Department of Mathematics & Statistics

Calculus of Several Variables and Differential Geometry

Mid-Semester Examination

Marks: 35 Time: 120 minutes

---

1. **True or False**

   (a) Let $n \geq 2$ and U be an open subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable function. Then, for every point x in U, there exists a non-zero vector $v \in \mathbb{R}^n$ such that $df_x(v) = 0$. [2 marks]

   Answer: TRUE
   Reason: We know that, for every $x \in U$, the derivative $df_x\colon \mathbb{R}^n \to \mathbb{R}$ is a linear map. Hence by rank-nullity theorem, there exists a non-vector $v \in \mathbb{R}^n$ such that $df_x(v) = 0$.

   (b) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) := \int_x^{x+y} g(t)dt$. Then the gradient of the function f at any point (x, y) is $\nabla f(x, y) = (g(x) - g(x + y), g(x + y))$. [2 marks]

   Answer: FALSE
   Reason: At any point $(x, y) \in \mathbb{R}^2$, the gradient $\nabla f(x, y) = (g(x + y) - g(x), g(x + y))$.

2. **Fill in the blanks**

   (a) Let n, $m \geq 2$, and A, B be two matrices of order $n \times m$. Let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ be the function defined by $f(x, y) := \langle Ax, By\rangle$. Then, for $(h, k) \in \mathbb{R}^m \times \mathbb{R}^m$, $df_{(x,y)}(h, k) := \ldots\ldots.$ [3 marks]

   Answer: For every point $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, the derivative $df_{(x,y)}(h, k) = \langle Ah, By\rangle + \langle Ax, Bk\rangle$.
   Reason: $f(x+h, y+k) = \langle A(x + h), B(y + k)\rangle = \langle Ax, By\rangle + \langle Ah, By\rangle + \langle Ax, Bk\rangle + \langle Ah, Bk\rangle$. Further $\langle Ah, By\rangle + \langle Ax, Bk\rangle$ is linear in (h, k) and $\langle Ah, Bk\rangle \leq \| A \|\| B \|\| h \|\| k \| \leq \| A \|\| B \|\| (h, k) \|^2$.

   (b) Let $f(u, v) := (u^2 + v^2, u^2 - v^2, 2uv)$ for (u, v) in $\mathbb{R}^2$ and $g(x, y, z) := x^2 + y^2 - z^2$ for (x, y, z) in $\mathbb{R}^3$. If $h : \mathbb{R}^2 \to \mathbb{R}$ is the function defined by $h(u, v) := g(f(u, v))$, then $\nabla h(u, v) = \ldots\ldots.$ [3 marks]

   Answer: $\nabla h(u, v) = 8(u^2 - v^2)(u, -v)$.
   Reason: For (u, v) in $\mathbb{R}^2$, we have $h(u, v) = (u^2+v^2)^2+(u^2-v^2)^2-4u^2v^2$. Therefore $\nabla h(u, v) = \left(\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}\right) = \left(8u(u^2 - v^2), 8v(v^2 - u^2)\right)$

   (c) Let B be a $k \times n$ matrix and C a $k \times k$ matrix with real entries. Let $A = [B, C]$ be a $k \times (n+k)$ matrix. Assume that C is invertible. Given $z \in \mathbb{R}^k$, let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^k$ be a column vector such that $A \begin{pmatrix} x \\ y \end{pmatrix} = z$. Then $y = \ldots\ldots.$ [3 marks]

Answer: $y = C^{-1}(z - Bx)$

Reason: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^k$ be a vector in $\mathbb{R}^n \times \mathbb{R}^k$. Then we can write $A\begin{pmatrix} x \\ y \end{pmatrix} = z$ as $Bx + Cy = z$. This means that $Cy = z - Bx$. Since C is invertible, it follows that $y = C^{-1}(z - Bx)$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $\|f(x) - f(y)\| \leq \|x - y\|^2$. Which of the following statement(s) is(are) **TRUE**? [2 marks]

   (a) f is differentiable only at 0 and $df_0 \equiv 0$.

   (b) f is differentiable at all points in $\mathbb{R}^n$ and $df_x = 0$.

   (c) f is not differentiable at 0.

   (d) f is differentiable at all points in $\mathbb{R}^n$ and df_x need not be zero.

   Answer: Only (b) is TRUE

   Reason: Observe that for $x \neq y$ in $\mathbb{R}^n$, we have $\frac{\|f(x)-f(y)\|}{\|x-y\|} \leq \|x - y\|$. Therefore if we take $df_x \equiv 0$, then $\frac{\|f(x+h)-f(x)-df_x(h)\|}{\|h\|} \leq \|h\|$ and it goes to zero as $\|h\| \to 0$.

4. For the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$, which of the following statement(s) is(are) **TRUE**? [5 marks]

   (a) f is continuous at $(0, 0)$.

   (b) f is not continuous at $(0, 0)$ but the partial derivatives exist at $(0, 0)$.

