Recall: (M,d) and (M,ρ):

- Sequential equivalence $(d \sim_s \rho)$: $d(x_n,x) \to 0 \iff \rho(x_n,x) \to 0$.
  $\iff I : (M,d) \to (M,\rho)$ is a homeomorphism.
  I & I^{-1} are continuous functions

- (Strongly) equivalent $(d \sim \rho)$: $\exists\ C_1, C_2 > 0$ s.t. $C_1 d(x,y) \le \rho(x,y) \le C_2 d(x,y)$ $\forall\ x,y \in M$.
  I & I^{-1} are Lipschitz cts. functions

Def^n: Uniformly equivalent $(d \sim_u \rho)$: $I : (M,d) \to (M,\rho)$ and $I^{-1} : (M,\rho) \to (M,d)$ are uniformly cts.

(HW) $d \sim \rho \Rightarrow d \sim_u \rho \Rightarrow d \sim_s \rho$.

$\overset{?}{\Leftarrow}$ (HW) $\quad \overset{?}{\Leftarrow}$ (HW)

→ (M,d) compact metric space
Let ρ be another metric on M.
Then, $d \sim_s \rho \iff d \sim_u \rho$. (Since cts. function on compact metric space is unif. cts.)

- (M,d) compact metric space and ρ another metric on M.
  Then $d \sim_u \rho \not\Rightarrow d \sim \rho$.
  not necessarily

- If $I : (M,d) \to (M,\rho)$ and $I^{-1} : (M,\rho) \to (M,d)$ are Lipschitz cts., then $d \sim \rho$.
  (as $\rho(x,y) \le C_1 d(x,y)$ and $d(x,y) \le C_2 \rho(x,y)$)

(Optional) → Normed linear spaces and linear maps between them.

$(V, \|\cdot\|)$ and $(W, |||\cdot|||)$ are nls. and $T : V \to W$ is a linear map. Then TFAE:

(i) T is Lipschitz cts.

(ii) T is uniformly cts.

(iii) T is cts. on V.

(iv) T is cts. at $0 \in V$.

(v) $\exists\ 0 < C < \infty$ st. $|||Tx||| \le C \|x\|$, $\forall\ x \in V$.

Corollary: $(V, \|\cdot\|)$ and $(V, |||\cdot|||)$ Then $\|\cdot\| \sim_s |||\cdot||| \iff \|\cdot\| \sim |||\cdot|||$.

Corollary: $(\mathbb{R}^n, \|\cdot\|)$ and $(\mathbb{R}^n, |||\cdot|||)$ Then $\|\cdot\| \sim |||\cdot|||$.

Corollary: Any linear map on a finite-dim. normed vector space is "automatically" cts.

Q: What if $\dim V = \infty$ (i.e, infinite-dimensional normed vector spaces)?

Self-Read! Interesting / Surprising: Last paragraph of Notes and Remarks in Chapter 8.

Functional Analysis Course (: