcal{G}_{\mathrm{Tr}(-,F)}$ of the form

$$\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$$

with $E \in \mathcal{X}$ and $f\colon A \longrightarrow F(E)$ a morphism in $\mathcal{C}$.

We define the function φ as above, whereas now, in order to define an inverse ψ of φ, we notice that

$$\left(A, \begin{tikzcd} \mathcal{C}\downarrow A \arrow[r, "f\circ -"] \arrow[rrd, "\mathrm{dom}"'] & \mathcal{C}\downarrow F(E) \arrow[r, "\alpha_E^{\beta_1}"] \arrow[rd, "\mathrm{dom}"] & \mathcal{X} \arrow[d, "F"] \\ & & \mathcal{C} \end{tikzcd}\right) = f^*\beta_1(E)$$

with respect to $\int \mathrm{Tr}\,(-,F)$.

But $f^*\beta_1(E) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))(A \xrightarrow{f} F(E)$ and we might recall axiom (CoAlg2*) of coalgebras for the comonad Ω (from Remark 4.3.2), which implies that

$$\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E)) = \beta_1 \circ \alpha_E^{\beta_1}.$$

Then, starting from a morphism in $\mathcal{X}$

$$q\colon X \longrightarrow (\varepsilon_F)_1\left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right),$$

that is

$$q\colon X \longrightarrow \alpha_E^{\beta_1}(A \xrightarrow{f} F(E)),$$

we need to construct a morphism in $\mathcal{G}_{\mathrm{Tr}(-,F)}$

$$\psi(q)\colon \beta_1(X) \longrightarrow \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) = \beta_1(\alpha_E^{\beta_1}(A \xrightarrow{f} F(E))).$$

And thus it is natural to define

$$\psi(q) := \beta_1(q)$$

(that is, we define ψ as the function on morphisms given by the functor β_1).

Remark 4.3.7. Now we would like to prove that, given $X \in \mathcal{X}$ and $\left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \in \mathcal{G}_{\mathrm{Tr}(-,F)}$, with $E \in \mathcal{X}$ and $f\colon A \to F(E)$ a morphism in $\mathcal{C}$, we have that the functions φ and ψ defined in Construction 4.3.6 are inverses of each other and yields a bijection

$$\mathrm{Hom}\left(\beta_1(X), \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)\right) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}\left(X, (\varepsilon_F)_1\left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)\right).$$

We surely have, given $q\colon X \longrightarrow (\varepsilon_F)_1\left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$ a morphism in $\mathcal{X}$, that

$$\varphi(\psi(q)) = \varphi(\beta_1(q)) = (\varepsilon_F)_1(\beta_1(q)) = q$$

by axiom (CoAlg1) of coalgebra for the comonad Ω (from Remark 4.3.2).

Let now $(a, \lambda)\colon \beta_1(X) \longrightarrow \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$ be a morphism in $\mathcal{G}_{\mathrm{Tr}(-,F)}$. We would need to show that $\psi(\varphi(a, \lambda)) = (a, \lambda)$ (because then we would conclude by arbitrariness of q and (a, λ). Using axiom (CoAlg2*) of coalgebra for the comonad Ω (from Remark 4.3.2) we get that

$$\psi(\varphi(a,\lambda)) = \psi((\varepsilon_F)_1(a,\lambda)) = \psi\left(\alpha_E^{\beta_1}(f\circ -)\left(\begin{tikzcd} F(X) \arrow[d, "a"'] \arrow[dr, "a"] & \\ A \arrow[r, "\mathrm{id}"'] & A \end{tikzcd}\right)\circ \lambda_{(F(X)\overset{\mathrm{id}}{\to}F(X))}\right) =$$

$$= \beta_1\alpha_E^{\beta_1}\left(\begin{tikzcd} F(X) \arrow[d, "a"'] \arrow[dr, "fa"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right)\circ \beta_1\left(\lambda_{(F(X)\overset{\mathrm{id}}{\to}F(X))}\right) =$$

$$= \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} F(X) \arrow[d, "a"'] \arrow[dr, "fa"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right)\circ \beta_1\left(\lambda_{(F(X)\overset{\mathrm{id}}{\to}F(X))}\right)$$

But here we find an issue, since

$$\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} F(X) \arrow[d, "a"'] \arrow[dr, "fa"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right)$$

cannot give us anything related to λ and then we would need to recover the whole information of λ starting from $\lambda_{(F(X)\overset{\mathrm{id}}{\to}F(X))}$ and applying β_1.

