On the Distinction Between Topological Space and Physical Space in SAF:

In other words, what are topo-relations and dof-relations?

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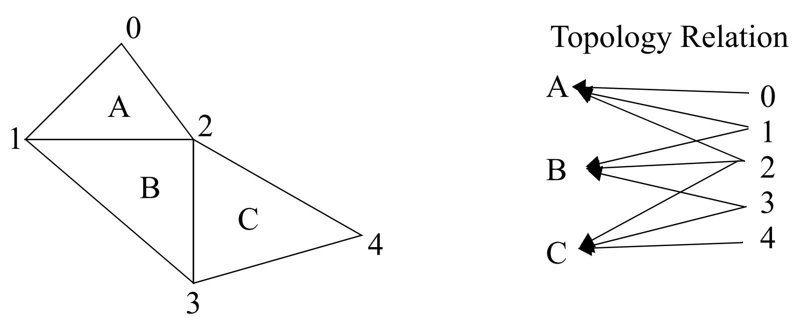
1 Topology relations and dof relations

A focused description of topology and geometry in SAF is essential to understand how to correctly model fields with SAF. This short paper is aimed at persons with some familiarity with the SAF data model and application programing interface who wish to address issues with modeling high order as well as sub-parametric and super-parametric fields[1].

1.1 Topology Relations

SAF currently supports an object called the “topo-relation” object. It is a topology relation. The purpose of a topology relation is to define how various elements of a discretized base-space (e.g. domain of a field) are knitted together to form something akin to a mesh. Often, but not always, the topology relation is related to “connectivity” information. For example, for the mesh consisting of 3 triangles, A, B, C in Figure 1, the topology relation enumerates for each triangle, which points are contained in that triangle (note the object at the tail of the arrow is contained in the object at the head).

Figure 1: Example Topology Relation



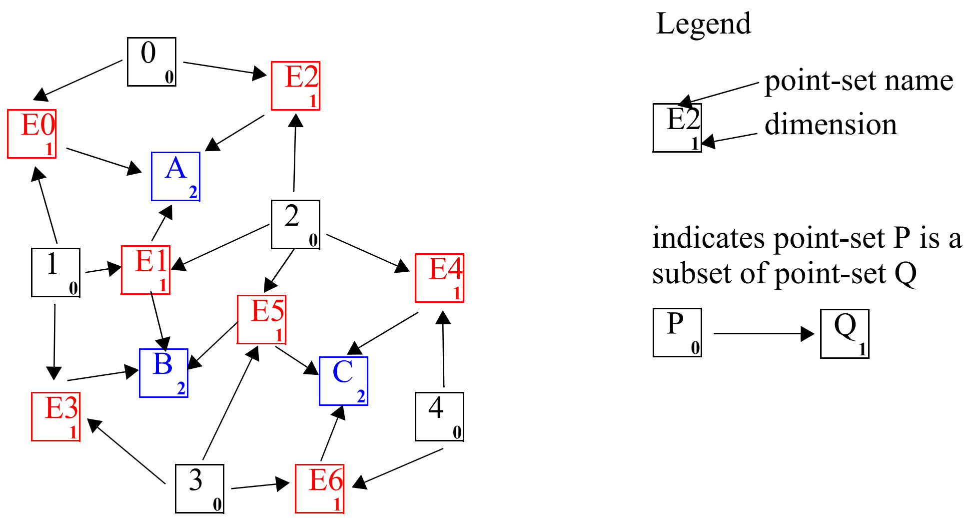
It is vital to understand that the topology relation is a statement solely about point-sets. Each triangle is a 2D point set and has some important subsets, namely the edges and the corner-points. While the triangle itself as well as its edges are point-sets containing an infinite number of points, the corner-points are point-sets containing only a single point. The edges and the corner-points are important subsets of a triangle because we name them. That is, we typically give each one a name consisting of a unique number. Sometimes we don’t give the edges names. However,

1. Note, it is common practice to refer to high-order “elements” or sub- or super-parametric “elements” as though it is an element that is embelished with high-order-ness or sub- or super-parametric-ness. This is simply not so. An element represents an elemental piece of the discretized base-space. It is a point-set, pure topology. The entity that has high-order-ness or sub- or super-parametric-ness is the evaluation scheme used to represent a field over an elemental piece of base-space.

by virtue of a canonical numbering scheme for the corner-points over common kinds of elements, we are able to infer relationships between edges (and faces in 3D) without explicitly representing them.

There is a serious problem with using the diagram in Figure 1 to illustrate what a topology relation really is. It is far too suggestive of geometric relationships and creates confusion between topology (topological space) and geometry (physical space). A much more accurate characterization of the meaning of and information conveyed by a topology relation is given by the diagram in Figure 2.

Figure 2: Better diagrammitic representation of a topology relation



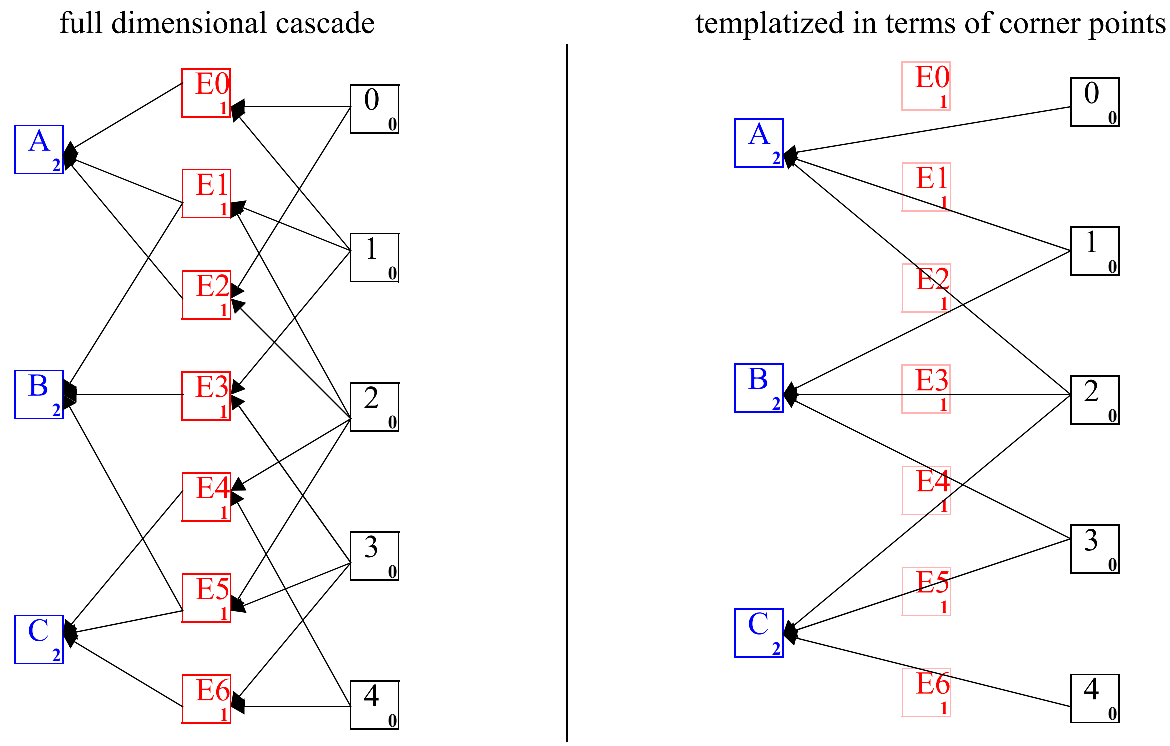
In Figure 2, we illustrate only point sets and the relationships between those point-sets. Each set is represented by a box with a name. In the lower, right corner of each box is a number indicating the topological dimension of the set. We have also drawn sets of the same dimension in a common color to help distinguish them in the diagram. If you study the diagram in Figure 2 long enough, you will begin to see how it relates to Figure 1. In fact the names of the point-sets used in the boxes in Figure 2 are the names of the same entities in Figure 1.

The value in illustrating a topology relation using the diagram in Figure 2 is that it is not suggestive of any geometry. And, this is precisely what a topology relation is. It is a statement about point-sets and their inter-relationships. No more, no less. From the topology relation, we know for example that point-sets “A” and “B” share a common subset, “E1”, and common corner-point subsets, “1” and “2”.

The strategy we used to draw the diagram in Figure 2 was to place the point sets in close relative proximity to their geometric-centric counterparts in Figure 1. However, this is not necessary and it makes for a messy diagram. We can re-arrange the sets on the page and produce a much more readable diagram. That is given in Figure 3.

We have split Figure 3 into two halves. In the left half, we have re-drawn exactly the same diagram as in Figure 2. This representation of topology is called a *full dimensional cascade* in which each point-set is characterized in terms of its boundary sets at the next lower dimension; 2D sets are characterized in terms of 1D sets and 1D sets are characterized in terms of 0D corner-point sets. In an arbitrarily connected mesh, a full dimensional cascade is the only way to specify topology. However, in cases where there are large numbers of topologically identical elemental pieces, it is more efficient to templatize the topology by piece-type and then enumerate only the 0D subsets of any given piece. This is precisely what we are doing when we define a *zoo* of shape types (e.g. tet4, wedge7, hex8, etc.) and specify a canonical numbering of corner-point subsets for each type. The right half of Figure 3 demonstrates this form of the topology relation. Note that because we do not explicitly represent the 1D subsets in this case, there is a substantial savings in storage[1]. This effect is even more pronounced in meshes of higher topological dimension.

Figure 3: More readable diagram of a topology relation



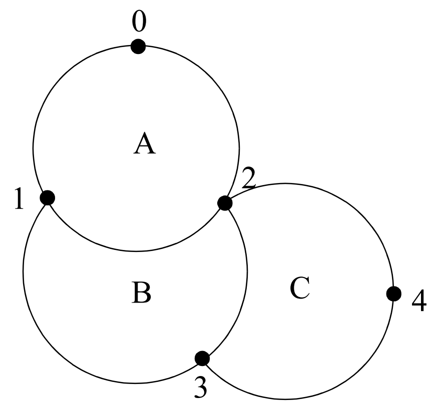
To emphasize the key point that a topology relation is a statement solely about point-sets, as well as the confusion that can be introduced when we “draw” them, we show a diagram equivalent to that of Figure 1 in Figure 5. In this case, we have intentionally drawn the “elements” so that they look like circles with curved “edges.” Of course, none of these features of the diagram are actually relevant. They simply serve to confuse the meaning of the diagram which is only to illustrate the relationships among the sets.

Note that there is really no sense of shape of any of the point-sets. In topological terms, we cannot say that point-sets “A”, “B” or “C” are triangles. We can only say they contain 3, 1D point-sets and 3, 0D point-sets. There is no context in which to talk about any of the 1D point-sets as being either curved or straight.

1. While it may be possible to avoid explicitly representing certain classes of subsets in a topology relation of a given mesh, it is impossible to avoid them entirely. Many algorithms that operate on the mesh will need to know about these subsets even if they are not explicitly represented. A contouring algorithm is a good example. This means that while a database system can get away with not storing them, it still has to provide enough information to a client so that the client can construct them when necessary.

Figure 4:

Figure 5: Equivalent diagram to Figure 1



1.2 Value Of Topology Relation in the Mathematical Formalism of *Numerical Fields*

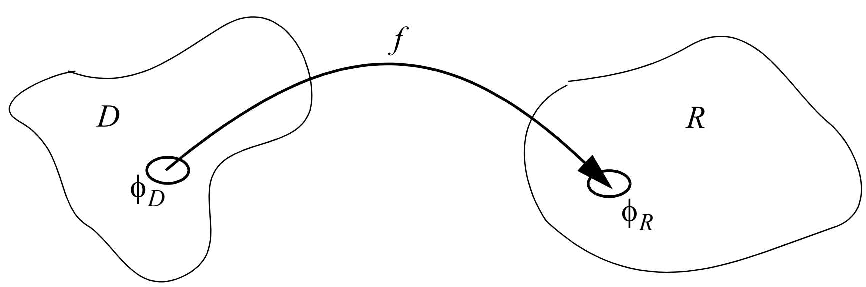
The value in representing explicit topology information (we will show later that it can often be inferred implicitly from something called a "dof-relation") in the form of a topology relation is that you cannot completely define continuity in a field without it.

To define continuity in a field you need the ability to talk about neighborhoods of points that are known to be *near* each other in the base-space of the field. For example, given a *function*, *f(a)*, which map points in *Domain* set, *D*, to points in *Range* set, *R*, we have

$$D \xrightarrow{f} R \qquad \text{(EQ 1)}$$

a non-rigorous definition of continuity in f means that points that are near each other in the domain set, *D*, map to points that are near each other in the range set, *R*. This concept is illustrated in Figure 6.

