24/July

What is Several Variable Calculus?

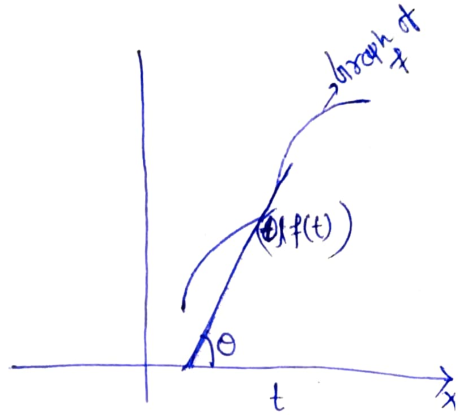
→ An extension of single Variable Calculus to the Calculus of functions of multiple Variables.

- Differentiation
- Integration of function of several variable.

Example. $f: \mathbb{R} \to \mathbb{R}$.
$t \to f(t)$.

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \quad \text{if exist.}$$

→ Derivative of f at t.



$f'(t)$ is the slope of the tangent line to the graph of f. at the pt. $(t, f(t))$

$\tan\theta = f'(t)$

Ques. What is the notion of derivative of $f: \mathbb{R}^n \to \mathbb{R}^n$?

→ Need generalization of derivative.

Tool. → linear Transformation and Norm.

Diffeos and Local diffeos.

Example-2 $f: \mathbb{R} \to \mathbb{R}$.
$f(t) = t^2$.

— differentiable, derivative function is cts.
→ Continuously differentiable function

f is Not injective. is Not surjective } In particular, f is uninvertible.

$f|_U : U \to f(U)$.
$ \| $
$ (16, 36)$

$0 \qquad t_0 = 5$

$\epsilon = 1$

$U = (t_0 - \epsilon, t_0 + \epsilon)$

$U = (4, 6)$

$f|_U$ is bijective $\Rightarrow$ invertible.

$g = (f|_U)^{-1} = (16, 36) \to (4, 6)$.

$g(t) = \sqrt{t}$ continuously differentiable.

Def:- i) A continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is called local diffeos at a point $t_0 \in \mathbb{R}$ if

$\exists\, t_0 \in \underset{\text{open}}{U \subseteq \mathbb{R}}$ s.t. $\qquad t_0 \in \underset{\text{open}}{U \subseteq \mathbb{R}}$.

$f|_U : U \to f(U)$ is

i) invertible and

ii) $(f|_U)^{-1}$ is continuously differentiable.

ii) if f is a local diffeos at every point $t_0 \in \mathbb{R}$, then it is called a local diffeos on $\mathbb{R}$.

iii) A strictly diffeos function $f : \mathbb{R} \to \mathbb{R}$ is a diffeos if f is i) invertible and

ii) f^{-1} is also continuously differentible.

Example: The function in eg ② is local diffeos at every $t_0 (\neq 0) \in \mathbb{R}$. $\quad 0 < t < |t_0|$

f is not local diffeos at $t_0 = 0$.

Remark: $f'(t_0) \neq 0$ if $t_0 \neq 0$.
$\qquad\qquad\quad 0$ if $t_0 = 0$.

Exercise: Show that a Continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$ is local diffeos at $t_0 \in \mathbb{R}$. iff $f'(t_0) \neq 0$.

Question-3. When a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a local diffeos?

Ans. Inverse function Theorem.

Major Results:

Ques 4: What do we study in several Variable Calculus?

Example- $f: \mathbb{R} \to \mathbb{R}$ $\quad$ It is locally diffeomorphism but
$f(t) = t^2$. $\quad$ not diffeomorphism.

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$.
$f(t) = t^2$.
$\longrightarrow$ locally diffeos.

Ex. A function $f: \mathbb{R} \to \mathbb{R}$ is a local diffeos iff it is a diffeos ?

$f'(t_0) \neq 0 \quad \forall\, t_0 \in \mathbb{R}$.

f' cts.

$\Rightarrow f' > 0$ or $f' < 0$
$\Downarrow$
$\Rightarrow f$ is strictly monotonic

Goal :- Study diff. geometry of curves and surfaces.

For the purpose, we study a few result in several variable calculus which are essential.

i) chain rule of derivatives
ii) inverse function theorem
iii) Implicit function theorem
iv) Taylor's formula
v) extremum.
vi) Lagrange multipliers etc.

Application of several variable calculus in differential geometry

1) Curves 1) $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ $n \geq 2$.

- tangent vectors.
- tangent lines
- Space
- curvature
- Torsion etc.

2) Surfaces $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ $n \geq 3$.

- Surface area.
- Curvature
- Surface integral, etc.

3) Scalar fields $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

- directional derivative
- extrema
- Lagrange multipliers.

4) vector field $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

- Any operation in vector calculus including gradient, divergence.

Differential geometry

Ques 5: What is differential geometry?

- Study of geometric properties using differential calculus and integral calculus.

  ↓

  (Mostly used)

Ques 6. What is geometric properties?

- The properties of geometric objects which are invariant under congruence are called geometric properties.

Example:-
1) Length of curve.
2) Angles
3) area etc.

Non-examples.
1) Slope.
2) Coordinates.
3) Ordered basis of $\mathbb{R}^2$.

Ques 7. What we study in differential geometry?

- Geometric properties of curves and surfaces

  classified into two classes.

1) Local properties
2) Global properties

Example of local properties →
- tangent vector
- tangent line
- Speed
- Curvature

Global properties: $\alpha : [0,1] \longrightarrow \mathbb{R}^2$

$$\mathbb{R}^2 \mid \text{Image}(\alpha) = \underset{\substack{\uparrow \\ \text{Bounded}}}{D_1} \sqcup D_2$$

$\text{area}(\alpha) = \text{area}(D_1)$

Ques 8 Among all simple closed curves of length l, which one bounds max. area? $\rightarrow$ Given by isoperimetric inequality

$4\pi \, \text{area}(\alpha) \le \text{length}(\alpha)^2$

Ques 9. When two surfaces are diffeomorphic?

1/Aug/2022.

Outline

1) Norm and its properties
2) Innerproduct and its properties
3) Matrix of linear transformation

1) Norm

$$\mathbb{R}^n = \left\{ (x_1, \ldots, x_n) \,\middle|\, \begin{matrix} x_i \in \mathbb{R} \\ 1 \le i \le n \end{matrix} \right\}$$

- $n=1$, $\mathbb{R}^1$ — line
- $n=2$, $\mathbb{R}^2 \rightarrow$ plane
- $n=3$, $\mathbb{R}^3$ — space, etc.

Distance function

for $x, y \in \mathbb{R}^n$

$$d_E(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}, \text{ where}$$

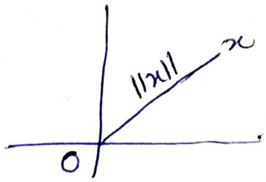
$x = (x_1, \ldots, x_n)$

$y = (y_1, \ldots, y_n)$

— Called Euclidean distance between x and y.

$(\mathbb{R}^n, d_E)$ — Called Euclidean space.

Norm : A f^n $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ defined by



$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$ — called the Euclidean norm of x.

Examples.

1) $X_\theta = (\cos\theta, \sin\theta)$

$\|X_\theta\| = 1$

2) $e_i = (0, 0, \ldots 0, 1, \ldots 0)$
$\downarrow$
i^{th} position.

$\|e_i\| = 1$.

Properties of Norm

Let $X = \{x_1, \ldots, x_n)$
$Y = \{y_1, \ldots, y_n) \in \mathbb{R}^n$, $a \in \mathbb{R}$.

Theorem 1 : i) (+)ve definiteness : $\|X\| \geq 0$. equality hold with $\Leftrightarrow x = 0$

2) $\left| \sum_{i=1}^{n} x_i y_i \right| \leq \|x\| \cdot \|y\|$ {Cauchy Schwarz's inequality}
(equality) with $\Leftrightarrow$ x and y are LD

3) $\|X + Y\| \leq \|X\| + \|Y\|$ {Triangle inequality}

4) $\|aX\| = |a| \cdot \|X\|$. {Scalar multiplication}.

Proof :

$\|X\|^2 = \sum_{i=1}^{n} x_i^2$

→ Sum of non (+)ve reals is non-negative.

$\Rightarrow \|X\|^2 \geq 0$.

$\Rightarrow \|X\| \geq 0$

if $a \in \mathbb{R}$.

$\sqrt{a^2} = |a|$

$\|X\| = 0$

$\Rightarrow \sum_{i=1}^{n} x_i^2 = 0 \quad \Rightarrow x_i = 0$

$X \neq \lambda y$

$X - \lambda y$

$\|X - \lambda y\|^2 \geq 0$

$\sum_{i=1} (x_i - \lambda y_i)^2 \geq 0$

2) Assume X and Y are L.I.

Then $X - \lambda Y \neq 0, \forall \lambda \in \mathbb{R}$.

$\Rightarrow \|X - \lambda Y\|^2 > 0 \quad \forall \lambda \in \mathbb{R}$ {By (1)}.

$\Rightarrow \sum_{i=1}^{n} (x_i - \lambda y_i)^2 > 0$

$\Rightarrow \sum_{i=1}^{n} (x_i^2 - 2x_i\lambda y_i + \lambda^2 y_i^2) > 0 \quad \forall \lambda \in \mathbb{R}$

$\Rightarrow \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2 > 0 \quad \forall \lambda \in \mathbb{R}$.

$\Rightarrow \|x\|^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \|y\|^2 > 0 \quad \forall \lambda \in \mathbb{R}$. How do we know this

L.H.S. is a quadratic equation with no real root.

{Use determinant concept $b^2 - 4ac$)

$\Rightarrow \left(-2 \sum_{i=1}^{n} x_i y_i\right)^2 - 4\|X\|^2 \|Y\|^2 < 0.$

$\Rightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 < \|X\|^2 \|Y\|^2$

$\Rightarrow \left|\sum_{i=1}^{n} x_i y_i\right| < \|X\| \cdot \|Y\|$

$X = \lambda Y$

$X - \lambda Y = 0$

$\|X - \lambda Y\|^2 = 0$

$\sum (X_i - \lambda y_i)^2 = 0$

$\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \|x\|^2 \|y\|^2$

$\sum_{i=1}^{n} x_i y_i = \|x\| \|y\|$

Assume X and Y are L.D.

We have $X = \lambda Y$

$Y = \lambda X$ for some $\lambda \in \mathbb{R}$.

Assume $X = \lambda Y$

Ex. Complete the proof.

3) $\|X+Y\|^2$

$$= \sum_{i=1}^{n} (x_i+y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$$

$$= \|X\|^2 + 2\sum_{i=1}^{n} x_i y_i + \|Y\|^2$$

$$\leq \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2$$

{By using 2}.

$$\Rightarrow \|X+Y\|^2 \leq (\|X\| + \|Y\|)^2$$

$$\Rightarrow \|X+Y\| \leq \|X\| + \|Y\|$$

Ex: When $=$ holds in (3) ?

Proof (4) $\|aX\| = \left(\sum_{i=1}^{n} (ax_i)^2\right)^{1/2}$

$$= (a^2)^{1/2} \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$\sqrt{a^2} = |a|$.

$$= |a| \cdot \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \quad \square. \quad = |a| \cdot \|x\|.$$

INNER PRODUCT

Defn.: $\langle \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

— Called the Euclidean inner product.

Properties of I.P.

Theorem: $X, Y, X_1, X_2 \in \mathbb{R}^n$. $a \in \mathbb{R}$.

1) Commutativity: $\langle X, Y \rangle = \langle Y, X \rangle$

2) Bilinearity: $\langle X_1 + X_2, Y \rangle = \langle X_1, Y \rangle + \langle X_2, Y \rangle$

$\langle aX, Y \rangle = a \langle X, Y \rangle$

3) (+)ve definite: $\langle X, X \rangle \geq 0$ with equality $\Leftrightarrow X = 0$.

4) $\|X\| = \langle X, X \rangle^{1/2}$

5) $\langle X, Y \rangle = \frac{1}{4}\left[\|X+Y\|^2 - \|X-Y\|^2 \right] \longrightarrow \frac{1}{4}\left[\|X\|^2 + \|Y\|^2 + 2\|X\|\|Y\| \ldots\right.$

Proof:

1) $a, b \in \mathbb{R}$. then $ab = ba$.

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \langle Y, X \rangle.$$

2) $X_1 = (a_1, \ldots, a_n)$

$X_2 = (b_1, \ldots, b_n)$

$$\begin{aligned}
\langle X_1 + X_2, Y \rangle &= \sum_{i=1}^{n} (a_i + b_i) y_i \\
&= \sum_{i=1}^{n} (a_i y_i + b_i y_i) \\
&= \sum_{i=1}^{n} a_i y_i + \sum_{i=1}^{n} b_i y_i \\
&= \langle X_1, Y \rangle + \langle X_2, Y \rangle
\end{aligned}$$

$$\begin{aligned}
\langle aX, Y \rangle &= \sum_{i=1}^{n} a x_i y_i \\
&= a \sum_{i=1}^{n} x_i y_i \\
&= a \langle X, Y \rangle
\end{aligned}$$

Ex. For $X, Y_1, Y_2 \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

Show that

$$\langle X, aY_1 + Y_2 \rangle = a\langle X, Y_1 \rangle + \langle X, Y_2 \rangle.$$

3) follows from (+)ve definiteness of norm

4) follows from definition of norm.

5) Straightforward, Begin with R.H.S.

Matrix representation of L.T.

$0 = (0, 0, \ldots, 0) \in \mathbb{R}^n$

$e_i = (0, \ldots, \underset{\substack{\uparrow \\ i^{th} \text{ position}}}{1}, \ldots, 0) \in \mathbb{R}^n$

$B = \underbrace{\{e_1, e_2, \ldots, e_n\}}_{\text{standard ordered basis of } \mathbb{R}^n} \subseteq \mathbb{R}^n$

Recall, $T: \mathbb{R}^n \to \mathbb{R}^m$ is L.T.

$$T(\lambda X + Y) = \lambda T(X) + T(Y)$$
$$\forall\, X, Y \in \mathbb{R}^n \;\&\; \lambda \in \mathbb{R}.$$

$$T(e_j) = (a_{1j}, \ldots, a_{mj}) \in \mathbb{R}^m.$$

define a matrix A, where

j^{th} column is $(a_{1j}\ a_{2j}\ \cdots\ a_{mj})^t$

i.e.,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & & \vdots \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

Defn: A is called the matrix of the L.T T, wrt the standard ordered basis

denote it by $[T] = A$.

Example: $T : \mathbb{R}^3 \to \mathbb{R}^4$

$$T(x,y,z) = (3x + 2y + z,\ x + y + z,\ x - 3y,\ 2x + 3y + z)$$

$$T(1,0,0) = (3,1,1,2)$$

$$T(0,1,0) = (2,1,-3,3)$$

$$T(0,0,1) = (1,1,0,1)$$

$$[T] = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{pmatrix}_{4\times 3}$$

Ex. 1) ?

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^m \arrow[r, "T_2"] & \mathbb{R}^l \end{tikzcd}$$

Compute $[T_2 \circ T_1]$ in terms of $[T_1]$ and $[T_2]$.

