Some important properties of probability f^n

(i) $P(\phi) = 0$

Let $A_1 = \Omega$ and $A_i = \phi$ for $i = 2, 3, \ldots$

Then $P(A_1) = 1$

Note that $A_1 = \bigcup_{i=1}^{\infty} A_i$ and $A_i \cap A_j = \phi \ \forall i \neq j$

$$\Rightarrow 1 = P(A_1) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\text{i.e. } 1 = 1 + \sum_{i=2}^{\infty} P(\phi)$$

$$\text{i.e. } 0 = \sum_{i=2}^{\infty} P(\phi)$$

$$\Rightarrow P(\phi) = 0$$

(ii) $A_1, \ldots, A_n \in \mathcal{F}$ and $A_i \cap A_j = \phi \ \forall i \neq j$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Take $A_i = \phi$ for $i = n+1, n+2, \ldots$ then

$A_1, \ldots \ldots \ni A_i \cap A_j = \phi \ \forall i \neq j$

$\& \ P(A_i) = 0 \ \forall i \geq n+1$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$= \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i)$$

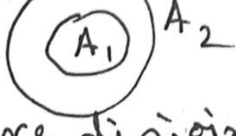
(iii) $\forall A \in \mathcal{F}$, $P(A^c) = 1 - P(A)$

Note that $\Omega = A \cup A^c$; $A \cap A^c = \phi$

$$\Rightarrow 1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$$

$$\Rightarrow P(A^c) = 1 - P(A)$$

(iv) $\forall A_1, A_2 \in \mathcal{F} \ni A_1 \subseteq A_2$



$A_2 = A_1 \cup (A_1^c A_2)$; A_1 & $A_1^c A_2$ are disjoint

$\Rightarrow P(A_2) = P(A_1) + P(A_1^c A_2)$

Now $A_1^c A_2 \in \mathcal{F} \Rightarrow P(A_1^c A_2) \geq 0$

$\Rightarrow P(A_2) \geq P(A_1)$ – monotonicity property

(v) $\forall A \in \mathcal{F}$, $0 \leq P(A) \leq 1$

Note that $\phi \subseteq A \subseteq \Omega$

$\Rightarrow P(\phi) \leq P(A) \leq P(\Omega)$

i.e. $0 \leq P(A) \leq 1$

(vi) $\forall A_1, A_2 \in \mathcal{F}$

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$

Note that $A_1 \cup A_2 = A_1 \cup (A_1^c A_2)$

$P(A_1 \cup A_2) = P(A_1) + P(A_1^c A_2)$

Also $A_2 = \underbrace{A_1 A_2 \cup A_1^c A_2}_{\text{disjoint}}$

$\Rightarrow P(A_2) = P(A_1 A_2) + P(A_1^c A_2)$

$\Rightarrow P(A_1^c A_2) = P(A_2) - P(A_1 A_2)$

$\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$

<u>Note</u>: $A_1, A_2, A_3 \in \mathcal{F}$

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3)$

Inclusion - Exclusion formula

$(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \ldots A_n \in \mathcal{F}$

$n \in \mathbb{N}$, $n \geq 2$

Let $p_{1,n} = \sum_{i=1}^{n} P(A_i)$

$p_{2,n} = \sum_{1 \leq i < j < n} P(A_i A_j)$

$p_{3,n} = \sum_{i<j<k} P(A_i A_j A_k)$ $\qquad p_{l,n} = \sum_{j_1 < j_2 < \ldots < j_l} P(A_{j_1} \ldots A_{j_l})$

$\vdots$

$p_{n,n} = P(A_1 \ldots A_n)$

Then $P\left(\bigcup_{i=1}^{n} A_i\right) = p_{1,n} - p_{2,n} + p_{3,n} - \cdots + (-1)^{n-1} p_{n,n}$

Pf: Note that for $n=2$

$p_1 = P(A_1) + P(A_2)$

$p_2 = P(A_1 A_2)$

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$

$= p_{1,2} - p_{2,2}$

result holds true for $n=2$ and also for $n=3$

Suppose it holds for $n = 2, 3, \ldots m$, i.e.