Thus, we won't have the bijection we have been searching for, and we have to change approach. Instead of applying the functions ψ and φ to an arbitrary morphism v which fits well into diagram (4.30) in the place of the morphism in violet, which is the analogue of what we did in the proof of Theorem 3.3.8 (in the monad case), we apply the functors β_1 and $(\varepsilon_F)_1$ (which gave us the functions ψ and φ) to the whole upper triangle of diagram (4.30) and use in the middle step the uniqueness given by the fact that (f,id) is a cartesian morphism (with respect to $\int \mathrm{Tr}\,(-,F)$). This strategy will be successful since what we have seen in Construction 4.3.6 will ensure that the domain and the codomain of the arrows will be correct, and if we consider the particular morphism

$$(f,\mathrm{id})\colon \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \longrightarrow \left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right)$$

in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, we have that

$$\beta_1((\varepsilon_F)_1(f,\mathrm{id})) = (f,\mathrm{id})$$

as we shall promptly prove, since the issue we have just described in this remark disappear taking $\lambda = \mathrm{id}$ (but be careful with the fact that domain and codomain will be different).

We are now ready to prove that the structure of an Ω-coalgebra (F, β_1) defines a structure of a split fibration over $\mathcal{C}$ for F. We will say that an Ω-coalgebra is a split fibration over $\mathcal{C}$.

Theorem 4.3.8. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

be an Ω-coalgebra. Then the assignment

$$\operatorname{Cart}(f, E) := (\varepsilon_F)_1(f, \mathrm{id})\colon (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X} \right) \longrightarrow E \tag{4.33}$$

for every diagram

$$\begin{tikzcd} & E \arrow[d, mapsto, "F"] \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$$

with $E \in \mathcal{X}$ and f a morphism in $\mathcal{C}$, which we have produced in Construction 4.3.5, defines a splitting cleavage for F (which thus makes F into a split fibration over $\mathcal{C}$).

Proof. We prove that the assignment in equation (4.33) defines a cleavage for F. After what we have shown in Construction 4.3.5, it remains to prove that $(\varepsilon_F)_1\left(a, \lambda^{\beta_1,k}\right)$ is the only morphism which fits well into diagram (4.30), which we write again below, in the place of the morphism in violet:

$$\begin{tikzcd} X \arrow[rrr, bend left, "k"] \arrow[dr, dashed, "(\varepsilon_F)_1\left(a,\lambda^{\beta_1,k}\right)"] \arrow[dd, mapsto, "F"'] & & & \\ & (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X} \right) \arrow[rr, "(\varepsilon_F)_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & & E \arrow[d, mapsto, "F"] \\ F(X) \arrow[r, "a"'] \arrow[rrr, bend left, "fa"] & A \arrow[rr, "f"'] & & F(E) \end{tikzcd} \tag{4.34}$$

In fact we will then conlcude by arbitrariness of k and a that $\operatorname{Cart}(f, E) = (\varepsilon_F)_1(f, \mathrm{id})$ is a cartesian morphism.

Let then $q\colon X \longrightarrow (\varepsilon_F)_1 \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X} \right)$ be another morphism which fits well into diagram (4.34) in the place of the morphism in violet, that is, such that

$$
\begin{tikzcd}
X \arrow[r, dashed, "q"] \arrow[rr, bend left, "k"] \arrow[d, mapsto, "F"'] & (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right) \arrow[r, "(\varepsilon_F)_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{4.35}
$$

Recalling what we have said in Remark 4.3.7, we apply (Id, β_1) to diagram (4.35), and we get

$$
\begin{tikzcd}
\left(F(X), \mathcal{C}\downarrow F(X) \xrightarrow{\alpha^{\beta_1}_X} \mathcal{X}\right) \arrow[r, dashed, "\beta_1(q)"] \arrow[rr, bend left, "(fa, \lambda^{\beta_1, k})"] \arrow[d, mapsto, "\int \mathrm{Tr}(-,F)"'] & \beta_1\left((\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right)\right) \arrow[r, "\beta_1((\varepsilon_F)_1(f,\mathrm{id}))"'] \arrow[d, mapsto, "\int \mathrm{Tr}(-,F)"] & \left(F(E), \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right) \arrow[d, mapsto, "\int \mathrm{Tr}(-,F)"] \\
F(X) \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}
\tag{4.36}
$$

Notice that diagram (4.36) is commutative since it is image of a commutative diagram under a functor, and that the upper triangle is above the lower triangle since β_1 is the structure morphism of the Ω-coalgebra (F, β_1).

