12. Algebra of Least Squares

Fall 2023

Matthew Blackwell

Gov 2002 (Harvard)

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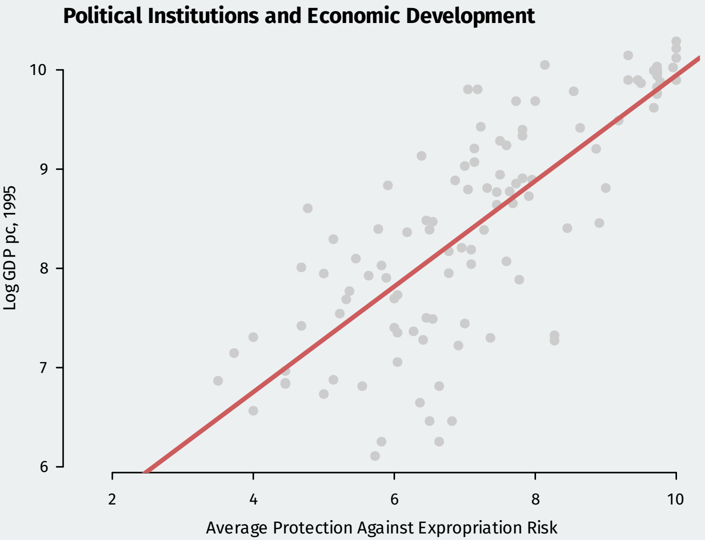
- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

Acemoglu, Johnson, and Robinson (2001)



1/ Deriving the OLS estimator

Samples vs population

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    - Weakening i.i.d. usually complicates notation but can be done.

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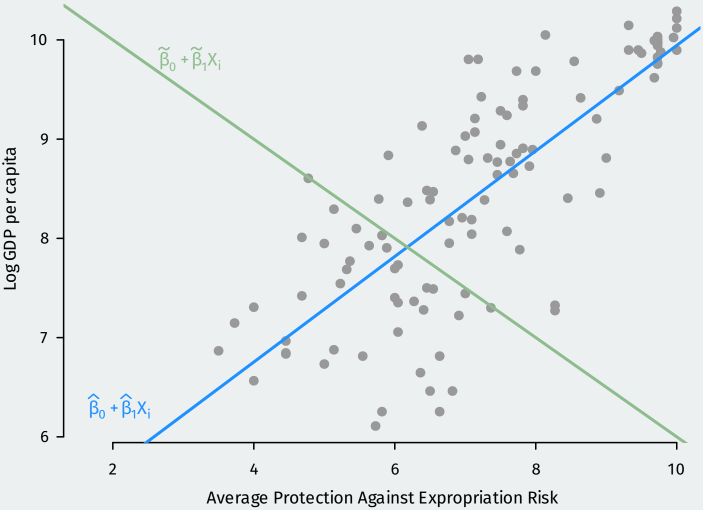

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- $\hat{S}(\mathbf{b})$ is an estimator of the expected squared error, $S(\mathbf{b})$.

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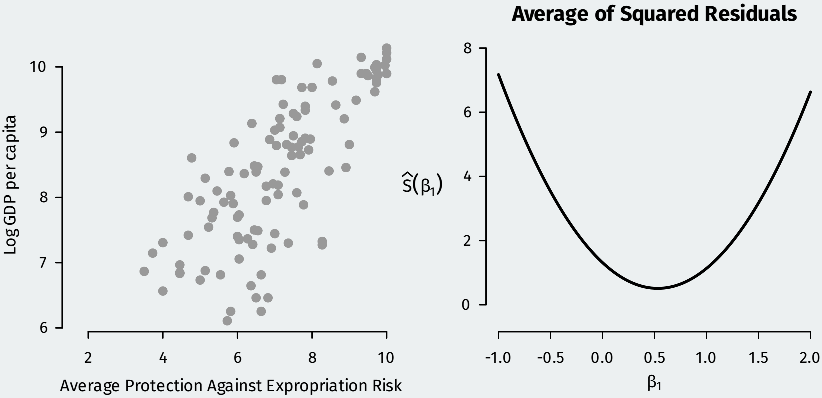
- We can show the OLS estimator of the slope is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\mathbb{V}}[X]}$$

