计算方法 B

Homework #3

2020.4.12

PB17000297 罗晏宸

Question 1

求满足下表数据以及边界条件 $S''(-2) = S''(2) = 0$ $(n = 3)$ 的三次样条插值函数 $S(x)$，并计算 $S(0)$ 的值。注意：这里的 n 为小区间个数。

x	-2.00	-1.00	1.00	2.00
$f(x)$	-4.00	3.00	5.00	10.0

在三个子区间上分别构造三次多项式

$$S(x) = \begin{cases} S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0, & x \in [-2.00, -1.00] \\ S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1, & x \in [-1.00, 1.00] \\ S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2, & x \in [1.00, 2.00] \end{cases}$$

共有 12 个未知数，需要等量的条件。由表中数据、插值函数在每个内点的关系以及边界条件有

$$\begin{cases} S_0(-2.00) = -4.00 \\ S_0(-1.00) = 3.00 \\ S_1(-1.00) = 3.00 \\ S_1(1.00) = 5.00 \\ S_2(1.00) = 5.00 \\ S_2(2.00) = 10.00 \\ S_0'(-1.00) = S_1'(-1.00) \\ S_1'(1.00) = S_2'(1.00) \\ S_0''(-1.00) = S_1''(-1.00) \\ S_1''(1.00) = S_2''(1.00) \\ S_0''(-2.00) = 0 \\ S_2''(2.00) = 0 \end{cases}$$

Question 1 continued on next page. . .

$$
\Rightarrow \begin{cases}
(-2.00)^3 a_0 + (-2.00)^2 b_0 - 2.00 c_0 + d_0 = -4.00 \\
(-1.00)^3 a_0 + (-1.00)^2 b_0 - 1.00 c_0 + d_0 = 3.00 \\
(-1.00)^3 a_1 + (-1.00)^2 b_1 - 1.00 c_1 + d_1 = 3.00 \\
1.00^3 a_1 + 1.00^2 b_1 + 1.00 c_1 + d_1 = 5.00 \\
1.00^3 a_2 + 1.00^2 b_2 + 1.00 c_2 + d_2 = 5.00 \\
2.00^3 a_2 + 2.00^2 b_2 + 2.00 c_2 + d_2 = 10.00 \\
3 \times (-1.00)^2 a_0 + 2 \times (-1.00) b_0 + c_0 = 3 \times (-1.00)^2 a_1 + 2 \times (-1.00) b_1 + c_1 \\
3 \times 1.00^2 a_1 + 2 \times 1.00 b_1 + c_1 = 3 \times 1.00^2 a_2 + 2 \times 1.00 b_2 + c_2 \\
6 \times (-1.00) a_0 + 2 b_0 = 6 \times (-1.00) a_1 + 2 b_1 \\
6 \times 1.00 a_1 + 2 b_1 = 6 \times 1.00 a_2 + 2 b_2 \\
6 \times (-2.00) a_0 + 2 b_0 = 0 \\
6 \times 2.00 a_2 + 2 b_2 = 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
-8a_0 + 4b_0 - 2c_0 + d_0 = -4 \\
-a_0 + b_0 - c_0 + d_0 = 3 \\
-a_1 + b_1 - c_1 + d_1 = 3 \\
a_1 + b_1 + c_1 + d_1 = 5 \\
a_2 + b_2 + c_2 + d_2 = 5 \\
8a_2 + 4b_2 + 2c_2 + d_2 = 10 \\
3a_0 - 2b_0 + c_0 = 3a_1 - 2b_1 + c_1 \\
3a_1 + 2b_1 + c_1 = 3a_2 + 2b_2 + c_2 \\
-6a_0 + 2b_0 = -6a_1 + 2b_1 \\
6a_1 + 2b_1 = 6a_2 + 2b_2 \\
-12a_0 + 2b_0 = 0 \\
12a_2 + 2b_2 = 0
\end{cases}
$$

$$\Rightarrow \begin{cases} a_0 = -\frac{11}{8} \\ b_0 = -\frac{33}{4} \\ c_0 = -\frac{65}{8} \\ d_0 = \frac{7}{4} \\ a_1 = \frac{5}{4} \\ b_1 = -\frac{3}{8} \\ c_1 = -\frac{1}{4} \\ d_1 = \frac{35}{8} \\ a_2 = -\frac{9}{8} \\ b_2 = \frac{27}{4} \\ c_2 = -\frac{59}{8} \\ d_2 = \frac{27}{4} \end{cases}$$

