Lax normal conical 2-limits and the Grothendieck construction

Luca Mesiti

University of Leeds

YaMCATS 29

15/12/2022

Weighted enriched limits

$F\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small

An ordinary cone is a natural transformation

$$\Delta U \Rightarrow F\colon A \to \mathcal{C}$$

Weighted enriched limits

$F\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small

An ordinary cone is a natural transformation

$$\Delta U \Rightarrow F\colon A \to \mathcal{C}$$

$$\Delta 1 \Rightarrow \mathcal{C}\left(U, F(-)\right) : \mathcal{A} \to \mathcal{S}et$$

But 1 will not be sensitive enough now.

Weighted enriched limits

$F\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small

An ordinary cone is a natural transformation

$$\Delta U \Rightarrow F\colon A \to \mathcal{C}$$

$$\Delta 1 \Rightarrow \mathcal{C}\,(U, F(-)) : \mathcal{A} \to \mathcal{S}et$$

But 1 will not be sensitive enough now.

Consider a weight $W\colon \mathcal{A} \to \mathcal{V}$

$$\mathcal{C}\left(U, \lim^W F\right) \cong [\mathcal{A}, \mathcal{V}]\,(W, \mathcal{C}\,(U, F(-)))$$

We call it **conical** when $W = \Delta 1$.

Conicalization of weighted $\mathcal{V}$-limits

Example 1 (Every presheaf is a colimit of representables).

$W\colon \mathcal{A} \to \mathcal{V}$ with $\mathcal{A}$ small. Consider $\mathrm{y}\colon \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, \mathcal{V}]$

$$W \cong \operatorname{colim}^W \mathrm{y}$$

Conicalization of weighted $\mathcal{V}$-limits

Example 1 (Every presheaf is a colimit of representables).

$W : \mathcal{A} \to \mathcal{V}$ with $\mathcal{A}$ small. Consider $\mathrm{y} : \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, \mathcal{V}]$

$$W \cong \operatorname{colim}^{W} \mathrm{y}$$

Lemma 2 (Continuity in the weight).

$S : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}, \quad Z : \mathcal{B} \to [\mathcal{A}, \mathcal{V}], \quad F : \mathcal{A} \to \mathcal{C}$

Suppose that $\operatorname{colim}^{S} Z$ and each $\lim^{Z(B)} F$ with $B \in \mathcal{B}$ exist. Then

$$\lim^{\operatorname{colim}^{S} Z} F \cong \lim^{S} \left(\lim^{Z(-)} F \right)$$

Conicalization of weighted 2-limits

$W: \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small. We search for $H: \mathcal{B} \to [\mathcal{A}, \mathcal{CAT}]$

$$\frac{\varphi: W \Rightarrow [\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(-), U)}{\widetilde{\varphi}: \Delta 1 \xRightarrow[\text{relaxed}]{} [\mathcal{A}, \mathcal{CAT}]\,(H(-), U) : \mathcal{B}^{\mathrm{op}} \to \mathcal{CAT}}$$

Conicalization of weighted 2-limits

$W\colon \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small. We search for $H\colon \mathcal{B} \to [\mathcal{A}, \mathcal{CAT}]$

$$\frac{\varphi\colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(-), U)}{\widetilde{\varphi}\colon \Delta 1 \underset{\text{relaxed}}{\Longrightarrow} [\mathcal{A}, \mathcal{CAT}]\,(H(-), U) : \mathcal{B}^{\mathrm{op}} \to \mathcal{CAT}}$$

As we will want to apply the lemma of continuity in the weight, we search for H of the form

$$\left(\int^{\mathrm{op}} W\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}(W)^{\mathrm{op}}} \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{y}} [\mathcal{A}, \mathcal{CAT}].$$

And we want $\widetilde{\varphi}$ to be at least a lax natural transformation.

