Problem & Solutions on Probability & Statistics

Problem Set-1

[1] A coin is tossed until for the first time the same result appear twice in succession.

To an outcome requiring n tosses assign a probability2^{-n}. Describe the sample space. Evaluate the probability of the following events:

(a) A= The experiment ends before the 6th toss.
(b) B= An even number of tosses are required.
(c) A∩ B, $A^c \cap B$

[2] Three tickets are drawn randomly without replacement from a set of tickets numbered 1 to 100. Show that the probability that the number of selected tickets are in (i) arithmetic progression is 1/ 66 and (ii) geometric progression is $\frac{105}{\binom{100}{3}}$.

[3] Three players A, B and C play a series of games; none of which can be drawn and their probability of wining any game are equal. The winner of each game scores 1 point and the series is won by the player who first scores 4 points. Out of the first three games A won 2 games and B won 1 game. Find the probability that C will win the series.

[4] A point P is randomly placed in a square with side of 1 cm. Find the probability that the distance from P to the nearest side does not exceed x cm.

[5] Let there be n people in a room and p denote the probability that there are no common birth days. Find an approximate value of p for n= 10.

[6] Suppose a lift has 3 occupants A, B and C and there are three possible floors (1, 2 and 3) on which they can get out. Assuming that each person acts independently of the others and that each person has an equally likely chance of getting off at each floor, calculate the probability that exactly one person will get out on each floor.

[7] If n men, among whom are A and B, stand in a row, what is the probability that there will be exactly r men between A and B ?

[8] In a town of n+ 1 inhabitants, a person tells a rumor to a second persons, who in trun tells it to a third persons, and so on. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without

(a) returning to the originator,

(b) being repeated to any persons.

Do the same problem when at each step the rumor is told to a gathering of N randomly chosen people.

[9] 2 points are taken at random and independently of each other on a line segment of length m. Find the probability that the distance between 2 points is less than m/3.

[10] n points are taken at random and independently of one another inside a sphere of radius R. What is the probability that the distance from the centre of the sphere to the nearest point is not less than r ?

[11] A car is parked among N cars in a row, not at either end. On his return, the owner finds that exactly r of the N places are still occupied. What is the probability that both neighboring places are empty?

[12] 3 points X, Y, Z are taken at random and independently of each other on a line segment AB. What is the probability that Y will be between X and Z?

[13] The coefficients of the equation $ax^2 + bx + c = 0$ are determined by throwing an ordinary die. Find the probability that the framed equation will have real roots.

[14] Let Ω= {1, 2, 3, 4}. Check whether any of the following is a σ-field of subsets of Ω

$$\mathcal{F}_1 = \{\phi, \{1,2\}, \{3,4\}\}$$

$$\mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2,3,4\}, \{3,4\}\}$$

$$\mathcal{F}_3 = \{\phi, \Omega, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}\}$$

[15] Prove that if F_1 and F_2 are σ-fields of subsets of Ω, then $F_1 \cap F_2$ is also a σ-field. Give a counter example to show that similar result for union σ- fields does not hold.

[16] Let F be a σ- field of subsets of the sample space Ω and let $A \in F$ be a fixed. Show that $F_A = \{C: C = A \cap B, B \in F\}$ is a σ-field of subsets of A.

Solution Set-1

1) Ω= {HH, TT, HTT, THH, HTHH, THTT, ….}

$$P(HH) = \frac{1}{4} = P(TT)$$

$$P(HTT) = P(THH) = \frac{1}{2^3}.$$

a) $P(A) = \sum_{i=2}^{5} P(\text{exp}\, ends\ in\ i\ tosses)$

$$= P(\text{exp}\, ends\ in\ 2\ tosses) + P(ends\ in\ 3) + P(ends\ in\ 4) + P(\dots 5)$$

$$= 2 \times \frac{1}{2^2} + 2 \times \frac{1}{2^3} + \cdots$$

b) P(B)= $2\sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \cdots$

c) P (A∩ B)= P(exp ends in 2 tosses) + P(exp ends in 4 tosses)

= ….

$P(A^c \cap B) = 2\sum_{i=3}^{\infty} \frac{1}{2^{2i}} =$…..

2) Total # of cases : $\binom{100}{3}$

(i) No. in AP Common diff 1, 2, ... , 49

# of cases 98, 96, …, 2

$\Longrightarrow$Total # of favorable cases 98+ 96 …+ 2

$= 2\left(\frac{49\times 50}{2}\right) = 49\times 50$

Reqd prob= $\frac{49\times 50}{\binom{100}{3}} = \frac{1}{66}$

1) No in GP. Common ratio can be integer or fraction

Case 1: C. V. integer

c. r.	# f tav cases	total #
2 $\longrightarrow$	(1, 2, 4), ……….. (25, 50, 100) $\longrightarrow$	25
3 $\longrightarrow$	(1, 3, 9), …. (11, 33, 99) $\longrightarrow$	11
4 $\longrightarrow$	(1, 4, 6), …. (6, 24, 96) $\longrightarrow$	6
5 $\longrightarrow$	(1, 5, 25), …. (4, 20, 100) $\longrightarrow$	4
6 $\longrightarrow$	(1, 6, 36), (2, 12, 72) $\longrightarrow$	2
7 $\longrightarrow$	(1, 7, 49), (2, 14, 98) $\longrightarrow$	2
8 $\longrightarrow$	(1, 8, 64), $\longrightarrow$	1
9 $\longrightarrow$	(1, 9, 81) $\longrightarrow$	1
10 $\longrightarrow$	(1, 10, 100) $\longrightarrow$	1

Total 53

Case 2: c. r. fractional

1st #	c. r.	tav cases	total #
1	$\frac{3}{2}$	$\longrightarrow (4,6,9),(8,12,18),\ldots.(44,66,99) \longrightarrow$	11
	$\frac{5}{2}$	$\longrightarrow (4,10,25),(8,20,50),(12,30,75)(16,40,100) \longrightarrow$	4
	$\frac{7}{2}$	$\longrightarrow (4,14,49),(8,28,98) \longrightarrow$	2

$\frac{9}{2} \longrightarrow (4, 18, 81) \longrightarrow$ 1

F

$9 \longrightarrow \left(\frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}\right) \longrightarrow \quad 6 + 4 + 2 + 1 + 1$

$6 \longrightarrow \left(\frac{5}{4}, \frac{7}{4}, \frac{9}{4}\right) \longrightarrow 4 + 2 + 1$

$5 \longrightarrow \left(\frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}\right) \longrightarrow 2 + 2 + 1 + 1$

$6 \longrightarrow \frac{7}{6} \longrightarrow 2$

$19 \longrightarrow \left(\frac{8}{7}, \frac{9}{7}, \frac{10}{7}\right) \longrightarrow 1 + 1 + 1$

$4 \longrightarrow \frac{9}{8} \longrightarrow 1$

:1 $\longrightarrow \frac{10}{9} \longrightarrow 1$

Total 52

$\Longrightarrow$ reqd prob. $\frac{53+52}{\binom{100}{3}}$.

3)C can win in exactly 4 or 5 or 6 or 7 additional games

Case 1: 4 additional games

C wins all $\longrightarrow$ prob $\left(\frac{1}{3}\right)^4$. ____________(i)

Case 2: 5 additional games.

C wins 3 out of 1st 4 & the 5th game prob $\longrightarrow \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) \times \frac{1}{3}$ __________(ii)

Case 3: 6 additional games

C wins 3 out of 1st 5 & 6th game and either (i) B wins 2 and A wins on one or (ii) B wins 1, A wins 1

Prob $\longrightarrow \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{1}{3}\right)^2 \times \frac{1}{3} + \binom{5}{3}\binom{2}{1}\left(\frac{1}{3}\right)^3 \frac{1}{3} \cdot \frac{1}{3} \times \frac{1}{3}$ ____________(iii)

Case 4 : 7 additional games

Out of 1st 6 games $\begin{cases} A \text{ wins } 1 \\ B \ldots\ldots..2 \\ C \ldots\ldots\ldots.3 \end{cases}$

$prob\ \binom{6}{3}\binom{3}{1}\left(\frac{1}{3}\right)^3\frac{1}{3}\left(\frac{1}{3}\right)^2\ \left(\frac{1}{3}\right)$ __________ (iv)

Reqd. prob = (i) + (ii) + (iii) + (iv) ← m. e. ways.

(4) The pt P must lie in the shaded region so that the distance from P to the nearest side does not exceed x cm.

If $x \geq \frac{1}{2}, then\ prob = 1$

$If\ 0 < x < \frac{1}{2}, then\ area\ of\ the\ shaded\ region = 1 - (1-2x)^2$

⟹ reqd. prob.= 1-$(1-2x)^2$.

5) Reqd prob.= $\frac{365\times364\times\ldots\times(365-(n-1))}{365^n} = \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\ldots\left(1-\frac{n-1}{365}\right) = p\ say$

$$\log_e p = \sum_{k=1}^{n-1}\log_e\left(1-\frac{k}{365}\right) \approx \sum_{k=1}^{n-1}\left(-\frac{k}{365}\right) = -\frac{1}{365}.\frac{n(n-1)}{2}$$

For n= 10 $\log_e p \approx -\frac{1}{365}\frac{10\times9}{2} = \cdots$

$\Longrightarrow p \approx \cdots$

6) Total # of possible outcomes: 3^3

Tavarable # of outcomes : 3! = 6

Reqd. prob.= $\frac{6}{27}$.

7) Total # of ways in which n men can stand in a row ⟶ n!

# of possible positions for A & B ∋ there are exactly r positions available between them

= 2! × (n – r - 1)

↗ ↖

permutation possible positions

among A&B ({1, r+ 2}, {2, r+ 3}, …., {(n- r-1), n} for A&B)

= Further # of ways that r persons can be chosen to stand between A& B= $\binom{n-2}{r}$

Favarable # of cases

$\left(2! \times (n-r-1)\right) \times \binom{n-2}{r} \times r! \quad \times (n-r-2)!$

↙ ↘

Perm of r perm of (n- r- 2) men excluding A< B and r men in

Men betn A & B between

$\Longrightarrow$ reqd. prob.

$$\frac{2! \times (n-r-1)! \times r! \times \binom{n-2}{r}}{n!}$$

8)

(a) Total # of of ways n^r

$$\left.\begin{array}{l} originator \longrightarrow n\, ways \\ 2^{nd}\ person \longrightarrow (n-1) ways \\ \vdots \\ r^{th}\ person \longrightarrow (n-1) ways \end{array}\right\} \longrightarrow n(n-1)^{r-1}$$

Reqd. prob.= $\frac{n(n-1)^{r-1}}{n^r}$

(b) Originator $\longrightarrow$ n options

2^{nd} person $\longrightarrow$ n- 1

3^{rd} person $\longrightarrow$ n- 2

$\vdots$

$r^{th}\ person \longrightarrow$ (n- r+ 1)

Reqd. prob. = $\frac{n(n-1)\dots(n-r+1)}{n^r}$

<u>Second part</u>:- Total # of cases $\binom{n}{N}^r$.

Case tavarable to 1^{st} event $\binom{n}{N}\binom{n-1}{N}^{r-1}$

$$Reqd\ prob = \frac{\binom{n}{N}\binom{n-1}{N}^{r-1}}{\binom{n}{N}^r}$$

Apply for 2^{nd} event

$$Reqd\ prob = \frac{\binom{n}{N}\binom{n-N}{N}\binom{n-2N}{N}\dots\binom{n-(r-1)N}{N}}{\underbrace{\binom{n}{N}^r \qquad \nearrow}_{with\ obvions\ \ assumption}}$$

9) Let the distance of 2 randomly chosen pts from a fixed pt A on the line segment be denoted as x& y

Reqd. condition is |x- y|< $\frac{m}{3}$.

$$i.e. -\frac{m}{3} < x - y < \frac{m}{3}$$

Note that x, y ∈ [0, m]

Inside the rect bounded by x axis, y axis, x= m and y= m, the area favorable to |x- y|< $\frac{m}{3}$ is clearly the region OABCDE

Area of OABCDE = $m^2 - \left(\frac{2}{3}\right)^2 m^2$

$$\left(= m^2 - 2\left(\frac{1}{2}\frac{2m}{3} \times \frac{2m}{3}\right)\right)$$

$\Rightarrow reqd\ prob = \frac{\left(m^2 - \frac{4}{9}m^2\right)}{m^2} = \frac{5}{9}.$

10) n pt must lie on or outside a sphere of raidus r, having same centre as the original sphere of raidus R.

For any of the n pts,

P(lie inside the smaller sphere)= $\frac{volm\ \ of\ \ sph\ with\ ra\ r}{volm\ \ of\ \ sph\ with\ ra\ R} = \frac{r^3}{R^3}$.

$\Rightarrow$ P(lie on or outside the smaller sphere)= $\left(1 - \frac{r^3}{R^3}\right)$

As the pts are taken independently, the reqd. prob.= $\left(1 - \frac{r^3}{R^3}\right)^n$.

11) Owner's car can be in any of the (N- 2) places (leaving 2 ends)

Remaining (r- 1) cars in (N- 1) remaining place

$\Rightarrow$ Total # of cases= (N- 2) $\binom{N-1}{r-1}$

Favorable # of cases:

Owner's car in any of the (N- 2) places and 2 neighboring places are empty

$\Rightarrow$remaining (r-1) cars can be in (N- 3) remaining places.

$\Rightarrow$favorable # of cases: (N- 2) $\binom{N-3}{r-1}$.

Reqd. prob. = $\frac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}}$.

12) Let x, y, 3 be the distances of X, Y, Z from a fixed pt P on the line segment the six possibilities are

x < y < 3 ; x < 3 < y ; 3 < x < y;

y < x < 3 ; y < 3 < x ; 3 < y < x;

The above 6 possibilities are equally likely due to random draws

Y lie between X & Z in 2 cases (x < y < 3 & 3 < y < x)

Reqd. prob. $=\frac{2}{6}$

13) Coefficient a, b or c can take any of the values 1, 2, …

Total # of (a, b, c) combinations $6\times 6 \times 6 = 216$

Real roots ⟶ requirement $b^2 \geq 4ac$

Listing of favorable # of cases

ac	(a , c)	4ac	$b \ni b^2 \geq 4ac$	# of cases
1	(1 , 1)	4	2, 3, 4, 5, 6	5
2	(1, 2) (2, 1) ⟶	8	3, 4, 5, 6	$2\times 4 = 8$
3	(1 , 3) (3, 1) ⟶	12	4, 5, 6	$2\times 3 = 6$
4	(1, 4) (4, 1) ⟶ (2, 2)	16	4, 5, 6	$3\times 3 = 9$
5	(1, 5) (5, 1) ⟶	20	5, 6	$2\times 2 = 4$
6	(1, 6) (6, 1) (2, 3) ⟶ (3, 2)	24	5 ; 6	$4\times 2 = 8$

_____ not possible to obtain ac= 7

	(2, 4) (4, 2) ⟶	32	6	$2\times 1 = 2$

ac values higher than 9 will not have any b∋ $b^2 \geq 4ac$

$\Longrightarrow$ # of favorable case for $b^2 \geq 4ac = (5 + 8 + 6 + 9 + 4 + 8 + 2 + 1) = 43$

$$reqd\ prob = \frac{43}{216}.$$

14) (i) $\phi^c = \Omega \in \mathcal{F}_1 \Longrightarrow \mathcal{F}_1\ is\ not\ a\ \sigma - field$

(ii) {1} ∪ {3, 4} = {1, 3, 4} $\notin \mathcal{F}_2$

$\mathcal{F}_2$ is not closed under union $\Longrightarrow \mathcal{F}_2$ is not a σ-field

Or {1, 2}∩{2, 3, 4}= {2} $\in \mathcal{F}_2$

$\Longrightarrow \mathcal{F}_2$ is not σ- field.

$(iii)\mathcal{F}_3$ contain Ω and is closed under complementation and union $\Longrightarrow \mathcal{F}_3$ is a σ-field.

15) $\Omega \in \mathcal{F}_1, \mathcal{F}_2 \Longrightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$_______(i)

Let A $\in \mathcal{F}_1 \cap \mathcal{F}_2, then\ A \in \mathcal{F}_1\ \&\ A \in \mathcal{F}_2$

$$\Longrightarrow A^c \in \mathcal{F}_1 \& A^c \in \mathcal{F}_2$$

$\Longrightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$_______(ii)

If $A_1, A_2, \dots \in \mathcal{F}_1 \cap \mathcal{F}_2, then$

$$A_1, A_2, \dots \in \mathcal{F}_1 \ \& \ \Longrightarrow \cup A_i \in \mathcal{F}_1$$

$$A_1, A_2, \dots \in \mathcal{F}_2 \ \& \ \Longrightarrow \cup A_i \in \mathcal{F}_2$$

$\Longrightarrow \cup A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$__________(iii)

(i), (ii) & (iii) $\Longrightarrow \mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field.

<u>Counter example</u>

Ω= {1, 2, 3}

i.e. $\mathcal{F}_1 = \{\phi, \Omega, \{1\}, \{2,3\}\} \longrightarrow \sigma - field$

$\mathcal{F}_2 = \{\phi, \Omega, \{2\}, \{1,3\}\} \longrightarrow \sigma - field$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2\}, \{1,3\}, \{2,3\}\}$

$\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field ($\{1\} \cup \{2\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$)

16)(i) A$\in \mathcal{F}$ & A ∩ A= A $\Longrightarrow$A$\in \mathcal{F}_A$.

(ii) Let C $\in \mathcal{F}_A\ then\ C = A \cap B\ for\ B \in \mathcal{F}$

$$C^{C_A}(complement\ w.r.t.A) = A - C$$

$$= A - A \cap B$$

$$= A \cap B^C \in \mathcal{F}_A \ (as\ B^C \in \mathcal{F}_A\ )$$

(iii)Let $C_1, C_2, \ldots \in \mathcal{F}_A, then$

$$C_i = A \cap B_i, i = 1, 2, \ldots .for\ B_i \in \mathcal{F}$$

$$\cup C_i = \underset{i}{U}(A \cap B_i) = A \cap \left(\underset{i}{U} B_i\right) \in \mathcal{F}_A (as\ UB_i \in \mathcal{F})$$

$$\Rightarrow \mathcal{F}_A\ is\ a\ \sigma - field\ of\ subsets\ of\ A.$$

Problem Set -2

[1] Let Ω= {0, 1, 2, …}. If for an event A,

(a) P(A)= $\sum_{x \in A} \frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0.$

(b) P(A) = $\sum_{x \in A} p(1-p)^x, 0 < p < 1.$

(c) P(A)= $\begin{cases} 1\ if\ the\ number\ of\ elements\ in\ A\ is\ finite \\ \qquad\qquad 0\ otherwise. \end{cases}$

Determine in each of the above cases whether P is a probability measure.

In case where your answer is in the affirmative, determine P(E), P(F), P(G), P(E∩F), P(E ∪F), P(F∪G), P(E∩G) and P(F∩G) , where E= {x ∈Ω : x > 2},

F= {x ∈ Ω : 0< x < 3} and G= {x ∈ Ω: 3 < x < 6}.

[2] Let Ω= $\mathfrak{R}$. In each of the following cases determine whether P(.) is a probability measure. For an interval I,

(a) P(I)= $\int_I \frac{1}{2} e^{|x|}\, dx$

(b) P(I)= $\begin{cases} \qquad 0, \ \ if\ I \subset (-\infty, 0), \\ \int_I \ 2xe^{-x^2} dx\ \ , if\ I \subset [0, \infty). \end{cases}$

(c) P(I)= $\begin{cases} 1\ \ if\ length\ of\ I\ is\ finite \\ \qquad 0\ otherwise \end{cases}$

[3] Show that the probability of exactly one of the events A or B occurring is P(A)+ P(A)- 2P(A ∩B).

[4] Prove that

P(A ∩B)- P(A) P(B)= P(A)P(B^c)- P(A∩ B^c)

$$= P(A^c)P(B) - P(A^c \cap B)$$

$$= P[(A \cup B^c)] - P(A^c)P(B^c)$$

[5] For events $A_1, A_2, \ldots\ldots, A_n$ $show\ that$ $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n-1} P(\cap_{i=1}^{n} A_i)$

[6] Consider the sample space Ω= {, 1, 2, ….}and $\mathcal{F}$ the σ-field of subsets of Ω. To the elementary event {j} assign the probability

P({j})= $C\frac{2^j}{j!}, j = 0, 1, 2, \ldots..$

(a) Determine the constant c.
(b) Define the events A, B and C by A= {j :2≤j ≤ 4}, B= {j : j ≥ 3} and C= {j : j is an odd integer}.

Evaluate P(A), P(B), P(C), P(A∩B), P(A∩C), P(C∩B), P(A∩B∩C) and verify the formula for P(A∪B∪C).

[7] Each packet of a certain cereal contains a small plastic model of one of the five different dinosaurs; a given packet is equally likely to contain any one of the five dinosaurs. Find the probability that someone buying six packets of the cereal will acquire models of three favorite dinosaurs.

[8] Suppose n cards numbered 1, 2, …, n are laid out at random in a row. Let A_i denote the event that 'card I appears in the ith position of the row', which is termed as a match. What is the probability of obtaining at least one match?

[9] A man addresses n envelopes and writes n cheques for payment of n bills.

(a) If the n bills are placed at random in the n envelopes, what would be the probability that eaxch bill would be placed in the wrong envelope?

(b) If the bills and n cheques are placed at random in the n envelopes, one bill and one cheque in each envelope, what would be the probability that in no instance would the enclosures be completely correct?

[10] For events A, B and C such that P(C) > 0, prove that

(a) P(A∪B| c)= P(A|C)+ P(B|C)- P(AB|C)

(b) P($A^C|C$)= 1- P(A|C).

[11] Let A and B be two events such that 0 < P(A) < 1. Which of the following statements are true?

(a) P(A|B) + P($A^C|B$)= 1; (b) P(A|B) + P(A|B^C)= 1; (c) P(A|B)+ P($A^C|\ B^C$)= 1

[12] Consider the two events A and B such that P(A)= $\frac{1}{4}$, P(B|A)= ½ and P(A|B)=$\frac{1}{4}$.

Which of the following statements are true?

(a) A and B are mutually exclusive events,

(b) A ⊂ B,

(c) P($A^C|B^C) = \frac{3}{4}$,

(d) P(A|B) + P(A| B^C)= 1

[13] Consider an urn in which 4 balls have been placed by the following scheme. A fair coin is tossed, if the coin comes up heads, a white ball is placed in the urn otherwise a red ball is placed in the urn.

(a) What is the probability that the urn will contain exactly 3 white balls?

(b) What is the probability that the urn will contain exactly 3 white balls, given that the first ball placed in the urn was white?

[14] A random experiment has three possible outcomes, A, B and C, with probabilities$p_A, p_B, and\ p_C$. What is the probability that, in independent performances of the experiment, A will occur before B?

[15] a system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component I, independent of other components, functions with probability$p_i, i = 1(1)n$, what is the probability that the system functions?

[16] A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, the student needs to answer correctly all the three questions. What is the probability that the student will pass the examination if he remembers correctly answer to 90 questions o the list?

[17] A person has three coins in his pocket, two fair coins (heads and tails are equally likely) but the third one is baised with probability of heads 2/3. One coin selected at random drops on the floor, landing heads up. How likely is it that it is one of the fair coins?

[18] A slip of paper is given to A, who marks it with either a+ or a- sign, with a probability 1/3 of writing a+ sign. A passes the slip to B, who may either leave it unchanged or change the sign before passing it to C. C in turn passes the slip to D after perhaps changing the sign; finally D passes it to a referee after perhaps changing the sign. It is further known that B, C and D each change the sign with probability 2/3. Find the probability that A originally wrote a+ given that the referee sees a+ sign o the slip.

[19] Each of the three boxes A, B, and C, identical in appearance, has two drawers. Box A contains a gold coin in each drawer, Box B contains a silver coin in each drawer and box C contains a gold coi in one drawer and silver coin in the other. A box is chosen at random and one of its drawers is then chosen at random and opened, and a gold coin is found. What is the probability that the other drawer of this box contains a silver coin?

[20] Each of four persons fires one shot at a target. Let C_k denote the event that the target is hit by person k, k= 1, 2, 3, 4. If the events C_1, C_2, C_3, C_4 are independent and if P(C_1)= P(C_2)= 0.7,

P(C_3)= 0.9 and P(C_4)= 0.4, compute the probability that : (a) all of them hit the target; (b) no one hits the target; (c) exactly one hits the target; (d) at least one hits the target.

[21] Let $A_1, A_2, \ldots\ldots, A_n$ be n independent events. Show that $P\left(\bigcap_{i=1}^{n} {A_i}^C\right) \leq \exp(-\sum_{i=1}^{n} P(A_i))$.

[22] Give a counter example to show that pair wise independence of a set of events $A_1, A_2, \ldots\ldots, A_n$ does not imply mutual independence.

[23] We say that B carries negative information about event A if P(A|B) < P(A). Let A, B and C be three events such that B carries negative information about A and C carries negative information about B. Is it true C carries negative information about A ? Prove your assertion.

[24] Suppose in a class there are 5 boys and 3 girl students. A list of 4 students, to be interviewed, is made by choosing 4 students at random from this class. If the first student selected at random from the list, for interview, is a girl, then find the conditional probability of selecting a boy next from among the remaining 3 students in the list.

[25] During the course of an experiment with a particular brand of a disinfectant on files, it is found that 80% are killed in the first application. Those which survive develop a resistance, so that the percentage of survivors killed in any later application is half of that in the preceding application. Find the probability that (a) a fly will survive 4 applications; (b) it will survive 4 applications, given that it has survived the 1st one.

[26] An art dealer receives a group of 5 old paintings and on the basis of past experience, the thinks that the probabilities are, 0.76, 0.09, 0.02, 0.01, 0.02 and 0.10 that 0, 1, 2, 3, 4 or all 5 of them, respectively, are forgeries. The art dealer sends one randomly chosen (out of 5) paintings for authentication. If this painting turns out to be a forgery, then what probability should he now assign to the possibility that the other 4 are also forgeries?

Solution Key

(1) (a) Ω= {0, 1, 2, ….}

P(A)= $\sum_{x\in A} \frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0.$

P(A)≥ 0 obv ____(i)

Define $A_i = \{x \in \Omega : i-1 < x < i+1\} = \{i\}$

$P(\cup_1^\infty A_i) = P\{1, 2, \ldots\} = 1 - P(X=0)$

$\sum_1^\infty P(A_i) = P(A_1) + \cdots = 1 - P(X=0)$

$P(\cup_1^\infty A_i) = \sum_1^\infty P(A_i)$ _________(iii)

⟹ P is prob. Measure.

(b)Similar to (a).

(c) P(A)≥ 0

P(Ω)= 0 (∵Ω is infinite).

P(Ω)= $\sum_{x\in\Omega} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_0^\infty \frac{\lambda^x}{x!} = 1$ _______(i)

In general, $A_1, A_2, \ldots.$

$$\partial A_i \cap A_j = \phi(i \,\forall i = j)$$

$$P\left(\bigcup_i A_i\right) = \sum_{x\in UA_i} P(\{x\})$$

$$= \sum_i \sum_{x\in A_i} P(\{x\}) = \sum_i P(A_i)$$

P is not prob. Measure.

Note:- Also take, $A_i = \{i\}$ i= 1, 2, …

Then $UA_i = \{1, 2, ...\}$ -infinite # of elements

$$P\left(\bigcup_{1}^{\infty} C_i\right) = 0 \neq \sum_{1}^{\infty} P(C_i) = \sum_{1}^{\infty} 1$$

2nd part

(a) P(E)= $\sum_3^\infty \frac{e^{-\lambda}\lambda^x}{x!} = 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!}\right)$

$$P(F) = \sum_{1}^{2} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left(\lambda + \frac{\lambda^2}{2!}\right)$$

$$P(E \cup F) = \sum_{1}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

$others.$

(2) Ω= R

$(a) P(I) = \int_I \frac{1}{2} e^{|x|}\, dx \geq 0 \;\forall\, I$

P(Ω)= ½ $\int_{-\infty}^{\infty} e^{|x|}\, dx = \frac{1}{2}\left(\int_{-\infty}^{0} e^x\, dx + \int_0^\infty e^{-x}\, dx\right) = 1$

$I_1 \cap I_2 = \phi : P(I_1 \cup I_2) = \frac{1}{2}\left[\int_{I_1} + \int_{I_2} \quad \right] = P(I_1) + P(I_2) extended$ .... P is prob measure

$(b) Similar\ to\ (a)$

(c) P(Ω)= P($\Re$)= $0 \neq 1$ P is not prob measure.

(3) P(exactly me of A or B)

$$= P((A \cap B^c) \cup (A^c \cap B)$$

$= P(A) + P(B) - 2P(A \cap B)$ ___ on simplification.

$P(AB) - P(A)P(B) = P(A)P(B^c) - P(AB^c)$. ______ 1st equation.

$[Using\ P(A) = P(AB) + P(AB^c).]$

2nd & 3rd equation is can be proved in a similar way.