By Construction 4.3.6, we know that

$$
\beta_1\left((\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right)\right) = \left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right),
$$

since $\beta_1(q) = \psi(q) \colon \beta_1(X) \to \left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha^{\beta_1}_E} \mathcal{X}\right)$.

Now we want to prove that

$$
\beta_1((\varepsilon_F)_1(f, \mathrm{id})) = (f, \mathrm{id})
$$

in order to be able to use the universal property of cartesian morphisms.
As we did in Remark 4.3.7, using axiom (CoAlg2*) of coalgebra for the comonad Ω (from Remark 4.3.2) we get that

$$\beta_1((\varepsilon_F)_1(f,\mathrm{id})) = \beta_1\left(\alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)\circ \mathrm{id}_{(A\xrightarrow{\mathrm{id}}A)}\right) =$$

$$= \beta_1\left(\alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)\right) = \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)$$

But by construction

$$\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id"'] & F(E) \end{tikzcd}\right)$$

is the unique morphism w which fits well into the diagram

$$\begin{tikzcd}
\left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[rr, "{(f,\mathrm{id})}"] \arrow[dr, dashed, "w"] \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] & & \left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[dr, bend left, "\mathrm{id}"] \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] & \\
& \left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[rr, "\mathrm{id}"] \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] & & \left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] \\
A \arrow[rr, "f"'] \arrow[dr, bend right, "f"'] & & F(E) \arrow[dr, "\mathrm{id}"] & \\
& F(E) \arrow[rr, "\mathrm{id}"'] & & F(E)
\end{tikzcd}$$

in the place of the morphism in violet. Since (f,id) is an acceptable morphism w which fits well into the diagram above in the place of the morphism in violet, we obtain that

$$\mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) = (f,\mathrm{id})$$

and thus that

$$\beta_1((\varepsilon_F)_1(f,\mathrm{id})) = (f,\mathrm{id}).$$

Then diagram (4.36) is equal to the diagram

$$
\begin{tikzcd}
\left(F(X), \mathcal{C}{\downarrow}F(X) \xrightarrow{\alpha_X^{\beta_1}} \mathcal{X}\right) \arrow[rrd, bend left, "{(fa,\lambda^{\beta_1,k})}"] \arrow[rd, dashed, "\beta_1(q)"] \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] & & \\
& \left(A, \mathcal{C}{\downarrow}A \xrightarrow{f\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "{(f,\mathrm{id})}"] \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] & \left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[dd, mapsto, "\int \mathrm{Tr}(-,F)"] \\
F(X) \arrow[rd, "a"'] \arrow[rrd, bend left, "fa"] & & \\
& A \arrow[r, "f"'] & F(E)
\end{tikzcd}
$$

Therefore we have shown that $\beta_1(q)$ is an acceptable morphism which fits well into diagram (4.29) in the place of the morphism in violet, whence we get that

$$\beta_1(q) = \left(a, \lambda^{\beta_1,k}\right).$$

But then, using axiom (CoAlg1) of Ω-coalgebra (from Remark 4.3.2) we get that

$$q = (\varepsilon_F)_1(\beta_1(q)) = (\varepsilon_F)_1\left(a, \lambda^{\beta_1,k}\right),$$

whence we obtain the desired uniqueness of the morphism in violet of diagram (4.34).

Therefore we have proved that the assignment in equation (4.33) defines a cleavage for F. We show that it makes F into a split fibration (and then F, with this cleavage, will be a split fibration over $\mathcal{C}$).

We first see that F, with the constructed cleavage, is a normal fibration. We need to show that for every $E \in \mathcal{X}$ we have that

$$\mathrm{Cart}\left(\mathrm{id}_{F(E)}, E\right) = \mathrm{id}_E \colon E \to E.$$

But this holds because if $E \in \mathcal{X}$ then

$$\mathrm{Cart}\left(\mathrm{id}_{F(E)}, E\right) = \left((\varepsilon_F)_1(\mathrm{id},\mathrm{id}) \colon (\varepsilon_F)_1\left(F(E), \mathcal{C}{\downarrow}F(E) \xrightarrow{\mathrm{id}\circ -} \mathcal{C}{\downarrow}F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \longrightarrow E\right) =$$
$$= ((\varepsilon_F)_1(\mathrm{id},\mathrm{id}) \colon (\varepsilon_F)_1(\beta_1(E)) \longrightarrow E) = (\mathrm{id} \colon E \longrightarrow E),$$

by axiom (CoAlg1) of Ω-coalgebra (from Remark 4.3.2) and by the fact that $(\varepsilon_F)_1$ is a functor.