Conicalization of weighted 2-limits

$$\frac{\varphi\colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(-), U)}{\widetilde{\varphi}\colon \Delta 1 \underset{\text{relaxed}}{\Longrightarrow} [\mathcal{A}, \mathcal{CAT}]\,((\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})\,(-), U) : \int^{\mathrm{op}} W \to \mathcal{CAT}}$$

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism

$$\varphi_A(X)\colon\ \mathrm{y}(A) \to U,$$

so we need an object $(A, X) \in \int^{\mathrm{op}} W$ such that $\mathcal{G}(W)\,(A, X) = A$.

Conicalization of weighted 2-limits

$$\frac{\varphi\colon W \Rightarrow [\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(-), U)}{\widetilde{\varphi}\colon \Delta 1 \underset{\text{relaxed}}{\Longrightarrow} [\mathcal{A}, \mathcal{CAT}]\,((\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})\,(-), U) : \int^{\mathrm{op}} W \to \mathcal{CAT}}$$

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism

$$\varphi_A(X)\colon\ \mathrm{y}(A) \to U,$$

so we need an object $(A, X) \in \int^{\mathrm{op}} W$ such that $\mathcal{G}(W)\,(A, X) = A$.

For every $\alpha\colon X \to X'$ in $W(A)$, we encode the 2-cell $\varphi_A(\alpha)$ as

$$\begin{array}{c} (A, X) \\ \downarrow(\mathrm{id}_A,\alpha) \\ (A, X') \end{array} \qquad \begin{tikzcd} 1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\widetilde{\varphi}_{(\mathrm{id}_A,\alpha)}}{\Swarrow}"] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \arrow[d, equal, "- \circ \mathrm{y}(\mathcal{G}(W)^{\mathrm{op}}(\mathrm{id}_A,\alpha))"] \\ 1 \arrow[r, "\widetilde{\varphi}_{(A,X')}"'] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \end{tikzcd}$$

In order to encode the naturality of φ into the relaxed naturality of $\widetilde{\varphi}$, for every $f\colon A \to A'$ in $\mathcal{A}$ and $X \in W(A)$,

$$\begin{tikzcd}
(A,X) \arrow[d, "\underline{f}^X=(f,\mathrm{id})"] \\
(A', W(f)(X))
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\varphi}_{\underline{f}^X}"] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \arrow[d, "-\circ \mathrm{y}(f)"', "-\circ \mathrm{y}\left(\mathcal{G}(W)^{\mathrm{op}}(\underline{f}^X)\right)"] \\
1 \arrow[r, "\widetilde{\varphi}_{(A',W(f)(X))}"'] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A'), U)}
\end{tikzcd}$$

Conicalization of weighted 2-limits

In order to encode the naturality of φ into the relaxed naturality of $\widetilde{\varphi}$, for every $f\colon A \to A'$ in $\mathcal{A}$ and $X \in W(A)$,

$$\begin{tikzcd} (A,X) \arrow[d, "\underline{f}^X=(f,\mathrm{id})"] \\ (A', W(f)(X)) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\varphi}_{\underline{f}^X} \,/\!\!/"] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \arrow[d, "-\circ \mathrm{y}(f)"', "-\circ \mathrm{y}\left(\mathcal{G}(W)^{\mathrm{op}}(\underline{f}^X)\right)"] \\ 1 \arrow[r, "\widetilde{\varphi}_{(A', W(f)(X))}"'] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A'), U)} \end{tikzcd}$$

$$\underline{f}^{X'} \circ (\mathrm{id}_A, \alpha) = (\mathrm{id}_{A'}, W(f)(\alpha)) \circ \underline{f}^X$$

so that we recover the naturality of φ on morphisms, that is

$$\varphi_A(\alpha) \circ \mathrm{y}(f) = \varphi_{A'}(W(f)(\alpha)).$$

At this point, every finite composition is reduced to (f, α) defined as

$$(A,X) \xrightarrow{\underline{f}^X} (A', W(f)(X)) \xrightarrow{(\mathrm{id}_{A'}, \alpha)} (A', X')$$

