Upwind and Lax-Wendroff methods for the linear advection equation

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October 2018

1 Problem

We consider the linear advection equation

Hyperbolic equation 1D: linear advection equation

Find u : $[0,2] \times [0,T] \to \mathbb{R}$ solution of

$$\frac{\partial u(x,t)}{\partial t} + a(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \quad (x,t) \in [0,2] \times [0,T] \tag{1}$$

where

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4} \tag{2}$$

$$u(0,t) = 0, \quad \forall t > 0 \tag{3}$$

$$u(x,0) = u_0(x), \quad \forall 0 \le x \le 2 \tag{4}$$

with initial condition

$$u(x,0) = 1, \quad x \in [0.2, 0.4], \quad u(x,0) = 0, \quad x \notin [0.2, 0.4] \tag{5}$$

We want to get numerical approximations to the solution applying the upwind scheme and Lax-Wendroff scheme. We choose as final time $T = 0.1$, $T = 0.5$ and $T = 1$.
We consider a uniform mesh both in x and t variables with size Δx and Δt respectively. We take $\Delta x = \Delta t$.
Let us denote by Nx the number of nodes in x variable and by Nt the number of nodes in the t variable.
Let $(x_i)_0^{Nx}$ the regular discretization of the interval $[0,1]$ in Nx+1 points :

$$x_i = i.\Delta x, \quad \forall i \in [\![0, Nx]\!], \quad with \quad \Delta x = \frac{1}{Nx}$$

and $(t_n)_0^{Nt}$ the regular discretization of the interval $[0,T]$ in Nt+1 points :

$$t_n = n.\Delta t, \quad \forall n \in [\![0, Nt]\!], \quad with \quad \Delta t = \frac{T}{Nt}$$

We will

- Make a Matlab code to approximate the solution using the upwind scheme and plot the numerical approximation and the true solution for $\Delta x = 0.02$ and $\Delta x = 0.01$ at times $T = 0.1$, $T = 0.5$ and $T = 1$.
- Make a Matlab code to approximate the solution using the Lax-Wendroff scheme and plot the numerical approximation and the true solution for $\Delta x = 0.02$ and $\Delta x = 0.01$ at times $T = 0.1$, $T = 0.5$ and $T = 1$.

N.B : The file Upwind001.m (resp. Upwind002.m) plot the numerical solution and true solution for $\Delta x = 0.01$ (resp. $\Delta x = 0.02$) and the file LaxWendroff001.m (resp. LaxWendroff002.m) plot the numerical solution and true solution for $\Delta x = 0.01$ (resp. $\Delta x = 0.02$) at times $T = 0.1$, $T = 0.5$ and $T = 1$.

Hyperbolic equation 1D: linear advection equation, formulation at the points of discretization

Find $u(x_i, t_n) \in \mathbb{R}$, $\forall n \in [\![0, Nt]\!]$, $\forall i \in [\![0, Nx]\!]$ such as

$$\frac{\partial u(x_i, t_n)}{\partial t} + a(x_i, t_n)\frac{\partial u(x_i, t_n)}{\partial x} = 0, \quad \forall n \in]\!]0, Nt[\![, \quad \forall i \in]\!]0, Nx[\![\tag{6}$$

where

$$a(x_i, t_n) = \frac{1 + x_i^2}{1 + 2x_i t_n + 2x_i^2 + x_i^4} \tag{7}$$

$$u(0, t_n) = 0, \quad \forall n \in [\![0, Nt]\!] \tag{8}$$

$$u(x_i, 0) = u_0(x_i), \quad \forall i \in [\![0, Nx]\!] \tag{9}$$

with initial condition

$$u(x_i, 0) = 1, \quad x_i \in [0.2, 0.4], \quad u(x_i, 0) = 0, \quad x_i \notin [0.2, 0.4] \tag{10}$$

2 Theoretical solution

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + a(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \quad (x,t) \in [0, L] \times [0, T] \\ u(0, t) = 0, \quad \forall t > 0 \\ u(x, 0) = u_0(x), \quad \forall 0 \le x \le L \end{cases} \tag{11}$$

We search for the characteristic curves which are the solutions of the ordinary differential equation:

$$\frac{dX}{dt} = a(X(t), t) \tag{12}$$

Along these curves we have:

$$\frac{d}{dt}(u(X(t), t)) = \frac{\partial u}{\partial t}(X(t), t) + \frac{\partial u}{\partial x}(X(t), t)\frac{dX}{dt} = \left(\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x}\right)(X(t), t) = 0 \tag{13}$$

This means that the solutions are constant along the characteristics, it is enough then take into account the initial condition. The solution in point (x_0, t_0) is equal to the value of $u_0(x^*)$ where is the value for t = 0 of the characteristic that go through point (x_0, t_0) in other words:

$$u(x_0, t_0) = u_0(x^*), \quad x^* = X(0) \tag{14}$$

And $X(t)$ is solution of

$$\begin{cases} \frac{dX}{dt} = a(X(t), t) \\ X(x_0) = t_0 \end{cases} \tag{15}$$

Now you have to integrate (15) to find $X(t)$ and therefore we can find the exact solution of the problem.
Study of $a(x,t)$: we know

$$\forall 0 \le x \le, \quad \forall t > 0, \quad 0 \le 2xt + x^2 + x^4 \iff 1 + x^2 \le 1 + 2xt + x^2 + x^2 + x^4 \tag{16}$$

We check that$1 + 2xt + 2x^2 + x^4 \neq 0, \forall t > 0$, so

$$0 \le a(x,t) = \frac{1 + x^2}{1 + 2xt + 2x^2 + x^4} \le 1 \tag{17}$$

3 The Upwind scheme

The first simple scheme we are intersted in belongs to the class of so-called *upwind* methods – numerical discretization schemes for solving hyperbolic PDEs. According to such a scheme, the spatial differences are skewed in the "upwind" direction,i.e., the direction from which the advecting flow originates.