Now, in order to prove that F is split, given $E \in \mathcal{X}$ and given $f \colon A \to F(E)$ and $a \colon A' \to A$ in $\mathcal{C}$, we prove that

$$\mathrm{Cart}\,(f, E) \circ \mathrm{Cart}\,(a, f^*E) = \mathrm{Cart}\,(fa, E).$$

Then we need to prove the equality in the diagram

$$\begin{tikzcd}
(\varepsilon_F)_1\left(A', \mathcal{C}\downarrow A' \xrightarrow{(fa)\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[d, equal] \arrow[ddrr, bend left, "(\varepsilon_F)_1(fa,\mathrm{id})"] & & \\
(\varepsilon_F)_1\left(A', \mathcal{C}\downarrow A' \xrightarrow{a\circ -} \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[dr, "(\varepsilon_F)_1(a,\mathrm{id})"'] \arrow[dd, mapsto, "F"] & & \\
 & (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \arrow[r, "(\varepsilon_F)_1(f,\mathrm{id})"] \arrow[d, mapsto, "F"] & E \arrow[d, mapsto, "F"] \\
A' \arrow[r, "a"'] \arrow[rr, bend left, "fa"] & A \arrow[r, "f"'] & F(E)
\end{tikzcd}$$

and verify that it is a commutative diagram. But the equality is trivial and the commutativity follows by the fact that $(\varepsilon_F)_1$ is a functor from $\mathcal{G}_{\mathrm{Tr}(-,F)}$ to $\mathcal{X}$ and by the fact that

$$(f, \mathrm{id}) \circ (a, \mathrm{id}) = (fa, \mathrm{id})$$

in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, by Construction 2.3.9. $\square$

Recall from Remark 4.3.4 that aiming at proving the comonadicity of split fibrations over $\mathcal{C}$, now that we have shown that an Ω-coalgebra is a split fibration over $\mathcal{C}$, it remains to show that 1-morphisms and 2-morphisms of coalgebras are respectively 1-morphisms and 2-morphisms of split fibrations over $\mathcal{C}$ (that is, of cloven fibrations over $\mathcal{C}$).

Proposition 4.3.9. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

be a 1-morphism of coalgebras between two Ω-coalgebras

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
\mathcal{X} \arrow[r, "\gamma_1"] \arrow[d, "G"'] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Consider F and G as split fibrations over $\mathcal{C}$ by using Theorem 4.3.8. Then H_1 is a cloven cartesian functor over $\mathcal{C}$.

Proof. We need to prove that for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$

$$H_1(\mathrm{Cart}_F(f,E)) = \mathrm{Cart}_G(f, H_1(E)).$$

Given $E \in \mathcal{X}$ and $f\colon A \to F(E)$ in $\mathcal{C}$, we have that

$$\mathrm{Cart}_F(f,E) = (\varepsilon_F)_1(f, \mathrm{id}_E)\colon (\varepsilon_F)_1\left(A, \mathcal{C}\!\downarrow\! A \xrightarrow{f\circ -} \mathcal{C}\!\downarrow\! F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \longrightarrow E$$

by construction (see Theorem 4.3.8 and Construction 4.3.5). Then

$$H_1(\mathrm{Cart}_F(f,E)) = H_1((\varepsilon_F)_1(f,\mathrm{id}_E)) = H_1\alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) \circ \mathrm{id}\,.$$

But H_1 is a 1-morphism of Ω-coalgebras and then it satisfies axiom (MCoAlg) (from Remark 4.3.2), which implies (looking at what it means on objects, in Remark 4.3.2) that

$$H_1\alpha_E^{\beta_1} = \alpha_{H_1(E)}^{\gamma_1}.$$

Therefore we get that

$$H_1(\mathrm{Cart}_F(f,E)) = H_1\alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) = \alpha_{H_1(E)}^{\gamma_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) = \mathrm{Cart}_G(f, H_1(E))$$

(using also that $G \circ H_1 = F$). $\square$

It now remains to consider 2-morphisms of Ω-coalgebras and prove that they are 2-morphisms of split fibrations over $\mathcal{C}$.

