Chapter 4, Section 2. Exercises 1, 2, and 4 through 9

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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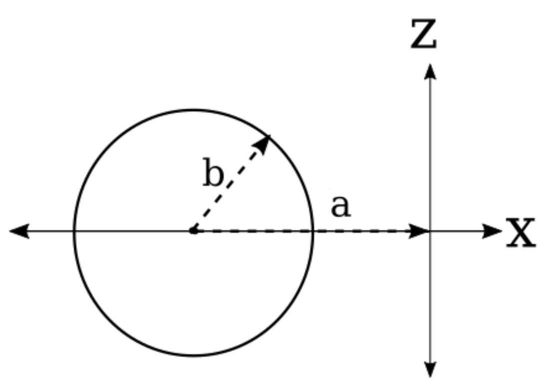
Exercise 4.2.5

A *torus* is obtained by rotating a circle C in a plane Π around a straight line l in Π that does not intersect C. Take Π to be the xz-plane, l to be the z-axis, $a > 0$ the distance of the centre of C from l, and $b < a$ the radius of C. Show that the torus is a smooth surface with parametrization

$$\sigma(\theta, \phi) = (\ (a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta\ )$$

Initial observations

Looking at the intersection of σ with the xz-plane when $\phi \approx 0$ (meaning, the xz-plane would have not quite begun rotating yet), will give us the following image:



Pictured is circle C (not labeled), non-zero distance a between axis of rotation z and the center of C, and radius b, which is less than a in length.

When $\phi = \pi$, we will have two circles in the xz-plane; one a mirror image of C, on the other size of the z-axis.

Demonstrating that σ is a smooth surface:

First, partial differentiation:

$$\sigma_\theta = (-b\cos\phi\sin\theta,\ -b\sin\phi\sin\theta,\ b\cos\theta)$$

$$\sigma_\phi = (-a\sin\phi - b\sin\phi\cos\theta,\ a\cos\phi + b\cos\phi\cos\theta,\ 0)$$

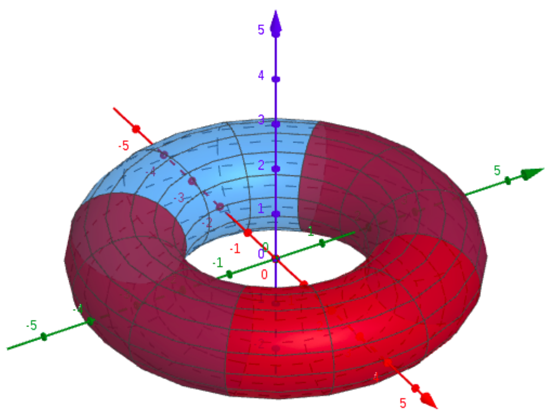
Because $\sigma \in C^\infty$, we can say it is smooth.

Now, to create an atlas for σ; its patches will be σ_u and $\sigma_{\tilde{u}}$, and they will both map U and $\widetilde{U}$ to torus S, respectively.

Let U and $\widetilde{U}$ be defined as follows:

$$U = \{\theta, \phi \mid 0 < \theta < 2\pi, \frac{\pi}{4} < \phi < \frac{7\pi}{4}\}$$

$$\widetilde{U} = \{\theta, \phi \mid 0 < \theta < 2\pi, -\frac{3\pi}{4} < \phi < \frac{3\pi}{4}\}$$



In the image above, the overlap/intersection of σ_u (blue) and $\sigma_{\tilde{u}}$ (red) takes on a purple color (or almost-purple).