Derivation of Iterative Risk Sensitive Control for Nonlinear Systems with Imperfect Observations

Bilal Hammoud, Armand Jordana, Ludovic Righetti

October 2021

Abstract

This document provides detailed derivations for the theorems presented in the paper "iRiSC: Iterative Risk Sensitive Control for Nonlinear Systems with Imperfect Observations".

1 Problem Definition

The iterative problem is to find an optimal policy that optimizes the expectation of an exponential of a quadratic $\mathbb{E}_\pi\left[e^{-\sigma\delta\mathcal{L}}\right]$ where

$$\delta x_{t+1} = \underbrace{f_t^x \delta x_t + f_t^u \delta u_t + \bar{f}_{t+1}}_{z_t} + \omega_{t+1} \tag{1}$$

$$\delta y_{t+1} = h_t^x \delta x_t + \gamma_{t+1} \tag{2}$$

$$\delta l_t = \frac{1}{2}\begin{bmatrix} 1 \\ \delta x_t \\ \delta u_t \end{bmatrix}^T \begin{bmatrix} 0 & l_t^{x^T} & l_t^{u^T} \\ l_t^x & l_t^{xx} & l_t^{xu} \\ l_t^u & l_t^{ux} & l_t^{uu} \end{bmatrix} \begin{bmatrix} 1 \\ \delta x_t \\ \delta u_t \end{bmatrix} \tag{3}$$

2 Future Stress

2.1 The Optimal Control

Theorem 2.1. *Assume that $\mathcal{F}(x_{t+1})$ is given by.*

$$\mathcal{F}(x_{t+1}) = \frac{1}{2}x_{t+1}^T V_{t+1} x_{t+1} + x_{t+1}^T v_{t+1} + \Delta\bar{v}_{t+1} \tag{4}$$

Then the optimal control is given by

$$\delta u_t^\star = -\underbrace{Q_t^{uu^{-1}} Q_t^u}_{k_t} - \underbrace{Q_t^{uu^{-1}} Q^{ux}}_{K_t} \delta x_t \tag{5}$$

$$\textit{where} \quad Q_t^{uu} = l_t^{uu} + f_t^{u^T} M_t f_t^u \tag{6}$$

$$Q_t^{ux} = l_t^{ux} + f_t^{u^T} M_t f_t^x \tag{7}$$

$$Q_t^u = l_t^u + f_t^{u^T} N_t + f_t^{u^T} M_t \bar{f}_{t+1} \tag{8}$$

Proof. We start by future stress optimization as

$$\mathcal{F}(x_t) = \min_{u_t} \underset{x_{t+1}}{ext}\ l_t + \frac{\sigma^{-1}}{2}(x_{t+1} - z_t)^T \Omega_{t+1}^{-1}(x_{t+1} - z_t) + \frac{1}{2}x_{t+1}^T V_{t+1} x_{t+1} + x_{t+1}^T v_{t+1} + \Delta\bar{v}_{t+1} \tag{9}$$

$$= \min_{u_t}\left(l_t(x_t, u_t) + \frac{\sigma^{-1}}{2} z_t^T \Omega_{t+1}^{-1} z_t + \Delta\bar{v}_{t+1} + \underset{x_{t+1}}{ext}\, g(x_{t+1})\right) \tag{10}$$

Since l_t is only a function of x_t and u_t it can be completely ignored in the integration over x_{t+1} part

$$g(x_{t+1}) = \frac{1}{2} x_{t+1}^T \underbrace{\left(\sigma^{-1}\Omega_{t+1}^{-1} + V_{t+1}\right)}_{W_{t+1}} x_{t+1} + x_{t+1}^T \left(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t\right) \tag{11}$$

$$\tag{12}$$

this is a quadratic form in x_{t+1}; hence, the optimal solution $x_{t+1}^\star$ is given as:

$$x_{t+1}^\star = -W_{t+1}^{-1}(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t) \tag{13}$$

$$g(x_{t+1}^\star) = -\frac{1}{2}(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t)^T W_{t+1}^{-1}(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t) \tag{14}$$

Now, let's write: $q(z_t) = \frac{\sigma^{-1}}{2} z_t^T \Omega_{t+1}^{-1} z_t + \Delta\bar{v}_{t+1} + \underset{x_{t+1}}{ext}\, g(x_{t+1})$ as a quadratic form in z_t:

$$q(z_t) = \frac{\sigma^{-1}}{2} z_t^T \Omega_{t+1}^{-1} z_t + \Delta\bar{v}_{t+1} - \frac{1}{2}(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t)^T W_{t+1}^{-1}(v_{t+1} - \sigma^{-1}\Omega_{t+1}^{-1} z_t) \tag{15}$$

Let's identify M_t, N_t, $\bar{g}_{t+1}$ where:

$$q(z_t) = \frac{1}{2} z_t^T M_t z_t + z_t^T N_t + \bar{g}_{t+1} \tag{16}$$

We find that:

$$M_t = \sigma^{-1}\Omega_{t+1}^{-1} - \sigma^{-2}\Omega_{t+1}^{-1} W_{t+1}^{-1} \Omega_{t+1}^{-1} \tag{17}$$

$$N_t = \sigma^{-1}\Omega_{t+1}^{-1} W_{t+1}^{-1} v_{t+1} \tag{18}$$

$$\bar{g}_{t+1} = -\frac{1}{2} v_{t+1}^T W_{t+1}^{-1} v_{t+1} + \Delta\bar{v}_{t+1} \tag{19}$$

