

13. Linear Model

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Gov 2002 (Harvard)

Where are we? Where are we going?

- Learned about the CEF in general, iterated expectation, etc.
- Now: focusing on when the CEF is (and isn't) linear.
- Linear model is ubiquitous but poorly understood. Lots of subtlety here.

Regression derivatives and partial effects

- Goal of regression: how mean of Y changes with X .
- For continuous regressors, we can use the partial derivative:

$$\frac{\partial \mu(x_1, \dots, x_k)}{\partial x_1}$$

- For binary X_1 , we can use the difference in conditional expectations:

$$\mu(1, x_2, \dots, x_k) - \mu(0, x_2, \dots, x_k)$$

- “Partial effect” of X_1 holding other included variables constant
- Exact form will depend on the functional form of $\mu(x)$.
 - How do we decide what form $\mu(x)$ should take?

Estimating the CEF for discrete covariates

- To motivate function form, useful to think about estimation.
- How do we estimate $\mu(x) = \mathbb{E}[Y|X = x]$ for binary X ?
- **Subclassification:** calculate sample averages with levels of X_i :

$$\hat{\mu}(1) = \frac{1}{n_1} \sum_{i=1}^n Y_i X_i$$

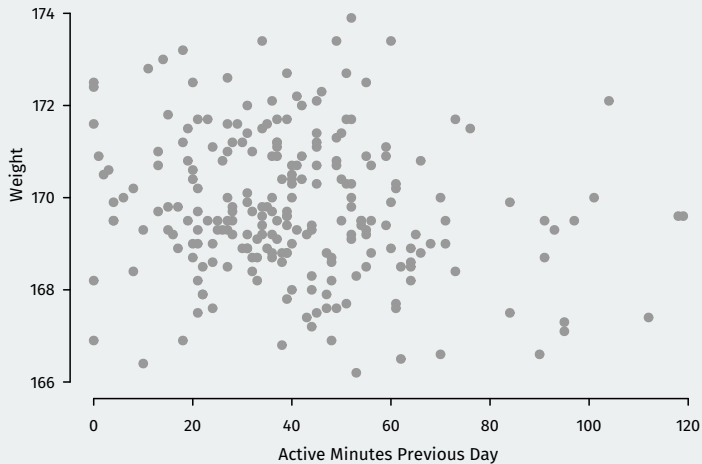
- $n_1 = \sum_{i=1}^n X_i$ is the number of units with $X_i = 1$ in the sample.
- More generally for any discrete X_i :

$$\hat{\mu}(x) = \frac{\sum_{i=1}^N Y_i \mathbb{I}(X_i = x)}{\sum_{i=1}^N \mathbb{I}(X_i = x)}$$

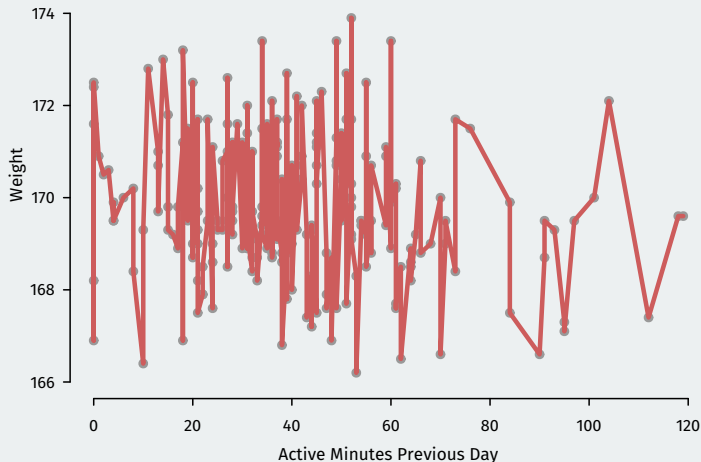
Continuous covariates

- What if X is continuous? Subclassification fall apart.
 - Each i has a unique value: $\sum_{i=1}^N \mathbb{I}(X_i = x) = 1$
 - Very noisy estimates
 - What about any x not in the sample?
- **Stratification:** bin X_i into categories and treat like as discrete.
 - Every x in the same bin gets the same conditional expectation.
 - Depends on arbitrary bin cutoffs/sizes.
- Example:
 - Personal data science: I wear an activity tracker and have a smart scale.
 - Relationship between my weight and active minutes in the previous day.

