Unicoherence in locales

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Unicoherence

Unicoherence is a connectedness property which was first defined by Kuratowski in 1926. It has then been extensively studied, for example by Stone, Wallace, Eilenberg, Borsuk, Dickman and Rubin.

Intuitively, a space is unicoherent if it is connected and does not contain any holes.

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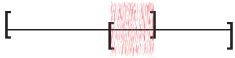
Definition.

A ***continuum*** in a topological space X is a non-empty closed connected subspace $F \subseteq X$.
A topological space X is ***unicoherent*** if it is connected and whenever $X = F \cup G$ with $F, G \subseteq X$ continua, we have that $F \cap G$ is a continuum.

Examples

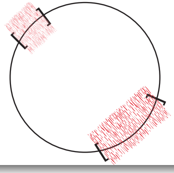
Example.

All euclidean spaces are unicoherent.



Example.

The circle S^1 is not unicoherent.



Example (Kuratowski's motivating example).

The 2-dimensional sphere S^2 is unicoherent.

Locales

Definition.

A ***locale (or frame)*** L is a complete lattice which satisfies the infinite distributive law

$$x \wedge \bigvee T = \bigvee \{x \wedge t \mid t \in T\}, \quad \text{for all } x \in L \text{ and } T \text{ subset of } L.$$

The top element of L is denoted by 1 and the bottom by 0.

If X is a topological space, then $\mathcal{O}X = \{U \subseteq X \mid U \text{ is open}\}$ is a locale with partial order $\subseteq$ and intersection as a binary meet.

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Definition.

An element x in a locale L is ***connected*** if whenever $x = b \vee c$ with $b \wedge c = 0$ we have either $b = 0$ or $c = 0$. ***A locale L is connected*** if its top element 1 is connected and is ***locally connected*** if every element in L can be written as a join of connected elements.

Sublocales

Definition.

A subset S of a locale L is called a ***sublocale*** if S is closed under arbitrary meets and if for $x \in L$ and $s \in S$ we have that $(x \to s) \in S$.

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The ***open sublocale*** associated with any $a \in L$ is

$$\mathfrak{o}(a) = \{x \in L \mid a \to x = x\}.$$

The ***closed sublocale*** associated with any $a \in L$ is

$$\mathfrak{c}(a) = \uparrow a = \{x \in L \mid a \leq x\}.$$

Closure, interior and ***boundary*** $\mathrm{bd}\,(S) = \overline{S} \setminus \mathrm{Int}\,(S)$ can be defined.

Unicoherence in locales

Definition.

A ***continuum*** in a locale L is a non-void closed connected sublocale $\mathfrak{c}(u) \subseteq L$.

A locale L is ***unicoherent*** if whenever $L = H \vee K$ with $H, K \subseteq L$ continua, we have that $H \cap K$ is a continuum.

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Proposition.

Let X be a topological space. ***The locale $\mathcal{O}X$ of opens of X is unicoherent precisely when X is unicoherent.***

Example.

The locale of reals is a unicoherent locale (under mild assumptions).

Characterizations of unicoherence

Definition.

A ***region*** in a locale L is a non-void open connected sublocale $\mathfrak{o}(u) \subseteq L$.

L is ***open unicoherent*** if whenever $L = A \vee B$ with $A, B \subseteq L$ regions, we have that $A \cap B$ is a region.

Characterizations of unicoherence

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Definition.

Let $S \subseteq L$ be a complemented sublocale. S ***separates L*** if $L \setminus S$ is not connected. S ***separates elements x and y*** in L if $L \setminus S$ admits a separation $L \setminus S \subseteq \mathfrak{c}(a) \vee \mathfrak{c}(b)$ such that $x \in (L \setminus S) \cap \mathfrak{c}(a)$ and $y \in (L \setminus S) \cap \mathfrak{c}(b)$. S ***separates sublocales X and Y*** of L if S separates every element $x \in X$ from every element $y \in Y$.

Main results

Theorem.

Let L be a connected and locally connected locale. TFAE:

(*I*) *Whenever $X, Y \subseteq L$ non-void and $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are such that $\mathfrak{c}(u) \cap \mathfrak{c}(v) = \emptyset$ and neither $\mathfrak{c}(u)$ nor $\mathfrak{c}(v)$ separate any element of X from any element of Y in L, we have that $\mathfrak{c}(u) \vee \mathfrak{c}(v)$ does not separate X and Y in L;*

(*II*) ***(Brouwer Property)*** *If $\mathfrak{c}(u) \subseteq L$ is a continuum, then every component D of $L \setminus \mathfrak{c}(u) = \mathfrak{o}(u)$ has as boundary $\mathrm{bd}\,(D)$ a continuum;*