Now recall the matrix inversion lemma: : $A^{-1} - A^{-1}(A^{-1} + B)^{-1}A^{-1} = (A + B^{-1})^{-1}$ and that $W_{t+1} = V_{t+1} + \sigma^{-1}\Omega_{t+1}^{-1}$. Then we can write M_t as

$$M_t = \left(\sigma\Omega_{t+1} + V_{t+1}^{-1}\right)^{-1}$$

We can also play with N_t

$$\begin{aligned} N_t &= \sigma^{-1}\Omega_{t+1}^{-1} \left(V_{t+1} + \sigma^{-1}\Omega_{t+1}^{-1}\right)^{-1} v_{t+1} \\ &= \sigma^{-1}\Omega_{t+1}^{-1} \left[\sigma\Omega_{t+1} - \sigma\Omega_{t+1} \left(\sigma\Omega_{t+1} + V_{t+1}^{-1}\right)^{-1} \sigma\Omega_{t+1}\right] v_{t+1} \\ &= v_{t+1} - \sigma M_t \Omega_{t+1} v_{t+1} \end{aligned}$$

The future stress optimization problem is now reduced to

$$\mathcal{F}(x_t) = \min_{u_t} \quad l_t + \frac{1}{2} z_t^T M_t z_t + z_t^T N_t + \bar{g}_{t+1} \tag{20}$$

we can provide an alternative writing in this case as a quadratic in u_t where

$$\mathcal{F}(x_t) = \min_{u_t} \frac{1}{2} u_t^T Q_t^{uu} u_t + u_t^T Q_t^u + u_t^T Q_t^{ux} x_t + \frac{1}{2} x_t^T Q_t^{xx} x_t + x_t^T Q_t^x + q_t \tag{21}$$

where

$$\bar{q}_t = \bar{g}_{t+1} + \bar{f}_{t+1}^T M_t \bar{f}_{t+1} + \bar{f}_{t+1}^T N_t - \frac{1}{2}\sigma^{-1} \ln |2\pi\sigma^{-1} W_{t+1}^{-1}| \tag{22}$$

$$Q_t^x = l_t^x + f_t^{x^T} M_t \bar{f}_{t+1} + f_t^{x^T} N_t \tag{23}$$

$$Q_t^u = l_t^u + f_t^{u^T} N_t + f_t^{u^T} M_t \bar{f}_{t+1} \tag{24}$$

$$Q_t^{uu} = l_t^{uu} + f_t^{u^T} M_t f_t^u \tag{25}$$

$$Q_t^{ux} = l_t^{ux} + f_t^{u^T} M_t f_t^x \tag{26}$$

$$Q_t^{xx} = l_t^{xx} + f_t^{x^T} M_t f_t^x \tag{27}$$

and the optimal control is given by

$$\delta u_t^\star = -k_t - K_t \delta x_t \tag{28}$$

$$\textit{where} \quad k_t = Q_t^{uu^{-1}} Q_t^u \tag{29}$$

$$K_t = Q_t^{uu^{-1}} Q^{ux} \tag{30}$$

This achieves the control optimization. The term in red, i.e. $-\frac{1}{2}\sigma^{-1} \ln |2\pi\sigma^{-1} W_{t+1}^{-1}|$ is the value of the integration over the random variable ω_{t+1}, keep in mind that the optimization over x_{t+1} is nothing but an integration of an exponential of a quadratic in x_{t+1}. The reader can refer to Appendix A in the paper for a proof. □

2.2 The Future Stress Recursion

Lemma 2.2. *The value function recursion is given by*

$$\mathcal{F}(x_t) = \frac{1}{2} x_t^T V_t x_t + x_t^T v_t + \Delta\bar{v}_t \tag{31}$$

where

$$V_t = Q_t^{xx} + K_t^T Q_t^{uu} K_t - K_t^T Q_t^{ux} - Q_t^{xu} K_t \tag{32}$$

$$v_t = Q_t^x + K_t^T Q_t^{uu} k_t - K_t^T Q_t^u - Q_t^{xu} k_t \tag{33}$$

$$\Delta\bar{v}_t = \bar{q}_t + k_t^T Q_t^u + \frac{1}{2} k_t^T Q_t^{uu} k_t \tag{34}$$

Proof. The terms containing the optimal control can be expanded as

$$\begin{aligned}
&\frac{1}{2}\left(k_t + K_t x_t\right)^T Q_t^{uu} \left(k_t + K_t x_t\right) - \left(k_t + K_t x_t\right)^T Q_t^u - \left(k_t + K_t x_t\right)^T Q_t^{ux} x_t \\
&= \frac{1}{2} k_t Q_t^{uu} k_t + \frac{1}{2} x_t^T K_t^T Q_t^{uu} K_t x_t + k_t^T Q_t^{uu} K_t x_t - \left(k_t + K_t x_t\right)^T Q_t^u - \left(k_t + K_t x_t\right)^T Q_t^{ux} x_t \\
&= \frac{1}{2} x_t^T \left(K_t^T Q_t^{uu} K_t - 2K_t^T Q_t^{ux}\right) x_t + x_t^T \left(K_t^T Q_t^{uu} k_t - K_t^T Q_t^u - Q_t^{xu} k_t\right) + \frac{1}{2} k_t Q_t^{uu} k_t - k_t^T Q_t^u
\end{aligned} \tag{35}$$

which concludes the proofs for the control recursions. !! □

3 Past Stress & Estimation

Since the cost is approximated as a quadratic, and the discrepancy parameters d_t are quadratic in the random variables, then the past stress will be a quadratic of the form stated below.

