A posteriori error estimation

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1 Problem

Let us consider the following boundary value problem:

$$-\epsilon u'' + u' = 1, \quad u(0) = u(1) = 0. \tag{1}$$

Let us denote by u_h the finite element approximation based on linear elements on a uniform partition of the interval $[0, 1]$. We want to get an posteriori error estimator of the error $\|u - u_h\|_{L^2(0,1)}$.

Goal of the exercise :

- Write a code using MATLAB that computes the finite elements approximation u_h.
- Plot different approximations for different values of h and compare the plot of the exact solution with the plot of the numerical approximations. Make a study of the behaviour of the approximations for decreasing values of ϵ.
- Write a code that computes the residual based a posterori estimator for the L^2 norm.
- Plot the a posteriori estimator against h for the values of h for the values of h in the table. Plot in the same figure the errors of the table.
- Description of the finite element approximation, a priori error analysis in the H^1 norm, values of the constants in the analysis depending on ϵ, a priori error analysis in the L^2 norm by duality, description of the a posteriori error estimator.

n = 1/h	$\|\|u - u_h\|\|_{L^2(0,1)}$
10	0.0151
20	0.0039
40	9.7229e-4
80	2.4342e-4

2 Variational formulation/Weak formulation

Step 1: Establishment of a variational formulation.

The first step in the finite element method is to multiply our differential equation by a test function v and to integrate on the domain equation. Note that the function test v must be in the same Hilbert space as u in relation to its initial conditions.

It follows that

$$\int_0^1 -\epsilon u''(x)v(x) + u'(x)v(x)\,\mathrm{d}x = \int_0^1 v(x)\,\mathrm{d}x, \quad \forall v \in H_0^1([0,1]) \tag{2}$$

and, by partially integrating the term left considering the initial conditions,

$$\int_0^1 \epsilon u'(x)v'(x) + u'(x)v(x)\,\mathrm{d}x = \int_0^1 v(x)\,\mathrm{d}x, \quad \forall v \in H_0^1([0,1]) \tag{3}$$

In conclusion, the proposed variational formulation for (1) is :
Find $u \in H_0^1(0,1)$ such that,

$$\int_0^1 \epsilon u'(x)v'(x) + u'(x)v(x)\,\mathrm{d}x = \int_0^1 v(x)\,\mathrm{d}x, \quad \forall v \in H_0^1([0,1]) \tag{4}$$

Step 2: Resolution of the variational formulation.
In this second step we verify that the variational formulation (4) admits a unique solution. For this we use the *Lax-Milgram* theorem whose hypotheses we check with the notation

$$a(u,v) = \int_0^1 \epsilon u'(x)v'(x) + u'(x)v(x)\,\mathrm{d}x, \quad L(v) = \int_0^1 v(x)\,\mathrm{d}x \tag{5}$$

It's easy to show that a is a bilinear form and L is a linear form. This property follows directly from the linearity of integration and differentiation. Let's show that a is continuous. Since it is bilinear, it suffices to show that there exists $C > 0$ such that

$$|a(u,v)| \leq C||u||_{H_0^1(0,1)}||v||_{H_0^1(0,1)} \tag{6}$$

We have

$$|a(u,v)| \leq |\epsilon| \int_0^1 |u'(x)v'(x)|\,\mathrm{d}x + \int_0^1 |u'(x)v(x)|\,\mathrm{d}x, \quad \textit{with} \;\; \epsilon > 0 \tag{7}$$

As we know u',$v' \in L^2(0,1)$, we have, according to Hölder's theorem $u'v' \in L^1(0,1)$ and

$$||u'v'||_{L^1(0,1)} \leq ||u'||_{L^2(0,1)}||v'||_{L^2(0,1)} \tag{8}$$

By Cauchy-Schwarz we have,

$$\int_0^1 |u'(x)v(x)|\,\mathrm{d}x \leq ||u'||_{L^2(0,1)}||v||_{L^2(0,1)} \tag{9}$$

Using (8) and (9) in the inequality (7) we get,

$$|a(u,v)| \leq \epsilon||u'||_{L^2(0,1)}||v'||_{L^2(0,1)} + ||u'||_{L^2(0,1)}||v||_{L^2(0,1)} \tag{10}$$