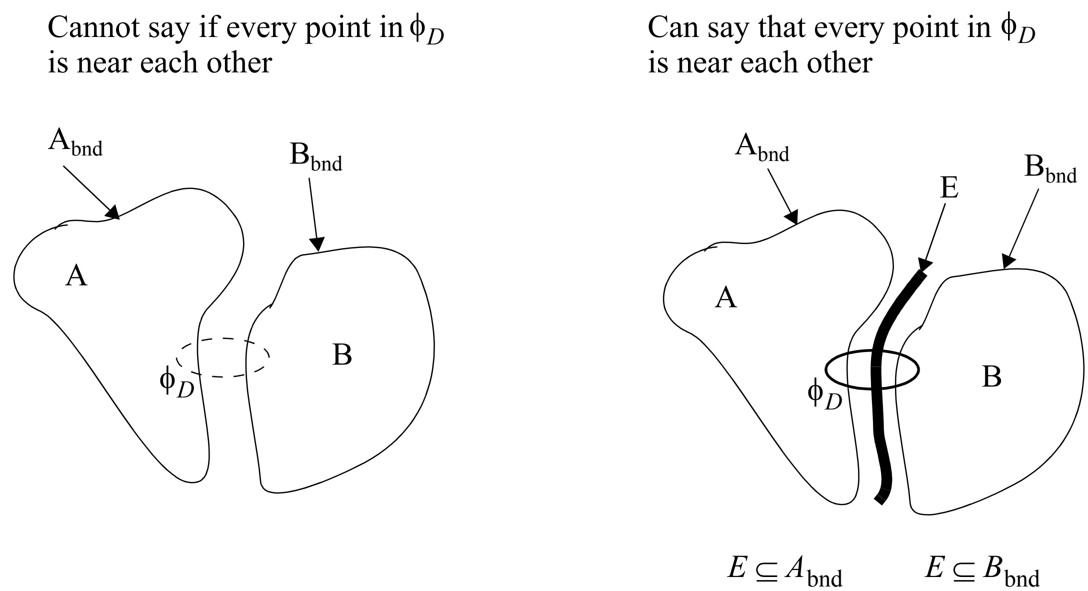
Figure 6: Example of points near each other in *D* mapping to points near each other in *R*



In Figure 6, we illustrate a subset of *D*, ϕ_D, and the mapping (or image) of that subset through the function, *f*, into *R*, ϕ_R Consequently, if we want to talk about the continuity in the function, we need to be able to talk about the *nearness* of points in *both* the domain set, *D*, and the range set, *R*.

Defining continuity for a *field* is complicated by the fact that the would-be domain of the field is not defined by a single point-set.. For a field, the would-be domain, called the *base-space*, is really a discrete collection of point-sets knitted together. Consequently, definitions of continuity run into problems on the boundaries between point-sets in the base-space. Such a situation is illustrated in Figure 7.

Figure 7: Example of trouble spot when defining continuity for a field



In Figure 7, we wish to define the continuity in a field for the points in subset ϕ_D which spans a boundary between two elemental pieces of the descretization, *A* and *B*. If we know only that *A* and *B* are point-sets but do not also know that *E* is a shared boundary between them, then it is impossible to define continuity in a field across the two elements because we cannot say whether or not the points in ϕ_D are near each other. However, once we have an explicit statement that the boundaries of both *A* and *B*, A_{bnd} and B_{bnd}, share the point-set, *E*, we can indeed say the points in ϕ_D are near each other. This is precisely the information a topo-relation provides us. Then, once we know that points in ϕ_D are near each other, we can begin to ask what happens to points in ϕ_D under the mapping to its image, ϕ_R.

1.3 Summary of Topology Relation

The preceding discussion is meant to help define, precisely what is meant by a topology relation in SAF. By definition, a topology relation enumerates all of the relationships between elemental subsets of the discretization. It is solely a statement about point-sets and their interrelationships. For this reason, a topology relation alone provides no context for talking about shape-centric notions such as curved or straight edges or even for that matter, triangle or quad elements.