$$\sigma\mathcal{P}(x_t, W_t) = \frac{1}{2}\left(x_t - \hat{x}_t\right)^T P_t^{-1} \left(x_t - \hat{x}_t\right) + \dots \tag{36}$$

where $\hat{x}_t$ is some cost biased Kalman filter estimate which will appear later, this cost biasing reflects pessimism and optimism in estimating the state depending on the task/cost desired to optimize. Also " ... " resembles the terms independent of x_t. In order to obtain P_t and $\hat{x}_t$ we have to solve the forward recursion given by

$$\sigma \mathcal{P}\left(x_t, W_t\right) = \underset{x_{t-1}}{ext} \quad d_t + \sigma l_{t-1} + \sigma \mathcal{P}\left(x_{t-1}, W_{t-1}\right) \tag{37}$$

For now we will follow Whittle's logic by first creating a perfect square in the cost terms etc

$$\begin{aligned}
\sigma l_{t-1} + \sigma \mathcal{P}\left(x_{t-1}\right) &= \frac{1}{2}\sigma \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix}^T \begin{bmatrix} l^{xx}_{t-1} & l^{ux^T}_{t-1} \\ l^{ux}_{t-1} & l^{uu}_{t-1} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \sigma \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix}^T \begin{bmatrix} l^{x}_{t-1} \\ l^{u}_{t-1} \end{bmatrix} + \frac{1}{2}\left(x_{t-1} - \hat{x}_{t-1}\right)^T P^{-1}_{t-1}\left(x_{t-1} - \hat{x}_{t-1}\right) + \dots \\
&= \frac{1}{2} x_T \left(P^{-1}_{t-1} + \sigma l^{xx}_{t-1}\right) x_{t-1} - \left(P^{-1}_{t-1}\hat{x}_{t-1} - \sigma l^{ux^T}_{t-1} u_{t-1} - \sigma l^{x}_{t-1}\right)^T x_{t-1} \qquad (38)
\end{aligned}$$

If we define:

$$\tilde{P}_{t-1} = \left(P^{-1}_{t-1} + \sigma l^{xx}_{t-1}\right)^{-1} \tag{39}$$

$$\tilde{x}_{t-1} = \tilde{P}_{t-1}\left(P^{-1}_{t-1}\hat{x}_{t-1} - \sigma l^{ux^T}_{t-1} u_{t-1} - \sigma l^{x}_{t-1}\right) \tag{40}$$

We find that:

$$\sigma l_{t-1} + \sigma \mathcal{P}\left(x_{t-1}\right) = \frac{1}{2}\left(x_{t-1} - \tilde{x}_{t-1}\right)^T \tilde{P}^{-1}_{t-1}\left(x_{t-1} - \tilde{x}_{t-1}\right) \tag{41}$$

ok the stress optimization now takes the form

$$\sigma \mathcal{P}\left(x_t, W_t\right) = \underset{x_{t-1}}{ext} \quad d_t + \frac{1}{2}\left(x_{t-1} - \tilde{x}_{t-1}\right)^T \tilde{P}^{-1}_{t-1}\left(x_{t-1} - \tilde{x}_{t-1}\right) + \dots \tag{42}$$

Theorem 3.1 (Whittle)**.** *The solution to the recursion:*

$$\frac{1}{2}\left(x_t - \hat{x}_t\right)^T D^{-1}_t \left(x_{t-1} - \hat{x}_t\right) = \underset{x_{t-1}}{ext} \quad d_t + \frac{1}{2}\left(x_{t-1} - \hat{x}_{t-1}\right)^T D^{-1}_{t-1}\left(x_{t-1} - \hat{x}_{t-1}\right) + \dots \tag{43}$$

is:

$$\hat{x}_{t+1} = f^x_t \hat{x}_t + f^u_t u_t + \bar{f}_{t+1} + G_t\left(y_{t+1} - H_t\hat{x}_t\right) \tag{44}$$

$$G_t = f^x_t\left(H^T_t \Gamma^{-1}_{t+1} H_t + D^{-1}_t\right)^{-1} H^T_t \Gamma^{-1}_{t+1} \tag{45}$$

$$D_{t+1} = \Omega_{t+1} + f^x_t\left(H^T_t \Gamma^{-1}_{t+1} H_t + D^{-1}_t\right)^{-1} f^{x^T}_t \tag{46}$$

And this is suppose to be equivalent to the Kalman filter.

