Lax normal conical 2-limits and the Grothendieck construction

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 $F \colon \mathcal{A} \to \mathcal{C}$ with \mathcal{A} small

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$$\Delta U \Rightarrow F: A \rightarrow C$$

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But 1 will not be sensitive enough now.

Consider a weight $W\colon \mathcal{A} o \mathcal{V}$

$$\mathcal{C}\left(U,\lim^{W}F\right)\cong\left[\mathcal{A},\mathcal{V}\right]\left(W,\mathcal{C}\left(U,F(-)\right)\right)$$

We call it **conical** when $W = \Delta 1$.



Conicalization of weighted \mathcal{V} -limits

Example 1 (Every presheaf is a colimit of representables).

$$W\colon \mathcal A o \mathcal V$$
 with $\mathcal A$ small. Consider y: $\mathcal A^{\operatorname{op}} o [\mathcal A,\mathcal V]$
$$W\cong\operatorname{colim}^W{\operatorname{y}}$$

Conicalization of weighted ${\mathcal V}$ -limits

Example 1 (Every presheaf is a colimit of representables).

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Lemma 2 (Continuity in the weight).

 $S \colon \mathcal{B}^{\mathsf{op}} o \mathcal{V}$, $Z \colon \mathcal{B} o [\mathcal{A}, \mathcal{V}]$, $F \colon \mathcal{A} o \mathcal{C}$

Suppose that $\operatorname{colim}^S Z$ and each $\operatorname{lim}^{Z(B)} F$ with $B \in \mathcal{B}$ exist. Then

$$\lim^{\operatorname{colim}^S Z} F \cong \lim^S \left(\lim^{Z(-)} F \right)$$



 $W: \mathcal{A} \to \mathcal{CAT}$ with \mathcal{A} small. We search for $H: \mathcal{B} \to [\mathcal{A}, \mathcal{CAT}]$

$$\varphi \colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}](y(-), U)$$

$$\widetilde{\varphi} \colon \Delta 1 \Longrightarrow_{\mathsf{relaxed}} [\mathcal{A}, \mathcal{CAT}] (\mathcal{H}(-), \mathcal{U}) \colon \mathcal{B}^\mathsf{op} \to \mathcal{CAT}$$

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$$\frac{\varphi \colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}] (\mathsf{y}(-), U)}{\widetilde{\varphi} \colon \Delta 1 \xrightarrow[\mathsf{relaxed}]{} [\mathcal{A}, \mathcal{CAT}] (\mathcal{H}(-), U) \colon \mathcal{B}^\mathsf{op} \to \mathcal{CAT}}$$

As we will want to apply the lemma of continuity in the weight, we search for H of the form

$$\left(\int^{\mathsf{op}} W\right)^{\mathsf{op}} \xrightarrow{\mathcal{G}(W)^{\mathsf{op}}} \mathcal{A}^{\mathsf{op}} \xrightarrow{\mathsf{y}} [\mathcal{A}, \mathcal{CAT}].$$

And we want $\widetilde{\varphi}$ to be at least a lax natural transformation.

$$\varphi \colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}](y(-), U)$$

$$\overline{\widetilde{\varphi} \colon \Delta 1 \xrightarrow[\mathsf{relaxed}]{} \left[\mathcal{A}, \mathcal{CAT} \right] \left(\left(\mathsf{y} \circ \mathcal{G} \left(W \right)^{\mathsf{op}} \right) \left(- \right), U \right) \colon \int^{\mathsf{op}} \! W \to \mathcal{CAT}}$$

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism

$$\varphi_A(X)\colon y(A)\to U,$$

so we need an object $(A, X) \in \int^{op} W$ such that $\mathcal{G}(W)(A, X) = A$.

$$\frac{\varphi \colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}] (\mathsf{y}(-), U)}{\widetilde{\varphi} \colon \Delta 1 \xrightarrow[\mathsf{relaxed}]{} [\mathcal{A}, \mathcal{CAT}] ((\mathsf{y} \circ \mathcal{G}(W)^{\mathsf{op}}) (-), U) \colon \int^{\mathsf{op}} W \to \mathcal{CAT}}$$

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism

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: $y(A) \to U$,

so we need an object $(A, X) \in \int^{op} W$ such that $\mathcal{G}(W)(A, X) = A$.

