14. Algebra of Least Squares

Spring 2021

Matthew Blackwell

Gov 2002 (Harvard)

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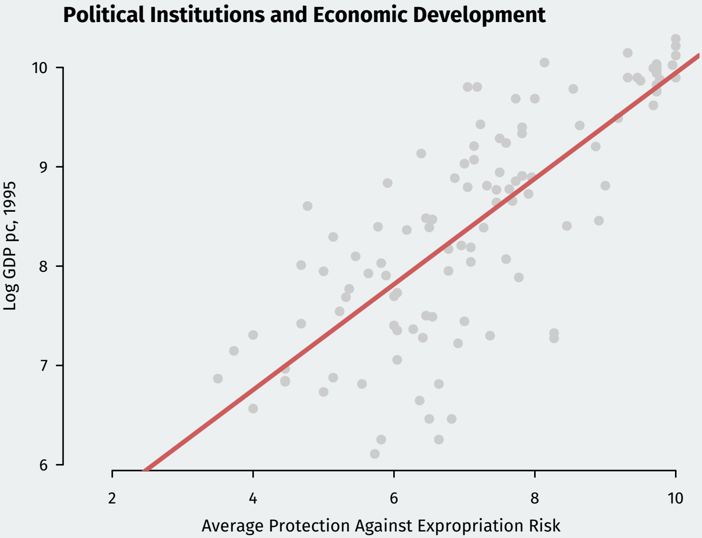
- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

Acemoglu, Johnson, and Robinson (2001)



Samples vs population

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The variables $\{(Y_1, \mathbf{X}_1), \dots, (Y_i, \mathbf{X}_i), \dots, (Y_n, \mathbf{X}_n)\}$ are i.i.d. draws from a common distribution F.

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- Violations include time-series data and clustered sampling.
  - Weakening i.i.d. usually complicates notation but can be done.

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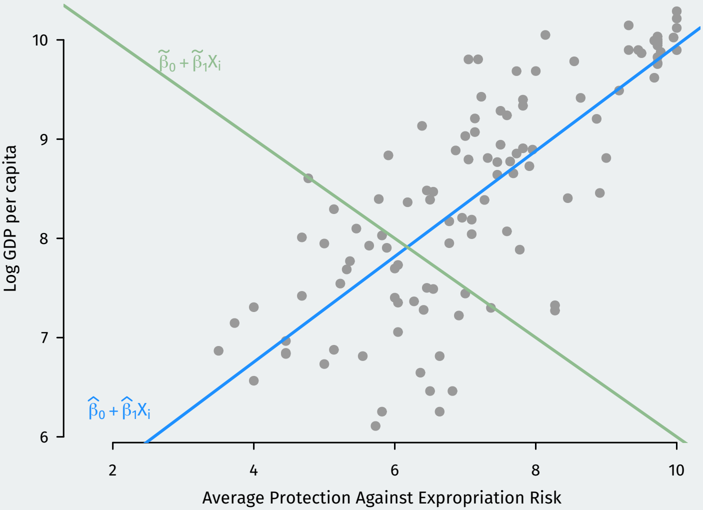

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- We can show the OLS estimator of the slope is:

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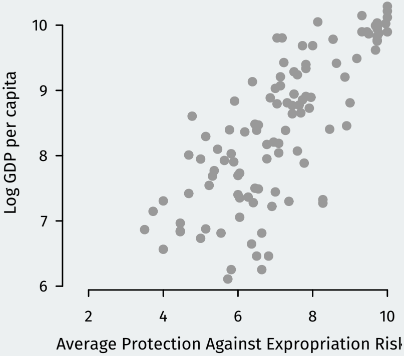
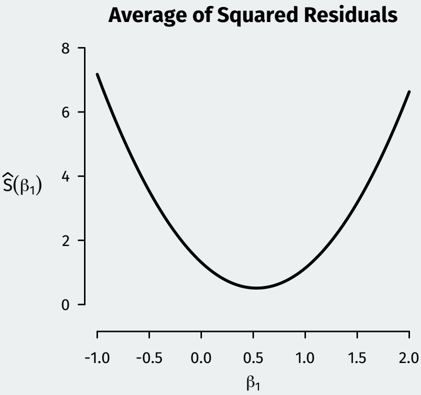
Visualizing OLS