We use this theorem with $\hat{x}_{t-1}$ replaced with $\tilde{x}_{t-1}$ and D_{t-1} replaced with $\tilde{P}_{t-1}$. The Transformed Kalman Gain and curvature then takes the form

$$G_t = f^x_t\left(P^{-1}_t + H^T_t \Gamma^{-1}_{t+1} H_t + \sigma l^{xx}_t\right)^{-1} H^T_t \Gamma^{-1}_{t+1} \tag{47}$$

$$P_{t+1} = \Omega_{t+1} + f^x_t\left(P^{-1}_t + H^T_t \Gamma^{-1}_{t+1} H_t + \sigma l^{xx}_t\right)^{-1} f^{x^T}_t \tag{48}$$

the state prediction can be written as

$$\begin{aligned}
\hat{x}_{t+1} &= f^x_t \tilde{x}_t + f^u_t u_t + \bar{f}_{t+1} + G_t\left(y_{t+1} - H_t\tilde{x}_t\right) & (49)\\
&= f^x_t \tilde{x}_t + f^u_t u_t + \bar{f}_{t+1} + G_t\left(y_{t+1} - H_t\tilde{x}_t\right) + f^x_t\hat{x}_t - f^x_t\hat{x}_t + G_t H_t\hat{x}_t - G_t H_t\hat{x}_t & (50)\\
&= f^x_t \hat{x}_t + f^u_t u_t + \bar{f}_{t+1} + G_t\left(y_{t+1} - H_t\hat{x}_t\right) + \left(f^x_t - G_t H_t\right)\left(\tilde{x}_t - \hat{x}_t\right) & (51)
\end{aligned}$$

Using that: $I-(U+V)^{-1}U=(U+V)^{-1}V$ with $U=H_t^T\Gamma_{t+1}^{-1}H_t$ and $V=P_t^{-1}+\sigma l_t^{xx}$

$$\begin{aligned}
A_t - G_t H_t &= A_t - A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1} H_t^T\Gamma_{t+1}^{-1}H_t && (52)\\
&= A_t\left(I-\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1} H_t^T\Gamma_{t+1}^{-1}H_t\right) && (53)\\
&= A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1}\left(P_t^{-1}+\sigma l_t^{xx}\right) && (54)
\end{aligned}$$

Hence:

$$\begin{aligned}
\left(A_t - G_t H_t\right)\left(\tilde{x}_t-\hat{x}_t\right) &= A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1}\left(P_t^{-1}\hat{x}_t-\sigma l_t^{ux^T}u_t-\sigma l_t^x\right) && (55)\\
&\quad - A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1}\left(P_t^{-1}+\sigma l_t^{xx}\right)\hat{x}_t && (56)
\end{aligned}$$

$$\left(A_t - G_t H_t\right)\left(\tilde{x}_t-\hat{x}_t\right) = -\sigma A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1}\left(l_t^{xx}\hat{x}_t+l_t^{ux^T}u_t+l_t^x\right) \quad (57)$$

Finally,

$$\hat{x}_{t+1}=A_t\hat{x}_t+f_t^u u_t+\bar{f}_{t+1}+G_t\left(y_{t+1}-H_t\hat{x}_t\right)-\sigma A_t\left(P_t^{-1}+H_t^T\Gamma_{t+1}^{-1}H_t+\sigma l_t^{xx}\right)^{-1}\left(l_t^{xx}\hat{x}_t+l_t^{ux^T}u_t+l_t^x\right) \quad (58)$$

4 Approximation of the risk sensitive cost

Once the optimal control updates are obtained, we desire to evaluate the expectation of the nonlinear cost, since this is general intractable, we evaluate the $\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right]$ subject to

$$\delta x_{t+1} = \underbrace{f_t^x \delta x_t + \bar{f}_{t+1}}_{z_t} + \omega_{t+1} \tag{59}$$

$$\delta x_0 = x_0 - \hat{x}_0 \tag{60}$$

with:

$$\mathcal{L} = \sum_{t=0}^{T} \underbrace{l(x_t^n, u_t)}_{\bar{l}_t} + \underbrace{\frac{1}{2}\delta x_t^T l_t^{xx} \delta x_t + l_t^x \delta x_t}_{\delta l_t} \tag{61}$$

we know that:

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\kappa}\int \exp\left(-\sigma\sum_{t=0}^{T}\bar{l}_t + \delta l_t - \sum_{t=0}^{T} d_t\right) dx_{0:T} \tag{62}$$

$$\begin{aligned} \textit{where} \quad d_0 &= \frac{1}{2}\delta x_0^T \chi_0^{-1}\delta x_0 \\ d_t &= \frac{1}{2}\omega_t^T \Omega_t^{-1}\omega_t \\ \kappa &= |2\pi\chi_0|^{\frac{1}{2}} \prod_{t=1}^{T} |2\pi\Omega_t|^{\frac{1}{2}} \end{aligned}$$

Theorem 4.1.

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\kappa}\int \exp\left(-\sigma V_t(\delta x_t) - \sigma\sum_{k=0}^{t-1}\bar{l}_k + \delta l_k - \sum_{k=0}^{t} d_k\right) d\delta x_{0:t} \tag{63}$$

where $V_t(x_t) = \frac{1}{2}\delta x_t^T V_t \delta x_t + \delta x_t^T v_t + \bar{v}_t$ *with:*

$$V_T = l_T^{xx} \tag{64}$$

$$v_T = l_T^x \tag{65}$$

$$\bar{v}_T = \bar{l}_T \tag{66}$$

where:

$$V_t = l_{t+1}^{xx} + ... \tag{67}$$

$$v_t = l_{t+1}^x + ... \tag{68}$$

$$\bar{v}_t = \bar{l}_{t+1} + ... \tag{69}$$

Proof. We prove this by induction. The initialization is trivial. Let's assume the property true at time $t+1$:

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\kappa}\int \exp\left(-\sigma V_{t+1}(\delta x_{t+1}) - \sigma\sum_{k=0}^{t}\bar{l}_k + \delta l_k - \sum_{k=0}^{t+1} d_k\right) d\delta x_{0:t+1} \tag{70}$$

$$= \frac{1}{\kappa}\int \exp\left(-\sigma\sum_{k=0}^{t-1}\bar{l}_k + \delta l_k - \sum_{k=0}^{t} d_k\right) \exp\left(-\sigma(\bar{l}_t + \delta l_t)\right) \mathcal{G}_t d\delta x_{0:t} \tag{71}$$

$$\tag{72}$$

where:

$$\mathcal{G}_t = \int \exp\left(-\sigma V_{t+1}(\delta x_{t+1}) - d_{t+1}\right) d\delta x_{t+1} \tag{73}$$

$$\tag{74}$$

We have:

$$\begin{aligned} V_{t+1}(\delta x_{t+1}) + \sigma^{-1} d_{t+1} &= \frac{1}{2}\delta x_{t+1}^T V_{t+1}\delta x_{t+1} + \delta x_{t+1}^T v_{t+1} + \bar{v}_{t+1} + \frac{1}{2}\sigma^{-1}(\delta x_{t+1} - z_t)^T \Omega_{t+1}^{-1}(\delta x_{t+1} - z_t) \\ &= \frac{1}{2}(\delta x_{t+1}^T - \delta x_{t+1}^\star)^T W_{t+1}(\delta x_{t+1}^T - \delta x_{t+1}^\star) + c_{t+1} \end{aligned}$$

where:

$$W_{t+1} = V_{t+1} + \sigma^{-1}\Omega_{t+1}^{-1} \tag{75}$$

$$\delta x_{t+1}^\star = W_{t+1}^{-1}(\sigma^{-1}\Omega_{t+1}^{-1} z_t - v_{t+1}) \tag{76}$$

$$c_{t+1} = \bar{v}_{t+1} + \frac{1}{2}\sigma^{-2} z_t^T \Omega_{t+1}^{-1} z_t - \frac{1}{2}\delta x_{t+1}^\star{}^T W_{t+1}\delta x_{t+1}^\star \tag{77}$$

where:

$$\begin{aligned} \delta x_{t+1}^\star{}^T W_{t+1}\delta x_{t+1}^\star &= (\sigma^{-1}\Omega_{t+1}^{-1} z_t - v_{t+1})^T W_{t+1}^{-1}(\sigma^{-1}\Omega_{t+1}^{-1} z_t - v_{t+1}) \\ &= \sigma^{-2} z_t^T \Omega_{t+1}^{-1} W_{t+1}^{-1}\Omega_{t+1}^{-1} z_t + v_{t+1}^T W_{t+1}^{-1} v_{t+1} - 2\sigma^{-1} v_{t+1}^T W_{t+1}^{-1}\Omega_{t+1}^{-1} z_t \end{aligned}$$

We define :

$$M_t = \left(\sigma\Omega_{t+1} + V_{t+1}^{-1}\right)^{-1}$$

and thanks to the inversion lemma, we have:

$$c_{t+1} = \bar{v}_{t+1} - \frac{1}{2} v_{t+1}^T W_{t+1}^{-1} v_{t+1} + \sigma^{-1} v_{t+1}^T W_{t+1}^{-1}\Omega_{t+1}^{-1} z_t + \frac{1}{2} z_t^T M_t z_t$$

In the end, we have:

$$V_t = l_t^{xx} + f_t^x{}^T M_t f_t^x \tag{78}$$

$$v_t = l_t^x + f_t^x{}^T M_t \bar{f}_{t+1} + \sigma f_t^x{}^T \Omega_{t+1}^{-1} W_{t+1}^{-1} v_{t+1} \tag{79}$$

$$\bar{v}_t = \bar{l}_t + \bar{v}_{t+1} - \frac{1}{2} v_{t+1}^T W_{t+1}^{-1} v_{t+1} + v_{t+1}^T W_{t+1}^{-1}\Omega_{t+1}^{-1}\bar{f}_{t+1} + \frac{1}{2}\bar{f}_{t+1}^T M_t \bar{f}_{t+1} - \frac{1}{2}\sigma^{-1}\ln|2\pi\sigma^{-1}W_{t+1}^{-1}| \tag{80}$$

We notice that:

$$\begin{aligned} \sigma^{-1}\Omega_{t+1}^{-1} W_{t+1}^{-1} v_{t+1} &= \sigma^{-1}\Omega_{t+1}^{-1}\left[\sigma\Omega_{t+1} - \sigma\Omega_{t+1}\left(\sigma\Omega_{t+1} + V_{t+1}^{-1}\right)^{-1}\sigma\Omega_{t+1}\right] v_{t+1} \\ &= v_{t+1} - \sigma M_t \Omega_{t+1} v_{t+1} = N_t \end{aligned}$$

Therefore:

$$V_t = l_t^{xx} + f_t^x{}^T M_t f_t^x \tag{81}$$

$$v_t = l_t^x + f_t^x{}^T M_t \bar{f}_{t+1} + f_t^x{}^T N_t \tag{82}$$

$$\bar{v}_t = \bar{l}_t + \bar{v}_{t+1} - \frac{1}{2} v_{t+1}^T W_{t+1}^{-1} v_{t+1} + \bar{f}_{t+1}^T N_t + \frac{1}{2}\bar{f}_{t+1}^T M_t \bar{f}_{t+1} - \frac{1}{2}\sigma^{-1}\ln|2\pi\sigma^{-1}W_{t+1}^{-1}| \tag{83}$$

□

Theorem 4.2.