Conicalization of weighted 2-limits

In order to encode the 2-naturality of φ, that is

for every $\delta\colon f \Rightarrow g\colon A \to A'$ in $\mathcal{A}$ and every $X \in W(A)$

$$\varphi_{A'}(W(\delta)_X) = \varphi_A(X)\,\mathrm{y}(\delta)\,,$$

we need a 2-cell $\underline{\delta}^X$ with $\mathcal{G}(W)(\underline{\delta}^X) = \delta$

$$\underline{\delta}^X\colon (f, W(\delta)_X) \Rightarrow \underline{g}^X\colon (A, X) \to (A', W(g)(X))$$

$$\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \arrow[d, "-\circ\mathrm{y}(f)"] \arrow[dl, phantom, "\swarrow \widetilde{\varphi}_{(f, W(\delta)_X)}"] \\
1 \arrow[r, "\widetilde{\varphi}_{(A', W(g)(X))}"'] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A'), U)}
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A), U)} \arrow[d, bend right, "-\circ\mathrm{y}(g)"'] \arrow[d, bend left, "-\circ\mathrm{y}(f)"] \arrow[d, phantom, "\overset{-\ast\mathrm{y}(\mathcal{G}(W)^{\mathrm{op}}(\underline{\delta}^X))}{\Longleftarrow}"] \arrow[dl, phantom, "\swarrow \widetilde{\varphi}_{\underline{g}^X}"] \\
1 \arrow[r, "\widetilde{\varphi}_{(A', W(g)(X))}"'] & {[\mathcal{A}, \mathcal{CAT}]\,(\mathrm{y}(A'), U)}
\end{tikzcd}$$

Conicalization of weighted 2-limits

$$\underline{\delta}^{X'}(\mathrm{id}_A, \alpha) = (\mathrm{id}_{A'}, W(g)(\alpha))\underline{\delta}^X$$

because they represent the same axiom. At his point, every horizontal composition of 2-cells is reduced to a whiskering

$$\delta := (\mathrm{id}, \beta)\underline{\delta}^X : (f, \beta \circ W(\delta)_X) \Rightarrow (g, \beta).$$

Conicalization of weighted 2-limits

$$\underline{\delta}^{X'}(\mathrm{id}_A, \alpha) = (\mathrm{id}_{A'}, W(g)(\alpha))\underline{\delta}^X$$

because they represent the same axiom. At his point, every horizontal composition of 2-cells is reduced to a whiskering

$$\delta := (\mathrm{id}, \beta)\underline{\delta}^X \colon (f, \beta \circ W(\delta)_X) \Rightarrow (g, \beta).$$

We call $\mathcal{G}(W)$ (or sometimes also just $\int^{\mathrm{op}} W$) the 2-$\mathcal{S}et$-**enriched Grothendieck construction** of $W \colon \mathcal{A} \to \mathcal{CAT}$.

A **lax normal natural transformation** $\alpha \colon M \underset{\mathrm{lax}^{\mathrm{n}}}{\Longrightarrow} N \colon \int^{\mathrm{op}} W \to \mathcal{D}$ is a lax natural transformation such that its structure 2-cell on every

$$\underline{f}^X \colon (A, X) \to (B, W(f)(X))$$

in $\int^{\mathrm{op}} W$ is the identity.

Conicalization of weighted 2-limits

Definition 3 ((Op)lax normal conical 2-(co)limit).

$W\colon \mathcal{A} \to \mathcal{CAT}$ and $F\colon \int^{\mathrm{op}} W \to \mathcal{C}$ with $\mathcal{A}$ small.

