Chapter 3, Section 1. Exercise 1 only

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

Matthew Connelly

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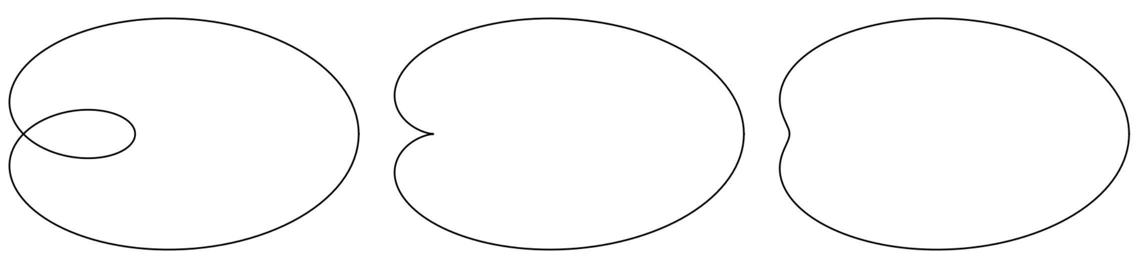
Exercise 3.1.1

Show that

$$\gamma(t) \;=\; ((1 + a\ cost)cost,\ (1 + a\ cost)sint)$$

where a is a constant, is a simple closed curve if $|a| < 1$, but that if $|a| > 1$ its complement is the disjoint union of three connected subsets of $\mathbb{R}^2$, two of which are bounded and one is unbounded.
What happens if $a = \pm 1$?



(a) $|a| > 1$ (b) $|a| = 1$ (c) $|a| < 1$

A limaçon (a) will degrade to a cardioid (b).

When $|a| > 1$, γ will have a self intersection. By definition, a simple closed curve is not to have any self-intersections.

Proof of self-intersection when $|a| > 1$:

For some $t_0,\ t_1 \in [0, T)$, where γ is T-periodic, γ will have the same position twice if γ has a self-intersection. Meaning,

$$\gamma(t_0) \ = \ \gamma(t_1)$$

but $t_0 \neq t_1$.

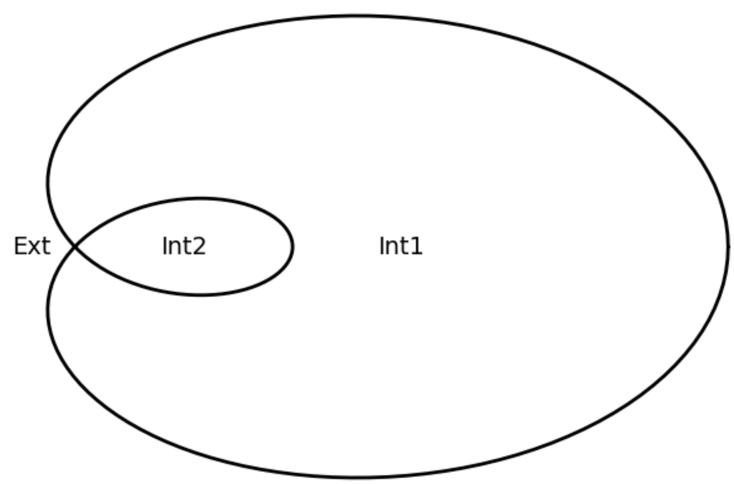
Solving $\gamma(t_0) = \gamma(t_1)$ for t_1 and t_0, we will arrive at

$$t_0 = cos^{-1}\left(-\frac{1}{a}\right)$$
$$t_1 = 2\pi - t_0$$

At t_0 and t_1, our position will be at the self-intersection and potential cusp of the curve if it were a cardioid; meaning, when $|a| = 1$.
This should prove that when $|a| > 1$, γ cannot be a simple closed curve.

γ's complement to disjoint sets $Int(\gamma)$ and $Ext(\gamma)$:

When $|a| > 1$, γ has a self-intersection, and therefore two sets of $Int(\gamma)$, as well as a single $Ext(\gamma)$ set; these sets are disjoint and complement γ as the union of $\sim \gamma$.



Int_2 is disjoint from Int_1, which is disjoint from Ext.

We can then classify these sets as follows, starting from the inside of γ, using t_0 and t_1 from the earlier proof of self-intersection:

$$\forall (x, y) \in \mathbb{R}^2,$$
$$Int_2(\gamma) = (x, y) < [\gamma(t_0), \gamma(t_1)]$$
$$Int_1(\gamma) = [\gamma(t_0), \gamma(t_1)] < (x, y) < [\gamma(0), \gamma(t_0)] \cup [\gamma(t_1), \gamma(2\pi)]$$
$$Ext(\gamma) = [\gamma(0), \gamma(2\pi)] < (x, y)$$

These sets can be written in the following inequality:

$$Int_2(\gamma) < Int_1(\gamma) < Ext(\gamma)$$

and summarized as the following equation:

$$\sim \gamma = Int_2(\gamma) \cup Int_1(\gamma) \cup Ext(\gamma)$$

When $|a| = 1$:

We will officially have a limaçon when $|a| = 1$, which means that

$$\gamma(t_0) \;=\; \gamma(t_1)$$

and $t_0 = t_1$; there is no self-intersection. The point $\gamma(t_n)$ will be the cusp singularity of the cardioid where our self-intersection would be on a cardioid. This curve will be simple and closed.