Ex. 2) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a L.T.

Show that $\exists\ M \in \mathbb{R}_{\geq 0}$

Such that $\|T(x)\| \leq M\|x\|$.

3 Aug

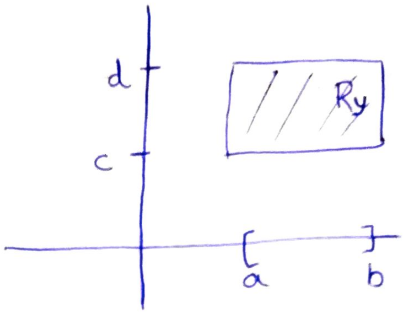
Goal: To characterise Compact sets in $\mathbb{R}^n$.

Outline: 1) open, closed and compact sets in $\mathbb{R}^n$.

2) Heine - Borel Theorem.

3) Tube lemma.

Definitions :-



$R = [a,b] \times [c,d]$ – closed Rectangle in $\mathbb{R}^2$.

$\mathring{R} = (a,b) \times (c,d)$ — open rectangle in $\mathbb{R}^2$.

Defn → 1) $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$.

$a_i < b_i \in \mathbb{R}$.

— a closed rectangle in $\mathbb{R}^n$.

2) $\mathring{R} = (a_1, b_1) \times \cdots \times (a_n, b_n)$

— an open rectangle in $\mathbb{R}^n$.

Defn : 1) A subset $\Omega \subseteq \mathbb{R}^n$ is called open if for every $x \in \Omega$, $\exists$ an open rectangle $\underbrace{x \in \mathring{R}}_{\{\text{open nbd of } x\}} \subseteq \Omega$

2) $K \subseteq \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus K$ is open in $\mathbb{R}^n$.

open :- if $x = (x_1, \dots, x_n) \in \Omega$, then $\exists\, \epsilon_i > 0$

$\Omega \subseteq \mathbb{R}^n$ is open.

if for every $x = (x_1, \dots, x_n) \in \Omega$

$\exists\, \epsilon_i > 0$, $i = 1, 2, \dots, n$ such that

$$(x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2) \times \dots \times (x_n - \epsilon_n, x_n + \epsilon_n) \subseteq \Omega$$

Ex. $\Omega \subseteq \mathbb{R}^n$ is open $\Leftrightarrow \exists\, \epsilon > 0$ s.t.
$(x_1 - \epsilon, x_1 + \epsilon) \times \dots \times (x_n - \epsilon, x_n + \epsilon) \subseteq \Omega$.

Open Ball.

for $x \in \mathbb{R}^n$ and $r > 0$

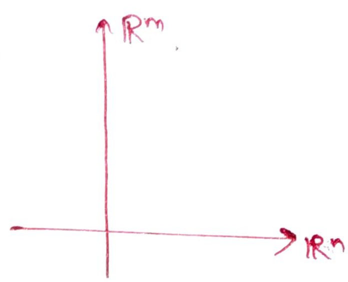
$$B_r(x) = \{ y \in \mathbb{R}^n \mid \|x - y\| < r \}.$$

— open ball with centre at x and radius r.

Ex. Show that $\Omega \subseteq \mathbb{R}^n$ is open $\Leftrightarrow$ for every $x \in \Omega$,
$\exists\, r > 0$ s.t. $B_r(x) \subseteq \Omega$.

Ex. if $U \subseteq \mathbb{R}^n$ is open and $V \subseteq \mathbb{R}^m$ is open.

Then $U \times V \subseteq_{open} \mathbb{R}^n \times \mathbb{R}^m$.
$= \mathbb{R}^{n+m}$



Ques. (Converse part)

$\Omega \subseteq_{open} \mathbb{R}^{n+m}$,

$\exists$? $U \subseteq_{open} \mathbb{R}^n$, $V \subseteq_{open} \mathbb{R}^m$,

s.t. $\Omega = U \times V$?

Not true globally.

Ex. Let $(x,y) \in \Omega \subseteq_{\text{open}} \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

Then show that $\exists$

$x \in U \subseteq_{\text{open}} \mathbb{R}^n$, $y \in V \subseteq_{\text{open}} \mathbb{R}^m$.

such that $U \times V \subseteq \Omega$.

Ex. Show that

1) Union of open sets is open.

2) Intersection of two (hence finitely many) open sets is open.

3) Infinite intersection of open sets is NOT open.

Definition : $A \subseteq \mathbb{R}^n$

1) $\mathcal{O} = \{U_\alpha \subseteq_{\text{open}} \mathbb{R}^n\}_{\alpha \in \Lambda}$ is called an open cover of A.

if $A \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$.

2) if $\mathcal{O}' \subseteq \mathcal{O}$ and is a cover of A, then it is called a subcover.

3) if $\mathcal{O}'$ is finite, then it is called a finite subcover.

Definition : $K \subseteq \mathbb{R}^n$ is called compact if every open cover of K admits a finite subcover.

Eg — 1) Finite sets are compact.

2) Union of two compact sets is compact.

3) Intersection of compact sets is compact.

More examples!

Theorem : [Heine Borel Theorem]

$[a,b]$ is compact.

Proof - Let $\mathcal{O}$ be an open cover of $[a,b]$

To show, $\mathcal{O}$ admits a finite subcover covering $[a,b]$.

$$A := \left\{ x \in [a,b] \;\middle|\; \begin{array}{l} [a,x] \text{ is covered by} \\ \text{finitely many elements in } \mathcal{O} \end{array} \right\}$$

claim 1: $b \in A$.

$$[a,b] \subseteq \bigcup_{\Omega \in \mathcal{O}} \Omega$$

$\Rightarrow \exists\, a \in \Omega_a$.

$\Rightarrow [a, a] \subseteq \Omega_a$.

$\Rightarrow a \in A \neq \phi$.

A is bounded above by b.

By LUB property $\Rightarrow \alpha = \sup A$ exists.

claim 2. $\alpha = b \in A$.

$\alpha \leq b$.

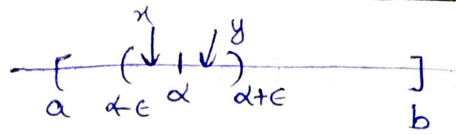
Proof is by contradiction.

Assume $a \leq \alpha < b$

$\alpha \in [a,b]$

$\Rightarrow \exists \quad \alpha \in \underset{\subseteq \text{open } \mathbb{R}}{\Omega_\alpha} \in \mathcal{O}$

$\Rightarrow \exists\, \epsilon > 0$ s.t. $(\alpha - \epsilon, \alpha + \epsilon) \subseteq \Omega_\alpha$



$\Rightarrow$ Pick x and y satisfying

$a \leq x < \alpha \leq y < b$.

$\Rightarrow \alpha$ is supremum

$\Rightarrow x \in A$

$\Rightarrow [a, x]$ is covered by $\Omega_1, \ldots, \Omega_k \in G$

$\Rightarrow [a, y]$ is covered by $\Omega_1, \ldots, \Omega_k, \Omega_\alpha$

$\Rightarrow y \in A$

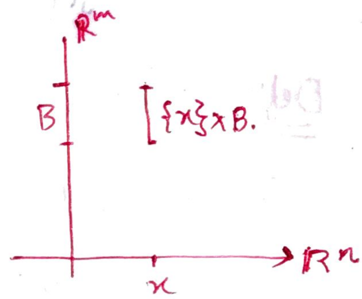
$\alpha < y \quad \Rightarrow \Leftarrow$

(This completes the proof)

<u>Ex.</u> Let $x \in \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be compact

Then show that

$\{x\} \times B$ is compact in $\mathbb{R}^{n+m}$.



<u>Theorem</u>: (Tube Lemma)

If $B \subseteq_{cpt} \mathbb{R}^m$ and G is an open cover of $\{x\} \times B \subseteq \mathbb{R}^{n+m}$. Then, $\exists x \in U \subseteq_{open} \mathbb{R}^n$ such that

$U_x \times B$ is covered by finitely many open sets in G

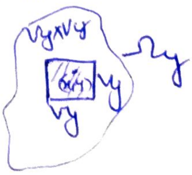
<u>Proof</u>:

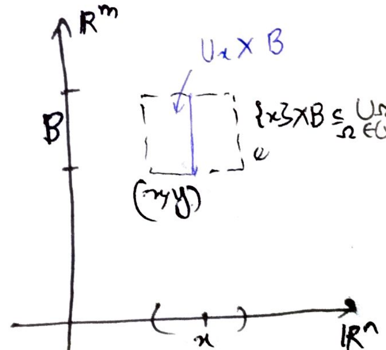
for $(x, y) \in \{x\} \times B$

$\exists (x, y) \in \Omega_y \in G$

$\Rightarrow \exists \underset{x \in}{V_y} \subseteq_{open} \mathbb{R}^n, \underset{y \in}{V_y} \subseteq_{open} \mathbb{R}^m$

s.t. $V_y \times V_y \in \Omega$





$\mathcal{A} = \{V_y \mid y \in B\}$

— an open cover of B.

Compactness of B $\Rightarrow \exists$ finite sub-cover

$V_{y_1}, V_{y_2}, \ldots V_{y_K}$.

$$U = \bigcap_{i=1}^{K} U_{y_i}$$

$U \times V_{y_i} \subseteq U_{y_i} \times V_{y_i} \subseteq \Omega_{y_i}$ {By construction}.

$\cancel{B \subseteq \bigcup_{i=1}^{k} V_{y_i}}$

$$U \times B \subseteq \bigcup_{i=1}^{k} U \times V_{y_i} \subseteq \bigcup_{i=1}^{k} \Omega_{y_i}$$

$\{\Omega_{y_i} \mid i = 1, \ldots k\}$ – finite subcover of $U \times B$. □

Defn $U_x \times B$ is called a tube about $\{x\} \times B$.

Corollary: If $A \subseteq_{cpt} \mathbb{R}^n$ and $B \subseteq_{cpt} \mathbb{R}^m$.

Then, $A \times B \subseteq \mathbb{R}^{n+m}$ is cpt.

Proof: $\mathcal{O}$ is an open cover of $A \times B$.

$x \in A$, $\{x\} \times B$ is covered by $\mathcal{O}$.

Apply tube lemma $\Rightarrow \exists U_x \subseteq_{open} \mathbb{R}^n$ such that $U_x \times B$ is covered by $\mathcal{O}_x \subseteq_{finite} \mathcal{O}$.

$\{U_x \mid x \in A\}$ an open cover of A.

$\Rightarrow$ it has a finite subcover say $U_{x_1}, \ldots, U_{x_k}$.

$U_{x_i} \times B$ is covered by $\mathcal{O}_{x_i} \subseteq_{finite} \mathcal{O}$

$\mathcal{O}' = \bigcup_{i=1}^{k} \mathcal{O}_{x_i} \subseteq_{finite} \mathcal{O}$ is finite subcover of $A \times B$.

□

Assignment-1

Q6. Let $A \subseteq \mathbb{R}^n$ be Compact and $B \subseteq_{closed} \mathbb{R}^n$. Show that if $B \subseteq A$, then $B \subseteq_{cpt.} \mathbb{R}^n$.

Let O be an open cover of B.

Then, $O \cup \underbrace{\{\mathbb{R}^n \setminus B\}}_{\text{open set}}$ is an open cover of A, and has a finite subcover 'O''

$O' / \{\mathbb{R}^n \setminus B\}$ is closed.

Q7. $A \subseteq \mathbb{R}^n$. Show that A is Compact $\Leftrightarrow$ A is closed + bounded.

—x—

Corollary of tube Lemma.

if $A_i \subseteq_{cpt.} \mathbb{R}^{n_i}$ $i = 1, \dots, k$.

Then, $A_1 \times \dots \times A_k \subseteq_{cpt.} \mathbb{R}^N$

$N = \sum_{i=1}^{k} n_i$

Corollary. $[a_1, b_1] \times \dots \times [a_n, b_n] \subseteq_{cpt.} \mathbb{R}^n$.

Assignment-1

Ques 7.

$(\Leftarrow)$ A is closed and bounded

$d(x, 0) \leq M$

$[-M, M] \times [-M, M]$.

A is closed and subset of

by Q6) A is compact.

$(\rightarrow)$ Suppose A is not bounded.

Now, $O = \{B_n(a) : n \in \mathbb{N}\}$

finite subcover of O covering A.

$B_{i_1}(0), B_{i_2}(0), \dots, B_{i_n}(0)$.

A is closed.

$p = A^c$, $r_q = \frac{1}{4} d(p, q) \;\forall\; q \in A$.

$B_{r_q}(p) \cap B_{r_q}(q) = \phi \;\forall\; q \in A$.

$O = \{ B_{r_q}(q) \;\forall\; q \in A \}$.

is open cover for A.

$B_{r_1}(q_1) \cup B_{r_2}(q_2) \cdots B_{r_n}(q_n) \supset A$.

$B_{r_1}(p) \cap B_{r_2}(p) \cdots B_{r_n}(p) \subseteq A^c$.

$\Rightarrow A^c$ is open.

Ques 5. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ L.T.

Then,

show that $\exists\; M \in \mathbb{R}_{\geq 0}$ such that

$\|T(x)\| \leq M \|x\|$.

Sol. $\|T(x)\| = \|T(x_1, x_2, \ldots, x_n)\|$

$= \|T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)\|$

where $e_i = (0, \ldots 0, 1, \ldots 0)$

$= \|x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)\|$

$\leq \|x_1 T(e_1)\| + \|x_2 T(e_2)\| + \cdots \|x_n T(e_n)\|$

$= \|x_1\| \cdot \|T(e_1)\| + \|x_2\| \|T(e_2)\| + \cdots + \|x_n\| \|T(e_n)\|$

$m = \max \{ \|T(e_i)\| \mid i = 1, \ldots, n \}$.

$\leq m(|x_1| + |x_2| + \cdots + |x_n|)$

$|x_1| \leq \|x\|$ $\Rightarrow |x_1| + \cdots + |x_n| \leq n\|x\|$

$\vdots$

$|x_n| \leq \|x\|$

$\leq mn \|x\| = M\|x\|$

where $M = mn$.

Matrix representation

$$[T] = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times n}$$

$$R_i = (a_{i1}, \cdots a_{in})$$

$$\|T(x)\| = \|(\langle R_1, x\rangle, \cdots, \langle R_m, x\rangle)\|$$

$$= \left(\sum_{i=1}^{m} |\langle R_i, x\rangle|^2\right)^{1/2} \leq \left(\sum_{i=1}^{m} (\|R_i\| \cdot \|x\|)^2\right)^{1/2}$$

$$\|T(x)\| \leq \|x\| \cdot \underbrace{\left(\sum_{i=1}^{m} \|R_i\|^2\right)^{1/2}}_{M}$$

$$\leq M \cdot \|x\|.$$

#

5/Aug/22.

1) cts fn $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$
2) Ch. Theorem of cts fn
3) Action of cts fn on compact sets.
4) Derivative of $f: \Omega \subseteq \mathbb{R}^n \to$, Ω open

Defn: $U_x \times B$ is called a tube about $\{x\} \times B$.

