Quiz-II
MTH-204, MTH-204A
ABSTRACT ALGEBRA
Spring-2023
Date: 18th April 2023

Time Allowed: 30 mins (6.15-6.45 PM) Max. Marks: 15

Write your answer in the space provided and explain all the major steps

1. List up to isomorphism all abelian groups of order 2100. Among them determine which are cyclic. [3]

Ans: $2100 = 3 \times 7 \times 5^2 \times 2^2$. So the number of non-isomorphic abelian groups of order 2100 is $p(2) \times p(2) = 4$ where $p(2)$ is the number of partitions of 2.

So the groups are

(1) $\mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$.

(2) $\mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

(3) $\mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

(4) $\mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$.

Among the above list only $\mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_4 \times \mathbb{Z}_{25}$ is cyclic as we know that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ iff $(m, n) = 1$.

2. Find a subnormal series of $GL_2(\mathbb{R})$. Does it have a composition series ? Justify your answer. [4]

Ans: $\{I\} \lhd SL_2(\mathbb{R}) \lhd GL_2(\mathbb{R})$ is a subnormal series of $GL_2(\mathbb{R})$.

We know that if $GL(2, \mathbb{R})$ has a composition series, then every normal subgroup also has a composition series. However the subgroup generated by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,

$$N := \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$$

is a normal subgroup of $GL(2, \mathbb{R})$ and is isomorphic to $\mathbb{Z}$. Since $\mathbb{Z}$ doesn't have a composition series, we conclude that $GL(2, \mathbb{R})$ has no composition series.

3. Are the principal ideals $I = (7)$ and $J = (13)$ maximal in $\mathbb{Z}[i]$? Explain your answer. [3]

Ans: Note that 7 is irreducible as for $7 = (a + ib)(c + id)$, we have $49 = 7.7 = (a^2 + b^2)(c^2 + d^2)$, implies that either $a + ib$ or $c + id$ is a unit. On the other hand $13 = (2 + 3i)(2 - 3i)$ is not irreducible. Since $\mathbb{Z}[i]$ is a PID the ideal $I = \langle 7 \rangle$ is maximal but $J = \langle 13 \rangle$ is not maximal.

4. Show that there is no commutative ring with the identity whose additive group is isomorphic to $\frac{\mathbb{Q}}{\mathbb{Z}}$. [5]

Ans: Let R be a ring with identity such that its additive group is isomorphic to $\frac{\mathbb{Q}}{\mathbb{Z}}$ and let f be the group isomorphism between them. Since every element of $\frac{\mathbb{Q}}{\mathbb{Z}}$ is of finite order, order of $f(1)$ is n say. Then $n.f(1) = f(n.1) = 0$ and since f is injective we have $n.1 = 0$. Then $n.f(a) = f(n.a) = f(n.(1.a)) = f((n.1).a) = f(0.a) = 0$. So it says that every element of $\frac{\mathbb{Q}}{\mathbb{Z}}$ is of order less than n. Which is a contradiction.

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