

## App. A: Sequences and difference equations

Hans Petter Langtangen<sup>1,2</sup>

Simula Research Laboratory<sup>1</sup>  
University of Oslo, Dept. of Informatics<sup>2</sup>

Sep 24, 2014

## Sequences

*Sequences* is a central topic in mathematics:

$$x_0, x_1, x_2, \dots, x_n, \dots,$$

Example: all odd numbers

$$1, 3, 5, 7, \dots, 2n+1, \dots$$

For this sequence we have a formula for the  $n$ -th term:

$$x_n = 2n + 1$$

and we can write the sequence more compactly as

$$(x_n)_{n=0}^{\infty}, \quad x_n = 2n + 1$$

## Other examples of sequences

$$1, 4, 9, 16, 25, \dots \quad (x_n)_{n=0}^{\infty}, \quad x_n = n^2$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \quad (x_n)_{n=0}^{\infty}, \quad x_n = \frac{1}{n+1}$$

$$1, 1, 2, 6, 24, \dots \quad (x_n)_{n=0}^{\infty}, \quad x_n = n!$$

$$1, 1+x, 1+x+\frac{1}{2}x^2, 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3, \dots \quad (x_n)_{n=0}^{\infty}, \quad x_n = \sum_{j=0}^n \frac{x^j}{j!}$$

## Finite and infinite sequences

- Infinite sequences have an infinite number of terms ( $n \rightarrow \infty$ )
- In mathematics, infinite sequences are widely used
- In real-life applications, sequences are usually finite:  $(x_n)_{n=0}^N$
- Example: number of approved exercises every week in INF1100  
 $x_0, x_1, x_2, \dots, x_{15}$
- Example: the annual value of a loan  
 $x_0, x_1, \dots, x_{20}$

## Difference equations

- For sequences occurring in modeling of real-world phenomena, there is seldom a formula for the  $n$ -th term
- However, we can often set up one or more equations governing the sequence
- Such equations are called difference equations
- With a computer it is then very easy to generate the sequence by solving the difference equations
- Difference equations have lots of applications and are very easy to solve on a computer, but often complicated or impossible to solve for  $x_n$  (as a formula) by pen and paper!
- The programs require only loops and arrays

## Modeling interest rates

### Problem:

Put  $x_0$  money in a bank at year 0. What is the value after  $N$  years if the interest rate is  $p$  percent per year?

### Solution:

The fundamental information relates the value at year  $n$ ,  $x_n$ , to the value of the previous year,  $x_{n-1}$ :

$$x_n = x_{n-1} + \frac{p}{100} x_{n-1}$$

How to solve for  $x_n$ ? Start with  $x_0$ , compute  $x_1, x_2, \dots$

## Simulating the difference equation for interest rates

### What does it mean to simulate?

Solve math equations by repeating a simple procedure (relation) many times (boring, but well suited for a computer!)

### Program for $x_n = x_{n-1} + (p/100)x_{n-1}$ :

```
from scipytools.std import *
x0 = 100                                # initial amount
p = 5                                   # interest rate
N = 4                                   # number of years
index_set = range(N+1)
x = zeros(len(index_set))

# Solution:
x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (p/100.0)*x[n-1]
print x
plot(index_set, x, 'ro', xlabel='years', ylabel='amount')
```

We do not need to store the entire sequence, but it is convenient for programming and later plotting

- Previous program stores all the  $x_n$  values in a NumPy array
- To compute  $x_n$ , we only need one previous value,  $x_{n-1}$

Thus, we could only store the two last values in memory:

```
x_old = x0
for n in index_set[1:]:
    x_new = x_old + (p/100.0)*x_old
    x_old = x_new # x_new becomes x_old at next step
```

However, programming with an array  $x[n]$  is simpler, safer, and enables plotting the sequence, so we will continue to use arrays in the examples

## Daily interest rate

- A more relevant model is to add the interest every day
- The interest rate per day is  $r = p/D$  if  $p$  is the annual interest rate and  $D$  is the number of days in a year
- A common model in business applies  $D = 360$ , but  $n$  counts exact (all) days

Just a minor change in the model:

$$x_n = x_{n-1} + \frac{r}{100}x_{n-1}$$

How can we find the number of days between two dates?

```
>>> import datetime
>>> date1 = datetime.date(2007, 8, 3) # Aug 3, 2007
>>> date2 = datetime.date(2008, 8, 4) # Aug 4, 2008
>>> diff = date2 - date1
>>> print diff.days
367
```