Lax normal conical 2-limit of F:

$$\mathcal{C}\left(U, \mathrm{lax}^{\mathrm{n}}\text{-}\mathrm{lim}^{\Delta 1} F\right) \cong \left[\int^{\mathrm{op}} W, \mathcal{CAT}\right]_{\mathrm{lax}^{\mathrm{n}}} (\Delta 1, \mathcal{C}(U, F(-)))$$

$W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ and $F\colon \int W \to \mathcal{C}$ with $\mathcal{A}$ small.

Oplax normal conical 2-colimit of F:

$$\mathcal{C}\left(\mathrm{oplax}^{\mathrm{n}}\text{-}\mathrm{colim}^{\Delta 1} F, U\right) \cong \left[\left(\int W\right)^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}^{\mathrm{n}}} (\Delta 1, \mathcal{C}(F(-), U))$$

Conicalization of weighted 2-limits

Theorem 4 (Street[3], but with a new proof and more detail).

Lax normal conical 2-limits are particular weighted 2-limits, weighted by

$$\begin{array}{rccc} \mathrm{W}^{\mathrm{lax}^{\mathrm{n}}}\colon & \int^{\mathrm{op}} Z & \longrightarrow & \mathcal{CAT} \\ & (B, X') & \mapsto & Z(B)/_{X'}. \end{array}$$

Conicalization of weighted 2-limits

Theorem 4 (Street[3], but with a new proof and more detail).

Lax normal conical 2-limits are particular weighted 2-limits, weighted by

$$\begin{aligned} \mathrm{W}^{\mathrm{lax^n}}\colon \quad {\textstyle\int^{\mathrm{op}}} Z &\longrightarrow \mathcal{CAT} \\ (B, X') &\mapsto Z(B)/_{X'}. \end{aligned}$$

Given $W\colon \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small, there is a 2-natural isomorphism

$$[W, [\mathrm{y}(-), U]] \cong \left[{\textstyle\int^{\mathrm{op}}} W, \mathcal{CAT}\right]_{\mathrm{lax^n}} (\Delta 1, [(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})(-), U])$$

$$W \cong \mathrm{colim}^{W} \mathrm{y} \cong \mathrm{lax^n}\text{-}\mathrm{colim}^{\Delta 1} (\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}}).$$

Whence every weighted 2-limits is essentially conicalizable:

$$\lim{}^{W} F \cong \lim{}^{\mathrm{W}^{\mathrm{lax^n}}} \left(\lim{}^{(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})(-)} F\right) \cong \mathrm{lax^n}\text{-}\lim{}^{\Delta 1} (F \circ \mathcal{G}(W))$$

Conicalization of weighted 2-limits

Example 5.

$W\colon \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small.

$$W \cong \operatorname{colim}^{W} \mathrm{y} \cong \mathrm{lax}^{\mathrm{n}}\text{-}\operatorname{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}}).$$

The universal lax normal cocone is given by

$$\forall \quad \begin{array}{c} (A, X) \\ \big\downarrow (f,\alpha) \\ (A', X') \end{array} \ \text{ in } \int^{\mathrm{op}} W \qquad \begin{tikzcd} \mathrm{y}(A) \arrow[r, "\lceil X \rceil"] \arrow[dr, phantom, "\Downarrow\lceil\alpha\rceil"] & W \\ \mathrm{y}(A') \arrow[u, "\mathrm{y}(f)"] \arrow[ur, bend right, "\lceil X' \rceil"'] & \end{tikzcd}$$

Taking $\mathcal{A} = \mathit{1}$, we obtain that 1 is "lax normal conical dense" in $\mathcal{CAT}$.

Proposition 6.

$F\colon \mathcal{B} \to \mathcal{CAT}$. Consider its 2-$\mathcal{S}et$-enriched Grothendieck construction. There is a lax normal natural transformation λ of the form

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\mathrm{lax}^{\mathrm{n}}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{CAT}
\end{tikzcd}
\qquad
\begin{aligned}
\lambda_{(A,X)} &= X \\
\lambda_{(f,\alpha)} &= \alpha
\end{aligned}$$

Proposition 6.

