Contents

Chapter 1

The Field of Complex Numbers

We assume that the reader has already been acquainted with the set $\mathbb{C}$ of complex numbers, the arithmetical operations on $\mathbb{C}$. The aim of this chapter is essentially to put these concepts on a rigorous footing and give a few exercises to deal with $\mathbb{C}$.

1.1 Algebra on $\mathbb{C}$

The readers must have learnt algebraic manipulations of the symbols $a+ib$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. We make those manipulations rigorous by giving a meaning to these symbols.

Definition 1.1.1. The set $\mathbb{C}$ of complex numbers is the set $\mathbb{R}^2$ with the following rules of addition and multiplication:

(A) $(x, y) + (u, v) := (x + u, y + v)$.

(M) $(x, y)(u, v) := (xu - yv, xv + yu)$.

Observe that our rule for multiplication was motivated by the fact that if we identify (x, y) with $x + iy$, (u, v) with $(u + iv)$ and carry out the multiplication $(x + iy)(u + iv)$ as we have been doing all along and identify the resulting expression with an element of $\mathbb{R}^2$ we get (M). Any element of $\mathbb{C}$ is called a complex number.

Theorem 1.1.2. *$\mathbb{C}$ is a field with respect to the addition and multiplication defined as above. That is, the rules of addition and multiplication satisfy the following properties:*

(A) *$(\mathbb{C}, +)$ is an abelian group with $(0, 0)$ as the additive identity.*

(B) *The set $\mathbb{C}^* := \mathbb{C} \setminus \{(0, 0)\}$ is an abelian group under multiplication with $(1, 0)$ as the multiplicative identity.*

(D) *The distribution law holds: for all (a, b), (x, y), $(u, v) \in \mathbb{C}$, we have*

$$(a, b)[(x, y) + (u, v)] = (a, b)(x, y) + (a, b)(u, v).$$

Proof. The element $(-x, -y)$ is the additive inverse of $(x, y) \in \mathbb{C}$. The element $(x/(x^2 +$

$y^2), -y/(x^2+y^2))$ is the multiplicative inverse of $(x,y) \in \mathbb{C}^*$. The rest are routine verifications and hence left to the reader. □

Remark 1.1.3. If you are intrigued how one could have guessed the inverse of a nonzero element, you may refer to Theorem 1.1.10.

Notation. We identify $\mathbb{R}$ with $\{(x,0) : x \in \mathbb{R}\}$ via the field isomorphism h of $\mathbb{R}$ into $\mathbb{C}$ given by $h(x) := (x,0)$. Hereafter, whenever we say a real number $x \in \mathbb{C}$ we mean $h(x) = (x,0)$. Note that the element $(0,1)$ is such that $(0,1)(0,1) = (-1,0) = -1$, that is, $(0,1)$ is a square root of -1. We denote this element by i. We can now assign a meaning to the symbol iy if $y \in \mathbb{R}$ — it is the product of the two complex numbers $(0,1)$ and $(y,0)$! Hence $x+iy$ also makes sense: $x+iy$ stands for the complex number $(x,0)+(0,1)(y,0)$. We shall also use the time-honoured notation $x+iy$ for the element $(x,y) \in \mathbb{C}$. The number x (respectively, y) is called the real (respectively, imaginary) part of the complex number $x+iy$. They are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively. When we say $z \in \mathbb{C}$ we mean that $z = x+iy$ for some $x, y \in \mathbb{R}$. For $z = x+iy = (x,y) \in \mathbb{C}$, the complex conjugate $\overline{z}$ of z is the complex number $x - iy = (x,-y) \in \mathbb{C}$. Note that $\operatorname{Re} z = (z+\overline{z})/2$ and $\operatorname{Im} z = (z-\overline{z})/2i$. A complex number $z = x+iy \in C$ is said to be real if $\operatorname{Im} z = 0$. It is said to be purely imaginary if $\operatorname{Re} z = 0$. The set of nonzero complex numbers is denoted by $\mathbb{C}^*$.

Ex. 1.1.4. Two complex numbers are equal iff their real parts are the same and their imaginary parts are the same.

Proposition 1.1.5. *For all complex numbers $z, w \in \mathbb{C}$,*
(i) $\overline{z+w} = \overline{z} + \overline{w}$.
(ii) $\overline{zw} = \overline{z} \cdot \overline{w}$.
(iii) $\overline{\overline{z}} = z$.
(iv) *z is real iff $\overline{z} = z$.*
(v) *z is purely imaginary iff $z = -\overline{z}$.*
(vi) $\overline{(1/z)} = 1/\overline{z}$ *for all $z \in \mathbb{C}^*$.*

Proof. The proof is a routine verification and hence is left to the reader. □

Proposition 1.1.6. *Every complex number $\alpha = a+ib$ has a square root in $\mathbb{C}$.*

Proof. If $z = x+iy$ is such that $z^2 = \alpha$, then we get $x^2 - y^2 = a$ and $2xy = b$. Hence $b^2 = 4x^2y^2 = 4x^2(x^2-a)$ so that $(2x^2-a)^2 = a^2+b^2$. From this x and hence y can be solved.

Since $a^2+b^2 \geq 0$, we let $\sqrt{a^2+b^2}$ denote the unique nonnegative square root of a^2+b^2. Thus we have $2x^2 - a = \sqrt{a^2+b^2}$, or $2x^2 = a + \sqrt{a^2+b^2}$. We now observe that $\sqrt{a^2+b^2} \geq |a|$ and hence $a + \sqrt{a^2+b^2} \geq 0$. Therefore we get the values of x and y as below:

$$x = \frac{\sqrt{\sqrt{a^2+b^2}+a}}{\sqrt{2}}, \qquad y = \frac{\sqrt{\sqrt{a^2+b^2}-a}}{\sqrt{2}}.$$

Did you note that we needed real analysis to prove this innocuous looking result? □

Theorem 1.1.7. *Every quadratic polynomial with complex coefficients has roots in $\mathbb{C}$.*

Proof. Let $p(z) := az^2 + bz + c$ be the polynomial, with $a \neq 0$. Proceed as in the real case. Completing the squares, we get

$$az^2 + bz + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

Hence $z + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$ or $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. □

Definition 1.1.8. We define the modulus $|z|$ of $z = x+iy \in \mathbb{C}$ by $|z| = \sqrt{x^2+y^2}$. Note that $|z|^2 = z\overline{z}$ so that $|z| = \sqrt{z\overline{z}}$. If we think of $z = x+iy$ as a vector in $\mathbb{R}^2$, then $|z| = \|(x,y)\|$, the norm of the vector with respect to the dot product on $\mathbb{R}^2$. (See Ex. 1.1.9 below.)

Ex. 1.1.9. The dot product on $\mathbb{R}^2$ is given by $z \cdot w = \operatorname{Re} z\overline{w}$, where the vector (x,y) (respectively, (u,v)) is identified with the complex number $z = x+iy$ (respectively, $u+iv$).

Theorem 1.1.10. *The following are true for all complex numbers z, z_i, $1 \le i \le 2$.*
(i) $|z|^2 = z\overline{z}$.
(ii) $1/z = \overline{z}/|z|^2$, *for all* $z \neq 0$.
(iii) $|z| = |\overline{z}| = |-z|$.
(iv) $|z_1z_2| = |z_1|\,|z_2|$.
(v) $-|z| \le \operatorname{Re} z \le |z|$ *for* $z \in \mathbb{C}$. *The equality holds in the first (respectively, in the second) inequality iff z is real and nonpositive (respectively, iff z is real and nonnegative).*

Proof. As a sample, let us prove v). If we write $z = x+iy$, then $|z| = \sqrt{x^2+y^2}$ so that $|z| \ge |x|$. Hence v) follows. The equality $-|z| = \operatorname{Re} z$ holds iff $\operatorname{Re} z \le 0$ and $\operatorname{Im} z = 0$, that is, z is a nonpositive real. The other equality is treated similarly. □

Theorem 1.1.11 (Triangle Inequality). *For any $z, w \in \mathbb{C}$, we have*
(i) $|z+w| \le |z|+|w|$. *Equality holds iff $z = 0$ or $w/z \ge 0$ (i.e. w is a nonnegative multiple of z).*
(ii) $||z|-|w|| \le |z-w|$. *The equality holds iff $z = 0$ or $w/z \ge 0$.*

Proof. We have $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re} z\overline{w}$. Using v) of the last theorem, we deduce

$$\begin{aligned}|z+w|^2 &\le |z|^2 + |w|^2 + 2\,|zw| \\ &= (|z|+|w|)^2.\end{aligned}$$

Equality in i) holds iff in the above chain of inequalities, we have $\operatorname{Re} z\overline{w} = |z|\,|\overline{w}|$. By the equality part of v) of the last theorem, this happens iff $z\overline{w} \ge 0$. Assuming $z \neq 0$, we write $z\overline{w} = t$ for some $t \geq 0$. Hence $\overline{z}w = t$, or what is the same $w = t\frac{z}{z\overline{z}} = sz$ where $s = \frac{t}{|z|^2}$. (i) follows.

Write $z = (z-w)+w$ and apply i) to get $|z| \leq |z-w|+|w|$. Hence $|z|-|w| \leq |z-w|$. Interchanging z and w in this we get $|w|-|z| \leq |z-w|$. Hence the inequality ii) follows. The equality case is left to the reader. □

Ex. 1.1.12. Verify the following: (a) $(1+i)^4 = -4$, (b) $\frac{26}{i(1-i)(3-2i)} = 5-i$.

Ex. 1.1.13. Compute (1) i^n, (2) $(\frac{1-i}{1+i})^n$ for $n \in \mathbb{N}$.

Ex. 1.1.14. $\operatorname{Re} z > 0$ iff $\operatorname{Re} 1/z > 0$ for $z \neq 0$.

Ex. 1.1.15. For $z, w \in \mathbb{C}$ with $w \neq 0$ solve the equation (in p): $z = pw$.

Ex. 1.1.16. Describe the relative positions of z, $-z$, $\overline{z}$, $\overline{-z}$, $1/z$, $1/\overline{z}$ and $-1/\overline{z}$ if $z \neq 0$ on the complex plane.

Ex. 1.1.17. Express the following quotients in the form of $a+ib \in \mathbb{C}$. (i) $z+(1/z)$, $z \neq 0$; (ii) $\frac{1+z}{1-z}$ for $z \neq 1$.

Ex. 1.1.18. Prove that for $z \in \mathbb{C}$, $z \neq 1$ we have $1+z+\cdots+z^n = \frac{1-z^{n+1}}{1-z}$, for any $n \in \mathbb{Z}_+$. Hence deduce that

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \alpha\beta^{n-2} + \beta^{n-1}, \qquad \text{for } \alpha \neq \beta.$$

Hint: Take $z = \alpha/\beta$ in the first part, if $\beta \neq 0$. If $\beta = 0$, the result is obvious.

Ex. 1.1.19. Prove the parallelogram law:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Ex. 1.1.20. Find the maximum of the modulus or the absolute value of z^2+1 on the set $\{z \in \mathbb{C} : |z| \leq 1\}$.

Ex. 1.1.21. Find the least upper bound of the set of real numbers $\{\operatorname{Im}(iz^2+1) : |z| < 3\}$.

Ex. 1.1.22. Compute the maximum of $|z^n + a|$ as z varies over $\{z \in \mathbb{C} : |z| \leq 1\}$.

Ex. 1.1.23. Prove that the equation $z^4 + z + 4 = 0$ has no solutions in $\{z : |z| < 1\}$.

Ex. 1.1.24. Prove that $|z+1| > |z-1|$ iff $\operatorname{Re} z > 0$. *Hint:* Draw a picture.

Ex. 1.1.25. If $\operatorname{Im} a > 0$ and $\operatorname{Im} b > 0$, then $\left|\frac{a-b}{a-\overline{b}}\right| < 1$. *Hint:* Draw a picture.

Ex. 1.1.26. Let $c \in C$ be such that $|c| < 1$. Prove that $|z+c| \leq |1+\overline{c}z|$ iff $|z| \leq 1$, with equality holding iff $|z| = 1$.

Ex. 1.1.27. If $|\alpha| < 1$, show that $|z| \leq 1$ iff $\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| \leq 1$. Show also that equality holds iff $|z| = 1$. *Hint:* This is a variant of the last exercise.

Ex. 1.1.28. If $|z| = 1$, prove that $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1$, for all complex numbers a, b.

Ex. 1.1.29. Under what conditions is $z^2 = |z|^2$?

Ex. 1.1.30. If $z_1 + z_2$ and $z_1\overline{z_2}$ are both real, then either (i) z_1 and z_2 are both real or (ii) $z_1 = -z_2$.

Ex. 1.1.31. If $z_1 + z_2$ and $z_1 z_2$ are both real, then either (i) z_1 and z_2 are both real or (ii) $z_1 = \overline{z_2}$.

Ex. 1.1.32. If $z + \frac{1}{z}$ is real either $\operatorname{Im} z = 0$ or $|z| = 1$.

Ex. 1.1.33. Prove that $\left|\frac{1-z}{\bar{z}-1}\right| = 1$ provided $z \neq 1$.

Ex. 1.1.34. If $|a| = 1$ and $b \neq a$ then $\left|\frac{a-b}{1-\bar{b}a}\right| = 1$. Where did you use the fact that $a \neq b$?

Ex. 1.1.35. If $a \in \mathbb{C}$ is such that $|a| < 1$, then $1 + a$ lies to the right of the y-axis. (Do you understand the meaning of this?)

Ex. 1.1.36. Prove that the equation of a circle in the complex plane is given by

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0, \qquad a \in \mathbb{C}, b \in \mathbb{R} \text{ and } |a|^2 > b. \tag{1.1}$$

Hint: The circle with centre at α and radius r is described by $|z - \alpha| = r$.

Ex. 1.1.37. Let p and q be distinct points of $\mathbb{C}$. Let $\lambda \neq 1$ be a positive real number. Then the set $\{z \in \mathbb{C} : |z - p| = \lambda\, |z - q|\}$ is a circle.

Ex. 1.1.38. Let $\lambda \in \mathbb{C}$ be such that $0 < |\lambda| < 1$. Prove the following:

(i) $\{z \in \mathbb{C} : |z - \lambda| < |1 - \bar{\lambda}z|\} = \{z \in \mathbb{C} : |z| < 1\}$.

(ii) $\{z \in \mathbb{C} : |z - \lambda| = |1 - \bar{\lambda}z|\} = \{z : |z| = 1\}$.

(iii) $\{z \in \mathbb{C} : |z - \lambda| > |1 - \bar{\lambda}z|\} = \{z : |z| > 1\}$.

Ex. 1.1.39. Let $\alpha > 0$ and Γ be the set of points z satisfying $|z - \alpha| = cx$, $z = x + iy$. Show that Γ is (i) an ellipse if $0 < c < 1$; (ii) a parabola if $c = 1$; (iii) a hyperbola if $1 < c < \infty$.

Ex. 1.1.40. Give a geometric description of the set of points z satisfying the following equations or inequalities:

(1)	$\operatorname{Im} z > 0$	(2)	$\lvert z - i\rvert < \lvert z + i\rvert$
(3)	$1 < \lvert z\rvert < 2$	(4)	$\lvert z\rvert < 1$ & $\operatorname{Im} z > 0$

(5) $|z|^2 = \operatorname{Im} z$ (6) $|z-2| > |z-3|$

(7) $\left|\dfrac{z-1}{z+1}\right| = 1$ (8) $\dfrac{1}{z} = \overline{z}$

(9) $|z - 2 + 3i| < 5$ (10) $\operatorname{Re}[(2+3i)z] > 0$

(11) $\operatorname{Im} z \geq \operatorname{Re} z$ (12) $\operatorname{Re} z = |z-2|$

(13) $|z-i| = |z+1|$ (14) $|z-1| + |z+1| \leq 4$

(15) $|z-2| - |z+2| > 3$

Ex. 1.1.41. Give a geometric description of the set of points z satisfying the following inequalities:

$$|z + (2/3)| < 1/3 \text{ and } |z + (1/3)| < |z+1|$$

Ex. 1.1.42. Describe the set $|z|^2 > z + \overline{z}$.

1.2 Polar Form and de Moivre's Theorem

A more rigorous treatment of this section will be carried out after we study the exponential map. (See Chapter 4.)

If $z = x + iy \in \mathbb{C}^*$ is identified with $(x, y) \in \mathbb{R}^2$, then we have the polar coordinates r and θ of (x, y). They are called the polar coordinates of z: $\operatorname{Re} z = |z| \cos\theta$ and $\operatorname{Im} z = |z| \sin\theta$. We write this in a shorthand: $z = |z| \operatorname{cis}\theta$ where $\operatorname{cis}\theta := \cos\theta + i\sin\theta$. This is called the polar form/representation of z. Notice that θ is not unique: If $z = |z|\operatorname{cis}\theta = |z|\operatorname{cis}\varphi$, then $\theta - \varphi$ is an integral multiple of 2π.

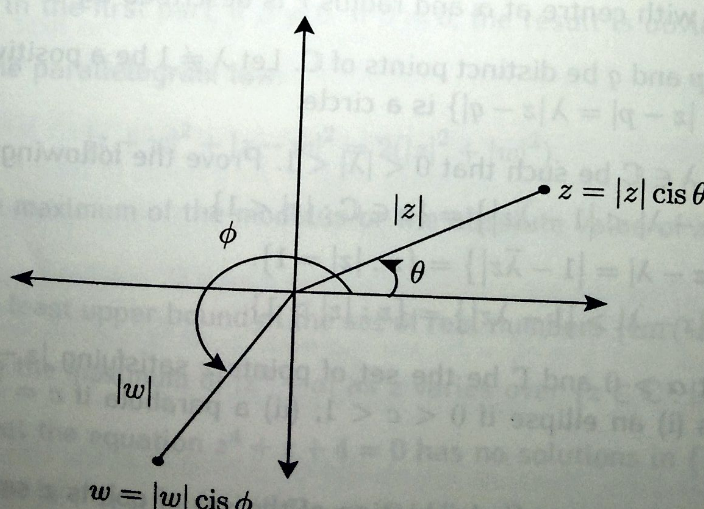

Figure 1.1: Polar representation of complex numbers

Definition 1.2.1. We define an *argument* $\arg z$ of any nonzero complex number to be any $\theta \in \mathbb{R}$ such that $z = |z| \operatorname{cis} \theta$. The *principal argument* $\mathbf{A}(z)$ of z is defined to be that argument of z which lies in $(-\pi, \pi]$.

The relation between the $\arg(z)$ and $\mathbf{A}(z)$ is given by

$$\arg(z) = \mathbf{A}(z) + 2n\pi, \qquad \text{for some } n \in \mathbb{Z}.$$

Note that excepting the obvious cases, $\tan(\mathbf{A}(z)) = y/x$.

Ex. 1.2.2. Prove the following: for any nonzero $z \in \mathbb{C}$, we have

$$\mathbf{A}\, z := \begin{cases} \sin^{-1}(y/|z|) & \text{if } x \geq 0 \\ \pi - \sin^{-1}(y/|z|) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi - \sin^{-1}(y/|z|) & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

Example 1.2.3. Let us find the polar form of $-1 + i$. $|z| = \sqrt{2}$ and $\tan\theta = y/x = -1$ so that $\theta = 3\pi/4$. Thus $-1 + i = \sqrt{2} \operatorname{cis}(3\pi/4)$.

Ex. 1.2.4. Prove the following:
1) $1 + i = \sqrt{2}\operatorname{cis}(\pi/4)$;
2) $\sqrt{3} + i = 2\operatorname{cis}(\pi/6)$;
3) $-1 + \sqrt{3}i = 2\operatorname{cis}(2\pi/3)$;
4) $-2 - 2i = 2\sqrt{2}\operatorname{cis}(-3\pi/4)$;
5) $\frac{1+i}{1-i} = \operatorname{cis}(\pi/2) = i$
6) $\mathbf{A}(1 - i) = -\pi/4$.
7) $\mathbf{A}(-\pi) = \pi$.

Ex. 1.2.5. Let $z = r \operatorname{cis}\theta$ and $w = R \operatorname{cis}\varphi$, $0 < r < R$. Prove that

$$\operatorname{Re}\left(\frac{z+w}{z-w}\right) = \frac{|w|^2 - |z|^2}{|w - z|^2} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}.$$

In terms of the polar forms, the multiplication of complex numbers admits a neat form. Let $z = |z| \operatorname{cis}\theta$ and $w = |w| \operatorname{cis}\varphi$. Using the standard trigonometric identities, we see

$$\begin{aligned} zw &= |zw| \left[(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\right] \\ &= |z|\,|w| \left[\cos(\theta + \varphi) + i\sin(\theta + \varphi)\right] \\ &= |z|\,|w| \operatorname{cis}(\theta + \varphi). \end{aligned}$$

Example 1.2.6. Let $z = -1 + i$ and $w = i$. Then $zw = -1 - i$. Note that $\mathbf{A}\, z = 3\pi/4$, $\mathbf{A}\, w = \pi/2$ so that $\arg zw = 5\pi/4$ but $\mathbf{A}\, zw = -3\pi/4$.

Theorem 1.2.7 (de Moivre). *For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.*

Proof. Use induction and the interpretation of the complex multiplication in polar form. □

de Moivre's formula is quite useful in computations.

Ex. 1.2.8. Show that (1) $(1+i\sqrt{3})^{99} = -2^{99}$; (2) $((1+i)/\sqrt{2})^{10} = ?$

Ex. 1.2.9. Use the polar form to solve Ex. 1.1.32.

Ex. 1.2.10. Show that $|z+w| = |z| + |w|$ iff either (i) one of z, w is zero or (ii) $\arg(z) = \arg(w)$. *Hint:* Use Theorem 1.1.11.

Ex. 1.2.11. Let z_k, $1 \le k \le n$ be complex numbers. Prove that $|\sum_{k=1}^n z_k| \le \sum_{k=1}^n |z_k|$ and equality holds iff there exists j such that each z_k is a positive multiple of z_j.

Example 1.2.12. Find the 4-th roots of unity, that is, the set of $z \in \mathbb{C}$ such that $z^4 = 1$. If $z = |z| \operatorname{cis}\theta$, then $z^4 = |z|^4 \operatorname{cis} 4\theta = \operatorname{cis} 0$, we see that $|z| = 1$ and that 4θ is an integral multiple of 2π. Hence we choose $\theta = 2k\pi/4$ and see the roots are, in fact, $2k\pi/4$ for $0 \le k \le 3$, thanks to the periodicity of the trigonometric functions.

More generally, the n-th roots of unity ($n \in \mathbb{N}$) are given by $\operatorname{cis}(2k\pi/n)$ for $0 \le k \le n-1$. If we let $\omega := \operatorname{cis}(2\pi/n)$, then the n-th roots of unity are given by $\{\omega^k : 0 \le k \le n-1\}$.

Ex. 1.2.13. With the notation just introduced, show that $\{\omega^k : 0 \le k \le n-1\}$ has n elements and that they form the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$ in $\mathbb{C}$. *Hint:* If $\omega^r = \omega^s$ for some $0 \le r < s \le n-1$, then $\omega^{s-r} = 1$ so that $(s-r)$ is divisible by n.

Ex. 1.2.14. Let $\alpha \neq 0$ be a complex number and $n \in \mathbb{N}$. The n distinct roots of the equation $z^n = \alpha = |\alpha| \operatorname{cis}\theta$ are given by

$$z_k = |\alpha|^{1/n} \operatorname{cis}(\theta/n)\operatorname{cis}(2k\pi/n) = |\alpha|^{1/n} \operatorname{cis}([\theta + 2k\pi]/n), \qquad 0 \le k \le n-1.$$

Ex. 1.2.15. Compute $i^{3/2}$. Answer: $\pm(-1+i)/\sqrt{2}$.

Ex. 1.2.16. Show that

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin(\theta/2)} \quad \& \quad \sum_{k=0}^{n} \sin k\theta = \frac{\cos(\theta/2) - \cos(\frac{n+1}{2})\theta}{2\sin(\theta/2)}.$$

Hint: Use Ex. 1.1.18 and separate real and imaginary parts.

Ex. 1.2.17. Let $\omega := \operatorname{cis}(2\pi/n)$ for an integer $n > 1$. Show that $\sum_{k=1}^n \omega^k = 0$. Why is this geometrically clear if n is even?

Ex. 1.2.18. Show that the n-th roots of unity other than 1 satisfy the cyclotomic equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$.

Ex. 1.2.19. Consider the $n-1$ diagonals of a regular n-gon inscribed in the unit circle in $\mathbb{C}$ obtained by connecting one vertex with all others. Show the products of the lengths of these diagonals is n.

In particular, if we choose the vertex at $z=1$ to connect to all other vertices, then the product of the complex numbers $\prod_{k=1}^{n-1}(\omega^k - 1)$ is $(-1)^{n-1}n$ where $\omega = \operatorname{cis}(2\pi/n)$.

Ex. 1.2.20. Describe the set $\{z : |z^2 - 1| < 1\}$.

Order in the Complex Field

Recall that an order (or a field order) in a field F is a subset P such that (1) F is the disjoint union of P, $-P := \{-x : x \in P\}$ and $\{0\}$ and (2) whenever $x, y \in P$, then $x+y$ and xy are in P. The subset P is called the set of positive elements of F with respect to the order P. Using this P we say that $x < y$ if $y - x \in P$ and $x \le y$ if $y - x \in P \cup \{0\}$. Note that if $x \in F$ is non-zero, then either $x \in P$ or $-x \in P$ but not both.

We claim that there is no (field) order on $\mathbb{C}$. Let, if possible, such an order exist with P as the set of positive elements in $\mathbb{C}$. Now either $i \in P$ or $-i \in P$. In either case, $-1 = i^2 \in P$ and hence $1 = (-1)^2 \in P$. This means that both $\pm 1 \in P$. Hence $0 = 1 + (-1) \in P$, a contradiction, since $\mathbb{C}$ is the disjoint union of $P, -P$ and 0.

Remark 1.2.21. However it should be noted that there exists a total order on $\mathbb{C}$, namely, the so-called dictionary order or the lexicographic order on $\mathbb{C}$. It is defined as follows: we say that $z_1 < z_2$ if either $\operatorname{Re} z_1 < \operatorname{Re} z_2$ or $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 < \operatorname{Im} z_2$. One easily verifies that this is a partial order on $\mathbb{C}$. Also, clearly, given $z_1, z_2 \in \mathbb{C}$, then either they are equal or one is 'less than' the other. That is, if $z_1 \ne z_2$, then either $z_1 < z_2$ or $z_2 < z_1$. Hence this partial order is a total order on $\mathbb{C}$.

Many beginners get confused with these. So, we suggest that the reader spends some time to understand them well.

Ex. 1.2.22 (Complex Numbers as 2×2 matrices). Let $\mathcal{A} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. Show that $\mathcal{A}$ is a field with respect to the standard matrix addition and multiplication. Show that the map

$$\varphi \colon a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is a field isomorphism and that $|a+ib| = (\det(\varphi(a+ib)))^{1/2}$.

Chapter 2

Sequences and Series of Complex Numbers

2.1 Sequences of Complex Numbers

We assume that the reader is familiar with elementary concepts and results from the theory of metric spaces. A brief account is given in the appendix. For a more detailed account with a lot of examples and exercises, the reader is referred to [17].

We start with an important notation. If we let $d(z,w) := |z-w|$, then d is a metric on $\mathbb{C}$. Hence all the concepts of metric spaces such as sequences, their convergence, continuity can be talked about in $\mathbb{C}$. For $z \in \mathbb{C}$ and $r > 0$, we let

$$B(z,r) := \{w \in \mathbb{C} : |z-w| < r\} \text{ and } B[z,r] := \{w \in \mathbb{C} : |z-w| \leq r\}.$$

They are respectively called the open disk and closed disk with centre at z and radius r. In $\mathbb{R}^2$ notation, the set $B(z,r)$ is nothing but the set of points (u,v) enclosed by the circle $(x-u)^2 + (y-v)^2 = r^2$, where $z = x + iy$

Definition 2.1.1. A sequence in $\mathbb{C}$ is a function $f\colon \mathbb{N} \to \mathbb{C}$. We let $z_n := f(n)$ and call it the n-th term of the sequence. As is the practice, we shall use (z_n) to denote the sequence f. We say a sequence (z_n) converges to $z \in \mathbb{C}$ if given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|z - z_n| < \varepsilon$, that is, $z_n \in B(z,\varepsilon)$ for all but finitely many n. We write this as $\lim_n z_n = z$ or $z_n \to z$. We say that a sequence (z_n) is convergent if there exists $z \in \mathbb{C}$ such that $z_n \to z$. . It is easy to show that if $z_n \to z$ and $z_n \to w$ then $z = w$ (Ex. 2.1.2). We call this unique number z as the limit of the sequence (z_n).

If the sequence (z_n) is not convergent, we say that (z_n) is divergent.

Ex. 2.1.2. Let $z_n \to z$ and $z_n \to w$. Show that $z = w$. *Hint:* Given $\varepsilon > 0$

$$|z-w| \leq |z-z_n| + |z_n - w| < \varepsilon + \varepsilon, \quad \text{for all } n \text{ sufficiently large.}$$

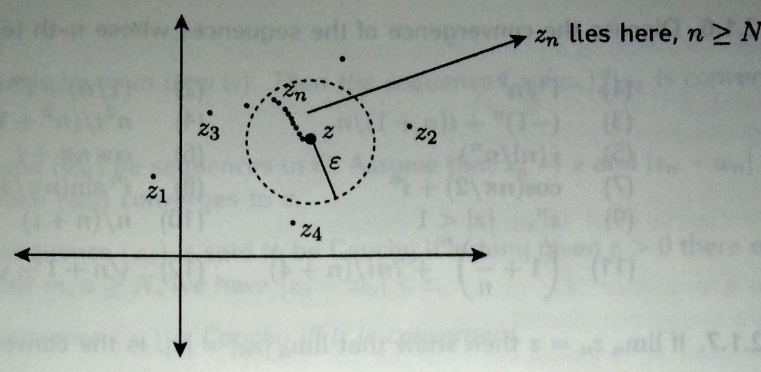

Figure 2.1: Convergence of a sequence

A more geometric proof is as follows. If $z \neq w$, then $d(z,w) > 0$. Choose $\varepsilon < d(z,w)/2$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that we have $z_n \in B(z,\varepsilon)$ for $n \geq n_1$ (respectively, $z_n \in B(w,\varepsilon)$ for $n \geq n_2$). If we take $N = \max\{n_1, n_2\}$, then $z_N \in B(z,\varepsilon) \cap B(w,\varepsilon)$. This is a contradiction, since our choice of ε implies that $B(z,\varepsilon) \cap B(w,\varepsilon) = \emptyset$.

Proposition 2.1.3. *Let* (z_n) *be a sequence in* $\mathbb{C}$. *Then* $z_n \to z$ *iff* $\operatorname{Re} z_n \to \operatorname{Re} z$ *and* $\operatorname{Im} z_n \to \operatorname{Im} z$.

Proof. Let $z_n \to z$. Then $|\operatorname{Re} z_n - \operatorname{Re} z| = |\operatorname{Re}(z_n - z)| \leq |z - z_n|$ from which the convergence $\operatorname{Re} z_n \to \operatorname{Re} z$ follows. Similarly, $\operatorname{Im} z_n \to \operatorname{Im} z$.

Conversely, if $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$, then the inequality

$$|z_n - z| \leq |\operatorname{Re} z_n - \operatorname{Re} z| + |\operatorname{Im} z_n - \operatorname{Im} z|$$

shows that $z_n \to z$. (The reader is encouraged to write down an ε-n_0 argument.)

We used the following observation: If $z = x + iy$, then $|z| \leq |x| + |y|$. Justify this. □

Example 2.1.4. Let $z_n = (1/n) + i(n-1)/n$. Then $\operatorname{Re} z_n \to 0$ and $\operatorname{Im} z_n \to 1$ so that $\lim z_n = i$.

Example 2.1.5. The sequence (i^n) is divergent. For, we have

$$i^n = \begin{cases} 1 & n \equiv 0 (\text{mod } 4) \\ i & n \equiv 1 (\text{mod } 4) \\ -1 & n \equiv 2 (\text{mod } 4) \\ -i & n \equiv 3 (\text{mod } 4). \end{cases}$$

Hence, looking at the real and imaginary parts of i^n, we see that neither of these real sequences converge.

Ex. 2.1.6. Discuss the convergence of the sequences whose n-th terms are:

(1)	i^n/n^2	(2)	$(1/n)+i^n$
(3)	$(-1)^n + i(n+1)/n$	(4)	$n^2 i/(n^2+1)$
(5)	$i\,(n!/n^n)$	(6)	$\cos n\pi + i$
(7)	$\cos(n\pi/2) + i^n$	(8)	$i^n \sin(n\pi/4)$
(9)	$z^n,\quad \lvert z\rvert < 1$	(10)	$n/(n+i)$
(11)	$\left(1+\frac{4}{n}\right)^n + 7ni/(n+4)$	(12)	$\sqrt{n+1}-\sqrt{n}+3i.$

Ex. 2.1.7. If $\lim_n z_n = z$ then show that $\lim_n |z_n| = |z|$. Is the converse true?

Ex. 2.1.8. Prove that $\lim_n z_n = 0$ iff $\lim_n |z_n| = 0$.

Ex. 2.1.9. We collect some of the most important facts from real analysis which will be needed later.(Refer to Section 2.5 of [2].)
(i) Establish Bernoulli's inequality: For $n \in \mathbb{N}$ and $t > -1$, we have $(1+t)^n \geq 1 + nt$. (Use induction.)
(ii) Show that if $|z| < 1$, then $\lim_n z^n = 0$. *Hint:* Write $1/|z| = 1 + t$ and apply the Bernoulli's inequality to $(1+t)^n$.
(iii) Use the binomial theorem to prove the following for $t \geq 0$ and $n \in \mathbb{N}$: (a) $(1+t)^n \geq 1 + nt$ and (b) $(1+t)^n \geq \binom{n}{2} t^2$.
(iv) Apply (iii) to show that $\lim nt^n = 0$ for $0 \leq t < 1$. *Hint:* Go through the proof of (ii)!
(v) Let $r > 0$. Then $\lim_n r^{1/n} = 1$. *Hint:* Let $r > 1$. Let $a_n := r^{1/n}$. Then $a_n = 1 + h_n$ with $h_n > 0$. Enough to show that $h_n \to 0$. If $0 < r < 1$, consider $1/r$ and apply the earlier result.

Ex. 2.1.10. Let $z \in \mathbb{C}$ be such that $z \neq 1$ but $|z| = 1$. Then $\lim_{n\to\infty} \frac{\sum_{k=1}^n z^k}{n} = 0$.

Definition 2.1.11. A sequence (z_n) in $\mathbb{C}$ is said to be *bounded* if there exists $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.

Ex. 2.1.12. If a sequence (z_n) is convergent, then it is bounded but the converse is not true.

We have the following theorem on algebra of limits whose proof is exactly analogous to the results in real theory and hence the proofs are left as an exercise to the reader.

Theorem 2.1.13 (Algebra of Limits). *Let (z_n), (w_n) be sequences in $\mathbb{C}$. Assume that $\lim z_n = z$ and $\lim w_n = w$. Then the following are true:*
(i) $\lim_n (z_n + w_n) = z + w$.
(ii) $\lim_n z_n w_n = zw$.
(iii) *If $\alpha \in \mathbb{C}$, then $\lim_n \alpha z_n = \alpha z$.*
(iv) *The set of convergent sequences in $\mathbb{C}$ is a vector space over $\mathbb{C}$.*
(v) *Let $\lim w_n = w$ and assume that $w \neq 0$. Then there exists $N \in \mathbb{N}$ such that $w_n \neq 0$*

for $n \geq N$.

(vi) *Let the hypothesis be as in item* (v). *Then the sequence* $(z_n/w_n)_{n=N}^{\infty}$ *is convergent with limit* z/w. □

Ex. 2.1.14. Let (z_n) and (w_n) be sequences in $\mathbb{C}$. Assume that $z_n \to z$ and $|z_n - w_n| \to 0$. Show that the sequence (w_n) converges to z.

Definition 2.1.15. A sequence (z_n) is said to be Cauchy if for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|z_n - z_m| < \varepsilon$.

Theorem 2.1.16. *A sequence* (z_n) *is Cauchy iff it is convergent.*

Proof. (z_n) is Cauchy iff $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are Cauchy. Use the real analogue of the theorem and Proposition 2.1.3 to complete the proof. □

Definition 2.1.17. Let $f\colon \mathbb{N} \to \mathbb{C}$ be a sequence. A *subsequence* is the restriction g of f to an infinite subset A of $\mathbb{N}$. Using the well ordering property of $\mathbb{N}$, we can write A as $\{n_1 < n_2 < \cdots\}$. In such a case, we denote the subsequence g by (z_{n_k}).

Ex. 2.1.18. If (z_{n_k}) is a subsequence of (z_n) and $\lim z_n = z$, then $\lim_k z_{n_k} = z$.

Theorem 2.1.19 (Bolzano-Weierstrass Theorem). *Let* (z_n) *be a bounded sequence in* $\mathbb{C}$. *Then there is a convergent subsequence* (z_{n_k}).

Proof. Let $x_n := \operatorname{Re} z_n$ and $y_n := \operatorname{Im} z_n$. We note that (x_n) and (y_n) are bounded sequences in $\mathbb{R}$. By Bolzano-Weierstrass theorem for real sequences, there is a subsequence (x_{n_k}) which is convergent, say, to x in $\mathbb{R}$. We once again appeal to Bolzano-Weierstrass Theorem to the bounded real sequence (y_{n_k}) to find a subsequence $(y_{n_{k_r}})$, converging to, say y. Then the subsequence $(z_{n_{k_r}})$ is convergent to $x + iy$.

> Reason: Since $(x_{n_{k_r}})$ is a subsequence of the convergent sequence (x_{n_k}), the former converges to the limit of the latter, namely, x. By Proposition 2.1.3, the sequence of complex numbers $z_{n_{k_r}} = x_{n_{k_r}} + iy_{n_{k_r}}$ converges to $x + iy$. As a subsequence of a subsequence of a sequence is a subsequence of the original sequence, the result follows.

□

Proposition 2.1.20. *Let* (z_n) *be a bounded but non-convergent sequence in* $\mathbb{C}$. *Then there exist two convergent subsequences with different limits.*

Proof. By Bolzano-Weierstrass theorem, there exists (z_{n_k}), a convergent subsequence with limit, say, α. Since (z_n) is not convergent, in particular, not to α, given any $\varepsilon > 0$ and given any $r \in \mathbb{N}$, there exists (by an induction on r) $n_r \in \mathbb{N}$ such that $n_r \geq r$, $n_r \geq n_{r-1}$ and such that $|z_{n_r} - \alpha| \geq \varepsilon$. Now (z_{n_r}) is a bounded sequence and hence it has a convergent subsequence, say, $(z_{n_{r_s}})$ with limit β. This is a subsequence of (z_n) and by construction, $\alpha \neq \beta$. (Why?) □

The following is an immediate corollary of the proposition.

Theorem 2.1.21. *Let (z_n) be a bounded sequence in $\mathbb{C}$. If there exists $z \in \mathbb{C}$ such that every convergent subsequence of (z_n) has the same limit z, then $\lim z_n = z$.* $\square$

2.2 Series of Complex Numbers

Given a sequence (a_n) in $\mathbb{C}$, let $s_n := \sum_{k=1}^{n} a_k$. Then the sequence (s_n) is called the *infinite series* associated with the sequence (a_n) and is denoted by the symbol $\sum_{n=1}^{\infty} a_n$. The number s_n is called the n-th *partial sum* s_n of the series. Thus, an *infinite series* of the complex numbers is a formal expression of the form $\sum_{n=1}^{\infty} a_n$, where (a_n) is a sequence of complex numbers and n-th partial sum of the series is defined by $s_n := \sum_{k=1}^{n} a_k$.

We say that the series is *convergent* if the sequence (s_n) of its partial sums is convergent. If (s_n) is not convergent, then we say that the series $\sum_{n=1}^{\infty} a_n$ is *divergent.* If $\alpha := \lim s_n$ we then write $\sum_{n=1}^{\infty} a_n = \alpha$ and call α the sum of the series. The series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the infinite series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Notation: In the sequel, we may use the notation such as $\sum_n a_n$ or $\sum a_n$ to denote $\sum_{n=1}^{\infty} a_n$. In the case of a finite sum, it will always be denoted by a notation of the form $\sum_{k=1}^{n} a_k$.

Example 2.2.1. Geometric Series. This is the most important example. Let $z \in \mathbb{C}$ be such that $|z| < 1$. Consider the infinite series $\sum_{n=0}^{\infty} z^n$. We claim that the series converges to $\alpha := 1/(1-z)$ for $|z| < 1$. Its n-th partial sum s_n is given by

$$s_n := \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

Now, $|\alpha - s_n| = \left|\frac{z^{n+1}}{1-z}\right| = C\,|z|^{n+1}$ which converges to 0 as $n \to \infty$ (by Exercise 2.1.9). Hence the claim.

Also note that if $|z| > 1$ then the n-th term does not go to 0, so that the series cannot be convergent in this case (Ex. 2.2.3).

Ex. 2.2.2. Let a_n be nonnegative real numbers. Show that $\sum_{n=1}^{\infty} a_n$ is convergent iff the sequence (s_n) of partial sums is bounded above. Hence conclude that if $\sum_{n=1}^{\infty} a_n$ is convergent, then its sum s is given by $s := \sup\{s_n : n \in \mathbb{N}\}$.

Ex. 2.2.3. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_n a_n = 0$. *Hint:* The sequence (s_n) of partial sums is convergent and, in particular, $|s_n - s_m| \to 0$ as $m, n \to \infty$.

Ex. 2.2.4. Prove that $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=1}^{\infty} (z^n/n^2)$ both diverge if $|z| > 1$. *Hint:* Previous exercise!

Ex. 2.2.5. Show that $\sum_{n=1}^{\infty} \frac{1}{2^k}(1+i)^{2k}$ is divergent.

Ex. 2.2.6. Let $\sum_n a_n$ be absolutely convergent. Prove that $\sum_n a_n$ is convergent. *Hint:* Let s_n (respectively, σ_n) denote the n-th partial sum of $\sum_n a_n$ (respectively, of $\sum_n |a_n|$). Observe that $|s_n| \leq \sigma_n$ and hence conclude that (s_n) is Cauchy.

Theorem 2.2.7. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series. Assume that $\sum_{n=1}^{\infty} a_n = \alpha$ and $\sum_{n=1}^{\infty} b_n = \beta$. Then*

(i) *$\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent and its sum is $\alpha + \beta$.*

(ii) *For any $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$, we have*

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda\alpha + \mu\beta.$$

(iii) *If $\sum_n |z_n|$ is convergent then $\sum_n z_n$ is convergent.*

(iv) *$\sum_n z_n$ is convergent iff $\sum_n \operatorname{Re} z_n$ and $\sum_n \operatorname{Im} z_n$ are convergent. Furthermore, the sum of the infinite series $\sum_n z_n$ is $x + iy$ if x and y are respectively the sums of $\sum \operatorname{Re} z_n$ and $\sum \operatorname{Im} z_n$.*

(v) *Let $\sum_n a_n$ converge with sum α. Then $\sum_n \overline{a}_n$ converges and its sum is $\overline{\alpha}$.*

Proof. The proofs of all the statements are straight forward. As a sample, we shall prove (iii).

To prove (iii), let s_n and σ_n denote the partial sums of $\sum a_n$ and $\sum |a_n|$ respectively. Then, for $n > m$,

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| = \sigma_n - \sigma_m,$$

which converges to 0, as (σ_n) is convergent. Hence (s_n) is Cauchy in $\mathbb{C}$. Now apply Theorem 2.1.16. □

Theorem 2.2.8 (Tests for Convergence).

(1) Comparison Test. *Let $\sum a_n$ and $\sum_n b_n$ be series of positive reals. Assume that $a_n \leq b_n$ for all n. Then (i) if $\sum b_n$ is convergent, then so is $\sum a_n$, and (ii) if $\sum a_n$ is divergent, so is $\sum b_n$.*

An extension of comparison test. *Let $\sum_n b_n$ be a series of positive reals. Assume that $\sum_n b_n$ is convergent and that $|a_n| \leq b_n$ for all n. Then $\sum_n a_n$ is absolutely convergent and hence convergent.*

(2) Ratio Test. *Let $\sum_n c_n$ be a series of positive reals. Assume that*

$$\lim_n c_{n+1}/c_n = r.$$

Then the series $\sum_n c_n$ is
(a) *convergent if $0 \le r < 1$,*
(b) *divergent if $r > 1$.*
The test fails if $r = 1$.

(3) Root Test. *Let $\sum_n a_n$ be a series of positive reals. Assume that $\lim_n a_n^{1/n} = a$. Then the series $\sum_n a_n$ is convergent if $0 \le a < 1$, divergent if $a > 1$ and the test fails if $a = 1$.*

Proof. (1) Assume that $\sum_n b_n$ is convergent. Let s_n and σ_n denote the n-th partial sums of $\sum_n a_n$ and $\sum_n b_n$ respectively. Since $a_n, b_n \ge 0$, we see that $s_n \le s_{n+1}$ and $\sigma_n \le \sigma_{n+1}$. Since $a_n \ge 0$ and $a_n \le b_n$, it follows that $0 \le s_n \le \sigma_n$. Since $\sum_n b_n$ is convergent, (σ_n) is convergent and converges to $\sigma := \sup\{\sigma_n : n \in \mathbb{N}\}$. Therefore, $s_n \le \sigma_n \le \sigma$. Thus we have shown that (s_n) is an increasing sequence of real numbers bounded above. Hence (s_n) is convergent.

If $\sum a_n$ is divergent, then (s_n) is an increasing sequence not bounded above. Consequently, since $\sigma_n \ge s_n$, the increasing sequence (σ_n) is not bounded above. We therefore conclude that $\sum_n b_n$ is divergent.

In the case of extension of the comparison test, we compare the series $\sum_n b_n$ and $\sum_n |a_n|$ to conclude that $\sum_n a_n$ is absolutely convergent.

(2) If $r < 1$, choose an s such that $r < s < 1$. Then there exists $N \in \mathbb{N}$ such that $c_{n+1} \le s c_n$ for all $n \ge N$.

> Reason: Take $\varepsilon := s - r$. By the convergence of $x_n := c_{n+1}/c_n$ to r, there exists $N \in \mathbb{N}$ such that $x_n \in (r - \varepsilon, r + \varepsilon)$ for all $n \ge N$. This implies that $c_{n+1}/c_n < r + \varepsilon = s$ for $n \ge N$.

By induction on k, we see that $c_{N+k} \le s^k c_N$, for $k \in \mathbb{N}$. The convergence of $\sum c_n$ follows.

> Reason: We compare the series $\sum_{N+1}^{\infty} c_n$ and the series $c_N \sum_{N+1}^{\infty} s^n$. By what we saw above, $c_n \le c_N s^{n-N}$ for $n \ge N$. The series $\sum_{n=0}^{\infty} s^n$ is a convergent geometric series as $0 < s < 1$. By the comparison test, it follows that $\sum_{N+1}^{\infty} c_n$ is convergent and hence the series $\sum_{n=1}^{\infty} c_n$ is also convergent.

If $r > 1$ then arguing as above, there exists $N \in \mathbb{N}$ such that $c_n \ge c_N$ for all $n \ge N$. Hence $\sum c_n$ is divergent as the n-th term does not go to 0. The failure of the test when $r = 1$ follows from looking at the examples $\sum_n 1/n$ and $\sum_n 1/n^2$. For, in each of the cases, $r = 1$ whereas the first series is divergent while the second is convergent. (See Example 2.2.10 below.)

(3) If $a < 1$, then choose α such that $a < \alpha < 1$. Then, arguing as in (2), we find that there exists $N \in \mathbb{N}$ such that $a_n < \alpha^n$ for $n \ge N$. Hence by comparing with the geometric series $\sum_{n \ge N} \alpha^n$, the convergence of $\sum_n a_n$ follows. If $a > 1$, then $a_n \ge 1$ for all large n and hence n-th term does not approach zero. The examples $\sum_n 1/n$ and $\sum 1/n^2$ illustrate the failure of the test when $r = 1$. $\square$

Remark 2.2.9. All convergence tests are the offspring of comparison test and geometric series. Most often, analysts, when they want to settle convergence questions, resort to these tricks directly if no standard tests are of use for the problems on hand. As a simpler example, see Example 2.2.11 below. (Another important test is the integral test, which is also very useful. We refer the reader to books on real analysis such as [2], [29], [3].)

Example 2.2.10. Harmonic Series. The series $\sum_{n=1}^{\infty} n^{-\alpha}$ is convergent if $\alpha > 1$ and is divergent if $\alpha \leq 1$.

This fact is quite useful and most often used in conjunction with the comparison test. The trick is to estimate both from above and from below the 2^k-terms starting from 2^k+1-th term to 2^{k+1}-th term. (We suggest that the reader works out the case when $\alpha = 1$ to start with.) Observe that

$$\frac{2^k}{2^{\alpha(k+1)}} < \sum_{2^k+1}^{2^{k+1}} \frac{1}{n^\alpha} < \frac{2^k}{2^{\alpha k}}.$$

Reason: If $n > 2^k$, then $n^\alpha > 2^{\alpha k}$. Therefore, each of terms in the finite sum is less than $2^{-\alpha k}$. There are 2^k terms in the sum. Hence we get the inequality on the right most side. The other inequality is derived similarly.

If $\alpha > 1$, then $2/2^\alpha < 1$. The given series is now compared with the convergent geometric series $\sum_k \left(\frac{2}{2^\alpha}\right)^k$. If $\alpha \leq 1$, the harmonic series is compared with the divergent series $2^{-\alpha} \sum \left(\frac{2}{2^\alpha}\right)^k$.

Example 2.2.11. Exponential Series. We show that $\sum_{n=0}^{\infty}(z^n/n!)$ is absolutely convergent for any $z \in \mathbb{C}$. This follows from the ratio test applied to $\sum_{n=0}^{\infty} \frac{|z^n|}{n!}$. One can also see this more directly as follows.

Given z, we select $N \in \mathbb{N}$ such that $|z| < N/2$. Then for $k \in \mathbb{N}$, one shows easily that $\left|\frac{z^{N+k}}{(N+k)!}\right| \leq \frac{1}{2^k}\frac{|z|^N}{N!}$ as follows:

$$\begin{aligned}
\left|\frac{z^{N+k}}{(N+k)!}\right| &= \frac{|z|}{1}\frac{|z|}{2}\cdots\frac{|z|}{N}\frac{|z|}{N+1}\cdots\frac{|z|}{N+k} \\
&\leq \frac{|z|^N}{N!}\frac{|z|}{N}\cdots\frac{|z|}{N} \\
&\leq \frac{|z|^N}{N!}(1/2)\cdots(1/2) \\
&\leq \frac{1}{2^k}\frac{|z|^N}{N!}.
\end{aligned}$$

Hence we have

$$\sum_{n\geq N}\left|\frac{z^n}{n!}\right| \leq \frac{|z|^N}{N!}\sum_{n\geq N}\frac{1}{2^{n-N}} \leq 2\frac{|z|^N}{N!}.$$

The result follows by the extension of comparison test. The sum of the series is denoted by $\exp(z)$ and the series is called the exponential series.

Ex. 2.2.12. Find which of the following series are absolutely convergent:

(1) $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ (2) $\sum_{n=1}^{\infty} \frac{i^n \log n}{n^2+1}$

(3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \log n}{n}$ (4) $\sum_{n=1}^{\infty} \frac{(1+i)^n}{n}$

(5) $\sum_{n=1}^{\infty} \frac{i^n+1}{n^{3/2}}$ (6) $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}, \quad |z|<1$

(7) $\sum_{n=1}^{\infty} \frac{i}{n^2+i}$ (8) $\sum_{n=1}^{\infty} \frac{i}{n+i}$

(9) $\sum_{n=1}^{\infty} \frac{(1+i)^n}{n!}$ (10) $\sum_{n=1}^{\infty} (n+1)\frac{(1+i)^n}{n!}$.

Ex. 2.2.13. Determine *all* $z \in \mathbb{C}$ for which the series below are absolutely convergent:

(1) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ (2) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$ (4) $\sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{1}{z+1}\right)^n$

(5) $\sum_{n=1}^{\infty} \left(\frac{z-1}{z+1}\right)^n$ (6) $\sum_{n=0}^{\infty} \frac{1}{z^n}$.

Ex. 2.2.14. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent and that $|z|<1$. Show that $\sum_{n=1}^{\infty} a_n z^n$ is absolutely convergent.

Ex. 2.2.15. If $\sum |a_n|$ converges then $\sum a_n^2$ is absolutely convergent.

Ex. 2.2.16. If $\sum_n |a_n|^2$ and $\sum_n |b_n|^2$ are convergent, then $\sum_n a_n b_n$ is absolutely convergent. *Hint:* To start with, assume a_n and b_n are nonnegative reals. Observe that $(a_n - b_n)^2 \geq 0$.

Ex. 2.2.17. Let $\sum_n a_n$ be convergent. Show that $|\sum_n a_n| \leq \sum_n |a_n|$ whether or not the latter series is convergent.

Ex. 2.2.18. Let $\ell^1 := \{(a_n) : a_n \in \mathbb{C}, \sum_n |a_n| < \infty\}$ be the set of all complex sequences such that the series $\sum_n |a_n|$ is convergent. Show that ℓ^1 is a vector space over $\mathbb{C}$.

Definition 2.2.19. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, their *Cauchy product* is a series $\sum_{n=1}^{\infty} c_n$ where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$. It is motivated by the product of polynomials and power series. For instance, if we let $p(z) := \sum_{k=0}^{m} a_k z^k$ and $q(z) := \sum_{k=0}^{n} b_k z^k$, then the product of polynomials is given by $pq(z) = \sum_{r=0}^{m+n} c_r z^r$, where $c_r := \sum_{k+l=r} a_k b_l$.

Theorem 2.2.20 (Multiplication of Series). *Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. Define $c_n := \sum_{k=0}^{n} a_k b_{n-k}$. If $A := \sum_n a_n$ and $B := \sum_n b_n$, then $\sum_n c_n$ is convergent and we have $\sum_n c_n = AB$.*

Proof. Using an obvious notation, we let A_n, B_n and C_n denote the partial sums of the three series. Let $D_n := B_n - B$.

$$
\begin{aligned}
C_n &= \sum_{k=0}^{n} c_k \\
&= \sum_{k=0}^{n}\sum_{r=0}^{k} a_r b_{k-r} \\
&= \sum_{r+s\leq n} a_r b_s \\
&= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\
&= a_0(-B + B_n) + a_1(-B + B_{n-1}) + \cdots + a_n(-B + B_0) + B\left(\sum_{k=0}^{n} a_k\right) \\
&= A_n B - R_n, \qquad (2.1)
\end{aligned}
$$

where $R_n := a_0 D_n + a_1 D_{n-1} + \cdots + a_n D_0$. Let $\alpha := \sum_n |a_n|$. Since $D_n \to 0$, (D_n) is bounded, say by D: $|D_n| \leq D$ for all n. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n\geq N} |a_n| < \varepsilon$ and $|D_n| \leq \varepsilon$. For all $n \geq 2N$, we have

$$
\begin{aligned}
|R_n| &\leq (|a_0| + \cdots + |a_{n-N}|)\varepsilon + (|a_{n-N+1}| + \cdots + |a_n|)D \\
&\leq (\alpha + D)\varepsilon.
\end{aligned}
$$

Hence $R_n \to 0$. Since $A_n B \to AB$, the result follows from (2.1). $\square$

Example 2.2.21. We know that $\sum_{n=0}^{\infty} z^n = 1/(1-z)$ for $|z| < 1$. If we take $a_n = z^n = b_n$ in the theorem, we get $\sum_{n=1}^{\infty} nz^{n-1} = 1/(1-z)^2$ for $|z| < 1$.

Ex. 2.2.22. Show that $(\sum_{n=0}^{\infty} \frac{z^n}{n!})(\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}) = 1$ for all z. In particular, $\exp(z) \neq 0$ for any $z \in \mathbb{C}$.

More generally, show that the Cauchy product of the series for $\exp(z)$ and $\exp(w)$ converges to $\exp(z + w)$.

Alternating Series Test

Proposition 2.2.23 (Alternating Series Test). *Let $\sum_n (-1)^{n+1} x_n$ be an infinite series with $x_n \geq 0$. Assume that (x_n) is decreasing and that $\lim x_n \to 0$. Then $\sum_n (-1)^{n+1} x_n$ is convergent.* □

This is an easy corollary of Abel's test. See Ex. 2.2.23

Theorem 2.2.24 (Abel's Test). *Let (c_n) be a monotone decreasing sequence of nonnegative reals with $\lim c_n = 0$. Let (b_n) be a bounded sequence in $\mathbb{C}$. Then*

$$\sum_n (b_{n+1} - b_n) c_n$$

is convergent.

Proof. Assume that $|b_n| \leq M$ for all n. Let s_n be the n-th partial sum of the series. We have for $n \geq m$,

$$\begin{aligned}
s_n - s_m &= \sum_{m+1}^{n} (b_{k+1} - b_k) c_k \\
&= (b_{m+2} - b_{m+1}) c_{m+1} + (b_{m+3} - b_{m+2}) c_{m+2} + \cdots \\
&\quad + (b_n - b_{n-1}) c_{n-1} + (b_{n+1} - b_n) c_n \\
&= -b_{m+1} c_{m+1} + b_{m+2}(c_{m+1} - c_{m+2}) + \cdots + b_n (c_{n-1} - c_n) + b_{n+1} c_n \\
&= -b_{m+1} c_{m+1} + \sum_{k=m+2}^{n} b_k (c_{k-1} - c_k) + b_{n+1} c_n.
\end{aligned}$$

Hence taking modulus, we get

$$\begin{aligned}
&|s_n - s_m| \\
&= \left| -b_{m+1} c_{m+1} + \sum_{k=m+2}^{n} b_k (c_{k-1} - c_k) + b_{n+1} c_n \right| \\
&\leq |b_{m+1}| \, |c_{m+1}| + \sum_{k=m+2}^{n} |b_k| \, |(c_{k-1} - c_k)| + |b_{n+1}| \, |c_n| \\
&\leq M \left(c_{m+1} + \sum_{m+2}^{n} (c_{k-1} - c_k) + c_n \right) \\
&= M \left(c_{m+1} + (c_{m+1} - c_{m+2}) + (c_{m+2} - c_{m+3}) + \cdots + (c_{n-1} - c_n) + c_n \right) \\
&\leq 2M c_{m+1}.
\end{aligned}$$

This implies that (s_n) is Cauchy, since c_n decreases to 0. Note that in the set of equations above, we have used the fact that the sequence (c_n) is nonnegative and decreasing. Do you see where exactly it was used? □

Ex. 2.2.25. Deduce the alternating series test from Abel's test. *Hint:* Take $b_n = (-1)^n/2$ in Abel's test.

Example 2.2.26. $\sum_n (-1)^n/n$ is convergent.

Ex. 2.2.27. Let $\sum_n a_n R^n$ be convergent where $a_n \geq 0$ and $R > 0$. Show that $\sum a_n z^n$ is convergent for all $z \in \mathbb{C}$ with $|z| \leq R$. Show that the result is false if the hypothesis $a_n \geq 0$ for all n is removed.

2.3 Power Series

Definition 2.3.1. A *power series* is an expression of the form $\sum_{k=0}^{\infty} a_k(z-a)^k$ where $a_k, a, z \in \mathbb{C}$. We do not assume that the series converges.

Example 2.3.2. Consider the three power series:
(1) $\sum_{n=1}^{\infty} n^n z^n$,
(2) $\sum_{n=0}^{\infty} z^n$
(3) $\sum_{n=0}^{\infty} (z^n/n!)$.
We claim that if $z \neq 0$ then the first series does not converge. For, if $z \neq 0$ is a complex number, choose $N \in \mathbb{N}$ so that $1/N < |z|$. Then for all $n \geq N$, we have $|(nz)^n| > 1$ and hence the series is not convergent. We have already seen that the second series converges absolutely for all z with $|z| < 1$ whereas the third series converges absolutely for all $z \in C$.

Theorem 2.3.3. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series. There is a unique extended real number R, $0 \leq R \leq \infty$, such that the following hold:*
(i) for all z with $|z-a| < R$, the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely,
(ii) for all z with $|z-a| > R$, the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ diverges.

Proof. Assume $a = 0$. Let $E := \{|z| : \sum_{n=0}^{\infty} a_n z^n \text{ is convergent.}\}$ and $R := \sup E$. Then $\sum_{n=0}^{\infty} a_n z^n$ is divergent if $|z| > R$, by the very definition.