Ex. Show that in tube lemma compactness of B is essential

Corollary: 1) $A \subseteq_{cpt.} \mathbb{R}^n$, $B \subseteq_{cpt.} \mathbb{R}^m$

$\Rightarrow A \times B \subseteq_{cpt.} \mathbb{R}^{n+m}$

2) $A_i \subseteq_{cpt.} \mathbb{R}^{n_i}$, $i = 1, \dots, k$

$\Rightarrow A_1 \times \dots \times A_k \subseteq_{cpt.} \mathbb{R}^{\sum_{i=1}^{k} n_i}$

3) Closed rectangle in $\mathbb{R}^n$ are compact.

Theorem: (Characterization Theorem of compact sets in $\mathbb{R}^n$)

$A \subseteq_{cpt.} \mathbb{R}^n \Leftrightarrow A$ is closed and bounded.

Definitions:-

1) Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

for $C \subseteq \mathbb{R}^m$,

$f^{-1}(C) = \{x \in \Omega \mid f(x) \in C\}$

— preimage of C under f.

2) Graphs of function:

$f: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$.

$G(f) = \{(x, y, f(x,y)) \mid (x,y) \in \Omega\}$

— graph of fn.

Similarly, one can define graph of

$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$.

3) Component function

$f : \Omega \to \mathbb{R}^m$

The f^ns ~~iff~~ $\exists$ $f_i : \Omega \to \mathbb{R}$ $i = 1, \ldots, m$ s.t.

$f(x) = (f_1(x), \ldots, f_m(x))$

Then,

f_i's are called Component function of f

f_i's is ith Component function of f.

4) $\pi_i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ defined by $\pi(x_1, \ldots, x_n) = x_i$ called i^{th} projection function.

$$\begin{tikzcd} \Omega \arrow[r, "f"] \arrow[rr, bend right, "\pi_i \circ f"'] & \mathbb{R}^m \arrow[r, "\pi_i"] & \mathbb{R} \end{tikzcd}$$

Ex. Show that $\pi_i \circ f$ is the i^{th} component function of f.

Def^n 1) Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$\lim_{x \to a} f(x) = b$ if for given

$\epsilon > 0$, $\exists$ $\delta > 0$ s.t.

$\|f(x) - b\| < \epsilon$ for all

$x \in \Omega$, $0 < \|x - a\| < \delta$.

2) Let $a \in \Omega$, f is cts. at a if $\lim_{x \to a} f(x) = f(a)$ i.e.,
f is cts. at a if for given $\epsilon > 0$, $\exists$ $\delta > 0$ s.t.
$\|f(x) - f(a)\| \leq \epsilon$ $\forall$ $x \in \Omega$ s.t. $\|x - a\| < \delta$.

3) f is cts. on Ω if it is cts. at every $a \in \Omega$.

Examples:-

1) $b \in \mathbb{R}^m$

$\Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = b \quad \forall\, x \in \Omega.$

is cts. on Ω.

2) $id : \mathbb{R}^n \to \mathbb{R}^n \qquad id(x) = x \quad \forall\, x \in \mathbb{R}^n$

is cts. on $\mathbb{R}^n$.

3) $i : \mathbb{R}^n \to \mathbb{R}^m \qquad m \geq n$

$i(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots x_n, 0 \dots 0)$

is cts.

{inclusive function}.

4) $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ cts.

$\Rightarrow f \pm g : \Omega \to \mathbb{R}^m.$

$(f \pm g)(x) = f(x) \pm g(x)$ is cts.

5) $f, g : \Omega \to \mathbb{R}$ cts. and

$g(x) \neq 0 \;\forall\; x \in \Omega$

$\frac{f}{g} : \Omega \to \mathbb{R}.$

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is cts.

6) (Projection function)

$\pi_i \,;\, i = 1, \dots, m$ is cts.

Theorem : A function $f : \Omega \to \mathbb{R}^m$ is cts. in Ω iff for every $U \subseteq_{open} \mathbb{R}^m$, $\exists\ V \subseteq_{open} \mathbb{R}^n$ such that

$f^{-1}(U) = V \cap \Omega$.

Proof:- Assume f is cts and $U \subseteq_{open} \mathbb{R}^m$ given

Let $a \in f^{-1}(U)$

$\Rightarrow f(a) \in U \subseteq_{open} \mathbb{R}^m$

By defn of open set,

$\exists\ \epsilon_a > 0$ such that $B_{\epsilon_a}(f(a)) \subseteq U$

As f is cts at $a \in \Omega$

For this $\epsilon_a > 0$, $\exists\ \delta_a > 0$ such that

$$\| f(x) - f(a) \| < \epsilon_a$$
$$\forall\ x \in \Omega,\ \|x - a\| < \delta_a$$

$$\equiv \forall\ x \in \Omega \cap B_{\delta_a}(a)$$

$$V \equiv \bigcup_{a \in f^{-1}(U)} B_{\delta_a}(a)$$ — this will work.

Ex. Show that $V \cap \Omega = f^{-1}(U)$.

Conversely,

$a \in \Omega$

Let $\epsilon > 0$ be given.

$f(a) \in B_\epsilon(f(a)) = U \subseteq_{open} \mathbb{R}^m$.

$\Rightarrow \exists\ V \subseteq_{open} \mathbb{R}^n$ s.t.

$f^{-1}(U) = V \cap \Omega$.

$a \in$

$a \in V \subseteq_{open} \mathbb{R}^n$

By defn of open subsets in $\mathbb{R}^n$.

$\Rightarrow \exists\ \delta > 0$ such that $B_\delta(a) \subseteq V$.

Ex. Show that $\|f(x)-f(a)\| < \epsilon \ \forall \ x \in \Omega$, with $\|x-a\| < \delta$.

Action of cts function on Compact sets.

Theorem : Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $A \subseteq_{cpt.} \mathbb{R}^n$, then $f(A) \subseteq_{cpt.} \mathbb{R}^m$.

Proof :- Let $\mathcal{O} = \{ U_\alpha \subseteq_{open} \mathbb{R}^m \mid \alpha \in \Lambda \}$ be cover of $f(A)$.

By characterization theorem.

$$\mathcal{O}' = \{ f^{-1}(U_\alpha) \subseteq_{open} \mathbb{R}^n \mid \alpha \in \Lambda \}$$
— open cover of A.

A is compact

$\Rightarrow \exists$ a finite subcover i.e.

$\{ f^{-1}(U_{\alpha_i}), i = 1, \ldots, k \}$.

$\Rightarrow \{ U_{\alpha_i} \mid i = 1, \ldots, k \}$ = cover of $f(A)$.

Recall,

Let $f : \mathbb{R} \to \mathbb{R}$.

$a \in \mathbb{R}$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exist} = f'(a). \quad \text{—(1)}.$$

Note that Eq(1) doesn't making sense for a fn.

$f : \mathbb{R}^n \to \mathbb{R}^m$

$T: \mathbb{R} \to \mathbb{R}$ L.T.

$T(h) = f'(a) \cdot h.$ $\longrightarrow$ linear

$$(1) \equiv \lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0. \leftarrow$$

<u>Defn</u>: A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$. if $\exists$ L.T. $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\|h\|} = 0$$

<u>Outline</u>

1) Differentiation of $f: \Omega \underset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ and uniqueness theorem.

2) Chain rule.

3) Properties of derivatives.

Recall, $f: \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is diff. at a pt. $a \in \mathbb{R}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

<u>Defn.</u> A function $f: \Omega \subseteq_{\text{open}} \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \Omega$, if $\exists$ $T: \mathbb{R}^n \to \mathbb{R}^m$ linear s.t.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\|h\|} = 0.$$

Notation:

$$df_a = T \qquad Df_a \quad , \quad f'(a).$$

Theorem ① (Uniqueness)

Let $f : \Omega \subseteq_{\text{open}} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be diff^n at $a \in \Omega$

Then, $\exists$ unique $T : \mathbb{R}^n \to \mathbb{R}^m$ linear such that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|} = 0$$

Proof: Let $\exists$ another $\tilde{T} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying Theorem ①.

$$T(h) - \tilde{T}(h) = \| [f(a+h) - f(a) - \tilde{T}(h)] - [f(a+h) - f(a) - T(h)] \|$$

$$\leq \| f(a+h) - f(a) - \tilde{T}(h) \| + \| f(a+h) - f(a) - T(h) \|.$$

$$\Rightarrow \frac{\| T(h) - \tilde{T}(h) \|}{\| h \|} \leq \lim_{h \to 0} \frac{\| f(a+h) - f(a) - \tilde{T}(h) \|}{\| h \|} - \lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|}$$

$$\Rightarrow \lim_{h \to 0} \frac{\| T(h) - \tilde{T}(h) \|}{\| h \|} = 0 \quad \text{—— (2)}$$

Now, $x \neq 0 \in \mathbb{R}^n$

$$(2) \Rightarrow \lim_{t \to 0} \frac{\| T(tx) - \tilde{T}(tx) \|}{\| tx \|} = 0$$

$$\Rightarrow \| T(x) - \tilde{T}(x) \| = 0$$

$$\Rightarrow T(x) = \tilde{T}(x) \,//$$

$\square$

Eg:- $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \cos x$.

$df_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}$. 1×2

$$\begin{bmatrix} -\sin x & 0 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$$

$df_{(a,b)}(h,k) = -\sin a \cdot h$

Ex. Show that, if $f: \Omega \subseteq$ open $\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \Omega$, then f is cts. at a.

Theorem -(2)

Chain rule.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^l \end{tikzcd}$$

$a \to f(a) = b$

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^l$ be diff^{n}ble at a and $f(a) = b$ respectively. Then, $g\circ f: \mathbb{R}^n \to \mathbb{R}^l$ is diff^{n}ble at a and $d(g\circ f)_a = dg_b \circ df_a$.

Proof. Let $T_f = df_a$, $T_g = dg_b$.

Define

$$\varphi(x) = f(x) - f(a) - T_f(x-a). \quad \forall\ x \in \mathbb{R}^n. \quad \text{— (1)}$$

$$\psi(y) = g(y) - g(b) - T_g(y-b) \quad \forall\ y \in \mathbb{R}^m. \quad \text{— (2)}$$

$$\mathcal{J}(x) = gf(x) - gf(a) - T_g \circ T_f(x-a) \quad \text{— (3)}$$

$$\lim_{x\to a} \frac{\|\varphi(x)\|}{\|x-a\|} = 0 \quad \text{— (4)}$$

$$\lim_{y\to b} \frac{\|\psi(x)\|}{\|y-b\|} = 0 \quad \text{— (5)}$$

To show.

$$\lim_{x\to a} \frac{\|\mathcal{J}(x)\|}{\|x-a\|} = 0$$

$$J(x) = g(f(x)) - g(b) - T_g(f(x) - f(a) - \varphi(x))$$
$$= g(f(x)) - g(b) - T_g(f(x) - b) + T_g(\varphi(x))$$

$$\psi(f(x)) + T_g(\varphi(x))$$

$$\Rightarrow \frac{\|J(x)\|}{\|x-a\|} \leq \frac{\|\psi(f(x))\|}{\|x-a\|} + \frac{\|T_g(\varphi(x))\|}{\|x-a\|}$$

<u>claim</u> 1: $\lim_{x \to a} \frac{\|\psi(f(x))\|}{\|x-a\|} = 0.$

<u>claim</u> 2: $\lim \frac{\|T_g(\varphi(x))\|}{\|x-a\|} = 0$

<u>Proof of claim 2</u>

$$\frac{\|T_g(\varphi(x))\|}{\|x-a\|} \leq M_g \frac{\|\varphi(x)\|}{\|x-a\|}$$

Eqⁿ (4) $\Rightarrow$ claim (2).

Let $\epsilon > 0$

(5) $\Rightarrow \exists\ \delta' > 0$

s.t. $\frac{\|\psi(f(x))\|}{\|f(x) - b\|} < \epsilon$ when $\|f(x) - f(a)\| < \delta'$

for $\delta' > 0$, $\exists\ \delta$ s.t.

$\|x-a\| < \delta \Rightarrow \|f(x) - f(a)\| < \delta'$

$\Rightarrow \|\psi(f(x))\| < \epsilon \| f(x) - b\|$

$\|\psi(f(x))\| < \epsilon \|\varphi(x) + T_f(x-a)\|$

$\leq \epsilon \|\varphi(x)\| + \epsilon \|T_f(x-a)\|$

$\Rightarrow \frac{\|\psi \cdot f(x)\|}{\|x-a\|} < \epsilon \cdot \frac{\|\varphi(x)\|}{\|x-a\|} + \epsilon \cdot M_f.$

$\frac{\|T_f(x-a)\|}{\|x-a\|} \leq M_f.$

$\Rightarrow \lim_{x \to a} \frac{\|\psi f(x)\|}{\|x-a\|} = 0.$

Theorem: $f : \Omega \subseteq_{\text{open}} \mathbb{R}^n \to \mathbb{R}^m$.

$$f(x) = b \quad \forall\ x \in \Omega$$

$$df_a = 0.$$

Theorem: if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, then

$$dT_a = T.$$

Corollary: 1) $d\pi_i = \pi_i$

2) $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$.

$$S(x, y) = x + y$$

$$dS_{(a,b)} = S.$$

Theorem: Let $f : \Omega \subseteq_{\text{open}} \mathbb{R}^n \to \mathbb{R}^m$.

$$f(x) = (f_1(x), \ldots, f_m(x)).$$

f is diff^{n}ble at $a \in \Omega$

$\Leftrightarrow$.

then $f_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, m$ are diff'ble at a.

Proof

($\Rightarrow$) Assume f is diff. at $a \in \Omega$.

$$f_i = \pi_i \circ f.$$

($\Leftarrow$) Conversely, f_i, $i = 1, \ldots, m$ are diff^{n}ble at $a \in \Omega$

$$T = \left((df_1)_a, \ldots, (df_m)_a \right) : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(h) = \left((df_1)_a (h), \ldots, (df_m)_a (h) \right)$$

$$\| f(a+h) - f(a) - T(h) \|$$

$$\Rightarrow \| f_1(a+h) - f_1(a) + (df_1)_a(h), \ldots, f_m(a+h) - f_m(a) - (df_m)_a(h) \|$$

$$= \sum_{i=1}^{m} |f_i(a+h) - f_i(a) - (df_i)_a(h)|$$

$$= \lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|} \leq \sum_{i=1}^{m} \lim_{h \to 0} \frac{|f_i(a+h) - f_i(a) - (df_i)_a(h)|}{\| h \|}$$

$$= 0$$

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be diff^{n}ble at $a \in \mathbb{R}^n$

1) $d(f \pm g)_a = df_a \pm dg_a$.

2) $d(f \cdot g)_a = f(a) \cdot dg_a + g(a) \cdot df_a$.

3) if $g(a) \neq 0$.

$$d\left(\frac{f}{g}\right)_a = \frac{g(a)\, df_a - f(a) \cdot dg_a}{[g(a)]^2}$$

4) Compute derivative of $\langle , \rangle$. $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

$$(x, y) \longrightarrow \langle x, y \rangle.$$

Partial derivatives :-

$f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$.

Defn If $\lim_{h \to 0} \frac{f(a+he_j) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{F(a_1, \ldots, a_j, a_j + h, a_{j+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

exists.

