Limit superior and Limit inferior

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1 Limit Superior and Limit Inferior of a Sequence

Definition 1.1. *Let $\{x_n\}$ be a sequence of real numbers. Consider $y_n = \sup\{x_n, x_{n+1}, \dots\}$. Clearly $\{y_n\}$ is a decreasing sequence i.e. $y_n \geq y_{n+1}$. Define*

$$\limsup_{n\to\infty} x_n = \begin{cases} \lim\limits_{n\to\infty} y_n & if\ x_n \text{if is bounded} \\ +\infty & if x_n \text{ is not bounded above} \\ -\infty & if\ x_n \text{ is not bounded below.} \end{cases}$$

Similarly we can define limit inferior as follows

Definition 1.2 (Limit Inferior)**.** *Let $\{x_n\}$ be a sequence of real numbers. Consider $z_n = \inf\{x_n, x_{n+1}, \dots\}$. Clearly $\{z_n\}$ is an increasing sequence i.e. $z_n \leq z_{n+1}$. Define*

$$\liminf_{n\to\infty} x_n = \begin{cases} \lim\limits_{n\to\infty} z_n & if\ x_n \text{if is bounded} \\ +\infty & if x_n \text{ is not bounded above} \\ -\infty & if\ x_n \text{ is not bounded below.} \end{cases}$$

Example 1.3. *1. Let $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$. Then $\lim \sup\limits_{n\to\infty} = 1$ and $\lim \inf\limits_{n\to\infty} = -1$.*

We can easily check the following.

Theorem 1.4. *Let $\{x_n\}$ and $\{y_n\}$ be real sequences. Then*

1. $\inf\ x_n \leq \lim \inf\limits_{n\to\infty} x_n \leq \lim \sup\limits_{n\to\infty} x_n \leq \sup\ x_n$.

2. $\lim \sup\limits_{n\to\infty} (x_n + y_n) \leq \lim \sup\limits_{n\to\infty} x_n + \lim \sup\limits_{n\to\infty} y_n$.
   and
   $\lim \inf\limits_{n\to\infty} (x_n + y_n) \geq \lim \inf\limits_{n\to\infty} x_n + \lim \inf\limits_{n\to\infty} y_n$.

3. *Let $\alpha > 0$. Then* $\lim \sup\limits_{n\to\infty} (\alpha x_n) = \begin{cases} \alpha \lim \sup\limits_{n\to\infty} x_n & if\ \alpha \geq 0 \\ \alpha \lim \inf\limits_{n\to\infty} x_n & if\ \alpha < 0. \end{cases}$.

4. *If $x_n \leq y_n$ then*
   $$\lim \sup_{n\to\infty} x_n \leq \lim \sup_{n\to\infty} y_n \text{ and } \lim \inf_{n\to\infty} x_n \leq \lim \inf_{n\to\infty} y_n.$$

Proposition 1.1. *Let $\{x_n\}$ be a sequence and $l \in \mathbb{R}$. The following are equivalent:*

1. $\limsup\limits_{n\to\infty} x_n = l$
2. *For any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $x_n < l + \epsilon; \forall n \geq N$ and there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = l.$$

Proof. Let $y_n = \sup\limits_{k\geq n} x_k$. So we have $\lim\limits_{n\to\infty} y_n = l$. So for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$l - \epsilon < y_n < l + \epsilon, \ \forall n \geq N.$$

Clearly, $x_n < y_n < l + \epsilon$. For $\epsilon = 1$ we get N_1 such that $l - 1 < y_n < l + 1, \ \forall n \geq N_1$. Thus there exists a n_1 such that $l - 1 < a_{n_1} < l + 1$. Similarly for $\epsilon = \frac{1}{2}$ we get n_2 such that $l - \frac{1}{2} < a_{n_2} < l + \frac{1}{2}$. Continuing like this we get for every $k \in \mathbb{N}$ we get n_k such that

$$l - \frac{1}{k} < a_{n_k} < l + \frac{1}{k}.$$

Clearly, $\lim\limits_{k\to\infty} a_{n_k} = l$.

For the converse, we are given that for $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $x_n < l + \epsilon, \ \forall n \geq N$. Thus, $y_n < l + \epsilon$ for all $n \geq N$. We want to show the other side. Also, we are given that there exists a subsequence x_{n_k} which converges to l as k tends to ∞. As $n_k \geq k$ we have $y_k \geq a_{n_k}$. Hence, the result.

□

Similar result can be proved for limit inferior.

Proposition 1.2. *Let $\{x_n\}$ be a sequence and $l \in \mathbb{R}$. The following are equivalent:*

1. $\liminf\limits_{n\to\infty} x_n = l$
2. *For any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $x_n > l - \epsilon; \forall n \geq N$ and there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = l.$$

For the above propositions we have the following corollaries, which we prefer to call it as a theorem.

Theorem 1.5. *Let $\{x_n\}$ be a sequence. Then*

$$\lim_{n\to\infty} = l$$

if and only if

$$\liminf_{n\to\infty} x_n = l = \limsup_{n\to\infty} x_n.$$

Example 1.6. *1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty}(2x_{n+1}-x_n) = l$. Prove that $\lim_{n\to\infty} x_n = l$.*

Soln: *We first claim that x_n bounded. Let $M \in \mathbb{N}$ such that $|a_1| \le M$ and $|2x_{n+1}-x_n| \le M$, $\forall n$. Let $|x_n| \le M$.*

$$\begin{aligned} |x_{n+1}| &= \left|\frac{x_n + (2x_{n+1} - x_n)}{2}\right| \\ &\le \frac{1}{2}(|x_n| + |2x_{n+1} - x_n|) \le M. \end{aligned}$$

So,

$$\limsup_{n\to\infty} x_{n+1} \le \frac{\limsup_{n\to\infty} x_n + l}{2}.$$

This gives $\limsup_{n\to\infty} x_n \le l$. Similarly, we can get $\liminf_{n\to\infty} x_n \ge l$. Thus we get $\lim_{n\to\infty} x_n = l$.