Chapter 4, Section 2. Exercises 1, 2, and 4 through 9

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

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Exercise 4.2.4

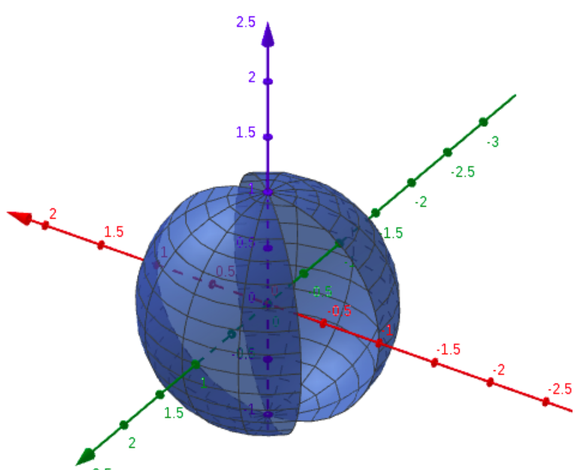
Show that the ellipsoid

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1,$$

where p, q and r are non-zero constants, is a smooth surface.

Let S equal our ellipsoid; S can be parametrized as the following rotational solid:

$$\sigma = (p \;\cos u \;\cos t, q \cos u \;\sin t, r \sin u)$$



Demonstrating smoothness via partial differentiation

If σ is smooth, then $\sigma \in C^n$; it is differentiable continuously to order n. The first partial derivatives are:

$$\sigma_t = (-p\cos u \ \sin t, q\cos u \ \cos t, 0)$$
$$\sigma_u = (-p\cos t \ \sin u, -q\sin t \ \sin u, r\cos u)$$

Since sin and cos $\in C^\infty$, it is then implied that $\sigma \in C^\infty$, making σ smooth.

Showing that σ is a surface via its atlas

To be a surface, σ must have an atlas.

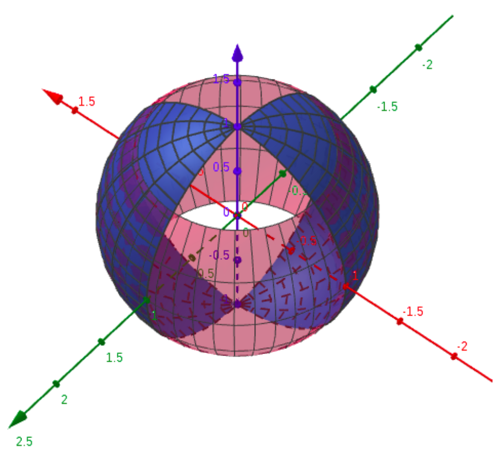
Let the atlas of σ consist of the patches σ_U and $\sigma_{\widetilde{U}}$, which map U and $\widetilde{U}$ to S:

$$\sigma_U : U \to S$$
$$\sigma_{\widetilde{U}} : \widetilde{U} \to S$$

where U and $\widetilde{U}$ are defined as follows:

$$U = \{t, u \mid 0 < t < 2\pi, -\frac{\pi}{2} < u < \frac{\pi}{2}\}$$

$$\widetilde{U} = \{t, u \mid 0 < t < \pi, 0 < u < 2\pi\}$$



Pictured above is this atlas of S. As σ_u (in red) closes about the north and south poles of S (not visible), $\sigma_{\tilde{u}}$ (in blue) closes like a shell.