Chapter 4, Section 4. Exercises 1, 2, and 4.

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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Exercise 4.4.1

Find the equation of the tangent plane of each of the following surface patches at the indicated points:

$$(i)\ \sigma(u,v) = (u, v, u^2 - v^2),\ (1,1,0)$$
$$(ii)\ \sigma(r,\theta) = (r\cosh\theta, r\sinh\theta, r^2),\ (1,0,1)$$

Preliminaries

The equation of a tangent plane to a surface $z = f(x,y)$ at point (x_0, y_0, z_0) is as follows:

$$z = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + z_0$$

Solutions

(i):
We are given $\sigma(u,v) = (u, v, u^2 - v^2)$ and point $(1,1,0)$.
The partial derivatives of σ are

$$\sigma_u = (1, 0, 2u),\ \sigma_v = (0, 1, -2v).$$

Computing σ_u and σ_v at $(1,1,0)$ gives

$$\sigma_u(1,1,0) = (1,0,2),\quad \sigma_v(1,1,0) = (0,1,-2).$$

Crossing these partials gives us our normal:

$$\sigma_u \times \sigma_v = (-2, 2, 1)$$

Our tangent plane will then be

$$-2(x-1)+2(y-1)+(z)=0$$

$$(-2x+2)+(2y-2)+(z)=0$$

$$-2x+2y+z=0$$

(ii):
Same procedure as for (i).
We are given $\sigma(r,\theta)=(r\cosh\theta, r\sinh\theta, r^2)$ and point $(1,0,1)$.
The partial derivatives of σ are

$$\sigma_r(\cosh\theta, \sinh\theta, 2r),\ \ \sigma_\theta=(r\sinh\theta, r\cosh\theta, 0)$$

From here, we'll solve for r, θ:

$$1=r\cosh\theta$$

$$0=r\sinh\theta$$

$$1=r^2$$

Solving the above yields $r=1,\ \theta=0$.

Finding the normal:

$$\sigma_r\times\sigma_\theta=(-2r^2\cosh\theta, 2r^2\sinh\theta, r)$$

Using found r and θ values, the normal is $(-2,0,1)$.
Our tangent plane is then

$$-2(x-1)+0(y-0)+1(z-1)=0$$

$$-2x+z+1=0$$