## Program for daily interest rate

```
from scipytools.std import *
x0 = 100                                # initial amount
p = 5                                   # annual interest rate
r = p/360.0                             # daily interest rate
import datetime
date1 = datetime.date(2007, 8, 3)
date2 = datetime.date(2011, 8, 3)
diff = date2 - date1
N = diff.days
index_set = range(N+1)
x = zeros(len(index_set))

# Solution:
x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (r/100.0)*x[n-1]
print x
plot(index_set, x, 'ro', xlabel='days', ylabel='amount')
```

## But the annual interest rate may change quite often...

### Varying $p$ means $p_n$ :

- Could not be handled in school (cannot apply  $x_n = x_0(1 + \frac{p}{100})^n$ )
- A varying  $p$  causes no problems in the program: just fill an array  $p$  with correct interest rate for day  $n$

Modified program:

```
p = zeros(len(index_set))
# fill p[n] for n in index_set (might be non-trivial...)

r = p/360.0
x = zeros(len(index_set)) # daily interest rate

x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (r[n-1]/100.0)*x[n-1]
```

## Payback of a loan

- A loan  $L$  is paid back with a fixed amount  $L/N$  every month over  $N$  months + the interest rate of the loan
- $p$ : annual interest rate,  $p/12$ : monthly rate
- Let  $x_n$  be the value of the loan at the end of month  $n$

The fundamental relation from one month to the next:

$$x_n = x_{n-1} + \frac{p}{12 \cdot 100}x_{n-1} - \left(\frac{p}{12 \cdot 100}x_{n-1} + \frac{L}{N}\right)$$

which simplifies to

$$x_n = x_{n-1} - \frac{L}{N}$$

( $L/N$  makes the equation *nonhomogeneous*)

### How to make a living from a fortune with constant consumption

- We have a fortune  $F$  invested with an annual interest rate of  $p$  percent
- Every year we plan to consume an amount  $c_n$  ( $n$  counts years)
- Let  $x_n$  be our fortune at year  $n$

A fundamental relation from one year to the other is

$$x_n = x_{n-1} + \frac{p}{100}x_{n-1} - c_n$$

Simplest possibility: keep  $c_n$  constant, but inflation demands  $c_n$  to increase...

### How to make a living from a fortune with inflation-adjusted consumption

- Assume  $l$  percent inflation per year
- Start with  $c_0$  as  $q$  percent of the interest the first year
- $c_n$  then develops as money with interest rate  $l$

$x_n$  develops with rate  $p$  but with a loss  $c_n$  every year:

$$x_n = x_{n-1} + \frac{p}{100}x_{n-1} - c_{n-1}, \quad x_0 = F, \quad c_0 = \frac{pq}{10^4}F$$
$$c_n = c_{n-1} + \frac{l}{100}c_{n-1}$$

This is a coupled system of two difference equations, but the programming is still simple: we update two arrays, first  $x[n]$ , then  $c[n]$ , inside the loop (good exercise!)

### The mathematics of Fibonacci numbers

No programming or math course is complete without an example on Fibonacci numbers:

$$x_n = x_{n-1} + x_{n-2}, \quad x_0 = 1, \quad x_1 = 1$$

#### Mathematical classification

This is a *homogeneous difference equation of second order* (second order means three levels:  $n, n-1, n-2$ ). This classification is important for mathematical solution technique, but not for simulation in a program.

Fibonacci derived the sequence by modeling rat populations, but the sequence of numbers has a range of peculiar mathematical properties and has therefore attracted much attention from mathematicians.

