A quadratic-time algorithm for the cardinality-constrained college application problem

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Abstract

This paper considers a novel portfolio optimization problem called the college application problem. We show that the objective function is a nondecreasing submodular set function and provide quadratic-time solution algorithm. We discuss the properties of the optimal solution and their implications for student welfare.

Keywords: combinatorial optimization, submodular maximization, portfolio optimization

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1 Introduction

Due to the widespread perception that career earnings depend on college admissions outcomes, the solution of (1) holds monetary value. In the American college consulting industry, students pay private consultants an average of $200 per hour for assistance in preparing application materials, estimating their admissions odds, and identifying target schools (Sklarow 2018). In Korea, an important revenue stream for admissions consulting firms such as Megastudy (megastudy.net) and Jinhak (jinhak.com) is "mock application" software that claims to use artificial intelligence to optimize the client's application strategy.

Most quantitative research on college admissions to date has been descriptive in nature. A standard benchmark for logistic regression techniques, for example, involves estimating students' admissions probabilities from information about their past academic performance (Acharya et al. 2019; Lim 2013). When professional admissions consultants advise their clients on where to apply to college, however, they but typically rely on qualitative heuristics, such as balancing applications between safety and reach schools, rather than optimization models (Peck 2021). (Admissions consultants refer to schools with low utility and high admissions probability as *safety schools*, and those with high utility and low admissions probability as *reach schools.*)

In the present study, we formulate the student's college application decision as the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & v(\mathcal{X}) = \mathrm{E}\Big[\max\big\{t_0, \max\{t_j Z_j : j \in \mathcal{X}\}\big\}\Big] \\ \text{subject to} \quad & \mathcal{X} \subseteq \mathcal{C}, \quad |\mathcal{X}| \leq H \end{aligned} \tag{1}$$

Here $\mathcal{C} = \{1 \dots m\}$ is an index set; $h > 0$ is the number of students the student can apply to; for $j = 1 \dots m$, Z_j is a random, independent Bernoulli variable with probability f_j; and for $j = 0 \dots m$, $t_j \geq 0$ is a utility parameter.

Consider a single prospective student in a college market with m colleges, and let each t_j-value indicate the utility she associates with attending school j, where her utility is t_0 if she does not attend college. Let h denote the number of schools to which the student can afford to apply. Lastly, let f_j denote the student's probability of being admitted to school j if she applies, so that Z_j equals one if she is admitted and zero if not. It is appropriate to assume that the Z_j are statistically independent as long as f_j are probabilities estimated specifically for this student (as opposed to generic acceptance rates). Then the student's objective is to maximize the expected utility associated with the best school she is admitted to. Therefore, her optimal college application strategy is given by the solution $\mathcal{X}$ to the problem above, where $\mathcal{X}$ represents the set of schools to which she applies.

We focus on the special case in which students are constrained in the *number* of schools to which they can apply, and show that the optimal portfolios in this case are nested in the budget constraint, implying the optimality of the greedy algorithm that adds schools one at a time according to how much they increase the objective value.

1.1 Literature review

The objective function is a monotonic submodular function in the sense first described by Nemhauser et al. (1978). They showed that the greedy algorithm is asymptotically $(1-1/e)$-optimal for optimizing a monotonic submodular function over a cardinality constraint, and a subsequent result of Nemhauser and Wolsey (1978) proved that this approximation ratio is the best achievable by a polynomial-time algorithm. In the special case of college application, however, we show that the same algorithm is exact.

The proof of the optimality of the greedy algorithm rests on the nested structure of the portfolios, a property that is of independent interest. As Rozanov and Tamir (2020) articulate, the knowledge that the optima are nested aids not only in computing the optimal solution, but in the implementation thereof under uncertain information. For example, in the United States, many college applications are due at the beginning of November, and it is typical for students to begin working on their applications during the prior summer because colleges reward students who tailor their essays to the target school. However, students may not know how many schools they can afford to apply to until late October. The nestedness property—or equivalently, the validity of a greedy algorithm—implies that even in the absence of complete budget information, students can begin to carry out the optimal application strategy by writing essays for schools in the order that they enter the optimal portfolio.

Fu (2014) considered a problem that is reducible to ours: since application is at cost rather than a constraint, the objective function is separable in the cost and therefore must be optimal over portfolios of the given size. Thus, solving her problem reduces to solving m instances of (1).

