Lambert Universal Variable Algorithm

Article *in* ARABIAN JOURNAL FOR SCIENCE AND ENGINEERING · January 2003

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LAMBERT UNIVERSAL VARIABLE ALGORITHM

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الخلاصـــة

تم فى هذا البحث تشييد خوارزمية بسيطة وفعالة لحل مسألة لمبرت الشاملة. وقد اعتمدت هذه الخوارزمية على مخطط تكراري متقارب لجميع أنواع الحركة المخروطية (إهليجية – مكافئ - زائدية) ، إلى جانب ذلك فقد احتوت الخوارزمية على طريقة فعالة لدوال استنف (Stumpff) المتسامية ، وذلك باستخدام المتسلسلات والصيغ التعاودية. واشتمل البحث أيضاً على بعض التطبيقات العددية للخوارزمية .

ABSTRACT

In this paper, an efficient and simple algorithm is developed for the solution of universal Lambert's problem. The algorithm based on iterative scheme which converges for all conic motion (elliptic, parabolic, or hyperbolic). Moreover, the involved transcendental Stumpff's functions are evaluated efficiently using series and recursion formulae. Applications to the algorithm are also given.

Key words:

Universal Lambert problem, orbital boundary value problem, orbit determination.

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LAMBERT UNIVERSAL VARIABLE ALGORITHM

1. INTRODUCTION

Lambert problem of space researches is concerned with the determination of an orbit from two position vectors and the time of flight [1]. It has very important applications in the areas of rendezvous, targeting, guidance [2], and interplanetary mission [3].

Solutions to Lambert's problem abound in the literature, as they did even in Lambert's time shortly after his original formulation in 1716. Examples are Lambert's original geometric formulation, which provides equations to determine the minimum-energy orbit, and the original Gaussian formulation, which gives geometrical insight into the problem.

Up to the year 1965, a fairly comprehensive list of references on Lambert's problem are given in references [4–6]. In 1969, Lancaster and Blanchard [7] (also Mansfield [8]) established unified forms of Lambert's problem; in 1990 Gooding [9] developed a procedure for the solution; and in 1995, Thorne and Bain [10] developed a direct solution using the series inversion technique.

Each of the above methods is characterized primarily by: (1) a particular form of the time of flight equation; and (2) a particular independent variable to be used in an iteration algorithm to determine the orbital elements.

One of the most compact and computationally efficient forms of Lambert's problem is that of Battin (cited in reference [11]). In this form, the time of flight equation is universal (*i.e.*, includes elliptic, parabolic, and hyperbolic orbits) as a well-behaved function of a single, physically significant, independent variable.

In the present paper, we aim at building an algorithm for universal Lambert's problem on the basic approach of Battin, extending it to include the following important points: (1) An iterative scheme which converges for all orbit types; and (2) an efficient procedure to evaluate the Stumpff's functions. Applications of the algorithm are also given.

2. BASIC FORMULATIONS

2.1. Two-Body Formulations

- The equation describing the relative motion of the two bodies of masses m_1 and m_2 in rectangular coordinates is:

$$\frac{d}{dt}\mathbf{v} = \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}\ , \tag{2.1}$$

where μ is the gravitational parameter (universal gravitational constant times the sum of the two masses), $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity vectors given in components as:

$$\mathbf{r} = x\ \mathbf{i}_x + y\ \mathbf{i}_y + z\ \mathbf{i}_z\ , \tag{2.2}$$

$$\mathbf{v} = \dot{x}\ \mathbf{i}_x + \dot{y}\ \mathbf{i}_y + \dot{z}\ \mathbf{i}_z\ , \tag{2.3}$$

$\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are the unit vectors along the coordinate axes x, y, and z respectively, and

$$r = (x^2+y^2+z^2)^{1/2}\ . \tag{2.4}$$

Equation (2.1) is unchanged if we replace $\mathbf{r}$ with $-\mathbf{r}$. Thus Equation (2.1) gives the motion of the body of mass m_2 relative to the body of the mass m_1, or the mass m_1 relative to m_2. Also, if we replace t with $-t$, Equation (2.1) remains unchanged.