### Program for generating Fibonacci numbers

```
N = int(sys.argv[1])
from numpy import zeros
x = zeros(N+1, int)
x[0] = 1
x[1] = 1
for n in range(2, N+1):
    x[n] = x[n-1] + x[n-2]
    print n, x[n]
```

### Fibonacci numbers can cause overflow in NumPy arrays

Run the program with  $N = 50$ :

```
2 2
3 3
4 5
5 8
6 13
...
45 1836311903
Warning: overflow encountered in long_scalars
46 -1323762223
```

Note:

- Changing `int` to `long` or `int64` for array elements allows  $N \leq 91$
- Can use `float96` (though  $x_n$  is integer):  $N \leq 23600$

### No overflow when using Python int types

- Best: use Python scalars of type `int` - these automatically changes to `long` when overflow in `int`
- The `long` type in Python has arbitrarily many digits (as many as required in a computation!)
- Note: `long` for arrays is 64-bit integer (`int64`), while scalar `long` in Python is an integer with as "infinitely" many digits

## Program with Python's int type for integers

The program now avoids arrays and makes use of three int objects (which automatically changes to long when needed):

```
import sys
N = int(sys.argv[1])
xnm1 = 1 # "x_n minus 1"
xnm2 = 1 # "x_n minus 2"
n = 2
while n <= N:
    xn = xnm1 + xnm2
    print 'x_%d = %d' % (n, xn)
    xnm2 = xnm1
    xnm1 = xn
    n += 1
```

Run with  $N = 200$ :

```
x_2 = 2
x_3 = 3
...
x_198 = 173402521172797813159685037284371942044301
x_199 = 280671172992510140037611932413038677189525
x_200 = 453973694165307953197296969697410619233826
```

Limitation: your computer's memory

## New problem setting: exponential growth with limited environmental resources

The model for growth of money in a bank has a solution of the type

$$x_n = x_0 C^n \quad (= x_0 e^{n \ln C})$$

Note:

- This is exponential growth in time ( $n$ )
- Populations of humans, animals, and cells also exhibit the same type of growth as long as there are unlimited resources (space and food)
- Most environments can only support a maximum number  $M$  of individuals
- How can we model this limitation?

## Modeling growth in an environment with limited resources

Initially, when there are enough resources, the growth is exponential:

$$x_n = x_{n-1} + \frac{r}{100} x_{n-1}$$

The growth rate  $r$  must decay to zero as  $x_n$  approaches  $M$ . The simplest variation of  $r(n)$  is a linear:

$$r(n) = \rho \left(1 - \frac{x_n}{M}\right)$$

Observe:  $r(n) \approx \rho$  for small  $n$  when  $x_n \ll M$ , and  $r(n) \rightarrow 0$  as  $x_n \rightarrow M$  and  $n$  is big

Logistic growth model:

$$x_n = x_{n-1} + \frac{\rho}{100} x_{n-1} \left(1 - \frac{x_{n-1}}{M}\right)$$

(This is a *nonlinear* difference equation)

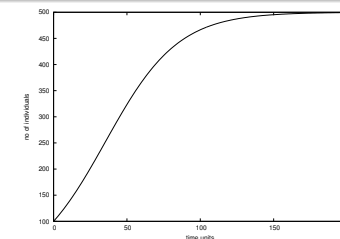
## The evolution of logistic growth

In a program it is easy to introduce logistic instead of exponential growth, just replace

$$x[n] = x[n-1] + p/100.0 * x[n-1]$$

by

$$x[n] = x[n-1] + (rho/100.0) * x[n-1] * (1 - x[n-1]/float(M))$$



## The factorial as a difference equation

The factorial  $n!$  is defined as

$$n(n-1)(n-2) \cdots 1, \quad 0! = 1$$

The following difference equation has  $x_n = n!$  as solution and can be used to compute the factorial:

$$x_n = n x_{n-1}, \quad x_0 = 1$$

## Difference equations must have an initial condition

- In mathematics, it is much stressed that a difference equation for  $x_n$  must have an initial condition  $x_0$
- The initial condition is obvious when programming: otherwise we cannot start the program ( $x_0$  is needed to compute  $x_n$ )
- However: if you forget  $x[0] = x_0$  in the program, you get  $x_0 = 0$  (because  $x = \text{zeros}(N+1)$ ), which (usually) gives unintended results!

## Have you ever though about how $\sin x$ is really calculated?

- How can you *calculate*  $\sin x$ ,  $\ln x$ ,  $e^x$  without a calculator or program?
- These functions were originally defined to have some desired mathematical properties, but without an algorithm for how to evaluate function values
- Idea: approximate  $\sin x$ , etc. by polynomials, since they are easy to calculate (sum, multiplication), but how??

## Would you expect these fantastic mathematical results?

Amazing result by Gregory, 1667:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Even more amazing result by Taylor, 1715:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k}{dx^k} f(0) \right) x^k$$

For "any"  $f(x)$ , if we can differentiate, add, and multiply  $x^k$ , we can evaluate  $f$  at any  $x$  (!!!)

