计算方法 B

Homework #6

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Question 1

分别计算下列矩阵的 $\|\bullet\|_1$，$\|\bullet\|_\infty$ 范数：

(1)

$$A = \begin{pmatrix} 5 & -2 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -6 \end{pmatrix}$$

$$\begin{aligned} \|A\|_1 &= \max_{1\leqslant j\leqslant 3}\left\{\sum_{i=1}^{3}|a_{ij}|\right\} \\ &= \max\{7,\,7,\,8\} \\ &= 8 \\ \|A\|_\infty &= \max_{1\leqslant i\leqslant 3}\left\{\sum_{j=1}^{3}|a_{ij}|\right\} \\ &= \max\{8,\,5,\,9\} \\ &= 9 \end{aligned}$$

(2)

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -2 & 3 & 1 \\ -1 & 1 & 6 \end{pmatrix}$$

Question 1 continued on next page. . .

$$
\begin{aligned}
\|A\|_1 &= \max_{1\leqslant j\leqslant 3}\left\{\sum_{i=1}^{3}|a_{ij}|\right\}\\
&= \max\{7,\,5,\,8\}\\
&= 8\\
\|A\|_\infty &= \max_{1\leqslant i\leqslant 3}\left\{\sum_{j=1}^{3}|a_{ij}|\right\}\\
&= \max\{6,\,6,\,8\}\\
&= 8
\end{aligned}
$$

Question 2

分别计算矩阵 $A=\begin{pmatrix} 4 & -2\\ -1 & 1\end{pmatrix}$ **的谱半径及其** $\|\bullet\|_2$ **范数。**

设 λ 为 A 的特征值，则有

$$
\begin{aligned}
& & \det(A-\lambda\mathbf{I}) &= 0\\
&\Rightarrow & \begin{vmatrix} 4-\lambda & -2\\ -1 & 1-\lambda\end{vmatrix} &= 0\\
&\Rightarrow & (4-\lambda)(1-\lambda)-(-1)\times(-2) &= 0\\
&\Rightarrow & \lambda^2-5\lambda+2 &= 0
\end{aligned}
$$

解得

$$
\lambda_1=\frac{5-\sqrt{17}}{2},\ \lambda_2=\frac{5+\sqrt{17}}{2}
$$

进而有 A 的谱半径

$$
\begin{aligned}
\rho(A) &= \max_i|\lambda_i|\\
&= \max\frac{5-\sqrt{17}}{2},\,\frac{5+\sqrt{17}}{2}\\
&= \frac{5+\sqrt{17}}{2}
\end{aligned}
$$

设 μ 是矩阵 $A^{\mathbf{T}}A$ 的特征值，则有

$$
\begin{aligned}
& & \det(A^{\mathbf{T}}A-\mu\mathbf{I})=0\\
\Rightarrow & & \det\left(\begin{pmatrix}4 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & -2\\ -1 & 1\end{pmatrix}-\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix}\right)=0\\
\Rightarrow & & \begin{vmatrix}17-\mu & -9\\ -9 & 5-\mu\end{vmatrix}=0\\
\Rightarrow & & (5-\mu)(17-\mu)-(-9)\times(-9)=0\\
\Rightarrow & & \mu^2-22\mu+24=0
\end{aligned}
$$

解得

$$\mu_1=11-3\sqrt{13},\ \mu_2=11+3\sqrt{13}$$

由此矩阵 A 的 $\|\bullet\|_2$ 范数计算如下

$$
\begin{aligned}
\|A\|_2 &= \sqrt{\max_{1\leqslant i\leqslant n}\{|\mu_i|\}}\\
&= \sqrt{\max\{11-3\sqrt{13},11+3\sqrt{13}\}}\\
&= \sqrt{11+3\sqrt{13}}
\end{aligned}
$$

Question 3

用 Doolittle 分解法解下列线性方程组（请给出详细的解题过程，包括矩阵分解）

$$
\begin{cases}
5x_1+\ \ x_2+2x_3 &= 10\\
\ \ x_1+3x_2-\ \ x_3 &= 5\\
2x_1+\ \ x_2+5x_3 &= 20
\end{cases}
$$

以矩阵形式表示题设线性方程组为 $Ax=b$，其中

$$A=\begin{pmatrix}5 & 1 & 2\\ 1 & 3 & -1\\ 2 & 1 & 5\end{pmatrix},b=\begin{pmatrix}10\\ 5\\ 20\end{pmatrix}$$

设

$$A=LU=\begin{pmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{pmatrix}$$

Question 3 continued on next page. . .

首先计算 U 的第一行元素有

$$a_{1j}=\sum_{r=1}^{3}l_{1r}u_{rj}=\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}u_{13}\\u_{23}\\u_{33}\end{pmatrix}=u_{1j},\quad j=1,2,3$$

得到

$$u_{11}=a_{11}=5,\ u_{12}=a_{12}=1,\ u_{13}=a_{13}=2,$$

其次计算 L 的第一列元素有

$$a_{i1}=\sum_{r=1}^{3}l_{ir}u_{r1}=\begin{pmatrix}l_{i1} & \cdots & l_{i,i-1} & 1 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}u_{11}\\0\\\vdots\\0\end{pmatrix}=l_{i1}u_{11},\quad i=2,3$$

得到

$$l_{21}=\frac{a_{21}}{u_{11}}=\frac{1}{5},\ l_{31}=\frac{a_{31}}{u_{11}}=\frac{2}{5}$$

然后由

$$a_{2i}=\begin{pmatrix}l_{21} & 1 & 0\end{pmatrix}\begin{pmatrix}u_{1i}\\u_{2i}\\u_{3i}\end{pmatrix}=l_{21}u_{1i}+u_{2i},\quad i=2,3$$

得到

$$u_{22}=a_{22}-l_{21}u_{12}=3-\frac{1}{5}\times 1=\frac{14}{5},\ u_{23}=a_{23}-l_{21}u_{13}=-1-\frac{1}{5}\times 2=-\frac{7}{5}$$

再由

$$a_{32}=\begin{pmatrix}l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{12}\\u_{22}\\0\end{pmatrix}=l_{31}u_{12}+l_{32}u_{22}$$

得到

$$l_{32}=\frac{a_{32}-l_{31}u_{12}}{u_{22}}=\frac{1-\frac{2}{5}\times 1}{\frac{14}{5}}=\frac{3}{14}$$

最后由

$$a_{33}=\begin{pmatrix}l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{13}\\u_{23}\\u_{33}\end{pmatrix}=l_{31}u_{13}+l_{32}u_{23}+u_{33}$$

得到

$$u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}=5-\frac{2}{5}\times 2-\frac{3}{14}\times(-\frac{7}{5})=\frac{9}{2}$$

Question 3 continued on next page...

综上得到

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{14}{5} & -\frac{7}{5} \\ 0 & 0 & \frac{9}{2} \end{pmatrix}$$

解方程 $Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix}$$

解方程 $Ux = y$

$$\begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{14}{5} & -\frac{7}{5} \\ 0 & 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{63} \\ \frac{25}{9} \\ \frac{215}{63} \end{pmatrix} \approx \begin{pmatrix} 0.0794 \\ 2.7778 \\ 3.4127 \end{pmatrix}$$