得到

$$S(x) = \begin{cases} S_0(x) = -\frac{11}{8}x^3 - \frac{33}{4}x^2 - \frac{65}{8}x + \frac{7}{4}, & x \in [-2.00, -1.00] \\ S_1(x) = \frac{5}{4}x^3 - \frac{3}{8}x^2 - \frac{1}{4}x + \frac{35}{8}, & x \in [-1.00, 1.00] \\ S_2(x) = -\frac{9}{8}x^3 + \frac{27}{4}x^2 - \frac{59}{8}x + \frac{27}{4}, & x \in [1.00, 2.00] \end{cases}$$

同时

$$S(0) = S_1(0) = \frac{35}{8} = 4.375$$

Question 2

利用下面的函数值表，构造分段线性插值函数，并计算 $f(1.075)$ 和 $f(1.175)$ 的近似值（保留 4 位小数）。

x	1.05	1.10	1.15	1.20
$f(x)$	2.00	2.20	2.17	2.35

Question 2 continued on next page. . .

在三个子区间上作 $f(x)$ 以区间端点为节点的线性插值，有

$$p(x) = \begin{cases} p_0(x) = \dfrac{x-1.10}{1.05-1.10} \times 2.00 + \dfrac{x-1.05}{1.10-1.05} \times 2.20, & x \in [1.05, 1.10] \\ p_1(x) = \dfrac{x-1.15}{1.10-1.15} \times 2.20 + \dfrac{x-1.10}{1.15-1.10} \times 2.17, & x \in [1.05, 1.10] \\ p_2(x) = \dfrac{x-1.20}{1.15-1.20} \times 2.17 + \dfrac{x-1.15}{1.20-1.15} \times 2.35, & x \in [1.05, 1.10] \end{cases}$$

$$= \begin{cases} p_0(x) = 4.00x - 2.20, & x \in [1.05, 1.10] \\ p_1(x) = -0.60x + 2.86, & x \in [1.10, 1.15] \\ p_2(x) = 3.60x - 1.97, & x \in [1.15, 1.20] \end{cases}$$

计算近似值有

$$f(1.075) \approx p_0(1.075) = 4.00 \times 1.075 - 2.20 = 2.1000$$
$$f(1.175) \approx p_2(1.175) = 3.60 \times 1.175 - 1.97 = 2.2600$$

Question 3

设 $f(x) = 10x^3 + 3x + 2020$，求 $f[1,2]$ 和 $f[1,2,3,4]$。

$$\begin{aligned} f[1,2] &= \frac{f(2)-f(1)}{2-1} \\ &= (10 \times 2^3 + 3 \times 2 + 2020) - (10 \times 1^3 + 3 \times 1 + 2020) \\ &= 53 \end{aligned}$$

$$\begin{aligned} f[1,2,3,4] &= \frac{f^{(3)}(\xi)}{3!}, \quad \xi \in [1,4] \\ &= \frac{60}{6} \\ &= 10 \end{aligned}$$

Question 4

设 $\{l_i(x)\}_{i=0}^6$ 是以 $\{x_i = 2i\}_{i=0}^6$ 为结点的 6 次 Lagrange 插值基函数，试求 $\sum_{i=0}^{6} (x_i^3 + x_i^2 + 1) l_i(x)$ 和 $\sum_{i=0}^{6} (x_i^3 + x_i^2 + 1) l_i'(x)$（结果需化简）。

Question 4 continued on next page. . .

形式上

$$\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)$$

给出了函数 x^3+x^2+1 以 $\{x_i=2i\}_{i=0}^{6}$ 为插值结点横坐标的 6 次插值多项式，由插值多项式的存在唯一性有

$$\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)=x^3+x^2+1$$

同理有

$$\begin{aligned}\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i'(x)&=\left(\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)\right)'-\sum_{i=0}^{6}(3x_i^2+2x_i+x_i)l_i(x)\\&=(x^3+x^2+1)-(3x^2+2x)\\&=x^3-2x^2-2x+1\end{aligned}$$