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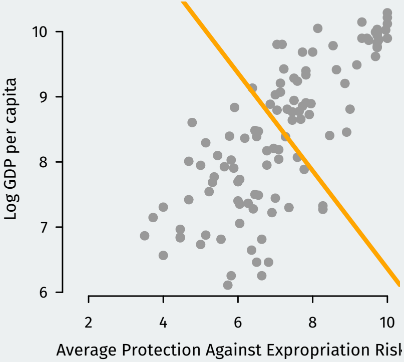

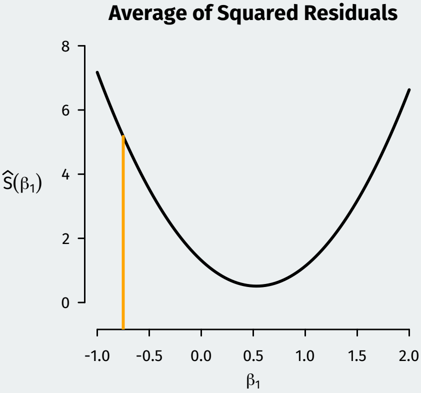

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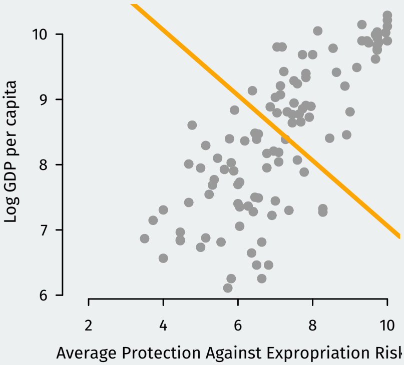

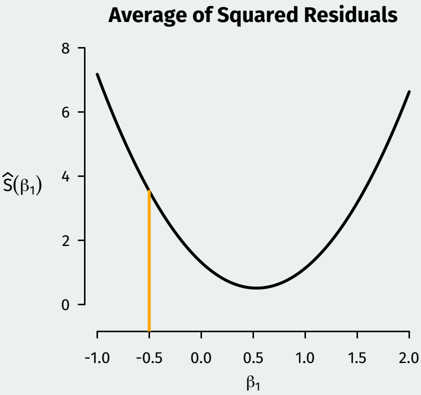

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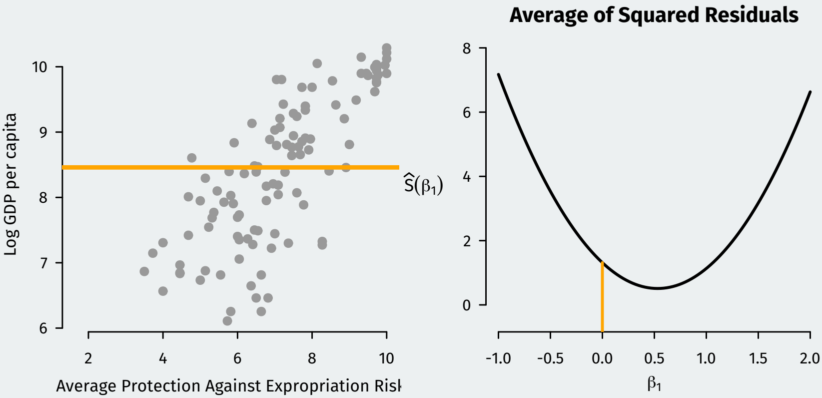

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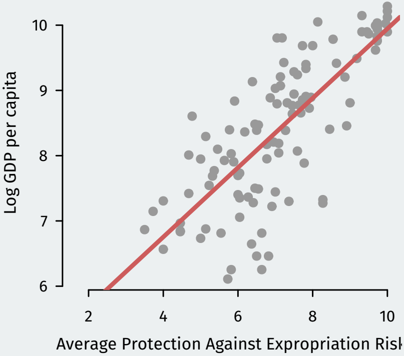

