Expectation vector, covariance matrix

$\underset{\sim}{X}_{p\times 1}$: random vector

$$E\,\underset{\sim}{X} = \begin{pmatrix} EX_1 \\ \vdots \\ EX_p \end{pmatrix} = \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$$

Covariance:

$$\mathrm{cov}(X_i, X_j) = E(X_i - \mu_i)(X_j - \mu_j) \qquad i \neq j$$

Note that $\mathrm{cov}(X_i, X_i) = E(X_i - \mu_i)^2 = V(X_i)$

$$\mathrm{cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j).$$

In general, for $k_1, \ldots, k_p$ non-negative integers, we can define joint moment as

$$E\left(X_1^{k_1} X_2^{k_2} \cdots X_p^{k_p}\right) = \mu'_{k_1, \ldots, k_p}.$$

joint moment of order $k_1 + \cdots + k_p$.

Correlation:

$$\rho = \rho_{X_i, X_j} = \frac{\mathrm{cov}(X_i, X_j)}{\left[V(X_i)\, V(X_j)\right]^{1/2}}$$

Note that $\mathrm{cov}(X_i, X_j) = 0$ is referred to as uncorrelated. If X_i & X_j are indep then $\mathrm{cov}(X_i, X_j) = 0$; but the converse is not true

Cauchy-Schwarz inequality

For any 2 r.v.s X & Y

$$E^2(XY) \leq E(X^2)\, E(Y^2)$$

(provided X & Y have finite 2^{nd} moment)

[Pf: Let $h(t) = E(tX - Y)^2 \geq 0 \quad \forall t$

$$E(tX - Y)^2 = t^2 E(X^2) + E(Y^2) - 2t\,E(XY)$$

If $h(t) > 0 \quad \forall t$, then roots of $h(t)$ are not real

i.e. $4\left(E(XY)^2 - E(X^2)\,E(Y^2)\right) < 0.$

i.e $\left(E(XY)\right)^2 < E(X^2)\, E(Y^2)$

If $h(t) = 0$ for some t, say t^*, then

$$E(t^* X - Y)^2 = 0 \Rightarrow P(t^* X = Y) = 1$$

and we have equality in C-S inequality]

Remark:

Take $X = (X_i - EX_i)$ & $Y = (X_j - EX_j)$ in C-S inequality

$$\Rightarrow \text{Cov}(X_i, X_j)^2 \le V(X_i)\, V(X_j)$$

i.e. $|\rho| \le 1$

Covariance matrix

$$\text{Cov}(\underset{\sim}{X}) = \underset{p\times p}{\Sigma} = E(\underset{\sim}{X} - \underline{\mu})(\underline{X} - \underline{\mu})'$$

a symmetric matrix

$$= \begin{pmatrix} V(X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_p) \\ & V(X_2) & \cdots & \text{Cov}(X_2, X_p) \\ \vdots & & \ddots & \\ & & & V(X_p) \end{pmatrix}$$

$\underset{\sim}{X}_{p\times 1}$, $\underset{\sim}{Y}_{q\times 1}$

$$\text{Cov}(\underset{\sim}{X}, \underset{\sim}{Y}) = E(\underset{\sim}{X} - \underset{\sim}{\mu}_X)(\underset{\sim}{Y} - \underset{\sim}{\mu}_Y)' = \underset{p\times q}{\Sigma_{XY}}$$

$$= \begin{pmatrix} \text{Cov}(X_1, Y_1) & \text{Cov}(X_1, Y_2) & \cdots & \text{Cov}(X_1, Y_q) \\ \text{Cov}(X_2, Y_1) & \text{Cov}(X_2, Y_2) & \cdots & \text{Cov}(X_2, Y_q) \\ - & - & - & - \\ \text{Cov}(X_p, Y_1) & \text{Cov}(X_p, Y_2) & \cdots & \text{Cov}(X_p, Y_q) \end{pmatrix}$$

