计算方法 B

Homework #6

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Question 1

分别计算下列矩阵的 $\|\bullet\|_1$，$\|\bullet\|_\infty$ 范数：

(1)

$$A = \begin{pmatrix} 5 & -2 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -6 \end{pmatrix}$$

$$\begin{aligned}
\|A\|_1 &= \max_{1\leqslant j\leqslant 3}\left\{\sum_{i=1}^{3}|a_{ij}|\right\} \\
&= \max\{7,\, 7,\, 8\} \\
&= 8 \\
\|A\|_\infty &= \max_{1\leqslant i\leqslant 3}\left\{\sum_{j=1}^{3}|a_{ij}|\right\} \\
&= \max\{8,\, 5,\, 9\} \\
&= 9
\end{aligned}$$

(2)

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -2 & 3 & 1 \\ -1 & 1 & 6 \end{pmatrix}$$

Question 1 continued on next page...

$$
\begin{aligned}
\|A\|_1 &= \max_{1\leqslant j\leqslant 3}\left\{\sum_{i=1}^{3}|a_{ij}|\right\}\\
&= \max\{7,\,5,\,8\}\\
&= 8\\
\|A\|_\infty &= \max_{1\leqslant i\leqslant 3}\left\{\sum_{j=1}^{3}|a_{ij}|\right\}\\
&= \max\{6,\,6,\,8\}\\
&= 8
\end{aligned}
$$

Question 2

分别计算矩阵 $A = \begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$ **的谱半径及其** $\|\bullet\|_2$ **范数。**

设 λ 为 A 的特征值，则有

$$
\begin{aligned}
& & \det(A-\lambda\mathbf{I}) &= 0\\
&\Rightarrow & \begin{vmatrix} 4-\lambda & -2 \\ -1 & 1-\lambda \end{vmatrix} &= 0\\
&\Rightarrow & (4-\lambda)(1-\lambda)-(-1)\times(-2) &= 0\\
&\Rightarrow & \lambda^2-5\lambda+2 &= 0
\end{aligned}
$$

解得

$$
\lambda_1 = \frac{5-\sqrt{17}}{2},\ \lambda_2 = \frac{5+\sqrt{17}}{2}
$$

进而有 A 的谱半径

$$
\begin{aligned}
\rho(A) &= \max_i |\lambda_i|\\
&= \max\left\{\frac{5-\sqrt{17}}{2},\,\frac{5+\sqrt{17}}{2}\right\}\\
&= \frac{5+\sqrt{17}}{2}
\end{aligned}
$$

Question 2 continued on next page. . .

设 μ 是矩阵 $A^{\mathbf{T}}A$ 的特征值，则有

$$\begin{aligned}
& & \det(A^{\mathbf{T}}A-\mu\mathbf{I}) &= 0 \\
\Rightarrow & & \det\left(\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}-\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\right) &= 0 \\
\Rightarrow & & \begin{vmatrix} 17-\mu & -9 \\ -9 & 5-\mu \end{vmatrix} &= 0 \\
\Rightarrow & & (5-\mu)(17-\mu)-(-9)\times(-9) &= 0 \\
\Rightarrow & & \mu^2-22\mu+24 &= 0
\end{aligned}$$

解得

$$\mu_1 = 11-3\sqrt{13},\ \mu_2 = 11+3\sqrt{13}$$

由此矩阵 A 的 $\|\bullet\|_2$ 范数计算如下

$$\begin{aligned}
\|A\|_2 &= \sqrt{\max_{1\leqslant i\leqslant n}\{|\mu_i|\}} \\
&= \sqrt{\max\left\{11-3\sqrt{13}, 11+3\sqrt{13}\right\}} \\
&= \sqrt{11+3\sqrt{13}}
\end{aligned}$$

Question 3

用 Doolittle 分解法解下列线性方程组（请给出详细的解题过程，包括矩阵分解）

$$\begin{cases} 5x_1 + x_2 + 2x_3 &= 10 \\ x_1 + 3x_2 - x_3 &= 5 \\ 2x_1 + x_2 + 5x_3 &= 20 \end{cases}$$

以矩阵形式表示题设线性方程组为 $Ax=b$，其中

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{pmatrix}, b = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$$

Question 3 continued on next page. . .

设

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

首先计算 U 的第一行元素有

$$a_{1j} = \sum_{r=1}^{3} l_{1r} u_{rj} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = u_{1j}, \quad j = 1, 2, 3$$

得到

$$u_{11} = a_{11} = 5,\ u_{12} = a_{12} = 1,\ u_{13} = a_{13} = 2,$$

其次计算 L 的第一列元素有

$$a_{i1} = \sum_{r=1}^{3} l_{ir} u_{r1} = \begin{pmatrix} l_{i1} & \cdots & l_{i,i-1} & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = l_{i1} u_{11}, \quad i = 2, 3$$

得到

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{1}{5},\ l_{31} = \frac{a_{31}}{u_{11}} = \frac{2}{5}$$

然后由

$$a_{2i} = \begin{pmatrix} l_{21} & 1 & 0 \end{pmatrix} \begin{pmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{pmatrix} = l_{21} u_{1i} + u_{2i}, \quad i = 2, 3$$

得到

$$u_{22} = a_{22} - l_{21} u_{12} = 3 - \frac{1}{5} \times 1 = \frac{14}{5},\ u_{23} = a_{23} - l_{21} u_{13} = -1 - \frac{1}{5} \times 2 = -\frac{7}{5}$$

再由

$$a_{32} = \begin{pmatrix} l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \\ 0 \end{pmatrix} = l_{31} u_{12} + l_{32} u_{22}$$

得到

$$l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}} = \frac{1 - \frac{2}{5} \times 1}{\frac{14}{5}} = \frac{3}{14}$$

最后由

$$a_{33} = \begin{pmatrix} l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = l_{31} u_{13} + l_{32} u_{23} + u_{33}$$

Question 3 continued on next page. . .

得到

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 5 - \frac{2}{5} \times 2 - \frac{3}{14} \times (-\frac{7}{5}) = \frac{9}{2}$$

综上得到

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{14}{5} & -\frac{7}{5} \\ 0 & 0 & \frac{9}{2} \end{pmatrix}$$

解方程 $Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix}$$

解方程 $Ux = y$

$$\begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{14}{5} & -\frac{7}{5} \\ 0 & 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{63} \\ \frac{25}{9} \\ \frac{215}{63} \end{pmatrix} \approx \begin{pmatrix} 0.0794 \\ 2.7778 \\ 3.4127 \end{pmatrix}$$