Fundamental Theorem of Finite Abelian Groups

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1 Introduction

In this paper, our goal is to apply the fundamental theorem of finite abelian groups on a defined abelian group. The group we are using in this paper is: $U_{(432)}$. $U_{(432)}$ is the group of all classes, each of which is a positive number smaller than *432* and relatively prime to *432*, i.e. $U_{(432)}$ is the group of all positive i's smaller than *432* where $\gcd_{432}^{i} = 1$ (*gcd* is the abbreviation of Greastest Common Divisor). Furthermore, $U_{(432)}$is a group under the operation multiplication modulo *432*. Having our group defined, in *sections 2* & *3*, we will first introduce the underlying theories of the fundamental theorem of finite abelian groups. We will then use our pre-defined group to put the fundamental theorem of finite abelian groups to practice. The process is done by first finding the order of the group and subsequently finding the elements of $U_{(432)}$(*section 4*). Accordingly, in *section 5*, we indicate how many isomorphisms are there possible for $U_{(432)}$. In *section 6* we find the order of all elements in $U_{(432)}$and write them in table 1. By looking at table 1 and the orders of the elements, we can cancel out the unaccepted isomorphisms. Moreover, in *section 7* we find the distinctive cyclic subgroups generated by the elements of $U_{(432)}$. Finally in *section 8*, we will show examples of how to write $U_{(432)}$as the direct product of it's cyclic sbgroups.

2 Fundamental Theorem of Finite Abelian Groups

The theorem is not complicated itself stating that every finite abelian group can be written as the direct product of cyclic groups. Note that the proof of the fundamental theory is quite complex and can be found in several textbooks [1] but what we are focusing on is how to use this theory. By using the theory, we can divide a large abelian group to the product of it's cyclic subgroups which would allow us to discover the characteristics of our initial group much easier by looking at the smaller cyclic groups.

3 Direct Product

We have defined the fundamental theorem of finite abelian groups using *direct products*, but what does it mean? What we mean by *direct product* is that if we have n groups: $G_1, G_2, ..., G_n$, then their direct product which is written as $G_1 \times G_2 \times G_3 \times ... \times G_n$ is the group of all $(a_1, a_2, a_3, ..., a_n)$ where $a_i \in G_i$, for i=*1,2,...,n*. [1]

4 Step 1: Finding the Order

To find the order of this group we must find the number of all positive integers smaller than 432 which are relatively prime to 432. To do so, we first write *432* as the multiplication of its prime factors:

$432 \div 2 = 216$, $216 \div 2 = 108$, $108 \div 2 = 54$, $54 \div 2 = 27$, $27 \div 3 = 9$, $9 \div 3 = 3$, $3 \div 3 = 1$.

$$432 = 2^4 \times 3^3 \tag{1}$$

Now based on equation 1, any number smaller that *432* which is nor a multiple of *2* nor a multiple of *3* is in the group $U_{(432)}$. we should mention that getting this group by hand will not be hard, but will be inefficient. Instead to get the group what we did is use the below python codes. That way even if we have larger numbers, calculating the group and its order will be less time consuming.

Listing 1: Finding the Elements of U432

```
U=[]
for i in range (1, 433):
        if i%2!=0 and i%3!=0:
                l.append(i)
print (f"the group of U_432 is\n{U}")
print (f"\nthe order of group U is {len(U)}")
```

Therefore, this group is introduced as following:

$$U_{432} = \left\{ \begin{array}{llllllllllll} 1, & 5, & 7, & 11, & 13, & 17, & 19, & 23, & 25, & 29, & 31, & 35 \\ 37, & 41, & 43, & 47, & 49, & 53, & 55, & 59, & 61, & 65, & 67, & 71 \\ 73, & 77, & 79, & 83, & 85, & 89, & 91, & 95, & 97, & 101, & 103, & 107 \\ 109, & 113, & 115, & 119, & 121, & 125, & 127, & 131, & 133, & 137, & 139, & 143 \\ 145, & 149, & 151, & 155, & 157, & 161, & 163, & 167, & 169, & 173, & 175, & 179 \\ 181, & 185, & 187, & 191, & 193, & 197, & 199, & 203, & 205, & 209, & 211, & 215 \\ 217, & 221, & 223, & 227, & 229, & 233, & 235, & 239, & 241, & 245, & 247, & 251 \\ 253, & 257, & 259, & 263, & 265, & 269, & 271, & 275, & 277, & 281, & 283, & 287 \\ 289, & 293, & 295, & 299, & 301, & 305, & 307, & 311, & 313, & 317, & 319, & 323 \\ 325, & 329, & 331, & 335, & 337, & 341, & 343, & 347, & 349, & 353, & 355, & 359 \\ 361, & 365, & 367, & 371, & 373, & 377, & 379, & 383, & 385, & 389, & 391, & 395 \\ 397, & 401, & 403, & 407, & 409, & 413, & 415, & 419, & 421, & 425, & 427, & 431 \end{array} \right\}$$

As we can see the number of elements in $U_{(432)}$is *144*, showing that there are 144 numbers smaller than 432 which are relatively prime to 432. As a result, the order of $U_{(432)}$ is *144*.

5 Step 2: Finding the Isomorphic groups

We know that any positive integer such as n can be written in the form of $n = {p_1}^{a_1} \times {p_2}^{a_2} \times {p_3}^{a_3} \times ... \times {p_k}^{a_k}$, where all p_i's are distinct prime numbers and their powers are positive integers. This is called the *prime factorization* of n. *The fundamental theorem of finite abelian groups* states that if we have G as an abelian group of order n, then G can be written as the direct product of its p_i- Sylow subgroups [1]. A p_i- Sylow subgroup is a subgroup of the order p_i from the group G.

As we have seen in section 4 the order of $U_{(432)}$ is *144*. Therefore, we have:

$$144 = 2^4 \times 3^2 \tag{2}$$

Lasltly, in ${p_i}^{a_i}$, there are as many groups of order ${p_i}^{a_i}$ as there are partitions of a_i. So to find the number of nonisomorphic abelian groups where each of them are isomorphic to the group G, we must find the total number of paritions.

For $U_{(432)}$we have found that *O(*$U_{(432)}$*)*$=144 = 2^4 \times 3^2$ so now we have to find the number of ways to partition *4* and *2*, which are the powers of *2* and *3* recpectively.

We have to find out how many ways there are to write a number as the sum of positive integers? To get an overview we will explain the process step by step. Take *4*; we can write it as the sum of *1, 2, 3, 4* positive integers. In other words, we cannot write *4* as the sum of *five or more* positive integers. (why?) because the minimum amount for each positive number is *1*, so the minimum amount for the sum of *5* positive integers will be *5* and thus, cannot be *4*.

1. If we want to write *4* as the sum of *1* integer, there is only one way to do so. $4 = 4$
2. If we want to write *4* as the sum of *2* integers, there is two ways to do so. $4 = 2 + 2 = 3 + 3$
3. If we want to write *4* as the sum of *3* integers, there is one way to do so. $4 = 2 + 1 + 1$
4. If we want to write *4* as the sum of *4* integers, there is one way to do so. $4 = 1 + 1 + 1 + 1$

- So the total ways to write *4* as the sum of positive integers is *5*

Now for *2* we can only write it as the sum of *1* or *2* positive integers. Take note that the sum of three or more positive integers will be atleast 3. Therefore, we will have the following options:

1. If we want to write *2* as the sum of *1* positive integer, there is only one way to do so. $2 = 2$

2. If we want to write *2* as the sum of *2* positive integers, there is only one way to do so. $2 = 1 + 1$

- For *2*, without the need of the mentioned process it was obvious there is only *2* ways to write it as the sum of positive inetegers.

