Department of Mathematics & Statistics

MTH305a

Quiz-II

Name: .... Roll No: ....

Marks: 10 Time: 25 minutes

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1. Let $\gamma\colon (a,b) \to \mathbb{R}^2$ be a smooth regular curve such that, for all $t \in (a,b)$, the normal line through $\gamma(t)$ pass through a given point. Show that the trace of the curve γ is contained in a circle. [4 marks]

   For a point $t \in (a,b)$, the normal line to the curve γ at $\gamma(t)$ is the line given by $\ell(\gamma(t), N(t)) = \{\gamma(t) + sN(t) : s \in \mathbb{R}\}$, where $N(t)$ is the unit normal to the curve γ for $t \in (a,b)$. [1 mark]

   If $p \in \mathbb{R}^2$ is a point such that all the normal lines to γ pass through p, then, for $t \in (a,b)$, there exists a real number $s(t)$ such that $\gamma(t) + s(t)N(t) = p$. Therefore, for every $t \in (a,b)$, $\gamma(t) - p = s(t)N(t)$. [1 mark]

   If we now show that $s(t)$ is a constant, say C, for all $t \in (a,b)$, then it would follow that $\|\gamma(t) - p\| = |s(t)|\|N(t)\| = C$. This will prove that the trace of the curve is contained in a circle centered at the point p. [1 mark]

   Let $f(t) = \|\gamma(t) - p\|^2$ for $t \in (a,b)$. Then the function f is differentiable and $f'(t) = 2\langle\gamma'(t), \gamma(t) - p\rangle = 2s(t)\langle\gamma'(t), N(t)\rangle = 0$. Hence the function f is constant. [1 mark]

2. Let $a, b > 0$ and $\gamma(t) = (a\cos t, a\sin t, bt)$ for $t \in \mathbb{R}$. Find the arc length parametrization of the curve γ. [2 marks]

   For the curve $\gamma(t) = (a\cos t, a\sin t, bt)$, the tangent vector at a point $t \in \mathbb{R}$ is $\gamma'(t) = (-a\sin t, a\cos t, b)$ and $\|\gamma'(t)\| = \sqrt{a^2+b^2}$. Therefore $s(t) = \int_0^t \|\gamma'(t)\|dt = \sqrt{a^2+b^2}\,t$ and this shows that $t = \frac{s}{\sqrt{a^2+b^2}}$. [1 mark]

   Hence the arc length parametrization of the curve γ is given by $\sigma(s) = (a\cos(\frac{s}{\sqrt{a^2+b^2}}), a\sin(\frac{s}{\sqrt{a^2+b^2}}), b\frac{s}{\sqrt{a^2+b^2}})$. [1 mark]

3. Find the curvature and torsion of the curve defined by $\gamma(t) = e^t(\cos t, \sin t, 1)$ for $t \in \mathbb{R}$. [4 marks]

   For the curve $\gamma(t) = e^t(\cos t, \sin t, t)$, we have

   $$\begin{aligned} \gamma'(t) &= e^t(\cos t - \sin t, \cos t + \sin t, 1) \\ \gamma''(t) &= e^t(-2\sin t, -2\cos t, 1) \text{ and} \\ \gamma'''(t) &= e^t(-2(\cos t + \sin t), 2(\cos t - \sin t), 1) \end{aligned} \qquad [1 \text{ mark}]$$

   Further $\gamma'(t) \times \gamma''(t) = e^{2t}(\sin t - \cos t, -(\cos t + \sin t), 2)$ and $\langle\gamma'(t) \times \gamma''(t), \gamma'''(t)\rangle = 2e^{3t}$. [1 mark]

   Observe that $\|\gamma'(t)\|^3 = e^{3t}3^{3/2}$ and $\|\gamma'(t) \times \gamma''(t)\|^2 = 6e^{4t}$. [1 mark]

   Hence for very point $t \in \mathbb{R}$, the curvature $\kappa(t) = \frac{\|\gamma'(t)\times\gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{\sqrt{2}}{3}e^{-t}$ and the torsion $\tau(t) = \frac{\langle\gamma'(t)\times\gamma''(t), \gamma'''(t)\rangle}{\|\gamma'(t)\times\gamma''(t)\|^2} = \frac{e^{-t}}{3}$. [1 mark]

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