Assignment 2: Several variables calculus & differential geometry (MTH305A)

(1) Use definition to check the differentiability of the function $f(x,y) = x(y+1)$ at $(1,0)$.

(2) Identify $\mathbb{R}^4$ with the set $M_2(\mathbb{R})$ of 2 real matrices . Define $F : \mathbb{R}^4 \to \mathbb{R}^4$ by $F(A) = A^2$ (Matrix multiplication of A with itself). Show that F is differentiable and what is its derivative?

(3) Let g be a continuous function on the unit circle $\{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ such that $g(1,0) = g(0,1)$ and $g(-x) = -g(x)$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} \|x\| g\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) If $x \in \mathbb{R}^2$ and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

(b) Show that f is NOT differentiable at $(0,0)$ unless $g = 0$.

(c) Show that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is not differentiable at $(0,0)$.

(4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq \|x\|^2$. Show that f is differentiable at 0.

(5) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Find f' using chain rule for the following:

(a) $f(x,y) = \int_a^{x+y} g$.

(b) $f(x,y) = \int_a^{xy} g$.

(c) $f(x,y,z) = \int_{x^y}^{\sin(x \sin(y \sin z))} g$.

(6) Let $E_i, i = 1, \ldots, k$, be Euclidean spaces of various dimensions. A function

$$f : E_1 \times \cdots \times E_k \to \mathbb{R}^p$$

is called multilinear if for each choice of $x_j \in E_j$, $i \neq j$, the function $g : E_i \to \mathbb{R}^p$, defined by

$$g(x) = f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k)$$

is a linear transformation.

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, \ldots, h_k)$ with $h_l \in E_l$, we have

$$\lim_{h \to 0} \frac{\|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)\|}{\|h\|} = 0.$$

(b) Show that $Df(a_1, \ldots, a_k)(x_1, \ldots, x_k) = \sum_{i=1}^{k} f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k)$.

(7) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is a C^∞ function.

(8) Let $f : \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$ the limit

$$\lim_{t \to 0} \frac{f(a + tx) - f(a)}{t}$$

if exists, denoted by $D_x(f)(a)$, is called the directional derivative of a in the direction x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$.

(b) Show that $D_{tx} f(a) = a D_x f(a)$.

(c) If f is differentiable at a, show that

$$S_x f(a) = Df(a)(x)$$

and therefore

$$D_{x+y} f(a) = D_x f(a) + D_y f(a).$$

(9) Give an example of a function where all the directional derivative exist at a point but not differentiable at that point.

(10) Give an example of a function where all partial derivative exist but not all directional derivatives.