(5) For n = 2

$$P(A_1 \cup A_2) = P\big(A_1 \cup ({A_1}^c A_2)\big)$$

$$= P(A_1) + P({A_1}^c A_2)$$

$= P(A_1) + [P(A_2) - P(A_1 A_2)]$ ___true for n= 2

$Proof$ by indention

$Arum$ that it is true for n= m, then

$$P\left(\bigcup_1^{m+1} A_k\right) = P\left(\left(\bigcup_1^{m} A_k\right) \cup A_{m+1}\right)$$

$$= P\left(\bigcup_1^{m} A_k\right) + P(A_{m+1}) - P\left(\left(\bigcup_1^{m} A_k\right) \cap A_{m+1}\right)$$

$$r.h.s = \left[\sum_1^n P(A_k) - \sum_{k_1<k_2} P\big(A_{k_1} \cap A_{k_2}\big) + \sum_{k_1<k_2<k_3} P\big(A_{k_1}A_{k_2}A_{k_3}\big) - \cdots + (-1)^{m-1} P\left(\bigcap_1^m A_k\right)\right] + P(A_{m+1}) - P\left(\bigcup_1^m (A_k\ A_{m+1})\right) \text{\_\_\_\_\_(1)}$$

$$P\left(\bigcup_1^m A_k\ A_{m+1}\right) = \sum_1^n P(A_k A_{m+1}) - \sum_{k_1<k_2} P\big(A_{k_1}A_{m+1}\big) \cap \big(A_{k_2}A_{m+1}\big) + \sum_{k_1<k_2<k_3} P\big(A_{k_1}A_{m+1} \cap A_{k_2}A_{m+1} \cap A_{k_3}A_{m+1}\big) - \cdots + (-1)^{m-1} P\left(\bigcap_1^m A_k A_{m+1}\right) \text{\_\_\_(2)}$$

$Using$ $(2) in$ (1) gives

$$P\left(\bigcup_1^{m+1} A_k\right) = \sum_1^{m+1} P(A_k) - \sum_{k_1<k_2}^{m+1} P\big(A_{k_1}A_{k_2}\big) + \sum_{k_1<k_2<k_3} P\big(A_{k_1}A_{k_2}A_{k_3}\big) - \cdots + (-1)^{m-1} P\left(\bigcap_1^m A_k\right).$$

6) Ω= {0, 1, 2, ….}. P({j})= $c\frac{2^j}{j!} = Ce^2 \implies C = e^{-2}$

$$(b)P(A) = \sum_2^4 e^{-2}\frac{2^j}{j!}; P(B) = \sum_3^{\propto} e^{-2}\frac{2^j}{j!}\ ; P(C) = \sum_0^{\propto} e^{-2}\frac{2^{2j+1}}{(2j+1)!}$$

$$P(B \cap C) = P(\{3\}) + P(\{5\}) + \cdots = \cdots$$

Other probs can be completed in a similar manner

(7) Favorite models of dinosaurs numbered 1, 2, 3 (say) define events

A_i= model # I not found in 6 pockets.

$$i = 1, 2, 3$$

Reqd. prob. = $P({A_1}^c \cap {A_2}^c \cap {A_3}^c) = 1 - P({A_1}^c \cap {A_2}^c \cap {A_3}^c)^c = 1 - P(A_1 \cup A_2 \cup A_3)$

$= 1 - [P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_1A_3) + P(A_1A_2A_3)]$ _______(1)

$$\left.\begin{aligned} P(A_i) &= \left(\tfrac{4}{5}\right)^6\ \forall\, i \\ \text{Note that } P(A_iA_j) &= \left(\tfrac{3}{5}\right)^6\ \forall\, i \neq j \\ P(A_1\, A_2\, A_3) &= \left(\tfrac{2}{5}\right)^6 \end{aligned}\right\} \text{\_\_\_\_(2)}$$

Use (2) in (1) get the desired prob.

(8) A_i : match at position i

P(at least one match)

= P($A_1 \cup A_2 \cup ... \cup A_n) = \sum_1^n P(A_i) - \sum_{i<j} P(A_i - A_j) + \cdots + (-1)^{n-1}P(\bigcap_1^n A_i)$

$$P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_r} = \frac{(n-r)!}{n!}; 1 \leq i_1 < i_2 < \cdots < i_r \leq n\ \ r = 1(1)n$$

$$reqd\ prob = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{n!}$$

(9)

(a) A_i : event that ith bill goes to ith envelope (i= 1(1) n)

$reqd.$ Prob.=$P(\bigcap_{i=1}^n {A_i}^c) = 1 - P(\bigcup A_i)$

$= 1 - \left[\sum P(A_i) - \sum_{i<j} P(A_iA_j) + \cdots + (-1)^{n-1}P(A_1, \dots, A_r)\right] =$
$1 - \sum P(A_i)\,\llcorner Q_1\ \sum_{i<j} P(A_iA_j)\,\llcorner Q_2 + \cdots + (-1)^n P(A_1, \dots, A_r)\llcorner Q_n$

$$In\ Q_i \longrightarrow \binom{n}{i}\ terms\ each\ equal\ to\ \frac{(n-i)!}{n!}\left(=\frac{1}{(n)_i}\right)$$

$$\Rightarrow P\left(\bigcap A_i{}^c\right) = 1-\binom{n}{1}\frac{1}{(n)_1}+\binom{n}{2}\frac{1}{(n)_2}-\cdots+(-1)^n\binom{n}{n}\frac{1}{(n)_n}$$

$$= 1-\frac{n!}{1!\,(n-1)!}.\frac{(n-1)!}{n!}+\frac{n!}{2!\,(n-2)!}.\frac{(n-2)!}{n!}\ldots+(-1)^n.\frac{1}{n!}$$

$$= 1-\frac{1}{1!}+\frac{1}{2!}-\cdots+\ (-1)^n\frac{1}{n!} = \sum_{i=0}^{n}(-1)^i\frac{1}{n!}$$

(b) B_i: ith envelope get ith bill & ith cheque

$reqd.$ Prob. $P(\cap B_i{}^c) = 1-P(\cup B_i) = 1-\ \sum P(B_i)+\sum_{i<j}P\big(B_iB_j\big)-\cdots+(-1)^nP(B_1\ldots B_n)$

$\leftrightarrow$ $\sqcup$ $\downarrow$

R_1 R_2 R_n

In $R_i \longrightarrow \binom{n}{i}\ term\ each\ equal\ to\ \left(\frac{(n-i)!}{n!}.\frac{(n-i)!}{n!}\right)$.

$\Rightarrow P(\cap_i B_i{}^c) = 1-P(\cup_i B_i) = 1-\sum_i P(B_i)+\sum_{i<j}P\big(B_iB_j\big)-\cdots+(-1)^nP(B_1\ldots B_n)$

$= 1-R_1+R_2-R_3+\cdots+(-1)^nR_n$

$$R_i = \binom{n}{i}\frac{(n-i)!}{n!}\times\frac{(n-i)!}{n!}$$

$$= \frac{n!}{i!\,(n-i)!}.\frac{(n-i)!}{n!}.\frac{(n-i)!}{n!} = \frac{1}{i!}.\frac{1}{(n)_i}\ \ |(n)_i = \frac{n!}{(n-i)!}$$

$$\Rightarrow P\left(\bigcap_i B_i{}^c\right) = \sum_{i=0}^{n}(-1)^i\frac{1}{i!\,(n)_i}.$$

(10) (i) P(A∪ B | C)= $\frac{P(A\cup B)\cap C}{P(C)} = \frac{P(AC\cap BC)}{P(C)}$

$$= P(A|C)+P(B|C)-\ P(AB|C)$$

(ii) $P(A^c|C) = \frac{P(A^cc)}{P(C)} = \frac{P(C)-P(AC)}{P(C)} = 1-P(A|C)$

(11) (a) true – $P(A|B)+\ P(A^c|B) = \frac{P(AB)}{P(B)}+\frac{P(A^cB)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(b) $P(A|B) = \frac{P(AB)}{P(B)}; P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{P(A)-P(AB)}{1-P(B)}$

$$Take\ A \subset B, P(A) > \ 0, P(B - A) > 0$$

$$P(A|B) + \ P(A|B^c) = \frac{P(AB)}{P(B)} + \frac{P(AB^c)}{P(B^c)} = \frac{P(A)}{P(B)} < 1 \ \ false.$$

(c) Take A⊂B, i.e. $B^c \subset A^c$

$$P(A|B) + \ P(A^c|B^c).$$

$$= \frac{P(AB)}{P(B)} + \frac{P(A^c\ B^c)}{P(B^c)} > 1 \ \ false.$$

(12) (a) $P(A|B) = \frac{1}{4} \Longrightarrow P(AB) \neq 0$

↖(i.e. AB ≠ 0)

(b) $ACB \ \Longrightarrow P(AB) = P(A)$

$$Given\ P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2} \ \ \Longrightarrow P(AB) \neq P(A)\, false.$$

(c) $P(A^c|B^c) = \frac{P(A^c B^c)}{P(B^c)} = \frac{1-P(A)- P(B)+P(AB)}{1-P(B)}$ ____(*)

$$P(B|A) = \frac{1}{2}$$

& $P(A) = \frac{1}{4}, \Longrightarrow P(AB) = \frac{1}{8}$

$$\& \, P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{8}}{P(B)} = \frac{1}{4} \Longrightarrow P(B) = \frac{1}{2}$$

(*) $\Longrightarrow P(A^c|B^c) = \frac{1-\frac{1}{4}-\frac{1}{2}+\frac{1}{8}}{\frac{1}{2}} = \frac{3}{4}$.

(13) (a) P(exactly 3 white balls, out of 4)

$= \binom{4}{3}\left(\frac{1}{2}\right)^3 . \frac{1}{2} = \cdots$

(b)A : first ball placed is white

$$P(A) = \frac{1}{2}$$

$$B\!: urn\ contain\ exactly\ 3\ white\ balls\ P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{2}\cdot\binom{3}{2}\left(\frac{1}{2}\right)^3}{\frac{1}{2}} = \frac{3}{8}.$$

(14) D: A occurs before B

$$P(D) = p_A + p_A p_C + p_A {p_C}^2 + \cdots = \frac{p_A}{1-p_C} = \frac{p_A}{p_A+p_B}$$

$$A_i: component\ i\ function$$

P(system function)= $1 - P(\cap {A_i}^c) = 1 - (1-p)^n$

(16) A_i: Question I is among the 90 questions that the student can answer correctly.

Reqd. prob. $P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$

$$= \frac{90}{100}.\frac{89}{99}.\frac{88}{98}....$$

(17) Apply bayes theorem

Reqd. prob. = $\frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3}} = \cdots$

(18) A^+, B^+, C^+, D^+-events that A, B, C, D passes the paper with (t) sign

bayes theorem

Reqd. prob. =P$(A^+|D^+) = \frac{P(A_+)P(D_+|A_+)}{P(D^+)}$

$$P(D^+|A^+) = \left(\frac{1}{3}\right)^3 + \binom{3}{2}\left(\frac{2}{3}\right)^2.\frac{1}{3} = \frac{13}{27}$$

$$also\ P(D^+) = P(D^+|A^+)P(A^+) + P(D^+|{A_+}^c)P({A_+}^c)$$

$$P(D^+|{A_+}^c) = P(D\ passes\ with\ +|A\ passes -).$$

$$= \binom{3}{1}\frac{2}{3}\left(\frac{1}{3}\right)^2 + \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{14}{27}$$

$$P(D^+) = \frac{13}{27}.\frac{1}{3} + \frac{14}{27}.\frac{2}{3} = \frac{41}{81}$$

$\Rightarrow P(A^+|D^+) = \frac{\frac{13}{27}.\frac{1}{3}}{\frac{41}{81}} = \frac{13}{41}$.

(19) silver coin in other

P(h| Gold coin in one drawer).

Bayes thm

$= \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times 1+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 0} = \frac{1}{3}$.

(20) (a) $P(C_1C_2C_3C_4) = \prod_1^4 P(C_i) = \cdots$

(b) $P({C_1}^c \cap {C_2}^c \cap {C_3}^c \cap {C_4}^c) = \prod_1^4 P({C_i}^c)$ $[explain\ why\ \ is\ so\ C_1, C_2, C_3, C_4\ indepen\ \Longrightarrow$ ${C_1}^c, {C_2}^c, {C_3}^c, {C_4}^c\ are\ also\ indepen.]$

(c) $P(C_1{C_2}^c{C_3}^c{C_4}^c) + P({C_1}^cC_2{C_3}^c{C_4}^c) + P({C_1}^c{C_2}^cC_3{C_4}^c) + P({C_1}^c{C_2}^c{C_3}^c{C_4}^c)$

$$= P(C_1)\prod_2^4 P\left({C_i}^3\right) + \cdots$$

(d) P(at least one hits)= $1 - P(no\ one\ hits)$

$$= 1 - P({C_1}^c{C_2}^c{C_3}^c{C_4}^c) = \cdots$$

(21) $P(\bigcap_1^n {A_i}^c) = \prod_1^n P({A_i}^c) = \prod_1^n\left(1 - P(A_i)\right) \le \prod_1^n \exp\left(-P(A_i)\right)$ $[0 < x < 1,\ 1 - x < e^{-x}\ ]$

$i.e. P(\bigcap_1^n {A_i}^c) \le \exp(-\sum_1^n P(A_i))$.

(22) Ω= {1, 2, 3, 4} 7: Proper set

$$P(\{i\}) = \frac{1}{4} \quad i = 1, 2, 3, 4$$

A= {1, 4} , B= {2, 4}, C= {3, 4}.

P(A)= P(B)= P(C)= ½

$$P(AB) = P(AC) = P(BC) = \frac{1}{4}; P(ABC) = \frac{1}{4}$$

$$\Longrightarrow P(AB) = P(A)P(B), P(AC) = P(A)P(C) \& P(BC) = P(B)P(C)$$

i.e. A, B, C are pair wise independent

but P(ABC)= $\frac{1}{4} \ne P(A).P(B).P(C) = \frac{1}{8}$

$$\Longrightarrow A, B, C\ not\ mutuals\ indep.$$

(23) Counter example

In prv prob set up take

A= {1, 2}, B= {3, 4}, C= {1}

P(A| B) < P(A)

P(B| C) < P(B) but P(A|C) > P(A)

$\Longrightarrow$ C does not carry negative information about A.

(24) A_i: I gives are in the list i= 0, 1, 2, 3

B : 1st student is girl.

C : 2nd student is boy:- to obtain P(C|B)

P(B)= $P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) = \frac{\binom{3}{1}\binom{5}{3}}{\binom{8}{4}} \times \frac{1}{4} + \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} \times \frac{2}{4} + \frac{\binom{3}{3}\binom{5}{1}}{\binom{8}{4}} \times \frac{3}{4}$

$$P(B) = \frac{105}{4 \times \binom{8}{4}}.$$

$$P(C|B) = \frac{P(CB)}{P(B)} = \frac{P\left(C \cap (\cup_1^3 A_i B)\right)}{P(B)} = \frac{P\left((\cup_1^3 CA_i B)\right)}{P(B)}$$

$$= \sum_1^3 P(C|A_i B)\ P(A_i|B)$$

$$= 1 \times P(A_1|B) + \frac{2}{3} \times P(A_2|B) + \frac{1}{3} \times P(A_3|B)$$

$$\begin{bmatrix} P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)} = \frac{2}{7} \\ P(A_2|B) = \frac{P(A_2)P(B|A_1)}{P(B)} = \frac{4}{7} \quad P(A_3|B) = \frac{1}{7} \end{bmatrix}$$

$$\Longrightarrow P(C|B) = 1 \times \frac{2}{7} + \frac{2}{3} \times \frac{4}{7} + \frac{1}{3} \times \frac{1}{7} = \cdots$$

(25)

A_i : event that a fly survives ith application i= 1, 2, 3, 4

Note that $A_4 \subset A_3 \subset A_2 \subset A_1$

$$\Longrightarrow A_4 = A_1 \cap A_2 \cap A_3 \cap A_4$$

(a) Reqd. prob. = P(a fly survives 4 applications)

= P$(A_1\ A_2\ A_3\ A_4) = P(A_4)$

$= P(A_1)P(A_2|A_1)P(A_3|A_1A_2)P(A_4|A_1\ A_2\ A_3)$

↓ ↓ ↓ ↓

= (1- 0.8) (1- 0.4) (1- 0.2) (1- 0.1)

(for the given conditions)

= 0.2 $\times$ 0.6 $\times$ 0.8 $\times$ 0.9

(b) $P(A_4|A_1) = \frac{P(A_4 \cap A_1)}{P(A_1)} = \frac{P(A_4)}{P(A_1)}$

$= 0.6 \times 0.8 \times 0.9$

(26) B_i: event that I of the paintings are forgeries i= 0(1) 5

P$(B_0) = 0.76, P(\ B_1) = 0.09, P(\ B_2) = 0.02, P(\ B_3) = 0.01, P(\ B_4) = 0.02$ & $P(\ B_5) =$ $0.1\ (given\ conditios)$

A : event that the painting sent for authentication turns out to be a forgery.

Reqd. prob. $= P(B_5|A) = \frac{P(\ B_5)P(A|\ B_5)}{\sum_{i=0}^{5} P(\ B_i)P(A|\ B_i)}$

↑

Bayes them

$$P(A) = \sum_{i=0}^{5} P(\ B_i)P(A|\ B_i)$$

$$= 0.76 \times 0 + 0.09 \times \frac{1}{5} + 0.02 \times \frac{2}{5} + 0.01 \times \frac{3}{5} + 0.02 \times \frac{4}{5} + 0.10 \times 1$$

$$= \cdots$$

$$P(B_5|A) = \frac{0.10 \times 1}{P(A)} = \cdots$$

Problem Set-3

[1] Let X be a random variable defined on $(\Omega, \mathfrak{I}, P)$.Show that the following are also random variables; (a) |X|, (b) $X^2\ \ and\ (c)\sqrt{X},$ given that {x< 0}=ϕ.

[2] Let Ω= [0, 1] and $\mathfrak{I}$ be the Boral σ-field of subsets of Ω. Define X on Ω as follows:

$$X(\omega) = \begin{cases} \omega\ \ if\ 0 \leq \omega \leq \dfrac{1}{2} \\ \omega - \dfrac{1}{2}\ \ if \dfrac{1}{2} < \ \omega \leq 1 \end{cases}$$

Show that X defined above is a random variable.

[3] Let Ω= {1, 2, 3, 4} and $\Im$= {ϕ, Ω, {1}, {2, 3, 4}} be a σ-field of subsets of Ω. Verify whether

$X(\omega) = \omega + 1;\ \forall\ \omega \in \Omega$, is a random variable with respect to $\Im$.

[4] Let a card be selected from an ordinary pack of playing cards. The outcome ω is one of these 52 cards. Define X on Ω as :

$$X(\omega) = \begin{cases} 4\ if\ \omega\ is\ an\ ace \\ 3\ if\ \omega\ is\ a\ king \\ 2\ if\ \omega\ is\ a\ queen \\ 1\ if\ \omega\ is\ a\ jack \\ 0\ therwise. \end{cases}$$

Show that X is a random variable. Further, suppose that P(.) assigns a probability of 1/52 to each outcome ω. Derive the distribution function of X.

[5] Let F(x)= $\begin{cases} 0\ \ if\ x < -1 \\ \frac{(x+2)}{4}\ \ if -1\ \leq x < 1 \\ 1\ \ if\ x\ \geq 1 \end{cases}$

Show that F(.) is a distribution function. Sketch the graph of F(x) and compute the probabilities P(- ½ < X $\leq \frac{1}{2})$, $P(X = 0), P(X = 1) and P(-1 \leq x < 1)$. Further, obtain the decomposition F(x)= $\alpha\ F_d(x) + (1-\alpha)F_c(x)$; where $F_d(x) and F_c(x)$ are purely discrete and purely continuous distribution functions, respectively.

[6] Which of the following functions are distribution function?

(a) F(x)= $\begin{cases} 0,\ x < 0 \\ x,\ 0 \leq x \leq \frac{1}{2} \\ 1,\ x > \frac{1}{2} \end{cases}$; (b) $F(x) = \begin{cases} 0,\ \ x < 0 \\ 1 -\ e^x,\ \ \ x \geq \end{cases}$; (c)$F(x) = \begin{cases} 0\ \ x \leq 1 \\ 1 - \frac{1}{x}\ \ \ x > 1 \end{cases}$

[7]$Let\ F(x) = \begin{cases} 0\ \ \ if\ x \leq 0 \\ 1 - \frac{2}{3}e^{-x/3} - \frac{1}{3}e^{-[x/3]}\ \ if\ x > 0 \end{cases}$

Where, [x] is the largest integer$\leq$ x. show that F(.) is a distribution function and compute P(X > 6), P(X = 5) and P(5 $\leq X \leq$ 8).

[8] The distribution function of a random variable X is given by

F(x)= $\begin{cases} 0,\ \ x < -2, \\ \frac{1}{3},\ \ -2 \leq x < 0, \\ \frac{1}{2},\ \ 0 \leq x < 5, \\ \frac{1}{2} + \frac{(x-5)^2}{2},\ \ 5 \leq x < 6 \\ 1,\ \ x \geq 6 \end{cases}$

$Find\ P(-2 \leq X < 5), P(0 < X < 5.5) and\ P(1.5 < X \ \leq 5.5\ |X > 2).$

[9] Prove that if $F_1(.), \dots., F_n(.) are\ n\ distribution\ functions, then\ F(x) = \sum_{i=1}^{n} \alpha_i F_i(x)$ is also a distribution function for any $(\alpha_1, \dots, \alpha_n)$, such that $\alpha_i \geq 0$and $\sum_{i=1}^{n} \alpha_i = 1$.

[10] Suppose $F_1\ and\ F_2$ are distribution functions. Verify whether G(x) = $F_1(x) + F_2(x)$is also a distribution function.

[11] Find the value of α and k so that F given by

F(x)= $\begin{cases} 0 \ \ if\ x \ \leq 0 \\ +ke^{-x^2/2}\ \ if\ x > 0 \end{cases}$

Is distribution function of a continuous random variable.

[12] Let F(x)= $\begin{cases} 0 \ \ if\ x < 0 \\ \frac{(x+2)}{8} \ \ if\ 0 \ \leq x < 1 \\ \frac{(x^2+2)}{8} \ \ if\ 1 \leq x < 2 \\ \frac{(2x+c)}{8} \ \ if\ 2 \leq x \leq 3 \\ 1 \ \ if\ x > 3 \end{cases}$

Find the value of c such that F is a distribution function. Using the obtained value of c, find the decomposition F(x)= $\alpha\ F_d(x) + (1 - \alpha)F_c(x)$; where $F_d(x) and F_c(x)$ are purely discrete and purely continuous distribution functions, respectively.

[13] Suppose F_X is the distribution function of a random variable X. Determine the distribution function of a $X^+ and\ (b)|X|$. Where

$$X^+ = \begin{cases} X\ if\ X \ \geq 0 \\ 0\ if\ X < 0 \end{cases}$$

[14]The convolution F of two distribution functions $F_1\ and\ F_2$ is defined as follows;

F(x)= $\int_{-\infty}^{\infty} F_1(x - y)\, dF_2(y); x \ \in \mathbb{R}$

And is denoted by F= $F_1 * F_2$. Show that is F is also a distribution function.

[15] Which of the following functions are probability mass functions?

(a) f(x)= $\begin{cases} \frac{(x-2)}{2} \ \ if\ x = 1, 2, 3, 4 \\ 0 \ \ therwise. \end{cases}$; $(b) f(x) = \begin{cases} \frac{(e^{-\lambda}\lambda^x)}{x!} \ \ if\ x = 0, 1, 2, 3, 4, \dots \ where, \lambda > 0 \\ 0 \ \ otherwise \end{cases}$; $(c) f(x) =$ $\begin{cases} \frac{(e^{-\lambda}\lambda^x)}{x!} \ \ if\ x = \ 1, 2, 3, 4, \dots \ where, \lambda > 0 \\ 0 \ \ otherwise \end{cases}$

[16] Find the value of the constant c such that f(x)= (1- c)c^x; x= 0, 1, 2,3, … defines a probability mass function.

[17] Let X be a discrete random variable taking values in $\mathcal{X} = \{-3, -2, -1, 0, 1, 2, 3\}$ *such that* $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$

And P(X < 0)= P(X= 0)= P(X > 0). Find the distribution function of X.

[18] A battery cell is labeled as good if it works for at least 300 days in a clock, otherwise it is labeled as bad. Three manufacturers, A, B, and C make cells with probability of marking good cells as 0.95, 0.90 and 0.80 respectively. Three identical clocks are selected and cells made by A, B, and C are used in clock numbers 1, 2 and 3 respectively. Let X be the total number of clocks working after 300 days. Find the probability mass function of X and plot the corresponding distribution function.

[19] Prove that the function $f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x} & if\ x>0 \\ 0 & otherwise \end{cases}$

Defines a probability density function for $\theta > 0$. Find the corresponding distribution function and hence compute P(2 < X < 3) and P(X > 5).

[20] Find the value of the constant c such that the following function is probability density function.

$$f_\lambda(x) = \begin{cases} c(x+1)e^{-\lambda x} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

Where λ > 0. Obtain the distribution function of the random variable associated with probability density function$f_\lambda(x)$.

[21] Show that f(x)= $\begin{cases} \frac{x^2}{18} & if -3 < x < 3 \\ 0 & otherwise \end{cases}$

Defines a probability density function. Find the corresponding distribution function and hence find P(|X|< 1) and P($x^2 < 9$)

Solution Key

(1) (a) y= |X| $\forall x \in \mathbb{R}$ $y^{-1}(-\infty, x] = \{\omega: y(\omega) \leq x\} = \{\omega: |X(\omega)| \leq x\}$

$$= \{\omega: -x \leq X(\omega) \leq x\}$$

$$= X^{-1}[-x, \infty) \cap X^{-1}(-\infty, x]$$

As X is a.r.v.

$$X^{-1}[-x, \infty) \cap X^{-1}(-\infty, x] \in \boldsymbol{\mathcal{F}}$$

[X is a. r. v. $\Rightarrow X^{-1}(B) \in \mathcal{F} \forall\ B \in \mathfrak{B}$

$\Rightarrow X^{-1}[-x, \infty) \in \mathcal{F}$ &$X^{-1}(-\infty, x] \in \mathcal{F}]$

$$\Rightarrow y^{-1}(-\infty, x] \in \mathcal{F}\ \forall\ x \in \mathbb{R}$$

$$\Longrightarrow \text{y} = |\text{X}| is\ a.r.v.$$

(b)y= X^2; $\forall\ x \in \mathbb{R}\ y^{-1}(-\infty, x] = \{\omega: y(\omega) \leq x\}$

$$= \{\omega: X^2(\omega) \leq x\}$$

$$= \{\omega: -\sqrt{x} \leq X(\omega) \leq \sqrt{x}\}$$

$$= X^{-1}\left[-\sqrt{x}, \infty\right) \cap X^{-1}\left(-\infty, \sqrt{x}\right] \in \mathcal{F}$$

$$\Longrightarrow y = X^2\ is\ a.r.v.$$

Sly part (c).

(2) Ω= [0, 1] $\mathcal{F}$: Boral σ-field of student of Ω

X(ω)= $\begin{cases} \omega,\ 0 \leq \omega \leq \frac{1}{2} \\ \omega - \frac{1}{2},\ \frac{1}{2} < \omega \leq 1 \end{cases}$

$$X^{-1}(-\infty, x] = \begin{cases} \phi\ \in \mathcal{F}, \qquad x < 0 \\ [0, x] \cup \left(\frac{1}{2}, \frac{1}{2} + x\right]\ \ 0 \leq x < \frac{1}{2} \\ \in \mathcal{F} \\ \Omega \in \mathcal{F}\ , x\ \geq \frac{1}{2} \end{cases}$$

$$\Longrightarrow\ X^{-1}(-\infty, x] \in \mathcal{F}\ \forall\ x\ \in \mathbb{R}\ \Longrightarrow \text{X is a.r.v.}$$

(3) Ω= {1, 2, 3, 4}

$\mathfrak{I}$= {ϕ, Ω, {1}, {2, 3, 4}}

X(ω)= ω+1 → 2, 3, 4, 5

$$X^{-1}(-\infty, x] = \begin{cases} \phi\ (\in \mathfrak{I})\ x < 2 \\ \{1\}(\in \mathfrak{I})\ 2\ \leq x < 3 \\ \{1, 2\} \notin \mathfrak{I}\ \ 3 \leq x < 4 \end{cases}$$

↓

$$\Longrightarrow X\ is\ not\ a.r.v.$$

(4) Ace king

↓ ↓

Ω= {1 H, ……, k H, 1 S, ….., k S, 1D, …., k D, 1C, …. K C }

↔ ↔ ↔ ↔

All hearts Spades Diamond club

$$X(\omega)=\begin{cases} 4 \ \ ace \ \ \mathfrak{I} = parallel \\ 3 \ \ king \\ 2 \ \ queen \\ 1 \ \ jack \\ 0 \ \ otherwise \end{cases}$$

$$X^{-1}(-\infty, x] = \begin{cases} \phi \ \ IF \ X < 0 \\ \{2H, \dots 10H, 2S, \dots 10S, 2D, \dots 10\underset{\leftrightarrow}{D}, 2C, \dots 10C\} if \ 0 \leq x < 1 \\ \{2H, J\underset{\leftrightarrow}{H}, 2S, \dots J\underset{\leftrightarrow}{S}, 2D, \dots J\underset{\leftrightarrow}{D}, 2C, \dots J\underset{\leftrightarrow}{C}\} if \ 1 \leq X < 2 \\ \{2H, \dots Q\underset{\leftrightarrow}{H}, 2S, \dots Q\underset{\leftrightarrow}{S}, 2D, \dots Q\underset{\leftrightarrow}{D}, 2C, \dots, Q\underset{\leftrightarrow}{C}, \} 2 \leq X < 3 \\ \{2H, \dots K\underset{\leftrightarrow}{H}, 2S, \dots K\underset{\leftrightarrow}{S}, 2D, \dots K\underset{\leftrightarrow}{D}, 2C, \dots K\underset{\leftrightarrow}{C}\} 3 \leq X < 4 \\ \Omega \ \ X \geq 4 \end{cases}$$

$$\Longrightarrow X^{-1}(-\infty, x] \in \mathcal{F} \forall \ x \ \in \mathbb{R}$$

$$\Longrightarrow \text{X is a. r. v.}$$

(5)

(i) F(.) is non decreasing
(ii) F(.) is right contain everywhere
(iii) F(-∞)= 0 & F(∞)= 1

⟹ F(.) is a d. f.

P(- ½ <X ≤ ½)= $F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right)$

$$= \frac{5}{8} - \frac{3}{8} = \frac{2}{8}$$

$$P(X = 0) = F(0) - \ F(0 -) = 0$$

$$P(X = 1) = F(1) - F1 — F(-1 -)$$

$$= \frac{3}{4} - 0 = \frac{3}{4}$$

(6) (a) F(x) is not right continuous at x = ½

⟹ F(.) is not a d. f.