Then, we say that f is partial diff. at a w.r.t the j^{th} variable.

Denote it by $D_j f(a)$ or $\frac{\partial f}{\partial x_j}(a)$.

Geometrical Interpretation

Define $g_j = (a_j - \epsilon, a_j + \epsilon) \longrightarrow \mathbb{R}$ by

$g_j(x) = f(a_1, \ldots a_{j-1}, x, a_{j+1}, \ldots, a_n)$

$\frac{dg_j}{dx}(a_j) = D_j f(a)$.

Graph of g_j is $\underbrace{\{(x, f(x)) \mid x \in \Omega\}}_{\subseteq \mathbb{R}^{n+1}} \cap \left\{(\overset{\Omega}{\downarrow}{x}, \overset{\mathbb{R}}{\downarrow}{y}) \,\middle|\, \begin{array}{l} x = (x_1, \ldots, x_n) \\ x_i = a_i \ \forall i \neq j \\ y, x_j \in \mathbb{R}. \end{array}\right\}$

$D_j f(a)$ is ~~is~~ the slope ~~of~~ to the curve at the point $(a, f(a))$.

Suppose $f : \Omega \subseteq_{\text{open}} \mathbb{R}^n \to \mathbb{R}$.

Such that $D_j f(a)$ exists for every $a \in \Omega$.

$D_j f : \Omega \to \mathbb{R}$ is defined

$D_i(D_j f)(a)$ is similarly defined

$D_{j,i} f(a)$ - 2^{nd} order partial derivative

Higher order partial derivatives defined similarly.

Defn:

1) Suppose $f : \Omega \subseteq_{open} \mathbb{R}^n \to \mathbb{R}$.

Such that $D_j f$ exists and cts. in an open nbd of a.

Then, we say that f is ~~cts~~ ~~cts~~ C^1 or cts. diff'ble at a.

2) If the partial derivatives of all order exists and cts. on Ω, then f is called smooth or C^∞ function.

3) A function $f : \Omega \to \mathbb{R}^m$ is smooth if for all $i = 1, \ldots, m$, $f_i : \Omega \to \mathbb{R}$ are smooth.

Example. 1) Constant function.

2) Polynomial function

3) $f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} \exp\left(\frac{-1}{1-x^2}\right) & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$\underbrace{\operatorname{Supp}(f)}_{\text{Support of } f} = \overline{\{x \mid f(x) \neq 0\}} = [-1, 1]$$

Theorem: Let $f : \Omega \subseteq_{open} \mathbb{R}^n \to \mathbb{R}$ is C^1

if f is max (or min) at $a \in \Omega$, then

$$D_j f(a) = 0 \quad \forall j \in \{1, \ldots, n\}$$

Proof: If $a \in \Omega$

$\exists \epsilon > 0$ such that $(a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon) \subseteq \Omega$

for $1 \leq j \leq n$, define $g_j : (a_j - \epsilon, a_j + \epsilon) \to \mathbb{R}$

by $g_j(x) = f(a_1, \ldots a_{j-1}, x, a_{j+1}, \ldots, a_n)$

has max (or min) at a_j.

$\frac{dg_j}{dx}(a_j) = 0 \quad \Rightarrow \quad D_j f(a) = 0$

Ques: What about the Converse of Theorem (1) ?

NOT TRUE.

Eg (1) $f(x) = x^3$

Eg. (2) $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$g(x, y) = x^2 - y^2$.

$f : \Omega \longrightarrow \mathbb{R}^m$

$x \longmapsto f(x) = (f_1(x), \ldots, f_m(x))$

Notation $f'(a)$ is the matrix of df_a w.r.t. standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Recall:

Theorem: f is diff^n. at a, iff f_i, $i = 1, \ldots, m$ are diff^n ble at a

$$df_a = \left((df_1)_a, \ldots, (df_m)_a\right).$$

$(df_i)_a : \mathbb{R}^n \longrightarrow \mathbb{R}$. linear

$df_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad (df_a)(h) = \left((df_1)_a(h), \ldots, (df_m)_a(h)\right)$

$$f'(a) = \begin{pmatrix} f_1'(a) \\ \vdots \\ f_m'(a) \end{pmatrix}$$

Theorem: if $f : \Omega \longrightarrow \mathbb{R}^m$

$x \longmapsto f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$

is differentiable at a. Then $D_j f_i(a)$ exists for every $1 \le i \le m$ and $1 \le j \le n$ and $f'(a) = (D_j f_i(a))_{m \times n}$

Proof: Consider $m = 1$. s.t. $f : \Omega \longrightarrow \mathbb{R}$. $(a \in \Omega)$

Define $h : (a_j - \epsilon, a_j + \epsilon) \longrightarrow \cancel{\mathbb{R}^n}, \Omega \xrightarrow{f} \mathbb{R}$

$h(x) = (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n)$

$f \circ h(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n)$.

$$D_j f(a) = (f \circ h)'(a_j) = f'(h(a_j))\, h'(a_j)$$
$$= f'(a) \cdot e_j^t \qquad \square$$

Theorem: (Sufficient Condition for differentiability)

Let $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$

Then, df_a exists if

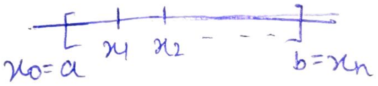
$D_j f_i$, $1 \le i \le m$, $1 \le j \le n$ exist and continuous on open nbd of a.

Remark: Converse of this theorem is not true.

Assignment-1

Ex-2 $$\left| \int_a^b f \cdot g \right| \le \left(\int_a^b f^2 \right)^{1/2} \cdot \left(\int_a^b g^2 \right)^{1/2}$$

i.e., $$\left| \int_a^b f \cdot g \right|^2 \le \left(\int_a^b f^2 \right) \cdot \left(\int_a^b g^2 \right)$$



$a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$.

$I_i = [x_{i+1}, x_i]$, $i = 1, \ldots, n$

Tagged partition, $t_i \in I_i$

$\{P, t_1, \ldots, t_n\} \to$ tagged partition

$\sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i)$ — Reimann sum

$U(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in I_i} f(x)$ Reimann upper sum

$L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in I_i} f(x)$ Lower reimann sum

$$\overline{\int_a^b} f = \inf_P U(f,P) \qquad \underline{\int_a^b} f = \sup_P L(f,P)$$

Defn : f is reimann integrable if

$$\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f .$$

$$L(f,P) \leq \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i) \leq U(f,P).$$

Now,

$$\sum_{i=1}^{n} \left| (x_i - x_{i-1}) f(t_i) g(t_i) \right| = \sum_{i=1}^{n} \left| \sqrt{x_i - x_{i-1}}\, f(t_i) . \sqrt{x_i - x_{i-1}}\, g(t_i) \right|$$

$$\leq \left(\sum_{i=1}^{n} f(t_i)^2 (x_i - x_{i-1}) \right)^{1/2} \left(\sum_{i=1}^{n} g(t_i)^2 (x_i - x_{i-1}) \right)^{1/2}$$

using Cauchy Schwarz inequality

$$\leq \left(\int_a^b f^2 \right)^{1/2} . \left(\int_a^b g^2 \right)^{1/2}$$

Assignment- 2

Q7. $f(x) = \begin{cases} e^{-1/x^2} , & x \neq 0 \\ 0 , & x = 0 . \end{cases}$

(#) $P : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$P(x,y) = xy$

Show that $dP(a,b) : \mathbb{R}^2 \to \mathbb{R}.$

$$dP(a,b)(h,k) = bh + ak$$

we want to prove,

$$\lim_{(h,k)\to(0,0)} = \frac{P(x+h,\ y+k) - P(x,y) - \lambda(h,k)}{\sqrt{h^2+k^2}}$$

$$= \frac{(x+h)(y+k) - xy - hy - kx}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} \qquad \frac{|h|}{\sqrt{h^2+k^2}} \leq 1$$

$$0 \leq \frac{|hk|}{\sqrt{h^2+k^2}} \leq \cancel{|k|}\ 0.$$

(#) $f : E_1 \times \cdots \times E_k \to \mathbb{R}^p$

where $E_i = \mathbb{R}^{n_i}$ for some $n_i \in \mathbb{N}$ is called multilinear

if for every $x_i \in E_i$, $i \neq j$.

$$g(x) = f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_k).$$

– linear for every $j = 1, \ldots, k$.

$g : E_j \to \mathbb{R}^p.$

$(f_1, f_2, \ldots f_m)$

Lemma: Let $f : U \to \mathbb{R}^m$ be C^1.

$U \subseteq \mathbb{R}^n$ — a closed rectangle.

if $\exists M \in \mathbb{R}$ s.t. $|D_j f_i(x)| \leq M \ \forall x \in U.$

$1 \leq i \leq m, \quad 1 \leq j \leq n$, then

$$\|f(x) - f(y)\| \leq M \cdot m \cdot n \ \|x-y\|$$

Inverse Function Theorem :

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open nbd of a and $\det(f'(a)) \neq 0$, then

$\exists\, V \subseteq_{open} \mathbb{R}^n$ containing a & $W \subseteq_{open} \mathbb{R}^n$ containing $f(a)$ such that

1) $f : V \to W$ is bijective (invertible)

2) $f^{-1} : W \to V$ is C^1.

3) $(f^{-1})'(y) = \left[f'(f^{-1}(y))\right]^{-1}$

Proof of Lemma :- $f = (f_1, \dots f_m)$

$$f_i(y) - f_i(x)$$

$$= f_i(y_1, \dots y_n) - f_i(x_1, x_2, \dots, x_n)$$

$$= f_i(y_1, \dots, y_{n}) - f_i(y_1, \dots, y_{n-1}, x_n)$$

$$+ f_i(y_1, \dots y_{n-1}, x_n) - f_i(y_1, \dots y_{n-2}, x_{n-1}, x_n)$$

$$+ f_i(y_1, x_2, \dots x_n) - f_i(x_1, \dots, x_n)$$

$$\Rightarrow |f_i(y) - f_i(x)| \leq \sum | f_i(y_1, \dots y_{j-1}, y_j, x_{j+1}, \dots x_n) - f_i(y_1, \dots y_{j-1}, x_j, \dots, x_n)|$$

(used MVT)

$$= \sum_{j=1}^{n} |y_j - x_j| \, | D_j f_i(z_{i,j})|$$

where $z_{i,j} = (y_1, \dots y_{j-1}, d_j, x_{j+1}, \dots x_n)$

$d_j \in (y_j, x_j)$ or (x_j, y_j).

$$\leq M \sum_{j=1}^{n} |y_j - x_j|$$

$$\leq M \cdot n \, \|y - x\|$$

$$|f_i(y) - f_i(x)| \leq M \cdot n \, \|y - x\|$$

Since for each j $|y_j - x_j| \leq \|y - x\|$

$$\Rightarrow \|f(y) - f(x)\| \leq M \cdot m \cdot n \, \|y - x\|$$

Proof of IFT.: $T = f'(a)$

then IFT holds true for $f \Leftrightarrow$ IFT holds true for $T^{-1} \circ f$. Consider $T = I_d$.

finding V and W.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|} = 0$$

If $f(a+h) = f(a)$ for $h \in \mathbb{R}^n \setminus \{0\}$

$$= \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|} = \frac{\| h \|}{\| h \|} = 1.$$

$\Rightarrow \exists$ closed rectangle U about a such that $f(a+h) \neq f(a)$ $\forall$ $h \in \mathbb{R}^n \setminus \{0\}$.

s.t. $a+h \in U$

$\det(f'(a)) \neq 0$

Can consider U such that $\det(f_i'(x_i)) \neq 0$ $\forall x \in U$.

$$|D_j f_i(x) - D_j f_i(a)| \leq \frac{1}{2n^2} \quad \forall x \in U$$

Step

(1) $T = I_d$

(2) $a \in U \subseteq$ open rect. $\mathbb{R}^n$.

(a) $f(a) \neq f(a+h)$ $\forall$ $h \in \mathbb{R}^n \setminus \{0\}$ with $a+h \in \overline{U}$

(b) $\det(f'(x)) \neq 0$ $\forall x \in U$.

(c) $|D_j f_i(x) - D_j f_i(a)| \leq \frac{1}{2n^2}$ $\forall x \in U$.

(d) $V = f^{-1}(W) \cap U$.

Claim-1: f is injective on U.

Apply lemma on $g : U \to \mathbb{R}^n$, $g(x) = f(x) - x$

$$|D_j g_i(x)| = |D_j f_i(x) - \delta_{ij}|$$

$$= |D_j f_i(x) - D_j f_i(a)| \le \frac{1}{2n^2} \ \forall x \in U.$$

$$\|g(x) - g(y)\| \le \frac{1}{2n^2} \cdot n^2 \cdot \|x-y\|$$

$$\Rightarrow \|g(x) - g(y)\| \le \frac{\|x-y\|}{2}.$$

$$\|f(x) - x - f(y) + y\| \le \frac{\|x-y\|}{2}$$

$$\le$$

$$\|x-y\| - \|f(x) - f(y)\| \le \frac{\|x-y\|}{2}$$

$$\Rightarrow \frac{\|x-y\|}{2} \le \|f(x) - f(y)\|$$

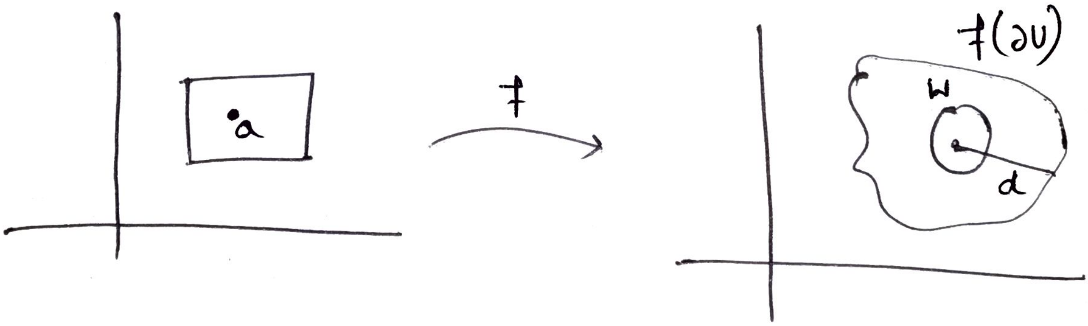
$\Rightarrow f$ is injective on U.

from (1) $\Rightarrow$.

$f(a) \neq f(x) \quad \forall x \in \partial U$ cpt. in $\mathbb{R}^n$.

$\Rightarrow \exists\, d > 0$ s.t.

$\|f(x) - f(a)\| \ge d$



$$W = B_{\frac{d}{2}}(f(a)) = \{ y \in \mathbb{R}^n \mid \|f(a) - y\| < \tfrac{d}{2} \}$$

claim 2: for every $y \in W$, $\exists$ unique $x \in U$, $\ni$ $f(x) = y$

$\|f(x) - y\| \geq \|f(a) - y\|$.

Consider, $h : \bar{U} \longrightarrow \mathbb{R}$.

$$h(x) = \|f(x) - y\|^2 = \sum_{i=1}^{n} \left(f_i(x) - y_i\right)^2.$$

Let the point of min. is $x_y \in \bar{U}$.