For every $\alpha \colon X \to X'$ in W(A), we encode the 2-cell $\varphi_A(\alpha)$ as

$$\begin{array}{ccc} (A,X) & 1 & \xrightarrow{\widetilde{\varphi}_{(A,X)}} & [\mathcal{A},\mathcal{CAT}] (\mathsf{y}(A),\mathcal{U}) \\ & \downarrow (\mathsf{id}_{A},\alpha) & \parallel & \widehat{\varphi}_{(\mathsf{id}_{A},\alpha)} & \parallel & -\circ\mathsf{y}(\mathcal{G}(W)^{\mathsf{op}}(\mathsf{id}_{A},\alpha)) \\ (A,X') & 1 & \xrightarrow{\widetilde{\varphi}_{(A,X')}} & [\mathcal{A},\mathcal{CAT}] (\mathsf{y}(A),\mathcal{U}) \end{array}$$

In order to encode the naturality of φ into the relaxed naturality of $\widetilde{\varphi}$, for every $f: A \to A'$ in \mathcal{A} and $X \in W(A)$,

$$\begin{array}{ccc} (A,X) & 1 \xrightarrow{\widetilde{\varphi}_{(A,X)}} [\mathcal{A},\mathcal{CAT}] (\mathsf{y}(A),\mathcal{U}) \\ & \downarrow \underline{f}^X = (f,\mathsf{id}) & \left\| \xrightarrow{\widetilde{\varphi}_{\underline{f}^X}} & -\mathsf{oy}(f) \middle\downarrow -\mathsf{oy} \big(\mathcal{G}(\mathcal{W})^\mathsf{op} (\underline{f}^X) \big) \\ (A',\mathcal{W}(f)(X)) & 1 \xrightarrow{\widetilde{\varphi}_{(A',\mathcal{W}(f)(X))}} [\mathcal{A},\mathcal{CAT}] \left(\mathsf{y}(A'),\mathcal{U} \right) \end{array}$$

In order to encode the naturality of φ into the relaxed naturality of $\widetilde{\varphi}$, for every $f: A \to A'$ in \mathcal{A} and $X \in W(A)$,

$$\underline{f}^{X'} \circ (\mathrm{id}_A, \alpha) = (\mathrm{id}_{A'}, W(f)(\alpha)) \circ \underline{f}^X$$

so that we recover the naturality of φ on morphisms, that is

$$\varphi_{A}(\alpha) \circ \mathsf{y}(f) = \varphi_{A'}(W(f)(\alpha)).$$

At this point, every finite composition is reduced to (f, α) defined as

$$(A,X) \xrightarrow{\underline{f}^X} (A',W(f)(X)) \xrightarrow{(\mathrm{id}_{A'},\alpha)} (A',X')$$

In order to encode the 2-naturality of φ , that is for every $\delta \colon f \Rightarrow g \colon A \to A'$ in $\mathcal A$ and every $X \in W(A)$

$$\varphi_{A'}(W(\delta)_X) = \varphi_A(X) y(\delta),$$

we need a 2-cell $\underline{\delta}^X$ with $\mathcal{G}(W)(\underline{\delta}^X) = \delta$

$$\underline{\delta}^X : (f, W(\delta)_X) \Rightarrow \underline{g}^X : (A, X) \to (A', W(g)(X))$$

$$\underline{\delta}^{X'}(\mathsf{id}_{\mathcal{A}},\alpha) = (\mathsf{id}_{\mathcal{A}'},W(g)(\alpha))\underline{\delta}^X$$

because they represent the same axiom. At his point, every horizontal composition of 2-cells is reduced to a whiskering

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We call $\mathcal{G}(W)$ (or sometimes also just $\int^{op} W$) the 2-Set-enriched Grothendieck construction of $W: \mathcal{A} \to \mathcal{CAT}$.

