Assignment 3

Infinite Series and Infinite Products, MTH 301A, 2022

1 Infinite Series

1. Let $|a| < 1$ and set $S_n = \sum_{k=0}^{n} a^k$ and $T_n = \sum_{n=0}^{n} (k+1)a^k$.

   (a) Show that $S_n^2 = \sum_{k=0}^{n} (k+1)a^k + \sum_{k+1}^{n} (n+1-k)a^{n+k}$.

   (b) Show that $|T_n - S_n^2| \leq \frac{n(n+1)}{2}|a|^{n+1}$.

   (c) Show that $\lim_{n\to\infty} T_n = (\lim_{n\to\infty} S_n)^2$. Hence obtain a formula for this sum.

   (d) Evaluate $\sum_{k=0}^{\infty} \frac{n+1}{3^n}$.

2. Suppose $a_n \geq a_{n+1} \geq 0$. Prove that the series $\sum a_n$ converges if and only is $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. As an application show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha}$, $\alpha > 1$ converges.

3. (**To be submitted**) Let $a_n > 0$ and let $s_n = a_1 + \cdots + a_n$. Prove

   (a) if $\sum a_n$ converges then $\sum \frac{a_n}{s_n}$ converges.[Hint: compare $\frac{a_n}{s_n}$ with $\frac{a_n}{\lim s_n}$]

   (b) if $\sum a_n$ diverges then $\sum \frac{a_n}{s_n}$ diverges and $\sum \frac{a_n}{{s_n}^2}$ converges.

4. Let $a_n > 0$ and let $r_n = \sum_{k=n+1}^{\infty} a_n$. Suppose $\sum a_n$ converges then prove that

   (a) $\sum \frac{a_n}{r_n}$ diverges. [Hint: for $m > n$ $\frac{a_n}{r_n} + \cdots + \frac{a_m}{r_m} > 1 - \frac{r_m}{r_n}$.]

   (b) $\sum \frac{a_n}{\sqrt{r_n}}$ converges.[Hint: $\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$.]

5. Let $a_n > 0$ and $\sum a_n$ converges then $\sum a_n^p$ converges for $1 < p < \infty$. Also show that $\sum \sqrt{a_n a_{n+1}}$ and $\sum \frac{\sqrt{a_n}}{n}$ converges.

6. (*Pringsheim's Theorem*) If $a_n \geq a_{n+1} > 0$ and $\sum a_n$ converges then $na_n \to 0$ as $n \to \infty$.

7. (*Integral Test*) Let f be a positive, monotone decreasing function on $[1, \infty)$. Show that the sequence $\{f(n)\}$ is summable if and only if $\int_1^\infty f(x)dx < \infty$.

8. (**To be submitted**)

   Definition 1.1. *Consider the series $\sum a_n$. Denote $s_n = \sum_{k=1}^{n} a_k$ and $\sigma_N = \frac{s_1+s_2+\cdots+s_N}{N}$. The quantity σ_N is called N-th Cesàro mean. If σ_N converges to a limit then we say that the series $\sum a_n$ is Cesàro summable.*

   Definition 1.2. *A series $\sum a_n$ is said to be Abel summable to a if for every $0 \leq r < 1$, the series $A(r) = \sum a_n r^n$ converges and $\lim_{r\to 1} A(r) = a$.*

The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.

(a) Show that if the series $\sum a_n$ converges to a then it is Abel summable to a. [Hint: Enough to show for $a = 0$. $\sum\limits_{n=1}^{N} a_n r^n = (1-r)\sum\limits_{n=1}^{N} s_n r^n + S_N r^{N+1}$]

(b) Show that there exists a series that is Abel summable but do not converge. [Hint: Try $a_n = (-1)^n$]

(c) Show that if $\sum a_n$ is Cesàro summable then it is Abel summable.

(d) Give an example of a series Abel summable but not Cesàro summable. [Hint: Try $a_n = (-1)^n n$.]

The result above can be summarized as

$$\text{convergent} \;\Rightarrow\; \text{Cesaro summable} \;\Rightarrow\; \text{Abel summable} \tag{1}$$

and none of the arrows can be reversed.

9. Show that a conditionally convergent series has a rearrangement converging to $+\infty$.

2 Infinite Product

Definition 2.1. *Given a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, let*

$$\begin{aligned} p_1 &= a_1 \\ p_2 &= a_1 a_2 \\ &\vdots \\ p_n &= \prod_{k=1}^{n} a_k \end{aligned}$$

p_n is called n^{th} partial product.

(i) If infinitely many a_n are zero then we say product diverges to zero.

(ii) If no a_n is zero we say product converges to p if and only if $p_n \to p$ as $n \to \infty$ we write $p = \prod\limits_{n=1}^{\infty} a_n$.

If $p_n \to 0$ then we say product diverges to 0.

(iii) If there exists a N such that $0 \neq a_n$, $\forall n > N$ and $\prod\limits_{n=N+1}^{m} a_n$ converges as $m \to \infty$ then value of the product is $a_1 \ldots a_N \prod\limits_{n=N+1}^{\infty} a_n$.

The value of a convergent infinite product can be zero if and only if a finite number of a_n is zero.

1. (Cauchy Criterion) Show that the product $\prod\limits_{n=1}^{\infty} a_n$ converges if and only if given any $\epsilon > 0$ there exists an integer N_0 such that

$$\left| \prod_{k=n+1}^{m} a_n - 1 \right| < \epsilon \tag{2}$$

whenever $m > n \geq N_0$.

2. Show that if $a_n > 0$, $\forall n \geq 1$, then the infinite product $\prod\limits_{n=1}^{\infty} (1 + a_n)$ converges if and only if the infinite series $\sum\limits_{n=1}^{\infty} a_n$ converges.

   In particular $\prod\limits_{n=1}^{\infty} (1 + a_n)$ converges absolutely if and only if $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

   (a) Give example of $\{a_n\}$ such that conditionally convergent but $\prod\limits_{n=1}^{\infty} (1 + a_n)$ diverges.

   (b) Let $a_{2n-1} = \frac{-1}{\sqrt{n}}$ and $a_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n}$. Show that $\prod\limits_{n=1}^{\infty} (1 + a_n)$ converges but $\sum\limits_{n=1}^{\infty} a_n$ diverges.

3. If $a_n \geq 0$ for all $n \geq 1$ then the infinite product $\prod\limits_{n=1}^{\infty} (1 - a_n)$ converges if and only if $\sum\limits_{n=1}^{\infty} a_n$ converges.