By the definition of the standard H_0^1 we have $||u||^2_{H_0^1(0,1)} = ||u||^2_{L^2(0,1)} + ||u'||^2_{L^2(0,1)}$ and so,

$$||u'||_{L^2(0,1)} \leq ||u||_{H_0^1(0,1)} \;\; \text{and} \;\; ||u||_{L^2(0,1)} \leq ||u||_{H_0^1(0,1)} \tag{11}$$

So,

$$|a(u,v)| \leq \epsilon||u||_{H_0^1(0,1)}||v||_{H_0^1(0,1)} + ||u||_{H_0^1(0,1)}||v||_{H_0^1(0,1)} \tag{12}$$

We get then for $u, v \in H_0^1(0,1)$,

$$|a(u,v)| \leq C_\epsilon||u||_{H_0^1(0,1)}||v||_{H_0^1(0,1)} \;\; \text{with} \;\; C_\epsilon = 1 + \epsilon \tag{13}$$

So a is bilinear continuous on $H_0^1(0,1)$.
It remains to show that L is continuous. Let's show that L is a continuous application. Since it is linear, it suffices to show that there exists $C > 0$ such that

$$|L(v)| \leq C||v||_{H_0^1(0,1)} \tag{14}$$

We have,

$$|L(v)| \leq \int_0^1 |v(x)|\,\mathrm{d}x \leq ||v||_{L^2(0,1)} \tag{15}$$

According to (11), we get

$$|L(v)| \leq ||v||_{H_0^1(0,1)} \tag{16}$$

So L is linear continuous on $H_0^1(0,1)$.
Let's show finally that the application a is coercive. it suffices to show that there exists $\nu > 0$ such that

$$a(u,u) \geq \nu ||u||^2_{H_0^1(0,1)} \quad \forall u \in H_0^1(0,1) \tag{17}$$

We have,

$$a(u,u) = |\epsilon| \int_0^1 |u'(x)|^2 + u'(x)u(x)\,\mathrm{d}x \tag{18}$$

If $u \in H_0^1(0,1)$ and $v \in H_0^1(0,1)$ then $uv \in W^{1,1}(0,1)$ and $(u,v)' = u'v + uv'$. We use this result for $u^2 \in W^{1,1}(0,1)$,

$$\int_0^1 u'(x)u(x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 (u(x)^2)'\,\mathrm{d}x = \frac{1}{2}\left(u(1)^2 - u(0)^2\right) = 0 \tag{19}$$

So, we have

$$a(u,u) = |\epsilon| \int_0^1 |u'(x)|^2\,\mathrm{d}x \tag{20}$$

Therefore,

$$a(u,u) = \epsilon \int_0^1 |u'(x)|^2\,\mathrm{d}x = \epsilon ||u'||^2_{L^2(0,1)} \quad \text{with} \quad \epsilon > 0 \tag{21}$$

We can use the inequality of Poincaré in $H_0^1(0,1)$

$$||u||_{L^2(0,1)} \leq C_p ||u'||_{L^2(0,1)} \quad \forall u \in H_0^1(0,1) \quad \text{with} \quad C_p > 0 \tag{22}$$

then,

$$||u||_{L^2(0,1)} + ||u'||_{L^2(0,1)} \leq (1 + C_p)||u'||_{L^2(0,1)} \quad \forall u \in H_0^1(0,1) \tag{23}$$

So,

$$a(u,u) = \epsilon ||u'||^2_{L^2(0,1)} \geq \frac{\epsilon}{1+C_p}\left(||u||_{L^2(0,1)} + ||u'||_{L^2(0,1)}\right) \tag{24}$$

Hence,

$$a(u,u) \geq \nu_{\epsilon,p} ||u||_{H_0^1(0,1)} \quad \text{with} \quad \nu_{\epsilon,p} = \frac{\epsilon}{1+C_p} > 0 \tag{25}$$

