Homework 7.

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913

1. b) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \dots + n\cdot\sqrt[n]{e^n}}{n^2} =$

$\lim\limits_{n\to\infty} \dfrac{e^{\frac{1}{n}} + 2\cdot e^{\frac{2}{n}} + \dots + n\cdot e^{\frac{n}{n}}}{n^2} =$

$\lim\limits_{n\to\infty} \dfrac{n\cdot\left(\frac{1}{n}\cdot e^{\frac{1}{n}} + \frac{2}{n}\cdot e^{\frac{2}{n}} + \frac{n}{n}\cdot e^{\frac{n}{n}}\right)}{n^2} =$

$\lim\limits_{n\to\infty} \dfrac{1}{n}\cdot\left(\sum\limits_{k=1}^{n} \dfrac{k}{n}\cdot e^{\frac{k}{n}}\right) = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{n}\cdot f\left(\dfrac{k}{n}\right) \ominus$

$f(x) = x\cdot e^x \qquad 0 = \frac{1}{n} < \frac{2}{n} < \dots < \frac{n}{n} = 1$

$\ominus \int_0^1 x\cdot e^x\,dx = \int_0^1 x\cdot(e^x)'\,dx = x\cdot e^x\Big|_0^1 - \int_0^1 e^x\,dx$

$= x\cdot e^x\Big|_0^1 - e^x\Big|_0^1$

$= e - e + 1 = 1$

d) $\lim_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \cdot \sin\frac{3\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n}} =$

$$\lim_{n\to\infty} \left(\sin\frac{\pi}{2n} \cdot \sin\left(\frac{\pi}{2}\cdot\frac{2}{n}\right) \cdot \ldots \cdot \sin\left(\frac{\pi}{2}\cdot\frac{(n-1)}{n}\right)\right)^{\frac{1}{n}} =$$

$$e^{\lim_{n\to\infty} \frac{1}{n} \cdot \ln\left(\sin\frac{\pi}{2} \cdot \sin\frac{\pi}{2}\cdot\frac{2}{n} \cdot \ldots \cdot \sin\frac{\pi}{2}\cdot\frac{(n-1)}{n}\right)} =$$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \left(\ln\sin\frac{\pi}{2} + \ln\sin\frac{\pi}{2}\cdot\frac{2}{n} \cdot \ldots \cdot \sin\frac{\pi}{2}\cdot\frac{(n-1)}{n}\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \ln\sin\frac{k}{n}\frac{\pi}{2}$$

$$f(x) = \ln\sin x \qquad 0 = \frac{1}{n}\cdot\frac{\pi}{2} < \frac{2}{n}\cdot\frac{\pi}{2} < \ldots < \frac{(n-1)}{n}\frac{\pi}{2} = \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \ln\sin x \, dx =$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$\Rightarrow$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2}-x\right)\right) dx$$

$u = \frac{\pi}{2} - x \quad ||'$

$du = -dx$

$x = 0 \rightarrow u = \frac{\pi}{2}$

$x = \frac{\pi}{2} \rightarrow u = 0$

$$\int_{\frac{\pi}{2}}^{0} -\ln(\cos u)\, du = \int_0^{\frac{\pi}{2}} \ln(\cos u)\, du \Rightarrow$$

write x instead of u

$$\underbrace{\int_0^{\frac{\pi}{2}} \ln \cos x \, dx}_{J} = \underbrace{\int_0^{\frac{\pi}{2}} \ln \sin x \, dx}_{J}$$

$$2J = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx + \int_0^{\frac{\pi}{2}} \ln \sin x \, dx$$

$$2J = \int_0^{\frac{\pi}{2}} \ln(\sin x \cdot \cos x)\, dx$$

$$2J = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin 2x}{2}\right) dx$$

$$2J = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \int_0^{\frac{\pi}{2}} \ln 2 \, dx$$

$$2J = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \frac{\pi}{2} \cdot \ln 2$$