Therefore, based on the *rule of product*, we can write $2 \times 5 = 10$ isomorphic groups for $U_{(432)}$.

$$
\begin{aligned}
&U_{432} \simeq Z_{16} \times Z_9\\
&U_{432} \simeq Z_8 \times Z_2 \times Z_9\\
&U_{432} \simeq Z_4 \times Z_4 \times Z_9\\
&U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9\\
&U_{432} \simeq Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_9\\
&U_{432} \simeq Z_{16} \times Z_3 \times Z_3\\
&U_{432} \simeq Z_8 \times Z_2 \times Z_3 \times Z_3\\
&U_{432} \simeq Z_4 \times Z_4 \times Z_3 \times Z_3\\
&U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_3 \times Z_3\\
&U_{432} \simeq Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3
\end{aligned} \tag{3}
$$

6 Step 3: Finding the Orders of Elements

Now if we want to write $U_{(432)}$as the product of it's cyclic subgroups, we have to find the order of it's elements. To do so , I have written the below python codes which will ease our process.

Listing 2: Order of elements

```
U= [1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35,
37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67,
71, 73, 77, 79, 83, 85, 89, 91, 95, 97, 101,
103, 107, 109, 113, 115, 119, 121, 125, 127,
131, 133, 137, 139, 143, 145, 149, 151, 155,
157, 161, 163, 167, 169, 173, 175, 179, 181,
185, 187, 191, 193, 197, 199, 203, 205, 209,
211, 215, 217, 221, 223, 227, 229, 233, 235,
239, 241, 245, 247, 251, 253, 257, 259, 263,
265, 269, 271, 275, 277, 281, 283, 287, 289,
293, 295, 299, 301, 305, 307, 311, 313, 317,
319, 323, 325, 329, 331, 335, 337, 341, 343,
347, 349, 353, 355, 359, 361, 365, 367, 371,
373, 377, 379, 383, 385, 389, 391, 395, 397,
401, 403, 407, 409, 413, 415, 419, 421, 425,
427, 431]

Order={}

for i in U:
        order=1
        while True:
                if ((i**order)%432)==1:
                        Order[i]=order
                        break
                order+=1
print (Order)
```

In the above lines we will go through each element of $U_{(432)}$. We will find the number of times for which when we muliply the element by itself and divide the result by *432*, the remainder will be 1. This is called the *order* of the element. The order of each element will look like the following:

O(1)=1	O(5)=36	O(7)=18	O(11)=36	O(13)=36	O(17)=6
O(19)=12	O(23)=18	O(25)=18	O(29)=36	O(31)=18	O(35)=12
O(37)=12	O(41)=18	O(43)=36	O(47)=18	O(49)=9	O(53)=4
O(55)=2	O(59)=36	O(61)=36	O(65)=18	O(67)=36	O(71)=6
O(73)=6	O(77)=36	O(79)=18	O(83)=36	O(85)=36	O(89)=6
O(91)=12	O(95)=18	O(97)=9	O(101)=36	O(103)=18	O(107)=4
O(109)=4	O(113)=18	O(115)=36	O(119)=18	O(121)=18	O(125)=12
O(127)=6	O(131)=36	O(133)=36	O(137)=18	O(139)=36	O(143)=6
O(145)=3	O(149)=36	O(151)=18	O(155)=36	O(157)=36	O(161)=2
O(163)=4	O(167)=18	O(169)=18	O(173)=36	O(175)=18	O(179)=12
O(181)=12	O(185)=18	O(187)=36	O(191)=18	O(193)=9	O(197)=12
O(199)=6	O(203)=36	O(205)=36	O(209)=18	O(211)=36	O(215)=2
O(217)=2	O(221)=36	O(223)=18	O(227)=36	O(229)=36	O(233)=6
O(235)=12	O(239)=18	O(241)=9	O(245)=36	O(247)=18	O(251)=12
O(253)=12	O(257)=18	O(259)=36	O(263)=18	O(265)=18	O(269)=4
O(271)=2	O(275)=36	O(277)=36	O(281)=18	O(283)=36	O(287)=6
O(289)=3	O(293)=36	O(295)=18	O(299)=36	O(301)=36	O(305)=6
O(307)=12	O(311)=18	O(313)=18	O(317)=36	O(319)=18	O(323)=4
O(325)=4	O(329)=18	O(331)=36	O(335)=18	O(337)=9	O(341)=12
O(343)=6	O(347)=36	O(349)=36	O(353)=18	O(355)=36	O(359)=6
O(361)=6	O(365)=36	O(367)=18	O(371)=36	O(373)=36	O(377)=2
O(379)=4	O(383)=18	O(385)=9	O(389)=36	O(391)=18	O(395)=12
O(397)=12	O(401)=18	O(403)=36	O(407)=18	O(409)=18	O(413)=12
O(415)=6	O(419)=36	O(421)=36	O(425)=18	O(427)=36	O(431)=2

