计算方法 B

Homework #10

2020.5.26

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Question 1

试推导例题 7.4 （第 3 版教材 151-152 页）中的差分格式

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right]$$

的局部截断误差，即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

依微分方程

$$\begin{cases} y'(x) = f(x,y) \\ y(a) = y_0 \end{cases} \qquad a \leqslant x \leqslant b$$

有

$$y_{n+1} = y(x_{n-1}) + \frac{h}{3}\left[7y'(x_n) - 2y'(x_{n-1}) + y'(x_{n-2})\right]$$

将此式在 x_n 处作 Taylor 展开，有

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\begin{aligned}
y_{n+1} = &\sum_{k=0}^{\infty} \frac{y^{(k)}(x_n)}{k!}(x_{n-1} - x_n)^k + \frac{h}{3}\Bigg[7y'(x_n) \\
&- 2\sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_n)}{k!}(x_{n-1} - x_n)^k + \sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_n)}{k!}(x_{n-2} - x_n)^k\Bigg] \\
= &\sum_{k=0}^{\infty} \frac{y^{(k)}(x_n)}{k!}(-h)^k + \frac{7h}{3}y'(x_n) \\
&+ \frac{h}{3}\Bigg[-2\sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_n)}{k!}(-h)^k + \sum_{k=0}^{\infty} \frac{y^{(k+1)}(x_n)}{k!}(-2h)^k\Bigg] \\
= &\sum_{k=0}^{\infty} \frac{(-1)^k h^k}{k!} y^{(k)}(x_n) + \frac{7h}{3}y'(x_n) \\
&+ \frac{h}{3}\Bigg[-2\sum_{k=1}^{\infty} \frac{(-1)^{k-1} k h^{k-1}}{k!} y^{(k)}(x_n) + \sum_{k=1}^{\infty} \frac{(-2)^{k-1} k h^{k-1}}{k!} y^{(k)}(x_n)\Bigg] \\
= &y(x_n) + \left(-h + \frac{7h}{3} + \frac{h}{3} \times (-2+1)\right) y'(x_n) + \sum_{k=2}^{\infty} \frac{\left(3 - (2^{k-1} - 2)k\right)(-1)^k h^k}{3k!} y^{(k)}(x_n) \\
= &y(x_n) + hy'(x_n) + \sum_{k=2}^{\infty} \frac{\left(3 - (2^{k-1} - 2)k\right)(-1)^k h^k}{3k!} y^{(k)}(x_n)
\end{aligned}$$

而 $y(x_{n+1})$ 在 x_n 处的 Taylor 展开式为

$$y(x_{n+1}) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_n)}{k!}(x_{n+1} - x_n)^k = y(x_n) + hy'(x_n) + \sum_{k=2}^{\infty} \frac{h^k}{k!} y^{(k)}(x_n)$$

故有

$$\begin{aligned} T_{n+1} &\equiv y(x_{n+1}) - y_{n+1} \\ &= \sum_{k=2}^{\infty} \left(\frac{h^k}{k!} - \frac{\left(3 - (2^{k-1} - 2)k\right)(-1)^k h^k}{3k!} \right) y^{(k)}(x_n) \\ &= \frac{1}{3} h^4 y^{(4)}(x_{n-1}) + O(h^5) \end{aligned}$$

Question 2

试用线性多步法构造 $p = 1$, $q = 2$ 时的隐式差分格式，求该格式局部截断误差的误差主项并判断它的阶（即精度），最后为该隐式格式设计一种合适的预估-校正格式。