MTH 442: Time Series Analysis
Problem Set # 6

[1] Suppose $\{X_t\}$ is a stationary time series, with $\lim_{h\to\infty} \gamma(h) = 0$, prove that $\lim_{n\to\infty} Cov(\bar{X}_n, X_n) = 0$.

[2] Let $\{X_t\}$ be a stationary time series with mean 0 and autocovariance function $\gamma(.)$. Show that, if $\lim_{h\to\infty} \gamma(h) = 0$ then $\lim_{n\to\infty} Var\left(\bar{X}_n\right) = 0$. Give a counter example to show that the converse is not true.

[3] Suppose that for a sample of size 100 $(x_1, x_2, ..., x_{100})$ from an AR(1) process with unknown mean μ; we obtain $\bar{x}_{100} = 0.157$, $\hat{\phi} = 0.6$ and $\hat{\sigma}^2 = 2$. Perform an asymptotic test, at 5% level of significance, for the hypothesis $H_0 : \mu = 0$ against the alternative $H_1 : \mu \neq 0$ and construct an approximate asymptotic 95% confidence interval for μ.

[4] Let $\{X_t\}$ be a process given by $X_t - \theta = \varepsilon_t + 0.5\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2}$, where θ is unknown and $\varepsilon_t \sim WN(0,1)$. Find an asymptotic $100(1-\alpha)\%$ confidence interval for θ based on a random sample of size n.

[5] Let $\{X_t\}$ be a linear stationary time series with mean μ and ACVF

$$\gamma(h) = (0.6)^{|h|} + 2(0.3)^{|h|} + (0.1)^{|h|}$$

(a) Find $\lim_{n\to\infty} n\,Var\left(\bar{X}_n\right)$ and hence derive the asymptotic distribution of $\bar{X}_n$.

(b) Further using the asymptotic distribution, find the smallest n such that $P\left(\bar{X}_n - 0.49 \le \mu \le \bar{X}_n + 0.49\right) \ge 0.95$.

[6] Consider two uncorrelated stationary AR(1) processes $\{X_t\}$ and $\{Y_t\}$; $X_t = \mu + \frac{1}{2} X_{t-1} + \varepsilon_t$, $Y_t = \mu + \frac{1}{3} Y_{t-1} + \delta_t$. $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independent $WN(0,1)$ processes. Let $Z_t = X_t + Y_t$, using asymptotic (large sample) distribution of $\bar{Z}_n$, find the smallest n such that $P\left(\left|\bar{Z}_n - 7\mu/2\right| \le 0.098\right) \ge 0.95$. ($\tau_{0.025} = 1.96$)

[7] Let $X_1, \ldots, X_n$ be a sample from $\{X_t\}$ and $Y_1, \ldots, Y_n$ be a sample from $\{Y_t\}$. Where, $\{X_t\}$ is an AR(1), $X_t = \mu + \phi\, X_{t-1} + \varepsilon_t$, $|\phi| < 1$ and $\{Y_t\}$ is MA(1), $Y_t = \delta + \eta_t + \theta\,\eta_{t-1}$, $|\theta| < 1$. $\{\varepsilon_t\}$ and $\{\eta_t\}$ are independent $WN\left(0, \sigma^2\right)$ sequences. Define a new time series $Z_t = X_t + Y_t$. Verify whether $\bar{Z}_n \xrightarrow{m.s.} E(Z_1)$.

[8] Suppose $Z, X_1, X_2, \ldots\ldots, X_n$ be a set of random variables with zero mean, known finite variance and covariances. Let $P_{(X_1,\ldots,X_n)} Z$ denote the BLP of Z based on $X_1, X_2, \ldots\ldots, X_n$. Prove that

$$Var\left(Z - P_{(X_1,\ldots,X_n)} Z\right) = Var(Z) - Var\left(P_{(X_1,\ldots,X_n)} Z\right).$$

[9] Let $\{X_t\}$ be a stationary time series defined by $X_t = \begin{cases} Z_1 & \text{if } t \text{ is even} \\ Z_2 & \text{if } t \text{ is odd} \end{cases}$

where, Z_1 and Z_2 are independent r.v.s with $E(Z_i)=0;\ V(Z_i)=\sigma^2; i=1,2$. Obtain the BLP of X_{h+1} based on $(X_h, X_{h-1}, \ldots, X_2)$ for h even.