Hyperbolic equation 1D: linear advection equation, Upwind scheme for a(x,t)>0

Find $u(x_i, t_n) \in \mathbb{R}$, $\forall n \in [\![0, Nt]\!]$, $\forall i \in [\![0, Nx]\!]$ such as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0, \quad \forall n \in]\!]0, Nt[\![, \quad \forall i \in]\!]0, Nx[\![\tag{18}$$

$$u(0, t_n) = 0, \quad \forall n \in [\![0, Nt]\!] \tag{19}$$

$$u(x_i, 0) = u_0(x_i), \quad \forall i \in [\![0, Nx]\!] \tag{20}$$

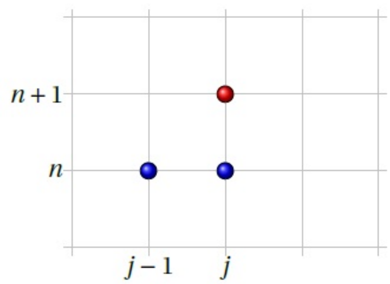

Figure 1: Stencil of the Upwind scheme

3.1 Von Neumann Stability Analysis : CFL conditions

To find the CFL constraint, we conduct the von Neumann stability analysis. We substitute $u_i^n = A^n e^{jkx_i}$ which yields

$$(A^{n+1} - A^n)e^{jkx_i} + \mu A^n(e^{jkx_i} - e^{jkx_{i-1}}) = 0, \quad with \quad \mu = \frac{a\Delta t}{\Delta x} \tag{21}$$

So,

$$A^{n+1} = A^n \left(1 - \mu(1 - e^{jk\Delta x})\right) = A^n \left(1 - \mu + \mu e^{jk\Delta x}\right) \tag{22}$$

The growth factor for the case when, $a \geq 0$ is :

$$G = 1 - \mu + \mu e^{jk\Delta x} = 1 - \mu(1 - \cos(k\Delta x)) - i\mu\sin(k\Delta x) \tag{23}$$

Let , $\theta = k\Delta x$

$$G(\theta) = 1 - \mu(1 - \cos(\theta)) - j\mu\sin(\theta) \tag{24}$$

So, we have

$$Re(G(\theta)) = 1 - \mu(1 - \cos(\theta)) \quad and \quad Im(G(\theta)) = -\mu\sin(\theta) \tag{25}$$

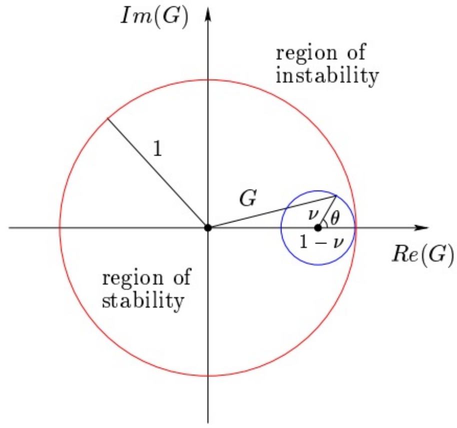

Figure 2: Region of stability/instability : Stability restriction $|G(\theta)| \leq 1$ means that G must lie within the unit circle in the complex plane.

with magnitude,

$$|G(\theta)|^2 = (1 - \mu(1 - cos(\theta))^2 + \mu^2 sin(\theta)^2 = (1-\mu)^2 + 2(1-\mu)\mu cos(\theta) + \mu^2 = 1 - 2(1-\mu)\mu(1 - cos(\theta)) \quad (26)$$

So , if $(1 - \mu) > 0$ (i.e., $\mu \leq 1$) or $\Delta x \leq \frac{\Delta x}{a}$ we have :

$$|G(\theta)| \leq 1 \quad (27)$$

Note that no NBC is needed for upwind scheme, and there is no severe time step restriction, since $\mu \leq 1$. If $a = a(x, t)$ is a variable function that does not change the sign (which is our case), the CFL condition is,

$$0 < \frac{max|a(x,t)|\Delta t}{\Delta x} \leq 1 \quad (28)$$

However, the upwind scheme is first-order in time and in space, and there are some high-order schemes.

3.2 Error analysis of the upwind scheme

3.2.1 Case where the solution is smooth

To evaluate the order of the method we study the local truncation error of the schema defined by :

$$\epsilon_i^n = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + a(x_i, t_n)\frac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x} \quad (29)$$

According to the Taylor-Young formula, we have:

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \Delta t \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(\Delta t^3) \quad (30)$$

so,

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} = \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i, t_n) + O(\Delta t^2) \quad (31)$$

and,

$$u(x_{i-1}, t_n) = u(x_i, t_n) - \Delta x \frac{\partial u}{\partial x}(x_i, t_n) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i, t_n) + O(\Delta x^3) \quad (32)$$

so,

$$\frac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x} = \frac{\partial u}{\partial x}(x_i, t_n) + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x_i, t_n) + O(\Delta x^2) \quad (33)$$

Thus,

$$\epsilon_i^n = \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i, t_n) + a(x_i, t_n)\left(\frac{\partial u}{\partial x}(x_i, t_n) + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x_i, t_n)\right) + O(\Delta x^2) + O(\Delta t^2) \quad (34)$$

Therefore,

$$\epsilon_i^n = \frac{\partial u}{\partial t}(x_i,t_n) + a(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n) + a(x_i,t_n)\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_n) + O(\Delta x^2) + O(\Delta t^2) \quad (35)$$

Hence,

$$\epsilon_i^n = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n) + a(x_i,t_n)\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_n) + O(\Delta x^2) + O(\Delta t^2) = O(\Delta x) + O(\Delta t) \quad (36)$$

Hence, the *upwind* scheme is first-order in time and in space, and there are some high-order schemes.