(b)& (c) the n. s. c. holds for these 2 and hence they are d. f. s.

(7) F(.) is non decreasing

F(∞)= 1 & F(-∞)= 0

F(x) is right contain ∀ $x \ \in \mathbb{R}$

$\Longrightarrow$ F(.) is a. d. f.

$$P(X > 6) = 1 - P(X \leq 6) = 1 - F(6) = 1 - \left(1 - \frac{2}{3}e^{-\frac{6}{3}} - \frac{1}{3}e^{-\left[\frac{6}{3}\right]}\right)$$

$$= 1 - \left(1 - \frac{2}{3}e^{-2} - \frac{1}{3}e^{-2}\right) = \ e^{-2}$$

$$P(X = 5) = F(5) - F(5-)$$

$$= \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-\left[\frac{5}{3}\right]}\right) - \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-\left[\frac{5}{3}\right]}\right)$$

$$= \ \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-1}\right) - \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-1}\right) = 0$$

$$P(5 \leq X \leq 8) = F(8) - F(5-)$$

$$= \cdots$$

(8) (6) (a) F(x) is not right continuous at x = ½

$\Longrightarrow$ F(.) is not a d. f.

(b)& (c) the n. s. c. holds for these 2 and hence they are d. f. s.

(7) F(.) is non decreasing

F(∞)= 1 & F(-∞)= 0

F(x) is right contain $\forall \ x \ \in \mathbb{R}$

$\Longrightarrow$ F(.) is a. d. f.

$$P(X > 6) = 1 - P(X \leq 6) = 1 - F(6) = 1 - \left(1 - \frac{2}{3}e^{-\frac{6}{3}} - \frac{1}{3}e^{-\left[\frac{6}{3}\right]}\right)$$

$$= 1 - \left(1 - \frac{2}{3}e^{-2} - \frac{1}{3}e^{-2}\right) = \ e^{-2}$$

$$P(X = 5) = F(5) - F(5-)$$

$$= \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-\left[\frac{5}{3}\right]}\right) - \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-\left[\frac{5}{3}\right]}\right)$$

$$= \ \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-1}\right) - \left(1 - \frac{2}{3}e^{-\frac{5}{3}} - \frac{1}{3}e^{-1}\right) = 0$$

$$P(5 \leq X \leq 8) = F(8) - F(5-)$$

$$= \cdots$$

(8) P(-2≤ X < 5)= F(5-)- F(-2 -)

$= \frac{1}{2} - 0 = \frac{1}{2}$

P(0< X < 5.5)= F(5.5-) – F(0)

$= \left(\frac{1}{2} + \frac{1}{8}\right) - \frac{1}{2} = \frac{1}{8}$

P(1.5 < X ≤ 5.5| X > 2)= $\frac{P(2<X \leq 5.5)}{P(X>2)}$

$$= \frac{F(5.5) - F(2)}{1 - F(2)}$$

$$= \frac{\left(\frac{1}{2} + \frac{1}{8}\right) - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{4}$$

(9) for $x_1 < x_2$

$F(x_2) - F(x_1) = \sum_1^n \alpha_i \left(F_i(x_2) - F_i(x_1)\right) \geq 0 \ \geq 0 \forall\, i$

⟹ F(x) is non-decreases.

$$F(-\infty) = \sum \alpha_i F_i(-\infty) = 0 \ \&\ F(\infty) = 1$$

$$F(x_+) = \lim_{z \downarrow x} F(z) = \lim_{z \downarrow x} \sum \alpha_i F_i(z)$$

$$= \sum \alpha_i F_i(x_+) = \sum \alpha_i F_i(x) = F(x)$$

⟹ F(.) is a d. f.

(10) $G(\infty) = F_1(\infty) + F_2(\infty) = 2 \neq 1$

$$\Longrightarrow G(.) is\ not\ a\ d.f.$$

(11) Right cont at x= 0 ⟹ F(0)= F(0+)

⟹ 0= α+ k _________(i)

$$F(\infty) = 1 \Longrightarrow \alpha = 1 \Longrightarrow k = -v$$

(12) Right cont at x= 3 ⟹ F(3)= F(3+)

$\Longrightarrow \frac{6+C}{8} = 1 \Longrightarrow C = 2$

$$f^n\ F(.) is\ having\ jump\ at\ x = 0\ only \left(magnitude\ \frac{2}{8}\right)$$

Discrete part of d. f.

$\alpha F_d(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{8} & x \geq 0 \end{cases}$

$$cont\ part$$

$$(1-\alpha)F_c(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{8} & 0 \leq x < 1 \\ \frac{x^2}{8} & 1 \leq x < 2 \\ \frac{2x}{8} & 2 \leq x \leq 3 \\ \frac{6}{8} & if\ x > 3 \end{cases}$$

$$\Rightarrow \alpha = \frac{2}{8} = \frac{1}{4}\ \&\ (1-\alpha) = \frac{3}{4}$$

$$F_d(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}\ \&\ F_c(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{6} & 0 \leq x < 1 \\ \frac{x^2}{6} & 1 \leq x < 2 \\ \frac{x}{3} & 2 \leq x \leq 3 \\ 1 & if\ x > 3 \end{cases}$$

(13) Y= X^+

P(Y ≤ y)= 0 if y < 0

If y= 0 $P(Y \leq 0) = P(X^+ \leq 0) = P(X^+ = 0) = P(X \leq 0) = F(0)$

If y> 0 $P(Y \leq y) = P(X^+ \leq y) = P(X \leq y) = F(y)$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F(y) & y \geq 0 \end{cases}$$

$$Z = |X|$$

$$F_Z(\mathfrak{z}) = P(1 \times 1 \leq \mathfrak{z})$$

$$= P(-\mathfrak{z} \leq X \leq \mathfrak{z})$$

$$= \begin{cases} F(\mathfrak{z}) - F(-\mathfrak{z}-) & if\ \mathfrak{z} \geq 0 \\ 0 & if\ \mathfrak{z} < 0 \end{cases}$$

(14) for $x_1 < x_2$

$$F(x_2) - F(x_1) = \int_{-\infty}^{\infty} F_1(x_2 - y)\, dF_2(y) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(x_1 - y)\, dF_2(y)$$

$$= \int_{-\infty}^{\infty} \left(F_1(x_2 - y) - F_1(x_1 - y)\right) dF_2(y)$$

$$\geq 0 \ \ \forall x_1 < x_2$$

$$F(\alpha) = \int_{-\infty}^{\infty} F_1\,(\infty) d\, F_2(y) = F_2(\infty) - F_2(-\infty) = 1$$

$$F(-\infty) = 0$$

$$F(X_+) = \int_{-\infty}^{\infty} F_1((X_+) - y)\, dF_2(y) = \int_{-\infty}^{\infty} F_1(x - y)\, dF_2(y) = F(x)$$

$\Longrightarrow$ F(.) as defined is a d. f.

(15) (a) f(1) < 0 $\Longrightarrow$ f(.) is not a p. m. f.

(b) f(x) ≥ 0 ∀ x

$$\sum_x f(x) = e^{-\lambda} \sum_{\alpha}^{\infty} \frac{\lambda^x}{x!} = 1$$

$\Longrightarrow$ f(.) is a p. m. f.

(c) $\sum_x f(x) \neq 1 \Longrightarrow f(.) is\ not\ a\ p.m.f.$

(16) f(x)≥ 0 ∀ C∈[0, 1]

$\sum_x f(x) = 1\ also$ ∀ C∈[0, 1]

$\Longrightarrow$ ∀ C → 0≤C ≤ 1; f(.) is p. m .f.

(17) Suppose

P(X= -3)= P(X= -2) = P(X= -1)= P(X= 1) = P(X= 2) = P(X= 3)= p

$\Longrightarrow$ P(X < 0)= 3p = P(X > 0)= P(X = 0)

P(X < 0) + P(X = 0) + P(X > 0)= 1

$\Longrightarrow p = \frac{1}{9}$

p. m. f.

X= x	-3	-2	-1	0	1	2	3
P(X=x)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

$$F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{1}{9} & -3 \le x < -2 \\ \frac{2}{9} & -2 \le x < -1 \\ \frac{3}{9} & -1 \le x < 0 \\ \frac{6}{9} & 0 \le x < 1 \\ \frac{7}{9} & 1 \le x < 2 \\ \frac{8}{9} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

(18) X: r. v. denoting total # of clocks working after 300days

X can take values in {0, 1, 2, 3}

$$P(X = 0) = P(A^c B^c C^c) = P(A^c)P(B^c)P(C^c)$$

$$= (1 - 0.95)(1 - 0.9)(1 - 0.8) = p_0, say$$

$$P(X = 1) = P(AB^c C^c \cup A^c BC^c \cup A^c B^c C\ )$$

$$= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C)$$

$$= \cdots$$

$$= p_1\ say$$

$$P(X = 2) = P(ABC^c \cup A\ B^c C \cup A^c BC)$$

$$= P(A)P(B)P(C^c) + P(A)P(B^c)P(C) + P(A^c)P(B)P(C)$$

$$= \cdots$$

$$= p_2\ say$$

$$P(X = 3) = P(ABC) = P(A)P(B)P(C) = 0.95 \times 0.9 \times 0.8 = \ p_3\ say$$

p. m. f.

X= x	0 1 2 3

P(X= x)	p_0 p_1 p_2 p_3

d. f.

(19) f(x) ≥0∀ x

$$\int_{-\infty}^{\infty} f(x)\,dx = \theta^2 \int_0^{\infty} xe^{-\theta x}\,dx = \theta^2 . \frac{\Gamma 2}{\theta^2} = 1$$

⟹ f(.) is a p. d. f.

P(2 < X < 3)= F(3)- F(2)

& P(X> 5)= 1- F(5)

Here, F(x)= $\int_{-\infty}^{x} f(x)\,dx = \theta^2 \int_0^x xe^{-\theta x}\,dx \;\; if\; x > 0$

$= 0 \;\; if\; x \le 0$

↗

Integration by parts.

(20) If C ≥ 0 then f(x) ≥ 0 ∀ x

$$\int_0^{\infty} f(x)\,dx = 1$$

$$\Longrightarrow C\int_0^{\infty} (x+1)\,e^{-\lambda x}\,dx = 1$$

i.e. $C\left[\frac{\Gamma 2}{\lambda^2} + \frac{\Gamma 1}{\lambda}\right] = 1 \Longrightarrow C = \frac{\lambda^2}{1+\lambda}$

d. f. F(x)= $\int_0^x f(x)\,dx = \frac{\lambda^2}{1+\lambda}\left[\int_0^x (1+x)\,e^{-\lambda x}\,dx\right] if x \ge 0$

$= 0 \;\; if\; x < 0$

↑

By parts.

(21) f(x) ≥ 0 ∀ x

$$\int_{-\infty}^{x} f(x)\,dx = \int_{-3}^{3} \frac{x^2}{18} dx = \frac{1}{18}\frac{x^3}{3} \Big|_{-3}^{3} = \frac{1}{18}.18 = 1$$

$\Longrightarrow$ f(.) is a p. d. f.

$$F_X(x) = \begin{cases} 0 \;\; if\ x < -3 \\ \dfrac{x^3+27}{54} \;\; -3 \le x \le 3 \\ 1 \;\;\; x > 3 \end{cases}$$

$$P(|X| < 1) = F(1) - F(-1) = \frac{28}{7} - \frac{26}{7}$$

$$P(X^2 < 9) = 1$$

Problem Set -4

[1] Find the expected number of throws of a fair die required to obtain a 6.

[2] Consider a sequence of independent coin flips, each of which has a probability p of being heads. Define a random variable X as the length of the urn (of either heads or tails) started by the first trial. Find E(X).

[3] Verify whether E(X) exists in the following cases:

(a) X has the p. m. f. P(X= x)= $\begin{cases} \left(x(x+1)\right)^{-1}, \;\; if\ x = 1, 2, \ldots \\ 0 \;\; otherwise. \end{cases}$

$$(b) X\ has\ the\ p.d.f.\ f(x) = \begin{cases} (2x^2)^{-1}, \;\; if\ |X| > 1 \\ 0, \;\;\; otherwise. \end{cases}$$

(c)X (Cauchy r. v.) has the p. d. f. f(x)= $\frac{1}{\pi}\frac{1}{1+x^2}; -\infty < x < \infty$.

[4] Find the mean and variance of the distributions having the following p. d. f. / p. m. f.

(a) $f(x) = ax^{a-1}, 0 < x < 1, a > 0$

$$(b) f(x) = \frac{1}{n}; x = 1, 2, \ldots, n; n > 0\ is\ an\ integer$$

$$(c) f(x) = \frac{3}{2}(x-1)^2; 0 < x < 2$$

[5] Find the mean and variance of the Weibull random variable having the p. d. f.

f(x)= $\begin{cases} \frac{c}{a}\left(\frac{x-\mu}{a}\right)^{c-1} \exp\left\{-\left(\frac{x-\mu}{a}\right)^{c}\right\} \; if\ x > \mu \\ 0 \;\; otherwise. \end{cases}$ $where, c > 0, a > 0\ and\ \mu \in (-\infty, \infty).$

[6] A median of a distribution is a value m such that P(X≥ m)≥ ½ and P(X≤ m)≥ ½ , with equality for a continuous distribution. Find the median of the distribution with p. d. f. f(x)= $3x^2, 0 < x < 1; =$ 0, otherwise.

[7] Let X be a continuous, nonnegative random variable with d. f. F(x). Show that

E(X)= $\int_0^{\infty}(1 - f(x))dx.$

[8] A target is made of three concentric circles of radii $\frac{1}{\sqrt{3}}, 1, \sqrt{3}$ feet. Shots within the inner circle give 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target give 0. Let X be the distance of the hit from the centre (in feet) and let the p. d. f. of X be

f(x)= $\begin{cases} \frac{2}{\pi(1+x^2)} & x > 0 \\ 0 & otherwise. \end{cases}$

What is the expected vale of the score in a single shot?

[9] Find the moment generating function (m. g. f.) for the following distributions

(a) X is a (Binomial r. v.) discrete random variable with p. m. f.

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \dots, n \\ 0, \ otherwise. \end{cases}$$

n is a positive integer.

(b)X is a (Poisson r. v.) continuous random variable with p. d. f.

$$f_x(x) = \begin{cases} \dfrac{e^{-x/\beta} x^{\alpha-1}}{\Gamma \alpha \beta^{\alpha}}, x > 0 \\ 0, \ otherwise \end{cases}$$

Find E(X) and V(X) from the m. g. f. s.

[10] The m. g. f. of a random variable X is given by

$$M_X(t) = \frac{1}{2}e^{-5t} + \frac{1}{6}e^{4t} + \frac{1}{8}e^{5t} + \frac{5}{24}e^{25t}$$

Find the distribution function of the random variable.

[11] Let X be a random variable with P(X ≤ 0)= 0 and let μ= E(X) exists. Show that P(X≥ 2μ) ≤ 0.5.

[12] Let X be a random variable with E(X)= 3 and E(X^2)= 13, determine a lower bound for P(-2 < X < 8).

[13] Let x be a random variable with p. m. f.

$$P(X=x)=\begin{cases}\frac{1}{8} & x=-1,1\\ \frac{6}{8} & x=0\\ 0 & otherwise.\end{cases}$$

Using the p. m. f. , show that the bound for Chebychev' s inequality cannot be improved.

[14] A communication system consists of n components, each of which will independently function with probability p. The system will be able to operate effectively if al least one half of its components function.

(a) For what value of a p a 5-component system is more likely to operate effectively than a 3-component system?

(b) In general, when is a (2k + 1) –component system better than a (2k- 1) –component system?

[15] An interviewer is given a list of 8 people whom he can attempt to interview. He is required to interview exactly 5 people. If each person(independently) agrees to be interviewed with probability 2/3, what is the probability that his list will enable him to complete his task?

[16] A pipe-smoking mathematician carries at all times 2 match boxes, 1 in his left-hand pocket and 1 in his right- hand pocket. Each time he needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches in the other box, k= 0, 1, …, N?

Solution Key

(1) Let X denote the # of throws reqd. to get a 6

$$\mathfrak{X}=\{1,2,3,\dots\}$$

$$P(X=x)=\left(\frac{5}{6}\right)^{x-1}\frac{1}{6}\ x\in\mathfrak{X}$$

$$=0\ otherwise$$

$$E(X)=\sum_{1}^{\infty}x\left(\frac{5}{6}\right)^{x-1}\frac{1}{6}=\frac{1}{6}\sum_{1}^{\infty}x\left(\frac{5}{6}\right)^{x-1}=\left(1+2\left(\frac{5}{6}\right)+3\left(\frac{5}{6}\right)^{2}+\cdots\right)\frac{1}{6}$$

$$=\frac{1}{6}\left[\left(1+\frac{5}{6}+\left(\frac{5}{6}\right)^{2}+\cdots\right)+\frac{5}{6}\left(1+\frac{5}{6}+\left(\frac{5}{6}\right)^{2}+\cdots\right)+\left(\frac{5}{6}\right)^{2}\left(1+\frac{5}{6}+\left(\frac{5}{6}\right)^{2}+\cdots\right)+\cdots\right]$$

$$=\frac{1}{6}\left[\frac{1}{1-\frac{5}{6}}+\frac{5}{6}\frac{1}{1-\frac{5}{6}}+\left(\frac{5}{6}\right)^{2}\frac{1}{1-\frac{5}{6}}+\ \dots\right]$$

$$=\frac{1}{6}\left[6\left(1+\frac{5}{6}+\left(\frac{5}{6}\right)^{2}+\cdots\right)\right]=\ 6$$

(2) X: length of run of heads or tails starting with trial 1

$\mathfrak{X}$= {1, 2, …}

$$P(X = x) = (1-p)^x p + p^x(1-p)$$

$$\uparrow \qquad\qquad \uparrow$$

Run of x T_s run of x H

$$E(X) = \sum_1^\infty x\big((1-p)^x p + p^x(1-p)\big) = p(1-p)\left(\sum_1^\infty x\,(1-p)^{x-1} + \sum_1^\infty x\,p^{x-1}\right)$$

$$= p(1-p)\left(\frac{1}{p^2} + \frac{1}{(1-p)^2}\right) \leftarrow as\ in\ (1)$$

$$= \frac{1-2p+2p^2}{p(1-p)}.$$

(3) (a) $E(|X|) = \sum_1^\infty |x| \frac{1}{x(x+1)} = \sum_1^\infty \frac{1}{x+1}\ not\ convergent$

$$\Longrightarrow E(X) does\ not\ exist$$

$$(b) E(|X|) = \int_{|x|>1} |x| \frac{1}{2x^2} dx = \infty \Longrightarrow E(x) does\ not\ exist$$

(d) $E(|X|) = \int_{-\infty}^{\infty} \frac{|x|}{\pi}\frac{1}{1+x^2} dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2} dx = \frac{1}{\pi}(\log(1+x^2))\big|_0^\infty = \infty$
$\Longrightarrow E(X) does\ not\ exist.$

(4) Trial calculations

(a) e. g. $E(X) = a\int_0^1 x^a\, dx = \frac{a}{a+1}; E(X^2) = \frac{a}{a+2}$

$$V(X) = E(X^2) - (E(X))^2 = \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$$

$$= \cdots$$

(5) $E(X) = \frac{c}{a}\int_\mu^\infty x\left(\frac{x-\mu}{a}\right)^{c-1} e^{-\left(\frac{x-\mu}{a}\right)^c} dx$

$$y = \left(\frac{x-\mu}{a}\right)^c \quad dy = \frac{c}{a}\left(\frac{x-\mu}{a}\right)^{c-1}$$

$$\Longrightarrow E(X) = \int_0^\infty \left(ay^{\frac{1}{c}} + \mu\right) e^{-y}\, dy = a\,\Gamma\frac{1}{c} + 1 \ + \mu$$

$$E(X^2) = \frac{c}{a}\int_{\mu}^{\infty} x^2 \left(\frac{x-\mu}{a}\right)^{c-1} e^{-\left(\frac{x-\mu}{a}\right)^c} dx$$

$$as\ in E(X) = \int_0^{\infty} \left(ay^{\frac{1}{c}} + \mu\right)^2 e^{-y}\, dy$$

$$= a^2 \sqrt{\frac{2}{c} + 1} + 2a\mu \sqrt{\frac{1}{c} + 1} + \mu^2$$

$$V(X) = EX^2 - (EX)^2$$

$$= \left(a^2 \sqrt{\frac{2}{c} + 1} + 2a\mu \sqrt{\frac{1}{c} + 1} + \mu^2\right) - \left(a\sqrt{\frac{1}{c} + 1} + \mu\right)^2$$

$$= \cdots$$

(6)$\int_0^m 3x^2\, dx = \int_0^1 3\, x^2 dx = \frac{1}{2} \Longrightarrow m = \cdots$

(7) $\int_0^{\infty}\big(1 - F(x)\big)\, dx = \int_0^{\infty}\int_x^{\infty} f_x\,(y) dy\ \ dx$

$$0 < x < y < \infty$$

$$= \int_0^{\infty} f_x\,(y) \int_0^{y} dx\ \ dy = \int_0^{\infty} y\, f_x(y) dy = E(X)$$

(8) Let Z denote the r. v. ∋

Z: Score in a shot $\mathfrak{X}_z = \{0, 1, 2, 3\ 4\}$

$$P\ (Z =) => \sqrt{3}) = \frac{2}{\pi}\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2} dx = \frac{2}{\pi} \tan^{-1} x \,|\, \frac{\infty}{\sqrt{3}} = \frac{1}{3}$$

$$P(Z = 2) = P\big(1 < X < \sqrt{3}\big) \frac{2}{\pi}\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{1}{6}$$

P(Z=3) = P$\left(\frac{1}{\sqrt{3}} < x < 1\right) = \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2} dx = \frac{1}{6}$

$$P(z = 4) = P\left(0 < x < \frac{1}{\sqrt{3}}\right) = \frac{2}{\pi}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+x^2} = \frac{1}{3}$$

Expendent score,

E(Z)=0 $\times \frac{1}{3} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4\,.\frac{1}{3} = \frac{1}{3} + \frac{1}{2} + \frac{4}{3} = \cdots$

(9)(a) $M_X(t) = E(e^{tX}) = \sum_0^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_0^n \binom{n}{x} (pe^t)^x (1-p)^{n-x}$

$$= (1 - p + pe^t)^n$$

$$q = 1 - p$$

$$\frac{d}{dt} M_X(t) \bigg|_{t=0} n(q + pe^t)^{n-1} pe^t \bigg| t = 0$$

$$= np = E(x)$$

$$\frac{d^2}{dt^2} M_X(t) = n(n-1)(q + pe^t)^{n-2}(pe^t)^2 + n(q + pe^t)^{n-1} pe^t$$

$$\frac{d^2 M_X(t)}{dt^2} |_{t=0} = n(n-1)p^2 + np = \mu_2{}^1 = E(X^2)$$

$$V(X) = EX^2 - (EX)^2 = n(n-1)p^2 + np - n^2p^2 = np\,(1-p) = npq$$

Sly (b)

(10) (i) $X \sim G(\alpha, \beta)$

$$M_X(t) = E(e^{tX}) = \frac{1}{\Gamma\alpha\beta^\alpha} \int_0^\infty e^{tX}\, x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$= \frac{1}{\Gamma\alpha\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} dx \ \left(region\ of\ existance\ t < \frac{1}{\beta}\right)$$

$$= \frac{\Gamma\alpha}{\Gamma\alpha\beta^\alpha} . \frac{1}{\left(\frac{1}{\beta} - t\right)^\alpha} = \left(\frac{1}{\beta} - t\right)^\alpha$$

$$E(X) = \frac{d}{dt} M_X(t) | t = 0 = -\alpha(1 - \beta t)^{-\alpha-1}(-\beta)|_{t=0}$$

$$= \alpha\beta$$

$$E(X^2) = \frac{d^2 M_X(t)}{dt^2} |_{t=0}$$

$$= \alpha\beta(-(\alpha+1)(1-\beta t)^{-\alpha-2}(-\beta))|_{t=0}$$

$$= \alpha(\alpha+1)\beta^2$$

$$V(X) = E(X^2) - ((EX))^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

Sly (e)

(ii)$M_X(t) = e^{-5t}\frac{1}{2} + e^{4t}\frac{1}{6} + e^{5t}\frac{1}{8} + \frac{5}{24}e^{25t}$

A 4 pt distn.

p. m. f.

$X = x$	-5	4	5	25
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{5}{24}$

$$\text{d. f. } F_X(x) = \begin{cases} 0 & x < -5 \\ \frac{1}{2} & -5 \le x < 4 \\ \frac{1}{2}+\frac{1}{6} & 4 \le x < 5 \\ \frac{1}{2}+\frac{1}{6}+\frac{1}{8} & 5 \le x < 25 \\ \frac{1}{2}+\frac{1}{6}+\frac{1}{8}+\frac{5}{24} = 1 & x \ge 25 \end{cases}$$

(11) X is (+) ve values r. v. , by Markov's inequality

$$P(X \ge 2\mu) = P(|X| \ge 2\mu\,) \le \frac{E((X))}{2\mu} = \frac{1}{2}$$

(12) $P(-2 < X < 8) = P\left(\frac{-2-3}{2} < \frac{X-E(X)}{\sqrt{V(X)}} < \frac{8-3}{2}\right)$

$$= P\left(-\frac{5}{2} < \frac{X-E(X)}{\sqrt{V(X)}} < \frac{5}{2}\right) = P\left(|X-\mu| \le \frac{5}{2}\sqrt{V(X)}\right)$$

$$= 1 - P\left(|X-\mu| \ge \frac{5}{2}\sqrt{V(X)}\right)$$

$$\ge 1 - \frac{V(X)}{\frac{25}{4}.V(X)} \quad (chebyshev's\ sinequal)$$

$$= 1 - \frac{4}{25} = \frac{21}{25}$$

(13) $E(X) = -\frac{1}{8}+\frac{1}{8} = 0 = \mu; V(X) = EX^2 = \frac{1}{8}+\frac{1}{8} = \frac{1}{4} = \sigma^2$

By chebyshev's inequality

$\forall\, t > 0 \;\; P(|X-\mu| \ge t) \le \frac{\sigma^2}{t^2}$

$$i.e. P(|X| \ge t) \le \frac{1}{4t^2}$$

$$Also, P(|X| \geq t) = \begin{cases} \frac{1}{8} + \frac{1}{8} = \frac{1}{4} & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$$

$$i.e. P(|X| \geq t) = \frac{1}{4} \forall\, t \ni 0 < t \ \leq 1$$

$\Longrightarrow$ for t= 1, bound from chebyshev's inequality is attained exactly and hence cannot be improved

(14) Let X be the r. v. denoting the # of components functioning X~ B(n, p)

P(system works effectively)= P(X≥ [n/2] +1).

(a) $P(5\ comp. system\ works) = \ p_5$

$$i.e. p_5 = p_5(X \geq 3) = \ \binom{5}{3}\ p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + p^5$$

$$\&\ p_3 = \binom{3}{2} p^2(1-p) + p^3 = P(3\ comp. system\ works)$$

$$p_5 > \ p_3$$

$$if\ \binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + p^5 > \binom{3}{2} p^2(1-p) + p^3$$

$$simplyfy\ to\ get\ the\ condition\ as\ p > \frac{1}{2}$$

$$\begin{aligned}(b) P_{2k+1}(X \geq k+1) &= p_{2k+1} \\ &= \ P_{2k-1}(X \geq k+1) + P_{2k-1}(X = k)P_2(X \geq 1) + P_{2k-1}(X = k-1)P_2(X = 2)\end{aligned}$$

$$i.e. p_{2k+1} = P_{2k+1}(X \geq k+1) + P_{2k+1}(X = k)(1-(1-p)^2) + P_{2k-1}(X = k-1)p^2$$

$$Further\ p_{2k-1} = \ P_{2k-1}(X \geq k) = \ P_{2k-1}(X = k) + P_{2k-1}(X \geq k+1)$$

$$since\ p_{2k-1} = \ P_{2k-1} + P_{2k-1}(X = k) - P_{2k-1}(X = k)(1-p)^2 + P_{2k-1}(X = k-1)p^2$$

$$\Longrightarrow p_{2k+1} = \ p_{2k-1} - P_{2k-1}(X = k)(1-p)^2 + P_{2k-1}(X = k-1)p^2$$

$$\Longrightarrow \ p_{2k+1} > \ p_{2k-1}\ if$$

$$P_{2k-1}(X = k)(-(1-p)^2) + P_{2k-1}(X = k-1)p^2 > 0$$

$$i.e.\ \binom{2k-1}{k} p^k(1-p)^{k-1}(-1-p^2+2p) + \binom{2k-1}{k-1} p^{k-1}(1-p)^k p^2 > 0$$

$$i.e. p^k(1-p)^{k-1}(-1-p^2+2p+p-p^2) > 0$$

$$i.e. -1-2p^2+3p > 0$$

$$i.e. (2p-1)(1-p) > 0$$

$$i.e. p > \frac{1}{2} \; reqd\ condition.$$

$(15) X$: # of intervals attempts to get 5 interviews,

$$P(X=x) = \binom{x-1}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{x-5} \times \frac{2}{3}; x = 5, 6, \ldots$$

$$reqd\ prob\ P(X \le 8) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$$

$$= \binom{4}{4}\left(\frac{2}{3}\right)^5 + \binom{5}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + \cdots + = \cdots$$

(16) P(selecting Box 1)= P(selecting box 2)= ½

Suppose Box 2 is found empty, then Box 2 has been chosen (n+1)th times, at it is time Box 1 contains k matches if it has been chosen n-k times

Chosing Box 2≡ success.