If $R > 0$ choose r such that $0 < r < R$. Since R is the least upper bound for E and $r < R$, there exists $z_0 \in E$ such that $|z_0| > r$ and $\sum a_n z_0^n$ is convergent. Hence $\{a_n z_0^n\}$ is bounded (why?), say, by M:

$$|a_n z_0^n| \leq M \text{ for all } n.$$

Now, if $|z| \leq r$, then

$$|a_n z^n| \leq |a_n| r^n \leq |a_n z_0^n| (r/|z_0|)^n \leq M(r/|z_0|)^n.$$

But the ("essentially geometric") series $M \sum (r/|z_0|)^n$ is convergent. Hence by comparison test, the series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for z with $|z| \leq r$. □

Definition 2.3.4. The extended real number R of Theorem 2.3.3 is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$. The open disk $B(a,R)$ is called the disk of convergence of the power series. We let $B(a,R) = \mathbb{C}$ if $R = \infty$.

Ex. 2.3.5. Go through the proof of Theorem 2.3.3 and observe that we have indeed established the following statement:

Let $\sum_n a_n(z-a)^n$ be convergent for $|z-a| < R$, with $0 < R \leq \infty$. Let the sum of the series $\sum_n a_n(z-a)^n$ be $f(z)$ for $z \in B(a,R)$. Let $0 \leq r < R$ be given. Let $f_n(z) := \sum_{k=0}^{n} a_k(z-a)^k$. Given $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$|f(z) - f_n(z)| < \varepsilon, \quad \text{for all } z \in B[a,r].$$

We shall understand the significance of this statement later, see Theorem 3.2.6.

Ex. 2.3.6. Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$. Show that the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$ is at least R.

Remark 2.3.7. Nothing can be said about the convergence of the power series on the circle of convergence $S(a,R)$. Using the ratio test, one easily shows that the radius of convergence of each of the following series is 1: i) $\sum_0^{\infty} z^n$, ii) $\sum_1^{\infty}(z^n/n^2)$ and iii) $\sum_1^{\infty}(z^n/n)$.

Then the following are easily seen:

The series i) diverges for all z on the circle of convergence, as the n-th terms do not approach 0.

The series ii) converges absolutely on the circle of convergence.

The series iii) diverges when $z = 1$ and converges when $z = -1$.

Ex. 2.3.8. In fact, the third series above converges for all z with $|z| = 1$ and $z \neq 1$. *Hint:* Use the following: Abel's test (Theorem 2.2.24) and the fact: $\sum_{k=1}^{n} \cos k\theta$ and $\sum_{k=1}^{n} \sin k\theta$ are bounded provided $0 < \theta < 2\pi$.

We shall adopt the following convention in the extended real number system: $0^{-1} = \infty$ and $\infty^{-1} = 0$.

The following theorem is quite useful in practice, as it gives two simple formulae to determine the radius of convergence of a power series.

Theorem 2.3.9. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be given. Assume that one of the following limits exists as an extended real number.*

(i) $\lim |a_{n+1}/a_n| = \rho$

(ii) $\lim |a_n|^{1/n} = \rho$.

Then the radius of convergence of the power series is given by $R = \rho^{-1}$.

Proof. (i) follows from the ratio test and (ii) from the root test. For instance, if (i) holds and if z is fixed, then

$$\left|\frac{a_{n+1}(z-a)^{n+1}}{a_n(z-a)^n}\right| = |z-a|\left|\frac{a_{n+1}}{a_n}\right| \to |z-a|\,\rho.$$

By the ratio test, we know that the *numerical* series $\sum_n a_n(z-a)^n$ is convergent if $|z-a|\,\rho < 1$. Thus the radius of convergence of the series is at least $1/\rho$. By the same ratio test, we know that if $|z-a|\,\rho > 1$, then the series is divergent. Hence we conclude that the radius of convergence of the given series is $1/\rho$.

The proof of (ii) is similar to that of (i) and is left to the reader. □

There is a formula for the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ in terms of the coefficients a_n. To state this we need the notion of lower limit, lim inf, and the upper limit, lim sup, of a sequence of real numbers.

Definition 2.3.10. Given a bounded sequence (a_n) of real numbers, the sequences

$$s_n := \inf\{a_k : k \geq n\} \text{ and } t_n := \sup\{a_k : k \geq n\}$$

are monotone. The sequence (s_n) is an increasing sequence of reals while (t_n) is a decreasing sequence of reals. Let

$$\liminf a_n := \lim s_n \equiv \sup\{s_n\} \text{ and } \limsup a_n := \lim t_n \equiv \inf\{t_n\}.$$

In case, the sequence (a_n) is not bounded above, then its lim sup is defined to be $+\infty$. Similarly, the lim inf of a sequence not bounded below is defined to be $-\infty$.

Lemma 2.3.11. *Let (a_n) be a bounded sequence of real numbers. Then*
(i) *If $\alpha > \limsup a_n$, then there exists $N \in \mathbb{N}$ such that $a_n < \alpha$ for $n \geq N$.*
(ii) *If $\lambda < \limsup a_n$, then $\lambda < a_n$ for infinitely many n.*

Proof. (i) Note that $\limsup a_n = \inf t_n$ in the notation used above. Since $\alpha > \inf t_n$, that is, α is greater than the greatest lower bound of (t_n), α is not a lower bound for t_n's. Hence there exists $N \in \mathbb{N}$ such that $\alpha > t_N$. Since t_N is the least upper bound for $\{x_n : n \geq N\}$, it follows that $\alpha > x_n$ for all $n \geq N$.

(ii) Let $\lambda < \limsup a_n$. Then λ is less than the greatest lower bound of t_n's and hence is certainly a lower bound for t_n's. Hence, for any $k \in \mathbb{N}$, λ is less than t_k, the least upper bound of $\{a_n : n \geq k\}$. Therefore, λ is not an upper bound for $\{a_n : n \geq k\}$. Thus, there exists $n_k \geq k$ such that $a_{n_k} > \lambda$. We have therefore found infinitely many n's, namely n_k's as required in (b). □

Ex. 2.3.12. State and prove analogous results for $\liminf$.

Theorem 2.3.13 (Hadamard formula for the radius of convergence). *The radius of convergence ρ of $\sum_{n=0}^{\infty} c_n(z-a)^n$ is given by*

$$\frac{1}{\rho} = \limsup |c_n|^{1/n} \text{ and } \rho = \liminf |c_n|^{-1/n}.$$

Proof. Let $\frac{1}{\beta} := \limsup |c_n|^{1/n}$. We wish to show that $\rho = \beta$.

If z is given such that $|z-a| < \beta$, choose μ such that $|z-a| < \mu < \beta$. Then $\frac{1}{\mu} > \frac{1}{\beta}$ and hence there exists N (by the last lemma) such that $|c_n|^{1/n} < \frac{1}{\mu}$ for all $n \geq N$. It follows that $|c_n| \mu^n < 1$ for $n \geq N$. Hence $(|c_n| \mu^n)$ is bounded, say, by M. Hence, $|c_n| \leq M\mu^{-n}$ for all n. Consequently,

$$|c_n(z-a)^n| \leq M\mu^{-n}|z-a|^n = M\left(\frac{|z-a|}{\mu}\right)^n.$$

Since $\frac{|z-a|}{\mu} < 1$, the convergence of $\sum c_n(z-a)^n$ follows.

Let $|z-a| > \beta$ so that $\frac{1}{|z-a|} < \frac{1}{\beta}$. Then $\frac{1}{|z-a|} < |c_n|^{1/n}$ for infinitely many n. Hence $|c_n||z-a|^n \geq 1$ for infinitely many n so that the series $\sum c_n(z-a)^n$ is divergent. We therefore conclude that $\rho = \beta$.

The other formula for the radius of convergence is proved similarly. □

Ex. 2.3.14. Find the radius of convergence of the power series $\sum_n a_n z^n$, whose n-th coefficient a_n is given below:

(1)	$1/(n^2+1)$	(2)	$2^n - 1$
(3)	$e^{in\pi}/n$	(4)	$(1+i)^n$
(5)	n^k, k fixed	(6)	n^n
(7)	$n^k/n!$	(8)	$(1+\frac{2}{n^2})^{n^2}$
(9)	$(\log n^n)/n!$	(10)	$\frac{2^n n!}{(2n)!}$
(11)	$(\frac{n}{\log n})^n$	(12)	$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$
(13)	$\frac{n^2}{2^{2n}}$	(14)	$\frac{1}{5}n^4(n^3+1)$
(15)	$\frac{(2n)!}{n!}$	(16)	$\frac{n!}{(2n)!}$
(17)	$\frac{n!}{n^n}$	(18)	$\frac{n^{2n}}{(n!)^2}$
(19)	$[1+(-1)^n 3]^n$	(20)	$(a^n+b^n), a > b > 0$
(21)	$\frac{1}{n^p}$, $p > 0$	(22)	$4^{n(-1)^n}$.

Ex. 2.3.15. Why do $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n z^{n+1}$ have the same radius of convergence?

Ex. 2.3.16. If $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have finite radii of convergence R_1 and R_2 respectively, what can you say about the radius of convergence of (1) $\sum_{n=0}^{\infty}(a_n \pm b_n)z^n$, (2) $\sum_{n=0}^{\infty} a_n b_n z^n$ and (3) $\sum_{n=0}^{\infty}(a_n/b_n)z^n$?

Ex. 2.3.17. Find the radius of convergence of each of the following series. (Caution: Note that $a_k = 0$ for infinitely many k.)

$$(1)\ \sum_{n=0}^{\infty} 2nz^{2n} \qquad (2)\ \sum_{n=0}^{\infty} n!z^{2n+1}$$
$$(3)\ \sum_{n=0}^{\infty} \frac{1}{n!}z^{2n+1} \qquad (4)\ \sum_{n=0}^{\infty} z^{n!}$$
$$(5)\ \sum_{n=0}^{\infty} n^2 z^{n^n} \qquad (6)\ \sum_{n=0}^{\infty} 5^n z^{3n}.$$

Ex. 2.3.18. If $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are both convergent for $|z| < R$ with sums $f(z)$ and $g(z)$ respectively, then their Cauchy product $\sum_{n=0}^{\infty} c_n z^n$ is convergent with sum $f(z)g(z)$.

Ex. 2.3.19. Let P be a nonzero polynomial. Show that the radius of convergence of $\sum_{n=0}^{\infty} P(n)z^n$ is 1. *Hint:* Observe that $P(n+1)/P(n) \to 1$ as $n \to \infty$.

Ex. 2.3.20. Let R be the radius of the power series $\sum_{n=0}^{\infty} a_n z^n$. Show that each of the series $\sum_{n=1}^{\infty} na_n z^n$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n$ have R as the radius of convergence. *Hint:* Given $z \in B(0,R)$, since $|a_n z^n| \to 0$, there exists N such that $|a_n| \le \frac{1}{|z|^n}$. Choose r so that $0 < |z| < r < R$.

Ex. 2.3.21. Let $\sum_{n=0}^{\infty} a_n z^n$ be convergent in $B(0,R)$. Let its sum be denoted by $f(z)$ for $z \in B(0,R)$. Let $p(z) := \sum_{k=0}^{m} a_k z^k$. Show that $\sum_{k=m+1}^{\infty} a_k z^k$ is convergent in $B(0,R)$ and its sum is $f(z) - p(z)$.

Ex. 2.3.22 (Abel). Let the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be 1. Assume that $\sum_{n=0}^{\infty} a_n$ is convergent. If $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for $z \in B(0,1)$, show that

$$\lim_{x \to 1_-} f(x) = \sum_{n=0}^{\infty} a_n. \tag{2.2}$$

Hint: Show first that it is sufficient to consider the case in which $\sum a_n = 0$. Let $s_n = \sum_0^n a_k$. Show that

$$f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n, \qquad z \in B(0,1).$$

Given $\varepsilon > 0$, choose N such that $|s_n| < \varepsilon$ for $n \ge N$. Now split the series into two parts and deduce the result.

Chapter 3

Continuity

3.1 Continuous Functions

We shall assume that the reader has acquired some basic knowledge of metric spaces. For a quick introduction, the reader may refer to the appendix. For a more thorough knowledge, we refer the reader to [17].

Definition 3.1.1. Let (X, d) and (Y, d) be metric spaces and $x \in X$. A function $f \colon X \to Y$ is said to be *continuous* at x if for **every** sequence $(x_n) \to x$ in X we have $f(x_n) \to f(x)$.

We say that f is continuous on X if it is continuous at all points of X.

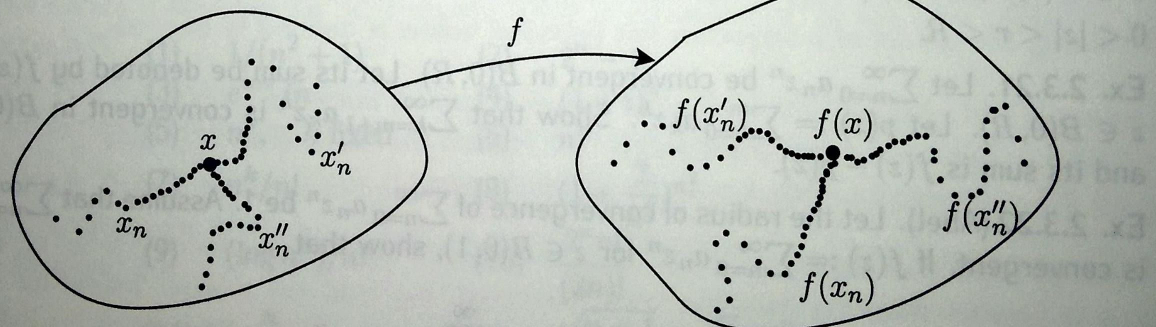

Figure 3.1: Continuity of f at x

If A is a subset of X and $f \colon A \to Y$ is a function, we say that f is continuous at $a \in A$ if f is continuous as a function from the metric space (A, d) to (Y, d). This is equivalent to the requirement that whenever (a_n) is a sequence in A converging to $a \in A$, $(f(a_n))$ converges to $f(a)$ in Y.

Example 3.1.2. Any constant function $f \colon X \to Y$ is continuous on X.

Example 3.1.3. Let f be the identity function of X into itself. Then f is continuous on X.

Example 3.1.4. Fix $a \in X$. Define $f\colon X \to \mathbb{R}$ by setting $f(x) := d(a,x)$. We show that f is continuous. Let $x \in X$ be arbitrary. Let (x_n) be a sequence such that $x_n \to x$. Then we need to show that $f(x_n) \to f(x)$ in $\mathbb{R}$ as $n \to \infty$. That is, $|f(x_n) - f(x)| \to 0$ as $n \to \infty$. But note that

$$|f(x_n) - f(x)| = |d(a,x_n) - d(a,x)| \le d(x,x_n) \to 0, \text{ as } n \to \infty.$$

Ex. 3.1.5. The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) := |z|$ is continuous.

Ex. 3.1.6. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given. Write $f(z) := u(z) + iv(z)$. Then u and v are called the real and imaginary parts of f. Show that f is continuous iff the real valued functions $u, v\colon \mathbb{C} \to \mathbb{R}$ are continuous.

Ex. 3.1.7. Show that a function $f\colon \mathbb{C} \to \mathbb{C}$ is continuous iff $\overline{f}\colon \mathbb{C} \to \mathbb{C}$ is continuous, where $\overline{f}(z) := \overline{f(z)}$.

Theorem 3.1.8. *Let $f, g\colon X \to \mathbb{C}$ be continuous at $x \in X$. Then*

(i) *$f + g$ is continuous at x.*

(ii) *fg is continuous at x.*

(iii) *αf is continuous at x for any $\alpha \in \mathbb{C}$.*

(iv) *If g is continuous at x and $g(x) \neq 0$, then there exists $r > 0$ such that $g(x') \neq 0$ for all $x' \in B(x,r)$.*

(iv) *If $g(x) \neq 0$, let r be as in (4). Then f/g is defined on $B(x,r)$ and $f/g\colon B(x,r) \to \mathbb{C}$ continuous at x.*

Proof. Follows from theorem on algebra of limits (Theorem 2.1.13) of sequences. □

An immediate application of the foregoing is

Corollary 3.1.9. *Let f be a polynomial with complex coefficients: $f(z) := a_0 + a_1 z + \cdots + a_n z^n$, $a_k \in \mathbb{C}$ for $0 \le k \le n$. Then $f\colon \mathbb{C} \to \mathbb{C}$ is continuous.* □

Ex. 3.1.10. Let f be of the form p/q where p and q are polynomials. Discuss the domain of f and the points of continuity of f. Such functions f are called *rational functions.*

Theorem 3.1.11. *Let X, Y and Z be metric spaces. Assume that $f\colon X \to Y$ is continuous at x and $g\colon Y \to Z$ is continuous at $y := f(x)$. Then $g \circ f\colon X \to Z$ is continuous at x.*

Proof. If $x_n \to x$, then by continuity of f at x, $f(x_n) \to f(x)$. By continuity of g at $y = f(x)$, we see that $g(f(x_n)) \to g(f(x))$. □

Theorem 3.1.12. *Let $f\colon X \to Y$ be given. Then f is continuous at a iff the following condition holds:*

> *Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in B(a,\delta)$, we have $f(x) \in B(f(a),\varepsilon)$, that is, if $d(x,a) < \delta$, then $d(f(x), f(a)) < \varepsilon$.*

Proof. Let f be continuous at a. Let $\varepsilon > 0$ be given. Assume that there exists no δ as stipulated in the theorem. Then for each $n \in \mathbb{N}$, if we take $\delta := 1/n$, there exists $x_n \in B(a, 1/n)$ but $f(x_n) \notin B(f(a), \varepsilon)$. By our choice, $d(x_n, a) < 1/n$ and hence we conclude that $x_n \to a$. Since f is continuous at a, it follows that $f(x_n) \to f(a)$. But this is impossible, since $d(f(x_n), f(a)) \geq \varepsilon$ for all n. This contradiction shows that our assumption was wrong.

Let f satisfy the condition of the theorem. Let (x_n) be a sequence converging to a. We need to show that $f(x_n) \to f(a)$. Let $\varepsilon > 0$ be given. For this $\varepsilon > 0$, by hypothesis, there exists $\delta > 0$ such that whenever $d(x, a) < \delta$, we have $d(f(x), f(a)) < \varepsilon$. Since $x_n \to a$, for the δ as above, there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \delta$ for $n \geq N$. Hence we see that $d(f(x_n), f(a)) < \varepsilon$ for $n \geq N$, that is, $f(x_n) \to f(a)$. □

Ex. 3.1.13. Let $f\colon (X, d) \to \mathbb{C}$ be continuous at $a \in X$. Assume that $f(a) \neq 0$. Show that there exists $r > 0$ such that $|f(x)| > |f(a)|/2$ for all $x \in B(a, r)$.

Example 3.1.14. Recall the exponential map: $\exp(z) := \sum_{n=0}^{\infty}(z^n/n!)$ defined in Example 2.2.11. We claim that it is continuous on $\mathbb{C}$. Fix $z \in \mathbb{C}$. Let $\varepsilon > 0$ be given. Choose δ such that $0 < \delta < 1$ and $\delta \exp(1 + |z|) < \varepsilon$. Let w be such that $|w - z| < \delta$. Then $|w| < 1 + |z|$. We compute

$$\begin{aligned}
|\exp(z) - \exp(w)| &\leq \sum_{n=1}^{\infty} \frac{|z^n - w^n|}{n!} \\
&= \sum_{n=1}^{\infty} \frac{|(z-w)(z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1})|}{n!} \\
&\leq \sum_{n=1}^{\infty} |z - w| \frac{|z|^{n-1} + |z|^{n-2}|w| + \cdots + |w|^{n-1}}{n!} \\
&\leq |z - w| \sum_{n=1}^{\infty} \frac{(1 + |z|)^{n-1}}{(n-1)!} \\
&\leq |z - w| \exp(1 + |z|).
\end{aligned}$$

Example 3.1.15. In fact, any power series function $f(z) := \sum_{n=0}^{\infty} a_n(z-a)^n$ is continuous on its disk of convergence. Assume $a = 0$. Let $R > 0$ be the radius of convergence and $|z| < R$. Choose $\delta > 0$ such that $|z| + 2\delta < R$. From the identity, $w^n - z^n = (w - z)[\sum_{k=0}^{n-1} w^k z^{n-k-1}]$ we obtain the inequality

$$|(z+h)^n - z^n| \leq |h| n(|z| + \delta)^{n-1}, \text{ for } |h| < \delta. \tag{3.1}$$

From the binomial expansion for $[(|z| + \delta) + \delta]^n$ we obtain

$$(|z| + 2\delta)^n \geq n(|z| + \delta)^{n-1}\delta. \tag{3.2}$$

We use these below:

$$
\begin{aligned}
|f(z+h)-f(z)| &\leq \sum_{n=1}^{\infty} |a_n| \left|(z+h)^n - z^n\right| \\
&\leq \sum_{n=1}^{\infty} |a_n| \left(n\,|h|\,(|z|+\delta)^{n-1}\right) \qquad \text{from (3.1)} \\
&\leq |h| \sum_{n=1}^{\infty} |a_n| \frac{(|z|+2\delta)^n}{\delta} \qquad \text{from (3.2)} \\
&\leq \frac{|h|}{\delta} M,
\end{aligned}
$$

where $M := \sum_{n=1}^{\infty} |a_n|\,(|z|+2\delta)^n < \infty$. The RHS goes to zero as $h \to 0$.

Example 3.1.16. Let A be a nonempty subset of a metric space (X, d). Define $f(x) := \inf\{d(x,a) : a \in A\}$. Then f is continuous on X. For $x, y \in X$, we have $d(x,a) \leq d(x,y) + d(y,a)$. We therefore have

$$f(x) \leq d(x,a) \leq d(x,y) + d(y,a), \ \forall a \in A.$$

Thus $f(x)$ is a lower bound for the set $\{d(x,y) + d(y,a) : a \in A\}$. Hence, $f(x) \leq \inf\{d(x,y) + d(y,a) : a \in A\}$. We arrive at $f(x) \leq d(x,y) + f(y)$ or $f(x) - f(y) \leq d(x,y)$. Interchanging x and y in the above, we get $|f(x) - f(y)| \leq d(x,y)$, from which the continuity follows.

The function f will usually be denoted by d_A. The function $d_A(x)$ is called the distance of x to A and is often denoted by $d(x, A)$.

Ex. 3.1.17. Let $A := B[0,1]$, $B := \{z \in \mathbb{C} : |z| = 1\}$ and $C := \{z \in \mathbb{C} : \operatorname{Im} z \cdot \operatorname{Re} z = 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ be subsets of $\mathbb{C}$. Find expressions for d_A, d_B and d_C.

Ex. 3.1.18. Let $f\colon (X,d) \to \mathbb{C}$ be continuous at $p \in X$. Show that there exists $\delta > 0$ and $M > 0$ such that $|f(x)| \leq M$ for all $x \in B(p, \delta)$.

Definition 3.1.19. Let (X, d) be a metric space. A subset U of X is said to be *open* in X if for every $x \in U$, there exists a real number $r_x > 0$ such that $B(x, r_x) \subset U$.

Example 3.1.20. An open ball $B(x,r)$ is open in X. For, if $y \in B(x,r)$, let $r_y := r - d(x,y) > 0$. Then $B(y, r_y) \subset B(x,r)$. Indeed, let $z \in B(y, r_y)$. We have $d(z,x) \leq d(z,y) + d(y,x) < r_y + d(y,x) = r$.

Example 3.1.21. The set $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is open in $\mathbb{C}$. Let $z \in \mathbb{H}$. Let $r := \operatorname{Im} z > 0$. We show that $B(z,r) \subset \mathbb{H}$. Let $w \in B(z,r)$. From the inequalities

$$\operatorname{Im} w - \operatorname{Im} z \leq |\operatorname{Im} w - \operatorname{Im} z| \leq |w - z| < r, \ \text{for } w \in B(z,r),$$

we get $-r < \operatorname{Im} w - \operatorname{Im} z < r$. In particular, $\operatorname{Im} w > -r + \operatorname{Im} z = 0$.

Ex. 3.1.22. Show that (1) the punctured plane $\mathbb{C}^*$ of nonzero complex numbers and (2) any punctured disk $B(z,r) \setminus \{z\}$ are open in $\mathbb{C}$.

Theorem 3.1.23. *A function $f\colon (X,d) \to (Y,d)$ is continuous on X iff for every open set $V \subset Y$ the inverse image $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is open in X.*

Proof. Let f be continuous and V be open in Y. If $V = \emptyset$, then $f^{-1}(V) = \emptyset$ and hence the result is true. So, we assume that $V \neq \emptyset$. Let $x \in U := f^{-1}(V)$ be arbitrary. Since V is open and $y := f(x) \in V$, there is an $\varepsilon > 0$ such that $B(y,\varepsilon) \subset V$. For this $\varepsilon > 0$ there is $\delta > 0$ by continuity of f at x. If $x_1 \in B(x,\delta)$, then $f(x_1) \in B(y,\varepsilon) \subset V$. Hence $x_1 \in f^{-1}(V)$. Since $x_1 \in B(x,\delta)$ is arbitrary, it follows that $B(x,\delta) \subset f^{-1}(V)$. Since $x \in f^{-1}(V)$ is arbitrary, we have shown that for every $x \in f^{-1}(V)$, there is a $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(V)$. Hence, $f^{-1}(V)$ is open.

To see the converse, let $x \in X$ and $\varepsilon > 0$ be given. Since $B(f(x),\varepsilon)$ is open in Y, its inverse image $U := f^{-1}(B(f(x),\varepsilon))$ is open in X. As $x \in U$, there is a $\delta > 0$ such that $B(x,\delta) \subset U$. Hence, if $x_1 \in B(x,\delta)$, then $x_1 \in U$, and hence $f(x_1) \in B(f(x),\varepsilon)$. This establishes the ε-δ definition of continuity of f at x. □

Remark 3.1.24. This criterion for continuity in terms of open sets can be used to show certain subsets are open, as the following couple of items show.

Example 3.1.25. Let $\mathbb{H}$ be as in Example 3.1.21. The function $f\colon \mathbb{C} \to \mathbb{R}$ given by $f(z) = \operatorname{Im} z$ is continuous. If we take $V := (0,\infty)$, then V is open in $\mathbb{R}$ and hence $f^{-1}(V) = \mathbb{H}$ is open in $\mathbb{C}$.

Ex. 3.1.26. Show that $U := \{z \in \mathbb{C} : |z|^2 > \operatorname{Im} z\}$ is open in $\mathbb{C}$.

Ex. 3.1.27. Show that the family $\mathcal{T}$ of open sets in a metric space (X,d) has the following properties:
(1) $\emptyset, X \in \mathcal{T}$.
(2) If $\{U_\alpha : \alpha \in I\}$ is a collection of open sets then $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$. That is, arbitrary union of open sets is open.
(3) If U_k, $1 \leq k \leq n$, is a finite collection of open sets then $\cap_{k=1}^n U_k$ is open.

Ex. 3.1.28. Any open subset of a metric space is a union of open balls.

Ex. 3.1.29. Let U be open in X. When is the characteristic function χ_U is continuous? Recall that a characteristic function χ_A of a subset $A \subset X$ is defined to be

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.1.30. A subset $F \subset X$ of a metric space is said to be *closed* if its complement $X \setminus F$ is open in X.

Example 3.1.31. Any closed ball $B[x,r]$ in a metric space is closed. For, let $y \notin B[x,r]$. If we take $s := d(x,y) - r > 0$, then $B(y,s) \cap B[x,r] = \emptyset$. For, if z lies in the intersection, then $d(x,y) \le d(x,z) + d(z,y) < r + s = d(x,y)$ — a contradiction.

Example 3.1.32. Any finite set F in a metric space is closed. Let $F = \{x_k : 1 \le k \le n\}$. If $x \ne x_k$ for any k, then $d(x,x_k) > 0$ for $1 \le k \le n$. If we let $r := \min\{d(x,x_k) : 1 \le k \le n\}$, then $r > 0$. We claim that $B(x,r) \subset X \setminus F$. That is, $x_k \notin B(x,r)$ for any k. For, otherwise, $d(x,x_j) < r \le d(x,x_j)$ for some j, a contradiction. Hence $B(x,r) \subset X \setminus F$.

Theorem 3.1.33. *A function $f\colon (X,d) \to (Y,d)$ is continuous on X iff for every closed set $K \subset Y$ the inverse image $f^{-1}(K) := \{x \in X : f(x) \in K\}$ is closed in X.*

Proof. The result follows by taking "complements" in the proof of Theorem 3.1.23.

Reason: Recall that taking inverse images behaves well with set theoretic operations. We use this fact below. Let $V := Y \setminus K$. Then V is open in Y. Therefore,

$$f^{-1}(V) = f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$$

is open in X. This shows that $f^{-1}(K)$ is closed in X.

$\square$

Ex. 3.1.34. What is the analogue of Remark 3.1.24?

Ex. 3.1.35. Show that the set $\{z \in \mathbb{C} : \operatorname{Re} z^3 = \operatorname{Im} z^2\}$ is closed in $\mathbb{C}$.

Ex. 3.1.36. Show that $\mathbb{R}$ is closed in $\mathbb{C}$.

Ex. 3.1.37. Show that the family $\mathcal{F}$ of closed sets in a metric space (X,d) has the following properties:
(1) $\emptyset$, $X \in \mathcal{F}$.
(2) If $\{F_\alpha : \alpha \in I\}$ is a collection of closed sets then $\cap_{\alpha \in I} F_\alpha \in \mathcal{F}$. That is, arbitrary intersection of closed sets is closed.
(3) If $\{F_k : 1 \le k \le n\}$, is a finite collection of closed sets then $\cup_{k=1}^n F_k$ is closed.

Ex. 3.1.38. Let X and Y be metric spaces. Let $f,g\colon X \to Y$ be continuous. Then show that $\{x \in X : f(x) \ne g(x)\}$ is open in X.

Ex. 3.1.39. Let $f,g\colon X \to \mathbb{R}$ be continuous. Then $\{x \in X : f(x) > g(x)\}$ is open in X.

Ex. 3.1.40. For each of the following subsets of $\mathbb{C}$ state whether the set is open, closed or neither.

(1)	$\{z : 0 \le \operatorname{Re} z \le 1,\ 0 < \operatorname{Im} z < 1\}$	(2)	$\{z : \lvert z\rvert = 1\}$
(3)	$\{z : a < \lvert z\rvert < b\}$	(4)	$\{z : z = m + in, m, n \in \mathbb{Z}\}$
(5)	$\{z : z = x + iy, x, \in \mathbb{Q}\}$	(6)	$\{z : z = x + iy, x \in \mathbb{Q}, y \in \mathbb{R}\}$
(7)	$\{z : \operatorname{Re} z \ge 2 \text{ and } \operatorname{Im} z \le 5\}$	(8)	$\{z : \operatorname{Re}(z^2) = 2\}$
(9)	$\{z : \lvert z-1\rvert < 1 \text{ or } \lvert z+1\rvert < 1\}$	(10)	$\{z : \lvert z+1\rvert \ge 1 \text{ and } x < 0\}$

Ex. 3.1.41. $U \subset \mathbb{C}$ is open (respectively closed) iff $V := \{\bar{z} : z \in U\}$ is so.

Ex. 3.1.42. Patching or Gluing Lemma. Let X and Y be metric spaces. Let $X = A \cup B$. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. Assume further that $f = g$ on $A \cap B$ (which is *vacuously* true if $A \cap B = \emptyset$). Define $F\colon X \to Y$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Then F is well-defined. Prove that

(1) If A and B are open, then F is continuous.
(2) If A and B are closed, then F is continuous.
(3) In general, F need not be continuous, for arbitrary subsets A and B.

Investigate the validity of the above results when the two sets are replaced by a family of subsets.

Another way of stating the lemma is as follows: Let $X = A \cup B$, where both A and B are open (or both of them closed). Let $F\colon X \to Y$ be such that its restrictions to A and B are continuous. Then F is continuous on X.

Definition 3.1.43. A subset A of a metric space X is said to be *dense* in X if for every nonempty open set U in X, we have $U \cap A \neq \emptyset$. This is equivalent to demanding that for each $x \in X$ and for every $r > 0$, we have $B(x,r) \cap A \neq \emptyset$.

Let $Y \subset X$. We say that a subset $D \subset Y$ is dense in Y, if D is dense in the metric space (Y,d) where d is the restriction to Y. This is equivalent to saying that for each $y \in Y$ and $r > 0$, the set $B(y,r) \cap Y \cap D$ is nonempty.

Example 3.1.44. The set $\mathbb{Q}$ of rationals in $\mathbb{R}$ is dense in $\mathbb{R}$. (See any book on a first course in real analysis, e.g., see Theorem 1.3.13 in [2].) If we take $A := \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$, then A is dense in $\mathbb{C}$.

More generally the set $\{z \in U : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$ is dense in U.

Ex. 3.1.45. Let D be dense in (X,d) and $r > 0$. Then show that $X = \cup_{x \in D} B(x,r)$.

Definition 3.1.46. Let $\emptyset \neq A \subset X$ be given. A point $x \in X$ is said to be a *limit point* of A if $B(x,r) \cap A \neq \emptyset$ for every $r > 0$.

A point x is said to be a *cluster point* or an *accumulation point* of A if $(B(x,r) \setminus \{x\}) \cap A \neq \emptyset$ for every $r > 0$.

In almost all books, what we call as cluster point is defined as limit point. However, we shall adhere[1] to our definitions. See the example and a couple of exercises below it to understand the difference between these notions.

Example 3.1.47. Note that any point $x \in A$ is a limit point of A, but it may not be a cluster point of A.

[1] *Pun intended!*

If x is a cluster point of A, then x is a limit point of A, but not conversely. *Hint:* If $A := \{1/n : n \in \mathbb{N}\}$, then for each $n \in \mathbb{N}$, $1/n$ is a limit point of A but not a cluster point.

Beware: In most text-books, there is no distinction between limit points and cluster points.

Ex. 3.1.48. Show that x is a limit point of A iff there exists a sequence (a_n) in A such that $\lim a_n = x$.

Ex. 3.1.49. Show that x is a limit point of A iff $d_A(x) = 0$. (The notation is as in Example 3.1.16.)

Ex. 3.1.50. Show that any point in $B(z,r) \subset \mathbb{C}$ is a cluster point of $B(z,r)$.

Ex. 3.1.51. Let X be a metric space and $A \subseteq X$. A subset A is closed in X iff all the limit points of A lie in A.

Ex. 3.1.52. Redo Ex. 3.1.36.

Ex. 3.1.53. Let X be a metric space and $A \subseteq X$. Then show that A is dense in X iff every point of X is a limit point of A.

Theorem 3.1.54. *Let X and Y be metric spaces. Let $f, g \colon X \to Y$ be continuous. Let A be dense in X. Assume that $f(a) = g(a)$ for all $a \in A$. Then $f = g$ on X.*

Proof. If $x \in X$, then there exists a sequence (a_n) in A such that $a_n \to x$. By continuity $f(a_n) \to f(x)$ and $g(a_n) \to g(x)$. By the uniqueness of limit of a sequence in a metric space, we conclude that $f(x) = g(x)$. □

A very special case of the theorem worth mentioning is the following exercise.

Ex. 3.1.55. Let X be a metric space. Let $f \colon X \to \mathbb{C}$ be continuous. Assume that $f = 0$ on a dense subset of X. Show that $f = 0$ on X.

3.2 Uniform Convergence and Continuity

Let X be any nonempty set. Let $f_n \colon X \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of functions. We say that f_n *converges uniformly* on X to a function $f \colon X \to \mathbb{C}$ if for any given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and for **all** $x \in X$, we have $|f_n(x) - f(x)| < \varepsilon$.

Theorem 3.2.1. *Let (X,d) be a metric space. Let $f_n \colon X \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of continuous functions. Assume that there exists $f \colon X \to \mathbb{C}$ such that f_n converges to f uniformly on X. Then f is continuous.*

Proof. Let $x_0 \in X$. We show that f is continuous at x_0. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in X$. For the same ε, by the continuity of f_N at x_0, there exists a $\delta > 0$. For $y \in B(x_0, \delta)$, we have

$$\begin{aligned}|f(y) - f(x_0)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

The first and third terms are less than ε by the uniform convergence while the middle one by the continuity of f_N at x_0. Thus f is continuous at x_0. Since $x_0 \in X$ is arbitrary, the continuity of f follows. □

Ex. 3.2.2. Let $f_n, g_n : X \to \mathbb{C}$ be a sequence of functions such that f_n (resp. g_n) converge uniformly on X to f (resp. g). Assume that f and g are bounded on X. Then $f_n g_n$ converges to fg uniformly on X. *Hint:* Mimic the proof of $a_n b_n \to ab$ with care.

Definition 3.2.3. Given a series $\sum_{n=1}^{\infty} f_n$ of functions $f_n \colon X \to \mathbb{C}$, we say that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on X if the sequence (s_n) of partial sums, where $s_n := \sum_{k=1}^{n} f_k$, converges uniformly on X to a function $f \colon X \to \mathbb{C}$. We then say that $\sum_n f_n$ is uniformly convergent (on X) to f and write $f = \sum_n f_n$.

Proposition 3.2.4. *If $f_n \colon X \to \mathbb{C}$ are continuous for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on X to f then f is continuous.*

Proof. Note that the n-th partial sum s_n is continuous for all n. Since the sequence of partial sums $s_n \to f$ uniformly, the result follows from Theorem 3.2.1. □

The next theorem gives the most useful sufficient condition for uniform convergence of a series of functions.

Theorem 3.2.5 (Weierstrass M-Test). *Let X be any set, and $f_n \colon X \to \mathbb{C}$ be a sequence of functions such that $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$. Assume that the series $\sum M_n$ is convergent. Then the series $\sum f_n$ is uniformly convergent on X.*

Proof. If we fix $x \in X$, then the numerical series $\sum |f_n(x)|$ is convergent as seen by comparing it with the series $\sum M_n$. Hence the series $\sum_n f_n(x)$ is convergent. Let $f(x) := \sum f_n(x)$. Thus we get a function $f \colon X \to \mathbb{C}$. We wish to show that $\sum f_n$ is uniformly convergent to f on X. Fix $x \in X$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $\sum_{n>N} M_n < \varepsilon$. For all $n > N$,

$$\left| \sum_{N+1}^{n} f_k(x) \right| \leq \sum_{N+1}^{n} |f_k(x)| \leq \sum_{N+1}^{n} M_k < \varepsilon.$$

Hence $|\sum_{n>N} f_n(x)| \leq \varepsilon$. Hence $|f(x) - s_N(x)| = |\sum_{N+1}^{\infty} f_n(x)| \leq \varepsilon$. Since $x \in X$ is arbitrary, this shows that the series is uniformly convergent to f. □

Theorem 3.2.6. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be given and $0 < R \leq \infty$ be its radius of convergence. Then for any r with $0 \leq r < R$, the power series is uniformly convergent on the closed ball $B[a, r]$.*

Proof. This is established during the course of our proof of Theorem 2.3.3. (See the boxed statement in Ex. 2.3.5 on page 22.) Note that for $|z| \leq r$, we showed that $|a_n z^n| \leq M(|r/z_0|)^n$. Thus we can invoke the M-test to arrive at the result. □

Remark 3.2.7. However it should be noted that the power series need not converge uniformly on its disk of convergence. For example, the geometric series $\sum_{k=0}^{\infty} z^n$ does not converge uniformly on $B(0, 1)$.

Reason: Suppose it did. We choose an ε such that $0 < \varepsilon < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{1-z} - \frac{1-z^{n+1}}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right| < \varepsilon, \qquad \text{for all } z \in B(0,1), n \geq N.$$

In particular, $|z|^{N+1} < \varepsilon\,|1-z|$ for $z = z_k := 1 - \frac{1}{k}$, $k \in \mathbb{N}$. This is clearly absurd. For, as $k \to \infty$, $z_k \to 1$ and hence $|z_k|^{N+1} \to 1$. Whereas, on the right side, the term $\varepsilon\,|1 - z_k| \to 0$. Thus we arrive at the contradiction that $1 < 0$.

Another way of seeing this would be as follows. The partial sums $s_n \to 1/(1-z)$ uniformly on $B(0, 1)$ iff the maximum $M_n := \max_{|z|<1} |s_n(z) - 1/(1-z)|$ goes to zero as $n \to \infty$. Let us fix $0 \leq r < 1$ and $z = re^{it}$. Then we have

$$|1 - z| \leq 1 + |z| = 1 + r \text{ so that } \left|\frac{1}{1-z}\right| \geq \frac{1}{1+r}.$$

Hence, we have

$$M_n \geq \left|s_n(z) - \frac{1}{1-z}\right| \geq \frac{|z^{n+1}|}{|1-z|} \geq \frac{r^{n+1}}{1+r}.$$

Since this is true for every $0 < r < 1$, we see that $M_n \geq 1/2 = \lim_{r\to 1-} \frac{r^{n+1}}{1+r}$, for each $n \in \mathbb{N}$.

Corollary 3.2.8. *Any power series with positive radius of convergence is continuous in its disk of convergence.*

Proof. Note that this is already proved in Example 3.1.15.

Let R be the radius of convergence of the power series $f(z) := \sum_{n=0}^{\infty} a_n z^n$. Then for any $z \in B(0, R)$, we find $0 < r < R$ such that $z \in B(0, r)$. Now by the last result, the series is uniformly convergent on $B[0, r]$ and hence f is continuous on $B[0, r]$, in particular, at z. □

Corollary 3.2.9 (Uniqueness Theorem for Power Series). *Let $f(z) := \sum_n a_n(z-a)^n$ and $g(z) := \sum b_n(z-a)^n$ for all $z \in B(a, R)$ with $R > 0$. Assume that there exists a sequence (z_n) in $B(a, R)$ such that* (i) *$z_n \neq a$ for all n,* (ii) *$\lim z_n = a$ and* (iii) *$f(z_n) = g(z_n)$ for all n. Then $a_n = b_n$ for all n.*

Proof. For convenience, assume that $a = 0$. Since f and g are continuous at $a = 0$, $f(0) = \lim f(z_n) = \lim g(z_n) = g(0)$ and hence $a_0 = b_0$. Now proceed by induction. By induction hypothesis, we have $a_k = b_k$ for $0 \le k \le n$. We let $\varphi(z) := a_{n+1} + a_{n+2}z + \cdots$ and $\psi(z) := b_{n+1} + b_{n+2}z + \cdots$. Note that

$$\varphi(z) = \begin{cases} \frac{f(z) - \sum_{k=0}^{n} a_k z^k}{z^{n+1}} & \text{for } z \neq 0, |z| < R \\ a_{n+1} & \text{at } z = 0 \end{cases}$$

Similar remark applies to ψ. It follows that φ and ψ are continuous on $B(0, R)$. Consequently, $\varphi(z_r) = \psi(z_r)$ for all $r \in \mathbb{N}$. Hence we have

$$\varphi(0) = \lim \varphi(z_r) = \lim \psi(z_r) = \psi(0).$$

Since $\varphi(0) = a_{n+1}$ and $\psi(0) = b_{n+1}$, the result follows. □

Ex. 3.2.10. Corollary 3.2.9 says that if two power series functions agree on a set $\{z_n\}$ with a as a cluster point in $B(a, R)$, then they agree on all of $B(a, R)$.

Compare this with Theorem 3.1.54.

Note, in particular, that if two power series convergent on $B(a, R)$ agree on a non-empty open set or even on a line segment containing a, they are identical on $B(a, R)$.

Remark 3.2.11. One should understand this kind of results in a large context. Given two functions $f, g \colon X \to Y$ between two sets, then $f = g$ as functions iff for all $x \in X$, we have $f(x) = g(x)$. Theorem 3.1.54 says that if the functions are continuous, we need only check $f(x) = g(x)$ for x in a dense subset of X, which may be less arduous. Similarly, Corollary 3.2.9 says if we wish to show that two functions defined by a power series are equal, we need only establish their equality on a subset which has a cluster point in the disk of convergence.

Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be two polynomials. Let N be the maximum of the degrees of f and g. Then $f = g$ as functions on $\mathbb{C}$ if $f(z_j) = g(z_j)$ for $N + 1$ distinct points z_j. (Why?)

If $f, g \colon V \to W$ are linear maps between vector spaces over the same field, then $f = g$ is true if $f(z) = g(z)$ for $z \in B$, a basis of V.

Chapter 4

Exponential Function and its Associates

In this chapter we study the important properties of the exponential function in depth. Using this we derive the properties of the trigonometric functions sine and cosine. We also introduce the real number π and establish the periodicity of the exponential and trigonometric functions. All the facts that the reader has been using about the trigonometric functions will be put on a rigorous footing in this chapter.

4.1 Exponential Function

Recall that we have defined $\exp(z) := \sum_{n=0}^{\infty} z^n/n!$ for $z \in \mathbb{C}$. The series converges for all z either by the ratio test or by comparison with $\sum(|z|/N)^n$ where $N > |z|$. (We have already seen this in Example 2.2.11.) The map $z \mapsto \exp(z)$ is denoted by $\exp\colon \mathbb{C} \to \mathbb{C}$ and is called the exponential function.

Theorem 4.1.1. *The exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *has the following properties:*
(i) $\exp(z+w) = \exp(z) \cdot \exp(w)$, *for all* $z, w \in \mathbb{C}$.
(ii) *For* $x \in \mathbb{R}$ *with* $x > 0$, *we have* $\exp(x) > 1 + x > x$.
(iii) $\exp$, *when restricted to* $\mathbb{R}$, *is positive and increasing:* $\exp(x) > 0$ *for* $x \in \mathbb{R}$ *and if* x *and* y *are real numbers with* $x < y$ *then* $\exp(x) < \exp(y)$.
(iv) $\exp$ *is continuous on* $\mathbb{C}$.
(v) $\exp(z) \neq 0$ *and* $\exp(-z) = 1/\exp(z)$.
(vi) $\exp(\overline{z}) = \overline{\exp(z)}$ *for all* $z \in \mathbb{C}$.
(vii) *If* x *is real then* $|\exp(ix)| = 1$.
(viii) $|\exp(z)| = \exp(\mathrm{Re}\,(z)) \leq \exp(|z|)$ *for* $z \in \mathbb{C}$.
(ix) *For any* $\alpha \in \mathbb{R}$ *with* $\alpha > 0$, *there is a unique* $x \in \mathbb{R}$ *such that* $\exp(x) = \alpha$.
(x) *For any* $n \in \mathbb{N}$, $x^{-n}\exp(x) \to \infty$ *as* $x \to \infty$, *i.e. given* $M > 0$, *there exists* $R > 0$ *such*

that for all $x > R$, we have $x^{-n}\exp(x) > M$.

(xi) *For each $n \in \mathbb{N}$, $x^n \exp(-x) \to 0$ as $x \to \infty$, i.e. given $\varepsilon > 0$, there exists $R > 0$ such that if $x > R$, then $0 \le x^n \exp(-x) < \varepsilon$.*

Proof. (i) The series for $\exp(z)$ and $\exp(w)$ are absolutely convergent and hence their Cauchy product converges to $\exp(z)\exp(w)$:

$$\begin{aligned}\exp(z)\exp(w) &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{z^j}{j!}\frac{w^{n-j}}{(n-j)!}\right)\\ &= \sum_{n=0}^{\infty}(z+w)^n/n! = \exp(z+w).\end{aligned}$$

(ii) If $x > 0$, then note that the series for $\exp(x)$ is a series of positive terms and hence $\exp(x) := \sup\{\sum_{k=0}^{n} x^k/k!\}$ by Ex. 2.2.2. Clearly, $\sup\{\sum_{k=0}^{n} x^k/k!\} > 1 + x > x$.

(iii) From (i), we have $\exp(x)\exp(-x) = \exp(0) = 1$. If x is any nonzero real number, then either $x > 0$ or $-x > 0$. Hence either $\exp(x) > 0$ or $\exp(-x) > 0$. From $\exp(x)\exp(-x) = 1$, it follows that $\exp(-x) = \frac{1}{\exp(x)}$ and hence for any $x \in \mathbb{R}$, we have $\exp(x) > 0$. Also, if $x < y$, then

$$\exp(y) = \exp((y-x)+x) = \exp(y-x)\exp(x) > (1 + (y-x))\exp(x) > \exp(x).$$

Hence exp is increasing in $\mathbb{R}$.

(iv) is proved already in Example 3.1.14.

(v) From (i), $\exp(z)\exp(-z) = \exp(0) = 1$. Therefore, $\exp(z)^{-1} = \exp(-z)$ and hence (4) follows.

(vi) Follows from the fact that $\sum z_n$ converges and its sum is z iff $\sum \overline{z}_n$ converges and its sum is $\overline{z}$.

(vii) For $x \in \mathbb{R}$, we have, using (vi) and (i),

$$|\exp(ix)|^2 = \exp(ix)\overline{\exp(ix)} = \exp(ix)\exp(-ix) = \exp(0) = 1.$$

(xiii) If $z = x+iy$, then $\exp(z) = \exp(x+iy) = \exp(x)\exp(iy)$ by (1). Hence $|\exp(z)| = |\exp(x)|\,|\exp(iy)| = \exp(x)$, thanks to (vii) and (iii). Since $x \le |x| \le |z|$, using (iii), we see that $|\exp(z)| \le \exp(|z|)$.

(ix) From (ii), $\exp(\alpha) > \alpha$ and hence

$$\exp(-1/\alpha) = 1/(\exp(1/\alpha)) < \frac{1}{1/\alpha} = \alpha < \exp(\alpha).$$

exp is continuous on $[-1/\alpha, \alpha]$ and we have just seen that $\exp(-1/\alpha) < \alpha$ and $\exp(\alpha) > \alpha$. Hence by the intermediate value theorem, there exists $x \in [-1/\alpha, \alpha]$ such that $\exp(x) = \alpha$.

(x) Fix n. Observe that, for $x > 0$, $\exp(x) > \frac{x^{n+1}}{(n+1)!}$ (why?) so that $x^{-n}\exp(x) > \frac{x}{(n+1)!}$.

Given $M > 0$, then for any $x > (n+1)!M$, we have

$$x^{-n}\exp(x) > \frac{x}{(n+1)!} > \frac{(n+1)!M}{(n+1)!} = M.$$

(xi) is proved similarly. Or, it can be deduced from (x). □

Loosely speaking, (x) of the last theorem says that $\exp(x)$ goes to infinity as $x \to \infty$ much faster than any power of x and hence much faster than any polynomial in x. Similar remark applies to (xi). $\exp(-x)$ goes to zero much faster than $1/x^n$ for any $n \in \mathbb{N}$ as x goes to infinity.

Ex. 4.1.2. Show that $|\exp(z)|^2 = \exp(2\mathrm{Re}\, z)$.

Ex. 4.1.3. Given $\alpha > 0$, let $x \in \mathbb{R}$ be such that $\exp(x) = \alpha$. Show that $\exp(x/n)$ is the unique positive n-th root of α.

Ex. 4.1.4. Let α and x be as in the last exercise. For $t \in \mathbb{R}$, define $\alpha^t := \exp(tx)$. Prove the following: (i) $\alpha^t > 0$, (ii) $\alpha^t\alpha^s = \alpha^{s+t}$, (iii) $(\alpha^t)^s = \alpha^{st}$ and (iv) $\alpha^{-t} = \frac{1}{\alpha^t} = (\frac{1}{\alpha})^t$.

Ex. 4.1.5. Prove that $\exp(z) = \exp(\alpha)\sum_{n=0}^{\infty}\frac{(z-\alpha)^n}{n!}$, for any $\alpha \in \mathbb{C}$.

Ex. 4.1.6. Show that, for any $z \in \mathbb{C}$, the following inequality holds: $|\exp(z) - 1| \leq |z|\exp(|z|)$.

4.2 Trigonometric Functions

We define the cosine and sine functions on $\mathbb{C}$ by

$$\cos(z) := \frac{\exp(iz) + \exp(-iz)}{2}$$

$$\sin(z) := \frac{\exp(iz) - \exp(-iz)}{2i}$$

for $z \in \mathbb{C}$. For brevity, we write $\cos z$ for $\cos(z)$ etc.

Theorem 4.2.1. *For all $z, w \in \mathbb{C}$, the following are true:*

(i) $\cos$ *and* $\sin$ *are continuous on* $\mathbb{C}$.

(ii) $\exp(iz) = \cos z + i \sin z$.

(iii) $\cos^2 z + \sin^2 z = 1$.

(iv) $\cos(z+w) = \cos z \cos w - \sin z \sin w$. *In particular,* $\cos(2z) = \cos^2 z - \sin^2 z$.

(v) $\sin(z+w) = \sin z \cos w + \cos z \sin w$. *In particular,* $\sin(2z) = 2\sin z \cos z$.

(vi) $\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$

(vii) $\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$.

Proof. Most of the statements are straight forward verifications.

To prove (iii) observe that, using (ii), we get

$$1 = \exp(iz)\exp(-iz) = (\cos z + i\sin z)(\cos z - i\sin z) = \cos^2 z + \sin^2 z.$$

We offer a second proof of (iii). First note that both sides of the identity in (iii) are power series. (Why?) Secondly, item (vi) of Theorem 4.1.1 tells us that these power series functions agree at each $x \in \mathbb{R}$. Hence, by the uniqueness theorem for power series, the result follows.

(iv) and (v) are the so-called addition formulas. To prove (iv) and (v), observe that

$$\begin{aligned}
&\exp(i(z+w)) \pm \exp(-i(z+w)) \\
&= \exp(iz)\exp(iw) \pm \exp(-iz)\exp(-iw) \\
&= (\cos z + i\sin z)(\cos w + i\sin w) \pm (\cos z - i\sin z)(\cos w - i\sin w).
\end{aligned}$$

Expanding the right side and simplifying yields (vi) and (v). □

Ex. 4.2.2. If $x \in \mathbb{R}$, then $\sin x$ and $\cos x$ are real. *Hint:* Use (vi) and (vii) of the last theorem. (Still the result is not obvious. You need Ex. 3.1.36.) Show also that $|\cos x| \leq 1$ and $|\sin x| \leq 1$.

Ex. 4.2.3. The function cos is not bounded on $\mathbb{C}$. What can you say about sin on $\mathbb{C}$? *Hint:* If $y > 0$, then $\cos(iy) \geq (1 + y^2)/2$.

Ex. 4.2.4. True or false? $|\cos z| \to \infty$ as $|z| \to \infty$, that is, for any given $M > 0$, there exists $R > 0$ such that for all z with $|z| > R$, we have $|\cos z| > M$.

Ex. 4.2.5. Let $f(z) := z\sin(1/z)$ for $z \neq 0$ and $f(0) = 0$. Is f continuous at 0?

Ex. 4.2.6. Prove de Moivre's theorem: For all positive integers n, $(\cos z + i\sin z)^n = \cos nz + i\sin nz$.

Ex. 4.2.7. Show that $\cos nz$ can be written as a polynomial in $\cos z$ with integer coefficients. In particular, establish that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Ex. 4.2.8. Show that $|\sin(nx)| \leq n\,|\sin(x)|$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Is a similar result true if x is a nonreal complex number?

Ex. 4.2.9. Prove the following for $x, y, z, w \in \mathbb{C}$:
(i) $\sin z \sin w = [\cos(z - w) + \cos(z + w)]/2$ and
(ii) $\sin(y + x)\sin(y - x) = \sin^2 y - \sin^2 x$.

Periodicity

Lemma 4.2.10. *Any zero of the sine function is real: if* $\sin z = 0$ *then* $z \in \mathbb{R}$.

Proof. If $\sin z = 0$, then $\exp(iz) - \exp(-iz) = 0$ so that $\exp(2iz) = 1$. Hence $\exp(-2y) = |\exp(2iz)| = 1$ where $z = x + iy$. Since exp is strictly increasing on $\mathbb{R}$, it is one-one on $\mathbb{R}$. It follows that $y = 0$. □

Lemma 4.2.11. *There exists* $t \in (0,1)$ *such that* $\cos t - \sin t = 0$.

Proof. If $0 < x < 2$, we have

$$\begin{aligned} |\sin x - x| &\le \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ &\le \frac{x^3}{3!}\left(1 + \frac{x^2}{20} + \left(\frac{x^2}{20}\right)^2 + \cdots\right) \\ &\le \frac{x^3}{3!}[1 + (1/5) + (1/5)^2 + \cdots] \\ &< x^3/4. \end{aligned}$$

That is, $\sin x \in (x - (x^3/4), x + (x^3/4))$ and hence

$$\sin x > \frac{x(4-x^2)}{4} > 0, \qquad 0 < x < 2. \tag{4.1}$$

From (4.1), we find that $\sin 1 > 3/4$. Since $\cos^2 + \sin^2 = 1$, it follows that $\cos 1 < 3/4$. Thus the continuous function $\cos - \sin$ is such that $(\cos - \sin)(0) = 1$ while $(\cos - \sin)(1) < 0$. By the intermediate value theorem, there exists a $t \in [0,1]$ such that $\cos t - \sin t = 0$. □

Lemma 4.2.12. *The sine function has at least one zero in* $(0,4)$.

Proof. Let $t \in (0,1)$ be such that $\cos t = \sin t$. Hence $\cos 2t = \cos^2 t - \sin^2 t = 0$ and hence $\sin 4t = 2\cos 2t \sin 2t = 0$. Hence sin has at least one zero in $(0,4)$. □

Definition 4.2.13. Let $E := \{x > 0 : \sin x = 0\}$. Then $E \neq \emptyset$, by the last lemma. Define $\pi := \inf E$. From (4.1) and the lemma, we see that $2 \le \pi < 4$. By continuity of sine, $\sin \pi = 0$: For each $n \in \mathbb{N}$, $\pi + 1/n$ is greater than the greatest lower bound π of E and hence there exists $x_n \in E$ with $\pi \le x_n < \pi + (1/n)$. Since $\sin(x_n) = 0$ for all n and since $x_n \to \pi$, it follows from the continuity of the sine function that $\sin(\pi) = 0$.

Observations:

(1) $\sin \pi/2 = 1$. For, $\sin \pi = 2 \sin \pi/2 \cos \pi/2$. Since $0 < \pi/2 < 2$, $\sin \pi/2 > 0$. Hence $\cos \pi/2 = 0$. Since $\cos^2 + \sin^2 = 1$, the result follows. Note also that $\cos(\pi) = -1$:

$$\cos(\pi) = \cos(2\frac{\pi}{2}) = \cos(\pi/2)\cos(\pi/2) - \sin(\pi/2)\sin(\pi/2) = -1 \cdot 1.$$

(2) $\sin x > 0$ for $x \in (0, \pi)$. For, $\sin(x) > 0$ for $x \in (0, \pi/2)$ from (4.1). Observe that $\sin(x + \pi/2) = \sin(-x + \pi/2)$. (Expand both sides using the addition formula for sine.) If $x \in (0, \pi/2)$, the right side of this identity is positive.

One can also prove this using the definition of π and continuity of sine. If $\sin(x) = 0$ for $x \in (0, \pi)$, this is a contradiction to the definition of π. If $\sin x < 0$ for some $x \in (0, \pi)$, then $\sin(y) = 0$ for some $y \in (0, x) \subset (0, \pi)$ by intermediate value theorem. This violates the definition of π.

(3) For $z \in \mathbb{C}$, we have $\sin(z + \pi) = -\sin(z)$ and hence $\sin(z + 2\pi) = \sin(z)$. This follows from (1) and the addition formula for sine. Note also that $\cos(z + 2\pi) = \cos(z)$ for all $z \in \mathbb{C}$. (Why?)

(4) cos is decreasing on $[0, \pi]$ so that $\cos\colon [0, \pi] \to [-1, 1]$ is a continuous bijection. First of all, we prove that cos is one-one on $[0, \pi]$. Let $0 \leq a < b \leq \pi$ be such that $\cos a = \cos b$. Then $\sin^2 a = \sin^2 b$ so that $\exp(ia) = \exp(ib)$ or $\exp(ia) = \exp(-ib)$. Hence we conclude that

$$\exp(i(b - a)) = 1 \text{ or } \exp(i(b + a)) = 1.$$

We claim that the second case does not arise. Since $\exp(i(b+a)) = 1$, it follows that (since a and b are real) $\sin(b + a) = 0$. Since $0 \leq a < b \leq \pi$, we see that $0 < b + a \leq 2\pi$. This along with $\sin(b + a) = 0$ implies that $b + a = \pi$. But then $\exp(i(b + a)) = \exp(i\pi) = -1$, a contradiction.

So we are left with the case $\exp(i(b - a)) = 1$. Hence $\sin(b - a) = 0$ so that either $b - a = 0$ or $b - a = \pi$. The latter possibility is ruled out, for then we have $\exp(i(b - a)) = \cos \pi = -1$. Thus $b = a$.

We now show that cos is decreasing on $[0, \pi]$. (This follows from a standard exercise in real analysis. See Exercise 4.2.15 below.) Let, if possible, $\cos a \leq \cos b$ for $0 \leq a < b \leq \pi$. Since $-1 = \cos \pi \leq \cos a \leq \cos b$, by the intermediate value theorem, there exists $a' \in (b, \pi)$ such that $\cos a' = \cos a$. This contradicts the one-one nature of cos on $[0, \pi]$.