Proposition 4.3.10. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, bend left, "H_1"] \arrow[rr, bend right, "K_1"'] \arrow[rr, phantom, "\Downarrow\lambda_1"] \arrow[dd, "F"'] & & \mathcal{Y} \arrow[dd, "G"] \\ & & \\ \mathcal{C} \arrow[rr, equal, bend right] & & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[rr, bend left, "H_1"] \arrow[dd, "F"'] & & \mathcal{Y} \arrow[dd, "G"] \\ & & \\ \mathcal{C} \arrow[rr, equal, bend left] \arrow[rr, equal, bend right] \arrow[rr, phantom, "\Downarrow\mathrm{Id}"] & & \mathcal{C} \end{tikzcd}$$

be a 2-morphism of Ω-coalgebras, where F and G have structures of coalgebra given by

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{X} \arrow[r, "\gamma_1"] \arrow[d, "G"'] & \mathcal{G}_{\mathrm{Tr}(-,G)} \arrow[d, "\int \mathrm{Tr}(-,G)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Consider F and G as split fibrations over $\mathcal{C}$ by using Theorem 4.3.8, and consider H_1 and K_1 as cloven cartesian functors over $\mathcal{C}$ by using Proposition 4.3.9. Then λ_1 is a 2-morphisms of split fibrations over $\mathcal{C}$.

Proof. The proof is trivial since we have defined a 2-morphism of split fibrations over $\mathcal{C}$ to be just a 2-cell in $\mathcal{CAT}\downarrow\mathcal{C}$. □

Remark 4.3.11. Notice that the proof of Proposition 4.3.10 is trivial because the structure of a split fibration over $\mathcal{C}$ is related to 1-morphisms only.

Proposition 4.3.12. *Theorem 4.3.8, Proposition 4.3.9 and Proposition 4.3.10 jointly yield a 2-functor*

$$W\colon \Omega-\mathcal{CoAlg} \longrightarrow \mathcal{SpFIB}_{\mathcal{C}}$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] & \arrow[r, mapsto] & & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} & & & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] & \arrow[r, mapsto] & & \mathcal{X} \arrow[r, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} & & & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend right, equal] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "K_1"'] \arrow[r, phantom, "\Downarrow\lambda_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend right, equal] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "H_1"] \arrow[d, "F"'] & \mathcal{Y} \arrow[d, "G"] \\
\mathcal{C} \arrow[r, bend left, equal] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \mathcal{C}
\end{tikzcd}$$

where $\begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$ *has the splitting cleavage defined by*

$$\mathrm{Cart}\,(f,E) := (\varepsilon_F)_1(f,\mathrm{id})\colon (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \longrightarrow E$$

for every $E \in \mathcal{X}$ and every $f\colon A \to F(E)$ in $\mathcal{C}$.

Proof. The proof is trivial. □

Theorem 4.3.13. *The comparison 2-functor $\overline{L}\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \Omega\text{–}\mathcal{CoAlg}$ associated to 2-functor $L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$ and to the 2-comonad Ω*

$$\begin{tikzcd} \mathcal{SpFIB}_{\mathcal{C}} \arrow[r, dashed, "\overline{L}"] \arrow[dr, "L"'] & \Omega-\mathcal{CoAlg} \arrow[d, "U_\Omega"] \\ & \mathcal{CAT}\downarrow\mathcal{C} \end{tikzcd}$$

is an isomorphism of categories. In particular, $L\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT}\downarrow\mathcal{C}$ *is comonadic.*

Proof. We prove that the 2-functor

$$W\colon \Omega-\mathcal{CoAlg} \longrightarrow \mathcal{SpFIB}_{\mathcal{C}}.$$

produced in Proposition 4.3.12 is an inverse functor of

$$\overline{L}\colon \mathcal{SpFIB}_{\mathcal{C}} \longrightarrow \Omega-\mathcal{CoAlg}.$$

We first prove that $\overline{L}$ and W are inverses of each other on objects.

Given a split fibration $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{smallmatrix}$ over $\mathcal{C}$, with cleavage denoted $\mathrm{Cart}_p(-,\cdot)$, we prove that

$$W\left(\overline{L}\left(\begin{matrix}\mathcal{E}\\ \Big\downarrow p\\ \mathcal{C}\end{matrix}\right)\right) = \begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}$$

We have that

$$W\left(\overline{L}\left(\begin{matrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{matrix}\right)\right) = W\left(\begin{tikzcd} \mathcal{E} \arrow[r, "V_p"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,p)} \arrow[d, "\int \mathrm{Tr}(-,p)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}\right)$$

by Remark 4.3.4, and

$$W\left(\begin{tikzcd} \mathcal{E} \arrow[r, "V_p"] \arrow[d, "p"'] & \mathcal{G}_{\mathrm{Tr}(-,p)} \arrow[d, "\int \mathrm{Tr}(-,p)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}\right)$$

coincides with $\begin{smallmatrix}\mathcal{E}\\ \downarrow p\\ \mathcal{C}\end{smallmatrix}$ equipped with the splitting clevage defined by

$$\mathrm{Cart}\,(u,E) := (\varepsilon_p)_1(u,\mathrm{id})\colon (\varepsilon_p)_1\Big(B, \mathcal{C}\downarrow B \xrightarrow{u\circ -} \mathcal{C}\downarrow p(E) \xrightarrow{\alpha_E^{V_p}} \mathcal{X}\Big) \longrightarrow E$$

for every $E \in \mathcal{E}$ and every $u\colon B \to p(E)$ in $\mathcal{C}$, by Proposition 4.3.12.