Thus, a is coercive.
All the hypotheses of the Lax-Milgram theorem are verified, the problem (1) **admits a unique solution**.
Remark : *Study of the behaviour of the approximations for decreasing values of* ϵ.
Theorem Lemma of Cea. Let $V_h \subset V$ and assume the conditions of the Theorem of Lax–Milgram. Then there is a unique solution of the problem to find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \tag{26}$$

and it holds the error estimate

$$||u - u_h||_V \leq \frac{C}{\nu} \inf_{v_h \in V_h} ||u - v_h||_V \tag{27}$$

where u is the unique solution of the continuous problem (4). and the constants are defined in (13) as well as (25).
In this case, one can apply the *lemma of Cea* and we obtain the error estimation

$$||u - u_h||_V \leq \frac{C_\epsilon}{\nu_{\epsilon,p}} \inf_{v_h \in V_h} ||u - v_h||_V \tag{28}$$

So,

$$||u-u_h||_{H_0^1(0,1)} \leq \frac{1+\epsilon}{\frac{\epsilon}{1+C_p}} \inf_{v_h \in V_h} ||u-v_h||_{H_0^1(0,1)} \tag{29}$$

Hence,

$$||u-u_h||_{H_0^1(0,1)} \leq \frac{(1+\epsilon)(1+C_p)}{\epsilon} \inf_{v_h \in V_h} ||u-v_h||_{H_0^1(0,1)} \tag{30}$$

So , we can write (30) as

$$\boxed{||u-u_h||_{H_0^1(0,1)} \leq \frac{(1+\epsilon)(1+C_p)}{\epsilon} \inf_{v_h \in V_h} ||u-v_h||_{H_0^1(0,1)}} \longrightarrow \infty \quad \text{when} \quad \epsilon \longrightarrow 0 \tag{31}$$

In the singularly perturbed case $\epsilon << 1$ the factor of this estimate becomes very large. Thus, from this error estimate one cannot expect that the Galerkin finite element solution is accurate. On uniformly refined grids, the best approximation error becomes very small only if the dimension of V_h becomes very large.

3 Finite element in $\mathbb{P}^1$

The finite element method $\mathbb{P}^1$ is based on the discrete space of globally continuous and affine functions on each mesh.

$$V_h = \{v \in C([0,1]) \;\; \text{such that} \;\; v|_{[x_i,x_{i+1}]} \in \mathbb{P}^1, \;\; \forall 0 \leq j \leq N\} \tag{32}$$

and on his subspace

$$V_{0h} = \{v \in V_h \;\; \text{such that} \;\; v(0) = v(1) = 0\} \tag{33}$$

The variational formulation of the internal approximation becomes here
Find $u_h \in V_{0h}$ such that,

$$\int_0^1 \epsilon u_h'(x) v_h'(x) + u_h'(x) v_h(x)\,\mathrm{d}x = \int_0^1 v_h(x)\,\mathrm{d}x, \quad \forall v_h \in V_{0h} \tag{34}$$

or

$$u_h(x) = \sum_{j=1}^{N} u_h(x_j)\phi_j(x), \quad u_h'(x) = \sum_{j=1}^{N} u_h(x_j)\phi_j'(x) \tag{35}$$

so,

$$\epsilon \int_0^1 \sum_{j=1}^{N} u_h(x_j)\phi_j'(x)\phi_i'(x) + \int_0^1 \sum_{j=1}^{N} u_h(x_j)\phi_j'(x)\phi_i(x) = \int_0^1 \phi_i(x)\mathrm{d}x \tag{36}$$

therefore,

$$\epsilon \sum_{j=1}^{N} u_h(x_j) \int_0^1 \phi_j'(x)\phi_i'(x) + \sum_{j=1}^{N} u_h(x_j) \int_0^1 \phi_j'(x)\phi_i(x) = \int_0^1 \phi_i(x)\mathrm{d}x \tag{37}$$

A simple calculation shows that

$$\int_0^1 \phi_j'(x)\phi_i'(x) = \begin{cases} -\frac{1}{h} & \text{if} \;\; j = i-1 \\ \frac{2}{h} & \text{if} \;\; j = i \\ -\frac{1}{h} & \text{if} \;\; j = i+1 \\ 0 & \text{if not} \end{cases} \tag{38}$$

and,

$$\int_0^1 \phi_j'(x)\phi_i(x) = \begin{cases} -\frac{1}{2} & \text{if } \; j = i-1 \\ 0 & \text{if } \; j = i \\ \frac{1}{2} & \text{if } \; j = i+1 \\ 0 & \text{if not} \end{cases} \tag{39}$$