- On any of the two bodies' orbits (elliptic, parabolic, or hyperbolic) we have:

$$\mathbf{r} = F\mathbf{r_0} + G\mathbf{v_0}\,, \tag{2.5}$$

$$\mathbf{v} = \dot{F}\mathbf{r}_0 + \dot{G}\mathbf{v}_0\,, \tag{2.6}$$

where ($\mathbf{r_0}$, $\mathbf{v_0}$) are the position and velocity vectors at time t_o, while ($\mathbf{r}$, $\mathbf{v}$) are the corresponding vectors at another time t. The coefficients F and G are functions of $\Delta t = t - t_o$ and known as the Lagrange F and G functions; $\dot{F}$ and $\dot{G}$ are their time derivatives such that

$$F\dot{G} - G\dot{F} = 1\,. \tag{2.7}$$

Equation (2.7) implies that, given any three of the four functions F, G, $\dot{F}$, or $\dot{G}$, we can solve for the remaining one.

There are different forms of F, …, $\dot{G}$ [12], but what important to us are their expressions in terms of the change $\Delta f = f - f_0$ in the true anomaly, which are

$$F = 1 - \frac{r}{p}[1 - \cos(\Delta f)]\,, \tag{2.8}$$

$$G = \frac{1}{\sqrt{\mu p}} r\, r_0 \sin(\Delta f)\,, \tag{2.9}$$

$$\dot{F} = \sqrt{\frac{\mu}{p}}\left[\frac{1-\cos(\Delta f)}{\sin(\Delta f)}\right]\left[\frac{1-\cos(\Delta f)}{p} - \frac{1}{r} - \frac{1}{r_0}\right], \tag{2.10}$$

$$\dot{G} = 1 - \frac{r_0}{p}[1 - \cos(\Delta f)]\,, \tag{2.11}$$

where p is the semi-latus rectum of the orbit, r and r_0 are the magnitudes of the position vectors $\mathbf{r}$ and $\mathbf{r}_0$ respectively.

- The total energy constant for the two-body problem is $-\mu/2a$, where a is the semi-major axis of the orbit. It is convenient to write α for $1/a$, so that, at the time $\mathbf{t}_0$ (say), the constant α is

$$\alpha = \alpha_0 \equiv \frac{1}{a} = \frac{2}{r_0} - \frac{\upsilon_0^{\,2}}{\mu}\,, \tag{2.12}$$

where ν_0 is the magnitude of the velocity vector $\mathbf{v}_0$. Depending on the sign of α, or the value of the eccentricity e, the type of the orbit is determined such that: $\alpha > 0$ (or $e < 1$) for elliptic orbits; $\alpha = 0$ (or $e = 1$) for parabolic orbits and $\alpha < 0$ (or $e > 1$) for hyperbolic orbits.

2.2. Universal Fomulations of the Two-Body Problem

- Different formulations for the various two-body orbits can be unified by using (1) a time transformation formula; and (2) a new family of transcendental functions.

  Regarding the first point, we shall use Sundman's time transformation [12] defined by:

$$\sqrt{\mu}\,\frac{dt}{d\chi} = r\,, \tag{2.13}$$

where χ is to be considered as a new independent variable — a kind of generalized anomaly. It could be shown [12] that when χ is used as the independent variable instead of the time t, then the non-linear equations of motion can be converted into linear, constant coefficients, differential equations.

For t_0 and t, the variable χ can be related to the classical anomalies by:

$$\chi = \begin{cases} \sqrt{a}\left(E - E_0\right) & \text{for} \quad \alpha > 0 \\ \sqrt{a}\left(\tan\frac{1}{2}f - \tan\frac{1}{2}f_0\right) & \text{for} \quad \alpha = 0 \\ \sqrt{-a}\left(H - H_0\right) & \text{for} \quad \alpha < 0 \end{cases}, \tag{2.14}$$

where E and H are respectively, the elliptic eccentric anomaly and the hyperbolic eccentric anomaly.