Visualizing OLS



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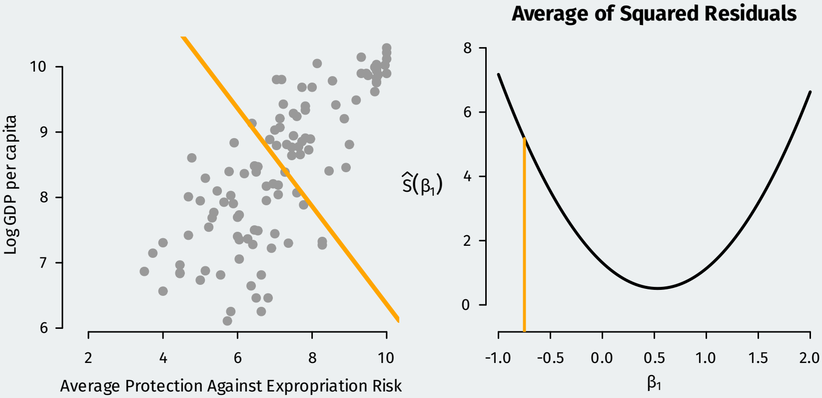

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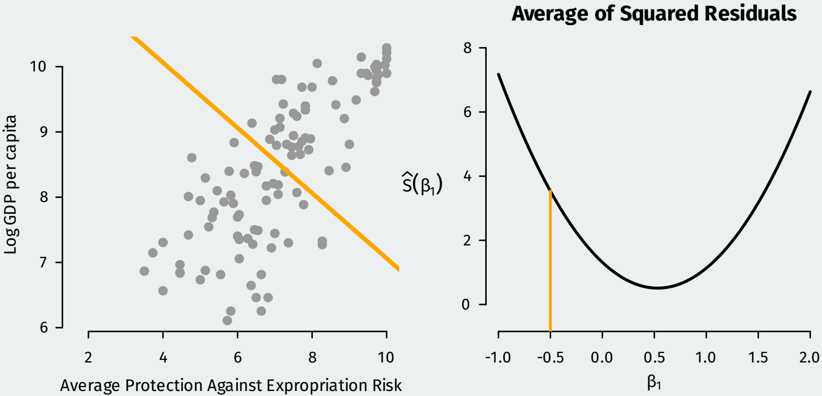

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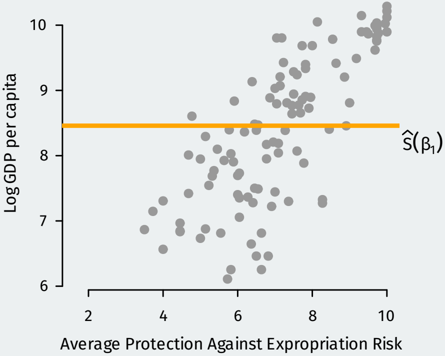

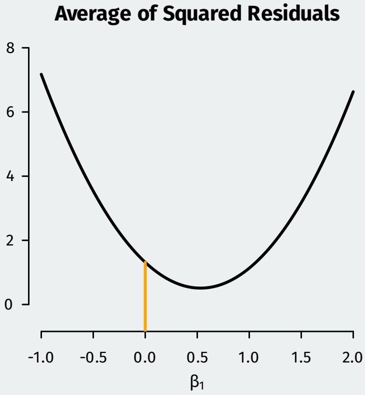

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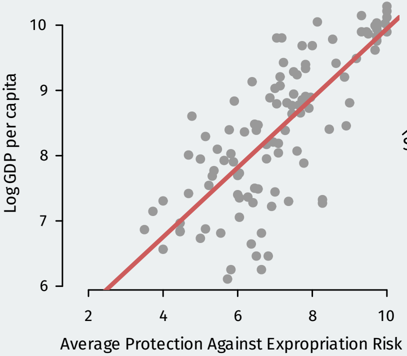