$\left(\text{As } \|f(x) - y\| \geq \|f(a) - y\|\right)$

$\Rightarrow x_y \in U$

$\Rightarrow D_j h(x_y) = 0$

$\Rightarrow \sum_{i=1}^{n} 2\left(f_i(x_y) - y_i\right) D_j f_i(x_y) = 0$

$\Rightarrow f'(x_y)\left(f(x_y) - y\right) = 0$

$\Rightarrow f(x_y) = y$. [Since $f'(x_y) \neq 0$]

Ex. Show uniqueness.

17/Aug/2022

Outline :

1) A sufficient Condition for differentiability of $f : \Omega \subseteq_{open} \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

2) Inverse function theorem.

Theorem: Let $f = (f_1, \dots, f_m) : \Omega \subseteq_{open} \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \Omega$. f is diff'ble at a, if $D_j f_i$ $1 \le i \le m$, $1 \le j \le n$. exist in an open ~~nbhd~~ nbhood of a and are continuous at a.

Proof: f is diff'ble at $a \Leftrightarrow f_i : \Omega \to \mathbb{R}$, $i = 1, \dots, m$ are diff'ble at a.

Consider $m = 1$, so that $f : \Omega \to \mathbb{R}$.

$a = (a_1, \dots, a_n)$.
$h = (h_1, \dots, h_n)$.

$$f(a+h) - f(a) = f(a_1+h_1, a_2, \dots, a_n) - f(a_1, a_2, \dots, a_n)$$
$$+ \left(f(a_1+h_1, a_2+h_2, a_3, \dots, a_n) - f(a_1+h_1, a_2, \dots, a_n) \right)$$
$$\vdots$$
$$f(a_1+h_1, a_2+h_2, a_3+h_3 + \dots + a_n+h_n) -$$
$$f(a_1+h_1, \dots a_{n-1}+h_{n-1}, a_n).$$

$$= h_1 \cdot D_1 \underbrace{(c_1, a_2, \dots, a_n)}_{f_1} + h_2 \cdot D_2 f(a_1+h, \underbrace{c_2, a_3, \dots, a_n}_{f_2})$$

$$+ \dots + h_n \cdot f(a_1+h, \dots, \underbrace{a_{n-1}+h_{n-1}, c_n}_{f_n})$$

where $a_j \le c_j \le a_j + h_j$.

$h \to 0, \Rightarrow d_j \to a.$

$$f(a+h) - f(a) = \sum_{j=1}^{n} h_j D_j f(d_j).$$

$$\left| f(a+h) - f(a) - \sum_{j=1}^{n} D_j f(a) h_j \right|$$

$$= \left| \sum_{j=1}^{n} h_j (D_j f(d_j) - D_j (f(a)) \right|.$$

$$\leq \sum_{j=1}^{n} \left| D_j f(d_j) - D_j f(a) \right| |h_j|.$$

$$\Rightarrow \lim_{h \to 0} \frac{\left| f(a+h) - f(a) - \sum_{j=1}^{n} D_j f(a) - h_j \right|}{\|h\|} \leq \sum_{j=1}^{n} \lim_{h \to 0} \left| D_j f(d_j) - D_j f(a) \right| = 0$$

continuous.

Note : Converse of the above theorem is not true.

Eg :- $f(x,y) = \begin{cases} (x^2+y^2) \sin \left(\frac{1}{\sqrt{x^2+y^2}} \right) ; & x,y \neq (0,0) \\ 0 & , (x,y) = (0,0). \end{cases}$

Lemma :- Let $f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ be C^1 and $M = \sup \{ | D_j f_i (x) | \mid x \in U \}$. Then,

$$\| f(x) - f(y) \| \leq M \, m \cdot n \, . \, \| x - y \| .$$

Theorem (IFT) Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open nbhood of a and $\det(f'(a)) \neq 0$.

Then, $\exists\, V \subseteq_{open} \mathbb{R}^n$ ($a \in V$) and $W \subseteq_{open} \mathbb{R}^n$ ($f(a) \in W$) such that

1) $f: V \to W$ is invertible.
2) $f^{-1}: W \to V$ is C^1.
3) for $y \in W$, $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$.

$f \cdot f^{-1} = Id.$

Proof. $T = f'(a)$

IFT holds for $f \Leftrightarrow$ IFT holds for $T^{-1} \circ f$

Consider $T = I_{n \times n}$

finding V and W.

$a \in U \subseteq_{open} \mathbb{R}^n$.
rectangle

1) $f(a) \neq f(a+h)$ $\forall\, h \in \mathbb{R}^n$, $a+h \in \bar{U}$

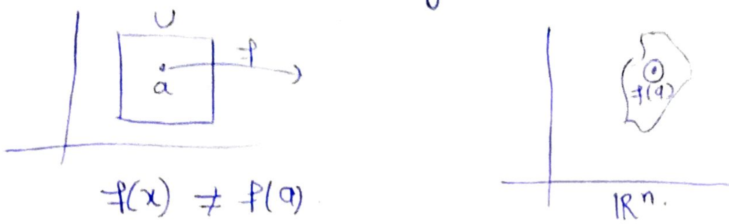
2) $\det(f'(x)) \neq 0$ $\forall\, x \in U$
$\Downarrow$
contn.

3) $|D_j f_i(x) - D_j f_i(a)| \leq \frac{1}{2n^2}$ $\forall\, x \in U$

Claim 1. f is injective on U.

To prove claim, apply lemma (1) on $g(x) = f(x) - x$.

$$\|x-y\| \leq 2\,\|f(x)-f(y)\|$$
$$\forall x,y \in U.$$



$f(x) \neq f(a)$

$\forall x \in \partial U.$

$\exists\, d>0,\ \|f(x)-f(a)\| \geq d$

$$W = B_{\frac{d}{2}}(f(a))$$

$\left\{ \begin{array}{l} \text{for every } y \in W, \exists \text{ unique } x \in U \\ \text{Such that } f(x) = y. \end{array} \right.$

$$V = f^{-1}(W) \cap U.$$

By Construction $f : V \to W$ is invertible.

$f^{-1} : W \to V. = f^{-1}(W) \cap U.$

We will prove,

1) f^{-1} is cts.

2) f^{-1} is C^1.

Proof of (1)

Let $y_1, y_2 \in W$.

$f^{-1}(y_1) = x_1 \quad \Rightarrow \quad f(x_1) = y_1$

$f^{-1}(y_2) = x_2 \qquad\quad f(x_2) = y_2$

$\|x-y\| \leq 2\,\|f(x)-f(y)\| \quad \forall\, x,y \in U.$

$\|x_1 - x_2\| \leq 2\,\|f(x_1) - f(x_2)\| \ \forall\, x_1, x_2 \in V.$

$\Rightarrow \|f^{-1}(y_1) - f^{-1}(y_2)\| \leq 2\,\|y_1 - y_2\|$

$\Rightarrow f^{-1}$ is cts. on W.

Proof of (2).

Let $x \in V$, $f'(x) = A$
- invertible

$\phi(h) = f(x+h) - f(x) - A(h)$

$$\lim_{h \to 0} \frac{\|\phi(h)\|}{\|h\|} = 0$$

Now,

$f(x+h) - f(x) = \phi(h) + A(h)$

$$A^{-1}(\underbrace{f(x+h)}_{y_1} - \underbrace{f(x)}_{y}) = h + A^{-1}\phi(h)$$

$$= (x+h) - x + A^{-1}\phi(x+h-x).$$

$$A^{-1}(y_1 - y_1) = f^{-1}(y_1) - f^{-1}(y) + A^{-1}\phi(f^{-1}(y_1) - f^{-1}(y)).$$

$$\Rightarrow \|f^{-1}(y_1) - f^{-1}(y) - A^{-1}(y_1 - y)\| = \|\underbrace{A^{-1}}_{\text{linear}}\phi(f^{-1}(y_1) - f^{-1}(y))\|$$

$$= M.\|\phi(f^{-1}(y_1) - f^{-1}(y))\|$$

$$\frac{\|f^{-1}(y_1) - f^{-1}(y) - A^{-1}(y_1 - y)\|}{\|y_1 - y\|}$$

$$\leq \frac{M.\|\phi(f^{-1}(y_1) - f^{-1}(y))\|}{\|f^{-1}(y_1) - f^{-1}(y)\|} \cdot \frac{\|f^{-1}(y_1) - f^{-1}(y)\|}{\|y_1 - y\|}$$

$$\therefore \|f^{-1}(y_1) - f^{-1}(y)\| \leq 2\|y_1 - y\|$$

$$\leq 2M. \frac{\| \phi(f^{-1}(y_1) - f^{-1}(y))\|}{\| f^{-1}(y_1) - f^{-1}(y)\|}$$

$$\Rightarrow \lim_{y_1 \to y} \frac{\| f^{-1}(y_1) - f^{-1}(y) - A^{-1}(y_1 - y)\|}{\| y_1 - y\|}$$

$$\leq 2M. \lim \frac{\| \phi(f^{-1}(y_1) - f^{-1}(y))\|}{\| f^{-1}(y_1) - f^{-1}(y)\|}$$

$\lim_{h \to 0} \frac{\|\phi(h)\|}{\|h\|} = 0$ and f^{-1} is cts.

$\Rightarrow y_1 \to y \Rightarrow \| f^{-1}(y_1) - f^{-1}(y)\| \Rightarrow 0.$

$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$

Date 22/08/22

Implicit function Theorem

Motivating equations

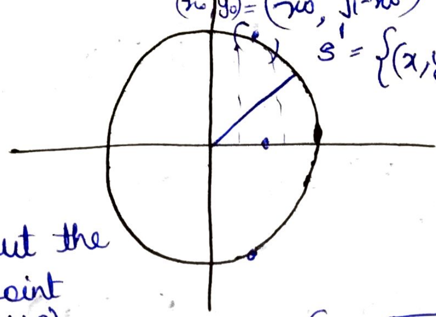
1) $x^2+y^2=1$ —— (1)

$F(x,y) = x^2+y^2-1.$ $\quad (F: \mathbb{R}^2 \to \mathbb{R})$

$1) \equiv \quad F(x,y) = 0.$

$(x_0, y_0) = (x_0, \sqrt{1-x_0^2})$

$S' = \{(x,y) \in \mathbb{R}^2 \mid F(x,y)=0\}$



S' is not the graph of a function globally, but the graph locally at every point $(x_0,y_0) \neq (\pm 1, 0)$.

$(x_0, -\sqrt{1-x_0^2})$

if $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0,y_0) = 0$

if $-1 < x_0 < 1$

$\Rightarrow \exists\, \delta > 0, \quad h: (x_0-\delta, x_0+\delta) \longrightarrow \mathbb{R}.$

$h(x) = \sqrt{1-x^2}$

$F(x, h(x)) = 0 \quad \forall\, x \in (x_0-\delta, x_0+\delta).$

$y = h(x)$ near every point $(x_0, y_0) \neq (\pm 1, 0)$

$$h(x) = \begin{cases} \sqrt{1-x^2} & \text{if } y_0 > 0 \\ -\sqrt{1-x^2} & \text{if } y_0 < 0. \end{cases}$$

Eg-2 $\quad y^5+y^3+y+x = 0.$

$F(x,y) = y^5+y^3+y+x.$

$\exists\, x = \phi(y) = -(y^5+y^3+y)$

such that

$F(\phi(y), y) = 0.$

Q. Can we solve the equation for y in terms of x?

$(x_0, y_0) \in \mathbb{R}^2$ s.t. $F(x_0, y_0) = 0$.

y_0 is a sol of $y^5 + y^3 + y + x_0 = 0$

$$P(y) = y^5 + y^3 + y + x_0$$

$$P'(y) = 5y^4 + 3y^2 + 1 > 0$$

$\Rightarrow$ P is strictly ↑ing

$\lim_{y \to \infty} P(y) = \infty$ if $\lim_{y \to -\infty} P(y) = -\infty$.

$\exists$ unique y_0 s.t. $P(y_0) = 0$.

$$\exists\ y = \psi(x).$$

$$F(x, \underset{\substack{\uparrow \\ \text{cts. differentiable?}}}{\psi(x)}) = 0$$

Ques 1. Let $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

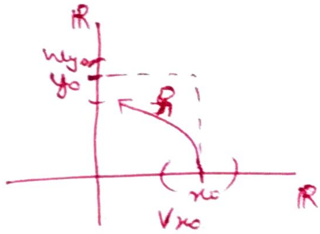
$(x, y) \longmapsto F(x, y)$

Consider $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ s.t. $F(x_0, y_0) = 0$.

~~$F(x_0, y_0)$~~ $F(x, y) = 0$

$\Downarrow$

can we write any implicit function



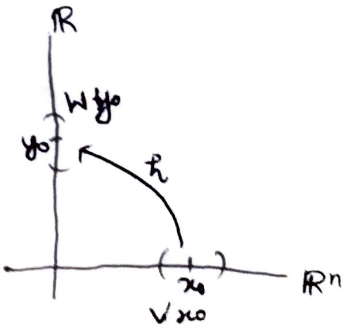
Does $\exists\ h : V_{x_0} \to W_{y_0}$, $h(x_0) = y_0$

s.t. $F(x, h(x)) = 0 \ \forall\ x \in V_{x_0}$?

Ques 2. $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

$(x, y) \longmapsto F(x, y)$

$(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$.



$\exists ?\ h : V_{x_0} \to W_{y_0}$

$h(x_0) = y_0$

Such that $F(x, h(x)) = 0 \ \forall \ x \in V_{x_0}$?

generalised.

$$F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(x, y) \mapsto F(x, y)$$

$(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$

$\exists \ h : V_{x_0} \subseteq_{open} \mathbb{R}^n \longrightarrow W_{y_0} \subseteq_{open} \mathbb{R}^m.$

$h(x_0) = y_0.$

Such that $F(x, h(x)) = 0 \ \forall x \in V_{x_0}$?

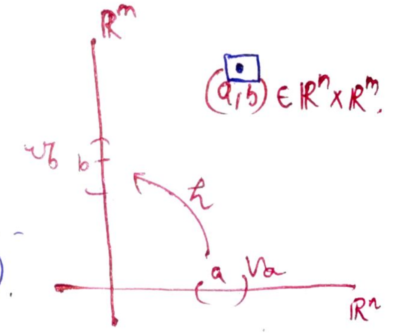
Theorem: Let $f = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be C^1 function in an open set containing $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$, where $f(a, b) = 0$.

Suppose $M = \left(D_{n+j} f_i(a, b)\right)_{m \times m}$

if $\det(M) \neq 0$. Then, $\exists \ \underset{a \in}{V_a} \subseteq_{open} \mathbb{R}^n$, $\underset{b \in}{W_b} \subseteq_{open} \mathbb{R}^m$,

and $h : V_a \rightarrow W_b$ such that $h(a) = b$ and

$f(x, h(x)) = 0 \ \forall \ x \in V_x.$



Proof: Define $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$F(x, y) = (x, f(x, y)) = (x_1, x_2, \ldots, x_n, f_1(x, y), \ldots, f_m(x, y)).$

$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$

$$F'(a, b) = \left[\begin{array}{c|c} I & 0 \\ \hline * & M \end{array}\right]_{(n+m) \times (n+m)}$$

$$\det F'((a, b)) = \det(M) \neq 0.$$

Apply IFT, $\exists$ $\underset{(a,b)\in}{A\times B} \subseteq_{\text{open}} \underset{\text{rectangle}}{\mathbb{R}^n \times \mathbb{R}^m}$ and

$$\underset{F(a,b)}{\overset{W}{\in}} \subseteq_{\text{open}} \mathbb{R}^n \times \mathbb{R}^m$$

Such that $F : A\times B \to W$ is diffeomorphism.