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\kappa}\exp\left(-\sigma\left(\bar{v}_0 - \frac{1}{2}v_0^T W^{-1} v_0 - \frac{1}{2}\sigma^{-1}\ln|2\pi\sigma^{-1}W^{-1}|\right)\right) \tag{84}$$

where $W = (V_0 + \sigma^{-1}\chi_0^{-1})$

Proof.

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\kappa}\int \exp\left(-\sigma\left(\frac{1}{2}\delta x_0^T V_0 \delta x_0 + \delta x_0^T v_0 + \bar{v}_0\right) - \frac{1}{2}\delta x_0^T \chi_0^{-1}\delta x_0\right) d\delta x_0 \tag{85}$$

$$\frac{1}{2}\delta x_0^T V_0 \delta x_0 + \delta x_0^T v_0 + \bar{v}_0 + \frac{1}{2}\sigma^{-1}\delta x_0^T \chi_0^{-1}\delta x_0 = \frac{1}{2}(\delta x_0 - x_\star)^T W(\delta x_0 - x_\star) + c \tag{86}$$

where

$$W = (V_0 + \sigma^{-1}\chi_0^{-1}) \tag{87}$$

$$x_\star = -W^{-1}v_0 \tag{88}$$

$$c = \bar{v}_0 - \frac{1}{2}v_0^T W^{-1} v_0 \tag{89}$$

□

Theorem 4.3. *This can be alternatively written as:*

$$\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right] = \frac{1}{\alpha}\exp\left(-\sigma(\bar{v}_0 - \frac{1}{2}v_0^T W^{-1} v_0)\right) \tag{90}$$

where:

$$V_t = l_t^{xx} + f_t^{x\,T} M_t f_t^x \tag{91}$$

$$v_t = l_t^x + f_t^{x\,T} M_t \bar{f}_{t+1} + f_t^{x\,T} N_t \tag{92}$$

$$\bar{v}_t = \bar{l}_t + \bar{v}_{t+1} - \frac{1}{2}v_{t+1}^T W_{t+1}^{-1} v_{t+1} + \bar{f}_{t+1}^T N_t + \frac{1}{2}\bar{f}_{t+1}^T M_t \bar{f}_{t+1} \tag{93}$$

and where $W = (V_0 + \sigma^{-1}\chi_0^{-1})$.
with

$$\alpha = |I + \sigma V_0 \chi_0|^{\frac{1}{2}} \ \Pi_{t=1}^T |I + \sigma V_t \Omega_t|^{\frac{1}{2}} \tag{94}$$

Proof. We notice that:

$$\frac{|2\pi\sigma^{-1}W_t^{-1}|}{|2\pi\Omega_t|} = \frac{|\sigma^{-1}(V_t + \sigma^{-1}\Omega_t^{-1})^{-1}|}{|\Omega_t|} \tag{95}$$

$$= \frac{1}{|\sigma(V_t + \sigma^{-1}\Omega_t^{-1})\Omega_t|} \tag{96}$$

$$= \frac{1}{|I + \sigma V_t \Omega_t|} \tag{97}$$

□

Theorem 4.4.

$$\mathcal{J} = -\sigma^{-1}\ln\left(\mathbb{E}\left[e^{-\sigma\mathcal{L}}|u_{0:T}\right]\right) = -\sigma^{-1}\ln(\frac{1}{\alpha}) + \bar{v}_0 - \frac{1}{2}v_0^T W^{-1} v_0 \tag{98}$$

$$= \sigma^{-1}\ln(\alpha) + \bar{v}_0 - \frac{1}{2}v_0^T W^{-1} v_0 \tag{99}$$

5 Future Expectation of Observations Gives a Prediction

In section 4 of the paper, we claim that the expectation over future observations yields a prediction of the form $\delta t_{t+1} = h_t^x \delta x_t$, in what follows we provide an alternative proof to what Whittle presented, our proof uses induction to prove this recursively on time step at a time.

Theorem 5.1.

$$\begin{aligned}&\mathbb{E}_{\pi^\star}\left[\exp\left(-\sigma\mathcal{L}\right)|W_t\right]p(W_t)\\&=\int\exp\left(-\sigma F(x_t)-\sigma\mathcal{Z}_t\right)dx_{0:t}\end{aligned}\tag{100}$$

$$F(x_t) = \frac{1}{2}x_tV_tx_t + v_t^Tx_t + \bar{v}_t \tag{101}$$

$$\mathcal{Z}_t = \sigma^{-1}\sum_{k=0}^{t} d_k + \sum_{k=0}^{t-1} l_k \tag{102}$$

And terminal condition:

$$V_T = l_T^{xx} \tag{103}$$

$$v_T = l_T^x \tag{104}$$

$$\bar{v}_T = \sigma^{-1}\ln\kappa \tag{105}$$

Proof. Let's show by backward induction that $\forall h \geq t$, we have:

$$\begin{aligned}&\mathbb{E}_{\pi^\star}\left[\exp\left(-\sigma\mathcal{L}\right)|W_t\right]p(W_t)\\&=\min_{u_{t:h-1}}\int\exp\left(-\sigma F_h(x_h)-\sigma\mathcal{Z}_h\right)dx_{0:h}dy_{t+1:h}\end{aligned}\tag{106}$$

