Def^n: $(\Omega, \mathcal{F}, P)$ be prob space

A real valued f^n $X : \Omega \to \mathbb{R}$ defined on the sample space Ω is called a random variable.

Remark: A more advanced textbook on prob would define r.v. as.

A real valued f^n $X : \Omega \to \mathbb{R}$ is called a r.v. if the inverse images under X of all Borel sets in $\mathbb{R}$ are events, i.e. if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}. \quad -(*)$$

Further, to check whether a real valued f^n on $(\Omega, \mathcal{F})$ is a r.v., it is not necessary to check $(*)$ $\forall$ Borel sets $B \in \mathcal{B}$. It suffices to verify $(*)$ for any class of subsets of $\mathbb{R}$ that generates $\mathcal{B}$; e.g. we can take the class of subsets as semiclosed intervals $(-\infty, x]$, $x \in \mathbb{R}$ or $(-\infty, x)$, $x \in \mathbb{R}$. In such a case, we would say

X is a r.v. iff $\forall x \in \mathbb{R}$

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \leq x\} \in \mathcal{F}.$$

Ex: $\Omega = \{HH, TH, HT, TT\}$; $\mathcal{F}$: power set of Ω

$X(\omega)$: # of Hs in $\{\omega\}$ $\qquad X(\omega) = \begin{cases} 0, & TT \\ 1, & TH, HT \\ 2, & HH \end{cases}$

To show that X is r.v., we look at

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \leq x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \leq x < 1 \\ \{TT, HT, TH\} & 1 \leq x < 2 \\ \Omega & x \geq 2 \end{cases}$$

$\Rightarrow \downarrow \in \mathcal{F} \quad \forall x \in \mathbb{R}$

$\Rightarrow X$ is a r.v.

Induced probability space

$(\Omega, \mathcal{F}, P)$: prob space

$X : \Omega \to \mathbb{R}$ a r.v.

$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}$

Define a set f^n $P_X : \mathcal{B} \to [0,1]$

$$P_X(B) = P(\omega \in \Omega : X(\omega) \in B) = P(X^{-1}(B))$$

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X)$$

This is a prob space with $P_X(.)$ as a probability measure, referred to as the induced prob measure

$(\mathbb{R}, \mathcal{B}, P_X)$ is the induced prob space, induced by X.

Distribution function of a random variable

Def^n: Let X be a r.v. defined on a prob space $(\Omega, \mathcal{F}, P)$ and let $(\mathbb{R}, \mathcal{B}, P_X)$ be the prob space induced by X. Define $F_X : \mathbb{R} \to \mathbb{R}$ as

$$F_X(x) = P(\omega : X(\omega) \leq x) = P_X(-\infty, x]$$

$F_X(.)$ is called the cumulative dist^n f^n or just dist^n f^n of r.v. X

Remark: As ~~intervals~~ class of intervals of the type $(-\infty, x]$ generates $\mathcal{B}$, c.d.f $F_X(.)$ determine the $P_X(.)$ uniquely.

Thus to study the random behavior of r.v. X it suffices to study it's c.d.f F.

<u>Examples</u>

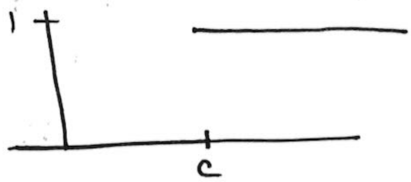
(1) $(\Omega, \mathcal{F}, P)$

$$X(\omega) = c \quad \forall\ \omega \in \Omega$$

$$P(X = c) = P(\omega : X(\omega) = c) = P(\Omega) = 1$$

$$F(x) = P(X \le x) = P(\omega : X(\omega) \le x)$$

$$= \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$



Note that

$$\left.\begin{array}{l} F(-\infty) = 0 \ ; \ F(\infty) = 1 \\ F \text{ is non-decreasing} \\ F \text{ is right continuous} \end{array}\right\} - (*)$$

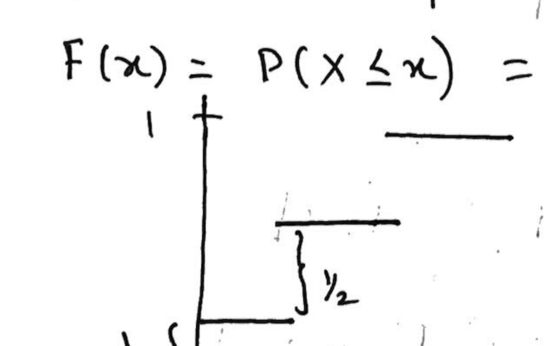
$F(\cdot)$ have 1 pt of jump discontinuity

(2) $\Omega = \{HH, HT, TH, TT\}$ example

$X(\omega)$: # of heads in ω

$$P(X=0) = \frac{1}{4} \ ; \ P(X=1) = \frac{1}{2} \ ; \ P(X=2) = \frac{1}{4}$$

$$F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{4} + \frac{1}{2}, & 1 \le x < 2 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, & x \geq 2 \end{cases} = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$



Once again $(*)$ is satisfied by the above $F(\cdot)$

$F(\cdot)$ has 3 pts of jump discontinuity

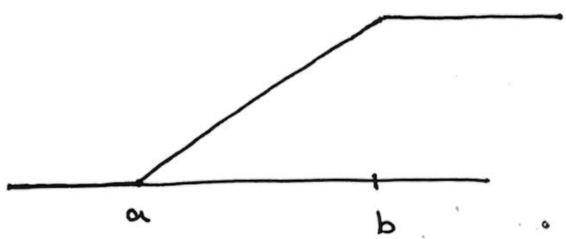
Example 3:

$$\Omega = [a, b]$$

For every $I \subset \Omega$; $P(I) = \dfrac{\text{length of } I}{b-a}$

Define. $X(\omega) = \omega$; $\omega \in \Omega$

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ \dfrac{b-a}{b-a} = 1, & x \ge b \end{cases}$$



$F(.)$ satisfies (*)

$F(.)$ is continuous everywhere

Result: Let $F(.)$ be the d.f. of a r.v. X defined on a prob space $(\Omega, \mathcal{F}, P)$. Then

(i) F is non-decreasing

(ii) F is right continuous

(iii) $F(-\infty) = \lim\limits_{n \uparrow \infty} F(-n) = 0$ and

$$F(\infty) = \lim_{n \uparrow \infty} F(n) = 1$$

Pf:

(a) Let $-\infty < x < y < \infty$, then

$$(-\infty, x] \subseteq (-\infty, y]$$

$$\Rightarrow P_X(-\infty, x] \le P_X(-\infty, y]$$

i.e. $P(\omega : X(\omega) \le x) \le P(\omega : X(\omega) \le y)$

$$\Rightarrow F(x) \le F(y)$$

$\Rightarrow F(.)$ is non-decreasing

(b) $$F(x+) = \lim_{h \downarrow 0} F(x+h)$$

$$= \lim_{n \to \infty} F\left(x + \frac{1}{n}\right)$$

$$= \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right)$$

Realize that $A_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, \ldots$ is $\ni A_n \downarrow$

and $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$

$$\Rightarrow \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right) = P_X\left(\lim_{n \to \infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} \left(-\infty, x + \frac{1}{n}\right]\right)$$

$$= P_X\left((-\infty, x]\right) = F(x)$$

$\Rightarrow F(x+) = F(x)$; i.e. $F(\cdot)$ is right continuous