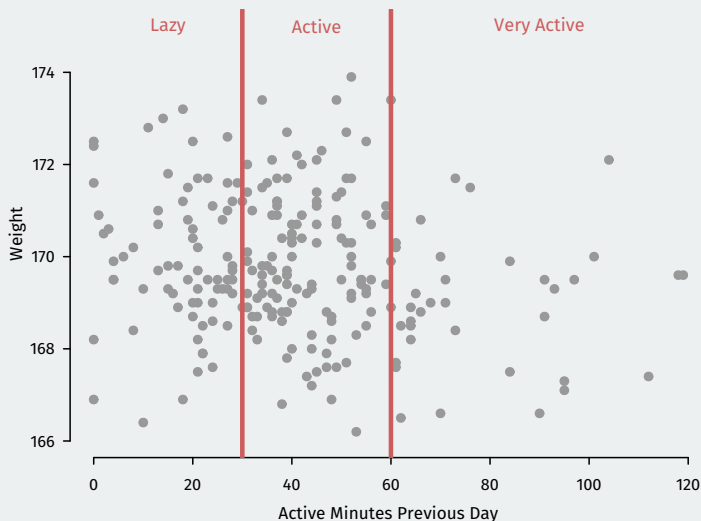
Continuous covariate example



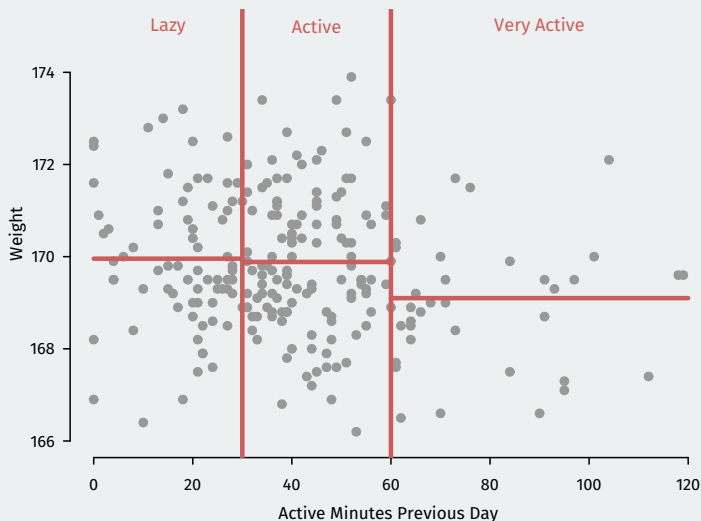
Continuous covariate CEF: interpolation



Continuous covariate CEF: stratification



Continuous covariate CEF: stratification



Linear CEFs

- Stratification requires lots of choices/hidden assumptions.
 - Number of categories, cutoffs for the categories, constant means within strata, etc.
- Alternative: assuming that the CEF is **linear**:

$$\mu(x) = \mathbb{E}[Y_i | X_i = x] = \beta_0 + \beta_1 x$$

- **Intercept**, β_0 : the condition expectation of Y_i when $X_i = 0$
- **Slope**, β_1 : change in the CEF of Y_i given a one-unit change in X_i

Why is linearity an assumption?