   (c) f is continuous at $(0, 0)$, the directional derivative exist along all (u, v) in $\mathbb{R}^2$ and the directional derivative at (u, v) is 0.

   (d) f is continuous at $(0, 0)$, the directional derivative exist along all (u, v) in $\mathbb{R}^2$ and the directional derivative at (u, v) is $\frac{u^3-v^3}{u^2+v^2}$.

   Answer: (a) and (d) are TRUE.

   Reason: For $(0, 0) \neq (x, y) \in \mathbb{R}^2$, we have $|x^3 - y^3| = |x - y||x^2 + xy + y^2| \leq |x - y|(|x^2 + y^2| + |xy|) \leq |x - y|(|x^2 + y^2| + |x^2 + y^2|)$. Therefore for $(0, 0) \neq (x, y) \in \mathbb{R}^2$, $|f(x, y) - f(0, 0)| = |\frac{x^3-y^3}{x^2+y^2}| \leq 2|x - y|$. This proves that f is continuous at $(0, 0)$.

   Let $(0, 0) \neq (u, v) \in \mathbb{R}^2$. Then the directional derivative of f at $(0, 0)$ along (u, v), if it exists is equal to $\lim_{t\to 0} \frac{f(tu,tv)}{t} = \frac{t^3(u^3-v^3)}{t^3(u^2+v^2)} = \frac{u^3-v^3}{u^2+v^2}$. In fact one could have directly proved that the function is continuous by the arguments we have used here.

5. If we let $S := \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is local minima or maxima } \}$ for the function $f(x, y) = (x^2 - y^2)e^{-(\frac{x^2+y^2}{2})}$, then $S = \ldots\ldots$ [5 marks]

Answer: $S = \{(\pm\sqrt{2}, 0), (0, \pm\sqrt{2})\}$.

Reason: For the function f, the partial derivatives $\frac{\partial f}{\partial x} = x\left[2 - (x^2 - y^2)\right] e^{-\frac{x^2+y^2}{2}}$ and $\frac{\partial f}{\partial y} = -y\left[2 + (x^2 - y^2)\right] e^{-\frac{x^2+y^2}{2}}$. Therefore the critical points satisfy the equations

$x = 0$ or $2-(x^2-y^2) = 0$ and $y = 0$ or $2+(x^2-y^2) = 0$. [1 mark]

Observe that f is non-negative along x-axis and non-positive along y axis. Hence $(0, 0)$ is neither a maximum nor a minimum. [1 mark]
The other possibilities are $(\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{2})$. To check the behavior of f around these points, we compute the second derivatives of f.

(a) $\frac{\partial^2 f}{\partial x^2} = \left[2 - (x^2 - y^2) - 2x^2 - x^2(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}}$.

(b) $\frac{\partial^2 f}{\partial y^2} = \left[-2 - (x^2 - y^2) + 2y^2 + y^2(2 + (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}}$.

(c) $\frac{\partial^2 f}{\partial x \partial y} = \left[2xy - xy(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}}$ [1 mark]

and

(a) $\frac{\partial^2 f}{\partial x^2} = \left[2 - (x^2 - y^2) - 2x^2 - x^2(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = -4$ at $(\pm\sqrt{2}, 0)$,

(b) $\frac{\partial^2 f}{\partial y^2} = \left[-2 - (x^2 - y^2) + 2y^2 + y^2(2 + (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = 4$ at $(\pm\sqrt{2}, 0)$ and

(c) $\frac{\partial^2 f}{\partial x \partial y} = \left[2xy - xy(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = 0$ at $(\pm\sqrt{2}, 0)$.

Therefore $(\pm\sqrt{2}, 0)$ are points of maximum for the function f. [1 mark]
Similarly

(a) $\frac{\partial^2 f}{\partial x^2} = \left[2 - (x^2 - y^2) - 2x^2 - x^2(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = 4$ at $(0, \pm\sqrt{2})$,

(b) $\frac{\partial^2 f}{\partial y^2} = \left[-2 - (x^2 - y^2) + 2y^2 + y^2(2 + (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = 4$ at $(0, \pm\sqrt{2})$ and

(c) $\frac{\partial^2 f}{\partial x \partial y} = \left[2xy - xy(2 - (x^2 - y^2))\right] e^{-\frac{x^2+y^2}{2}} = 0$ at $(0, \pm\sqrt{2})$.

This shows that $(0, \pm\sqrt{2})$ are points of minimum for the function f. [1 mark]
Hence $S = \{(\pm\sqrt{2}, 0), (0, \pm\sqrt{2})\}$.

6. Let Let $n \geq 1$ and $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. Show that the function $F \colon \mathbb{R}^n \to \mathbb{R}$ defined by $F(x) := \langle f(x), x\rangle$ is differentiable using $\varepsilon - \delta$ method. Further find its gradient $\nabla F(x)$ at any point $x \in \mathbb{R}^n$. [5 marks]

Let $a \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$. Then

$$\begin{aligned} F(a+h) - F(a) &= \langle f(a+h), a+h\rangle - \langle f(a), a\rangle \\ &= \langle f(a+h) - f(a) - df_a(h), a\rangle + \langle f(a+h) - f(a), h\rangle \\ &\quad + \langle df_a(h), a\rangle + \langle f(a), h\rangle \end{aligned}$$ [1 mark]

Let $\varepsilon > 0$ be given.
Observe that by Cauchy-Schwarz inequality $|\langle f(a+h) - f(a) - df_a(h), a\rangle| \leq \|f(a+h) - f(a) - df_a(h)\|\|a\|$.