$2x = u \quad |'$

$2dx = du$

$dx = \frac{1}{2} du$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{2} \rightarrow u = \pi$

$$2S = \int_0^{\pi} \frac{1}{2} \ln(\sin u)\, du - \frac{\pi}{2}\ln 2$$

$$2S = \frac{1}{2}\int_0^{\pi} \ln(\sin u)\, du - \frac{\pi}{2}\ln 2$$

$$2S = \frac{1}{2}\int_0^{\pi} \ln \sin x \, dx - \underbrace{\frac{\pi}{2}\ln 2}_{S}$$

$$\underbrace{\frac{1}{2}\int_0^{\pi} \ln(\sin x)\, dx}_{2\cdot\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx} = \underbrace{\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx}$$

From $0 \to \frac{\pi}{2}$ the graph is symetrical with the one for $\frac{\pi}{2} \to \pi$

$$\Rightarrow 2S = S - \frac{\pi}{2}\ln 2$$

$$S = -\frac{\pi}{2}\cdot\ln 2$$

$$\Leftrightarrow e^{-\frac{\pi}{2}\ln 2} = \frac{1}{e^{\frac{\pi}{2}\cdot\ln 2}}$$

3. d) $\overbrace{\int_0^\infty e^{-x}\cdot \sin x \, dx}^{J} = \int_0^\infty (-e^{-x})' \cdot \sin x \, dx =$

$-e^{-x}\cdot \sin x + \int_0^\infty e^{-x}\cdot \cos x \, dx =$

$-e^{-x}\cdot \sin x + \int_0^\infty (-e^{-x})' \cdot \cos x \, dx =$

$-e^{-x}\cdot \sin x + -e^{-x}\cdot \cos x - \underbrace{\int_0^\infty e^{-x}\cdot \sin x \, dx}_{J} =$

$2J = (-e^{-x}\cdot \sin x)\Big|_0^\infty + (-e^{-x}\cdot \cos x)\Big|_0^\infty$

$2J = -e^{-x}(\sin x + \cos x)\Big|_0^\infty$ $\qquad -2 \le \sin x + \cos x \le 2$

$2J = -\frac{1}{e^x}(\sin x + \cos x)\Big|_0^\infty$

$2J = \underbrace{-\underbrace{\frac{1}{e^\infty}}_{0}(\underbrace{\sin\infty + \cos\infty}_{(-2,2)})}_{0} + 1$

$\Rightarrow 2J = 1 \Rightarrow J = \frac{1}{2}$

```python
import matplotlib.pyplot as plt
import numpy as np

EULER = 2.7182818284590452353602874713527

def f(x: float) -> float:
    return EULER ** (-(x ** 2))

def aproximate_area(rect_len: float, y: np.array) -> float:
    return np.sum(y * rect_len)

num_points = 50
a = 5

x = np.linspace(-a, a, num_points)
y = f(x)


for i in (1, 2, 5, 10):
    x = np.linspace(-a, a, num_points * i)
    y = f(x)
    rect_len = x[1] - x[0]
    area = aproximate_area(rect_len, y)

    plt.figure(figsize=(16, 10))
    plt.plot(x, y, label=f'Number of points={num_points * i}', color='red', linewidth=3, linestyle='-')
    plt.bar(x - rect_len/2, y, width=rect_len, alpha=0.5, align='edge', label=f'Rectangle width={rect_len}')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.title('Gaussian function')
    plt.legend()
    plt.show()

    print(f'Number of points: {num_points * i}')
    print(f'area={area}, error={abs(area - np.sqrt(np.pi))}, error%={abs(area - np.sqrt(np.pi)) / np.sqrt(np.pi) * 100}, rect_len={rect_len}, sqrt(pi)={np.sqrt(np.pi)}')
    print()
```

