3: Random Variables

Fall 2023

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Gov 2002 (Harvard)

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    - What is the true Biden approval rate in the US?
- Today: given a probability distribution, what data is likely?
    - If we knew the true Biden approval, what samples are likely?

Roadmap

1. Random variables
2. Famous distributions
3. Cumulative distribution functions
4. Functions of random variables
5. Independent random variables

1/ Random variables

What are random variables?

Definition

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- Randomness comes from the randomness of the experiment.

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- Usually abstract away from the underlying sample space fairly quickly.

Types of r.v.s

- Two main types of r.v.s: discrete and continuous. Focus on discrete now.

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A r.v. X is **discrete** the values it takes with positive probability is finite ($X \in \{x_1, \dots, x_k\}$) or countably infinite ($X \in \{x_1, x_2, \dots\}$).

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- The **support** of X is the values x such that $\mathbb{P}(X = x) > 0$.

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- Often there are many ways to express a distribution.

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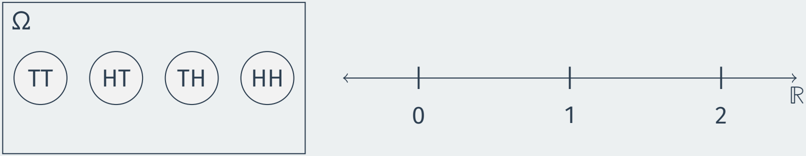

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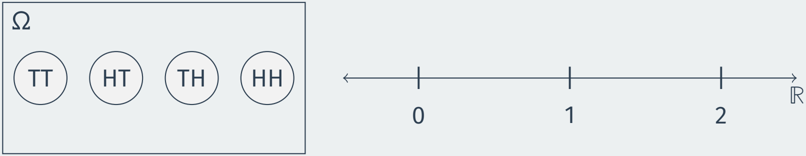

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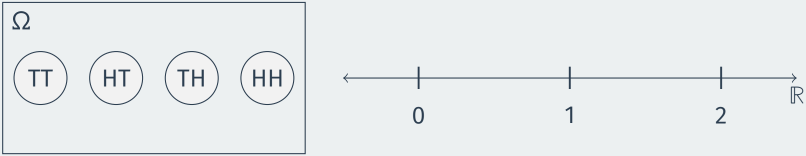

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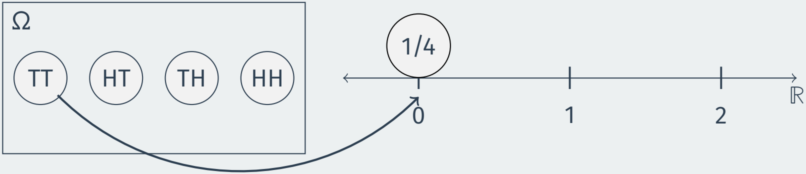

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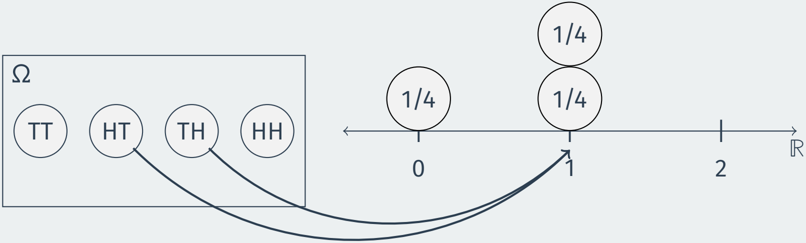

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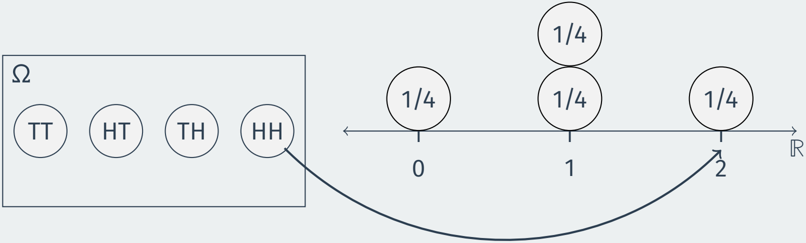