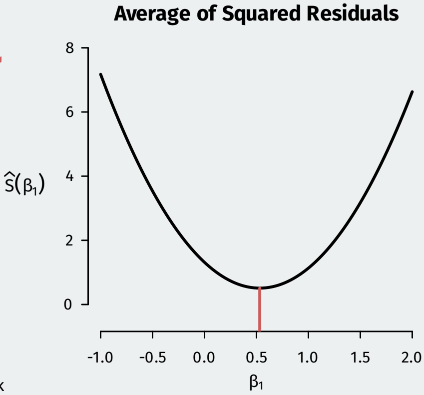

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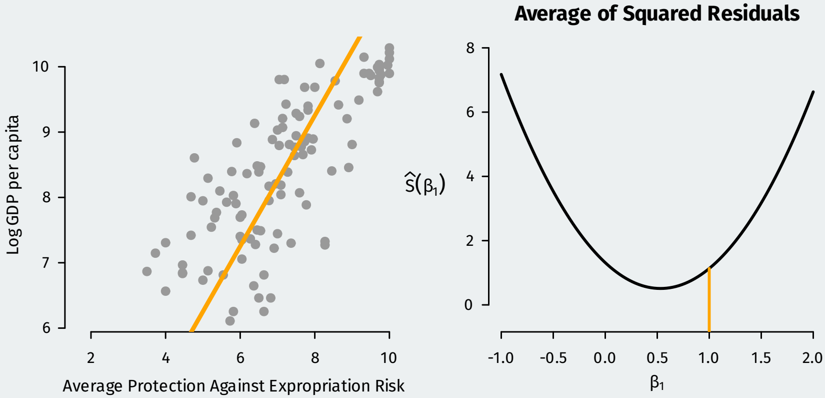

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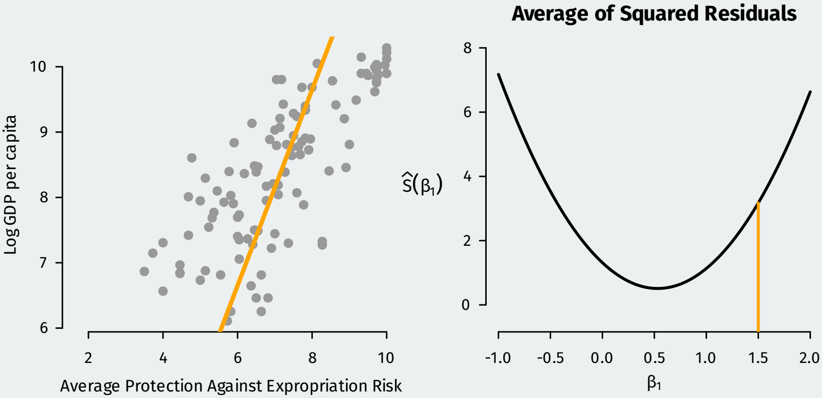

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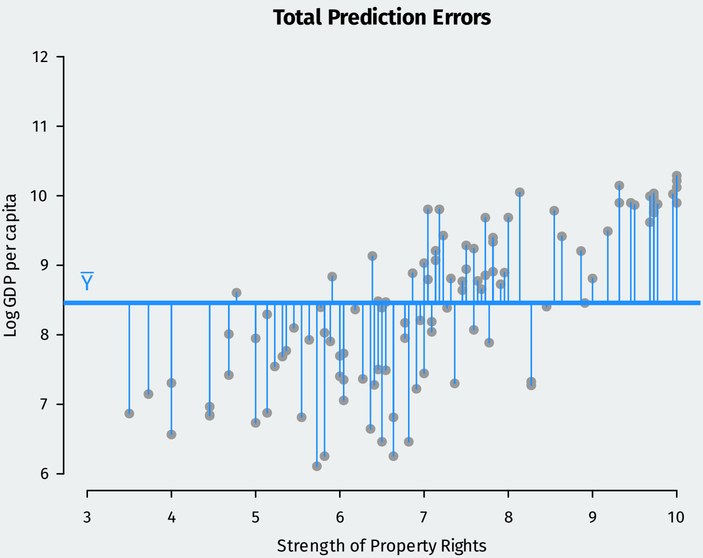

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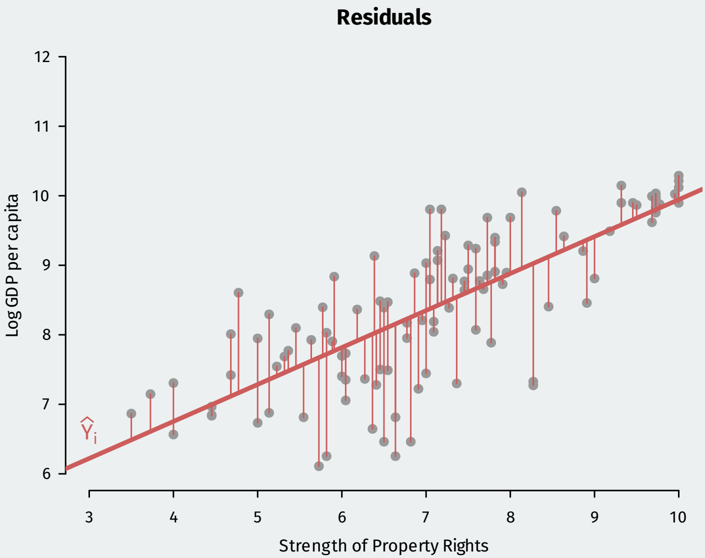