Regarding the second point mentioned above, we shall consider for the family of transcendental functions, the generating functions:

$$C_n\left(\mathrm{z}\right) = \sum_{k=0}^{\infty}\left(-1\right)^k \frac{z^k}{\left(2k+n\right)!}, \qquad n = 0, 1, 2, \ldots \tag{2.15}$$

and are known as Stumpff functions [1]; they are well defined for a complex variable z, since the power series is convergent at any point in the complex plane. They are real values for real z. Note also that:

$$C_n(0) = \frac{1}{n!}. \tag{2.16}$$

What concerns us in the subsequent analysis are the following relations between C_2 and C_3 of arguments ψ and 4ψ:

$$C_3(4\psi) = [C_2(\psi) + C_3(\psi) - \psi C_3(\psi) C_2(\psi)]/4 , \tag{2.17}$$

$$C_2(4\psi) = \tfrac{1}{2}\{1 - \psi C_3(\psi)\}^2 . \tag{2.18}$$

- The universal form of the $F, \ldots, \dot{G}$ functions which are valid for any type of conic motions are given as [11]:

$$F = 1 - \frac{\chi^2}{r_0} C_2(\alpha_0 \chi^2), \tag{2.19}$$

$$G = t - t_0 - \frac{\chi^3}{\sqrt{\mu}} C_3(\alpha_0 \chi^2), \tag{2.20}$$

$$\dot{F} = \frac{\sqrt{\mu}}{r r_0} \chi \left\{\alpha_0 \chi^2 C_3(\alpha_0 \chi^2) - 1\right\}, \tag{2.21}$$

$$\dot{G} = 1 - \frac{\chi^2}{r} C_2(\alpha_0 \chi^2). \tag{2.22}$$

2.3. Universal Lambert Problem

The formulations of the universal Lambert problem could easy be obtained [11] using the two sets of Equations (2.8) to (2.11) and Equations (2.19) to (2.22), and summarized in stepwise form as follows:

- Equating Equations (2.8) and (2.19) and then solve for the universal variable χ we get

$$\chi = \sqrt{\frac{r\, r_0}{pC_2}(1-\cos\Delta f)}\,, \tag{2.23}$$

where $C_2 = C_2(\alpha_0\chi^2)$. Also let $C_3 = C_3(\alpha_0\chi^2)$ below.

- Substituting Equation (2.23) into Equation (2.21) and then equate it with $\dot{F}$ of Equation (2.10) we get

$$B = r_0 + r + \frac{A(\psi C_3 - 1)}{\sqrt{C_2}}\,, \tag{2.24}$$

where

$$A = \sin\Delta f \sqrt{\frac{r\, r_0}{1-\cos\Delta f}}\,, \tag{2.25}$$

$$B = \frac{r\, r_0}{p}(1-\cos\Delta f)\,, \tag{2.26}$$

$$\psi = \alpha_0\chi^2\,. \tag{2.27}$$

Also,

$$\chi = \sqrt{\frac{B}{C_2}}\,, \tag{2.28}$$

$$\cos\Delta f = \frac{1}{r_0 r}(\mathbf{r}_0 . \mathbf{r})\,, \tag{2.29}$$

$$\sin\Delta f = \frac{1}{r_0 r}\left|\mathbf{r}_0 \times \mathbf{r}\right|. \tag{2.30}$$

- Solve for $\Delta t = t - t_0$ from Equations (2.20) and (2.9) we get

$$\sqrt{\mu}\,\Delta t = \chi^3 C_3 + A\sqrt{B}\,. \tag{2.31}$$

- Finally, the unknown velocity vectors $\mathbf{v}_0$ and $\mathbf{v}$ are obtained from

$$\mathbf{v}_0 = \frac{1}{G}\left(\mathbf{r} - \mathbf{r}_0 F\right), \tag{2.32}$$

$$\mathbf{v} = \frac{1}{G}(\dot{G}\mathbf{r} - \mathbf{r}_0)\,, \tag{2.33}$$

where from Equations (2.25)

$$F = 1 - \frac{B}{r_0}\,, \tag{2.34}$$

$$G = A\sqrt{\frac{B}{\mu}}\,, \tag{2.35}$$

$$\dot{G} = 1 - \frac{B}{r}\,. \tag{2.36}$$

3. COMPUTATIONAL DEVELOPMENTS

3.1. Computations of Stumpff's Functions $C_2(\psi)$ and $C_3(\psi)$

An efficient algorithm was developed recently [13] for computing $C_j(\psi)$; $j = 0, 1, 2, 3$. In what follows, this algorithm will be modified to compute $C_2(\psi)$ and $C_3(\psi)$ only.