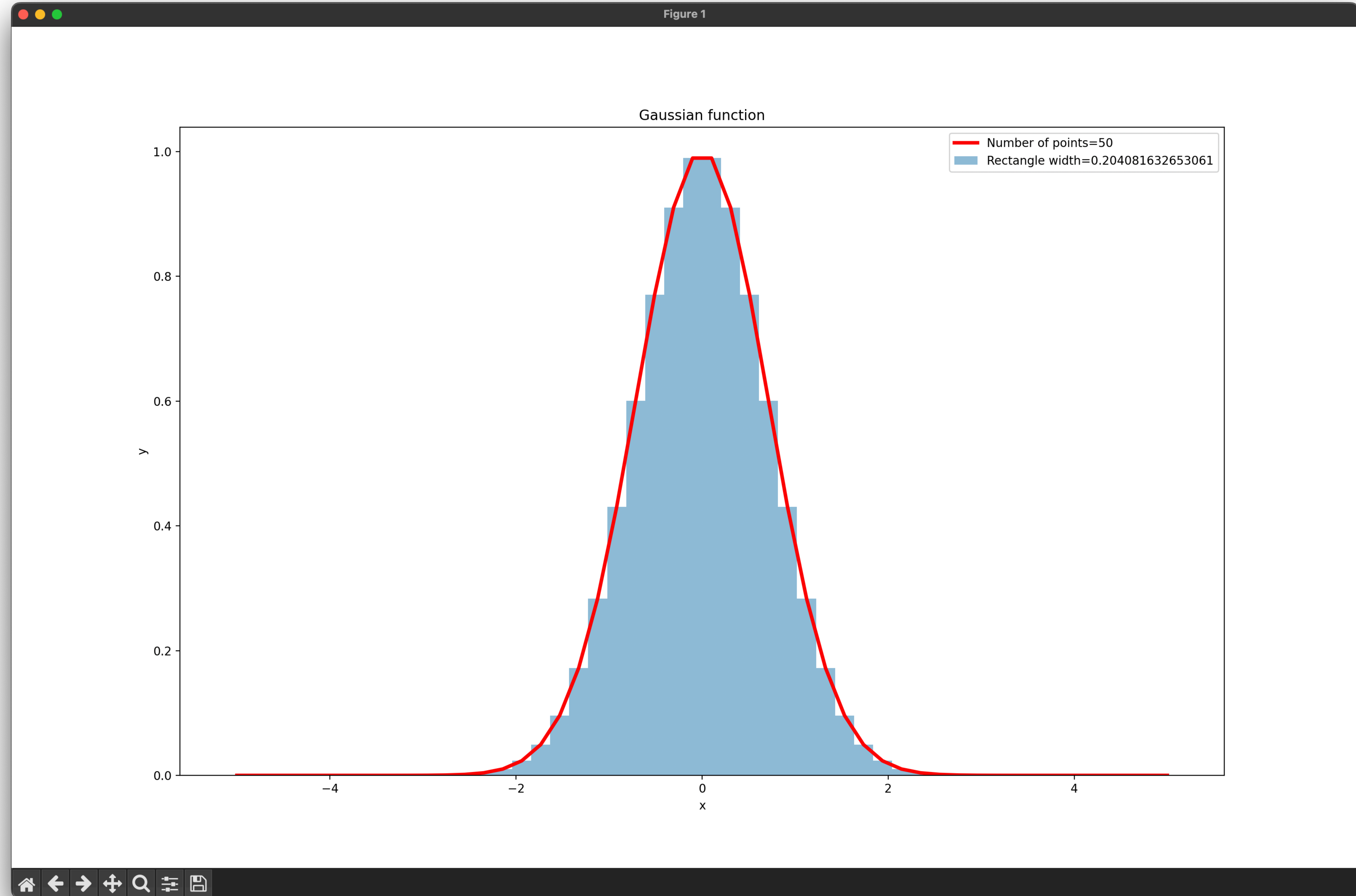
Figure 1
Gaussian function
Number of points=50
Rectangle width=0.204081632653061
1.0
0.8
0.6
0.4
0.2
0.0
y
−4
−2
0
2
4
x