## Taylor polynomials

Practical applications works with a truncated sum:

$$f(x) \approx \sum_{k=0}^N \frac{1}{k!} \left( \frac{d^k}{dx^k} f(0) \right) x^k$$

$N = 1$  is very popular and has been essential in developing physics and technology

Example:

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &\approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\ &\approx 1 + x \end{aligned}$$

## Taylor polynomials around an arbitrary point

The previous Taylor polynomials are most accurate around  $x = 0$ . Can make the polynomials accurate around any point  $x = a$ :

$$f(x) \approx \sum_{k=0}^N \frac{1}{k!} \left( \frac{d^k}{dx^k} f(a) \right) (x - a)^k$$

## Taylor polynomial as one difference equation

The Taylor series for  $e^x$  around  $x = 0$  reads

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Define

$$e_n = \sum_{k=0}^{n-1} \frac{x^k}{k!} = \sum_{k=0}^{n-2} \frac{x^k}{k!} + \frac{x^{n-1}}{(n-1)!}$$

We can formulate the sum in  $e_n$  as the following difference equation:

$$e_n = e_{n-1} + \frac{x^{n-1}}{(n-1)!}, \quad e_0 = 0$$

## More efficient computation: the Taylor polynomial as two difference equations

Observe:

$$\frac{x^n}{n!} = \frac{x^{n-1}}{(n-1)!} \cdot \frac{x}{n}$$

Let  $a_n = x^n/n!$ . Then we can efficiently compute  $a_n$  via

$$a_n = a_{n-1} \frac{x}{n}, \quad a_0 = 1$$

Now we can update each term via the  $a_n$  equation and sum the terms via the  $e_n$  equation:

$$\begin{aligned} e_n &= e_{n-1} + a_{n-1}, \quad e_0 = 0, \quad a_0 = 1 \\ a_n &= \frac{x}{n} a_{n-1} \end{aligned}$$

See the book for more details

## Nonlinear algebraic equations

Generic form of any (algebraic) equation in  $x$ :

$$f(x) = 0$$

Examples that can be solved by hand:

$$ax + b = 0$$

$$ax^2 + bx + c = 0$$

$$\sin x + \cos x = 1$$

- Simple numerical algorithms can solve "any" equation  $f(x) = 0$
- Safest: Bisection
- Fastest: Newton's method
- Don't like  $f'(x)$  in Newton's method? Use the Secant method
- Secant and Newton are difference equations!

## Newton's method for finding zeros

Newton's method

Simpson (1740) came up with the following general method for solving  $f(x) = 0$  (based on ideas by Newton):

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad x_0 \text{ given}$$

Note:

- This is a (nonlinear!) difference equation
- As  $n \rightarrow \infty$ , we hope that  $x_n \rightarrow x_s$ , where  $x_s$  solves  $f(x_s) = 0$
- How to choose  $N$  when what we want is  $x_N$  close to  $x_s$ ?
- Need a slightly different program: simulate until  $f(x) \leq \epsilon$ , where  $\epsilon$  is a small tolerance
- Caution: Newton's method may (easily) diverge, so  $f(x) \leq \epsilon$  may never occur!

## A program for Newton's method

Quick implementation:

```
def Newton(f, x, dfdx, epsilon=1.0E-7, max_n=100):
    n = 0
    while abs(f(x)) > epsilon and n <= max_n:
        x = x - f(x)/dfdx(x)
        n += 1
    return x, n, f(x)
```

Note:

- $f(x)$  is evaluated twice in each pass of the loop - only one evaluation is strictly necessary (can store the value in a variable and reuse it)
- $f(x)/dfdx(x)$  can give integer division
- It could be handy to store the  $x$  and  $f(x)$  values in each iteration (for plotting or printing a convergence table)

## An improved function for Newton's method

Only one  $f(x)$  call in each iteration, optional storage of  $(x, f(x))$  values during the iterations, and ensured float division:

```
def Newton(f, x, dfdx, epsilon=1.0E-7, max_n=100,
          store=False):
    f_value = f(x)
    n = 0
    if store: info = [(x, f_value)]
    while abs(f_value) > epsilon and n <= max_n:
        x = x - float(f_value)/dfdx(x)
        n += 1
        f_value = f(x)
        if store: info.append((x, f_value))
    if store:
        return x, info
    else:
        return x, n, f_value
```