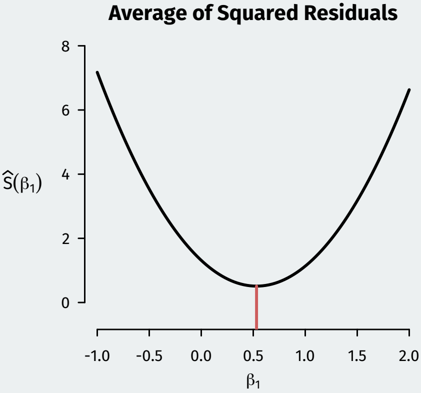

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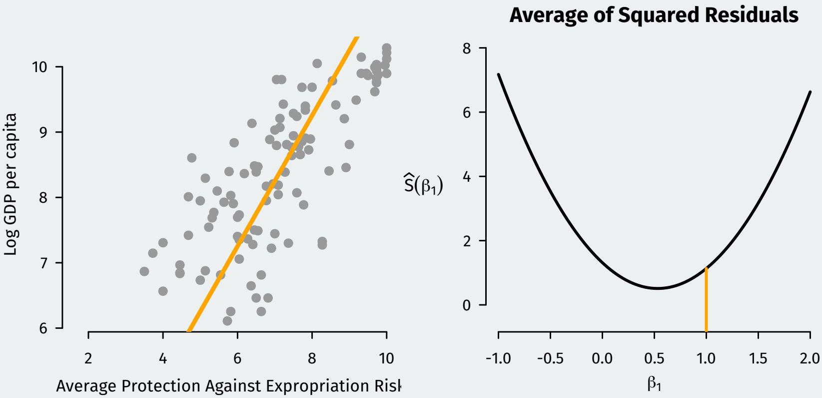

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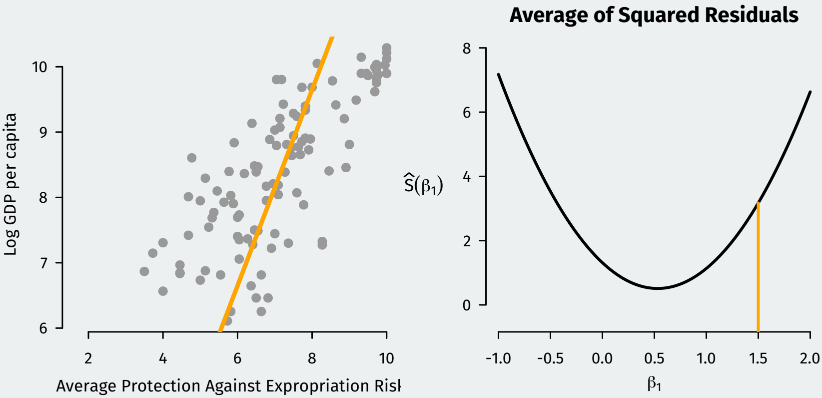

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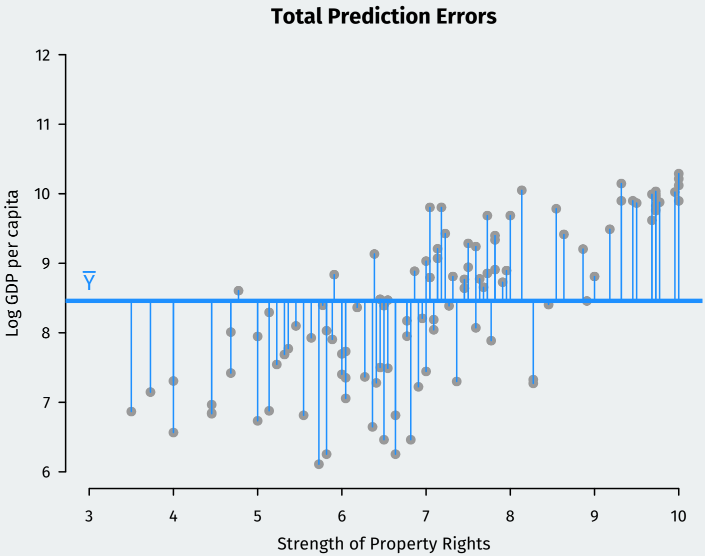