$P\left(\bigcup_{i=1}^{m} A_i\right) = p_{1,m} - p_{2,m} + p_{3,m} - \cdots + (-1)^{m-1} p_{m,m}$

Then $P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}\right)$

$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right)$

(using $n=2$ result)

$$= \sum_{j=1}^{m} (-1)^{j-1} p_{j,m} + P(A_{m+1}) - P\left(\bigcup_{i=1}^{m} A_i A_{m+1}\right)$$

Let $B_i = A_i A_{m+1}$

Then $$P\left(\bigcup_{i=1}^{m} B_i\right) = p_{1,m}^{(B)} - p_{2,m}^{(B)} + \cdots + (-1)^{m-1} p_{m,m}^{(B)}$$

$$= \sum_{j=1}^{m} (-1)^{j-1} p_{j,m}^{(B)}.$$

where $$p_{1,m}^{(B)} = \sum_{i=1}^{m} P(A_i A_{m+1})$$

$$p_{2,m}^{(B)} = \sum_{i<j} P(A_i A_j A_{m+1})$$

$$p_{m,m}^{(B)} = P(A_1 A_2 \cdots A_m A_{m+1}) = p_{m+1,m+1}$$

Thus

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = \sum_{j=1}^{m} (-1)^{j-1} p_{j,m} + P(A_{m+1}) - \sum_{j=1}^{m} (-1)^{j-1} p_{j,m}^{(B)}$$

$$= \left(p_{1,m} + P(A_{m+1})\right) - \left(p_{2,m} + p_{1,m}^{(B)}\right) + \left(p_{3,m} + p_{2,m}^{(B)}\right) - \cdots + (-1)^{m} p_{m,m}^{(B)}$$

Note that

$$p_{1,m} + P(A_{m+1})$$
$$= \sum_{i=1}^{m} P(A_i) + P(A_{m+1}) = \sum_{i=1}^{m+1} P(A_i) = p_{1,m+1}.$$

$$p_{2,m} + p_{1,m}^{(B)} = \sum_{1 \le i < j \le m} P(A_i A_j) + \sum_{i=1}^{m} P(A_i A_{m+1})$$
$$= \sum_{1 \le i < j \le m+1} P(A_i A_j) = p_{2,m+1}$$

and so on hence

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = p_{1,m+1} - p_{2,m+1} + \cdots + (-1)^m p_{m+1,m+1}$$

Result follows by induction

Boole's inequality

$$P\left(\bigcup_1^n A_i\right) \le \sum P(A_i)$$

For $n=2$ $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$
$$\le P(A_1) + P(A_2)$$

result holds for $n=2$

Suppose the result holds for $n=m$, then

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\left(\bigcup_1^m A_i\right) \cup A_{m+1}\right)$$
$$\le P\left(\bigcup_1^m A_i\right) + P(A_{m+1})$$
$$\le \sum_{i=1}^{m} P(A_i) + P(A_{m+1})$$
$$= \sum_{i=1}^{m+1} P(A_i)$$

Bonferroni's inequality

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Realize that

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcap_{i=1}^{n} A_i\right)^c$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$$

$$\geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

$$= 1 - \sum_{i=1}^{n} \left(1 - P(A_i)\right)$$

$$= 1 - n + \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) - (n-1)$$

Also $P\left(\bigcap_{i=1}^{n} A_i\right) \geq 0$

$\Rightarrow P\left(\bigcap_1^n A_i\right) \geq \max\left(0, \sum_1^n P(A_i) - (n-1)\right)$

Inequalities from Inclusion-exclusion formula

Borel's inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) = p_{1,n}$$

By induction, one can show that

$$p_{1,n} - p_{2,n} \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq p_{1,n}$$

$$p_{1,n} - p_{2,n} + p_{3,n} - p_{4,n} \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq p_{1,n} - p_{2,n} + p_{3,n}$$

Conditional Probability

Conditional prob of A given B : $P(A|B) = \dfrac{P(AB)}{P(B)}$; $A, B \in \mathcal{F}$

Intuitive interpretation thro' rel freq : $\dfrac{f_N(AB)}{f_N(B)}$

Defn: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B \in \mathcal{F}$ be such that $P(B) > 0$. For any arbitrary $A \in \mathcal{F}$

$$Q(A) = P(A|B) = \frac{P(AB)}{P(B)}$$ is the conditional prob of A given B.

Result: $Q(.)$ is a probability measure

Pf: (i) $\forall A \in \mathcal{F}$

$$Q(A) = P(A|B) = \frac{P(AB)}{P(B)} \geq 0$$

(ii) $$Q(\Omega) = P(\Omega|B) = \frac{P(\Omega B)}{P(B)} = 1$$

(iii) If $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_i^{\infty} A_i | B\right) = \frac{P\left(\left(\bigcup_i^{\infty} A_i\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_i^{\infty} A_i B\right)}{P(B)} = \sum_i^{\infty} \frac{P(A_i B)}{P(B)} = \sum_i^{\infty} P(A_i|B)$$

$$= \sum_i^{\infty} Q(A_i)$$

($A_i B$ s are disjoint)

Note: $P(A \cap B) = P(A) P(B|A) = P(B) P(A|B)$ if $P(A), P(B) > 0$

Note: We can interpret cond prob by restricting the sample space to B

$\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}$ - σ-field of subsets of B