Table 1: table of orders

Based on equation 2, we know that *144* has two primes *2* and *3* in it's prime factorization. Firstly, we'll focus on *2* and we will start looking for it's maximal order which will be $2^4 = 16$. If we do not find any elements in $U_{(432)}$with an order of *16*, we will search for the next maximal order $2^3 = 8$ and then $2^2 = 4$. Finally, if *no* elements of order 2^4, 2^3 or 2^2 are found in $U_{(432)}$, we will search for elements of order 2. Watching table 1, we can aboserve that no element in the list has an *order* of *16* or *8*. To make sure, we run the below lines of codes and we confirm that there is no element of *order 16* or *8* in the group $U_{(432)}$.

Listing 3: Checking order of 8 or 16

```
#Order is a dictionary where the keys are
#elements of U432.
#The values are the orders of these elements
for i in Order:
        if Order[i]==16 or Order[i]==8:
                print (i)
```

Maybe for *144* elements just looking at the table 1 would have sufficed to determine whether we have elements of order *8* or *16*. However, if the number of elements exceeds using the codes written in Listing 3 will be handy.

Obviously there are plenty elements with an *order* of *4* in the group $U_{(432)}$, as seen in table 1. So from the isomorphisms written in equation 3 we exclude the options where we have Z_{16} or Z_8 and also we exclude those isomorphisms where Z_4 is not seen. Doing the process will limit our options to the following:

$$\begin{aligned}
&(4.1)\quad U_{432} \simeq Z_4 \times Z_4 \times Z_9 \simeq Z_4 \times Z_{36} \\
&(4.2)\quad U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9 \simeq Z_2 \times Z_4 \times Z_{18} \simeq Z_2 \times Z_2 \times Z_{36} \\
&(4.3)\quad U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_3 \times Z_3 \simeq Z_2 \times Z_2 \times Z_3 \times Z_{12} \\
&\simeq Z_2 \times Z_4 \times Z_3 \times Z_6 \simeq \times Z_4 \times Z_6 \times Z_6 \simeq Z_2 \times Z_6 \times Z_{12} \\
&(4.4)\quad U_{432} \simeq Z_4 \times Z_4 \times Z_3 \times Z_3 \simeq Z_{12} \times Z_{12} \simeq Z_3 \times Z_4 \times Z_{12}
\end{aligned} \tag{4}$$

As we can see in the orders list, $U_{(432)}$has an element of order *36*, thus isomorphisms (4.3), and (4.4) above cannot be accepted. Another way to cancel out isomorphisms (4.3) and (4.4) is to look at the elements of *order 3*. In table 1 we can find two elements of *order 3*; namely, *145* & *289*. However, the cycle of element *145* is the group of $145, 289, 1$. So we cannot define two *disjoint* cyclic subgroups inside $U_{(432)}$which are of order *3*. Therefore, the only remaining acceptable isomorphisms would be:

$$\begin{aligned}
&U_{432} \simeq Z_4 \times Z_4 \times Z_9 \simeq Z_4 \times Z_{36} \\
&U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9 \simeq Z_2 \times Z_4 \times Z_{18} \simeq Z_2 \times Z_2 \times Z_{36}
\end{aligned} \tag{5}$$

We cannot write any other options (isomorphisms). Take note that $U_{(432)}$does not have an element of order greater than 36. In other words, the maximum order of the cylic subgroups of $U_{(432)}$is 36.

Before we continue our process we will have one intermediate step.

7 Step 4: Distinctive Cyclic Subgroups of order 2, 4, 9, 18, 36

This step is where we find subgroups of order *2, 4, 9, 18, 36* from $U_{(432)}$so that we can write $U_{(432)}$as the product of its cyclic subgroups.

Looking at table 1, we can easily find the elements of the mentioned orders:

7.1 Subgroups of Order 2

The group of elements of *Order 2*:

$$\{[55], [161], [215], [217], [271], [377], [431]\} \tag{6}$$

As we can see we have *7* elements of *order 2* in $U_{(432)}$.

