Name:
Roll No:

MSO201A: Probability & Statistics
Quiz 2: Full Marks 20

[1] Let $\boldsymbol{X} = (X_1, X_2, X_3)^T \sim N_3(0, \Sigma)$, where $\Sigma = \begin{pmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 0.5 \\ -0.5 & 0.5 & 1 \end{pmatrix}$ and p.d.f. of $\boldsymbol{X}$ is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}\boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x}};\ \boldsymbol{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3.$$

(a) Find the joint p.d.f. of $X_1 + X_2 + X_3$ and $X_2 - X_3$.
(b) Prove or disprove "$X_1 + X_2 + X_3$ and $X_2 - X_3$ are independently distributed".
(c) Find the distribution of $Z = (X_1 - 2X_2 + 3X_3)^2$.

10 (4+3+3) marks

[2] Let $\{X_n\}$ be a sequence of i.i.d. random variables with p.d.f.

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S_n = \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\,\bar{X}_n^2\right)$.
(a) Prove or disprove " $\lim_{n\to\infty} P(\bar{X}_n \leq 3) = 1$ ".
(b) Prove or disprove "$\frac{1}{n}\sum_{i=1}^n e^{-\frac{2X_i}{3}} \xrightarrow{p} e^{-2}$ as $n \to \infty$".
(c) Find β such that $\left(S_n e^{-\bar{X}_n}\right) \xrightarrow{p} \beta$ as $n \to \infty$.

10 (4+3+3) marks

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(1) $\underset{\sim}{X} \sim N_3(\underset{-}{0}, \Sigma)$ $\quad \Sigma = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{pmatrix}$

(a) $\underset{\sim}{y} = \begin{pmatrix} X_1 + X_2 + X_3 \\ X_2 - X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \underset{\sim}{X} = A\underset{\sim}{X}$

$A\underset{\sim}{X} \sim N_2(\underset{-}{0}, A\Sigma A')$ (2) as $\forall\, \underset{-}{\alpha} \in \mathbb{R}^2$, $\underset{-}{\alpha}' A \underset{\sim}{X} \sim N_1$

$$A\Sigma A' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1/2 \\ 1/2 & 1 \end{pmatrix} = \Sigma^* \text{ say}$$

jt p.d.f. of $\underset{\sim}{y}$

Give full marks if it's written as $N_2(\underset{-}{0}, A\Sigma A')$ without calculating Σ^*

$$f_{\underset{\sim}{y}}(\underset{\sim}{y}) = (2\pi)^{-1} |\Sigma^*|^{-1/2} e^{-\frac{1}{2}\underset{\sim}{y}'\Sigma^{*-1}\underset{\sim}{y}} \text{ (2)} ; \quad \underset{\sim}{y} \in \mathbb{R}^2$$

(b) $\begin{pmatrix} X_1 + X_2 + X_3 \\ X_2 - X_3 \end{pmatrix} \sim N_2(\underset{-}{0}, \Sigma^*)$

$Y_1 = X_1 + X_2 + X_3 \sim N_1(0, 3)$

$Y_2 = X_2 - X_3 \sim N_1(0, 1)$ (3)

$f_{\underset{\sim}{y}}(\underset{\sim}{y}) \neq f_{Y_1} f_{Y_2} \Rightarrow X_1 + X_2 + X_3$ & $X_2 - X_3$ are not indep.

Alternately, one can argue that $Cov(X_1 + X_2 + X_3, X_2 - X_3) \neq 0$

$\Rightarrow X_1 + X_2 + X_3$ & $X_2 - X_3$ are not indep.

Give full marks for this also

(c) $X_1 - 2X_2 + 3X_3 \sim N_1$

$E(X_1 - 2X_2 + 3X_3) = 0$

$$V(X_1 - 2X_2 + 3X_3) = V(X_1) + 4V(X_2) + 9V(X_3) - 4\,\mathrm{Cov}(X_1, X_2) + 6\,\mathrm{Cov}(X_1, X_3) - 12\,\mathrm{Cov}(X_2, X_3)$$

$$= 1 + 4 + 9 - 0 + 6\left(-\tfrac{1}{2}\right) - 12\left(\tfrac{1}{2}\right)$$

$$= 14 - 3 - 6 = 5$$

i.e. $X_1 - 2X_2 + 3X_3 \sim N(0, 5)$ — (1)

$$\Rightarrow \frac{(X_1 - 2X_2 + 3X_3)}{\sqrt{5}} \sim N(0, 1)$$

$$\Rightarrow \left(\frac{X_1 - 2X_2 + 3X_3}{\sqrt{5}}\right)^2 \sim \chi^2_1$$

(2)

i.e. $\dfrac{Z}{5} \sim \chi^2_1$

$\Rightarrow Z \sim 5\chi^2_1$

(2) $\{X_n\}$ i.i.d. seq of $\exp(3)$

$E(X_i) = 3$; $V(X_i) = 9$ $\forall i = 1, \ldots, n$

(a)

By CLT $\sqrt{n}(\bar{X}_n - 3) \xrightarrow{L} N(0, 9)$ — (2)

$$P(\bar{X}_n \le 3) = P(\bar{X}_n - 3 \le 0)$$

$$= P\left(\frac{\sqrt{n}(\bar{X}_n - 3)}{3} \le 0\right)$$

$$\longrightarrow \Phi(0) = \frac{1}{2} \ne 1 \quad (2)$$

as $n \to \infty$ by CLT.

(b) $X_1, \ldots, X_n$ i.i.d. with $f(x) = \frac{1}{3} e^{-x/3}$ $x>0$

Let $\underline{Y_i = e^{-\frac{2X_i}{3}}}$; $EY_i = \frac{1}{3}\int_0^\infty e^{-\frac{2}{3}x} e^{-x/3}\,dx$

$$= \frac{1}{3}\int_0^\infty e^{-x}\,dx = \frac{1}{3}$$

$\Rightarrow$ $\underline{Y_1, \ldots, Y_n \text{ are i.i.d. with } EY_1 = \frac{1}{3}}$ (1)

By WLLN $\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{p} \frac{1}{3}$

i.e. $\frac{1}{n}\sum_{i=1}^n e^{-\frac{2X_i}{3}} \xrightarrow{p} \frac{1}{3}$ —(2)

* There can be alternate solutions; give partial marks accordingly.

(c) By WLLN $\bar{X}_n \xrightarrow{p} EX_1 = 3$

$$e^{-\bar{X}_n} \xrightarrow{p} e^{-3} \quad \text{—(1)}$$

$$S_n = \frac{n}{n-1}\frac{1}{n}\sum X_i^2 - \frac{n}{n-1}\bar{X}_n^2$$

By WLLN $\frac{1}{n}\sum X_i^2 \xrightarrow{p} EX_1^2 = 2\times 3^2 = 18$

$$\bar{X}_n^2 \xrightarrow{p} 3^2 = 9$$

$\underline{S_n \xrightarrow{p} 18 - 9 = 9}$ (1)

& $e^{-\bar{X}_n} \xrightarrow{p} e^{-3}$

$\Rightarrow$ $\underline{S_n e^{-\bar{X}_n} \xrightarrow{p} 9e^{-3}}$ (1)

Note: Deduct only $(\frac{1}{2})$ marks if the in answer the (by mistake) student takes $\frac{1}{n-1}(\sum X_i^2 - n\bar{X}_n^2) = S_n^2$ and does the calculations (otherwise) conceptually correct and gets to $S_n e^{-\bar{X}_n} \xrightarrow{p} \underline{\underline{3}}\, e^{-3}$