Chosing Box 1≡ failure} Bernoulli trial p= ½

⟹ Box 2 found empty with k matches left in Box 1

≡ N- k failures preceding (N+ 1)th success

$prob = \binom{N+(N-k)}{N}\left(\frac{1}{2}\right)^N\left(\frac{1}{2}\right)^{N-k} \times \frac{1}{2} = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}$

sly Box 1 found empty with k matches in Box 2

$$prob = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}$$

$$\Longrightarrow reqd\ prob = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}$$

Problem Set-5

[1] A machine contains two belts of different lengths. These have times to failure which are exponentially distributed, with means α and 2α. The machine will stop if either belt fails. The failure of the belts are assumed to be independent. What is the probability that the system performs after time α from the start?

[2] Let X be a normal random variable with parameters μ= 10 and $\sigma^2 = 36. Compute$

$$(a) P(X > 5), (b) P(4 < X < 16), (c) P(X < 8).$$

[3] Let $X \sim N(\mu, \sigma^2). If\ P(X \leq 0) = 0.5\ and\ P(-1.96 \leq X \leq 1.96) = 0.95, find\ \mu\ and\ \sigma^2$.

[4] It is assumed that the lifetime of computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters μ= $1.4 \times 10^6\ and\ \sigma^2 = 3 \times 10^5$ hours. What is the approximate probability that a batch of 10 chips will contain at least 2 chips whose lifetime are less than 1.8×10^6 hours?

[5] Let X be a normal random variable with mean 0 and variance 1 i.e. N (0, 1). Prove that

$$P(|X| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}; \forall\ t > 0$$

[6] Show that if X is a discrete random variable with values 0, 1, 2, … then

$E(X) = \sum_{k=0}^{\infty}(1 - F(k))$, where F(x) is the distribution function of the random variable X.

[7] The cumulative distribution function of a random variable X defined over $0 \leq x < \infty\ is\ F(x) = 1 - e^{-\beta x^2}, where\ \beta > 0$. Find the mean, median and variance of X.

[8] Show that for any x > 0,$1 - \phi(x) \leq \frac{\phi(x)}{x}$, where ϕ(x) is the c. d. f. and ϕ(x) is the p. d. f. of standard normal distribution.

[9] A point m_0 is said to mode of a random variable X, if the p. m. f. or the p. d. f. of X has a maximum at m_0. For the distribution given in problem [7], if m_0 denotes the mode; μ and σ^2, the variance of the corresponding random variable, then show that

$$m_0 = \mu \sqrt{\frac{2}{\pi}}\ \ and\ 2{m_0}^2 - \mu^2 = \sigma^2.$$

[10] Let X be a poison random variable with parameter λ. Find the probability mass function of Y= $X^2 - 5$.

[11] Let X be a Binomial random variable with parameters n and p. Find the probability mass function of Y= n- X.

[12] Consider the discrete random variable X with the probability mass function

$$P(X = -2) = \frac{1}{5}, P(X = -1) = \frac{1}{6}, P(X = 0) = \frac{1}{5},$$

$$P(X = 1) = \frac{1}{15}, P(X = 2) = \frac{10}{30}, P(X = 3) = \frac{1}{30}.$$

Find the probability mass function of Y= X^2.

[13] The probability mass function of the random variable X is given by

$$P(X = x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x & x = 0, 1, 2, \dots \\ 0 & otherwise. \end{cases}$$

Find the distribution of Y= X/ (X +1).

[14] Let X be a random variable with probability mass function

$$P(X = x) = \begin{cases} e^{-1}, & x = 0 \\ \frac{e^{-1}}{2(|X|)!}, & x \in \{\pm 1, \pm 2, \dots\} \\ 0 & otherwise. \end{cases}$$

Find the p. m. f. and distribution of the random variable Y= |X|.

……………………………………………………………………………………………………..

Useful data

ϕ (1/3)= 0.6293, ϕ(5/6)= 0.7967, ϕ(1)= 0.8413, ϕ(4/3) 0.918

……………………………………………………………………………………………………..

Solution Key

(1) Belt 1

X~ Exp with mean $\alpha \sim \frac{1}{\alpha} e^{-x/\alpha}; x > 0$

Belt 2

Y~ Exp with mean 2α $\sim \frac{1}{2\alpha} e^{-x/2\alpha}; x > 0$

P(system works beyond α)

P(X >α∩ Y> α)= P(X > α) P(Y > α)

$= \left(\int_\alpha^\infty \frac{1}{\alpha} e^{-\frac{x}{\alpha}} dx\right) * \left(\int_\alpha^\infty \frac{1}{2\alpha} e^{-\frac{x}{2\alpha}} dx\right) = e^{-1} \times e^{-1/2} = e^{-3/2}$

(1) (a) $P(X > 5) = P\left(\frac{X-10}{6} > \frac{5-10}{6}\right) = P\left(Z > -\frac{5}{6}\right);\ Z \sim N(0,1)$

$$= 1 - \phi\left(-\frac{5}{6}\right) = 1 - \left(1 - \phi\left(\frac{5}{6}\right)\right)$$

$$= \phi\left(\frac{5}{6}\right) = 0.7967$$

$$(b)\ P(4 < X < 16) = P\left(\frac{4-10}{6} < Z < \frac{16-10}{6}\right) = P(-1 < Z < 1)$$

$$= \phi(1) - \phi(-1) = 2\phi(1) - 1$$

$$= \cdots$$

$$(c) P(X < 8) = P\left(Z < \frac{8-10}{6}\right) = \phi\left(-\frac{1}{3}\right) = 1 - \phi\left(\frac{1}{3}\right) = \cdots$$

(3) $P(X \leq 0) = \frac{1}{2} = P(X \geq 0) \Longrightarrow \mu = 0$

$$P(-1.96 \leq X \leq 1.96) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \leq \frac{X}{\sigma} \leq \frac{1.96}{\sigma}\right) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \leq Z \leq \frac{1.96}{\sigma}\right) = 0.95; Z \sim N(0,1)$$

$$2\phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.5$$

$$\phi\left(\frac{1.96}{\sigma}\right) = 0.975$$

$$\Longrightarrow \frac{1.96}{\sigma} = \phi^{-1}(0.975) = 1.96 \Longrightarrow \sigma = 1$$

(4) X : lifetime r. v.

X$\sim N(\mu, \sigma^2)$

= $1.4 \times 10^6\ hrs$

$$\sigma^2 = 3 \times 10^5 hrs$$

$$P(X < 1.8 \times 10^6)$$

$$= \left(\frac{X - 1.4 \times 10^6}{3 \times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5}\right)$$

$$= P\left(Z < \frac{4}{3}\right) \ [Z \sim N(0,1)]$$

$$= \phi\left(\frac{4}{3}\right) = 0.918$$

Y: r. v. denoting # of chips that have lifetime < $1.8 \times 10^6 hr$

Y~ Bin(10, 0.918)

$\Longrightarrow$ P(Y≥ 2)= 1- P(Y< 2)

= 1-P(Y= 0) – P(Y = 1)

= 1-$\binom{10}{0}(.918)^{\circ}(1-.918)^{10} - \binom{10}{1}(0.918)^1(1-.918)^9$

$$= \cdots$$

(5) $X \sim \mathrm{N}(0,1)$

$$\forall\ \mathrm{t} > 0 \ \ P(|X| \geq t) = 1 - \mathrm{P}(|X| < t)$$

$$= 1 - \mathrm{P}(-\mathrm{t} < X < t)$$

$$= 1 - [\phi(\mathrm{t}) - \phi(-\mathrm{t})]$$

$$= 1 - [2\phi(\mathrm{t}) - 1]$$

$$= 1 - \left[2\big(1 - \mathrm{p}(\mathrm{X} > t)\big) - 1\right]$$

$$= 2 - 2 + 2\,\mathrm{P}(\mathrm{X} > t) = 2\,\mathrm{P}(\mathrm{X} > t)$$

$$\mathrm{P}(\mathrm{X} > t) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{t}}^{\infty} \mathrm{e}^{\frac{-\mathrm{x}^2}{2}} \mathrm{dx} \ \leq \frac{1}{\sqrt{2\pi}} \int_{\mathrm{t}}^{\infty} \frac{\mathrm{x}}{\mathrm{t}} \mathrm{e}^{\frac{-\mathrm{x}^2}{2}} \mathrm{dx} \ \ [\mathrm{t} < x < \infty]$$

$$\mathrm{y} = \frac{\mathrm{x}^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{t}} \int_{\frac{\mathrm{t}^2}{2}}^{\infty} \mathrm{e}^{-\mathrm{y}}\, \mathrm{dy} = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{\frac{-\mathrm{t}^2}{2}}}{t}$$

$$\Longrightarrow \mathrm{P}(|\mathrm{X}| \geq \mathrm{t}) \leq 2\, \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{\frac{-\mathrm{t}^2}{2}}}{t} = \sqrt{\frac{2}{\pi}}\, \frac{\mathrm{e}^{\frac{-\mathrm{t}^2}{2}}}{t}$$

(6) $X = x$	0	1	2 …….
$P(X = x)$	p_0	p_1	p_2 ……….

$$\sum_{k=0}^{\infty}\big(1-F(k)\big) = \sum_{k=0}^{\infty} P(X>k) = P(X>0) + P(X>1) + P(X>2) + \cdots$$

$$= (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + \cdots) + (p_3 + p_4 + \cdots)$$

$$= p_1 + 2p_2 + 3p_3 + \cdots$$

$$= \sum_{i=1}^{\infty} i\, p_i = \sum_{i=0} i\, P(X=i) = E(X)$$

(7) d. f.

F(x)= $\begin{cases} 0\ , & x<0 \\ 1-e^{-\beta x^2}, & x \geq 0 \end{cases}$

$$\beta > 0$$

$$p.d.f.\, f(x) = \begin{cases} 2\beta\, x e^{-\beta x^2}, & x \geq 0 \\ 0 \ \frac{o}{w} & \end{cases}$$

$$E(X) = 2\beta \int_0^{\infty} x^2 e^{-\beta x^2}\, dx$$

$$y = x^2$$

$$= \beta \int_0^{\infty} y^{\frac{1}{2}} e^{-\beta y} = \beta.\frac{\Gamma \frac{3}{2}}{\beta^{\frac{3}{2}}} = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{3}} = \mu$$

$$EX^2 = 2\beta \int_0^{\infty} x^3 e^{-\beta x^2}\, dx = \beta \int_0^{\infty} y\, e^{-\beta y}\, dy = \beta \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}$$

$$V(X) = E(X^2) - (EX)^2 = \frac{1}{\beta} - \mu^2 = \frac{1}{\beta} - \frac{\pi}{4\beta}$$

$$median : m_0$$

$$m_0 \Longrightarrow F(m_0) = \frac{1}{2} = 1 - F(m_0)$$

$$i.e.\, 2\beta \int_0^{m_0} x\, e^{-\beta x^2} dx = 2\beta \int_{m_0}^{\infty} x e^{-\beta x^2}\, dx = \frac{1}{2}$$

$$i.e.\, 1 - e^{-\beta {m_0}^2} = \frac{1}{2}$$

$$\Longrightarrow m_0 = \cdots$$

(8) $1-\ \phi(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-y^2}{2}}\,dy$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_x^\infty \frac{1}{y}\left(ye^{\frac{-y^2}{2}}\right)dy\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{y}.\left(-e^{\frac{-y^2}{2}}\right)\Bigg|_x^\infty-\int_x^\infty\left(-\frac{1}{y^2}\right)\left(-e^{\frac{-y^2}{2}}\right)dy\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{x}e^{\frac{-x^2}{2}}-\int_0^\infty\frac{1}{y^2}e^{\frac{-y^2}{2}}dy\right]$$

$$\geq 0$$

$$\Longrightarrow 1-\phi(x)\leq\frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}=\frac{\phi(x)}{x}.$$

(9)

Mode- pt at which f(x) is maximum.

$$f^{'}(x)=2\beta\left(xe^{-\beta x^2}\left(-2\beta x+e^{-\beta x^2}\right)\right)$$

$$f^{'}(x)=0\Longrightarrow 2\beta x^2=1\Longrightarrow x=\frac{1}{\sqrt{2\beta}}$$

$$f^{''}(x)=2\beta\frac{d}{dx}\left(e^{-\beta x^2}\left(_{1-2\beta x^2}\right)\right)$$

$$=2\beta\left(e^{-\beta x^2}(-4\beta x)+(1-2\beta x^2)e^{-\beta x^2}(-2\beta x)\ \right)$$

$$f^{''}(x)|_{x=\frac{1}{\sqrt{2\beta}}}\ =2\beta\left(e^{-\frac{1}{2}}\left(-4\sqrt{\frac{\beta}{2}}\right)\right)<0$$

$$\beta>0$$

$$\Longrightarrow m^*, the\ mode\ of\ the\ distn\ is\ at\ \frac{1}{\sqrt{2\beta}}.$$

$$m^*=\frac{1}{\sqrt{2\beta}}$$

$$\mu=E(X)=\frac{\sqrt{\pi}}{2}.\frac{1}{\sqrt{\beta}}=\left(\frac{\sqrt{\pi}}{2}\right)\sqrt{2}\ \ m^*$$

$$i.e.\ \mu = \frac{\sqrt{\pi}}{2} m^*$$

$$\& \ 2{m^*}^2 - \mu^2 = 2\left(\frac{2}{\pi}\mu^2\right) - \mu^2$$

$$= \frac{4}{\pi}\mu^2 - \mu^2 = \frac{\cancel{4}}{\cancel{\pi}}.\frac{\cancel{\pi}}{\cancel{4}}.\frac{1}{\beta} - \mu^2$$

$$i.e.\ 2{m^*}^2 - \mu^2 = \sigma^2 = \frac{1}{\beta} - \mu^2$$

$$= EX^2 - \mu^2 = V(X).$$

(10) X~ P(λ)

p. m. f. $P(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \dots. \\ 0, \quad otherwise. \end{cases}$

$$Y = X^2 - 5 \Longrightarrow range\ space\ of\ Y = \{-5, -4, -1, 4, 11, \dots\} = \mathscr{Y}$$

$$P(Y = y) = P(X^2 - 5 = y) = P(X^2 = y + 5)$$

$$p.m.f.of\ Y: P(Y = y) = P\left(X = \sqrt{y+5}\right) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{\sqrt{y+5}}}{\left(\sqrt{y+5}\right)!}, & y \in \mathscr{Y} \\ 0, & otherwise. \end{cases}$$

(11) X~ B(n, p)

p. m. f. $P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \dots, n \\ 0, \ otherwise \end{cases}$

$$Y = n - x \Longrightarrow \mathscr{Y} = \{0, 1, \dots, n\}.$$

$$P(Y = y) = P(n - X = y) = P(X = n - y)$$

$$\Longrightarrow p.m.f.of\ Y$$

$$P(Y = y) = \begin{cases} \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}; y = 0, 1, \dots, n \\ 0, \ otherwise \end{cases}$$

$$= \begin{cases} \binom{n}{y} (1-p)^y p^{n-y}, y = 0, 1, \dots, n \\ 0, \quad otherwise. \end{cases}$$

$$\Longrightarrow Y \sim B(n, 1-p).$$

(12) $Y = X^2 \rightarrow range\ rp = \{0, 1, 4, 9\}$

$$p.m.f.\,P(Y=y)=\begin{cases} P(X=0) & y=0 \\ P(X=-1)+P(X=1) & y=1 \\ P(X=-2)+P(X=2) & y=4 \\ P(X=3) & y=9 \end{cases}$$

$$=\begin{cases} \frac{1}{5}, & y=0 \\ \frac{1}{6}+\frac{1}{15}, & y=1 \\ \frac{1}{5}+\frac{1}{3}, & y=4 \\ \frac{1}{30}, & y=9 \end{cases}$$

(13) $P(X=x)=\begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{x}, & x=0,1,2,\ldots.. \\ 0, & otherwise \end{cases}$

$$Y=\frac{X}{X+1}\Longrightarrow X=\frac{Y}{1-Y}$$

$$range\ space\ of\ Y=\left\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\right\}$$

$$P(Y=y)=P\left(\frac{X}{X+1}=y\right)=\ P\left(X=\frac{y}{1-y}\right)$$

$$=\begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, y=0,\frac{1}{2},\frac{2}{3},\ldots \\ 0 \quad otherwise. \end{cases}$$

(14) $p.m.f\ of\ X$

$$P(X=x)=\begin{cases} e^{-1}, & x=0 \\ \frac{e^{-1}}{2(|X|)!}, & x\in\{\pm1,\pm2,\ldots\} \\ 0, & otherwise \end{cases}$$

$$Y=|X| \quad \mathcal{Y}=\{0,1,2,\ldots\}$$

$$P(Y=0)=P(X=0)=e^{-1}$$

$$P(Y=1)=P(X=\ -1)+P(X=1)$$

$$=\frac{e^{-1}}{2}+\frac{e^{-1}}{2}=\ e^{-1}$$

$$P(Y=2)=P(X=\ -2)+P(X=2)$$

$$= \frac{e^{-1}}{2.2!} + \frac{e^{-1}}{2.2!} = \frac{e^{-1}}{2}$$

$$sly\ for\ k = 1, 2, \ldots.$$

$$P(Y = k) = P(X = -k) + P(X = k)$$

$$= \frac{e^{-1}}{2.k!} + \frac{e^{-1}}{2.k!} = \frac{e^{-1}}{k!}$$

$$p.m.f.of\ Y$$

$$P(Y = y) = \begin{cases} \dfrac{e^{-1}}{y!}, & y = 0, 1, 2, \ldots. \\ 0, & oterwise \end{cases}$$

$$Y \sim P(1).$$

Problem Set-6

[1] The probability density function of the random variable X is

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

$i.e. X \sim U(0, 1)$. Find the distribution of the following functions of X

(a) $Y = \sqrt{X}$; $(b) Y = X^2$; $(c) Y = 2X + 3$; $(d) Y = -\lambda \log X; \lambda > 0$.

[2] Let X be a random variable with $U(0, \theta), \theta > 0$ $distribution. Find\ the\ distribution\ of\ Y =$ min $\left(X, \frac{\theta}{2}\right)$.

[3] The probability density function of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{2} & -\dfrac{1}{2} \leq x \leq \dfrac{3}{2} \\ 0 & otherwise \end{cases}$$

Find the distribution of Y=X^2 .

[4] The probability density function of X is by

$$f_X(x) = \begin{cases} k\dfrac{x^{p-1}}{(1+x)^{p+q}} & x > 0 \\ 0 \quad therwise, \end{cases}$$

$p, q > 0$. Derive the distribution of Y= $(1+X)^{-1}$.

[5] The probability density function of X is by

$$f_X(x) = \begin{cases} k\, x^{\beta-1}\exp\left(-\alpha x^{\beta}\right) & x > 0 \\ 0 \quad otherwise, \end{cases}$$

$$\alpha, \beta > 0. Derive\ the\ distribution\ of\ Y = X^{\beta}$$

[6] According to the Maxwell-Boltzmann law of theoretical physics, the probability density function of V, the velocity of a gas molecule, is

$$f_V(v) = \begin{cases} k\, v^2\exp(-\beta v^2)\ v > 0 \\ 0 \quad otherwise, \end{cases}$$

$where\ \beta$> 0 is a constant which depends on the mass and absolute temperature of the molecule and k > 0 is a normalizing constant. Derive the distribution of the kinetic energy E= $mV^2/2$.

[7] The probability density function of the random variable X is

$$f_X(x) = \begin{cases} \dfrac{3}{8}(x+1)^2 & -1 < x < 1 \\ 0 \quad otherwise. \end{cases}$$

Find the distribution of the following functions of Y= 1- X^2.

[8] Let X be a random variable with U(0, 1) distribution. Find the distribution function of Y= min (X, 1- X) and the probability density function of Z= (1- Y)/ Y.

[9] Suppose X~ N$(\mu, \sigma^2), \mu \in \Re, \sigma \in \Re^+$. Find the distribution of 2X – 6.

[10] Let X be a continuous random variable on (a, b) with p. d. f. f and c. d. f. F. Find the p. d. f. of Z= -log (F(X)).

[11] Let X be a continuous r. v. having the following p. d. f.

$$f(x)= \begin{cases} 6x\,(1-x) \quad if\ 0 \le x \le 1 \\ 0 \quad if\ otherwise. \end{cases}$$

Derive the distribution function of X and hence find the p. d. f. of Y= $X^2(3-2X)$.

[12] Let X be a distributed as double exponential with p. d. f. f(x) $=\frac{1}{2}e^{-|x|}; x \in \Re. Find\ the\ p.d.f.of\ Y = |X|$

[13] 3 balls are placed randomly in 3 boxesB_1, B_2 $and B_3$. Let N be the total number of boxes which are occupied an X_i be the total number of balls in the boxB_i, i= 1, 2, 3. Find the joint p. m. f. of (N, X_1) and (X_1, X_2). Obtain the marginal distributions of N, $X_1 and$ X_2 from the joint p. m. f. s.

[14] The joint p. m. f. of X and Y is given by

p(x, y)= $\begin{cases} c\ xy\ \ if\ (x, y) \in \{(1, 1), (2, 1), (2, 2), (3, 1)\} \\ \quad\quad 0 \quad otherwise. \end{cases}$

Find the constant c, the marginal p. m. f. of X and Y and the conditional p. m. f. of X given Y= 2.

[15] The joint p. m. f. of X and Y is given by

p(x, y)= $\begin{cases} \frac{(x+2y)}{18} \quad if\ (x, y) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\} \\ \quad\quad 0 \quad otherwise \end{cases}$

(a) Find the marginal distributions.
(b) Verify whether X and Y are independent random variables.
(c) Find P(X< Y), P(X+ Y > 2).
(d) Find the conditional p. m. f. of Y given X= x, x= 1, 2.

[16] 5 cards are drawn at random without replacement from a deck of 52 playing cards. Let the random variables X_1, X_2, X_3 denote the number of spades, the number of hearts, the number of diamonds, respectively, that appear among the five cards. Find the joint p. m. f. of X_1, X_2, X_3. Also determine whether the 3 random variables are independent.

[17] Consider a sample of size 3 drawn with replacement from an urn containing 3 white , 2 black and 3 red balls. Let the random variables$X_1,$ and X_2 denote the number of white balls and number of black balls in the sample, respectively. Determine whether the two random variables are independent.

[18] Let $X = (X_1, X_2, X_3)^T$ $be\ a\ random\ vector\ with\ joint\ p.m.f.$

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{4}\ (x_1, x_2, x_3) \in \\ 0\ \ otherwise. \end{cases}$$

X= {(1, 0, 0), (0, 1, 0), (0, 0, 1),(1, 1, 1)}. Show that X_1, X_2, X_3 are pair wise independent but are not mutually independent.

Solution Key

(1) X~ U(0, 1)

$$f_X(x) = \begin{cases} 0 \quad x < U \\ x \quad 0 \leq x \leq 1 \\ 1 \quad x > 1 \end{cases}$$

$$solution\ usins\ d.f.method$$

$$(a) Y = \sqrt{X}$$

$$d.f.of\ Y: f_Y(y) = P\left(\sqrt{X} \le y\right)$$

$$= P(X \le y^2) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

$$p.d.f.of\ Y is\ f_Y(y) = \begin{cases} 0, & y < 0 \\ 2y, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases} = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & otherwise. \end{cases}$$

$$(b) Y = X^2$$

$$d.f.of\ Y: f_Y(y) = P(X^2 \le y) = \begin{cases} 0, & y < 0 \\ P\left(-\sqrt{y} \le x \le \sqrt{y}\right), & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

$$for\ 0 \le y \le 1$$

$$P\left(-\sqrt{y} \le X \le \sqrt{y}\right) = P\left(0 \le X \le \sqrt{y}\right) = F_X\left(\sqrt{y}\right) = \sqrt{y}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

$$p.d.f\ of\ Y: f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & otherwise. \end{cases}$$

$$(c)\ Y = 2X + 3 \rightarrow (3,5)$$

$$d.f.of\ Y: F_Y(y) = P(2X + 3 \le y)$$

$$= P\left(X \le \frac{y-3}{2}\right) = \begin{cases} 0, & y < 3 \\ \frac{y-3}{2}, & 3 \le y \le 5 \\ 1, & y > 5 \end{cases}$$

$$p.d.f.f_Y(y) = \begin{cases} \frac{1}{2}, & 3 \le y \le 5 \\ 0 & otherwise. \end{cases} \Rightarrow Y \sim U(3,5)$$

$$(d)\ Y = -\lambda \log X \rightarrow (0, \infty)$$

$$F_Y(y) = P(Y \le y) = P(-\lambda \log X \le y) = P\left(X > e^{-\frac{y}{\lambda}}\right)$$

$$= 1 - P\left(X \le e^{-\frac{y}{\lambda}}\right)$$

$$i.e. F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\frac{y}{\lambda}} & y \geq 0 \end{cases}$$

$$p.d.f. of\ Y: f_Y(y) = \begin{cases} \frac{1}{\lambda} e^{-\frac{y}{\lambda}}, y \geq 0 \\ 0 \quad otherwise \end{cases}$$

$$i.e. Y \sim Exp(\lambda)(scale\ \lambda)$$

(2) X~$U(0,\theta)$

$$Y = \min\left(X, \frac{\theta}{2}\right) \rightarrow \left(0, \frac{\theta}{2}\right) \leftarrow range\ sp\ of\ Y$$

$$F_Y(y) = P(Y \leq y) = P\left(\min\left(X, \frac{\theta}{2}\right) \leq y\right)$$

$$= 1 - p\left(\min\left(X, \frac{\theta}{2}\right) > y\right)$$

$$= 1 - p\left(X > y \cap \frac{\theta}{2} > y\right)$$

$$now\ P\left(X > y, \frac{\theta}{2} > y\right) = 1\ \ if\ y < 0$$

$$= 0 \quad if\ y \geq \frac{\theta}{2}$$

$$for\ 0 \leq y < \frac{\theta}{2}; P\left(X > y, \frac{\theta}{2} > y\right) = P(X > y) = \frac{1}{\theta}\int_y^\theta dx = \frac{\theta - y}{\theta}$$

$$\Longrightarrow F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{\theta}, & 0 \leq y < \frac{\theta}{2} \\ 1, & y \geq \frac{\theta}{2} \end{cases}$$

$$Note: - F_Y(y)\ has\ a\ jump\ discontiuity\ at\ \frac{\theta}{2}$$

(3) $f_X(x) = \begin{cases} \frac{1}{2}, & \frac{1}{2} \leq x \leq \frac{3}{2} \\ 0 & otherwise \end{cases}\ X \sim U\left(-\frac{1}{2}, \frac{3}{2}\right)$

$$Y = X^2 \rightarrow y \in \left(0, \frac{9}{4}\right)$$

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$for\ u < 0;\ F_Y(y) = 0$$

$$\& \ y > \frac{9}{4};\ F_Y(y) = 1$$

$$for, 0 \le y \le \frac{1}{4};\ F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx = \sqrt{y}$$

$$for, \frac{1}{4} < y < \frac{9}{4};\ F_Y(y) = \int_{-\sqrt{y}}^{-\frac{1}{2}} 0.dx + \int_{-\frac{1}{2}}^{\sqrt{y}} \frac{1}{2}\ dx = \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right)$$

$$= \left(\frac{1}{4} + \frac{\sqrt{y}}{2}\right)$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le \frac{1}{4} \\ \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right), & \frac{1}{4} \le y \le \frac{9}{4} \\ 1, & y \ge \frac{9}{4} \end{cases}$$

$$p.d.f.\ f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 \le y \le \frac{1}{4} \\ \frac{1}{4\sqrt{y}}, & \frac{1}{4} < y \le \frac{9}{4} \end{cases}$$