- Example: Y_i is income, X_i is years of education.
 - β_0 : average income among people with 0 years of education.
 - β_1 : expected difference in income between two adults that differ by 1 year of education.
- Why is linearity an assumption?

$$\begin{aligned}\mathbb{E}[Y_i|X_i = 12] - \mathbb{E}[Y_i|X_i = 11] &= \mathbb{E}[Y_i|X_i = 16] - \mathbb{E}[Y_i|X_i = 15] \\ &= \beta_1\end{aligned}$$

- Effect of HS degree is the same as the effect of college degree.
- Put another way: average partial effects are constant $\frac{\partial \mu(x)}{\partial x} = \beta_1$

Linear CEF with nonlinear effects

- What if we think the effect is nonlinear?
- We can include nonlinear transformations:

$$\mu(x) = \beta_0 + x\beta_1 + x^2\beta_2$$

- Partial effect now varies: $\partial\mu(x)/\partial x = \beta_1 + 2x\beta_2$
- **Linear** means linear in the parameters $\beta = (\beta_1, \dots, \beta_k)$, not \mathbf{X} .
- We can also include **interactions** between covariates:

$$\mu(x_1, x_2) = \beta_0 + x_1\beta_1 + x_2\beta_2 + x_1x_2\beta_3$$

- Average partial effect of X_1 depends on X_2 : $\partial\mu(x_1, x_2)/\partial x_1 = \beta_1 + x_2\beta_3$

Linear CEF with a binary covariate

- Wait-times (Y_i) and race ($X_i = 1$ for white, $X_i = 0$ for POC)
 - Two possible values of the CEF: μ_1 for whites and μ_0 for POC.
- Can write the CEF as follows:

$$\mu(x) = x\mu_1 + (1 - x)\mu_0 = \mu_0 + x(\mu_1 - \mu_0) = \beta_0 + x\beta_1$$

- No assumptions, just rewriting! Interpretations:
 - $\beta_0 = \mu_0$: expected wait-time for POC
 - $\beta_1 = \mu_1 - \mu_0$: diff. in avg. wait times between whites and POC.
- > 2 categories: dummies for all but category and everything is linear.

Linear CEF with multiple binary covariates

- What if we have two binary covariates, X_1 (race) and X_2 (1 urban/0 rural):

$$\mu(x_1, x_2) = \begin{cases} \mu_{00} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \text{ (POC, rural)} \\ \mu_{10} & \text{if } x_1 = 1 \text{ and } x_2 = 0 \text{ (white, rural)} \\ \mu_{01} & \text{if } x_1 = 0 \text{ and } x_2 = 1 \text{ (POC, urban)} \\ \mu_{11} & \text{if } x_1 = 1 \text{ and } x_2 = 1 \text{ (white, urban)} \end{cases}$$

- Can rewrite this without assumptions as a linear CEF with interaction:

$$\mu(x_1, x_2) = \beta_0 + x_1\beta_1 + x_2\beta_2 + x_1x_2\beta_3$$

- Interpretations:
 - $\beta_0 = \mu_{00}$: average wait times for rural POC.
 - $\beta_1 = \mu_{10} - \mu_{00}$: diff. in means for rural whites vs rural POC.
 - $\beta_2 = \mu_{01} - \mu_{00}$: diff. in means for urban POC vs rural POC.
 - $\beta_3 = (\mu_{11} - \mu_{01}) - (\mu_{10} - \mu_{00})$: diff. in urban racial diff. vs rural racial diff.
- Generalizes to p binary variables if **all interactions included (saturated)**

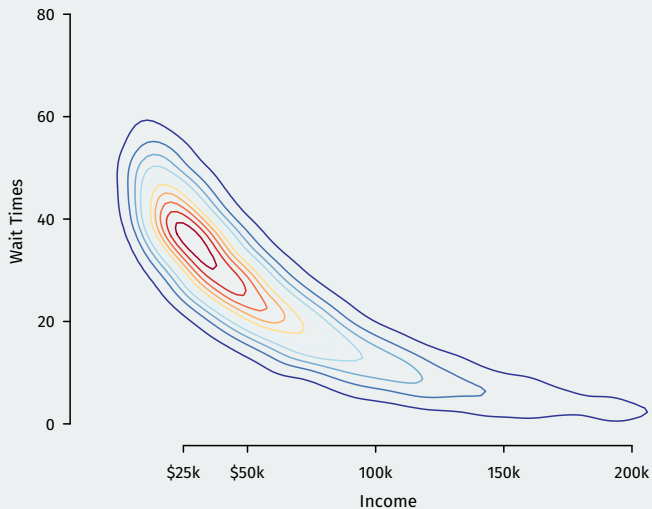
Linear approximation

- Outside of saturated discrete settings, CEF almost never truly linear.
- Alternative goal: find **best linear predictor** of Y given X .
- Formally, linear function of X that **minimizes squared prediction errors**:

