Connectedness and Continuity

(Continuous functions carry information about the metric spaces.)

→ Criterion for disconnectedness:

(M,d) is disconnected iff $\exists$ a cts. function $f: M \xrightarrow[\text{onto}]{} \{0,1\}$.
↑ with discrete topology.

Pf. Suppose $f: M \xrightarrow[\text{onto}]{} \{0,1\}$ cts. Then $f^{-1}(0)$ and $f^{-1}(1)$ are disjoint nonempty open sets in M. Hence $f^{-1}(0) \cup f^{-1}(1) = M$.

Conversely, if $\exists$ $A, B \neq \phi$, open sets s.t. $A \cup B = M$.

Define $f: M \xrightarrow[\text{onto}]{} \{0,1\}$ as $f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B. \end{cases}$

f is cts. because $f^{-1}(0) = A$ (open), $f^{-1}(1) = B$ open, and $f^{-1}\{0,1\} = M$ (open).

HW. → If $f: (M,d) \to (N,\rho)$ continuous.

$E \subset M$ connected,

then $f(E)$ is connected in N.

Consequently, if I is an interval in $\mathbb{R}$ and $f: I \to (\mathbb{R}, |\cdot|)$ is cts., then $f(I)$ is connected and hence $f(I)$ is an interval.

(In other words, for $a, b \in I$ s.t. $f(a) \neq f(b)$, $\forall$ c with $f(a) < c < f(b)$, $\exists$ $x \in I$ s.t. $f(x) = c$. (Intermediate Value Thm))

HW: Suppose $f: \mathbb{R} \to \mathbb{R}$ has the IVP, i.e., if $x < y$ with $f(x) < f(y)$, then f assumes every value between $f(x)$ and $f(y)$ on the interval (x,y).

Furthermore, assume the graph of f, i.e., $G(f) := \{(x, f(x)) \in \mathbb{R}^2\}$ is closed in $\mathbb{R}^2$.

Then, f is cts. on $\mathbb{R}$.

(We say $G(f)$ is closed if $(x_n, f(x_n)) \to (a,b)$ s.t. $x_n \to a$, $f(x_n) \to b$, then $f(a) = b$.)

→ $(a,b) \overset{\text{homeo}}{\not\approx} [a,b) \overset{\text{homeo}}{\not\approx} [a,b]$

If $(a,b) \overset{f}{\approx} [a,b)$ | $[a,b) \overset{f}{\approx} [a,b]$

$f(c) = a$ | If $[a,b) \xrightarrow{f} [a,b]$

$(a,c) \cup (c,b) \overset{f}{\approx} (a,b)$ | disconnected $\quad (a,b)$: connected.

disconnected $\qquad$ connected.

→ If A and B are connected, then $A \times B$ is also connected.

$(A, d) \qquad (B, \rho) \qquad$ "the" product metric space w.r.t d_∞.

Pf: Recall that M is disconnected iff $\exists$ a cts. function $f : M \xrightarrow[\text{onto}]{} \{0,1\}$.

In order to show that $A \times B$ $(= M)$ is connected, we need to show that every cts. function $f : A \times B \to \{0,1\}$ is constant.

Let $f : A \times B \to \{0,1\}$ is cts.

Fix $(a_1, b_1) \in A \times B$.

claim: $f(a,b) = f(a_1, b_1)$, $\forall (a,b) \in A \times B$.

Pf of claim:

Define $f^{(1)} : A \to \{0,1\}$, $\quad f^{(2)} : B \to \{0,1\}$

as $f^{(1)}(a) := f(a, b_1) \qquad f^{(2)}(b) := f(a_1, b)$.

Since A and B are connected, $f^{(1)}$ and $f^{(2)}$ are constant functions.

Hence, $\forall a \in A$, $f(a, b_1) = f(a_1, b_1)$

$\forall b \in B$, $f(a_1, b) = f(a_1, b_1)$ — (*)

Take $(a', b') \in A \times B$ and repeat the above argument. One obtains,

$\forall a \in A$, $f(a, b') = f(a', b')$

in particular, $f(a_1, b') = f(a', b')$.
Using (*), $f(a_1, b') = f(a_1, b_1)$
$\Rightarrow f(a', b') = f(a_1, b_1)$

∎

$\to$ $[0,1]\times[0,1]$ is connected.

claim: $[0,1] \overset{\text{homeo}}{\not\sim} [0,1]\times[0,1]$.

Suppose $[0,1] \overset{\text{homeo}}{\sim} [0,1]\times[0,1]$.

$[0,1]\setminus\{\tfrac{1}{2}\} \overset{\text{homeo}}{\sim} [0,1]\times[0,1]\setminus\{f(\tfrac{1}{2})\}$ where $[0,1]\xrightarrow{f}[0,1]\times[0,1]$ is the homeo.

Note that $[0,1]\times[0,1]\setminus\{(a,b)\}$ is still connected.

(Easier to prove this using the notion of path-connected metric spaces.)

A set E is path-connected if $\forall\ x,y\in E$, $\exists$ a cts. function $f:[0,1]\to E$ s.t. $f(0)=x$ and $f(1)=y$.

HW: Try to show that $[0,1]\times[0,1]\setminus\{(a,b)\}$ is path-connected!

$\to$ $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$, if $n>1$.

Suppose $\mathbb{R} \overset{\text{homeo}}{\sim} \mathbb{R}^n$.

$\mathbb{R}\setminus\{0\} \overset{\text{homeo}}{\sim} \mathbb{R}^n\setminus\{f(0)\}$

disconnected   connected.

$\to$ $\mathbb{R}^n \overset{\text{homeo}}{\not\sim} \mathbb{R}^m$, for $n,m>1$.