(4) $f_X(x) = \begin{cases} k\frac{x^{p-1}}{(1+x)^{p+q}}, & x > 0 \\ 0 & otherwise. \end{cases}$

$$Y = \frac{1}{1+X} \Rightarrow X = \frac{1-Y}{Y} = g^{-1}(Y); Y \in (0,1)$$

$$J = \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right) |J|, & 0 \le y \le 1 \\ 0 & otherwise. \end{cases}$$

$$= \begin{cases} k.\left(\frac{1-y}{y}\right)^{p-1} \left(\frac{1}{y^{-1}}\right)^{p+q} .\frac{1}{y^2}, & 0 \le y \le 1 \\ 0, & otherwise. \end{cases}$$

$$i.e.\ f_Y(y) = \begin{cases} k.y^{q-1}(1-y)^{p-1}, & 0 \le x \le 1 \\ 0, & otherwise. \end{cases}$$

$$k = \left(Beta(q,p)\right)^{-1}$$

$$\Rightarrow Y \sim Beta\ (q,p)$$

(5) $f_X(x) = \begin{cases} k\ x^{\beta-1}e^{-\alpha x^{\beta}}, & x > 0 \\ 0, otherwise \end{cases}$

$$Y = x^{\beta}\ \ J = \frac{dx}{dy} = \frac{1}{\beta} y^{\frac{1}{\beta}-1} \left(x = y^{\frac{1}{\beta}} = g^{-1}(y)\right)$$

$$f_Y(y) = \begin{cases} f_X g^{-1}(y)|J|, & y > 0 \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} k.\left(y^{\frac{1}{\beta}}\right)^{\beta-1} e^{-\alpha y}\left(\frac{1}{\beta}.y^{\frac{1}{\beta}-1}\right), & y > 0 \\ 0, & otherwise. \end{cases}$$

$$= \begin{cases} ky^{1-\frac{1}{\beta}}e^{-\alpha y}\,\frac{1}{\beta}.y^{\frac{1}{\beta}-1}, & y > 0 \\ 0 & otherwise. \end{cases}$$

$$= \begin{cases} k\,\frac{e^{-\alpha y}}{\beta}, & y > 0 \\ 0, & otherwise \end{cases}$$

$$k = \alpha\beta \Longrightarrow Y \sim Exp\left(\frac{1}{\alpha}\right).$$

(6) $f_V(v) = \begin{cases} kv^2e^{-\beta v}, & v > 0 \\ 0\ , & otherwise \end{cases}$

$$E = \frac{1}{2}mV^2;\ V^2 = \frac{2E}{m}.$$

$$\frac{\partial e}{\partial v} = mv$$

$$J = \frac{\partial v}{\partial e} = \frac{1}{\sqrt{2me}}.$$

$$f_E(e) = \begin{cases} k.\left(\frac{2e}{m}\right)e^{-\beta\left(\frac{2e}{m}\right)}.\frac{1}{\sqrt{2me}}; & e > 0 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} c.e^{\frac{1}{2}}\exp\left(-\frac{2\beta}{m}e\right), & e > 0 \\ 0, & otherwise \end{cases}$$

$$c\ is\ \ni \int_0^\infty C.e^{\frac{1}{2}}\exp\left(-\frac{2\beta}{m}e\right)de = 1$$

$$i.e.\,C.\frac{\Gamma\frac{3}{2}}{\left(\frac{2\beta}{m}\right)^{\frac{3}{2}}} = 1 \Longrightarrow C = \frac{\left(\frac{2\beta}{m}\right)^{\frac{3}{2}}}{\Gamma\frac{3}{2}}$$

$$\Longrightarrow E \sim Gamma\ (\ldots)$$

(7) $f_X(x) = \begin{pmatrix} \frac{3}{8}(x+1)^2\ , & -1<x<1 \\ 0, & otherwise \end{pmatrix}$

$$Y = 1 - X^2, y \in (0,1)$$

$$x^2 = 1 - y \Longrightarrow x = \pm\sqrt{1-y}$$

$$x \in (-1,0) \to x = -\sqrt{1-y} = g^{-1}{}_1(y) \to \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{1-y}}$$

$$x \in (0,1) \to x = \sqrt{1-y} = g_2{}^{-1}(y) \to \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{1-y}}$$

$$(-1,0)\quad(-1,0)\qquad(0,1)\qquad(-1,0)$$

$$\downarrow\qquad\downarrow\qquad\qquad\downarrow\qquad\qquad\downarrow$$

$$f_Y(y) = f_X\left(g_1{}^{-1}(y)\right)\left|\frac{dx}{dy}\right| + f_X\left(g_2{}^{-1}(y)\right)\left|\frac{dx}{dy}\right|. \qquad 0 < y < 1$$

$$= \frac{3}{8}\left(1-\sqrt{1-y}\right)^2 . \frac{1}{2\sqrt{1-y}} + \frac{3}{8}\left(1+\sqrt{1-y}\right)^2 . \frac{1}{2\sqrt{1-y}}$$

$$= \frac{3}{16\sqrt{1-y}}\left(\left(1-\sqrt{1-y}\right)^2 + \left(1+\sqrt{1-y}\right)^2\right)$$

$$= \frac{3}{16\sqrt{1-y}}\left(2\left(1+(1-y)\right)\right)$$

$$i.e. f_Y(y) = \begin{cases} \frac{3}{8}\left((1-y)^{-\frac{1}{2}} + (1-y)^{\frac{1}{2}}\right), & 0 < y < 1 \\ 0, & otherwise. \end{cases}$$

(8) Y= min (X, 1- X) → range of Y(0, ½)

$$F_Y(y) = P(Y \le y) = P(\min(X, 1-X) \le y) = 1 - P(\min(X, 1-X) > y)$$

$$= 1 - P(X > y, 1 - X > y)$$

$$= 1 - P(X > y, 1 - y > X)$$

$$= 1 - P(y < X < 1 - y)$$

$$P(y < x < 1 - y) = \begin{cases} 1 & y \le 0 \\ \int_y^{1-y} dx & if\ 0 < y < \frac{1}{2} \\ 0 & y \ge \frac{1}{2} \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y \le 0 \\ 2y & 0 < y < \frac{1}{2} \\ 1 & y \ge \frac{1}{2} \end{cases} \Rightarrow p.d.f. f_Y(y) = \begin{cases} 2, & 0 < y < \frac{1}{2} \\ 0, & otherwise \end{cases}$$

$$Z = \frac{1-Y}{Y} = \frac{1}{Y} - 1 \rightarrow range\ of\ Z\ is\ (1, \infty)$$

$$F_Z(3) = P(Z \le 3)$$

$$if\ 3 \le 1, then\ F_Z(3) = 0$$

$$if\ 3 > 1, then\ P(Z \le 3) = P\left(\frac{1}{Y} - 1 \le 3\right) = P\left(\frac{1}{Y} \le 3 + 1\right)$$

$$= P\left(Y \ge \frac{1}{3+1}\right) = 1 - P\left(y < \frac{1}{3+1}\right)$$

$$= 1 - \frac{2}{3+1} (using\ d.f. of\ Y)$$

$$\Rightarrow F_Z(3) = \begin{cases} 0, & if 3 \le 1 \\ 1 - \dfrac{2}{3+1}, & if\ 3 > 1 \end{cases}$$

$$\Rightarrow p.d.f.of\ Z\ is\ f_Z(3) = \begin{cases} \dfrac{2}{(3+1)^2}, & 3 > 1 \\ 0 & otherwise \end{cases}$$

(9) X~ N(μ, σ^2)

$$Y = 2X - 6; y \ \in (-\infty, \infty)$$
$$F_Y(y) = P(Y \le y) = P(2X - 6 \le y)$$
$$= P\left(X \le \frac{y+6}{2}\right)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{\frac{y+6}{2} - \mu}{\sigma}\right) = \ \phi\left(\frac{y+6-2\lambda}{2\sigma}\right)$$
$$f_Y(y) = \ \phi\left(\frac{y+6-2\lambda}{2\sigma}\right).\frac{1}{2\sigma}.y \ \in (-\infty, \infty)$$
$$= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}} \left(\frac{y-(2\mu-6)}{2\sigma}\right)^2 . \frac{1}{2\sigma} = \frac{1}{\sqrt{2\pi}(2\sigma)} \exp\left(-\frac{1}{2(4\sigma^2)}\big(y-(2\mu-6)\big)^2\right)$$
$$\Rightarrow Y \sim N(2\mu - 6, 4\sigma^2)$$

(10)

$$X \sim f_X(x)$$
$$Z = \ -\log F(X); \quad 3 \in (0, \infty)$$
$$3 = \ -\log F(X) \Rightarrow x = \ F^{-1}(e^{-z}) \left|\frac{\partial z}{\partial x}\right| = \left(\frac{f(x)}{F(x)}\right) \Rightarrow |J| = \frac{F(x)}{f(x)}$$
$$p.d.f.of\ Z = f\big(F^{-1}(e^{-z})\big).\frac{F\big(F^{-1}(e^{-z})\big)}{f\big(F^{-1}(e^{-z})\big)}$$
$$= F\big(F^{-1}(e^{-z})\big) = e^{-z};$$
$$\Rightarrow \ f_Z(z) = \begin{cases} e^{-3}, & 3 > 0 \\ 0, & otherwise \end{cases}$$

(11)

$$f_X(x) = \ \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0 & therwise \end{cases}$$

$$F_X(x) = P(X \le x) = \ \begin{cases} 0, & x < 0 \\ \displaystyle\int_0^x (6y - 6y^2)\, dx, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ x^2(3-2x), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$F_X(X) = X^2(3 - 2X)$$

If X $\sim F_X(x)\ disth\ function, them\ Y = F(X) \sim U(0,1)$

↓

[X $\sim f_X(x)\ p.d.f.\&\ d.f.F.$]

↑

Cont r. v.   Y= F(X)   →y ∈ (0, 1)[General result as in prob. 10]

X= $F^{-1}(y)$

$$\frac{dy}{dx} = f(x) \quad \left|\frac{dx}{dy}\right| = \left|\frac{1}{f(x)}\right|$$

$$\Longrightarrow p.d.f.of\ Y : f_Y(y) = \begin{cases} \dfrac{f_X\left(F^{-1}(y)\right)}{f_X\left(F^{-1}(y)\right)} = 1 \quad if\ 0 < y < 1 \\ 0 \quad otherwise \end{cases}$$

$$\Longrightarrow Y \sim U(0,1)$$

$$\Longrightarrow For\ the\ given\ distn\ X^2(3-2X) = F(X) \sim U(0,1)$$

(12) X~ Double exponential

$$f_X(x) = \frac{1}{2}e^{-|x|};\ -\infty < x < \infty$$

$$Y = |X| range\ of\ Y : (0, \infty)$$

$$x \in (-\infty, 0) \to x = -y \to \left|\frac{dx}{dy}\right| = 1$$

$$x \in (0, \infty) \to x = y \to \left|\frac{dx}{dy}\right| = 1$$

$$\Longrightarrow f_Y(y) = \underset{\substack{\uparrow \\ \text{In } (-\infty, 0)}}{f_X\left(g_1{}^{-1}(y)\right)|J|} + f_X\left(g_2{}^{-1}(y)\right)|J|$$

$$i.e. f_Y(y) = \begin{cases} \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y}, \quad 0 < y < \infty \\ 0, \quad otherwise \end{cases}$$

$$\therefore p.d.f.of\ Y : f_Y(y) = \begin{cases} e^{-y}, \quad 0 < y < \infty \\ 0 \quad otherwise. \end{cases}$$

(13) $X_i = 0, 1, 2, 3\ for\ i = 1, 2, 3$

$$N = 1, 2, 3$$

Possible configurations with 3 boxes and 3 balls

B_1	B_2	B_3
3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0

→

→ each with prob. = $\frac{1}{\binom{3+3-1}{3}} = \frac{1}{10}$

↓N	X_1	X_2	X_3
1	3	0	0
1	0	3	0
1	0	0	3
2	2	1	0
2	2	0	1
2	1	2	0
2	0	2	1
2	1	0	2
2	0	1	2
3	1	1	1

0	2	1
1	0	2
0	1	2
1	1	1

$jt\ p.m.f.of\ (N, X_1)$

N \ X_1	0	1	2	3
1	$\frac{2}{10}$	0	0	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	0
3	0	$\frac{1}{10}$	0	0
	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

Marginal of X_1

$\frac{1}{10}$

$jt\ p.m.f.of\ (N, X_1)$

X_1 \ X_2	0	1	2	3
1	$\frac{2}{10}$	0	0	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	0
3	0	$\frac{1}{10}$	0	0

$\frac{4}{10}$ $\frac{3}{10}$ $\frac{2}{10}$

Marginal of X_2

(14)

$$\sum_{(x,y)} p(x,y) = C\sum_{(x,y)}(xy) = 1$$

$$\Longrightarrow C[(1,1) + (2,1) + (2,2) + (3,1)] = 1$$

$$\Longrightarrow C = \frac{1}{10}$$

$jt\ p.m.f.$

X \ Y	1	2	
1	$\frac{1}{10}$	0	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{6}{10}$ } marginal of X
3	$\frac{3}{10}$	0	$\frac{3}{10}$
	$\frac{6}{10}$	$\frac{4}{10}$	

Marginal of Y

Conditional p. m. f. of X given Y= 2, $\frac{p(x,2)}{p_Y(2)} = 1 \quad if\ x = 2$

$= 0 \quad otherwise.$

(15) $jt\ p.m.f.$

Y \ X	1	2	
1 2	$\frac{3}{18}$ $\frac{4}{18}$	$\frac{5}{18}$ $\frac{6}{18}$	$\frac{8}{18}$ $\frac{10}{18}$ } marginal of X
	$\frac{7}{18}$	$\frac{11}{18}$	

$\leftrightarrow$

Marg of Y

(b)P(X= 1, Y=1)= $\frac{3}{18} \neq P(X=1), P(Y=1) = \frac{8}{18} . \frac{7}{18}$

$\Longrightarrow X\ \&\ Y\ maginal$

$$(c) P(X < Y) = P(X = 1, Y = 2) = \frac{5}{18}$$

$$P(X + Y > 2) = P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 2, Y = 2) = \frac{15}{18}$$

$$(d) marg\ of\ X{:}\ p_X(x) = P(X = x) = \frac{x+3}{9}; x = 1, 2$$

$$p_{Y|X=x} = \frac{\frac{1}{18}(x + 2y)}{\frac{1}{18}(2x + 6)} = \frac{x + 2y}{2x + 6}; y = 1, 2$$

(16) $if\ p.m.f.of\ X_1, X_2, X_3$

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{\binom{13}{x_1}\binom{13}{x_2}\binom{13}{x_3}\binom{13}{5-x_1-x_2-x_3}}{\binom{52}{5}};\ x_i \geq 0\ \&\ \sum_1^3 x_i \leq 5$$

$$p_{X_i}(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}\ x = 0, 1, 2, 3, 4, 5$$

$p_{X_1}(x_1)p_{X_2}(x_2)p_{X_3}(x_3) \neq p_{X_1,X_2,X_3}(x_1, x_2, x_3)$

$\Longrightarrow (X_1, X_2, X_3)\ are\ not\ indep.$

(17) $X_1{:}\ \#\ of\ white\ balls$

$X_2{:}\ \#\ of\ black\ balls.$

$W, 2\ B\ , 1\ R - 7$

$$= p_{X_1,X_2}(x_1,x_2) = \frac{3!}{x_1!\,x_2!\,(3-x_1-x_2)!}\left(\frac{3}{8}\right)^{x_1}\left(\frac{2}{8}\right)^{x_2}\left(\frac{3}{8}\right)^{3-x_1-x_2} ;\ x_i \geq 0, x_1 + x_2 \leq 3$$

$$(X_1, X_2) \sim Mult\left(3, \frac{3}{8}, \frac{2}{8}\right)$$

$$p_{X_1}(x_1) = \binom{3}{x_1}\left(\frac{3}{8}\right)^{x_1}\left(\frac{5}{8}\right)^{3-x_1} ; x_1 = 0, 1, 2, 3$$

$$p_{X_2}(x_2) = \binom{3}{x_2}\left(\frac{2}{8}\right)^{x_1}\left(\frac{6}{8}\right)^{3-x_2} ; x_2 = 0, 1, 2, 3$$

$$i.e.\ X_1 \sim B\left(3, \frac{3}{8}\right);\ X_2 \sim B\left(3, \frac{2}{8}\right)$$

$$p_{X_1}(x_1)p_{X_2}(x_2) \neq p_{X_1,X_2}(x_1,x_2)$$

$$\Longrightarrow\ X_1 \& X_2\ are\ not\ indep.$$

(18) From the jt p. m. f. of X_1, X_2

$$P(X_1 = 0, X_2 = 0) = P(X_1 = 0, X_2 = 1) = P(X_1 = 1, X_2 = 0) = P(X_1 = 1, X_2 = 1) = \frac{1}{4}$$

$$Further\ (X_1, X_2) \equiv (X_1, X_3) \equiv (X_2, X_3)$$

$$\&\ P(X_i = 0) = \frac{1}{2} = P(X_i = 1); i = 1, 2, 3$$

$$\Longrightarrow\ X_1, X_2, X_3\ are\ pair\ wise\ indep.$$

$$But\ P(X_1 = 0, X_2 = 0, X_3 = 0) = \frac{1}{4} \neq P(X_1 = 0)P(X_2 = 0)P(X_3 = 0) = \frac{1}{8}$$

$$\Longrightarrow\ X_1, X_2, X_3\ are\ not\ indep.$$

Problem Set-7

[1] The joint p. d. f. of (X, Y) is given by f(x, y)= $\begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise. \end{cases}$

Find the probability p. d. f. s and verify whether the random variables are independent. Also find

P(0<X< ½ , ¼ <Y < 1), P(X+ Y <1)

[2] If the joint p. d. f. of (X, Y) f(x, y)= $\begin{cases} e^{-(x+y)} & 0 < x, y < \infty \\ 0 & otherwise, \end{cases}$

Show that X and Y are independent.

[3] If the joint p. d. f. of (X, Y) is f(x, y)= $\begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & otherwise, \end{cases}$

Show that X and Y are not dependent.

[4] Show that the random variables X and Y with joint p. d. f.

f(x, y)= $\begin{cases} 12xy(1-y) & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise. \end{cases}$

Are independent.

[5] Suppose the joint p. d. f. of (X, Y) is f(x, y)= $\begin{cases} cx^2y & 0<x<y<1 \\ 0 & otherwise. \end{cases}$

Find (a) the value of the constant c, (b) the marginal p. d. f. s of X and Y and (c) P(X+ Y ≤ 1).

[6] The joint p. d. f. of (X, Y) is given by f(x, y)= $\begin{cases} 6(1-x-y) & x>0, y>0, x+y<1 \\ 0 & otherwise. \end{cases}$

Find the marginal p. d. f. of X and Y and P(2X + 3Y < 1).

[7] The joint p. d. f. of (X, Y) is f(x, y)= $\begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & otherwise. \end{cases}$

Find the conditional distribution of Y given X= x, 0< x<1; the conditional mean and conditional varience of the conditional distribution.

[8] Suppose the conditional p. d. f. of X given Y = y is f(x|y)= $\begin{cases} \frac{cx}{y^2} & 0<x<y \\ 0 & otherwise. \end{cases}$

Further, the marginal distribution of Y is g(y)= $\begin{cases} dy^4 & 0<y<1 \\ 0 & otherwise. \end{cases}$

(a) Find the constants c and d.
(b) The p. d. f. of (X, Y).
(c) P(0.25 < X < 0.5) and P(0.25 < X < 0.5 |Y= 0.625)

[9] Let f(x) and g(y) be two arbitrary p. d. f. s with corresponding distribution functions F(x) and G(y) respectively. Suppose the joint p. d. f. of X and Y is given by

h(x, y)= f(x) g(y)[1+$\alpha\{2F(x)-1\}\{2G(y)-1\}$], $|\alpha| \leq 1$.

Show that the marginal p. d. f. of X and Y are f(x) and g(y), respectively. Does there exist a value of α for which the random variables X and Y are independent?

[10] Suppose the marginal density of the random variable is $f_X(x) = \begin{cases} 4x(1-x^2), & 0<x<1 \\ 0, & otherwise. \end{cases}$

And the conditional density of the random variable Y given X = x is

$$f_{Y|X=x}(y|x) = \begin{cases} \frac{2y}{(1-x^2)}, & x<y<1, 0<x<1 \\ 0, & otherwise. \end{cases}$$

Find the conditional p. d. f. of X given Y= y, E(X|Y= ½) and Var (X| Y = ½).

[11] The joint p. d. f. of (X, Y) f(x, y)= $\begin{cases} e^{-(x+y)} & 0<x,y<\infty \\ 0 & otherwise. \end{cases}$

Find the joint m. g. f. of (X, Y) and the m. g. f. of Z= X+ Y are have hence V(Z).

[12] Derive the joint m. g. f. of
$(X_1, X_2) \sim N_2(\mu_1, \mu_2, {\sigma_1}^2, {\sigma_2}^2, \rho) and\ using\ the\ joint\ m.g.f.find\ \rho(X_1, X_2)$.

[13] Let the joint p. d. f. of (X, Y) be f(x, y)= $\begin{cases} 2 & 0<x<y<1 \\ 0 & otherwise, \end{cases}$

Find the conditional mean and conditional variance of X given Y= y and that of Y given X= x. Compute further $\rho(X, Y)$.

[14] Let X, Y and Z be three random variables and a and b be two scalar constants. Prove that (a) Cov(X, b)= Cov(Y, b)= $Cov(Z, b) = 0$; $(b) Cov(X, aY + b) = a\, Cov(X, Y)$; $(c) Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$; $(d) \rho(X, aY + b) = \rho(X, Y)$ for a>0.

[15] Let $X_1, X_2, and X_3$ be three independent random variables each with a variance $\sigma^2. Define\ the\ new\ random\ variables$

$$W_1 = X_1, W_2 = \frac{\sqrt{3}-1}{2} X_1 + \frac{3-\sqrt{3}}{2} X_2 \ and\ W_3 = \left(\sqrt{2}-1\right) X_2 + \left(2-\sqrt{2}\right) X_3. Find\ \rho(W_1, W_2), \rho(W_1, W_3) and\ \rho\ (W_2, W_3).$$

[16] Let (X, Y)~$N_2(3, 1, 16, 25, 0.6)$. Find (a)P(3< Y< 8); (b) P(3 < y< 8| X= 7); (c) P(-3 <X <3) and (d) P(-3 <X < 3| Y= 4).

[17] Let (X, Y) ~ $N_2(5, 10, 1, 25, \rho) with\ \rho > 0. If\ it\ is\ given\ that\ P(4 < y < 16 | X = 5) = 0.954\ and\ \phi(2) = 0.977$, find the value of ρ.

[18] Let $X_1, X_2, \dots, X_{20}$ be independent random variables with identical distributions, each with a mean 2 and variance 3. Define Y= $\sum_{i=1}^{15} X_i\ \ and\ Z = \sum_{i=11}^{20} X_i$. Find E(Y), E(Z), V(Y), V(Z) and ρ(Y, Z).

[19] Let X and Y be a jointly distributed random variables with E(X)= 15, E(Y)= 20, V(X)= 25, V(Y)= 100 and ρ(X, Y)= -0.6. Find ρ(X- Y, 2X – 3Y).

[20] suppose that the lifetime of light bulbs of a certain kind follows exponential distribution with p. d. f.

$$f_X(x) = \begin{cases} \frac{1}{50} e^{-\frac{x}{50}} & x > 0 \\ 0 & otherwise. \end{cases}$$

Find the probability that among 8 such bulbs, 2 will last less that 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere 60 and 80 hours and 1 will last for more than 80 hours. Find the expected number of bulbs in a lot of 8 bulbs with lifetime between 60 and 80 hours and also the expected number of bulbs in a lot of 8 with lifetime 60 and 80 hours, given that the number of bulbs with lifetime anywhere between 40 and 60 hours is 2.

[21] Let the random variables X and Y have the following joint p. m. f. s

(a) P(X= x, Y= y)= 1/3, if (x, y)∈{(0, 0), (1, 1), (2, 2)} and 0 otherwise.

(b) P(X= x, Y= y)= 1/3 , if (x, y)∈ {(0, 2), (1, 1), (2, 0)} and 0 otherwise.

(c) P(X= x, Y= y)= 1/3 , if (x, y) ∈ {(0, 0), (1, 1), (2, 0)} and 0 otherwise.

In each of the above cases find the coefficient of correlation between X and Y.

[22] The joint p. m. f. of (X, Y) is

P(X= x, Y= y)= xy/10, if (x, y)∈ {(1, 1), (2, 1), (2, 2), (3, 1)} and 0 otherwise.

Find the joint m. g. f. of X and Y and the coefficient of correlation between X and Y. Using the joint m. g. f. , find the p. m. f. Z= X+ Y.

[23] Let $M_{X,Y}(u,v)$ *denote the joint m.g.f.* (X,Y) *and* $\Psi(u,v) = \log\left(M_{X,Y}(u,v)\right)$. *Show that*

$\frac{\partial \Psi(u,v)}{\partial u}|_{u=v=0}, \frac{\partial \Psi(u,v)}{\partial v}|_{u=v=0}, \frac{\partial^2 \Psi(u,v)}{\partial u^2}|_{u=v=0}, \frac{\partial^2 \Psi(u,v)}{\partial v^2}|_{u=v=0}$ *and* $\frac{\partial^2 \Psi(u,v)}{\partial u \partial v}|_{u=v=0}$ yeilds the mean, the variance and the covariance of the two random variables.

$f_{X,Y}(x,y) = \frac{1}{2}\left(f_\rho(x,y) + f_{-\rho}(x,y)\right); \; -\infty < x, y < \infty$

Where, $f_\rho(x,y)$ is the probability density function of $N_2(0,0,1,1,\rho)$ *and* $f_{-\rho}(x,y)$ is the probability density function of $N_2(0,0,1,1,-\rho)$. Find the marginal p. d. f. s of X and Y, the correlation coefficient between X and Y. Are the 2 variables independent?

[25] Let the joint p. d. f. of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} k, & if -x < y < x; 0 < x < 1 \\ 0, & otherwise. \end{cases}$$

Find the value of the constant k and obtain the conditional expectations E(X|Y= y) and E(Y| X= x). Verify whether the 2 random variables are independent and / or uncorrelated.

[26] The joint moment generating function of X and Y is given by

$$M_{X,Y}(s,t) = \{a(e^{s+t}+1) + b(e^s + e^t)\}, a, b > 0; a + b = \frac{1}{2}.$$

Find the correlation coefficient between X and Y.

[27] Let X and Y be jointly distributed random variables with

E(X)= E(Y)= 0, E$(X^2) = E(Y^2) = 2$ *and* $\rho(X,Y) = \frac{1}{3}$

$$Find\ \rho\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right).$$

Solution Key

(1) $jt\ p.d.f.of\ (X,Y)$

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise. \end{cases}$$

$$Marginal\ of\ X:$$

$$f_X(x) = \int_0^1 4xy\,dy = 2x \quad 0 < x < 1$$

$$= 0 \quad otherwise.$$

$$Sly\ f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & otherwise \end{cases}$$

$$observe\ that f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$\Longrightarrow X\ \&\ Y\ arew\ indep.$$

$$P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < 1\right) = P\left(0 < X < \frac{1}{2}\right) P\left(\frac{1}{4} < Y < 1\right)$$

$$= \left(\int_0^{\frac{1}{2}} 2x\,dx\right)\left(\int_{\frac{1}{4}}^{1} 2y\,dy\right) = \cdots$$

$$P(X+Y<1) = \int_0^1 P(X<1-y)\,f_Y(y)dy \;\rightarrow X\;\&\;Y\;are\;indep.$$

$$= \int_0^1 \left[\int_0^{1-y} 2x\,dx\right] 2y\,dy$$

$$= \cdots$$

(2) $f_X(x) = \int_0^\infty e^{-x}e^{-y}\,dy = e^{-x} \quad x > 0$

$$= 0 \quad otherwise$$

$$f_Y(y) = e^{-y} \quad y > 0$$

$$= 0 \quad otherwise$$

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$$

$$\Longrightarrow X\;\&\;Y\;are\;indep.$$

(3) $f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy$

$$= 2e^{-x}\,e^{-x} = 2e^{-2x} \quad x > 0$$

$$= 0 \quad otherwise$$

$$sly\;f_Y(y) = 2\int_0^y e^{-y}\;e^{-x}dx = 2\,e^{-y}(1-\,e^{-y}) \;\; y > 0$$

$$= 0 \quad otherwise$$

$$f(x,y) \neq f(x)f(y)$$

$$\Longrightarrow X\;\&\;Y\;are\;not\;idep.$$

(4) $f_X(x) = 12x\,\int_0^1 (y-y^2)\,dy = 12x\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_0^1$

$$\Longrightarrow f_X(x) = \begin{cases} 2x & 0<x<1 \\ 0 & otherwise \end{cases}$$

$$f_Y(y) = 12y(1-y)\int_0^1 x\,dx = 6\,y(1-y) \quad 0<y<1$$

$$= 0 \;\; otherwise.$$

$$f(x,y) = f(x)f(y)$$

$$\Longrightarrow X\;\&\;Y\;are\;indep.$$

(5) $\int_0^1\int_x^1 f(x,y)\,dy\,dx = 1, i.e.\, C\int_0^1 x^2\int_x^1 y\,dy\,dx = 1$

$$\Longrightarrow C\int_0^1 x^2\,\frac{1}{2}(1-x^2)dx = 1$$

$$\Longrightarrow \frac{c}{2}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]\begin{matrix}1\\0\end{matrix} = 1 \Longrightarrow C = 15$$

