

计算方法 B

Homework #10

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Question 1

试推导例题 7.4（第 3 版教材 151-152 页）中的差分格式

$$y_{n+1} = y_{n-1} + \frac{h}{3} [7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})]$$

的局部截断误差，即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

对于差分格式，若 $y_n = y(x_n), y_{n-1} = y(x_{n-1}), y_{n-2} = y(x_{n-2})$ ，则有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3} [7f(x_n, y(x_n)) - 2f(x_{n-1}, y(x_{n-1})) + f(x_{n-2}, y(x_{n-2}))]$$

依微分方程

$$\begin{cases} y'(x) = f(x, y) \\ y(a) = y_0 \end{cases} \quad a \leq x \leq b \quad (1)$$

有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3} [7y'(x_n) - 2y'(x_{n-1}) + y'(x_{n-2})]$$

将此式在 x_{n-1} 处作 Taylor 展开，有

$$\begin{aligned} y_{n+1} &= y(x_{n-1}) + \frac{h}{3} \left[7 \sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_{n-1})}{k!} (x_n - x_{n-1})^k - 2y'(x_{n-1}) \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_{n-1})}{k!} (x_{n-2} - x_{n-1})^k \right] \\ &= y(x_{n-1}) - \frac{2h}{3} y'(x_{n-1}) + \frac{7}{3} \sum_{k=1}^{\infty} \frac{h^k}{(k-1)!} y^{(k)}(x_{n-1}) + \frac{1}{3} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} h^k}{(k-1)!} y^{(k)}(x_{n-1}) \\ &= y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty} \frac{((-1)^{k-1} + 7) h^k}{3(k-1)!} y^{(k)}(x_{n-1}) \end{aligned}$$

而 $y(x_{n+1})$ 在 x_{n-1} 处的 Taylor 展开式为

$$\begin{aligned} y(x_{n+1}) &= \sum_{k=0}^{\infty} \frac{y^{(k)}(x_n)}{k!} (x_{n+1} - x_{n-1})^k \\ &= y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty} \frac{(2h)^k}{k!} y^{(k)}(x_{n-1}) \end{aligned} \quad (2)$$

故有

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1}$$

$$\begin{aligned}
&= \sum_{k=2}^{\infty} \left(\frac{(2h)^k}{k!} - \frac{((-1)^{k-1} + 7) h^k}{3(k-1)!} \right) y^{(k)}(x_{n-1}) \\
&= (2h^2 - 2h^2) y''(x_{n-1}) + \left(\frac{4h^3}{3} - \frac{4h^3}{3} \right) y^{(3)}(x_{n-1}) + \left(\frac{2h^4}{3} - \frac{h^4}{3} \right) y^{(4)}(x_{n-1}) \\
&\quad + \sum_{k=5}^{\infty} \frac{(3 \times 2^k - k((-1)^{k-1} + 7)) h^k}{3k!} y^{(k)}(x_{n-1}) \\
&= \frac{1}{3} h^4 y^{(4)}(x_{n-1}) + O(h^5)
\end{aligned}$$

Question 2

试用线性多步法构造 $p = 1, q = 2$ 时的隐式差分格式，求该格式局部截断误差的误差主项并判断它的阶（即精度），最后为该隐式格式设计一种合适的预估-校正格式。

取积分区间 $[x_{n-1}, x_{n+1}]$ ，积分节点为 $\{x_{n+1}, x_n, x_{n-1}\}$ 。构造格式

$$y_{n+1} = y_{n-1} + h [\beta_0 f(x_{n+1}, y_{n+1}) + \beta_1 f(x_n, y_n) + \beta_2 f(x_{n-1}, y_{n-1})]$$

则由数值积分公式，有

$$\begin{aligned}
\beta_0 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x - x_n)(x - x_{n-1})}{(x_{n+1} - x_n)(x_{n+1} - x_{n-1})} dx = \frac{h}{3} \\
\beta_1 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x - x_{n+1})(x - x_{n-1})}{(x_n - x_{n+1})(x_n - x_{n-1})} dx = \frac{4h}{3} \\
\beta_2 h &= \int_{x_{n-1}}^{x_{n+1}} \frac{(x - x_{n+1})(x - x_n)}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} dx = \frac{h}{3}
\end{aligned}$$

得到格式

$$y_{n+1} = y_{n-1} + \frac{h}{3} [f(x_{n+1}, y_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})]$$

若 $y_{n+1} = y(x_{n+1}), y_n = y(x_n), y_{n-1} = y(x_{n-1})$ ，则有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3} [f(x_{n+1}, y(x_{n+1})) + 4f(x_n, y(x_n)) + f(x_{n-1}, y(x_{n-1}))]$$

依微分方程(1)，有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3} [y'(x_{n+1}) + 4y'(x_n) + y'(x_{n-1})]$$

将此式在 x_{n-1} 处作 Taylor 展开，有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3} \left[\sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_{n-1})}{k!} (x_{n+1} - x_{n-1})^k \right]$$

$$\begin{aligned}
& + 4 \sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_{n-1})}{k!} (x_n - x_{n-1})^k + y'(x_{n-1}) \Big] \\
& = y(x_{n-1}) + \frac{h}{3}(1+4+1)y'(x_{n-1}) + \sum_{k=2}^{\infty} \frac{(2^{k-1}+4)h^k}{3(k-1)!} y^{(k)}(x_{n-1}) \\
& = y(x_{n-1}) + 2hy'(x_{n-1}) + \sum_{k=2}^{\infty} \frac{(2^{k-1}+4)h^k}{3(k-1)!} y^{(k)}(x_{n-1})
\end{aligned}$$

而 $y(x_{n+1})$ 在 x_{n-1} 处的 Taylor 展开式为(2), 故有

$$\begin{aligned}
T_{n+1} & \equiv y(x_{n+1}) - y_{n+1} \\
& = \sum_{k=2}^{\infty} \left(\frac{(2h)^k}{k!} - \frac{(2^{k-1}+4)h^k}{3(k-1)!} \right) y^{(k)}(x_{n-1}) \\
& = (2h^2 - 2h^2) y''(x_{n-1}) + \left(\frac{4h^3}{3} - \frac{4h^3}{3} \right) y^{(3)}(x_{n-1}) + \left(\frac{2h^4}{3} - \frac{2h^4}{3} \right) y^{(4)}(x_{n-1}) \\
& \quad \left(\frac{4h^5}{15} - \frac{5h^5}{18} \right) y^{(5)}(x_{n-1}) + \sum_{k=6}^{\infty} \frac{(3 \times 2^k - k(2^{k-1}+4)) h^k}{3k!} y^{(k)}(x_{n-1}) \\
& = -\frac{1}{90} h^5 y^{(5)}(x_{n-1}) + O(h^6)
\end{aligned}$$

因此该格式是四阶的。

隐格式的计算, 可以使用显格式作为初值估计, 得到预估-校正公式

$$\begin{cases} \bar{y}_{n+1} = y_{n-1} + \frac{h}{3} [7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})] \\ y_{n+1} = y_{n-1} + \frac{h}{3} [f(x_{n+1}, y_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})] \end{cases}$$