Initialization

$$\begin{aligned}&\mathbb{E}_{\pi^\star}\left[\exp\left(-\sigma\mathcal{L}\right)|W_t\right]p(W_t)\\&=p(W_t)\min_{u_{t:T-1}}\int\exp\left(-\sigma\mathcal{L}\right)p(x_{0:T},y_{t+1:T}|W_t)dx_{0:T}dy_{t+1:T}\\&=\min_{u_{t:T-1}}\int\exp\left(-\sigma\mathcal{L}\right)p(x_{0:T},y_{t+1:T},W_t)dx_{0:T}dy_{t+1:T}\\&=\min_{u_{t:T-1}}\int\frac{1}{\kappa}\exp\left(-\sigma\sum_{k=0}^{T}(l_k+\sigma^{-1}d_k)\right)dx_{0:T}dy_{t+1:T}\\&=\min_{u_{t:T-1}}\int\exp\left(-\sigma(\mathcal{Z}_T+l_T(x_T)+\bar{v}_T)\right)dx_{0:T}dy_{t+1:T}\\&=\min_{u_{t:T-1}}\int\exp\left(-\sigma(F_T(x_T)+\mathcal{Z}_T)\right)dx_{0:T}dy_{t+1:T}\end{aligned}\tag{107}$$

Now, assuming the property true for $h+1$, let's show the property at time h:

$$\begin{aligned}&\mathbb{E}_{\pi^\star}\left[\exp\left(-\sigma\mathcal{L}\right)|W_t\right]p(W_t)\\&=\min_{u_{t:h}}\int e^{-\sigma(F_{h+1}(x_{h+1})+\mathcal{Z}_{h+1})}dx_{0:h+1}dy_{t+1:h+1}\\&=\min_{u_{t:h-1}}\int\mathcal{G}_h dx_{0:h}dy_{t+1:h}\end{aligned}\tag{108}$$

$$\begin{aligned}\mathcal{G}_h &= \min_{u_h} \int \exp\left(-\sigma(F_{h+1}(x_{h+1}) + \mathcal{Z}_{h+1})\right) dx_{h+1} dy_{h+1} \\ &= \exp\left(-\sigma \mathcal{Z}_h\right) \min_{u_h} \int \exp\left(-G\right) d\omega_{h+1} d\gamma_{h+1}\end{aligned}$$

where

$$\begin{aligned}G =& \frac{1}{2}\sigma x_{h+1}^T V_{h+1} x_{h+1} + \sigma v_{h+1}^T x_{h+1} + \sigma \bar{v}_{h+1} \\ &+ \frac{1}{2}\omega_{h+1}^T \Omega_{h+1}^{-1} \omega_{h+1} + \frac{1}{2}\gamma_{h+1}^T \Gamma_{h+1}^{-1} \gamma_{h+1} + \sigma l_h(x_h, u_h)\end{aligned} \tag{109}$$

$\frac{1}{2}\gamma_{h+1}^T \Gamma_{h+1}^{-1} \gamma_{h+1}$ is a perfect square so we can integrate other γ_{h+1}:

$$\mathcal{G}_h = \exp\left(-\sigma \mathcal{P}_h + \ln(|2\pi \Gamma_{h+1}|)\right) \min_{u_h} \int \exp\left(-\tilde{G}\right) d\omega_{h+1}$$

Let's use that:

$$x_{h+1} = \underbrace{f_h^x x_h + f_h^u u_h + \bar{f}_{h+1}}_{z_h} + \omega_{h+1} \tag{110}$$

We have that:

$$\begin{aligned}\tilde{G} =& \frac{1}{2}\sigma(z_h + w_{h+1})^T V_{h+1}(z_h + w_{h+1}) + \sigma v_{h+1}^T (z_h + w_{h+1}) \\ &+ \sigma \bar{v}_{h+1} + \frac{1}{2} w_{h+1}^T \Omega_{h+1}^{-1} w_{h+1} + \sigma l_h(x_h, u_h)\end{aligned} \tag{111}$$

Next step is to create a perfect square of the form $S(\omega_{h+1}) = \frac{1}{2}\left(\omega_{h+1} - \omega_{h+1}^*\right)^T W_{h+1}^{-1}\left(\omega_{h+1} - \omega_{h+1}^*\right)$, clearly this leads to

$$W_{h+1}^{-1} = \Omega_{h+1}^{-1} + \sigma V_{h+1} \tag{112}$$

$$\omega_{h+1}^T W_{h+1}^{-1} \omega_{h+1}^* = -\sigma w_{h+1}^T (V_{h+1} z_h + v_{h+1}) \tag{113}$$

$$\omega_{h+1}^* = -\sigma W_{h+1}\left(V_{h+1} z_h + v_{h+1}\right) \tag{114}$$

Now the integral we need to evaluate is of the form

$$\int_{-\infty}^{+\infty} \exp\left(-\sigma c_{h+1} - S(\omega_{h+1})\right) d\omega_{h+1} \tag{115}$$

$$\begin{aligned}c_{h+1} =& \frac{1}{2} z_h^T \left(V_{h+1} - \sigma V_{h+1}^T W_{h+1} V_{h+1}\right) z_h \\ &+ z_h^T \left(v_{h+1} - \sigma V_{h+1}^T W_{h+1} v_{h+1}\right) \\ &+ \bar{v}_{h+1} - \frac{\sigma}{2} v_{h+1}^T W_{h+1} v_{h+1} + l_h(x_h, u_h)\end{aligned} \tag{116}$$