3.2.2 Case where the solution has discontinuous derivatives

The analysis of truncation error is only valid for solutions which are sufficiently smooth, while this problem has a discontinuous solution. In fact the maximum error in this problem is $O\left((\Delta x)\right)^{1/2}$ for the upwind scheme.

3.3 Fourier analysis of the upwind scheme

Because hyperbolic equations often describe the motion and development of waves, Fourier analysis is of great value in studying the accuracy of methods as well as their stability. The modulus of $G(\theta)$ describes the *damping* and the argument describes the *dispersion* in the scheme, i.e., the extent to which the wave speed varies with the frequency. We must, for the present and for a strict analysis, assume that a is a positive function. The Fourier mode

$$u(x,t) = e^{i(kx+\omega t)} \quad (37)$$

is then an exact solution of the differential equation (1) provided that ω and k satisfy the dispersion relation

$$\omega = -a(x,t)k \quad (38)$$

The phase of the numerical mode is given by

$$arg(G) = -tan^{-1}\left(\frac{\mu sin\theta}{(1-\mu)+\mu cos\theta}\right) =_{\theta\to 0} -\mu\theta(1-\frac{1}{6}(1-\mu)(1-2\mu)\theta^2...)... \quad (39)$$

we have found that the upwind scheme always has an amplitude error which, from (26), is of order $\theta^2 = (k\Delta x)^2$ in one time step, corresponding to a global error of order θ, and from (39) it has a relative phase error of order θ^2.

3.4 Maximum principle

The *upwind* scheme satisfies the principle of the maximum under the condition of stability

$$0 < \frac{max|a(x,t)|\Delta t}{\Delta x} \leq 1 \quad (40)$$

It suffices to show that the condition

$$\alpha \leq u_i^n \leq \beta \quad i \in \mathbb{N}, n \ \text{ fixed} \quad (41)$$

leads

$$\alpha \leq u_i^{n+1} \leq \beta \quad \forall i \in \mathbb{N} \quad (42)$$

then reason on n by recurrence. Or the equation (16) can be written

$$u_i^{n+1} = (1-\mu)u_i^n + \mu u_i^{n-1} \quad with \quad \mu = \frac{a_i^n \Delta t}{\Delta x} \leq 1 \quad (43)$$

Hence u_i^{n+1} is a **convex linear combination** of u_i^n and u_i^{n-1}. Hypothesis (35) leads to both $(1-\mu) \geq 0$ and $\mu > 0$ as well as we have $(1-\mu)+\mu = 1$ and therefore by multiplication (36) by $(1-\mu)$ (for the index i) and μ (for the index i-1), so we deduce the relation (37).