Ex. 4.2.14. The sine function is increasing on $[0, \pi/2]$. *Hint:* Mimic the proof in Observation (4). Or, use the identity:

$$\sin(y + x)\sin(y - x) = \sin^2 y - \sin^2 x.$$

For, if $\sin x = \sin y$ for $x, y \in [0, \pi/2]$ with, say, $x < y$, then the left side of the identity, namely, $\sin(y+x)\sin(y-x) = 0$. Since $0 \leq x < y \leq \pi/2$, we see that $0 < y - x < \pi/2$ and hence $\sin(y - x) \neq 0$. It follows that $\sin(y + x) = 0$. Since $0 < x + y < \pi$, this is a contradiction to the definition of π. We are forced conclude that $x = y$.

Ex. 4.2.15. Let $f\colon J \to \mathbb{R}$ be a continuous one-one function on an interval J. Show that f is either strictly decreasing or strictly increasing on J.

Theorem 4.2.16. *Let $S := \{z \in \mathbb{C} : \sin(z) = 0\}$. Then*
(i) $S = \{n\pi : n \in \mathbb{Z}\}$.

(ii) $\{z \in \mathbb{C} : \cos(z) = 0\} = S + (\pi/2)$.
(iii) $\exp(z) = 1$ *iff* $z = n2\pi i$ *for some* $n \in \mathbb{Z}$. *Hence* $\exp(z) = \exp(w)$ *iff* $z - w = k2\pi i$ *for some* $k \in \mathbb{Z}$.

Proof. We know that if $\sin(z) = 0$ then $z \in \mathbb{R}$. Since $\sin(-x) = -\sin(x)$ and sine is strictly increasing in $[0, \pi/2]$, the only zeros of sin in $[-\pi, \pi]$ are $-\pi, 0, \pi$. Let $J := (-\pi, \pi]$. Then $\mathbb{R} = \cup_{n\in\mathbb{Z}} J + 2n\pi$ is a disjoint union. If $x \in \mathbb{R}$, then there exists $x_0 \in J$ and $n \in \mathbb{Z}$ such that $x = x_0 + 2n\pi$. Now, if we further assume that $\sin(x) = 0$, then $0 = \sin(x) = \sin(x_0 + n2\pi) = \sin(x_0)$. We hence conclude that x_0 is either 0 or π so that $x = 2n\pi$ or $(2n+1)\pi$. Thus (i) is proved. (ii) is a consequence of $\sin(z + \pi/2) = \cos(z)$.

To prove (iii), let $z = x + iy$ be such that $\exp(z) = 1$. Then $|\exp(z)| = \exp(x) = 1$ so that $x = 0$. Hence $\cos(y) = 1$ and $\sin(y) = 0$. From (i) and (ii), we infer that $y = n2\pi$. □

Definition 4.2.17. Let $f : \mathbb{C} \to \mathbb{C}$ be given. A complex number p is called a *period* of f if for every $z \in \mathbb{C}$, we have $f(z + p) = f(z)$. If f has a nonzero period p, then f is said to be periodic. Note that if p is a period of f, so is np for any $n \in \mathbb{Z}$.

It follows from the above discussions that exp, sine and cos are periodic with the set of periods $\{n2\pi i : n \in \mathbb{Z}\}$, $\{n2\pi : n \in \mathbb{Z}\}$ and $\{n2\pi : n \in \mathbb{Z}\}$ respectively. (Prove these.)

Ex. 4.2.18. Prove that $\sin(\pi/6) = 1/2$ and $\sin(\pi/4) = 1/\sqrt{2} = \cos(\pi/4)$. Compute $\sin(\pi/3)$.

Ex. 4.2.19. Under what conditions is $\sin z = \sin w$?

Ex. 4.2.20. Show that $\cos \geq 0$ on $(0, \pi/2)$.

Hyperbolic Functions

The hyperbolic functions cosh and sinh are defined as follows:

$$\cosh(z) = [\exp(z) + \exp(-z)]/2$$
$$\sinh(z) = [\exp(z) - \exp(-z)]/2.$$

The next proposition lists all the basic facts about the hyperbolic functions.

Proposition 4.2.21. *The following hold:*
(i) $\cosh(z) = \cos(iz)$ *and* $\sinh(z) = -i\sin(iz)$ *for* $z \in \mathbb{C}$. *Hence* $\cosh^2 z - \sinh^2 z = 1$.
(ii) $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ *and* $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ *for* $z \in \mathbb{C}$.
(iii) cosh *and* sinh *are positive and strictly increasing on the set of positive reals.*
(iv) $\cosh(-z) = \cosh(z)$ *and* $\sinh(-z) = -\sinh(z)$ *for* $z \in \mathbb{C}$.
(v) $\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$ *for* $z = x + iy \in \mathbb{C}$.
(vi) $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ *for* $z = x + iy \in \mathbb{C}$.
(vii) *For* $z = x + iy \in \mathbb{C}$ *we have*

$$\sinh(y) \leq |\cos(x \pm iy)| \leq \cosh(y)$$
$$\sinh(y) \leq |\sin(x \pm iy)| \leq \cosh(y).$$

Proof. Most of the statements are straightforward verification.

We indicate a proof for a part of (vii). We have $\cos(x+iy) = [\exp(-y+ix)+\exp(y-ix)]/2$. We use the fact that $|\exp(u+iv)| = \exp(u)$ and the triangle inequality to estimate above:

$$\begin{aligned}\frac{1}{2}|\exp(-y+ix)+\exp(y-ix)| &\le \frac{1}{2}|\exp(-y+ix)| + \frac{1}{2}|\exp(y-ix)|\\ &\le \frac{1}{2}(e^{-y}+e^{y})\\ &= \cosh(y).\end{aligned}$$

The other set of inequalities is proved in an analogous way. □

Ex. 4.2.22. If $y \neq 0$, show that $\cosh(x+iy)$ is real iff x is an integral multiple of π. What are the values of z for which $\sin z$ is real? Show that if $\cos z$ and $\sin z$ are real, then z is real.

Ex. 4.2.23. Show that sine maps the strip $\{z := x+iy : 0 \le x \le \pi/2, 0 \le y < \infty\}$ bijectively onto the whole of the first quadrant, mapping the boundary of the strip onto that of the first quadrant. Hence it maps $\{-\pi/2 < x < \pi/2, y > 0\}$ onto the upper half-plane $\{\mathrm{Im}\,(z) > 0\}$.

Ex. 4.2.24. Show that $\sin\colon \mathbb{C} \to \mathbb{C}$ maps the lines parallel to the x-axis into ellipses and the lines parallel to the y-axis into hyperbolas.

Ex. 4.2.25. Show that exp is 1-1 on the strip $S := \{x+iy : |y| \le \pi/2\}$ and maps S onto $U := \{w := s+it : s \ge 0, w \ne 0\}$. Show that exp maps the boundary of S onto all of the boundary of U except $w = 0$. What happens to each of the horizontal lines $\{\mathrm{Im}\,(z) = \pi/2\}$ and $\{\mathrm{Im}\,(z) = -\pi/2\}$?

Ex. 4.2.26. Find all zeros and periods of cosh.

Ex. 4.2.27. Find all solutions of $\sin z = 2$. Ans. $z = \frac{4k+1}{2}\pi \pm i\cosh^{-1}(2)$, $k \in \mathbb{Z}$.

Ex. 4.2.28. Let three subsets of $\mathbb{C}$ be defined as follows:

$$\begin{aligned}U_1 &:= \{z \in \mathbb{C} : |\mathrm{Re}\, z| < \pi/2\},\\ U_2 &:= \{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < \pi\},\\ V &:= \mathbb{C} \setminus \{(-\infty,-1] \cup [1,\infty)\}.\end{aligned}$$

Then sine (respectively, cos) maps U_1 (respectively, U_2) bijectively onto V.

4.3 Arguments of a Nonzero Complex Number

Definition 4.3.1. Let $z \in \mathbb{C}$ be nonzero. We define $\mathbf{A}\,(z)$ to be the set $\{t \in \mathbb{R} : z = |z|\exp(it)\}$. Any $t \in \mathbf{A}\,(z)$ is called an argument of z.

Proposition 4.3.2. *Let $z \in \mathbb{C}$ be nonzero. Then there exists a unique $\theta \in [0, 2\pi)$ such that $z = |z| \exp(i\theta)$. Also, $\mathbf{A}(z) = \{\theta + n2\pi : n \in \mathbb{Z}\}$.*

Proof. Assume that z lies in the closed upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. Let $z/|z| = u + iv$ so that $u^2 + v^2 = 1$. Note that $-1 \leq u \leq 1$ and $0 \leq v \leq 1$. Since $\cos\colon [0, \pi] \to [-1, 1]$ is a bijection, there exists a unique $t \in [0, \pi]$ such that $\cos t = u$. Then $\sin^2 t = v^2$. But, for $t \in [0, \pi]$, $\sin t \geq 0$. Hence $\sin t = v$. Thus, $z = |z| \exp(it)$.

If $\operatorname{Im} z < 0$, then $-z$ lies in the upper half-plane so that $-z = |z| \exp(it)$ for some $t \in (0, \pi)$. It follows that $z = |z| \exp(i(t + \pi))$ with $t + \pi \in (\pi, 2\pi)$.

If $z = |z| \exp(is) = |z| \exp(it)$ for $s, t \in [0, 2\pi)$, then $\exp(i(s - t)) = 1$ so that $s - t$ is an integral multiple of 2π, say, $s - t = k \cdot 2\pi$ for some $k \in \mathbb{Z}$. Since $s, t \in [0, 2\pi)$, $|s - t| < 2\pi$. We therefore have $|k|\, 2\pi = |s - t| < 2\pi$ and hence conclude that $k = 0$ and $s = t$.

Assume that $z = |z|\, e^{i\theta}$ for $\theta \in [0, 2\pi)$. Let $z = |z|\, e^{is}$ for some $s \in \mathbb{R}$. Hence $e^{i\theta} = e^{is}$ or $e^{i(s-\theta)} = 1$. By (iii) of Theorem 4.2.16, $s - \theta = n2\pi$. □

Ex. 4.3.3. Show that $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is onto.

Example 4.3.4. Let $z = x + iy \in \mathbb{C}$. Then $\exp(z) = \exp(x)\exp(iy)$. Since $|\exp(z)| = \exp(x)$, we see that $\mathbf{A}(\exp(z)) = \{y + n2\pi : n \in \mathbb{Z}\}$.

Ex. 4.3.5. Let z and w be nonzero complex numbers. Let $\theta \in \mathbf{A}(z)$ and $\varphi \in \mathbf{A}(w)$. Then
(1) $-\theta \in \mathbf{A}(z^{-1})$.
(2) $\theta + \varphi \in \mathbf{A}(zw)$.
(3) $\theta - \varphi \in \mathbf{A}(z/w)$.

Example 4.3.6. Let $z = x + iy$ be in the first quadrant. Let $\theta \in \mathbf{A}(z)$. Then $x + iy = |z|(\cos\theta + i\sin\theta)$ so that $\cos\theta = x/|z|$ and $\sin\theta = y/|z|$. Thus our definition of the sine and cosine functions coincides with that of the high school.

Notation. Hereafter, we often write e^{it} for $\exp(it)$.

We now prove a result which says that we cannot assign argument on $\mathbb{C}^*$ in a continuous manner. Note that if we can assign argument to points on $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ in a continuous manner, we can assign continuous argument on $\mathbb{C}^*$. For, $z = |z|\,\theta(z/|z|)$, where $\theta\colon S^1 \to \mathbb{R}$ is a continuous function such that $z = \exp(i\theta(z))$ for $z \in S^1$.

Lemma 4.3.7. *It is impossible to make a continuous choice $\theta(z) \in \mathbf{A}(z)$ on $\mathbb{C}^*$. That is, there is no continuous map $\theta\colon \mathbb{C}^* \to \mathbb{R}$ such that $z = |z| \exp(i\theta(z))$ for $z \in \mathbb{C}^*$.*

Proof. Assuming such a θ exists, consider $f\colon [0, 2\pi] \to \mathbb{R}$ by setting

$$f(t) := [\theta(e^{it}) + \theta(e^{-it})]/2\pi.$$

Then f is a real valued continuous function on $[0, 2\pi]$. Since $2\pi f(t)$ is a choice of $\mathbf{A}(e^{it}e^{-it})$ and hence of $\mathbf{A}(1)$, it is integer valued continuous function on the interval

$[0, 2\pi]$. By intermediate value theorem, it is a constant. In particular, $f(0) = f(\pi)$. This implies that $[\theta(1)+\theta(1)]/2\pi = [\theta(-1)+\theta(-1)]/2\pi$. This implies that $\theta(1) = \theta(-1)$, which is impossible. For, $\theta(1) \in \mathbf{A}(1)$, which is the set of even integral multiples of π while $\theta(-1) \in \mathbf{A}(-1)$ which is the set of odd integral multiples of π.

Or, one may argue as follows. As earlier, assume that θ exists: $z = |z|\exp(i\theta(z))$ for $z \in \mathbb{C}^*$. Then we claim that θ is one-one on $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. For, if $\theta(z_1) = \theta(z_2)$, then $z_1 = \exp(i\theta(z_1)) = \exp(i\theta(z_2)) = z_2$. Therefore, $\theta(z) - \theta(-z) \neq 0$ for $z \in S^1$. Now consider the continuous function $f\colon S^1 \to \{\pm 1\}$ given by

$$f(z) = \frac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}.$$

Since $f(-z) = -f(z)$, we see that f is onto. This is a contradiction, since S^1 is connected but its image under f, $\{\pm 1\}$ is no connected. If you are not familiar with the notion of connected spaces, we may proceed as follows. Let $\varphi\colon [0, 2\pi] \to S^1$ be the continuous map $\varphi(t) = e^{it}$. Then the composition $f \circ \varphi\colon [0, 2\pi] \to \mathbb{R}$ is a continuous function which takes only the values ± 1. This contradicts the intermediate value theorem. $\square$

If we are ready to ignore a closed half-line L starting from 0, we can get a continuous choice of $\mathbf{A}(z)$ for $z \notin L$.

Let $\alpha \in \mathbb{R}$ and let $L_\alpha := \{te^{i\alpha} : t \geq 0\}$. Note that a nonzero $z \in \mathbb{C} \setminus L_\alpha$ iff $\alpha \notin \mathbf{A}(z)$. For $z \notin L_\alpha$, there is a unique choice of $\mathbf{A}(z)$ in the interval $(\alpha, \alpha + 2\pi)$. We denote this choice by $\arg_\alpha\colon \mathbb{C} \setminus L_\alpha \to (\alpha, \alpha + 2\pi)$.

Theorem 4.3.8. $\arg_\alpha$ *is continuous on* $\mathbb{C} \setminus L_\alpha$.

Proof. Let $\theta := \arg_0$. Then $\mathbb{C} \setminus L_0 = H_0 \cup H_1 \cup H_2$ where $H_0 := \{x + iy : y > 0\}$, $H_1 := \{x + iy : x < 0\}$ and $H_2 := \{x + iy : y < 0\}$. We show that θ is continuous on each of the open half-planes H_i, $0 \leq i \leq 2$.

If $z \in H_0$, then $\operatorname{Im}(z) = |z|\sin(\theta(z))$. So, $\sin(\theta(z)) > 0$. Since θ takes values only in $(0, 2\pi)$, this means that $\theta(z) \in (0, \pi)$ for such z. Now, $\cos\colon (0, \pi) \to (-1, 1)$ is strictly decreasing. Hence it has a continuous inverse $\cos^{-1}\colon (-1, 1) \to (0, \pi)$. (See Ex. 4.3.9 below.) Hence

$$\theta(z) = \cos^{-1}(\cos(\theta(z))) = \cos^{-1}(\operatorname{Re}(z)/|z|), \qquad z \in H_0.$$

Since the RHS is continuous, it follows that the restriction of θ to H_0 is continuous.

Similar reasoning leads us to the following expressions for θ restricted to H_1 and H_2:

$$\theta(z) = \pi - \sin^{-1}\left(\frac{\operatorname{Im} z}{|z|}\right), \qquad z \in H_1$$

$$\theta(z) = 2\pi - \cos^{-1}\left(\frac{\operatorname{Re} z}{|z|}\right) \qquad z \in H_2.$$

The continuity of the restrictions of θ to H_1 and H_2 follows from these expressions. Since the union of H_0, H_1, H_2 is $\mathbb{C} \setminus L_0$, an appeal to the gluing lemma (Ex. 3.1.42) establishes the continuity of θ on $\mathbb{C} \setminus L_0$.

General case: Consider the map $f(z) := z\exp(-i\alpha)$ which maps $\mathbb{C} \setminus L_\alpha$ onto $\mathbb{C} \setminus L_0$. Then one shows that $\arg_\alpha(z) = \alpha + \theta(f(z))$. □

An explicit formula for a continuous argument for $\arg_0$ can be found in [19].

Ex. 4.3.9. Let $f\colon I \to \mathbb{R}$ be strictly increasing on the interval I. Then the inverse function $f^{-1}\colon f(I) \to I$ is continuous. (For those who know topology: In particular, any continuous strictly monotone function on an interval I maps I homeomorphically onto $f(I)$.) *Hint:* Let $y_0 = f(x_0)$ and $\varepsilon > 0$ be given. Then choose $\delta > 0$ so that

$$f(x_0 - \varepsilon) < y_0 - \delta < y_0 = f(x_0) < y_0 + \delta < f(x_0 + \varepsilon).$$

Then f^{-1} maps $f(I) \cap (y_0 - \delta, y_0 + \delta)$ into $(x_0 - \varepsilon, x_0 + \varepsilon)$. Draw a picture.

Ex. 4.3.10. For each nonzero $z \in \mathbb{C}$, there is a continuous choice of $\mathbf{A}$ on $B(z, |z|)$. *Hint:* Observe that $B(z, |z|) \subset \mathbb{C} \setminus L_\alpha$ where $\alpha = t + \pi$ for some $t \in \arg(z)$. See Figure 4.1.

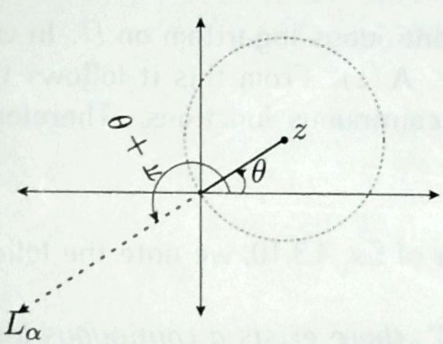

Figure 4.1: Continuous choice of argument

4.4 Logarithms and Power Functions

If $x \in \mathbb{R}$ is positive, we have already seen that there exists a unique $y \in \mathbb{R}$ such that $x = \exp(y)$. We let $y := \ln(x)$. Thus, we have a function $\ln\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $\exp(\ln(x)) = x$ for all $x \in \mathbb{R}_+$. Since exp is a (continuous) increasing function, ln is increasing and continuous (by Ex. 4.3.9).

Ex. 4.4.1. Let x and y be positive reals. Then $\ln(xy) = \ln(x) + \ln(y)$ and $\ln(1/x) = -\ln(x)$.

Ex. 4.4.2. Show that $\ln(x) < x$ for $x > 0$.

Definition 4.4.3. Given $z \in \mathbb{C}^*$, any complex number w such that $\exp(w) = z$ is called a *logarithm* of z. We let $\operatorname{Log}(z)$ stand for the set of all logarithms of a nonzero $z \in \mathbb{C}$. This set is nonempty by Ex. 4.3.3. See also the example below.

Example 4.4.4. If $z \in \mathbb{C}^*$, then $w := \ln(|z|) + it$ where $t \in \mathbf{A}(z)$ is a logarithm of z. Furthermore, any logarithm of z is of this form.

Definition 4.4.5. Given an open set $U \subset \mathbb{C}$, we say that there exists a continuous argument on U if there exists a continuous function $\theta \colon U \to \mathbb{C}$ such that $z = |z|\, e^{i\theta(z)}$ for all $z \in U$.

Similarly, we say that there exists a continuous logarithm on U if there exists a continuous function $F \colon U \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all $z \in U$.

Lemma 4.3.7 says that there is no continuous argument on $\mathbb{C}^*$.

Lemma 4.4.6. *Let $U \subset \mathbb{C}^*$ be open. Then there is a continuous argument on U iff there is a continuous logarithm on U. In particular, there is no continuous logarithm on $\mathbb{C}^*$.*

Proof. Let θ be a continuous argument on U. Then $z \mapsto \ln|z| + i\theta(z)$ is a continuous logarithm on U.

Conversely, let F be a continuous logarithm on U. In view of Example 4.4.4, $F(z) = \ln|z| + i\theta(z)$ for some $\theta(z) \in \mathbf{A}(z)$. From this it follows that $z \mapsto i\theta(z)$ is continuous, being the difference of two continuous functions. Therefore, there exists a continuous argument on U. $\square$

As a special case, in view of Ex. 4.3.10, we note the following:

Corollary 4.4.7. *Given $z \in \mathbb{C}^*$, there exists a continuous logarithm on $B(z, |z|)$.*

Ex. 4.4.8. Let $w_i \in \operatorname{Log}(z_i)$ for $i = 1, 2$. Then $w_1 + w_2 \in \operatorname{Log}(z_1 z_2)$ and $-w_1 \in \operatorname{Log}(1/z_1)$.

Ex. 4.4.9. Find the values of $\operatorname{Log}(-1)$, $\operatorname{Log}(i)$ and $\operatorname{Log}(\exp(z))$.

Remark 4.4.10. Note that for $x > 0$, $\ln(x) \in \operatorname{Log}(x)$. Thus, even here, $\operatorname{Log}(x)$ is the infinite set $\ln(x) + \{n2\pi i : n \in \mathbb{Z}\}$.

Recall the definitions $L_\alpha := \{t\exp(i\alpha) : t \geq 0\}$ for $\alpha \in \mathbb{R}$ and the argument $\arg_\alpha \colon \mathbb{C} \setminus L_\alpha \to (\alpha, \alpha + 2\pi)$. We define a logarithm $\log_\alpha \colon \mathbb{C} \setminus L_\alpha \to \mathbb{C}$ by setting

$$\log_\alpha(z) := \ln(|z|) + i\arg_\alpha(z), \qquad z \in \mathbb{C} \setminus L_\alpha. \tag{4.2}$$

Then $\log_\alpha$ is continuous on its domain. Since exp maps $\{z = x + iy : \alpha < y < \alpha + 2\pi\}$ onto $\mathbb{C} \setminus L_\alpha$, $\log_\alpha$ is its continuous inverse.

Power Functions

Definition 4.4.11. Given $x > 0$ and $\lambda \in \mathbb{C}$, we define $x^\lambda := \exp(\lambda \log(x))$.

Ex. 4.4.12. Show that the following laws of exponents are valid: (i) $a^z a^w = a^{z+w}$ for $a > 0$ and $z, w \in \mathbb{C}$. (ii) $(ab)^z = a^z b^z$ for $a > 0$, $b > 0$ and $z \in \mathbb{C}$.

In analogy with the above definition, we make the following

Definition 4.4.13. For $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^*$, we define z^λ by setting

$$z^\lambda := \exp(\lambda \mathrm{Log}\,(z)) := \{\exp(\lambda w) : w \in \mathrm{Log}\,(z)\},$$

a subset of $\mathbb{C}$.

Example 4.4.14. Let us find i^i. Since $\mathbf{A}\,(i) = \{(\pi/2) + 2n\pi\}$, we have $\mathrm{Log}\,(i) := \ln(1) + i\mathbf{A}\,(i) = i\mathbf{A}\,(i)$. Hence

$$i^i = \{\exp(i[i(2n\pi + (\pi/2))]) : n \in \mathbb{Z}\} = \{\exp(-[2n\pi + (\pi/2)]) : n \in \mathbb{Z}\},$$

a subset of $\mathbb{R}$!

To get a single-valued function, we proceed as follows. For $\lambda \in \mathbb{C}$ and $z \neq 0$, we define $p_\alpha^\lambda(z) := \exp(\lambda \log_\alpha(z))$, for $z \in \mathbb{C} \setminus L_\alpha$. It is immediate from the definition that p_α^λ is continuous on its domain.

Ex. 4.4.15. What is the relation between the two definitions (do you understand which two definitions are we talking about?) of $z \mapsto z^n$ if $n \in \mathbb{Z}$? What about z^r for $r \in \mathbb{Q}$?

Ex. 4.4.16. Show that $(-1)^i = \{\exp([2n+1]\pi) : n \in \mathbb{Z}\}$.

Ex. 4.4.17. Find all solutions of $z^{1+i} = 4$.

Ex. 4.4.18. Let $e := \exp(1)$. Show that $\exp(z) \in e^z$. (Here e^z is **not** a shorthand notation for $\exp(z)$! It stands for the z-th power of e.)

Remark 4.4.19. Why cannot we work with the set valued maps z^λ? Consider $f(z) := z^{1/2}$. Then $f + f \neq 2f$. For, at $z = 1$, the LHS is $\{-2, 0, 2\}$ while the RHS is $\{\pm 2\}$.

Example 4.4.20. For $\lambda = \frac{1}{2}$, we have

$$(p_\alpha^{\frac{1}{2}}(z))^2 = \exp\left(\frac{1}{2}\log_\alpha(z) + \frac{1}{2}\log_\alpha(z)\right) = z, \qquad z \neq 0.$$

Thus each $p_\alpha^{\frac{1}{2}}$ is a square root function. Note that

$$p_{\alpha+2\pi}^{\frac{1}{2}}(z) = \exp(\frac{1}{2}\log_{\alpha+2\pi}(z)) = \exp(\frac{1}{2}\log_\alpha(z) + \pi i) = -p_\alpha^{\frac{1}{2}}(z).$$

Thus each of $\pm p_\alpha^{\frac{1}{2}}$ is a square root.

Chapter 5

Differentiation

5.1 Limits

We shall be brief in this section.

Definition 5.1.1. Let $A \subset X$ be a subset of a metric space. Let $x_0 \in X$ be a cluster point of A, not necessarily in A. Let $f\colon A \to \mathbb{C}$ be a function. We say that $\lim_{x \to x_0} f(x)$ exists if there exists $\ell \in \mathbb{C}$ such that for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x \in A \text{ and } 0 < d(x, x_0) < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

It is easy to show that ℓ with this property, if exists, is unique. (This is where we need the fact that x_0 is a cluster point of A.) In such a case, we write $\lim_{x \to x_0} f(x) = \ell$. Note that f need not be defined at x_0.

The study of this concept may be reduced to that of sequences and their limits because of the following exercise.

Ex. 5.1.2. With the above notation, $\lim_{x \to x_0} f(x) = \ell$ iff for every sequence (x_n) in A (with $x_n \neq x_0$ for $n \in \mathbb{N}$) such that $\lim x_n = x_0$ we have $\lim f(x_n) = \ell$. *Hint:* Go through the proof of the equivalence of the definitions of continuity in terms of sequences and $\varepsilon - \delta$. (Refer to Theorem 3.1.12.)

Proposition 5.1.3. *Let* $\lim_{x \to x_0} f(x) = \alpha$ *and* $\lim_{x \to x_0} g(x) = \beta$. *Then*
(i) $\lim_{x \to x_0}(af + bg)(x) = a\alpha + b\beta$ *for* $a, b \in \mathbb{C}$.
(ii) $\lim_{x \to x_0}(fg)(x) = \alpha\beta$.
(iii) $\lim_{x \to x_0}(f/g)(x) = \alpha/\beta$ *if* $\beta \neq 0$.

Proof. Use the above exercise Ex. 5.1.2 and the algebra of limits theorem. □

Ex. 5.1.4. Show that $\lim_{z \to i} \frac{z^4 - 1}{z - i} = -4i$.

Ex. 5.1.5. Let $f: U \subset \mathbb{C} \to \mathbb{C}$ be given. Show that f is continuous at $z_0 \in U$ iff $\lim_{z\to z_0} f(z) = f(z_0)$.

Definition 5.1.6. Let $f: \mathbb{C} \to \mathbb{C}$ be given. We use the notation $\lim_{z\to\infty} f(z) = \ell$ to mean that if given $\varepsilon > 0$ there exists an $R > 0$ such that for all z with $|z| > R$, we have $|f(z) - \ell| < \varepsilon$.

We use the notation $\lim_{z\to z_0} |f(z)| = \infty$ to say that if for a given $M > 0$, there exists $r > 0$ such that $|f(z)| > M$ for all $z \in B(z_0, r) \setminus \{z_0\}$.

Note that we have not yet introduced the point at infinity.

Ex. 5.1.7. Prove that $\lim_{z\to\infty}(1/z^n) = 0$ for any $n \in \mathbb{N}$.

Ex. 5.1.8. Prove or disprove that $\lim_{z\to\infty} |e^{-z}| = 0$.

Ex. 5.1.9. Let $p(z) := \sum_{k=0}^{n} a_k z^k$ be a nonconstant polynomial. Show that there exists $R > 1$ such that $|p(z)| > \frac{1}{2}|a_n||z|^n$ for $|z| \geq R$. In particular, $|p(z)| \to \infty$ as $|z| \to \infty$.
Hint: Observe that

$$\left|\frac{p(z)}{z^n}\right| \geq |a_n| - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|}, \qquad \text{for } |z| \geq 1. \tag{5.1}$$

5.2 Functions from $\mathbb{R}$ to $\mathbb{C}$

We deal with differentiation of functions from $\mathbb{R}$ to $\mathbb{C}$.

Definition 5.2.1. Let $f: [a,b] \to \mathbb{C}$ be given. We say that f is *differentiable* function at $t \in [a,b]$ if there exists a complex number α such that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 \neq h \in \mathbb{R},\ |h| < \delta \ \&\ t + h \in [a,b] \Rightarrow \left|\frac{f(t+h) - f(t)}{h} - \alpha\right| < \varepsilon.$$

That is, f is differentiable at t iff $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists. If the limit exists and if we denote it by α, then α is unique and called the derivative of f at t. It is denoted by $f'(t)$.

Ex. 5.2.2. With the above notation, write $f := u + iv$ where $u(t) := \operatorname{Re}(f(t))$ and $v(t) := \operatorname{Im}(f(t))$. Then f is differentiable at $t \in [a,b]$ iff the real valued functions $u, v: [a,b] \to \mathbb{R}$ are differentiable at t. Furthermore, $f'(t) = u'(t) + iv'(t)$. Also, if f is differentiable at t, then it is continuous at t.

Remark 5.2.3. If f is as above, we may think of f as a differentiable curve in the complex plane: $f(t) = u(t) + iv(t)$ which corresponds to $(u(t), v(t)) \in \mathbb{R}^2$. And $f'(t) \equiv (u'(t), v'(t))$ is thought of as the tangent vector to the curve at t. In physical terms, we may think of $f(t)$ as the position of a particle at time t and $f'(t)$ as the "velocity".

Example 5.2.4. Let $z, w \in \mathbb{C}$. Let $\gamma\colon [0,1] \to \mathbb{C}$ be given by $\gamma(t) := z + t(w-z)$. Then γ is the line segment $[z,w]$ joining z and w and $\gamma'(t) = w - z$, since $\frac{\gamma(t+h)-\gamma(t)}{h} - (w-z) = 0$, for $h \neq 0$.

Example 5.2.5. Let $a \in \mathbb{C}$, $R > 0$. Consider the map $\sigma\colon [0, 2\pi] \to \mathbb{C}$ given by $\sigma(t) := a + R(\cos t + i \sin t)$. Then σ is the circle with centre at a and radius R. Also, $\sigma'(t) = R(-\sin t + i \cos t)$.

Remark 5.2.6. The analogue of mean value theorem is false for complex valued functions of a real variable. For, consider $f\colon [0, 2\pi] \to \mathbb{C}$ given by $f(t) := \cos t + i \sin t$. Then $f(2\pi) - f(0) = 0$ while $|f'(t)| = 1$ for all $t \in [0, 2\pi]$.

Ex. 5.2.7. Let $h\colon [a,b] \to \mathbb{C}$ be differentiable with $h'(t) = 0$. Then h is a constant. *Hint:* Invoke the result from real analysis to the real and imaginary parts of h.

5.3 Differentiable Functions on $\mathbb{C}$

Definition 5.3.1. Let $U \subset \mathbb{C}$ be open. A function $f\colon U \to \mathbb{C}$ is said to be *differentiable* at $z \in U$ if there exists an $\alpha \in \mathbb{C}$ such that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z+h) - f(z)}{h} - \alpha \right| < \varepsilon, \quad \text{for } 0 < |h| < \delta.$$

The number α is unique (Exercise!) and called the *derivative* of f at z. It is denoted by $f'(z)$.

Note that f is differentiable at $z \in U$ iff the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

If f is differentiable at each $z \in U$, then f is said to be *holomorphic* on U. Let $H(U)$ denote the set of functions holomorphic on U.

If $f \in H(\mathbb{C})$, then f is called an *entire* function.

If $f\colon U \to \mathbb{C}$ is differentiable and if $g := f'\colon U \to \mathbb{C}$ is differentiable at $z \in U$, we denote $f''(z) = g'(z)$. f'' is called the second derivative of f at z. More generally, we define inductively the n-th derivative of f at z by setting $f^{(0)}(z) := f(z)$ and $f^{(n)}(z) := (f^{(n-1)})'(z)$, the derivative of $f^{(n-1)}$ at z. In general, the first, second and third derivatives are denoted by f', f'' and f''' respectively. If $f^{(n)}(z)$ exists for all $n \in \mathbb{N}$ and for all $z \in U$, we say that f is infinitely differentiable on U.

Proposition 5.3.2. *Let $f\colon U \to \mathbb{C}$ be given and $z \in U$. Then f is differentiable at z iff there exists a function $f_1\colon U \to \mathbb{C}$, continuous at z and such that $f(w) = f(z) + (w-z) f_1(w)$. Furthermore, $f'(z) = f_1(z)$.*

Proof. Let f be differentiable at z. Define $f_1(w) := \frac{f(w)-f(z)}{w-z}$ for $w \neq z$ and $f_1(z) = f'(z)$. To check the continuity of f_1 at z, let $\varepsilon > 0$ be given. Since f is differentiable at z, for this ε there exists a δ such that if $0 < |w-z| < \delta$, then $\left|\frac{f(w)-f(z)}{w-z} - f'(z)\right| < \varepsilon$, that is, $|f_1(w) - f_1(z)| < \varepsilon$ for $|w-z| < \delta$. Hence f_1 is continuous at z.

Conversely, if f_1 exists as specified, observe that $f_1(w)$ must equal $\frac{f(w)-f(z)}{w-z}$ for $w \neq z$. We claim that f is differentiable at z and $f'(z) = f_1(z)$: By continuity of f_1 at z, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |w-z| < \delta$, then $\left|\frac{f(w)-f(z)}{w-z} - f_1(z)\right| < \varepsilon$. This precisely means that f is differentiable at z and $f'(z) = f_1(z)$. □

Corollary 5.3.3. *If f is differentiable at z, then f is continuous at z.*

Proof. We use the notation of the last proposition. We have

$$f(w) = f(z) + (w-z)f_1(w) \text{ for } w \in U.$$

Observe that the right side is a finite sum of a constant function $f(a)$, the product of f_1 (which is continuous at z) and a continuous function $9w - z$). Hence it is continuous and so is the right side. □

Example 5.3.4. Let $f\colon U \to \mathbb{C}$ be a constant. Then f is holomorphic on U and $f'(z) = 0$ for all $z \in U$.

Example 5.3.5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^n$, for $n \in \mathbb{N}$. We show that f is differentiable at all z and $f'(z) = nz^{n-1}$.

$$(z+h)^n - z^n = h\left(nz^{n-1} + \sum_{k\geq 2}\binom{n}{k}h^{k-1}z^{n-k}\right).$$

The RHS is of the form $[(z+h)-z)]f_1$, where $f_1(z+h) = \left(nz^{n-1} + \sum_{k\geq 2}\binom{n}{k}h^{k-1}z^{n-k}\right)$ which is continuous at z. Also, $f_1(z) = nz^{n-1}$.

Example 5.3.6. Let $f(z) = \overline{z}$, for $z \in \mathbb{C}$. We claim that f is not differentiable at any $z \in \mathbb{C}$. This follows from the following observations:

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} &= \frac{\overline{z}+\overline{h}-\overline{z}}{h} \\
&= \frac{\overline{h}}{h} = 1, \quad \text{if } h \in \mathbb{R} \\
&= \frac{-it}{it} = -1 \quad \text{if } h = it,\ t \in \mathbb{R}.
\end{aligned}$$

We conclude that $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ does not exist. Hence f is not differentiable at z.

Ex. 5.3.7. Show that $f(z) := |z|$ is not differentiable at 0.

Ex. 5.3.8. Show that $f(z) := |z|^2$ is differentiable at $z = 0$ and nowhere else. Thus, f is differentiable at 0 but not holomorphic in any open disk containing of 0.

Ex. 5.3.9. Show that $f(z) := x^2 - iy^3$ is differentiable only at $z = 0$.

Theorem 5.3.10. *Let $f, g\colon U \to \mathbb{C}$ be differentiable at $z \in U$. Let $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g$, fg and f/g if $g(z) \neq 0$ are differentiable at z and we have*

$$(\alpha f + \beta g)'(z) = \alpha f'(z) + \beta g'(z)$$
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$
$$(f/g)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof. The strategy for the proof is simple. We make use of Proposition 5.3.2 and make an educated guess for the "f_1" in each of the cases.

We employ an obvious notation. Observe the following:

$$(\alpha f + \beta g)(w) = (\alpha f + \beta g)(z) + (w - z)[\alpha f_1(w) + \beta g_1(w)] \tag{5.2}$$
$$(fg)(w) = f(z)g(z) + (w - z)[f(z)g_1(z) + g(z)f_1(z) + (w - z)f_1(w)g_1(w)] \tag{5.3}$$
$$f(w)/g(w) = f(z)/g(z) + (w - z)\left[\frac{g(z)f_1(w) - f(z)g_1(w)}{g(w)g(z)}\right]. \tag{5.4}$$

In the last we have used the fact that $g(w) \neq 0$ for $w \in B(z, r)$ for some $r > 0$ provided that $g(z) \neq 0$. The stated results follow from the above.

Now (5.2) shows that $\alpha f + \beta g$ is differentiable at z and the derivative is given by $\alpha f_1(z) + \beta g_1(z)$, that is, $\alpha f'(z) + \beta g'(z)$.

Equation (5.3) shows that fg is differentiable at z and the derivative is given by

$$[f(z)g_1(z) + g(z)f_1(z) + (z - z)f_1(z)g_1(z)],$$

that is, $f(z)g'(z) + g(z)f'(z)$.

Similarly, (5.4) says that the derivative of f/g is given by $\left[\frac{g(z)f_1(w) - f(z)g_1(w)}{g(w)g(z)}\right]$ evaluated at $w = z$. Hence the derivative of f/g is $\frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$. □

Ex. 5.3.11. Any polynomial function $f(z) := \sum_{k=0}^{n} a_k z^k$ is holomorphic on $\mathbb{C}$.

Ex. 5.3.12. Any rational function $r(z) := p(z)/q(z)$, where p, q are polynomials, is holomorphic on the complement of the set of zeros of q, which is an open set in $\mathbb{C}$.

Theorem 5.3.13 (Chain Rule). *(i) Let $\gamma\colon [a, b] \to U \subset \mathbb{C}$ and $f\colon U \to \mathbb{C}$ be differentiable. Then the map $g := f \circ \gamma\colon [a, b] \to \mathbb{C}$ is differentiable and $g'(t) = f'(\gamma(t)) \cdot \gamma'(t)$.*

(ii) Let U and V be open sets in $\mathbb{C}$. Let $f\colon U \to V$ be differentiable at $a \in U$ and $g\colon V \to \mathbb{C}$ be differentiable at $b := f(a) \in V$. Then $g \circ f\colon U \to \mathbb{C}$ is differentiable at a and $(g \circ f)'(a) = g'(b)f'(a)$.

Proof. The strategy is the same as that adopted in Theorem 5.3.10.

Observe that

$$(f \circ \gamma)(t+h) - (f \circ \gamma)(t) = [\gamma(t+h) - \gamma(t)]f_1(\gamma(t+h)).$$

As $h \to 0$, we see that $\frac{g(t+h)-g(t)}{h} = \frac{\gamma(t+h)-\gamma(t)}{h} f_1(\gamma(t+h))$ goes to $\gamma'(t)f'(\gamma(t))$.

(ii) Using standard notation of ours, we have

$$\begin{aligned} g(t) &= g(b) + (t-b)g_1(t) \\ f(w) &= f(a) + (w-a)f_1(w). \end{aligned}$$

In this set of equations, we take $t = f(w)$, $b = f(a)$ and get

$$\begin{aligned} (g \circ f)(w) = g(f(w)) &= g(b) + (f(w) - f(a))g_1(f(w)) \\ &= g \circ f(a) + (w-a)f_1(w)g_1(f(w)). \end{aligned}$$

This suggests us to consider $h_1(w) := f_1(w)g_1(f(w))$. Clearly, h_1 is continuous at a and we see that $h := g \circ f$ is differentiable at a. □

Ex. 5.3.14. Let $U := B(0,1) \cup B(2,1)$. Let

$$f(z) = \begin{cases} 0 & \text{if } z \in B(0,1) \\ 2 & \text{if } z \in B(2,1) \end{cases}.$$

Then f is differentiable on the open set U and $f'(z) = 0$. However f is not a constant on U.

Definition 5.3.15. A function $f\colon (X,d) \to (Y,d)$ is said to be *locally constant* if for every $x \in X$ there exists an $r > 0$ such that $f(z) = f(x)$ for all $z \in B(x,r)$.

Ex. 5.3.16. Any locally constant function from a metric space to another is continuous.

Ex. 5.3.17. Show that any function from $\mathbb{C}$ to a discrete metric space is locally constant. Hence conclude that it is continuous.

Lemma 5.3.18. *Let $f\colon U \to \mathbb{C}$ be differentiable. Assume that $f'(z) = 0$ for all $z \in U$. Then f is locally constant.*

Proof. Given z choose $r > 0$ such that $B(z,r) \subset U$. Fix any $w \in B(z,r)$. Consider $\gamma(t) := z + t(w-z)$ for $t \in [0,1]$. Check that $\gamma(t) \in B(z,r)$ for $t \in [0,1]$. Then $g := f \circ \gamma\colon [0,1] \to \mathbb{C}$ is differentiable and $g'(t) = f'(\gamma(t))\gamma'(t) = 0(w-z) = 0$. Hence g is a constant (by Ex. 5.2.7) and $f(w) = g(1) = g(0) = f(z)$. □

The concept which will allow us to conclude the constancy of f from the fact that $f' = 0$ is that of connectedness.

5.4 Connectedness

Definition 5.4.1. We say that a metric space X is *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X. If X is not connected, then there exist nonempty open sets U and V in X such that $X = U \cup V$ and $U \cap V = \emptyset$. We say that the pair (U, V) is a *disconnection* of X.

Let $A \subset X$ be a subset of a metric space X. We say that A is connected iff the metric space $(A, d|_A)$ is connected. Note that this is equivalent to saying that there exists no pair of disjoint nonempty open subsets U and V of X such that (1) $A \cap U \neq \emptyset \neq A \cap V$, (2) A is not a subset of either U or V and (3) $A \subset U \cup V$.

The proof of the following result is a typical use of connectedness argument.

Theorem 5.4.2. *Let X be a metric space. Then X is connected iff any locally constant function from X to any metric space Y is a constant.*

Proof. Assume that X is connected. Let f be a locally constant map from X to Y. Fix an $x \in X$. Consider the set $U := \{z \in X : f(z) = f(x)\}$. Then $x \in U$. Since any locally constant function is continuous and since any singleton set in a metric space is closed, we see that U is closed: $U := f^{-1}\{f(x)\}$. It is open, since if $z \in U$, then by hypothesis, there is an $r > 0$ such that $f(w) = f(z)$ for all $w \in B(z, r)$. But then $f(w) = f(z) = f(x)$ for all $w \in B(z, r)$. Hence $B(z, r) \subset U$. Thus U is a nonempty set which is closed as well as open. We conclude that U must be all of X.

The converse is proved by contradiction. If X is not connected, then there exists a subset A which is both open and closed and such that $\emptyset \neq A \neq X$. Then $B := X \setminus A$ enjoys the same properties. We now take f to be the characteristic function χ_A of the set A, defined as $f : X \to \mathbb{R}$ such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ if $x \in B$. It is clear that f is a locally constant function which is not a constant. □

Definition 5.4.3. A subset $U \subset \mathbb{C}$ is called a *region* if U is open and connected.

Ex. 5.4.4. Let U be a region in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be differentiable with $f'(z) = 0$ for $z \in U$. Then f is a constant.

Ex. 5.4.5. Let U be a region. Assume that f' is a constant on U. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$ and $z \in U$.

Ex. 5.4.6. With the above notation, if $f^{(n)} = 0$ on U, then f is a polynomial in z of degree at most $n - 1$.

The following is a most useful analytical characterization of connected spaces.

Proposition 5.4.7. *A metric space X is connected iff any continuous function from X to $\{0,1\} \subset \mathbb{R}$ is a constant.*

Proof. Assume that X is connected. Let, if possible, $f: X \to \{0,1\}$ be continuous and onto. Then $U := f^{-1}((-1/2,1/2))$ and $V := f^{-1}((1/2,3/2))$ are nonempty open subsets of X such that $U \cup V = X$ and $U \cap V = \emptyset$. Thus (U,V) is a disconnection of X. This contradiction shows that there could be no such f.

Conversely, if X is not connected, there exist U and V which provide a disconnection. Define

$$f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}.$$

Then f is continuous on X and onto $\{0,1\}$. □

Ex. 5.4.8. Show that a metric space X is connected iff any continuous function $f: X \to \mathbb{Z}$, ($\mathbb{Z}$ being considered as a subset of $\mathbb{R}$ with the induced metric) is a constant.

Ex. 5.4.9. Let $X := [a,b] \subset \mathbb{R}$. Then X is connected. *Hint:* If $f: [a,b] \to \{0,1\}$ is continuous and onto, what does the intermediate value theorem say?

Proposition 5.4.10. *The following are true:*
(i) *Let $f: X \to Y$ be continuous and X connected. Then $f(X)$ is connected.*
(ii) *Let E_i be connected subsets of a metric space X, for $i \in I$, an index set. Assume that $\cap_i E_i \neq \emptyset$. Then $\cup_i E_i$ is connected.*
(iii) *Let $A \subset X$ be connected. Assume that $A \subseteq B \subseteq \overline{A}$. Then B is connected. (Here $\overline{A}$ is the union of A and the set of limit points of A. It is called the closure of A.)*
(iv) *Let X and Y be connected. Then $X \times Y$ (with product metric) is connected.*

Proof. All are easy applications of Proposition 5.4.7. For details, refer to Chapter 5 of [17]. □

Ex. 5.4.11. Show that any circle $\{z \in \mathbb{C} : |z-a| = r\}$ is connected.

Ex. 5.4.12. Show that the punctured disk $B'(a,r) := \{z \in \mathbb{C} : 0 < |z-a| < r\}$ is connected.

Ex. 5.4.13. Let $A, B \subset \mathbb{C}$ be such that $d(A,B) := \inf\{d(z,w) : z \in A, w \in B\} > 0$. Then $A \cup B$ is not connected.

We shall introduce some more topological concepts which will be needed in the course.

Definition 5.4.14. We say that a space X is *path connected* if for any two $p, q \in X$, there exist $a, b \in \mathbb{R}$ with $a < b$ and a continuous map $c: [a,b] \to X$ such that $c(a) = p$ and $c(b) = q$. We then say that c is a path from p to q.

Most often, we may assume that the domain of the path is $[0,1]$. There is no loss of generality in this. For given c as above, consider $\gamma: [0,1] \to X$ defined by $\gamma(t) = c(a + (b-a)t)$. Then γ is a path connecting p and q.

Note that if $\gamma\colon [0,1] \to X$ is a path connecting p to q, that is, $p = \gamma(0)$ and $q = \gamma(1)$, then the path $\tilde{\gamma}\colon [0,1] \to X$ defined by setting $\tilde{\gamma}(t) := \gamma(1-t)$ is a path connecting q to p. (See also Definition 6.2.30.)

Example 5.4.15. Let $X := [a,b] \subset \mathbb{R}$. Then X is path connected. For, if $x, y \in [a,b]$, then $c\colon [0,1] \to [a,b]$ given by $c(t) := x + t(y-x)$ is path from x to y.

Example 5.4.16. Any disk $B(a,r) \subset \mathbb{C}$ is path connected. For, if $z, w \in B(a,r)$, consider $c(t) := z + t(w-z)$ for $0 \le t \le 1$. Check that c lies in $B(a,r)$ and is a path from z to w.

Lemma 5.4.17. *Any path connected space is connected.*

Proof. The strategy is to apply Proposition 5.4.7.

Let X be path connected and $f\colon X \to \{0,1\}$ be continuous. Fix $p \in X$. Let $x \in X$ be arbitrary. Since X is path connected, there exists a path $c\colon [0,1] \to X$ connecting $p = c(0)$ and $x = c(1)$. Consider the composite $f \circ c\colon [0,1] \to \{0,1\}$. It is continuous and by the intermediate value theorem (or by the connectedness of $[0,1]$), $f \circ c$ is a constant. In particular, $f(p) = f(c(0)) = f(c(1)) = f(x)$. We have thus shown that $f(x) = f(p)$ for any $x \in X$. □

Lemma 5.4.18. *If x and y are connected by a path and y and z are connected by a path in X, then x and z are connected by a path in X.*

Proof. Let $\alpha\colon [0,1] \to X$ be a path such that $\alpha(0) = x$ and $\alpha(1) = y$. Let $\beta\colon [0,1] \to X$ be a path connecting y and z. The strategy is to move along $[0,1/2]$ at twice the speed via α and then do the same with β on $[1/2,1]$. Consider $\gamma\colon [0,1] \to X$ defined by

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Then γ is continuous: for the restriction of γ to $[0,1/2]$ is continuous, being the composite of two continuous functions: $t \mapsto 2t \mapsto \alpha(2t)$. Similarly, the restriction of γ to $[1/2,1]$ is continuous. Also, $\alpha(1/2) = \beta(1/2)$. Hence, by the gluing lemma (Ex. 3.1.42), γ is continuous on $[0,1]$. Clearly, $\gamma(0) = x$ and $\gamma(1) = z$. □

Ex. 5.4.19. Show that X is path connected iff there exists $p \in X$ such that there is a path connecting p to any $x \in X$.

Definition 5.4.20. A subset E of $\mathbb{C}$ is said to be *convex* if for every $z, w \in E$, the line segment $[z,w] := \{z + t(w-z) : 0 \le t \le 1\}$ is contained in E.

We say that $A \subset \mathbb{C}$ is *star-shaped* at $z \in A$ if the line segment $[z,w] \subset A$ for each $w \in A$. The point z is called a star of A. It is clear that a set C is convex iff it is star-shaped at any point of C.

Note that A in Figure 5.1 is star-shaped but not convex whereas B is convex (and hence star-shaped). Any point in the shaded region of A is a star of A.

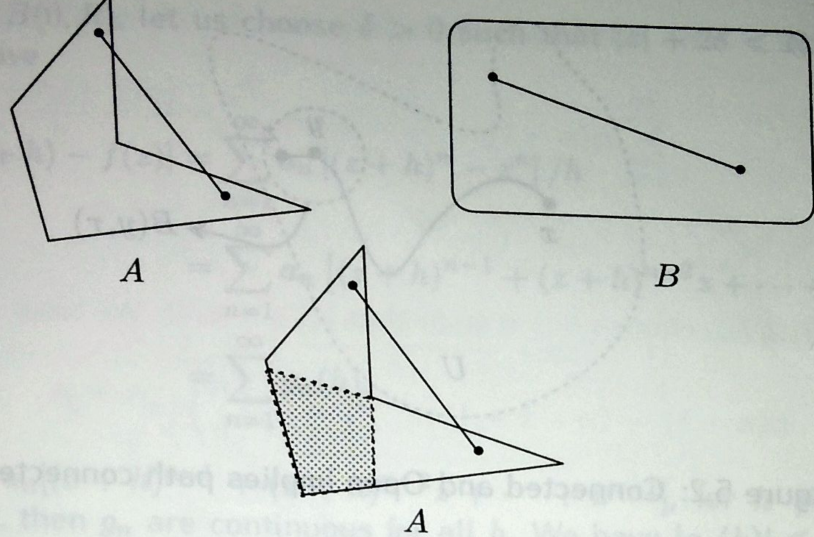

Figure 5.1: Star-shaped and convex sets

Ex. 5.4.21. Any disk $B(a,r)$ in $\mathbb{C}$ is convex.

Ex. 5.4.22. $E \subset \mathbb{C}$ is convex iff it is star-shaped at each of its points.

Ex. 5.4.23. Any star-shaped subset of $\mathbb{C}$ is path-connected and hence connected.

Ex. 5.4.24. Show that $\mathbb{C} \setminus L_\alpha$ is star-shaped at each z where z is such that $-z \in L_\alpha$.

Another typical application of connectedness is the following

Proposition 5.4.25. *Let U be an open subset of $\mathbb{C}$. Then U is connected iff it is path connected.*

Proof. Fix $x \in U$. In view of Ex. 5.4.19, it suffices to show that there is a path from x to any $y \in U$. This prompts us to consider

$$E := \{y \in U : \text{ there is a path from } x \text{ to } y\}.$$

Then E is open in U: If $y \in E$, choose $r > 0$ such that $B(y,r) \subset U$. If $z \in B(y,r)$, then y and z are connected by the line segment $[y,z]$. Since x and y are path connected, x and z are path connected (by Lemma 5.4.18). Hence $z \in E$. See Figure 5.2.

E is closed in U: For, if $a \in U$ is a limit point of E and if $r > 0$ is such that $B(a,r) \subset U$, then there exists $z \in E \cap B(a,r)$. Now, x is path connected to z and z is path connected to a via the line segment $[z,a]$. Thus, x is path connected to a by Lemma 5.4.18. Hence $a \in E$.

Since $x \in E$, E is a nonempty subset of U which is open and closed in U. Hence it is U. Now path-connectedness of U follows from Ex. 5.4.19. ∎

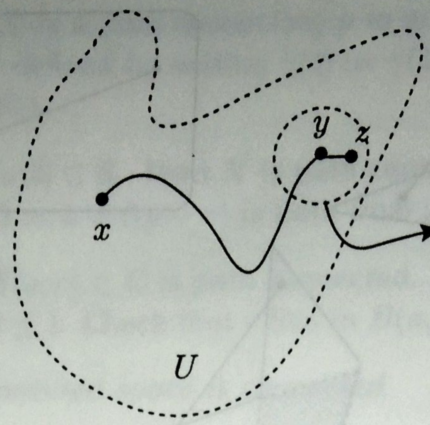

Figure 5.2: Connected and Open implies path connected

Ex. 5.4.26. Let $U \subset \mathbb{C}$ be open and connected. Show that given any two points $z, w \in U$, there exists a path from z to w in U consisting of line segments which are parallel to the axes. *Hint:* Mimic the proof of Proposition 5.4.25 after suitably defining E.

Proposition 5.4.27. *Let $\theta \colon U := \mathbb{C} \setminus L_\alpha \to \mathbb{R}$ be a continuous argument on U. Then there exists an integer n such that $\theta - \arg_\alpha = n2\pi$.*

Proof. Note that U is star-shaped and hence path connected whence connected. Let $\arg_\alpha$ be the continuous argument on $\mathbb{C} \setminus L_\alpha$ (as in Theorem 4.3.8). Now for each $z \in U$, $\theta(z), \arg_\alpha(z) \in \mathbf{A}(z)$ and hence their difference $\theta(z) - \arg_\alpha(z) \in \{2\pi n : n \in \mathbb{Z}\}$. The continuous function $(\theta - \arg_\alpha)/2\pi$ is a constant on the connected set U, by Ex. 5.4.8. The proof is complete.. □

Remark 5.4.28. This proposition yields another proof of non-existence of a continuous argument on $\mathbb{C}^*$. For, if θ is one such then it must coincide with $\theta_0 + 2n\pi$ on $\mathbb{C} \setminus L_0$ for some n. But then we can easily show that $\theta_0 + 2n\pi$ does not extend continuously to $\mathbb{C}^*$.

5.5 Power Series and Analytic Functions

We need the following fact from Ex 2.1.9: If $0 \le r < 1$, then $nr^n \to 0$ as $n \to \infty$. We also use Ex. 1.1.18 in the proof below.

Theorem 5.5.1. *Let $f(z) := \sum_{n=0}^\infty a_n z^n$ be a power series with radius of convergence $R > 0$. Then f is differentiable on $B(0, R)$. Furthermore, $f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$ for $z \in B(0, R)$. That is, a power series can be differentiated term by term in the disk of convergence.*

Proof. Given $z \in B(0, R)$, let us choose $\delta > 0$ such that $|z| + 2\delta < R$. For $h \in \mathbb{C}$ with $0 < |h| < \delta$, we have

$$\begin{aligned}\frac{1}{h}[f(z+h) - f(z)] &= \sum_{n=1}^{\infty} a_n \left[(z+h)^n - z^n\right]/h \\ &= \sum_{n=1}^{\infty} a_n \left[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1}\right] \\ &= \sum_{n=1}^{\infty} g_n(h),\end{aligned}$$

where $g_n(h) := a_n[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1}]$, for $h \neq 0$. If we define $g_n(0) := na_n z^{n-1}$, then g_n are continuous for all h. We have $|g_n(h)| \leq n\,|a_n|\,(|z| + \delta)^{n-1}$ for $|h| \leq \delta$. If we show that the sum $\sum g_n(h)$ defines a function, say, f_1, continuous at 0, then we have

$$f(z+h) - f(z) = hf_1(h).$$

It will follow that f is differentiable at z with derivative $f_1(0)$. We shall show that the series $\sum_n g_n$ is uniformly convergent in $B(z, \delta)$.

Since $nt^n \to 0$ if $0 < t < 1$, if we take $t = (|z| + \delta)/(|z| + 2\delta)$ and $\varepsilon = |z| + \delta$, we see that there exists N such that if $n > N$, $nt^n \leq |z| + \delta$. Hence $n(|z| + \delta)^{n-1} \leq (|z| + 2\delta)^n$. Since $|z| + 2\delta < R$, $\sum_{n=0}^{\infty} |a_n|\,(|z| + 2\delta)^n$ is convergent and hence $\sum_{n=1}^{\infty} n\,|a_n|\,(|z| + \delta)^{n-1}$ is convergent. By M-test, we see that $\sum_{n\geq 1} g_n$ is uniformly convergent on $B(0, \delta)$ and hence is a continuous function, say f_1 on the disk $B(0, \delta)$. From this it follows that

$$f(z+h) - f(z) = hf_1(z+h), \qquad \text{for } |h| < \delta.$$

Hence f is differentiable and its derivative is $f_1(0) = \sum na_n z^{n-1}$. □

We indicate another proof of this result which again teaches a good trick. This proof is 'similar' to the proof of continuity of the power series function on the disk of convergence. Also, recall the proofs of $\lim t^n = 0$ and $\lim nt^n = 0$ for $0 < t < 1$. The estimates we needed were respectively, $(1+h)^n \geq nh$ and $(1+h)^n \geq \binom{n}{2}h^2$. The inequality (5.5) below is the analogue of the second estimate in the present situation. Keep these remarks in mind while reviewing the second proof for better understanding.

Ex. 5.5.2. Show that the radii of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} na_n z^{n-1}$ are the same. *Hint:* If $|z| < r < s < R$, argue as in the above proof to show that $|a_n|\, nr^{n-1} \leq |a_n|\, s^n$. Or, you may use Hadamard's formula for the radius of convergence. Or, see the next lemma.

Ex. 5.5.3. Show that the radii of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_n \frac{z^{n+1}}{n+1}$ are the same. If the functions defined by them (on their common disk of convergence) are denoted by f and g, what is the relation between them?

Lemma 5.5.4. *Let $R > 0$ be the radius of convergence of $f(z) := \sum_n a_n z^n$. Then for any $k \in \mathbb{N}$, the series*

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k}$$

will converge for $|z| < R$.

Proof. Given $z \in B(0, R)$, choose $\delta > 0$ such that $|z| + \delta < R$. We have

$$\begin{aligned} n(n-1)\cdots(n-k+1)\left|z^{n-k}\right| &= \frac{k!}{\delta^k}\binom{n}{k}|z|^{n-k}\delta^k \\ &\leq k!\frac{(|z|+\delta)^n}{\delta^k}. \end{aligned}$$

Since $|z|+\delta \in B(0, R)$, the series $\sum_k^\infty a_n(|z|+\delta)^n$ is convergent. By comparing the series $\sum_{n=k}^\infty n(n-1)\cdots(n-k+1)|a_n||z|^{n-k}$ with the convergent series $\frac{k!}{\delta^k}\sum_k^\infty |a_n|(|z|+\delta)^n$, the result follows. □

Lemma 5.5.5. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $\delta > 0$. Then for all $h \in \mathbb{C}$ with $0 < |h| < \delta$, we have*

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \leq \frac{|h|}{\delta^2}(|z|+\delta)^n. \tag{5.5}$$

Proof. If $0 < |h| < \delta$, we have

$$\begin{aligned} \delta^2\left|(z+h)^n - z^n - nz^{n-1}h\right| &= \delta^2\left|\sum_{k=2}^{n}\binom{n}{k}z^{n-k}h^k\right| \\ &\leq \delta^2|h|^2\sum_{k=2}^{n}\binom{n}{k}|z|^{n-k}|h|^{k-2} \\ &\leq |h|^2\sum_{k=2}^{n}\binom{n}{k}|z|^{n-k}\delta^k \\ &\leq |h|^2(|z|+\delta)^n. \end{aligned}$$

On dividing both sides by $|h|\,\delta^2$, we obtain (5.5). □

Alternative proof of Theorem 5.5.1. Let z with $|z| < R$ be given. We prove directly that $f'(z) = \sum na_n z^{n-1}$. According to Ex. 5.5.2 or Lemma 5.5.4, the series $\sum_n na_n z^{n-1}$ is convergent.