In the end, we find that:

$$\mathcal{G}_h = \exp\left(-\sigma \mathcal{P}_h - \underset{u_t}{ext}\ \sigma \mathcal{Q}_t\right) \tag{117}$$

$$\mathcal{Q}_t = \frac{1}{2}\begin{bmatrix}1 \\ x_h \\ u_h\end{bmatrix}^T \begin{bmatrix}\bar{q} & Q_x^T & Q_u^T \\ Q_x & Q_{xx} & Q_{ux}^T \\ Q_u & Q_{ux} & Q_{uu}\end{bmatrix}\begin{bmatrix}1 \\ x_h \\ u_h\end{bmatrix} \tag{118}$$

if we define $M_{h+1} = I - \sigma V_{h+1}^T W_{h+1}$ then we can further simplify this by writing

$$Q_x = l_h^x + f_h^{x^T} M_{h+1} v_{h+1} + f_h^{x^T} M_{h+1} V_{h+1} \bar{f}_{h+1} \tag{119a}$$
$$Q_u = l_h^u + f_h^{u^T} M_{h+1} v_{h+1} + f_h^{u^T} M_{h+1} V_{h+1} \bar{f}_{h+1} \tag{119b}$$
$$Q_{ux} = l_h^{ux} + f_h^{u^T} M_{h+1} V_{h+1} f_h^x \tag{119c}$$
$$Q_{xx} = l_h^{xx} + f_h^{x^T} M_{h+1} V_{h+1} f_h^x \tag{119d}$$
$$Q_{uu} = l_h^{uu} + f_h^{u^T} M_{h+1} V_{h+1} f_h^u \tag{119e}$$

where:

$$\begin{aligned}\bar{q} = {} & \bar{f}_{h+1}^T M_{h+1} V_{h+1} \bar{f}_{h+1} + 2 \bar{f}_{h+1}^T M_{h+1} v_{h+1} \\ & + 2\bar{v}_{h+1} - v_{h+1}^T W_{h+1} v_{h+1} \\ & - 2\sigma^{-1} \ln \sqrt{|2\pi W_{h+1}|} - 2\sigma^{-1} \ln \sqrt{|2\pi \gamma_{h+1}|}\end{aligned} \tag{120}$$

The optimal control minimizing the Hamiltonian is then given by

$$u_h^* = \arg\min_{u_h} Q = \underbrace{-Q_{uu}^{-1} Q_u}_{k_h} \underbrace{-Q_{uu}^{-1} Q_{ux}}_{K_h} x_h \tag{121}$$

and we have:

$$V_h = Q_{xx} + Q_{ux}^T Q_{uu}^{-1} Q_{ux} \tag{122a}$$
$$v_h = Q_x - Q_{ux}^T Q_{uu}^{-1} Q_u \tag{122b}$$
$$\bar{v}_h = \bar{q} + Q_u^T Q_{uu}^{-1} Q_u \tag{122c}$$

This completes the proof of the induction. Property (106) is hence true for all $h \geq t$. $h = t$ proves the theorem. □

Theorem 5.2.

$$\begin{aligned}& \mathbb{E}_{\pi^\star} \left[\exp\left(-\sigma \mathcal{L}\right)\right] \\ & \quad = - \int_{x_t} \exp\left(-\sigma(F(x_t) + P(x_t, W_t))\right) dx_t\end{aligned} \tag{123}$$

where:

$$P(x_t, W_t) = \sigma^{-1} (x_t - \hat{x}_t)^T P_t (x_t - \hat{x}_t) + \bar{p}_t \tag{124}$$

Where:

$$\hat{x}_t = \textit{Risk sensitive EKF} \tag{125}$$

Proof. In this proof, we do not compute constants, therefore, we use the symbol $\propto$ to denote proportionality. Let's show by induction that for all $h \leq t$:

$$\begin{aligned}& \mathbb{E}_{\pi^\star} \left[\exp\left(-\sigma \mathcal{L}\right) | W_t\right] \\ & \quad = - \int \exp\left(-\sigma(F(x_t) + P(x_h, W_h) + \mathcal{R}_h)\right) dx_{h:t}\end{aligned} \tag{126}$$

where:

$$\mathcal{R}_h = \sigma^{-1} \sum_{k=h}^{t} \frac{1}{2} d_k + \sum_{k=h}^{t-1} l_k \tag{127}$$

Initialisation
The property is true for $t = 0$ if we take $P_0 = \chi$ and $\hat{x}_0 = W_0$
Now, assuming the property true for $h-1$, let's show the property at time h:

$$\begin{aligned}
&\mathbb{E}_{\pi^\star}\left[\exp\left(-\sigma\mathcal{L}\right)|W_t\right] \\
&\quad = -\int \exp\left(-\sigma(F(x_t) + P(x_{h-1}, W_{h-1}) + \mathcal{R}_{h-1})\right) dx_{h-1:t} && (128)\\
&\quad = -\int \exp\left(-\sigma(F(x_t) + \mathcal{R}_h)\right) \mathcal{A}_h dx_{h:t} && (129)
\end{aligned}$$

where:

$$\mathcal{A}_h = \int \exp\left(-\sigma(P(x_{h-1}, W_{h-1}) + l_{h-1} + \sigma^{-1} d_h)\right) dx_{h-1} \tag{130}$$

This completes the proof of the induction. Property (126) is hence true for all $h \leq t$. $h = t$ proves the theorem.

$\square$