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- Probability of a set of values $S \subset \{x_1, x_2, \ldots\}$:

$$\mathbb{P}(X \in S) = \sum_{x \in S} p_X(x)$$

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- Let X be the number of treated units:

$$X = \begin{cases} 0 & \text{if } (C, C, C) \\ 1 & \text{if } (T, C, C) \text{ or } (C, T, C) \text{ or } (C, C, T) \\ 2 & \text{if } (T, T, C) \text{ or } (C, T, T) \text{ or } (T, C, T) \\ 3 & \text{if } (T, T, T) \end{cases}$$

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- Use independence and fair coins:

$$\mathbb{P}(C, T, C) = \mathbb{P}(C)\mathbb{P}(T)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

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- What's $\mathbb{P}(X = 4)$? 0!

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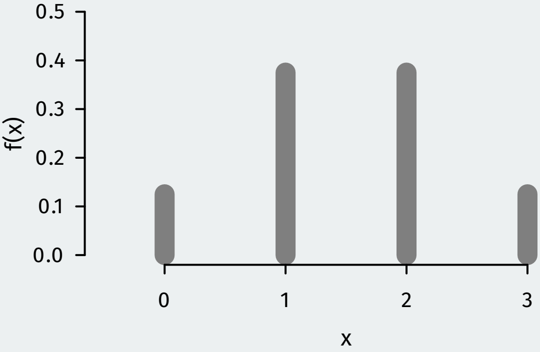

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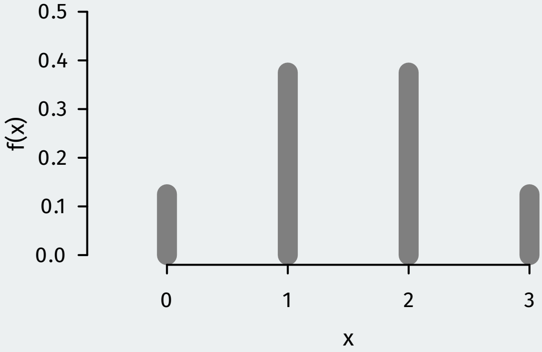

- **Question**: Does this seem like a good way to assign treatment? What is one major problem with it?

2/ Famous distributions

Bernoulli distribution

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- Actually a **family** of distributions indexed by p.
- Any event A has an associated Bernoulli r.v.: **indicator variable**:

$$\mathbb{I}(A) \sim \text{Bern}(p) \text{ with } p = \mathbb{P}(A)$$

Binomial distribution

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Let X be the number of successes in n independent Bernoulli trials all with success probability p. Then X follows the **binomial distribution** with parameters n and p, which is written $X \sim \text{Bin}(n, p)$.

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  - $\text{Bin}(1, p) \sim$ Bern(p).
  - If $X \sim \text{Bin}(n, p)$, then $n - X \sim \text{Bin}(n, 1 - p)$.

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If $X \sim \text{Bin}(n, p)$, then the p.m.f. of X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

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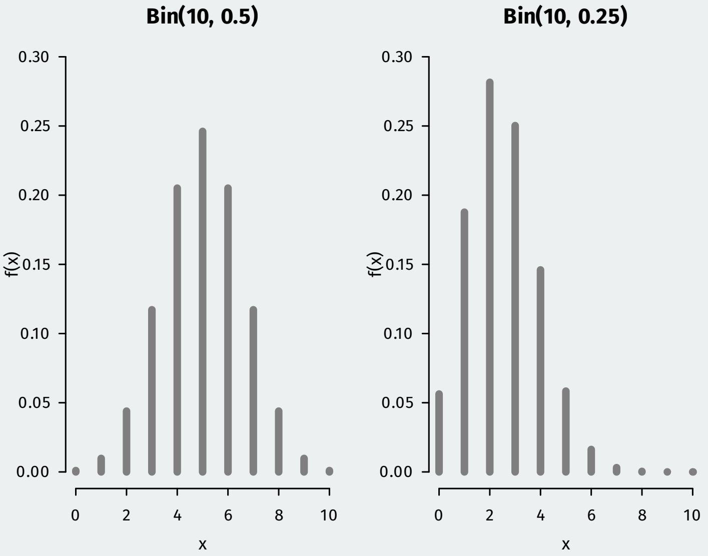
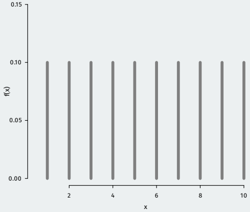
Some binomials