Consider a positive real $\delta < R - |z|$. If $0 < |h| < \delta$, then

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - \sum_1^\infty n a_n z^{n-1}\right| &= \left|\frac{1}{h}\left(\sum_0^\infty a_n(z+h)^n - \sum a_n z^n\right) - \sum_1^\infty n a_n z^{n-1}\right| \\
&= \left|\sum_1^\infty a_n \left(\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right)\right| \\
&\le \sum_1^\infty |a_n| \left|\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right| \\
&\le \frac{|h|}{\delta^2} \sum_1^\infty |a_n| (|z| + \delta)^n.
\end{aligned}$$

The last series is convergent since $|z| + \delta < R$, and its sum does not involve h. Hence the inequality proves that

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_1^\infty n a_n z^{n-1}.$$

This completes the proof. □

Example 5.5.6. Consider $\exp(z) := \sum_{n=0}^\infty (z^n/n!)$. Then we have $\exp'(z) = \exp(z)$ for $z \in \mathbb{C}$. Hence, $\sin' = \cos$ and $\cos' = -\sin$ on $\mathbb{C}$. More generally, if $f(z) := \exp(\alpha z)$ for some fixed $\alpha \in \mathbb{C}$, then $f' = \alpha f$.

Ex. 5.5.7. Prove the following:

(1) $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = 1$.

(2) $\lim\limits_{z\to 0} \dfrac{\sinh z}{iz} = -i$.

(3) $\lim\limits_{z\to 0} \dfrac{e^{iz}-1}{z} = i$.

Hint: Recall that $\frac{f(z)-f(0)}{z} \to f'(0)$.

Proposition 5.5.8. *Any power series function $f(z) := \sum a_n z^n$ is infinitely differentiable on its disk of convergence. In fact, we have*

$$f^{(k)}(z) = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1) a_n z^{n-k}.$$

In particular, we have

$$a_n := \frac{f^{(n)}(0)}{n!}. \tag{5.6}$$

Proof. Repeated application of Theorem 5.5.1 yields the result. □

Ex. 5.5.9. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ with $R > 0$ as its radius of convergence. Assume that $f' = f$ on $B(0,R)$ and that $f(0) = 1$. Find a_n explicitly and hence f. *Hint:* Proposition 5.5.8. Do you see that you need the result on uniqueness of power series?

Ex. 5.5.10. Let $f(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}$. Show that f satisfies the differential equation

$$z^2 f''(z) + z f'(z) = 4z^2 f(z), \qquad z \in \mathbb{C}.$$

Definition 5.5.11. Let $U \subset \mathbb{C}$ be open. A function $f\colon U \to \mathbb{C}$ is said to be *analytic* in U if for each $a \in U$, there exists a sequence (a_n) of complex numbers and an $r > 0$ such that $B(a,r) \subset U$ and such that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for all $z \in B(a,r)$. The sequence (a_n) depends on a.

Example 5.5.12. Let $U = \mathbb{C}^*$ and $f(z) = 1/z$. Then for any $a \neq 0$, we have

$$f(z) = \frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{(z-a)}{a}\right)^n$$

for all $z \in B(a, |a|)$.

How did we get the power series? Look at the following:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{(z-a)+a} \\ &= \frac{1}{a}\frac{1}{1+\frac{z-a}{a}}. \end{aligned}$$

Thus, if we assume $|z-a| < |a|$, we can expand this as a geometric series.

Thus, f is analytic in U.

Ex. 5.5.13. $\exp\colon \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$.

Ex. 5.5.14. Analytic functions on U are holomorphic in U. In fact, analytic functions are infinitely differentiable.

Remark 5.5.15. It is a central result of Cauchy theory, the theme of Chapter 7, that the converse is also true. This is remarkable since this implies that any function which is once differentiable *on an open set* is infinitely differentiable. This should be contrasted with the existence of differentiable functions on intervals in $\mathbb{R}$ whose derivatives are not even continuous, leave alone being differentiable.

Our next result, Theorem 5.5.18 will show that any power series function on its disk of convergence is analytic on its domain. Note that this is not as obvious as you would like to think. If f is analytic in $B(a,R)$, given any $z_0 \in B(a,R)$, we need to show that there exist $r>0$ and $c_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$ such that $f(z) = \sum_n c_n(z-z_0)^n$ for $z \in B(z_0,r) \subset B(a,R)$. We need a preliminary result on iterated sum of a double series.

The next lemma gives us a sufficient condition under which the order of summation can be interchanged in a double series. First some notation. Let $a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ be a function. We let $a_{ij} = a(i,j)$ for $(i,j) \in \mathbb{N}\times\mathbb{N}$. We want to investigate conditions under which

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

First of all notice that the displayed equation means a host of things:

(1) $\sum_{j=1}^{\infty} a_{ij}$ is convergent, say, with sum α_i and that $\sum_{i=1}^{\infty}\alpha_i$ is convergent,

(2) the series $\sum_{i=1}^{\infty} a_{ij}$ converges to, say, α'_j and $\sum_{j=1}^{\infty}\alpha'_j$ is convergent and

(3) $\sum_i \alpha_i = \sum_j \alpha'_j$.

Note that we have neither defined the sum or convergence of a double series nor have shown that under the stated conditions the double series is convergent.

Lemma 5.5.16. *Let (a_{ij}) be a double sequence in $\mathbb{C}$, i.e. a function $a\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ so that $a(i,j) = a_{ij}$. Assume that $\sum_{j=1}^{\infty}|a_{ij}| = b_i$ for $i \in \mathbb{N}$ and that $\sum_i b_i$ is convergent. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. We claim that $\sum_i a_{ij}$ is absolutely convergent for every j fixed. Note that $|a_{ij}| \le \sum_j |a_{ij}| \le b_i$, and by hypothesis, the series $\sum_i b_i$ is convergent. Hence by the comparison test, it follows that $\sum_i a_{ij}$ is absolutely convergent and hence convergent. Let $\sum_i a_{ij} = \alpha_j$.

Fix $n \in \mathbb{N}$. We claim that

$$\sum_{j=1}^{n}\sum_{i} a_{ij} = \sum_{i}\left(\sum_{j=1}^{n} a_{ij}\right). \tag{5.7}$$

In fact, we show that both the sides are equal to $\alpha_1 + \cdots + \alpha_n$. Let us set out to prove this.

$$\alpha_1 + \cdots + \alpha_n = \sum_i a_{i1} + \cdots + \sum_i a_{in} = \sum_{j=1}^{n}\sum_i a_{ij}. \tag{5.8}$$

On the other hand, we have

$$
\begin{aligned}
\alpha_1 + \cdots + \alpha_n &= \sum_i a_{i1} + \cdots + \sum_i a_{in} \\
&= \sum_i (a_{i1} + \cdots + a_{in}) \quad \text{(by algebra of convergent series)} \\
&= \sum_i \left(\sum_{j=1}^{n} a_{ij} \right). \qquad (5.9)
\end{aligned}
$$

(5.7) follows from (5.8)–(5.9).

Consider the metric space $X := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric. For each $i \in \mathbb{N}$, we define a function $f_i \colon X \to \mathbb{C}$ by setting

$$
f_i(1/n) = \sum_{j \leq n} a_{ij} \text{ and } f_i(0) = \sum_{j=1}^{\infty} a_{ij}.
$$

It follows from hypothesis that f_i is continuous at 0 and that $\sum_i f_i$ converges uniformly on X to a function $g \colon X \to \mathbb{C}$ by the M-test. Hence g is continuous at 0. We have

$$
\begin{aligned}
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_i f_i(0) = g(0) &= \lim_{n \to \infty} g(1/n) \\
&= \lim_{n\to\infty} \sum_i f_i(1/n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} \quad \text{(by (5.7))} \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.
\end{aligned}
$$

This completes the proof. □

The next elementary exercise will be used in the proof of the next theorem.

Ex. 5.5.17. Show that if the order of a summation in a double series of the form $\sum_{j=0}^{\infty} \sum_{i=0}^{j} a_{ij}$ is reversed we get a double series of the form $\sum_{i=0}^{\infty} \sum_{j=i}^{\infty} a_{ij}$. *Hint:* Plot the terms a_{ij} at the points (i, j) of the lattice $\mathbb{Z}_+ \times Z_+$ in $\mathbb{R}^2$. Observe that the first iterated sum means to sum horizontally and then vertically. See Figure 5.3.

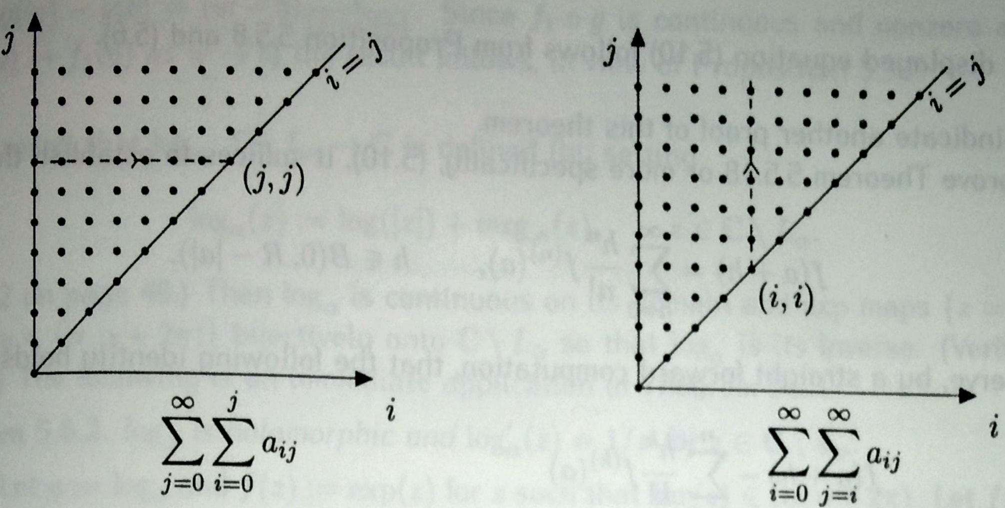

Figure 5.3: Change of order for double sum

Theorem 5.5.18. *Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series for $z \in B(z_0, R)$ with $R > 0$. Then f is analytic in $B(z_0, R)$.*

In fact, for any $a \in B(z_0, R)$ and $z \in B(a, R-|a|)$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n. \tag{5.10}$$

Proof. Assume $z_0 = 0$. Let $a \in B(0, R)$. We note that we need only show that f admits a power series expansion in powers of $(z-a)$. The explicit representation in (5.10) follows from (5.6) (Proposition 5.5.8) and the uniqueness theorem for the power series (Corollary 3.2.9).

To write f in powers of $(z-a)$, we do the obvious thing. We have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n((z-a)+a)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z-a)^k a^{n-k} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} a_n a^{n-k} \right) (z-a)^k \end{aligned}$$

The only thing is to justify the last equality, i.e. to justify the interchange of the order of the sums.

By Lemma 5.5.16, this is valid, if $\sum_{n=0}^{\infty}\sum_{k=0}^{n} \left|a_n \binom{n}{k} a^{n-k}(z-a)^k\right|$ is convergent. But this series is dominated by $\sum_{n=0}^{\infty} |a_n| (|z-a|+|a|)^n$ which is convergent if $|z-a|+|a| < R$.

The displayed equation (5.10) follows from Proposition 5.5.8 and (5.6). □

We indicate another proof of this theorem.
To prove Theorem 5.5.18 or more specifically, (5.10), it suffices to establish that

$$f(a+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(a), \qquad h \in B(0, R - |a|).$$

We observe, by a straight forward computation, that the following identity holds:

$$\begin{aligned} f(a+h) - \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(a) \\ = \sum_{n+1}^{\infty} a_k[(a+h)^k - (a^k + ka^{k-1}h + \cdots + \binom{k}{n} a^{k-n}h^n)]. \end{aligned}$$

It follows that

$$\begin{aligned} \left| f(a+h) - \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(a) \right| \\ \leq \sum_{n+1}^{\infty} |a_k| \left((|a| + |h|)^k + |a|^k + \cdots + \binom{k}{n} |h|^n |a|^{k-n} \right) \\ \leq 2 \sum_{n+1}^{\infty} |a_n| (|a| + |h|)^k. \end{aligned}$$

Since $|h| < R - |a|$, we see that $|a| + |h| < R$ and hence the sum $\sum_0^\infty |a_n| (|a| + |h|)^k$ is convergent. Therefore its tail $\sum_{n+1}^\infty |a_n| (|a| + |h|)^k$ goes to zero as $n \to \infty$. The result follows. □

5.6 Inverse Functions

Theorem 5.6.1 (Differentiability of Inverse Functions). *Let U and V be open in $\mathbb{C}$. Let $f: U \to V$ be a bijection with $g := f^{-1}$. Let $a \in U$ and $b := f(a)$. Assume that*
(a) f is differentiable at a and $f'(a) \neq 0$ and
(b) g is continuous at b.
Then g is differentiable at b and $g'(b) = (f'(a))^{-1}$.

Proof. Let $f(z) - f(a) = (z-a)f_1(z)$, the notation being as in Proposition 5.3.2. Let $w = f(z) \in V$. We substitute $g(w)$ for z in the above equation and obtain

$$w - b = [g(w) - g(b)]f_1(g(w)).$$

Hence $g(w) - g(b) = (w-b)\frac{1}{(f_1 \circ g)(w)}$. Since $f_1 \circ g$ is continuous and nonzero at b, (as $f_1 \circ g(w) \to f_1(a)$ as $w \to b$) the result follows, in view of Proposition 5.3.2. □

We recall that $\log_\alpha : \mathbb{C} \setminus L_\alpha \to \mathbb{C}$ is defined (by setting

$$\log_\alpha(z) := \log(|z|) + i\arg_\alpha(z), \qquad z \in \mathbb{C} \setminus L_\alpha.$$

(See 4.2 on page 48.) Then $\log_\alpha$ is continuous on its domain and exp maps $\{z = x+iy : x \in \mathbb{R}, y \in (\alpha, \alpha + 2\pi)\}$ bijectively onto $\mathbb{C} \setminus L_\alpha$ so that $\log_\alpha$ is its inverse. (Verify these claims.) The following is an immediate application of Theorem 5.6.1.

Theorem 5.6.2. $\log_\alpha$ *is holomorphic and* $\log_\alpha'(z) = 1/z$ *for* $z \in \mathbb{C} \setminus L_\alpha$.

Proof. Let $g := \log_\alpha$ and $f(z) := \exp(z)$ for z such that $\mathrm{Im}\,(z) \in (\alpha, \alpha + 2\pi)$. Let $f(z) = w$. Then $g'(w) = (1/f'(z)) = (1/\exp(z)) = 1/w$. □

Ex. 5.6.3. We say a continuous function $g \colon U \to \mathbb{C}$ is **a** logarithm on U if we have $\exp(g(z)) = z$ for all $z \in U$. Show that if g is a logarithm on U, then $g'(z) = 1/z$ for $z \in U$.

Example: If $U := B(1+i, 1)$ and if we take as g the restriction of any one of $\log_0$, $\log_{\pi/2}$, $\log_\pi$ and $\log_{3\pi/2}$ to U, then g is a logarithm on U. The notation is as in and around (4.2) on page 48.

Theorem 5.6.4. *If* $|z| < 1$, *then* $\log_\pi(1+z) = \sum_{n=1}^\infty (-1)^{n-1} z^n/n$.

Proof. Let $f(z) := \log_\pi(1+z)$ and $g(z) := \sum_{n=1}^\infty (-1)^{n-1} z^n/n$ for $|z| < 1$. Then by Theorem 5.6.2, $f'(z) = 1/(1+z)$ and $g'(z) = \sum_{n=0}^\infty (-1)^n z^n = 1/(1+z)$ by Theorem 5.5.1. Hence $f - g$ is a constant on $B(0,1)$. Since $f(0) = 0 = g(0)$, the result follows. □

Ex. 5.6.5. For any $z \in \mathbb{C}$, show that $n \log_\pi(1 + z/n)$ is defined for all large values of n and that it tends to z as $n \to \infty$. Hence deduce that $(1 + z/n)^n \to \exp(z)$ as $n \to \infty$.

5.7 Cauchy-Riemann Equations

Now going back to the first definition and assuming that f is complex differentiable at z, are there any special directions we want to investigate the existence of the limit of the difference quotients? Look at Figure 5.4. We want h to approach 0 either completely from the real axis or from the imaginary axis. We first look at $\frac{f(z+h)-f(z)}{h}$ as $h \in \mathbb{R}$ goes to 0. We have

$$\begin{aligned} f'(z) &= \lim_{h\in\mathbb{R},\, h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\in\mathbb{R}, h\to 0} \frac{[u(x+h,y) + iv(x+h,y)] - [u(x,y) + iv(x,y)]}{h} \\ &= u_x(x,y) + iv_x(x,y). \end{aligned} \tag{5.11}$$

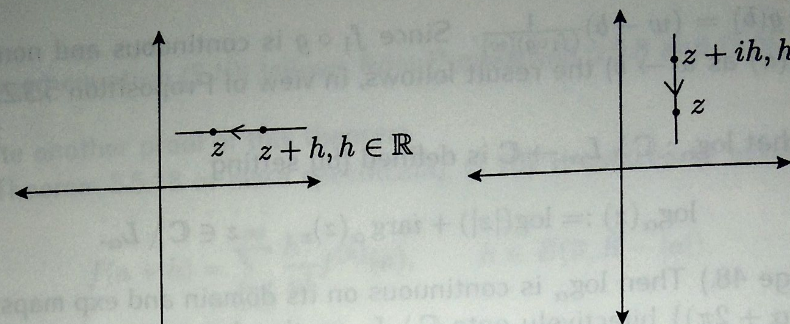

Figure 5.4: Cauchy Riemann Equations

We consider increments ih with $h \in \mathbb{R}$ and proceed in a similar way to get

$$f'(z) = \frac{1}{i}(u_y + iv_y) = -iu_y + v_y. \tag{5.12}$$

From (5.11) and (5.12), using the uniqueness of the limits, we get the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{5.13}$$

□ In a similar way, the reader can show that if f is differentiable at z, then $\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z) = f'(z)$. We use the "new variables" z and $\overline{z}$. Then $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$. Applying chain rule formally, we get

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \Rightarrow \frac{\partial f}{\partial z} = [f_x - if_y]/2$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} \Rightarrow \frac{\partial f}{\partial \overline{z}} = [f_x + if_y]/2.$$

This allows us to define two first order partial differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Apart from the Cauchy-Riemann equations in classical form

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},} \tag{5.14}$$

we have them in a couple of other forms too:

$$\boxed{\begin{aligned} f'(z) &= \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} && (5.15)\\ \frac{\partial f}{\partial \bar z} &= 0. && (5.16)\end{aligned}}$$

Ex. 5.7.1. This is a compendium of standard applications of the Cauchy-Riemann equations (5.14). We assume that U is open and connected. Let $f\colon U \to \mathbb{C}$ be given.
(1) Assume that f is differentiable on U and that $f' = 0$ on U. Show that f is a constant using (5.14) and Ex. 5.2.7.
(2) If f is holomorphic on U and $\operatorname{Re} f$ (resp. $\operatorname{Im} f$) is a constant on U, then f is a constant.
(3) If f is holomorphic on U and assumes only real (resp. purely imaginary) values, then f is a constant.
(4) If f and $\overline{f}$ are both holomorphic on U, then f is a constant.
(5) If f is holomorphic on U and if $|f|$ is a constant, then f is a constant. *Hint:* Show that $\overline{f}$ is holomorphic.
(6) Let f be holomorphic on U. Assume that there exists a constant $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|\, e^{i\alpha}$ for all $z \in U$. Then f is a constant.

Ex. 5.7.2. If the range of a holomorphic function on a connected open set lies in either a straight line or a circle, then f is a constant. *Hint:* Let the straight line be given by $\operatorname{Re}(aw + b) = 0$ for some $a, b \in \mathbb{C}$. Consider $g(z) := af(z) + b$. (This result is true for an arbitrary open set U. You need a bit more topology to reduce to the special case.)

Ex. 5.7.3. Use C-R equations to show that $\log_0$ is holomorphic in $\mathbb{C} \setminus L_0$.

Ex. 5.7.4. The C-R equations are not sufficient for the differentiability. Consider

$$f(z) := \begin{cases} z^5\,|z|^{-4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Then f satisfies the CR equations at $z = 0$ but it is not differentiable at 0. As another example, consider $f(x + iy) = \sqrt{|xy|}$.

Lemma 5.7.5. *Let $u\colon U \to \mathbb{R}$ have continuous partial derivatives u_x and u_y in U. Fix $(x, y) \in U$. Then for all (ξ, η) near $(0, 0)$ we have*

$$u(x + \xi, y + \eta) - u(x, y) = \xi u_x(x, y) + \eta u_y(x, y) + \xi\varepsilon_1 + \eta\varepsilon_2,$$

where $\varepsilon_i \to 0$ as $(\xi, \eta) \to 0$.

Proof. We use mean value theorem of one variable calculus:

$$\begin{aligned}
u(x+\xi, y+\eta) - u(x,y) &= u(x+\xi, y+\eta) - u(x, y+\eta) + u(x, y+\eta) - u(x,y) \\
&= \xi u_x(x+\theta_1\xi, y+\eta) + \eta u_y(x, y+\theta_2\eta) \\
&= \xi[u_x(x,y) + (u_x(x+\theta_1\xi, y+\eta) - u_x(x,y))] \\
&\quad + \eta[u_y(x,y) + (u_y(x, y+\theta_2\eta) - u_y(x,y))] \\
&= \xi u_x(x,y) + \eta u_y(x,y) + \varepsilon_1\xi + \varepsilon_2\eta,
\end{aligned}$$

in an obvious notation. It is clear that $\varepsilon_i \to 0$ as $(\xi,\eta) \to 0$ thanks to the continuity of u_x and u_y. □

Proposition 5.7.6. *Let $f\colon U \to \mathbb{C}$. Let $f = u + iv$. Assume that the partial derivatives u_x, u_y, v_x and v_y exist and are continuous and further that the C-R equations (5.14) hold. Then f is differentiable on U. In fact, $f' = u_x + iv_x$ on U.*

Proof. We use the lemma for u and v. Let $h := \xi + i\eta$.

$$\begin{aligned}
f(z+h) - f(z) &= [u(x+\xi, y+\eta) - u(x,y)] + i[v(x+\xi, y+\eta) - v(x,y)] \\
&= \xi u_x + \eta u_y + \xi\varepsilon_1 + \eta\varepsilon_2 + i[\xi v_x + \eta v_y + \xi\varepsilon_3 + \eta\varepsilon_4] \\
&= h(u_x + iv_x) + (\varepsilon_1\xi + \varepsilon_2\eta) + i(\xi\varepsilon_3 + \eta\varepsilon_4).
\end{aligned} \tag{5.17}$$

In the above, we made use of C-R equations (5.14). Now, it follows from (5.17) that

$$\frac{f(z+h) - f(z)}{h} = (u_x + iv_x) + \frac{\varepsilon_1\xi + \varepsilon_2\eta}{h} + i\frac{\xi\varepsilon_3 + \eta\varepsilon_4}{h},$$

and each of second and third terms on the right goes to 0, thanks to the last lemma, as $\left|\frac{\xi}{h}\right| \leq 1$ and $\left|\frac{\eta}{h}\right| \leq 1$. □

Ex. 5.7.7. If $f\colon U \to \mathbb{C}$ is holomorphic, then $|f'(z)|^2 = u_x^2 + v_x^2$.

Ex. 5.7.8. Let $f\colon U \to \mathbb{C}$ be holomorphic. Let $\overline{U} := \{\overline{z} : z \in U\}$. Define $g(z) := \overline{f(\overline{z})}$ for $z \in \overline{U}$. Then g is holomorphic on $\overline{U}$.

Ex. 5.7.9. Let $f(x,y)$ be a polynomial in x and y with coefficients in $\mathbb{C}$. Show that f is holomorphic on $\mathbb{C}$ iff it is a polynomial in z alone. In fact, $f = \sum_{k=0}^{n} a_k z^k$ where $a_k = \frac{1}{k!}\frac{\partial^k f}{\partial x^k}(0)$, $0 \leq k \leq n$. *Hint:* Use either (5.15) or (5.16).

5.8 Geometric Meaning of C-R Equations

There is a geometric meaning underlying the Cauchy-Riemann equations (5.13). This involves the "complex structure" on $\mathbb{R}^2$, i.e., the scalar multiplication by i on $\mathbb{C} \ni z = (x,y) \in \mathbb{R}^2$. The following exercises are in fact a series of remarks which will explicate this. (We chose to mot write all the details, as it only smothers the real understanding. For complete details, we refer the reader to [21].)

Ex. 5.8.1. Consider $\mathbb{C}$ as a vector space over itself. Show that the $\mathbb{C}$ (or complex) linear maps from $\mathbb{C}$ to $\mathbb{C}$ are of form $\varphi_\lambda(z) = \lambda z$ for a unique $\lambda \in \mathbb{C}$.

Ex. 5.8.2. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. We fix once and for all $\{1, i\}$ as an ordered basis of $\mathbb{C}$ over $\mathbb{R}$. Show that with respect to to this basis, the matrix representation of φ_λ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $\lambda = a + ib$.

In particular, φ_i corresponds to $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Ex. 5.8.3. Conversely, a real linear map $\varphi\colon \mathbb{C} \to \mathbb{C}$ whose matrix representation $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is linear over $\mathbb{C}$ iff $AJ = JA$ (matrix multiplication). As a consequence, we have $a = d$ and $b = -c$ and hence $\varphi = \varphi_{a-ib}$. *Hint:* A is $\mathbb{C}$-linear iff $A(iz) = iA(z)$ for $z \in \mathbb{C}$.

Ex. 5.8.4. Let $z = x + iy \in \mathbb{C}$. Then $|z| = \|(x, y)\|$, the Euclidean norm on $\mathbb{R}^2$.

For $f\colon U \subset \mathbb{C} \to \mathbb{C}$ we let $f = u + iv$. Then $u, v\colon U \subset \mathbb{R}^2 \to \mathbb{R}$. We let $F(x, y) := (u(x, y), v(x, y))$ for $(x, y) \in U$. Note that under the real linear isomorphism $z = x + iy \mapsto (x, y)$, the basis element 1 (resp. i) goes to e_1 (resp. e_2).

Ex. 5.8.5. Let $f\colon U \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z \in U$. Then there exists $\lambda = a + ib$ (in fact $\lambda = f'(z)$) such that for $\varepsilon > 0$ there exists $\delta > 0$ with the property that $|h| < \delta$ implies that $|f(z + h) - f(z) - \varphi_\lambda(h)| < \varepsilon |h|$. This is equivalent to saying that

$$\left\| F((x, y) + (h_1, h_2)) - F(x, y) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\| < \varepsilon \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|,$$

whenever $\|(h_1, h_2)^t\| < \delta$. Thus F is differentiable as map from $U \subset \mathbb{R}^2$ to $\mathbb{R}^2$ with total derivative $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

But if F is differentiable, then we know that its matrix representation with respect to the standard basis $\{e_1, e_2\}$ is given by its Jacobian matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. Hence we deduce the C-R equations.

Ex. 5.8.6. Conversely, if $F = (u, v)\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) with u and v satisfying the C-R equations, then the map $f(z) := u(x, y) + iv(x, y)$ for $z = x + iy$ is differentiable at $z_0 = x_0 + iy_0$ with $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Remark 5.8.7. Compare the result in the last exercise with that in Proposition 5.7.6. The later is weaker than the one in the exercise. For, let $F = (u, v)$. Assume that u, v

are continuously differentiable and satisfy the C-R equations, then F is continuously differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and u, v satisfy the C-R equations. But in Ex. 5.8.6 above, we need only assume that F is differentiable. Thus, the result of the exercise is stronger than the conventional result in Proposition 5.7.6.

5.9 Cauchy Riemann Equations in Polar Coordinates

Let $f := u + iv : U \to \mathbb{C}$ be holomorphic. Our aim in this section is to derive the C-R equations in polar coordinates:

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.} \tag{5.18}$$

Let $U \subset \mathbb{C} \setminus L_\alpha$, $\alpha \in \mathbb{R}$, be an open set. Then any $z \in U$ can be written in the polar form $z = |z|\, e^{i\theta_\alpha(z)}$. In the sequel we shall let $\theta(z)$ stand for $\theta_\alpha(z)$. We then say $r := |z|$ and $\theta(z)$ as polar coordinates of z. If we write $z = x + iy$, then $x = r\cos(\theta(z))$ and $y = r\sin(\theta(z))$. Then by chain rule of (multi-variable calculus), we have, for any differentiable function u of (x, y),

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \frac{\partial u}{\partial r}\left(\frac{1}{\cos\theta}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-1}{r\sin\theta}\right) \\
&= \frac{1}{\cos\theta}u_r - \frac{1}{r\sin\theta}u_\theta,
\end{aligned} \tag{5.19}$$

where to simplify typing, we have written u_r for $\frac{\partial u}{\partial r}$ etc. Similarly, for a differentiable function $v = v(x, y)$, we have

$$v_y = \frac{1}{\sin\theta}v_r + \frac{1}{r\cos\theta}v_\theta. \tag{5.20}$$

Let $f : U \to \mathbb{C}$ be any differentiable function. Write $f = u + iv$ as usual. Then $u_x = v_y$ iff

$$\frac{1}{\cos\theta}u_r - \frac{1}{r\sin\theta}u_\theta = \frac{1}{\sin\theta}v_r + \frac{1}{r\cos\theta}v_\theta. \tag{5.21}$$

Proceeding as above, we show that $u_y = -v_x$ iff

$$\frac{1}{\sin\theta}u_r + \frac{1}{r\cos\theta}u_\theta = \frac{1}{\cos\theta}v_r - \frac{1}{r\sin\theta}v_\theta. \tag{5.22}$$

Multiplying (5.21) and (5.22) by $r\cos\theta\sin\theta$ and adding the resulting equations we obtain

$$(ru_r - v_\theta)(\sin^2\theta + \cos^2\theta) = 0, \text{ i.e., } (ru_r - v_\theta) = 0.$$

Multiplying (5.21) and (5.22) by $r\cos\theta\sin\theta$ and subtracting the resulting equations we obtain

$$(u_\theta + rv_r)(\sin^2\theta + \cos^2\theta) = 0, \text{ i.e., } (u_\theta + rv_r) = 0.$$

Thus we have derived (5.18).

Chapter 6

Complex Integration

6.1 Integration of functions from $\mathbb{R}$ to $\mathbb{C}$

Let $I = [a,b] \subseteq \mathbb{R}$ and $f : I \to \mathbb{C}$ be continuous. Then we define the integral of f by

$$\boxed{\int_a^b f \equiv \int_a^b f(t)\,dt := \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.}$$

Here the integrals are Riemann and they exist since $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are continuous.

Ex. 6.1.1. Let $f : [0, 2\pi] \to \mathbb{C}$ be given by $f(t) = e^{int}$ for some $n \in \mathbb{Z}$. Compute $\int_0^{2\pi} f(t)\,dt$.

Ex. 6.1.2. Let $C([a,b],\mathbb{C})$ denote the complex vector space of continuous functions on $[a,b]$. Show that the map $f \mapsto \int_a^b f(t)\,dt$ is a complex linear map from $C([a,b],\mathbb{C})$ to $\mathbb{C}$.

Theorem 6.1.3 (Fundamental Theorem of Calculus).
(i) *Let $f : [a,b] \to \mathbb{C}$ be continuous. Define $F(x) := \int_a^x f(t)\,dt$. Then F is differentiable and $F'(x) = f(x)$ for $x \in [a,b]$.*
(ii) *Let $G : [a,b] \to \mathbb{C}$ be differentiable with $g := G'$ continuous. Then $\int_a^b g(t)\,dt = G(b) - G(a)$.*

Proof. The strategy is to make use of our definitions of derivative and the integral of a complex valued function of a real variable and make use the fundamental theorem of calculus for real valued functions of a real variable.

Let us write $f(t) = u(t) + iv(t)$. Then u and v are continuous and

$$F(x) = \int_a^x f(t)\,dt = \int_a^x u(t)\,dt + i \int_a^x v(t)\,dt.$$

Now F is differentiable iff $\int_a^x u(t)\,dt$ and $\int_a^x v(t)\,dt$ are so. By the fundamental theorem of calculus, it follows that these two indefinite integrals are differentiable and their derivatives are $u(x)$ and $v(x)$ respectively. Hence $F'(x)$ exists and is equal to $u(x) + iv(x) = f(x)$. This proves (i).

Proof of (ii) is similar and the reader should do it on his own.

Let $H(x) := \int_a^x g(t)\,dt$. Then, $H'(x) = g(x)$ by (i). so that $(H-G)' = 0$. Thus $H-G$ is a constant on $[a,b]$ whence $H(b)-G(b) = H(a)-G(a)$, or $H(b)-H(a) = G(b)-G(a)$. Since $H(a) = 0$, (ii) follows.

□

Ex. **6.1.4.** Do Ex. 6.1.1 now!

Ex. **6.1.5.** Do Ex. 5.2.7. *Hint:* $h(x) - h(a) = \int_a^x h'(t)\,dt$.

Ex. **6.1.6.** Let $\lambda = a + ib \in \mathbb{C}^*$. Evaluate $\int_0^t e^{\lambda s}\,ds$. Equating the real parts, show that

$$(a^2 + b^2)\int_0^t e^{as}\cos bs\,ds = e^{at}[a\cos bt + b\sin bt] - a.$$

Ex. **6.1.7.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a continuously differentiable map with $\gamma(t) \neq z_0$ for all $t \in [a,b]$. (We think of γ as a path not passing through z_0.) Let $g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$, for $t \in [a,b]$. Show that $h(t) := e^{-g(t)}[\gamma(t) - z_0]$ is a constant and hence deduce that

$$\exp(g(t)) = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}, \qquad \text{for } t \in [a,b].$$

What can you say about $g(b)$ if $\gamma(b) = \gamma(a)$? Take $z_0 = 0$ and do you see the meaning of g? We shall return to this exercise later. See Definition 10.1.1, Example 10.1.2 and Theorem 10.1.3.

Corollary 6.1.8 (Integration by parts). *Let $f, g\colon [a,b] \to \mathbb{C}$ be continuously differentiable. Then*

$$\int_a^b f(t)g'(t)\,dt = [f(t)g(t)]_a^b - \int_a^b f'(t)g(t)\,dt.$$

Proof. This follows from Theorem 6.1.3. Recall that $(fg)'(t) = f'(t)g(t) + f(t)g'(t)$. Both sides are continuous functions and so we can integrate them to obtain

$$\begin{aligned}\int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt &= \int_a^b (fg)'(t)\,dt \\ &= [f(t)g(t)]_a^b,\end{aligned}$$

by the fundamental theorem of calculus. The result follows from this. □

Proposition 6.1.9 (Change of Variable). *Let $h\colon [c,d] \to \mathbb{R}$ be continuously differentiable and $f\colon [a,b] \to \mathbb{R}$ be continuous. Assume that $h([c,d]) \subset [a,b]$. Then*

$$\int_c^d (f \circ h)(s) h'(s)\, ds = \int_{h(c)}^{h(d)} f(t)\, dt.$$

Proof. Consider $\varphi(s) := \int_{h(c)}^{h(s)} f(t)\, dt$. Then φ is the composition of the functions $s \mapsto h(s)$ and $x \mapsto \int_{h(c)}^{x} f(t)\, dt$. Hence $\varphi'(s) = f(h(s))h'(s)$ by chain rule and Theorem 6.1.3. Again, by the same theorem, we have

$$\int_{h(c)}^{h(d)} f(t)\, dt = \varphi(d) - \varphi(c) = \int_c^d \varphi'(s)\, ds = \int_c^d f(h(s))h'(s)\, ds.$$

(Why does the integrand on the left most side make sense?) □

Proposition 6.1.10. *Let $f\colon I \to \mathbb{C}$ be continuous. Then*

$$\boxed{\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt.}$$

Proof. If f is real valued, then $-|f(t)| \leq f(t) \leq |f(t)|$ so that $\int -|f| \leq \int f \leq \int |f|$ Hence $|\int f| \leq \int |f|$ in this case.

Let f be complex valued. Choose $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $\alpha \int_a^b f(t)dt \in \mathbb{R}$. (If $\int_a^b f(t)\, dt = re^{it}$ is a polar representation, then we may take $\alpha = e^{-it}$.) By linearity, $\alpha \int_a^b f = \int_a^b \alpha f$ so that

$$\int_a^b \mathrm{Re}\,(\alpha f(t))\, dt = \mathrm{Re}\,(\alpha \int_a^b f(t)\, dt) = \alpha \int_a^b f(t)\, dt. \tag{6.1}$$

Also, observe that

$$|\mathrm{Re}\,(\alpha f(t)| \leq |\alpha f(t)| = |\alpha|\, |f(t)| = |f(t)|\,. \tag{6.2}$$

Hence

$$\begin{aligned}
\left|\int_a^b f(t)dt\right| &= \left|\alpha \int_a^b f(t)dt\right| && \text{(by our choice of } \alpha\text{)}\\
&= \left|\int_a^b \mathrm{Re}\,(\alpha f(t))dt\right| && \text{(by 6.1)}\\
&\leq \int_a^b |\mathrm{Re}\,(\alpha f(t))|\, dt && \text{(by the real case)}\\
&\leq \int_a^b |f(t)|\, dt && \text{(by 6.2 and monotonicity of the integral).}
\end{aligned}$$

This completes the proof. □

Proposition 6.1.11. *If $f_n : [a, b] \to \mathbb{C}$ are continuous and they converge uniformly on $[a, b]$ to an $f : [a, b] \to \mathbb{C}$, then f is necessarily continuous and we have $\lim_n \int_a^b f_n(t)\, dt = \int_a^b f(t)\, dt$.*

Proof. Follows from the observation:

$$\left|\int_a^b f_n(t)\, dt - \int_a^b f(t)\, dt\right| = \left|\int_a^b [f_n(t) - f(t)]\, dt\right| \le \int_a^b |f_n(t) - f(t)|\, dt$$
$$\le \int_a^b \varepsilon_n = \varepsilon_n(b-a),$$

where $\varepsilon_n := \sup\{|f_n(t) - f(t)| : t \in [a, b]\}$. Since $f_n \to f$ uniformly on $[a, b]$, $\varepsilon_n \to 0$. The result follows.(Question: Where did we use the fact that f is continuous?) □

We give applications of this result which constitute some of the important results of Cauchy Theory in the next chapter.

Ex. 6.1.12. Let $f(z) := \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $z \in B(z_0, R)$. Then for $0 \le r \le R$, and $0 \le t \le 2\pi$, we have the Parseval identity:

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right|^2 dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \tag{6.3}$$

If $|f(z)| \le M(r)$ for $|z - z_0| = r$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M(r)^2. \tag{6.4}$$

Hint: Observe that $\int_0^{2\pi} |s_n(t)|^2\, dt = \sum_{k=0}^{n} |a_k|^2 r^{2k}$. Then use Ex. 3.2.2.

Ex. 6.1.13. Assume the notation of Ex. 6.1.12 but with $R = \infty$ and that $|f(z)| \le M$ for all $z \in \mathbb{C}$. Use (6.4) to show that f is a constant. (This is Liouville's theorem.)

Ex. 6.1.14. Keep the notation of Ex. 6.1.12. Let $M(r) := \sup\{|f(z)| : |z - z_0| = r\}$. Derive Cauchy's inequalities: $\left|f^{(n)}(z_0)\right| \le n! \frac{M(r)}{r^n}$ for $0 < r < R$. (See also Corollary 6.4.2. The point here is to use Ex. 6.1.12.)

Ex. 6.1.15. With the notation of Ex. 6.1.14, assume that $|f(z_0)| = M(r)$ for some $0 < r < R$. Then f is a constant in $B(z_0, R)$. (This is the maximum modulus principle.)

Ex. 6.1.16. Prove the fundamental theorem of algebra. *Hint:* If $p(z) \ne 0$ on $\mathbb{C}$, then $f(z) := \frac{1}{p(z)}$ is entire. It is bounded on $\mathbb{C}$, since $\lim_{|z|\to\infty} |p(z)| = \infty$.

We end this section with Leibnitz rule for differentiation under the integral sign. It will be used later.

Theorem 6.1.17 (Leibnitz Rule). *Let $f\colon [a,b]\times[c,d] \to \mathbb{C}$ be a continuous function. Assume that $\frac{\partial f}{\partial x}(t,x)$ exists for all $(t,x) \in [a,b]\times[c,d]$ and is continuous. Then $x \mapsto \int_a^b f(t,x)\,dt$ is differentiable and we have*

$$\boxed{\frac{d}{dx}\left(\int_a^b f(t,x)\,dt\right) = \int_a^b \frac{\partial f}{\partial x}(t,x)\,dt.}$$

Proof. We may assume that f is real valued. Let $g(x) := \int_a^b f(t,x)\,dt$. Consider

$$\begin{aligned}\left|\frac{g(x+h)-g(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(t,x)\,dt\right| &= \left|\int_a^b \left[\frac{f(t,x+h)-f(t,x)}{h} - \frac{\partial f}{\partial x}(t,x)\right] dt\right| \\ &= \left|\int_a^b \left[\frac{\partial f}{\partial x}(t,c_{t,h}) - \frac{\partial f}{\partial x}(t,x)\right] dt\right|,\end{aligned}$$

for some $c_{t,h} \in (x, x+h)$ by the mean value theorem. Since $\frac{\partial f}{\partial x}$ is continuous, it is uniformly continuous on $[a,b]\times[c,d]$. Thus given $\varepsilon > 0$, we choose $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial x}(t,x) - \frac{\partial f}{\partial x}(t_0,x_0)\right| < \frac{\varepsilon}{b-a},$$

if $|t-t_0| < \delta$ and $|x-x_0| < \delta$. Then for $0 < |h| < \delta$, we have

$$\begin{aligned}\left|\frac{g(x+h)-g(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(t,x)\,dt\right| &\le \int_a^b \left|\left[\frac{\partial f}{\partial x}(t,c_{t,h}) - \frac{\partial f}{\partial x}(t,x)\right]\right| dt \\ &< \frac{\varepsilon}{b-a}(b-a) = \varepsilon.\end{aligned}$$

□

6.2 Path Integrals

Definition 6.2.1. Let U be a region in $\mathbb{C}$. Let $z_0, z_1 \in U$. A continuous function $\gamma\colon [a,b] \to U \subseteq \mathbb{C}$ is called a *curve* from z_0 to z_1 where $\gamma(a) = z_0$ and $\gamma(b) = z_1$.

Let $\gamma\colon [a,b] \to U$ be continuous and assume that there exists a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ such that γ is continuously differentiable on the closed intervals $[t_j, t_{j+1}]$ for $0 \le j \le n-1$. We then say γ is piece-wise smooth on $[a,b]$. (The point here is that the right and left side derivatives at t_i may not be the same. See

Example 6.2.7 below.) A *path* in U is a piecewise smooth map γ from an interval to U and it is called a smooth path if γ is continuously differentiable on its domain.

$\gamma(a)$ is called the *initial point* and $\gamma(b)$ is called *terminal point* of γ. We say that γ is a closed path if the initial and terminal points are the same.

If $\gamma: [a,b] \to U$ is a path, we let $[\gamma]$ denote its image: $[\gamma] := \gamma([a,b])$. $[\gamma]$ is called the *trace* of the curve γ and is not to be confused with the curve. See Example 6.2.6 below.

Example 6.2.2. Constant paths or null paths: Let $\gamma: [a,b] \to U$ be given by $\gamma(t) = z_0$, a fixed point of U.

Example 6.2.3. Line segments: Let $z, w \in B(\alpha, r)$ and $\gamma: [0,1] \to B(\alpha, r)$ be defined by $\gamma(t) := z + t(w - z)$. Then γ is a path from z to w in $B(\alpha, r)$. We denote this path by $[z, w]$, or sometimes $\ell_{z,w}$.

Example 6.2.4. Circular paths: Let $a \in U$ and $r > 0$ be such that $B[a,r] \subset U$. Then $\gamma(t) := a + re^{it}$ $(0 \le t \le 2\pi)$ is a closed path. We denote this path by $S(a,r)$ or C_r, if a is understood.

Remark 6.2.5. Throughout this book, when we talk of a line segment or a circular arc or path, we shall assume that they are defined as in the last two examples. We shall call these as standard parametrization and we may not explicitly say this in the sequel.

Example 6.2.6. Let the notation be as above, but, let $\sigma(t) := a + re^{-it}$ and $\tau(t) := a + re^{2it}$. The images of σ, τ and γ are the same, namely, the circle centred at a and radius r. However, these maps are different and hence as paths they are not the same.

Example 6.2.7. Let $U = \mathbb{C}$ and

$$\gamma(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 + i(t-1) & 1 \le t \le 2. \end{cases}$$

Then γ is a (piecewise smooth) path.

As another example, consider the map

$$\gamma(t) = \begin{cases} e^{i(\pi - t\pi)} & 0 \le t \le 1 \\ t & 1 \le t \le 2. \end{cases}$$

These two are typical examples of paths which are not smooth. See Figure 6.1.

Example 6.2.8. Let $\gamma_k : [a_k, b_k] \to \mathbb{C}$, $k = 1,2$, be paths such that $\gamma_1(b_1) = \gamma_2(a_2)$, that is, the terminal point of γ_1 is the initial point of γ_2. We define a new path $\gamma: [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$ by setting

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [a_1, b_1] \\ \gamma_2(a_2 + t - b_1) & t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

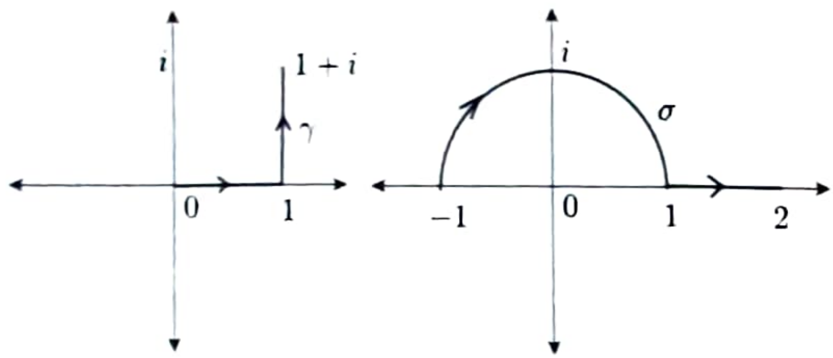

Figure 6.1: Piecewise smooth paths

There is no standard notation for this path and we shall denote it either by $\gamma_1 + \gamma_2$ or by $\gamma_2 \star \gamma_1$. We call γ the *juxtaposition* of the paths γ_1 and γ_2.

Warning. The notation for the juxtaposition should not be confused with another natural meaning for $\gamma_1 + \gamma_2$, namely, $\gamma_1 + \gamma_2$ is a function defined by $(\gamma_1 + \gamma_2)(t) = \gamma_1(t) + \gamma_2(t)$. The context should make it clear what is meant.

Definition 6.2.9. Let γ be smooth path in U and $f\colon [\gamma] \to \mathbb{C}$ be continuous. We define the *integral* of f *over* γ by setting

$$\boxed{\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.}$$

If γ is just a path, using the above notation, we set

$$\boxed{\int_\gamma f(z)\, dz := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\gamma'(t)\, dt.}$$

Note that the integrals on the RHS make sense, since γ' exists and is continuous on $[t_j, t_{j+1}]$. The complex number $\int_\gamma f$ is called the integral of f over the path γ.

Note also that if $\gamma(t) = t$, $t \in [a, b]$, then $\int_\gamma f(z)dz = \int_a^b f(t)dt$.

Example 6.2.10. We compute $\int_\gamma \cos z\, dz$ where γ is the line segment from $-\pi/2 + i$ to $\pi + i$. We have

$$\gamma(t) := i - (\pi/2) + t\,(\pi + i - [(-\pi/2) + i]) = t\frac{3\pi}{2} - \frac{\pi}{2} + i.$$

Also, $\cos(x+iy) = \cos x \cosh(y) - i \sin x \sinh y$. Hence,

$$\begin{aligned}\int_\gamma \cos z\, dz &= \int_0^1 \left[\cos\left(\frac{3\pi t}{2} - \frac{\pi}{2}\right)\cosh(1) - i \sin\left(\frac{3\pi t}{2} - \frac{\pi}{2}\right)\sinh(1)\right] \frac{3\pi}{2}\, dt \\ &= \cosh(1)\sin\left(\frac{3\pi t}{2} - \frac{\pi}{2}\right) + i\sinh(1)\cos\left(\frac{3\pi t}{2} - \frac{\pi}{2}\right)\Big|_0^1 \\ &= \cosh(1) - i\sinh(1).\end{aligned}$$

Example 6.2.11. Let $\gamma, \sigma, \tau \colon [0, 2\pi] \to \mathbb{C}^*$ be given by $\gamma(t) = e^{it}$, $\sigma(t) = e^{2it}$ and $\tau(t) = e^{-it}$. Note that $[\gamma] = [\sigma] = [\tau]$. Let $f(z) = z^{-1}$ for $z \in \mathbb{C}^*$. We have $\int_\gamma f = 2\pi i$, $\int_\sigma f = 4\pi i$ and $\int_\tau f = -2\pi i$.

Remark 6.2.12. The last example brings out a very important fact about path integrals, namely, the path integral depends on the path and not on the trace. See also the next exercise, especially the last sentence.

Ex. 6.2.13. Compute the following path-integrals $\int_\gamma f(z)\, dz$:
(1) $f(z) := |z|^2$ and γ is the line segment from 2 to $3+i$. Ans. $\frac{20(1+i)}{3}$.
(2) $f(z) := \mathrm{Re}\,(z)$ and γ is the line segment from 1 to $-i$. Ans. $\frac{-(1+i)}{2}$.
(3) $f(z) := \bar{z}$ and γ is the semicircle from 1 to -1 passing through i. Ans. πi.
(4) $f(z) := \bar{z}$ and γ is the semicircle from 1 to -1 passing through $-i$. Ans. $-\pi i$.
(5) $f(z) := z^2$ and γ is as in (3).
(6) $f(z) := z^2$ and γ as in (4).
(7) $f(z) := \exp(z)$ and γ is the line segment $[-1, 1]$.
(8) $f(z) := \log_\pi(z)$ and γ is the semicircle connecting $-i$ to i in the right half-plane $\mathrm{Im}\,(z) \geq 0$.

Compare the answers of (3) and (4) and similarly those of (5) and (6).

Ex. 6.2.14. Let $\gamma_k(t) := e^{ikt}$ for $k \in \mathbf{Z}$ and $0 \leq t \leq 2\pi$. Let $f_n(z) := z^n$ for $n \in \mathbf{Z}$ and $z \in \mathbb{C}^*$. Compute $\int_{\gamma_k} f_n$.

Ex. 6.2.15. Let γ be a path in $\mathbb{C}$. Show that the map $f \mapsto \int_\gamma f$ from $C([\gamma], \mathbb{C})$ (the complex vector space of continuous functions from $[\gamma]$ to $\mathbb{C}$) into the complex numbers is a complex linear map.

Ex. 6.2.16. Is it true that $\mathrm{Re}\,(\int_\gamma f\, dz) = \int_\gamma \mathrm{Re}\,(f)\, dz$?

Theorem 6.2.17. *Let $f \colon U \to \mathbb{C}$ be continuous. Assume that there exists an $F \colon U \to \mathbb{C}$ such that $F' = f$ on U. Let $\gamma \colon [a, b] \to U$ be any path. Then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, then $\int_\gamma f(z) = 0$.

Proof. This is a straightforward verification and the reader is encouraged to do it on his own.

Assume that γ is smooth. Using the chain rule and the fundamental theorem of calculus, we get

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F\circ\gamma)'(t)\,dt \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

To treat the general case of a piecewise smooth path γ, work with a sum of the form $\sum_{j=1}^n \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)\,dt$. □

Definition 6.2.18. Let $f\colon U\to\mathbb{C}$ be continuous. Assume that there exists an $F\colon U\to\mathbb{C}$ such that $F' = f$ on U. Then, F is called a *primitive* of f in U. Note that the difference of any two primitives is locally constant. (See Exercise 6.2.20.)

Corollary 6.2.19. *If $f\colon U\to\mathbb{C}$ is continuous and has a primitive in U, then $\int_\gamma f = 0$ for any closed path γ in U.* □

Ex. 6.2.20. If F and G are primitives of f on U, then how that $F-G$ is locally constant in U.

Ex. 6.2.21. Let $f(z) := \sum_n a_n z^n$ be a power series function with disk of convergence $B(0,R)$. Let γ be any closed path in $B(0,R)$. Show that $\int_\gamma f = 0$. *Hint:* Ex. 5.5.3 may be of help.

Ex. 6.2.22. Let U_1, U_2 be open sets in $\mathbb{C}$. Assume that every holomorphic function on U_j has a primitive in U_j, $j = 1,2$. Assume that $U_1\cap U_2$ is connected. Show that any holomorphic function on $U := U_1\cup U_2$ has a primitive.

Ex. 6.2.23. Let γ be any closed path lying entirely in $\mathbb{C}\setminus(-\infty,0]$. Then $\int_\gamma \frac{dz}{z} = 0$.

Ex. 6.2.24. Assume that $f\in H(U)$, $f'\in C(U)$ and $f(U)\subset\mathbb{C}\setminus L_0$. Show that $\int_\gamma (f'/f) = 0$ for any closed path γ in U. (The requirement that f' be continuous is superfluous, as will be seen later.)

Example 6.2.25. Let us compute an important path integral which will be used often in the sequel. Consider the path integral $\int_\gamma (z-a)^n dz$, where $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$.

$$\begin{aligned}\int_\gamma (z-a)^n dz &= \int_0^{2\pi} (re^{it})^n ire^{it} dt \\ &= r^{n+1} \int_0^{2\pi} ie^{i(n+1)t} dt.\end{aligned}$$

But $\frac{1}{n+1} e^{i(n+1)t}$ is a primitive of $ie^{i(n+1)t}$ if $n \neq -1$. Hence we get

$$\boxed{\int_\gamma (z-a)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1, \end{cases} \qquad \text{where } \gamma(t) = a + re^{it},\ 0 \le t \le 2\pi.}$$

Ex. 6.2.26. Let $z_0, z_1 \in U$. Assume that γ is a path in U from z_0 to z_1. Then

$$\int_\gamma z^n dz = \frac{1}{n+1}(z_1^{n+1} - z_0^{n+1}), \qquad n \in \mathbb{Z}_+.$$

Ex. 6.2.27. In Ex. 6.2.13, how many can be easily done with the help of Theorem 6.2.17? How about Example 6.2.10?

Ex. 6.2.28. Show that there is no continuous logarithm on $\mathbb{C}^*$ using results of this section. (Compare this with Ex. 4.4.6.)

Ex. 6.2.29. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. For $n \in \mathbb{N}$, show that

$$\frac{1}{2\pi} \int_\gamma (2\cos\theta)^{2n}\, d\theta = \frac{(2n)!}{n!n!}.$$

Hint: Evaluate $\int_\gamma (z + \frac{1}{z})^{2n} \frac{dz}{z}$ using binomial expansion. Among the terms of the form $\binom{2n}{k} z^{2n-k} z^{-k}$, which is the term that will contribute to the integral? Look at Example 6.2.25.

Definition 6.2.30. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Then $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ defined by $\tilde{\gamma}(t) := \gamma(a+b-t)$ (Why does this make sense?) is said to be *opposite* to γ and called the opposite or reverse path and it goes from $\gamma(b)$ to $\gamma(a)$.

Example 6.2.31. If $\gamma\colon [0,1] \to \mathbb{C}$ is $\gamma(t) = z + t(w-z)$ then $\tilde{\gamma}(t) = \gamma(1-t)$ is the line segment from w to z.

Example 6.2.32. Let $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$. Then γ is the circular path centred at a from $a+r$ to $a+r$ in the counter-clockwise direction. Its opposite, given by $\tilde{\gamma}(t) = a + re^{-it}$, is the circle traversed in the opposite direction, namely, the clockwise direction.

Proposition 6.2.33. *Let γ be a path and $f\colon [\gamma] \to \mathbb{C}$ be continuous. Then $\int_{\tilde{\gamma}} f\,dz = -\int_\gamma f\,dz$.*

Proof. The strategy is to unwind the definition of the path integral $\int_{\tilde{\gamma}} f\,dz$ and apply Proposition 6.1.9 to the function

$$\varphi(t) := f(\gamma(t))\gamma'(t), \qquad t \in [a,b]$$

with $h\colon [a,b] \to [a,b]$ given by $h(t) = a+b-t$. The reader is encouraged to work out the details on his own.

Details: Note that $\tilde{\gamma}$ is a composite of $t \mapsto a+b-t$ followed by γ. Hence $\tilde{\gamma}'(t) = -\gamma'(a+b-t)$, by the chain rule. We have

$$\begin{aligned}
\int_{\tilde{\gamma}} f\,dz &= \int_a^b (f\circ\tilde{\gamma})(t)\tilde{\gamma}'(t)\,dt \\
&= \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))\,dt \\
&= \int_b^a f(\gamma(s))\gamma'(s)\,ds \\
&= -\int_a^b f(\gamma(s))\gamma'(s)\,ds.
\end{aligned}$$

We have effected the change of variable $s := a+b-t$ in the last but one equation.

□

Definition 6.2.34. Let $\gamma\colon [a,b] \to \mathbb{C}$ and $\sigma\colon [c,d] \to \mathbb{C}$ be paths. We say that σ is a *reparametrization* of γ if there exists a bijective continuously differentiable function $h\colon [c,d] \to [a,b]$ such that $h' > 0$ and $\sigma(s) = \gamma(h(s))$ for $s \in [c,d]$.

Theorem 6.2.35. *Let σ and γ be as above. If $f\colon [\gamma] = [\sigma] \to \mathbb{C}$ is continuous, then $\int_\sigma f = \int_\gamma f$.*

Proof. The hint is to apply Proposition 6.1.9 to the function $\varphi(t) := f(\gamma(t))\gamma'(t)$. The reader should be able to write down the details by now. □

Proposition 6.2.36. *Let $f\colon U \to \mathbb{C}$ be continuous with a primitive F in U. Let γ_j, $j = 1,2$, be paths in U both having the same initial (resp. terminal) points z and w respectively. Then $\int_{\gamma_1} f = F(w) - F(z) = \int_{\gamma_2} f$. Thus, in this case $\int_\sigma f$ depends only on the end points of σ for any path σ in U.*

Proof. This is a direct application of Theorem 6.2.17.

Alternatively, we may proceed as follows. Let $\gamma = \tilde{\gamma}_2 * \gamma_1$. Since γ is closed, by Corollary 6.2.19, $\int_\gamma f = 0$. We now note that $\gamma|_{[b_1, b_1+b_2-a_2]}$ is a reparametrization of $\tilde{\gamma}_2$, via the map $h\colon [a_2, b_2] \to [b_1 + b_2 - a_2]$ defined by $h(t) = b_1 - a_2 + t$. Hence $0 = \int_\gamma f = \int_{\gamma_1} f - \int_{\gamma_2} f$. □

Ex. 6.2.37. Let γ_j be paths in U so that $\gamma_2 * \gamma_1$ is defined. (See Example 6.2.10.) Let $f \in C(U)$. Show that $\int_{\gamma_2 * \gamma_1} f = \int_{\gamma_1} f + \int_{\gamma_2} f$. *Hint:* Go through the second proof of the last proposition.

6.3 ML-Inequality and its Applications

Definition 6.3.1. If $\gamma\colon [a,b] \to \mathbb{C}$ is a smooth path, then the *length* $L(\gamma)$ of the path γ is defined by $L(\gamma) := \int_a^b |\gamma'(t)|\,dt$.

If γ is just a path, we then define, using an obvious notation

$$L(\gamma) := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\gamma'(t)|\,dt.$$

Remark 6.3.2. The length $L(\gamma)$ should not be confused with the length of its trace $[\gamma]$. See Example 6.3.4.

Example 6.3.3. If γ is the line segment $[z,w]$, then $L(\gamma) = |w-z|$.

Example 6.3.4. If γ is the circular path $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, then $L(\gamma) = 2\pi r$.

If σ is the path $\sigma(t) = a + re^{it}$, $0 \le t \le 4\pi$, then $L(\sigma) = 4\pi r$.

