计算方法 B

Homework #11

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Question 1

试推导如下 Runge-Kutta 公式的局部截断误差和精度。

$$\begin{cases} y_{n+1} = y_n + \dfrac{h}{4}(3k_1 + k_2) \\ \quad k_1 = f(x_n, y_n) \\ \quad k_2 = f(x_n + 2h, y_n + 2hk_1) \end{cases}$$

将 k_1 与 k_2 代入 $y_{n+1} = y_n + \dfrac{h}{4}(3k_1 + k_2)$，得到

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{4}\left[3f(x_n, y_n) + f(x_n + 2h, y_n + 2hf(x_n, y_n))\right] \\ &= y_n + h\left[\frac{3}{4}f(x_n, y_n) + \frac{1}{4}f(x_n + 2h, y_n + 2hf(x_n, y_n))\right] \end{aligned}$$

将此式在 x_n 处做 Taylor 展开，得到

$$\begin{aligned} y_{n+1} &= y_n + h\left[\frac{3}{4}f(x_n, y_n) + \frac{1}{4}f(x_n + 2h, y_n + 2hf(x_n, y_n))\right] \\ &= y_n + h\Bigg\{\frac{3}{4}f(x_n, y_n) + \frac{1}{4}\Big[f(x_n, y_n) \\ &\qquad + 2hf'_x(x_n, y_n) + 2hf(x_n, y_n)f'_y(x_n, y_n) + O(h^2)\Big]\Bigg\} \end{aligned}$$

而依微分方程

$$\begin{cases} y'(x) = f(x, y) \\ y(a) = y_0 \end{cases} \qquad a \leqslant x \leqslant b$$

$y(x_{n+1})$ 在 x_n 处的展开式为

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + O(h^3) \\ &= y(x_n) + hf(x_n, y_n) + \frac{h^2}{2!}\left[f'_x(x_n, y_n) + f'_y(x_n, y_n)f(x_n, y_n)\right] + O(h^3) \end{aligned}$$

所以局部截断误差

$$\begin{aligned} T_{n+1} &= y(x_{n+1}) - y_{n+1} \\ &= hf(x_n, y_n) + \frac{h^2}{2!}\left[f'_x(x_n, y_n) + f'_y(x_n, y_n)f(x_n, y_n)\right] + O(h^3) \\ &\quad - h\Bigg\{\frac{3}{4}f(x_n, y_n) + \frac{1}{4}\Big[f(x_n, y_n) \\ &\quad + 2hf'_x(x_n, y_n) + 2hf(x_n, y_n)f'_y(x_n, y_n) + O(h^2)\Big]\Bigg\} \\ &= \left(1 - \frac{3}{4} - \frac{1}{4}\right)hf(x_n, y_n) + \left(\frac{1}{2!} - \frac{1}{4} \times 2\right)h^2 f'_x(x_n, y_n) \end{aligned}$$

Question 1 continued on next page. . .

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\begin{aligned}
&+ h^2\left(\frac{1}{2!} - \frac{1}{4}\times 2\right) f(x_n,\, y_n) f'_y(x_n,\, y_n) + \left[O(h^3) - \frac{h}{4}O(h^2)\right] \\
&= O(h^3)
\end{aligned}
$$

所以题设公式是 2 阶的，即具有 2 阶精度。

Question 2

讨论梯形格式 $y_{n+1} = y_n + \dfrac{h}{2}\left[f(x_n,\, y_n) + f(y_{n+1},\, y_{n+1})\right]$ 的绝对稳定性（$h > 0$）。