To the best of our knowledge, the first systematic study of the college application problem was undertaken by Kapur in his master's thesis, which considers (1) as well as the more general case of a knapsack constraint. The present study extends Kapur's results with a further discussion of the application strategies recommended in the admissions consulting industry and how these compare to the optimal strategy.

1.2 Outline

This paper has five sections. In section 2, we derive a closed-form expression for the objective function of (1) and show that it is a monotonic submodular function. In section 3, we prove that the optimal portfolios are nested in the budget h, which implies the validity of the greedy algorithm. Further refinement reduces the computation time to $O(hm)$. In section 4, we examine alternative solution mechanisms used in the admissions consulting industry and their performance relative to the optimal solution. A brief conclusion follows in section 5.

2 Preliminaries

For the remainder of the paper, unless otherwise noted, we assume with trivial loss of generality that each $f_j \in (0, 1]$ and $< t_1 \leq \cdots \leq t_m$. Unless otherwise noted, we assume $t_0 = 0$, an assumption justified presently. Unless otherwise noted, we assume that $t_0 = 0$, a restriction that we justify presently.

2.1 The objective function

First we derive a closed-form expression for the objective function of (1).

We refer to the set $\mathcal{X} \subseteq \mathcal{C}$ of schools to which a student applies as her *application portfolio.* The expected utility the student receives from $\mathcal{X}$ is called its *valuation.* Given an application portfolio, let $p_j(\mathcal{X})$ denote the probability that the student attends school j. This occurs if and only if she *applies* to school j, is *admitted* to school j, and is *rejected* from any school she prefers to j; that is, any school with higher index. Hence, for $j = 0 \ldots m$,

$$p_j(\mathcal{X}) = \begin{cases} f_j \prod\limits_{\substack{i \in \mathcal{X}: \\ i > j}} (1 - f_i), & j \in \{0\} \cup \mathcal{X} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

The following proposition follows by computing $v(\mathcal{X}) = \sum_{j=0}^{m} t_j p_j(\mathcal{X})$.

Proposition 1 (Closed form of portfolio valuation function)**.**

$$v(\mathcal{X}) = \sum_{j=0}^{m} t_j p_j(\mathcal{X}) = \sum_{j \in \{0\} \cup \mathcal{X}} \Big(f_j t_j \prod_{\substack{i \in \mathcal{X}: \\ i > j}} (1 - f_i) \Big) \tag{3}$$

Next, we show that without loss of generality, we may assume that $t_0 = 0$.

Lemma 1. *For some $\gamma \leq t_0$, let $\bar{t}_j = t_j - \gamma$ for $j = 0 \ldots m$. Then $v(\mathcal{X}; \bar{t}_j) = v(\mathcal{X}; t_j) - \gamma$ for any $\mathcal{X} \subseteq \mathcal{C}$.*

Proof. By definition, $\sum_{j=0}^{m} p_j(\mathcal{X}) = \sum_{j \in \{0\} \cup \mathcal{X}} p_j(\mathcal{X}) = 1$. Therefore

$$\begin{aligned} v(\mathcal{X}; \bar{t}_j) &= \sum_{j \in \{0\} \cup \mathcal{X}} \bar{t}_j p_j(\mathcal{X}) = \sum_{j \in \{0\} \cup \mathcal{X}} (t_j - \gamma) p_j(\mathcal{X}) \\ &= \sum_{j \in \{0\} \cup \mathcal{X}} t_j p_j(\mathcal{X}) - \gamma = v(\mathcal{X}; t_j) - \gamma \end{aligned} \tag{4}$$

which completes the proof. $\square$

2.2 An elimination technique

Now, we present a variable-elimination technique that will prove useful throughout the paper.[1] Suppose that the student has already resolved to apply to school k, and the remainder of her decision consists of determining which *other* schools to apply to. Writing her total application portfolio as $\mathcal{X} = \mathcal{Y} \cup \{k\}$, we devise a function $w(\mathcal{Y})$ that orders portfolios according to how well they "complement" the singleton portfolio $\{k\}$. Specifically, the difference between $v(\mathcal{Y} \cup \{k\})$ and $w(\mathcal{Y})$ is the constant $f_k t_k$.