$F\colon \mathcal{B} \to \mathcal{CAT}$. Consider its 2-$\mathcal{S}et$-enriched Grothendieck construction. There is a lax normal natural transformation λ of the form

$$\begin{tikzcd} \int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & 1 \arrow[d, "1"] \arrow[dl, Rightarrow, "\mathrm{lax^n}"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{CAT} \end{tikzcd} \qquad \begin{array}{l} \lambda_{(A,X)} = X \\[1em] \lambda_{(f,\alpha)} = \alpha \end{array}$$

Notice that this diagram lives in 2-$\mathcal{CAT}_{\mathrm{lax}}$, that is a lax 3-category (Lambert[2]), where the interchange rule is lax. 2-$\mathcal{CAT}_{\mathrm{lax}} = $ 2-$\mathcal{S}et$-$\mathcal{CAT}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd} \qquad \beta_{\alpha_A}\colon K(\alpha_A) \circ \beta_{F(A)} \Rightarrow \beta_{G(A)} \circ H(\alpha_A)$$

The Grothendieck constr from an abstract point of view

Definition 7 (lax comma; M., refining Bird[1], Lambert[2]).

$$
\begin{tikzcd}
\mathcal{M} \arrow[rr, bend left, "P"] \arrow[dr, bend right, "Q"'] & & \mathcal{A} \arrow[d, "F"] \arrow[dl, phantom, "\Swarrow \forall\gamma"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{M} \arrow[dr, dashed, "\exists!\, V"] \arrow[drr, bend left, "P"] \arrow[ddr, bend right, "Q"'] & & \\
& F \mathbin{/\!/} G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\Swarrow \lambda"] & \mathcal{A} \arrow[d, "F"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{M} \arrow[d, "W"'] \arrow[dr, "V"] & & \\
F \mathbin{/\!/} G \arrow[dr, "\partial_1"'] & F \mathbin{/\!/} G \arrow[l, phantom, "\Swarrow \Delta"] \arrow[r, "\partial_0"] \arrow[d, "\partial_1"] \arrow[dr, phantom, "\Swarrow \lambda"] & \mathcal{A} \arrow[d, "F"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
\overset{\forall \Xi}{\Rrightarrow}
\begin{tikzcd}
\mathcal{M} \arrow[r, "V"] \arrow[d, "W"'] & F \mathbin{/\!/} G \arrow[dr, "\partial_0"] \arrow[d, phantom, "\Swarrow \Gamma"] & \\
F \mathbin{/\!/} G \arrow[rr, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[drr, phantom, "\Swarrow \lambda"] & & \mathcal{A} \arrow[d, "F"] \\
\mathcal{B} \arrow[rr, "G"'] & & \mathcal{C}
\end{tikzcd}
$$

$$\exists!\, \nu\colon V \underset{\text{lax}}{\Longrightarrow} W \ \text{ s.t. } \ \partial_0 \nu = \Gamma, \ \partial_1 \nu = \Delta, \ \lambda_\nu = \Xi.$$

Proposition 8.

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{CAT}
\end{tikzcd}$$

Or equivalently

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{CAT}_{\bullet,\mathrm{lax}} \arrow[r] \arrow[d, "\tau"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{CAT} \arrow[r, equal] & \mathcal{CAT}
\end{tikzcd}$$

$$\mathcal{G}(-) : [\mathcal{B}, \mathcal{CAT}] \to 2\text{-}\mathcal{CAT}/_{\mathcal{B}}.$$

Definition 9.

A 2-$\mathcal{S}et$-**opfibration over** $\mathcal{B}$ is a 2-functor $P\colon \mathcal{E} \to \mathcal{B}$ such that

(i) the underlying functor P_0 is an ordinary Grothendieck opfibration;

(ii) for every pair $X, Y \in \mathcal{E}$ the functor

$$P_{X,Y}\colon \mathcal{E}(X,Y) \to \mathcal{B}(P(X), P(Y))$$

is a discrete fibration.