By Construction 4.2.4, given $E \in \mathcal{E}$ and $u\colon B \to p(E)$ in $\mathcal{C}$, we have that

$$V_p(E) = \left(p(E), \begin{tikzcd} \mathcal{C}\downarrow p(E) \arrow[r, "\mathrm{Cart}_p(-,E)"] \arrow[dr, "\mathrm{dom}"'] & \mathcal{E} \arrow[r, "\mathrm{Id}_\mathcal{E}"] \arrow[d, "p"] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \right),$$

whence we get that $\alpha_E^{V_p} = \mathrm{Cart}_p(-,E)$.
We then have that

$$(\varepsilon_p)_1\left(B, \mathcal{C}\downarrow B \xrightarrow{u\circ -} \mathcal{C}\downarrow p(E) \xrightarrow{\alpha_E^{V_p}} \mathcal{X}\right) = (\varepsilon_p)_1\left(B, \mathcal{C}\downarrow B \xrightarrow{u\circ -} \mathcal{C}\downarrow p(E) \xrightarrow{\mathrm{Cart}_p(-,E)} \mathcal{X}\right) =$$
$$= \mathrm{Cart}_p(-,E)(B \xrightarrow{u} p(E)) = u^*E$$

and that

$$(\varepsilon_p)_1(u,\mathrm{id}) = \mathrm{Cart}_p(-,E)\left(\begin{tikzcd} B \arrow[d, "u"'] \arrow[dr, "u"] & \\ p(E) \arrow[r, "\mathrm{id}"'] & p(E) \end{tikzcd} \right) \circ \mathrm{id} = \mathrm{Cart}_p(u,E).$$

Therefore we have shown that the splitting cleavage $\mathrm{Cart}(-,\cdot)$ coincides with the original cleavage $\mathrm{Cart}_p(-,\cdot)$, whence

$$W\left(\overline{L}\left(\begin{tikzcd} \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \end{tikzcd}\right)\right) = \begin{tikzcd} \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \end{tikzcd}.$$

Now, given

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

an Ω-coalgebra, we prove that

$$\overline{L}\left(W\left(\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}\right)\right) = \begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

By Proposition 4.3.12,

$$W\left(\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}\right)$$

coincides with $\begin{matrix}\mathcal{X}\\ \downarrow F\\ \mathcal{C}\end{matrix}$ equipped with the splitting cleavage defined by

$$\operatorname{Cart}(f,E) := (\varepsilon_F)_1(f,\mathrm{id})\colon (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) \longrightarrow E \tag{4.37}$$

for every $E\in\mathcal{X}$ and every $f\colon A\to F(E)$ in $\mathcal{C}$. Then

$$\overline{L}\left(W\left(\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}\right)\right) = \begin{tikzcd} \mathcal{X} \arrow[r, "V_F"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

and it remains to prove that $V_F=\beta_1$.

On objects $E\in\mathcal{X}$, we have that

$$V_F(E) = \left(F(E), \mathcal{C}\downarrow F(E) \xrightarrow{\operatorname{Cart}(-,E)} \mathcal{X}\right),$$

by Construction 4.2.4.

By equation (4.37), if $(f\colon A\to F(E))\in\mathcal{C}\downarrow F(E)$

$$\operatorname{Cart}(-,E)(f) = (\varepsilon_F)_1\left(A, \mathcal{C}\downarrow A \xrightarrow{f\circ -} \mathcal{C}\downarrow F(E) \xrightarrow{\alpha_E^{\beta_1}} \mathcal{X}\right) = \alpha_E^{\beta_1}(f).$$

If we consider a morphism $\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}$ in $\mathcal{C}\downarrow F(E)$, we have that

$$\operatorname{Cart}(-,E)\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) = \operatorname{Cart}(f,E) = (\varepsilon_F)_1(f,\mathrm{id}) = \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)\circ \mathrm{id}$$