To construct $w(\mathcal{Y})$, let $\tilde{t}_j$ denote the expected utility the student receives from school j *given* that she has been admitted to school j and applied to school k. For $j < k$ (including $j = 0$), this is $\tilde{t}_j = (1 - f_k)t_j + f_k t_k$; for $j > k$, this is $\tilde{t}_j = t_j$. This means that

$$v(\mathcal{Y} \cup \{k\}) = \sum_{j \in \{0\} \cup \mathcal{Y}} \tilde{t}_j p_j(\mathcal{Y}). \tag{5}$$

The transformation to $\tilde{t}$ does not change the order of the t_j-values. Therefore, the expression on the right side of (5) is itself a portfolio valuation function. In the corresponding market, t is replaced by $\tilde{t}$ and $\mathcal{C}$ is replaced by $\mathcal{C} \setminus \{k\}$. To restore our convention that $t_0 = 0$, we obtain $w(\mathcal{Y})$ by taking $\bar{t}_j = \tilde{t}_j - \tilde{t}_0$ for all $j \neq k$ and letting

$$w(\mathcal{Y}) = \sum_{j \in \{0\} \cup \mathcal{Y}} \bar{t}_j p_j(\mathcal{Y}) = \sum_{j \in \{0\} \cup \mathcal{Y}} \tilde{t}_j p_j(\mathcal{Y}) - \tilde{t}_0 = v(\mathcal{Y} \cup \{k\}) - f_k t_k \tag{6}$$

where the second equality follows from Lemma 1. The validity of this transformation is summarized in the following theorem, where we write $v(\mathcal{X}; \bar{t})$ instead of $w(\mathcal{Y})$ to emphasize that $w(\mathcal{Y})$ is, in form, a portfolio valuation function.

Lemma 2 (Eliminate school k)**.** *For $\mathcal{X} \subseteq \mathcal{C} \setminus \{k\}$, $v(\mathcal{X} \cup \{k\}; t) = v(\mathcal{X}; \bar{t}) + f_k t_k$, where*

$$\bar{t}_j = \begin{cases} (1 - f_k)t_j, & t_j \leq t_k \\ t_j - f_k t_k, & t_j > t_k. \end{cases} \tag{7}$$

Proof. It is easy to verify that (7) is the composition of the two transformations (from t to $\tilde{t}$, and from $\tilde{t}$ to $\bar{t}$) discussed above. □

2.3 Submodularity of the objective

Now, we show that the portfolio valuation function is submodular. This result is primarily of taxonomical interest and may be safely skipped, as our subsequent results do not rely on submodular analysis.

Definition 1 (Submodular set function)**.** Given a ground set $\mathcal{C}$ and function $v : 2^{\mathcal{C}} \mapsto \mathbb{R}$, $v(\mathcal{X})$ is called a *submodular set function* if and only if $v(\mathcal{X}) + v(\mathcal{Y}) \geq v(\mathcal{X} \cup \mathcal{Y}) + v(\mathcal{X} \cap \mathcal{Y})$ for all $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$. Furthermore, if $v(\mathcal{X} \cup \{k\}) - v(\mathcal{X}) \geq 0$ for all $\mathcal{X} \subset \mathcal{C}$ and $k \in \mathcal{C} \setminus \mathcal{X}$, $v(\mathcal{X})$ is said to be a *nondecreasing* submodular set function.

Theorem 1. *The college application portfolio valuation function $v(\mathcal{X})$ is a nondecreasing submodular set function.*

[1] We thank Yim Seho for pointing out this useful transformation.

Proof. It is self-evident that $v(\mathcal{X})$ is nondecreasing. To establish its submodularity, we apply proposition 2.1.iii of Nemhauser and Wolsey (1978) and show that

$$v(\mathcal{X} \cup \{j\}) - v(\mathcal{X}) \geq v(\mathcal{X} \cup \{j,k\}) - v(\mathcal{X} \cup \{k\}) \tag{8}$$

for $\mathcal{X} \subset \mathcal{C}$ and $j \neq k \in \mathcal{C} \setminus \mathcal{X}$. By Lemma 2, we can repeatedly eliminate the schools in $\mathcal{X}$ according to (7) to obtain a portfolio valuation function $w(\mathcal{Y})$ with parameter $\bar{t}$ such that $w(\mathcal{Y}) = v(\mathcal{X} \cup \mathcal{Y}) + \text{const.}$ for any $\mathcal{Y} \subseteq \mathcal{C} \setminus \mathcal{X}$. Therefore, (8) is equivalent to