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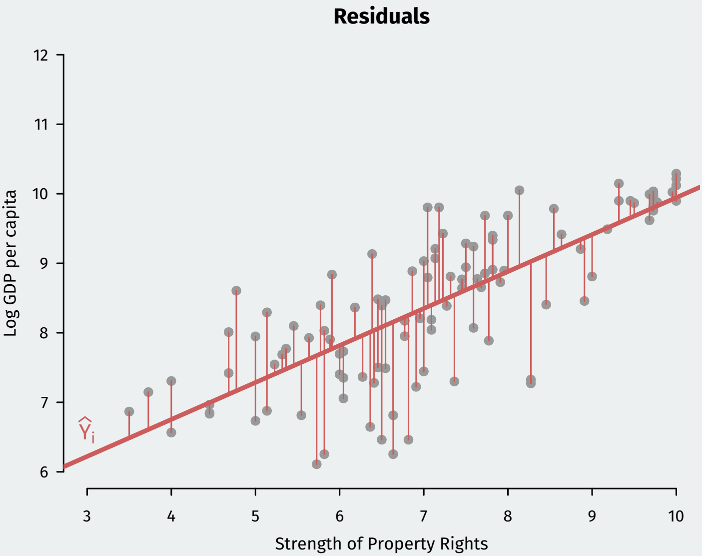

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- Mechanically increases with additional covariates (better fit measures exist)

Linear model in matrix form

- Linear model is a system of n linear equations:

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Y_1 &= \mathbf{X}_1'\boldsymbol{\beta} + e_1 \\
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- Model is now just:

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OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

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Least squares in matrix form

- OLS still minimizes sum of the squared residuals

$$\underset{\mathbf{b}\in\mathbb{R}^{k+1}}{\arg\min}\, \hat{\mathbf{e}}'\hat{\mathbf{e}} = \underset{\mathbf{b}\in\mathbb{R}^{k+1}}{\arg\min} (\mathbf{Y} - \mathbb{X}\mathbf{b})'(\mathbf{Y} - \mathbb{X}\mathbf{b})$$

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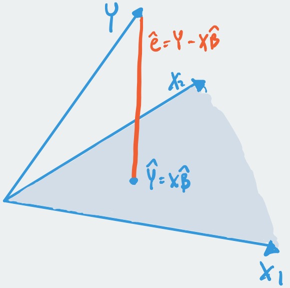
- We can write the covariate-residual orthogonality as $\mathbb{X}'\hat{\mathbf{e}} = 0$.

Projection



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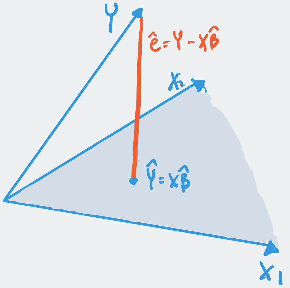

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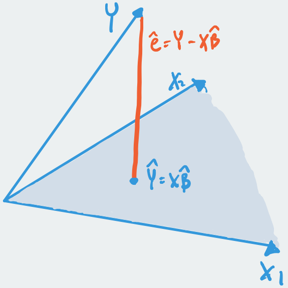

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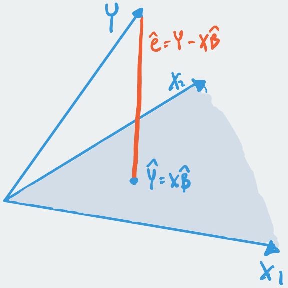

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- Intuition: $\hat{\boldsymbol{\beta}}$ defines the projection that gets is shortest distance between Y and prediction.