## Application of Newton's method

$$e^{-0.1x^2} \sin\left(\frac{\pi}{2}x\right) = 0$$

Solutions:  $x = 0, \pm 2, \pm 4, \pm 6, \dots$

Main program:

```
from math import sin, cos, exp, pi
import sys

def g(x):
    return exp(-0.1*x**2)*sin(pi/2*x)

def dg(x):
    return -2*0.1*x*exp(-0.1*x**2)*sin(pi/2*x) + \
        pi/2*exp(-0.1*x**2)*cos(pi/2*x)

x0 = float(sys.argv[1])
x, info = Newton(g, x0, dg, store=True)
print 'Computed zero:', x

# Print the evolution of the difference equation
# (i.e., the search for the root)
for i in range(len(info)):
    print 'Iteration %3d: f(%g)=%g' % (i, info[i][0], info[i][1])
```

## Results from this test problem

$x_0 = 1.7$  gives quick convergence towards the closest root  $x = 0$ :

```
zero: 1.999999999768449
Iteration 0: f(1.7)=0.340044
Iteration 1: f(1.99215)=0.00828786
Iteration 2: f(1.99998)=2.53347e-05
Iteration 3: f(2)=2.43808e-10
```

Start value  $x_0 = 3$  (closest root  $x = 2$  or  $x = 4$ ):

```
zero: 42.49723316011362
Iteration 0: f(3)=-0.40657
Iteration 1: f(4.66667)=0.0981146
Iteration 2: f(42.4972)=-2.59037e-79
```

## What happened here??

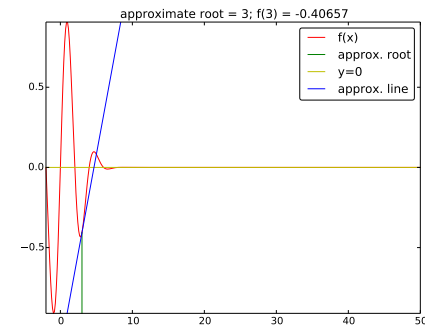
Try the demo program `src/diffeq/Newton_movie.py` with  $x_0 = 3$ ,  $x \in [-2, 50]$  for plotting and numerical approximation of  $f'(x)$ :

```
Terminal> python Newton_movie.py "exp(-0.1*x**2)*sin(pi/2*x)" \
         numeric 3 -2 50
```

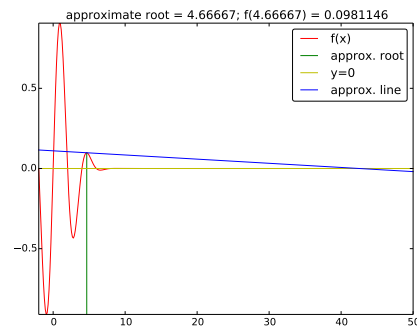
### Lesson learned:

Newton's method may work fine or give wrong results! You need to understand the method to interpret the results!

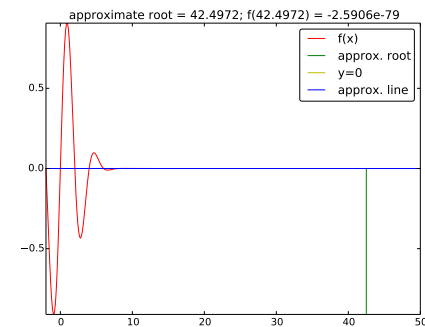
## First step: we're moving to the right ( $x = 4$ ?)



## Second step: oops, too much to the right...



## Third step: disaster since we're "done" ( $f(x) \approx 0$ )



## Programming with sound

### Tones are sine waves:

A tone **A** (440 Hz) is a sine wave with frequency 440 Hz:

$$s(t) = A \sin(2\pi f t), \quad f = 440$$

On a computer we represent  $s(t)$  by a discrete set of points on the function curve (exactly as we do when we plot  $s(t)$ ). CD quality needs 44100 samples per second.