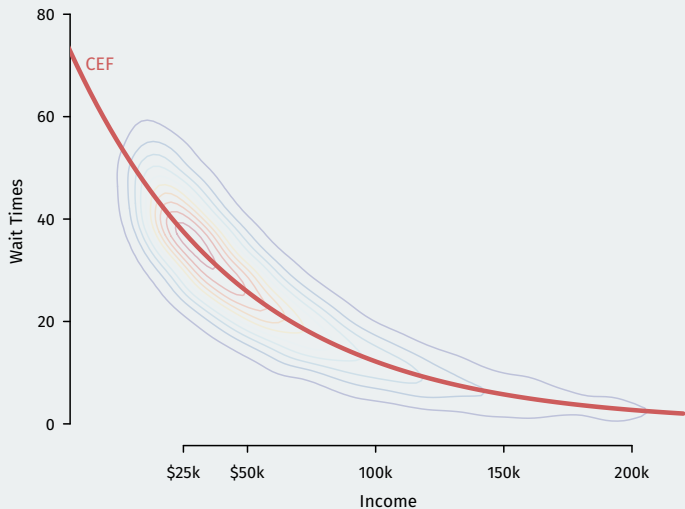
$$(\beta_0, \beta_1) = \arg \min_{(b_0, b_1)} \mathbb{E}[(Y - (b_0 + b_1 X))^2]$$

- $\mathbb{L}[Y | X] = \beta_0 + \beta_1 X$ is called the **linear projection** of Y onto X .
 - $\beta_1 = \text{Cov}(X, Y) / \mathbb{V}[X]$
 - $\beta_0 = \mu_Y - \mu_X \beta_1$, where $\mu_Y = \mathbb{E}[Y]$ and $\mu_X = \mathbb{E}[X]$
- In general, $\mathbb{L}[Y | X]$ distinct from the CEF:
 - CEF, $\mu(x)$ is the best predictor of Y_i among all functions.
 - Linear projection is best predictor among linear functions.

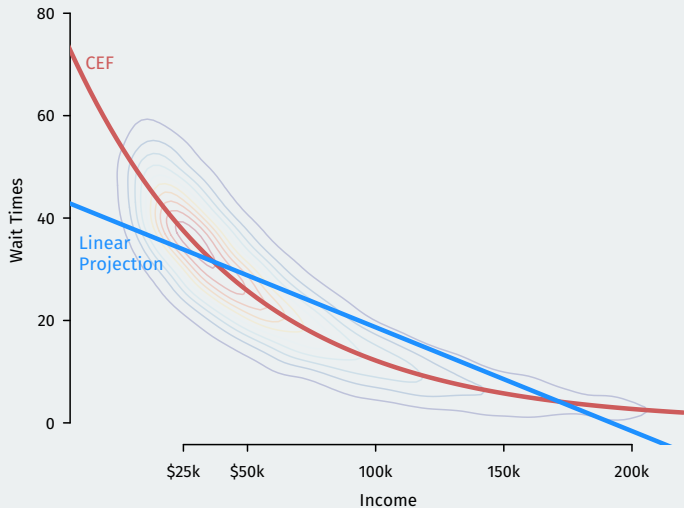
Linear approximation



Linear approximation



Linear approximation



Best linear predictor

- We'll almost always condition on a vector \mathbf{X} :

$$\mathbb{L}[Y \mid \mathbf{X}] = \mathbb{L}[Y \mid X_1, \dots, X_k] = X_1\beta_1 + \dots + X_k\beta_k = \mathbf{X}'\boldsymbol{\beta}$$

