Assignment 8: Several variables calculus & differential geometry (MTH305A)

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(1) Show that $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ admits an atlas consisting of two charts. Compute the transition function.

(2) Find an atlas for $S = \{(x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1\}$.

(3) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = (x + y + z - 1)^2$.

(a) What are the critical point and critical values of f?

(b) For which α, the set $\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = \alpha\}$ is a regular surface?

(c) Answer similar question for the function $f(x, y, z) = xyz^2$.

(4) Let $U = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi \text{ and } 0 < v < 2\pi\}$ and $\phi : U \to \mathbb{R}^3$ be defined by

$$\phi(x, y) = (a \sin x \cos y, b \sin x \sin y, c \cos x),$$

where $a, b, c \neq 0$. Show that (U, ϕ) is a local co-ordinate chart of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = constant$ on the ellipsoid.

(5) Let $S_1 = \{(x, 0, z) \mid x, z \in \mathbb{R}\}, S_2 = \{(\cos\theta, \sin\theta, t) \mid \theta, t \in \mathbb{R}\}$ and $S_3 = \{(\cos\theta, \sin\theta, t) \mid \theta, -1 < t < 1\}$.

(a) Show that $S_i, i = 1, 2, 3$, are regular surfaces.

(b) For which pairs (i, j), the surfaces S_i, S_j are diffeomorphic.

(c) Define a function $f : S_1 \to S_3$ by $f(x, 0, z) = (\cos x, \sin x, z)$. Show that f is a local diffeomorphism but not global.

(6) For $0 < a < b$, let us consider

$$T = \{(a \cos\theta + b) \cos\phi, (a \cos\theta + b) \sin\phi, a \sin\theta) \mid 0 \leq \theta, \phi \leq 2\pi\}.$$

(a) Show that T is a regular surface by constructing an atlas.

(b) Show that a single chart is not enough to construct an atlas for T.

(c) Does there exist an atlas of T consisting of two charts?

(d) Does there exist a map $f : \mathbb{R}^2 \to T$ which is a local diffeomorphism on to T?