(b) $f_X(x) = 15x^2\int_x^1 y\;\;dy = \begin{cases} \frac{15}{2}x^2(1-x^2), & 0<x<1 \\ \quad 0 & otherwise \end{cases}$

$$f_Y(y) = 15y\int_0^y x^2\,dx = \begin{cases} 5y^4, & 0<y<1 \\ 0, & otherwise \end{cases}$$

$$(c)P(X+Y\leq 1) = \iint_{\substack{x+y\leq 1 \\ x<y}} 15\,x^2y\,dy\,dx$$

$$= 15\int_0^{\frac{1}{2}} x^2\int_x^{1-x} y\;\;dy\,dx = 15\int_0^{\frac{1}{2}} x^2\,\frac{y^2}{2}\Big|\begin{matrix}1-x\\x\end{matrix}\,dx$$

$$= \cdots = \frac{15}{192}.$$

$$Alt\ P(X+Y \le 1) = \underset{\substack{x+y\le 1\\ x<y}}{\int\int} 15\, x^2 y\, dy\, dx$$

$$= 15\int_0^{\frac{1}{2}} y \int_0^{y} x^2\, dx\, dy + 15\int_{\frac{1}{2}}^{1} y \int_0^{1-y} x^2\ dx\, dy$$

$$= \cdots = \frac{15}{15\times 32} + \frac{15}{10\times 32} = \frac{15}{192}$$

(6) $f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\, dy = 6\int_0^{1-x} (1-x-y)\, dy = 6\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x}$

$$= \begin{cases} 3(1-x)^2, & 0<x<1 \\ 0, & otherwise \end{cases}$$

$$sly\ (by\ symmetry)$$

$$f_Y(y) = \begin{cases} 3(1-y)^2, & 0<y<1 \\ 0, & otherwise \end{cases}$$

$$P(2X+3Y<1) = 6\int_0^{\frac{1}{2}}\int_0^{\frac{1-2x}{3}} (1-x-y)\, dy\, dx$$

$$= 6\int_0^{\frac{1}{2}} \left[(1-x)y - \frac{y^2}{2}\right]_0^{\frac{1-2x}{3}} dx$$

$$= 6\int_0^{\frac{1}{2}} \left\{(1-x)\left(\frac{1-2x}{3}\right) - \frac{1}{2}\left(\frac{1-2x}{3}\right)^2\right\} dx$$

$$= 6\int_0^{\frac{1}{2}} \left(\frac{1+2x^2-3x}{3} - \frac{1+4x^2-4x}{18}\right) dx$$

$$= 6\int_0^{\frac{1}{2}} \frac{8x^2-14x+5}{18}\, dx$$

$$= \frac{6}{18}\left(8\frac{x^3}{3} - 14\frac{x^2}{2} + 5x\Big|_0^{\frac{1}{2}}\right)$$

$$= \frac{6}{18}\left(\frac{\cancel{8}}{3}.\frac{1}{\cancel{8}} - 7.\frac{1}{4} + \frac{5}{2}\right) = \frac{13}{36}$$

$$Alt\ \ P(2X+3Y<1) = 6\int_0^{\frac{1}{3}}\int_0^{\frac{1-3y}{2}} (1-y-x)\, dx\, dy$$

$$= 6\int_0^{\frac{1}{3}} \left((1-y)x - \frac{x^2}{2}\Big|_0^{\frac{1-3y}{2}}\right) dy$$

$$= 6\int_0^{\frac{1}{3}} \left[(1-y)\left(\frac{1-3y}{2}\right) - \frac{1}{2}\left(\frac{1-3y}{2}\right)^2\right] dy$$

$$= \cdots = \frac{13}{36}$$

(7) $f_X(x) = \int_0^1 f(x,y)\ dy = \int_0^1 (x+y)\, dy$

$$= \begin{cases} x+\dfrac{1}{2} & 0<x<1 \\ 0 & therwise \end{cases}$$

$$f_{Y|X} = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{(2x+1)} \quad 0<y<1$$

$$E(Y|X) = \int_0^1 y\frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (xy+y^2)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{2}+\frac{1}{3}\right) = \frac{2(3x+2)}{6(2x+1)} = \frac{3x+2}{6x+3}$$

$$E(Y^2|X) = \int_0^1 y^2\frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (y^2x+y^3)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{3}+\frac{1}{4}\right) = \frac{2(4x+3)}{12(2x+1)} = \frac{4x+3}{6(2x+1)}$$

$$V(Y|X) = E(Y^2|X) - E^2(Y|X)$$

$$= \frac{4x+3}{6(2x+1)} - \left(\frac{3x+2}{3(2x+1)}\right)^2 = \cdots$$

(8) f(x, y)= f(x| y) g(y)= c d x $y^2; 0<x<y,\ 0<y<1$

$$\int_0^1 g(y)\,dy = 1 \Rightarrow d\int_0^1 y^4\,dy = 1 \Rightarrow d = 5$$

$$\Rightarrow f(x,y) = 5\,c\,xy^2; 0<x<y<1$$

$$\Rightarrow 5c\int_0^1 y^2\int_0^y x\,dx\,dy = 1 \Rightarrow \frac{5c}{2}\int_0^1 y^4\,dy = 1$$

$$\Rightarrow c = 2$$

$$\Rightarrow f(x,y) = 10x\,y^2; 0<x<y<1$$

$$= 0 \quad otherwise.$$

$$f_X(x) = 10x\int_0^1 y^2\,dy \quad 0<x<1$$

$$f_X(x) = \begin{cases} \dfrac{10}{3}x(1-x^3) & 0<x<1 \\ 0 & otherwise \end{cases}$$

$$P(0.25<X<0.5) = \frac{10}{3}\int_{\frac{1}{4}}^{\frac{1}{2}}(x-x^4)\,dx = \cdots$$

$$P\left(\frac{1}{4}<X<\frac{1}{2}\middle|Y=0.625\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_{X|Y=y}\,dx$$

$$= 2\int_{\frac{1}{4}}^{\frac{1}{2}}\frac{x}{(0.625)^2}\,dx = \frac{2}{(0.625)^2}\frac{1}{2}\left(\left(\frac{1}{2}\right)^2-\left(\frac{1}{4}\right)^2\right) = \cdots$$

(9) Marginal of X from h (x, y)

$$f_X(x) = \int_{-\infty}^{\infty} h(x,y)\,dy = \int_{-\infty}^{\infty} f(x)\,g(y)\{1+\alpha[2\,F(x)-1][2G(y)-1]dy\}$$

$$f(x) = \int_{-\infty}^{\infty} g(y)\,dy + f(x)\alpha[2F(x)-1]\int_{-\infty}^{\infty} g(y)(2G(y)-1)\,dy$$

$$= f(x)\times 1 + f(x)\alpha[2F(x)-1]\int_{-\infty}^{\infty} g(y)(2G(y)-1)\,dy$$

$$\int_{-\infty}^{\infty} g(y)\,(2G(y)-1)dy \overset{?}{=} \int_0^1 (2u-1)du = 2/\frac{u^2}{\cancel{2}} - u\Big| \begin{matrix}1\\0\end{matrix} = 0$$

$$\Longrightarrow f_X(x) = f(x) + 0$$

$$sly\ f_Y(y) = \int_{-\infty}^{\infty} h(x,y)\,dx = g(y)$$

$$h(x,y) = f_X(x).f_Y(y) = f(x)g(y)\ \ iff\ \alpha = 0$$

(10)

$$f_{X,Y}(x,y) = f_{Y|X=x}(y|x)\, f_X(x)$$

$$= \begin{cases} 8xy, & 0<x<y<1 \\ 0 & otherwise \end{cases}$$

$$Marginal\ p.d.f.of\ Y$$

$$f_Y(y) = \begin{cases} 8y \int_0^y x\,dx = 4y^3, & 0<y<1 \\ 0, & otherwise \end{cases}$$

$$conditional\ p.d.f\ of\ X\ given\ Y$$

$$f_{X|Y=y}(x|y) = \begin{cases} \frac{8xy}{4y^3} = \frac{2x}{y^3}, & 0<x<y; 0<y<1 \\ 0, & otherwise \end{cases}$$

$$E(X|Y=y) = \frac{2}{y^2}\int_0^y x^2\,dx = \frac{2}{y^2} - \frac{y^3}{3} = \frac{2y}{3}$$

$$\Longrightarrow E\left(X\middle|Y=\frac{1}{2}\right) = \frac{1}{3}$$

$$E(X^2|Y=y) = \frac{2}{y^2}\int_0^y x^3\,dx = \frac{2}{y^2}.\frac{y^4}{4} = \frac{y^2}{2}$$

$$\Longrightarrow E\left(X^2\middle|Y=\frac{1}{2}\right) = \frac{1}{8}$$

$$V(X|Y=y) = E(X^2|Y=y) - E^2(X|Y=y)$$

$$= \frac{1}{8} - \frac{1}{9} = \frac{1}{72}.$$

(11) Jt . m. g. f.

$$M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1+t_2X_2}) = \int_0^{\infty}\int_0^{\infty} e^{t_1x_1+t_2x_2}\, e^{-(x_1+x_2)}dx_2dx_1$$

$$= \int_0^{\infty} e^{-x_2(1-t_1)}\,dx_1 \int_0^{\infty}\int_0^{\infty} e^{-x_2(1-t_2)}\,dx_2$$

$$= (1-t_1)^{-1}\,(1-t_2)^{-1}\ \ if t_1,t_2 < 1$$

$$Note: since\ \ X_1 \& X_2\ are\ indep, we\ write$$

$$M_{X_1,X_2}(t_1,t_2) = M_{X_1}(t_1)M_{X_2}(\,t_2)$$

$$m.g.f.of\ Z = X_1 + X_2$$

$$M_Z(t) = E\left(e^{t(X_1+X_2)}\right) = (1-t)^{-2}, t<1$$

$$E(z) = \frac{\partial M_Z(t)}{dt}\Big|_{t=0} = 2(1-t)^{-3}\Big|_{t=0} = 2$$

$$E(z^2) = \frac{\partial^2 M_Z(t)}{dt^2}\Big|_{t=0} = 6(1-t)^{-4}\Big|_{\,t=0} = 6 \implies V(z) = 2$$

(12)
$$M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1+t_2X_2})$$
$$= EE\left(e^{t_1X_1+t_2X_2}|_{X_1}\right) = E\left(e^{t_1X_1}E\left(e^{t_2X_2}|_{X_1}\right)\right)$$
$$since\ X_2 \Big| X_1 \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right), \sigma_2{}^2(1-\rho^2)\Big)$$
$$E\left(e^{t_2X_2}|_{X_1}\right) \rightarrow condition\ at\ m.g.f.of\ X_2\ given\ X_1$$
$$M_{X_1,X_2}(t_1,t_2) = \left(e^{t_1X_2}\left(e^{t_2\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)} + \frac{t_2{}^2}{2}\sigma_2{}^2(1-\rho^2)\right)\right)$$
$$= e^{t_2\mu_2+\frac{t_2{}^2}{2}\sigma_2{}^2(1-\rho^2)}E\left(e^{t_1X_1+t_2\rho\frac{\sigma_2}{\sigma_1}X_1}\right)e^{-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}$$
$$= e^{t_2\mu_2+\frac{t_2{}^2}{2}\sigma_2{}^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}XE\left(e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)X_1}\right)$$
$$= e^{t_2\mu_2+\frac{t_2{}^2}{2}\sigma_2{}^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)\mu_1} + \frac{\sigma_1{}^2}{2}\left(t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}\right)$$
$$= \exp\left(t_2\mu_2 + \frac{t_2{}^2}{2}\sigma_2{}^2(1-\cancel{\rho}^2) - t_2\rho\frac{\cancel{\sigma_2}}{\sigma_1}\mu_1 + t_1\mu_1 + t_2\rho\frac{\cancel{\sigma_2}}{\sigma_1}\right.$$
$$\left. + \frac{\sigma_1{}^2}{2}\left(t_1{}^2 + t_2{}^2\cancel{\rho}^2\frac{\sigma_2{}^2}{\sigma_1{}^2} + 2t_1\,t_2\rho\frac{\sigma_2}{\sigma_1}\right)\right)$$
$$= \exp\left(t_2\mu_2 + \frac{t_2{}^2}{2}\sigma_2{}^2 + t_1\mu_1 + \frac{t_1{}^2}{2}\sigma_1{}^2 + t_1\,t_2\rho\sigma_1\sigma_2\right)$$
$$= \exp\left(t_1\mu_1 + t_2\mu_2 + \frac{1}{2}(t_1{}^2\sigma_1{}^2 + t_2{}^2\sigma_2{}^2 + 2t_1\,t_2\sigma_1\sigma_2\rho)\right)$$
$$\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_1}\Big|_{t_1=0,t_2=0} = \mu_1\ sly\ \ \frac{\partial M_{X_1,X_2}}{\partial t_2}\Big|_{t_1=0,t_2=0} = \mu_2 \,\&\, V(X_1) = \sigma_1{}^2, V(X_2) = \sigma_2{}^2$$
$$E(X_1,X_2) = \frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1\,\partial t_2}\Big|_{t_1=0,t_2=0} = \rho\sigma_1\sigma_2 + \mu_1\mu_2$$
$$\implies Cov(X_1,X_2) = (\rho\sigma_1\sigma_2 + \mu_1\mu_2) - \mu_1\mu_2 = \rho\sigma_1\sigma_2$$
$$\implies corv\,(X_1,X_2) = \rho$$

(13)
$$\text{f(x, y)}=\begin{cases} 2, & 0 < x < y < 1 \\ 0, & otherwise \end{cases}$$
$$f_X(x) = \int_x^1 2\,dy = 2(1-x),\ \ 0 < x < 1$$
$$= 0\ \ otherwise$$
$$f_Y(y) = \int_0^y 2\,dx = 2y, 0 < y < 1$$
$$= 0\ \ otherwise$$
$$f_{Y|X=x} = \begin{cases} \dfrac{2}{2(1-x)}, & x < y < 1 \\ 0 & otherwise \end{cases}$$

$$f_{X|Y=y} = \begin{cases} \frac{2}{2y}, & 0 < x < y \\ 0 & otherwise \end{cases}$$

$$E(Y|X) = \int_x^1 y \frac{1}{1-x} dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

$$E(Y^2|X) = \frac{\int_x^1 y^2\, 1}{1-x} dy = \frac{1-x^3}{3(1-x)}.$$

$$\Longrightarrow V(Y|X) = E(Y^2|X) - E^2(Y|X) = \frac{1-x^3}{3(1-x)} - \frac{1+x}{2}$$

$$sly\ E(X|Y), E(X^2|Y) and\ hence\ V(X|Y).$$

(14) (a) $Cov(X,b) = E\big(X - E(X)\big)\big(b - E(b)\big) = 0 = Cov(X,b) = ZCov(Z,b)$

$$(b) Cov(X, aY+b) = E\big(X - E(X)\big)\big(aY + b - E(aY+b)\big)$$
$$= E\big(X - E(X)\big)\big(aY + \cancel{b} - aE(Y) - \cancel{b}\big)$$
$$= a\ Cov(X,Y)$$
$$(c) Cov(X, Y+Z) = E\big(X - E(X)\big)\big(Y + Z - E(Y) - E(Z)\big)$$
$$= E\big(X - E(X)\big)\big[Y - E(Y) + \big(z - E(z)\big)\big]$$
$$= Cov(X,Y) + Cov(X,z)$$
$$(d) Cov(X, aY+b) = a\ cov(X,Y)$$
$$cov\ (X, aY+b) = \frac{cov(X, aY+b)}{[V(X)\,V(aY+b)]^{\frac{1}{2}}} = \frac{a\ cov(x,Y)}{[V(X)a^2V(Y)]^{\frac{1}{2}}} = cov(X,Y)$$

(15) $cov(w_1, w_2) = Cov\left(X_1, \frac{\sqrt{3}-1}{2} X_1 + \frac{3-\sqrt{3}}{2} X_2\right) = \frac{\sqrt{3}-1}{2} V(X_1) + \frac{3-\sqrt{3}}{2} cov(X_1, X_2) = \frac{\sqrt{3}-1}{2} {\sigma_2}^2$

$$V(w_1) = \sigma^2 \,\&\, V(w_2) = \left(\frac{\sqrt{3}-1}{2}\right)^2 \sigma^2 + \left(\frac{3-\sqrt{3}}{2}\right)^2 \sigma^2 = \big(\sqrt{3}-1\big)^2 \sigma^2$$

$$\Longrightarrow \rho_{w_1,w_2} = \frac{1}{2}$$

$$sly\ \ \rho_{w_1,w_3} \,\&\, \rho_{w_2,w_3}$$
$$\downarrow$$
$$cov(w_1, w_3) = cov\big(X_1, \big(\sqrt{2}-1\big)X_2 + \big(2-\sqrt{2}\big)X_3\big) = 0$$
$$\Longrightarrow \rho_{w_1,w_3} = 0$$

(16) (a) $P(3 < Y < 8)\quad Y \sim N(1, 25)$

$$= P\left(\frac{3-1}{5} < \frac{Y-1}{5} < \frac{8-1}{5}\right) = \phi\left(\frac{7}{5}\right) - \phi\left(\frac{2}{5}\right)$$
$$= \cdots (from\ table)$$

$$(b) P(3 < Y < 8\ |X=7) \left[Y \middle| X \sim N\left(1 + P\frac{5}{4}(x-3), 25(1-p^2)\right)\right]$$
$$= P\left(\frac{3-4}{4} < \frac{Y-4}{4} < \frac{8-4}{4} \middle| X = 7\right)$$
$$= \phi(1) - \phi(-0.25)$$
$$= \cdots$$

$$(c) P(-3 < X < 3)\quad X \sim N(3, 16)$$
$$= P\left(\frac{-3-3}{4} < \frac{X-3}{4} < \frac{3-3}{4}\right) = \phi(0) - \phi\left(-\frac{6}{4}\right) = \cdots$$

$$(d) P(-3 < X < 3 | Y = 4) \quad \left[X \middle| Y \sim N\left(3 + p\frac{4}{5}(y-1), 16(1-p^2)\right)\right]$$

$$= P\left(\frac{-3-4.44}{3.2} < \frac{X-4.44}{3.2} < \frac{3-4.44}{3.2} \middle| Y = 4\right) = \phi\left(-\frac{1.44}{3.2}\right) - \phi\left(-\frac{7.44}{3.2}\right) = \cdots$$

(17)

$$(X, Y) \sim N_2(5, 10, 1, 25, \rho); \rho > 0$$

$$Y | X = 5 \sim N_2\left(10, 25(1-\rho^2)\right)$$

$$P(4 < Y < 16 | X = 5) = P\left(\frac{4-10}{5\sqrt{1-\rho^2}} < \frac{Y-10}{5\sqrt{1-\rho^2}} < \frac{16-10}{5\sqrt{1-\rho^2}} \middle| X = 5\right)$$

$$= \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \phi\left(-\frac{6}{5\sqrt{1-\rho^2}}\right)$$

$$= 2\phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1 = 0.954 \; (given\ condition)$$

$$\Rightarrow \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = 0.977 = \phi(2)$$

$$\Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow 1 - \rho^2 = 0.36 \Rightarrow \rho = 0.8 \; (as\ \rho > 0)$$

(18)

$$E(Y) = \textstyle\sum_1^{15} E(X_i) = 30$$

$$V(Y) = 15\, V(X_i) = 45\, ; V(Z) = 10 \times 3 = 30$$

$$cov(Y, Z) = cov\left(\sum_1^{15} X_i, \sum_{11}^{10} X_i\right) = 5V(X_i) = 15$$

$$\rho_{Y,Z} = \frac{15}{[45 \times 30]^{\frac{1}{2}}}$$

(19)

$$U = X - Y; \;\; V = 2X - 3Y$$

$$E(U) = -5$$

$$E(V) = 2 \times 15 - 3 \times 20 = -30$$

$$V(U) = V(X) + V(Y) - 2cov(X, Y)\; V(v) = 4V(X) + 9V(Y) - 12\, cov(X, Y)$$

$$Now \rho_{X,Y} = -0.6 = \frac{cov(X, Y)}{\sqrt{25 \times 100}} = \frac{cov(X, Y)}{50} \Rightarrow cov(X, Y) = -30$$

$$\Rightarrow V(U) = 185 \;\&\; V(v) = 100 + 900 + 360 = 1360$$

$$\Rightarrow cov(U, V) = cov(X - Y, 2X - 3Y)$$

$$= 2V(X) - 3\, cov(X, Y) - 2cov(Y, X) + 3V(Y) = 2 \times 25 - 5(-30) + 30 \times 100 = 500$$

$$\Rightarrow \rho_{U,V} = \frac{500}{\sqrt{185 \times 1360}}$$

(20)

X : r. v. denoting life time

$$X \sim Exp(50)\; p.d.f.\, f(x) = \begin{cases} \frac{1}{50} e^{-\frac{x}{50}}, & x > 0 \\ 0 & otherwise \end{cases}$$

$$F_X(x) = 1 - e^{-\frac{x}{50}}, \qquad x > 0$$

$$Y_1 : \#\ of\ buibs\ out\ of\ 8\ to\ have\ lifetime < 40$$

$$Y_2 : \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \geq 40 \& < 60$$

$$Y_3 : \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \geq 60 \;\&\; \leq 80$$

$$Y_4 : \ldots\ldots\ldots\ldots\ldots\ldots\ldots . > 80$$

$$P(X < 40) = F_X(40) = 1 - e^{-\frac{40}{50}} = p_1, say$$

$$P(40 \le X < 60) = F_X(60) - F_X(40) = e^{-\frac{40}{50}} - e^{-\frac{60}{50}} == p_2, say$$

$$P(60 \le X \le 80) = F_X(80) - F_X(60) = e^{-\frac{60}{50}} - e^{-\frac{80}{50}} = p_3, say$$

$$P(X > 80) = 1 - p_1 - p_2 - p_1 = e^{-\frac{80}{50}}$$

$$jt. p. m. f. of\ Y_1, Y_2, Y_3\ is\ maltinomial\ (8, p_1, p_2, p_3)$$

$$\Longrightarrow P(Y_1 = 2, Y_2 = 3, Y_3 = 2) = \frac{8!}{2!\,3!\,22!\,1!} {p_1}^2 {p_2}^3 {p_3}^2 (1 - p_1 - p_2 - p_3)$$

$$marginal\ distribution Y_3 \sim Bin\left(8, p_3 = e^{-\frac{60}{50}} - e^{-\frac{80}{50}}\right)$$

$$E(Y_3) = 8\left(e^{-\frac{60}{50}} - e^{-\frac{80}{50}}\right)$$

$$conditional\ Y_3 | Y_2 = y_2 \sim Bin\left(8 - y_2, \frac{p_3}{1 - p_2}\right)$$

$$\Longrightarrow E(Y_3 | Y_2 = 1) = (8 - 1)\left(\frac{e^{-\frac{60}{50}} - e^{-\frac{80}{50}}}{1 - e^{-\frac{40}{50}} - e^{-\frac{60}{50}}}\right).$$

(21) (a)

Y \ X	0	1	2	
0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
2	0	0	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$E(X) = 1 = E(Y)$$

$$V(X) = E(X^2) - 1$$

$$= \frac{5}{3} - 1 = \frac{2}{3} = V(Y)$$

$$E(XY) = (0 \times 0)\frac{1}{3} + (1 \times 1)\frac{1}{3} + (2 \times 2) \times \frac{1}{3} = \frac{5}{3}$$

$$cov(X, Y) = \frac{5}{3} - 1 = \frac{2}{3}$$

$$\rho_{X,Y} = 1$$

(b)

Y \ X	0	1	2
0	0	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0
2	$\frac{1}{3}$	0	0

$sly \implies \rho_{X,Y} = -1$

(c)

Y \ X	0	1	2
0	0	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0
2	$\frac{1}{3}$	0	0

$\rho_{X,Y} = 0.$

(22)

Y \ X	1	2	
1	$\frac{1}{10}$	0	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{6}{10}$
3	$\frac{3}{10}$	0	$\frac{3}{10}$
	$\frac{6}{10}$	$\frac{4}{10}$	

$$E(X) = \frac{1}{10} + 2\frac{6}{10} + 3\ \frac{3}{10} = \frac{22}{10}$$

$$E(Y) = \frac{6}{10} + \ 2\ \frac{4}{10} = \frac{14}{10}$$

$$E(X^2) = \frac{1}{10} + 4\ \frac{6}{10} + 9\ \frac{3}{10} = \frac{52}{10}$$

$$E(Y^2) = \frac{6}{10} + 4\ \frac{4}{10} = \frac{22}{10}$$

$$V(X) = \frac{52}{10} - \left(\frac{22}{10}\right)^2 = \cdots$$

$$V(Y) = \left(\frac{22}{10}\right)^2 - \left(\frac{14}{10}\right)^2 = \cdots$$

$$E(XY) = (1 \times 1)\ \frac{1}{10} + (2 \times 1)\frac{2}{10} + (2 \times 2)\frac{4}{10} + (3 \times 1)\frac{3}{10} = \frac{30}{10} = 3$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = 3 - \frac{22}{10}.\frac{14}{10} = \cdots$$

$$cov(X,Y) = \frac{cov(X,Y)}{[V(X)V(Y)]^{\frac{1}{2}}} = \cdots$$

$$jt\ m.g.f.of\ (X,Y)$$

$$M_{X,Y}(t_1,t_2) = \sum_{x,y} e^{t_1x+t_2y}\ \rho(X=x,Y=y)$$

$$= e^{t_1+t_2} \times \frac{1}{10} + e^{2t_1+t_2}\frac{2}{10} + e^{2(t_1+t_2)}\frac{4}{10} + e^{3t_1+t_2}.\frac{3}{10}.$$

(23)

$$M_{X,Y}(u,v) = E(e^{uX+vY})$$

$$= \Psi(u,v) = l0gM_{X,Y}(u,v)$$

$$\frac{\partial\Psi(u,v)}{\partial u} = \frac{1}{M_{X,Y}(u,v)}.\frac{\partial M_{X,Y}(u,v)}{\partial u}$$

$$\frac{\partial\Psi(u,v)}{\partial u}|_{u=0,v=0} = \frac{1}{M(0,0)}.E(X) = E(X)$$

$$sly\ \frac{\partial\Psi(0,0)}{\partial v} = \frac{\partial\Psi(u,v)}{\partial v}|_{u=0,v=0} = E(Y)$$

$$\frac{\partial^2\Psi(u,v)}{\partial u^2} = \frac{1}{M(u,v)}\ \frac{\partial^2 M(u,v)}{\partial u^2} + \left[\frac{-1}{\left(M(u,v)\right)^2}\frac{\partial M(u,v)}{\partial u}\right]\left(\frac{\partial M(u,v)}{\partial u}\right)$$

$$= \frac{1}{M(u,v)}\ \frac{\partial^2 M(u,v)}{\partial u^2} - \left(\frac{\partial M(u,v)}{\partial u}.\frac{1}{M(u,v)}\right)^2$$

$$\frac{\partial^2\Psi(u,v)}{\partial u^2}|_{u=0,v=0} = E(X^2) - E^2(X) = V(X) = \frac{\partial^2\Psi(0,0)}{\partial u^2}$$

$$sly\frac{\partial^2\Psi(u,v)}{\partial u^2}|_{u=0,v=0} = V(Y)$$

$$\frac{\partial^2\Psi(u,v)}{\partial v\ \partial u} = \frac{1}{M(u,v)}.\frac{\partial^2 M(u,v)}{\partial v\ \partial u} - \frac{1}{\left(M(u,v)\right)^2}.\frac{\partial M(u,v)}{\partial v}.\frac{\partial M(u,v)}{\partial u}$$

$$\frac{\partial^2\Psi(u,v)}{\partial v\ \partial u}|_{u=0,v=0} = E(XY) - E(X)E(Y)$$

$$i.e.\frac{\partial^2\Psi(u,v)}{\partial v\ \partial u}|_{u=0,v=0} = cov(X,Y).$$

(24)

$$Marginal\ p.d.f.of\ X$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}\ (x,y)dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f_\rho\ (x,y)dy + \frac{1}{2}\int_{-\infty}^{\infty} f_{-\rho}\ (x,y)dy$$

$$= \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x)\ [\phi(x)\ p.d.f.of\ N(0,1)]$$

$$= \phi(x) \Longrightarrow X \sim N(0,1)$$

$$sly\ f_Y(y) = \phi(y) \Longrightarrow Y \sim N(0,1)$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ \ f_{X,Y}(x,y)dx\ dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f_\rho(x,y)dx\ dy + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f_{-\rho}(x,y)dx\ dy$$

$$= \frac{1}{2}(\rho) + \frac{1}{2}(-\rho) = 0$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = 0 - 0.0 = 0$$

$$\rho(X,Y) = 0 \Longrightarrow X\ \&\ Y\ are\ uncrrelated$$

$$since, f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y).$$
$$X \,\&\, Y \text{ are not independent }.$$

(25) $$\int_0^1 \int_{-x}^{x} k\ dy\, dx = 1 \Longrightarrow k \int_0^1 2x\, dx = 1 \Longrightarrow k = 1$$

$$Marginal\ of\ X, f_X(x) = \int_{-x}^{x} k\, dy = \begin{cases} 2x, & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

$$Marginal\ of\ Y, f_Y(y) = \int_{|y|}^{1} dx = \begin{cases} 1-|y|, & -1 < y < 1 \\ 0 & otherwise \end{cases}$$

$$conditional\ distribution\ of\ Y|X=x; f_{Y|X=x} = \begin{cases} \frac{1}{2x}, & -x < y < x \\ 0, & otherwise \end{cases}$$

$$E(Y|X=x) = \int_{-x}^{x} y\ \frac{1}{2x}\, dy\ = 0$$

$$sly\ \ E(X|Y=y) = \int_{|y|}^{1} x\ \frac{1}{1-|y|}\, dx = \frac{1-y^2}{2(1-|y|)}.$$

$$f_{X|Y=y} = \begin{cases} (1-|y|)^{-1}, |y| < x < 1 \\ 0 \quad otherwise \end{cases}$$

$$f_{X,Y}(x,y) = 1 \neq\ f_X(x)\, f_Y(y)$$
$$\Longrightarrow X \,\&\, Y \text{ are not indep.}$$

$$E(XY) = \int_0^1 \int_{-\infty}^{\infty} xy\, dy\, dx = 0$$
$$E(Y) = E.E(Y|X) = 0$$
$$\Longrightarrow cov(X,y) =\ \rho_{X,Y} = 0$$
$$\Longrightarrow X \,\&\, Y \text{ are uncorrelated.}$$

(26) $$M_{X,Y}(s,t) = \{a(e^{s+t}+1) + b(e^s + e^t)\},\ \left(a, b > 0, a + b = \frac{1}{2}\right)$$

$$E(X) = \frac{\partial}{\partial s}(a(e^{s+t}+1) + b(e^s+e^t))|_{s=t=0}$$
$$= a\, e^t\, e^s + b\, e^s|_{t=s=0} = a + b = \frac{1}{2} = E(Y)$$
$$E(X^2) = \frac{\partial^2}{\partial s^2}\left(a(e^{s+t}+1) + b(e^s+e^t)\right)|_{s=t=0}$$
$$= a\, e^t\, e^s + b\, e^s|_{s=t=0} = a + b = \frac{1}{2} = E(Y^2)$$
$$V(X) = V(Y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$
$$E(XY) = \frac{\partial^2}{\partial t\, \partial s}\ \left(a(e^{s+t}+1) + b(e^s+e^t)\right)|_{s=t=0}$$
$$=\ a\, e^t\, e^s|_{s=t=0} = a$$
$$\therefore cov(X,Y) = a - \frac{1}{4} \Longrightarrow \rho_{X,Y} = \frac{a - \frac{1}{4}}{\frac{1}{4}} = 4a - 1.$$

(27) $$var\left(\frac{X}{3} + \frac{2Y}{3}\right)\left(= var\left(\frac{2X}{3} + \frac{Y}{3}\right)\right) \leftarrow \because V(X) = V(Y)$$
$$= \frac{1}{9}V(X) + \frac{4}{9}V(Y) +\ 2\, cov\left(\frac{X}{3}, \frac{2Y}{3}\right)$$
$$= \frac{2}{9} + \frac{8}{9} + \frac{4}{9} \times \frac{2}{3} = \frac{2}{9} + \frac{8}{9} + \frac{8}{27} = \frac{38}{27}$$

$$cov\left(\frac{X}{3}+\frac{2Y}{3},\frac{2X}{3}+\frac{Y}{3}\right)$$
$$=\frac{2}{9}V(X)+\frac{1}{9}cov(X,Y)+\frac{4}{9}cov(X,Y)+\frac{2}{9}V(Y)$$
$$=\frac{4}{9}+\frac{2}{27}+\frac{8}{27}+\frac{4}{9}=\frac{34}{27}$$
$$cov\ \left(\frac{X}{3}+\frac{2Y}{3},\frac{2X}{3}+\frac{Y}{3}\right)$$
$$=\frac{\frac{34}{27}}{\frac{38}{27}}=\frac{34}{38}.$$

Problem Set-8

[1] The joint probability mass function of the random variables X_1 and X_2 is $given$ by

$$P(X_1 = x_1, X_2 = x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2} \left(\frac{1}{3}\right)^{2-x_1-x_2} & if\ (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\ 0 \quad otherwise. \end{cases}$$

(a) Find the joint probability mass function of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.
(b) Find the marginal probability mass functions of Y_1 and Y_2.
(c) Verify whether Y_1 and Y_2 are independent.