If τ is the path $\tau(t) = a + re^{-it}$, $0 \le t \le 4\pi$, then $L(\tau) = 4\pi r$.

Ex. 6.3.5. Show that $L(\gamma)$ remains invariant under reparametrization, i.e. if σ is a reparametrization of γ, then $L(\gamma) = L(\sigma)$.

Remark 6.3.6. We now want to derive an upper bound for $\left|\int_\gamma f\right|$. Most often, for the students, the obvious guess for such an upper bound is $\int_\gamma |f|$. It is wrong as the following example shows. Consider γ to be the unit circle with the standard parametrization and $f(z) := 1/z$ for $z \in \mathbb{C}^*$. We have already shown that $\int_\gamma f = 2\pi i$ (see Example 6.2.25) that its modulus is 2π. However $|f| \equiv 1$ on the unit circle and it has a primitive $z \mapsto z$. Hence $\int_\gamma |f| = 0$!

Proposition 6.3.7 (ML Inequality). *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Let $f\colon [\gamma] \to \mathbb{C}$ be continuous with $|f(\gamma(t))| \le M$ for all $t \in [a,b]$. We have*

$$\boxed{\left|\int_\gamma f\,dz\right| \le ML(\gamma).} \tag{6.5}$$

Proof. The idea is to unwind the definition of the path integral and use the inequality $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$. For simplicity, we shall assume that the path is smooth.

Observe the following:

$$\left|\int_\gamma f\,dz\right| := \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\,dt$$
$$\le M\int_a^b |\gamma'(t)|\,dt$$
$$= ML(\gamma).$$

At the cost of repetition, we remind the reader that $L(\gamma)$ is the length of γ and not of its trace. □

This proposition is very frequently used in the sequel.

Ex. 6.3.8. Establish the following:
(1) $\left|\int_\gamma \frac{dz}{z^2+4}\right| \le \frac{\pi R}{(R^2-4)}$ where $\gamma(t) = Re^{it}$ for $0 \le t \le \pi$ and $R > 2$.
(2) $\left|\int_\gamma e^{-z}\,dz\right| \le 2$, where γ is the line segment from $-i$ to i.
(3) $\left|\int_\gamma \frac{dz}{z^4}\right| \le 4\sqrt{2}$ where $\gamma := [i, 1]$.
(4) $\left|\int_\gamma \frac{e^z}{z}\,dz\right| \le 2\pi e$ where γ is the unit circle with the standard parametrization.

Ex. 6.3.9. Let $\gamma(t) := e^{it}$ for $t \in [0, 2\pi]$ and $f(z) := 1/(4+3z)$. Show that $\left|\int_\gamma \frac{dz}{4+3z}\right| \le 2\pi$. Improve this estimate to $\left|\int_\gamma \frac{dz}{4+3z}\right| \le \frac{6}{5}\pi$. *Hint:* Estimate on the semicircles in the left and right half planes separately.

Ex. 6.3.10. (1) Let γ be as in the last exercise. Show that $\left|\int_\gamma \frac{dz}{12+5z}\right| \le \frac{20}{91}\pi$.
(2) Let $\gamma(t) := 2e^{it}$ $(0 \le t \le 2\pi)$. Show that $\left|\int_\gamma (2+3x)\,dz\right| \le 20\pi$.

Ex. 6.3.11. Let $f\colon B(z_0, r) \to \mathbb{C}$ be continuous. Let $\gamma_\varepsilon(t) := z_0 + \varepsilon e^{it}$ for $0 \le t \le 2\pi$. Show that

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\,dz = f(z_0).$$

Hint: Start with $\int_0^{2\pi}[f(z_0 + re^{it}) - f(z_0)]\,dt$.

Ex. 6.3.12. Let $f\colon \mathbb{C} \to \mathbb{C}$ be continuous and bounded. Let $\gamma_R(t) := Re^{it}$ for $0 \le t \le 2\pi$. Show that $\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^2}\,dz = 0$ for each z_0.

Ex. 6.3.13. Let γ be any path in $\mathbb{C}$. Show that the map $z \mapsto \int_\gamma \frac{dw}{w-z}$ is continuous on $\mathbb{C} \setminus [\gamma]$.

Ex. 6.3.14. Show that $\lim_{R\to\infty} \int_{S(0,R)} \frac{z\,dz}{z^5+9} = 0$.

Ex. 6.3.15. Show that $\lim_{R\to\infty} \int_{[-R,-R+i]} \frac{z^3\exp(z)}{z+3} = 0$.

We now give some theoretical applications of the ML inequality (6.5).

Proposition 6.3.16. *Let $f\colon U \to \mathbb{C}$ be continuous. Then f has a primitive in U iff $\int_\gamma f = 0$ for any closed path γ in U.*

Proof. One way is already done in Corollary 6.2.19. To see the other way, assume that U is connected and fix a point $z_0 \in U$. We define $F(z) := \int_\gamma f$ where γ is any path from z_0 to z. We claim that this is independent of the choice of the path and hence well-defined. If σ is another such path, then $\tau := \tilde{\sigma} * \gamma$ is a closed path and hence $\int_\tau f = 0$. Arguing as in Proposition 6.2.36, we see that $\int_\gamma f = \int_\sigma f$.

Let $\varepsilon > 0$ be given. Using the continuity of f at z_0, we choose a corresponding $\delta > 0$ such that $B[z,\delta] \subset U$. Now if $|h| < \delta$, then $F(z+h) - F(z) = \int_{[z,z+h]} f$. We therefore have

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{[z,z+h]} [f(w)-f(z)]\,dw\right| \le \varepsilon,$$

by the continuity of f at z for all w with $|w-z| < \delta$.

To treat the general case, we need a notion and a fact from topology: The maximal connected subsets, known as connected components, of an open set $U \subset \mathbb{C}$ are open. Hence the connected components of $U \subset \mathbb{C}$ are path connected. (See Proposition 5.4.25 and Remark 12.2.2.) If $\gamma\colon [a,b] \to U$ is any path, then its trace $[\gamma]$ is connected and lies in a connected component of U. Now we may proceed as earlier. □

Proposition 6.3.17. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Let $f_n\colon [\gamma] \to \mathbb{C}$ be continuous ($n \in \mathbb{N}$) and that f_n converge uniformly on $[\gamma]$ to a function $f\colon [\gamma] \to \mathbb{C}$. Then $\int_\gamma f_n \to \int_\gamma f$.*

Proof. First of all, note that $\int_\gamma f$ is defined, since $f \in C([\gamma])$. (Why?) Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[\gamma]$, there exists $N \in \mathbb{N}$ such that

$$|f_n(\gamma(t)) - f(\gamma(t))| < \varepsilon, \text{ for all } n \ge N \text{ and } t \in [a,b].$$

Now the result follows from an easy application of the ML inequality:

$$\begin{aligned}\left|\int_\gamma [f(z)-f_n(z)]\,dz\right| &= \left|\int_a^b [f(\gamma(t)) - f_n(\gamma(t))]\gamma'(t)\,dt\right| \\ &\le \int_a^b \left|[f(\gamma(t)) - f_n(\gamma(t))]\gamma'(t)\right|\,dt \\ &\le \varepsilon \int_a^b |\gamma'(t)|\,dt \\ &= L(\gamma)\varepsilon,\end{aligned}$$

if $n \ge N$. □

Corollary 6.3.18. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Let $f_n\colon [\gamma] \to \mathbb{C}$ be continuous ($n \in \mathbb{N}$) and that $\sum_n f_n$ converges uniformly on $[\gamma]$ to a function $f\colon [\gamma] \to \mathbb{C}$. Then $\sum_n \int_\gamma f_n = \int_\gamma f$.*

Proof. The hypothesis says that the sequence (s_n) of partial sums of the series $\sum_n f_n$ converges uniformly on $[\gamma]$ to f. Hence by the last result, $\int_\gamma s_n \to \int_\gamma f$. By the linearity of the path-integral, we have $\int_\gamma s_n = \sum_{k=1}^n \int_\gamma f_k$. Therefore, $\sum_{k=1}^n \int_\gamma f_k \to \int_\gamma f$. This shows that $\lim_{n\to\infty} \sum_{k=1}^n \int_\gamma f_k$ exists and is equal to $\int_\gamma f$. Hence we deduce that $\sum_{k=1}^\infty \int_\gamma f_k$ makes sense and we have the result $\sum_{n=1}^\infty \int_\gamma f_n = \int_\gamma f$. ☐

Ex. 6.3.19. Let $f(z) := \sum_{n=0}^\infty a_n(z-a)^n$ be a power series function with radius of convergence R. Show that f has a primitive in $B(0,R)$ and hence conclude $\int_\gamma f = 0$ for any closed path in $B(a,R)$. (Compare with Ex. 6.2.21.)

Remark 6.3.20. Go through your solution of the last exercise. Recall that the power series with radius of convergence $R > 0$ converges uniformly only on subdisks of the form $B(a,r)$ with $0 < r < R$. Since $[\gamma]$ is a compact subset of $B(a,R)$, there exists $0 < r < R$ such that $[\gamma] \subset B(0,r)$. Once this is observed then the rest of the argument is easy.

6.4 A Preview of Cauchy Theory

In spite of its simplicity, Proposition 6.3.17 and its corollary yield a lot of dividends which, when properly understood, constitute fundamental results of the Cauchy Theory. This section contains these results. These may be regarded as local versions of Cauchy theory which is the mainstay of Chapter 7. This section may be skipped without loss of continuity.

The major results of the Cauchy theory are (global) manifestations of the corresponding results about power series. This section gives a glimpse of such results whose analogues will be proved in the next chapter.

Proposition 6.4.1. *Let $f(z) := \sum_{n=0}^\infty a_n(z-z_0)^n$ on $B(z_0,R)$. Then for any $0 < r < R$ and $\gamma(t) := z_0 + re^{it}$, $0 \le t \le 2\pi$,*

$$\frac{1}{n!} f^{(n)}(z_0) = a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad n \in \mathbb{Z}_+. \tag{6.6}$$

Proof. The integral on the right side of the above equation is

$$\sum_{k=0}^\infty a_k \int_\gamma (z-z_0)^{k-n-1}\, dz = 2\pi i a_n,$$

by Example 6.2.25. Since $n!a_n = f^{(n)}(z_0)$ from (5.6), the result follows. ☐

Corollary 6.4.2 (Cauchy's Inequalities). *Keep the notation of the above proposition. Let*

$$M(r) := \sup\{|f(z)| : z \in [\gamma]\}.$$

Then

$$n!\,|a_n| = \left|f^{(n)}(z_0)\right| \leq n!\frac{M(r)}{r^n}, \qquad (0 < r < R) \tag{6.7}$$

Proof. (6.7) is an immediate consequence of (6.6). □

Remark 6.4.3. This corollary is interesting, since its statement does not involve any integration, even though to prove it we resorted to integration. The inequalities in (6.7) are called the Cauchy's inequalities and were derived in Ex. 6.1.14.

Ex. 6.4.4. Let $f\colon [a,b] \to \mathbb{C}$ be continuous. Then

$$F(z) := \int_a^b f(t)\exp(-tz)\,dt$$

is holomorphic on $\mathbb{C}$.

The following generalization of Example 6.2.25 will play a crucial role in the latter development. The reader should note the clever trick in the proof.

Proposition 6.4.5. *Let $\gamma(t) := z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. Then*

$$\int_\gamma \frac{dz}{z-a} = \begin{cases} 2\pi i & a \in B(z_0,r) \\ 0 & a \notin B[z_0,r]. \end{cases} \tag{6.8}$$

Proof. Note that this is not Example 6.2.25. The trick here is to write

$$\frac{1}{z-a} = \frac{1}{(z-z_0)-(a-z_0)} = \frac{1}{(z-z_0)[1-\frac{a-z_0}{z-z_0}]} = \frac{1}{z-z_0}\frac{1}{1-w},$$

where $w := \frac{a-z_0}{z-z_0}$. If $a \in B(z_0,r)$, then $|w| < 1$ and we can substitute the geometric series expansion for $\frac{1}{1-w}$ in the integral. Since the convergence is uniform on $[\gamma]$, by Corollary 6.3.18, we have

$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,dz &= \int_\gamma \frac{1}{z-z_0}\sum_{n=0}^\infty (a-z_0)^n(z-z_0)^{-n} \\ &= \sum_{n=0}^\infty (a-z_0)^n \int_\gamma (z-z_0)^{-n-1} \\ &= 2\pi i,\end{aligned}$$

where we have used Example 6.2.25 to each of the terms of the series in the last but one line.

If $a \notin B[z_0, r]$ we then write $\frac{1}{z-a} = -\frac{1}{a-z_0}\frac{1}{1-w}$, where $w = \frac{z-z_0}{a-z_0}$. Proceeding as above, we get

$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,dz &= \int_\gamma \frac{1}{a-z_0}\sum_{n=0}^{\infty}(a-z_0)^{-n}(z-z_0)^n \\ &= \sum_{n=0}^{\infty}(a-z_0)^{-n-1}\int_\gamma (z-z_0)^n \\ &= 0,\end{aligned}$$

again by Example 6.2.25. □

Ex. **6.4.6.** Let p be a polynomial in z. Let γ be any circle $|z - z_0| = r$ with standard parametrization. Then we have, for $a \notin [\gamma]$, the Cauchy integral formula:

$$\frac{1}{2\pi i}\int_\gamma \frac{p(z)}{z-a}\,dz = \begin{cases} p(a) & \text{if } |a-z_0| < r \\ 0 & \text{if } |a-z_0| > r. \end{cases}$$

Hint: $p(z) - p(a)$ is divisible by $z - a$.

Theorem 6.4.7. *Let γ be any path and $f\colon [\gamma] \to \mathbb{C}$ be continuous. Then $F(z) := \int_\gamma \frac{f(w)}{w-z}\,dw$ is differentiable on $\mathbb{C} \setminus [\gamma]$. In fact, F is analytic in $\mathbb{C} \setminus [\gamma]$.*

Proof. The trick employed in the proof of the last proposition is implemented again.

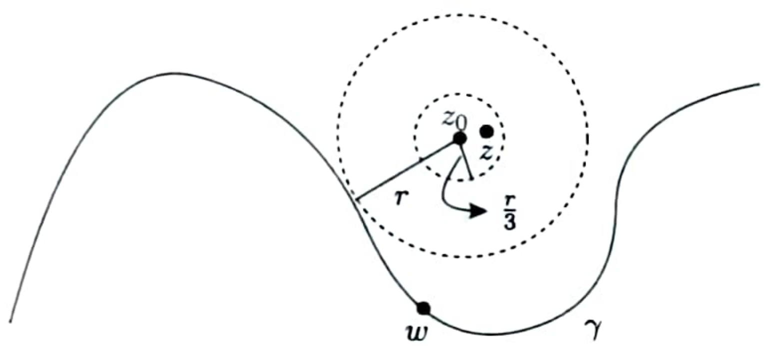

Figure 6.2: Illustration for Theorem 6.4.7

Note that $\mathbb{C}\setminus[\gamma]$ is open. (Why? For, $[\gamma]$ is compact and hence closed. Its complement $\mathbb{C} \setminus [\gamma]$ is therefore open.) If $z_0 \notin [\gamma]$, choose $r > 0$ such that $B(z_0, r) \cap [\gamma] = \emptyset$. If

$z \in B(z_0, r/3)$, then $\left|\frac{z-z_0}{w-z_0}\right| < 1/2$ for $w \in [\gamma]$. See Figure 6.2. Thus the series

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

of functions in w for fixed z and z_0 converges uniformly on $[\gamma]$. Therefore,

$$F(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where $a_n := \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$. □

Remark 6.4.8. Keep the notation of the last theorem. Note that in view of the relation between the n-th coefficients in a power series and the n-th derivative of the function defined by the power series, we arrive at Cauchy integral formula for the derivatives:

$$\boxed{F^{(n)}(z_0) = n!\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw.} \tag{6.9}$$

Remark 6.4.9. The last result shows us a way to generate a lot of analytic functions. The core of Cauchy theory which says that a function is holomorphic on an open set iff it is analytic is an immediate consequence of this once we establish Cauchy integral formula. See Theorem 8.1.3.

Compare the following result with Theorem 6.4.7.

Proposition 6.4.10. *Let $\gamma := a + re^{it}$ for $0 \le t \le 2\pi$. Let $g\colon [\gamma] \to \mathbb{C}$ be continuous and $n \in \mathbb{Z}_+$. Let*

$$\varphi(z) := \int_\gamma \frac{g(w)}{(w-z)^n}\,dw, \qquad z \in B(a,r).$$

Then φ is holomorphic in $B(a,r)$ and we have

$$\varphi'(z) = n\int_\gamma \frac{g(w)}{(w-z)^{n+1}}\,dw, \qquad z \in B(a,r). \tag{6.10}$$

Proof. Let $z_0 \in B(a,r)$. Choose $\rho < r$ such that $z_0 \in B(a,\rho)$. Let $z_1 \in B(a,\rho)$. Note the following estimates where $|w-a| = r$:

(i) $r = |w-a| \le |w-z_0| + |z_0-a| \le |w-z_0| + \rho$ so that $|w-z_0| \ge r-\rho$. Similarly, $|w-z_1| \ge r-\rho$.

(ii) $|w-z_0| \le |w-a| + |a-z_0| \le r+\rho \le 2r$. Similarly, $|w-z_1| \le 2r$.

We have

$$\begin{aligned}
&\left|\frac{1}{(w-z_1)^n} - \frac{1}{(w-z_0)^n}\right| \\
&\quad = \left|\frac{z_1 - z_0}{(w-z_0)^n (w-z_1)^n}\right| \\
&\qquad \times \left|(w-z_0)^{n-1} + (w-z_0)^{n-2}(w-z_1) + \cdots + (w-z_1)^{n-1}\right| \qquad (6.11)\\
&\quad \leq \frac{|z_1 - z_0|}{(r-\rho)^{2n}} n(2r)^{n-1}.
\end{aligned}$$

Choose M such that $|g(w)| \leq M$ for $w \in [\gamma]$. Then

$$\begin{aligned}
|\varphi(z_1) - \varphi(z_0)| &= \left|\int_\gamma \left[\frac{g(w)}{(w-z_1)^n} - \frac{g(w)}{(w-z_0)^n}\right] dw\right| \\
&\leq 2\pi r \frac{Mn(2r)^{n-1}}{(r-\rho)^{2n}} |z_1 - z_0|.
\end{aligned}$$

This proves the continuity of φ at z_0.

Using the algebraic manipulation that established (6.11), we now have

$$\frac{\varphi(z) - \varphi(z_0)}{z - z_0} = \sum_{j=1}^n \int_\gamma \frac{g(w)}{(w-z_0)^{n-j+1}(w-z)^j}\, dw. \tag{6.12}$$

Let $g_j(w) := \frac{g(w)}{(w-z_0)^{n-j+1}}$, $1 \leq j \leq n$. Note that g_j is continuous on $[\gamma]$. Then (6.12) can be rewritten as

$$\frac{\varphi(z) - \varphi(z_0)}{z - z_0} = \sum_{j=1}^n \int_\gamma \frac{g_j(w)}{(w-z)^j}\, dw = \sum_{j=1}^n \varphi_j(z), \text{ say.} \tag{6.13}$$

Arguing as in the discussion about the continuity of φ, we see that each of the summand integrals φ_j is continuous at z_0. Letting $z \to z_0$, the result (6.10) follows. □

Ex. 6.4.11. Go through the statements of Theorem 6.4.7 and Proposition 6.4.10. Which do you think is stronger? Which of the last two results do you prefer? Why? (This is a subjective question!)

Ex. 6.4.12. Let γ be any closed path in $\mathbb{C}$ and define $n_\gamma(z)\colon \mathbb{C} \setminus [\gamma] \to \mathbb{Z}$ by

$$n_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

Then n_γ is continuous by Proposition 6.4.10. Show that $n_\gamma' = 0$ on $\mathbb{C} \setminus [\gamma]$ and hence deduce that it is locally constant. What does this say in view of Ex. 6.1.7?

Chapter 7

Cauchy Theory

7.1 Cauchy's Theorem for Star-shaped Domains

The most important results of the Cauchy theory are (1) the existence of local primitives for holomorphic functions on an open set in $\mathbb{C}$ and (2) the fact that the class of holomorphic functions and that of analytic functions on an open set coincide. We attend to the first result in this chapter. The latter result is proved in the next. Note that the second result means that any holomorphic function on an open set admits a power series expansion around each point of its domain, hence is infinitely differentiable on the domain and it has local primitives.

We have already seen (in Corollary 6.2.19) that if $f: U \to \mathbb{C}$ has a primitive F in U, then $\int_\gamma f = 0$ for any closed path γ in U. Cauchy's theorem says that if f is holomorphic on U, then $\int_\gamma f = 0$ *under certain restrictions either on* U *or on* γ. (In this chapter, the restriction is on U.) As a by product, one shows that any $f \in H(U)$ has primitives locally, i.e. given $z \in U$, there exist $r > 0$ and $F \in H(B(z,r))$ with $F' = f$ on $B(z,r)$.

In real analysis, we can find a primitive for *any* continuous function $f: [a,b] \to \mathbb{R}$ by setting $F(x) := \int_a^x f(t)\,dt$. This motivates the following approach to find primitives of continuous $f: U \subset \mathbb{C} \to \mathbb{C}$: $F(z) := \int_{[z_0,z]} f$ if there exists $z_0 \in U$ such that $[z_0, z] \subset U$ for all $z \in U$. (This is possible if U is star-shaped at z_0.)

If F as defined above is a primitive of f on U then the following must be true: For w near z, $\int_{[w,z_0]} f + \int_{[z_0,z]} f = \int_{[w,z]} f$. See Figure 7.1. Or equivalently,

$$\int_{[z_0,z]} f + \int_{[z,w]} f + \int_{[w,z_0]} f = 0. \tag{7.1}$$

That is, loosely speaking, the integral of f over the boundary of a triangle must be zero.

Definition 7.1.1. If a, b, c are three (not necessarily non-collinear) points in $\mathbb{C}$, the triangle $\Delta(a,b,c) \equiv [a,b,c]$ is the set of points on the line segments of the form $[a,w]$, where

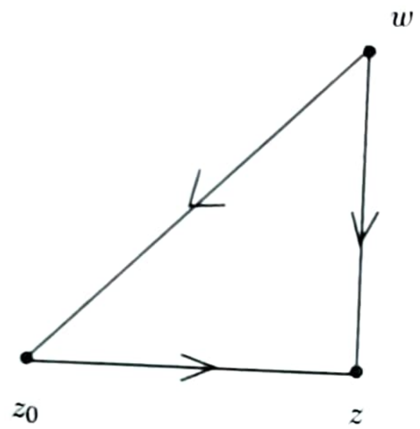

Figure 7.1: Illustration for (7.1)

$w \in [b, c]$. See Figure 7.2. One easily sees that

$$\Delta(a, b, c) = \{w \in \mathbb{C} : w = t_1 a + t_2 b + t_3 c, 0 \leq t_i \leq 1, t_1 + t_2 + t_3 = 1\}. \tag{7.2}$$

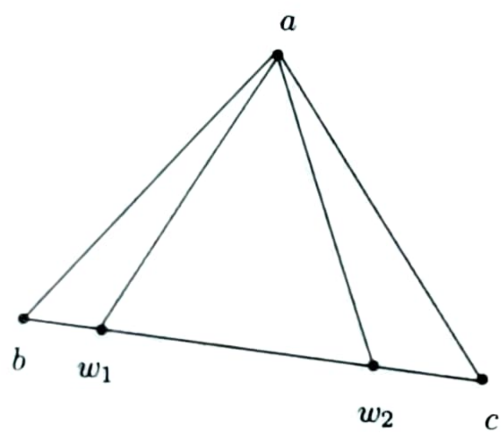

Figure 7.2: Illustration for (7.2)

One also shows that $\Delta(a, b, c)$ is the smallest convex set containing a, b, c.

Remark 7.1.2. The description of the triangle $\Delta(a, b, c)$ in (7.2) shows that it is independent of the choice of a in the definition. For, if we choose another vertex, say b, then the triangle $\Delta(a, b, c)$ would be the set of points on the line segments of the form $[b, w]$, where $w \in [a, c]$. The description in (7.2) is symmetric in a, b, c and hence we deduce that the second method will also lead to the same subset of $\mathbb{C}$.

The points a, b, c are called the vertices of $[a, b, c]$ and the line segments $[a, b]$ etc. are its sides. The side $[b, c]$ is said to be opposite to the vertex a. If we let $\Delta := \Delta(a, b, c)$, the (juxtaposed) path $[a, b] + [b, c] + [c, a]$ is called the boundary of Δ and is denoted by $\partial\Delta$ or $T(a, b, c)$. Note that while in the definition of $\Delta(a, b, c)$ the order of the vertices is not important, it is important for the definition of its boundary T (of course, up to a cyclic permutation of a, b, c.)

Theorem 7.1.3. *Let U be open and star-shaped at $a \in U$. Let $f \colon U \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for the boundary T of any triangle $\Delta \subset U$. Then f has a primitive in U.*

Proof. This is a repetition of the proof of Proposition 6.3.16.

Let $z_0 \in U$. Choose $r > 0$ such that $B(z_0, r) \subset U$. If $z \in B(z_0, r)$, then $[z_0, z] \subset U$. Since $[z_0, z] \subset U$ and U is star-shaped at a, it follows that $[a, w] \subset U$ for any $w \in [z_0, z]$. See Figure 7.3. We define $F(z) := \int_{[a,z]} f(w)\, dw$. It follows from the hypothesis that $F(z) = F(z_0) + \int_{[z_0,z]} f$.

Reason: In view of the definition of F, we need to show that $\int_{[a,z]} f = \int_{[a,z_0]} f + \int_{[z_0,z]} f$. By Ex. 6.2.37 and by Proposition 6.2.33, we see that proving the equality is equivalent to proving $\int_{[a,z]+[z,z_0]+[z_0,a]} f = 0$. Since the path $[a, z] + [z, z_0] + [z_0, a]$ is the boundary of the triangle $\Delta(a, z, z_0) \subset U$, the integral is zero by hypothesis.

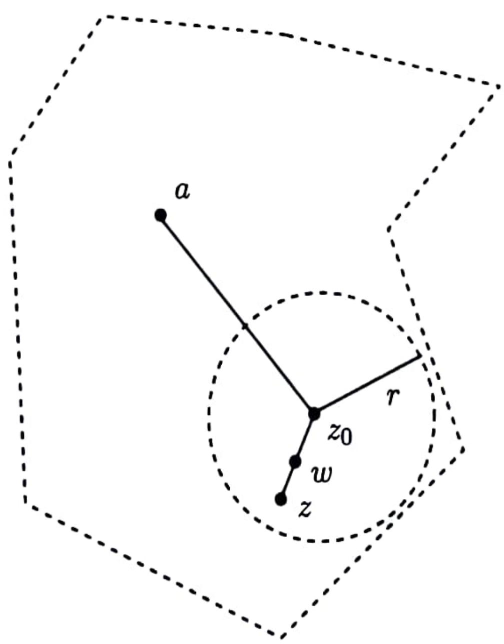

Figure 7.3: Illustration for Theorem 7.1.3

$$\begin{aligned}
F(z) - F(z_0) &= \int_{[z_0,z]} f \\
&= \int_{[z_0,z]} (f(w) - f(z_0) + f(z_0)) \\
&= \left(\int_{[z_0,z]} [f(w) - f(z_0)] \right) + (z - z_0) f(z_0).
\end{aligned}$$

Hence we find that

$$\frac{F(z) - F(z_0)}{z - z_0} = f(z_0) + \frac{1}{z - z_0} \int_{[z_0,z]} [f(w) - f(z_0)].$$

Given $\varepsilon > 0$, by continuity of f at z_0, we obtain $|f(w) - f(z_0)| < \varepsilon$ for $w \in B(z_0, \delta)$. We now show that the second term above goes to zero, by estimating it by ML-inequality.

$$\left| \frac{1}{z - z_0} \int_{[z_0,z]} [f(w) - f(z_0)] \right| < \frac{|z - z_0|}{|z - z_0|} \varepsilon = \varepsilon,$$

for all $z \in B(z_0, \delta)$. Hence F is differentiable at z_0 and $F'(z_0) = f(z_0)$. □

Ex. 7.1.4. If E is a subset of a metric space, then the diameter $\operatorname{diam}(E)$ of E is defined to be $\sup\{d(x, y) : x, y \in E\}$ if it exists. The diameter of the triangle $[a, b, c]$ is $\max\{d(a, b), d(b, c), d(a, c)\}$, that is, it is the largest side, $\max\{|b - a|, |c - a|, |c - b|\}$, and the length of its boundary $T(a, b, c)$ is $|a - b| + |b - c| + |c - a|$.

We need the concept of compact metric spaces and some of their properties. We have collected them in an appendix and refer the reader to it.

Ex. 7.1.5. The triangle $[a, b, c]$ is a compact subset of $\mathbb{C}$. *Hint:* It is a continuous image of $\{(t_1, t_2, t_3) : 0 \le t_j \le 1 \text{ with } t_1 + t_2 + t_3 = 1\} \subset \mathbb{R}^3$.

Ex. 7.1.6. Let a_1, b_1, c_1 be the mid points of sides opposite to the vertices a, b, c respectively and if we let Δ' denote the triangle $[a_1, b_1, c_1]$ and T' its boundary, then we have (1) $L(T') = L(T)/2$ and (2) the diameter of Δ', $\operatorname{diam}(\Delta') = \operatorname{diam}(\Delta)/2$. *Hint:* Draw pictures. Note that $|b_1 - c_1| = |b - c|/2$ etc.

Ex. 7.1.7 (Cantor Intersection Theorem). Let (K_n) be a sequence of nonempty compact sets in a metric space X such that $K_n \subset K_{n-1}$ and $\operatorname{diam}(K_n)$ decreases to zero. Then $\cap_n K_n$ consists of a single point.

Pick a point $z_n \in K_n$. We thus get a sequence (z_n). Note that $(z_k)_{k \ge n}$ is a sequence in K_n for $n \in \mathbb{N}$. Since K_1 is compact, the sequence (z_n) has a convergent subsequence (z_{n_r}) which converges to $z \in K_1$. Observe that the sequence $(z_{n_r})_{r \ge k}$ lies in K_k for each

$k \in N$. Since K_k is compact, it is closed and hence the limit z of the sequence $(z_{n_r})_{r \geq k}$ lies in K_k for each $k \in \mathbb{N}$. Hence $z \in \cap_n K_n$. Uniqueness is easy to see.

Theorem 7.1.8 (Cauchy-Goursat Integral Theorem). *Let U be any open subset of $\mathbb{C}$. Let $f \in H(U)$. Let $\Delta = [a, b, c] \subset U$ be a triangle with $T = \partial\Delta$. Then $\int_T f = 0$.*

> Strategy: The proof below should be reminiscent of the way the nested interval theorem is used to prove that a continuous function on a closed and bounded interval is bounded. Assuming that f is not bounded, we from a nested sequence (J_n) of closed and bounded intervals on each of which f is not bounded. If $c \in \cap J_n$, using the continuity of f at c, we see that f is bounded on an interval J around c. For $n \gg 0$,[1] $J_n \subset J$ and hence f is bounded on all such J_n's.

Proof. Let $\varepsilon > 0$ be given. We show that $\left|\int_T f\right| < C\varepsilon$ for some constant C which depends only on Δ. Since $\varepsilon > 0$ is arbitrary, it will follow that $\int_T f = 0$.

Let the notation be as in Ex. 7.1.6. Draw pictures to see what is happening. Consider the triangles $\Delta^1 := [a, c_1, b_1]$, $\Delta^2 := [c_1, b, a_1]$, $\Delta^3 := [a_1, c, b_1]$ and $\Delta^4 := [a_1, b_1, c_1]$. See Figure 7.4. Let T^j denote their boundary as specified above. Then $\int_T f = \sum_{j=1}^4 \int_{T^j} f$. (Check this.)

Reason:

$$\int_{T^1} f = \int_{[a,c_1]} f + \int_{[c_1,b_1]} f + \int_{[b_1,a]} f =: ac_1 + c_1b_1 + b_1a, \text{ say}$$

$$\int_{T^2} f = \int_{[c_1,b]} f + \int_{[b,a_1]} f + \int_{[a_1,c_1]} f =: c_1b + ba_1 + a_1c_1$$

$$\int_{T^3} f = \int_{[a_1,c]} f + \int_{[c,b_1]} f + \int_{[b_1,a_1]} f =: a_1c + cb_1 + b_1a_1$$

$$\int_{T^4} f = \int_{[a_1,b_1]} f + \int_{[b_1,c_1]} f + \int_{[c_1,a_1]} f =: a_1b_1 + b_1c_1 + c_1a_1.$$

From this, by grouping terms, we obtain

$$\begin{aligned}\sum_{j=1}^4 \int_{T^j} f &= [(ac_1 + c_1b) + (ba_1 + a_1c) + (cb_1 + b_1a)] \\ &\quad + [(c_1b_1 + b_1c_1)) + (a_1c_1 + c_1a_1) + (b_1a_1 + a_1b_1)] \\ &= \int_T f + [0 + 0 + 0].\end{aligned}$$

[1] $n \gg 0$ means that for large values of n; here it means that there exists N such that for $n \geq N$, we have $J_n \subset J$

Let $I := \left|\int_T f\right|$. Since $I \leq \sum_j \left|\int_{T^j} f\right|$, we see that there exists at least one j such that $\left|\int_{T^j} f\right| \geq 4^{-1}I$. Call this Δ^j as Δ_1. Repeat the above process for this triangle. Proceeding this way, we get a sequence of triangles (Δ_n) with the following properties:

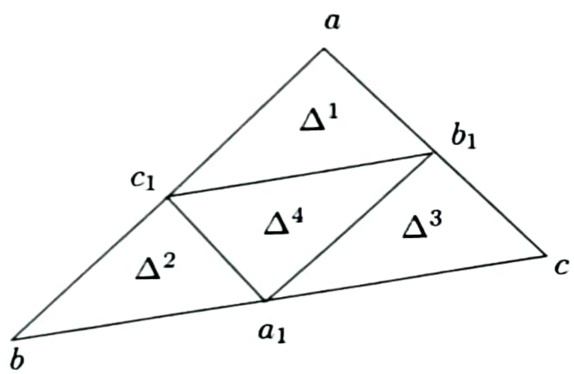

Figure 7.4: Illustration for Theorem 7.1.8

(1) The vertices of the Δ_n are the mid points of the sides of Δ_{n-1}, so that for all $n \in \mathbb{N}$,

(i) $L(T_n) = L(T_{n-1})/2$,
(ii) $\operatorname{diam}(\Delta_n) = 2^{-1}\operatorname{diam}(\Delta_{n-1})$ and
(iii) $\Delta_n \subset \Delta_{n-1}$, for $n \geq 1$, where $\Delta_0 := \Delta$.

Note that $L(T_n) = 2^{-n}L(T)$ and $\operatorname{diam}(\Delta_n) = 2^{-n}\operatorname{diam}(\Delta)$.

(2) $\left|\int_{T_n} f\right| \geq 4^{-1}\left|\int_{T_{n-1}} f\right|$ so that $\left|\int_{T_n} f\right| \geq 4^{-n}I$.

By Cantor intersection theorem (Ex. 7.1.7), $\cap_n \Delta_n \neq \emptyset$. Let $z_0 \in \cap\Delta_n$. Note that $z_0 \in U$. Since f is differentiable at z_0, for $\varepsilon > 0$ given above, there exists a $\delta > 0$ such that

$$\left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \leq \varepsilon\left|z - z_0\right| \text{ for } z \in B(z_0, \delta). \tag{7.3}$$

Since $\operatorname{diam}(\Delta_n) \to 0$, there exists n_0 such that $\Delta_n \subset B(z_0, \delta)$ for $n \geq n_0$.

Reason: For δ as above, we can choose n_0 such that $\operatorname{diam}(\Delta_n) < \delta$ for $n \geq n_0$. If $z \in \Delta_n$, then $d(z, z_0) \leq \operatorname{diam}(\Delta_n) < \delta$ and hence $z \in B(z_0, \delta)$.

Since the constant function $f(z_0)$ and the function $z \mapsto f'(z_0)(z - z_0)$ have primitives in U, their integrals over T_n are 0. Hence we have

$$\int_{T_n} f = \int_{T_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\, dz.$$

By ML inequality and (7.3), we have

$$\begin{aligned}\left|\int_{T_n} f(z)\,dz\right| &= \left|\int_{T_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,dz\right| \\ &\leq \sup\{|f(z) - f(z_0) - f'(z_0)(z - z_0)| : z \in T_n\}L(T_n) \\ &\leq \varepsilon\,|z - z_0|\,L(T_n) \\ &\leq \varepsilon \operatorname{diam}(\Delta_n)L(T_n) \\ &= \varepsilon 2^{-n}\operatorname{diam}(\Delta)2^{-n}L(T) \\ &= \varepsilon 4^{-n}\operatorname{diam}(\Delta)L(T).\end{aligned}$$

Since $\left|\int_T f\right| \leq 4^n \left|\int_{T_n} f\right|$, we deduce that $\left|\int_T f\right| \leq \operatorname{diam}(\Delta) \cdot L(T)\varepsilon$. □

Theorem 7.1.9 (Cauchy Theorem). *Let U be star-shaped and open. Let $f \in H(U)$. Then f has a primitive in U and hence $\int_\gamma f = 0$ for any closed path γ in U.*

Proof. Follows from Theorem 7.1.8 and Theorem 7.1.3. □

Our promised result now follows.

Corollary 7.1.10. *Let $U \subset \mathbb{C}$ be any open set. If $f \in H(U)$, then f has local primitives, that is, given any $z_0 \in U$, there exists $r > 0$ and an $F \in H(B(z_0, r))$ such that $B(z_0, r) \subset U$ and $F' = f$ on $B(z_0, r)$.*

Proof. Given $z \in U$, choose $r > 0$ so that $B(z, r) \subset U$ and apply the last result to $f \in H(B(z, r))$. □

7.2 Applications of Cauchy's Theorem

Before we proceed with theoretical applications, let us look at some typical applications of Cauchy's theorem.

Example 7.2.1. If $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, then $\int_\gamma f = 0$ where (i) $f(z) = e^z/(z - 2)$ and (ii) $f(z) = z^3/(z^2 + 4)$.

Example 7.2.2. Let $a \in \mathbb{C}$ be fixed and let L be a closed line joining a to ∞. For instance, if $a \neq 0$ and $a = re^{it}$, then $L := \{Re^{it} : R \geq r\}$. Let γ be any closed path that does not meet L. Then the integral $\int_\gamma \frac{dw}{w-a} = 0$. For, the integrand is holomorphic in the star-shaped set $\mathbb{C} \setminus L$. In particular, $\int_\sigma \frac{dw}{w-a}$ depends only on the endpoints of the path σ.

An application of this observation is that we can define a holomorphic logarithm on $\mathbb{C} \setminus (-\infty, 0]$ by setting $\operatorname{Log}(z) = \int_1^z \frac{dw}{w}$.

Ex. 7.2.3. Find $\int_\gamma f$ where $f(z) = 1/z$ and γ is any closed path lying in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

We shall assume the following fact:

$$\int_{\mathbb{R}} e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}. \tag{7.4}$$

Example 7.2.4. Let $f(z) := e^{-z^2}$ and $b > 0$ be fixed. Let $\gamma := \sum_{j=1}^4 \gamma_j$ be the path where $\gamma_1(t) = t$, $0 \le t \le R$, $\gamma_2(t) = R + it$, $0 \le t \le b$, $\gamma_3(t) = ib + (R - t)$, $0 \le t \le R$ and $\gamma_4(t) = i(b - t)$, $0 \le t \le b$ for $R \gg 0$, i.e. for all sufficiently large values of R. See Figure 7.5.

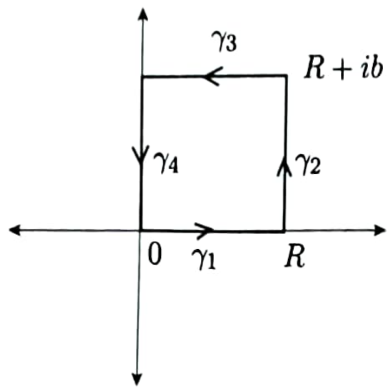

Figure 7.5: Illustration for Example 7.2.4

By Cauchy's theorem, $\int_\gamma f = 0$. Hence $\sum_j \int_{\gamma_j} f = 0$.
As $R \to \infty$, the first integral goes to $\sqrt{\pi}/2$ by (7.4).
The second integral is estimated below:

$$\begin{aligned}
\left|\int_0^b e^{-(R+it)^2}\, dt\right| &= \left|\int_0^b e^{-R^2} e^{t^2} e^{-2iRt}\right| dt \\
&\le \int_0^b \left|e^{-R^2} e^{t^2} e^{-2iRt}\right| dt \\
&\le e^{-R^2} \int_0^b e^{t^2}\, dt, \quad \text{since } \left|e^{-2iRt}\right| = 1 \\
&\le e^{-R^2} \int_0^b e^{b^2}\, dt \\
&= e^{-R^2} b e^{b^2}.
\end{aligned}$$

Hence the second integral goes to 0 as $R \to \infty$.

We make a change of variable $x := R - t$ in the third integral and see that it converges, as $R \to \infty$, to

$$-e^{b^2} \int_0^\infty e^{-x^2} [\cos 2bx - i \sin 2bx]\, dx.$$

The fourth integral is computed to be $-i\int_0^b e^{x^2}\,dx$.

Separating the real and imaginary parts in the equation $\int_\gamma f = 0$, we get

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = e^{-b^2}\sqrt{\pi}/2$$

$$\int_0^\infty e^{-x^2}\sin 2bx\,dx = e^{-b^2}\int_0^b e^{t^2}\,dt.$$

Example 7.2.5 (Fundamental Theorem of Algebra). If P is a nonconstant polynomial with complex coefficients, then P has at least one zero in $\mathbb{C}$, that is, there exists $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.

Write $P(z) = zQ(z) + a_0$, $n \geq 1$ and the coefficient of z^n, $a_n \neq 0$. Then

$$\frac{1}{z} = \frac{P(z)}{zP(z)} = \frac{zQ(z) + a_0}{zP(z)} = \frac{Q(z)}{P(z)} + \frac{a_0}{zP(z)}.$$

If $P(z) \neq 0$ for all $z \in \mathbb{C}$, then $1/P$ is entire in $\mathbb{C}$. Let $\gamma_R(t) := Re^{it}$ for $0 \leq t \leq 2\pi$ and $R > 0$. By Cauchy's theorem $\int_{\gamma_R} \frac{Q}{P} = 0$ so that

$$2\pi i = \int_{\gamma_R} \frac{1}{z} = \int_{\gamma_R} \frac{a_0}{zP(z)}.$$

Since $\lim_{z\to\infty} \frac{P(z)}{a_n z^n} = 1$ (why? see Ex. 5.1.9), we have $|P(z)| \geq \frac{1}{2}|a_n|\,|z|^n$ for large R. Hence it follows that

$$2\pi = \left|\int_{\gamma_R} \frac{a_0}{zP(z)}\,dz\right| \leq 2\pi R \frac{2|a_0|}{R|a_n|R^n} = \frac{4\pi|a_0|}{|a_n|R^n} \to 0,$$

as $R \to \infty$. Thus, we get the contradiction: $2\pi = 0$. We therefore conclude that our assumption that $P(z) \neq 0$ for all $z \in \mathbb{C}$ is wrong.

Example 7.2.6 (Fresnel's Integrals). For $a \in \mathbb{R}$ with $|a| < 1$, we have

$$\int_0^\infty e^{-(1+ia)^2t^2}\,dt = \frac{\sqrt{\pi}}{2}\left(\frac{1}{1+ia}\right). \tag{7.5}$$

Assume $0 \leq a < 1$. Define $f(z) = e^{-z^2}$. Let $\gamma := \gamma_1 + \gamma_2 - \gamma_3$ where $\gamma_1(t) = t$, $0 \leq t \leq R$, $\gamma_2(t) = R + it$, $0 \leq t \leq aR$ and $\gamma_3(t) = (1 + ia)t$, $0 \leq t \leq R$. See Figure 7.6. By Cauchy's theorem

$$\int_{\gamma_3} f = \int_{\gamma_1} f + \int_{\gamma_2} f. \tag{7.6}$$

Now,

$$\left|\int_{\gamma_2} f\right| \leq e^{-R^2}\int_0^{aR} e^{-t^2}\,dt \leq e^{-R^2}\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}e^{-R^2},$$

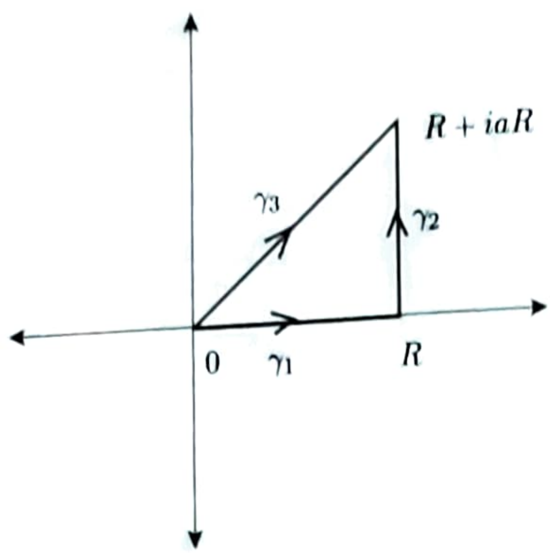

Figure 7.6: Illustration for Example 7.2.6

which tends to 0 as $R \to \infty$.

It follows from (7.6) that

$$(1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt = \lim_{R\to\infty}\int_{\gamma_3} f = \lim_{R\to\infty}\int_{\gamma_1} f = \int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2.$$

Taking the real and imaginary parts of (7.5), we obtain

$$\int_0^\infty e^{(a^2-1)t^2}\cos 2at^2\,dt = \frac{1}{2(1+a^2)}\sqrt{\pi}, \qquad 0 \le a < 1 \tag{7.7}$$

$$\int_0^\infty e^{(a^2-1)t^2}\sin 2at^2\,dt = \frac{a}{2(1+a^2)}\sqrt{\pi} \qquad 0 \le a < 1. \tag{7.8}$$

Note that (7.7)-(7.8) continue to be true for $|a| \le 1$. (Why?)

Taking $a = 1$ and $s := \sqrt{2}t$, we get the Fresnel's integrals:

$$\int_0^\infty \cos s^2\,ds = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{2}} = \int_0^\infty \sin s^2\,ds.$$

Ex. 7.2.7. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ be the path where $\gamma_1(t) = t$ for $0 \le t \le R$, $\gamma_2(t) = Re^{it}$ for $0 \le t \le \pi/4$ and $\gamma_3(t) = te^{i\pi/4}$ for $R \ge t \ge 0$. Integrate e^{iz^2} over γ to obtain the Fresnel's integrals. *Hint:* On the circular arc γ_2, you will need the inequality (called Jordan's inequality): $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/4$. To prove this inequality, consider the function $\sin x/x$ on $(0, \pi/2)$ and use calculus (twice!). (See also Ex. 15.2.10.)

Ex. 7.2.8. Prove that if $a \in \mathbb{R}$, the integral $I(a) := \int_{\mathbb{R}} e^{-(x+ia)^2}\,dx$ is independent of a.

Hint: Assume $a > 0$. By Cauchy's theorem, $\int_\gamma e^{-z^2}\, dz = 0$ where γ is the rectangle with vertices at $-b$, b, $b + ia$ and $-b + ia$. Note that

$$\left|\int_0^a e^{-(b+iy)^2}\, dy\right| \leq e^{-b^2} \int_0^a e^{y^2}\, dy.$$

Let $b \to \infty$.

Ex. 7.2.9. Notation as in the last exercise. Use the fact that $I(0) = \sqrt{\pi}$ to show that

$$\int_{-\infty}^{\infty} e^{-x^2/2} \cos ux\, dx = \sqrt{2\pi} e^{-u^2/2}.$$

Ex. 7.2.10. Let $f \in H(B(0,R))$ and $f \in C(B[0,R])$. Let $\gamma_R(t) := Re^{it}$ for $0 \leq t \leq 2\pi$. Show that $\int_{\gamma_R} f = 0$. *Hint:* $\int_{\gamma_R} f = \int_{\gamma_R} f - \int_{\gamma_r} f$ and f is uniformly continuous on $B[0,R]$.

Ex. 7.2.11. Criticize the following proof of Cauchy's theorem. Let $\varphi(t) := t \int_\gamma f(tz)\, dz$, $t \in [0,1]$. Then $\varphi \in C([0,1])$ and

$$\begin{aligned}\varphi'(t) &= \int_\gamma [f(tz) + tzf'(tz)]\, dz \\ &= \int_\gamma \frac{d}{dz}[zf(tz)]\, dz = 0.\end{aligned}$$

Hence $\int_\gamma f = \varphi(1) = \varphi(0) = 0$.

Ex. 7.2.12. Let $U := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Let f be continuous on $\overline{U}$ and $f \in H(U)$. Assume that there exists $M > 0$ such that $|f(z)| \leq \frac{M}{|z|^2}$ for $z \in \overline{U}$ with $|z| \geq 1$. Show that $\int_{-\infty}^{\infty} f(iy)\, dy = 0$.

7.3 An Extension of Cauchy's Theorem

We need an extension of Theorem 7.1.8

Theorem 7.3.1 (Extension of Cauchy-Goursat). *Let U be any open set, $p \in U$. Assume that $f : U \to \mathbb{C}$ is* $f \in C(U)$ and $H(U\setminus\{p\})$ ~~*continuous and holomorphic on $U \setminus$*~~ *$\{p\}$. Let $\Delta = [a,b,c]$ be a triangle in U with boundary T. Then $\int_T f = 0$.*

Proof. If $p \notin \Delta$, the result follows from Theorem 7.1.8.

If $p \in \Delta$, let us assume that p is one of the vertices, say, a. If a, b, c are collinear, then the result is true even for a continuous f.

> Reason: If they are collinear, then one of them must be in between the other two. Even though they are complex numbers, this makes perfect sense, since we can write, say, $c = (1-t)a + tb$ for some $t \in (0,1)$. Hence $\int_{[a,b]} + \int_{[b,c]} + \int_{[c,a]} = 0$.

Assume that a, b, c are not collinear. We choose a point $z \in [a, b]$ and $w \in [a, c]$ very near to a. For example, choose z and w so that $d(a, z) = 1/n = d(a, w)$. (Draw pictures.) Then

$$\int_T f = \int_{[a,z,w]} f + \int_{[z,b,w]} f + \int_{[b,c,w]} f.$$

The last two integrals are 0, since they do not contain $p = a$.

> Reason: The triangles $\Delta(z, b, w)$ and $\Delta(b, c, w)$ are subsets of $U \setminus \{p\}$ and by hypothesis $f \in H(U \setminus \{p\})$. Hence by Cauchy-Goursat theorem 7.1.8, the integral of f over the boundaries of these triangles are zero.

The first integral can be made arbitrarily small, since f is bounded on Δ.

> Reason: Since Δ is compact, and f is continuous on U, f is bounded on Δ, say, $|f(z)| \leq M$ for $z \in \Delta$. We now estimate $L(T(a, z, w))$:
>
> $$\begin{aligned} L(T(a, z, w)) &= |a - z| + |z - w| + |w - a| \\ &\leq 1/n + |w - a| + |a - z| + 1/n \\ &= 4/n. \end{aligned}$$
>
> Hence by ML inequality, $\left|\int_{T(a,z,w)}\right| \leq 4M/n$.

Thus, we have shown $\left|\int_{T(a,b,c)}\right| \leq 4M/n$. This being true for all n, we conclude that $\int_{T(a,b,c)} = 0$.

Assume that $p \in \Delta$ is in the interior. Draw pictures. Then by the argument of the last paragraph, the integrals over $[a, b, p]$, $[b, c, p]$ and $[c, a, p]$ are zero and their sum is $\int_T f$. □

Theorem 7.3.2 (Extended Cauchy's Theorem). *Let U be star-shaped and $f \in C(U)$ and $f \in H(U \setminus \{p\})$. Then (i) f has a primitive in U and (ii) for every closed path γ in U, we have $\int_\gamma f = 0$.*

Proof. Note that this strengthens Theorem 7.1.9. (i) follows from Theorem 7.3.1 and Theorem 7.1.3. (ii) follows from (i) and Corollary 6.2.19. □

Remark 7.3.3. Theorem 7.3.2 is the most significant result in Cauchy theory. It is better if we reflect on the path[2] that led us to this marvelous piece. Do not worry about the proofs but go though the following results in that order. This will help you get a global or an eagle's view and will make sure that you do not miss the wood for the trees: Theorem 7.1.3, Theorem 7.1.8, Theorem 7.1.9, Theorem 7.1.10 , Theorem 7.1.8 and Theorem 7.3.2. This is typical of what analysts call bootstrap argument — the results are interlaced as you would tie the shoe lace.

[2]Pun intended!

7.4 Green's Theorem and Cauchy's Theorem

In this section, we indicate briefly how to derive the simple version of Cauchy's theorem from Green's theorem of calculus.

We recall Green's theorem in the following form.

Theorem 7.4.1. *Let γ be a simple closed path in an open set $U \subset \mathbb{R}^2$. Let p, q be continuously differentiable functions on U. Let V be the region enclosed by γ. Then*

$$\int_\gamma p\,dx + q\,dy = \int_V \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy,$$

where γ is oriented in such a way that the region V lies to the left of γ. See Figure 7.7. □

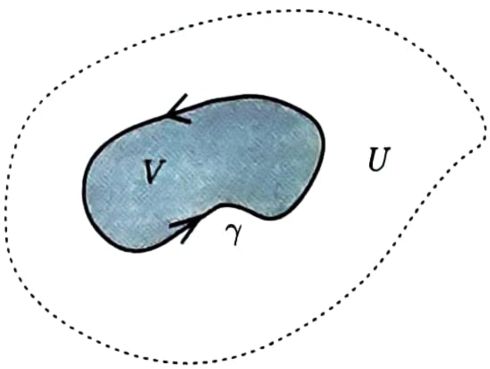

Figure 7.7: Region enclosed inside a positively oriented curve

We now assume that $U \subset \mathbb{C}$ is open and that $f \colon U \to \mathbb{C}$ is holomorphic and assume that $f' \colon U \to \mathbb{C}$ is continuous. Let $f = u + iv$. Let γ be a simple closed path in U. Let us write the coordinates of the path by $\gamma(t) := x(t) + iy(t)$ or in $\mathbb{R}^2$ as $\gamma(t) = (x(t), y(t))$ so that $\gamma'(t) = x'(t) + iy'(t) = \frac{dx}{dt} + i\frac{dy}{dt}$. Then

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= [u(\gamma(t)) + iv(\gamma(t))]\left[\frac{dx}{dt} + i\frac{dy}{dt}\right] \\ &= \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right) + i\left(u\frac{dy}{dt} + v\frac{dx}{dt}\right). \end{aligned}$$

If you recall that by definition of the line integral

$$\int_\gamma (p\,dx + q\,dy) = \int_a^b [p(x(t), y(t))\frac{dx}{dt} + q(x(t), y(t))\frac{dy}{dt}]\,dt$$

it follows that

$$\int_\gamma f = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).$$

Now the right side integrals can be evaluated using Green's theorem:

$$\begin{aligned}
&\int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx) \\
&= \int_V -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\int_V \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \\
&= 0,
\end{aligned}$$

by Cauchy-Riemann equations. Hence, we see that $\int_\gamma f = 0$ for any closed path γ and f continuously differentiable (a little more stringent than asking for holomorphicity alone) in the area enclosed by γ

By modern standards, there are several objections to the above derivation of Cauchy's theorem. First of all we need to assume that f' is continuous. Secondly, we have to assign a precise meaning to the term 'region enclosed by' γ. Finally we need to make the notion of 'the region lying to the left of the path as we traverse the path' rigorous. Our approach in the earlier sections, following Goursat, avoided all these technical objections.

Chapter 8

Cauchy Integral Formula & its Consequences

In this chapter, we derive a remarkable formula called the Cauchy integral formula (CIF, in short) and apply it to the study of holomorphic functions.

8.1 Cauchy Integral Formula

Theorem 8.1.1 (Cauchy Integral Formula). *Let U be an open set star-shaped at $p \in U$. Let $f \in H(U)$. Let γ be a closed path in U. Then for any $a \in U \setminus [\gamma]$, we have*

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw = f(a)\left[\frac{1}{2\pi i}\int_\gamma \frac{dw}{w-a}\right].} \tag{8.1}$$

Proof. Fix $z \in U \setminus [\gamma]$. Consider $g\colon U \to \mathbb{C}$ defined as follows:

$$g(w) := \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \in U \text{ and } w \neq z \\ f'(z), & \text{at } w = z \end{cases}$$

Then g is continuous on U and $g \in H(U \setminus \{z\})$. Hence by Theorem 7.3.2, $\int_\gamma g = 0$. But on $[\gamma]$, we have $g(w) = \frac{f(w)-f(z)}{w-z}$. Substituting this value of g in $\int_\gamma g = 0$ and separating the terms yields (8.1) with z in place of a. ☐

Corollary 8.1.2. *Let U be any open set and $B(z_0, R) \subset U$. Let $f \in H(U)$. Let $\gamma(t) := z_0 + re^{it}$, $0 < r < R$ and $0 \leq t \leq 2\pi$. Then we have*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}, \qquad z \in B(z_0, r). \tag{8.2}$$

Proof. The term inside the square brackets in (8.1) is 1 by Proposition 6.4.5. Now the result follows from Theorem 8.1.1, more specifically from (8.1). □

> As we go along, you will notice that all local properties of a holomorphic function follow from (8.1).

As immediate consequence of CIF and Theorem 6.4.7, we get the following

Theorem 8.1.3. *Any holomorphic function f on an open set U is an analytic function. In particular, f' is also holomorphic in U.*

Proof. At the cost of repetition, we indicate a proof of this important result.

Given $z \in B(z_0, r)$, choose ρ such that $0 < \rho < r$ and $z \in B(z_0, \rho)$. Let $\gamma(t) := z_0 + re^{it}$, $0 \leq t \leq 2\pi$. Then

$$\frac{1}{w-z} = \frac{1}{(w-z_0)+(z_0-z)} = \frac{1}{(w-z_0)[1-\frac{z-z_0}{w-z_0}]} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}, \tag{8.3}$$

the series being uniformly convergent on $[\gamma]$ for fixed z, z_0. Using Corollary 6.3.18, we see that we can substitute the value of $1/(w-z)$ from (8.3) in (8.2). We obtain $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, for $z \in B(z_0, \rho)$, where $a_n := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\, dw$. □

Thus, we have arrived at the heart of the Cauchy theory:

> $f\colon U \to \mathbb{C}$ is holomorphic in U iff it is analytic in U.

Remark 8.1.4. Revisit Remark 7.3.3. Now add the results of Theorem 8.1.1, Corollary 8.1.2 and Theorem 8.1.3 to the results in the cited remark. Now you have a summary of the development of Cauchy theory in a nutshell.

Theorem 8.1.3 allows us to establish many properties of holomorphic functions from the corresponding results for power series. For more on this, we refer the reader to Chapter 13.

There are a few points worth noting about Theorem 8.1.3.

Remark 8.1.5. From Theorem 8.1.3 it follows that if $f\colon U \to \mathbb{C}$ is differentiable on an open set U, then f is infinitely differentiable. This should be contrasted with the fact that for any $k \in \mathbb{N}$, there exist functions from $\mathbb{R}$ to $\mathbb{R}$ which are k-times differentiable but not $k+1$ differentiable.

In fact, there exists a real valued function which is infinitely differentiable on all of $\mathbb{R}$ but does not have a power series expansion at the point $x = 0$: The standard example is $f(x) = e^{-1/x^2}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. (See Example 4.5.3 in [2].) See also Remark 8.1.27.

Remark 8.1.6. The second point is that the coefficients a_n of the power series expansion of f around a are unique. This follows from the uniqueness theorem for power series (Corollary 3.2.9) or the integral representation for a_n (Proposition 6.4.1). Note that the integrals $\int_{\gamma_r} \frac{f(w)}{w-z}$ are all the same for $0 < r < R$. Here $\gamma_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

Remark 8.1.7. The third point is that the power series expansion of f around z_0 is valid in the largest possible disk $B(z_0, R) \subset U$. The only subtle point in the above proof is that the coefficients a_n, which apparently depend on the path γ (i.e. on ρ) are, in fact, independent. This is attended to in the last remark.

The most important fall-out from this is the following: If $f \in H(\mathbb{C})$, then there exists a power series expansion of f around 0 with infinite radius of convergence: $f(z) = \sum a_n z^n$ for $z \in \mathbb{C}$.

