Mathematical formulation of a time series

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be an index set

<u>Def^n</u>: A real valued time series is a real valued function $X(t,\omega)$ defined on $T \times \Omega$, $\ni$ for a fixed t, $X(t,\omega)\left(= X_t(\omega) = X_t, \text{ say}\right)$ is a random variable defined on $(\Omega, \mathcal{F}, P)$.

A time series is thus a collection $\{X_t : t \in T\}$ of random variables.

We can define joint distribution function of a finite set of random variables $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ from the collection $\{X_t : t \in T\}$ is

$$F_{X_{t_1},\ldots,X_{t_n}}(x_1,\ldots,x_n) = P(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n)$$

Concept of stationarity of time series: $\{X_t\}$ process is $\ni$ "statistical properties" of the process do not change over time, i.e. realizations come from a stable physical system which has achieved a "steady-state statistical equilibrium" mode.

<u>Remark</u>: Different forms of stationarity concept is defined under different paradigms of quantifying "statistical equilibrium".

Important definitions of stationarity

(I) <u>Strict stationary</u>: A process $\{X_t\}$ is said to be strict stationary or completely stationary, if for all $n \geq 1$, any admissible $t_1, t_2, \ldots, t_n$ and any k (integer), the joint dist^n of $(X_{t_1}, \ldots, X_{t_n})$ is identical with the joint dist^n of $(X_{t_1+k}, \ldots, X_{t_n+k})$.

i.e. $F_{X_{t_1}, \ldots X_{t_n}}(x_1, \ldots, x_n) = F_{X_{t_1+k}, \ldots X_{t_n+k}}(x_1, \ldots, x_n) \quad \forall n \geq 1, \forall t_1, \ldots t_n \ \& \ \forall k \geq 1$

i.e. the jt dist^n of any finite set of r.v.s is invariant w.r.t. time shift.

(II) <u>Stationarity upto order m</u>: A time series process $\{X_t\}$ is said to be stationary upto order m, if, for all $n \geq 1$, any admissible $t_1, t_2, \ldots, t_n$ and any integer k, all the joint moments upto order m of $\{X_{t_1}, \ldots, X_{t_n}\}$ exist and equal the corresponding joint moments upto order m of $\{X_{t_1+k}, \ldots, X_{t_n+k}\}$

i.e. $E\left(X_{t_1}^{m_1} X_{t_2}^{m_2} \cdots X_{t_n}^{m_n}\right) = E\left(X_{t_1+k}^{m_1} X_{t_2+k}^{m_2} \cdots X_{t_n+k}^{m_n}\right)$

$\forall n \geq 1$

$\forall k$ and $\forall$ integer $m_1, m_2, \ldots, m_n$ $(m_i \geq 0)$

$\ni \sum_{i=1}^{n} m_i \leq m$

Note: In particular, setting $m_2 = m_3 = \cdots = m_n = 0$, we get that for any t and $\forall m_1 \le m$

$$E\left(X_{t_1}^{m_1}\right) = E\left(X_{t_1+K}^{m_1}\right) \quad \forall K$$

$$\Rightarrow E\left(X_{t_1}^{m_1}\right) = \underbrace{E\left(X_0^{m_1}\right)}_{\text{const indep of } t} \quad \text{setting } K = -t$$

Also $$E\left(X_t^{m_1} X_s^{m_2}\right) = E\left(X_{t+K}^{m_1} X_{s+K}^{m_2}\right) \quad \forall K$$

$\forall m_1, m_2 \ni m_1 + m_2 \le m$

$$= \underbrace{E\left(X_0^{m_1} X_{s-t}^{m_2}\right)}_{\text{indep of } t \text{ and is a } f^n \text{ of } (s-t) \text{ only}} \quad \text{for } K = -t$$

Important special cases of "stationarity upto order m"

(i) Order 1 stationary

$$EX_t \text{ exists } \& \; EX_t = \mu \;\; \forall t$$

This is referred to as "mean stationarity"

Thus a time series $\{X_t\}$ is mean stationary if EX_t exists and is indep of t.

(ii) Order 2 stationary: EX_t^2 and EX_tX_s exist $\forall t, s$ and

(a) $EX_t = \mu \leftarrow$ const, indep of t

(b) $EX_t^2 = \mu_2'$ const; indep of t; hence

$V(X_t) = \sigma^2$ is also indep of t

& (c) $E(X_t X_s) = E(X_{t+h} X_{s+h}) = \underline{E(X_0 X_{s-t})}$ — (*)

f^n of $(s-t)$ only and indep of t

$(s-t)$: time difference

(*) $\Rightarrow$ $Cov(X_t, X_s) = \underline{E(X_t X_s) - \mu^2}$

f^n of time difference $(s-t)$ only and is indep of t

<u>Remark</u>: The order 2 stationarity is also referred to as covariance stationary (or weak stationary or stationary in the wide sense). This form of stationarity is the most widely used form of stationarity.

Remark: If $\{X_t\}$ is strict stationary, then $\{X_t\}$ is also covariance stationary provided moments upto order 2 exist for the joint distribution.

Pf: For $n=1$, defn of strict stationarity implies that X_t has the same distn for each t in the index set,

$$\text{i.e. } F_{X_{t_1}}(x) = F_{X_{t_1+k}}(x) \quad \forall k \qquad -(*)$$

If $EX_t^2 < \infty$, then $(*)$ in particular implies that EX_t & VX_t exists and are indep of t (as the distns are identical $\forall t$).

Further, take $n=2$, defn of strict stationarity implies that the jt distn of (X_{t_1}, X_{t_2}) and (X_{t_1+h}, X_{t_2+h}) are identical

$$\text{i.e. } (X_{t_1}, X_{t_2}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h})$$

$$\Rightarrow \operatorname{Cov}(X_{t_1}, X_{t_2}) = \operatorname{Cov}(X_{t_1+h}, X_{t_2+h}) \quad \forall t_1, t_2 \quad \forall h$$

$$= \operatorname{Cov}(X_0, X_{t_2-t_1}); \quad h = -t_1$$

fn of $(t_2 - t_1)$ only

$\Rightarrow \{X_t\}$ is covariance stationary.