1 $X_t = \epsilon_t + \theta \epsilon_{t-1}$: $|\theta| < 1$ $\epsilon_t \sim WN(0, \sigma^2)$

$\epsilon_t = X_t - \theta \epsilon_{t-1}$

$\Rightarrow X_t = \epsilon_t + \theta (X_{t-1} - \theta \epsilon_{t-2})$

$= \epsilon_t + \theta X_{t-1} - \theta^2 \epsilon_{t-2}$

$= \epsilon_t + \theta X_{t-1} - \theta^2 (X_{t-2} - \theta \epsilon_{t-3})$

$= \epsilon_t + \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 \epsilon_{t-3}$

K^{th} step $\vdots$

$X_t = \epsilon_t - \sum_{i=1}^{K} (-\theta)^i X_{t-i} - (-\theta)^{K+1} \epsilon_{t-(K+1)}$

$\Rightarrow E\left(X_t - \epsilon_t + \sum_{i=1}^{K} (-\theta)^i X_{t-i}\right)^2 = E\left((-\theta)^{2(K+1)} \epsilon^2_{t-K-1}\right)$

$= \sigma^2 (-\theta)^{2(K+1)}$

$\Rightarrow \lim_{K \to \infty} E\left(X_t - \epsilon_t + \sum_{i=1}^{K} (-\theta)^i X_{t-i}\right)^2 = 0$

2 $X_t = \phi X_{t-1} + \epsilon_t$ $|\phi| < 1$ $\epsilon_t \sim WN(0, \sigma^2)$

$= \phi (\phi X_{t-2} + \epsilon_{t-1}) + \epsilon_t$

$= \phi^2 X_{t-2} + \phi \epsilon_{t-1} + \epsilon_t$

N^{th} step $\vdots$

$= \phi^{N+1} X_{t-(N+1)} + \sum_{j=0}^{N} \phi^j \epsilon_{t-j}$

$\Rightarrow E\left(X_t - \sum_{j=0}^{N} \phi^j \epsilon_{t-j}\right)^2 = E\left(\phi^{2(N+1)} X^2_{t-N-1}\right)$

$= \phi^{2(N+1)} E(X^2_{t-N-1})$

As $\{X_t\}$ is covariance stationary, $E\left(X_{t-N-1}^2\right) < \infty$

$$\Rightarrow \lim_{N \to \infty} E\left(X_t - \sum_{j=0}^{N} \phi^j \epsilon_{t-j}\right)^2 = 0 \quad (\text{note that } |\phi| < 1)$$

(3) $Y_t = X_{2t}$

$X_t = \phi X_{t-1} + \epsilon_t ; \quad \epsilon_t \sim WN(0, \sigma^2)$

$$Y_t = X_{2t} = \phi X_{2t-1} + \epsilon_{2t}$$

$$= \phi(\phi X_{2t-2} + \epsilon_{2t-1}) + \epsilon_{2t}$$

i.e. $Y_t = \phi^2 X_{2(t-1)} + (\phi \epsilon_{2t-1} + \epsilon_{2t})$

i.e. $Y_t = \phi^2 Y_{t-1} + \eta_t$

$$\eta_t = \epsilon_{2t} + \phi \epsilon_{2t-1} \sim WN(0, \sigma^2(1+\phi^2))$$

$\Rightarrow Y_t = \delta Y_{t-1} + \eta_t$ is a stationary AR(1)

$$\delta = \phi^2 ; \quad |\delta| < 1$$

As $\{Y_t\}$ is covariance stationary, $\{Y_t\}$ is Causal also.

Causal representation of $\{Y_t\}$:

$$Y_t = \sum_{j=0}^{\infty} (\phi^2)^j \eta_{t-j}$$

(4) $X_t = Z_t Z_{t-1} + Z_{t-1} Z_{t-2} + \frac{1}{4} Z_{t-2} Z_{t-3}$

Let $\eta_t = Z_t Z_{t-1}$; η_t is $\ni E\eta_t = 0$

& $\text{cov}(\eta_t, \eta_s) = E(Z_t Z_{t-1} Z_s Z_{s-1})$

$$= \begin{cases} \sigma^4, & t = s \\ 0, & o/w \end{cases}$$

$\Rightarrow \eta_t \sim WN(0, \sigma^4)$

$\Rightarrow X_t = \eta_t + \eta_{t-1} + \frac{1}{4}\eta_{t-2}$ is MA(2)

$X_t = \theta(B)\eta_t$; $\theta(B) = 1 + B + \frac{1}{4}B^2$

roots of $\theta(z) = 0$; i.e. of $1 + z + \frac{1}{4} z^2 = \left(1 + \frac{1}{2} z\right)^2$

are $-2, -2$

$\Rightarrow X_t$ is invertible MA(2)