$h : F^{-1} ; W \to A\times B \xrightarrow{f} \mathbb{R}^m$ $\quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$h(x,y) = (x, k(x,y))$ where k be any function

$$h(x,y) = (h_1(x,y), h_2(x,y))$$

$$(x,y) = F \circ h(x,y) = F(h_1(x,y), h_2(x,y))$$

$$= h_1(x,y), f((h_1(x,y), h_2(x,y))$$

$$h_1(x,y) = x.$$

Consider.

$$\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m.$$

$$\pi(x,y) = y. \qquad f : \pi \circ F$$

$$f \circ h(x,y) = f(x, k(x,y))$$

$$\|$$

$$y = (\pi \circ F) \circ h(x,y) = f(x, k(x,y)).$$

Put $y = 0 \Rightarrow f(x, k(x,0)) = 0.$

$$\tilde{h}(x) = k(x,0).$$

$$= f(x, \tilde{h}(x)) = 0.$$

24/08/22

MVT

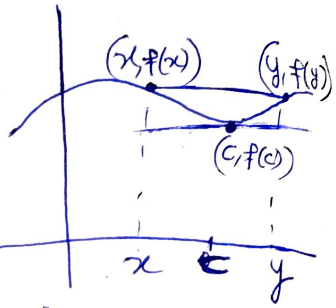
Recall,

Theorem (MVT). Let $f: \mathbb{R} \to \mathbb{R}$ be C^0, $x<y \in \mathbb{R}$.

f is cts. $[x,y]$ and diff'ble on (x,y)

Then, $\exists\ c \in (x,y)$ such that

$$f(y) - f(x) = f'(c)(y-x) \quad (1) \qquad \Rightarrow f'(c) = \frac{f(y)-f(x)}{y-x}.$$



Slope of the chord with end points at $((x, f(x))$ and $(y, f(y)))$ is $\dfrac{f(y)-f(x)}{y-x}$.

$f'(c)$ - slope of tangent line at $(c, f(c))$

Consider,

$f: \mathbb{R}^n \to \mathbb{R}^m$

$x, y \in \mathbb{R}^n$

$f(y) - f(x) \in \mathbb{R}^m$

$f'(c): \mathbb{R}^n \to \mathbb{R}^m$

$f'(c)(y-x) \in \mathbb{R}^m$

Eq (1) does make sense for $f: \mathbb{R}^n \to \mathbb{R}^m$ but not true, ingeneral.

Let us consider, $f: \mathbb{R}^1 \to \mathbb{R}^2$

$n=1,\ m=2$

$f(t) = (\cos t, \sin t)$

$x=0,\ y=2\pi$

$f(y) - f(x) = 0 \quad \Rightarrow \|f(y)-f(x)\| = 0.$

$$f(y) - f(x) = f'(c)(y-x) \quad \text{——} ①$$

$$f'(t) = (-\sin t, \cos t)$$

$$f'(c)(y-x) = (-\sin c, \cos c)\, 2\pi .$$

$$\| f'(c)(y-x) \| = \| (-\sin c, \cos c)\, 2\pi \| = 2\pi . \quad \forall\, c \in \mathbb{R}'$$

$$\cancel{\forall\, c \in t}$$

$$\Rightarrow f(y) - f(x) \neq f'(c)(y-x) \quad \forall\, c \in \mathbb{R}.$$

Let $a = (a_1, a_2) \in \mathbb{R}^2$

Consider $F : \mathbb{R} \to \mathbb{R}$.

$$F(t) = \langle a, f(t) \rangle = a \cdot f(t)$$

$$F(y) - F(x) = F'(c) \cdot (y-x) \text{ where } x < c < y .$$

$$\Rightarrow a\big(f(y) - f(x)\big) = a \cdot f'(c)(y-x) \quad \text{——} (2)$$

$y = 2\pi$ and $x = 0$.

$$\Rightarrow a \cdot 0 = a \cdot (-\sin c, \cos c)\, 2\pi$$

$$\Rightarrow -a_1 \sin c + a_2 \cos c = 0$$

$$\Rightarrow \tan c = \frac{a_2}{a_1}$$

$$\Rightarrow c = \tan^{-1}\left(\frac{a_2}{a_1}\right) \in (0, 2\pi)$$

Such that equality (2) holds.

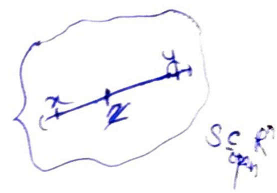
Defn: For $x, y \in \mathbb{R}^n$, the Euclidean line segment with end points at x and y denoted by $L(x,y)$, is defined by

$$L(x,y) = \{(1-t)x + ty \mid 0 \le t \le 1\}$$

$$= \{x + tu \mid 0 \le t \le 1\}. \quad \text{where } u = y - x.$$

Theorem (MVT) Let $S \subseteq_{open} \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ is differentiable on S. Let $x, y \in S$, such that $L(x,y) \subseteq S$. and $a \in \mathbb{R}^m$. Then, $\exists\, z \in L(x,y)$ such that

$$a \cdot (f(y) - f(x)) = a \cdot f'(z)(y-x)$$



Proof:

$\Rightarrow \exists\, \delta > 0$ such that $x + tu \in S$

$\forall\, t \in (-\delta, 1+\delta)$

$L(x,y) \subseteq S$

define $F : (-\delta, 1+\delta) \to \mathbb{R}$ by

$F(t) = a \cdot f(x + tu)$

— cts. on $[0,1]$

diff^{n}ble on $(0,1)$.

MVT for real valued single variable function.

$\Rightarrow \quad F(1) - F(0) = F'(\theta)(1-0) \qquad 0 < \theta < 1$

$\Rightarrow a \cdot f(y) - a\, f(x) = a\, f'(x + \theta u) \cdot u.$

$= \quad a\,(f(y) - f(x)) = a \cdot f'(z)\,(y - x).$

$z = x + \theta u, \; 0 < \theta < 1$

$\in L(x,y)$ $\square$.

Theorem MVT. → after recall section,

Proof :- Apply rolle's theorem on $g(x) = f(x) - \gamma x$.

where $\gamma \in \mathbb{R}$ such that $g(b) = g(a)$

$$\Leftrightarrow f(b) - \gamma b = f(a) - \gamma a.$$

$$\Leftrightarrow \gamma = \frac{f(b) - f(a)}{b - a}.$$

MIXED PARTIAL DERIVATIVES

$(f_1, \dots, f_m) f : \mathbb{R}^n \to \mathbb{R}^m$.

for $1 \le i \le n$.

$$D_i f(x) = (D_i f_1(x), \dots, D_i f_m(x)) \in \mathbb{R}^m$$

where

$$D_i f_\ell(x) = \lim_{t \to 0} \frac{f_\ell(x + te_i) - f_\ell(x)}{t}$$

$$D_i f : \mathbb{R}^n \to \mathbb{R}^m.$$

for $1 \le j \le n$

$D_j(D_i f) : \mathbb{R}^n \to \mathbb{R}^m$ is defined similarly.

$\quad \parallel$

$D_{ij} f$.

Eg: $f : \mathbb{R}^2 \to \mathbb{R}$.

$$f(x,y) = \begin{cases} \dfrac{xy(y^2 - x^2)}{x^2 + y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Compute $D_1 f(0,0)$.

$$D_1 f(x,y) = \begin{cases} \underline{\qquad\qquad} & , (x,y) \neq (0,0). \\ 0 & , (x,y) = (0,0) \end{cases}$$

$D_2 f(x,y) = \begin{cases} \\ 0 & ; (x,y) = (0,0). \end{cases}$

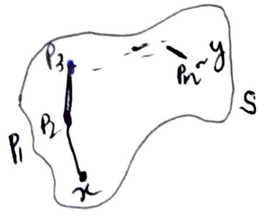
$D_{12}(0,0) \neq D_{(2,1)}(0,0)$

$f : \mathbb{R}^n \to \mathbb{R}^m$

Ques- When two partial derivatives $D_{i,j} f$ and $D_{j,i} f$ are equal?

Application of MVT.

Lemma: Let $S \underset{\substack{\text{open +} \\ \text{connected}}}{\subset} \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ differentiable on S if $f'(c) = 0 \ \forall \ c \in S$, then f is a constant function.



Proof $\exists$ a polygon path joining x and y

$\exists\ x = P_1, P_2, \ldots, P_N = y$

such that $L(P_i, P_{i+1}),\ i = 1, \ldots, N-1$
$\subseteq S$

for $a \in \mathbb{R}^m$

$L(P_i, P_{i+1})$

$$\sum_{i=1}^{N-1} a\left(f(P_{i+1}) - f(P_i)\right)$$

$$= \underbrace{a f'(z_i)}_{=0} \cdot (P_{i+1} - P_i) \quad z_i \in L(P_i, P_{i+1})$$

$$\Rightarrow a \cdot \left(f(y) - f(x)\right) = 0 \quad \forall a \in \mathbb{R}^m.$$

Constant $a = f(y) - f(x) \Rightarrow \|f(y) - f(x)\|^2 = 0$

$\Rightarrow f(y) - f(x).$

Assignment - 3

Ques 2. $D_1 f(x,y)$, $D_2 f(x,y)$ both are continuous ($\because f$ is C^1)

Suppose f is injective.

Now, if $\exists$ an open set $A \subset \mathbb{R}^2$ where $D_1 f(x,y) = 0$ $= f_x$
then, f would be independent of x on A violating the fact that f is 1-1.

So, assume $D_1 f(x,y) \neq 0 \ \forall (x,y) \in A$.

Define $g : A \to \mathbb{R}^2$

$$g(x,y) = (f(x,y), y)$$

So, g is also injective.

$$|(g'(x,y))| = \begin{vmatrix} f_x & f_y \\ 0 & 1 \end{vmatrix} = f_x \neq 0 \text{ on } A$$

So, $g(A) \subseteq \mathbb{R}^2$ is open (by Ex 1). — (1)

Now, claim $(f(x,y), \tilde{y}) \notin g(A)$.

$(f(x,y), \tilde{y}) = (f(a,b), b)$.

$\tilde{y} \in \mathbb{R}$ so that $y \neq \tilde{y}$ where $y \neq \tilde{y}$ & $(x,\tilde{y}) \in A$.

$\tilde{y} = b$ & $f(x,y) = f(a,b)$

$\Rightarrow y = b \Rightarrow \Leftarrow$.

if $(f(x,y), y) \in g(A)$, then, $\nexists$ no nbd

$U \ni (f(x,y), y) \subset U \subset g(A)$

$\Rightarrow g(A)$ is not open — (2).

(1) & (2) — Contradiction

f Can't be 1-1.

Ques: $\exists$ bijection $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\forall$ $m, n \in \mathbb{N}$
but not diffeomorphic.

Ques 3. (a) Suppose not, then $\exists$ $a<b$ such that $f(a) = f(b)$

By MVT. $\exists$ $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$

To show $f: \mathbb{R} \to f(\mathbb{R})$ is diffeomorphism.

$f^{-1}: f(\mathbb{R}) \longrightarrow \mathbb{R}$ — C^1.

| $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
if $\det(f'(a)) \neq 0$ $\forall$ $a \in \mathbb{R}^n$
$\exists$ $a \in V \subseteq_{open} \mathbb{R}^n$ &
$f(a) \in W \subseteq_{open} \mathbb{R}^n$
s.t. $f: V \to W$ C^1.

b)

$$\det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x} \neq 0$$

$f(x,0) = f(x, 2\pi)$. $\qquad f(1,0) = f(1, 2\pi)$.
Not injective.

Ques 6. a) $J: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$J(P_1, P_2) = (-P_2, P_1)$$

$$J \circ J = -(P_1, P_2)$$

$$= -id.(P_1, P_2) \quad \forall \; P_1, P_2 \in \mathbb{R}^2.$$

b) Inner product preserving and hence Norm preserving.

$p = (p_1, p_2)$, $q = (q_1, q_2) \in \mathbb{R}^2$

$$\langle p, q \rangle = p_1 q_1 + p_2 q_2$$

$$\langle J(p), J(q) \rangle = \langle (-p_2, p_1), (-q_2, q_1) \rangle$$

$$= p_2 q_2 + p_1 q_1$$

$$\langle p, q \rangle$$

$\Rightarrow J$ is IP.

J is linear hence it is Norm preserving.

c) $0 \neq p = (p_1, p_2) \in \mathbb{R}^2$. Show that $p \perp J(p)$.

$$\cos\left(\angle (p, J(p))\right) = \frac{\langle p, J(p) \rangle}{\|p\| \cdot \|J(p)\|}$$

$$= \frac{\langle (p_1, p_2) \cdot (-p_2, p_1) \rangle}{\|p\|^2} = 0$$

$$\angle (p, J(p)) = \frac{\pi}{2}.$$

<u>Ques (D)</u> $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$J(p_1, p_2) = (-p_2, p_1)$

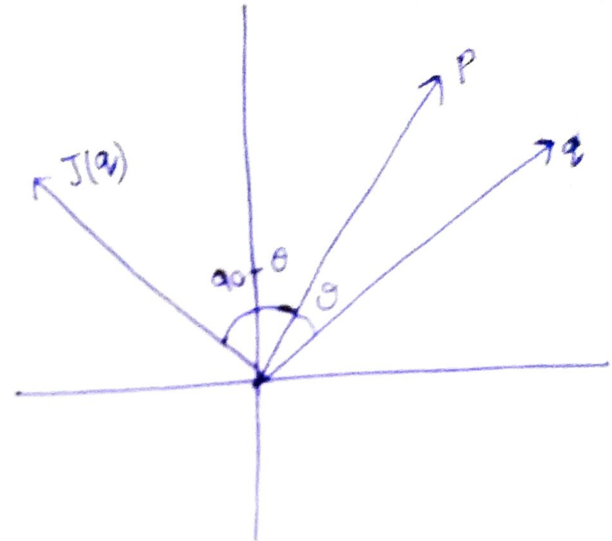
$\mathbb{R}^2 \equiv \mathbb{C}$ $\qquad (p_1, p_2) \equiv p_1 + i p_2$

$(-p_2, p_1) = -p_2 + i p_1 = i(p_1 + i p_2)$.