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  - Projecting $\mathbb{X}$ onto itself returns itself: $\mathbf{P}\mathbb{X} = \mathbb{X}$

Annihilator matrix

- **Annihilator matrix** projects onto the space spanned by the residual:

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    - Admits a nice expression for the residual vector: $\hat{\mathbf{e}} = \mathbf{Me}$

Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \ \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

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  - $\tilde{\mathbf{e}}_1 = \mathbf{M}_1\mathbf{Y}$ = residuals from regression of $\mathbf{Y}$ on $\mathbb{X}_1$.
- Then remembering that $\mathbf{M}_1$ is symmetric and idempotent:

$$
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  - Sample version of the results we saw for the linear projection.

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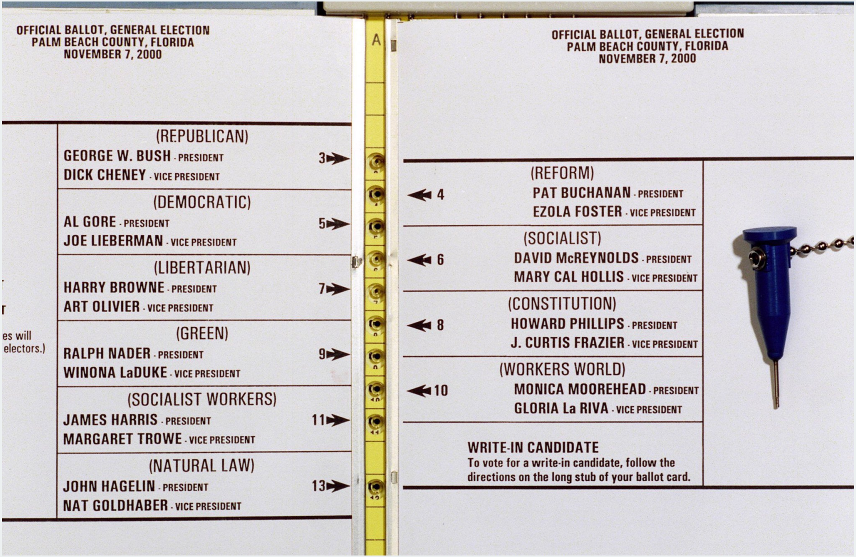
Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)



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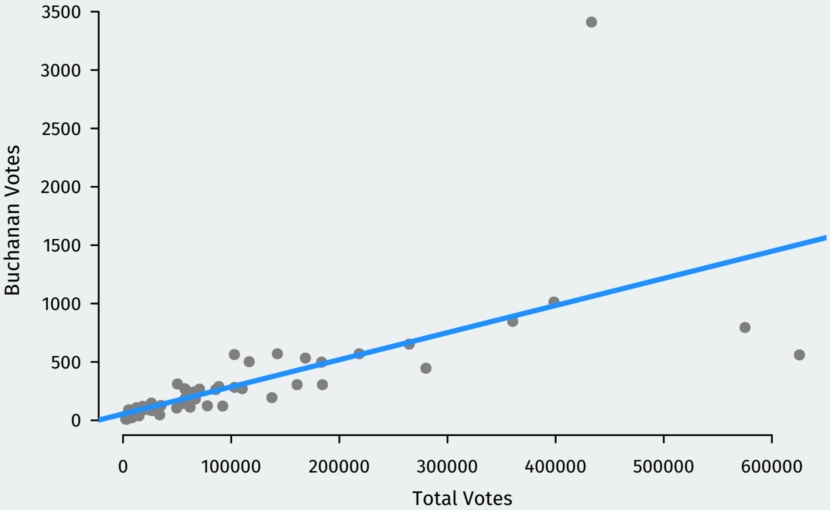