Obviously, all the cyclic subgroups of order *2* are disjoint. So we can use any two cyclic subgroups of order 2 with each other in the direct product.

7.2 Subgroups of Order 4

The group of elements of *Order 4*:

$$\{[163],[323],[325],[107],[269],[109],[53],[379]\} \tag{7}$$

As we can see we have *8* elements of *order 4* in $U_{(432)}$.

Throughout the elements with order 4 written in section 7.2, we will write the cyclic subgroups generated by each element:

$$\begin{array}{ll} [53]=\{53,217,269,1\}, & [107]=\{107,217,323,1\} \\ [109]=\{109,217,325,1\}, & [163]=\{163,217,379,1\} \\ [269]=\{269,217,53,1\}, & [323]=\{323,217,107,1\} \\ [325]=\{325,217,109,1\}, & [379]=\{379,217,163,1\} \end{array} \tag{8}$$

In the above cyclic subgroups of *order 4*, we can see [53]=[269], [109]=[325], [107]=[323], and [163]=[379]. Therefore, we have *4* distinctive cyclic subgroups, but not disjoint, of order *4*.

7.3 Subgroups of Order 9

The group of elements of *Order 9*:

$$\{[193],[97],[385],[337],[49],[241]\} \tag{9}$$

As we can see we have *6* elements of *order 9* in $U_{(432)}$.

we are now going to find distinctive cylcic subgroups of order 9. Throughout the elements with order 9 written in section 7.3, if we choose *49* and write out the cyclic subgroup generated by *49*, we will have:

$$[49]=\{49,241,145,193,385,289,337,97,1\} \tag{10}$$

As we can see this cycle consists of all other elements in $U_{(432)}$which are of order *9*. Therefore, we only have one distinctive cyclic subgroup of order 9 in $U_{(432)}$which is shown above.

7.4 Subgroups of Order 18

The group of elements of *Order 18*:

$$\left\{\begin{array}{lllllll} [257], & [7], & [263], & [137], & [265], & [391], & [401] \\ [23], & [151], & [25], & [281], & [407], & [409], & [31], \\ [167], & [295], & [41], & [169], & [425], & [47], & [175], \\ [311], & [185], & [313], & [191], & [319], & [65], & [329], \\ [79], & [335], & [209], & [247], & [95], & [223], & [353], \\ [103], & [239], & [367], & [113], & [119], & [121], & [383] \end{array}\right\} \tag{11}$$

As we can see we have *42* elements of *order 18* in $U_{(432)}$.

Now we want to find how many distinctive cycles of order *18* is generated from the elements of order *18*. The process is not as straight forward as in sections 7.1, 7.2 and 7.3. Here, by finding all the elements of order *18*, creating cyclic subgroups generated from them, comparing the cyclic subgroups, we find out that there is *7* distinctive cyclic subgroups of order *18*.

The algorithm discussed to find out these *7* distinctive cyclic subgroups of order 18 can be broken down into two parts. The first part, which is finding all the cyclic subgroups of order *18* can be written as the following python codes:

Listing 4: cyclic subgroups of order 18

```
cycles={}
for i in OrderOf18:
        temporary=[]
        for j in range (1,19):
                temporary.append(((i**j)%432))
        cycles[i]=temporary
print (cycles)
```

cycles is the dictionary where the *keys* are the representing elements in the class and the *values* are cyclic groups generated from the key.

We now apply the second part of the algorithm which is to find distinctive cycles of order 18. For this we must go through each cycle, in this case *value* in the dictionary *cycles*, and cancel out the similar ones. The below codes show the exact process.