$(B, \mathcal{F}_B, Q)$ as prob space

Note: If $P(B) > 0$ and $B \subseteq A$, then $P(A|B) = 1$

If $P(B) > 0$ and $A \cap B = \phi$, then $P(A|B) = 0$

Example: 5 cards drawn at random (WOR) from a pack of 52 cards

B: all are spades

A: at least 4 are spades

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(B)}{P(A)} \qquad B \subseteq A$$

$$= \frac{\binom{13}{5} \Big/ \binom{52}{5}}{\left(\binom{13}{4}\binom{39}{1} + \binom{13}{5}\right) \Big/ \binom{52}{5}} = \ldots$$

Multiplication law

(i) $P(AB) = P(A)\, P(B|A)$ If $P(A) > 0$

$= P(B)\, P(A|B)$ If $P(B) > 0$

(ii) $P(ABC) = P(AB)\, P(C|AB)$

$= P(A)\, P(B|A)\, P(C|AB)$

provided $P(AB) > 0$ (ensures that $P(A) > 0$)

chain in any order is possible

(iii) $P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1 A_2 \ldots A_n)$

$= \cancel{P(A_n A_2 \ldots A_{n-1} | A_n)}$

$= P(A_1, \ldots A_{n-1})\, P(A_n | A_1 \ldots A_{n-1})$

$= P(A_1, \ldots A_{n-2})\, P(A_{n-1} | A_1 \ldots A_{n-2})\, P(A_n | A_1, \ldots A_{n-1})$

$\vdots$

$= P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 A_2) \ldots P(A_n | A_1 \ldots A_{n-1})$

provided $P(A_1 \ldots A_{n-1}) > 0$ (ensures $P(A_1 \ldots A_i) > 0$, $i = 1, \ldots n-2$)

Theorem of total probability

Let $A_1, A_2 \ldots$ be mutually exclusive and exhaustive events $\in \mathcal{F}$ (i.e. $A_i \cap A_j = \phi \; \forall i \neq j$ & $\cup A_i = \Omega$).

Suppose B is any other event ($B \in \mathcal{F}$)

Then $B = B \cap \Omega = B \cap (\bigcup_i A_i) = \bigcup_1^{\infty} (A_i \cap B)$

$$P(B) = P(\bigcup_1^{\infty} A_i B) = \sum_1^{\infty} P(A_i B) \quad (A_i B\text{s are m.e})$$

$$= \sum_{i=1}^{\infty} P(A_i) P(B|A_i) \quad (\text{provided } P(A_i) > 0)$$

Bayes theorem

Suppose that $A_1, A_2, \ldots$ are mutually exclusive and exhaustive and B be any other event $\ni P(B) > 0$,

then

$$P(A_K|B) = \frac{P(A_K B)}{P(B)} = \frac{P(A_K) P(B|A_K)}{\sum_1^{\infty} P(A_K) P(B|A_K)}$$

$P(A_K)$: prior prob

$P(A_K|B)$: posterior prob.

$\rightarrow$ th^m of total probability

Independence of events

Def^n: Let $(\Omega, \mathcal{F}, P)$ be a prob space. $A, B \in \mathcal{F}$ are independent if

$$P(AB) = P(A) P(B)$$

Remark: Intuitively

$P(A|B) = P(A)$ $\qquad$ $P(B|A) = P(B)$ $\qquad$ with $P(A), P(B) > 0$

Multiplication rule: $P(AB) = P(A) P(B|A) = P(B) P(A|B)$

$= P(A) P(B) \qquad = P(A) P(B)$

The above def^n however does not require the $P(A), P(B) > 0$

<u>Result</u>: If A & B are indep, then

(i) A^c & B are indep

(ii) A & B^c are indep

(iii) A^c & B^c are indep

<u>Pf</u>: (i)

$$B = AB \cup A^c B$$

$$P(B) = P(AB) + P(A^c B)$$

i.e. $P(A^c B) = P(B) - P(AB)$

$$= P(B) - P(A)P(B) = P(B)\,P(A^c)$$

(ii) Sly $\quad A = AB \cup AB^c$

$$P(AB^c) = P(A) - P(AB)$$

$$= P(A)\,P(B^c)$$

(iii)

$$P(A^c B^c) = P(A \cup B)^c$$

$$= \cancel{B}\, 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(AB)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= P(A^c)\,P(B^c)$$

<u>Def^n</u>: Events $A_1, \ldots A_n$ are pairwise indep if

$$P(A_i A_j) = P(A_i)\,P(A_j) \quad \forall\, i \neq j$$

<u>Def^n</u>: Events $A_1, A_2, \ldots A_n$ are mutually indep if

(i) $P(A_i A_j) = P(A_i)\,P(A_j) \quad \forall\, i \neq j$

(ii) $P(A_i A_j A_k) = P(A_i)P(A_j)\,P(A_k) \quad \forall\, i \neq j \neq k$

$\vdots$

(n-1) $P(A_1 \ldots A_n) = P(A_1) \ldots P(A_n)$.