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- Regression will always improve in-sample fit: $TSS > SSR$
- How much better does using $\mathbf{X}_i$ do? **Coefficient of determination** or R^2:

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- **Common interpretation:** R^2 is the fraction of the variation in Y_i is "explained by" $\mathbf{X}_i$.
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- Mechanically increases with additional covariates (better fit measures exist)

3/ Geometry of OLS

Linear model in matrix form

- Linear model is a system of n linear equations:

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\begin{aligned}
Y_1 &= \mathbf{X}_1'\boldsymbol{\beta} + e_1 \\
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- We can write this more compactly using matrices and vectors:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \\ \vdots \\ \mathbf{X}_n' \end{pmatrix} = \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nk} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

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- Model is now just:

$$
\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \mathbf{e}
$$

OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

$$\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' = \mathbb{X}'\mathbb{X}$$

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Projection/hat matrix

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Annihilator matrix

- **Annihilator matrix** projects onto the space spanned by the residual:

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P} = \mathbf{I}_n - \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'$$

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- Allows the following orthogonal partition:

$$\mathbf{Y} = \mathbf{PY} + \mathbf{MY} = \text{projection} + \text{residual}$$

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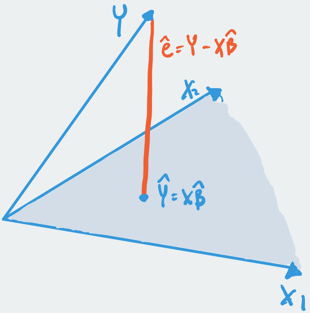
    - Find the point in $\mathcal{C}(\mathbb{X})$ that is closest to $\mathbf{Y}$

Projection



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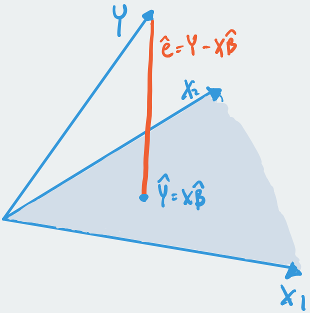

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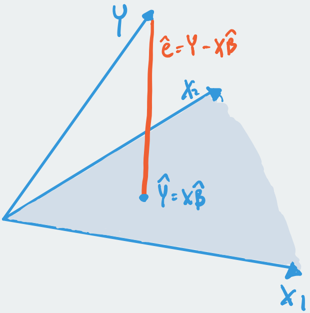

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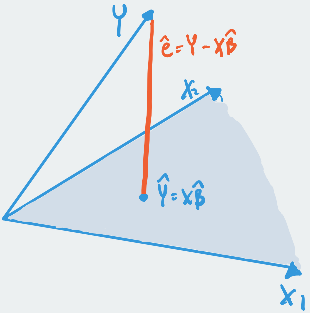

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- Residual vector $\hat{\mathbf{e}} = \mathbf{Y} - \mathbb{X}\hat{\boldsymbol{\beta}}$ is **orthogonal** to $\mathcal{C}(\mathbb{X})$

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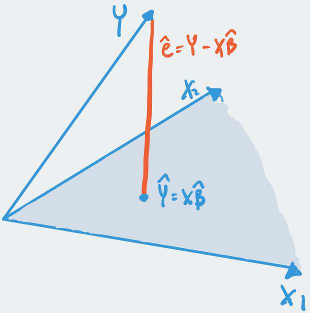

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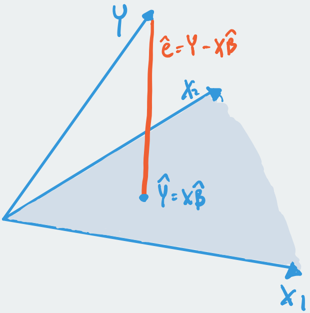
- Residual vector $\hat{\mathbf{e}} = \mathbf{Y} - \mathbb{X}\hat{\boldsymbol{\beta}}$ is **orthogonal** to $\mathcal{C}(\mathbb{X})$
  - Shortest distance from $\mathbf{Y}$ to $\mathcal{C}(\mathbb{X})$ is a straight line to the plane, which will be perpendicular to $\mathcal{C}(\mathbb{X})$.

Projection



- Finding closest point in $\mathcal{C}(\mathbb{X})$ to $\mathbf{Y}$ is called **projection**
- Example: $n = 3$ and $k = 2$: points in 3D space.
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  - Implies that $\mathbb{X}'\hat{\mathbf{e}} = 0$

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  - Including fixed effects for group and variables that do not vary within groups.

