Chapter 4, Section 2. Exercises 1, 2, and 4 through 9

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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Exercise 4.2.1

Show that, if $f(x, y)$ is a smooth function, its graph

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$$

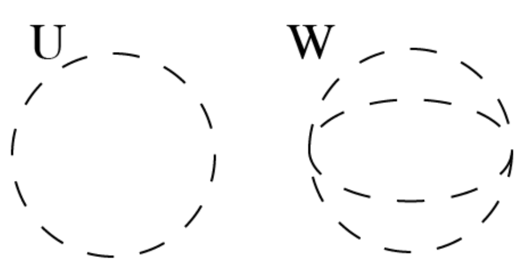
is a smooth surface with atlas consisting of the single regular surface patch

$$\sigma(u, v) = (u, v, f(u, v)).$$

In fact, every surface is "locally" of this type - see Exercise 5.6.4.

If $f(x, y)$ is smooth, that implies $f \in C^n$; f is continuously differentiable to the nth order.
If $z = f(x, y)$ is a smooth surface, then $z \in C^n$, as well.

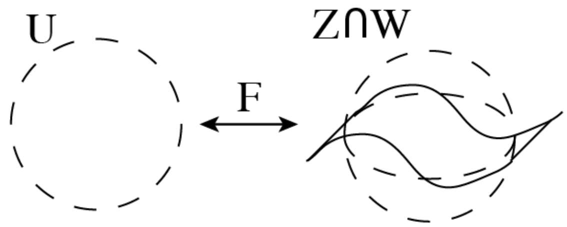
To show that z is a surface, let there be open sets U and W in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.



Then, let there be some mapping function F, that will map U to the intersection of z and W:

$$F : U \to z \cap W$$
$$F^{-1} : z \cap W \to U$$

Which could be loosely visualized as the following if doing so proves useful:



Meaning, F is a smooth, bijective map, and therefore a homeomorphism between $z \cap W$ and U.

z is now a surface because it can be made into surface patches (homeomorphic open subsets).

<u>z has a single regular surface patch:</u>

As stated in the problem, let z's surface patch be parametrized as the following:

$$\sigma(u, v) = (u, v, f(u, v)).$$

Then we can take for granted that $(u, v) \in U$ and $U \in \mathbb{R}^2$.

We can also use the aforementioned mapping function F to map $(u, v) \in U$ to $z \cap W$, but refer to it as σ (as it is referred to in the text).

Here will follow some information about z that points to its atlas having only a single patch:

Smoothness:

Because z is smooth (and smooth all over), there are no "holes" or other singularities for all u and v mapped to $z \cap W$ by σ.

No disjoint points:

Because $z \cap W$ and U are open, there are no disjoint points within either:

$$\{u, v \in U \mid ||v - u|| < r\}$$

Here, r is the radius of U as an open disc. Similarly, r may be the radius of $z \cap W$ as an open ball.

$\therefore$, we will be able to approach r in $z \cap W$ from any point and cover the entirety of the surface, showing that z's atlas consists solely of σ.