As a final note, observe that the notion of continuity requires us to be able to talk about the nearness of points in two fundamentally different contexts. The domain or base-space of a field and the range, or fiber-space of a field[1]. This is so because the notion of continuity has to do with what happens to points in the mapping from the one to the other. In particular, if you want to characterize continuity in the coordinate field of some scientific data, you cannot do it either from

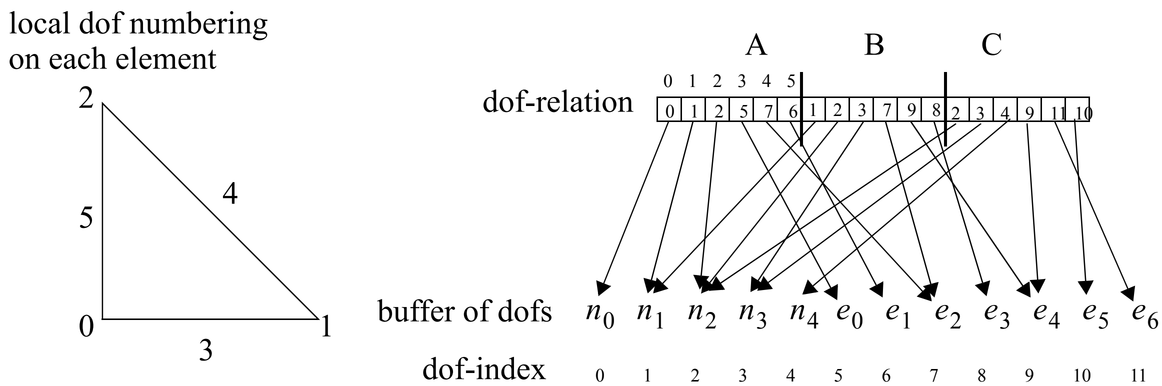
1. The former is the topological space of the field while the latter is the physical space

knowledge of the topology relation or from knowledge of the relationships between dofs alone. Both are required to define continuity in the coordinate field. Nonetheless, we will show that it often makes sense to infer the topology relation from the relationships between the dofs of the coordinate field (e.g. the dof-relation).

1.4 Dof-Relation (A new object to add to SAF)

SAF also supports another entity called a “dof-relation”. “dof” is short for *degree of freedom*[1] in the representation of a field. The purpose of a dof-relation is to, given a buffer of dofs (some problem sized array of floating point numbers representing a field), enumerate how those dofs are assigned to elements of the discretization. For example, to represent a piecewise quadratic field over the base-space discretized by the set of triangles in Figure 1, we would have the dof buffer and dof relation illustrated in Figure 8.

Figure 8: Example of dof-relation for quadratic field



Note that in this example we have organized the dof buffer so that dofs that control the field at the corners of each element come before dofs that control the field along the edges. The buffer of dofs is the data that gets written in a `saf_write_field()` call. The dof-relation is the data that would be written in a new `saf_write_dof_relation()` call to be added to SAF. Note also that the first three entries in the dof-relation for each element are the same as the topology relation. For each element, there are 6 entries in the dof relation. This is because the evaluation of the field on each element involves 6 dofs.

In the current implementation of SAF, the support for a dof-relation is limited. It is specified by the `assoc_ratio` and `assoc_cat` arguments in the `saf_declare_field()` call. In other words, dofs are n:1 associated with the members of the discretization in the base-space. For node-centered and zone-centered fields as well as numerous other classes of fields, this poor man’s dof-relation has worked splendidly because we always have node and zone entities defined in the base space. However, it is not sufficient in general, particularly for high order or sub- or super-parametric fields.

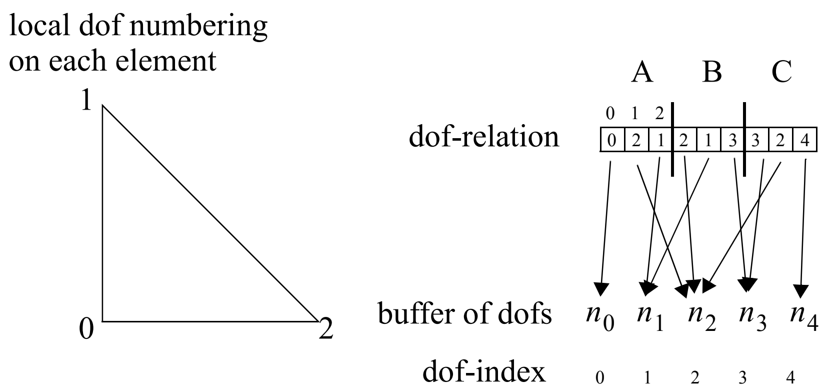
Both the topo-relation and dof-relation are arrays of integers. A topo-relation enumerates for each element, the point-sets in the discretized base-space that the element is composed of. A dof-relation enumerates for each element, the dofs that are used to interpolate a field on that ele-

1. Note, this is not to be confused with a degree of freedom in a system of equations involving the field. In SAF, we are concerned solely with representing a field

ment. Thus, while the domain of each relation is the same, the elemental pieces of the discretized base space, the ranges are very different. In a topology relation, the range is other point-sets in the base-space. In a dof-relation the range is a buffer-index into a the buffer of dofs.

