Chapter 2, Section 2. Exercises 1, 4-6

MTH 594, Prof. Mikael Vejdemo-Johansson
Differential Geometry Independent Study

Matthew Connelly

September 22, 2018

Exercise 2.2.4

[Part 1]
Let k be the signed curvature of a plane curve C expressed in terms of its arc-length. Show that, if C_a is the image of C under the dilation of $\mathbf{v} \to a\mathbf{v}$ of the plane (where a is a non-zero constant), the signed curvature of C_a in terms of *its* arc-length s is $\frac{1}{a}k(\frac{s}{a})$.

[Part 2]
A heavy chain suspended at its ends hanging loosely takes the form of a plane curve C. Show that, if s is the arc-length of C measured from its lowest point, ϕ the angle between the tangent of C and the horizontal, and T the tension in the chain, then

$$Tcos\phi = \lambda \ , \ Tsin\phi = \mu s$$

where λ, μ are non-zero constants (we assume that the chain has constant mass per unit length). Show that the signed curvature of C is

$$k_s = \frac{1}{a}\left(1 + \frac{s^2}{a^2}\right)^{-1}$$

where $a = \lambda/\mu$, and deduce that C can be obtained from the catenary in Example 2.2.4 by applying a dilation and an isometry of the plane.

[Part 1]
Showing that the signed curvature of C_a is $\frac{1}{a}k(\frac{s}{a})$.

First, relating the arc-length of C to that of C_a:

$C = (x,\ y) \qquad C_a = (ax,\ ay)$
k is the signed curvature of C in respect to its arc-length.

Let s_c be the arc-length of C, given s is the arc-length of C_a.

$\therefore\ \ s = a\ s_c$ C_a's arc-length = C's arc-length · a

Proof:

$s_c = \int_0^t \parallel C' \parallel dt$ C's arc-length

$s_c = \int_0^t \sqrt{(x')^2 + (y')^2}\ dt$

$s = \int_0^t \parallel C_a' \parallel dt$ C_a's arc-length

$s = \int_0^t \sqrt{(ax')^2 + (ay')^2}\ dt$

$s = \int_0^t \sqrt{a^2[\ (x')^2 + (y')^2\ ]}\ dt$

$s = \int_0^t \sqrt{a^2}\sqrt{(x')^2 + (y')^2}\ dt$

$s = \int_0^t a\sqrt{(x')^2 + (y')^2}\ dt$

$s = a\int_0^t \sqrt{(x')^2 + (y')^2}\ dt$ a is constant; commuted outside of integral expression

$s = a\ s_c$

Lastly, relating the curvature of C to that of C_a:

$$k = n_s k_s \qquad C\text{'s signed curvature = unit normal} \cdot \text{tangent's rate of turning}$$

C's rate of turning $k_s = \dot{\phi}$:

$$\phi = tan^{-1}\frac{y'}{x'} \qquad \text{Angle between } C\text{'s tangent and the horizontal}$$

$$\dot{\phi} = \frac{d\phi}{ds} = \frac{1}{1-(\frac{y'}{x'})^2} \cdot \frac{x'y'' - x''y'}{(x')^2} = k \qquad C\text{'s tangent's rate of turning}$$

C_a's rate of turning $\dot{\phi}_a$:

$$\phi_a = tan^{-1}\frac{ay'}{ax'} = tan^{-1}\frac{y'}{x'} = \phi \qquad \text{Angle between } C_a\text{'s tangent and the horizontal}$$

$$\therefore \qquad \dot{\phi}_a = \dot{\phi} = k_s \qquad C\text{'s rate of turning} = C_a\text{'s rate of turning}$$

To relate all of this to $\frac{1}{a}k(\frac{s}{a})$:

Let $n_s^a = (-a\ sin\phi,\ a\ cos\phi)$ be C_a's unit normal; a $\frac{\pi}{2}$ rotation of C_a's tangent $\mathbf{t}$.
n_s is C's unit normal.

$k(s_c) = k\left(\frac{s}{a}\right)$ Because $s = a \cdot s_c$ and C and C_a have the same turning rate

$n_s \cdot k_s = n_s^a \cdot k_s$ Expansion of k at s_c and $\frac{s}{a}$

$(-sin\phi,\ cos\phi) \cdot \dot{\phi} = (-a\ sin\phi,\ a\ cos\phi) \cdot \dot{\phi}$ Further expansion

$(-sin\phi,\ cos\phi) \cdot \dot{\phi} \cdot \frac{1}{a} = (-sin\phi,\ cos\phi) \cdot \dot{\phi}$ Division by constant of dilation

$k(s_c) \cdot \frac{1}{a} = k\left(\frac{s}{a}\right) \cdot \frac{1}{a}$ Simplification

[Part 2]

Show:

$$Tcos\phi = \lambda\ ,\ Tsin\phi = \mu s$$

Where λ, μ are non-zero constants.

Curve C is described as a heavy chain with tension T; a catenary.
$\therefore\ C = (t,\ cosht),\ C' = (1,\ sinht)$

Unit tangent vector: $\mathbf{t} = (cos\phi,\ sin\phi)$
Arc-length $s = sinht$, because $s = \int_0^t \sqrt{1 + cosh^2 t} = sinht$ for a catenary.

$$C' = (1,\ sinht)$$

$$C' = \mathbf{t} \qquad \text{Relating } C' \text{ to arc-length of } C$$

$$TC' = T\mathbf{t} \qquad \text{Scaling by constant of tension}$$

$$TC' = (Tcos\phi,\ Tsin\phi) \qquad \text{Expansion of } T\mathbf{t}$$

$$\therefore\ TC' = (\lambda,\ \mu s) \qquad \text{Expansion of } T\mathbf{t}$$

Also show:

$$k_s = \frac{1}{a}\left(1 + \frac{s^2}{a^2}\right)^{-1}$$

The signed curvature of C is k_s.

Given: $a = \frac{\lambda}{\mu}$
From earlier definitions of λ, μ:

$$\mu = \frac{Tsin\phi}{s}$$

$$\therefore\ a = \frac{Tcos\phi}{Tsin\phi} \cdot s$$

Which can be simplified:

$$a = \frac{cos\phi}{sin\phi} \cdot s \qquad \text{Cancellation of } T$$

$$a = \frac{1}{tan\phi} \cdot s$$

$$a = \frac{1}{tan\phi} \cdot sinh \qquad \text{Expansion of arc-length } s$$

$$\therefore\ a = \frac{sinh}{tan\phi}$$

However,
$tan\phi = sinh$, making $a = \frac{\lambda}{\mu} = 1$.

Justification:
$tan\phi = \frac{y'}{x'}$ tan of angle between $\mathbf{t}$ and x-axis
$tan\phi = sinh$

Now expanding a and s in k_s:

$$k_s = \frac{1}{(1 + sinh^2 t)}$$