3.5 Numerical results

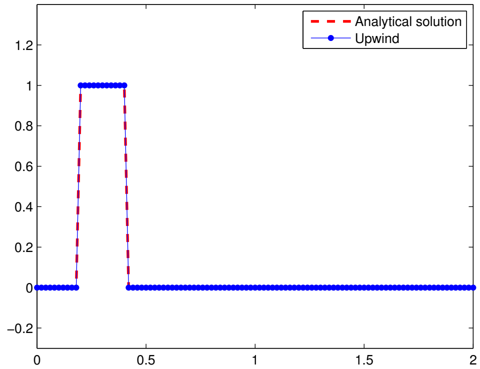

Figure 3: $\Delta x = 0.02$ and $t = 0$

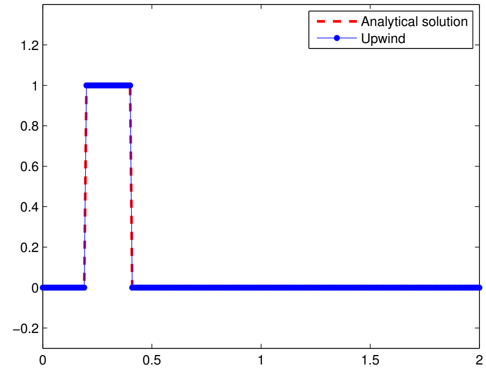

Figure 4: $\Delta x = 0.01$ and $t = 0$

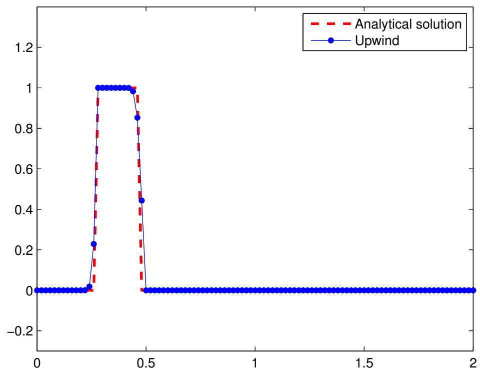

Figure 5: $\Delta x = 0.02$ and $t = 0.1$

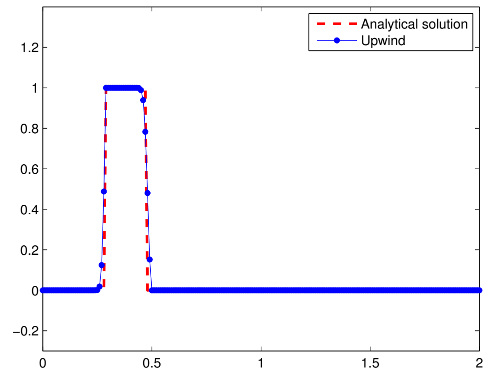

Figure 6: $\Delta x = 0.01$ and $t = 0.1$

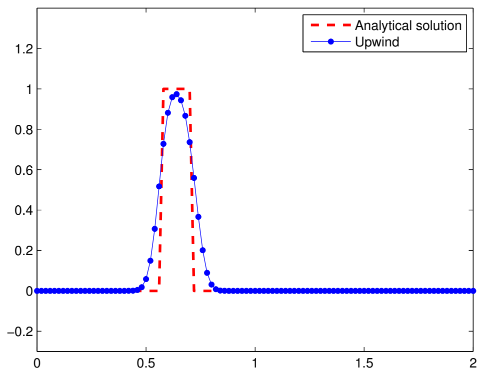

Figure 7: $\Delta x = 0.02$ and $t = 0.5$

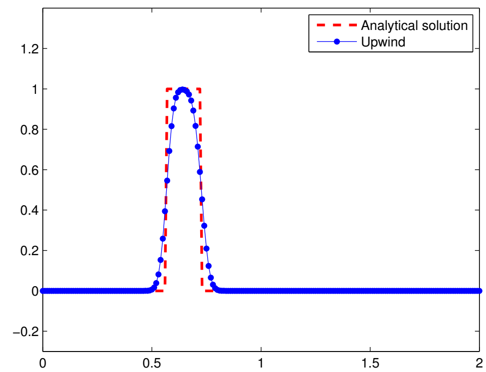

Figure 8: $\Delta x = 0.01$ and $t = 0.5$

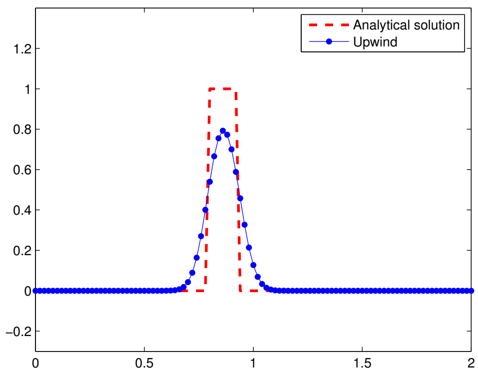

Figure 9: $\Delta x = 0.02$ and $t = 1$

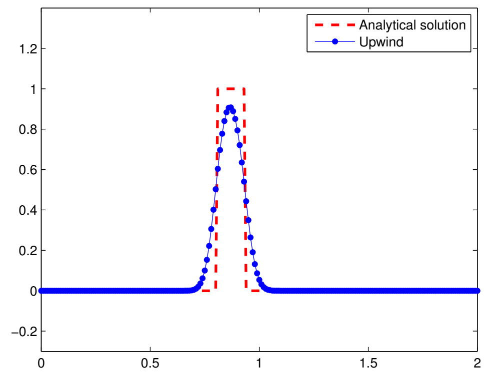

Figure 10: $\Delta x = 0.01$ and $t = 1$

- There are now no spurious oscillations generated at the sharp edges of the wave-form. On the other hand, the wave-form shows evidence of dispersion. Indeed, the upwind differencing scheme suffers from the same type of spurious dispersion problem as the Lax scheme. Unfortunately, there is no known differencing scheme which is both non-dispersive and capable of dealing well with sharp wave-fronts. In fact, sophisticated codes which solve the advection (or wave) equation generally employ an upwind scheme in regions close to sharp wave-fronts, or shocks, and a more accurate non-dispersive scheme elsewhere.
- The solution represents a square pulse moving to the right. It is clear from the figures how the damping of the high frequency modes has resulted in a substantial smoothing of the edges of the pulse, and a slight reduction of its height. However, the rather small phase error means that the pulse moves with nearly the right speed. The second set of results, with a halving of the mesh size in both co-ordinate directions, shows the expected improvement in accuracy, though the results are still not very satisfactory.

4 The Lax-Wendroff scheme

Hyperbolic equation 1D: linear advection equation, Lax-Wendroff scheme

Find $u(x_i, t_n) \in \mathbb{R}$, $\forall n \in [\![0, Nt]\!]$, $\forall i \in [\![0, Nx]\!]$ such as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{\Delta t}{2}\left(ap_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - a_i^n \frac{V_{i+\frac{1}{2}}^n - V_{i-\frac{1}{2}}^n}{\Delta x}\right) = 0, \quad \forall n \in]\!]0, Nt[\![, \quad \forall i \in]\!]0, Nx[\![\tag{44}$$

with,

$$V_i^n = a_i^n \frac{u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n}{\Delta x} \quad and \quad ap_i^n = \left[\frac{\partial a}{\partial t}\right]_i^n \tag{45}$$

$$u(0, t_n) = 0, \quad \forall n \in [\![0, Nt]\!] \tag{46}$$

$$u(x_i, 0) = u_0(x_i), \quad \forall i \in [\![0, Nx]\!] \tag{47}$$

We have,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{\Delta t}{2}\left(ap_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - a_i^n \frac{V_{i+\frac{1}{2}}^n - V_{i-\frac{1}{2}}^n}{\Delta x}\right) = 0 \tag{48}$$

Or, we have $V_i^n = a_i^n \frac{u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n}{\Delta x}$, so we can write

$$V_{i+\frac{1}{2}}^n - V_{i-\frac{1}{2}}^n = \frac{1}{\Delta x}\left(a_{i+\frac{1}{2}}^n (u_{i+1}^n - u_i^n) - a_{i-\frac{1}{2}}^n (u_i^n - u_{i-1}^n)\right) \tag{49}$$

so,

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a_i^n\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}+\frac{\Delta t}{2}\left(ap_i^n\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}-\frac{a_i^n}{(\Delta x)^2}\left(a_{i+\frac{1}{2}}^n(u_{i+1}^n-u_i^n)-a_{i-\frac{1}{2}}^n(u_i^n-u_{i-1}^n)\right)\right)=0 \tag{50}$$

which may be written,

$$u_i^{n+1}=u_i^n-\frac{a_i^n\Delta t}{2\Delta x}(u_{i+1}^n-u_{i-1}^n)-\frac{ap_i^n\Delta t^2}{4\Delta x}(u_{i+1}^n-u_{i-1}^n)+\frac{a_i^n\Delta t^2}{2\Delta x^2}\left(a_{i+\frac{1}{2}}^n(u_{i+1}^n-u_i^n)-a_{i-\frac{1}{2}}^n(u_i^n-u_{i-1}^n)\right) \tag{51}$$