A lax normal natural transformation $\alpha \colon M \Longrightarrow_{\mathsf{lax}^n} \mathsf{N} \colon \int^\mathsf{op} W \to \mathcal{D}$ is a lax natural transformation such that its structure 2-cell on every

$$\underline{f}^X \colon (A,X) \to (B,W(f)(X))$$

in $\int^{op} W$ is the identity.



Definition 3 ((Op)lax normal conical 2-(co)limit).

 $W \colon \mathcal{A} \to \mathcal{CAT}$ and $F \colon \int^{\mathsf{op}} W \to \mathcal{C}$ with \mathcal{A} small.

Lax normal conical 2-limit of F:

$$\mathcal{C}\left(U, \mathsf{lax}^{\mathsf{n}} - \mathsf{lim}^{\Delta 1} F\right) \cong \left[\int^{\mathsf{op}} W, \mathcal{CAT}\right]_{\mathsf{lax}^{\mathsf{n}}} (\Delta 1, \mathcal{C}\left(U, F(-)\right))$$

 $W \colon \mathcal{A}^{\mathsf{op}} \to \mathcal{CAT}$ and $F \colon \int W \to \mathcal{C}$ with \mathcal{A} small.

Oplax normal conical 2-colimit of F:

$$\mathcal{C}\left(\mathsf{oplax}^\mathsf{n}\operatorname{-colim}^{\Delta 1}F,U\right)\cong\left[\left(\int\!W\right)^\mathsf{op}\!,\mathcal{CAT}\right]_{\mathsf{oplax}^\mathsf{n}}\left(\Delta 1,\mathcal{C}\left(F(-),U\right)\right)$$

Theorem 4 (Street[3], but with a new proof and more detail).

Lax normal conical 2-limits are particular weighted 2-limits, weighted by

$$W^{\mathsf{lax}^{\mathsf{n}}}: \int^{\mathsf{op}} Z \longrightarrow \mathcal{CAT}$$

 $(B, X') \mapsto Z(B)/\chi'.$

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$$\begin{array}{cccc} W^{\mathsf{lax}^{\mathsf{n}}} : & \int^{\mathsf{op}} Z & \longrightarrow & \mathcal{CAT} \\ & (B, X') & \mapsto & Z(B)/\chi' \,. \end{array}$$

Given $W: \mathcal{A} \to \mathcal{CAT}$ with \mathcal{A} small, there is a 2-natural isomorphism

$$[W, [y(-), U]] \cong \left[\int^{op} W, \mathcal{CAT} \right]_{lax^{n}} (\Delta 1, [(y \circ \mathcal{G}(W)^{op})(-), U])$$

$$W \cong \operatorname{colim}^{W} y \cong \operatorname{lax}^{n} \operatorname{-colim}^{\Delta 1} (y \circ \mathcal{G}(W)^{op}).$$

Whence every weighted 2-limits is essentially conicalizable:

$$\mathsf{lim}^W F \cong \mathsf{lim}^{\mathsf{W}^{\mathsf{lax}^n}} \left(\mathsf{lim}^{(\mathsf{y} \circ \mathcal{G}(W)^{\mathsf{op}})(-)} F \right) \cong \mathsf{lax}^n \, \text{-} \, \mathsf{lim}^{\Delta 1} (F \circ \mathcal{G}(W))$$

Example 5.

 $W: \mathcal{A} \to \mathcal{CAT}$ with \mathcal{A} small.

$$W \cong \operatorname{colim}^W y \cong \operatorname{lax}^n \operatorname{-colim}^{\Delta 1}(y \circ \mathcal{G}(W)^{\operatorname{op}}).$$

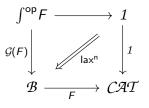
The universal lax normal cocone is given by

Taking $\mathcal{A}=1$, we obtain that 1 is "lax normal conical dense" in \mathcal{CAT} .