In applications with piecewise-linear representations for the coordinate field, the topo-relation and dof-relation of the coordinate field are one in the same. That is, they are the same array of integers and are easily confused. For example, consider the dof-relation for a coordinate field in Figure 9.

Figure 9: Dof Relation for Piecewise Linear Field



Note that we have chosen a local numbering for dofs over the triangle that is a permutation of the numbering for corner-points in the topology in Figure 1. This choice was arbitrary and primarily for the purpose of illustration. Typically, the numberings are identical. Nonetheless, it is easy to see that the topo-relation can be easily derived from the dof-relation of the coordinate field once the local dof-numbering for the field and local corner-point numbering for the topology are known. In general, the one is a simple permutation of the other. Why is this so?

The reason that the topo-relation can often be derived from the dof-relation of the coordinate field is that dofs are 1:1 with the corner-points of the topology. However, this only happens for piecewise linear interpolation schemes. Since a majority of finite element applications use piecewise linear interpolation schemes for the coordinate field, the topology relation and dof relation of the coordinate field are indistinguishable -- and hopelessly confused! To add to the confusion, it is common practice to use the word “node” to refer both to the corner-points of the topology as well as to refer to the dofs that interpolate the coordinate field at the corner-points. This is why we have taken great lengths in this text to use “corner-point” when referring to topology and “dof” when referring to a field’s interpolation scheme.

For coordinate field interpolation schemes other than piecewise linear, it is often still possible to derive the topology relation from the dof relation. For example, consider the piecewise-quadratic interpolation scheme in Figure 8. The topology relation can be derived from taking the first 3 entries from each element’s contribution to the dof relation. When we do this, what we are basically saying is that elements that share dofs are neighbors of each other topologically. In other words, referring to Figure 6,

2 When can’t the topology relation be derived from the dof relation of the coordinate field?

The preceding observations lead to an interesting question. Is there ever a situation in which the topology relation cannot be derived from the dof relation of the coordinate field? The answer is that in general, yes it is possible to have such situations. In this section, we illustrate several examples.

The basic premise behind deriving topology information from the dof relation of the coordinate field is that elements that share coordinate field dofs must be next to each other topologically. To be clear, we need to understand what it means for two elements to *share* a dof. Because we use a dof relation to enumerate which dofs are assigned to which elements, there are two potential ways in which we could define what it means for elements to *share* a dof. Either two elements refer to the same dof index or two elements refer to two different dof indices but with the same dof value.

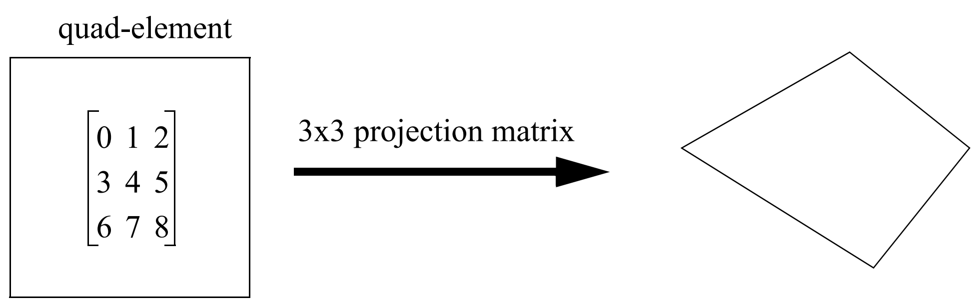
Consequently, in this section we will be concerned with examples of scientific data in which one of the following is true

1. Two elements share coordinate field dofs (either by index or by value) but are NOT topological neighbors of each other
2. Two elements DO NOT share coordinate field dofs but are topological neighbors of each other

2.1 Coordinate Field of a 2D Collection of Quads Expressed as Projection Transformations

In this example, we intentionally construct a representation for a coordinate field that is not based directly upon field samples at corner-points of the topology. The relationships between the topology of an element and its coordinate field dofs is illustrated in Figure 3.