Construction of the matrix C

```
e = ones(n-1,1);
A = spdiags([ (-1/h)*e (2/h)*e (-1/h)*e], -1:1, n-1, n-1);
B = spdiags([ (-0.5/epsilon)*e 0*e (0.5/epsilon)*e], -1:1, n-1, n-1);
C = A+B;
```

4 Numerical solution

4.1 Compulation for N = 40

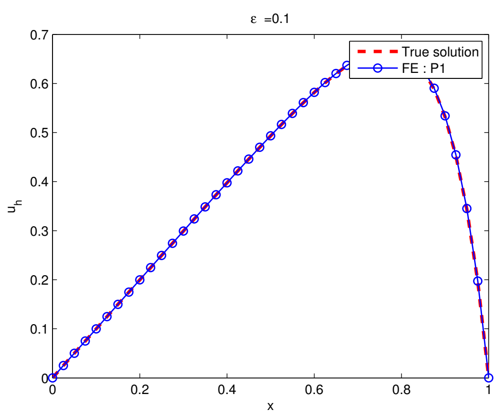

Figure 1: $\epsilon = 0.1$ and $N = 40$

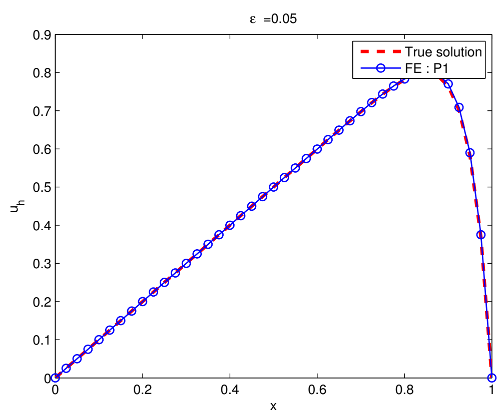

Figure 2: $\epsilon = 0.05$ and $N = 40$

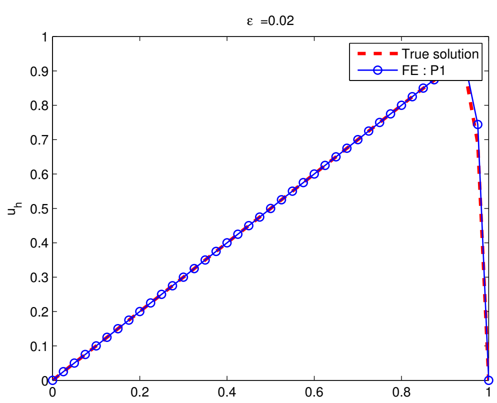

Figure 3: $\epsilon = 0.02$ and $N = 40$

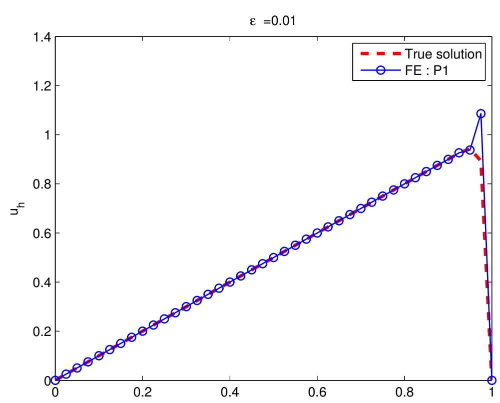

Figure 4: $\epsilon = 0.01$ and $N = 40$

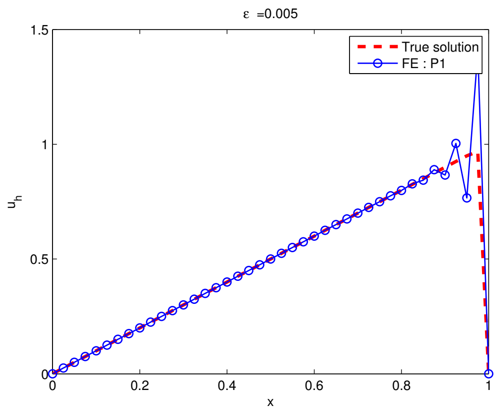

Figure 5: $\epsilon = 0.005$ and $N = 40$

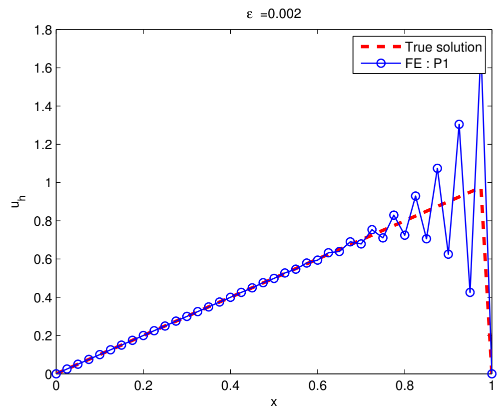

Figure 6: $\epsilon = 0.002$ and $N = 40$

4.2 Compulation for N = 20

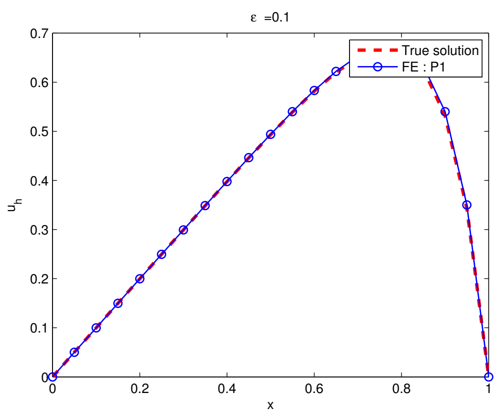

Figure 7: $\epsilon = 0.1$ and $N = 20$

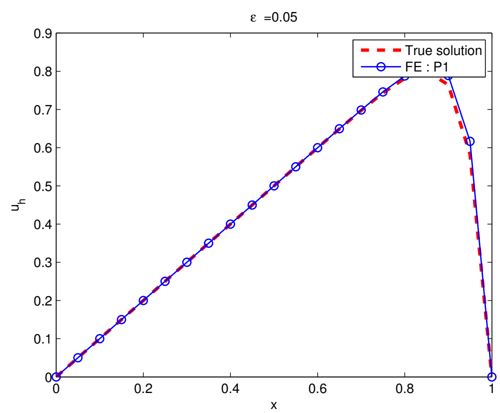

Figure 8: $\epsilon = 0.05$ and $N = 20$

