计算方法 B

Homework #10

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Question 1

试推导例题 7.4 （第 3 版教材 151-152 页）中的差分格式

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right]$$

的局部截断误差，即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

对于差分格式，若 $y_n = y(x_n), y_{n-1} = y(x_{n-1}), y_{n-2} = y(x_{n-2})$，则有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[7f(x_n, y(x_n)) - 2f(x_{n-1}, y(x_{n-1})) + f(x_{n-2}, y(x_{n-2}))\right]$$

依微分方程

$$\begin{cases} y'(x) = f(x, y) \\ y(a) = y_0 \end{cases} \qquad a \leqslant x \leqslant b \tag{1}$$

有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[7y'(x_n) - 2y'(x_{n-1}) + y'(x_{n-2})\right]$$

将此式在 x_{n-1} 处作 Taylor 展开，有

$$\begin{aligned}
y_{n+1} = y(x_{n-1}) + \frac{h}{3}\Bigg[&7\sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_n - x_{n-1})^k - 2y'(x_{n-1}) \\
&+ \sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_{n-2} - x_{n-1})^k\Bigg] \\
= y(x_{n-1}) &- \frac{2h}{3}y'(x_{n-1}) + \frac{7}{3}\sum_{k=1}^{\infty}\frac{h^k}{(k-1)!}y^{(k)}(x_{n-1}) + \frac{1}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h^k}{(k-1)!}y^{(k)}(x_{n-1}) \\
= y(x_{n-1}) &+ 2hy'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{\left((-1)^{k-1} + 7\right)h^k}{3(k-1)!}y^{(k)}(x_{n-1})
\end{aligned}$$

而 $y(x_{n+1})$ 在 x_{n-1} 处的 Taylor 展开式为

$$\begin{aligned}
y(x_{n+1}) &= \sum_{k=0}^{\infty}\frac{y^{(k)}(x_n)}{k!}(x_{n+1} - x_{n-1})^k \\
&= y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{(2h)^k}{k!}y^{(k)}(x_{n-1})
\end{aligned} \tag{2}$$

故有

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1}$$

$$
\begin{aligned}
&= \sum_{k=2}^{\infty}\left(\frac{(2h)^k}{k!} - \frac{\left((-1)^{k-1}+7\right)h^k}{3(k-1)!}\right)y^{(k)}(x_{n-1}) \\
&= \left(2h^2 - 2h^2\right)y''(x_{n-1}) + \left(\frac{4h^3}{3} - \frac{4h^3}{3}\right)y^{(3)}(x_{n-1}) + \left(\frac{2h^4}{3} - \frac{h^4}{3}\right)y^{(4)}(x_{n-1}) \\
&\quad + \sum_{k=5}^{\infty}\frac{\left(3\times 2^k - k\left((-1)^{k-1}+7\right)\right)h^k}{3k!}y^{(k)}(x_{n-1}) \\
&= \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)
\end{aligned}
$$

Question 2

试用线性多步法构造 $p = 1$, $q = 2$ 时的隐式差分格式，求该格式局部截断误差的误差主项并判断它的阶（即精度），最后为该隐式格式设计一种合适的预估-校正格式。

取积分区间 $[x_{n-1}, x_{n+1}]$，积分节点为 $\{x_{n+1}, x_n, x_{n-1}\}$。构造格式

$$y_{n+1} = y_{n-1} + h\left[\beta_0 f(x_{n+1}, y_{n+1}) + \beta_1 f(x_n, y_n) + \beta_2 f(x_{n-1}, y_{n-1})\right]$$

则由数值积分公式，有

$$
\begin{aligned}
\beta_0 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_n)(x-x_{n-1})}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})}\,\mathrm{d}x = \frac{h}{3} \\
\beta_1 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_{n+1})(x-x_{n-1})}{(x_n-x_{n+1})(x_n-x_{n-1})}\,\mathrm{d}x = \frac{4h}{3} \\
\beta_2 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_{n+1})(x-x_n)}{(x_{n-1}-x_{n+1})(x_{n-1}-x_n)}\,\mathrm{d}x = \frac{h}{3}
\end{aligned}
$$

得到格式

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f(x_{n+1}, y_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})\right]$$

若 $y_{n+1} = y(x_{n+1}), y_n = y(x_n), y_{n-1} = y(x_{n-1})$，则有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[f(x_{n+1}, y(x_{n+1})) + 4f(x_n, y(x_n)) + f(x_{n-1}, y(x_{n-1}))\right]$$

依微分方程(1)，有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[y'(x_{n+1}) + 4y'(x_n) + y'(x_{n-1})\right]$$

将此式在 x_{n-1} 处作 Taylor 展开，有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[\sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_{n+1}-x_{n-1})^k\right.$$

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$$
\begin{aligned}
&\quad + 4\sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_n - x_{n-1})^k + y'(x_{n-1})\Bigg] \\
&= y(x_{n-1}) + \frac{h}{3}(1+4+1)y'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{(2^{k-1}+4)h^k}{3(k-1)!}y^{(k)}(x_{n-1}) \\
&= y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{(2^{k-1}+4)h^k}{3(k-1)!}y^{(k)}(x_{n-1})
\end{aligned}
$$

而 $y(x_{n+1})$ 在 x_{n-1} 处的 Taylor 展开式为(2)，故有

$$
\begin{aligned}
T_{n+1} &\equiv y(x_{n+1}) - y_{n+1} \\
&= \sum_{k=2}^{\infty}\left(\frac{(2h)^k}{k!} - \frac{(2^{k-1}+4)h^k}{3(k-1)!}\right)y^{(k)}(x_{n-1}) \\
&= \left(2h^2 - 2h^2\right)y''(x_{n-1}) + \left(\frac{4h^3}{3} - \frac{4h^3}{3}\right)y^{(3)}(x_{n-1}) + \left(\frac{2h^4}{3} - \frac{2h^4}{3}\right)y^{(4)}(x_{n-1}) \\
&\quad \left(\frac{4h^5}{15} - \frac{5h^5}{18}\right)y^{(5)}(x_{n-1}) + \sum_{k=6}^{\infty}\frac{\left(3\times 2^k - k\left(2^{k-1}+4\right)\right)h^k}{3k!}y^{(k)}(x_{n-1}) \\
&= -\frac{1}{90}h^5y^{(5)}(x_{n-1}) + O(h^6)
\end{aligned}
$$