Listing 5: distinctive cyclic subgroups of order 18

```
#values is the list of all cyclic subgroups.
values=[]
for i in cycles:
        values.append(set(cycles[i]))

distinctive_cycles=[]
for i in values:
        if i in distinctive_cycles:
```

```
                continue
            distinctive_cycles.append(i)
print (distinctive_cycles)
```

The *distinctive_cycles* list is the list of all the *7* distinctive cyclic subgroups of order 18. specifically *distinctive_cycles* is as following:

$$\left(\begin{bmatrix}7, & 49, & 343, & 241, & 391, & 145,\\ 151, & 193, & 55, & 385, & 103, & 289,\\ 295, & 337, & 199, & 97, & 247, & 1\end{bmatrix}\\ \begin{bmatrix}23, & 97, & 71, & 337, & 407, & 289,\\ 167, & 385, & 215, & 193, & 119, & 145,\\ 311, & 241, & 359, & 49, & 263, & 1\end{bmatrix}\\ \begin{bmatrix}25, & 193, & 73, & 97, & 265, & 145,\\ 169, & 337, & 217, & 241, & 409, & 289,\\ 313, & 49, & 361, & 385, & 121, & 1\end{bmatrix}\\ \begin{bmatrix}41, & 385, & 233, & 49, & 281, & 289,\\ 185, & 241, & 377, & 337, & 425, & 145,\\ 329, & 97, & 89, & 193, & 137, & 1\end{bmatrix}\\ \begin{bmatrix}47, & 49, & 143, & 241, & 95, & 145,\\ 335, & 193, & 431, & 385, & 383, & 289,\\ 191, & 337, & 287, & 97, & 239, & 1\end{bmatrix}\\ \begin{bmatrix}31, & 97, & 415, & 337, & 79, & 289,\\ 319, & 385, & 271, & 193, & 367, & 145,\\ 175, & 241, & 127, & 49, & 223, & 1\end{bmatrix}\\ \begin{bmatrix}65, & 337, & 305, & 385, & 401, & 145,\\ 353, & 49, & 161, & 97, & 257, & 289,\\ 209, & 193, & 17, & 241, & 113, & 1\end{bmatrix}\right) \tag{12}$$

In other words, [7], [103], [151], [247], [295], [391] create the same cyclic subgroups of order *18*. Same for: [23], [119], [167], [263], [311], [407]. Similarly, [25], [121], [169], [265], [313], [409] classes create the same subgroups. As for classes of [31], [79], [175], [223], [319], [367]. Likewise, [41], [137], [185], [281], [329], [425] create the same cyclic subgroups of order *18*. The same goes for [47], [95], [191], [239], [335], [383]. Lastly, [65], [113], [209], [257], [353], [401] generate similar cyclic subgroups.

7.5 Subgroups of Order 36

The group of elements of *Order 36*:

$$\left\{ \begin{array}{llllllll} [131], & [259], & [5], & [133], & [389], & [11], & [139], & [13], \\ [275], & [403], & [149], & [277], & [155], & [283], & [157], & [29], \\ [419], & [293], & [421], & [43], & [299], & [173], & [301], & [427], \\ [59], & [187], & [61], & [317], & [67], & [203], & [331], & [77], \\ [205], & [83], & [211], & [85], & [347], & [221], & [349], & [227], \\ [355], & [101], & [229], & [365], & [115], & [371], & [245], & [373] \end{array} \right\} \tag{13}$$

As we can see we have *48* elements of *order 36* in $U_{(432)}$.

Again note that not all these elements create distinctive cyclic subgroups. If we repeat the process described in section 7.4 this time for elements of order *36*, we will get a total of *4* distinctive cyclic subgroups of order *36* from $U_{(432)}$ which are written below:

$$\left(\begin{array}{c} \begin{bmatrix} 5, & 25, & 125, & 193, & 101, & 73, \\ 365, & 97, & 53, & 265, & 29, & 145, \\ 293, & 169, & 413, & 337, & 389, & 217, \\ 221, & 241, & 341, & 409, & 317, & 289, \\ 149, & 313, & 269, & 49, & 245, & 361, \\ 77, & 385, & 197, & 121, & 173, & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 11, & 121, & 35, & 385, & 347, & 361, \\ 83, & 49, & 107, & 313, & 419, & 289, \\ 155, & 409, & 179, & 241, & 59, & 217, \\ 227, & 337, & 251, & 169, & 131, & 145, \\ 299, & 265, & 323, & 97, & 203, & 73, \\ 371, & 193, & 395, & 25, & 275, & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 13, & 169, & 37, & 49, & 205, & 73, \\ 85, & 241, & 109, & 121, & 277, & 145, \\ 157, & 313, & 181, & 193, & 349, & 217, \\ 229, & 385, & 253, & 265, & 421, & 289, \\ 301, & 25, & 325, & 337, & 61, & 361, \\ 373, & 97, & 397, & 409, & 133, & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 43, & 121, & 19, & 385, & 139, & 361, \\ 403, & 49, & 379, & 313, & 67, & 289, \\ 331, & 409, & 307, & 241, & 427, & 217, \\ 259, & 337, & 235, & 169, & 355, & 145, \\ 187, & 265, & 163, & 97, & 283, & 73, \\ 115, & 193, & 91, & 25, & 211, & 1 \end{bmatrix} \end{array} \right) \tag{14}$$