$$\begin{aligned} & w(\{j\}) - w(\varnothing) \geq w(\{j,k\}) - w(\{k\}) && (9)\\ \iff \quad & w(\{j\}) + w(\{k\}) \geq w(\{j,k\}) && (10)\\ \iff \quad & \mathrm{E}[\,\bar{t}_j Z_j\,] + \mathrm{E}[\,\bar{t}_k Z_k\,] \geq \mathrm{E}\big[\max\{\bar{t}_j Z_j, \bar{t}_k Z_k\}\big] && (11)\end{aligned}$$

which is plainly true. □

3 The greedy algorithm

3.1 The nestedness property

The optimality of the greedy algorithm for the college application problem rests on the fact the the optimal solution possesses a special structure: An optimal portfolio of size $h+1$ includes an optimal portfolio of size h as a subset.

Theorem 2 (Nestedness of optimal application portfolios)**.** *There exists a sequence of portfolios $\{\mathcal{X}_h\}_{h=1}^m$ satisfying the nestedness relation*

$$\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset \mathcal{X}_m \tag{12}$$

such that each $\mathcal{X}_h$ is an optimal application portfolio when the application limit is h.

Proof. By induction on h. Applying Lemma 1, we assume that $t_0 = 0$.

(Base case.) First, we will show that $\mathcal{X}_1 \subset \mathcal{X}_2$. To get a contradiction, suppose that the optima are $\mathcal{X}_1 = \{j\}$ and $\mathcal{X}_2 = \{k,l\}$, where we may assume that $t_k \leq t_l$. Optimality requires that

$$v(\mathcal{X}_1) = f_j t_j > v(\{k\}) = f_k t_k \tag{13}$$

and

$$\begin{aligned} v(\mathcal{X}_2) = f_k(1-f_l)t_k + f_l t_l > v(\{j,l\}) & \\ = f_j(1-f_l)t_j + (1-f_j)f_l t_l + f_j f_l \max\{t_j, t_l\} & \\ \geq f_j(1-f_l)t_j + (1-f_j)f_l t_l + f_j f_l t_l & \qquad (14)\\ = f_j(1-f_l)t_j + f_l t_l & \\ \geq f_k(1-f_l)t_k + f_l t_l = v(\mathcal{X}_2) & \end{aligned}$$

which is a contradiction.

(Inductive step.) Assume that $\mathcal{X}_1 \subset \cdots \subset \mathcal{X}_h$, and we will show $\mathcal{X}_h \subset \mathcal{X}_{h+1}$. Let $k = \arg\max\{t_k : k \in \mathcal{X}_{h+1}\}$ and write $\mathcal{X}_{h+1} = \mathcal{Y}_h \cup \{k\}$.

Suppose $k \notin \mathcal{X}_h$. To get a contradiction, suppose that $v(\mathcal{Y}_h) < v(\mathcal{X}_h)$ and $v(\mathcal{X}_{h+1}) > v(\mathcal{X}_h \cup \{k\})$. Then

$$\begin{aligned} v(\mathcal{X}_{h+1}) &= v(\mathcal{Y}_h \cup \{k\}) \\ &= (1-f_k)v(\mathcal{Y}_h) + f_k t_k \\ &\le (1-f_k)v(\mathcal{X}_h) + f_k \operatorname{E}\big[\max\{t_k, X_h\}\big] \\ &= v(\mathcal{X}_h \cup \{k\}) \end{aligned} \tag{15}$$

is a contradiction.

Now suppose that $k \in \mathcal{X}_h$. We can write $\mathcal{X}_h = \mathcal{Y}_{h-1} \cup \{k\}$, where $\mathcal{Y}_{h-1}$ is some portfolio of size $h-1$. It suffices to show that $\mathcal{Y}_{h-1} \subset \mathcal{Y}_h$. By definition, $\mathcal{Y}_{h-1}$ (respectively, $\mathcal{Y}_h$) maximizes the function $v(\mathcal{Y} \cup \{k\})$ over portfolios of size $h-1$ (respectively, h) that do not include k. That is, $\mathcal{Y}_{h-1}$ and $\mathcal{Y}_h$ are the optimal complements to the singleton portfolio $\{k\}$.

