Suppose $\{X_t\}$ is covariance stationary

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\mu_X = E(X_t) = E(\phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t) = \mu_X \phi_1 + \mu_X \phi_2$$

i.e. $\mu_X (1 - \phi_1 - \phi_2) = 0$

$1 - \phi_1 - \phi_2 \neq 0$ as $\{X_t\}$ is covariance stationary and hence root of $\phi(z) = 0$ can't be on unit circle

$$\Rightarrow \mu_X = 0.$$

$$\sigma_X^2 = V(X_t) = V(\phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t)$$

$$= \phi_1^2 \sigma_X^2 + \phi_2^2 \sigma_X^2 + 2\phi_1 \phi_2 \gamma_1 + \sigma^2$$

$$\left(cov(\epsilon_t, X_{t-1}) = 0 = Cov(\epsilon_t, X_{t-2}) \right)$$

i.e. $\sigma_X^2 = \phi_1^2 \sigma_X^2 + \phi_2^2 \sigma_X^2 + 2\phi_1 \phi_2 \rho_1 \sigma_X^2 + \sigma^2$

$$\Rightarrow \gamma_0 = \sigma_X^2 = \frac{\sigma^2}{1 - \phi_1^2 - \phi_2^2 - 2\phi_1 \phi_2 \rho_1}$$

Now $\gamma_1 = E(X_{t+1}, X_t)$

$$= E(\phi_1 X_t + \phi_2 X_{t-1} + \epsilon_{t+1}) X_t$$

i.e. $\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$

i.e. $\rho_1 = \phi_1 + \phi_2 \rho_1$

i.e. $\rho_1 = \phi_1 / 1 - \phi_2$

$$\Rightarrow \gamma_0 = \sigma_X^2 = \frac{\sigma^2}{1 - \phi_1^2 - \phi_2^2 - 2\phi_1 \phi_2 \left(\frac{\phi_1}{1 - \phi_2} \right)}$$

$$\gamma_0 = \frac{\sigma^2 (1 - \phi_2)}{(1 - \phi_1 - \phi_2)(1 - \phi_2 + \phi_1)(1 + \phi_2)}$$

Note: factors in γ_0 are corresponding to the region of stationarity conditions

AR Yule-Walker eqⁿ and ACF for AR(2)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\gamma_X(h) = \text{Cov}(X_{t+h}, X_t)$$

$\forall h > 0$; $\gamma_X(h) = \text{Cov}(\phi_1 X_{t+h-1} + \phi_2 X_{t+h-2} + \epsilon_{t+h}, X_t)$

i.e. $\gamma_X(h) = \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2) \quad \forall h > 0$

The ACVF/ACF satisfies the same 2^{nd} order diff eqⁿ as the data process.

Yule-Walker eqⁿ for ACF

$$\rho_X(h) = \phi_1 \rho_X(h-1) + \phi_2 \rho_X(h-2)$$

Yule-Walker eqⁿ can be used to express the ACF/ACVF seq in terms of ϕ_1 & ϕ_2

Take $h=1$ in Y-W; $\rho_1 = \phi_1 + \phi_2 \rho_1$

$h=2$ in Y-W; $\rho_2 = \phi_1 \rho_1 + \phi_2$

$$\Rightarrow \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\rho_2 = \phi_1 \frac{\phi_1}{1-\phi_2} + \phi_2 = \frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2}$$

$$\rho_3 = \phi_1 \left(\frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2} \right) + \phi_2 \left(\frac{\phi_1}{1-\phi_2} \right)$$

Conversely, Y-W eqⁿ also enables us to express ϕs in terms of ρs.

$$\phi_1 = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2} \quad \& \quad \phi_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}$$

The above suggest a way for parameter estimation from sample ACF sequence.

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

$$\phi(B) X_t = \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

AR(p) is stationary if roots of $\phi(z) = 0$, i.e. roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$ all lie outside the unit circle.

i.e. roots of $y^p - \phi_1 y^{p-1} - \cdots - \phi_p = 0$ all lie inside the unit circle.

Suppose $\{X_t\}$ is covariance stationary AR(p)

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

$$\mu_X = E(X_t) = E(\phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t)$$

$$= \phi_1 \mu_X + \cdots + \phi_p \mu_X$$

i.e. $\mu_X(1 - \phi_1 - \cdots - \phi_p) = 0$

$1 - \phi_1 - \cdots - \phi_p \neq 0$ (as $\{X_t\}$ is covariance stat)

$\Rightarrow \mu_X = 0$

ACVF structure of AR(p)

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$
$$= E(X_{t+h} X_t)$$
$$= E(\phi_1 X_{t+h-1} + \phi_2 X_{t+h-2} + \cdots + \phi_p X_{t+h-p} + \epsilon_{t+h}) X_t$$
$$= \begin{cases} \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2) + \cdots + \phi_p \gamma_X(h-p) + \sigma^2 & h=0 \\ \phi_1 \gamma_X(h-1) + \phi_2 \gamma_X(h-2) + \cdots + \phi_p \gamma_X(h-p), & h=1,2,\ldots \end{cases}$$

& $\gamma_X(-h) = \gamma_X(h)$

$$\rho_h = \phi_1 \rho_{h-1} + \cdots + \phi_p \rho_{h-p} \quad ; \quad h=1,2,\ldots$$

$h=1$; $\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 + \cdots + \phi_p \rho_{p-1}$

$h=2$; $\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 + \cdots + \phi_p \rho_{p-2}$

$\vdots$

$h=p$; $\rho_p = \phi_1 \rho_{p-1} + \cdots + \phi_p \rho_0$

i.e.
$$\underset{\sim}{\rho}_p = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_p \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{p-1} \\ & 1 & \cdots & \rho_{p-2} \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix}$$

i.e. $\underset{\sim}{\rho}_p = A_p \underset{\sim}{\phi} \Rightarrow \underset{\sim}{\phi} = A_p^{-1} \underset{\sim}{\rho}_p$

ARMA(p,q) process

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$\phi_p \neq 0$, $\theta_q \neq 0$; $\epsilon_t \sim WN(0, \sigma^2)$; $Cov(\epsilon_t, X_{t-j}) = 0 \; \forall j > 0$

$$\phi(B) X_t = \theta(B) \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

ARMA(p,q) process is stationary if AR part is stationary

i.e. ARMA(p,q) is stationary if the roots of $\phi(z) = 0$ all lie outside the unit circle

i.e. If the roots of $1 - \phi_1 z - \cdots - \phi_p z^p = 0$ all lie outside unit circle

i.e. If the roots of $y^p - \phi_1 y^{p-1} - \cdots - \phi_p = 0$ all lie inside the unit circle.

It is easy to see that

$E X_t = 0$ if $\{X_t\}$ is covariance stationary