Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Solution to Quiz -1 (MTH305A)

Semester: 2020-2021, I

Full Marks–30 Time - 60 Minutes

Question A. 6+4=10

(1) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} \exp\left(-\frac{1}{x}\right), & \text{if } 0 < x \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is smooth function on its domain of definition.

(2) Show that $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) = h(2 + x)h(2 - x), \quad \forall x \in \mathbb{R}$$

is compactly supported smooth function on $\mathbb{R}$.

Question B. 6+4=10

Suppose h and ϕ are the functions as in Question A.

(1) Show that the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x) = h(2 + x)h(2 - x) + h(x - 1) + h(-x - 1), \quad \forall x \in \mathbb{R},$$

is nowhere vanishing on $\mathbb{R}$ and the function

$$f(x) = \frac{\phi(x)}{\psi(x)}, \forall x \in \mathbb{R}$$

is smooth satisfying
- $f(x) = 1$, if $-1 \leq x \leq 1$ and
- $f(x) = 0$, if $|x| > 2$.

(2) Let $\epsilon, \delta \in \mathbb{R}$ such that $0 < \delta < \epsilon$. Construct a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that
(a) $0 \leq F(x) \leq 1, \quad \forall x \in \mathbb{R}$,
(b) $F(x) = 1$, if $|x| < \delta$ and
(c) $F(x) = 0$, if $|x| > \epsilon$.

Question C. 10

Let $B(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i| < r, i = 1, \ldots n\}$. Construct a smooth function

$$\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$$

such that

(1) $0 \leq \mathcal{F}(x) \leq 1, \quad \forall x \in \mathbb{R}^n$,
(2) $\mathcal{F}(x) = 1$, if $x \in B(\delta)$ and
(3) $\mathcal{F}(x) = 0$, if $x \in \mathbb{R}^n \setminus B(\epsilon)$.

Solution to Question A.

(1) For $x < 0$, we have $h^{(k)}(x) = 0, \forall k \in \mathbb{N}$.
Let us consider $x > 0$. Then

$$\begin{aligned} h'(x) &= \frac{1}{x^2} e^{-\frac{1}{x}} \\ h''(x) &= \frac{1}{x^4} e^{-\frac{1}{x}} - \frac{2}{x^3} e^{-\frac{1}{x}} \\ h'''(x) &= \frac{1}{x^6} e^{-\frac{1}{x}} - \frac{6}{x^5} e^{-\frac{1}{x}} + \frac{6}{x^4} e^{-\frac{1}{x}}. \end{aligned}$$

Thus $h^{(k)}(x)$, for $x > 0$, consists of sum of the terms of the form $\frac{c}{x^p} e^{-\frac{1}{x}}$, where $c \in \mathbb{Z}$ and $p \in \mathbb{N}$. We have

$$\begin{aligned} \lim_{x \to 0} \frac{c}{x^p} e^{-\frac{1}{x}} &= \lim_{t \to \infty} \frac{ct^p}{e^t} \quad \left(\text{letting } t = \frac{1}{x} \right) \\ &= 0. \end{aligned}$$

This follows that, for every $k \in \mathbb{N}$, the k^{th} derivative $h^{(k)}$ exists on $\mathbb{R}$ and is continuous. Thus we conclude that h is a smooth function.

(2) Consider $\phi(x) = h(2 + x)h(2 - x)$. If $x \geq 2$ or $x \leq -2$, one or the other factor is zero, and so the product is supported inside $[-2, 2]$ and thus compactly supported.
Now, $h(x + 2) = h \circ g_2(x)$, where $g_2(x) = x + 2$. Thus $x \mapsto h(x + 2)$ is smooth as a composition of two smooth functions. Similarly, $x \mapsto h(2 - x)$ is smooth. Now, applying the fact that product of two smooth function is smooth, we conclude that

$$\phi(x) = h(2 + x)h(2 - x), \quad \forall x \in \mathbb{R}$$

is a smooth function on $\mathbb{R}$.

Solution to Question B.

(1) Let $\psi(x) = h(2 + x)h(2 - x) + h(x - 1) + h(-x - 1)$. First of all $h(x) \geq 0, \forall x \in \mathbb{R}$.
- For $|x| > 1$, we have $h(x - 1) + h(-x - 1) > 0 \implies \psi(x) > 0$.
- For $|x| \leq 1$ we have $h(2 + x)h(2 - x) > 0 \implies \psi(x) > 0$.

Thus, $\psi(x) > 0, \forall x \in \mathbb{R}$.
The function $f(x) = \frac{\phi(x)}{\psi(x)}$ is smooth as a quotient of ϕ by ψ, which are smooth and $\psi \neq 0$.
For $-1 \leq x \leq 1$, we have $h(x - 1) + h(-x - 1) = 0$ which implies

$$f(x) = \frac{\phi(x)}{\psi(x)} = \frac{h(2 + x)h(2 - x)}{h(2 + x)h(2 - x) + h(x - 1) + h(-x - 1)} = \frac{h(2 + x)h(2 - x)}{h(2 + x)h(2 - x)} = 1.$$

Finally, for $|x| > 2$, we have $\phi(x) = 0 \implies f(x) = 0$.

(2) Consider

$$F(x) = \frac{h(\epsilon + x)h(\epsilon - x)}{h(\epsilon + x)h(\epsilon - x) + h(x - \delta) + h(-x - \delta)},$$

which is a required function.

Solution to Question C. Define $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{F}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F(x_i), \forall (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where F is the function as in Question B, (2). It is straightforward to check that $\mathcal{F}$ has the required properties.