ARMA(p, q)

$\{X_t\}$ is $\ni$

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$$\phi(B) X_t = \theta(B)\epsilon_t \qquad \epsilon_t \sim WN(0, \sigma^2)$$

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

(i) $\{X_t\}$ is said to be Causal if roots of $\phi(z) = 0$ all lie outside the unit circle. In such a case the MA(∞) causal representation is

$$X_t = \phi(B)^{-1}\theta(B)\epsilon_t = \psi(B)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

$$\Rightarrow \phi(B)^{-1}\theta(B) = \psi(B)$$

i.e. $\theta(B) = \phi(B)\psi(B)$

i.e. $(1 + \theta_1 B + \cdots + \theta_q B^q)$

$$= (1 - \phi_1 B - \cdots - \phi_p B^p)(\psi_0 + \psi_1 B + \cdots + \cdots)$$

Comparing coeffs of B^j, we get

$$\psi_s = \phi_1 \psi_{s-1} + \cdots + \phi_p \psi_{s-p} + \theta_s \quad \forall\, s \le q$$

$$\text{"} = \phi_1 \psi_{s-1} + \cdots + \phi_p \psi_{s-p} \quad \forall\, s > q$$

with $\psi_r = 0 \;\forall\, r < 0$ & $\psi_0 = 1$

(ii) $\{X_t\}$ is said to be invertible if it can be expressed as AR(∞), i.e. if roots of $\theta(z)=0$ all lie outside the unit circle. In such a case

$$\phi(B)X_t = \theta(B)\epsilon_t$$

$$\epsilon_t = \theta(B)^{-1}\phi(B)X_t$$

$$\epsilon_t = \psi(B)X_t = \sum_{j=0}^{\infty}\psi_j X_{t-j}$$

AR(∞)

We can use method of comparing coefficients to express ψ_s interms of $\theta_j s$ & $\phi_j s$ as done for the causal representation.

$$\theta(B)^{-1}\phi(B) = \psi(B)$$

$$\text{i.e. } \phi(B) = \theta(B)\psi(B)$$

Auto Covariance Generating Function (ACGF)

ACGF is a simple concept and usually easy to calculate. The auto covariances at different lags be determined through ACGF

If $\{X_t\}$ is a covariance stationary time series with ACVF $\gamma(\cdot)$, then it's ACGF is defined by

$$g_X(z) = \sum_{j=-\infty}^{\infty} \gamma(j) z^j \quad - (*)$$

provided the series converges for all z in some annulus $r^{-1} < |z| < r$ with $r > 1$.

Note: Coeff of z^j in $(*)$ is $\gamma(j)$, auto covariance at lag j.

MA(1) ACGF

$$X_t = \epsilon_t + \theta \epsilon_{t-1} ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$X_t = \theta(B) \epsilon_t$$

$$\gamma(j) = \begin{cases} \sigma^2(1+\theta^2), & j = 0 \\ \theta\sigma^2, & j = \pm 1 \\ 0, & o/w \end{cases}$$

ACGF: $g_X(z) = (\theta\sigma^2) z^{-1} + \sigma^2(1+\theta^2) z^0 + (\theta\sigma^2) z^1$

i.e. $g_X(z) = \sigma^2(\theta z^{-1} + (1+\theta^2) + \theta z)$

i.e $g_X(z) = \sigma^2 (1+\theta z)(1+\theta z^{-1})$

i.e. $g_X(z) = \sigma^2 \theta(z)\theta(z^{-1})$

MA(q) ACGF

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$$X(t) = \theta(B)\epsilon_t \qquad \epsilon_t \sim WN(0,\sigma^2)$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