Correlation matrix

$$\mathrm{Cov}(\underset{\sim}{X}) = \Sigma = ((\sigma_{ij}))$$

$$R = \mathrm{Corr}^n(\underset{\sim}{X}) = \begin{pmatrix} 1 & \mathrm{Corr}(X_1, X_2) & \cdots & \mathrm{Corr}(X_1, X_p) \\ & 1 & \cdots & \mathrm{Corr}(X_2, X_p) \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Let $D = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{pp})$

$$R = D^{-1/2} \Sigma D^{-1/2}$$

Linear combination of elements of $\underset{\sim}{X}$

$$\underset{\sim}{X}_{p\times 1} \longrightarrow Y = \underline{\alpha}' \underset{\sim}{X} \qquad \underset{\sim}{\alpha} \in \mathbb{R}^p$$

i.e. $Y = \sum_{i=1}^{p} \alpha_i X_i$; Y is a random variable

$$E(Y) = E\left(\sum_1^p \alpha_i X_i\right)$$
$$= \sum_1^p \alpha_i E X_i$$
$$= \sum_1^p \alpha_i \mu_i = \underline{\alpha}' \underline{\mu}$$

i.e. $E(\underline{\alpha}' \underline{X}) = \underline{\alpha}' E(\underline{X}) = \underline{\alpha}' \underline{\mu}$

$$V(\underline{\alpha}' \underline{X}) = E(\underline{\alpha}'\underline{X} - \underline{\alpha}'\underline{\mu})^2$$
$$= E(\underline{\alpha}'\underline{X} - \underline{\alpha}'\underline{\mu})(\underline{\alpha}'\underline{X} - \underline{\alpha}'\underline{\mu})'$$
$$= E\, \underline{\alpha}'(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'\underline{\alpha} = \underline{\alpha}' E(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'\underline{\alpha}$$
$$= \underline{\alpha}' \Sigma \underline{\alpha}$$

Multivariate normal

Def^n: A $p \times 1$ random vector $\underset{\sim}{X} = (X_1, \ldots, X_p)'$ with $E(\underline{X}) = \underset{\sim}{\mu}$ and $Cov(\underline{X}) = \Sigma$ is said to follow a multivariate normal, $N_p(\mu, \Sigma)$ iff $\forall\, \underline{\alpha} \in \mathbb{R}^p$ $(\underline{\alpha} \neq \underline{0})$, $\underline{\alpha}'\underset{\sim}{X}$ follows univariate normal (i.e. $\underset{\sim}{X} \sim N_p(\mu, \Sigma)$ iff $\underline{\alpha}'\underset{\sim}{X} \sim N_1$ $\forall\, \underline{\alpha} \in \mathbb{R}^p (\underline{\alpha} \neq \underline{0})$).

Note: Marginal dist^{n}s.

Marginal of X_i: $X_i \sim N(\mu_i, \sigma_{ii})$ where, $\Sigma = ((\sigma_{ij}))$

follows from the def^n of $N_p(\mu, \Sigma)$; take

$$\underline{\alpha} = (0, \ldots 0, \underset{\substack{\uparrow \\ i\text{th}}}{1}, 0 \ldots 0)'.$$

ii. marginal of any subset of $\underset{\sim}{X}$, say $X_1, \ldots X_q$ $(q < p)$

$$\underset{\sim}{Y}_{q \times 1} = \begin{pmatrix} X_1 \\ \vdots \\ X_q \end{pmatrix} \sim N_q(\underset{\sim}{\mu}_q, \Sigma_{11})$$

$\underset{\sim}{\mu}_q = E(\underline{Y})$ & $\Sigma_{11} = Cov(\underset{\sim}{Y})$.

follows from def^n of N_p as

$$\forall\, \underset{\sim}{\beta} \in \mathbb{R}^q; \quad \underline{\beta}'\underset{\sim}{Y} = (\underline{\beta}', \underset{\sim}{0}'_{p-q})\underset{\sim}{X} \sim N_1 \text{ as } \underset{\sim}{X} \sim N_p.$$

Note: If $\underset{p \times 1}{\underset{\sim}{X}} \sim N_p(\mu, \Sigma)$

$$\underset{\sim}{X} \to \underset{q \times 1}{\underset{\sim}{Y}} = \underset{q \times p}{A}\,\underset{\sim}{X} \sim N_q(A\mu, A\Sigma A')$$

$$\Big(\forall\, \underset{\sim}{\beta} \in \mathbb{R}^q; \ \underline{\beta}'\underset{\sim}{Y} = \underline{\beta}'A\underline{X} = \underline{\alpha}^{*\prime}\underset{\sim}{X} \sim N_1 \ (\text{as } \underline{\alpha}'\underline{X} \sim N_1 \ \forall\, \underline{\alpha} \in \mathbb{R}^p\Big)$$

$$\underline{\alpha}^* \in \mathbb{R}^p \implies \underline{Y} \sim N_q; \quad E(\underline{Y}) = A\mu, \quad Cov(\underline{Y}) = A\Sigma A'.$$