Note: For a countable collection $\{A_1, A_2, \ldots\}$; we say that this class is of indep events if every finite subclass of these events is mutually indep.

Note: Mutual indep $\Rightarrow$ pairwise indep

Converse is not true

Counter example

$\Omega = \{1, 2, 3, 4\}$; $\mathcal{F} = \mathcal{P}(\Omega) \rightarrow$ powerset of Ω

$(\Omega, \mathcal{F}, P)$: prob space

$P(\{i\}) = \frac{1}{4} \quad i = 1, 2, 3, 4$

$A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$

$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(AB) = \frac{1}{4} = P(AC) = P(BC) = P(\{4\})$

$P(ABC) = P(\{4\}) = \frac{1}{4}$

$P(AB) = P(A)P(B)$; $P(AC) = P(A)P(C)$; $P(BC) = P(B)P(C)$

$\Rightarrow A, B, C$ are pair wise indep

But $P(ABC) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$

$\Rightarrow A, B, C$ are not indep.

Note: if collection is not p.i then it can't be m.i.

$A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$

$P(AC) = 0 \neq P(A)P(C)$

A, B, C can't be m.i.

<u>Note</u> : If $A_1, \dots A_n$ are m.i. then collection of events and complementary events set would be indep

i.e for any $k \in \{1, 2, \dots n-1\}$ and $(\alpha_1, \dots \alpha_n)$ of $(1, \dots n)$ events $A_{\alpha_1}, \dots A_{\alpha_k}, A^c_{\alpha_{k+1}} \dots A^c_{\alpha_n}$ are indep.

Continuity of probability measure

<u>Def^n</u> : $(\Omega, \mathcal{F}, P)$: prob space

$A_1, \dots$ events $\{A_n : n = 1, 2, \dots\}$ seq of events in $\mathcal{F}$

(i) $A_n \uparrow$ If $A_n \subseteq A_{n+1}$, $n = 1, 2, \dots$

(ii) $A_n \downarrow$ If $A_{n+1} \subseteq A_n$, $n = 1, 2, \dots$

(iii) A_n is monotone If either $A_n \downarrow$ or $A_n \uparrow$

(iv) If $A_n \uparrow$, we define

$$\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

(v) If $A_n \downarrow$, we define

$$\lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Continuity of probability measure

Let $\{A_n : n = 1, 2, \dots\}$ be a sequence of monotone events in $(\Omega, \mathcal{F}, P)$, then

$$P\left(\lim_{n \to \infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

Random variable

$(\Omega, \mathcal{F}, P)$: prob space

In many situations, we may not be directly interested in the sample space Ω or the $\mathcal{F}$; rather we may be interested in some numerical aspect of Ω, i.e. assignment of numbers to elements of Ω.

e.g : Interested to know prop of defective items in a lot

Sample of size n is drawn

Sample space : 2^n elements of the form

$(a_1, a_2, \ldots, a_n)$; $a_i = D$ if item is def
$= N$ if item is non-def

$$\Omega \xrightarrow{X} \{0, 1, \ldots, n\}$$

$X(a_1, \ldots a_n) = r$ if r of a_is are D

e.g : fair coin tossed 2 times

$$\Omega = \{HH, HT, TH, TT\}$$

$$P(\{\omega\}) = \frac{1}{4} \quad \forall \omega \in \Omega$$

$X(\omega)$: # of heads in ω

$X : \Omega \to \mathbb{R}$

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = TT \\ 1, & \text{if } \omega = TH \text{ or } HT \\ 2, & \text{if } \omega = HH \end{cases}$$

$\omega : X(\omega) = 0$

$$P(X=0) = P(TT) = \frac{1}{4}$$

$$P(X=1) = P(TH \text{ or } HT) = \frac{1}{2}$$

$$P(X=2) = P(HH) = \frac{1}{4}$$

$$P(X \in \{0, 1, 2\}) = 1$$

Def^n: $(\Omega, \mathcal{F}, P)$ be prob space

A real valued f^n $X : \Omega \to \mathbb{R}$ defined on the sample space Ω is called a random variable.

Remark: A more advanced textbook on prob would define r.v. as.

A real valued f^n $X : \Omega \to \mathbb{R}$ is called a r.v. If the inverse images under X of all Borel sets in $\mathbb{R}$ are events, i.e. If

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}. \quad -(*)$$