Proposition 6.

 $F: \mathcal{B} \to \mathcal{CAT}$. Consider its 2-Set-enriched Grothendieck construction.

There is a lax normal natural transformation λ of the form



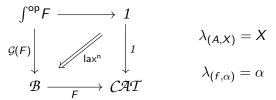
$$\lambda_{(A,X)} = X$$
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Proposition 6.

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Notice that this diagram lives in 2- \mathcal{CAT}_{lax} , that is a lax 3-category (Lambert[2]), where the interchange rule is lax. 2- $\mathcal{CAT}_{lax} = 2$ - \mathcal{Set} - \mathcal{CAT}

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\mathcal{W}_{K}} \mathcal{C} \qquad \beta_{\alpha_{A}} \colon K(\alpha_{A}) \circ \beta_{F(A)} \Rightarrow \beta_{G(A)} \circ H(\alpha_{A})$$

Definition 7 (lax comma; M., refining Bird[1], Lambert[2]).

$$\mathcal{M} \xrightarrow{P} \mathcal{A} = \mathcal{M} \xrightarrow{\exists \exists V} \mathcal{A} \\
\mathcal{B} \xrightarrow{G} \mathcal{C} \qquad \mathcal{A} \xrightarrow{F /\!\!/ G} \xrightarrow{\partial_0} \mathcal{A} \\
\mathcal{M} \xrightarrow{V} F /\!\!/ G \xrightarrow{\partial_0} \mathcal{A} \xrightarrow{\varphi \equiv} \mathcal{C}$$

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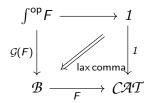
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Proposition 8.



Or equivalently

$$\int_{\mathcal{G}(F)}^{\mathsf{op}} F \longrightarrow \mathcal{CAT}_{\bullet,\mathsf{lax}} \longrightarrow 1$$

$$\downarrow_{\mathsf{lax} \, \mathsf{comm},\mathsf{lax}}^{\mathsf{lax}} \longrightarrow 1$$

$$\mathcal{B} \longrightarrow_{F} \mathcal{CAT} \Longrightarrow \mathcal{CAT}$$

$$\mathcal{G}(-) : [\mathcal{B}, \mathcal{CAT}] \to 2-\mathcal{CAT}/\mathcal{B}.$$

Definition 9.

A 2-Set-opfibration over $\mathcal B$ is a 2-functor $P\colon \mathcal E\to \mathcal B$ such that

- (i) the underlying functor P_0 is an ordinary Grothendieck opfibration;
- (ii) for every pair $X, Y \in \mathcal{E}$ the functor

$$P_{X,Y} \colon \mathcal{E}(X,Y) \to \mathcal{B}(P(X),P(Y))$$

is a discrete fibration.

Theorem 10 (Lambert[2]).

The essential image of the 2-functor

$$\mathcal{G}(-): [\mathcal{B}, \mathcal{CAT}] \rightarrow 2\text{-}\mathcal{CAT}/\mathcal{B}$$

is given by the split 2-Set-opfibrations with small fibres.

Theorem 11 (M., the first part is in Bird[1], Street[3]).

$$\begin{split} & [\mathcal{A}, \mathcal{CAT}]_{\mathsf{lax}}(F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{CAT} \right]_{\mathsf{lax}} (\Delta 1, U \circ \mathcal{G}(F)), \\ & [\mathcal{A}, \mathcal{CAT}]_{\mathsf{ps}}(F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{CAT} \right]_{\mathsf{sigma}} (\Delta 1, U \circ \mathcal{G}(F)) \\ & [\mathcal{A}, \mathcal{CAT}](F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{CAT} \right]_{\mathsf{lax}^{\mathsf{n}}} (\Delta 1, U \circ \mathcal{G}(F)) \end{split}$$