Theorem 10 (Lambert[2]).

The essential image of the 2-functor

$$\mathcal{G}(-) : [\mathcal{B}, \mathcal{CAT}] \to 2\text{-}\mathcal{CAT}\,/_{\mathcal{B}}$$

is given by the split 2-$\mathcal{S}et$-opfibrations with small fibres.

The Grothendieck constr from an abstract point of view

Theorem 11 (M., the first part is in Bird[1], Street[3]).

$$[\mathcal{A}, \mathcal{CAT}]_{\text{lax}}(F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT}\right]_{\text{lax}} (\Delta 1, U \circ \mathcal{G}(F)),$$

$$[\mathcal{A}, \mathcal{CAT}]_{\text{ps}}(F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT}\right]_{\text{sigma}} (\Delta 1, U \circ \mathcal{G}(F))$$

$$[\mathcal{A}, \mathcal{CAT}](F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT}\right]_{\text{lax}^{\text{n}}} (\Delta 1, U \circ \mathcal{G}(F))$$

The Grothendieck constr from an abstract point of view

Theorem 11 (M., the first part is in Bird[1], Street[3]).

$$[\mathcal{A}, \mathcal{CAT}]_{\mathrm{lax}}(F, U) \cong \left[\int^{\mathrm{op}} F, \mathcal{CAT}\right]_{\mathrm{lax}} (\Delta 1, U \circ \mathcal{G}(F)),$$

$$[\mathcal{A}, \mathcal{CAT}]_{\mathrm{ps}}(F, U) \cong \left[\int^{\mathrm{op}} F, \mathcal{CAT}\right]_{\mathrm{sigma}} (\Delta 1, U \circ \mathcal{G}(F))$$

$$[\mathcal{A}, \mathcal{CAT}](F, U) \cong \left[\int^{\mathrm{op}} F, \mathcal{CAT}\right]_{\mathrm{lax^n}} (\Delta 1, U \circ \mathcal{G}(F))$$

$$[\mathcal{A}, \mathcal{CAT}]_{\mathrm{lax}}(F, G) \cong \left[\int^{\mathrm{op}} F, \mathcal{CAT}\right]_{\mathrm{lax}} (\Delta 1, G \circ \mathcal{G}(F)) \cong 2\text{-}\mathcal{CAT}/_{\mathcal{A}} \left(\int^{\mathrm{op}} F, \int^{\mathrm{op}} G\right)$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}]_{\mathrm{lax}} \xrightarrow{\sim} 2\text{-}\mathcal{S}et\text{-}\mathcal{O}p\mathcal{F}ib(\mathcal{A})$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}]_{\mathrm{ps}} \xrightarrow{\sim} 2\text{-}\mathcal{S}et\text{-}\mathcal{O}p\mathcal{F}ib_{\mathrm{cart}}(\mathcal{A})$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}] \xrightarrow{\sim} 2\text{-}\mathcal{S}et\text{-}\mathcal{O}p\mathcal{F}ib_{\mathrm{clov}}(\mathcal{A})$$

A pointwise Kan extension result

Definition 12 (colimits in $2\text{-}\mathcal{CAT}_{\mathrm{lax}}$; M.).

$M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ (the marking) with $\mathcal{A}$ small,
$F\colon \int M \to \mathcal{C}$ (the diagram), $W\colon \left(\int M\right)^{\mathrm{op}} \to \mathcal{CAT}$ (the weight).
Oplax normal 2-colimit of F marked by M and weighted by W:

$$\mathcal{C}\left(\mathrm{oplax^n}_M\text{-}\mathrm{colim}^W F, U\right) \cong \left[\left(\int M\right)^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax^n}} (W, \mathcal{C}(F(-), U))$$

We say **opmarked** when the domain of F is expressed as $\int^{\mathrm{op}} M$ for some $M\colon \mathcal{A} \to \mathcal{CAT}$.

