Chapter 4, Section 1. Exercises 1, 2, 3 and 5

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

Matthew Connelly

October 9, 2018

Exercise 4.1.3

The *hyperboloid of one sheet* is

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}.$$

Show that, for every θ, the straight line

$$(x - z)cos\theta = (1 - y)sin\theta, \ (x + z)sin\theta = (1 + y)cos\theta$$

is contained in S, and that every point of the hyperboloid lies on one of these lines. Deduce that S can be covered by a single surface patch, and hence is a surface. (Compare the case of the cylinder in Example 4.1.3.)

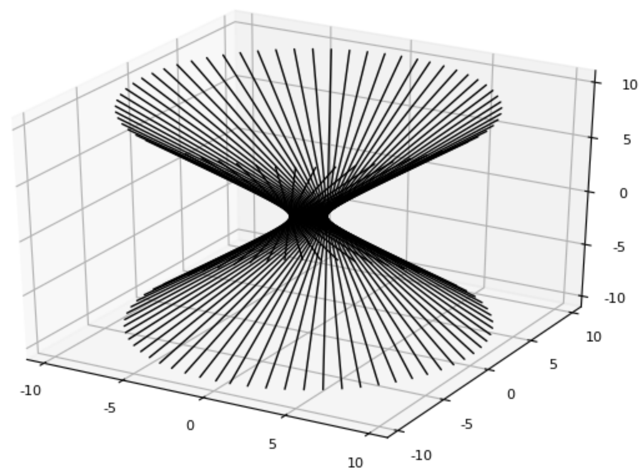
Find a second family of straight lines on S, and show that no two lines of the same family intersect, while every line of the first family intersects every line of the second family with one exception. One says that the surface S is doubly ruled.

Lines in hyperboloid S:

The line mentioned in the problem can be parametrized as

$$r = (cos\theta - sin\theta\ t, sin\theta + cos\theta\ t, t)$$

When rotated about a unit circle in the xy-plane, r will trace out a hyperboloid of one sheet.



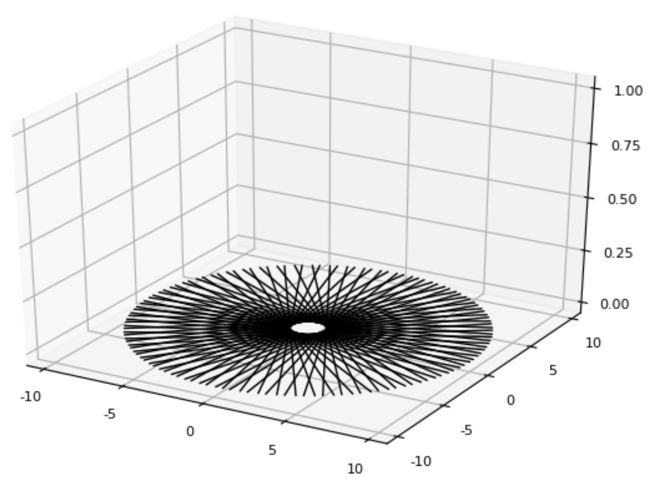
A collection of r lines at various points of rotation.

All points within S will lie on one of these lines after a single period of rotation $\theta \in [0, 2\pi)$. Then, for a homeomorphism σ: by restricting σ's domain to a open interval of $(0, 2\pi)$ (similar to a single period of r's rotation), σ will be made injective. Then, for an open disc $U \in \mathbb{R}^2$,

$$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}$$

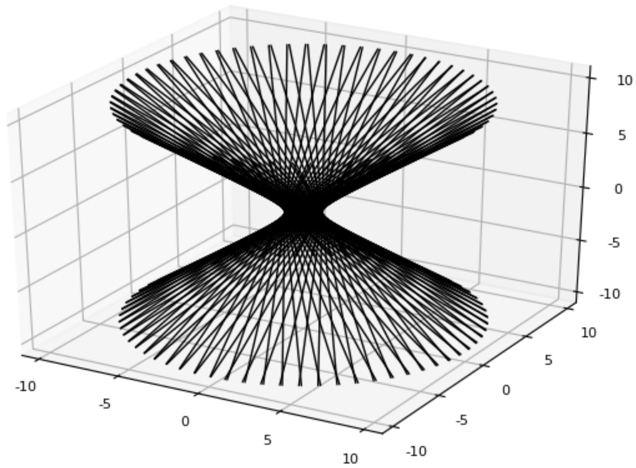
$$\sigma : U \to S$$

showing S to be coverable by a single surface patch σ.



Lines in U, rotated about $x^2 + y^2 = 1$, similar to r.

A second set of Lines in hyperboloid S:



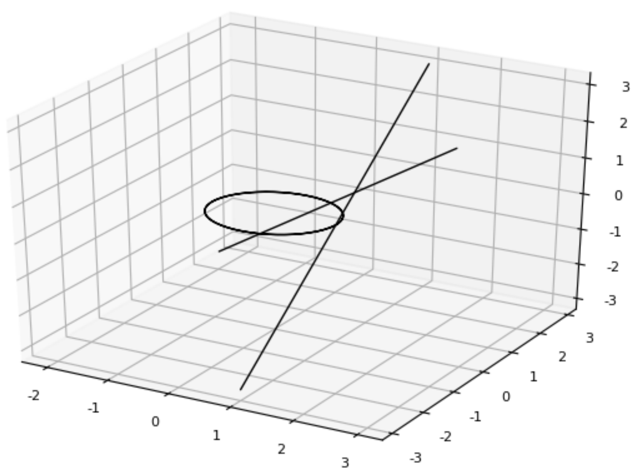
A collection of r_2 lines intersecting r lines.

Let r_2 be a second set of lines on S:

$$r_2 = (sin\theta + cos\theta \ t, cos\theta - sin\theta \ t, t)$$

No two lines of r will intersect each other, for about a point p on a unit circle C in $\mathbb{R}^2$, the line intersecting C at $p + \delta$ will be parallel to the line intersecting C at p. All lines in r will be skew to each other, unlike the lines in $U \in \mathbb{R}^2$, shown in the earlier graphic, which must always intersect and cannot be skew.

Lines in r_2 will always intersect lines at r, first at every point p on C, and then many other points as they are rotated about C and their distance over C increases.



An r line intersecting an r_2 line, both an arbitrary distance from each other on C.