[10] For a zero mean causal *AR(p)* process $Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots\ldots - \phi_p Y_{t-p} = \varepsilon_t;\ \ \varepsilon_t \sim WN(0,\sigma^2)$. Find the BLP of Y_{n+1} based on $(Y_n, \ldots., Y_1)$.

[11] Consider the causal AR(1) model $X_t - \phi X_{t-1} = \varepsilon_t; \varepsilon_t \sim WN(0,\sigma^2)$. Suppose that only every second value is possible to observe, i.e. we have $Y_t = X_{2t}$. Find the BLP of Y_{t+1} based on $Y_t, Y_{t-1}, \ldots, Y_1$. Also find the minimum mean square prediction error.

[12] Let $\{X_t\}$and $\{Y_t\}$ be two $AR(1)$ processes given by

$$\begin{aligned} X_t &= \phi_X X_{t-1} + Z_t \\ Y_t &= \phi_Y Y_{t-1} + Z_t + U_t \end{aligned}.$$

Where, $\{Z_t\} \sim WN(0,\sigma_Z^2), \{U_t\} \sim WN(0,\sigma_U^2)$; $\{Z_t\}$ and $\{U_t\}$ are independent; $|\phi_X| < 1$ and $|\phi_Y| < 1$. Derive the BLP of X_{t+1} based on Y_t only.

[13] $\{X_t\}$ is a stationary $AR(1)$ process; $X_t = 0.5 X_{t-1} + \varepsilon_t; \varepsilon_t \sim WN(0,1)$ and $Y_t = X_t + \eta_t$; $\eta_t \sim WN(0,\sigma^2)$, where ε_t and η_t are independently distributed. Show that the BLP of Y_3 based on Y_2 and Y_1 is given by $\dfrac{6(1+\sigma^2)Y_2 + 3\sigma^2 Y_1}{(4+3\sigma^2)^2 - 4}$.

[14] Let $\{X_t\}$ be a MA(1) process $X_t = \varepsilon_t + \theta\varepsilon_{t-1},\ \varepsilon_t \sim WN(0,\sigma^2)$. Find the BLP of a missing value X_2 based on X_1, X_3 and X_4.

[15] Let $X_1, \ldots, X_n$ be a sample from $\{X_t\}$ and $Y_1, \ldots, Y_n$ be a sample from $\{Y_t\}$. Where, $\{X_t\}$ is an AR(1), $X_t = \mu + \phi X_{t-1} + \varepsilon_t,\ |\phi| < 1$ and $\{Y_t\}$ is MA(1), $Y_t = \delta + \eta_t + \theta\eta_{t-1},\ \ |\theta| < 1$. $\{\varepsilon_t\}$ and $\{\eta_t\}$are independent $WN(0,\sigma^2)$ sequences. Derive the best linear predictor of X_{n+1} based on $Y_1, \ldots, Y_n$ and the corresponding mean square prediction error.

[16] $\{X_t\}$ is a covariance stationary $AR(1)$ process; $X_t = 0.5 X_{t-1} + \varepsilon_t; \varepsilon_t \sim WN(0,\sigma_\varepsilon^2)$ and $Y_t = X_t + \eta_t;\ \eta_t \sim WN(0,\sigma_w^2)$, ε_t and η_t are independently distributed.

(a) Find the BLP of X_{t+2} based on Y_t and Y_{t-1}.

(b) Find the PACF at lag 2 of $\{Y_t\}$.

[17] Let $\{X_t\}$ be an MA(1) process $X_t = \varepsilon_t + 2\varepsilon_{t-1} - \varepsilon_{t-2},\ \varepsilon_t \sim WN(0,1)$.

(a) Find the BLP of X_5 based on X_4 and X_3.

(b) Find the mean square prediction error corresponding to the BLP obtained in (a).

(c) Find the PACF of $\{X_t\}$ at lag 2.

(d) Find the relationship between the PACF at lag 2, obtained in (c) and the coefficient of X_3 in the BLP obtained in (a).