Theorem 13.

In $2\text{-}\mathcal{CAT}_{\mathrm{lax}}$ every trivially marked weighted 2-colimit can be equivalently expressed as a marked trivially weighted 2-colimit.

A pointwise Kan extension result

Definition 14 (M.).

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{\lambda}{\Swarrow}\,\mathrm{lax}^{\mathrm{n}}"] & \mathcal{C} \\ \mathcal{A} \arrow[ur, "\operatorname{Lan}_K F"'] & \end{tikzcd}$$

with $\mathcal{B}$ small and $K = \mathcal{G}(M)$ a 2-$\mathcal{S}et$-opfibration.

Pointwise left Kan extension of F along K:

$$(\operatorname{Lan}_K F)(A) \cong \mathrm{oplax}^{\mathrm{op\text{-}n}}{}_M\text{-}\mathrm{colim}^{\mathcal{A}(K(-),A)} F$$

with universal oplax normal cocylinder

$$\mathcal{A}(K(-),A) \xRightarrow{L} \mathcal{C}((L \circ K)(-), L(A)) \xRightarrow[\mathrm{oplax}^{\mathrm{n}}]{\mathcal{C}(\lambda_-,\mathrm{id})} \mathcal{C}(F(-), L(A))$$

$$\mathcal{C}(L(A), C) \cong [\mathcal{B}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}^{\mathrm{n}}}(\mathcal{A}(K(-),A), \mathcal{C}(F(-),C))$$

A pointwise Kan extension result

Theorem 15 (M.).

$F\colon \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small. Then the 2-$\mathcal{S}et$-enriched Grothendieck construction lax comma square exhibits

$$F = \mathrm{Lan}_{\mathcal{G}(F)}\, \Delta 1.$$

Proof.

$$\begin{aligned} \mu_{(B,X)}: \quad \mathcal{B}(B,A) &\longrightarrow \mathcal{CAT}(1, F(A)) \\ (B \xrightarrow{u} A) &\longmapsto F(u)(X) \end{aligned}$$

$$\left(\mu_{(g,\gamma)}\right)_u = F(u)(\gamma)\colon F(u \circ g)(X') \to F(u)(X)$$

is 2-universal:

Theorem 15 (M.).

$F\colon \mathcal{A} \to \mathcal{CAT}$ with $\mathcal{A}$ small. Then the 2-$\mathcal{S}et$-enriched Grothendieck construction lax comma square exhibits

$$F = \mathrm{Lan}_{\mathcal{G}(F)}\, \Delta 1.$$

Proof.

$$\begin{array}{rrcl} \mu_{(B,X)}: & \mathcal{B}(B,A) & \longrightarrow & \mathcal{CAT}(1, F(A)) \\ & (B \xrightarrow{u} A) & \longmapsto & F(u)(X) \end{array}$$

$$\left(\mu_{(g,\gamma)}\right)_u = F(u)(\gamma)\colon F(u \circ g)(X') \to F(u)(X)$$

is 2-universal: given $Q\colon \mathcal{A}(\mathcal{G}(F)(-), A) \underset{\text{oplax}^{\text{n}}}{\Longrightarrow} \mathcal{CAT}(\Delta 1(-), C)$,

$\exists! s\colon F(A) \to C$ such that $(s \circ -) \circ \mu = Q$.

$$s(X) = s\left(\mu_{(A,X)}(\mathrm{id}_A)\right) = Q_{(A,X)}(\mathrm{id}_A)$$

$$s(\alpha) = s\left(\left(\mu_{(\mathrm{id}_A,\alpha)}\right)_{\mathrm{id}_A}\right) = \left(Q_{(\mathrm{id}_A,\alpha)}\right)_{\mathrm{id}_A}.$$

A pointwise Kan extension result

Definition 16 (M.).

