Homework.   Mihai Crișan

2.

a) $\sum\limits_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum \ln\left(\frac{n^2-1}{n^2}\right) = \sum \ln\left(\frac{(n-1)(n+1)}{n\cdot n}\right)$

$\cancel{\sum\limits_{n\geq 2} \ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)}$

$\ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{2}{2}\right) + \ln\left(\frac{4}{2}\right) + \dots$

$\ln\left(\frac{1\cdot \cancel{3}}{2^2}\right)\left(\frac{\cancel{2}\cdot 4}{3^2}\right)\left(\frac{\cancel{3}\cdot 5}{4^2}\right)\dots\left(\frac{\cancel{(n-1)}(n+1)}{n^2}\right)$

$\ln\left(\frac{1}{2}\cdot\underbrace{\frac{n+1}{n}}_{1}\right) \Rightarrow \ln\left(\frac{1}{2}\right)$

b) $\sum\limits_{n\geq 1} \frac{n+1}{3^n} = x_n$

Ratio test:

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim \frac{\frac{n+2}{3^{n+1}}}{\frac{n+1}{3^n}} = \lim \underbrace{\frac{n+2}{n+1}}_{1}\cdot\frac{1}{3} = \frac{1}{3}$

$\frac{1}{3} < 1 \Rightarrow$ converges

c) $$\sum_{n \geq 1} \frac{n}{n^4+n^2+1} =$$

$$\cancel{\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{n+1}{(n+1)^4+(n+1)^2+1} \cdot \frac{n^4+n^2+1}{n} = 1}$$

$$n^4+n^2+1 = (n^2+1)^2 - n^2 = (n^2+1-n)(n^2+1+$$

$$\frac{n}{(n^2+1-n)(n^2+1+n)} = \frac{An+B}{n^2+1-n} + \frac{Cn+D}{(n^4+1+n)}$$

$$n = (An+B)(n^2+1+n) + (Cn+D)(n^2+1+n)$$

$$\cancel{n = An^3 + An + An^2 + Bn^2 + B + Bn + Cn^3 + Cn + Cn^2 + Dn^2 + D + D}$$

$$n = n^3\underbrace{(A+C)}_{0} + n^2\underbrace{(A+B+C+D)}_{0} + n\underbrace{(A+B+C+D)}_{1} + \underbrace{B+D}_{0}$$

$$\left.\begin{array}{l} A+B+C+D = 1 \\ B+D = 0 \end{array}\right\} \Rightarrow A+C = 1$$

$A+C$

$$n = (An+B)(n^2+1+n) + (Cn+D)(n^2-1+n)$$

$$n = An^3 + An + An^2 + Bn^2 + B + Bn + Cn^3 - Cn + Cn^2 + Dn^2 - D + Dn$$

$$n = n^3\underbrace{(A+C)}_{0} + n^2\underbrace{(A+B+C+D)}_{0} + n\underbrace{(A+B+D-C)}_{1} + \underbrace{B-D}_{0}$$

$$\left\{\begin{array}{l} A+C=0 \\ A+B+C+D=0 \\ A+B+D-C=1 \\ B-D=0 \end{array}\right.$$

$$n = (An+B)(n^2+1+n) + (Cn+D)(n^2+1-n)$$

$$n = An^3 + An + An^2 + Bn^2 + B + Bn + Cn^3 + Cn - Cn^2 + Dn^2 + D - Dn$$

$$n = n^3\underbrace{(A+C)}_{0} + n^2\underbrace{(A+B+D-C)}_{0} + n\underbrace{(A+B+C-D)}_{1} + \underbrace{B+D}_{0}$$

$A+C=0 \Rightarrow A=-C$

$A+B+D-C=0$

$A+B+C-D=1 \Rightarrow B-D=1$

$B+D=0 \qquad B+D=0$

A

$2B=1$

$\boxed{B=\frac{1}{2}}$

$B+D=0 \Rightarrow \boxed{D=-\frac{1}{2}}$

$$\left.\begin{array}{l} B+D=0 \\ A+B+D-C=0 \\ A+C=0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} A-C=0 \\ A+C=0 \end{array} \Bigg| \oplus$$

$$2A=0$$

$$\boxed{A=0 \Rightarrow C=0}$$

$$(A,B,C,D)=\left(0,\frac{1}{2},0,-\frac{1}{2}\right)$$

$$\Rightarrow \frac{n}{(n^2+1+n)(n^2+1-n)}=\frac{1}{2(n^2+1-n)}-\frac{1}{2(n^2+1+n)}$$

$$\sum_{n\geq 1}\frac{1}{2(n^2+1-n)}-\frac{1}{2(n^2+1+n)}=\frac{1}{2}\sum\frac{1}{n^2+1-n}-\frac{1}{n^2+1+n}$$

$$\cancel{\frac{1}{2}\left(\frac{1}{3}-1+\frac{1}{6}-\frac{1}{3}+\frac{1}{11}-\frac{1}{7}\right.}$$

$$\frac{1}{2}\left(1-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}-\cancel{\frac{1}{11}}+\ldots+\cancel{\frac{1}{n^2+1-n}}-\frac{1}{n^2+1+n}\right.$$

$$\frac{1}{2}\left(1-\frac{1}{n^2+n+1}\right)=\frac{1}{2}\cdot(1-0)=\frac{1}{2}$$