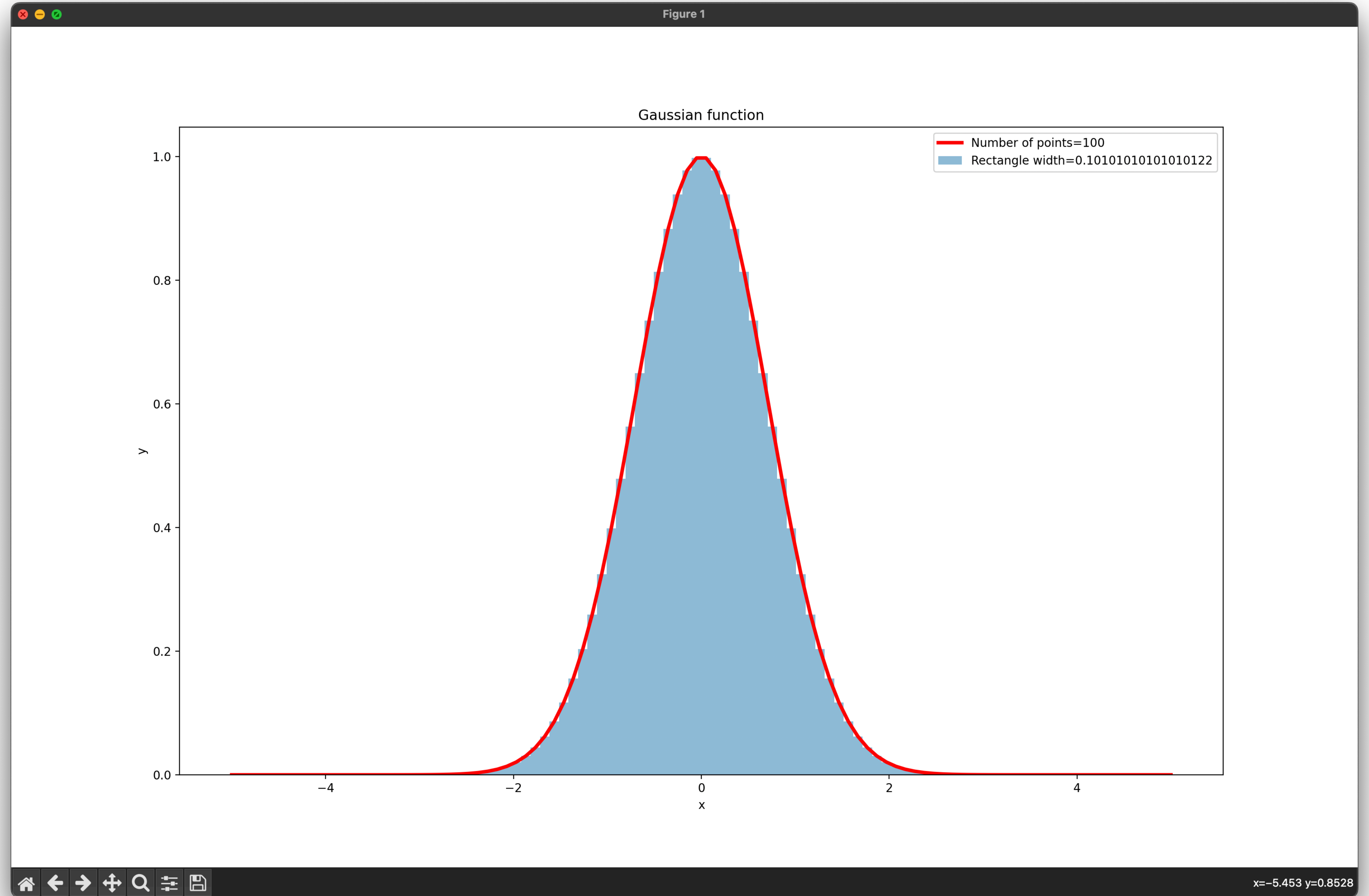
Figure 1
Gaussian function
Number of points=100
Rectangle width=0.10101010101010122
1.0
0.8
0.6
0.4
0.2
0.0
y
−4
−2
0
2
4
x
x=−5.453 y=0.8528