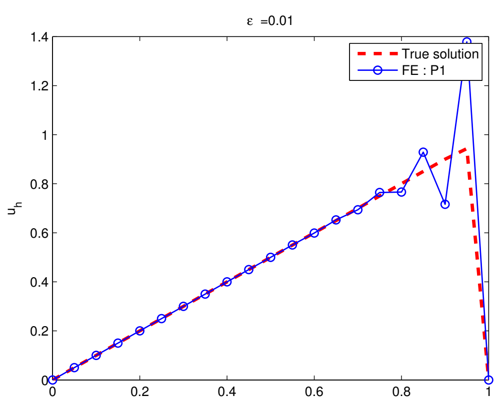

Figure 9: $\epsilon = 0.01$ and $N = 20$

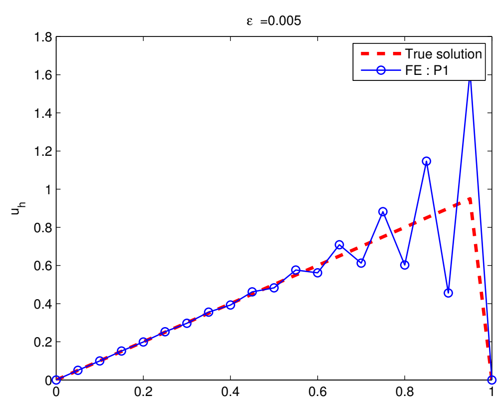

Figure 10: $\epsilon = 0.005$ and $N = 20$

5 A posterori estimator for the L^2 norm

The aim of the present section is, therefore, to derive a computable bound on the error, and to demonstrate how such a bound may be implemented into an adaptive mesh-refinement algorithm, capable of $u - u_h$ below a certain prescribed tolerance in an automated manner, without human intervention. The approach is based on seeking a bound on reducing the error $u - u_h$ in terms of the computed solution u_h rather than in terms of norms of the unknown analytical solution u. A bound on the error in terms of u_h is referred to as an a posteriori error bound, due to the fact that it becomes computable only after the numerical solution u_h has been obtained.
The computable *a posterori* error bound,

