Convergence of sequence of random variables

Modes of convergence:

Convergence in probability - to be covered in this course
Convergence in distribution - to be covered in this course
Convergence almost surely
Convergence in r^{th} mean

Convergence in probability

Let $\{X_n\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. $\{X_n\}$ is said to converge in probability to a random variable X (we write $X_n \xrightarrow{p} X$ as $n \to \infty$) if

$$P(|X_n - X| > \epsilon) \to 0 \quad \text{as } n \to \infty \quad \forall\, \epsilon > 0$$

Some important results

(i) If $X_n \xrightarrow{p} X$ and 'a' is a constant, then

$$aX_n \xrightarrow{p} aX$$

(ii) If $X_n \xrightarrow{p} X$ and $g(\cdot)$ is any continuous function, then

$$g(X_n) \xrightarrow{p} g(X)$$

(iii) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then

$$X_n \pm Y_n \xrightarrow{p} X \pm Y$$

$$X_n Y_n \xrightarrow{p} XY$$

$$\frac{X_n}{Y_n} \xrightarrow{p} \frac{X}{Y} \quad (\text{provided } P(Y=0)=0)$$

Remark: Approaches to verify convergence in prob

(i) Direct approach (by calculating limiting prob)

(ii) Using Chebyshev's inequality (provided 2nd order moment exists)

Examples

(1) $X_1, \ldots, X_n$ are i.i.d. Bernoulli $(1, \theta)$; $0 < \theta < 1$

Let $Z_n = \sum_{i=1}^{n} X_i \sim B(n, \theta)$

Consider the r.v. $Y_n = \frac{Z_n}{n}$

$$P(|Y_n - \theta| > \epsilon) \leq \frac{E(Y_n - \theta)^2}{\epsilon^2}$$

$$= \frac{E\left(\frac{Z_n}{n} - \theta\right)^2}{\epsilon^2} = \frac{E(Z_n - n\theta)^2}{n^2 \epsilon^2} = \frac{V(Z_n)}{n^2 \epsilon^2}$$

$$= \frac{n\theta(1-\theta)}{n^2 \epsilon^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\forall \epsilon > 0$

$\Rightarrow Y_n \xrightarrow{p} \theta$

i.e. $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \theta \; (= E(X_i))$

(2) $X_1, \ldots, X_n$ i.i.d $U(0, \theta)$ $\theta > 0$

$Y_n = \max\{X_1, \ldots, X_n\} = X_{(n)}$

$$F_{X_{(n)}}(x) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 \leq x \leq \theta \\ 1, & x > \theta \end{cases}$$

$$P(|X_{(n)} - \theta| > \epsilon) = 1 - P(|X_{(n)} - \theta| \leq \epsilon)$$

$$= 1 - P(\theta - \epsilon \leq X_{(n)} \leq \theta + \epsilon)$$

$$= 1 - \left(F_{X_{(n)}}(\theta + \epsilon) - F_{X_{(n)}}(\theta - \epsilon)\right)$$

$$= 1 - \left(1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n\right) \quad \forall \epsilon > 0$$

$$\to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad Y_n = X_{(n)} \xrightarrow{p} \theta$$ (can be proved through Chebyshev ineq also)

$$e^{X_{(n)}} \xrightarrow{p} e^{\theta}$$

$$\sqrt{X_{(n)}} \xrightarrow{p} \sqrt{\theta}$$

(3) Let $\{X_n\}$ be sequence of r.v.s with p.m.f.

$$P(X_n = 1) = \frac{1}{n} \text{ and } P(X_n = 0) = 1 - \frac{1}{n}$$

For $\epsilon > 0$,

$$P(|X_n| > \epsilon) = \begin{cases} P(X_n = 1) = \frac{1}{n}, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases}$$

$$\to 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0$$

$$\Rightarrow X_n \xrightarrow{p} 0$$

Changing support example

$$P(X_n = 0) = 1 - \frac{1}{n^r} \ \& \ P(X_n = n) = \frac{1}{n^r} \quad r > 0$$

$$P(|X_n| > \epsilon) = \begin{cases} P(X_n = n), & 0 < \epsilon < n \\ 0, & \epsilon \geq n \end{cases}$$

$$\to 0 \quad \text{as } n \to \infty$$

$$X_n \xrightarrow{p} 0$$

(4) $X_1, \ldots, X_n$ be i.i.d. with p.d.f. (or p.m.f) f_X

X_i with mean μ and variance $\sigma^2 (< \infty)$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad : \text{Sample mean r.v.}$$

$$E\bar{X}_n = \frac{1}{n} \sum_i EX_i = \mu$$

$$V(\bar{X}_n) = E(\bar{X}_n - \mu)^2$$

$$= E\left(\frac{1}{n}\sum_1^n X_i - \mu\right)^2$$

$$= E\left(\frac{1}{n}\sum (X_i - \mu)\right)^2$$

$$= \frac{1}{n^2} E\left(\sum (X_i - \mu)\right)^2$$

$$= \frac{1}{n^2}\sum_{i=1}^n E(X_i - \mu)^2 \quad (\because X_1, \dots X_n \text{ are indep})$$

$$= \frac{\sigma^2}{n}$$

$$P(|\bar{X}_n - \mu| > \epsilon) \le \frac{E(\bar{X}_n - \mu)^2}{\epsilon^2} = \frac{V(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \bar{X}_n \xrightarrow{p} \mu$$

Remark: This convergence is irrespective of the underlying distⁿ.

