Chapter 4, Section 2. Exercises 1, 2, and 4 through 9

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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Exercise 4.2.6

A *helicoid* is the surface swept out by an aeroplane propeller, when both the aeroplane and its propeller move at constant speed. If the aeroplane is flying along the z-axis, show that the helicoid can be parametrized as

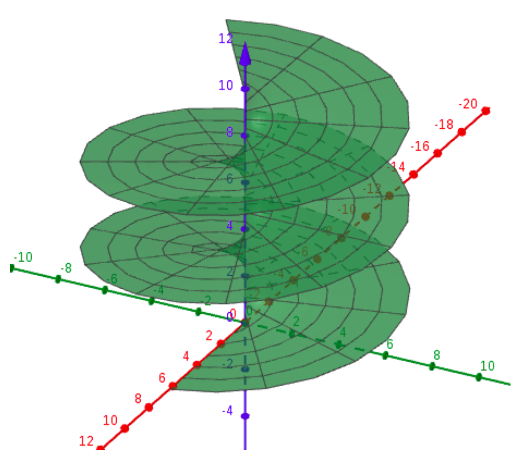
$$\sigma(u, v) = (v \cos u, v \sin u, \lambda u)$$

where λ is a constant. Show that the cotangent of the angle that the standard unit normal of σ at a point p makes with the z-axis is proportional to the distance of p from the z-axis.

In σ, v can be taken to represent the radius of the propeller of the plane (or just the length of a single blade, considering the length of propeller blades should be uniform).

u will then be the angle the propeller is making with some horizontal (x-axis) and represent how many rotations the propeller has made in radians cumulatively.

Lastly, λ will represent the distance the plane has traveled in the direction of z.



Finding σ's unit normal

σ's unit normal can be computed as follows:

$$\hat{n} = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$$

But first, its normal is:

$$\sigma_u \times \sigma_v = n = (-\lambda \sin u, \lambda \cos u, -v)$$

And the norm of the normal is:

$$\begin{aligned} ||n|| &= \sqrt{\lambda^2 \sin^2 u + \lambda^2 \cos^2 u + v^2} \\ &= \sqrt{\lambda^2 + v^2} \end{aligned}$$

$\therefore$, the unit normal to σ is:

$$\hat{n} = \left(\frac{-\lambda \sin u}{\sqrt{\lambda^2 + v^2}}, \frac{\lambda \cos u}{\sqrt{\lambda^2 + v^2}}, \frac{-v}{\sqrt{\lambda^2 + v^2}} \right)$$

Relating the angle that $\hat{n}$ makes with the z-axis to its *distance* from the z-axis:

To find the cotangent of the angle that $\hat{n}$ makes with the z-axis when $\hat{n}$ is at a point p, we'll need $\sin\theta$ and $\cos\theta$. We can find these values by dotting and crossing $\hat{n}$ with the standard unit vector $k =< 0, 0, 1 >$:

$$k \cdot \hat{n} = ||k||\ ||\hat{n}|| \cos\theta$$

and

$$||k \times \hat{n}|| = ||k||\ ||\hat{n}|| \sin\theta.$$

With $||k||$ and $||\hat{n}||$ both equal to 1, we then arrive at the following results:

$$k \cdot \hat{n} = \cos\theta = -\frac{v}{\sqrt{\lambda^2 + v^2}}$$

and

$$||k \times \hat{n}|| = \sin\theta = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}.$$

Compute the cotangent:

$$\begin{aligned}\cot\theta &= \frac{\cos\theta}{\sin\theta} \\ &= -\frac{v}{\sqrt{\lambda^2 + v^2}}\ \frac{\sqrt{\lambda^2 + v^2}}{\lambda} \\ &= -\frac{v}{\lambda}\end{aligned}$$

To relate this value to the distance of p from the z-axis, we first need to look at p as a point at, say, $\sigma(x_0, y_0)$:

$$p = \sigma(x_0, y_0) = (y_0 \cos x_0, y_0 \sin x_0, \lambda x_0)$$

Because we are only interested in distance from the z-*axis*, we can ignore the k-component of $\sigma(x_0, y_0)$ and treat it as a planar function:

$$p = \sigma(x_0, y_0) = (y_0 \cos x_0, y_0 \sin x_0)$$

Now to find the distance of p to the z-axis (or to the origin in the xy-plane):

$$\begin{aligned}||p|| &= \sqrt{y_0^2 \cos^2 x_0 + y_0^2 \sin^2 x_0} \\ &= \sqrt{y_0^2} \\ &= y_0\end{aligned}$$

We can see that the distance from p to the z-axis is equal to y_0 in this case. Knowing that $y_0 \in v$ as $\sigma(x_0, y_0) \in \sigma(u, v)$, looking at $\cot\theta$ again:

$$-\frac{v}{\lambda}$$

we could restate the above as:

$$-\frac{y_0}{\lambda}$$

which shows that the distance of p from the z-axis directly affects $\cot\theta \;\forall\; x, y \in u, v$.