计算方法 B

Homework #10

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Question 1

试推导例题 7.4 （第 3 版教材 151-152 页）中的差分格式

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right]$$

的局部截断误差，即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

依微分方程

$$\begin{cases} y'(x) = f(x, y) \\ y(a) = y_0 \end{cases} \qquad a \leqslant x \leqslant b$$

有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[7y'(x_n) - 2y'(x_{n-1}) + y'(x_{n-2})\right]$$

将此式在 x_{n-1} 处作 Taylor 展开，有

$$\begin{aligned} y_{n+1} =& y(x_{n-1}) + \frac{h}{3}\Bigg[7\sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_n - x_{n-1})^k - 2y'(x_{n-1}) \\ &+ \sum_{k=0}^{\infty}\frac{y^{(k+1)}(x_{n-1})}{k!}(x_{n-2} - x_{n-1})^k\Bigg] \\ =& y(x_{n-1}) - \frac{2h}{3}y'(x_{n-1}) + \frac{7}{3}\sum_{k=1}^{\infty}\frac{h^k}{(k-1)!}y^{(k)}(x_{n-1}) + \frac{1}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h^k}{(k-1)!}y^{(k)}(x_{n-1}) \\ =& y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{\left((-1)^{k-1} + 7\right)h^k}{3(k-1)!}y^{(k)}(x_{n-1}) \end{aligned}$$

而 $y(x_{n+1})$ 在 x_{n-1} 处的 Taylor 展开式为

$$\begin{aligned} y(x_{n+1}) =& \sum_{k=0}^{\infty}\frac{y^{(k)}(x_n)}{k!}(x_{n+1} - x_{n-1})^k \\ =& y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty}\frac{(2h)^k}{k!}y^{(k)}(x_{n-1}) \end{aligned}$$

故有

$$\begin{aligned} T_{n+1} \equiv& y(x_{n+1}) - y_{n+1} \\ =& \sum_{k=2}^{\infty}\left(\frac{(2h)^k}{k!} - \frac{\left((-1)^{k-1} + 7\right)h^k}{3(k-1)!}\right)y^{(k)}(x_{n-1}) \\ =& \left(2h^2 - 2h^2\right)y''(x_{n-1}) + \left(\frac{4h^3}{3} - \frac{4h^3}{3}\right)y^{(3)}(x_{n-1}) + \left(\frac{2h^4}{3} - \frac{h^4}{3}\right)y^{(4)}(x_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=5}^{\infty}\frac{\left(3\times 2^k-k\left((-1)^{k-1}+7\right)\right)h^k}{3k!}y^{(k)}(x_{n-1})\\
=&\frac{1}{3}h^4y^{(4)}(x_{n-1})+O(h^5)
\end{aligned}
$$

Question 2

试用线性多步法构造 $p=1$, $q=2$ 时的隐式差分格式，求该格式局部截断误差的误差主项并判断它的阶（即精度），最后为该隐式格式设计一种合适的预估-校正格式。