Chapter 4, Section 1. Exercises 1, 2, 3 and 5

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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October 9, 2018

Exercise 4.1.2

Define surface patches $\sigma^x_\pm : U \to \mathbb{R}^3$ for S^2 by solving the equation $x^2 + y^2 + z^2 = 1$ for x in terms of y and z:

$$\sigma^x_\pm(u, v) = (\pm\sqrt{1 - u^2 - v^2}, u, v)$$

defined on the open set $U = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$. Define $\sigma^y_\pm$ and $\sigma^z_\pm$ similarly (with the same U) by solving for y and z, respectively. Show that these six patches give S^2 the structure of a surface.

<u>Surface patch definition for S^2:</u>

The six surface patches for S^2 are

$$x = \pm\sqrt{1 - z^2 - y^2}$$

$$y = \pm\sqrt{1 - z^2 - x^2}$$

$$z = \pm\sqrt{1 - y^2 - x^2}$$

which can be parametrized, as a stated in the problem, as

$$\sigma^x_\pm(u, v) = (\pm\sqrt{1 - u^2 - v^2}, u, v)$$

$$\sigma^y_\pm(u, v) = (u, \pm\sqrt{1 - u^2 - v^2}, v)$$

$$\sigma^z_\pm(u, v) = (u, v, \pm\sqrt{1 - u^2 - v^2})$$

Then, for the open set U defined in the problem as $u^2 + v^2 < 1$, which also looks like $Int(x^2 + y^2 = 1)$,

$$\sigma : U \to S^2.$$

Proof that S^2 is a surface:

For any of the six patches , say, $x = +\sqrt{1 - z^2 - y^2}$, there will be a point $p \in U$ that can be mapped to S^2 by σ_+^x. The union of patches σ_+^x and σ_-^x, will then make a surface, as will the union of any other pair of homeomorphisms of opposing signs from the six patches.