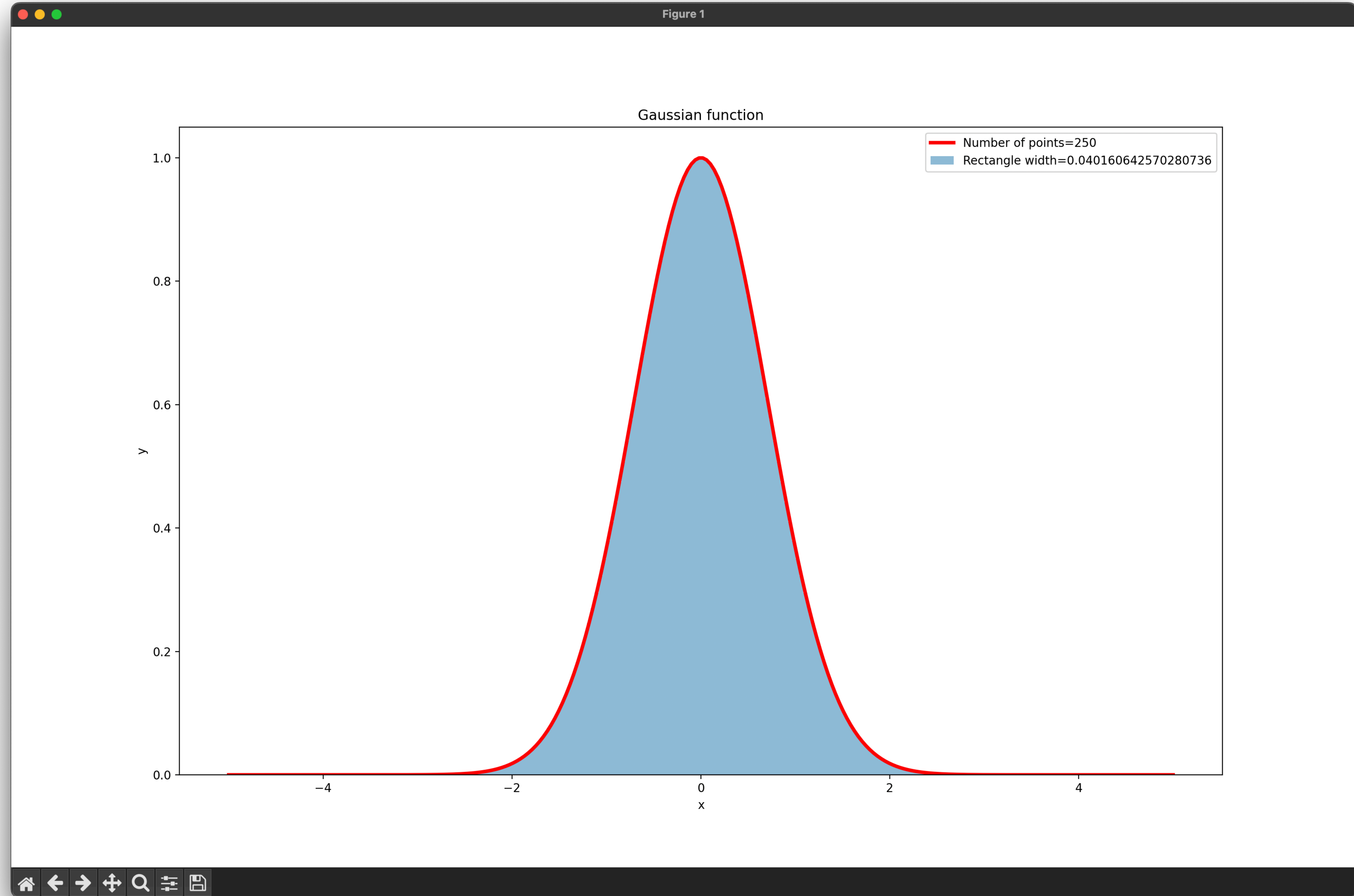

Figure 1
Gaussian function
Number of points=250
Rectangle width=0.040160642570280736
1.0
0.8
0.6
0.4
0.2
0.0
y
−4
−2
0
2
4
x