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$$\begin{split} & [\mathcal{A}, \mathcal{C}\!\mathcal{A}T]_{\mathsf{lax}}(F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{C}\!\mathcal{A}T\right]_{\mathsf{lax}}(\Delta 1, U \circ \mathcal{G}(F)), \\ & [\mathcal{A}, \mathcal{C}\!\mathcal{A}T]_{\mathsf{ps}}(F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{C}\!\mathcal{A}T\right]_{\mathsf{sigma}}(\Delta 1, U \circ \mathcal{G}(F)) \\ & [\mathcal{A}, \mathcal{C}\!\mathcal{A}T](F, U) \cong \left[\int^{\mathsf{op}} F, \mathcal{C}\!\mathcal{A}T\right]_{\mathsf{lax}^n}(\Delta 1, U \circ \mathcal{G}(F)) \end{split}$$

$$\begin{split} [\mathcal{A}, \mathcal{C}\!\mathcal{A}\mathcal{T}]_{\mathsf{lax}}(F, G) &\cong \left[\int^{\mathsf{op}} \! F, \mathcal{C}\!\mathcal{A}\mathcal{T}\right]_{\mathsf{lax}}(\Delta 1, G \circ \mathcal{G}(F)) \cong 2\text{-}\!\mathcal{C}\!\mathcal{A}\mathcal{T} /_{\mathcal{A}}\left(\int^{\mathsf{op}} \! F, \int^{\mathsf{op}} \! G\right) \\ \mathcal{G}(-) &: [\mathcal{A}, \mathcal{C}\!\mathcal{A}\mathcal{T}]_{\mathsf{lax}} \stackrel{\sim}{\to} 2\text{-}\!\mathit{Set}\text{-}\!\mathit{OpFib}\left(\mathcal{A}\right) \\ \mathcal{G}(-) &: [\mathcal{A}, \mathcal{C}\!\mathcal{A}\mathcal{T}]_{\mathsf{ps}} \stackrel{\sim}{\to} 2\text{-}\!\mathit{Set}\text{-}\!\mathit{OpFib}_{\mathsf{cart}}(\mathcal{A}) \end{split}$$

 $\mathcal{G}(-): [\mathcal{A}, \mathcal{CAT}] \stackrel{\sim}{\rightarrow} 2\text{-Set-OpFib}_{close}(\mathcal{A})$

Definition 12 (colimits in 2- \mathcal{CAT}_{lax} ; M.).

 $M \colon \mathcal{A}^{\mathsf{op}} \to \mathcal{CAT}$ (the marking) with \mathcal{A} small,

 $F \colon \int M \to \mathcal{C}$ (the diagram), $W \colon (\int M)^{\mathsf{op}} \to \mathcal{CAT}$ (the weight).

Oplax normal 2-colimit of F marked by M and weighted by W:

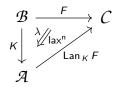
$$\mathcal{C}\left(\mathsf{oplax}^{\mathsf{n}}_{M}\operatorname{-colim}^{W}F,U\right)\cong\left[\left(\int\!M\right)^{\mathsf{op}},\mathcal{CAT}\right]_{\mathsf{oplax}^{\mathsf{n}}}\left(W,\mathcal{C}\left(F(-),U
ight)
ight)$$

We say **opmarked** when the domain of F is expressed as $\int^{op} M$ for some $M: \mathcal{A} \to \mathcal{CAT}$.

Theorem 13.

In 2- \mathcal{CAT}_{lax} every trivially marked weighted 2-colimit can be equivalently expressed as a marked trivially weighted 2-colimit.

Definition 14 (M.).



with \mathcal{B} small and $K = \mathcal{G}(M)$ a 2-Set-opfibration.