0

4. a.) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$

I. a. $x \geq 0$, $x<1$, $p\leq 1$

$\lim \frac{x^n \to 0}{n^p \to 0}$ Ratio test: $\lim \frac{x^{n+1}}{(n+1)^p} \cdot \frac{n^p}{x^n} = x < 1$

$\to$ converges

I b. $x>0$, $x<1$, $p>1$

$\lim \frac{x^n \to 0}{n \to \infty} \longrightarrow 0$ - converges

II a. $x=1$, $p\leq 1$

$\lim \frac{1}{n^p \to 0} = \infty \longrightarrow$ diverges

b $x=1$ $p>1$

$\lim \frac{1}{n^p \to \infty} \to$ converges

III a. $x>1$, $p\leq 1$

$\lim \frac{x^n \to \infty}{n^p \to 0} \longrightarrow \infty \longrightarrow$ diverges

b.) $x>1$, $p>1$

$\lim \frac{x^n \to \infty}{n^p \to \infty}$ Ratio test: $\lim \frac{\cancel{x^n} \cdot x}{(n+1)^p} \cdot \frac{n^p}{\cancel{x^n}} = x >$

$\Rightarrow$ diverges

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4. b) $\sum\limits_{n\geq 2} \frac{1}{\ln n^{\ln n}}$

Cauchy condensation test:

$$\sum_{n\geq 2} \frac{1}{\ln n^{\ln n}} \text{ "like" } \sum_{n\geq 1} \frac{2^n}{(\ln 2^n)^{(\ln 2)^n}}$$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{\left[\frac{2}{(\ln 2^n)^{\ln 2}}\right]^n} = \lim_{n\to\infty} \frac{2}{(\ln 2^n)^{\ln 2}}$$

$$= \lim_{n\to\infty} \frac{2}{(n \cdot \ln 2)^{\ln 2}} = \lim_{n\to\infty} \underbrace{\frac{2}{(\ln 2)^{\ln 2}}}_{\text{constant}} \cdot \frac{1}{n^{\ln 2}}$$

$$\ln 2 < 1 \implies \lim_{n\to\infty} \text{constant} \cdot \frac{1}{n^{\ln 2}} = \infty > 1$$

$$\implies \sum \frac{2^n}{(\ln 2^n)^{\ln 2^n}} \text{ is divergent}$$

$$\implies \sum \frac{1}{\ln n^{\ln n}} \text{ is divergent}$$

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4. c) $\sum_{n \geq 1} \left(\sqrt[n]{n} - 1\right)$

$$\sqrt[n]{n} - 1 = n^{\frac{1}{n}} - 1 = e^{\frac{1}{n} \cdot \ln n} - 1$$

$$\left.\begin{array}{l} \frac{\ln n}{n} \geq \frac{1}{n} \\ e^{\frac{\ln n}{n}} > \frac{\ln n}{n} \end{array}\right\} \Rightarrow$$

harmonic series $\to \infty$

$$e^{\frac{\ln n}{n}} > \frac{\ln n}{n} > \frac{1}{n} \quad \Big| -1$$

$$e^{\frac{\ln n}{n}} > \frac{\ln n}{n} > \frac{1}{n} \to \infty \quad \Big| -1$$

$$\Rightarrow e^{\frac{\ln n}{n}} - 1 \to \infty$$

(Comparison test)

$$\Rightarrow \sum_{n \geq 1} \left(\sqrt[n]{n} - 1\right) \text{ diverges}$$

Area:

Starting from the initial triangle having Area A to each side gets ~~add~~ added a triangle of size $\frac{1}{9}$ of A', where A' is the ~~size~~ are of the triangle from the last iteration

Iteration ~~zer~~ 0: $A_1 = A$

Iteration 1: $A_2 = A + (3 \cdot 4^0)(\frac{1}{9} \cdot A)$

Iteration 2: $A_3 = A + (3 \cdot 4^0 \cdot \frac{1}{9} \cdot A) + (3 \cdot 4^1 \cdot \frac{1}{9^2} \cdot A)$

⋮

Iteration n: $A_n = A + \underbrace{\sum_{i \geq 1}^{n} (3 \cdot 4^{i-1})}_{\text{number of sides from previous iteration}} \cdot \underbrace{\left(\frac{1}{9^i} \cdot A\right)}_{\text{area of the triangle that gets to be added to the total Area}}$

$$\Rightarrow \text{Area} = A + \sum_{i=1}^{n} (3 \cdot 4^{i-1}) \cdot \left(\frac{1}{9^i} \cdot A\right)$$ where A is the initial size of the triangle $A = \frac{1^2\sqrt{3}}{4}$ so we get:

$$\text{Area} = \frac{\sqrt{3}}{4} + \sum_{i=1}^{n} \frac{3 \cdot \sqrt{3}}{4} \cdot \frac{4^{i-1}}{9^i}$$

$$\text{Area} = \frac{\sqrt{3}}{4} + \frac{3 \cdot \sqrt{3}}{4} \cdot \sum_{i=1}^{n} \frac{4^{i-1}}{9^i}$$

$$\text{Area} = \frac{\sqrt{3}}{4} + \frac{3 \cdot \sqrt{3}}{4^2} \sum_{i=1}^{\infty} \left(\frac{4}{9}\right)^i \Rightarrow \frac{4}{5}$$

$$\text{Area} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4 \cdot \cancel{4}} \cdot \frac{\cancel{4}}{5} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20}$$

$\frac{8\sqrt{3}}{20}$ is the Area as an approximation, as a limit.

- Take into account the area cannot be infinite because the whole fractal can be included in a circumscribed circle.