Thus

$$u_i^{n+1}=u_i^n-v_1(u_{i+1}^n-u_{i-1}^n)-v_2(u_{i+1}^n-u_{i-1}^n)+v_3\left(a_{i+\frac{1}{2}}^n(u_{i+1}^n-u_i^n)-a_{i-\frac{1}{2}}^n(u_i^n-u_{i-1}^n)\right) \tag{52}$$

with

$$v_1=\frac{a_i^n\Delta t}{2\Delta x},\quad v_2=\frac{ap_i^n\Delta t^2}{4\Delta x}\quad and\quad v_3=\frac{a_i^n\Delta t^2}{2\Delta x^2} \tag{53}$$

so , we can write

$$u_i^{n+1}=u_{i+1}^n(-v_1-v_2+v_3a_{i+\frac{1}{2}}^n)+u_i^n\left(1-v_3(a_{i+\frac{1}{2}}^n+a_{i-\frac{1}{2}}^n)\right)+u_{i-1}^n(v_1+v_2+v_3a_{i-\frac{1}{2}}^n) \tag{54}$$

Hence,

$$\begin{cases}u_i^{n+1}=\alpha u_{i+1}^n+\beta u_i^n+\gamma u_{i-1}^n\\ \alpha=-v_1-v_2+v_3a_{i+\frac{1}{2}}^n\\ \beta=1-v_3(a_{i+\frac{1}{2}}^n+a_{i-\frac{1}{2}}^n)\\ \gamma=v_1+v_2+v_3a_{i-\frac{1}{2}}^n\end{cases} \tag{55}$$

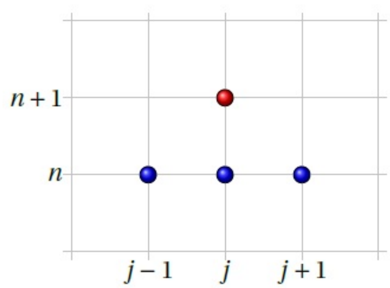

Figure 11: Stencil of the Lax-Wendroff scheme

4.1 Von Neumann Stability Analysis : CFL conditions

To find the CFL constraint, we conduct the von Neumann stability analysis. We substitute $u_i^n=A^ne^{jkx_i}$ which yields We have,

$$\begin{cases}u_i^{n+1}=\alpha u_{i+1}^n+\beta u_i^n+\gamma u_{i-1}^n\\ \alpha=-v_1-v_2+v_3a_{i+\frac{1}{2}}^n\\ \beta=1-v_3(a_{i+\frac{1}{2}}^n+a_{i-\frac{1}{2}}^n)\\ \gamma=v_1+v_2+v_3a_{i-\frac{1}{2}}^n\end{cases} \tag{56}$$

so,

$$A^{n+1}e^{jkx_i}=A^n\left(\alpha e^{jkx_{i+1}}+\beta+\gamma e^{jkx_{i-1}}\right) \tag{57}$$

by simplification (with e^{jkx_i}), we have

$$A^{n+1}=A^n\left(\alpha e^{jk\Delta x}+\beta+\gamma e^{-jk\Delta x}\right) \tag{58}$$

Let , $\theta = k\Delta x$ Or, we know

$$\alpha+\beta+\gamma = -v_1 - v_2 + v_3 a^n_{i+\frac{1}{2}} + 1 - v_3(a^n_{i+\frac{1}{2}} + a^n_{i-\frac{1}{2}}) + v_1 + v_2 + v_3 a^n_{i-\frac{1}{2}} \tag{59}$$

so, we have

$$\alpha+\beta+\gamma = 1 \tag{60}$$

so, we can simplify (53)

$$G(\theta) = \frac{A^{n+1}}{A^n} = \alpha e^{j\theta} + \beta + \gamma e^{-j\theta} = 1 + (cos(\theta)-1)(\alpha+\gamma) + j.sin(\theta)(\alpha-\gamma) \tag{61}$$

therefore,

$$G(\theta) = G_r + jG_i, \quad with \;\; G_r = 1 - 2sin(\frac{\theta}{2})^2(\alpha+\gamma) \;\; and \;\; G_i = sin(\theta)(\alpha-\gamma) \tag{62}$$

with magnitude,

$$|G(\theta)|^2 = \left(1 - 2sin(\frac{\theta}{2})^2(\alpha+\gamma)\right)^2 + (sin(\theta)(\alpha-\gamma))^2 \tag{63}$$

$$= 1 + 4sin(\frac{\theta}{2})^4(\alpha+\gamma)^2 - 4sin(\frac{\theta}{2})^2(\alpha+\gamma) + sin(\theta)^2(\alpha-\gamma)^2$$

$$\leq 1 + 4(\alpha+\gamma)^2 - 4(\alpha+\gamma) + (\alpha-\gamma)^2 \tag{64}$$

$$= 1 + 4(\alpha+\gamma)(\alpha+\gamma-1) + (\alpha-\gamma)^2 \tag{65}$$

$$= 1 - 4(1-\beta)\beta + (\alpha-\gamma)^2 \tag{66}$$

$$\tag{67}$$

so to have stability it is necessary that,

$$(\alpha-\gamma)^2 - 4(1-\beta)\beta \leq 0 \Longrightarrow (\alpha-\gamma)^2 \leq 4(1-\beta)\beta \tag{68}$$