Remark 8.1.8. The proof of Theorem 8.1.3 will not yield a single power series expansion for a holomorphic function f on a connected open set. This is not due to the inadequacy of the method. It is just not possible to do so. For instance, if we let $f(z) = 1/z$ on $\mathbb{C}^*$ and $a \in \mathbb{C}^*$, we have $f(z) = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n(\frac{z-a}{a})^n$ for all $z \in B(a, |a|)$. By the uniqueness of a_n's (Remark 8.1.6), this is the power series expansion of f around a. Clearly, its radius of convergence is $|a|$. Thus, it is impossible to find a single power series expansion for f on all of $\mathbb{C}^*$.

The upshot of this remark is that while the theorem says that any holomorphic function is analytic, it does **not** say that any holomorphic function is a power series function.

Before we look into theoretical applications of CIF, we shall look at some typical applications of CIF in the computation of integrals.

Example 8.1.9. Let us evaluate $\int_\gamma \frac{z^2+1}{z+1}$ where $\gamma(t) = 2e^{it}$. If we take $f(z) := z^2 + 1$, then the given integral is $\int_\gamma \frac{f(w)}{w-(-1)}$ so that the value is $2\pi i f(-1) = 4\pi i$.

Ex. 8.1.10. Evaluate $\int_\gamma \frac{3z-5}{z^2-2z-3}\, dz$, where $\gamma(t) := 2e^{it}$. *Hint:* Use partial fractions.

Ex. 8.1.11. Evaluate $\int_\gamma \frac{1}{(z-a)(z-b)}\, dz$ where $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ and (i) $|a|, |b| < 1$, (ii) $|a| < 1$ but $|b| > 1$ and (iii) $|a|, |b| > 1$.

Ex. 8.1.12. Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{az}}{z^2+1}\, dz = \sin a$$
$$\frac{1}{2\pi i}\int_\gamma \frac{ze^{az}}{z^2+1}\, dz = \cos a$$

Ex. 8.1.13. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Show that $\int_\gamma \frac{e^z}{z} = 2\pi i$. Hence, deduce that for any $k \in \mathbb{Z}$,

$$\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\, d\theta = 2\pi.$$

Ex. 8.1.14. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Evaluate $\int_\gamma \frac{\cos z}{z}\, dz$ and hence deduce that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta = 2\pi.$$

Ex. 8.1.15. Let P be a polynomial none of whose zeros lie on the path $\gamma_R(t) = Re^{it}$, $0 \leq t \leq 2\pi$. Show that $\frac{1}{2\pi i}\int_{\gamma_R} \frac{P'}{P}$ is the number of zeros (counted with multiplicity) of P in $B(0, R)$. *Hint:* Note that $\frac{P'(z)}{P(z)} = \sum_j \frac{1}{z - \alpha_j}$ where α_j are all the roots of P. (This is a special case of the argument principle, see Theorem 11.2.1.)

Ex. 8.1.16. Evaluate $\int_\gamma \frac{z^3}{z^4 - 1}\, dz$ where $\gamma(t) = 3e^{it}$, $0 \leq t \leq 2\pi$.

Ex. 8.1.17. Evaluate $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$. *Hint:* Observe that $\frac{1}{2 + \cos\theta} = \frac{2z}{z^2 + 4z + 1}$ for $z = e^{i\theta}$. Ans.. $2\pi/\sqrt{3}$.

Ex. 8.1.18. Let $0 < r < R$ and $\gamma(\theta) = re^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate $\int_\gamma \frac{R + z}{(R - z)z}\, dz$ and hence deduce that $\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}\, d\theta = 1$.

Ex. 8.1.19. Keep the notation of the last exercise. By integrating $1/(R - z)$ over γ and using the last exercise, show that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos\theta}{R^2 - 2Rr\cos\theta + r^2}\, d\theta = \frac{r}{R^2 - r^2}, \qquad 0 \leq r < R.$$

Ex. 8.1.20. Let $\mathcal{F} \subset H(U)$ be such that given any compact subset K of U, there exists $C_K > 0$ such that $|f(z)| \leq C_K$ for $z \in K$ and for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is equicontinuous at each point $a \in U$, that is, given $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in B(a, \delta)$, then $|f(z) - f(a)| < \varepsilon$ for all $f \in \mathcal{F}$. (Look at Definition 19.2.5.)

Ex. 8.1.21. Show that an odd entire function has only odd terms in its power series expansion. Is there an analogue of this for even functions?

Ex. 8.1.22. Let U be convex. If $f \in H(U)$ satisfies $|f'(z) - 1| < 1$ for $z \in U$, then f is one-one. *Hint:* Fundamental theorem of calculus applied to $1 - (f'(z) - 1)$.

In the rest of the chapter, we look at various applications of CIF to the study of properties of holomorphic functions. The next result will be referred to as CIF-D, Cauchy integral formula for derivatives.

Theorem 8.1.23 (CIF for Derivatives). *Let $f \in H(U)$ and $B(z_0, R) \subset U$. Let $\gamma_r(t) := z_0 + re^{it}$ for $0 < r < R$. We have*

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w - z)^{n+1}}\, dw, \quad z \in B(z_0, r).} \tag{8.4}$$

Proof. Recall from the proof of Theorem 8.1.3 that the coefficients a_n in the power series expansion of f around z_0 are given by $a_n := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}$. Now what is the relation between a_n and $f^{(n)}(a)$? But for some factorial, a_n is $f^{(n)}(a)$. See (5.6). CIF-D (8.4) follows from this.

Alternatively, the result is an immediate consequence of Proposition 6.4.10. □

Remark 8.1.24. In view of (8.4), we have the Cauchy's estimates (6.7) for the coefficients of the power series expansion about z_0.

$$|a_n| \le r^{-n}M(r), \text{ where } M(r) := \sup\{|f(z)| : |z - z_0| = r\}. \tag{8.5}$$

In particular, $\left|f^{(n)}(z_0)\right| \le n!r^{-n}M(r)$. Note that such an estimate is not possible in real analysis for the following reason. A theorem of E. Borel says that given *any* sequence (a_n) real numbers there exists an infinitely differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(0) = a_n$. (See Theorem 1.2.6 of [13], Theorem 1.5.4 of [25].)

Definition 8.1.25. Let f be a holomorphic function on an open set $U \subset \mathbb{C}$. The *Taylor series* of f at $a \in U$ is defined by $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$.

The following is an immediate corollary of Theorems 8.1.23 and 8.1.3.

Corollary 8.1.26. *Let $f \in H(U)$ and $B(a,R) \subset U$. Then the Taylor series of f converges to f absolutely and uniformly on compact subsets of $B(a,R)$.* □

Remark 8.1.27. The real analogue of this result is false. For instance, the function f of Remark 8.1.5 is infinitely differentiable and $f^{(n)}(0) = 0$. Thus, the Taylor series of f at 0 is the identically zero power series, hence does not converge to f. Hence, we conclude that for any $r > 0$, there is no holomorphic function g on $B(0,r)$ such that $g = f$ on $(-r,r)$.

Corollary 8.1.28. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ be power series with radii of convergence R_1 and R_2 respectively. Let $h(z) := \sum_{n=0}^{\infty} c_n z^n$ where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$. Then the radius of convergence of h is at least $R := \min\{R_1, R_2\}$ and we have $h(z) = f(z)g(z)$.*

Proof. In $B(0,R)$, both f and g are holomorphic and we know that $a_n = f^{(n)}(0)/n!$ and $b_n = g^{(n)}(0)/n!$. The product function fg is holomorphic in $B(0,R)$. Hence, it is represented in $B(0,R)$ by its Taylor series $f(z)g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ where

$$\alpha_n = \frac{(fg)^{(n)}(0)}{n!} = \sum_{r=0}^{n} \frac{1}{n!}\frac{n!}{r!(n-r)!} f^{(r)}(0)g^{(n-r)}(0) = \sum_{r=0}^{\infty} a_r b_{n-r} = c_n.$$

(We have used the Leibnitz formula for the n-th derivative of a product.) □

Remark 8.1.29. The proper setting (domain) to study the power series (or analytic functions) is the field of complex numbers. To illustrate this point, let us look at the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/(1+x^2)$. This function is infinitely differentiable and its Taylor series at $0 \in \mathbb{R}$ is $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ and it has radius of convergence 1. There is nothing in the behaviour of f in $\mathbb{R}$ that accounts for this restriction on the radius of convergence. However, if we try to extend the domain of the function to $\mathbb{C}$, we detect the reason for this. The points $\pm i$, at which the function is not defined, lie on the unit circle. We leave it to the reader to think over this.

As an application of (8.4), we prove the following theorem.

Theorem 8.1.30 (Weierstrass). *Let $f_n \in H(U)$ for $n \in \mathbb{N}$. Assume that f_n converges to f uniformly on compact subsets of U. Then $f \in H(U)$. Furthermore, for any $k \in \mathbb{N}$, $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact subsets of U.*

Proof. First of all note that f is continuous on U.

> Reason: Let $z \in U$. Choose $r > 0$ such that $B(z, 2r) \subset U$. Then the closed ball $B[z, r]$ is a closed and bounded subset of $\mathbb{C}$ and hence is compact. Hence $f_n \to f$ uniformly on $B[z, r]$. Hence f is continuous on $B[z, r]$ and in particular at z.

Let $z_0 \in U$ and $r > 0$ be such that $B[z_0, r] \subset U$. Let $z \in B(z_0, r)$. Let $\gamma(t) := z_0 + re^{it}$, $0 \le t \le 2\pi$. By (8.1), we have $f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{w-z}\,dw$. Since $[\gamma]$ is compact, $\varepsilon := d(z, [\gamma]) > 0$.

> Reason: Let $K \neq \emptyset$ be a compact subset of a metric space X and $C \neq \emptyset$ closed in X. Assume that $K \cap C = \emptyset$. Then $d(K, C) > 0$. For, the function $x \mapsto d(x, C)$ is continuous and hence assumes its minimum on the compact set K. Hence, if $d(K, C) = 0$, then there exists $x \in K$ such that $d(x, C) = 0$. Since C is closed, it follows that $x \in C$, a contradiction.
>
> Note that this is already seen in the footnote in the proof of Theorem 6.4.7.

Hence $\frac{f_n(w)}{w-z} \to \frac{f(w)}{w-z}$ uniformly on $[\gamma]$.

> Since $[\gamma]$ is compact, $f_n \to f$ uniformly on $[\gamma]$. Also, the function $(w - z)$ is bounded away from zero: $|w - z| \ge \varepsilon$ so that if $g := \frac{1}{(w-z)}$, then $|g| \le 1/\varepsilon$, and hence g is a bounded function. It is now easy to see that $f_n g \to fg$ uniformly on $[\gamma]$. (Or, you may let $g_n = g$ for $n \in \mathbb{N}$ and use Ex. 3.2.2.)

Hence $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$. Now, f is holomorphic on $B(z_0, r)$ either by Theorem 6.4.7 or by Proposition 6.4.10. (Or, repeat the proof of Theorem 8.1.3.)

As argued above, we see that

$$\frac{f_n(w)}{(w-z)^{k+1}} \to \frac{f(w)}{(w-z)^{k+1}},$$

uniformly on $[\gamma]$. Hence

$$\frac{n!}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^{k+1}} \to \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}.$$

From (8.4), it follows that $f_n^{(k)}(z) \to f^{(k)}(z)$ for every $z \in B(z_0, r)$. However the uniform convergence is not established. So we argue more carefully as follows.

By (8.4), we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}. \qquad z \in B(z_0, r).$$

If $z \in B[z_0, \rho]$ with $\rho < r$, then we have

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \le \frac{k!}{2\pi}\max_{z\in\gamma}|f_n(w) - f(w)|\,\frac{2\pi r}{(r-\rho)^{k+1}} \to 0 \text{ as } n \to \infty.$$

Thus $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on closed disks $B[z_0, r]$ and hence uniformly on compact subsets of U. (Why?)

> Reason: Note that if $f_n \to f$ uniformly on A and B, so does it on $A \cup B$. Hence, if K is compact subset of U, then there exist $z_i \in K$ and $r_i > 0$, $1 \le i \le n$, such that $B[z_i r_i] \subset U$ and such that $K \subset \cup_{i=1}^n B(z_i, r_i)$. Since we have already shown that $f_n \to f$ uniformly on each of $B[z_i, r_i]$, the result follows from the remark made in the beginning of this paragraph.

□

Remark 8.1.31. It is worthwhile to go through the last part of the proof again. What should strike you is that we were able to manage to get a uniform estimate while CIF is ready to yield pointwise estimates only!

8.2 Mean Value Property

Theorem 8.2.1 (Mean Value Property). *Let $f \in H(U)$ and $B(a, R) \subset U$ and $\gamma_r(t) = a + re^{it}$, $0 < r < R$. Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\, dt. \tag{8.6}$$

Proof. Note that the right side of the above equation may be considered as the "average" or "mean" of f on the circle γ_r. Thus the theorem says that the mean value of f on *any* circle γ_r centred at a (for $0 < r < R$) is $f(a)$.

We start with the Cauchy integral formula and unwind the definition of the path integral to obtain (8.6). The details are given below. Using the expression $\gamma_r(t) = a + re^{it}$, we see that $dw = ire^{it}$, $w - a = re^{it}$. Plugging these values in the path integral in CIF, we obtain

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} ire^{it} = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\, dt,$$

Thus we get (8.6). □

Remark 8.2.2. The analogous result for real valued differentiable functions of a real variable is false. If $f\colon (r, R) \to \mathbb{R}$ is differentiable, then the mean value property would be $f((a+b)/2) = [f(a)+f(b)]/2$ for all $r < a < b < R$. Consider $f(x) := x^2$ and $g(x) := x^3$ for $x \in (0, 1)$.

Ex. 8.2.3. Let f be holomorphic and nonconstant on $B(z_0, R)$. Assume that $f(z_0) = 0$. Show that on every circle $S(z_0, r)$, $0 < r < R$, $\operatorname{Re} f$ assumes both positive and negative values.

Theorem 8.2.4. *Let $f \in H(U)$, $B(z_0, R) \subset U$. Let $\gamma_r(t) = z_0 + re^{it}$, for any r that $0 < r < R$. We have the Parseval identity:*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right|^2 dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \tag{8.7}$$

Proof. This is essentially Ex. 6.1.12 and is included here for convenience. Observe that

$$\int_0^{2\pi} |s_n(z)|^2\, dt = 2\pi \sum_{k=0}^{n} |a_k|^2 r^{2k}.$$

Recall the fact: if f_n and g_n converge uniformly to f and g on X, and if f and g are bounded on X, then $f_n g_n$ converges uniformly on X to fg. Using this, we see that $\int_0^{2\pi} s_n(t)\overline{s_n(t)}\, dt$ converges to $\int_0^{2\pi} f(z_0+re^{it})\overline{f(z_0+re^{it})}\, dt$. But $\lim_n 2\pi\sum_{k=0}^{n} |a_k|^2 r^{2k} = 2\pi\sum_{k=0}^{\infty} |a_k|^2 r^{2k}$. In particular, this infinite series is convergent and its sum is $\int_0^{2\pi} f(z_0 + re^{it})\overline{f(z_0+re^{it})}\, dt$. □

Ex. 8.2.5 (Solid Mean Value Theorem). Let $f \in H(B(a, R))$. Let $dA = r dr\, d\theta$ be the area element of $\mathbb{R}^2$. Prove that for any $0 < r < R$, we have

$$f(a) = \frac{1}{\pi r^2}\int_{B(a,r)} f\, dA. \tag{8.8}$$

Start with the standard (Gauss) mean value theorem and multiply it by s, $0 < s < r$ and integrate with respect to s to obtain (8.8).

8.3 Liouville's Theorem

Theorem 8.3.1 (Liouville's Theorem). *Let f be a bounded entire function on $\mathbb{C}$, say, $|f(z)| \leq M$ for $z \in \mathbb{C}$. Then f is a constant.*

Proof. Let $C_R(t) := Re^{it}$, $0 \leq t \leq 2\pi$. Let $z \in \mathbb{C}$. Choose any $R > |z|$, we have

$$\begin{aligned} f(z) - f(0) &= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z)}\, dw - \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\, dw \\ &= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)z}{(w-z)w}\, dw. \end{aligned} \tag{8.9}$$

Note that for $w \in [C_R]$, $|w-z| \geq ||w| - |z|| = R - |z|$. Hence $|1/(w-z)| \leq 1/(R-|z|)$. We use this estimate and apply ML inequality to (8.9). We obtain

$$|f(z) - f(0)| \leq \frac{1}{2\pi}\frac{M\,|z|\,2\pi R}{R(R-|z|)} \to 0, \text{ as } R \to \infty.$$

Hence $f(z) = f(0)$ for any $z \in \mathbb{C}$. □

We indicate two more proofs of this result.

Proof 2. We invoke (8.4) now to conclude that $|f'(z)| \leq \frac{1}{2\pi} 2\pi R \frac{M}{R^2} \to 0$ as $R \to \infty$. Thus $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence f is locally constant. Since $\mathbb{C}$ is connected, we conclude that f is a constant.

Proof 3. Let $f(z) = \sum a_n z^n$ be the power series expansion of f with $R = \infty$. (See Remark 8.1.5.) The result follows from (8.7):

$$|a_n|^2 r^{2n} \leq \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^2 dt \leq M^2, \quad n \geq 0.$$

Hence $|a_n|^2 \leq M/r^{2n}$ for any r. Letting $r \to \infty$ we see that $|a_n| = 0$ for $n \geq 1$.

Remark 8.3.2. The function in Remark 8.1.5 shows that the real analysis analogue of Liouville's theorem is false.

Remark 8.3.3. Liouville's theorem does *not* say that $|f(z)| \to \infty$ as $|z| \to \infty$. Look at $f(z) = \exp(z)$ or $g(z) = \cos z$. Then $|f(it)| = 1$ for $t \in \mathbb{R}$ while $|g(z)| \leq 1$ for $z \in \mathbb{R}$. Also, look at Ex. 8.5.17, Corollary 9.1.38, Proposition 13.3.8.

Ex. 8.3.4. Prove the fundamental theorem of algebra. *Hint:* If $f(z) \neq 0$ for $z \in \mathbb{C}$, then $g(z) = 1/f(z)$ is holomorphic on $\mathbb{C}$. Recall the estimate (5.1) of Ex. 5.1.9 on page 51.

Ex. 8.3.5. Let f be entire with $|\mathrm{Re}\, f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is a constant. *Hint:* Consider e^f.

Ex. 8.3.6. Let f be entire such that $|f(z)| \leq M|z|^n$ for $|z| \gg 0$.[1] Then f is a polynomial of degree at most n. *Hint:* Use either (8.4) for $n+1$ or (8.7).

Contrast this with the following in the real case: $f(x) := x^{3/2}$ for $x \in \mathbb{R}$. Then $|f(x)| \leq |x|^2$ for x with $|x| \geq 1$.

Ex. 8.3.7. Let f be entire and $|f(z)| \leq A + B|z|^{5/2}$. Then f is a polynomial of degree at most 2.

Ex. 8.3.8. Let f be entire with $|f(z)| \leq R \cdot \varepsilon(R)$ on $S(0, R)$, where $\varepsilon(R) \to 0$ as $R \to \infty$. Then f is a constant.

Ex. 8.3.9. Let f be entire and $\frac{f(z)}{z} \to 0$ as $z \to \infty$. Then f is a constant.

Ex. 8.3.10. Let f be entire. Assume that there exists $a \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - a| > \varepsilon$ for all $z \in \mathbb{C}$. Then f is a constant. Deduce that if f is nonconstant, entire, then $f(\mathbb{C})$ is dense in $\mathbb{C}$. (A stronger result is Casorati-Weierstrass theorem 9.1.35. Still stronger result is Picard's theorem.)

Ex. 8.3.11. Let f be entire such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Then f is a constant.

Ex. 8.3.12. Let f be entire. If f is such that $\operatorname{Re} f$ or $\operatorname{Im} f$ has no zeros, then f is a constant. (To gain perspective, consider $f(z) := \exp(z)$.)

Ex. 8.3.13. Prove that the constant function $f = 0$ is the only holomorphic function which vanishes at infinity. (A function $f \colon X \to \mathbb{C}$ from a locally compact space X to $\mathbb{C}$ is said to *vanish at infinity* if for any $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.)

Ex. 8.3.14. Let f be entire and $f(1) = 2f(0)$. If $\varepsilon > 0$, then show that there exists z such that $|f(z)| < \varepsilon$.

Ex. 8.3.15. Let f be an entire function. Assume that there exist constants C_1, C_2 such that for each $z \in \mathbb{C}$, either $f(z) = C_1$ or $f'(z) = C_2$. Show that f is polynomial of degree at most 1.

Ex. 8.3.16. Assume that an entire function f is such that $|f(z)| \geq e^{\pi}$ for all $z \in \mathbb{C}$. What can you conclude about f?

8.4 Morera's Theorem

The main result of this sections gives us a very powerful method of checking whether a continuous function is holomorphic. Do not the simplicity of the argument fool you!

[1]The symbol $|z| \gg 0$ means that for all sufficiently large values of $|z|$. In our context it means that there exists $R > 0$ such that for all z with $|z| > R$, we have $|f(z)| \leq M|z|^n$.

Theorem 8.4.1 (Morera's Theorem). *Let $f: U \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for the boundary T of any triangle $\Delta \subset U$. Then f is holomorphic in U.*

Proof. Let $a \in U$ with $B(a, r) \subset U$. Then f has a primitive F in $B(a, r)$ by Theorem 7.1.3. Since F is holomorphic in $B(a, r)$, so is its derivative f. Since $a \in U$ is arbitrary, f is differentiable on all of U. □

We now give some typical applications of Morera's theorem.

1) Let $f: U \to \mathbb{C}$ be continuous. Assume that for any $\varepsilon > 0$ there exists a holomorphic function g on U such that $|f(z) - g(z)| < \varepsilon$. Then f is holomorphic on U. To see this, fix ε and let g be the corresponding holomorphic function. We write $f = (f - g) + g$. Then for any triangle $\Delta \subset U$ with $\partial\Delta = T$, we have $\int_T f = \int_T (f - g) + \int_T g$. The second term on the right is zero since g is holomorphic.

$$\left|\int_T f(z)\, dz\right| \leq \left|\int_T f - g\right| < L(T)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\left|\int_T f(z)\, dz\right| = 0$. Since Δ was any triangle in U, it follows from Morera's theorem that f is holomorphic.

The above reasoning proves the following more useful version. Let f_n be holomorphic in U and assume that f_n converge uniformly on compact subsets of U to a function f. Then f is holomorphic in U. This is weaker than Weierstrass theorem 8.1.30.

Let us remark that this is false in the case of differentiable functions on $\mathbb{R}$. If $f \in C([-1, 1])$ is the function $f(x) := |x|$, then f is not differentiable at 0. However, by Weierstrass approximation theorem, there exists a sequence p_n of real polynomials that converge to f uniformly on all of $[-1, 1]$.

2) Recall the extension of Cauchy-Goursat theorem 7.3.1. Let f be as in the theorem. Then, by Morera's theorem, it is holomorphic in U. (Later, we shall see that this is a special case of Riemann's theorem on removable singularity.) Thus, Theorem 7.3.1 follows from Cauchy's theorem..

Contrast this situation with the real case: The function $f(x) = |x|$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$.

3) Let g be continuous on [0,1] and define $F(z) := \int_0^1 g(t) \sin(tz)\, dt$. We claim that F is holomorphic in $\mathbb{C}$. It is easy to show that F is continuous. We now use Fubini's theorem for continuous functions on a compact set to see

$$\int_T F(z)\, dz = \int_T \left[\int_0^1 g(t) \sin(tz)\, dt\right] dz = \int_0^1 g(t) \left[\int_T \sin(tz)\, dz\right] dt = 0.$$

Ex. 8.4.2. Show that $f(z) := \int_0^1 \frac{1}{1-tz}\, dt$ is holomorphic in $B(0, 1)$. Find a power series expansion for f.

8.5 Identity Theorem

Let $p(z)$ be a polynomial. Recall that $\alpha \in \mathbb{C}$ is a *root* or *zero* of p if $p(\alpha) = 0$. When do we say that α is root of multiplicity m? We say that α is root of multiplicity m if we can write $p(z) = (z - \alpha)^m q(z)$ with $q(\alpha) \neq 0$. Note also that this is equivalent to saying that $p^{(m-1)}(\alpha) = 0$ but $p^{(m)}(\alpha) \neq 0$. This motivates the next definition.

Definition 8.5.1. Let $f\colon U \to \mathbb{C}$ be holomorphic with power series expansion $f(z) := \sum a_n(z - z_0)^n$ around $z_0 \in U$. We say that z_0 is a *zero* of f of *order* m iff $a_n = 0$ for all $n < m$, and $a_m \neq 0$. Thus, z_0 is a zero of f of order m iff $f^{(k)}(z_0) = 0$ for $0 \leq k \leq m - 1$ and $f^{(m)}(z_0) \neq 0$. Then $f(z) = (z - z_0)^m g(z)$ where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Note that g admits the following description:

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & \text{for } z \neq z_0 \\ a_m & \text{at } z = z_0. \end{cases}$$

It is clear that g is holomorphic in $U \setminus \{z_0\}$ and is continuous at z_0. Hence g is holomorphic. See the second application following Morera's theorem.

Notation. We let $m(f; z)$ denote the order of z as a zero of f. Note that $m(f; z) = 0$ iff $f(z) \neq 0$. Let $Z(f) := \{z \in B(z_0, R) : f(z) = 0\}$ denote the set of zeros of a function f.

Theorem 8.5.2 (Uniqueness theorem for power series). *Let $f(z) = \sum_{k=0}^{\infty} c_n(z - z_0)$ be convergent on $B(z_0, R)$, $R > 0$. Assume that z_0 is a cluster point of $Z(f)$. Then $f = 0$.*

Proof. Since z_0 is a cluster point of $Z(f)$, there exists a sequence (z_n) of distinct points in U such that $z_n \neq a$, $f(z_0) = 0$ and $z_n \to z_0$. We conclude that $f(z_0) = \lim f(z_n) = 0$. It follows that $c_0 = f(z_0) = 0$.

Assume by way of induction that $c_k = 0$ for $0 \leq k \leq n$. We claim $c_{n+1} = 0$. Observe that $f(z) = (z - z_0)^n g(z) = (z - z_0)^n \sum_{k=1}^{\infty} c_{n+k}(z - z_0)^k$. We have

$$g(z_0) = \lim_k g(z_k) = \lim_k \frac{f(z)}{(z_k - z_0)^n} = 0.$$

Since $c_{n+1} = g(z_0)$, the claim is proved. We have thus shown for all $n \in \mathbb{N}$, we have $c_n = 0$, that is, $f = 0$. ☐

Remark 8.5.3. Let $U \subset \mathbb{C}$ be nonempty open. We observe that any $z \in U$ is a cluster point of V. For, if $z \in B(z, r) \subset V$, then $z + t \in B(z, r)$ for any t such that $|t| < r$. In particular, $z + \frac{1}{n} \in B(z, r) \subset V$ for $n \gg 0$.

Theorem 8.5.4. *Let U be a connected open set in $\mathbb{C}$, and let $f\colon U \to \mathbb{C}$ be holomorphic. Suppose that the set of zeros $Z(f) := \{z \in U : f(z) = 0\}$ has a cluster point in U. Then $f = 0$ on U.*

Proof. Let $E := \{z \in U : z \text{ is a cluster point of zeroes of } f\}$. By hypothesis, $E \neq \emptyset$. We observe that for any $z \in U$, we have $f(z) = 0$. In particular, $E \subset Z(f)$. We shall use this observation often below. We show that E is both open and closed in U.

If $z_0 \in E$, let $r > 0$ be such that $B(z_0, r) \subset U$. We have the Taylor expansion of f in $B(z_0, r)$. Since $z_0 \in E$, there exists a sequence (z_n) of distinct points in $Z(f)$ such that $z_n \to z_0$. Therefore, $z_n \in B(z_0, r)$ for $n \gg 0$. By the uniqueness theorem for power series, it follows that $f = 0$ on $B(z_0, r)$. Since each $z \in B(z_0, r)$ is a cluster point of $B(z_0, r)$, as observed in Remark 8.5.3, and $f(w) = 0$ for each $w \in B(z_0, r)$ it follows that $z \in E$. Thus, $B(z_0, r) \subset E$. We therefore conclude that E is open in $\mathbb{C}$ and hence in U.

We claim that E is closed in U. Let $z_0 \in U$ be a cluster point of E. Hence there exists a sequence (z_n) of distinct points in E such that $z_n \to E$. Since $f(z_n) = 0$, it follows that z_0 is cluster point of $Z(f)$. That is, $z_0 \in E$. Hence $E = U$. Since $U = E \subset Z(f)$, we arrive at the result. □

Corollary 8.5.5 (Identity Theorem). *Let U be connected and open. Let A be a subset of U which has a cluster point in U. Let $f, g \colon U \to \mathbb{C}$ be holomorphic. Assume that $f(z) = g(z)$ for all $z \in A$. Then $f = g$ on U.*

Proof. This is an immediate consequence of the last theorem. The reader should prove it on his own.

Consider $h(z) = f(z) - g(z)$ for $z \in U$. Then h is holomorphic in U and its set of zeros contains A. Hence the set of zeros of h has a cluster point in U. The result now follows from the last theorem. □

A special but an important special case is the following.

Corollary 8.5.6. *Let $U \subset \mathbb{C}$ be connected and open. Assume that $f, g \in H(U)$. Assume that $f = g$ on a nonempty open subset $V \subset U$. Then $f = g$ on U.*

Proof. The result follows from the observation in Remark 8.5.3 and the last result. □

Remark 8.5.7. The cluster[2] of the last three results are collectively known as the Principle of Analytic Continuation. There is no analogue of such a result in real analysis. Consider the function of Remark 8.1.5.

Definition 8.5.8. Let E be a subset of a metric space X. We say that E is *discrete* if (1) E is closed and (2) if every point of E is isolated, that is, for each $x \in E$, there exists $r > 0$ such that $B(x, r) = \{x\}$.

Corollary 8.5.9. *Let $U \subset \mathbb{C}$ be connected and open. Let $f \in H(U)$ be non-constant. Then $Z(f)$, the set of zeros of f in U is a discrete subset of U.*

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[2] Pun!

Proof. $Z(f)$ is the inverse image of the closed set $\{0\}$ and hence $Z(f)$ is closed. We need only prove that every point of $Z(f)$ is isolated. If this is false, then there exists $z_0 \in Z(f)$ such that for any $r > 0$ the set $B(z_0, r) \cap Z(f)$ contains a point other than z_0. In such a case, z_0 is the cluster point of $Z(f)$. Since U is connected and open, by the uniqueness theorem, it follows that $f = 0$. But this contradicts the hypothesis that f is non-constant. $\square$

Example 8.5.10. Let f be any entire function such that $f(x) = e^x$ for $x \in \mathbb{R}$, then $f(z) = \exp(z)$.

Ex. 8.5.11. If we knew $\sin(2x) = 2\sin x \cos x$ for $x \in \mathbb{R}$, we then can conclude the same result for $z \in \mathbb{C}$.

Ex. 8.5.12. Let $f(z) = \sin(1/z)$ on $\mathbb{C}^*$. Show that 0 is a cluster points of zeros of f. Does this contradict the identity theorem?

Ex. 8.5.13. Is it possible to construct $f \in H(B(0,1))$ such that $f(1/n) = z_n$ where (i) $z_n = (-1)^n$, (ii) $z_n = n/(n+1)$, (iii) $z_n = 0$ if n is even and $z_n = 1/n$ when n is odd?

Ex. 8.5.14. Show that a nonconstant holomorphic function f on a connected open set U cannot be a constant on any open ball $B(a,r) \subset U$.

Ex. 8.5.15. *An often used observation.* Let U be connected and open and $f \in H(U)$ be nonconstant. Given $\alpha \in U$ and $\lambda \in \mathbb{C}$, there exists $r = r(\alpha) > 0$ such that $f(z) \neq \lambda$ for $0 < |z - \alpha| < r$. (The significant result here is when $\lambda = f(\alpha)$. If $\lambda \neq f(\alpha)$, mere continuity will do.)

Ex. 8.5.16. Let U be a connected open set. Let $f \in H(U)$ be nonconstant and $K \subset U$ be compact. Then, for any $\alpha \in \mathbb{C}$, the set $\{z \in K : f(z) = \alpha\}$ is finite.

Ex. 8.5.17. Let f be entire and $|f(z)| \to \infty$ as $|z| \to \infty$. Then f is a polynomial. (Compare this with the observation $|\exp(it)| = 1$ for $t \in \mathbb{R}$.) *Hint:* Choose $R \gg 0$ such that $|f(z)| > 1$ if $|z| > R$. Let α_i, $1 \leq i \leq N$ be the zeros of f with multiplicity m_j in $B[0, R]$. Consider $g(z) = \dfrac{f(z)}{\prod_{j=1}^N (z - \alpha_j)^{m_j}}$.

Ex. 8.5.18. Let U be connected and open. Then $H(U)$ is an integral domain under the operations of pointwise addition and pointwise multiplication. (The crucial thing to be shown is that if $fg = 0$ for some $f, g \in H(U)$, then either $f = 0$ or $g = 0$ on U.)

Is the converse true?

8.6 Maximum Modulus Theorem

Theorem 8.6.1 (Maximum Modulus Principle). *Let $U \subset \mathbb{C}$ be a connected open set. Let $f \in H(U)$. Let $a \in U$ and $R > 0$ be such that $B(a, R) \subset U$. Assume that $|f(a)| \geq |f(z)|$ for all $z \in B(a, R) \subset U$. Then f is a constant in U.* $\square$

This has the following geometric interpretation. If f is nonconstant and if we think of $|f|$ as a function from $U \subset \mathbb{R}^2$ to $\mathbb{R}$, then its graph in $\mathbb{R}^3$ has no peaks.

We offer five proofs of this result. Except the last proof, the first four make use of the following observation (Ex. 8.5.14): A nonconstant holomorphic function f on a connected open set U cannot be a constant on any open ball $B(a,r) \subset U$.

Proof 1. Let us assume that $a \in U$ is such that $|f(a)| \geq |f(z)|$ for all $z \in U$.

If $f(a) = 0$, then $f = 0$ on $B(a,R)$ and hence zero on U by the identity theorem. So we assume that $f(a) \neq 0$.

Choose r such that $0 < r < R$. We have, by the mean value property (8.6),

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta. \tag{8.10}$$

Hence we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{f(a)}\, d\theta = 1.$$

We consider $g(\theta) := \frac{f(a+re^{i\theta})}{f(a)}$ as a function of $\theta \in [0, 2\pi]$ and write $g(\theta) := u(\theta) + iv(\theta)$ as the sum of real and imaginary parts. Then $\int_0^{2\pi} (u + iv)\, d\theta = 2\pi$ so that $\int_0^{2\pi} u = 2\pi$ and $\int_0^{2\pi} v = 0$. Now $|u(\theta) + iv(\theta)| \leq 1$ for $\theta \in [0, 2\pi]$ by hypothesis. Hence, $|u(\theta)| \leq 1$ and $|v(\theta)| \leq 1$.

If $u(\theta_0) < 1$ for some θ_0, then by continuity $u(\theta) \leq \alpha < 1$ for $\theta \in (\theta_0 - \delta, \theta_0 + \delta) \cap [0, 2\pi]$. It follows that $\int_0^{2\pi} u(\theta)\, d\theta < 2\pi$, a contradiction.

Reason: Let us assume that $\theta_0 \in (0, 2\pi)$. Then

$$\begin{aligned}
\int_0^{2\pi} u(\theta)\, d\theta &= \int_0^{\theta_0 - \delta} u(\theta)\, d\theta + \int_{\theta_0 - \delta}^{\theta_0 + \delta} u(\theta)\, d\theta + \int_{\theta_0 + \delta}^{2\pi} u(\theta)\, d\theta \\
&< \int_0^{\theta_0 - \delta} 1\, d\theta + \int_{\theta_0 - \delta}^{\theta_0 + \delta} \alpha\, d\theta + \int_{\theta_0 + \delta}^{2\pi} 1\, d\theta \\
&= (\theta_0 - \delta) + (\theta_0 + \delta - (\theta_0 - \delta))\alpha + (2\pi - (\theta_0 + \delta)) \\
&< 2\pi.
\end{aligned}$$

If θ_0 is 0 or 2π, a similar argument can be carried out.

Hence $u(\theta) = 1$ for all θ. Since $|u + iv| \leq 1$, we infer $v = 0$. Since $0 < r < R$ is arbitrary, it follows that $f(a + re^{i\theta}) = f(a)$ for all θ and $0 < r < R$. That is, $f(z) = f(a)$ for $z \in B(a,R)$. By the uniqueness theorem, $f = f(a)$, a constant on U.

A slight variant of this proof runs as follows: If $|f(a)| \geq |f(z)|$ for all $z \neq a$ in $B(a,R) \subset U$, then we have for $0 < r < R$,

$$|f(a)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, dt = |f(a)|.$$

Thus we deduce that

$$\int_0^{2\pi} [|f(a)| - |f(a+re^{it})|] = 0.$$

Since the integrand is a nonnegative continuous function, we conclude that the integrand is zero. We therefore see that $|f(a)| = |f(a+re^{it})|$ for any $0 < r < R$ and $0 \le t \le 2\pi$. That is, $|f(a)| = |f(z)|$ for any $z \in B(a,R)$. By (v) of Ex. 5.7.1, $|f(z)|$ is a constant on $B(a,R)$ and hence f is a constant on $B(a,R)$ etc.

Proof 2 uses the power series expansion of f around a. In fact, we prove that if f is any nonconstant holomorphic function on a connected open set U and $a \in U$, then for all $r > 0$ such that $B(a,r) \subset U$, we can find $z \in B(a,r)$ so that $|f(z)| > |f(a)|$.

Consider the power series expansion of f in $B(a,R)$:

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots \qquad |z-a| < R.$$

Let $M := |f(a)|$ so that $|f(z)| \le M$ for all $z \in B(a,r)$ for some $0 < r < R$. We want to find $z \in B(a,r)$ such that $|f(z)| > |f(a)|$. If $c_0 = 0$, then $f(a) = 0$ and hence $M = 0$. This means that the function is identically zero. So, we assume that $c_0 \neq 0$. Assume $k \ge 1$ to be the first integer (if exists) such that $c_k \neq 0$.

> We suggest that the reader to assume $k = 1$ and go though the proof to get the idea of the proof.

Let $z = a + \delta e^{i\theta}$ for δ and θ to be chosen later. Then

$$\begin{aligned} |f(z)| &\ge \left|c_0 + c_k(z-a)^k\right| - \left|c_{k+1}(z-a)^{k+1} + c_{k+2}(z-a)^{k+2} + \cdots\right| \\ &\ge \left|c_0 + c_k\delta^k e^{ki\theta}\right| - \delta^{k+1}(|c_{k+1} + c_{k+2}(z-a) + \cdots|). \end{aligned} \tag{8.11}$$

Now, we choose θ so that c_0 and $c_k\delta^k e^{ki\theta}$ have the same argument. Let $c_0 = Me^{is}$ and $c_k = |c_k| e^{it}$. We choose θ so that $\theta = (s-t)/k$. Hence

$$\left|c_0 + c_k\delta^k e^{ki\theta}\right| = |c_0| + \left|c_k\delta^k\right|.$$

Now the second term in brackets in (8.11) is a convergent series which represents a continuous function, say, g in $B(a,R)$. Let $A > 0$ be such that $|g(z)| \le A$ for all $z \in B[a,r]$. Then we have

$$|f(z)| \ge M + |c_k|\delta^k - \delta^{k+1}A \ge M + \delta^k(|c_k| - A\delta) \qquad z \in B(a,r).$$

Choose δ sufficiently small so that $A\delta < |c_k|/2$. It then follows that $|f(z)| > |f(a)|$. ☐

Remark 8.6.2. The reader should probably recognize the argument of the second proof. For instance, the fundamental theorem of algebra can be proved directly using a similar argument. Let f be a nonconstant polynomial such that $f(z) \neq 0$ for any $z \in \mathbb{C}$. Let $R > 0$ be such that outside of $B(0, R)$, $|f(z)| \geq 1$. This is possible, since $|f(z)| \to \infty$ as $|z| \to \infty$. Now on the compact set $B[0, R]$, $|f|$ attains a minimum, say, α, say at a. By adapting the second proof, we can choose δ and θ so that the minimum is violated, that is, $|f(a + \delta e^{i\theta})| < |f(a)|$. The reader is strongly urged to write down the details of this proof.

Proof 3 uses the Parseval identity (8.7). Let $f(z) = \sum_0^\infty a_n (z - a)^n$ for $z \in B(a, R)$. Then we have the Parseval Identity:

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(a + re^{i\theta}) \right|^2 d\theta = \sum_0^\infty |a_n|^2 r^{2n}.$$

If a is a point of maximum for $|f|$ in $B[0, R]$, then LHS is less than or equal to $|f(a)|^2 = |a_0|^2$. This implies that

$$|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + \cdots \leq |a_0|^2,$$

for $0 < r < R$. Hence $a_k = 0$ for $k \geq 1$. Hence $f(z) = f(a)$ for $z \in B(a, R)$. □

The fourth proof is contained in the following theorem of Landau.

Theorem 8.6.3. *Let f be holomorphic in a connected open set U containing $B := B[\alpha, R]$. Let $M := \sup\{|f(z)| : z \in \partial B\}$. Then*
(i) *$|f(z)| \leq M$ for all $z \in B$.*
(ii) *If the maximum modulus is attained in the open ball $B(\alpha, R)$, then f is a constant on U, that is, if $a \in B(\alpha, R)$ is such that $|f(a)| = M$, then f is a constant in $B(\alpha, R)$ and hence in U.*

Proof. Let γ stand for the boundary ∂B with the standard parametrization: $\gamma(t) := \alpha + Re^{it}$. If $n \in \mathbb{N}$ and $a \in B$, we have by Cauchy integral formula,

$$2\pi i [f(a)]^n = \int_\gamma [f(z)]^n \frac{dz}{z - a}.$$

If $\delta := d(a, \partial B)$, then $\delta > 0$ and we have

$$|f(a)|^n \leq \frac{M^n \ell}{2\pi\delta},$$

where ℓ is the length of γ. Hence we see that

$$|f(a)| \leq M \left(\frac{\ell}{2\pi\delta} \right)^{1/n}.$$

As the left side is independent of n, we let $n \to \infty$ on both sides. By (v) of Ex. 2.1.9, $\left(\frac{\ell}{2\pi\delta}\right)^{1/n} \to 1$. Hence we deduce that $|f(a)| \leq M$. This proves (i).

To prove (ii), we plan to show that if a is a point of maximum of $|f|$ inside B, then all the derivatives of f at a are zero. Hence in the power series expansion of f near a, all the coefficients c_n for $n \geq 1$ are zero so that $f(z) = f(a) = c_0$ for all z near a. Since the domain is connected, the identity theorem will imply that f is a constant in the domain.

We employ Cauchy integral formula for the first derivative of f^n:

$$(2\pi i)nf'(a)[f(a)]^{n-1} = \int_\gamma [f(z)]^n \frac{dz}{(z-a)^2},$$

so that

$$n\,|f'(a)|\,|f(a)|^{n-1} \leq \frac{M^n \ell}{2\pi\delta^2}. \tag{8.12}$$

Now if a is such that $|f(a)| = M$ with $a \in B$, then (8.12) yields

$$|f'(a)| \leq \frac{M\ell}{2\pi\delta^2 n}.$$

Letting $n \to \infty$ in this, we get $|f'(a)| = 0$. We now proceed by induction. Assume that we have shown that $f^{(j)}(a) = 0$ for $1 \leq j < k$. We have $(\frac{d^k}{dz^k} f^n)(a) = nf^{(k)}(a)f^{n-1}(a) +$ terms involving $f^{(j)}(a)$ for $1 \leq j < k$. Hence by the Cauchy integral formula for the k-th derivative of f^n is given by

$$2\pi i n f^{(k)}(a)[f(a)]^{n-1} = (k!)\int_\gamma [f(z)]^n \frac{dz}{(z-a)^{k+1}},$$

so that as earlier we deduce that

$$\left|f^{(k)}(a)\right| \leq \frac{M\ell k!}{2\pi\delta^{k+1}n}.$$

Letting $n \to \infty$, it follows that $f^{(k)}(a) = 0$. □

Proof 5. Let p be a point of maximum of $|f|$ in U. Now, if $|f|$ is a constant, then f is a constant by CR equations. (See (v) of Ex. 5.7.1.)

If $|f|$ is not a constant, there exists $z_1 \in U$ such that $|f(z_1)| < M := |f(p)|$. Since U is connected, there exists a curve $\gamma\colon [0,1] \to U$ such that $\gamma(0) = p$ and $\gamma(1) = z_1$. Let $E := \{t \in [0,1] : |f(\gamma(t))| = M\}$. Since $0 \in E$, the set E is not empty. Let $t_0 = \sup E$. Then $0 \leq t_0 \leq 1$. Observe that $|f(\gamma(t_0))| = M$. For, there exists a sequence $t_n \in E$ such that $t_n \to t_0$ so that $\gamma(t_n) \to \gamma(t_0)$. Hence $|f(\gamma(t_0))| = M$. It follows that $t_0 < 1$ and that for $t > t_0$, $|f(\gamma(t))| < M$. Let $z_0 = \gamma(t_0)$. Choose $r > 0$, so small that $B[z_0, r] \subset U$ and $z_1 \notin B(z_0, r)$. There exists a $\tau > t_0$ such that $\gamma(\tau) \in S(z_0, r)$, that is, $|\gamma(\tau) - z_0| = r$. Let $\gamma(\tau) = z_0 + re^{i\varphi}$. Since $|f(\gamma(\tau))| < M$, there exists an arc parameterized by $A := (\varphi - \varepsilon, \varphi + \varepsilon)$, of $S(z_0, r)$ such that $|f(w)| < M$ for all $w \in A$.

Using the mean value property, we obtain

$$\begin{aligned} M = |f(z_0)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt \\ &= \frac{1}{2\pi}\left[\int_{\varphi-\varepsilon}^{\varphi+\varepsilon} |f(z_0+re^{it})|\,dt + \int_{[0,2\pi]\setminus A} |f(z_0+re^{it})|\,dt\right] \\ &< M, \end{aligned}$$

a contradiction. Hence we conclude that $|f|$ and hence f is a constant by Ex. 5.7.1. □

The case when U is bounded is often needed:

Corollary 8.6.4. *Let U be a bounded and connected open set in $\mathbb{C}$, $f \in H(U)$ and continuous on the closure $\overline{U}$ of U. Then either f is a constant or $|f|$ attains its maximum on the boundary of U.*

Proof. This is an easy consequence of the maximum modulus principle. We suggest that the reader works out a proof on his own.

If f is a constant, there is nothing to prove. So, we assume that f is not a constant. Let $M := \sup\{|f(z)| : z \in U\}$. Since $|f|$ is continuous on the closed and bounded set $\overline{U}$, it attains its maximum at some point $a \in \overline{U}$, that is, there exists $a \in \overline{U}$ such that $|f(a)| = M$. If $a \in U$, since U is a connected open set f is a constant on U by the maximum modulus principle. This contradicts our assumption. So $a \in \overline{U} \setminus U$, that is, a belongs to the boundary of U.

> Reason: The boundary of a subset A of a metric space X is the set of points p such that any open ball of the form $B(p,r)$ has points of A as well as points from $X \setminus A$. If A is open, then its boundary is $\overline{A} \setminus A$.

□

Even though this is of frequent use, we advise the reader to think of the formulation in Theorem 8.6.1. In this respect the following exercise is of significance.

Ex. 8.6.5. Let $U := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Consider $f : U \to \mathbb{C}$ given by $f(z) := e^z$. What is the boundary of U? What is the significance of this with respect to Corollary 8.6.4?

Remark 8.6.6. Neither the maximum modulus principle nor the above corollary have any analogue in real analysis. Consider $f(x) := 2 - x^2$ on $[-1, 1]$.

Can we have a minimum principle in an analogous manner? If we look at $f(z) = z$ on $B(0,1)$, we realize that we need to formulate it carefully.

Corollary 8.6.7 (Minimum Modulus Principle). *If f is a nonconstant holomorphic function on a connected open set U then $z \in U$ cannot be a relative local minimum of $|f|$ unless $f(z) = 0$.*

Proof. The reader should be able to supply his own proof.

If $f(z) = 0$ for some $z \in U$, then z is a point of minimum for $|f|$. So, let us assume that $f(z) \neq 0$ for any $z \in U$. Let us further assume, if possible, that $a \in U$ is such that $|f(a)| \leq |f(z)|$ for all $z \in U$. Then $g = 1/f$ is holomorphic on U and we have $|g(z)| \leq |g(a)|$ for all $z \in U$. Hence g is a constant by the maximum modulus principle. Consequently, f is a constant on U. ∎

Ex. 8.6.8. Let $f(z) = \frac{z^2}{z+2}$. Find the maximum of $|f|$ on $B[0,1]$.

Ex. 8.6.9. Let $f(z) = ze^z$. Find the maximum of $|f|$ on $\{z \in \mathbb{C} : |z|^2 \leq 4, \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$.

Ex. 8.6.10. Find the maximum and minimum moduli of $z^2 - z$ on $B[0,1]$.

Ex. 8.6.11. Let $U := \{z \in \mathbb{C} : |\operatorname{Re} z| < 1, |\operatorname{Im} z| < 1\}$. Let $f \in C(\overline{U}) \cap H(U)$ be such that $f(z) = 0$ if $\operatorname{Re} z = 1$. Show that $f = 0$ on $\overline{U}$. *Hint:* Consider $g(z) = f(z)f(-z)f(iz)f(-iz)$.

Ex. 8.6.12. Let f be continuous on $B[0,1]$ and holomorphic on $B(0,1)$. Assume that $f(z) = 0$ for z such that $\operatorname{Im} z \geq 0$ and $|z| = 1$. Then $f = 0$ on $B[0,1]$.

Ex. 8.6.13. Let $f \in C(B[0,1]) \cap H(B(0,1))$ be such that f vanishes on a nontrivial arc of the boundary. Then $f = 0$ on $B[0,1]$.

Ex. 8.6.14. If $f \in H(B(0,R))$ is nonconstant, then $M(r) := \max\{|f(z)| : |z| = r\}$ is a strictly increasing function of r on $(0,R)$.

Ex. 8.6.15. Let $f \in H(B(0,R))$ and $M(r) = r^n$ for $0 < r < R$. Show that $f(z) = cz^n$ for some c of unit modulus.

Ex. 8.6.16. Let U be a bounded open subset of $\mathbb{C}$. Let $f, f_n \in C(\overline{U}) \cap H(U)$. Assume that f_n converges uniformly to f on ∂U. Then f_n converges to f uniformly on $\overline{U}$.

Ex. 8.6.17. Let U be bounded, open and connected. Let $f \in H(U) \cap C(\overline{U})$ be nonconstant. Assume that $|f(z)| = 1$ for $z \in \partial U$. Then f vanishes at some point of U.

Ex. 8.6.18. Let U be bounded and open. Let $f \in C(\overline{U})$ be nonconstant, holomorphic on U. If $|f|$ is constant on ∂U, then $f(a) = 0$ for some $a \in U$.

Ex. 8.6.19. Let $f \in H(U)$ with U open and connected. Assume that $|f|$ is a constant. Then f is a constant. (We have already seen this by elementary means, see Ex. 3.2.2. On the other hand, one of our proofs, namely, the fifth one needed this result!)

Ex. 8.6.20. Derive the fundamental theorem of algebra using any of the results in this section.

Ex. 8.6.21. Find the maximum and minimum of $|f|$ where $f(z) := e^{z^2}$ on $B(0,1)$.

Ex. 8.6.22. Let U be connected, open. Let f be a nonconstant holomorphic function on U. Show that $\operatorname{Re} f$ cannot have relative maximum in U. How about an analogous result for relative minimum?

Ex. 8.6.23. Let U be bounded, open and connected. Assume f and g are continuous on $\overline{U}$ and holomorphic on U. Assume further $\operatorname{Re} f = \operatorname{Re} g$ on ∂U. Then there exists a real number c such that $f - g = ic$ on U.

Ex. 8.6.24. Let U be bounded and open. Let α_i, $1 \le i \le n$ be points of U. For $z \in \overline{U}$, define $d(z) := |z - \alpha_1| \cdots |z - \alpha_n|$. Show that d attains its maximum at some boundary point of U. Where are the minimum values attained?

Ex. 8.6.25. Let f and g be continuous on $\overline{U}$ and holomorphic on U. Assume that U is connected and that $f = g$ on ∂U. Then $f = g$ on U.

Ex. 8.6.26. If P is a polynomial of degree n show that the set $\{z : P(z) = \alpha\}$ can have at most n connected components for any $\alpha \in \mathbb{C}$.

Ex. 8.6.27. Let f be holomorphic on $B(a, 1)$ and $|f(z) - f(a)| \le k$ for $z \in B(a, 1)$. Show that

$$|f(z) - f(a)| \le k\,|z - a|, \qquad \text{for } z \in B(a, 1).$$

Ex. 8.6.28. For $|\alpha| < 1$, let $\varphi_\alpha(z) := \dfrac{z - \alpha}{1 - \overline{\alpha} z}$. Show that φ_α maps $B(0, 1)$ to itself.

Theorem 8.6.29 (Open Mapping Theorem). *The image of an open connected set U under a nonconstant holomorphic function is open.*

Proof. Let $V \subset U$ be open and $a \in V$. Assume without loss of generality that $f(a) = 0$. Then there exists an $r > 0$ such that $B[a, r] \subset V$ and such that $f(z) \neq 0$ on $B[a, r] \setminus \{a\}$ (Ex. 8.5.15).

> Reason: For, otherwise, $f = 0$ on the set $B(a, r)$ for any $r > 0$ with $B(a, r) \subset V$. The set $B(a, r)$ has cluster points in $V \subset U$ and hence by the identity theorem f is a constant on U – a contradiction.

Let $2\varepsilon := \min_{z \in S(a,r)} |f(z)|$. We claim that $f(B(a, r)) \supset B(0, \varepsilon)$.
Let $w \in B(0, \varepsilon)$ and consider $g(z) := f(z) - w$, for $z \in B(a, r)$. We have

$$|f(z) - w| \ge |f(z)| - |w| \ge 2\varepsilon - \varepsilon = \varepsilon, \qquad \text{for } z \in S(a, r)$$
$$|f(a) - w| = |-w| < \varepsilon.$$

Hence, $|g|$ attains its minimum somewhere in $B(a, r)$. By the minimum modulus principle g must be zero in $B(a, r)$. That is, there exists $z \in B(a, r)$ such that $w = f(z)$. Hence the claim is proved. Thus we have shown that for any arbitrary $a \in V$, there exists $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subset f(V)$. Therefore $f(V)$ is open. ☐

Ex. 8.6.30. Let f be an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$. Show that $f(\mathbb{C})$ is closed and hence conclude that f is onto.

Ex. 8.6.31. Prove the fundamental theorem of algebra using the open mapping theorem. *Hint:* Note that $P(\mathbb{C})$ is closed, since $|P(z)| \to \infty$ as $|z| \to \infty$.

Ex. 8.6.32. Deduce the maximum modulus principle from the open mapping theorem. (Later, we shall give a proof of the open mapping theorem which is independent of the maximum/minimum principles.)

Ex. 8.6.33. Consider $f(x) := x^2$ for $x \in \mathbb{R}$. What is its relevance to open mapping theorem?

Ex. 8.6.34. Let $U \subset \mathbb{C}$ be connected and open. Assume that $f, g \in H(U)$ and $\overline{f}g \in H(U)$. Show that either f is a constant or $g = 0$.

We shall close this chapter with one more application.

Theorem 8.6.35 (Schwarz Lemma). *Let $f\colon B(0,1) \to B(0,1)$ be holomorphic. Assume that $f(0) = 0$. Then*

(i) $|f(z)| \le |z|$ *for all* $z \in B(0,1)$ *and*
(ii) $|f'(0)| \le 1$.

Furthermore, equality holds in (i) for some nonzero z or equality holds in (ii) iff $f(z) = cz$ for some c with $|c| = 1$.

Proof. Consider

$$g(z) := \begin{cases} \dfrac{f(z)}{z} & z \neq 0,\ |z| < 1 \\ f'(0) & z = 0. \end{cases}$$

Then g is continuous on $B(0,1)$ and holomorphic on the punctured disk. Hence, by the second application of Morera's theorem (on page 119), g is holomorphic on $B(0,1)$. We have

$$|g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{r}, \qquad \text{for } |z| = r.$$

Now applying the maximum modulus principle for g on the closed disk $B[0,r]$, we see that $|g(z)| \le \frac{1}{r}$ for $z \in B[0,r]$. Since $B[0,r] \subset B[0,s]$ for $r < s$, we see that

$$|g(z)| \le \frac{1}{s}, \qquad z \in B[0,r], \quad \text{for all } s > r.$$

We thus deduce that $|g(z)| \le 1$ for $|z| < r$ and hence for all $z \in B(0,1)$.

Let $|g(z)| = 1$ for some z. By maximum modulus principle, then $g = c$, a constant. Necessarily, $|c| = 1$. ∎

Ex. 8.6.36. Let $f\colon B(0,1) \to B(0,1)$ be bijective and holomorphic with holomorphic inverse. Assume that $f(0) = 0$. Then there exists c such that $|c| = 1$ and $f(z) = cz$ for all z.

Ex. 8.6.37. Let φ_α be as in Ex. 8.6.28. Then φ_α is a bijection on $B(0,1)$ with inverse $\varphi_{-\alpha}$. Show that any bijective holomorphic map f on $B(0,1)$ with holomorphic inverse is of the form φ_α.

Ex. 8.6.38. We slightly modify our definition of φ_α as follows:

$$\phi_a(z) := \frac{a-z}{1-\bar{a}z}.$$

Prove the following:

(i) We have $\phi_a(0) = a$ and $\phi_a(a) = 0$.

(ii) $\phi_a^{-1} = \phi_a$.

(iii) We have $|\phi_a'(0)| = 1 - |a|^2$ and $|\phi_a'(a)| = \frac{1}{1-|a|^2}$.

Chapter 9

Isolated Singularities and Laurent Series

9.1 Isolated Singularities

Definition 9.1.1. Let U be open, $a \in U$. A function $f\colon U \setminus \{a\}$ is said to have an *isolated singularity* at $z = a$ if f is holomorphic on a deleted neighbourhood of a.

We let $B'(a, r)$ stand for the punctured disk $B(a, r) \setminus \{a\}$.

Example 9.1.2. Consider $\tan(\frac{1}{z})$, for $z \neq 0$. Then $z = 0$ is not an isolated singular point. Similarly, $z = 0$ is not an isolated singularity for the function $z \mapsto 1/\sin(1/z)$, for $z \neq 0$.

Example 9.1.3. Consider the three functions

1. $z \mapsto \frac{\sin z}{z}$, $z \neq 0$.

2. $z \mapsto \frac{\cos z}{z}$, $z \neq 0$.

3. $z \mapsto e^{\frac{1}{z}}$, $z \neq 0$.

For all these function $z = 0$ is an isolated singularity.

If we examine $z \mapsto \frac{\sin z}{z}$ more closely, we find that there is nothing wrong with it at $z = 0$. As $z \to 0$, $\frac{\sin z}{z} \to 1$. So the function

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0, \end{cases}$$

is continuous at $z = 0$.

From the extended version of Cauchy-Goursat theorem and Morera's theorem, we know that this latter function is holomorphic (See the second observation after Morera's theorem on p. 119). Thus by defining a suitable value for the function at 0, we have been able to remove the singularity. Accordingly, we call $z = 0$ as a *removable singularity* of the function $\frac{\sin z}{z}$ defined on $\mathbb{C} \setminus \{0\}$.

The singularities at $z = 0$ of $z \mapsto \frac{\cos z}{z}$, $z \mapsto e^{1/z}$ are not removable. If f has a singularity at $z = 0$, a necessary condition for the singularity to be removable is that $\lim_{z\to 0} f(z)$ exists.

$\cos 0 = 1$, hence by continuity there exists a neighborhood of $z = 0$ on which $|\cos z| > \frac{1}{2}$. If we take z sufficiently small, we can make $\left|\frac{\cos z}{z}\right|$ greater than any positive number. This means that $\left|\frac{\cos z}{z}\right|$ becomes arbitrarily large, as $z \to 0$. We usually write this as $\lim_{z\to 0} \frac{\cos z}{z} = \infty$. (See Definition 5.1.6.)

But on the other hand, $e^{1/z}$ does not have any limit as $z \to 0$. For $x > 0$, $x \to 0$, $e^{1/z} = e^{1/x} \to \infty$ and $-x > 0$, $x \to 0$, $e^{-1/x} \to 0$. For $z = iy$, $y \to 0$, $y \in \mathbb{R}$, $e^{1/z} = e^{1/iy}$ does not have a limit.

Definition 9.1.4. Let U be an open set in $\mathbb{C}$ and $a \in U$. Assume that $f\colon U \setminus \{a\} \to \mathbb{C}$ has an isolated singularity at a. If

1. $\lim_{z\to a} f(z)$ exists (as a finite complex number), then a is called a *removable singularity* of f.

2. $\lim_{z\to a} f(z) = \infty$, then a is called a *pole* of f.