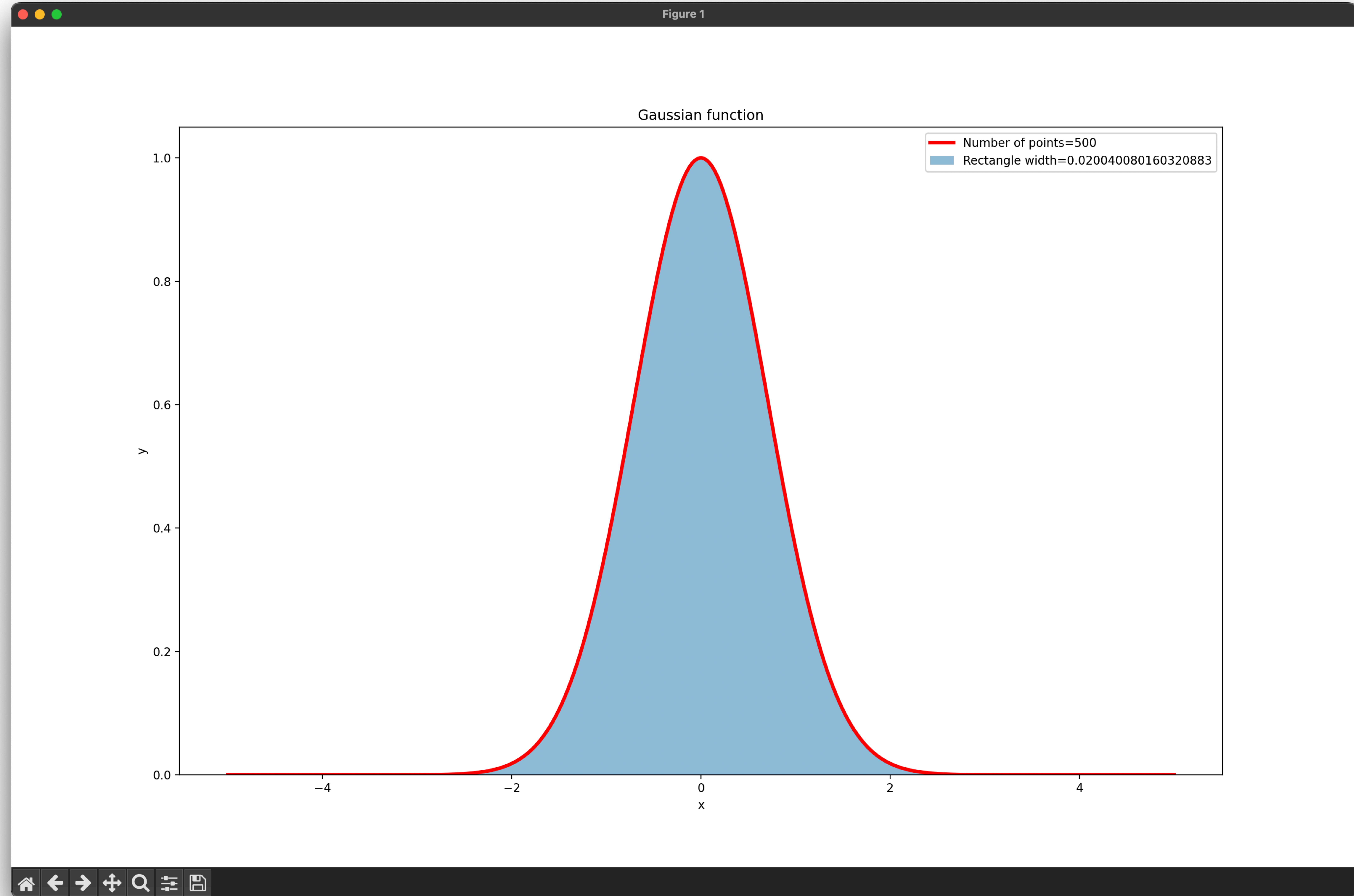
Figure 1
Gaussian function
Number of points=500
Rectangle width=0.020040080160320883
1.0
0.8
0.6
0.4
0.2
0.0
y
−4
−2
0
2
4
x

mihaicrisan@Mihais-Air:~/Developer/UBB/UBB-School-work/Analisys

```
 ~/Developer/UBB/UBB-School-work/Analisys  main !1 ?1                ✔  base  09:17:07
python homework7.py
Number of points: 50
area=1.7724538509047172, error=7.986944439153376e-13, error%=4.5061508569450116e-11, rect_len=0.204081632653061, sq
rt(pi)=1.7724538509055159

Number of points: 100
area=1.7724538509039653, error=1.5505374761914936e-12, error%=8.747970929676679e-11, rect_len=0.10101010101010122,
sqrt(pi)=1.7724538509055159

Number of points: 250
area=1.7724538509032945, error=2.2213422767700132e-12, error%=1.2532536328295219e-10, rect_len=0.040160642570280736
, sqrt(pi)=1.7724538509055159

Number of points: 500
area=1.7724538509030814, error=2.4344970483980433e-12, error%=1.373517875883934e-10, rect_len=0.020040080160320883,
 sqrt(pi)=1.7724538509055159


 ~/Developer/UBB/UBB-School-work/Analisys  main !1 ?1                ✔  5s  base  09:17:16
```