In other words [43], [67], [115], [139], [187], [211], [259], [283], [331], [355], [403], and [427] all create the same cyclic subgroup of order *36*. Moreover, [13], [61], [85], [133], [157], [205], [229], [277], [301], [349], [373], and [421] generate the same subgroup of order *36*. The same for [11], [59], [83], [131], [155], [203], [227], [275], [299], [347], [371], and [419]. Lastly, all classes of [5], [29], [77], [101], [149], [173], [221], [245], [293], [317], [365], and [389] generate the same subgroups of order *36*.

8 Step 5: Writing $U_{(432)}$as the Product of Cyclic Subgroups

This is the final step of our process. Now that we have our cyclic subgroups, we will see how to write $U_{(432)}$as the product of its disjoint cyclic subgroups. Recall the equation number 5. The first isomorphism that is $U_{432} \simeq Z_4 \times Z_4 \times Z_9 \simeq Z_4 \times Z_{36}$ will only exist if we have disjoint cyclic subgroups of order *4* and *36*. As we can see in section 7.5 all the cyclic subgroups of order *36* have the element *217*. Likewise, as seen in section 7.2 all the cyclic subgroups of order *4* have the element *217*. Therefore, we cannot find disjoint cyclic subgroups of order *4* and *36*. As a result, the isomorphism $U_{432} \simeq Z_4 \times Z_4 \times Z_9 \simeq Z_4 \times Z_{36}$ is not acceptable.

Only the following isomorphism remains:

$$U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9 \simeq Z_2 \times Z_4 \times Z_{18} \simeq Z_2 \times Z_2 \times Z_{36} \tag{15}$$

8.1 $U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9$

Take note that $Z_2 \times Z_2 \times Z_4 \times Z_9$ forms the group with elements (a, b, c, d) where $a \in Z_2$, $b \in Z_2$, $c \in Z_4$, $d \in Z_9$. In this group for instance $(1, 0, 1, 1)$ is in the order of 36 because $lcm(9, 4, 2) = 36$; lcm is the abbreviation of Least Common Multiple. Element (1,0,1,0) is in the order of *4* because $lcm(2, 4) = 4$. Applying the same reasoning we can calculate the order of other elements.

Having defined $Z_2 \times Z_2 \times Z_4 \times Z_9$ we want to write $U_{(432)}$as the product of disjoint cyclic subgroups of orders *9*, *4*, *2* and *2*. First of all because all the elements in the cyclic subgroup of order *9* are of order *9* or of order *3* (or *1*), so they cannot be seen in the cyclic subgroup of order *4* or *2* given that $gcd_2^9 = 1$. Therefore, the cyclic subgroups of order *9* and *4* are disjoint. Moreover, all the cyclic subgroups of order *4* have the element *217*. We can see from section 7.1 that *217* is an element of order *2*. Also noticed in sections 7.1 & 7.2, the cyclic subgroups of order *4* are disjoint with all cyclic subgroups of order *2* except for [217]. Therefore, we can use any cyclic subgroup of order *4* with any two cyclic subgroups of order *2*, but with the condition to not use the cyclic subgroup [217].

All in all, given the mentioned description the number of ways to write $U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9$; that is $U_{(432)}$as the product of two of its cyclic subgroup of order *2*, the cyclic subgroup of order *4* and the cyclic subgroup of order *9*, we have $C(6,2) \times 4 \times 1 = 60$ ways. Because we can pick any two cyclic subgroups of order 2 except for [217]. Furthermore, we have 4 distinctive cyclic subgroups of order *4* and one distinctive cycle of order *9*.