$G, H \colon \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$. **Oplax$^{\mathbf{n}}$-lax natural transformation** $\alpha \colon G \Rightarrow H$:

$$\alpha_{B,C} \colon G(B,C) \to H(B,C)$$

$\alpha_{-,C}$ oplax normal in B, $\alpha_{B,-}$ lax in C and compatibility

$$\begin{tikzcd}
& G(B,C) \arrow[rr, "\alpha_{B,C}"] \arrow[dr, "G(\mathrm{id},g)"] & \arrow[d, phantom, "\Downarrow \alpha_{B,g}"] & H(B,C) \arrow[dr, "H(\mathrm{id},g)"] & \\
G(B',C) \arrow[ur, "G(f,\mathrm{id})"] \arrow[dr, "G(\mathrm{id},g)"'] \arrow[rr, phantom, "/\!/"] & & G(B,C') \arrow[rr, "\alpha_{B,C'}"] \arrow[d, phantom, "\Downarrow \alpha_{f,C'}"] & & H(B,C') \\
& G(B',C') \arrow[ur, "G(f,\mathrm{id})"'] \arrow[rr, "\alpha_{B',C'}"'] & \, & H(B',C') \arrow[ur, "H(f,\mathrm{id})"'] &
\end{tikzcd}
=
\begin{tikzcd}
& G(B,C) \arrow[rr, "\alpha_{B,C}"] \arrow[d, phantom, "\alpha_{f,C} \Downarrow"] & & H(B,C) \arrow[dr, "H(\mathrm{id},g)"] & \\
G(B',C) \arrow[ur, "G(f,\mathrm{id})"] \arrow[dr, "G(\mathrm{id},g)"'] \arrow[rr, "\alpha_{B',C}"] & \, \arrow[d, phantom, "\alpha_{B',g} \Downarrow"] & H(B',C) \arrow[ur, "H(f,\mathrm{id})"'] \arrow[dr, "H(\mathrm{id},g)"'] \arrow[rr, phantom, "/\!/"] & & H(B,C') \\
& G(B',C') \arrow[rr, "\alpha_{B',C'}"'] & & H(B',C') \arrow[ur, "H(f,\mathrm{id})"'] &
\end{tikzcd}$$

Modifications between them are modifications in each variable.

A pointwise Kan extension result

Theorem 17 (The oplax$^{\mathrm{n}}$-lax param Yoneda lemma; M.).

$K\colon \mathcal{B} \to \mathcal{A}$ a 2-$\mathcal{S}et$-opfibration, $F\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{CAT}$.

There is a bijection between

$$\alpha_{B,A}\colon \mathcal{A}(K(B), A) \to F(B, A)$$

oplax$^{\mathrm{n}}$ - lax natural in $(B, A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$ and

$$\eta_B\colon 1 \to F(B, K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$.

Moreover this bijection extends to an isomorphism of categories.

A pointwise Kan extension result

Theorem 17 (The oplax$^{\mathsf{n}}$-lax param Yoneda lemma; M.).

$K\colon \mathcal{B} \to \mathcal{A}$ a 2-$\mathcal{Set}$-opfibration, $F\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{CAT}$.

There is a bijection between

$$\alpha_{B,A}\colon \mathcal{A}(K(B), A) \to F(B, A)$$

oplax$^{\mathsf{n}}$ - lax natural in $(B, A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$ and

$$\eta_B\colon 1 \to F(B, K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$.

Moreover this bijection extends to an isomorphism of categories.

Corollary 18 (M.).

Every pointwise left Kan extension in 2-$\mathcal{CAT}_{\mathrm{lax}}$ is also a weak one.

G. J. Bird.
Limits in 2-categories of locally-presented categories.
Ph.D. Thesis, 1984.

M. Lambert.
Discrete 2-fibrations, 2020.

R. Street.
Limits indexed by category-valued 2-functors.
J. Pure Appl. Algebra, 8:149–181, 1976.