ACGF:

$$\begin{aligned} g_X(z) = \sigma^2 \Big[& (\theta_0^2 + \theta_1^2 + \cdots + \theta_q^2) z^0 \\ & + (\theta_0\theta_1 + \cdots + \theta_{q-1}\theta_q) z^1 \\ & + (\theta_0\theta_1 + \cdots + \theta_{q-1}\theta_q) z^{-1} \\ & + (\theta_0\theta_2 + \cdots + \theta_{q-2}\theta_q) z^2 \\ & + (\theta_0\theta_2 + \cdots + \theta_{q-2}\theta_q) z^{-2} + \\ & \vdots \\ & + \theta_0\theta_q z^q + \theta_0\theta_q z^{-q} \Big] \end{aligned}$$

$\theta_0 = 1$

(using the already known auto covariance sequence)

i.e $g_X(z) = \sigma^2 (\theta_0 + \theta_1 z + \cdots + \theta_q z^q)$
$(\theta_0 + \theta_1 z^{-1} + \cdots + \theta_q z^{-q})$

$$g_X(z) = \sigma^2 \theta(z)\theta(z^{-1})$$

MA(∞) :

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \quad \text{with } \epsilon_t \sim WN(0,\sigma^2) \ \& \ \sum_j |\psi_j| < \infty$$

$$X_t = \psi(B)\,\epsilon_t$$

ACGF $\quad g_X(z) = \sigma^2 \psi(z)\psi(z^{-1}).$

AR(1) $\quad \{X_t\}$ is stationary AR(1)

$$(1-\phi B)X_t = \epsilon_t$$

$\theta_1 + \theta_1\theta_2$

$$\phi(B)X_t = \epsilon_t$$

$$X_t = \phi(B)^{-1}\epsilon_t \quad (\{X_t\} \text{ is causal})$$

$$= \psi(B)\,\epsilon_t$$

$|\phi z| < 1 \quad \left|\frac{\phi}{z}\right| < 1$

$|z| < \frac{1}{|\phi|} \quad |\phi| < |z|$

ACGF

$$g_X(z) = \sigma^2 \psi(z)\psi(z^{-1})$$

i.e. $$g_X(z) = \frac{\sigma^2}{\phi(z)\,\phi(z^{-1})} = \frac{\sigma^2}{(1-\phi z)(1-\phi z^{-1})}$$

i.e. $$g_X(z) = \sigma^2(1+\phi z + \phi^2 z^2 + \cdots)(1+\phi z^{-1} + \phi^2 z^{-2} + \cdots) \quad \text{— } (*)$$

Note that coeff of z^j in the r.h.s. of $(*)$ is

$$= \sigma^2(\phi^j + \phi^{j+1}\phi + \phi^{j+2}\phi^2 + \cdots)$$

$$= \frac{\phi^j \sigma^2}{1-\phi^2} = \gamma(j) \text{ as we expected from ACGF.}$$

Note: for AR(p) $\quad g_X(z) = \dfrac{\sigma^2}{\phi(z)\,\phi(z^{-1})}$; $\phi(B)X_t = \epsilon_t$

ARMA (p, q)

Suppose $\{X_t\}$ is covariance stationary ARMA(p, q)

$$\phi(B) X_t = \theta(B) \epsilon_t$$

Using the same causal representation (as used for AR, ACGF of ARMA(p, q) is

$$g_X(z) = \sigma^2 \frac{\theta(z)\,\theta(z^{-1})}{\phi(z)\,\phi(z^{-1})}.$$

Note: ACGF of WN is a constant.

ACGF of a filtered process

Let $\{X_t\}$ be a covariance stationary process with ACVF $\gamma(\cdot)$ and ACGF

$$g_X(z) = \sum_{j=-\infty}^{\infty} z^j \gamma(j)$$

Consider a linear filtered process $\{Y_t\}$ $\ni$

$$Y_t = \sum_{i=0}^{q} \theta_i X_{t-i} = \theta(B) X_t$$

$$\theta(B) = \theta_0 + \theta_1 B + \cdots + \theta_q B^q$$

$$\gamma_Y(h) = \mathrm{Cov}(Y_{t+h}, Y_t)$$

$$= \mathrm{Cov}\left(\sum_{i=0}^{q} \theta_i X_{t+h-i}, \sum_{j=0}^{q} \theta_j X_{t-j}\right)$$