Let , $m = max|a(x,t)|$ and $m' = max|\frac{\partial a}{\partial t}(x,t)|$. We replace α, β, γ by their values

$$0 < (v_1+v_2)^2 \leq 2v_3 m(1-2v_3 m) \Longrightarrow \begin{cases} 1 - 2v_3 m = 1 - \frac{m^2\Delta t^2}{\Delta x^2} \geq 0 \Longrightarrow 0 \leq \frac{m\Delta t}{\Delta x} \leq 1, \\ \text{the previous result simplifies the inequality, so} \\ \Longrightarrow 0 < (v_1+v_2)^2 \leq 1 \Longrightarrow 0 \leq \frac{\Delta t.max(m,m')}{2\Delta x} \leq 1 \end{cases} \tag{69}$$

Thus we see that the scheme is stable, if

$$\begin{cases} 0 < \frac{max|a(x,t)|\Delta t}{\Delta x} \leq 1 \\ 0 \leq \frac{\Delta t.max(m,m')}{2\Delta x} \leq 1 \end{cases} \tag{70}$$

4.2 Error analysis of the upwind scheme

4.2.1 Case where the solution is smooth

To evaluate the order of the method we study the local truncation error of the schema defined by :

$$\epsilon^n_i = \frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{\Delta t} + a(x_i,t_n)\frac{u(x_{i+1},t_n) - u(x_{i-1},t_n)}{2\Delta x} + \\ \frac{\Delta t}{2}\left(ap(x_i,t_n)\frac{u(x_{i+1},t_n) - u(x_{i-1},t_n)}{2\Delta x} - a(x_i,t_n)\frac{a(x_{i+\frac{1}{2}},t_n)(u(x_{i+1},t_n) - u(x_i,t_n)) - a(x_{i-\frac{1}{2}},t_n)(u(x_i,t_n) - u(x_{i-1},t_n))}{\Delta x^2}\right) \tag{71}$$

According to the Taylor-Young formula, we have:

$$u(x_i,t_{n+1}) = u(x_i,t_n) + \Delta t\frac{\partial u}{\partial t}(x_i,t_n) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n) + O(\Delta t^3) \tag{72}$$

so,

$$\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\Delta t}=\frac{\partial u}{\partial t}(x_i,t_n)+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n)+O(\Delta t^2) \tag{73}$$

we have,

$$u(x_{i+1},t_n)=u(x_i,t_n)+\Delta x\frac{\partial u}{\partial x}(x_i,t_n)+\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_n)+O(\Delta x^3) \tag{74}$$

and,

$$u(x_{i-1},t_n)=u(x_i,t_n)-\Delta x\frac{\partial u}{\partial x}(x_i,t_n)+\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i,t_n)+O(\Delta x^3) \tag{75}$$

so,

$$\frac{u(x_{i+1},t_n)-u(x_{i-1},t_n)}{2\Delta x}=\frac{\partial u}{\partial x}(x_i,t_n)+O(\Delta x^2) \tag{76}$$

We have,

$$\frac{a(x_{i+\frac{1}{2}},t_n)(u(x_{i+1},t_n)-u(x_i,t_n))-a(x_{i-\frac{1}{2}},t_n)(u(x_i,t_n)-u(x_{i-1},t_n))}{\Delta x^2}=\frac{V(x_{i+1/2},t_n)-V(x_{i-1/2},t_n)}{\Delta x} \tag{77}$$

so,

$$\frac{V(x_{i+1/2},t_n)-V(x_{i-1/2},t_n)}{\Delta x}=\frac{\partial V}{\partial x}(x_i,t_n)+O(\Delta x^2) \tag{78}$$

or,

$$V(x_i,t_n)=a(x_i,t_n)\frac{u(x_{i+1/2},t_n)-u(x_{i-1/2},t_n)}{\Delta x}=a(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)+O(\Delta x^2) \tag{79}$$

thus,

$$\frac{V(x_{i+1/2},t_n)-V(x_{i-1/2},t_n)}{\Delta x}=\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)(x_i,t_n)+O(\Delta x^2) \tag{80}$$

and we have,

$$ap(x_i,t_n)\frac{u(x_{i+1},t_n)-u(x_{i-1},t_n)}{2\Delta x}=ap(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)+O(\Delta x^2)=\frac{\partial a}{\partial t}(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)+O(\Delta x^2) \tag{81}$$

With (71), (70), (66) and (63) we can write

$$\epsilon_i^n=\frac{\partial u}{\partial t}(x_i,t_n)+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n)+a(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)+\frac{\Delta t}{2}\left(\frac{\partial a}{\partial t}(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)(x_i,t_n)\right) \tag{82}$$

therefore,

$$\epsilon_i^n=\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}(x_i,t_n)+\frac{\partial a}{\partial t}(x_i,t_n)\frac{\partial u}{\partial x}(x_i,t_n)-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)(x_i,t_n)\right)+O(\Delta x^2)+O(\Delta t^2) \tag{83}$$

or, we know that

$$\frac{\partial^2 u}{\partial t^2}=-\frac{\partial a}{\partial t}\frac{\partial u}{\partial x}+a\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) \tag{84}$$

Hence,

$$\epsilon_i^n=O(\Delta x^2)+O(\Delta t^2) \tag{85}$$

Hence, the *Lax-Wendroff* scheme is second-order in time and in space

4.2.2 Case where the solution has discontinuous derivatives

The analysis of truncation error is only valid for solutions which are sufficiently smooth, while this problem has a discontinuous solution. In fact the maximum error in this problem is $O\left((\Delta x)\right)^{2/3}$ for the Lax-Wendroff scheme.