3. $\lim_{z\to a} f(z)$ does not exist, then a is called an *essential singularity* of f.

Notice that these cases are mutually exclusive.

Remark 9.1.5. Let $z = a$ be a removable singularity. If we define $f(a) := \lim_{z\to a} f(z)$, then f is continuous on $B(a, r)$ for some $r > 0$ and holomorphic in $B'(a, r)$. Hence by the second observation after Morera's theorem on p. 119, f is holomorphic in $B(a, r)$. This explains the reason why the singularity is called removable.

Theorem 9.1.6 (Riemann's Theorem on Removable Singularity). *Let f be holomorphic in $B'(a, r)$. Assume that $\lim_{z\to a}(z - a)f(z) = 0$. Then $\lim_{z\to a} f(z)$ exists so that $z = a$ is a removable singularity of f.*

Proof. Consider the function g defined by

$$g(z) = \begin{cases} (z-a)^2 f(z), & z \neq a \\ 0, & z = a. \end{cases}$$

Then g is holomorphic at a. For, $g'(z)$ exists for $z \neq a$. We also have $g'(a) = 0$ for $\frac{g(z)-g(a)}{z-a} = (z - a)f(z) \to 0$ as $z \to a$, by hypothesis on f.

We now expand g into the Taylor series about a. The first two terms are zero, so

$$g(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n = (z-a)^2 h(z)$$

where h is holomorphic at a.

Hence $(z-a)^2h(z) = (z-a)^2f(z)$ for all $z \neq a$ so that $h(z) \equiv f(z)$ for $z \neq a$. Moreover, $\lim_{z\to a} h(z)$ exists and hence $\lim_{z\to a} f(z)$ exists. If we redefine $f(a)$ as this limit, we have $f(z) = h(z)$ and so f is holomorphic at a. □

Corollary 9.1.7. *Let $z = a$ be an isolated singularity of f. If either (i) f is bounded in a punctured disk centred at a or (ii) if f is continuous at a, then f has a removable singularity at $z = a$.* □

Remark 9.1.8. Let $f, g\colon B'(a,r) \to \mathbb{C}$ be functions. We say that $f = O(g)$ as $z \to a$ if there exist $C > 0$ and $\delta > 0$ such that for $z \in B'(a,\delta)$, we have $|f(z)| \leq C\,|g(z)|$. One reads $f = O(g)$ as f is big oh of g. This notation is due to Landau.

It is worth noting that the boundedness of $|f(z)|$ can be replaced by $f(z) = O(|z-a|^{-c})$, $0 \leq c < 1$.

Remark 9.1.9. Note that there could be no analogue of Riemann's theorem on removable singularity in real analysis. For instance, consider the function $f(x) := \frac{1}{\sqrt{|x|}}$ for $x \in \mathbb{R}$ with $x \neq 0$. Then $\lim_{x\to 0} xf(x) = 0$, but there is no way we could assign a value for f at $x = 0$ so that it becomes continuous at 0, leave alone being differentiable. Or, consider $g(x) := |x|$ for $x \in \mathbb{R}$. It is continuous on all of $\mathbb{R}$ and differentiable on $\mathbb{R}^*$. But the singularity is not removable.

Ex. 9.1.10. Let $U \subset \mathbb{C}$ be open and $a \in U$. Let a be a zero of order m for f. Then $g(z) := (z-a)^{-m}f(z)$ has a removable singularity at a.

Now let us have a closer look at functions with a pole.

If $|f(z)| \to \infty$ as $z \to a$, there will be a disk centered at a, in which $|f(z)| > 1 > 0$. Consider $g(z) = \frac{1}{f(z)}$ in this disk. In the punctured disk, g is holomorphic and also bounded. So g must have a removable singularity at a. This singularity is removed by defining $g(a) = 0$. Let the power series expansion for g at a be given by $g(z) = \sum_{k=0}^{\infty} c_k(z-a)^k = (z-a)^m \sum_{k=m}^{\infty} c_k(z-a)^{k-m}$, where m is the first positive integer k such that $c_k \neq 0$. We can therefore write $g(z) = (z-a)^m\phi(z)$ where $\phi(z)$ is holomorphic in the disk and $\phi(a) \neq 0$. Thus we have shown that if $z = a$ is a pole of f, then there exists a positive integer m and a holomorphic function $g \in H(B(a,\delta))$ for some $\delta > 0$ so that $g(a) \neq 0$ and $f(z) = \frac{g(z)}{(z-a)^m}$ for $z \in B'(a,\delta)$.

We claim such an expression is unique. That is, if $f(z) = \frac{g(z)}{(z-a)^m} = \frac{h(z)}{(z-a)^n}$ for some holomorphic functions g, h with $g(a) \neq h(a)$ and for all $z \in B'(a,\delta)$, then $m = n$ and $g = h$ on this disk. Let, if possible, $m < n$. Then we have

$$g(a) = \lim_{z\to a}(z-a)^m f(z) = \lim_{z\to a}(z-a)^{m-n}h(z) = \infty.$$

Hence $m \geq n$ and similarly, $n \geq m$. Let us summarize the above as a theorem.

Theorem 9.1.11. *Let U be open and $a \in U$. Let $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic. Then f has a pole at $z = a$ iff there exists a holomorphic function $g \in H(U)$ and a positive integer m such that $g(a) \neq 0$ and $f(z) = \frac{g(z)}{(z-a)^m}$ in a deleted open disk centred at a. The integer m is called the order of the pole a.* □

Ex. 9.1.12. Show that the conclusion of Theorem 9.1.11 is not valid if g has a zero at a. *Hint:* Consider the functions $\frac{1-\cos z}{z^2}$ and $\frac{\sin z}{z^2}$ at the point $z = 0$.

Ex. 9.1.13. Show that an isolated singularity $z = a$ of f is a pole of order m iff m is the smallest positive integer n such that $(z-a)^n f(z)$ is bounded in a deleted neighborhood of a.

Lemma 9.1.14. *Let f be holomorphic in $B'(a, R)$. Assume that f has a pole of order m at a. Let $g(z) := (z-a)^m f(z)$ in $B'(a, R)$. Then*

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{(z-a)} + h(z), \qquad z \in B'(a, R), \tag{9.1}$$

where $h \in H(B(a, R))$ and $a_k = \frac{g^{(m+k)}(a)}{(m+k)!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw$ for $\gamma = S(a, r)$, $0 < r < R$.

Proof. We claim that g has removable singularity at a. For, by the last theorem, there exists a holomorphic function, say φ on $B(a, R)$ such that $f(z) = \varphi(z)/(z-a)^m$ in $B'(a, R)$. Hence $g(z) = \varphi(z)$ on $B'(a, R)$. We conclude that g has removable singularity at a. We assume that $g(a)$ is suitably defined so that g is holomorphic around a. Using the power series expansion for g at a, we get

$$g(z) = \sum_{k=0}^{m-1} \frac{g^{(k)}(a)}{k!}(z-a)^k + (z-a)^m h(z). \tag{9.2}$$

Dividing (9.2) by $(z-a)^m$ and making use of CIF for derivatives for g yields (9.1). □

Remark 9.1.15. The series involving negative k's is called the *principal* or *singular* part of f at a. In the next chapter, we shall find an expansion for f analogous to (9.1) at any isolated singularity.

Proposition 9.1.16. *Let f have an isolated singularity at $z = a$. Then f has a pole of order m iff for some $r > 0$, there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 |z-a|^{-m} \leq |f(z)| \leq C_2 |z-a|^{-m}, \qquad z \in B'(a, r). \tag{9.3}$$

Proof. Let f have a pole of order m. We then write $f(z) = g(z)/(z-a)^m$ in $B'(a, R)$ for some $R > 0$ with $g \in H(B(a, R))$ and $g(a) \neq 0$. Hence if $0 < r < R$ is such that $g(z) \neq 0$ for $z \in B[a, r]$, we define $C_1 := \inf\{|g(z)| : z \in B[a, r]\}$ and $C_2 := \sup\{|g(z)| : z \in B[a, r]\}$. It is easy to see that (9.3) holds with these definitions.

Conversely, (9.3) shows that $|(z-a)^m f(z)| \leq C_2$ in $B'(a,r)$ and hence $g(z) := (z-a)^m f(z)$ has a removable singularity. But since $\left|(z-a)^{m-1} f(z)\right| \geq C_1 \left|z-a\right|^{-1}$, we deduce $h(z) := (z-a)^{m-1} f(z)$ has a pole at a. Now $g(a) \neq 0$, since otherwise, we have $\lim_{z\to a}(z-a)h(z) = \lim_{z\to a} g(z) = g(a) = 0$ so that h has a removable singularity at a, a contradiction. Hence $g(a) \neq 0$ and hence f has a pole of order m at a, by Theorem 9.1.11. □

Remark 9.1.17. There is no holomorphic function which approaches ∞ like a fractional power of $\frac{1}{z-a}$ in the neighborhood of an isolated singularity. For example, if f were holomorphic in a deleted neighborhood of 0 and satisfied $|f(z)| \leq \frac{1}{\sqrt{|z|}}$, then f would be bounded since the singularity will be removable.

Similarly, given that $|f(z)| \leq \frac{1}{|z|^{\frac{5}{2}}}$, we conclude that $z^2 f(z)$ has a removable singularity at 0. Hence f has a pole of order at most 2 and hence $|f(z)| \leq \frac{C}{|z|^2}$.

Ex. 9.1.18. Prove that f has a pole of order m at a iff $1/f$ has a zero of order m at a. (There is a subtle point here. Why does it make sense to talk of the order of zero of $1/f$? Recall that we have introduced such a notion only in the context of holomorphic functions!)

Ex. 9.1.19. f has a pole of order m at $z = a$ iff

$$\lim_{z\to a}(z-a)^m f(z) \neq 0 \text{ and } \lim_{z\to a}(z-a)^{m+1} f(z) = 0.$$

Ex. 9.1.20. Let $z = a$ be an isolated singularity of f. Then it is (1) a removable singularity iff $|f|$ is bounded in a punctured disk around a, (2) a pole iff $\lim_{z\to a}|f(z)| = \infty$ and (3) an essential singularity iff $\lim_{z\to a}|f(z)|$ does not exist.

Ex. 9.1.21. Let f be nonconstant and holomorphic in $B(a,R)$. Show that $1/f$ is either holomorphic in some neighbourhood of a or has pole at a.

Ex. 9.1.22. Prove the following version of l'Hospital's rule: Let f and g be holomorphic in $B(a,r)$. Assume that f (resp. g) has a zero of order m (resp. n) at a. Then f/g has an isolated singularity at a. It is (i) removable if $m \geq n$ and (ii) a pole of order $n-m$ if $n > m$. Furthermore, when $m = n$, we have

$$\lim_{z\to a}\frac{f(z)}{g(z)} = \lim_{z\to a}\frac{f^{(m)}(z)}{g^{(m)}(z)}.$$

Ex. 9.1.23. Decide which of the following functions have removable singularities at the indicated points.

(1) $\dfrac{\sin z}{z^2-\pi^2}$ at $z = \pi$ (2) $\dfrac{\sin z}{(z-\pi)^2}$ at $z = \pi$

(3) $\dfrac{1-\cos z}{z}$ at $z=0$ (4) $\dfrac{\sin z}{z^2+z}$ at $z=0$

(5) $z\cot z$ at $z=0$ (6) $z\cot z$ at $z=\pi$

(7) $\dfrac{\cos z}{1-\sin z}$ at $z=\dfrac{\pi}{2}$ (8) $\dfrac{e^z}{z}$ at $z=0$

(9) $\dfrac{(e^z-1)^2}{z^2}$ at $z=0$ (10) $\dfrac{z}{e^z-1}$ at $z=0$

Ex. 9.1.24. Decide which of the following points have poles at the indicated points. Find the order of the poles.

(1) $\dfrac{e^{z^2}-1}{z^3}$ at 0 (2) $\dfrac{\cos z-1}{z^2}$ at 0

(3) $\dfrac{z}{1-\sin z}$ at 0 (4) $\tan^2 z$ at $\dfrac{\pi}{2}$

(5) $\displaystyle\sum_0^\infty n(-1)^n z^n$ at -1 (6) $\dfrac{\cos z}{\frac{\pi}{2}-z}$ at $\dfrac{\pi}{2}$.

(7) $\dfrac{\cos z}{z^2}$ at 0 (8) $\dfrac{e^z-1}{z^2}$ at 0

Ex. 9.1.25. Find and classify the singularities of the following functions:

(1) $\dfrac{1-\cos z}{z}$ (2) $\dfrac{1-\cos z}{z^2}$

(3) $\dfrac{1-\cos z}{z^3}$ (4) $\dfrac{z}{1-\cos z}$

(5) $\dfrac{z^2}{1-\cos z}$ (6) $\dfrac{z^3}{1-\cos z}$

(7) $\tan z$ (8) $\cot z$

(9) $\dfrac{1}{e^z-1}$ (10) $\dfrac{e^z-1}{z}$

(11) $\dfrac{z^4-2z^2+1}{(z-1)^2}$ (12) $\dfrac{1}{z(e^z-1)}$

Ex. 9.1.26. Let $f_n(z) := \frac{1}{z^n[\exp(z)-1]}$ on $B'(0,2\pi)$. Then f_n has a pole of order $n+1$ at $z=0$.

Ex. 9.1.27. Let $f(z) := \exp(\frac{z+1}{z-1})$, $z\in\mathbb{C}\setminus\{1\}$. Show that f is bounded on $B(0,1)$ and has an essential singularity at 1.

Ex. 9.1.28. Show that $f(z) = e^{\sin z/z}$ has a removable singularity at $z=0$ and an essential singularity at $z=k\pi$ for $k\in\mathbb{Z}$, $k\neq 0$. *Hint:* Draw pictures and see what happens as real $z\to k\pi$ only along the real axis, $k\neq 0$.

Ex. 9.1.29. Let f be holomorphic in $B(0,R)$ and have a zero of order m at $z=0$. If g is holomorphic in $0<|z|<R$ and has a pole of order l and if $l\leq m$ at $z=0$, then fg has a removable singularity at $z=0$.

What can you conclude when $l > m$?

Definition 9.1.30. A function holomorphic in a region except for (isolated singularities which are) poles is said to be *meromorphic* in that region.

We shall have a closer look at the class of meromorphic functions in Section 9.4.

Example 9.1.31. A rational function f is of the form $f(z) = P(z)/Q(z)$ where P and Q are polynomials. Any rational function is meromorphic in $\mathbb{C}$.

Ex. 9.1.32. Show that $\tan z$, $\frac{\tan z}{z^2+1}$ are meromorphic in $\mathbb{C}$ but $\exp(1/z)$ is not.

Ex. 9.1.33. If f is an entire function which is not identically zero, then $1/f$ is meromorphic in $\mathbb{C}$.

Ex. 9.1.34. Let a be an isolated singularity of f in an open set U. For $r > 0$, sufficiently small, let

$$\begin{aligned} m(f,r) &= \min\{|f(z)| : |z-a| = r\} \\ M(f,r) &= \max\{|f(z)| : |z-a| = r\}. \end{aligned}$$

Then we have $m(f,r) \leq |f(z)| \leq M(f,r)$ for $z \in S(a,r)$.

If a is a pole of order m, then the functions $r \mapsto m(f,r)$ and $r \mapsto M(f,r)$ have the same limit behaviour as $r \to 0$:

$$\lim_{r\to 0} \frac{m(f,r)}{r^{-m}} = \lim_{r\to 0} \frac{M(f,r)}{r^{-m}}$$

so that $\lim_{z\to a} \frac{|f(z)|}{|z-a|^{-m}}$ exists and is finite.

Theorem 9.1.35 (Casorati-Weierstrass Theorem). *Let a be an essential singularity of f in U. Then $f(B'(a,r))$ is dense in $\mathbb{C}$, for any $r > 0$ with $B(a,r) \subset U$.*

Proof. If it were false, then there exists an $\alpha \in \mathbb{C}$, $\varepsilon > 0$ and an $r > 0$ such that $|f(z) - \alpha| > \varepsilon$ for all $z \in B'(a,r)$.

Since we assume that the result is false, it means that there exists $r > 0$ such that $f(B'(a,r))$ is not dense in $\mathbb{C}$. This, in turn, means that there exists a nonempty open set U such that which does not have any point in common with $f(B'(a,r))$. Since U is nonempty and open, if we take any $\alpha \in U$ and $\varepsilon > 0$ such that $B(\alpha,\varepsilon) \subset U$. So, we therefore see that $f'(B'(a,r)) \cap B(\alpha,\varepsilon) = \emptyset$.

Then, $\frac{1}{f(z)-\alpha}$ is bounded in the deleted neighborhood of a. We conclude that $\frac{1}{f(z)-\alpha}$ has a removable singularity at a. Thus $\frac{1}{f(z)-\alpha} \to \lambda$ for some λ as $z \to a$.

Now there are two possibilities: If $\lambda = 0$, then $|f(z) - \alpha| \to \infty$ as $z \to a$. Hence $|f(z)| \to \infty$ as $z \to a$. Thus f has a pole at a, a contradiction.

If $\lambda \neq 0$, then $f(z) - \alpha \to \frac{1}{\lambda}$ so f has a removable singularity at a, again a contradiction. $\square$

There is a much stronger theorem than the above which is stated below:

Theorem 9.1.36 (Picard's big theorem). *Let $U \subset \mathbb{C}$ be open. Let f have an essential singularity at $a \in U$. Then f assumes every value with at most one exception in every deleted neighbourhood of a.* □

Its proof is quite difficult. A proof of Picard's theorem is presented in Chapter ??.

Remark 9.1.37. This remark may be omitted on the first reading. We point out a significant application of Proposition 9.1.16 and Theorem 9.1.35.

Consider the problem of defining z^α for non-integral real α. We claim that there is no way of defining the function so as to make $z \mapsto z^\alpha$ differentiable on some $B'(0,r)$. Suppose we can do it. Then, if w is any logarithm of z, then $|z|^\alpha = e^{\alpha w}$. (Why? If we write $w = x + iy$, then $|z| = |e^w| = e^x$ and hence $|z|^\alpha = e^{\alpha x}$.) There is no integer n for which $|z|^{\alpha - n}$ is bounded away from 0 on any $B'(0,r)$. Nor does $|z|^\alpha$ come arbitrarily close to all the real values on any $B'(0,r)$. Therefore, there is no way of choosing logarithm so as to make $z \mapsto z^\alpha$ differentiable on any $B'(0,r)$.

Corollary 9.1.38. *Let f be a nonconstant entire function. Then f is not a polynomial iff $g(z) := f(1/z)$, $z \in \mathbb{C}^*$ has an essential singularity at $z = 0$.*

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If 0 is not an essential singularity for g, then for large $n \in \mathbb{N}$, $z^n g(z)$ must have a removable singularity at 0. For these values of n, by Cauchy's theorem,

$$0 = \int_{S(0,1)} w^n g(w)\, dw = \sum_k a_k \int_{S(0,1)} w^{n-k}\, dw = 2\pi i a_{n+1}.$$

Thus f is a polynomial of degree at most n. Converse is easy.

If $f(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial, then g has only a pole of order $\deg f$. Go through the proof of the first part and repeat the same argument, if you are not convinced!

□

Ex. 9.1.39. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Ex. 9.1.40. Let f be entire and not a polynomial. Then $f(\mathbb{C} \setminus B(0,R))$ is dense in $\mathbb{C}$ for any $R > 0$. (Intuitively, the point at infinity is an essential singularity of f. See Proposition 13.3.8.)

Ex. 9.1.41. Consider $f(z) = \sin(1/z)$ near $z = 0$. Find sequences (z_n) and (w_n) such that (i) $\lim z_n = 0 = \lim w_n$ and (ii) $\lim_n f(z_n) = \infty$ and $\lim f(w_n) = 0$. Thus $z = 0$ is an essential singularity of f.

Ex. 9.1.42. Let f be holomorphic in $B'(a,R)$. Assume that f does not take values in $B(b,\delta)$. Prove that a is a removable singularity. (Why cannot it be a pole?)

Ex. 9.1.43. Show that in each neighbourhood of $z = 0$, the equation $e^{1/z} = \alpha$ for $\alpha \in \mathbb{C}^*$ has infinitely many solutions. *Hint:* $e^{1/z} = \alpha$ for $|\alpha| < r$ is equivalent to $e^w = \alpha$ for $|w| > 1/r$. Use periodicity of exp.

Ex. 9.1.44. Keep the notation of Ex. 9.1.34. Let $f(z) = e^{1/z}$ for $z \neq 0$. Compute $m(f, r)$ and $M(f, r)$ explicitly. Do they have the same limit behaviour as $r \to 0$? Compare this with the conclusion of Ex. 9.1.34.

Ex. 9.1.45. Let a be an isolated singularity of a nonzero (that is, not identically zero) f in a connected open set U. Assume further that a is a cluster point of zeros of f in U. Show that a is an essential singularity of f.

9.2 Laurent Series

Definition 9.2.1. A series $\sum_{n=-\infty}^{\infty} a_n$ is convergent if there exists $s \in \mathbb{C}$ such that for a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|(a_{-m} + a_{-m+1} + \cdots + a_{-1} + a_0 + a_1 + \cdots + a_n) - s| < \varepsilon, \text{ for } m, n > N.$$

The complex number s is called the sum of the series $\sum_{-\infty}^{\infty} a_n$.

A series $\sum_{n=-\infty}^{\infty} f_n$ of functions defined on a set U is said to be uniformly convergent on U if there exists a function f on U such that for a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with the property that

$$|(f_{-m}(z) + f_{-m+1}(z) + \cdots + f_{-1}(z) + f_0(z) + f_1(z) + \cdots + f_n(z)) - f(z)| < \varepsilon,$$
$$\text{for } m, n > N, \text{ for all } z \in U.$$

Ex. 9.2.2. Let the notation be as in the above definition. Let $b_n := a_{-n}$. Show that the series $\sum_{-\infty}^{\infty} a_n$ is convergent to s iff the series $\sum_{n=1}^{\infty} b_n$ and the series $\sum_{n=0}^{\infty} a_n$ are convergent to sums s_1 and s_2 and we have $s = s_1 + s_2$.

State and prove an analogue for the series of functions.

Ex. 9.2.3. Show by an example that the statement $a_{-n} + \cdots + a_n \to s$ as $n \to \infty$ is not the same as $\sum_{n=-\infty}^{\infty} a_n = s$.

Lemma 9.2.4. *Let $R > 0$ be the radius of convergence of the power series $f(z) := \sum_{n=0}^{\infty} c_n z^n$. Then $\sum_{n=0}^{\infty} c_n (z-a)^{-n}$ is convergent for $|z - a| > 1/R$.*

In fact, the convergence is uniform on $\{z : |z - a| \geq r\}$ where $r > 1/R$. Let the sum be g in $|z| > 1/R$. We have $g'(z) = -\sum_{n=1}^{\infty} nc_n(z-a)^{-n-1}$, for $|z - a| > 1/R$.

Proof. Define $g(z) := f(\frac{1}{z-a})$, for $|z - a| > 1/R$. Given $\varepsilon > 0$, by the uniform convergence of $\sum c_n (z-a)^n$ in $B(a, \rho)$ for $0 < \rho < R$, there exists $N \in \mathbb{N}$ such that

$|f(z) - \sum_0^n c_k(z-a)^k| < \varepsilon$ for $n \geq N$ and $z \in B(a,\rho)$. Hence we deduce that

$$\left| g(z) - \sum_0^n c_k(z-a)^{-k} \right| < \varepsilon \text{ whenever } n \geq N \text{ and } |z-a| \geq 1/\rho.$$

Since $f'(z) = \sum_1^\infty nc_n(z-a)^{n-1}$ in $B(a,R)$, we have by the chain rule, for $|z-a| > 1/R$,

$$g'(z) = -(z-a)^{-2} f'\left(\frac{1}{(z-a)}\right) = -\sum_1^\infty nc_n(z-a)^{-n-1}.$$

□

Let f be holomorphic in a deleted neighbourhood $B'(a,R)$ of a. The aim of this section is to represent f as an infinite series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ in an annulus centred at a. The key to this expansion is a Cauchy integral formula for f in an annulus. See Lemma 9.2.7.

Definition 9.2.5. Let $0 < r < R \leq \infty$. The set

$$A(a;r,R) := \{z \in \mathbb{C} \mid r < |z-a| < R\}$$

is called an annulus. When $R = \infty$, note that $A(a;r,R)$ is $\mathbb{C} \setminus B[a,r]$.

Lemma 9.2.6. *Let $F\colon B'(a,R) \to \mathbb{C}$ be holomorphic. For $0 < r < R$, let $\gamma_r := a + re^{it}$, $0 \leq t \leq 2\pi$. Then the function $r \mapsto \int_{\gamma_r} F$ is a constant. In particular, if $0 < r,s < R$, $\int_{\gamma_r} F = \int_{\gamma_s} F$.*

Proof. Let $\varphi(r) := \int_{\gamma_r} F$, $r \in (0,R)$. We prove $\varphi'(r) = 0$. By Leibnitz rule of differentiation under the integral sign (Theorem 6.1.17 on page 80), we have

$$\begin{aligned}
\varphi'(r) = \int_0^{2\pi} \frac{\partial}{\partial r}[F(a+re^{it})ire^{it}]\,dt &= \int_0^{2\pi} [F'(a+re^{it})re^{2it} + F(a+re^{it})]ie^{it}\,dt \\
&= \int_0^{2\pi} \frac{\partial}{\partial t}[F(a+re^{it})e^{it}]\,dt = 0.
\end{aligned}$$

We conclude that F is a constant on $(0,R)$. □

Lemma 9.2.7 (Cauchy Integral Formula for an Annulus). *Let f be holomorphic in a punctured disk $B'(a,R)$. Let $z \in B'(a,R)$. Choose r_1 and r_2 such that $0 < r_1 < r_2 < R$. Let $\gamma_j(t) := a + r_j e^{it}$, $0 \leq t \leq 2\pi$. Then, for z such that $r_1 < |z-a| < r_2$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw. \tag{9.4}$$

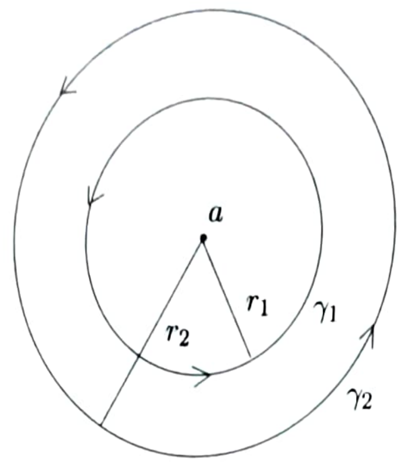

Figure 9.1: Annular region

Proof. Consider

$$F(w) := \begin{cases} \frac{f(w)-f(z)}{w-z} & w \in B'(a,R),\ w \neq z \\ f'(z) & w = z. \end{cases}$$

Then F is holomorphic in $B(a,R)$ by Riemann's theorem on removable singularity. Applying the last Lemma 9.2.6 to $\frac{1}{2\pi i}F$, we get

$$\frac{1}{2\pi i}\int_{\gamma_2} F = \frac{1}{2\pi i}\int_{\gamma_1} F. \tag{9.5}$$

Since $z \notin [\gamma_j]$, we see that

$$\frac{1}{2\pi i}\int_{\gamma_j} F \tag{9.6}$$

$$= \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)-f(z)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z}\,dw - f(z)\frac{1}{2\pi i}\int_{\gamma_j} \frac{1}{w-z}\,dw. \tag{9.7}$$

Note that $\frac{1}{2\pi i}\int_{\gamma_2}\frac{dw}{w-z} = 1$ whereas $\frac{1}{2\pi i}\int_{\gamma_1}\frac{dw}{w-z} = 0$. Using this observation and (9.7) in (9.5) yields the result. ∎

Theorem 9.2.8 (Laurent Expansion). *Let f be holomorphic on the annulus $A := A(a;r,R)$. Then f is given by*

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n, \quad z \in A,$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw$$

and $\gamma = \gamma_\rho$ *is* $\gamma_\rho(t) = a + \rho e^{it}$, $0 \le t \le 2\pi$, *for any* $r < \rho < R$. *The series converges absolutely and uniformly on compact subsets of* A.

Proof. Choose r_j such that $r < r_1 < |z-a| < r_2 < R$. From Lemma 9.2.7, we get

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}dw = f_2 + f_1, (\text{say}).$$

To find the series expansion for f_j's, we argue as in the power series expansion from Cauchy's Integral formula:

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-a+a-z}\\
&= \frac{1}{w-a-(z-a)}\\
&= \begin{cases} \dfrac{-1}{(z-a)\left(1-\dfrac{w-a}{z-a}\right)} & \text{if } w \in [\gamma_1]\\[2ex] \dfrac{1}{(w-a)\left(1-\dfrac{z-a}{w-a}\right)} & \text{if } w \in [\gamma_2]. \end{cases}
\end{aligned}$$

Observe that $\left|\frac{w-a}{z-a}\right| < 1$ for $w \in [\gamma_1]$ and $z \in A$ and that $\left|\frac{z-a}{w-a}\right| < 1$ for $w \in [\gamma_2]$ and $z \in A$. In each of the cases, we have a geometric series, uniformly convergent on the corresponding image $[\gamma_j]$. We therefore can interchange the order of integration and summation to get

$$\begin{aligned}
f(z) = &\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)^{n+1}}\,dw\right)(z-a)^n\\
&+\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)^{-k}}\,dw\right)(z-a)^{-k-1}.
\end{aligned}$$

In the second summation, we let $n = -k-1$. Then the second series is

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)^{n+1}}\,dw\right)(z-a)^n.$$

In view of Lemma 9.2.6, it follows that we can write the coefficients c_n of powers of $(z-a)^n$ as $c_n = \frac{1}{2\pi i}\int_{\gamma_\rho}\frac{f(w)}{(w-a)^{n+1}}\,dw$ for any $r < \rho < R$. ☐

The series of the theorem is called the Laurent series (or expansion) of f in the annulus.

There is a uniqueness assertion about the Laurent expansion. The precise statement is the following

Proposition 9.2.9. *Let f be holomorphic in an annulus $A(a; r, R)$. Let a series of the form $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ converge uniformly on compact subsets of the annulus to f. Then a_n are given by $a_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{(w-a)^{n+1}}\,dw$ for any $r < \rho < R$.*

Proof. This can be proved in a way similar to the proof of Proposition 6.4.1. □

Remark 9.2.10. The advantage of this proposition is that if we can find (by some algebraic means or other) a series as in the proposition, then it is necessarily the Laurent series of f. See examples below.

However, the reader should realize that the same function may have different Laurent expansions in different annuli. See (4) of Example 9.2.16.

Definition 9.2.11. If $f(z) = \sum a_k(z-a)^k$ is the Laurent expansion of f about an isolated singularity a,

(1) $\sum_{-\infty}^{-1} a_k(z-a)^k$ is called the *principal part* or *singular part* of f at a and

(2) $\sum_0^{\infty} a_k(z-a)^k$ is called the *analytic part* of f at a.

Ex. 9.2.12. The principal (resp. analytic) part of f holomorphic on $A(a; r_1, r_2)$ converges uniformly on compact subsets of $\mathbb{C} \setminus B[a, r_1]$ (resp. of $B(a, r_2)$).

Example 9.2.13. Let $f(z) = \frac{e^z}{z}$, $z \in \mathbb{C}^*$. To find the Laurent expansion of f about $z = 0$, we need to compute

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{e^z}{z^{n+2}}\,dz,$$

where $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. When $n+1 \ge 0$, the integral on the right side is the Cauchy integral formula for the $\frac{1}{(n+1)!} \times (n+1)$-th derivative of e^z at $z = 0$. Hence $c_n = \frac{1}{(n+1)!}$ for $n \ge -1$. If $n \le -2$, then the integrand is an entire function in the star domain $\mathbb{C}$ and hence by Cauchy's theorem, the integral vanishes. Hence the Laurent series of f about 0 is $\sum_{n=-1}^{\infty} \frac{z^n}{(n+1)!}$.

We could have derived this by a formal manipulation and an appeal to Proposition 9.2.9. Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, formally $f(z) = \sum_{n=0}^{\infty} \frac{z^{n-1}}{n!}$.

Example 9.2.14. Let $f(z) = \frac{1}{z(1-z)}$, $z \in \mathbb{C} \setminus \{0, 1\}$. We want to find the Laurent expansion about $z = 0$ in $0 < |z| < 1$. We can write

$$f(z) = \frac{1}{z} + \frac{1}{1-z} = \frac{1}{z} + \sum_{n=0}^{\infty} z^n,$$

which is valid since $|z| < 1$. Proposition 9.2.9 now show that this is the Laurent series of f in the punctured unit disk.

We may also compute the coefficients c_n for $n \geq -1$ as follows: Let $\gamma(t) = e^{it}/2$, $0 \leq t \leq 2\pi$.

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{1}{z^{n+2}(1-z)}\,dz = \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left(\frac{1}{1-z}\right)|_{z=0} = 1.$$

Example 9.2.15. Lest the reader believes that we can always compute the coefficients c_n of the Laurent series by its integral expansion, we present the following example. This illustrates that finding the Laurent series helps us evaluate some horrendous looking integrals!

Let $f(z) := \exp(z + z^{-1})$ for $z \in \mathbb{C}^*$. We show that the Laurent expansion of f around $z = 0$ is given by $f(z) = a_0 + \sum_{n=1} a_n(z^n + z^{-n})$ where

$$a_n = \frac{1}{\pi}\int_0^\pi e^{2\cos\theta}\cos n\theta\,d\theta = \sum_{k=0}^\infty \frac{1}{(n+k)!k!}.$$

Let the Laurent expansion be $f(z) = \sum_{-\infty}^\infty c_n z^n$. Letting $\gamma(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, we have

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\,dw \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \exp(e^{i\theta} + e^{-i\theta})e^{-in\theta}\,i d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}e^{-in\theta}\,d\theta. \end{aligned}$$

Also we have $\overline{c_n} = c_n$ so that c_n is real. For,

$$\begin{aligned} \overline{c_n} = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}e^{in\theta}\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos(2\pi-\theta)}e^{-in(2\pi-\theta)}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}e^{-in\theta}\,d\theta = c_n. \end{aligned}$$

Thus $c_{-n} = \overline{c_n} = c_n$. Now the integral representation for a_n follows. To get the series expansion, we take the Cauchy product of the series of $\exp(z)$ and $\exp(1/z)$:

$$f(z) = \left(1 + \sum_{k=1}^\infty \frac{z^k}{k!}\right)\left(1 + \sum_{k=1}^\infty \frac{1}{k!z^k}\right).$$

Since the Laurent series is unique, we obtain $a_n = \sum_{k=0}^\infty \frac{1}{(n+k)!k!}$, by comparing the coefficients.

Example 9.2.16. We look at some simple examples (without offering much of an explanation).

(1) Consider $f(z) = \frac{(z+1)^2}{z}$ in $|z| > 0$: $\frac{(z+1)^2}{z} = \frac{1}{z} + 2 + z$ for all $z \neq 0$.
(2) Consider $f(z) = \frac{1}{z^2(1-z)}$ in $0 < |z| < 1$:

$$\begin{aligned}\frac{1}{z^2(1-z)} &= \frac{1}{z^2}(1 + z + z^2 + \cdots)\\ &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots, \text{ for } 0 < |z| < 1.\end{aligned}$$

(3) Consider $\frac{1}{z^2(1-z)}$ in $0 < |z-1| < 1$:

$$\begin{aligned}\frac{1}{z^2(1-z)} = \frac{-1}{z^2(z-1)} &= \frac{-1}{[1+(z-1)]^2(z-1)}\\ &= \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - \cdots, \text{ for } 0 < |z-1| < 1.\end{aligned}$$

(4) Consider $f(z) = \frac{1}{z(z-1)(z-2)}$ in (a) $0 < |z| < 1$, (b) $1 < |z| < 2$ and (c) $|z| > 2$: By the method of partial fractions, we have $f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$. Hence,

$$\begin{aligned}f(z) &= \frac{1}{2z} + \sum_0^\infty (1 - 2^{-n-2})z^n, \quad 0 < |z| < 1\\ &= \frac{-1}{2z} - \sum_{n=2}^\infty \frac{1}{z^n} - \frac{1}{4}\sum_0^\infty 2^{-n}z^n, \quad 1 < |z| < 2\\ &= \frac{1}{z^3} + \frac{3}{z^4} + \cdots, \quad |z| > 2.\end{aligned}$$

Ex. 9.2.17. Let f be holomorphic in $B(a,R)$. What is its Laurent expansion in $B'(a,R)$?

Ex. 9.2.18. Find the Laurent expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ in (a) $0 < |z| < 1$, (b) $1 < |z| < 2$ and (c) $|z| > 2$. *Hint:* Use partial fractions.

Ex. 9.2.19. Find the Laurent expansion of the following functions in the indicated annulus:

$$\begin{array}{lll} (1) & \dfrac{2z-2}{(z+1)(z-3)}, & 1 < |z| < 3\\ (2) & \dfrac{1}{1-z}, & |z| > 1\\ (3) & \dfrac{1}{z(1-z)}, & 0 < |z| < 1\\ (4) & \dfrac{1}{z(1-z)}, & |z| > 1 \end{array}$$

(5) $\dfrac{1}{z(1-z)}$, $\quad 0 < |1-z| < 1$

(6) $\dfrac{1}{z(1-z)}$, $\quad |1-z| > 1$

(7) $\dfrac{z+1}{z}$, $\quad |z| > 0$

(8) $\dfrac{\sin z}{z^3}$, $\quad |z| > 0$

(9) $\dfrac{1}{z^2-3+2}$, $\quad 1 < |z| < 2$

(10) $\dfrac{1}{z}$, $\quad 1 < |z-i| < \infty$

(11) $\dfrac{1}{z^2}$, $\quad 1 < |z-i| < \infty$

Ex. 9.2.20. Find the Laurent expansion of the function $\dfrac{az+b}{cz+d}$ in the annulus $0 < |z+(d/c)| < \infty$.

Ex. 9.2.21. Let f be holomorphic in $U := \mathbb{C} \setminus B[0,R]$. Assume that $|f(z)| \leq M$ for $z \in U$. Let c_n be the coefficients of the Laurent series of f in $R < |z| < \infty$. Show that $c_n = 0$ for $n > 0$.

Ex. 9.2.22. Find the Laurent expansion of $e^{1/z}$ in $0 < |z| < \infty$ and hence show that

$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}, \qquad n \in \mathbb{Z}_+.$$

Ex. 9.2.23. Let $u \in \mathbb{C}$. Let f, g be defined by $f(z) = \exp(uz + \frac{u}{z})$, $g(z) = \exp(uz - \frac{u}{z})$ for $z \in \mathbb{C}^*$.

(1) Show that $f(z) = a_0 + \sum_{n=1}^{\infty} a_n(z^n + z^{-n})$ where

$$a_n = \frac{1}{\pi}\int_0^{\pi} \exp(2u\cos\theta)\cos n\theta\, d\theta = u^n \sum_{j=0}^{\infty} \frac{u^{2j}}{(n+j)!j!}.$$

(2) Show that $g(z) = b_0 + \sum_{n=1}^{\infty} b_n(z^n + (-z)^{-n})$, where

$$b_n = \frac{-1}{\pi}\int_0^{\pi} \cos(n\theta - 2u\sin\theta)\, d\theta = u^n \sum_{n=0}^{\infty} \frac{(-1)^j u^{2j}}{(n+j)!j!}.$$

Show also that $b_n(-u) = (-1)^n b_n(u)$. The function b_n, as a function of u, is called the Bessel function of order n.

Ex. 9.2.24. Use (9.4) on page 141 to prove Riemann's theorem on removable singularity, as suggested in Remark 9.1.8. *Hint:* Enclose the singular point, say, 0 by a circle γ_ε of very small radius and another circle γ_r. Then (9.1.8) tells us

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{w} - \frac{1}{2\pi i}\int_{\gamma_\epsilon} \frac{f(w)}{w}.$$

Our hypothesis on the behaviour of f near 0 allows us to show that the contribution of the second integral can be ignored, by obvious estimates. Hence f is represented by the first integral in $B(0,r)$ and hence is analytic etc.

9.3 Characterization of Singularities

The intimate connection between the nature of singularity of f at a and the Laurent expansion of f at a is given by the following

Theorem 9.3.1. *Let f be holomorphic in $B'(a,R)$. Let c_j, $j \in \mathbb{Z}$, be the coefficients of the Laurent series of f at a. Then*

(i) f has a removable singularity at a iff $c_j = 0$ for all $j < 0$, i.e. iff the principal part has no nonzero term.

(ii) f has a pole at a iff there exists $N \in \mathbb{N}$ such that $c_{-j} = 0$ for all $j > N$ and $c_{-N} \neq 0$.

(iii) f has an essential singularity at a iff infinitely many $c_{-j} \neq 0$ for $j > 0$, i.e. iff infinitely many terms in the principal part are nonzero.

Proof. Case (i). If $c_j = 0$ for all $j < 0$, then $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ on $B'(a,R)$. If we set $f(a) = c_0$, then f becomes holomorphic in $B(a,R)$. Converse follows from Ex. 9.2.17.

Case (ii). Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{j=1}^{N} \frac{c_{-j}}{(z-a)^j}$ with $c_{-N} \neq 0$. Then $(z-a)^N f(z)$ is bounded in a deleted neighbourhood of a but $(z-a)^{N-1} f(z)$ is not. Hence f has a pole of order N.

To see the converse, assume that f has a pole of order m at a. Let $g(z) := (z-a)^m f(z)$ in $B'(a,R)$. Then g has removable singularity at a. We assume that $g(a)$ is suitably defined so that g is holomorphic around a. Using the power series expansion for g at a, we get

$$g(z) = \sum_{k=0}^{m-1} \frac{g^{(k)}(a)}{k!}(z-a)^k + (z-a)^m h(z), \tag{9.8}$$

where h is holomorphic on $B(a,R)$. Dividing (9.8) by $(z-a)^m$ yields

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{(z-a)} + h(z), \qquad z \in B'(a,R). \tag{9.9}$$

This proves (ii).

Any isolated singularity which is neither removable nor a pole must be an essential singularity. Now if the coefficients c_{-j} do not satisfy the properties in (i) or (ii), then they have the property in (iii) and hence (iii) follows. $\square$

The next two exercises are the same as Corollary 9.1.38 and the exercise following it.

Ex. 9.3.2. Let f be an entire function which is not a polynomial. Then $f(\mathbb{C} \setminus B)$ is dense in $\mathbb{C}$ for any bounded set B. *Hint:* Consider the singularity at $z = 0$ of the function $g(z) = f(1/z)$.

Ex. 9.3.3. Let f be entire. Suppose that $|f(z)| \to \infty$ as $|z| \to \infty$. Then f is a polynomial. *Hint:* $g(z) := f(1/z)$ has a pole at 0. Compare this hint with the one in Ex. 8.5.17.

Ex. 9.3.4. Let f be nonconstant and entire. Then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Ex. 9.3.5. The Bernoulli numbers B_n are defined as the coefficients of the power series expansion of $\frac{z}{e^z-1} := \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$. Find an integral expression for B_n of the form $B_n = \int_0^{2\pi} g_n(\theta)\, d\theta$.

9.4 Meromorphic Functions

This section may be skipped on the first reading and may be taken up while reading Section 13.3.

We recall the definition and recast it in a more precise language.

Definition 9.4.1. Let $U \subset \mathbb{C}$ be open. A function f on U is said to be *meromorphic* if it is holomorphic at all points of U except for isolated singularities, which must be poles.

Since we wish to prove various facts about meromorphic functions, we recast it as follows: a function f is said to be meromorphic on U if there exists an open set $V \subset U$ such that f is holomorphic on V and every $z \in U \setminus V$ is a pole of f. We let P_f denote the set of poles of f in U. Hence $P_f = U \setminus V$.

We shall use this definition and the notation below without explicit mention.

Example 9.4.2. The most important class of examples are the rational function p/q where p, q are polynomials with no common factors and $q \neq 0$.

Ex. 9.4.3. How about functions of the form f/g where f and g are entire functions with $g \neq 0$?

Lemma 9.4.4. *If f is meromorphic on U and $K \subset U$ is compact, then the set of poles of f in K is finite.*

Proof. If the set $P_f \cap K$ of poles of f is infinite, then we can extract a sequence of distinct points, say, (a_n) from $P_f \cap K$. Since K is compact, there exists a subsequence converging to some $a \in K$. (Recall that any compact subset in a metric space is closed.) We shall assume that the original sequence itself converges $a_n \to a \in K$. Now where does a belong?

If $\alpha \in V$, then there exists an open disk $B(\alpha, r) \subset V$ on which f is holomorphic. But then by convergence assumption, all but finitely many $\alpha_n \in B(\alpha, r)$. Since α_n are poles, this is absurd.

Almost similar argument shows that $\alpha \notin P_f$. For, if $\alpha \in P_f$, being an isolated singularity, there exists a punctured open disk, say, $B'(\alpha, r)$ on which f is holomorphic. This means that $\alpha_n \in B(\alpha, r)$ for all but finitely many, must equal α. This contradicts our assumption of (α_n), namely that all terms are distinct.

Hence we are lead to conclude that $P_f \cap K$ is finite. □

Remark 9.4.5. Since any U can be 'exhausted' by means of an increasing sequence of compact subsets of U (see Ex. 19.2.15), it follows that we can list the set P_f of poles of a meromorphic function as a sequence.

Theorem 9.4.6 (Identity Theorem for Meromorphic Functions). *Let f be a meromorphic function on a connected open subset of $\mathbb{C}$. Assume that the set Z_f of zeros of f in U has an accumulation point in U. Then f is identically zero.*

Proof. Let V, as usual, denote the set on which f is holomorphic. As we do not know as yet whether V is connected, we cannot apply the identity theorem (for holomorphic functions) to the pair (f, V). Let $z, w \in V$. Since U is an open connected subset in $\mathbb{C}$, it is path connected. Let γ be a path that connects z to w. It may happen that γ may pass through points in P_f. However, the set $[\gamma]$, being a continuous image of a compact set $[0, 1]$, is compact. Therefore γ can pass through at most finitely many points of P_f. We can then divert the path near them so that the resulting path does not pass through any point of P_f. See Figure 16.1 on 227.

> If $\alpha \in P_f$ lies on γ, there exists a punctured disk $B'(\alpha, r)$ on which f is holomorphic. Now, by shrinking r if necessary, assume that $B'[\alpha, r] \subset V$. This means that there is an entry point, say, $\gamma(t_1) \in S(\alpha, r)$ and an exit point, say, $\gamma(t_2) \in S(\alpha, r)$. The part of the curve that lies in $B(\alpha, r)$ can be replaced by the arc joining $\gamma(t_1)$ to $\gamma(t_2)$. This way, the modified path avoids P_f.

Hence V is path connected and hence connected. We conclude that $f = 0$ on V. It is easy to see that $P_f = \emptyset$.

> For, if $\alpha \in P_f$, then there exists a punctured disk $B'(\alpha, r) \subset V$. In a Euclidean space, any point of an open set is an accumulation point of the set. Hence, there exists a sequence $z_n \in B(\alpha, r)$ such that $z_n \to \alpha$. But then $\lim_{z_n \to \alpha} f(z_n) = 0$. This contradicts the fact that for a pole, $\lim_{z \to \alpha} f(z) = \infty$.

Hence $V = U$ and $f = 0$ on U. □

We want to show that the set $M(U)$ of meromorphic functions is a field with respect to 'pointwise operations'. (Note the quotes.) Let $f_1, f_2 \in M(U)$. Let V_1, V_2 be the open sets associated with them. We define

$$(f_1 + f_2)(z) := f_1(z) + f_2(z),\ z \in V_1 \cap V_2$$
$$(f_1 f_2)(z) := f_1(z) f_2(z), \quad z \in V_1 \cap V_2.$$

We see easily that $f_1 + f_2$ and $f_1 f_2$ are meromorphic on U. Note that we cannot assert that $U \setminus (V_1 \cap V_2)$ is the set of poles of $f_1 + f_2$ or of $f_1 f_2$. What can be said is this set consists of poles of the sum (or the product) *along with* removable singularities.

Ex. 9.4.7. Give some simple examples to illustrate the point made at the end of the last paragraph.

Theorem 9.4.8. *$M(U)$ is a field with respect to sum and product defined above.*

Proof. Note that the constant function $f \equiv 1$ serves as the multiplicative identity.

The only point that merits any check is to verify that if $f \in M(U)$ is nonzero, then $1/f \in M(U)$.

It is clear that Z_f, the set of zeros of f, is closed in V so that $V \setminus Z_f$ is open in V, hence in U and hence in $\mathbb{C}$. Equally clear is the fact that $1/f$ is holomorphic on $V \setminus Z_f$.

Now we attend the points of the set P_f and Z_f.

If $a \in P_f$, then $\lim_{z\to a} f(z) = \infty$ so that $\lim_{z\to a} \frac{1}{f(z)} = 0$. Thus, $1/f$ has a removable singularity at $z = a$.

If $a \in Z_f$, then in a disk $B(a,r)$, the function is of the form $(z-a)^m g(z)$, where g is holomorphic in $B(a,r)$ and $g(a) \neq 0$. (See Definition 8.5.1.) By shrinking r, if necessary, we may assume that $g(z) \neq 0$ for all $z \in B(a,r)$. So, on the punctured disk $B'(a,r)$, the function $1/f$ is of the form $(z-a)^m(1/g(z))$. Thus, $1/f$ has an isolated singularity at $z = a$, namely, a pole of order m.

To summarize, we have shown that $V_{1/f} = V \cup P_f$ and Z_f is the set of poles of $1/f$ in U. Thus, $1/f \in M(U)$. $\square$

Let $f\colon U \to \mathbb{C}$ be meromorphic with an isolated singularity at a. We denote the principal or singular part of f at a by $P(f;a)$.

Lemma 9.4.9. *Let $f \in M(U)$ and $a_1, \dots, a_n \in P_f$ be distinct. Then $f - \sum_{j=1}^n P(f;a_j)$ is holomorphic around each a_j, $1 \le j \le n$.*

Furthermore, if $P_f = \{a_j : 1 \le j \le n\}$, then

$$f = g + \sum_{j=1}^n P(f;a_j),$$

for some $g \in H(U)$.

Proof. The argument below is a repeat of the one in the proof of the residue theorem.

First of all, we note that $P(f;a_j)$ is holomorphic in $\mathbb{C}\setminus\{a_j\}$ and that $f-P(f;a_j)$ is holomorphic around a_j, for $1\le j\le n$. This proves the result if $n=1$. Let $n>1$. Assume that we have shown that $f_k := f-\sum_{j=1}^{k} P(f;a_j)$ is holomorphic at a_j, $1\le j\le k$. Since f_k is holomorphic in a neighbourhood of a_{k+1}, we see that $P(f_k,a_{k+1}) = P(f,a_{k+1})$. Hence

$$f_k - P(f_k,a_{k+1}) = f - \sum_{j=1}^{k+1} P(f;a_j).$$

Since $f_k - P(f,a_{k+1})$ is holomorphic around each of a_j, $1\le j\le k+1$, it follows that the same is true of $f-\sum_{j=1}^{k+1} P(f;a_j)$.

If $P_f = \{a_j : 1\le j\le n\}$, and if we define $g := f-\sum_{j=1}^{n} P(f;a_j)$, then $g\in H(U)$ and $f = g+\sum_{j=1}^{n} P(f;a_j)$.

□

Corollary 9.4.10 (Partial fraction expansion of rational functions). *Let $f = p/q$ be a rational function with p and q having no common factors. Then $f = P+\sum_{j=1}^{n} P(f,a_j)$, where P is a polynomial.*

Proof. We assume that $\deg p < \deg q$. Assume that the distinct zeros of q are a_j, $1\le j\le r$, with multiplicity m_j. Then $P_f = \{a_j : 1\le j\le r\}$. As the "domain" of the rational function is $\mathbb{C}$, by the last lemma, we can write $p/q = g+\sum_{j=1}^{r} P(f,a_j)$ where g is an entire function. Now, $\lim_{z\to\infty} f(z) = 0$, as $\deg p<\deg q$. Also, $\lim_{z\to\infty} P(f,a_j) = 0$, since it is a finite sum of the terms of the from $(z-a_j)^{-l}$ for some $l\in\mathbb{N}$. Hence we conclude that $\lim_{z\to\infty} g(z) = 0$. Consequently, g is bounded.

> Given $\varepsilon = 1$, there exists $R>0$ such that $|g(z)|\le 1$ for $|z|>R$. On the compact set $B[0,R]$, the continuous function g is bounded, say, by M. Then $\max\{M,1\}$ is a bound for g.

By Liouville's theorem, g is a constant. This constant has to be zero, as $\lim_{z\to\infty} g(z) = 0$. We have therefore shown that $f(z) = \sum_{j=1}^{r} P(f;a_j)$ when $\deg p<\deg q$.

In the general case, if $\deg p\ge\deg q$, by division algorithm, we can write $p = Pq+p_1$ where $0\le\deg p_1<\deg q$ so that $f = P+p_1/q$. We now apply our earlier result to p_1/q and deduce the result as stated.

□

Chapter 11

Residue Theorem and its Applications

In this chapter, we prove the residue theorem which may be considered as a far-reaching generalization of the Cauchy integral formulas for the function as well as its derivatives. (See Ex. 11.1.14.)

11.1 Residue Theorem

Definition 11.1.1. Let $U \subset \mathbb{C}$ be open and $a \in U$. Let f be holomorphic on $U \setminus \{a\}$. Let $B(a, R) \subset U$ and $0 < r < R$. The value of the integral $\frac{1}{2\pi i}\int_\gamma f(z)dz$, where $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$ is independent of r and is called the *residue* of f at a. It is denoted by $\operatorname{Res}(f; a)$. Thus,

$$\boxed{\operatorname{Res}(f; a) := \frac{1}{2\pi i}\int_\gamma f(z)\,dz \text{ where } \gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi, 0 < r < R.}$$

Note that $\operatorname{Res}(f; a)$ is c_{-1}, the coefficient of $(z-a)^{-1}$ in the Laurent expansion of f at a.

Example 11.1.2. If f is holomorphic at a, then $\operatorname{Res}(f; a) = 0$. Show by means of an example that the converse is not true.

Example 11.1.3. Let f have a simple pole at a. Then $f(z) = \frac{c_{-1}}{z-a} + g(z)$ where g is the holomorphic part of the Laurent expansion of f at a. Then we have

$$\operatorname{Res}(f; a) = \lim_{z \to a}(z-a)f(z), \qquad a \text{ is a simple pole.} \tag{11.1}$$

As a concrete example, let us compute the residue of $f(z) = \frac{e^{-iz}}{z^2+1}$ at $z = i$. By (11.1),

$$\mathrm{Res}\,(f; i) = \lim_{z \to i}(z-i)\frac{e^{-iz}}{(z+i)(z-i)} = \frac{e}{2i}.$$

Example 11.1.4. Let $f(z) = \frac{g(z)}{h(z)}$, g, h holomorphic around a. Assume that a is a simple zero of h and $g(a) \neq 0$. Then, we have,

$$\mathrm{Res}\,(f; a) = \frac{g(a)}{h'(a)}, \quad \text{where } a \text{ is a simple zero of } h \text{ and } g(a) \neq 0. \tag{11.2}$$

To prove this we use (11.1):

$$\mathrm{Res}\,(f; a) = \lim_{z \to a}\frac{g(z)}{\left(\frac{h(z)-h(a)}{z-a}\right)} = \frac{g(a)}{h'(a)}.$$

As a specific example, consider $f(z) = \frac{1}{z^6+1}$. Then f has six simple poles. They are the sixth roots of -1: $\zeta_1 = e^{i\pi/6}$, $\zeta_2 = e^{i\pi/2}$, $\zeta_3 = e^{i5\pi/6}$, $\zeta_4 = \overline{\zeta}_1$, $\zeta_5 = \overline{\zeta}_2$ and $\zeta_6 = \overline{\zeta}_3$. By (11.2),

$$\mathrm{Res}\,(f; \zeta_1) = \frac{1}{6z^5}\Big|_{z=\zeta_1} = \frac{1}{6}e^{-i5\pi/6}.$$

One can also do this using (11.1) by observing that $z^6 + 1 = \prod_{k=1}^{6}(z - \zeta_k)$ so that

$$\mathrm{Res}\,(f; \zeta_1) = \prod_{k=2}^{6}(\zeta_1 - \zeta_k).$$

Ex. 11.1.5. The result in (11.2) remains true if $z = a$ is a zero of g and a simple pole of h.

Example 11.1.6. Let f have a pole of order m at $z = a$ and its Laurent expansion be

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c_{-1}}{z-a} + g(z),$$

g being the holomorphic part. Then

$$(z-a)^m f(z) = c_{-m} + \cdots + c_{-1}(z-a)^{m-1} + (z-a)^m g(z).$$

It follows that

$$c_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]\big|_{z=a}\,. \tag{11.3}$$

We use (11.3) to find the residue of $f(z) = e^{3z}(z-2)^{-3}$ at $z = 2$. In this case, $(z-1)^3 f(z) =$

so that

$$\operatorname{Res}(f;a) = \frac{1}{2!}\frac{d^2}{dz^2}e^{3z}\bigg|_{z=2} = \frac{9e^6}{2}.$$

Ex. 11.1.7. Find the residues at the singularities of

$$(a)\quad \tan z \qquad\qquad (b)\quad \frac{(z^2+1)^{2n}}{z^{2n+1}}$$

$$(c)\quad \frac{z^{2n}}{(z^2+1)^n} \qquad\qquad (d)\quad \frac{1}{z(z-1)}$$

$$(e)\quad \frac{z}{z^4+1} \qquad\qquad (f)\quad \frac{ze^{iz}}{(z-\pi)^2}$$

$$(g)\quad \frac{1}{(z-1)^2(z+1)} \qquad\qquad (h)\quad \frac{1}{az^2+bz+c}.$$

Ex. 11.1.8. Find the residue of $f(z) := \frac{e^z}{\sin^2 z}$ at $z = k\pi$. *Hint:* Use the Laurent series expansion at $k\pi$.

We now prove a powerful generalization of the Cauchy integral formula. We restrict ourselves to star-shaped open sets, which will be adequate for most of the applications. We shall later indicate a more general version. See Corollary 16.1.17.

Theorem 11.1.9 (Residue Theorem.). *Let U be a star-shaped open set. Let $z_1, \dots, z_m \in U$ be m distinct points in U. Let f be holomorphic on $U \setminus \{z_1, \dots, z_m\}$. Let γ be any closed path in U such that $z_j \notin [\gamma]$, $1 \le j \le m$. Then*

$$\boxed{\int_\gamma f = 2\pi i \sum_{j=1}^{m} \operatorname{Res}(f; z_j) \cdot n(\gamma, z_j).} \tag{11.4}$$

Proof. Choose $\delta > 0$ such that $2\delta < \min\{d(z_r, [\gamma]), d(z_j, z_k) : 1 \le r \le m, 1 \le j < k \le m\}$ and such that $B[z_j, \delta] \subset U$ for $1 \le j \le m$. Note that $B(z_j, \delta) \cap [\gamma] = \emptyset$ and $z_k \notin B(z_j, \delta)$ for $k \ne j$. See Figure 11.1.

Let $f_j = \sum_{n=1}^{\infty} c^j_{-n}(z - z_j)^{-n}$ be the singular part of f at z_j. Then the infinite series is uniformly convergent on the compact subsets of $\mathbb{C} \setminus B[z_j, \delta]$, in particular, on $[\gamma]$. The function f_j is holomorphic around z_k for $k \ne j$. Since $f - f_j$ is holomorphic at z_j and $\sum_{k=1, k\ne j}^{k=m} f_k$ is holomorphic at z_j, we conclude that $g := f - \sum_{k=1}^{m} f_k$ is holomorphic at z_j for $1 \le j \le m$. Hence g is holomorphic in U. Since U is star-shaped, by Cauchy's

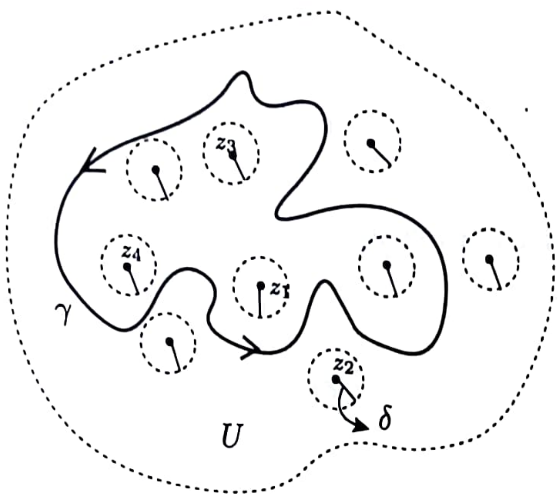

Figure 11.1: Illustration of Residue Theorem

theorem $\int_\gamma g = 0$. Therefore,

$$\begin{aligned}
\int_\gamma f &= \sum_{j=1}^{m} \int_\gamma f_j \\
&= \sum_{j=1}^{m} \int_\gamma \left(\sum_{n=1}^{\infty} c^j_{-n}(z-z_j)^{-n}\right) \\
&= \sum_{j=1}^{m} \sum_{n=1}^{\infty} c^j_{-n} \int_\gamma (z-z_j)^{-n} \\
&= \sum_{j=1}^{m} c^j_{-1}(2\pi i) n(\gamma, z_j),
\end{aligned}$$

since $(z-z_j)^{-n}$ for $n \geq 2$ has a primitive in U. This completes the proof of the theorem. □

Remark 11.1.10. A path $\gamma \colon [a,b] \to \mathbb{C}$ is said to be a *simple positively oriented* closed path if it is closed, γ is 1-1 on (a,b) and $n(\gamma, z)$ is either 0 or 1 for all $z \notin [\gamma]$. Any point z with $n(\gamma, z) = 1$ is said to be *inside* of γ. If $n(\gamma, z) = 0$ then z is said to be *outside* of γ. Look at the Figure 11.2. The shaded region is inside of γ whereas the non-shaded region is outside γ.

