Assignment-11 (Discussion/solution)

1. $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) := \frac{x}{n}$. Then $f_n \to 0$ ptwise.

On $[a,b]$, $|f_n(x)| = \frac{|x|}{n} \leq \frac{|b-a|}{n}$ .... $f_n \to 0$ uniformly on $[a,b]$.

$\to$ (f_n) not unif. Cauchy. Show that $\exists\ \varepsilon > 0$ s.t. $\left|\frac{x}{m} - \frac{x}{n}\right| > \varepsilon$ for some $x \in \mathbb{R}$.

2. (a) n: even, n: odd . . $f_n \to 0$ ptwise on $(-1,1)$ / on $(-1,1]$. where $f(x) = \begin{cases} 0, & x \in (-1,1) \\ 1, & x = 1. \end{cases}$

$f_n \not\to f$ uniformly.

(b) $f_n(x) = n^2 x (1-x^2)^n$ $\quad f_n \to 0$ ptwise. on $[0,1]$

$f_n \not\to 0$ unif.

(c) $\frac{nx}{1+nx}$ on $[0,\infty)$ $\quad f_n(x) \to 1$ ptwise for $x \neq 0$

$f_n(0) \to 0$.

Suppose (f_n) cvg. uniformly. Then the limit function must be cts. - - - -

(d) HW

(e) xe^{-nx} on $[0,\infty)$ $\quad xe^{-nx} \to 0$ ptwise. (b/c. $e^{-nx} < 1$ for $x > 0$)

Note that $1+nx \leq e^{nx}$ for $x \in [0,\infty)$.

$\frac{x}{e^{nx}} \leq \frac{x}{1+nx} < \frac{x}{nx}$ $\quad$ Now, for $x>0$, $\frac{x}{e^{nx}} < \frac{1}{n}$ and also for $x=0$ $\frac{x}{e^{nx}} = 0$

Hence, $\frac{x}{e^{nx}} < \frac{1}{n}$ $\forall x \in [0,\infty)$. HW.

3. $f_n : \mathbb{R} \to \mathbb{R}$ cts. and $f_n \to f$ unif. on $[a,b]$.

f is cts. on every closed & bdd. interval,

$x_0 \in \mathbb{R}$ Consider $[x_0 - \delta, x_0 + \delta]$ - - - -

4. (f_n) s.t. $f_n \to f$ unif. where $f_n \in C[0,1]$.

Claim: $\int_0^{1-1/n} f_n \to \int_0^1 f$.

Pf. $\left| \int_0^{1-1/n} f - \int_0^1 f \right| = \left| \int_0^{1-1/n} f + \int_{1-1/n}^1 f - \int_{1-1/n}^1 f - \int_0^1 f \right|$

$$\leq \left| \int_0^1 f_n - \int_0^1 f \right| + \left| \int_{1-1/n}^1 f_n \right|$$

$$\leq \underbrace{\int_0^1 |f_n - f|}_{\substack{f_n \to f \text{ unif.} \\ \downarrow \\ 0}} + \int_{1-1/n}^1 |f_n|$$

Since $f_n \to f$ unif. (f_n) is unif. bdd. i.e. $\|f_n\|_\infty \leq M$, $\forall n \geq 1$

$\int_{1-1/n}^1 |f_n| \leq M \cdot \frac{1}{n} \to 0$ as $n \to \infty$.

5. $\sum_{n=1}^\infty \frac{x^2}{(1+x^2)^n}$ cvgs. for $|x| \leq 1$.

For each $x \neq 0$: $\sum_{n=1}^\infty \frac{x^2}{(1+x^2)^n} = 1$.

So, $\sum_{n=1}^m \frac{x^2}{(1+x^2)^n} \to 1$ ptwise. for $x \neq 0$ & $|x| \leq 1$

For $x = 0$: $\sum_{n=1}^m \frac{x^2}{(1+x^2)^n} \to 0$. Since the limit function is not cts, so $\sum_1^m (\ldots) \not\to \sum_1^\infty$ unif.

6. $\sum_{n=1}^\infty \frac{n}{e^{nx}}$ (HW) For $x = 0$ and $x < 0$, $\sum_{n=1}^\infty \frac{n}{e^{nx}}$ does not converge.

For $x > 0$: $\sum_{n=1}^{\infty} \frac{n}{e^{nx}} < \infty$ (Ratio test ...)

So, ptwise $\sum_{n=1}^{m} \frac{n}{e^{nx}}$ cvgs for $x \in (0,\infty)$.

Consider $(0,1)$: Not unif. Cauchy.

$$\left| \sum_{n=k+1}^{m} \frac{n}{e^{nx}} \right| \overset{?}{<} \varepsilon \quad \forall\, x \in (0,1) \ ?$$

Since $\frac{m}{e^{mx}} < \sum_{n=k+1}^{m} \frac{n}{e^{nx}} \quad \forall\, x \in (0,1)$, take $x = \frac{1}{m}$

$$\frac{m}{e} < \sum_{n=k+1}^{m} \frac{n}{e^{n/m}}$$

$\sum_{n=1}^{m} \frac{n}{e^{nx}}$ does not cvgs. unif. on $(0,1)$.

Consider $[1,\infty)$: Recall: (Taylor's thm:) for $x \in [1,\infty)$.

$$e^{nx} \geq \frac{n^3 x^3}{6}$$

$nx = y$

$x > 0$ ξ $[0,x]$

$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + C(\xi)$ ≥ 0

$$\frac{n}{e^{nx}} \leq 6 \cdot \frac{n}{n^3 x^3} = \frac{6}{x^3} \cdot \frac{1}{n^2} < 6 \cdot \frac{1}{n^2} \quad \cdots\cdots$$

$\sum_{n=1}^{m} \frac{n}{e^{nx}}$ cvgs. unif. on $[1,\infty)$.