Discrete uniform distribution

Definition

Let C be a finite, nonempty set of numbers. If X is the number chosen randomly with all values equally likely, we say it follows the **discrete uniform** distribution.



- p.m.f. for a discrete uniform r.v.:

$$p_X(x) = \begin{cases} 1/|C| & \text{for } x \in C \\ 0 & \text{otherwise} \end{cases}$$

3/ Cumulative distribution functions

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- For discrete r.v.: $F_X(x) = \sum_{x_j \leq x} p_X(x_j)$

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- Remember example where X is the number of treated units:

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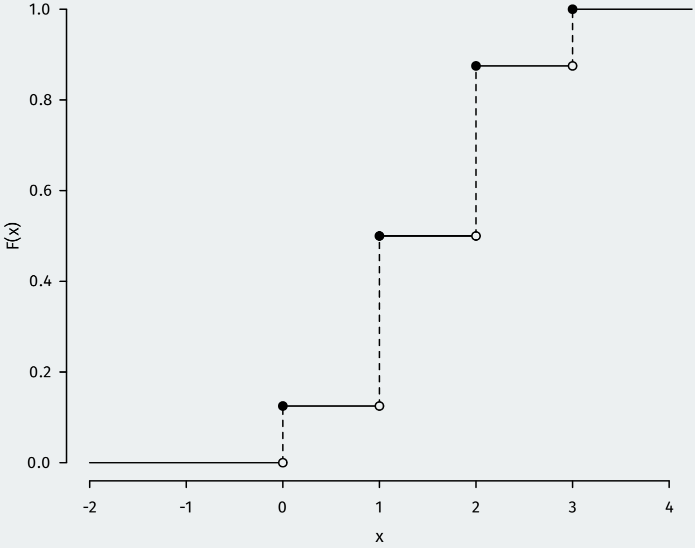

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3. **Right continuous**: no jumps when we approach a point from the right:

$$F(a) = \lim_{x \to a^+} F(x)$$

4/ Functions of random variables

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- If all x_j values map to a single y_j value ("one-to-one"), then we have:

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- If there are redundencies, we have to add those probabilities together:

$$\mathbb{P}(Y = y_j) = \mathbb{P}(g(X) = y_j) = \sum_{x_i : g(x_i) = y_j} \mathbb{P}(X = x_i)$$

Sum vs mean vs any

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- $Z = \mathbb{I}(X > 0)$: any successes (not one-to-one)

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2	3/8
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y	$\mathbb{P}(Y = y)$
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z	$\mathbb{P}(Z = z)$
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1	3/8 + 3/8 + 1/8 = 7/8

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- A few common examples:
  - If X and Y have the same distribution $\not\Rightarrow \mathbb{P}(X = Y) = 1$
  - Scaling an r.v. doesn't scale the p.m.f., so $Y = 2X$ does not have $p_Y(y) \neq 2p_X(x)$

5/ Independent random variables

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- For discrete r.v.s (not continuous), we can write this as:

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

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- **Theorem:** If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ with X and Y independent, then $X + Y \sim \text{Bin}(n + m, p)$.

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  - $\overline{X} = (1/n) \sum_i X_i$ is our estimate of p. Properties?