Applying Lemma 2, we eliminate school k, transform the remaining t_j-values to $\bar{t}_j$ according to (7), and obtain a function $w(\mathcal{Y}) = v(\mathcal{Y} \cup \{k\}) - f_k t_k$ that grades portfolios $\mathcal{Y} \subseteq \mathcal{C} \setminus \{k\}$ according to how well they complement $\{k\}$. Since $w(\mathcal{Y})$ is itself a portfolio valuation funtion and $\bar{t}_0 = 0$, the inductive hypothesis implies that $\mathcal{Y}_{h-1} \subset \mathcal{Y}_h$, which completes the proof. $\square$

3.2 Quadratic-time solution

Applying Theorem 2 yields an efficient greedy algorithm for the optimal portfolio: Start with the empty set and add schools one at a time, maximizing $v(\mathcal{X} \cup \{k\})$ at each addition. Sorting t is $O(m \log m)$. At each of the h iterations, there are $O(m)$ candidates for k, and computing $v(\mathcal{X} \cup \{k\})$ is $O(h)$ using (3); therefore, the time complexity of this algorithm is $O(h^2 m + m \log m)$.

We reduce the computation time to $O(hm)$ by taking advantage of the transformation from Lemma 2. Once school k is added to $\mathcal{X}$, we eliminate it from the set $\mathcal{C} \setminus \mathcal{X}$ of candidates, and update the t_j-values of the remaining schools according to (7). Now, the *next* school added must be the optimal singleton portfolio in the modified market. But the optimal singleton portfolio consists simply of the school with the highest value of $f_j \bar{t}_j$. Therefore, by updating the t_j-values at each iteration according to (7), we eliminate the need to compute $v(\mathcal{X})$ entirely. Moreover, this algorithm does not require the schools to be indexed in ascending order by t_j, which removes the $O(m \log m)$ sorting cost.

Algorithm 1: Optimal portfolio algorithm for the college application problem.

Input: Utility values $t \in (0,\infty)^m$, admissions probabilities $f \in (0,1]^m$, application limit $h \le m$.

$\mathcal{C} \leftarrow \{1 \dots m\}$;
`X`, `V` $\leftarrow$ empty h-arrays;
for $i = 1 \dots h$ **do**
    $k \leftarrow \arg\max_{j \in \mathcal{C}} \{f_j t_j\}$;
    $\mathcal{C} \leftarrow \mathcal{C} \setminus \{k\}$;
    `X`$[i] \leftarrow k$;
    if $i = 1$ **then** `V`$[i] \leftarrow f_k t_k$ **else** `V`$[i] \leftarrow$ `V`$[i-1] + f_k t_k$;
    for $j \in \mathcal{C}$ **do**
        if $t_j \le t_k$ **then** $t_j \leftarrow (1-f_k)t_j$ **else** $t_j \leftarrow t_j - f_k t_k$;
    end
end
return `X`, `V`

Theorem 3 (Validity of Algorithm 1)**.** *Algorithm 1 produces an optimal application portfolio for the cardinality-constrained college application problem in $O(hm)$ time.*

Proof. Optimality follows from the proof of Theorem 2. At each of the h iterations of the main loop, finding the top school costs $O(m)$, and the t_j-values of the remaining $O(m)$ schools are each updated in unit time. Therefore, the overall time complexity is $O(hm)$. $\square$

By running the algorithm with $h = m$, one can obtain the optimal portfolio for *all* h by taking $\mathcal{X}_h = \{\mathtt{X}[1], \dots, \mathtt{X}[h]$. Thus, Fu (2014)'s variant of the college application problem

$$\text{maximize} \quad v(\mathcal{X}) - c(|\mathcal{X}|)$$

(where $c(|\mathcal{X}|$ is an arbitrary increasing cost function) can be solved in $O(m^2)$-time by a trivial modification of Algorithm 1: Simply compute

$$h^* = \underset{h=1\dots m}{\arg\max} \big\{ \mathtt{V}[h] - c(h) \big\}$$

and return $\mathcal{X}_{h^*}$.

4 Discussion

5 Conclusion

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