[2] Let the joint probability mass function of X_1 and X_2 be

$$P(X_1 = x_1, X_2 = x_2) = \begin{cases} \frac{x_1 x_2}{36} & if\ x_1 = 1, 2, 3;\ x_2 = 1, 2, 3; \\ 0\ otherwise. \end{cases}$$

(a) Find the joint probability mass function of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.
(b) Find the marginal probability mass functions of Y_1
(c) Find the probability mass function of Z=$X_1 + X_2$.

[3] (a) Let X$\sim$ $Bin(n_1, p) and\ Y \sim Bin(n_2, p) be\ independent\ random\ variables. Find\ the\ conditional\ distribution\ of\ X\ given\ X + Y = t, t \in \{0, 1, \dots, \min(n_1, n_2)\}$.

(b) Let X$\sim Bin\left(n_1, \frac{1}{2}\right) and Y \sim Bin\left(n_2, \frac{1}{2}\right)$ be independent random variables.

Find the distribution of Y=$X_1 - X_2 + n_2$.

[4] Let X$\sim$ Poisson $(\lambda_1) and\ Y \sim Poisson(\lambda_2)\ be\ independent\ random\ variables. Find\ the\ conditional\ distribution\ of\ X\ given\ X + Y = t, t \in \{0, 1, \dots\}$.

[5] Let X_1, X_2, X_3 and X_4 be four mutually independent random variables each having probability density function

f(x)= $\begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & otherwise. \end{cases}$

Find the probability density functions of Y= min($X_1, X_2, X_3, X_4) and\ Z = (X_1, X_2, X_3, X_4)$.

[6] Suppose $X_1, \dots, X_n$ are n independent random variables, where $X_i (i = 1, \dots, n)$ has the exponential distribution Exp(α_1), $with\ probability\ density\ function$

$$f_{X_i}(x) = \begin{cases} \alpha_i e^{-\alpha_i x} & x > 0 \\ 0 & otjherwise. \end{cases}$$

Find the probability density functions of Y= $\min(X_1, \ \ldots., X_n)$ $and\ Z = \max(X_1, \ \ldots., X_n)$.

[7] Let X and Y be the respective arrival times of two friends A and B who agree to meet at a spot and wait for the other only for t minutes. Supposing that X and Y are i. i. d. Exp (λ). Show that the probability of A and B meeting each other is $1 - e^{-\lambda t}$.

[8] Let $X_1\ and\ X_2\ be\ i.i.d.U(0,1)$. Define two new random variables as $Y_1 = X_1 + X_1 and\ Y_2 = X_2 - X_1$. Find the joint probability density function of $Y_1\ and\ Y_2$ are also the marginal probability density functions of $Y_1\ and\ Y_2$.

[9] Let X and Y be i. i. d. N(0, 1). Find the probability density function of Z= X/ Y.

[10] Let X and Y be independent random variables with probability density functions

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha_1 - 1}}{\Gamma \alpha_2 \theta^{\alpha_2}} e^{-y/\theta} & y > 0 \\ 0 & otherwise. \end{cases}$$

Find the distributions of $U = X + Y\ and\ V = \frac{X}{(X+Y)}$ and also that they are independently distributed.

[11] Let X and Y be i.i.d. random variables with common probability density function

f(x)= $\begin{cases} \frac{c}{1+x^4} & -\infty < x < \infty \\ 0 & otherwise, \end{cases}$

Where, c is a normalizing constant. Find the probability density function of Z= X / Y.

[12] Let X and Y be i. i. d. N(0, 1), define the random variables R and$\Theta\ by\ X = R\cos\Theta\,, Y = R\ sin\Theta$,

(a) show that R and $\Theta\ are\ independent\ with \frac{R^2}{2} \sim Exp\ (1) and\ \Theta \sim U(0, 2\pi)$

$$(b) show\ that\ X^2 + Y^2\ and \frac{X}{Y}\ are\ independently\ distributed.$$

[13] Let $U_1 and U_2$ be i.i.d. U(0, 1) random variables. Show that

$X_1 = \sqrt{-2lnU_1}\cos(2\pi U_2)\ and\ X_2 = \sqrt{-2lnU_1}\sin(2\pi U_2)\ are\ i.i.d.N(0,1)$ random variables.

[14] Let $X_1, X_2, and\ X_3$ be i. i. d. with probability density function

f(x)= $\begin{cases} e^{-x} & x > 0 \\ 0 & otherwise. \end{cases}$

Find the probability density function of Y_1, Y_2, Y_3; where

$$Y_1 = \frac{X_1}{X_1 + X_2}; Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}; Y_3 = X_1 + X_2 + X_3$$

[15] Let $X_1, X_2, and\ X_3$ be three mutually independent chi-square random variables with n_1, n_2, n_3 degrees of freedom respectively; i.e. $X_1 \sim {\lambda_{n_1}}^2, X_2 \sim {\lambda_{n_2}}^2\ and\ X_3 \sim {\lambda_{n_3}}^2$ and they are independent.

(a) Show that $Y_1 = \frac{X_1}{X_2}\ and Y_2 = X_1 + X_2 are\ independent\ and\ that\ Y_2$ is chi-square random variable with $n_1 + n_2$ degree of freedom.

(b) Find the probability density functions of

$$Z_1 = \frac{X_1/n_1}{X_2/n_2}\ and\ Z_2 = \frac{X_3/n_3}{(X_1 + X_2)/(n_1 + n_2)}$$

[16] Let X and Y be independent random variables such that X$\sim$ N(0, 1) and Y$\sim {\lambda_{n_2}}^2$.

Find the probability density function of $T = \frac{X}{\sqrt{Y/n}}$.

[17] Let $X_1, \ldots, X_n$ be a random sample from N(0, 1) distribution. Find the m.g. f. of Y= $\sum_{i=1}^{n} {X_i}^2$ and identify its distribution. Further, suppose X_{n+1} is another random sample from N(0, 1) independent of $X_1, \ldots, X_n$. Derive the distribution of $\left(\frac{X_{n+1}}{\sqrt{\frac{Y}{n}}}\right)$.

[18] X and Y are i. i. d. random variables each having geometric distribution with the following p. m. f.

P(X= x)= $\begin{cases} (1-p)^x p, & x = 0, 1, \ldots \\ 0 & otherwise. \end{cases}$

$Identify\ the\ distribution\ of\ \frac{X}{X+Y}. Further\ find\ the\ p.m.f.of\ Z = \min(X, Y).$

Solution Key

$$(1)\ P(Y_1 = y_1, Y_2 = y_2) = P(X_1 - X_2 = y_1, X_1 + X_2 = y_2)$$
$$= P\left(X_1 = \frac{y_1 + y_2}{2}, X_2 = \frac{y_1 - y_2}{2}\right)$$
$$= \begin{cases} \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{2-0} & if\ \frac{y_1 + y_2}{2} = 0, \frac{y_1 - y_2}{2} = 0, i.e.\ y_1 = 0, y_2 = 0 \\ \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} & if\ \frac{y_1 + y_2}{2} = 1, \frac{y_1 - y_2}{2} = 0, i.e.\ y_1 = 1, y_2 = 1 \\ \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} & if\ \frac{y_1 + y_2}{2} = 0, \frac{y_1 - y_2}{2} = 1, i.e.\ y_1 = -1, y_2 = 1 \\ \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{2-2} & if\ \frac{y_1 + y_2}{2} = 1, \frac{y_1 - y_2}{2} = 1, i.e.\ y_1 = 0, y_2 = 2 \end{cases}$$
$$i.e.$$

$$P(Y_1 = y_1, Y_2 = y_2) = \begin{cases} \frac{1}{9}\ if\ (y_1, y_2) = (0,0) \\ \frac{2}{9} if\ (y_1, y_2) = (-1,1), (1,1) \\ \frac{4}{9}\ if\ (y_1, y_2) = (0,2) \\ 0\ \ otherwise \end{cases}$$

$Y_1 \backslash Y_2$	0	1	2
0	$\frac{1}{9}$	0	$\frac{4}{9}$
-1	0	$\frac{2}{9}$	0
1	0	$\frac{2}{9}$	0

$$P(Y_1 = y_1) = \begin{cases} \frac{5}{9} & y_1 = 0 \\ \frac{2}{9} & y_1 = -1 \\ \frac{2}{9} & y_1 = 1 \\ 0 & otherwise \end{cases} \quad P(Y_2 = y_2) = \begin{cases} \frac{1}{9} & y_2 = 0 \\ \frac{4}{9} & y_2 = 1 \\ \frac{4}{9} & y_2 = 2 \\ 0 & otherwise \end{cases}$$

$$since\ P(Y_1 = y_1, Y_2 = y_2) \neq P(Y_1 = y_1)P(Y_2 = y_2) \forall (y_1, y_2)$$

$$Y_1\ \&\ Y_2\ are\ not\ indep.$$

(2) $Y_1 = X_1X_2$; $Y_2 = X_2$

$$jt\ p.m.f.$$

$$P(Y_1 = y_1, Y_2 = y_2) = P(X_1X_2 = y_1, X_2 = y_2) = P\left(X_1 = \frac{y_1}{y_2}, X_2 = y_2\right)$$

$$= \begin{cases} \frac{y_1}{36} \quad if\ \frac{y_1}{y_2} = 1,2,3;\ y_2 = 1,2,3 \\ 0 \quad otherwise. \end{cases}$$

$$Possible\ value\ of\ Y_1\ in\ \{1,2,3,4,6,9\}.$$

$$P(Y_1 = y_1) = \begin{cases} P(X_1 = 1, X_2 = 1) = \frac{1}{36} & y_1 = 1 \\ P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) = \frac{2}{36} + \frac{2}{36} = \frac{4}{36} & y_1 = 2 \\ P(X_1 = 1, X_2 = 3) + P(X_1 = 3, X_2 = 1) = \frac{3}{36} + \frac{3}{36} = \frac{6}{36} & y_1 = 3 \\ P(X_1 = 2, X_2 = 2) = \frac{4}{36} & y_1 = 4 \\ P(X_1 = 2, X_2 = 3) + P(X_1 = 3, X_2 = 2) = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} & y_1 = 6 \\ P(X_1 = 3, X_2 = 3) = \frac{9}{36} & y_1 = 9 \\ 0 \quad otherwise. \end{cases}$$

$$Z = X_1 + X_2 \rightarrow possible\ value\ of\ Z\ in\ \{2,3,4,5,6\}$$

$$P(z = 3)$$

$$= P(X_1 + X_2 = 3)$$

$$= \begin{cases} P(X_1 = 1, X_2 = 1) = \dfrac{1}{36} \ \mathfrak{J} = 2 \\ P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) = \dfrac{4}{36} \ \mathfrak{J} = 3 \\ P(X_1 = 1, X_2 = 3) + P(X_1 = 2, X_2 = 2) + P(X_1 = 3, X_2 = 1) = \dfrac{10}{36} \ \mathfrak{J} = 4 \\ P(X_1 = 2, X_2 = 3) + P(X_1 = 3, X_2 = 2) = \dfrac{12}{36} \ \mathfrak{J} = 5 \\ P(X_1 = 3, X_2 = 2) = \dfrac{9}{36} \ \mathfrak{J} = 6 \\ 0 \ otherwise. \end{cases}$$

(3) $P(X = x|X + Y = t) = \frac{P(X=x,X+Y=t)}{P(X+Y=t)} = \frac{P(X=x,Y=t-x)}{P(X+Y=t)}$

$$= \frac{P(X = x)P(Y = t - x)}{P(X + Y = t)} = \frac{\binom{n_1}{x}p^x(1-p)^{n-x}\binom{n_2}{t-x}p^{t-x}(1-p)^{n_2-(t-x)}}{\binom{n_1+n_2}{t}p^t(1-p)^{n_1+n_2-t}}$$

$$= \frac{\binom{n_1}{x}\binom{n_2}{t-x}}{\binom{n_1+n_2}{t}} ; \ \ 0 \le x \le n_1 \ \ 0 \le t - x \le n_2$$

$$\uparrow hyper\ geometric\ (n_1, n_2)$$

(4) $P(X = x|X + Y = t) = \frac{P(X=x,Y=t-x)}{P(X+Y=t)} [X \sim P(\lambda_1), Y \sim P(\lambda_2); X + Y \sim P(\lambda_1 + \lambda_2)]$

$$= \frac{P(X = x)P(Y = t - x)}{P(X + Y = t)} = \frac{\left(\dfrac{e^{-\lambda_1}{\lambda_1}^x}{x!}\right)\left(\dfrac{e^{-\lambda_2}{\lambda_2}^{t-x}}{(t-x)!}\right)}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^t}{t!}}$$

$$= \binom{t}{x}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^1$$

$$i.e. X|\ X + Y = t\ Bin\left(t, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

(5) $f_X(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & otherwise. \end{cases}$ $F_X(x) = \begin{cases} 0 & x < 0 \\ 3\int_0^x (1-t)^2 dt & 0 \le x < 1 = 1 - (1-x)^3 \\ 1 & x \ge 1 \end{cases}$

$$Y = Min(X_1, X_2, X_3, X_4); Z = Max(X_1, X_2, X_3, X_4)$$

$$X_1, X_2, X_3, X_4\ i.i.d. from\ f_X(x)$$

$$d.f. of\ Y = F_Y(y) = P(Y \le y) = 1 - P(Y > y)$$

$$= 1 - \prod_1^4 P(X_i > y)$$

$$= 1 - \left(1 - P(X \le y)\right)^4$$

$$= 1 - (1 - \{1 - (1 - y)^2\})^4 = 1 - (1 - y)^{12} \ \ 0 < y < 1$$

$$f_Y(y) = 12(1 - y)^{11} \ \ 0 < y < 1$$

$$= 0 \ \ otherwise.$$

$$d.f. og\ Z{:}\ f_Z(3) = P(Z \le 3) = \prod_1^4 P(X_i \le 3) = [P(X \le 3)]^4$$

$$= (1-(1-\mathfrak{z})^3)^4 \ 0<\mathfrak{z}<1$$

$$f_Z(\mathfrak{z}) = \begin{cases} 12(1-\mathfrak{z})^2(1-(1-\mathfrak{z})^3)^3 & 0<\mathfrak{z}<1 \\ 0 & otherwise. \end{cases}$$

(6) Similar to (5) .

(7) X: arrival time of A

Y : arrival time of B

X & Y i. i. d. Exp(λ)- p. d. f.

$$f(x) = \lambda e^{-\lambda x} \quad x>0$$

$$reqd\ prob = P(X<Y, Y-X\le t) + P(Y<X, X-Y\le t)$$

$$= P(Y-t\le X\le Y) + P(X-t\le Y\le X)$$

$$= P(X\le Y\le X+t) + P(Y\le X\le Y+t)\left[jt\ p.d.f\ of\ X,Y \to \lambda^2 e^{-\lambda(x+y)}\ x>0, y>0\right]$$

$$= \int_0^\infty \int_x^{x+t} \lambda^2 e^{-\lambda(x+y)}\ dy\ dx + \int_0^\infty \int_x^{y+t} \lambda^2 e^{-\lambda(x+y)}\ dx\ dy$$

$$= 2\lambda^2 \int_0^\infty e^{-\lambda x} \int_x^{x+t} e^{-\lambda t}\ dy\ dx$$

$$= 2\lambda^{\cancel{2}} \frac{1}{\cancel{\lambda}}\left(1-e^{-\lambda t}\right) \int_0^\infty e^{-2\lambda x}\ dx = \left(1-e^{-\lambda t}\right).$$

(8) $X_1, X_2 \sim U(0,1)$

$$Y_1 = X_1 + X_2 \Longrightarrow \frac{1}{|J|} = \left\| \begin{matrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \right\| = 2$$

$$Y_2 = X_2 - X_1 \qquad |J| = \frac{1}{2}$$

$$f_{X_1,X_2}(x_1,x_2) = 1; \quad 0<x_1<1, 0<x_2<1$$

$$\Longrightarrow f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2}; \ \ 0<y_1+\ y_2<2, 0<y_1-\ y_2<2$$

$$Range\ unconditionally\ 0<y_1<2\ \&-1<y_2<1$$

$$i_2 = \frac{y_1+y_2}{2}, i_1 = \frac{y_1-y_2}{2}\ Also\ 0<x_1<1\ ;\Longrightarrow 0<\frac{y_1-y_2}{2}<1$$

$$\Longrightarrow 0<y_1-\ y_2<2$$

$$y_2<\ y_1<2+y_2\ \&\ y_1-2<\ y_2<\ y_1\}\_\_\_(1)$$

$$Also\ 0<\ x_2<1\ ; 0<\frac{y_1+y_2}{2}<1$$

$$0<y_1+y_1<2$$

$$-y_2<\ y_1<2-y_2\ \&-y_1<\ y_2<2-y_1\}\_\_\_(2)$$

$$Combining\ (1)\&\ (2)$$

$$\max(y_2, -y_2) < \ y_1 < \min(2+y_2, 2-y_2)$$
$$\& \max(y_1 - 2, -y_1) < y_2 < \min(y_1, 2-y_1)) \_\_\_(3)$$
$$If -1 < \ y_2 < 0 \ then\ from\ (3) - y_2 < \ y_1 < 2+y_2 \ \&\ if\ 0 < \ y_2 < 1 \ then\ from\ (3) y_2 < \ y_1 < 2-y_2\} \_\_\_\_(4)$$
$$Alternatively\ if\ 0 < \ y_1 < 1 \ then\ from\ (3) - y_1 < \ y_2 < \ y_1 \ \&\ if\ 1 < \ y_1 < 2 \ then\ from\ (3) y_1 - 2 < \ y_2 < 2 - y_1\} \_\_\_(5)$$
$$\Longrightarrow Marg\ of\ Y_1$$
$$f_{Y_1}(y_1) = \frac{1}{2}\int_{-y_1}^{y_1} dy_2 = \ y_1 \quad if\ 0 < y_1 < 1$$
$$\big(Using\ (5)\big) \rightarrow = \frac{1}{2}\int_{y_1-2}^{2-y_1} dy_2 = 2 - y_1 \ \ if\ 1 < y_1 < 2$$
$$\&\ Marg\ of\ Y_2$$
$$f_{Y_2}(y_2) = \frac{1}{2}\int_{-y_2}^{2+y_2} dy_1 = (1+y_2) \ \ if -1 < \ y_2 < 0$$
$$\big(using(4)\big) \rightarrow = \frac{1}{2}\int_{y_2}^{2-y_2} dy_1 = (1-y_2) \ \ if\ 0 < y_2 < 1$$

(9) X ~N(0, 1)

Y~ N(0, 1)   > ind

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}(x^2+y^2)\right)$$
$$U = Y\}\ X = UZ\ |J| = \left\|\begin{matrix} z & u \\ 1 & 0 \end{matrix}\right\| = \ |u|$$
$$f_{U,Z}(u,z) = \ \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(u^2 3^2 + u^2)\right\}|u|; \ -\infty < u < \infty, -\infty < z < \infty$$
$$f_Z(3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |u| \exp\left(-\frac{1}{2}u^2(1+3^2)\right) du$$
$$= \frac{1}{\pi}\int_0^{\infty} u \exp\left(-\frac{u^2}{2}(1+3^2)\right) du = \ \frac{1}{\pi}.\frac{1}{1+3^2}; \ -\infty < 3 < \infty$$
$$\big(i.e.\ Z \sim Cauchy\ distn(0,1)\big)$$
$$In\ general\ X \sim Cauchy\ (\mu,\theta) \rightarrow f_X(x) = \frac{\theta}{\pi}\frac{1}{1+(x-\mu)^2}; \ -\infty < x < \infty$$

(10) $f_{X,Y}(x,y) = \frac{1}{\Gamma\alpha_1\,\Gamma\alpha_2\theta^{\alpha_1+\alpha_2}} x^{\alpha_1-1}y^{\alpha_2-1}e^{-\frac{x+y}{\theta}}, x > 0, y > 0$

$$= 0 \quad otherwise.$$
$$U = X + Y\}\ X = UV$$
$$V = \frac{X}{X+Y}\}\ Y = U(1-V)$$
$$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = \ -u$$
$$Range\ u > 0, 0 < v < 1$$
$$f_{U,V}(u,v) = \frac{1}{\Gamma\alpha_1\,\Gamma\alpha_2\theta^{\alpha_1+\alpha_2}}(uv)^{\alpha_1-1}\big(u(1-v)\big)^{\alpha_2-1}e^{-\frac{u}{\theta}}.u \quad u > 0, 0 < v < 1$$
$$= 0 \quad otherwise.$$

$i.e. f_{U,V}(u,v)$

$$= \begin{cases} \frac{1}{\Gamma\alpha_1+\alpha_2\ \theta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\theta}} \times \frac{1}{B(\alpha_1,\alpha_2)} v^{\alpha_1-1}(1-v)^{u^{\alpha_2-1}} & u>0, 0<v<1 \\ 0 \quad otherwise. \end{cases}$$

$$\Longrightarrow f_U(u) = \frac{1}{\Gamma\alpha_1+\alpha_2\ \theta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\theta}}\ u>0$$

$$U \sim Gamma. = 0 \quad otherwise$$

$$f_V(v) = \frac{1}{B(\alpha_1,\alpha_2)} v^{\alpha_1-1}(1-v)^{u^{\alpha_2-1}} 0<v<1$$

$$V \sim Beta. \quad = 0 \quad otherwise.$$

$$\Longrightarrow U \,\&\, V\ are\ indep.$$

(11) $f_{X,Y} = \frac{C^2}{(1+x^4)(1+y^4)} - \infty < x < \infty, -\infty < y < \infty$

$$U_1 = \frac{X}{Y}, U_2 = Y\} \begin{matrix} X = U_1U_2 \\ Y = U_2 \end{matrix}\} J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

$$Range - \infty < u_1 < \infty, -\infty < u_2 < \infty$$

$$f_{U_1,U_2}(u_1,u_2) = \frac{C^2|u_2|}{(1+{u_1}^4{u_2}^4)(1+{u_2}^4)} - \infty < u_1 < \infty, -\infty < u_2 < \infty$$

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} f_{U_1,U_2}(u_1,u_2)\, du_2 = 2C\int_0^{\infty} \frac{u_2}{(1+{u_1}^4{u_2}^4)(1+{u_2}^4)} du_2$$

$$= \frac{C\pi}{2} . \frac{1}{1+{u_1}^2} (an\ integrating).$$

$$\int_{-\infty}^{\infty} f_{U_1}(u_1) du_1 = 1 \Longrightarrow C = \frac{2}{\pi^2}$$

$$\Longrightarrow f_{U_1}(u_1) = \frac{2}{\pi} . \frac{1}{1+{u_1}^2} - \infty < u_1 < \infty.$$

$$\uparrow Cauchy\ distn.$$

(12) $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad - \infty < x < \infty, -\infty < y < \infty$

$$X = R\cos\Theta$$

$$Y = R\sin\Theta$$

$$J = \begin{vmatrix} cos\theta & -r\ sin\theta \\ sin\theta & r\ cos\theta \end{vmatrix} = r$$

$$Range\ r \geq 0, 0 < \theta < 2\pi$$

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} e^{-\frac{r^2}{2}} r, \qquad r > 0, 0 < \theta < 2\pi$$

$$= 0 \quad otherwise$$

$$f_R(r) = re^{-\frac{r^2}{2}} r > 0$$

$$= 0 \quad otherwise$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \quad 0 < \theta < 2\pi \quad \Theta \sim U(0, 2\pi)$$

$$= 0 \quad otherwise$$

$$\Longrightarrow R\ \&\Theta\ are\ indep.$$

$$Define\ Y = \frac{R^2}{2} \quad y > 0$$

$$R = \sqrt{2}\sqrt{y} \quad \frac{dr}{dy} = \frac{1}{\sqrt{2y}}$$

$$f_Y(y) = \frac{1}{\sqrt{2y}} \sqrt{2}ye^{-y} \quad y > 0$$

$$= 0 \quad otherwise$$

$$i.e. f_Y(y) = \quad e^{-y} \quad y > 0$$

$$= 0 \quad otherwise$$

$$\Longrightarrow \frac{R^2}{2} \sim Exp(1).$$

$$U = X^2 + Y^2 = R^2 - f^n \ of \ r.v. \ k$$

$$V = \frac{X}{Y} = cot\Theta - f^n \ of \ r.v. \Theta$$

$$since \ R \ \& \ \Theta \ are \ indep, U \ \& \ V \ are \ also \ indep.$$

$$i.e. X^2 + Y^2 \ \& \frac{X}{Y} \ are \ indep.$$

(13) $U_1 \sim U(0,1)$

$$-lnU_1 \sim Exp(1) - straight\ forward$$

$$U_2 \sim U(0,1)$$

$$2\pi U_2 \sim U(0,2\pi) - straight\ forward.$$

$$\Longrightarrow -lnU_1 \sim Exp(1) \ \& \ 2\pi U_2 \sim U(0,2\pi) \ and \ are \ indep. by \ problem \ \# \ (12)$$

$$jt \ distn \ of \ (-lnU_1, 2\pi U_2) is \ same \ as \ jt \ distn \ of \ \left(\frac{R^2}{2}, \Theta\right)$$

$$i.e. (-lnU_1, 2\pi U_2) \stackrel{\text{def}}{=} \left(\frac{R^2}{2}, \Theta\right)$$

$$i.e. (-2lnU_1, 2\pi U_2) \stackrel{\text{def}}{=} (R^2, \Theta)$$

$$i.e. \left(\sqrt{-2lnU_1}\cos(2\pi U_2), \sqrt{-2lnU_1}\sin(2\pi U_2)\right) \stackrel{\text{def}}{=} (R\ cos\Theta, R\ sin\Theta)$$

$$i.e. (X_1, X_2) \stackrel{\text{def}}{=} (R\ cos\Theta, R\ sin\Theta)$$

$$\Longrightarrow X_1 and \ X_2 \ are \ i.i.d. N(0,1) r.v.s.$$

$$Divert \ method \ U_1, U_2 \ i.i.d. U(0,1).$$

$$f_{U_1,U_2}(u_1,u_2) = 1 \ ; 0 < u_1 < 1, 0 < u_2 < 1$$

$$= 0 \quad otherwise$$

$$X_1 = \sqrt{-2lnU_1}\cos(2\pi U_2)$$

$$X_2 = \sqrt{-2lnU_1}\sin(2\pi U_2)$$

$$Range \ of \ X_1; \ -\infty < x_1 < \infty, sly - \infty < x_2 < \infty$$

$$X_1{}^2 + X_2{}^2 = -2lnU_1$$

$$\frac{X_2}{X_1} = tan(2\pi U_2)$$

$$U_1 = \exp\left(-\frac{1}{2}\left(X_1{}^2 + X_2{}^2\right)\right)$$

$$U_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{X_2}{X_1}\right)$$

$$J = \begin{vmatrix} \exp\left(-\frac{1}{2}\left({X_1}^2 + {X_2}^2\right)\right)(-X_1) & \exp\left(-\frac{1}{2}\left({X_1}^2 + {X_2}^2\right)\right)(-X_2) \\ -\frac{X_2}{2\pi\left({X_1}^2 + {X_2}^2\right)} & \frac{X_1}{2\pi\left({X_1}^2 + {X_2}^2\right)} \end{vmatrix}$$

$$J = \exp\left(-\frac{1}{2}\left({X_1}^2 + {X_2}^2\right)\right)\left(-\frac{1}{2\pi}\right)$$

$$|J| = \frac{\exp\left(-\frac{1}{2}\left({X_1}^2 + {X_2}^2\right)\right)}{2\pi}$$

$$\Longrightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left({x_1}^2 + {x_2}^2\right)\right);\ -\infty < x_1 < \infty, -\infty < x_2 < \infty \downarrow$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}{x_1}^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}{x_2}^2}\right)$$

$$\Longrightarrow X_1 \& X_2\ are\ indep\ N(0,1) r.v.s.$$

(14) $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = e^{-(x_1,x_2,x_3)};\ x_1 > 0, x_2 > 0, x_3 > 0$

$$Y_1 = \frac{X_1}{X_1 + X_2};\ Y_2 = \frac{X_1}{X_1 + X_2 + X_3}; Y_3 = X_1 + X_2 + X_3$$

$$i.e. X_1 = Y_1\,Y_2\,Y_3$$

$$X_2 = Y_2\,Y_3(1 - Y_1)$$

$$X_3 = Y_3(1 - Y_2)$$

$$X_1 + X_2 = Y_2\,Y_3, X_1 = Y_1\,Y_2\,Y_3, X_2 = Y_2\,Y_3$$

$$J = \begin{vmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ -y_2y_3 & y_3(1-y_1) & y_2(1-y_1) \\ 0 & -y_3 & (1-y_2) \end{vmatrix} = y_2{y_3}^2$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = y_2{y_3}^2\,e^{-y_3}; 0 < y_1 < 1, y_3 > 0$$

$$f_{Y_1}(y_1) = \int_0^1 y_2\,dy_2 \int_0^\infty {y_3}^2\,e^{-y_3}\,dy_3 = 1 \quad 0 < y_1 < 1$$

$$i.e. Y_1 \sim U(0,1)$$

$$f_{Y_2}(y_2) = y_2 \times 1 \times 2 \quad 0 < y_2 < 1$$

$$i.e. Y_2 \sim Beta(2,1)\left[X \sim Beta(m,n) f_X(x) = \frac{\Gamma m + n}{\Gamma m\,\Gamma n} x^{m-1}(1-x)^{n-1}\right]$$

$$\& f_{Y_3}(y_3) = \left(\int_0^1 dy_1 \int_0^1 dy_2\right){y_3}^2\,e^{-y_3}$$

$$= \frac{1}{2} e^{-y_3}{y_3}^2 \quad 0 < y_3 < \infty$$

$$y_2y_3\left(y_3(1-y_1)(1-y_2) + y_2y_3(1-y_1)\right)$$

$$\begin{matrix} y_1y_3 & y_2y_3 & y_1y_3 & y_1y_2 \\ & 0 & y_3 & y_2 \\ & 0 & -y_3 & 1-y_2 \end{matrix}$$