$J : \mathbb{C} \to \mathbb{C}$

$J(p_1 + ip_2) = i(p_1 + ip_2)$

$J(z) = iz$



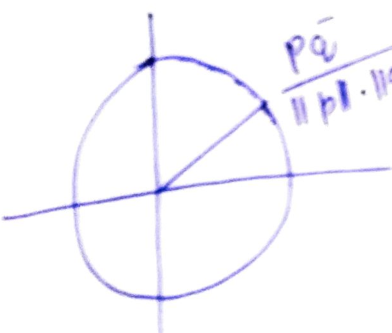
$$\frac{\langle p, q\rangle}{\|p\|\cdot\|q\|} + i\,\frac{\langle p, J(q)\rangle}{\|p\|\cdot\|q\|} \in S^1$$

$$\frac{p\cdot\bar{q}}{\|p\|\cdot\|q\|} = \quad p\cdot\bar{q} = (p_1 + ip_2)\cdot(q_1 - iq_2)$$

$$= p_1q_1 + p_2 q_2 + i(p_2q_1 - p_1q_2)$$

$$= \langle p, q\rangle + i\,\langle (p_1, p_2), (-q_2, q_1)\rangle$$

$$= \langle p, q\rangle + i\,\langle p, J(q)\rangle$$



$$= \frac{\langle p, q\rangle + i\,\langle p, J(q)\rangle}{\|p\|\cdot\|q\|} \in S^1$$

$$= e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$= \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{\langle p, q\rangle}{\|p\|\cdot\|q\|}$$

$$\sin\theta = \frac{\langle p, J(q)\rangle}{\|p\|\,\|q\|}$$

Date 26 | 08 | 22

Recall :

$f : \mathbb{R}^n \to \mathbb{R}^n$

$(f_1, \dots, f_m)$; for $1 \le i \le n$.

$D_i f : \mathbb{R}^n \to \mathbb{R}^m$

$x \mapsto D_i f(x) = (D_i f_1(x), \dots, D_i f_m(x))$

where.

$$D_i f_k(x) = \lim_{t \to 0} \frac{f_k(x + te_i) - f_k(x)}{t}$$

for

$1 \le j \le n$,

$D_j(D_i f) = D_{ij} f : \mathbb{R}^n \to \mathbb{R}^m$ is ~~denote~~ defined similarly.

2) Eg.

$f : \mathbb{R}^2 \to \mathbb{R}$.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & ; (x,y) \neq (0,0). \\ 0 & (x,y) \neq (0,0) \end{cases}$$

$$D_1 f(x,y) = \begin{cases} \dfrac{y(x^4 + 4x^2y^2 + y^4)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$D_2 f(x,y) = \begin{cases} \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2} & ; (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$D_2(D_1 f)(x,y) = D_{1,2}(x,y) = \begin{cases} \dfrac{x^6+y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}, & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0) \end{cases}$$

$$D_1(D_2 f)(x,y) = D_{2,1} f(x,y) = \begin{cases} D_{1,2} f(x,y) & (x,y)\neq(0,0) \\ -1, & (x,y)=(0,0) \end{cases}$$

$$D_{1,2} f(0,0) \neq D_{2,1} f(0,0).$$

Ques $f : \mathbb{R}^n \to \mathbb{R}^m$

$1 \le i,j \le n$, $c \in \mathbb{R}^n$.

under what Condition, $D_{i,j} f(c) = D_{j,i} f(c)$?

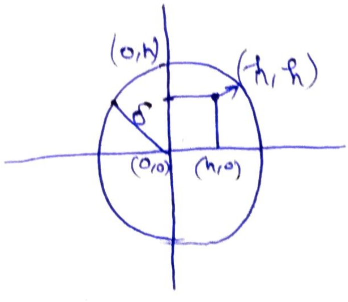
Theorem: if both $D_i f$ and $D_j f$ exist in $B_\delta(c)$ and both are diff'ble at c, then $D_{i,j} f(c) = D_{j,i} f(c)$.

Consider $m=1$, $n=1$ and for simplicity $c=(0,0)$

Proof

$$\left\{ \begin{array}{l} f = (f_1, \dots, f_m) \\ D_{i,j} f(c) = (D_{i,j} f_1(c), \dots, D_{i,j} f_m(c)) \\ \quad \| \\ D_{j,i} f(c) = (D_{j,i} f_1(c), \dots, D_{j,i} f_m(c)) \end{array} \right\}$$

$f : \mathbb{R}^2 \to \mathbb{R}$



Define $\varphi(h) = f(h,h) - f(h,0) - f(0,h) + f(0,0)$, $h>0$.

Claim 1. $\displaystyle\lim_{h\to 0} \frac{\varphi(h)}{h^2} = D_{1,2} f(0,0).$

Claim 2. $\displaystyle\lim_{h\to 0} \frac{\varphi(h)}{h^2} = D_{2,1} f(0,0).$

Define

$G(x) = f(x,h) - f(x,0)$. $\qquad G'(x) = D_1 f(x,h) - D_1 f(x,0)$

$G(h) = f(h,h) - f(h,0)$

$G(0) = f(0,h) - f(0,0)$

$G(h) - G(0) = f(h,h) - f(h,0) - f(0,h) + f(0,0) = \varphi(h)$

By MVT, $\quad \varphi(h) = h\, G'(x_1) \qquad 0 < x_1 < h$

$= h\left[D_1 f(x_1,h) - D_1 f(x_1,0)\right]$

Recall, f is diff. at (h,k) then

$$\lim_{(h,k)\to 0} \frac{|f(h,k) - f(0,0) - D_1 f(0,0)h - D_2 f(0,0)k|}{\|(h,k)\|} = 0$$

Replace f by $D_1 f$. As $D_1 f$ and $D_2 f$ are all diff^{n}ble at c.

$$\lim_{(h,k)\to 0} \frac{\{D_1 f(h,k) - D_1 f(0,0) - D_1 f(0,0)h - D_2 f(0,0)k\}}{\|(h,k)\|} = \epsilon_1(h,k)$$

$\left\{ \lim_{(h,k)\to 0} \epsilon_1(h,k) = 0. \right.$

Bcs f^n is diff^{n}ble

$\Rightarrow D_1 f(h,k) = D_1 f(0,0) + D_{1,1} f(0,0)h + D_{1,2} f(0,0)k + \epsilon_1(h,k).\|h,k\|$

$D_1 f(x,h) = D_1 f(0,0) + D_{1,1} f(0,0)x_1 + D_{1,2} f(0,0)h + G(x,h)\|(x,h)\|$

$D_1 f(x,0) = D_1 f(0,0) + D_{1,1} f(0,0)x + D_{1,2} f(0,0).0 + \epsilon_1(x,0)\|(x,0)\|$

~~$\epsilon(x_1,h) - \|(x_1,h)\| \epsilon(x,h) - \|(x,0)\|\epsilon$~~

$\epsilon(x,h) = \|(x_1,h)\|\,\epsilon_1(x_1,h) - \|(x,0)\|\,\epsilon_1(x,0)$

$\varphi(h) = h\, G'(x_1) \qquad 0 < x_1 < h$

$= h\left[D_1 f(x,h) - D_1 f(x,0)\right]$

$= h\left[D_{1,2} f(0,0)h + \epsilon(x_1,h)\right]$

$$\frac{\varphi(h)}{h^2} = D_{1,2} f(0,0) + \frac{\epsilon(x_1, h)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\varphi(h)}{h^2} = D_{1,2} f(0,0).$$

Proof of claim 2 :- $\lim_{h\to 0} \frac{\varphi(h)}{h^2} = D_{2,1} f(0,0)$.

Apply same procedure.

$H(y) = f(h,y) - f(0,y)$

Taylor's formula $f : \mathbb{R}^n \to \mathbb{R}$

Recall,

Theorem : Let f be a f^n. having n^{th} derivative everywhere in (a,b). and $f^{(n-1)}$ is continuous. on $[a,b]$.

Assume $c \in [a,b]$. Then for every $x \in [a,b] \setminus \{c\}$, $\exists\ x_1 \in I$.

where I is the open interval with endpoints at x and c

Such that

$$f(x) = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f^{(2)}(c)(x-c)^2}{2!} + \cdots$$

$$+ \frac{f^{(n-1)}(c)(x-c)^{n-1}}{(n-1)!} + \frac{f^n(x_1)(x-c)^n}{n!}$$

$$= \sum_{k=0}^{n-1} \frac{f^k(c)(x-c)^k}{k!} + \frac{f^n(x_1)(x-c)^n}{n!}$$

Proof: $F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f^2(t)(x-t)^2}{2!} + \cdots + \frac{f^{n-1}(t)(x-t)^{n-1}}{(n-1)!}$

To show $F(c) = \frac{f^n(x_1)}{n!}(x-c)^n$ for some $x_1 \in I$.

Proof. $F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f^{(2)}(t)(x-t)^2}{2!} + \dots + \frac{f^{(n-1)}(t)(x-t)^{n-1}}{(n-1)!}$

We show. $F(c) = \frac{f^n(x_1)(x-c)^n}{n!}$ for some $x_1 \in I$.

$F'(t) = -f'(t) + f^{(1)}(t) - \frac{f^2(t)(x-t)}{2!} + \dots$

$+ \frac{f^{n-1}(t)(x-t)^{n-2}}{(n-2)!} - \frac{f^n(t)(x-t)^{n-1}}{(n-1)!}$

$F'(t) = \frac{-f^n(t)(x-t)^{n-1}}{(n-1)!}$

$g(t) = F(t) - \frac{(x-t)^n}{(x-c)^n} F(c).$ $(x \neq c)$

Applying Rolle's theorem on g.

$g(c) = F(c) - F(c) = 0$

$g(x) = F(x) - 0 = F(x) = 0$

Apply Rolle's theorem on g

$\exists\ x_1 \in I$ such that $g'(x_1) = 0.$

$\Rightarrow \left. F'(t) + n\frac{(x-t)}{(x-c)^n} F(c) \right|_{t=x_1} = 0$

$\Rightarrow -\frac{f^n(x_1)(x-x_1)^{n-1}}{(n-1)!} \cancel{\neq} -n\frac{(x-x_1)^{n-1}}{(x-c)^n} F(c) = 0$

$\Rightarrow F(c) = \frac{f^n(x_1)\cdot(x-c)^n}{n!}$

Date
29/08/22

1) Defn : $f(c) + \frac{f'(c)(x-c)}{1!} + \dots + \frac{f^k(c)}{K!}(x-c)^k = P_{K,c}(x)$

— Called K^{th} order T polynomial about c.

2) $R_{K,c}(x) = f(x) - P_{K,c}(x)$

— the k^{th} order remainder

Remark : 1) $P_{K,c}(x) \to$ polynomial of deg $\leq K$.

2) $P^{i}_{K,c}(c) = f^{i}(c)$.

3) for $k=1$

Taylor's polynomial

$P_{1,c}(x) = f(c) + f'(c)(x-c)$ [linear function]

↳ The best linear approximation of f at c.

— The tangent line to the graph of f at $(c, f(c))$

Ex. Suppose f is K-times diff^{n}ble on (a,b) and $c \in (a,b)$. show that $\lim_{x \to c} \frac{R_{K,c}(x)}{x-c} = 0$

Observation $\quad f : \mathbb{R}^n \to \mathbb{R}$

Defn — (1) if all the partial derivative of f at $x \in \mathbb{R}^n$ exist Then,

$$f'(x,t) = \sum_{i=1}^{n} D_i f(x) \cdot t_i \quad , \; t = (t_1, \cdots, t_n) \in \mathbb{R}^n .$$

2) if all the second order mixed partial derivatives of f at $x \in \mathbb{R}^n$ exist. Then.

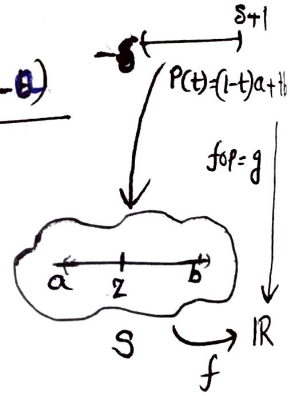
$$f^2(x,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(x)\, t_i t_j .$$

3) $\quad$ Ily, $f^3(x,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{K=1}^{n} D_{i,j,K} f(x)\, t_i t_j t_K .$

& $f^n(x,t) = \sum \cdots \sum D_{i_1, i_2, \cdots i_n} f(x)\, t_{i_1} t_{i_2} \cdots t_{i_n}$ are defined.

Theorem: Assume f and all its partial derivatives of order $\leq m$ are diff^n^ble at each point of $S \subseteq$ open $\mathbb{R}^n$. If $a, b \in S$ with $L(a,b) \subseteq S$, then $\exists$ $z \in L(a,b)$ such that

$$f(b) - f(a) = \sum_{K=1}^{m-1} \frac{f^K(a; b-a)}{K!} + \frac{f^{(m)}(z, b-a)}{m!}$$



Generalization

$\delta > 0$ sufficiently small.

$$g : (-\delta, 1+\delta) \to \mathbb{R}.$$

$$g(t) = f \circ p(t) = f\big((1-t)a + tb\big)$$

$$g(1) - g(0) = \sum_{K=1}^{m-1} \frac{g^K(0)}{K!} + \frac{g^m(\theta)}{m!} \quad ; 0 < \theta < 1 \text{ —(1)}$$

$g(1) = f(b)$
$g(0) = f(a)$

Now, we calculate $g^{(k)}(t)$.

$p(t) = (1-t)a + tb$
$= a + t(b-a)$
$= (a_1 + t(b_1 - a_1), \ldots$
$\ldots a_n + t(b_n - a_n))$

$k=1$

$g'(t) = f'(p(t)) \cdot p'(t)$

$= \sum_{i=1}^{n} D_i f(p(t)) \cdot (b_i - a_i)$

$= f^{(1)}(p(t) ; b-a)$

at $t=0$, $g'(0) = f^{(1)}(a, b-a)$.

$g^2(t) = \sum_{i,j=1}^{n} D_{i,j} f(p(t)) \cdot (b_j - a_j)(b_i - a_i)$

$= f^{(2)}(p(t), b-a)$. {By using chain rule}

$\vdots$

$|||^{ly}$ $g^k(0) = f^{(k)}(a, b-a)$

Putting all the equation in (1), we get the required result.

$0 < \theta < 1$

$f^m(p(\theta); b-a)$.

$p(\theta) = (1-\theta)a + \theta b \in L(a, b)$
$0 < \theta < 1$.

Extrema: (of real valued single variable f^n's).

Recall,

Theorem: for some ~~m~~ $n \geq 1$, let f has its n^{th} derivative on (a,b). Suppose that for some $c \in (a,b)$.

$$f^{(1)}(c) = 0,\ f^{(2)}(c) = 0,\ ---\ f^{(n-1)}(c) = 0,$$

but $f^{(n)}(c) \neq 0$.

Then, we have the following,

Case 1) if n is even

a) if $f^n(c) > 0$, then f has a local min. at c.

b) if $f^n(c) < 0$, then f has a local max. at c.

Case 2 n-odd,

neither max. nor min. at c.

Example-1 1) $f(x) = x^2$, $f'(0) = 0$

$f''(0) = 2 > 0$

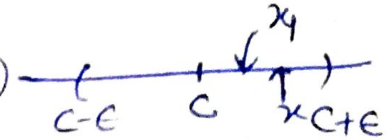
2) $f(x) = x^3$

$f'(0) = 0 = f^2(0)$

$f^3(0) = 6 > 0$

1. Let $f^{(n)}(c) \neq 0 \Rightarrow \exists\, \epsilon > 0$ s.t. $\underbrace{f^{n}(x) \neq 0}_{\downarrow} \ \forall\, x \in (c-\epsilon, c+\epsilon)$.