$$||u - u_h||_{L^2(0,1)} \leq K_0 \left(\sum_{i=1}^{N} h_i^4 ||R(u_h)||^2_{L^2(x_i,x_{i-1})} \right) \tag{40}$$

where $K_0 = \frac{K}{\pi^2}$, where $K = 1 + \frac{1}{\sqrt{2}}||b||_{L_\infty(0,1)} + \frac{1}{2}||c - b'||_{L_\infty(0,1)}$. In our case we have,

$$K = 1 + \frac{1}{\sqrt{2}\epsilon}, \quad ||R(u_h)||^2_{L^2(x_i,x_{i-1})} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{\epsilon} - \frac{u_h'}{\epsilon} \right)^2 \mathrm{d}x \tag{41}$$

Let's start to simplify $||R(u_h)||^2_{L^2(x_i,x_{i-1})}$.

$$||R(u_h)||^2_{L^2(x_i,x_{i-1})} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{\epsilon} - \frac{u_h'}{\epsilon} \right)^2 \mathrm{d}x = \frac{1}{\epsilon^2} \int_{x_{i-1}}^{x_i} 1 - 2u_h'(x) + (u_h'(x))^2 \, \mathrm{d}x \tag{42}$$

$$= \frac{1}{\epsilon^2} \left[(x_i - x_{i-1}) - 2.(u_h(x_i) - u_h(x_{i-1})) + \int_{x_{i-1}}^{x_i} (u_h'(x))^2 \, \mathrm{d}x \right] \tag{43}$$

We have,

$$u_h'(x) = \sum_{j=1}^{N} u_h(x_j) \phi_j'(x) \tag{44}$$

So,

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} (u_h'(x))^2 \, \mathrm{d}x &= \int_{x_{i-1}}^{x_i} u_h'(x) u_h'(x) \, \mathrm{d}x = \int_{x_{i-1}}^{x_i} \sum_{j=1}^{N} u_h(x_j)\phi_j'(x) \sum_{p=1}^{N} u_h(x_p)\phi_p'(x) \, \mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} \sum_{j=1}^{N} \sum_{p=1}^{N} u_h(x_j) u_h(x_p) \phi_j'(x) \phi_p'(x) \mathrm{d}x \\
&= \sum_{j=1}^{N} \sum_{p=1}^{N} u_h(x_j) u_h(x_p) \int_{x_{i-1}}^{x_i} \phi_j'(x) \phi_p'(x) \mathrm{d}x \\
&= u_h(x_i)^2 \int_{x_{i-1}}^{x_i} \phi_i'(x)\phi_i'(x) \mathrm{d}x + u_h(x_i) u_h(x_{i-1}) \int_{x_{i-1}}^{x_i} \phi_i'(x) \phi_{i-1}'(x) \mathrm{d}x \\
&+ u_h(x_{i-1}) u_h(x_i) \int_{x_{i-1}}^{x_i} \phi_{i-1}'(x) \phi_i'(x) \mathrm{d}x + u_h(x_{i-1})^2 \int_{x_{i-1}}^{x_i} \phi_{i-1}'(x) \phi_{i-1}'(x) \mathrm{d}x \\
&= u_h(x_i)^2 \int_{x_{i-1}}^{x_i} \phi_i'(x)^2 \mathrm{d}x + 2 u_h(x_i) u_h(x_{i-1}) \int_{x_{i-1}}^{x_i} \phi_i'(x) \phi_{i-1}'(x) \mathrm{d}x \\
&+ u_h(x_{i-1})^2 \int_{x_{i-1}}^{x_i} \phi_{i-1}'(x)^2 \mathrm{d}x
\end{aligned}$$

Or,

$$\int_{x_{i-1}}^{x_i} \phi_i'(x)^2 \mathrm{d}x = \frac{1}{h}, \quad \int_{x_{i-1}}^{x_i} \phi_i'(x)\phi_{i-1}'(x)\mathrm{d}x = -\frac{1}{h} \quad \int_{x_{i-1}}^{x_i} \phi_{i-1}'(x)^2 \mathrm{d}x = \frac{1}{h} \tag{45}$$