Figure 3: Deformation of a quad element by 3x3 projection transformation



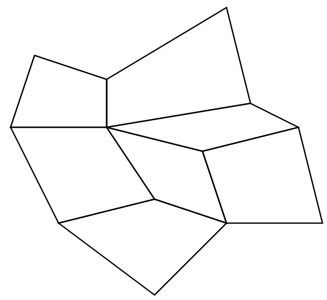
In this example, the coordinate field over each quad element is interpolated by the 3x3 plane geometric transformation called a *projection* transformation. The coordinate field, *(x,y)* for any point *(u,v)* over a quad element is given by Eq.2.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \qquad \text{(EQ 2)}$$

Such a transformation permits translation, rotation, scaling and perspective. Note that according to the givien field representation, there is no way to associate any dof governing the coordinate field with any specific set in the topology. For example, there is no way to associate any of the 9 entries in the projection transformation matrix with any of the corner-points or edges of the topology.

Without a lot of imagination, one can see that any collection of quadrilateral elements in the plane, such as the on illustrated in Figure 4, can be created by assigning a unique 3x3 transformation matrix to each quad. Thus, there are 9 dofs for each element.[1]

Figure 4: Example of mesh created by transforming a bunch of quads



Thus, the dof-relation for the coordinate field for such a mesh consists of *N*9* integers where *N* is the number of quad elements in the mesh. Now, can the topology relation be derived from this array of *N*9* integers? The answer is, no. The reason is that there is no sharing of dofs between two elements that neighbor each other in physical space. This is unlike the situations illustrated in Figure 8 and in Figure 9 where dofs are shared between neighboring elements. In this situation, it is necessary for the topology relation to be explicitly specified by the application.

2.2 An Electrical Circuit with No Coordinate Field

Consider a simulation in which we represent the voltages and currents in an electric circuit. An electric circuit is uniquely characterized by a graph of the electrical components comprising the circuit. That is strictly topological information. To simulate the voltages and currents, there is no requirement to specify a coordinate field for the circuit. In fact, a given circuit can be realized using a number of different coordinate fields. Consequently, in this example, there is topology but no geometry. If we based our knowledge of topology entirely on relationships between dofs of a non-existent coordinate field, we would be unable to represent this kind of data.

2.3 A Look-Up Table Approach to Compression of Highly Quantized Coordinate Fields

3 So, what is the topology relation good for?

In the previous example, we demonstrate that there are cases in which it is not possible, in general, to derive the topology relation from the dof relation of the cooridinate field. In this section, we answer the question why do we even need to know the topology relation to begin with?

The topology relation is essential to answer questions such as

- what elements are my neighbors?
- what elements do I share a corner-point, edge, or face with?
- are there discontinuities in a given field?

The fact is, the topology relation gives us explicit statements about which points in the discretized base-space are *next* to each other.

The previous sections indicate that this information is often derivable from knowledge of

1. Note that there is a linear dependence involved in a projection transformation such that only 8 of the 9 entries in the matrix are independent. And, those 8 independent entries, if derived, would be the same values as the 4 pairs of (x,y) values we would ordinarily use in a piecewise-linear representation of a quad element.

the coordinate field. The argument there is that points that are next to each other spatially are also next to each other topologically. However, in general, this is not always the case. Consider what happens when two objects collide or slide against each other. They are next two each other spatially but do not share points in the base-space. They are distinct objects. Likewise, suppose a structural failure is occuring where an object is tearing. The elements on each side of the tear are not spatially next to each other. There is a discontinuity in the coordinate field between them. However, we cannot know about this discontinuity without having an independent statement saying that they are indeed supposed to be *next* to each other. This is the information a topology relation provides.

Suppose one wanted to track a particle through the mesh in Section 2? As the particle moves from one element to the next, it is necessary to determine which elements are next to a given element. Given only the coordinate field dofs and the dof-relation, it is *very* difficult to do this. However, with an explicit topology relation, it would be very easy.

4 Conclusions

In many instances, the topology relation is deriveable from the dof-relation of the coordinate field. However, in general, it is not. So, it needs to be explicitly supported. We also need to support situations where it is derivable and a SAF client stores only the dof-relation assuming any client that needs the topology relation can compute it from the dof-relation of the coordinate field.

We also need to support dof relations in SAF.