(*III*) ***(Unicoherence)*** *L is unicoherent;*

(*IV*) *If $\mathfrak{c}(u) \subseteq L$ is non-void and D_1, D_2 are disjoint components of $L \setminus \mathfrak{c}(u) = \mathfrak{o}(u)$ such that $\mathrm{bd}\,(D_1) = \mathrm{bd}\,(D_2)$ then $\mathrm{bd}\,(D_1)$ is a continuum;*

Main results

(V) If C and D are disjoint complemented connected sublocales of L such that $\mathrm{bd}\,(C) \subseteq \mathrm{bd}\,(D)$, then $\mathrm{bd}\,(C)$ is connected;

(VI) If $C \subseteq L$ is simple, then $\mathrm{bd}\,(C)$ is connected;

(VII) If R is a simple region, then $\mathrm{bd}\,(R)$ is connected;

($VIII$) If A and B are disjoint regions such that $\mathrm{bd}\,(A) = \mathrm{bd}\,(B)$, then $\mathrm{bd}\,(A)$ is connected;

(IX) If A and B are regions such that $\mathrm{bd}\,(A) \cap \mathrm{bd}\,(B) = \emptyset$, then $A \cap B$ is connected;

(X) **(Open unicoherence)** L is open unicoherent.

Challenges

- Singletons are not sublocales;
- An element of a join of sublocales need not be contained in one of them;
- Classical proofs are highly non-constructive;
- Not every element of a sublocale is contained in a component;
- Local connectedness does not behave as in topological spaces.

Local connectedness

Theorem.

For a topological space X, *the following three are equivalent characterizations of local connectedness:*

(*i*) *for every $x \in X$, every neighborhood of $x \in X$ contains a connected open neighborhood of x;*

(*ii*) *X has a basis made of connected opens;*

(*iii*) *the components of every open subspace of X are all open.*

Theorem.

For a locale L, *the following two are equivalent characterizations of local connectedness:*

(*i*) *every element of L can be written as a join of connected elements;*

(*ii*) *for every open sublocales $\mathfrak{o}(a)$ of L, its components are all open and their join is $\mathfrak{o}(a)$.*

Strong local connectedness

Definition.

A locale L is ***strongly locally connected*** if for every $o(u) \subseteq L$ open and for every $x \in o(u)$ there exists $o(v) \subseteq o(u)$ open connected sublocale with $x \in o(v)$.

Proposition.

Every strongly locally connected locale is locally connected.
Moreover, ***if L is strongly locally connected and $o(u) \subseteq L$ is an open sublocale, then every $x \in o(u)$ belongs to a component of $o(u)$.***

Other characterizations

Theorem.

Let L be a connected and strongly locally connected locale. TFAE:

(I^+) *Whenever $x, y \in L$ with $x \neq 1 \neq y$ and $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are such that $\mathfrak{c}(u) \cap \mathfrak{c}(v) = \emptyset$ and neither $\mathfrak{c}(u)$ nor $\mathfrak{c}(v)$ separate x and y in L, we have that $\mathfrak{c}(u) \vee \mathfrak{c}(v)$ does not separate x and y in L;*

(I) *Whenever $X, Y \subseteq L$ with $X \neq \emptyset \neq Y$ and $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are such that $\mathfrak{c}(u) \cap \mathfrak{c}(v) = \emptyset$ and neither $\mathfrak{c}(u)$ nor $\mathfrak{c}(v)$ separate any element of X from any element of Y in L, we have that $\mathfrak{c}(u) \vee \mathfrak{c}(v)$ does not separate X and Y in L;*

(I') ***(Phragmen-Brouwer Property)*** *Whenever $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are such that $\mathfrak{c}(u) \cap \mathfrak{c}(v) = \emptyset$ and neither $\mathfrak{c}(u)$ nor $\mathfrak{c}(v)$ separate L, we have that $\mathfrak{c}(u) \vee \mathfrak{c}(v)$ does not separate L.*

Other characterizations

Theorem.

Let L be a normal, connected and strongly locally connected locale. TFAE:

(N^+) *If $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are disjoint and $x \in \mathfrak{c}(u)$ and $y \in \mathfrak{c}(v)$ are such that $x \neq 1 \neq y$, then there exists a continuum $\mathfrak{c}(w) \subseteq L \setminus (\mathfrak{c}(u) \vee \mathfrak{c}(v)) = \mathfrak{o}(u \wedge v)$ which separates x and y in L;*

(N) *If $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are disjoint and $X \subseteq \mathfrak{c}(u)$ and $Y \subseteq \mathfrak{c}(v)$ are such that $X \neq \emptyset \neq Y$, then there exists a continuum $\mathfrak{c}(w) \subseteq L \setminus (\mathfrak{c}(u) \vee \mathfrak{c}(v))$ which separates an element of X from an element of Y in L;*

(III) *L is unicoherent.*