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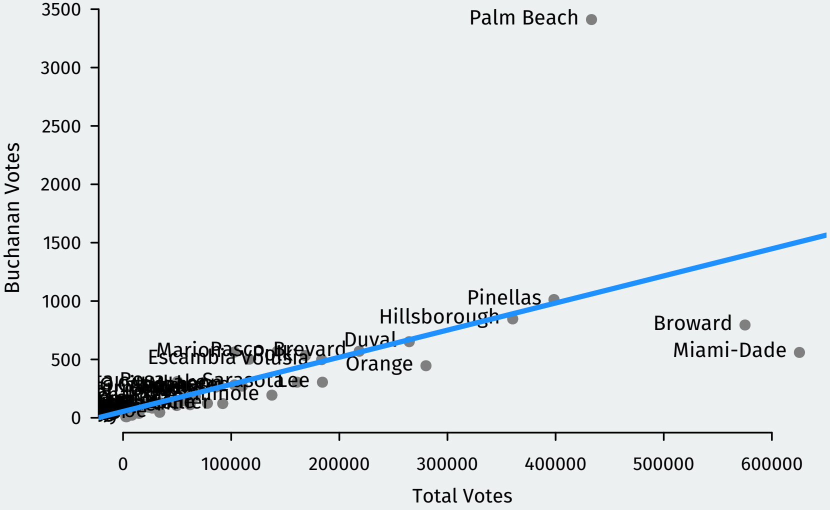
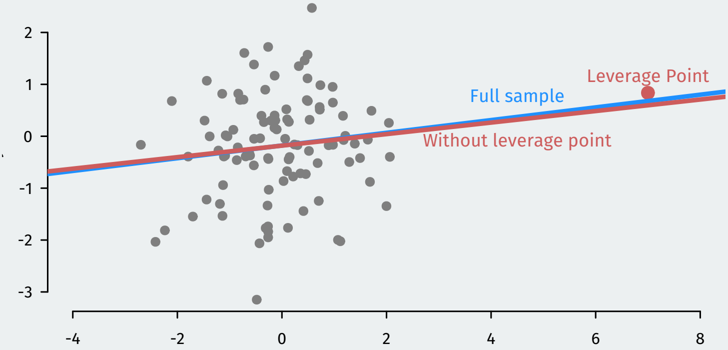


Example: Buchanan votes

```r
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 54.22945   49.14146    1.10     0.27
## edaytotal    0.00232    0.00031    7.48  2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463,  Adjusted R-squared:  0.455
## F-statistic:   56 on 1 and 65 DF,  p-value: 2.42e-10
```

Leverage point definition



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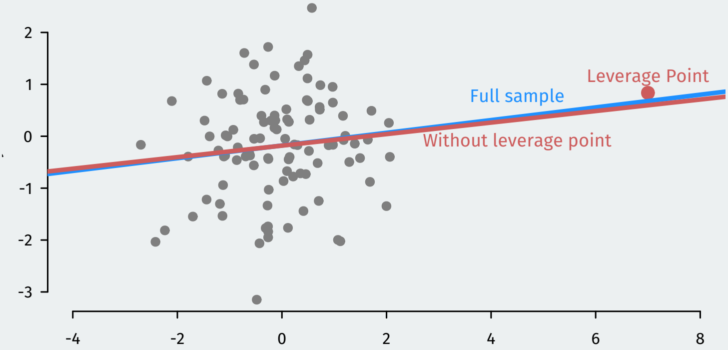

- Values that are extreme in the X dimension
- That is, values far from the center of the covariate distribution

Leverage values

- Let h_{ij} be the (i, j) entry of $\mathbf{P}$. Then:

$$\widehat{\mathbf{Y}} = \mathbf{P}\mathbf{Y} \qquad \Longrightarrow \qquad \widehat{Y}_i = \sum_{j=1}^{n} h_{ij} Y_j$$