Hence,

$$\int_{x_{i-1}}^{x_i} (u_h'(x))^2 \, \mathrm{d}x = \frac{1}{h}\left(u_h(x_i)^2 - 2u_h(x_i)u_h(x_{i-1}) + u_h(x_{i-1})^2\right)\tag{46}$$

Therefore

$$\boxed{\begin{aligned} ||R(u_h)||^2_{L^2(x_i,x_{i-1})} &= \frac{1}{\epsilon^2}(x_i - x_{i-1}) - \frac{2}{\epsilon^2}.(u_h(x_i) - u_h(x_{i-1})) \\ &+ \frac{1}{\epsilon^2 h}\left(u_h(x_i)^2 - 2u_h(x_i)u_h(x_{i-1}) + u_h(x_{i-1})^2\right)\end{aligned}} \tag{47}$$

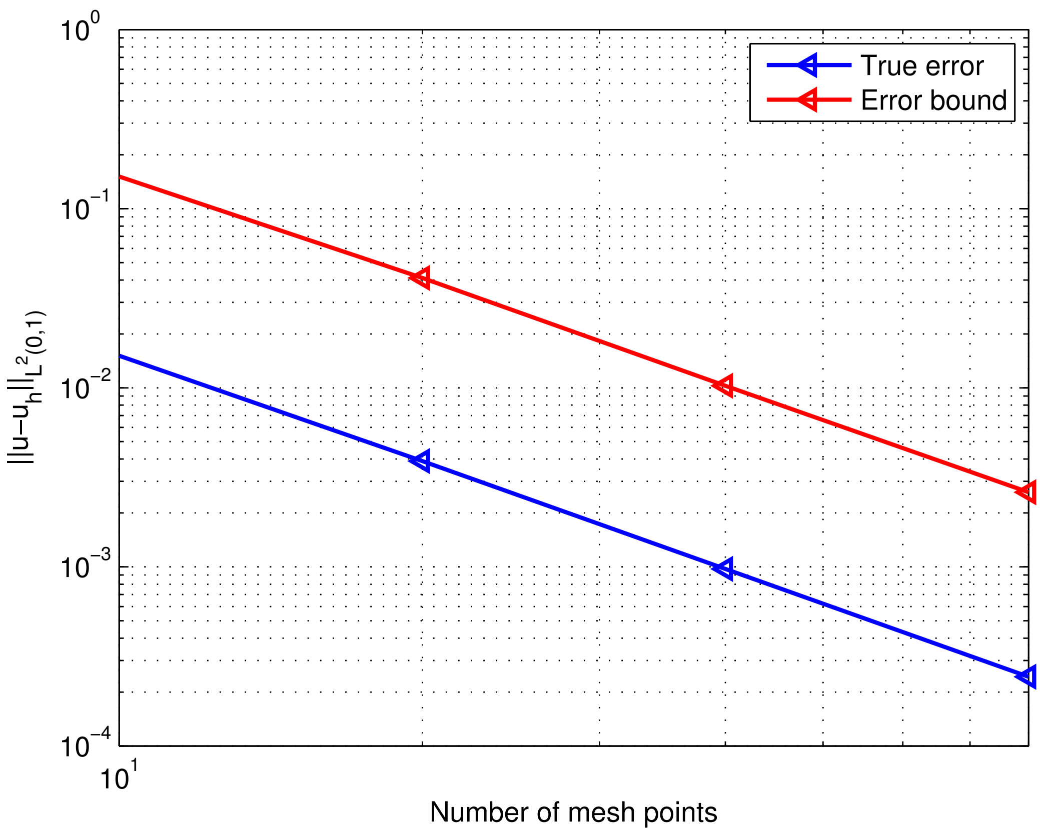

Figure 11: Comparison of the true error $||u - u_h||_{L^2(0,1)}$ with the a posteriori error bound delivered by the adaptive algorithm with $TOL = 10^{-4}$