{Not change sign bcs of continuity}



Now, Taylor's formula

$\Rightarrow$

$$f(x) = f(c) + \underbrace{\sum_{k=1}^{n-1} \frac{f^{k}(c)}{k!}(x-c)^{k}}_{= 0} + \frac{f^{n}(x_1)}{n!}(x-c)^{n}.$$

$$\Rightarrow f(x) - f(c) = \frac{f^{n}(x_1)(x-c)^{n}}{n!}$$

Case - 1 n even, $f^{n}(c) > 0$.

$f^{(n)}(x_1) > 0$.

$f(x) - f(c) > 0 \quad \forall\, x(\neq c) \in (c-\epsilon, c+\epsilon)$.

$\Rightarrow f(x) > f(c) \quad \forall\, x(\neq c) \in (c-\epsilon, c+\epsilon)$.

$\Rightarrow$ c is a pt. of local minima.

Ex. Complete the theorem.

Date
31/08/2022

Recall,

Theorem: for some $n \geq 1$, let f have a cts. n^{th} derivative in an open interval (a,b). Suppose for some $c \in (a,b)$.

- $f^{K}(c) = 0 \quad \forall \ K = 1, \dots, n-1$.

and

- $f^{n}(c) \neq 0$

Then,

Case-1 $\quad n$ - even.

a) If $f^{n}(c) > 0$, then f has a local min. at c.

b) If $f^{n}(c) < 0$, then f has a local max. at c.

Case-2 $\quad n$ - odd

f has neither local max. nor local min. at c.

Proof:

- $f^{(n)}$ — cts. at c, $f^{(n)}(c) \neq 0$

$\Rightarrow \exists \ \epsilon > 0$ such that

$$f^{n}(x) \neq 0 \quad \forall \ x \in (c-\epsilon, c+\epsilon)$$

Moreover, continuity of f

$$\Rightarrow \ \text{Sign}\left(f^{(n)}(x)\right) = \text{Sign}\left(f^{(n)}(c)\right).$$

$$\forall \ x \in (c-\epsilon, c+\epsilon)$$

- $x \in (c-\epsilon, c+\epsilon)$

$\exists \ x_1 \in x$ such that.

$$f(x) = f(c) + \underbrace{\sum_{K=1}^{n-1} \frac{f^{K}(c).(x-c)^{K}}{K!}}_{= 0} + \frac{f^{n}(x_1)(x-c)^{n}}{n!}$$

$$\Rightarrow f(x) - f(c) = \frac{f^{n}(x_1)}{n!}(x-c)^n$$

$$x_1 \in I\{x, c\} \subseteq (c-\epsilon, c+\epsilon).$$

<u>Case-1</u> n - <u>even</u>.

$f^{(n)}(c) > 0$, then $f^{n}(x_1) > 0$.

$f(x) - f(c) > 0 \quad \forall\, x \in (c-\epsilon, c+\epsilon) \setminus \{c\}$.

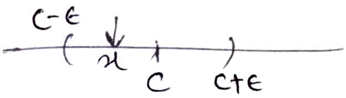
$\Rightarrow$ Case (1), (a).

Case-1, b) Similar.

<u>Case-2.</u> $f(x) - f(c) = \dfrac{f^{n}(x_1)}{n!}(x-c)^n$.

$f^{n}(x) \neq 0$
$\forall\, x \in (c-\epsilon, c+\epsilon)$

Suppose $f^{(n)}(x) < 0$, $\forall\, x \in (c-\epsilon, c+\epsilon)$



choose x from left hand side of c.

$x - c < 0$

$(x-c)^n < 0$ bcs n is odd.

$f(x) - f(c) > 0$.

$(y-c)^n > 0$.

$f(y) - f(c) < 0$,

$\Rightarrow \begin{cases} f(x) > f(c) \\ f(y) < f(c) \end{cases} \Rightarrow$ At c, f has neither max. nor minimum.

Extrema for $f: \mathbb{R}^n \to \mathbb{R}$

Fact-1
If at $c \in \mathbb{R}^n$, f has a local max/min.
then $f'(c) = 0 \Leftrightarrow D_i f(c) = 0 \;\; \forall\, i = 1, \dots, n$
provided, f is differentiable.

Also, the converse is not true.

Eg. $f(x,y) = x^2 - y^2$

Directional derivative of f.

$$D_u f(c) = \lim_{t \to 0} \frac{f(c+tu) - f(c)}{t}$$

$$= Df_c(u) = f'(c) \cdot u \cdot \left\{ \begin{array}{l} \text{Provided } f \text{ is} \\ \text{differentiable at } c. \end{array} \right.$$

$\exists\, f$ where $D_i f(c)$ – exist $\forall\, i = 1, \dots, n$ but directional derivative doesn't exist.

~~Ques. Suppose $D_u f(c)$ – exist $\forall\, u (\neq 0) \in \mathbb{R}^n$~~

Fact-2 if $f: \mathbb{R}^n \to \mathbb{R}$ has a local extremum at c.
Then $D_u f(c) = 0 \;\; \forall\, u \neq 0 \in \mathbb{R}^n$.

Ques. What about the converse?
Ans. Converse is not true.

Defn: Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$.
1) $\nabla f(a) = (D_1 f(a), \dots, D_n f(a))$ — gradient of f at a.

$$D_u f(a) = f'(a) \cdot (u) = D_1 f(a) \cdot u_1 + \dots + D_n f(a) \cdot u_n$$

$$= \langle \nabla f(a), u \rangle = \nabla f(a) \cdot u.$$

$u \in S^{n-1}$
Set of unit vectors

- Rate and direction of fastest increase of f (geometrical meaning of gradient).
if $\nabla f(a) \neq 0$.

2) If $\nabla f(a) = 0$, then a is called a stationary point.

Stationary point are classified as.

a - stationary point

1) pt. of max.

2) pt. of min.

3) Saddle point.

3) A stationary point a if $f: \mathbb{R}^n \to \mathbb{R}$ is called a saddle point if for every $\epsilon > 0$

$B_\epsilon(a)$ Contains x, y such that.

$f(x) - f(a) > 0$

$f(y) - f(a) < 0$.

Eg 1) $f(x,y) = x^2 - y^2$

$(0,0)$

f has neither maximum nor minimum,

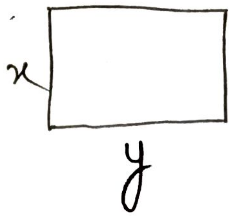
$(0,0)$ - Saddle point.

2) $f(x,y) = (y - x^2)(y - 2x^2)$

$(0,0)$ - Saddle point.

Ques. Let $a \in \mathbb{R}^n$, be a stationary point of $f: \mathbb{R}^n \to \mathbb{R}$. determine if it a point of max./ min./ saddle point.

Prob. R be a rectangle with perimeter 100. what is the max. possible area of R.



$$2x+2y=100$$

To maximise $f(x,y) = xy$

Subject to constraints: $x+y=50$

Basically, the equivalent problem is,

To study $\text{Sign}(f(x)-f(a))$ locally at a.

$B_\delta(a)$.

$$f(x)-f(a) > 0 \quad \forall x \in B_\delta(a) \setminus \{a\}$$

$\Downarrow$

pt. of min.

Tool. Taylor's theorem.

21 Sept 2022

Problem Study extrema of $f: \mathbb{R}^n \to \mathbb{R}$.

Recall,

Def^n s. $a \in \mathbb{R}^n$.

1) $\nabla f(a) = (D_1 f(a), \ldots D_n f(a))$ — gradient of f at a.

∇ — Vector differential ~~equati~~ operator — gives the direction and rate of fastest increase of f.

$\nabla \cdot f(a) \neq 0$.

$a \xrightarrow{} u$

$$u \in S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

$$D_i f(a) = \nabla f(a) \cdot u$$
$$\leq \|\nabla f(a)\| \text{ = holds if } u = \lambda \cdot \nabla f(a).$$
$$\lambda > 0, \quad \lambda = \frac{1}{\|\nabla f(a)\|}.$$

2) if $\nabla \cdot f(a) = 0$. Then, a is called stationary point of f.

3) A stationary point a of f is called a saddle point if for every $\epsilon > 0$ $\exists (x, y) \in B_\epsilon(a)$ such that

$f(x) > f(a)$ and

$f(y) < f(a)$.

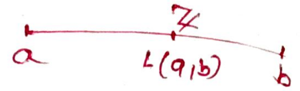
$$\text{Sign}\{f(t) - f(a)\} = \{\pm\}$$
$$t \in B_\epsilon(a).$$

if a is stationary point of f, then it is

1) Point of maximum or,
2) Point of minimum or,
3) Saddle point.

Ques- Let $a \in \mathbb{R}^n$ be a stationary point of $f: \mathbb{R}^n \to \mathbb{R}$. Determine, if a is a point of max. or min. or saddle point.

- Use Taylor's theorem



Recall, $$f(b) - f(a) = \sum_{K=1}^{m-1} \frac{f^{K}(a)(b-a)}{K!} + \frac{f^{m}(z; b-a)}{m!}$$

for, $m=2$.

$$f(b) - f(a) = \frac{f^{(1)}(a, b-a)}{1!} + \frac{f^{(2)}(z, b-a)}{2}$$

or,

$$f(a+t) - f(a) = f'(a, t) + \frac{f^{2}(z, t)}{2}.$$

$$f'(a,t) = \sum_{i=1}^{n} D_i f(a) t_i$$

$$= \nabla f(a) \cdot t = 0$$

$$= f(a+t) - f(a) = \frac{1}{2} f^{(2)}(z, t) \quad \text{—— (1)}$$

$$\Rightarrow \quad f(a+t) - f(a) = \frac{1}{2} f^{(2)}(a,t) + \|t\|^2 E(t)$$

where,

$$\|t\|^2 E(t) = \frac{1}{2}\left[f^{(2)}(z; t) - \frac{1}{2} f^{(2)}(a; t)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} D_{ij} f(z) t_i t_j - \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} f(a) t_i t_j\right]$$

$$= \frac{1}{2} \sum\sum \left[D_{ij} f(z) - D_{ij} f(a)\right] t_i t_j.$$

$$\Rightarrow \quad \left|\|t\|^2 E(t)\right| \le \frac{1}{2} \sum\sum \left|\left[D_{ij} f(z) - D_{ij} f(a)\right] t_i t_j\right|$$

{By Triangle inequality}

$$\leq \frac{1}{2} \|t\|^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left| D_{ij} f(z) - D_{ij} f(a) \right|$$

$$\Rightarrow |E(t)| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| D_{ij} f(z) - D_{ij} f(a) \right|$$

when $D_{ij} f$ are Continuous at a.

$\Rightarrow E(t) \to 0$ as $t \to 0$.

Theorem:- Assume that $D_{ij} f$ exist and are Continuous ($1 \leq i, j \leq n$) at a, where a is a stationary point of f. Define,

Quadratic symmetric $\rightarrow$ $Q(t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} f(a) \cdot t_i t_j$.

a) if $Q(t) > 0 \ \forall \ t \in \mathbb{R}^n \setminus \{0\}$, then f has a local minimum at a.

b) if $Q(t) < 0 \ \forall \ t \in \mathbb{R}^n \setminus \{0\}$, then f has a local max at a.

c) if $Q(t)$, takes both (+)ve and (−)ve values, then f has a saddle point at a.

Proof: Assume 1). $Q(t) > 0 \ \forall \ t \in \mathbb{R}^n \setminus \{0\}$.

$$\Rightarrow Q(t) > 0 \ \forall \ t \in \underbrace{S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}}_{\text{Compact}} = \partial B_1(0)$$

{ Boundary of Ball }

$m = \inf \{Q(t) \mid t \in S^{n-1}\} > 0$

$$t \neq 0, \quad Q(t) = Q\left(\|t\| \cdot \frac{t}{\|t\|}\right) \qquad Q(\lambda t) = \lambda^2 Q(t).$$

$$= \|t\|^2 \cdot \underbrace{Q\left(\frac{t}{\|t\|}\right)}_{\geq m} \geq \|t\|^2 \cdot m$$

$Q : S^{n-1} \to \mathbb{R}$

$Q(S^{n-1})$ compact $[a, b]$

$Q(u) = a$, > 0, $a > 0$

$$f(a+t) - f(a) = Q(t) + \|t\|^2 \cdot E(t).$$

$$\begin{cases} E(t) \to 0 \\ \text{as } t \to 0 \end{cases}$$

$\Rightarrow \exists\, \epsilon > 0 \quad |E(t)| < \frac{m}{2} \ \forall\, t \in \mathbb{R}^n.$

$0 < \|t\| < \epsilon.$

$$f(a+t) - f(a) > \|t\|^2 \cdot m - \|t\|^2 \cdot \frac{m}{2}$$

$$= \|t\|^2 \cdot \frac{m}{2} > 0.$$

<u>Proof (2)</u>. as similar as (1).

<u>OR</u>. Apply part (1) as f to get (2).

3) $f(a+\lambda t) - f(a) = Q(\lambda t) + \|\lambda t\|^2 \cdot E(\lambda t)$

$= \lambda^2 Q(t) + \lambda^2 \|t\|^2 \cdot E(\lambda t)$ where $\lambda \neq 0 \in \mathbb{R}$.

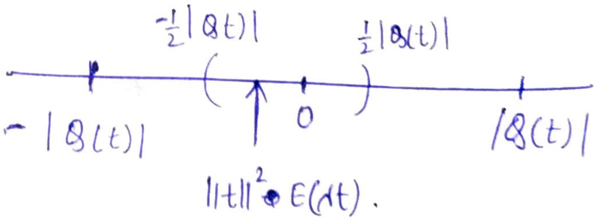
$= \lambda^2 \left[Q(t) + \|t\|^2 \cdot E(\lambda t) \right]$

Let $Q(t) \neq 0$.

$E(y) \to 0$ as $y \to 0 \Rightarrow \exists\ r > 0$ if $0 < \lambda < r$

$\Rightarrow \|t\|^2 \cdot |E(\lambda t)|$

$< \frac{1}{2} |Q(t)|$



$\Rightarrow Q(t) + \|t\|^2 \cdot E(\lambda t)$. has the same sign in $Q(t)$.

$Q(t)$ has both (+)ve and (−)ve

$\Rightarrow f(a+\lambda t) - f(a)$ has both (+)ve and (−)ve.

$\Rightarrow$ it has a saddle point.

Q: How to determine a quadratic form is (+)ve or (-)ve definite?

Theorem: Suppose ~~Let~~ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $D_{i,j}(f)$ exists and are cts. at a stationary point 'a'

$$A = D_{11} f(a)$$

$$B = D_{12} f(a) = D_{21} f(a).$$

$$C = D_{22} f(a)$$

$$\Delta = \det \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

1) if $\Delta > 0$, $A > 0$,

$\Rightarrow$ At 'a' f has a local min.

2) if $\Delta > 0$, $A < 0$

$\Rightarrow$ At 'a' f has a local max.

3) if $\Delta < 0$, At 'a', f has a saddle point.

Proof Exercise.