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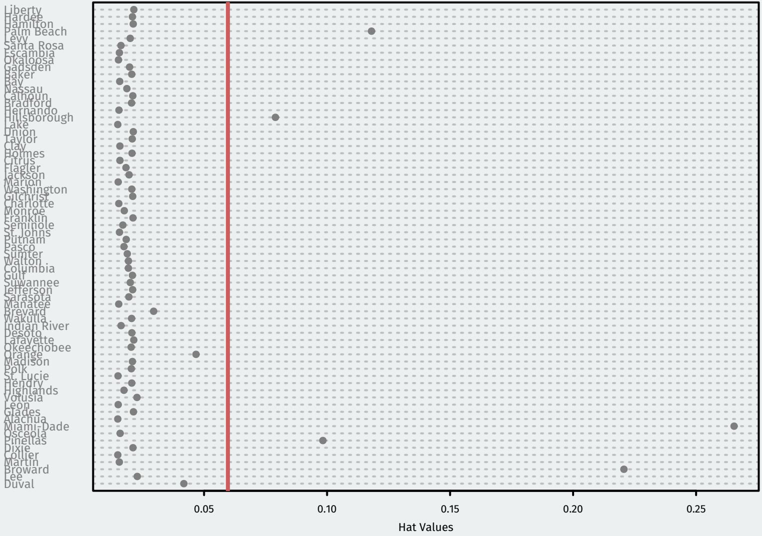
  - $\rightsquigarrow$ how far i is from the center of the X distribution
- **Rule of thumb:** examine hat values greater than $2(k+1)/n$

