$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = \frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}} = \frac{1}{n+1}\cdot\frac{1}{\ln\frac{n+1}{n}}$$

$$= 1$$

Homework 2

1. $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

Let $\varepsilon > 0$. We want to find $N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon$

$$\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| = \left|\frac{\cancel{2n}+\cancel{2}-\cancel{2n}-3}{4n+6}\right| = \left|\frac{-1}{4n+6}\right| = \frac{1}{4n+6} < \varepsilon$$

$$\frac{1}{4n+6} < \varepsilon \iff \frac{1}{\varepsilon} < 4n+6 \iff \frac{1}{\varepsilon} - 6 < 4n$$

$$\frac{1-6\varepsilon}{\varepsilon} < 4n \iff \frac{1-6\varepsilon}{4\varepsilon} < n$$

Take $N_\varepsilon = \left[\frac{1-6\varepsilon}{4\varepsilon}\right] + 1 > \frac{1-6\varepsilon}{4\varepsilon}$

Then $\forall n \geq N_\varepsilon > \frac{1-6\varepsilon}{4\varepsilon} \iff \frac{n+1}{2n+3} < \varepsilon$, $\forall n \geq N_\varepsilon$

2. c) $x_n = \frac{\sin(n)}{n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

$$\frac{x_{n+1}}{x_n} = \frac{\sin(n+1)}{n+1} \cdot \frac{n}{\sin(n)} = \frac{\sin(n+1)}{\sin(n)} \cdot \underbrace{\frac{n}{n+1}}_{<1}$$

$$\left.\begin{array}{l} -1 \leq \sin(n) \leq 1 \\ -1 \leq \sin(n+1) \leq 1 \end{array}\right\} \Rightarrow \begin{array}{l} \cancel{\frac{\sin(n+1)}{\sin(n)}} -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \\ -\frac{1}{n} \leq \frac{\sin(n+1)}{n+1} \leq \frac{1}{n} \end{array}$$

$$\left.\begin{array}{l} \dfrac{-1}{n+1} \le \dfrac{\sin(n+1)}{n+1} \le \dfrac{1}{n+1} \\[2ex] \dfrac{-1}{n} \le \dfrac{\sin(n)}{n} \le \dfrac{1}{n} \end{array}\right\} \Rightarrow x_n \to 0$$

$$(-)$$

$$\frac{-1}{n+1} + \frac{1}{n} \le \frac{\sin(n+1)}{n+1} - \frac{\sin(n)}{n} \le \frac{1}{n+1} - \frac{1}{n}$$

$$\frac{1}{n(n+1)} \le x_{n+1} - x_n \le \frac{-1}{n(n+1)}$$

$x_n \in (-\infty, \infty)$ is bounded ~~below~~ (-1 is the lowest value x_n can get, and is not ∞)

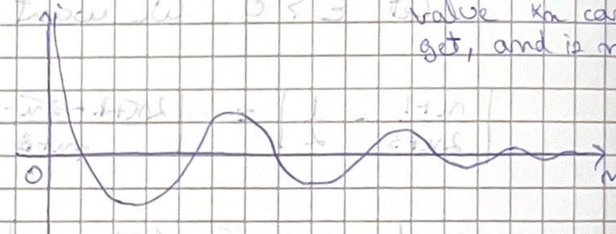
$x_0 = \infty$ (limit)

$x_1 = \sin 1$

$x_2 = \sin 2/2$

$\vdots$

$x_n \to 0$



It does converge to 0, but I do not think x_n is monotone because of $\sin(n)$

3. a.) $\lim\limits_{n \to \infty} \sqrt[n]{1+2+3+\dots+n}$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} = \lim_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}}$$

$$e^{\lim\limits_{n\to\infty} \frac{1}{n} \cdot \ln\left(\frac{n(n+1)}{2}\right)} \; (=)$$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n(n+1)}{2}\right)}{n} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{n\to\infty} \frac{\cancel{2}}{n(n+1)} \cdot \frac{n+n+1}{\cancel{2}} = 0$$

$$(=)\; e^{0} = 1$$

5. c) $\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln n}\right)^n = \lim_{n\to\infty} \left(\ln((n+1)-n)\right)^n$

$\lim_{n\to\infty} \left(\ln(n+1-n)\right)^n = \lim_{n\to\infty} (\ln 1)^n = 0$

c.) $\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\ln(n+1)-\ln n}{\ln(n)}\right)^n$

$e^{\lim_{n\to\infty} \left[\left(1 + \frac{\ln\left(\frac{n+1}{n}\right)}{\ln n}\right)^{\frac{\ln n}{\ln\left(\frac{n+1}{n}\right)}}\right]^{\frac{\ln\left(\frac{n+1}{n}\right)}{\ln n} \cdot \frac{n}{1}}}$

$e^{\lim_{n\to\infty} \frac{\overbrace{\ln\left(\frac{n+1}{n}\right)}^{\to 0}}{\underbrace{\ln n}_{\to\infty}} \cdot \overbrace{n}^{\to\infty}} = e^0 = 1$