While if we consider a morphism $\begin{tikzcd} A' \arrow[d, "a"'] \arrow[dr, "f'"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}$ in $\mathcal{C}\downarrow F(E)$, then both

$$\operatorname{Cart}(-,E)\left(\begin{tikzcd} A' \arrow[d, "a"'] \arrow[dr, "f'"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right) \quad \text{and} \quad \alpha_E^{\beta_1}\left(\begin{tikzcd} A' \arrow[d, "a"'] \arrow[dr, "f'"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right)$$

are acceptable morphisms which fit well into the diagram

$$\begin{tikzcd}
\alpha_E^{\beta_1}(f') \arrow[rr, "\mathrm{Cart}(f',E)"] \arrow[dd, mapsto, "F"] \arrow[dr, dashed, "v"] & & E \arrow[dr, bend left, "\mathrm{id}_E"] \arrow[dd, mapsto] & \\
 & \alpha_E^{\beta_1}(f) \arrow[rr, "\mathrm{Cart}(f,E)"] \arrow[dd, mapsto, "F"] & & E \arrow[dd, mapsto, "F"] \\
A' \arrow[rr, "f'"] \arrow[dr, bend right, "a"'] & & F(E) \arrow[dr, "\mathrm{id}_{F(E)}"] & \\
 & A \arrow[rr, "f"'] & & F(E)
\end{tikzcd}$$

in the place of the morphism in violet v, since the argument above shows that

$$f^*E = \alpha_E^{\beta_1}(f) \quad \text{and} \quad \mathrm{Cart}\,(f,E) = \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)$$

and analogously for f' instead of f.
Then we conclude that

$$\mathrm{Cart}\,(-,E)\left(\begin{tikzcd} A' \arrow[d, "a"'] \arrow[dr, "f'"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right) = \alpha_E^{\beta_1}\left(\begin{tikzcd} A' \arrow[d, "a"'] \arrow[dr, "f'"] & \\ A \arrow[r, "f"'] & F(E) \end{tikzcd}\right)$$

and it follows that $\mathrm{Cart}\,(-,E) = \alpha_E^{\beta_1}$ (as functors), whence V_F and β_1 coincide on objects.

Let now $e\colon E \to E'$ a morphism in $\mathcal{X}$. By Construction 4.2.4 we have that $V_F(e) = (F(E), \lambda^e)$, with λ^e defined to be such that for every $(f\colon A \to F(E)) \in \mathcal{C}{\downarrow}F(E)$ the morphism λ_f^e is the unique morphism which fits well into the diagram

$$\begin{tikzcd}
f^*E \arrow[rr, "\mathrm{Cart}(f,E)"] \arrow[dd, mapsto, "F"] \arrow[dr, dashed, "\lambda_f^e"] & & E \arrow[dr, bend left, "e"] \arrow[dd, mapsto] & \\
 & (F(e)f)^*E' \arrow[rr, "\mathrm{Cart}(F(e)f,E')"] \arrow[dd, mapsto, "F"] & & E' \arrow[dd, mapsto, "F"] \\
A \arrow[rr, "f"] \arrow[dr, bend right, equal] & & F(E) \arrow[dr, "F(e)"] & \\
 & A \arrow[rr, "F(e)f"'] & & F(E')
\end{tikzcd} \tag{4.38}$$

in the place of the morphism in violet.
Since $\beta_1(e)$ is of the form $\left(F(e), \lambda^{\beta_1,e}\right)$, by the fact that β_1 is the structure morphism of the Ω-coalgebra (F, β_1), in order to prove that $V_F(e) = \beta_1(e)$ it remains to prove that $\lambda^e = \lambda^{\beta_1,e}$. Let then $(f\colon A \to F(E)) \in \mathcal{C}{\downarrow}F(E)$. It suffices to prove that $\lambda_f^{\beta_1,e}$ fits well into diagram (4.38) in the place of the morphism in violet.

Since $\beta_1(e)$ is a morphism in $\mathcal{G}_{\mathrm{Tr}(-,F)}$, we get that

$$F(\lambda^{\beta_1,e_f}) = \mathrm{id}_{\mathrm{dom}(f)} = \mathrm{id}_A\,.$$

It remains to show that the square

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{Cart}(f,E)"] \arrow[d, "\lambda_f^{\beta_1,e}"'] & E \arrow[d, "e"] \\ (F(e)f)^*E' \arrow[r, "\mathrm{Cart}(F(e)f,E')"'] & E' \end{tikzcd}$$

commutes. But the argument above shows that

$$\mathrm{Cart}\,(f,E) = \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right) \quad \text{and} \quad \mathrm{Cart}\,(f,E) = \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "F(e)f"'] \arrow[dr, "F(e)f"] & \\ F(E') \arrow[r, "\mathrm{id}"'] & F(E') \end{tikzcd}\right).$$