It is most efficient to compute C_2 and C_3 by series (Equation (2.15)). On the other hand, it is inefficient to use the series except for small values of the argument; but if the input argument is too large, it can be reduced by using Equations (2.17) and (2.18) as illustrated in what follows.

Now, suppose that the series for C_2 and C_3 are only to be used for values of $|\psi| \le \psi_m < 1$. The series might be found using a loop that would include a test to terminate it when the necessary accuracy was achieved according to a given tolerance ε (say). Thus, if we write $C_2(\psi)$ or $C_3(\psi)$ as

$$C(\psi) = \sum_{k=0}^{N} H_k \, , \tag{3.1}$$

then, it is easy to show from Equation (2.15) that, the H's coefficients satisfy the recurrence formula

$$H_{k+1} = -\frac{\psi}{2(k+2)(2k+\eta)} H_k \, , \tag{3.2}$$

where (η, H_0) is (3, 1/2) for computing $C_2(\psi)$ and (5, 1/6) for computing $C_3(\psi)$, while the positive integer N is determined by the condition that

$$|H_N| \le \varepsilon. \tag{3.3}$$

The computations of $C_2(\psi)$ or $C_3(\psi)$ for $|\psi| \le \psi_m < 1$ are illustrated by the following algorithm.

3.1.1. Computational Algorithm 1

- *Purpose:* To compute $C_2(\psi)$ or $C_3(\psi)$ using Equations (3.1) and (3.2), together with the corresponding value of N.
- *Input:* ψ, H_0, η, ε, M (large positive integer).
- *Computational Sequence:*

1. Set

    $H = H_0$

    $C = H_0$

    $K = 2$

    $L = \eta$

2. For $i = 1, 2, \ldots, M$, do:

    Set $H = -\dfrac{\psi}{2 * K * L} * H$

    If $|H| \le \varepsilon$ go to step 3

    $C = C + H$

    $K = K + 1$

    $L = L + 2$

3. $N = i$
4. End

Having obtained C_2 and C_3 for a value of ψ such that $|\psi| \le \psi_m < 1$, it remains to compute their values for any given ψ. The following algorithm illustrates these computations.

3.1.2. Computational Algorithm 2

- *Purpose:* To compute $C_2(\psi)$ and $C_3(\psi)$ for any ψ using series and recursion.
- *Input:* ψ, ψ_m.
- *Computational Sequence:*

  1. Set $K = 0$
  2. If $|\psi| \le \psi_m$ go to step 5
  3. Set $K = K + 1$, $\psi = \psi/4$
  4. Go to step 2
  5. Compute C_2 and C_3 using algorithm 1
  6. If $K = 0$ go to step 11
  7. Set $K = K - 1$
  8. Compute C_3 and C_2 from Equations (2.17) and (2.18) respectively
  9. Set $\psi = 4*\psi$
  10. Go to step 6
  11. End

3.2. Implementing the Universal Lambert Problem

Bate, Mueller, and White [14] present a Newton iterative scheme to find ψ. Newtonian iteration works fine on most problems but fails to converge on many difficult (hyperbolic) orbits. To overcome this difficulty, Vallado and McClain [15] suggested a bisection technique which works well on all orbit types but is only about 5% slower. In this case, the object is to find the value of ψ that corresponds to the given change in time (Δt). This could be done by bounding the correct value of ψ and picking a trial value of ψ halfway between these bounds. Subsequent iterations determine the upper and lower bounds and successively readjust them until the interval for ψ is tight enough to locate the correct value of ψ.

Finally, it should be noted that, traveling between two specified points can take the long way or the short way. For the long way, the change Δf in the true anomaly exceeds 180°, while for the short way $\Delta f < 180°$.