## Making a sound file with single tone (part 1)

- $r$ : sampling rate (samples per second, default 44100)
- $f$ : frequency of the tone
- $m$ : duration of the tone (seconds)

Sampled sine function for this tone:

$$s_n = A \sin\left(2\pi f \frac{n}{r}\right), \quad n = 0, 1, \dots, m \cdot r$$

Code (we use descriptive names: frequency  $f$ , length  $m$ , amplitude  $A$ , sample\_rate  $r$ ):

```
import numpy
def note(frequency, length, amplitude=1,
        sample_rate=44100):
    time_points = numpy.linspace(0, length,
                                length*sample_rate)
    data = numpy.sin(2*numpy.pi*frequency*time_points)
    data = amplitude*data
    return data
```

## Making a sound file with single tone (part 2)

- We have data as an array with float and unit amplitude
- Sound data in a file should have 2-byte integers (int16) as data elements and amplitudes up to  $2^{15} - 1$  (max value for int16 data)

```
data = note(440, 2)
data = data.astype(numpy.int16)
max_amplitude = 2**15 - 1
data = max_amplitude*data
import scitools.sound
scitools.sound.write(data, 'Atone.wav')
scitools.sound.play('Atone.wav')
```

## Reading sound from file

- Let us read a sound file and add echo
- Sound = array s[n]
- Echo means to add a delay of the sound

```
# echo: e[n] = beta*s[n] + (1-beta)*s[n-b]

def add_echo(data, beta=0.8, delay=0.002,
             sample_rate=44100):
    newdata = data.copy()
    shift = int(delay*sample_rate) # b (math symbol)
    for i in xrange(shift, len(data)):
        newdata[i] = beta*data[i] + (1-beta)*data[i-shift]
    return newdata
```

Load data, add echo and play:

```
data = scitools.sound.read(filename)
data = data.astype(float)
data = add_echo(data, beta=0.6)
data = data.astype(int16)
scitools.sound.play(data)
```

## Playing many notes

- Each note is an array of samples from a sine with a frequency corresponding to the note
- Assume we have several note arrays data1, data2, ...:

```
# put data1, data2, ... after each other in a new array:
data = numpy.concatenate((data1, data2, data3, ...))
```

The start of "Nothing Else Matters" (Metallica):

```
E1 = note(164.81, .5)
G = note(392, .5)
B = note(493.88, .5)
E2 = note(659.26, .5)
intro = numpy.concatenate((E1, G, B, E2, B, G))
...
song = numpy.concatenate((intro, intro, ...))
scitools.sound.play(song)
scitools.sound.write(song, 'tmp.wav')
```

## Summary of difference equations

- Sequence:  $x_0, x_1, x_2, \dots, x_n, \dots, x_N$
- Difference equation: relation between  $x_n$ ,  $x_{n-1}$  and maybe  $x_{n-2}$  (or more terms in the "past") + known start value  $x_0$  (and more values  $x_1, \dots$  if more levels enter the equation)

Solution of difference equations by simulation:

```
index_set = <array of n-values: 0, 1, ..., N>
x = zeros(N+1)
x[0] = x0
for n in index_set[1:]:
    x[n] = <formula involving x[n-1]>
```

Can have (simple) systems of difference equations:

```
for n in index_set[1:]:
    x[n] = <formula involving x[n-1]>
    y[n] = <formula involving y[n-1] and x[n]>
```

Taylor series and numerical methods such as Newton's method can be formulated as difference equations, often resulting in a good way of programming the formulas

## Summarizing example: music of sequences

- Given a  $x_0, x_1, x_2, \dots, x_n, \dots, x_N$
- Can we listen to this sequence as "music"?
- Yes, we just transform the  $x_n$  values to suitable frequencies and use the functions in `scitools.sound` to generate tones

We will study two sequences:

$$x_n = e^{-4n/N} \sin(8\pi n/N)$$

and

$$x_n = x_{n-1} + q x_{n-1} (1 - x_{n-1}), \quad x = x_0$$

The first has values in  $[-1, 1]$ , the other from  $x_0 = 0.01$  up to around 1

Transformation from "unit"  $x_n$  to frequencies:

$$y_n = 440 + 200x_n$$

(first sequence then gives tones between 240 Hz and 640 Hz)

## Module file: soundeq.py

- Three functions: two for generating sequences, one for the sound
- Look at files/soundeq.py for complete code

Try it out in these examples:

```
Terminal> python soundeq.py oscillations 40
Terminal> python soundeq.py logistic 100
```

Try to change the frequency range from 200 to 400.