4/ Partitioned regression and partial regression

Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \; \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

$$\mathbf{Y} = \mathbb{X}_1\boldsymbol{\beta}_1 + \mathbb{X}_2\boldsymbol{\beta}_2 + \mathbf{e}$$

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    - With exactly orthogonal covariates, multivariate OLS is the same as univariate OLS.
    - Only holds in balanced, designed experiments.

Adding the intercept

- Consider the OLS slope with an intercept:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} - \overline{Y}\mathbf{1} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle}$$

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  2. Regress $\mathbf{Y}$ on residuals from step 1, $\mathbf{X} - \overline{X}\mathbf{1}$

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  2. Regress $\mathbf{Y}$ on residual from

Visualizing orthogonalization

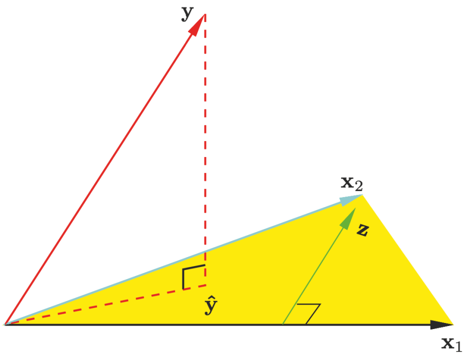

FIGURE 3.4. *Least squares regression by orthogonalization of the inputs. The vector $\mathbf{x}_2$ is regressed on the vector $\mathbf{x}_1$, leaving the residual vector $\mathbf{z}$. The regression of $\mathbf{y}$ on $\mathbf{z}$ gives the multiple regression coefficient of $\mathbf{x}_2$. Adding together the projections of $\mathbf{y}$ on each of $\mathbf{x}_1$ and $\mathbf{z}$ gives the least squares fit $\hat{\mathbf{y}}$.*

Why does residual regression work?

- We can find $\hat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\hat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left(\min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1 \boldsymbol{\beta}_1 - \mathbb{X}_2 \boldsymbol{\beta}_2\|^2 \right)$$

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$$\widehat{\boldsymbol{\beta}}_1 = \left(\mathbb{X}_1'\mathbf{M}_2\mathbb{X}_1\right)^{-1}\left(\mathbb{X}_1'\mathbf{M}_2\mathbf{Y}\right)$$

- When will $\widehat{\boldsymbol{\beta}}_1$ will be the same regardless of whether $\mathbb{X}_2$ is included?

  - If $\mathbb{X}_1$ and $\mathbb{X}_2$ are orthogonal so $\mathbb{X}_2'\mathbb{X}_1 = 0$ so $\mathbf{M}_2\mathbb{X}_1 = \mathbb{X}_1$

Residual regression

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  - Sample version of the results we saw for the linear projection.

5/ Influential observations

Outliers, leverage points, and influential observations

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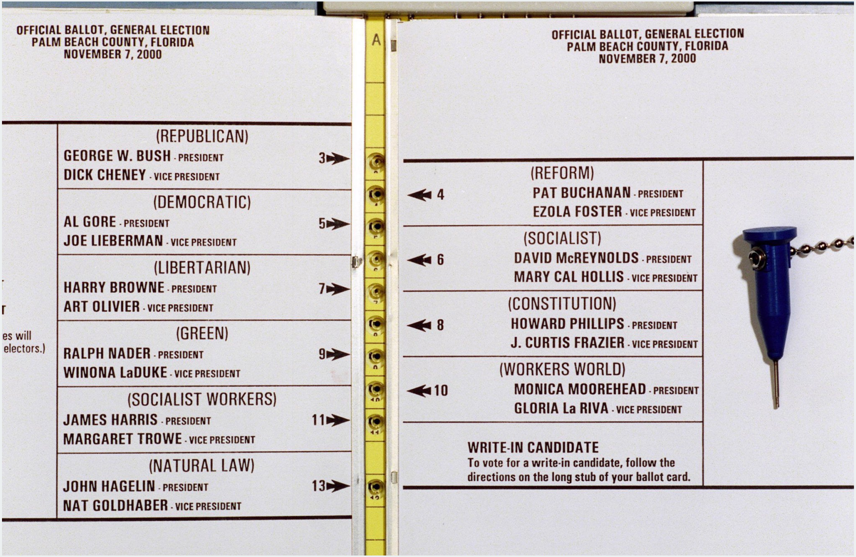
1. **Leverage point**: extreme in one X direction
2. **Outlier**: extreme in the Y direction
3. **Influence point**: extreme in both directions

Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)