The solution of a Lambert problem using the bisection technique is illustrated in the following algorithm.

3.2.1. Computational Algorithm 3

- *Purpose:* To solve the universal Lambert problem for the velocity vectors v_0 and $\mathbf{v}$.
- *Input:* $\mathbf{r}_0 \equiv (x_0, y_0, z_0)$;
  $\mathbf{r} \equiv (x, y, z)$; Δt;
  ψ_0 (initial value); ψ_u (upper value); ψ_L (lower value);
  t_m ($t_m = +1$ for short way transfers, $t_m = -1$ for long way transfers);
  M (large positive integer), Tol (specified tolerance) and μ.

- *Computational Sequence:*

1. $r_0 = (x_0^2 + y_0^2 + z_0^2)^{1/2}$.
2. $r = (x^2 + y^2 + z^2)^{1/2}$.
3. $\gamma = (x^* x_0 + y^* y_0 + z^* z_0)/(r^* r_0)$.
4. $\beta = t_m (1 - \gamma^2)^{1/2}$.
5. $A = t_m [r^* r_0^*(1 + \gamma)]^{1/2}$.
6. If $A = 0$, we cannot calculate the orbit, then go to step 11.
7. For i: = 1 to M do

   begin$\{i\}$

   $\psi = \psi_0$

   Compute $C_2(\equiv C_2(\psi))$ and $C_3(\equiv C_3(\psi))$ using algorithm 2.

   $$B = r_0 + r + \frac{1}{\sqrt{C_2}}\left\{A * \left(\psi * C_2 - 1\right)\right\}$$

   If $A > 0.0$ and $B < 0.0$, then readjust ψ_L until $B > 0.0$

   $$\chi = \sqrt{\frac{B}{C_2}}$$

   $$\Delta\tilde{t} = \frac{1}{\sqrt{\mu}}\left(\chi^3 * C_3 + A * \sqrt{B}\right)$$

   If $\left|\Delta t - \Delta\tilde{t}\right| < Tol$, go to step 8

   If $\Delta\tilde{t} \le \Delta t$ set $\psi_L = \psi$, go to step 7.1

   Set $\psi_u = \psi$

   7.1. $\psi_1 = \frac{1}{2}(\psi_u + \psi_L)$

   $\psi_0 = \psi_1$

   End $\{i\}$

8. Compute F, G, and $\dot{G}$ from Equations (2.34), (2.35), and (2.36) respectively.
9. $\dot{x}_0 = \frac{1}{G}(x - x_0 F)$; $\dot{y}_0 = \frac{1}{G}(y - y_0 F)$; $\dot{z}_0 = \frac{1}{G}(z - z_0 F)$
10. $\dot{x} = \frac{1}{G}(\dot{G}x - x_0)$; $\dot{y} = \frac{1}{G}(\dot{G}y - y_0)$; $\dot{z} = \frac{1}{G}(\dot{G}z - z_0)$
11. End

3.2.2. Comments:

- The initial limits of ψ are chosen to let the most common solutions converge.
- For very eccentric (hyperbolic) orbits, one should expand the lower negative limit.
- To reduce the size of numbers involved in calculations, to make operations more mathematically stable, and also to speed up the algorithm, we used canonical units, such that, for the heliocentric system, length, mass, and time are expressed in astronomical units, solar masses, and days, respectively and the Gaussian gravitational constant given exactly [16] by

  $k = k_{\otimes} = 0.01720209895$.

  While for the geocentric system, length, mass, and time are expressed in Earth's radii, Earth's masses, and minutes, respectively. By analogy with the heliocentric system, we shall assume a fundamental geocentric gravitational constant given by

  $k = k_{\oplus} = 0.074366864$.

  In both systems the gravitational parameter $\mu = 1$.