4.2.3 Case where the solution has discontinuous derivatives

The phase of the numerical mode is given by

$$arg(G) = -tan^{-1}\left(\frac{sin(\theta)(\alpha-\gamma)}{1-2sin(\frac{\theta}{2})^2(\alpha+\gamma)}\right) =_{\theta\to 0} -\mu\theta(\alpha-\gamma)(1-\frac{1}{6}(1-(\alpha+\gamma)^2)\theta^2...)... \tag{86}$$

we have proved,

$$|G(\theta)| = 1 + 4sin(\frac{\theta}{2})^4(\alpha+\gamma)^2 - 4sin(\frac{\theta}{2})^2(\alpha+\gamma) + sin(\theta)^2(\alpha-\gamma)^2 \tag{87}$$

by making an approximation of α, γ we have the amplitude error in one time step is now of order θ^4 when θ is small, compared with order θ^2 for the upwind scheme; this is a substantial improvement. Both the schemes have a relative phase error of order θ^2.

4.3 Maximum principle

The Lax-Wendroff scheme does not satisfy the principle of the maximum. The Lax-Wendroff scheme has this defect: it can create oscillations (gibbs phenomenon) if the solution has a strong gradient. We have,

$$\begin{cases} u_i^{n+1} = \alpha u_{i+1}^n + \beta u_i^n + \gamma u_{i-1}^n \\ \alpha = -v_1 - v_2 + v_3 a_{i+\frac{1}{2}}^n \\ \beta = 1 - v_3(a_{i+\frac{1}{2}}^n + a_{i-\frac{1}{2}}^n) \\ \gamma = v_1 + v_2 + v_3 a_{i-\frac{1}{2}}^n \end{cases} \tag{88}$$

we can easily check that,

$$\alpha + \beta + \gamma = 1 \tag{89}$$

now, we have to check if $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$

$$\begin{cases} -v_1 - v_2 + v_3 a_{i+\frac{1}{2}}^n \geq 0 : (L_1) \\ 1 - v_3(a_{i+\frac{1}{2}}^n + a_{i-\frac{1}{2}}^n) \geq 0 : (L_2) \\ v_1 + v_2 + v_3 a_{i-\frac{1}{2}}^n \geq 0 : (L_3) \end{cases} \quad (L_3) \leftarrow (L_3) + (L_1) \Longrightarrow \begin{cases} -v_1 - v_2 + v_3 a_{i+\frac{1}{2}}^n \geq 0 : (L_1) \\ 1 - v_3(a_{i+\frac{1}{2}}^n + a_{i-\frac{1}{2}}^n) \geq 0 : (L_2) \\ v_3(a_{i+\frac{1}{2}}^n + a_{i-\frac{1}{2}}^n) \geq 0 : (L_3) \end{cases} \tag{90}$$

$$(L_2) \leftarrow (L_2) + (L_3) \Longrightarrow \begin{cases} -v_1 - v_2 + v_3 a_{i+\frac{1}{2}}^n \geq 0 : (L_1) \\ 1 \geq 0, \quad \text{always true} \quad (L_2) \\ v_3(a_{i+\frac{1}{2}}^n + a_{i-\frac{1}{2}}^n) \geq 0 : (L_3) \end{cases} \Longrightarrow \begin{cases} -v_1 - v_2 + v_3 m \geq 0 \\ v_3 \geq 0 \end{cases} \tag{91}$$

so,

$$-\frac{m\Delta t}{2\Delta x} - \frac{m'\Delta t^2}{4\Delta x} + \frac{m^2\Delta t^2}{2\Delta x^2} \geq 0 \Longrightarrow \frac{m^2\Delta t}{\Delta x} \geq m + \frac{m'\Delta t}{2} \Longrightarrow \frac{m\Delta t}{\Delta x} \geq 1 + \frac{m'\Delta t}{2m} \tag{92}$$

It is impossible because $\frac{m\Delta t}{\Delta x} \leq 1$ according to (64). Hence u_i^{n+1} is **not** a **convex linear combination** of u_i^n and u_i^{n-1}.

The oscillations in the figures arise because the Lax–Wendroff scheme does not satisfy a maximum principle.