8. $\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} = x_n$

$\frac{x_{n+1}}{x_n} = \frac{n^{n+1}}{1+2^2+\dots+n^n+n^{n+1}} \cdot \frac{1+2^2+\dots+n^n+n^{n+1}}{n^n}$

$= \frac{n\cdot(1+2^2+\dots+n^n+n^{n+1})}{(1+2^2+\dots+n^n+(n+1)^{n+1})}$

$1^1 \le n^n$

$2^2 \le n^n$

$3^3 \le n^n$

$\vdots$

$n^n \le n^n$

(+)

$1+2^2+3^2+\dots+n^n \le n\cdot n^n$

$1+2^2+3^2+\dots+n^n \le n^{n+1}$

In this case: $\lim_{n\to\infty} \frac{n^n}{n^{n+1}} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{n^n}{n^n} = 1$, but we have $\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\ldots+n^n}$

So when we are talking about limit we can use the reasoning from before and apply it to our limit. And so the result for the original limit is 0.

~~10.~~

8. $\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\ldots+n^n} \overset{S-C}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}}$

$\lim_{n\to\infty} 1 - \frac{n^n}{(n+1)^{n+1}} = 1 - 0 = 1$

$\lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$

$\lim_{n\to\infty} \frac{1}{n+1} \cdot \underbrace{\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]}_{e}{}^{\frac{-1}{n+1}\cdot n} = \frac{1}{\infty} \cdot \frac{1}{e} = 0$

$\cancel{\lim_{n\to\infty} \frac{1}{n+1} \cdot e\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}}}$

10. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $x_1 = 1$, $a > 1$

$x_1 = 1$

$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = \frac{1}{2}(1+a) = \frac{1+a}{2} > 1$

$\vdots$

$x_n =$

$\Rightarrow x_n \in [1, \infty) \rightarrow$ bounded

6. $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$

decreasing?

bounded?

$x_{n+1} - x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n+1} - \ln(n+1) - 1 - \dots - \frac{1}{n} + \ln n$

$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln(n)$

$= \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right)$

$x_{n+1} - x_n < 0$

$\frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) < 0$

$\frac{1}{n+1} < -(\ln n - \ln(n+1))$

$\frac{1}{n+1} < \ln(n+1) - \ln(n)$?

Consider $f(x) = \ln(x)$, ~~$f: \{\mathbb{N}^*\} \to \{\mathbb{R}\}$~~

$f: \mathbb{N}^* \to \mathbb{R}$

Let $a, b \in \mathbb{N}^*$, $a < b$

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$\Rightarrow \frac{f(b) - f(a)}{b - a} = f'(c)$, $c \in (a, b)$

$f(b) - f(a) = f'(c)(b - a)$

$\left.\begin{array}{l} a = n \\ b = n+1 \end{array}\right\} \Rightarrow f(n+1) - f(n) = f'(c)$

$\ln(n+1) - \ln(n) = f'(c)$

$\ln(n+1) - \ln(n) = \frac{1}{c}$, $c \in (n, n+1)$

$$c \in (n, n+1) \Rightarrow \frac{1}{c} \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

$$\frac{1}{n+1} < \frac{1}{c} < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$n=1$ $$\frac{1}{2} < \ln(2) - \ln(1) < 1$$

$n=2$ $$\frac{1}{3} < \ln(3) - \ln(2) < \frac{1}{2}$$

$\vdots$

n $$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$(+)$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} < \ln(n+1) - 0 < 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} < \ln(n+1) < 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

$\boxed{1}$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} - \ln(n+1) < 0 \qquad |+1$$

$$\underbrace{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} - \ln(n+1)}_{x_{n+1}} < \cancel{0}\, 1$$

$$\longrightarrow x_{n+1} < 1 \rightarrow x_n < 1$$

$\Rightarrow$ x_n bounded above (1)

$\boxed{2}$

$$1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n+1) > 0$$

$$\underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n)}_{x_n} > \underbrace{\ln(n+1) - \ln n}_{>0} > 0$$

$\Rightarrow x_n > 0 \Rightarrow x_n$ bounded below (2)

(1), (2) $\rightarrow x_n \in (0,1)$

From Langrages Theorem we have:

$$\underbrace{\frac{1}{n+1} < \ln(n+1) - \ln(n)} < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln(n+1) - \ln(n)$$

$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right)$$

$$\frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right) < 0$$

$$\frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) < 0$$

$$x_{n+1} - x_n < 0$$

$\Rightarrow x_n$ is decreasing

10. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $x_1 = 1$, $a > 1$

I assume x_n is convergent, and in order for this to stand I need the limit of x_n to be finite.

Let l be the limit of x_n

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right) \quad \Big| \cdot 2$$

$$2\ell = \ell + \frac{a}{\ell} \quad \Big| -\ell$$

$$\ell = \frac{a}{\ell}$$

$$\ell^2 = a \quad , \quad a > 1$$

$$\Rightarrow \ell = \sqrt{a}$$

$\sqrt{a}$ is finite $\Rightarrow$ x_n is convergent