Now, using axiom (CoAlg1) (on morphisms) and axiom (CoAlg2*) (on objects) of Ω-coalgebra (from Remark 4.3.2), we see that

$$e \circ \mathrm{Cart}\,(f,E) = (\varepsilon_F)_1\,(\beta_1\,(e \circ \mathrm{Cart}\,(f,E))) = (\varepsilon_F)_1\left(\beta_1(e) \circ \beta_1\alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)\right) =$$

$$= (\varepsilon_F)_1\left(\beta_1(e) \circ \mathrm{Cart}_{\int \mathrm{Tr}(-,F)}(-,\beta_1(E))\left(\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ F(E) \arrow[r, "\mathrm{id}"'] & F(E) \end{tikzcd}\right)\right) = (\varepsilon_F)_1\,(\beta_1(e) \circ (f,\mathrm{id})) =$$

$$= (\varepsilon_F)_1\left(\left(F(e),\lambda^{\beta_1,e}\right) \circ (f,\mathrm{id})\right) = (\varepsilon_F)_1\left(F(e)f,\lambda^{\beta_1,e}(f\circ -)\right) =$$

$$= \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "F(e)f"'] \arrow[dr, "F(e)f"] & \\ F(E') \arrow[r, "\mathrm{id}"'] & F(E') \end{tikzcd}\right) \circ \left(\lambda^{\beta_1,e}(f\circ -)\right)_{(A\xrightarrow{\mathrm{id}}A)} =$$

$$= \alpha_E^{\beta_1}\left(\begin{tikzcd} A \arrow[d, "F(e)f"'] \arrow[dr, "F(e)f"] & \\ F(E') \arrow[r, "\mathrm{id}"'] & F(E') \end{tikzcd}\right) \circ \lambda^{\beta_1,e}_{(A\xrightarrow{f}F(E))}.$$

Then we conclude that

$$\lambda_f^e = \lambda_f^{\beta_1,e}$$

by the fact that $\operatorname{Cart}(F(e)f, E')$ is a cartesian morphism.
Therefore we have proved that $V_F = \beta_1$ (as functors) and hence that

$$\overline{L}\left(W \left(\begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \right) \right) = \begin{tikzcd} \mathcal{X} \arrow[r, "\beta_1"] \arrow[d, "F"'] & \mathcal{G}_{\mathrm{Tr}(-,F)} \arrow[d, "\int \mathrm{Tr}(-,F)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd},$$

whence we get that $\overline{L}$ and W are inverses of each other on objects.

But on 1-morphisms and on 2-morphisms both $\overline{L}$ and W are essentially defined as the identity, by Remark 4.3.4 and Proposition 4.3.12. Therefore we immediatly get that

$$W \circ \overline{L} = \mathrm{Id}_{\mathcal{S}p\mathcal{FIB}_{\mathcal{C}}} \qquad \text{and} \qquad \overline{L} \circ W = \mathrm{Id}_{\Omega\text{-}\mathcal{C}o\mathcal{A}lg},$$

which means that the 2-functor $\overline{L}$ is an isomorphism of 2-categories with inverse 2-functor W.

In particular, it trivially follows that the comparison 2-functor $\overline{L}$ is an equivalence of 2-categories, that is, that the 2-functor $L \colon \mathcal{S}p\mathcal{FIB}_{\mathcal{C}} \longrightarrow \mathcal{CAT} \downarrow \mathcal{C}$ is comonadic. $\square$

Bibliography

[CHP03] Eugenia Cheng, Martin Hyland, and John Power. Pseudo-distributive laws. *Electronic Notes in Theoretical Computer Science*, 83, 12 2003.

[Jac99] Bart Jacobs. *Categorical Logic and Type Theory*. Studies in Logic and the Foundations of Mathematics; v. 141. Elsevier Science, Amsterdam ; New York, 1st edition, 1999.

[Kel82] Max Kelly. *Basic Concepts of Enriched Category Theory*. London Mathematical Society lecture note series 64. Cambridge University Press, 1982.

[ML71] Saunders Mac Lane. *Categories for the Working Mathematician*. SPRINGER-VERLAG, 6 edition, 1971.

[Sch19] Daniel Schäppi. Monads and their applications. Notes on a course held in Regensburg, 2019.

[Str18] Thomas Streicher. Fibered categories à la jean bénabou. arXiv 1801.02927 [Category Theory], 2018.

[Tro19] Davide Trotta. Existential completion and pseudo-distributive laws: an algebraic approach to the completion of doctrines. 11 2019.