3.3. Numerical Applications

We used three Earth test bodies with position vectors $\mathbf{r}_0 \equiv (x_0, y_0, z_0)$ and $\mathbf{r} \equiv (x, y, z)$ listed in Table 1 together with the eccentricities e and their types. The position vectors are expressed in the geocentric canonical unit ER (ER = 6378.1363 km). Computational algorithm 3 is then applied for each orbit with $\mu = 1$, $Tol = 10^{-7}$, $M = 50$, $t_m = 1$ (short way transfer), and $\Delta t = 5$ TU (unit of time TU = $1/k_{\oplus}$ = 13.446849 solar min). The initial limits of ψ are chosen for each orbit so as to secure the convergence of the solution. The values of these limits are listed for each orbit in the second, third, and fourth columns of Table 2, while the other columns of the table are devoted to the final values of B, χ, and ψ_1.

The resulting velocity vectors $\mathbf{v}_0 \equiv (v_{x_0}, v_{y_0}, v_{z_0})$ and $\mathbf{v} \equiv (v_x, v_y, v_z)$ are listed in Table 3 and expressed in canonical units ER/TU(ER/TU = 7.905367km/sec).

Table 1. Position Vectors for the Test Orbits.

	Orbit 1	Orbit 2	Orbit 3
x_o	1.01566 ER	−0.253513 ER	−0.668461 ER
y_o	0.0 ER	1.21614 ER	−2.05807 ER
z_o	0.0 ER	−1.20916 ER	−1.9642 ER
x	0.387926 ER	−0.434366 ER	3.18254 ER
y	0.183961 ER	4.92818 ER	2.08111 ER
z	0.551884 ER	0.0675545 ER	−4.89447 ER
e	0.9114797	1.0	4.2100249
Type	Elliptic	Parabolic	Hyperbolic

Table 2. The Initial Limits of ψ and the Final values of B, χ, and ψ_1.

Orbit	ψ_o	ψ_u	ψ_L	B	χ	ψ_1
1	0.8	$4\pi^2$	$-4\pi^2$	1.91549	3.3704	11.6446
2	0.8	$4\pi^2$	$-4\pi^2$	1.27779	1.59862	0.0
3	−0.1	$4\pi^2$	−2	0.990575	1.30274	−1.88569

Table 3. Solution of the Universal Lambert Problem for the Velocity Vectors v_0 and v.

	Orbit 1	Orbit 2	Orbit 3
v_{x_0}	0.885477 ER/TU	−0.0851362 ER/TU	0.788746 ER/TU
v_{y_0}	0.126493 ER/TU	1.06699 ER/TU	0.748957 ER/TU
v_{z_0}	0.379481 ER/TU	0.0892477 ER/TU	−0.782571 ER/TU
v_x	−1.16237 ER/TU	−0.0159003 ER/TU	0.727364 ER/TU
v_y	−0.220033 ER/TU	0.564771 ER/TU	0.828386 ER/TU
v_z	−0.660101 ER/TU	0.291558 ER/TU	−0.467453 ER/TU

In concluding the present paper, an efficient and simple algorithm for the solution of the universal Lambert's problem of the space dynamics has been developed. Its efficiency is due to several factors; of these some are the following:

(1) Based on iterative scheme that could be made to cover all conic motion. The scheme uses a bisection technique; although it is somewhat slow, it usually bounds one root on the interval. On the other hand, Newton iteration [14] works fine on most problems but fails to converge on many difficult (hyperbolic) orbits. There is full agreement that the Newton method is extremely sensitive to the initial guess [17]. Moreover, in many cases the initial guess may lead to drastic situations between divergent and very slowly convergent solutions. Such discouraging facts aside, recent progress in numerical solution of nonlinear equations has given us techniques whose convergence properties are more global than for the Newton method. These techniques are known as homotopy continuation methods [18], and have been applied recently in space dynamics [15, 19].

(2) Our algorithm includes a very fast procedure to evaluate the Stumpff's functions, which are the basic functions for all forms of the universal Lambert's problem.

(3) The accuracy of the algorithm was confirmed by using $(\mathbf{r}_0, \mathbf{v}_0)$ of each orbit as initial values at $t = 0$ for the Runge–Kutta fourth order method (with fixed step size) to integrate the two-body equations of motion (Equation (2.1)). The resulting components of $\mathbf{v}$ at the given value of $t (t = 5)$ are in full agreement with those obtained from our algorithm 3.

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Paper Received 21 November 2001; Revised 2 April 2002; Accepted 23 October 2002.