Weak Law of Large Numbers (WLLN)

Defⁿ: Let $\{X_n\}$ be a seq of r.v.s. We say that $\{X_n\}$ satisfies WLLN if $\exists$ constants $\{a_n\}$ and $\{b_n\}$ where $b_n > 0$ and $b_n \uparrow \infty$ $\ni$

$$\frac{S_n - a_n}{b_n} \xrightarrow{p} 0 \quad \text{as } n \to \infty$$

where, $S_n = \sum_{i=1}^n X_i$

Khintchine's WLLN

If $\{X_n\}$ is $\ni$ $X_1, X_2, \dots$ is i.i.d. seq of r.v.s with $E|X_i| < \infty$, then WLLN holds and

$$\frac{1}{n}\sum X_i \xrightarrow{p} \mu = E(X_1) \quad \left(\begin{array}{c} S_n = \sum_1^n X_i \\ a_n = n\mu \\ b_n = n \end{array}\right)$$

Remark: Khintchine's WLLN does not require existence of 2^{nd} moment

Applications

(i) $X_1, \ldots, X_n$ i.i.d. $B(1,\theta)$; θ unknown $0<\theta<1$

$E X_i$ exists ; $E X_i = \theta$

Khintchine's WLLN $\Rightarrow \frac{1}{n}\sum X_i \xrightarrow{p} \theta$ as $n \to \infty$

(ii) $X_1, \ldots, X_n$ i.i.d. with p.d.f. / p.m.f. f_X having mean θ and variance σ^2

by WLLN sample mean $\frac{1}{n}\sum_1^n X_i \xrightarrow{p} \theta$ (pop^n mean)

Further, let

$$S^2 = \frac{1}{n-1}\sum (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n-1}\left(\sum X_i^2 - n\bar{X}_n^2\right)$$

$$= \frac{1}{n-1}\sum X_i^2 - \frac{n}{n-1}\bar{X}_n^2$$

Now $\bar{X}_n \xrightarrow{p} \theta \Rightarrow \bar{X}_n^2 \xrightarrow{p} \theta^2$

Note that $X_1, \ldots, X_n$ i.i.d. with mean θ & var σ^2

$\Rightarrow X_1^2, \ldots, X_n^2$ i.i.d. with mean $(\sigma^2 + \theta^2) < \infty$

by WLLN

$$\frac{1}{n}\sum_1^n X_i^2 \xrightarrow{p} E X_1^2 = \sigma^2 + \theta^2$$

$$\Rightarrow \frac{1}{n-1}\sum_1^n X_i^2 = \frac{n}{n-1}\,\frac{1}{n}\sum X_i^2 \xrightarrow{p} \theta^2 + \sigma^2$$

$$\Rightarrow S^2 = \frac{1}{n-1}\sum X_i^2 - \frac{n}{n-1}\bar{X}_n^2 \xrightarrow{p} (\theta^2+\sigma^2) - \theta^2 = \sigma^2$$

$\Rightarrow S^2 \xrightarrow{p} \sigma^2$; i.e. sample variance random variable converges in prob to corresponding population variance.

Note: $S_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2 = \frac{(n-1)s^2}{n} \xrightarrow{p} \sigma^2$

Remark: WLLN for non i.i.d. setup

Suppose $X_1, X_2, \ldots$ be a seq of uncorrelated r.v.s with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$; $i = 1, 2, \ldots$

If $\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 \to 0$ as $n \to \infty$, then WLLN holds for $\{X_n\}$.

Take $a_n = \sum_{i=1}^{n}\mu_i$ & $b_n = n$,

then

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \epsilon\right) = P\left(\left|\frac{1}{n}\sum_{1}^{n}(X_i - \mu_i)\right| > \epsilon\right)$$

$$\leq \frac{E\left(\frac{1}{n}\sum(X_i - \mu_i)\right)^2}{\epsilon^2} = \frac{E\left(\sum(X_i - \mu_i)\right)^2}{n^2\epsilon^2}$$

$$= \frac{\sum_{1}^{n}\sigma_i^2}{n^2\epsilon^2} \quad (\because \text{ of uncorrelatedness})$$

$$\to 0 \text{ as } n \to \infty$$

$$\Rightarrow \frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

i.e. WLLN holds for $\{X_n\}$.