Cheatsheet

Transformation of Random Variables

To find the PDF of a transformed random variable $Y = g(X)$, where g is a differentiable and strictly monotone function, we use:

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|.$$

Here's the process:

1. Identify the function $Y = g(X)$.
2. Solve for X in terms of Y to get $X = g^{-1}(Y)$.
3. Compute $\frac{d}{dy} g^{-1}(y)$.
4. Substitute $g^{-1}(y)$ into $f_X(x)$ and multiply by $\left|\frac{d}{dy} g^{-1}(y)\right|$.

Exercise Statement

Let $X \sim N(0, 1)$, a standard normal random variable. Find the probability density function (PDF) of $Y = |X|$.

Step 1: Transformation of a Random Variable

Theory:
To find the PDF of $Y = g(X)$, we use the following steps when $g(X)$ is not strictly monotonic over the entire domain:

1. Divide the domain of X into monotonic segments.
2. Compute the PDF of Y by summing contributions from all segments.

Here:

$$Y = |X| = \begin{cases} X, & X \geq 0, \\ -X, & X < 0. \end{cases}$$

The segments are:
• $X \geq 0$, where $Y = X$.
• $X < 0$, where $Y = -X$.

The corresponding inverse transformations are:

- $X = Y$ for $X \geq 0$,
- $X = -Y$ for $X < 0$.

Step 2: Derive the PDF of Y

The PDF of Y is obtained by summing over both segments:

$$f_Y(y) = f_X(y) \cdot \left| \frac{d}{dy} y \right| + f_X(-y) \cdot \left| \frac{d}{dy}(-y) \right|,$$

where $f_X(x)$ is the standard normal PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Since $\frac{d}{dy} y = 1$ and $\frac{d}{dy}(-y) = -1$, we have:

$$f_Y(y) = f_X(y) \cdot 1 + f_X(-y) \cdot 1.$$

Substitute $f_X(x)$ into the equation:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(-y)^2/2}.$$

Since $(-y)^2 = y^2$:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Combine the terms:

$$f_Y(y) = \frac{2}{\sqrt{2\pi}} e^{-y^2/2}, \quad y \geq 0.$$

Step 3: Domain of Y

Since $Y = |X|$, and $X \sim N(0, 1)$, $Y \geq 0$.

The PDF of Y is:

$$f_Y(y) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-y^2/2}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$