(5) $X_t = \phi X_{t-1} + \epsilon_t$; $\phi = \frac{1}{4}$ $\quad \epsilon_t \sim WN(0,1)$ indep

$Y_t = \delta_t - \delta_{t-1}$; $\quad \delta_t \sim WN(0,1)$

Note that $X_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$

$Y_t = \delta_t - \delta_{t-1}$ $\Rightarrow \{X_t\}$ & $\{Y_t\}$ are indep

$$E\, Z_t = \left(E(1 - X_t)\right)\left(E(1 + Y_t)\right) = 1$$

$\text{cov}(Z_t, Z_{t+h}) = E(Z_t Z_{t+h}) - 1$

$E(Z_t Z_{t+h}) = E(1 - X_t)(1 - X_{t+h})\; E(1 + Y_t)(1 + Y_{t+h})$

$$E(Z_t Z_{t+h}) = (1+\gamma_X(h))(1+\gamma_Y(h))$$

$$\Rightarrow Cov(Z_t, Z_{t+h}) = \gamma_X(h)\,\gamma_Y(h) + \gamma_X(h) + \gamma_Y(h)$$
$$\neq 0 \quad \forall\, h \neq 0$$

$\{Z_t\}$ is not a WN

Remark: Note that $E Z_t \neq 0$ and WN is a 0 mean process

So one can simply say that $E Z_t = 1 \Rightarrow Z_t \nsim WN$

(6) $$\phi(B) X_t = \theta(B) \epsilon_t$$
$$\phi(B) = 1 - .5B \; ; \; \theta(B) = 1 + .4B$$

Root of $\phi(z) = 0$ is $\frac{1}{.5} \Rightarrow \{X_t\}$ is stationary & causal

Root of $\theta(z) = 0$ is $-\frac{1}{.4} \Rightarrow \{X_t\}$ is invertible

AR(∞) representation of $\{X_t\}$

$$\theta(B)^{-1}\phi(B) X_t = \epsilon_t$$

$$\Psi(B) X_t = \epsilon_t \; ; \; \Psi(B) = \Psi_0 + \Psi_1 B + \cdots$$

$$\Rightarrow \theta(B)^{-1}\phi(B) = \Psi(B)$$

i.e. $\phi(B) = \theta(B)\Psi(B)$

$$(1 - .5B) = (1 + .4B)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$$

Comparing coeffs of B^j we get:

$$\Psi_0 = 1$$
$$\Psi_1 = -(.5 + .4)$$
$$\Psi_2 = -.4\,\Psi_1 = -.4\,(-(.5+.4))$$
$$\Psi_3 = (-.4)^2(-(.5+.4))$$

$\vdots$

$$\psi_j = (-.4)^{j-1}\left(-(.5+.4)\right)$$

$$\sum_{j=0}^{\infty} \psi_j X_{t-j} = \epsilon_t$$

$$X_t = -\sum_{j=1}^{\infty} \psi_j X_{t-j} + \epsilon_t \leftarrow AR(\infty)$$

Sly the causal $MA(\infty)$ representation can be obtained

(7) $X_t \sim MA(q)$

$$X_t = \theta(B)\delta_t \quad - \quad \delta_t \sim WN(0, \sigma^2),$$

$$Y_t = \sum_{j=0}^{\infty} a_j X_{t-j} + \epsilon_t : \quad \sum_j |a_j| < \infty$$

$$Y_t = P_t + \epsilon_t \text{, say}$$

$$P_t = \sum_{j=0}^{\infty} a_j X_{t-j} = a(B) X_t \leftarrow \text{filtered process}$$

$$a(B) = a_0 + a_1 B + a_2 B^2 + \cdots$$

Note that since ϵ_t & $\{X_t\}$ are indep, P_t & ϵ_t are indep

Also, $\gamma_y(h) = E\, Y_t Y_{t+h}$

$$= E\left(\sum_{l=0}^{\infty} a_l X_{t-l}\right)\left(\sum_{k=0}^{\infty} a_k X_{t+h-k}\right) + E\,\epsilon_t \epsilon_{t+h}$$

$$= \sum_l \sum_k a_l a_k \gamma_X(h-k+l) + \gamma_\epsilon(h)$$

i.e. $\gamma_y(h) = \gamma_p(h) + \gamma_\epsilon(h)$

ACGF of $\{Y_t\}$

$$g_y(z) = \sum_{j=-\infty}^{\infty} z^j \gamma_y(j)$$

$$= \sum_{j=-\infty}^{\infty} z^j \left(\gamma_p(j) + \gamma_\epsilon(j) \right)$$

$$= g_p(z) + g_\epsilon(z)$$

$$= a(z)\, a(z^{-1})\, g_x(z) + g_\epsilon(z)$$

$$\Rightarrow g_y(z) = \left(\sum_l a_l z^l\right)\left(\sum_k a_k z^{-k}\right) \theta(z)\, \theta(z^{-1})\, \sigma^2 + 1$$