- Random vector ($k \times 1$) of covariates: $\mathbf{X} = (X_1, \dots, X_k)'$
 - May contain nonlinear transformations/interactions of “real” variables.
 - Typically, $X_1 = 1$ and is the intercept/constant.
- Assumptions (“Regularity conditions”):
 1. $\mathbb{E}[Y^2] < \infty$ (outcome has finite mean/variance)
 2. $\mathbb{E}\|\mathbf{X}\|^2 < \infty$ (\mathbf{X} has finite means/variances/covariances)
 3. $\mathbf{Q}_{\mathbf{X}\mathbf{X}} = \mathbb{E}[\mathbf{X}\mathbf{X}']$ is positive definite (columns of \mathbf{X} are linearly independent)

Linear Projection

- How to find β ? Minimize squared prediction error!

$$\beta = \arg \min_{\mathbf{b} \in \mathbb{R}^k} \mathbb{E} [(Y - \mathbf{X}'\beta)^2]$$

- After some calculus:

$$\beta = \mathbf{Q}_{\mathbf{X}\mathbf{X}}^{-1} \mathbf{Q}_{\mathbf{X}Y} = (\mathbb{E}[\mathbf{X}\mathbf{X}'])^{-1} \mathbb{E}[\mathbf{X}Y]$$

- $\mathbb{E}[\mathbf{X}\mathbf{X}']$ is $k \times k$ and $\mathbb{E}[\mathbf{X}Y]$ is $k \times 1$
- Notes about the $\mathbb{L}[Y \mid \mathbf{X}]$:
 - β is a population quantity and possible quantity of interest.
 - Well-defined under very mild assumptions!
 - Not necessarily a conditional mean nor a causal effect!

Projection errors

- Projection error: $e = Y - \mathbf{X}'\boldsymbol{\beta}$
- Decomposition of Y into the linear projection and error: $Y = \mathbf{X}'\boldsymbol{\beta} + e$
- Properties of the projection error:
 - $\mathbb{E}[\mathbf{X}e] = 0$
 - $\mathbb{E}[e] = 0$ when \mathbf{X} contains a constant.
 - Together, implies $\text{Cov}(X_j, e) = 0$ for all $j = 1, \dots, k$
- Distinct from CEF errors: $u = Y - \mu(\mathbf{X})$ which had the additional property: $\mathbb{E}[u \mid \mathbf{X}] = 0$
 - Zero conditional mean is stronger: CEF errors are 0 at every value of \mathbf{X}
 - $\mathbb{E}[\mathbf{X}e] = 0$ just says they are uncorrelated.

Regression coefficients

- Sometimes useful to separate the constant:

$$Y = \beta_0 + \mathbf{X}'\boldsymbol{\beta} + e$$

where \mathbf{X} doesn't have a constant.

- Solution for $\boldsymbol{\beta}$ more interpretable here:

$$\boldsymbol{\beta} = \mathbb{V}[\mathbf{X}]^{-1}\text{Cov}(\mathbf{X}, Y), \quad \beta_0 = \mu_Y - \boldsymbol{\mu}'_{\mathbf{X}}\boldsymbol{\beta}$$

Partitioned Regression

$$\mathbb{L}(Y \mid \mathbf{X}, \mathbf{Z}) = \mathbf{X}'\boldsymbol{\beta} + \mathbf{Z}'\boldsymbol{\gamma}$$

- Can we get an expression for just $\boldsymbol{\beta}$? With some tricks, yes!
- Population residuals from projection of \mathbf{X} on \mathbf{Z} : $\mathbf{R} = \mathbf{X} - \mathbb{L}(\mathbf{X} \mid \mathbf{Z})$.
 - \mathbf{R} is now **orthogonal** to \mathbf{Z} .
- Project Y onto these residuals gives $\boldsymbol{\beta}$ as coefficient: $\mathbb{L}(Y \mid \mathbf{R}) = \mathbf{R}'\boldsymbol{\beta}$

$$\boldsymbol{\beta} = (\mathbb{E}[\mathbf{R}\mathbf{R}'])^{-1} \mathbb{E}[\mathbf{R}Y]$$