Important: All these 60 ways are the same isomorphism but they have different elements as representatives.
Below we can see some examples:

$$\begin{aligned} U_{432} &\simeq [161] \times [55] \times [163] \times [193] \simeq [271] \times [55] \times [323] \times [193] \\ &\simeq [377] \times [431] \times [109] \times [193] \simeq [161] \times [215] \times [53] \times [193] \\ &\simeq [161] \times [431] \times [269] \times [193]... \end{aligned} \tag{16}$$

Therefore, we have shown that $U_{(432)}$has *60* different representatives for the the isomorphism $U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9$!

8.2 $U_{432} \simeq Z_2 \times Z_4 \times Z_{18}$

As we know from equation 15 all the products of cycles in the form of $U_{432} \simeq Z_2 \times Z_4 \times Z_{18}$ are isomorphic to products seen in section 8.1. Again these are different representatives but the same isomorphism. In finding an example we have to take into account that cyclic subgroups of order *18* can conatin elements which have a cycle of *2*, thus might not be disjoint.

For each of the *7* distinctive cyclic subgroups of order *18* shown in section 7.4 we must find a disjoint cyclic subgroup of order *2* and *4* to write our initial group $U_{(432)}$as the product of it's cyclic subgroups.

Looking at the cycles, we write a few examples of $U_{(432)}$as the product of three cyclic subgroups of orders *2, 4, and 18*:

$$\begin{aligned} U_{432} &= [161] \times [53] \times [7] \\ U_{432} &= [161] \times [109] \times [23] \end{aligned} \tag{17}$$

8.3 $U_{432} \simeq Z_2 \times Z_2 \times Z_{36}$

Again these are isomorphisms to the products written in sections 8.1 and 8.2. For an example, let's write the cyclic subgroup of order *36*. To do so, we will write it using the smallest element which has an order of *36*. Looking at table 1, we have *5* which has an order of *36*. The cyclic subgroup generated by *5* is:

$$[5] = \{5, 25, 125, 193, 101, 73, 365, 97, 53, 265, 29, 145, 293, 169, 413,$$
$$337, 389, 217, 221, 241, 341, 409, 317, 289, 149, 313, 269, 49, 245, 361,$$
$$77, 385, 197, 121, 173, 1\}$$

from section 7.1 we can pick the two classes, [55] and [161] which are disjoint from [5]. Thus, we can write:

$$U_{432} = [55] \times [161] \times [5] \tag{18}$$

9 Conclusions

We have applied *the fundamental theorem of finite abelian groups* on our abelian group: $U_{(432)}$; i.e. we have shown that $U_{(432)}$can be written as the direct product of its cyclic subgroups. To do so we have found $U_{(432)}$a group of order *144*. Likewise, it has *10* possible isomorphisms, out of which only one written as $U_{432} \simeq Z_2 \times Z_2 \times Z_4 \times Z_9 \simeq Z_2 \times Z_4 \times Z_{18} \simeq Z_2 \times Z_2 \times Z_{36}$ is acceptable. Furthermore, the maximum order of elements in $U_{(432)}$is *36*. By finding the orders of the elements of the group $U_{(432)}$, we have found the cyclic subgroups generated from these elements. Finally by applying *the fundamental theorem of finite abelian groups* we have written $U_{(432)}$as the direct product of it's cyclic subgroups of order 2, 2, 4, and 9. Also we can write $U_{(432)}$as the direct product of cyclic subgroups of order 2, 4, and 18. Ultimately we have shown that $U_{(432)}$can be written as the direct product of it's cyclic subgroups of order 2, 2, and 36. For further research we can explore that whether all groups of order 144 have the single isomorphism $Z_2 \times Z_2 \times Z_4 \times Z_9 \simeq Z_2 \times Z_4 \times Z_{18} \simeq Z_2 \times Z_2 \times Z_{36}$? Do we have groups of order *144* with other acceptable isomorphisms? What makes the difference?

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