Take $q = 2$, $a_0 = 1$, $a_1 = 2$ & $a_j = 0 \ \forall j > 1$

$$X_t = (1 + \theta_1 B + \theta_2 B^2)\, \epsilon_t$$

$$g_y(z) = (1 + 2z^{-1})(1 + 2z)\, g_x(z) + 1$$

$$= (1 + 2z^{-1} + 2z + 4)\, g_x(z) + 1$$

where,

$$g_x(z) = (1 + \theta_1 z + \theta_2 z^2)(1 + \theta_1 z^{-1} + \theta_2 z^{-2})\, \sigma^2$$

$$= \Big[(1 + \theta_1^2 + \theta_2^2) + (\theta_1 + \theta_1\theta_2)(z + z^{-1}) + \theta_2 (z^2 + z^{-2}) \Big] \sigma^2$$

$\gamma_y(2)$ = Coeff of z^2 in $g_y(z)$

...

8

$$X_t = \frac{1}{2} X_{t-1} + \epsilon_t$$

$$Y_t = \delta_t - \delta_{t-1} - \delta_{t-2}$$

$$\left.\begin{array}{l} \epsilon_t \sim WN(0, \sigma^2) \\ \delta_t \sim WN(0, \sigma^2) \end{array}\right\} \text{ indep.}$$

Note that $X_t = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \epsilon_{t-j}$ – f^n of $\{\epsilon_t\}$ seq

& $Y_t = \delta_t - \delta_{t-1} - \delta_{t-2}$ – f^n of $\{\delta_t\}$ seq

$\Rightarrow$ X_t & Y_t are indep

$$Z_t = X_t - X_{t-1} + Y_t \quad \text{— } (*)$$

$$\gamma_Z(h) = Cov(X_t - X_{t-1} + Y_t,\ X_{t+h} - X_{t+h-1} + Y_{t+h})$$

$$= 2\gamma_X(h) - \gamma_X(h-1) - \gamma_X(h+1) + \gamma_Y(h)$$

ACGF of $\{Z_t\}$:

$$g_Z(y) = \sum_{-\infty}^{\infty} y^h \gamma_Z(h)$$

$$= \sum_{-\infty}^{\infty} y^h \left(2\gamma_X(h) - \gamma_X(h-1) - \gamma_X(h+1) + \gamma_Y(h)\right) y^h$$

$$= 2 g_X(y) - y\, g_X(y) - y^{-1} g_X(y) + g_Y(y)$$

i.e. $g_Z(y) = (1-y)(1-y^{-1}) g_X(y) + g_Y(y)$

Note that can be obtained from $(*)$ directly by using filtering argument

9

$$X_t = \epsilon_t + \epsilon_{t-1}; \quad \epsilon_t \sim WN(0, \sigma^2); \quad |\theta| > 1$$

$$Y_t = \sum_{j=0}^{\infty} \left(-\frac{1}{\theta}\right)^j X_{t-j}$$

$$\Rightarrow \left(1 + \frac{1}{\theta} B\right) Y_t = X_t \qquad \left|\frac{1}{\theta}\right| < 1 \Rightarrow \left(1 + \frac{1}{\theta}\right)^{-1} \text{exists}$$

ACGF of L.h.s.

$$\left(1 + \frac{z}{\theta}\right)\left(1 + \frac{z^{-1}}{\theta}\right) g_y(z)$$

$$= \text{ACGF of r.h.s} = (1 + \theta z)(1 + \theta z^{-1}) \sigma^2$$

$$\Rightarrow g_y(z) = \sigma^2 \frac{(1+\theta z)(1+\theta z^{-1})}{\left(1 + \frac{z}{\theta}\right)\left(1 + \frac{z^{-1}}{\theta}\right)}$$

$$= \sigma^2 \frac{(1+\theta z)(1+\theta z^{-1})}{(z+\theta)(z\theta+1)/\theta^2 z}$$

$$= \sigma^2 \frac{(1+\theta z)(z+\theta)/z}{(z+\theta)(z\theta+1)/\theta^2 z} = \theta^2 \sigma^2$$

i.e. $g_y(z) = \theta^2 \sigma^2 \leftarrow$ const

$$\Rightarrow Y \sim WN(0, \theta^2 \sigma^2)$$