- Also holds if we get residuals from projection of Y on \mathbf{Z} :
 $V = Y - \mathbb{L}(Y \mid \mathbf{Z})$.

$$\mathbb{L}(V \mid \mathbf{R}) = \mathbf{R}'\boldsymbol{\beta}$$

Omitted variable bias

- Consider two projections/regressions with and without some Z :

$$\mathbb{L}[Y \mid \mathbf{X}, \mathbf{Z}] = \mathbf{X}'\boldsymbol{\beta} + Z\gamma, \quad \mathbb{L}[Y_i \mid \mathbf{X}_i] = \mathbf{X}_i'\boldsymbol{\delta}$$

- $\mathbb{L}[Y \mid \mathbf{X}, \mathbf{Z}]$ is the long regression, $\mathbb{L}[Y \mid \mathbf{X}]$ is the short regression.
- How do $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ relate? Use law of iterated projections:

$$\mathbb{L}[Y \mid \mathbf{X}] = \mathbb{L}\{\mathbb{L}[Y \mid \mathbf{X}, \mathbf{Z}] \mid \mathbf{X}\} = \mathbb{L}[\mathbf{X} \mid \mathbf{X}]\boldsymbol{\beta} + \mathbb{L}[Z \mid \mathbf{X}]\gamma$$

- First regress/project Z on \mathbf{X} : $\mathbb{L}[Z \mid \mathbf{X}] = \mathbf{X}'\boldsymbol{\pi}$ and so:

$$\mathbb{L}[Y \mid \mathbf{X}] = \mathbf{X}'(\boldsymbol{\beta} + \boldsymbol{\pi}\gamma), \quad \boldsymbol{\delta} = \boldsymbol{\beta} + \boldsymbol{\pi}\gamma$$

- $\boldsymbol{\delta} - \boldsymbol{\beta} = \boldsymbol{\pi}\gamma$ is the “bias” but this is misleading.
 - $\boldsymbol{\beta}$ not necessarily “correct”, we’re just relating two projections
 - Difference is (coef of excluded) \times (effect of included on excluded)

Best linear approximation

- What is the relationship between $\mathbb{L}[Y | \mathbf{X}]$ and $\mu(\mathbf{X}) = \mathbb{E}[Y | \mathbf{X}]$?
 - If $\mu(\mathbf{X})$ is linear, then $\mu(\mathbf{X}) = \mathbb{L}[Y | \mathbf{X}]$.
 - But $\mu(\mathbf{X})$ could be nonlinear, what then?
- Linear projection justification: best linear approximation to $\mu(\mathbf{X})$:

$$\boldsymbol{\beta} = \arg \min_{\mathbf{b} \in \mathbb{R}^K} \mathbb{E} [(\mu(\mathbf{X}) - \mathbf{X}'\boldsymbol{\beta})^2]$$

- Linear projection is best linear approximation to Y and $\mathbb{E}[Y | X]$.
- Limitations:
 - If nonlinearity of $\mu(\mathbf{X})$ is severe, $\mathbb{L}[Y | X]$ can only be so good.
 - $\mathbb{L}[Y | \mathbf{X}]$ can be sensitive to the marginal distribution of \mathbf{X} .

$$Y = \mathbf{X}'\boldsymbol{\beta} + e$$

- “The Linear Model”: is this an assumption?
- Depends on what we assume about the error, e
 - If $\mathbb{E}[e \mid \mathbf{X}] = 0$, then we are assuming the CEF is linear, $\mathbb{E}[Y \mid X] = \mathbf{X}'\boldsymbol{\beta}$
 - If just $\mathbb{E}[\mathbf{X}e] = 0$, then this is just a linear projection.
 - First is very strong, second is very mild.
- Why do we care? Affects the properties of OLS.
 - Some finite-sample properties of OLS (unbiasedness) require linear CEF
 - Asymptotic results (consistency, asymptotic normality) apply to both.
 - OLS will consistently estimate something, but maybe not what you want.