4.4 Numerical results

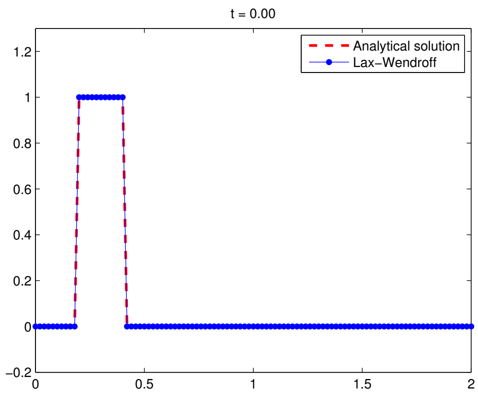

Figure 12: $\Delta x = 0.02$ and $t = 0$

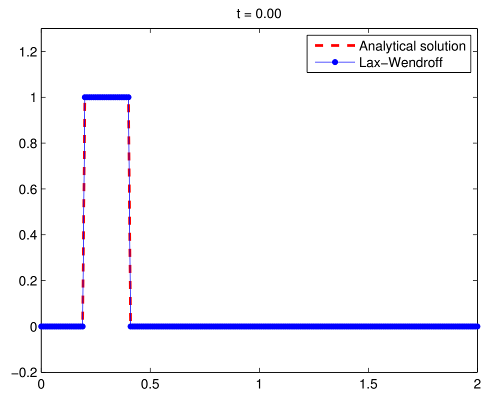

Figure 13: $\Delta x = 0.01$ and $t = 0$

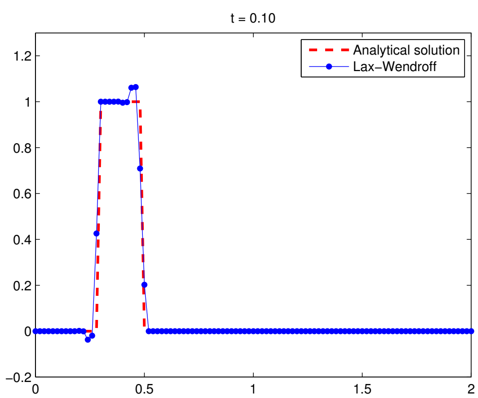

Figure 14: $\Delta x = 0.02$ and $t = 0.1$

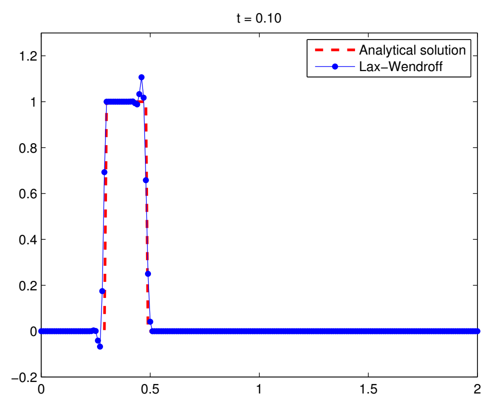

Figure 15: $\Delta x = 0.01$ and $t = 0.1$

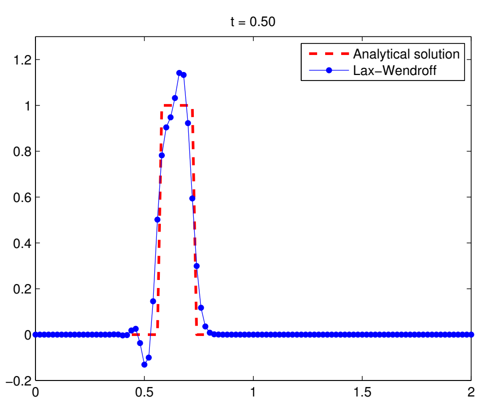

Figure 16: $\Delta x = 0.02$ and $t = 0.5$

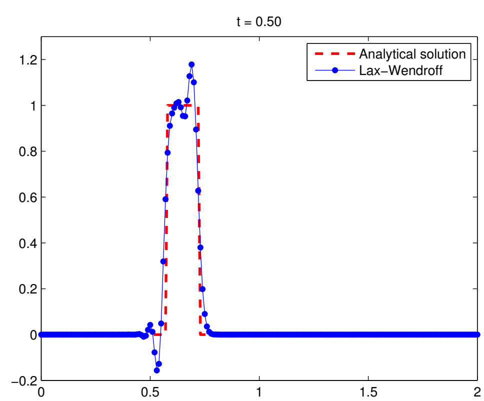

Figure 17: $\Delta x = 0.01$ and $t = 0.5$

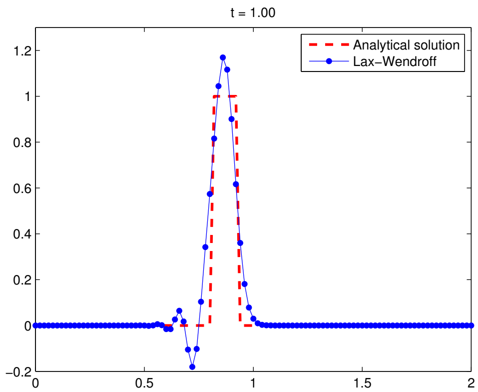

Figure 18: $\Delta x = 0.02$ and $t = 1$

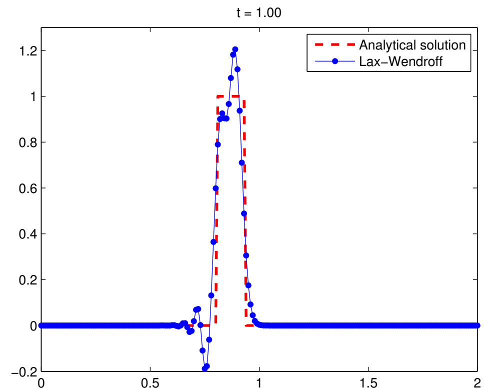

Figure 19: $\Delta x = 0.01$ and $t = 1$

5 Code Matlab for the both scheme : Upwind & Lax-Wendroff

5.1 Upwind

```
function [ u,x,t ] = UpWind(T,L,delta_t,delta_x,a)
    Nt = T/delta_t;
    Nx = L/delta_x;
    x = 0:delta_x:L;
    t = 0:delta_t:T;
    u = zeros(Nx+1,Nt+1);
    u(1:Nx+1,1) = u_0(x);
    for n=1:Nt
       for i=2:Nx+1
          v = a(x(i),t(n))*delta_t/delta_x;
          u(i,n+1) = (1-v)*u(i,n) + u(i-1,n)*v ;
       end
    end
end
```

5.2 Lax-Wendroff

```
function [ u,x,t ] = LaxWendroff(T,L,delta_t,delta_x,a,ap)
    Nt = T/delta_t;
    Nx = L/delta_x;
    x = 0:delta_x:L;
    t = 0:delta_t:T;
    u = zeros(Nx+1,Nt+1);
    u(1:Nx+1,1) = u_0(x);
    for n=1:Nt
       for i=2:Nx
          v1 = a(x(i),t(n))*delta_t/(2*delta_x);
          v2 = ap(x(i),t(n))*(delta_t^2)/(4*delta_x);
          v3 = a(x(i),t(n))*(delta_t^2)/(2*delta_x^2);
          alpha = -v1 -v2 + (v3 * a(x(i)+(delta_x/2),t(n)) ) ;
          beta  =  1 - (v3 *( a(x(i)+(delta_x/2),t(n)) + a(x(i)-(delta_x/2),t(n)) ) );
          gamma =  v1 + v2 + (v3 * a(x(i)-(delta_x/2),t(n)) );
          u(i,n+1) = u(i+1,n)*alpha + u(i,n)*beta + u(i-1,n)*gamma;
       end
    end
end
```