Pointwise left Kan extension of F along K:

$$(\operatorname{\mathsf{Lan}}_K F)(A) \cong \operatorname{\mathsf{oplax}}^{\operatorname{\mathsf{op-n}}}_M \operatorname{\mathsf{-colim}}^{\mathcal{A}(K(-),A)} F$$

with universal oplax normal cocylinder

$$\mathcal{A}(K(-),A) \stackrel{L}{\Longrightarrow} \mathcal{C}((L \circ K)(-),L(A)) \xrightarrow{\underbrace{\mathcal{C}(\lambda_{-},\mathrm{id})}_{\mathrm{oplax}^{n}}} \mathcal{C}(F(-),L(A))$$
$$\mathcal{C}(L(A),C) \cong [\mathcal{B}^{\mathrm{op}},\mathcal{CAT}]_{\mathrm{oplax}^{n}} (\mathcal{A}(K(-),A),\mathcal{C}(F(-),C))$$

Theorem 15 (M.).

 $F: \mathcal{A} \to \mathcal{CAT}$ with \mathcal{A} small. Then the 2-Set-enriched Grothendieck construction lax comma square exhibits

$$F = \operatorname{\mathsf{Lan}}_{\mathcal{G}(F)} \Delta 1.$$

Proof.
$$\mu_{(B,X)}: \mathcal{B}(B,A) \longrightarrow \mathcal{CAT}(1,F(A))$$
$$(B \xrightarrow{u} A) \longmapsto F(u)(X)$$
$$(\mu_{(g,\gamma)})_{u} = F(u)(\gamma): F(u \circ g)(X') \rightarrow F(u)(X)$$

is 2-universal:

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is 2-universal: given $Q: \mathcal{A}(\mathcal{G}(F)(-), A) \xrightarrow[\text{oplax}^n]{} \mathcal{CAT}(\Delta 1(-), C)$,

$$\exists !s \colon F(A) \to C \text{ such that } (s \circ -) \circ \mu = Q.$$

$$s(X) = s\left(\mu_{(A,X)}(\mathsf{id}_A)\right) = Q_{(A,X)}(\mathsf{id}_A)$$

$$s(\alpha) = s\left(\left(\mu_{(\mathsf{id}_A,\alpha)}\right)_{\mathsf{id}_A}\right) = \left(Q_{(\mathsf{id}_A,\alpha)}\right)_{\mathsf{id}_A}.$$

Definition 16 (M.).

 $G, H: \mathcal{B}^{op} \times \mathcal{C} \to \mathcal{E}$. Oplaxⁿ-lax natural transformation $\alpha: G \Rightarrow H$:

$$\alpha_{B,C}\colon G(B,C)\to H(B,C)$$

 $\alpha_{-,C}$ oplax normal in B, $\alpha_{B,-}$ lax in C and compatibility

$$G(B,C) \xrightarrow{\alpha_{B,C}} H(B,C)$$

$$G(G,\operatorname{id},\operatorname{g}) \xrightarrow{\alpha_{B,C}} H(\operatorname{id},\operatorname{g})$$

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Modifications between them are modifications in each variable.

Theorem 17 (The oplaxⁿ-lax param Yoneda lemma; M.).

 $K \colon \mathcal{B} \to \mathcal{A}$ a 2-Set-opfibration, $F \colon \mathcal{B}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{CAT}$.

There is a bijection between

$$\alpha_{B,A} \colon \mathcal{A}(K(B),A) \to F(B,A)$$

oplax $^{\mathsf{n}}$ - lax natural in $(B,A) \in \mathcal{B}^{\mathsf{op}} imes \mathcal{A}$ and

$$\eta_B \colon 1 \to F(B, K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$.

Moreover this bijection extends to an isomorphism of categories.

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Moreover this bijection extends to an isomorphism of categories.

Corollary 18 (M.).

Every pointwise left Kan extension in $2-\mathcal{CAT}_{lax}$ is also a weak one.

G. J. Bird.

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