Similarly $|\langle f(a+h)-f(a),h\rangle| \leq \|f(a+h)-f(a)\|\|h\|$.
Since f is differentiable at a, there exists $\delta_1 > 0$ such that for $h \in B(0,\delta_1)$, we have $\|f(a+h)-f(a)-df_a(h)\| < \frac{\varepsilon}{2(1+\|a\|)}\|h\|$. [1 mark]
Since f is differentiable it is also continuous. Hence there exists $\delta_2 > 0$ such that for $h \in B(0,\delta_2)$, $\|f(a+h)-f(a)\| < \varepsilon/2$. [1 mark]
Let $\delta = \min\{\delta_1,\delta_2\}$. Then for $h \in B(0,\delta)$,

$$\begin{aligned}
|F(a+h)-F(a)-\langle df_a(h),a\rangle-\langle f(a),h\rangle| &\leq |\langle f(a+h)-f(a)-df_a(h),a\rangle| \\
&\quad +|\langle f(a+h)-f(a),h\rangle| \\
&\leq \|a\|\|f(a+h)-f(a)-df_a(h)\| \\
&\quad +\|h\|\|f(a+h)-f(a)\| \\
&< \|a\|\frac{\varepsilon}{2(1+\|a\|)}\|h\| \\
&\quad +\frac{\varepsilon}{2}\|h\| \\
&< \varepsilon\|h\| \qquad \text{[1 mark]}
\end{aligned}$$

This proves that the fnction F is differentiable and

$$\begin{aligned}
dF_a(h)\ &\langle df_a(h),a\rangle + \langle f(a),h\rangle \\
&= \left\langle f(a)+df_a^t(a),h\right\rangle
\end{aligned}$$

If $f(x) = (f_1(x), f_2(x),\ldots,f_n(x))$ we can write $F(x) = \sum_{i=1}^n x_i f_i(x)$ for every point $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$.
If the gradient $\nabla F(x) = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \cdots, \frac{\partial F}{\partial x_n}\right)$, then $\frac{\partial F}{\partial x_j} = f_j(x) + \sum_{i=1}^n \frac{\partial f_i}{\partial x_j}(x)$ or $\nabla F(x) = f(x) + (df_x)^t(x)$. [1 mark]

7. Let $n \geq 2$ and W be a proper vector subspace of $\mathbb{R}^n$. Given a point $x \in \mathbb{R}^n \setminus W$, let z the unique point in W such that $\|x-z\| = \inf\{\|x-w\| : w \in W\}$. Using the method of Lagrange multiplier, show that the vector $x-z$ is orthogonal to W. [5 marks]

Let W be a proper subspace of $\mathbb{R}^n$.
Let $W^\perp = \{y \in \mathbb{R}^n : \langle y,w\rangle = 0 \text{ for all } w \in W\}$ be the orthogonal complement of dimension k. Let $a_1,a_2,\ldots,a_k$ be a basis of $W^\perp$ and for $1 \leq i \leq k$, $a_i = (a_{i1},a_{i2},\ldots,a_{in})$ with respect to the standard basis of $\mathbb{R}^n$. [1 mark]
Then for $1 \leq i \leq k$ and $x = (x_1,x_2,\ldots,x_n) \in W$, $\langle a_i,x\rangle = \sum_{j=1}^n a_{ij}x_j = 0$.
Let $g_i \colon \mathbb{R}^n \to \mathbb{R}$ be defined by $g_i(x) := \sum_{j=1}^n a_{ij}x_j$ for $1 \leq i \leq k$. Then $W = \{x \in \mathbb{R}^n : g_i(x) = 0 \text{ for } 1 \leq i \leq k\}$. [1 mark]
For the point $x \in \mathbb{R}^n \setminus W$ let $f \colon W \to \mathbb{R}$ be the function defined by $f(w) = \|x-w\|^2$. [1 mark]
Since $z \in W$ is such that $\|x-z\| = \inf\{\|x-w\| : w \in W\}$, by Lagrange multplier, there exist $\lambda_1,\lambda_2,\ldots,\lambda_k \in \mathbb{R}$ such that $\nabla f(z) = \lambda_1 g_1(z)+\lambda_2 g_2(z)+\cdots+\lambda_k g_k(z)$. [1 mark]
Since $\nabla g_i(z) = a_i$ is orthogonal to W for $1 \leq i \leq k$ and $\nabla f(z) = 2(x-z)$, it follows that $x-z = \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k)$. Hence the vector $x-z$ is orthogonal to W. [1 mark]