Buchanan hats

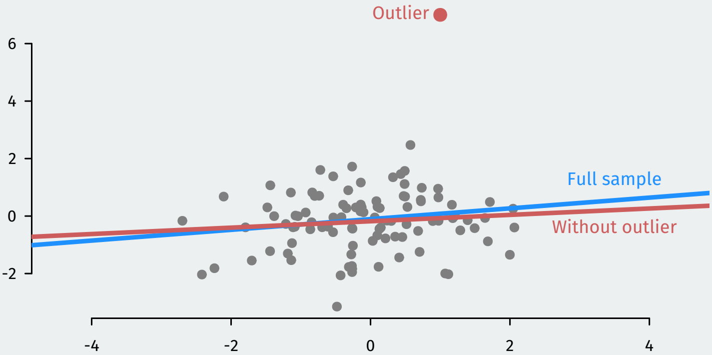
```r
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

Buchanan hats



Outlier definition



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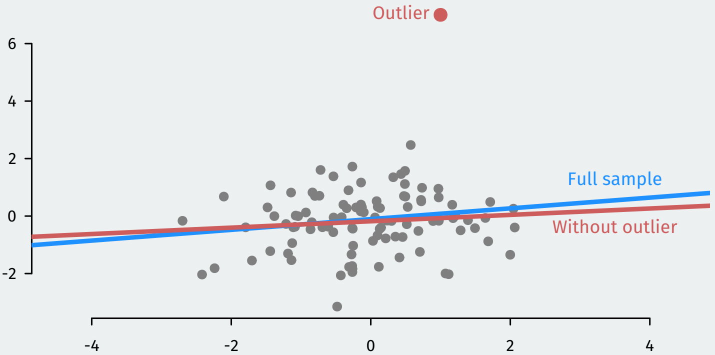

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- Intuitively: a point that would be poorly predicted by the regression.

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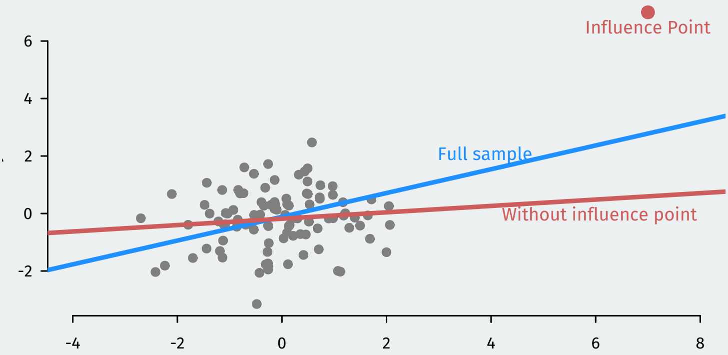
- Simple closed-form expressions:

$$\hat{\boldsymbol{\beta}}_{(-i)} = \hat{\boldsymbol{\beta}} - \left(\mathbb{X}'\mathbb{X}\right)^{-1}\mathbf{X}_i\tilde{e}_i \qquad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

Influence points



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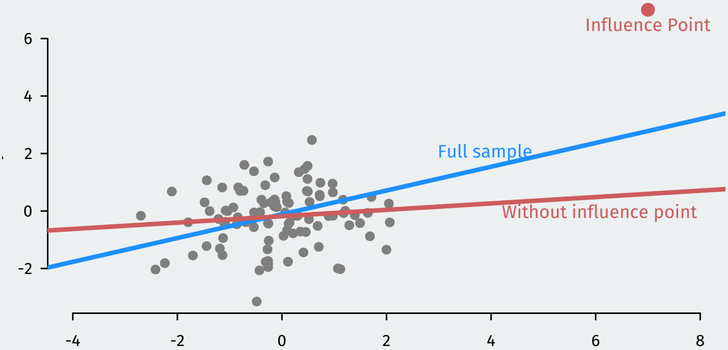

- An **influence point** is one that is both an outlier and a leverage point.
- Extreme in both the X and Y dimensions

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  - Does removing the point change a coefficient by a lot?

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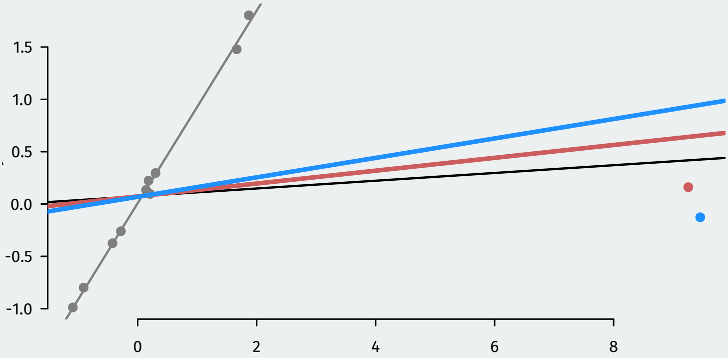

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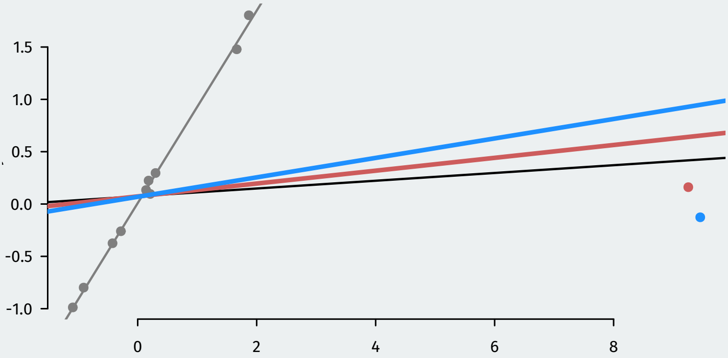

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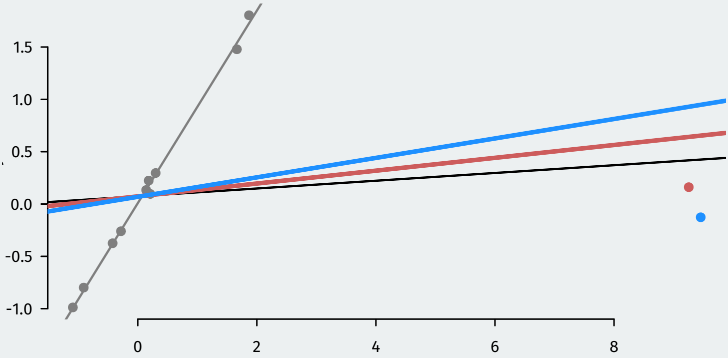

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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  - Use a method that is robust to outliers (robust regression, least absolute deviations)