Chapter 3, Section 2. Exercises 1 and 2

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

Matthew Connelly

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Exercise 3.2.2

By applying the isoperimetric inequality to the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

(where p and q are positive constants), prove that

$$\int_0^{2\pi} \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}\ dt\ \geq 2\pi\sqrt{pq}\ ,$$

with equality holding if and only if $p = q$.

Let $\gamma(t) = (p\ cost, q\ sint)$ be our ellipse.

Then, differentiate γ and find its norm:

$$\gamma'(t) = (-p\ cost, q\ cost)$$

$$||\gamma'(t)|| = \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}$$

We know the following definitions of $l(\gamma)$ and $A(\gamma)$:

$$A(\gamma) = \frac{1}{2}\int_0^T (xy' - yx')\, dt$$

$$l(\gamma) = \int_0^T ||\gamma'(t)||\ dt$$

γ is T-periodic.

We can expand these definitions using our current definition of γ:

$$A(\gamma) = \frac{1}{2}\int_0^{2\pi} \left(pq\ cos^2 t + pq\ sin^2 t\right) dt$$

$$l(\gamma) = \int_0^{2\pi} \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}\ \ dt$$

Simplifying $A(\gamma)$:

$$A(\gamma) = \frac{1}{2}\int_0^{2\pi} pq\ dt = \pi\ pq$$

Then we can use the isoperimetric inequality:

$$A(\gamma) \le \frac{l(\gamma)^2}{4\pi}$$

After expansion:

$$\pi\ pq\ \le\ \frac{\left(\int_0^{2\pi} \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}\ dt\right)^2}{4\pi}$$

$$4\pi^2\ pq\ \le\ \left(\int_0^{2\pi} \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}\ dt\right)^2$$

$$2\pi\ \sqrt{pq}\ \le\ \int_0^{2\pi} \sqrt{p^2\ sin^2 t + q^2\ cos^2 t}\ dt$$

When $p = q$:

Let radius $R = p = q$.

$$2\pi\ \sqrt{R^2}\ \le\ \int_0^{2\pi} \sqrt{R^2}\ dt$$

$$2\pi\ R\ \le\ R\int_0^{2\pi} 1\ dt \qquad R \text{ is constant.}$$

$$2\pi\ R\ =\ R\ 2\pi$$