The figure 7 shows that the a posteriori bound consistently overestimates the error $||u-u_h||_{L^2(0,1)}$ by about one order of magnitude. By comparing the slopes of the two curves in Figure 7, we also see that the error and the a posteriori error bound decay at approximately the same rate as the number of mesh points increases in the course of mesh adaptation. The slope of the red curve is -0.0016 and the slope of the blue curve is -0.000163.

n = 1/h	True error	Posteriori error bouned
10	0.0151	1.5113e-01
20	0.0039	4.1036e-02
40	9.7229e-4	1.0259e-02
80	2.4342e-4	2.6131e-03

Table 1: Comparison of the true error with the a posteriori error bound delivered by the adaptive algorithm with $TOL = 10^{-4}$.

6 A priori error analysis in the H^1 norm

Let's show that the solution u of satisfies a similar estimate of the form :

$$||u||_{H^2(0,1)} \leq C \tag{48}$$

We recall that if v is the solution of the variational formulation of the problem $-\Delta v = g$ on Ω with boundary conditions $v = 0$ on Γ and $g \in L^2(\Omega)$, we have the regularity estimate :

$$||u||_{H^2(\Omega)} \leq C_\Omega ||g||_{L^2(\Omega)} \tag{49}$$

with the constant C_Ω depends only on the domain Ω.
In our example we have $u'' = \frac{1}{\epsilon}(u' - 1)$, according to the previous reminder we have:

$$||u||_{H^2(0,1)} \leq C_\Omega ||\frac{1}{\epsilon}(u' - 1)||_{L^2(0,1)} \leq \frac{C_\Omega}{\epsilon}(||u'||_{L^2(0,1)} + 1) \leq \frac{C_\Omega}{\epsilon}(||u||_{H_0^1(0,1)} + 1) \tag{50}$$

Or, we have

$$\frac{\epsilon}{1 + C_p}||u||^2_{H_0^1(0,1)} \leq a(u,u) = L(u) \leq ||u||_{H_0^1(0,1)} \tag{51}$$

thus,

$$\frac{\epsilon}{1 + C_p}||u||^2_{H_0^1(0,1)} \leq ||u||_{H_0^1(0,1)} \Longrightarrow ||u||_{H_0^1(0,1)} \leq \frac{1 + C_p}{\epsilon} \tag{52}$$

therefore inequality (50) is written ($C_1 = C_\Omega$)

$$||u||_{H^2(0,1)} \leq \frac{C_1}{\epsilon}(\frac{1 + C_p}{\epsilon} + 1) \tag{53}$$

We have a series of regular meshes of $[0, 1]$ and we then have for $k+1 > d/2$ the existence of a constant C independent of h and v such that :

$$||u - r_h(v)||_{H^1(0,1)} \leq Ch^k ||v||_{H^{k+1}(0,1)} \tag{54}$$

where $r_h(v)$ is the interpolation operator of the finite elements $\mathbb{P}^k$ introduced in the course. For $k = 1$, According to *lemma of Céa*, we have

$$||u - u_h||_{H^1(0,1)} \leq \frac{(1+\epsilon)(1 + C_p)}{\epsilon} \inf_{v_h \in V_h} ||u - v_h||_{H^1(0,1)} \tag{55}$$

So,

$$||u - u_h||_{H^1(0,1)} \leq \frac{(1+\epsilon)(1 + C_p)}{\epsilon} ||u - r_h(v)||_{H^1(0,1)} \tag{56}$$

According to (54) and (53) , we get

$$||u - u_h||_{H^1(0,1)} \leq \frac{(1+\epsilon)(1 + C_p)}{\epsilon} C_2 h ||v||_{H^2(0,1)} \leq \frac{(1+\epsilon)(1 + C_p)}{\epsilon} C_2 h \frac{C_1}{\epsilon}(\frac{1 + C_p}{\epsilon} + 1) \tag{57}$$

Hence,

$$\boxed{||u - u_h||_{H^1(0,1)} \leq C_2 \frac{(1+\epsilon)(1 + C_p)}{\epsilon} \frac{C_1}{\epsilon} \left(\frac{1 + C_p}{\epsilon} + 1 \right) h} \quad \longrightarrow \infty \quad \text{when} \quad \epsilon \longrightarrow 0 \tag{58}$$