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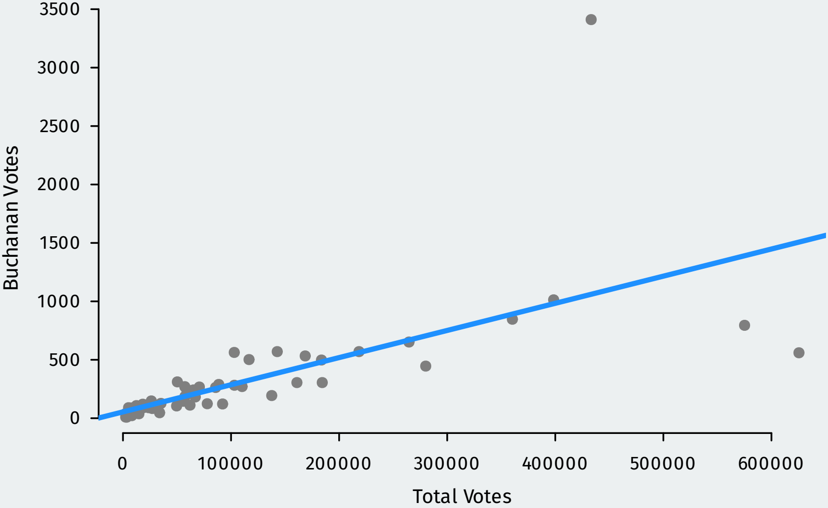

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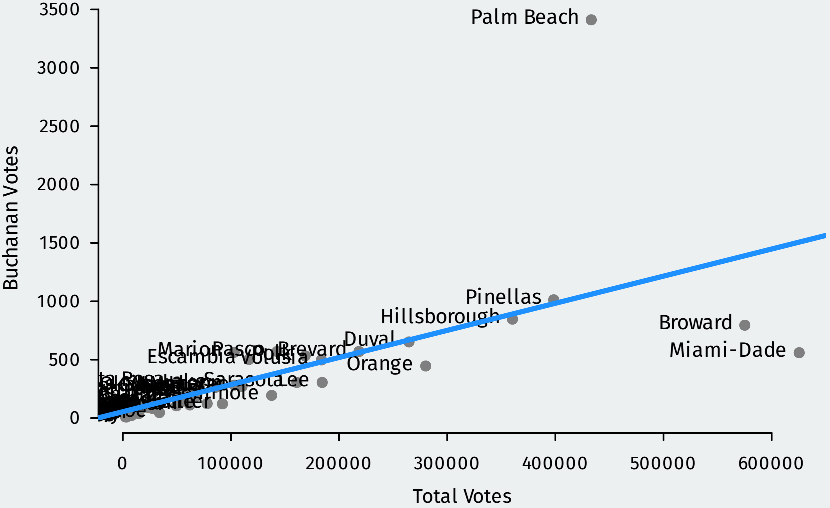
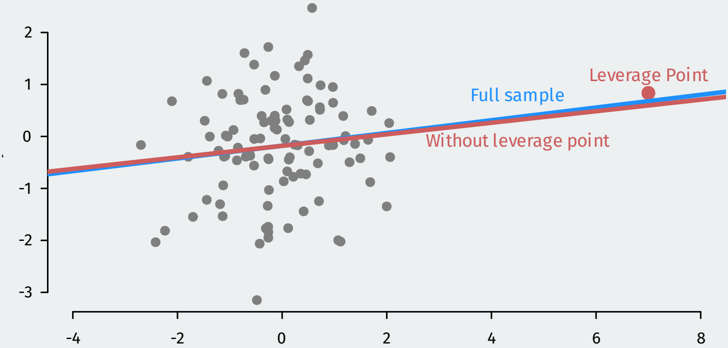


Example: Buchanan votes

```r
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 54.22945   49.14146    1.10     0.27
## edaytotal    0.00232    0.00031    7.48  2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463,  Adjusted R-squared:  0.455
## F-statistic:   56 on 1 and 65 DF,  p-value: 2.42e-10
```

Leverage point definition



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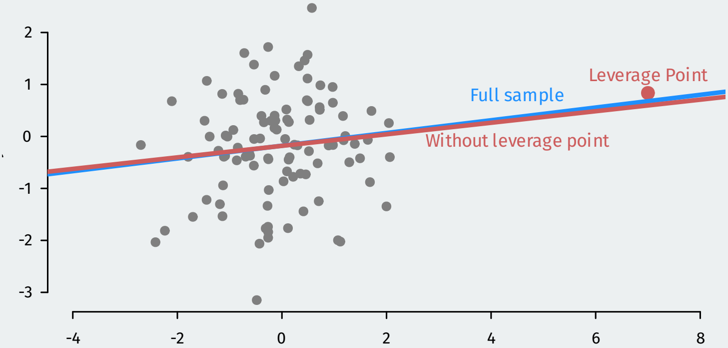