Examples of simple, positively oriented closed paths are $\gamma(t) = a + re^{it}$, $a \in \mathbb{C}$, $0 \leq t \leq 2\pi$. More generally, if $\gamma(t) = r(t)e^{it}$, $0 \leq t \leq 2\pi$ and $r(0) = r(2\pi)$, then γ is simple, positively oriented and closed. (See Example 10.1.10.)

A important observation is that if we assume that γ is a simple, positively oriented closed path, then (11.4) becomes $\int_\gamma f$ is $2\pi i \times$ (the sum of residues inside γ).

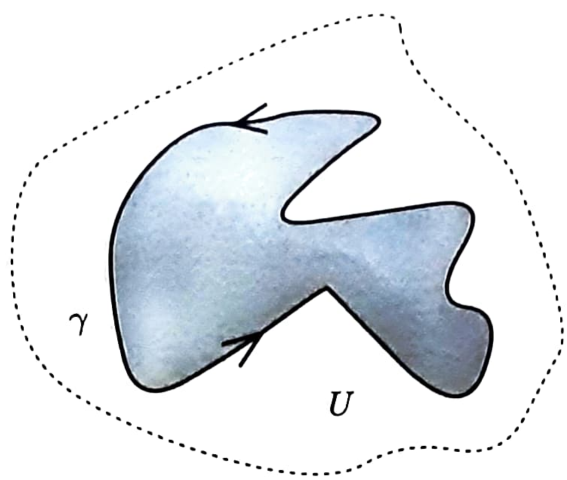

Figure 11.2: Inside and outside of a closed path

In the residue theorem, if we assume that γ is a simple, positively oriented closed path, then (11.4) becomes $\int_\gamma f$ is $2\pi i \times$ (the sum of residues inside γ).

Let us give some typical applications of the residue theorem.

Example 11.1.11. Let $f(z) = \frac{z^4}{z^2+1}$ and $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. Then f has simple poles at $z = \pm i$. Using (11.1), we find that the residues $\operatorname{Res}(f; \pm i) = \pm\frac{1}{2i}$. Hence $\int_\gamma f = 0$.

Ex. 11.1.12. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Show that $\int_\gamma \frac{e^z}{z^3}\,dz = \pi i$.

Ex. 11.1.13. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ and $f(z) = \frac{z^2+1}{z(z-\alpha)}$ for $0 < |\alpha| < 1$. Compute $\int_\gamma f$ and hence show that $\int_\gamma \frac{\operatorname{Re} z}{z-\alpha}\,dz = \pi\alpha i$.

Ex. 11.1.14. Deduce the Cauchy integral formulas from the residue theorem.

Ex. 11.1.15. Prove that $\int_\gamma \frac{e^{az}}{1+z^2} = 2\pi i \sin a$, where $\gamma(t) = re^{it}$ for $0 \le t \le 2\pi$ and $r > 1$.

Ex. 11.1.16. Show that $\int_\gamma \frac{e^{-iz}}{1+z^2} = \pi e$ where γ is any circle with standard parametrization containing i in its interior but not $-i$.

Ex. 11.1.17. Evaluate $\int_\gamma \sin(1/z)$ where γ is the unit circle with standard parametrization.

Ex. 11.1.18. Let $\gamma_r(t) = re^{it}$, $0 \le t \le 2\pi$. Show that

(a) $\displaystyle\int_{\gamma_1} \frac{\sin z}{z^4} = -\frac{\pi i}{3}$.

(b) $\displaystyle\int_{\gamma_1} \frac{(1+z^5)\sinh z}{z^6} = \frac{\pi}{60}$.

(c) $\int_{\gamma_2} \dfrac{z^2+4}{(z-i)(z+i)} = 0.$

(d) $\int_{\gamma_1} \dfrac{e^z}{z^3} = \pi i.$

Example 11.1.19. Let us compute $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ for $a > 1$ using the residue theorem. Let $z = e^{it}$. Then $\cos t = (z+\bar{z})/2 = \frac{z^2+1}{2z}$ and $dz = ie^{it}\,dt$. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. We have

$$\int_0^{2\pi} \frac{dt}{a+\cos t} = \int_\gamma \frac{2z}{z^2+2az+1}\frac{dz}{iz} = -2i\int_\gamma \frac{dz}{z^2+2az+1}.$$

The integrand f in the right most integral has simple poles at $-a \pm \sqrt{a^2-1}$. Of these, $-a-\sqrt{a^2-1}$ lies outside γ. Since the product of these poles is 1, the other pole $-a+\sqrt{a^2-1}$ lies inside γ. By (11.1), we find $\operatorname{Res}(f; -a+\sqrt{a^2-1}) = \frac{-i}{\sqrt{a^2-1}}$. Hence by the residue theorem, we have

$$\int_\gamma f = 2\pi i \times \operatorname{Res}(f; -a+\sqrt{a^2-1}) = \frac{2\pi}{\sqrt{a^2-1}}.$$

In Chapter 15, we shall develop a systematic way of computing some real integrals using the residue theorem. Till then, you may have practice with the following exercises.

Ex. 11.1.20. Show that $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$.

Ex. 11.1.21. Express the integral $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ as a path integral of a suitable function along the unit circle $S(0,1)$.

Lemma 11.1.22. *Let f be holomorphic on $\mathbb{C}$ except for a finite number of simple poles z_k, $1 \le k \le n$. Assume further that there exist constants M and R such that $|f(z)| \le M$ for $|z| \ge R$. Then for $z \ne z_k$, $1 \le k \le n$, we have*

$$f(z) = c + \sum_{k=1}^{n} \frac{\operatorname{Res}(f; z_k)}{z-z_k}, \tag{11.5}$$

where $c := \lim_{z\to\infty} f(z)$.

Proof. Let $g(z) := \sum_{k=1}^n \frac{\operatorname{Res}(f;z_k)}{z-z_k}$. Then $f-g$ is holomorphic on $\mathbb{C}$ (as seen in the residue theorem). Also, $f(z)-g(z)$ is bounded on $\mathbb{C}$. Hence, $f-g$ is a constant. Let c be this constant. Therefore, $f = c+g$.

Note that $\lim_{z\to\infty} g(z) = 0$. Hence

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} (f-g)(z) = c.$$

□

Ex. 11.1.23. Use (11.5) to express the following as partial fractions:

(1) $\dfrac{(z+1)^2}{(z-1)(z+2)(z+4)}$

(2) $\dfrac{(z-2)(2z+1)(3z+1)}{z(z^2-1)}$

(3) $\dfrac{(z+1)(z+2)(z+3)}{(z-1)(z-2)(z-3)}$.

11.2 Argument Principle and Rouché's Theorem

The results of this section are theoretical applications of the residue theorem. So, we shall assume that the open sets in the statements of the theorem of this section are star-shaped.

Let a be an isolated singularity of f. If a is a pole of order m, we write this order as $m(f,a)$. Similarly, if b is a zero of order m we denote it by $m(f,b)$.

Theorem 11.2.1 (Argument Principle). *Let U be a star-shaped open subset of $\mathbb{C}$. Let $f\colon U \to \mathbb{C}$ be holomorphic except for a finite number of poles. Assume $f \neq 0$ on $[\gamma]$. Let γ be a closed path in U. Assume that the zeros and poles of f in U are finite and that none of them lie on $[\gamma]$. Then we have*

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{a\in U} m(f,a)n(\gamma,a).} \tag{11.6}$$

Proof. Note that $\frac{f'}{f}$ is holomorphic except at the zeroes and poles of f. (Why?)

Hence by the residue theorem, $\int_\gamma \frac{f'}{f}$ is $2\pi i \sum n(\gamma,a)\mathrm{Res}\,(f'/f;a)$ where the sum is over those a which are either poles of f or zeros of f.

If f has a zero of order k at $z=a$, then we can write $f(z)=(z-a)^k g(z)$, with $g(a)\neq 0$ in $B'(a,r)$. Then $f'(z)=(z-a)^{k-1}kg(z)+(z-a)^k g'(z)$ has a zero of order $k-1$ at $z=a$.

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}.$$

To find $\mathrm{Res}\,(f;a)$ we integrate the above expression over $S(a,r)$. Hence at each zero of f of order k, $\frac{f'}{f}$ has a simple pole with residue k

Similarly, if $f(z)=(z-a)^{-k}g(z)$, then

$$\frac{f'(z)}{f(z)} = \frac{-k}{z-a} + \frac{g'(z)}{g(z)}$$

so that at each pole of order k, $\frac{f'}{f}$ has a simple pole with residue $-k$. The residue theorem now gives the result. □

The following is the most important special case.

Proposition 11.2.2. *Let f be holomorphic on U except possibly a finite numbers of poles. Let γ be a simple positively oriented closed path in U. If neither the zeros nor the poles of f lie on $[\gamma]$, then*

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = N - P,} \tag{11.7}$$

where N (resp. P) is the number of zeros (resp. poles) inside γ, the number being computed with multiplicity. (Here, if z_k is a zero of order r_k for $1 \le k \le m$ inside γ and p_j is a pole of order s_j for $1 \le j \le n$ inside γ then $N = \sum_k r_k$ and $P = \sum_j s_j$.) □

Example 11.2.3. We give an easy example to illustrate the power of the argument principle. Let $a, b, c \in \mathbb{C}$ be distinct. Let $R > \max\{|a|, |b|, |c|\}$. Let $f(z) = \frac{(z-a)^2(z-b)^3}{(z-c)^5}$. Let $C_R(t) = Re^{it}, 0 \le t \le 2\pi$. Then $\int_{C_R} \frac{f'(z)}{f(z)}\,dz = 0$. For, according to the argument principle the path integral is $2\pi i$ times the number of zeros minus the number of poles (counted with multiplicity) inside the region enclosed by C_R.

The following corollary is a special case of the proposition.

Corollary 11.2.4 (Argument Principle). *Let f be holomorphic on U. Let γ be a simple positively oriented closed path in U. Assume that f is never zero on γ. Then*

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{a \in U} m(f, a)\, n(\gamma, a).} \tag{11.8}$$

In particular, if γ is further assumed to be a simple, positively oriented closed path, then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$ is the number of zeros of f inside γ. □

Remark 11.2.5. We explain why this is called the argument principle.

If γ is given by $\gamma(t)$, $0 \le t \le 1$, we choose a logarithm of $f \circ \gamma$ on $[0, 1]$ (possible by Theorem 10.2.5). We compute:

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} &= \frac{1}{2\pi i}\left([\log f(\gamma(1)) - \log f(\gamma(0))\right] \\
&= \frac{1}{2\pi}\,\Delta\,\mathbf{A}\, f(z),
\end{aligned}$$

$\Delta\mathbf{A}$ being the difference in the arguments of $f(\gamma(1))$ and $f(\gamma(0))$ as $\gamma(t)$ travels around γ starting from $\gamma(0)$ to $\gamma(1)$.

Note also that the integral under consideration can be shown to be the integer $n(f \bullet \gamma, 0)$, whence the plausibility of the result. See also Theorem 10.4.7.

Example 11.2.6. We illustrate the use of the argument principle. Let $f(z) = \frac{z}{z^2+1}$ for $z \neq \pm i$ and $\gamma(t) = 2e^{it}$, $0 \leq t \leq 2\pi$. We can compute the integral $\int_\gamma f$ in two ways: one by residue theorem and the other by the argument principle. Note that $f = \frac{1}{2}\frac{g'}{g}$ where $g(z) = z^2 + 1$. Hence by (11.7), the integral is $\frac{1}{2}2\pi i\times$ the number of zeros of g inside γ counted with multiplicity. In this case, $\pm i$ are the only simple zeros of g inside γ. Hence $\int_\gamma \frac{z}{z^2+1} = 2\pi i$. The reader should compute the integral using the residue theorem.

Theorem 11.2.7 (Rouché). *Let U be a star-shaped open set in $\mathbb{C}$. Let γ be a simple, positively oriented closed path in U.*

Let f and g be holomorphic on U and $|f(z) - g(z)| < |f(z)|$ for $z \in [\gamma]$. Then f and g have the same number of zeroes inside γ.

Equivalently if we assume that $|g(z)| < |f(z)|$ on $[\gamma]$, then f and $f+g$ have the same number of zeros inside γ.

Proof. We work with the first hypothesis on f and g. Note that the assumption implies that f, g have no zeroes on γ. We have

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1 \text{ on } \gamma.$$

Thus the values of g/f are contained in $B(1,1)$. Let $F = g/f$. Then $F \circ \gamma$ is a closed path contained in that disc. As $0 \notin B(1,1)$, $n(F\circ\gamma, 0) = 0$ by Ex. 10.1.14). But then

$$\begin{aligned}
0 = n(F\circ\gamma, 0) &= \frac{1}{2\pi i}\int_{F\circ\gamma} \frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_a^b \frac{F'(\gamma(t))}{F(\gamma(t))}\gamma'(t)\,dt \\
&= \frac{1}{2\pi i}\int_\gamma \frac{F'}{F} \\
&= \frac{1}{2\pi i}\left(\int_\gamma \frac{g'}{g} - \int_\gamma \frac{f'}{f}\right).
\end{aligned}$$

The result follows from the argument principle.

The last part is deduced from the first as follows. Define $h := f + g$. Then the pair, (f,h) in place of the pair (f,g) satisfies the first hypothesis. Hence f and $h = f+g$ have the same number of zeros inside $[\gamma]$.

We give a second proof of Rouché's theorem with the second hypothesis that $|g(z)| < |f(z)|$ on $[\gamma]$. Consider

$$J(t) := \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + t(f'(z) - g'(z))}{f(z) + t(f(z) - g(z))}, \; (0 \leq t \leq 1).$$

Observe that the integrand makes sense, since $f(z) + tg(z) \neq 0$ on $[\gamma]$. For,

$$|f(z)| > |g(z)| \geq t\,|g(z)| \text{ on } [\gamma].$$

We claim that J is continuous on $[0,1]$. Since it takes values in $\mathbb{Z}$, it will follow that J is a constant. In particular, $J(0) = J(1)$, which is what we wanted. For $s, t \in [0,1]$, we have

$$\begin{aligned} J(s) - J(t) &= \frac{1}{2\pi i}\int_g \left(\frac{f'(z) + sg'(z)}{f(z) + sg(z)} - \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\right) dz \\ &= \frac{1}{2\pi i}\int_g \frac{(s-t)(f(z)g'(z) - f'(z)g(z))}{(f(z) + sg(z))(f(z) + tg(z))}\, dz. \end{aligned}$$

We need to estimate $|J(s) - J(t)|$. Note that, for $0 \leq s \leq 1$

$$|f(z) + sg(z)| \geq |f(z)| - s\,|g(z)| \geq |f(z)| - |g(z)|, \text{ on } [\gamma].$$

So, if we let

$$\begin{aligned} M &:= \max\{|f(z)g'(z) - f'(z)g(z)| : z \in [\gamma]\} \\ m &:= \min\{|f(z)| - |g(z)| : z \in [\gamma]\}, \end{aligned}$$

it follows that

$$|J(s) - J(t)| \leq \frac{|s-t|}{2\pi}\frac{M\ell(\gamma)}{m^2}.$$

Thus J is a Lipschitz continuous function on $[0,1]$. ☐

Remark 11.2.8 (Dog on a leash). There is a geometric interpretation of the result. If we assume that a man $f(\gamma(t))$ walks a dog which is on a leash $g(\gamma(t))$ whose length is less than the distance $|f(z)|$ of the man from the hydrant $z = 0$, then the man and the dog will go around the hydrant the same number of times.

We give some typical applications of Rouché's theorem below.

Example 11.2.9. Of course, the first application has to be the fundamental theorem of algebra! Let $g(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ and $f(z) = z^n$. Let $r > \max\{1, \sum_{k=0}^{n-1} |a_k|\}$. Then on $|z| = r$, we have

$$\begin{aligned} |g(z) - f(z)| = \left|\sum_{k=0}^{n-1} a_k z^k\right| &\leq \sum_{k=0}^{n-1} |a_k|\,|z|^k \\ &\leq \sum_{k=0}^{n-1} |a_k|\, r^{n-1}, \text{ since } r > 1, r^k \leq r^{n-1}, \\ &\leq r^n = |f(z)|. \end{aligned}$$

Hence by Rouche's theorem, f and g have the same number of zeros inside $B(0,r)$. Since f obviously has n zeros $z = 0$, counted with multiplicity, g also has n zeros in $B(0,r)$.

Example 11.2.10. We solve $e^{z-\lambda} = z$ in $B(0,1)$ for $\lambda > 1$. Let $f(z) = z$ and $g(z) = z - e^{z-\lambda}$. Then on $S^1 := S(0,1)$, we have

$$|f(z)| = 1 \qquad \text{and} \qquad |f(z) - g(z)| = \left|e^{z-\lambda}\right| = e^{\operatorname{Re} z - \lambda} \leq e^{1-\lambda} < 1.$$

Hence g has exactly one zero in $B(0,1)$. The solution lies in $(0,1)$, since $g(0) < 0$ and $g(1) > 0$.

Example 11.2.11. Let $\lambda \in \mathbb{R}$ be such that $\lambda > 1$. We claim that the equation $e^{-z} + z - \lambda = 0$ has exactly one solution in the right half plane $\operatorname{Re} z > 0$.

We take $f(z) = e^{-z} + z - \lambda$, $g = z - \lambda$ and the domain to be $\mathbb{C}$. We take γ to be the semicircle of radius R centred at 0 from $-iR$ to iR and then the line segment from iR to $-iR$, with the standard parametrizations. We now estimate $|f - g|$ and $|g|$ on the semicircle part of γ

$$|f(z) - g(z)| = \left|e^{-z}\right| = e^{-\operatorname{Re} z} \leq e^0 = 1.$$

On the same part of γ, we have

$$|g(z)| = |z - \lambda| \geq |z| - \lambda = R - \lambda > 1,$$

provided that we take $R > \lambda + 1$.

Now for the estimates on the line segment:

$$|g(z)| = |z - \lambda| = |iy - \lambda| = \sqrt{y^2 + \lambda^2} \geq \lambda > 1 = |f - g|.$$

Therefore, by Rouché's theorem, f and g have the same number of zeros inside γ. Since g has only one zero inside γ with $R > \lambda + 1$, it follows that f has only one zero inside γ for any $R > \lambda + 1$. By letting $R \to \infty$, the claim follows.

Ex. 11.2.12. Show directly that $2z^5 + 8z - 1$ has no roots in $|z| \geq 2$. Confirm this by Rouché's theorem. *Hint:* Take $f(z) = 2z^5$ and $g(z) = 8z - 1$ in one of the versions.

Show that the above equation has exactly one root in $B(0,1)$ and that this root is real and positive.

Show that the other four roots lie in $1 < |z| < 2$.

Ex. 11.2.13. Show that $z^4 + 3z - 1$ has precisely one zero in $(0,1)$ and exactly three zeros in $1 < |z| < 2$.

Ex. 11.2.14. Show that the zeros of $f(z) = z^3 + z + 1$ lie in $2/3 \leq |z| \leq 4/3$.

Ex. 11.2.15. Let $f(z) = 3z^7 + 5z - 1$. (i) Show that f has its zeros in $B(0,2)$. (ii) Show that f has one zero in $B(0,1)$ and it is real. (iii) All other zeros lie in $1 < |z| < 2$.

Ex. 11.2.16. Let $f(z) = z^4 + 6z + 3$. Show that all zeros of f lie in $B(0,2)$ and that three of them lie in $1 < |z| < 2$.

Ex. 11.2.17. If g is holomorphic in an open set containing $B[0,1]$ and $|g(z)| < 1$ for $|z| = 1$, then $g(z) = z$ has exactly one solution in $B(0,1)$.

Ex. 11.2.18. Let $|a| > e$. Show that the equation $aze^z = 1$ has exactly one solution in $B(0,1)$. If $a > e$, then this solution is real and positive.

Ex. 11.2.19. Show that the equation $ze^z - a$, $a \neq 0$, has infinitely many zeros.

Ex. 11.2.20. If $a > e$, show that $e^z = az^n$ has exactly n solutions in $B(0,1)$.

Ex. 11.2.21. Show that $az^n + be^z$ has n zeros in $B(0,1)$ if $|a| > |b|\, e$.

Ex. 11.2.22. Prove the following version of the maximum modulus principle. Let $f \in H(U)$. Assume that γ is a simple closed positively oriented path in U and that $|f(z)| \leq M$ for $z \in [\gamma]$. Then $|f(z)| \leq M$ on the inside of γ. *Hint:* If $|f(a)| > M$, consider the functions $f(a)$ and $f(z) - f(a)$.

Ex. 11.2.23. Show that $\exp(z) = 2z+1$ has exactly one solution in $B(0,1)$. *Hint:* Observe that $e^z - 1 = \int_0^z e^w\, dw$ so that $|e^z - 1| \leq e - 1$ on $B(0,1)$.

Ex. 11.2.24. Prove that for each $R > 0$, there exists an integer $N(R)$ such that $f_n(z) := \sum_{k=0}^n \frac{z^k}{k!}$ has no zeros in $|z| < R$ for $n \geq N(R)$.

Ex. 11.2.25. Let $f \in H(U)$ and $f(a) = 0$ for $a \in U$. Assume that $r > 0$ is such that $f(z) \neq 0$ for $z \in B'[z,r]$. Let $\varepsilon := \min\{|f(z)| : |z - a| = r\}$. If $g \in H(U)$ is such that $|f(z) - g(z)| < \varepsilon$ on $S(a,r)$, then g has a zero in $B(a,r)$.

The following result says that the roots of a polynomial depend continuously on its coefficients and is an interesting application of Rouché's theorem.

Theorem 11.2.26. *Let* $f(z) := \sum_{j=0}^n a_j z^j = a_n \prod_{k=1}^p (z - z_k)^{m_k}$, $a_n \neq 0$ *and* $g(z) := (a_0 + \varepsilon_0) + \cdots + (a_{n-1} + \varepsilon_{n-1})z^{n-1} + (a_n + \varepsilon_n)z^n$. *Let*

$$0 < r_k < \min_{l \neq k} \left| \frac{z_k - z_l}{2} \right|.$$

Then there exists an $\varepsilon > 0$ *such that if* $|\varepsilon_j| \leq \varepsilon$, $0 \leq j \leq n-1$, *then* g *has precisely* m_k *zeros in* $B_k := B(z_k, r_k)$ *for all* $1 \leq k \leq n$.

Proof. On ∂B_k, we have

$$\begin{aligned} |f(z)| &= |a_n| \prod_{l=1}^{p} |z - z_l|^{m_l} \\ &= |a_n| \, |z - z_k|^{m_k} \prod_{l \neq k} |z - z_l|^{m_l} \\ &\geq |a_n| \, r_k^{m_k} \prod_{l \neq k} (|z_l - z_k| - r_k)^{m_l} . \end{aligned}$$

Call the right hand side of the last inequality as δ_k. Let

$$\varepsilon < \min \left\{ \frac{\delta_k}{M_k}, 1 \leq k \leq n \right\},$$

where $M_k := \sum_j (|z_k| + r_k)^j$. Choose $0 < \varepsilon_k < \varepsilon$ for $1 \leq k \leq n$.

For $z \in \partial B_k$ we have

$$|g(z) - f(z)| \leq \sum_{j=1}^{n} \varepsilon_j \, (|z_k| + r_k)^j \leq M_k \varepsilon < \delta_k \leq |f(z)| .$$

This means by Rouché's theorem that f and g have the same number of zeroes in B_k. By our choice of r_k, the only zeroes of f in B_k is z_k with multiplicity m_k. Hence the result. ☐

Ex. **11.2.27.** Let f be a polynomial of degree n. Assume that z_k, $1 \leq k \leq p$ are the distinct zeros of f with orders m_k. Choose pairwise disjoint open disks $B_k = B(z_k, r_k)$ and let $\varepsilon = \min\{|f(z)| : z \in S(z_k, r_k), 1 \leq k \leq n\}$. Then $\varepsilon > 0$. Let $g(z) = f(z) - w$. Show that if $|w| < \epsilon$, then $f(z) = w$ has exactly m_k solutions in B_k. (Thus, the solutions of $f(z) = w$ vary continuously with w.)

Chapter 15

Computation of Definite Real Integrals

15.1 Preliminaries on Improper Integrals

Definition 15.1.1. Let $f\colon [a,\infty) \to \mathbb{R}$ be given. We say that $\int_a^\infty f$ exists if $\int_a^R f$ exists for all $R \geq a$ and if $\lim_{R\to\infty} \int_a^R f$ exists. If the latter limit is s, we then say $\int_a^\infty f$ converges to s and write $\int_a^\infty f = s$. We similarly assign a meaning to $\int_{-\infty}^a g$ for $g\colon (-\infty, a) \to \mathbb{R}$. Finally, we say that $\int_{-\infty}^\infty f$ exists and is s if $\int_{-\infty}^0 f = s_1$ and $\int_0^\infty f = s_2$ with $s = s_1 + s_2$.

The reader is advised to keep the analogous concept $\sum_{n=-\infty}^\infty a_n$ in the following.

Ex. 15.1.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that for any $\varepsilon > 0$, there exists $R > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x > R$ and $y > R$. Then $\lim_{x\to\infty} f(x)$ exists.

Proposition 15.1.3.
(1) *If f is continuous and if $\int_a^\infty |f|$ exists, then $\int_a^\infty f$ exists.*
(2) *Let f and g be continuous. Assume that $\int_a^\infty g$ exists and that there exists a positive constant such that $0 \leq f(x) \leq \lambda g(x)$ for $x \geq a$. Then $\int_a^\infty f$ exists.*
(3) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be given. Then $\int_{-\infty}^\infty f = s$ iff for any $\varepsilon > 0$, there exists $R > 0$ such that $\left|\int_{-a}^b f - s\right| < \varepsilon$ for $a \geq R$ and $b \geq R$.*

Proof. To prove a part of (3), let $\varepsilon > 0$ be given. Let R be chosen as per the condition. Then for $a, b > R$, we have

$$\left|\int_a^b f\right| = \left|\int_{-R}^b - \int_{-R}^a f\right| < 2\varepsilon.$$

Thus, $\int_0^\infty f$ converges, say, to s_1. Similarly $\int_{-\infty}^0 f$ converges, say, to s_2. It remains to show that $s_1 + s_2 = s$. Given $\varepsilon > 0$, by the other part of (3), there exists $R > 0$ such that if $A > R$, then $\left|\int_{-A}^A f - (s_1 + s_2)\right| < \varepsilon$. Hence $|s - (s_1 + s_2)| < \varepsilon$. $\square$

Ex. 15.1.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous such that there exist $R > 0$ and M such that $|f(x)| \le \frac{M}{|x|^2}$ for all x with $|x| \ge R$. Then $\int_{-\infty}^{\infty} f$ exists.

Definition 15.1.5. There is another way of assigning a meaning to $\int_{-\infty}^{\infty} f$. We say that the *Cauchy principal value* of the integral is L if $\int_{-R}^{R} f$ exists for all R and $\lim_{R\to\infty} \int_{-R}^{R} f = L$. In such a case, we write $P.V. \int_{-\infty}^{\infty} f = L$. The next few exercises clarify the relation between these two concepts.

Ex. 15.1.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be any odd continuous function: $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Then P.V. $\int_{-\infty}^{\infty} f = 0$. In particular, $\int_{-\infty}^{\infty} x$ is not convergent but its Cauchy principal value exists.

Ex. 15.1.7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be even: $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Assume that the principal value P.V.$\int_{-\infty}^{\infty} f$ exists. Then $\int_{-\infty}^{\infty} f$ exists. *Hint:* $\int_{-R}^{0} f = \int_{0}^{R} f = \frac{1}{2}\int_{-R}^{R} f$.

Ex. 15.1.8. Let $\int_{-\infty}^{\infty} f$ exist. Then P.V. $\int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f$.

Remark 15.1.9. Most often, we may first of all establish the convergence of $\int_{-\infty}^{\infty} f$ and then compute P.V. $\int_{-\infty}^{\infty} f$. In view of Ex. 15.1.8, we then would have computed $\int_{-\infty}^{\infty} f$.

There is another kind of improper integrals: these are the ones where the integrand becomes infinite at some points of the interval of integration. Let f be piecewise continuous on $[a, b]$, except, say, at $c \in [a, b]$. If the limits $\lim_{\varepsilon\to 0_+} \int_{a}^{c-\varepsilon} f(x)\,dx$ and $\lim_{\delta\to 0_+} \int_{c+\delta}^{b} f(x)\,dx$ exist, we then define

$$\int_{a}^{b} f(x)\,dx := \lim_{\varepsilon\to 0_+} \int_{a}^{c-\varepsilon} f(x)\,dx + \lim_{\delta\to 0_+} \int_{c+\delta}^{b} f(x)\,dx.$$

The integral is then called an improper integral.

Ex. 15.1.10. Define the analogous notion of principal value of such type of integrals.

Example 15.1.11. The integral $\int_{-1}^{1} \frac{1}{x}$ does not exist but the principal value

$$\text{P.V.} \int_{-1}^{1} \frac{1}{x} \text{ exists and is } 0.$$

Ex. 15.1.12. How do you define $\int_{-\infty}^{\infty} f(x)\,dx$ if f becomes infinite at some point?

15.2 Evaluation of Real Integrals

We illustrate some of the basic methods.

Type 1. Integrals of the form $\int_{-\infty}^{\infty} f$ where $|f(x)| \le \frac{C}{|x|^2}$ for $|x|$ very large. We start with a simplest example.

Example 15.2.1. Let us consider $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$. By Ex. 15.1.4, $\int_{\mathbb{R}} f$ exists. Let $f(z) := \frac{1}{1+z^2}$. The poles of f are at $z = \pm i$. Let C_R be the closed path: the line segment $[-R, R]$ followed by $\gamma_R(t) := Re^{it}$, $0 \le t \le \pi$. See Figure 15.1.

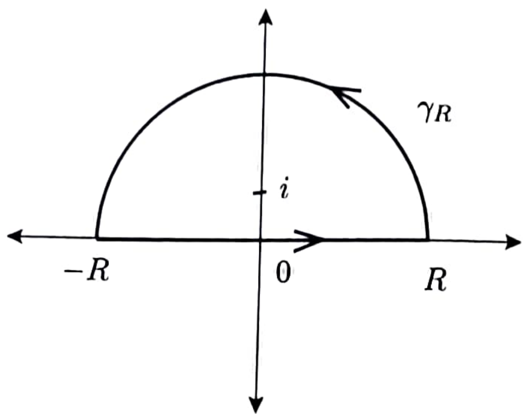

Figure 15.1: Illustration of Examples 15.2.1–15.2.2

If $R > 1$, then the pole $z = i$ is inside γ. Hence by the residue theorem we have

$$\int_{C_R} f = 2\pi i \mathrm{Res}\,(f; i) = 2\pi i \times \frac{1}{2i}.$$

Now, $\int_{C_R} f = \int_{-R}^{R} \frac{1}{1+x^2} + \int_{\gamma_R} f$. We claim that the second integral on the right goes to 0, as $R \to \infty$. The result then follows from Remark 15.1.9. Now, the claim follows from the estimate below:

$$\begin{aligned} \left| \int_{\gamma_R} f(z)\, dz \right| &= \left| \int_0^{\pi} \frac{iRe^{i\theta}}{1 + R^2 e^{2i\theta}}\, d\theta \right| \\ &\le \int_0^{\pi} \frac{R}{R^2 - 1}\, d\theta. \end{aligned}$$

Example 15.2.2. Consider $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\, dx$,where $a > 0$. We let $f(z) := \frac{z^2}{(z^2+a^2)^2}$. Then f has poles at $z = \pm ia$. Employing (11.3), we find $\mathrm{Res}\,(f; ia) = \frac{1}{4ia}$. If C_R is as in Eg. 15.2.1 with $R > a$, we then have

$$\int_{C_R} f = 2\pi i \mathrm{Res}\,(f; ia) = 2\pi i \times \frac{1}{4ia}.$$

It is easy to show that $\int_{\gamma_R} f \to 0$ as $R \to \infty$. We conclude that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} = \frac{\pi}{2a}$. (How? By Ex. 15.1.7. This can also be seen from Ex. 15.1.4 and Remark 15.1.9.)

These examples suggest the following result whose proof is left to the reader.

Theorem 15.2.3. Let f be holomorphic on $\mathbb{C}$ except at a finite number of points, none of which are real and that those in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are $z_1, \dots, z_n$. Assume that there exist positive constants R and M such that $|z^2 f(z)| \le M$ for $|z| \ge R$. Then

$$\int_{-\infty}^{\infty} f = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

□

Ex. 15.2.4. Prove the following:

(1) $\int_{-\infty}^{\infty} \frac{x^2+3}{(x^2+1)(x^2+4)} = \frac{5\pi}{6}$.

(2) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$.

(3) $\int_0^{\infty} \frac{x^2}{x^4+x^2+1} = \frac{\pi}{2\sqrt{3}}$.

(4) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = \frac{\pi(2n-2)!}{2^{2n-2}[(n-1)!]^2}$.

(5) $\int_0^{\infty} \frac{x^2}{x^6+1} = \frac{\pi}{6}$.

(6) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$, where $a > 0$ and $b > 0$.

(7) $\int_0^{\infty} \frac{2x^2-1}{x^4+5x^2+4} = \frac{\pi}{4}$.

Example 15.2.5. The integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}$ exists, for $a > 0$. The obvious choice $f(z) := \frac{\cos z}{z^2+a^2}$ is bad, as will be seen below. We consider $f(z) := \frac{e^{iz}}{z^2+a^2}$. Let C_R be as earlier. Then the only pole inside C_R is at $z = ia$, since $a > 0$. The residue $\operatorname{Res}(f; ia) = \frac{e^{-a}}{2ia}$. Proceeding as usual, we see that the given real integral converges to $\frac{\pi e^{-a}}{a}$, provided that we show that $\int_{\gamma_R} f \to 0$ as $R \to \infty$. On γ_R, we note that $|e^{iz}| = e^{-y} \le 1$ so that

$$\left| \int_{\gamma_R} \frac{e^{iz}}{z^2+a^2} \right| \le \pi R \frac{1}{R^2-a^2}, \qquad \text{for } R > a$$

which is what we wanted. Note that had we chosen cos in place of e^{iz}, we would be in trouble, since $|\cos z|$ becomes large when $\operatorname{Im} z$ is large.

Ex. 15.2.6. Prove that $\int_{-\infty}^{\infty} \frac{\cos 2x}{1+x^2} = \frac{\pi}{e^2}$ and $\int_{-\infty}^{\infty} \frac{\sin 2x}{1+x^2} = 0$. (To prove the second, do you need residue theory?)

Type II. We now consider the integrals of the form $\int_{-\infty}^{\infty} f(x) \cos ax$ etc. The following result says that the integral converges if $f(z) \to 0$ as $|z| \to \infty$ in the upper half plane. Note that we cannot invoke Ex. 15.1.4 here.

Theorem 15.2.7 (Jordan's Lemma). *Let f be holomorphic on $\mathbb{C}$ except at a finite number of points, none of which are real and that those in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are $z_1, \dots, z_n$. Assume that given $\varepsilon > 0$ there exists $R > 0$ such that $|f(z)| < \varepsilon$ whenever $\operatorname{Im} z > R$. Let $a > 0$. Then*

$$\boxed{\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(g; z_k), \text{ where } g(z) = f(z)e^{iaz}.} \tag{15.1}$$

Proof. Given $\varepsilon > 0$, choose $R > 0$ such that (i) $|z_k| < R$ for $1 \le k \le n$, (ii) $|f(z)| \le \varepsilon$ for z in the upper half plane with $\operatorname{Im} z > R$ and (iii) $te^{-at} \le 1$ for $t \ge R$. Let $a > R$, $b > R$ and $c := a + b$. Choose C to be the square with vertices at $-a$, b, $b + ic$ and $-a + ic$. See Figure 15.2.

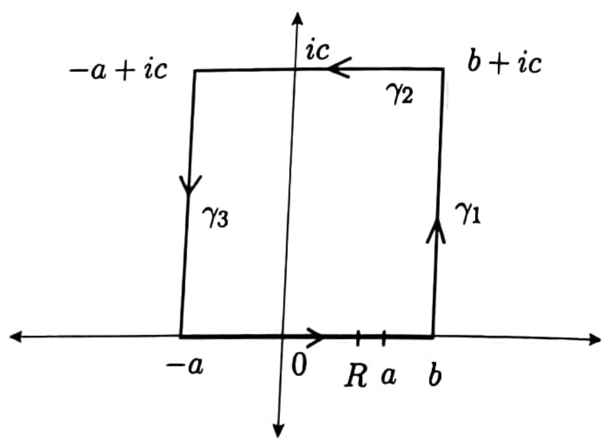

Figure 15.2: Illustration for Jordan's Lemma

Then by residue theorem,

$$\int_C f(z)e^{iz} = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(g, z_k).$$

Let the line segments $[b, b+ic]$, $[b+ic, -a+ic]$ and $[-a+ic, -a]$ be denoted by γ_1, γ_2 and γ_3 respectively. For z on γ_1 and γ_3, we have $|f(z)| \le \varepsilon$ and $\left|e^{iaz}\right| = e^{-ay}$ so that

$$\left|\int_{\gamma_j} g\right| \le \varepsilon \int_0^c e^{-ay}\,dy = \frac{\varepsilon}{a}(1 - e^{-ac}) \le \frac{\varepsilon}{a}, \qquad \text{for } j = 1, 3.$$

Since $L(\gamma_2) = c$, we have

$$\left|\int_{\gamma_2} g\right| \le c\varepsilon e^{-ac} \le \varepsilon.$$

Now the result follows from Proposition 15.1.3 (3). □

Remark 15.2.8. There is a more popular version of Jordan's lemma which is given below. Most often when it is used what we get is the Cauchy principal value of $\int_{-\infty}^{\infty}$. So one needs some standard test of convergence of improper integral to assert the existence of the integral under consideration.

Lemma 15.2.9 (Jordan). *Let $\gamma_R(t) := Re^{it}$, $0 \le t \le \pi$. Assume that*

(i) f is continuous on γ_R for $R \ge R_0$,

(ii) $|f(z)| \le M_R$ on γ_R, where $M_R \to 0$ as $R \to \infty$ and

(iii) $a > 0$.

Then

$$\boxed{\int_{\gamma_R} f(z)e^{iaz} \to 0, \qquad \text{as } R \to \infty.} \tag{15.2}$$

Proof. Proceeding as usual, we arrive at

$$\left|\int_{\gamma_R} f(z)e^{iaz}\right| \le R\int_0^{\pi} M_R e^{-aR\sin t}\,dt = 2R\int_0^{\pi/2} M_R e^{-aR\sin t}\,dt.$$

The crucial observation now is the following estimate (Ex. 15.2.10 below):

$$\sin t \ge \frac{2}{\pi}t, \qquad \text{for } 0 \le t \le \pi/2. \tag{15.3}$$

Using this, we get $\left|\int_{\gamma_R} f(z)e^{iaz}\right| \le \frac{\pi M_R}{a}(1 - e^{-aR})$. □

Ex. 15.2.10. Prove the Jordan's inequality:

$$\frac{2}{\pi} \le \frac{\sin t}{t} \le 1, \qquad \text{for } 0 \le t \le \pi/2.$$

Hint: Draw pictures of the functions involved. Show that $(\sin t)/t$ decreases on $(0, \pi/2)$ by taking derivative. Observe that $t\cos t - \sin t$ is decreasing on $(0, \pi/2)$, again by taking derivative. (Or, observe the concavity of the function $\sin t - (2t/\pi)$ on $(0, \pi/2)$.)

Example 15.2.11. We show that $\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^2+9} = \frac{\pi}{2e^9}$. Consider $f(z) = \frac{ze^{3iz}}{z^2+9}$. Then f has a simple pole at $z = 3i$ in the upper half plane. Its residue there is $\operatorname{Res}(f; 3i) = \frac{\pi i}{2e^9}$. The result follows from the theorem on separating the real and imaginary parts.

Ex. 15.2.12. Show the following:

(1) $\int_{-\infty}^{\infty} \dfrac{x\sin x}{x^2 + a^2} = \dfrac{\pi}{e^a}$.

(2) $\int_{-\infty}^{\infty} \dfrac{x^3\sin x}{(1 + x^2)^2} = \dfrac{\pi}{2e}$.

(3) $\int_{-\infty}^{\infty}(\frac{x^2-a^2}{x^2+a^2})(\frac{\sin x}{x}) = \pi(2e^{-a}-1)$, where $a>0$.

(4) $\int_{-\infty}^{\infty}\frac{\cos ax - \cos bx}{x^2} = \pi(b-a)$, where $a, b \geq 0$.

(5) $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2} = \pi$. *Hint:* Use (4). Or, see Ex. 15.1.4 (2) below.

Type III. These are the integrals of the form $\int_{-\infty}^{\infty} f$ where f has singularities on the real line.

Lemma 15.2.13. *Assume that f has a simple pole at a. Let $\gamma_r(t) := a + re^{it}$, $\alpha \leq t \leq \beta$. Then*

$$\int_{\gamma_r} f \to ia_{-1}(\beta - \alpha), \quad \text{as } r \to 0_+.$$

Remark 15.2.14. If the integration is performed on the full circle, then $\beta - \alpha = 2\pi$ and the residue theorem gives the value $2\pi i a_{-1}$, even without letting $r \to 0$. Thus the lemma indicates that the integration on an arc of the circle gives the corresponding fraction of $2\pi i a_{-1}$, provided that (i) $r \to 0$ and (ii) the pole is simple.

Proof. Without loss of generality, let us assume that $a = 0$.
Write $f(z) = a_{-1}z^{-1} + g(z)$ where g is holomorphic in a ball around a, say, $B[a, \varepsilon]$. Then $|g|$ is bounded on this compact set, say, by M. Thus for $0 < r < \varepsilon$, we have $\left|\int_{\gamma_r} g\right| \leq M(\beta - \alpha)r$ which goes to 0 as $r \to 0$. It is easily seen that $\int_{\gamma_r} a_{-1}z^{-1}\, dz = ia_{-1}(\beta - \alpha)$. The result follows. □

Example 15.2.15. Consider $\int_{-\infty}^{\infty}\frac{\sin x}{x}\, dx$. The integrand, when understood correctly, is continuous on $\mathbb{R}$ and as such it has no singularity anywhere. However, if we consider $f(z) := \frac{e^{iz}}{z}$, then f has a simple pole at $z = 0$. We use the path as in the Jordan's lemma (Theorem 15.2.7) but with a slight detour near the pole $z = 0$. More precisely, we replace the bottom side of the square C by the path γ which is made up of the following: the line segment $[-a, -r]$, followed by the semicircular arc (in the lower half plane) $\gamma_r(t) = re^{it}$, $\pi \leq t \leq 2\pi$, the line segment $[r, b]$. Let the new closed path be denoted by σ_R. See Figure 15.3.

Proceeding as in Jordan's lemma 15.2.7, we get:

$$\begin{aligned} 2\pi = 2\pi i \times \operatorname{Res}(f; 0) &= \int_{\sigma_R} f \\ &= \int_{\gamma} f + \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f \\ &\to \int_{-\infty}^{-r} f(x)\, dx + \int_{r}^{\infty} f(x)\, dx + \int_{\gamma_r} f(z)\, dz, \end{aligned}$$

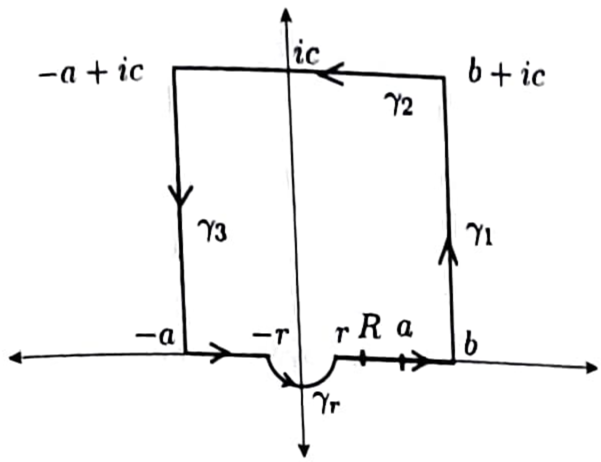

Figure 15.3: Illustration for Example 15.2.15

as $R \to \infty$. By Lemma 15.2.13, $\int_{\gamma_r} f \to \pi i \mathrm{Res}(f;0)$, as $r \to 0$. Hence we find that $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$.

Remark 15.2.16. This example is very interesting on at least two counts:

(i) It is not at all clear at the outset why the integral exists. There is no trouble at all if the limits of integration are 0 and any real $R > 0$. The problem is at the far end, as $\int_R^\infty \frac{|\sin x|}{x}$ is divergent. See Ex. 15.2.17 below.

(ii) In view of Ex.15.2.17, it is clear that the function $\frac{\sin x}{x}$ is not Lebesgue integrable on $(0,\infty)$ but its improper (Riemann) integral exists!

Ex. 15.2.17. Show that $\int_0^\infty \left|\frac{\sin x}{x}\right|$ does not exist. *Hint:* Observe the following:

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx > \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx \geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{k\pi}\right| dx$$
$$= \sum_{k=2}^{n} \frac{1}{k\pi} \int_0^\pi \sin x\, dx.$$

Ex. 15.2.18. Use Lemma 15.2.9 to find P.V. $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$. *Hint:* Consider the path consisting of upper semicircle $\gamma_R(t) = Re^{it}$, $0 \leq t \leq \pi$, followed by the line segment $[-R,-r]$, the upper semicircle γ_r, the line segment $[r,R]$.

Ex. 15.2.19. Prove the following:

(1) $\int_0^\infty \frac{1-\cos x}{x^2}\, dx = \pi/2$.

(2) $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$. *Hint:* $2\sin^2 x = 1 - \cos 2x$ and $f(z) = \frac{1-e^{2iz}}{z^2}$.

(3) $\int_0^\infty \frac{x - \sin x}{x^3}\, dx = \frac{\pi}{4}$. *Hint:* Consider $f(z) = \frac{z + ie^{iz} - i}{z^3}$.

(4) $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \pi \frac{\sin a}{a}$, $a > 0$.

Type IV. Trigonometric integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$. We have already seen examples of this kind (Eg. 11.1.19).

Example 15.2.20. Evaluate $\int_0^{\pi} \sin^{2n} t\, dt = \frac{1}{2}\int_0^{2\pi} \sin^{2n} t\, dt$. Since $\sin t = (e^{it} - e^{-it})/2i = (z - z^{-1})/2i$ for $z \in \gamma_1$, the unit circle, we have $\int_0^{2\pi} \sin^{2n} t\, dt = -i\int_{\gamma_1} f(z)$ where $f(z) = \frac{1}{z}(\frac{z - z^{-1}}{2i})^{2n}$. By binomial theorem,

$$\begin{aligned} f(z) &= z^{-1}\sum_{k=0}^{2n}\binom{2n}{k}(2i)^{-2n} z^k(-z)^{k-2n} \\ &= \sum_{k=0}^{2n}(-1)^{n-k}4^{-n}\binom{2n}{k} z^{2k-2n-1}. \end{aligned}$$

Thus the only singularity of f in $B(0,1)$ is a pole at the origin. From the above expression, we have $\operatorname{Res}(f;0) = 4^{-n}\binom{2n}{n}$. We find that $\int_0^{\pi} \sin^{2n} t\, dt = \pi 4^{-n}\binom{2n}{n}$.

It is possible to avoid the residue theorem in the above argument. Do you see how?

In general, given an integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)$, we can transform it into a path integral over the unit circle, by using the facts $\sin\theta = \frac{1}{2i}[z - z^{-1}]$ etc. on the unit circle. In fact, we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{\gamma_1} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{iz}.$$

Ex. 15.2.21. Show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}} = \int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta},$$

for $-1 < a < 1$.

Ex. 15.2.22. Show that

(1) $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}$, for $a > 1$.

(2) $\int_0^{\pi} \frac{a}{a^2 + \sin^2\theta}\, d\theta = \frac{\pi}{\sqrt{1 + a^2}}$, $a > 0$. *Hint:* Express $\sin^2 t$ in terms of $\cos 2t$.

(3) $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$, $n \in \mathbb{N}$. *Hint:* This is tricky. Evaluate the integral $\int_0^{2\pi} e^{\cos\theta}[\cos(n\theta - \sin\theta) - i\sin(n\theta - \sin\theta)]\, d\theta$.

As in integral calculus, there are just too many tricks (some bordering on ingenuity) in the employment of residue theorem to compute the real integrals. We give a few examples of this kind.

Example 15.2.23. We show that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n}\frac{1}{\sin(\pi/n)}$, for integers $0 < m < n$. Note that the integrals exist. We consider $f(z) = \frac{z^{m-1}}{1+z^n}$. The path under consideration is the sector formed of the line segment $[0, R]$, $R > 1$, followed by the arc $\gamma_R(t) = Re^{it}$, $0 \le t \le 2\pi/n$ and the line segment from $Re^{\frac{2\pi i}{n}}$ to 0. See Figure 15.4.

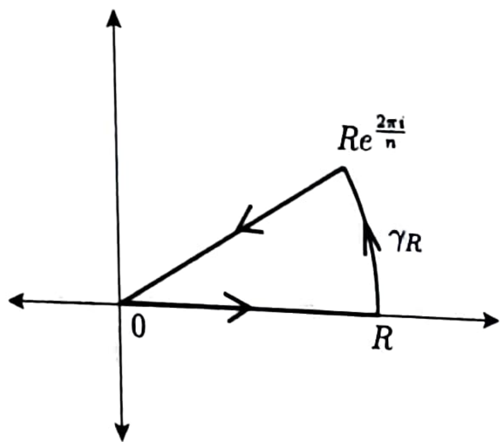

Figure 15.4: Illustration for Example 15.2.23

The point $p = e^{i\pi/n}$ is only one pole inside this closed path. We have $\operatorname{Res}(f; p) = -\frac{1}{n}p^m$. It is easy to see that the integral over the arc goes to zero as $R \to \infty$. We compute the integral over the line segment $[Rp^2, 0]$ as follows: Let $\gamma_3(t) = tp^2$, $0 \le t \le R$. Then $\int_{[Rp^2,0]} f = -\int_{\gamma_3} f$.

$$-\int_{\gamma_3} f = -\int_0^R \frac{t^{m-1}p^{2m-2}}{1+t^n p^{2n}} p^2\,dt = -p^{2m} \int_0^R f(x),$$

that is, a multiple of the original integral. One can now proceed as usual and establish the claim.

Ex. 15.2.24. Derive the following special cases:

(1) $\int_0^\infty \frac{x^{n-1}}{1+x^{2n}} = \frac{\pi}{2n}$.

(2) $\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n}\frac{1}{\sin(\pi/n)}$.

Ex. 15.2.25. For $0 < a < 1$, show that $\int_{-\infty}^\infty \frac{e^{ax}}{e^x+1} = \frac{\pi}{\sin \pi a}$ by integrating along the rectangle whose vertices are $-R$, R, $R+2\pi i$ and $-R+2\pi i$. *Hint:* The integrals along the vertical sides go to zero as $R \to \infty$ while the one on the top side is a multiple of the given integral. (Did you show that the integral exists?)

Ex. 15.2.26. Evaluate $\int_0^\infty \frac{t^{a-1}}{1+t}\,dt$. *Hint:* Make the substitution $t = e^x$. Use the last exercise.

Ex. 15.2.27. In general, integrals of the form $\int_0^\infty x^{-a-1} f(x)\,dx$ can be evaluated as follows: Let $x = e^t$ and the integral becomes $\int_{-\infty}^\infty e^{at} f(e^t)\,dt$. Now let $g(z) := \exp(az)f(e^z)$, $z \in \mathbb{C}$. Integrate g round the boundary of the rectangle with vertices at R, $R + 2\pi i$, $-T + 2\pi$ and $-T$ and take limits as $R, T \to \infty$. Use this method to show:

(i) $\int_0^\infty \dfrac{x^{a-1}}{1+x^b}\,dx = \dfrac{\pi}{b\sin(\frac{\pi a}{b})}$, $0 < a < b$.

(ii) $\int_0^\infty \dfrac{x^a}{1 + 2x\cos\theta + x^2}\,dx = \dfrac{\pi}{\sin(\pi a)}\dfrac{\sin a\theta}{\sin\theta}$, $-1 < a < 1$ and $-\pi < \theta < \pi$.

Ex. 15.2.28. The beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad p > 0, q > 0.$$

Show that $B(p, 1-p) = \frac{\pi}{\sin p\pi}$. *Hint:* Transform the integral to the one in Ex. 15.2.26 by a substitution.

Ex. 15.2.29. Evaluate the Fresnel's integrals

$$\int_0^\infty \cos^2 x\,dx \text{ and } \int_0^\infty \sin^2 x\,dx$$

by integrating e^{-z^2} along the boundary of the triangle with vertices 0, R and $Re^{i\pi/4}$ where $R > 0$, and letting $R \to \infty$. (See Example 7.2.6.)

Miscellaneous Exercises

Ex. 15.2.30. Compute the following integrals:

(1) $\int_{-\infty}^\infty \dfrac{e^{iax}}{x^6+1}\,dx$. (Ans. $\dfrac{2\pi}{6}$.)

(2) $\int_0^\infty \dfrac{x^2-1}{(x^2+1)^2}\,dx$.

(3) $\int_{-\infty}^\infty \dfrac{1}{(x^2+4)(x^2+9)}\,dx$.

(4) $\int_0^\infty \dfrac{x^2}{(x^2+1)(x^2+4)}\,dx = \pi/6$.

(5) $\int_0^\infty \dfrac{dx}{(x^2+1)^2} = \pi/4$.

(6) $\int_0^\infty \frac{x^2}{x^6+1} = \pi/6.$

(7) $\int_0^\infty \frac{2x^2-1}{x^4+5x^2+1} = \pi/4.$

(8) $\int_{-\infty}^\infty \frac{x^k}{1+x^{2n}}\,dx = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{\pi}{n\sin(\dfrac{(k+1)\pi}{2n})} & \text{if } k \text{ is even and } 0 \le k \le 2n-2. \end{cases}$

Ex. 15.2.31. Compute the following integrals:

(1) $\int_{-\infty}^\infty \frac{e^{\imath ax}}{1+x^2}\,dx$, $a \neq 0$. (Ans. πe^{-a} if $a > 0$ and πe^a if $a < 0$.)

(2) $\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a}.$

(3) $\int_0^\infty \frac{\cos ax}{x^2+1} = \frac{\pi}{2}e^{-a}$, $a > 0.$

(4) $\int_0^\infty \frac{x \sin ax}{x^2+4}\,dx = \frac{\pi}{2}e^{-a}\sin a.$ $a > 0.$

(5) $\int_{-\infty}^\infty \frac{\cos x}{(1+x^2)^2}\,dx = \pi e^{-1}.$

Ex. 15.2.32. Compute the following integrals:

(1) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2.$

(2) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi/2.$ *Hint:* Use (1).

Ex. 15.2.33. Compute the following integrals:

(1) $\int_{-\pi}^\pi \frac{1}{1+\cos^2\theta}\,d\theta = \pi\sqrt{2}.$

(2) $\int_0^\pi \cos^{2n}\theta\,d\theta = \pi\frac{(2n)!}{2^{2n}(n!)^2}$, $n \in \mathbb{N}.$

(3) $\int_0^\pi \sin^{2n}\theta\,d\theta = \pi\frac{(2n)!}{2^{2n}(n!)^2}$, $n \in \mathbb{N}.$

(4) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}.$

(5) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, where $a > |b|$.

(6) $\int_0^{2\pi} \frac{\cos m\theta}{5+3\cos\theta}\,d\theta = \frac{(-1)^m\pi}{2\cdot 3^m}$, $m \in \mathbb{N}.$

(7) $\int_0^{2\pi} \frac{d\theta}{r^2 - 2r\cos\theta + 1} = \frac{2\pi}{1-r^2}, \; 0 \le r < 1.$

(8) $\int_0^{2\pi} \frac{\cos 2t}{1 - 2k\cos t + k^2} = \frac{\pi k^2}{1-k^2}, \;$ for $-1 < k < 1.$

15.3 Summation of Infinite Series

Theorem 15.3.1. *Let f be holomorphic on $\mathbb{C}$ except at a finite number of points z_j, $1 \le j \le k$. Assume that none of the z_j is an integer. Suppose that there exist constants R and M such that $|z^2 f(z)| \le M$ for $|z| \ge R$. Let*

$$g(z) := \pi \frac{\cos \pi z}{\sin \pi z} f(z), \qquad h(z) := \frac{\pi}{\sin \pi z} f(z).$$

Then

$$\sum_{-\infty}^{\infty} f(n) = -\sum_{j=1}^{k} \operatorname{Res}(g; z_j), \tag{15.4}$$

$$\sum_{-\infty}^{\infty} (-1)^n f(n) = -\sum_{j=1}^{k} \operatorname{Res}(h; z_j). \tag{15.5}$$

Proof. We indicate a proof of Eq. 15.4. The hypothesis on f implies the convergence of $\sum |f(n)|$.

The function g is differentiable except at the integers and z_j's. If $f(n) \neq 0$, then g has a simple pole at $z = n$ with residue $\operatorname{Res}(g; n) = f(n)$. (If $f(n) = 0$, then g has a removable singularity at n.)

Let C_N be the square with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. Then by residue theorem, for all large N,

$$\int_{C_N} g = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(g; z_j) + \sum_{r=-N}^{N} f(r). \tag{15.6}$$

If we show that $\int_{C_N} g \to 0$ as $N \to \infty$, the result will follow. Lemma 15.3.2 below, in conjunction with the ML-inequality yields this assertion. $\square$

Lemma 15.3.2. *Let C_N be the square with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$, $N \in \mathbb{N}$. Then there exists M such that*

$$|\cot \pi z| \le M \qquad \text{for all } N \text{ and } z \in C_N.$$

Proof. On the horizontal sides $z = x \pm i(N + \frac{1}{2})$,

$$
\begin{aligned}
|\cot \pi z| &= \left| \frac{e^{i\pi[x \pm i(N+\frac{1}{2})]} + e^{-i\pi[x \pm i(N+\frac{1}{2})]}}{e^{i\pi[x \pm i(N+\frac{1}{2})]} - e^{-i\pi[x \pm i(N+\frac{1}{2})]}} \right| \\
&\le \frac{e^{\pi(N+\frac{1}{2})} + e^{-\pi(N+\frac{1}{2})}}{e^{\pi(N+\frac{1}{2})} - e^{-\pi(N+\frac{1}{2})}} \\
&= \coth \pi(N + \frac{1}{2}) \\
&\le \coth(3\pi/2), \qquad \text{(since coth is decreasing for } t \ge 0\text{)}.
\end{aligned}
$$

On the vertical sides $z = \pm(N + \frac{1}{2}) + iy$,

$$|\cot \pi z| = |\tan i\pi y| = |\tanh \pi y| \le 1.$$

□

Example 15.3.3. As a special case of the theorem, consider $f(z) = (z^2 - a^2)^{-1}$ for a fixed $a \notin \mathbb{Z}$. This is holomorphic except at $z = \pm a$. Then $\operatorname{Res}(f; a) = 1/2a$. Hence (15.4) yields $\sum_{n=-\infty}^{\infty} \frac{1}{a^2 - n^2} = \frac{\pi \cot \pi a}{2a}$, $a \notin \mathbb{Z}$.

Ex. 15.3.4. Prove the following:

(1) $\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)(3n+1)} = \frac{\pi}{\sqrt{3}}$ (2) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(3) $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh \pi}$ (4) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Ex. 15.3.5. Let $f(z) = \frac{1}{z(z-a)}$, $z \notin \mathbb{Z}$ and $g(z) := f(z) \cot \pi z$. Argue as in Theorem 15.3.1 to prove the following

$$\frac{1}{a} \cot \pi a = \frac{1}{\pi a^2} - \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{1}{r^2 - a^2}.$$

Hence deduce the following expansion

$$\frac{1}{z} \cot \pi z = \frac{1}{\pi z^2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 - z^2}, \qquad \text{for } z \notin \mathbb{Z}.$$