- Values that are extreme in the X dimension
- That is, values far from the center of the covariate distribution

Leverage values

- Let h_{ij} be the (i, j) entry of $\mathbf{P}$. Then:

$$\widehat{\mathbf{Y}} = \mathbf{P}\mathbf{Y} \qquad \Longrightarrow \qquad \widehat{Y}_i = \sum_{j=1}^{n} h_{ij} Y_j$$

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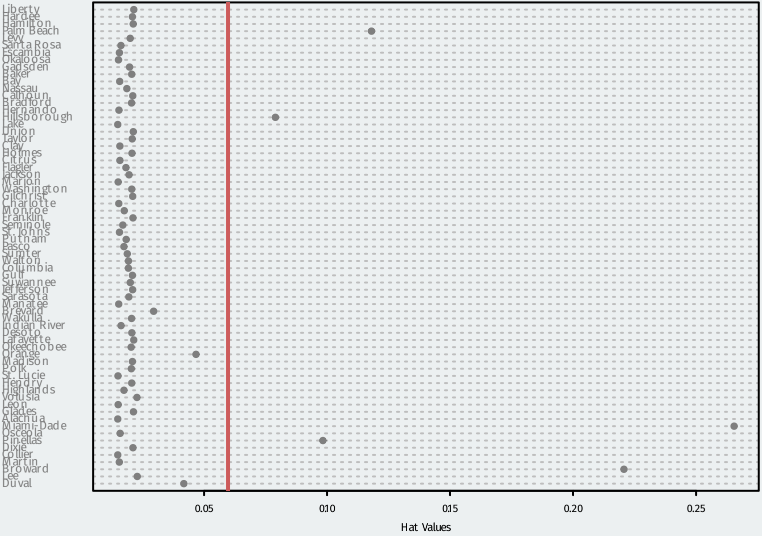
  - $\rightsquigarrow$ how far i is from the center of the X distribution
- **Rule of thumb:** examine hat values greater than $2(k+1)/n$

Buchanan hats

```r
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

Buchanan hats



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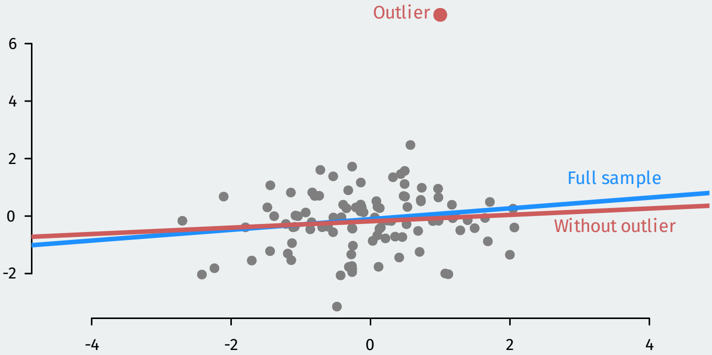

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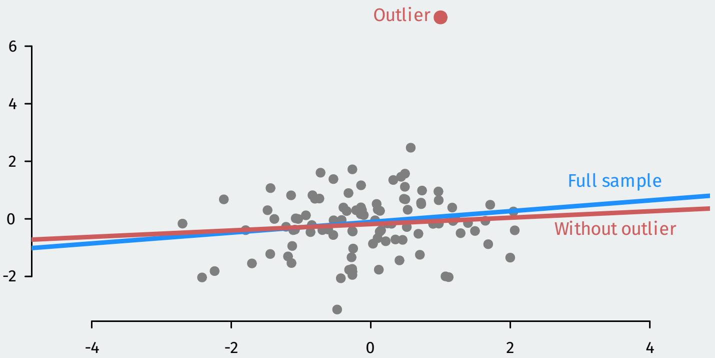
Outlier definition



- An **outlier** is far away from the center of the Y distribution.
- Intuitively: a point that would be poorly predicted by the regression.

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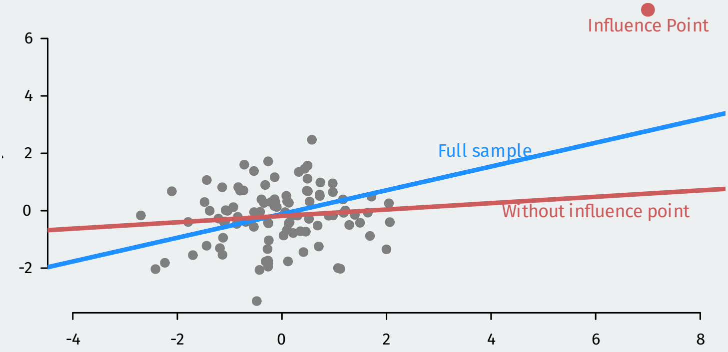
- Simple closed-form expressions:

$$\hat{\boldsymbol{\beta}}_{(-i)} = \hat{\boldsymbol{\beta}} - \left(\mathbb{X}'\mathbb{X}\right)^{-1}\mathbf{X}_i\tilde{e}_i \qquad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

Influence points



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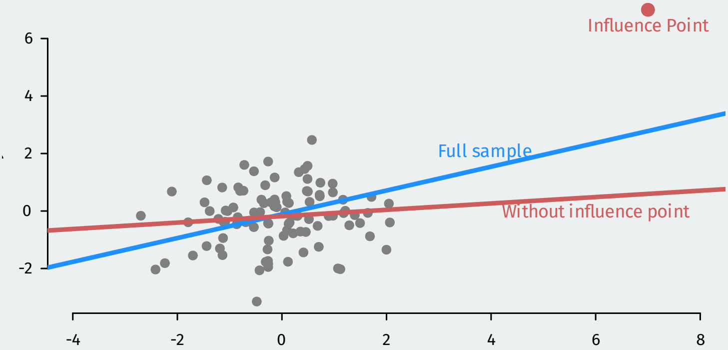

- An **influence point** is one that is both an outlier and a leverage point.
- Extreme in both the X and Y dimensions

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  - Does removing the point change a coefficient by a lot?

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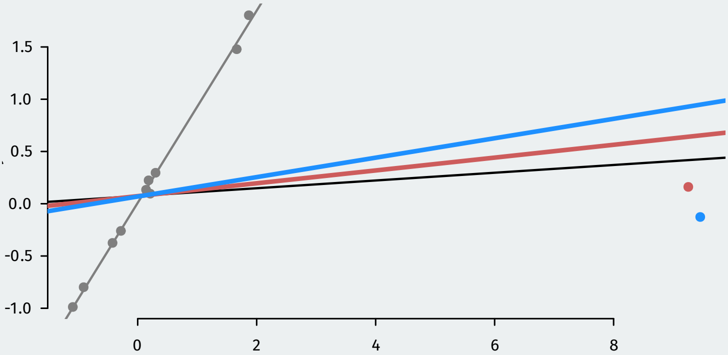

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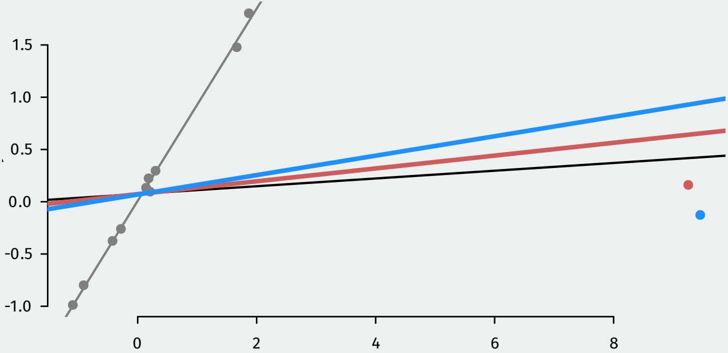

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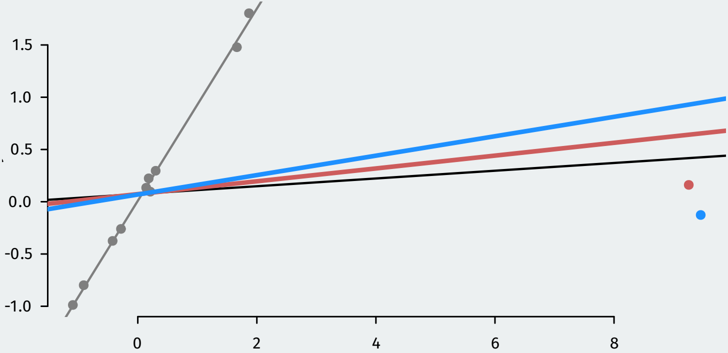

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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    - Transform the dependent variable ($\log(y)$)
    - Use a method that is robust to outliers (robust regression, least absolute deviations)