(15) (a) $f_{X_1,X_2}(x_1,x_2) = \prod_{i=1}^{2} \frac{1}{2^{\frac{n_i}{2}}\Gamma\frac{n_i}{2}}\ e^{-\frac{x_i}{2}}\ {x_i}^{\frac{n_i}{2}-1};\ x_i > 0$

$$= C \prod_{i=1}^{2} e^{-\frac{x_i}{2}} \; x_i^{\frac{n_i}{2}-1} \; ; \; x_i > 0$$

$$Y_1 = \frac{X_1}{X_2}; \; Y_2 = X_1 + X_2 \mid \begin{matrix} X_1 = \dfrac{Y_1 \, Y_2}{Y_1 + 1} \\ X_2 = \dfrac{Y_2}{Y_1 + 1} \end{matrix}$$

$$\frac{1}{J} = \begin{vmatrix} \dfrac{1}{x_2} & -\dfrac{x}{x_2{}^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x_2} + \frac{x}{x_2{}^2} = \frac{x_1 + x_2}{x_2{}^2} = \frac{y_2}{\left(\dfrac{y_2}{1+y_1}\right)^2} = \frac{(1+y_1)^2}{y_2}$$

$$|J| = \frac{y_2}{(1+y_1)^2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = C \; e^{-\frac{y_2}{2}} \left(\frac{y_1 \, y_2}{y_1+1}\right)^{\frac{n_1}{2}-1} \left(\frac{y_2}{1+y_1}\right)^{\frac{n_2}{2}-1} \frac{y_2}{(y_1+1)^2} \quad y_1 > 0, y_2 > 0$$

$$i.e. \, f_{Y_1,Y_2}(y_1, y_2) = \left(C_1 e^{-\frac{y_2}{2}} \, y_2^{\frac{n_1+n_2}{2}-1}\right) \downarrow$$

$$f_{Y_2} \;\; X \left(\frac{y_1^{\frac{n_2}{2}-1}}{(1+y_1)^{\frac{n_1+n_2}{2}}}\right) y_1 > 0, y_2 > 0$$

$$\downarrow$$

$$f_{Y_1}$$

$$\Longrightarrow \; Y_1 \; \& \; Y_2 \; are \; indep$$

$$\& \; f_{Y_2}(y_2) = \;\; C_1 e^{-\frac{y_2}{2}} \, y_2^{\frac{n_1+n_2}{2}-1} y_2 > 0$$

$$\int_0^\infty f_{Y_2}(y_2) \; dy_2 = 1 \Longrightarrow C_1 = \left(\Gamma \frac{n_1+n_2}{2} . \, 2^{\frac{n_1+n_2}{2}} \right)^{-1}$$

$$\Longrightarrow Y_2 \sim \lambda^2 with \; (n_1 + n_2) d.f.$$

$$(b) \; similar \; ta \; (a)$$

$$Z_1 = \frac{\frac{X_1}{n_1}}{\frac{X_2}{n_2}} \sim \; F_{n_1,n_2} \rightarrow F \; distn \; with \; (n_1, n_2) d.f. \&$$

$$Z_2 = \frac{X_3/n_3}{X_1 + X_2/\, n_1 + n_2} \; \sim \; F_{n_3, n_1 + \, n_2}$$

(16) X~ N(0, 1)

$$f_{X,Y} = \frac{1}{\sqrt{2x}} e^{-\frac{x^2}{2}} \frac{1}{2^{\frac{n}{2}} \ \Gamma\frac{n}{2}} e^{-\frac{y}{2}} y^{\frac{n}{2}-1}$$

$$T = \frac{X}{\sqrt{\frac{Y}{n}}} \ define\ dummy\ U = Y \ \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} T \\ U = Y \end{pmatrix}$$

$$\Longrightarrow X = T\sqrt{\frac{U}{n}}$$

$$Y = U$$

$$J = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{2\sqrt{n}\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

$$\Longrightarrow f_{T,U}(t,u) = \frac{1}{\sqrt{2\pi}\ 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}} \exp\left(-\frac{1}{2}\frac{t^2 u}{n}\right) \exp\left(-\frac{u}{2}\right), u^{\frac{n}{2}-\frac{1}{2}} \quad -\infty < t < \infty \quad u > 0$$

$$f_T(t) = \int_0^\infty f_{T,U}(t,u)\, du$$

$$= \frac{1}{\sqrt{2\pi}\ 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}} . \int_0^\infty u^{\frac{n}{2}-\frac{1}{2}} \exp\left(-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right)\ du$$

$$= \frac{\Gamma\frac{n+1}{2}}{\sqrt{2\pi}\ 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}} . \frac{1}{\left(\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right)^{\frac{n+1}{2}}} \ -\infty < t < \infty$$

$$= \cdots$$

(17) $$M_Y(t) = E(e^t Y) = E\left(e^{t\sum_1^n X_i{}^2}\right)$$

$$= \prod_{i=1}^n E\left(e^{tX_i{}^2}\right) = \prod_{i=1}^n M_{X_i{}^2}(t)$$

$$X_i{}^2 \sim \lambda_1{}^2 \to = \prod_{i=1}^n (1-2t)^{-\frac{1}{2}} = (1-2t)^{-\frac{n}{2}}$$

$$\Longrightarrow Y \sim \lambda_n{}^2$$

$$\left.\begin{matrix} X_{n+1} \sim N(0,1) \\ Y \sim \lambda_n{}^2 \end{matrix}\right> indep.$$

$$jt\ p.d.f.of\ Y\ \&\ X_{n+1}$$

$$f_{Y,X_{n+1}}(y,x) = \left(\frac{1}{2^{\frac{n}{2}} \ \Gamma\frac{n}{2}}\ e^{-\frac{y}{2}} y^{\frac{n}{2}-1}\right) \times \left(\frac{1}{\sqrt{2\pi}}\ e^{-\frac{x^2}{2}}\right)$$

$$\left.\begin{matrix} T = \frac{X_{n+1}}{\sqrt{\frac{Y}{n}}} \\ U = Y \end{matrix}\right\} \Longrightarrow X_{n+1} = T\sqrt{\frac{u}{n}}\ Y = U$$

$$J = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{2\sqrt{n}\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}.$$

$$jt\ p.d.f.\ of\ T\ \&\ U$$

$$f_{T,U}(t,u) = \left(2^{\frac{n}{2}} \Gamma\frac{n}{2}\sqrt{2\pi}\sqrt{n}\right)^{-1} \exp\left(-\frac{1}{2}\frac{t^2u}{n}\right)\exp\left(-\frac{u}{2}\right)\ u^{\frac{n}{2}-1};\ -\infty < t < \infty,\ \ u > 0$$

$$f_T(t) = \left(2^{\frac{n}{2}} \Gamma\frac{n}{2}\sqrt{2\pi}\sqrt{n}\right)^{-1} \int_0^1 u^{\frac{n}{2}-1} \exp\left(-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right)du$$

$$= \left(2^{\frac{n}{2}} \Gamma\frac{n}{2}\sqrt{2\pi}\sqrt{n}\right)^{-1} \frac{\Gamma\frac{n+1}{2}}{\left(\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right)^{\frac{n+1}{2}}};\ -\infty < t < \infty$$

$$= \frac{\Gamma\frac{n+1}{2}}{\sqrt{\pi}\ \Gamma\frac{n}{2}\sqrt{n}}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}\ ;\ -\infty < t < \infty$$

(18) Z= X + Y ; Z∈ {0, 1, …}

$$P(z = 3) = P(X+Y=3) = P\left(\bigcup_{x=0}^{3}(X = x\ \cap Y = 3-x)\right)$$

$$= \sum_{x=0}^{3} P(X = x\ \cap Y = 3-x)$$

$$= \sum_{x=0}^{3} P(X = x)\ P(Y = 3-x)$$

$$= \sum_{x=0}^{3} q^x p\ q^{3-x} p = p^2 \sum_{x=0}^{3} q^3$$

$$i.e. P(z = 3) = \begin{cases} p^2q^3(3+1), & 3 = 0,1,\dots \\ 0, & otherwise. \end{cases}$$

$$P(X = x, Z = 3) = P(X = x, Y = 3-x)$$

$$= \begin{cases} p^2q^3; x = 0,,1\dots,3;\ 3 = 0,1,\dots. \\ 0 \quad otherwise \end{cases}$$

Problem Set-9

[1] Let $\{X_n\}$ be a sequence of $N\left(\frac{1}{n}, 1-\frac{1}{n}\right), show\ that\ X_n \to Z, where\ Z \sim N(0,1).$

[2] Let $\{X_n\}$ be a sequence of i.i.d. random variables with E(X_i)=μ, $Var(X_i) = \sigma^2\ and\ E(X_i - \mu)^4 = \sigma^4 + 1. Find\ \lim_{n\to\infty} P[\sigma^2 - \frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2 + \dots + (X_n-\mu)^2}{n} \le \sigma^2 + \frac{1}{\sqrt{n}}]$.

[3] Let $X_1, X_2, \dots, X_n\ be\ i.i.d.\ B(1,p), S_n = \sum_{i=1}^{n} X_i\ . Find\ n\ which\ would\ guarantee$ $P\left(\left|\frac{S_n}{n} - p\right| \ge 0.01\right) \le 0.01$, no matter whatever the unknown p may be.

[4] Let $X_1, \ \dots, X_n$ be i.i.d. from a distribution with mean μ *and finite variance* σ^2. *Prove that* $\frac{\sqrt{n}\,(X_n - \mu)}{S_n} \rightarrow Z$, *where* $Z \sim N(0,1)$.

[5] The p. d. f. of a random variable X is f(x)= $\begin{cases} \frac{1}{x^2} \ x \geq 1 \\ 0 \ \ otherwise. \end{cases}$

Consider a random sample of size 72 from the distribution having the above p. d. f. compute, approximately, the probability that more than 50 of these observations are less than 3.

[6] Let $X_1, \ \dots, X_{100}$ be i. i. d. from poisson (3) distribution and let Y= $\sum_{i=1}^{100} X_i$. Using CLT, find an approximate value of P(100≤ Y ≤ 200).

[7] Let X~ Bin (100, 0.6). Find an approximate value of P(10 ≤X ≤ 16).

[8] The p. d. f. of X_n *is given by* $f_n(x) = \begin{cases} \frac{1}{\Gamma n} \ e^{-} x^{n-1} \ \ x > 0 \\ \ 0 \quad otherwise \end{cases}$

Find the limiting distribution of $Y_n = \frac{X_n}{n}$.

[9] Let X̅ denote the mean of a random sample OF SIZE 64 FROM THE Gamma distribution with density

$$f_n(x) = \begin{cases} \dfrac{1}{\Gamma p \alpha^p} \ e^{-\frac{x}{\alpha}} x^{n-1} \ \ x > 0 \\ \quad 0 \quad otherwise \end{cases}$$

With $\alpha = 2, p = 4$. Compute the approximate value of P(7 < X̅ < 9).

[10] $X_1, \ \dots, X_n$ is a random sample from U(0, 2). Let $Y_n = \overline{X_n}$, *show that* $\sqrt{n}\,(Y_n - 1) \rightarrow N\left(0, \frac{1}{3}\right)$.

Solution Set

(1) $F_{X_n}(x) = P(X_n \leq x) = P\left(\frac{X_n - \frac{1}{n}}{\sqrt{1 - \frac{1}{n}}} \leq \frac{x - \frac{1}{n}}{\sqrt{\left(1 - \frac{1}{n}\right)}} \right)$

$$= \Phi \frac{x - \dfrac{1}{n}}{\sqrt{\left(1 - \dfrac{1}{n}\right)}} \longrightarrow \Phi(x) as \ n \rightarrow \infty$$

$$\Longrightarrow X_n \longrightarrow X \sim N(0,1)$$

$$Alt \ m.g.f. of \ X_n$$

$$M_{X_n}(t) = \exp\left(\frac{t}{n} + \frac{t^2}{2}\left(1 - \frac{1}{n}\right) \right)$$

$$\longrightarrow e^{\frac{t^2}{2}} \leftarrow m.g.f.of\ N(0,1)$$

$$\Longrightarrow\ X_n \longrightarrow X \sim N(0,1).$$

(2) $Y_i = \ (X_i - \mu)^2$

$$E(Y_i) = E(X_i - \mu)^2 = \ \sigma^2$$

$$V(Y_i) = E((X_i - \mu)^2 - \ \sigma^2)^2$$

$$= E(X_i - \mu)^4 + \sigma^4 - 2\sigma^2 E(X_i - \mu)^2$$

$$= (\sigma^4 + 1) + \sigma^4 - \ 2\sigma^4 = 1$$

$$i.e.E(Y_i) = \ \sigma^2; V(Y_i) = 1\ \forall\ i\ \&\ Y_1 \dots\ Y_n\ i.i.d.$$

$$S_n = \sum Y_i$$

$$ES_n = n\sigma^2$$

$$V\,S_n = n$$

$$CLT \Longrightarrow \frac{S_n - E\ S_n}{\sqrt{V(S_n)}} \to N(0,1)$$

$$i.e.\frac{(X_1-\mu)^2 + \ \dots + \ (X_n-\mu)^2 - n\sigma^2}{\sqrt{n}} \to X \sim N(0,1).$$

$$\lim_{n\to\infty} P\left(\sigma^2 - \frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2 + \ \dots + \ (X_n-\mu)^2}{n} \le \sigma^2 + \frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} P\left(-\frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2 + \ \dots + \ (X_n-\mu)^2 - n\sigma^2}{n} \le \frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} P\left(-1 \le \frac{(X_1-\mu)^2 + \ \dots + \ (X_n-\mu)^2 - n\sigma^2}{\sqrt{n}} \le 1\right)$$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = \cdots$$

(3) $ES_n = np\ ;\ \ VS_n = npq$

$$P\left(\left|\frac{S_n}{n} - p\right| \ge t\right) \le \frac{E(S_n - np)^2}{t^2n^2} = \frac{np(1-p)}{n^2t^2} \le \frac{1}{4nt^2} \le 0.01\ (given)$$

$$\Longrightarrow n\ \ge \frac{1}{0.04t^2}\ for\ t = 0.01$$

$$n \ge \cdots$$

(4) $CLT \Longrightarrow \frac{\sqrt{n}\ (\overline{X_n}-\mu)}{\sigma} \longrightarrow Z\ \sim N(0,1)$

$$Also\ S_n{}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \longrightarrow \sigma^2$$

$(S_n{}^2 = \frac{1}{n}\sum X_i{}^2 - \ \bar{X}^2) \Longrightarrow S_n \longrightarrow \sigma,$

$\llcorner\!\lrcorner$

$\downarrow$p $\downarrow$p

$\sigma^2 + \mu^2$ $\mu^2\ \rightarrow$

$\llcorner\!\lrcorner$

$\downarrow$p

σ^2

Using slutsky's $\left(X_n \rightarrow X;\ Y_n \rightarrow c \frac{X_n}{Y_n} \rightarrow \frac{X}{c}\right)$

$$\frac{\frac{\sqrt{n}\ (\overline{X_n} - \mu)}{\sigma}}{\left(\frac{S_n}{\sigma}\right)} \rightarrow X \sim N(0,1)$$

$$i.e. \frac{\sqrt{n}\ (\overline{X_n} - \mu)}{S_n} \longrightarrow X \sim N(0,1).$$

(5) $X_1 \ldots X_{72}\ r.s. from\ f(x) = \begin{cases} \frac{1}{x^2} & x > 1 \\ 0 & otherwise. \end{cases}$

$$Define\ Y_i\ = \begin{cases} 1 & if\ X_i < 3 \\ 0 & otherwise \end{cases}$$

$$P(Y_i = 1) = \ P(X_i < 3) = \int_1^3 \frac{1}{x^2}\ \ dx = \frac{2}{3} = \ \theta\ say$$

$$Y_1, \ldots, Y_{72}\ are\ i.i.d.B(1,\theta)$$

$$Y = \sum_{i=1}^{72} Y_i \sim B\left(72, \theta = \frac{2}{3}\right)$$

$$CLT\ \Longrightarrow \frac{Y - 72 \times \frac{2}{3}}{\sqrt{72 \times \frac{2}{3} \times \frac{1}{3}}} \longrightarrow Z \sim N(0,1)$$

$$i.e. \frac{Y-48}{4} \longrightarrow Z \sim N(0,1)$$

$$P(Y > 50) = 1 - P(Y \leq 50) = 1 - P(Y \leq 50.5) \leftarrow$$

$$= 1 - P\left(\frac{Y-48}{4} \leq \frac{50.5-48}{4}\right)$$

$$\approx 1 - \Phi\left(\frac{2.5}{4}\right) = \cdots$$

(6) $X_1 \dots X_{100}$ $i.i.d$ $P(3)$

$$E(X_1) = 3; V(X_1) = 3 ; Y = \sum_{i=1}^{100} X_1 \sim P(300) \Longrightarrow E(Y) = V(y) = 3$$

$$CLT \Longrightarrow \frac{Y-300}{10\sqrt{3}}\left(= \frac{S_n - ES_n}{\sqrt{VS_n}}\right) \rightarrow N(0,1)$$

$$P(100 \leq Y \leq 200) = P(99.5 \leq Y \leq 200.5) \leftarrow cont\ correlation$$

$$= P\left(\frac{99.5-300}{10\sqrt{3}} \leq \frac{Y-300}{10\sqrt{3}} \leq \frac{200.5-300}{10\sqrt{3}}\right)$$

$$\approx \Phi\left(\frac{200.5-300}{10\sqrt{3}}\right) - \Phi\left(\frac{99.5-300}{10\sqrt{3}}\right).$$

(7) $X \sim bin(100, 0.6)$

$$CLT \Longrightarrow \frac{X - 100 \times 0.6}{\sqrt{100 \times 0.6 \times 0.4}} = \frac{X-60}{\sqrt{24}} \longrightarrow Z \sim N(0,1)$$

$\Longrightarrow P(10 \leq X \leq 16) = P(9.5 \leq X \leq 16.5)$

$$= P\left(\frac{9.5-60}{\sqrt{24}} \leq \frac{X-60}{\sqrt{24}} \leq \frac{16.5-60}{\sqrt{24}}\right)$$

$$\approx \Phi\left(\frac{16.5-60}{\sqrt{24}}\right) - \Phi\left(\frac{9.5-60}{\sqrt{24}}\right)$$

$$= \cdots$$

(8)X_n has p.d.f.

$$f_n(x) = \begin{cases} \dfrac{1}{\Gamma n} e^{-x} x^{n-1} & x > 0 \\ 0 & otherwise \end{cases}$$

m. g. f. of X_n

$$M_{X_n}(t) = \frac{1}{\Gamma n}\int_0^\infty e^{tx}\, e^{-x} x^{n-1}\, dx$$

$$= \frac{1}{\Gamma n}\int_0^\infty e^{-x(1-t)}\, x^{n-1}\, dx$$

$$= \frac{1}{(1-t)^n} = (1-t)^n$$

m. g. f. of $Y_n = \frac{X_n}{n}$ is

$$M_{Y_n}(t) = E\left(e^{t\,\frac{X_n}{n}}\right) = \left(1 - \frac{t}{n}\right)^{-n} \longrightarrow e^t\ as\, n \to \infty$$

$$\uparrow m.g.f.r.v.deng\ at\ \ x = 0$$

$$Y_n \longrightarrow X\ (degree\ at\ 1)$$

(9)

$$f_{X_n}(x) = \begin{cases} \Gamma p \frac{1}{\alpha^p}\, e^{-\frac{x}{\alpha}}\, x^{p-1} & x > 0 \quad \alpha = 2, p = 4 \\ 0 & otherwise \end{cases}$$

$$E(X_n) = \alpha p\,; V(X_n) = \alpha^2 p = 16 = \sigma^2$$

$$E(\bar{X}) = 8; V(\bar{X}) = \frac{16}{n} = \frac{1}{4}$$

By C LT $\frac{\sqrt{n}\ (\bar{X} - \alpha p)}{\alpha^2 p} \to Z \sim N(0,1)$

$$ie.\, 2(\bar{X} - 8) \longrightarrow Z \sim N(0,1)$$

$$P(7 < \bar{X} < 9) = P\big(2(7-8) < 2(\bar{X} - 8) < 2(9-8)\big)$$

$$\approx P(-2 < z < 2)$$

$$= \Phi(2) - \Phi(-2) = 2\Phi(2) - 1$$

(10)

$$X_i\ \sim U(0,2)\ \ EX_i = \frac{1}{2}\int_0^2 x\ dx = 1$$

$$EX_i{}^2 = \frac{1}{2}\int_0^2 x^2\ dx = \frac{4}{3}; V(X_i) = \frac{1}{3}$$

$X_1, \dots, X_n$ i. i. d. with E$X_1 = 1$ & $VX_1 = \frac{1}{3}$

$$By\ CLT\ \sqrt{n}(\overline{X_n}-1) \longrightarrow N\left(0, \frac{1}{3}\right).$$

$i.$ e. $\sqrt{n}(Y_n-1) \longrightarrow N\left(0, \frac{1}{3}\right)$.

Problem Set-10

[1] Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with p.d.f.

$$f_X(x)=\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right);\ x>0$$

Show that $\overline{X}= \sum_{i=1}^{n} X_i\,/n$ is an unbiased estimator ofβ.

[2] Let $X_1, X_2, \ldots, X_n$ be a random sample from U (0,); θ> 0, show that

$\frac{n+1}{n}\ X_{(n)}\ and\ 2\overline{X}$ are both unbiased estimators of θ.

[3] Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with p.d.f.

$$f(x)=\ \beta\exp(-\beta\ x)\,;x>0$$

Show that $\overline{X}$ is an unbiased estimator of $\frac{1}{\beta}$.

[4] Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0,\theta^2), \theta>0. Show\ that\ \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n(n+1)}\ \ and\ \frac{\sum_{i=1}^{n} {X_i}^2}{2n}$ are both unbiased estimators of θ^2.

[5] Let $X_1, X_2, \ldots, X_n$ be a random sample from P(θ); θ >0. Find an unbiased estimator of $\theta\ e^{-2\theta}$.

[6] Let $X_1, X_2, \ldots, X_n$ be a random sample from B(1, θ); 0≤ θ ≤1.

(a) Show that the estimator $T(X)=\frac{\frac{1}{2}\sqrt{n}+\sum_{i=1}^{n} X_i}{n+\sqrt{n}}$ is not unbiased θ?

(b) Show that $\lim_{n\to\infty} E\left(T(X)\right)=\ \theta$.

(An estimator satisfying the condition in (b) is said to be unbiased in the limit)

[7] $X_1, \ldots, X_n$ be a random sample from N($\mu, \sigma^2), \mu \in \Re, \sigma \in \Re^+$. Find unbiased estimators of $\frac{\mu}{\sigma^2}\ and\ \frac{\mu}{\sigma}$.

[8] Let $X_1, X_2, \ldots, X_n$ be a random sample from B(1, θ); 0 ≤ θ ≤1. Find an unbiased estimator of $\theta^2(1-\theta)$.

[9] Using Neyman Fisher Factorization Theorem, find a sufficient based on a random sample $X_1, X_2, \ldots, X_n$ from each of the following distributions

(a) $f_\alpha(x) = \begin{cases} \frac{1}{\alpha}\exp\left(-\frac{x}{\alpha}\right) & if\ x > 0 \\ 0 & otherwise. \end{cases}$

(b) $f_\beta(x) = \begin{cases} \exp\left(-(x-\beta)\right) & if\ x > \beta \\ 0 & otherwise. \end{cases}$

(c) $f_{\alpha,\beta}(x) = \begin{cases} \frac{1}{\alpha}\exp\left(-\frac{(x-\beta)}{\alpha}\right) & if\ x > \beta \\ 0 & otherwise. \end{cases}$

(d) $f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(log\ x_1-\mu)^2}{2\sigma^2}\right) & if\ x > 0 \\ 0 & otherwise \end{cases}$

(e) $f_\theta(x) = \begin{cases} \frac{1}{\theta} & -\frac{\theta}{2} \le x \le \frac{\theta}{2} \\ 0 & otherwise \end{cases}$

[10] Let $X_1\ and\ X_2$ be independent random samples with densities $f_1(x_1) = \theta e^{-\theta\, x_1}\ and\ f_2(x_2) = 2\theta e^{-2\theta\, x_2}$ as the respective p.d.f.s where θ >0 is an unknown parameter and 0<$x_1, x_2 < \infty$. Using Neyman Fisher Factorization Theorem find a sufficient stastic for θ.

[11]Let $X_1, \dots, X_n$ be a random sample with densities

$$f_{x_i}(x) = \begin{cases} \exp(i\theta - x) & if\ x \ge i\theta \\ 0 & otherwise. \end{cases}$$

Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[12] Let $X_1, X_2, \dots, X_n$ be a random sample from a Beta (α, β) distribution $(\alpha > 0, \beta > 0) with\ p.d.f.$

$$f(x) = \begin{cases} \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta}\, x^{\alpha-1}\,(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

Show that

(a) $\prod_{i=1}^n X_i$ is sufficient for α if β is known to be a given constant.
(b) $\prod_{i=1}^n (1-X_i)$ is sufficient for β if α is known to be given constant.
(c) $(\prod_{i=1}^n X_i\,, \prod_{i=1}^n (1-X_i))$ is jointly sufficient for (α, β) if both the parameters are unknown.

[13] Let T and T* be two statistic such that T= $\Psi(T^*)$, Show that if T is sufficient then T^* is also sufficient.

[14] $X_1, \dots, X_n$ be a random sample from U$\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$, $\theta \in \Re$. Find a sufficient statistic for θ.

[15] Let $X_1, \dots, X_n$ be independent random variables with $X_i (i = 1, 2, \dots, n)$ having the

$$f_i(x_i) = \begin{cases} i\theta\, e^{-i\theta\, x_i} & x_i > 0 \\ 0 & otherwise \end{cases}$$

Find a sufficient statistic for θ.

Solution Key

(1) $E(X) = \frac{1}{\beta}\int_0^\infty x \ \ e^{-\frac{\beta}{x}}dx = \ \beta$

$$\Rightarrow E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\sum E(X_i) = \ \beta$$
$$\Rightarrow \bar{X} \ is \ u.e.of \ \beta$$

(2) $f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} \ \ x^{n-1} & 0 < x < \theta \\ 0 & o/w \end{cases}$

$$E\left(X_{(n)}\right) = \frac{n}{\theta^n}\int_0^\theta x^n \ \ dx = \frac{n}{n+1}\theta$$
$$\Rightarrow E\left(\frac{n+1}{n} \ X_{(n)}\right) = \ \theta$$
$$\Rightarrow \frac{n+1}{n}X_{(n)} \ is \ u.e.of \ \theta.$$

Also $f_X(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & o/w \end{cases}$

$$E(X) = \frac{\theta}{2}$$
$$\Rightarrow E(2\bar{X}) = E\left(\frac{2}{n}\sum X_i\right)$$
$$= \frac{2}{n}\sum E(X_i) = \theta$$

$\Rightarrow$ 2 $\bar{\text{X}}$ is u. e. for θ

(3) $E(X) = \ \beta\int_0^\infty x \ \ e^{-\beta x} \ \ dx = \frac{1}{\beta}$

$$E(\bar{X}) = \ \beta$$
$$\Rightarrow \bar{X} is \ u.e.of \ \beta.$$
$$\left(E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\sum E(X_i)\right)$$

(4) $T_1 = \ \sum X_i \, , T_2 = \ \sum {X_i}^2$

$$E\left({T_1}^2\right) = V(T_1) + E^2(T_1)$$
$$= n\theta^2 + n^2\theta^2 = \ \theta^2 n(n+1)$$
$$\Rightarrow E\left(\frac{{T_1}^2}{n(n+1)}\right) = \ \theta^2$$
$$\Rightarrow \frac{{T_1}^2}{n(n+1)} \ is \ u.e.of \ \theta^2$$
$$E(T_2) = E\left(\sum {X_i}^2\right) = \ \sum E\left({X_i}^2\right)$$
$$= \ \sum\left(V(X_i) + E^2(X_i)\right)$$
$$= \ \sum(\theta^2 + \theta^2) = 2n\theta^2$$

$$\Rightarrow \frac{T_2}{2n} \quad is\, u.e. of\ \theta^2$$

(5) $g(\theta) = \theta e^{-2\theta}$

$$\delta_0(\underset{\sim}{X}) = \begin{cases} 1 & if\ X_1 = 0, X_2 = 0 \\ 0 & o/w \end{cases}$$

$$E\left(\delta_0(\underset{\sim}{X})\right) = 1.P(X_1 = 0, X_2 = 1)$$

$$= P(X_1 = 0)P(X_2 = 1)$$

$$= e^{-\theta}.\frac{e^{-\theta}\theta^1}{1!} = \theta e^{-2\theta}$$

$\Rightarrow \delta_0(\underset{\sim}{X})$ is u. e. of$\theta e^{-2\theta}$.

(6) $X_1, \dots, X_n$ i. i. d. B(1, θ)

$$\sum X_i \sim B(n, \theta)$$

$$E\left(T(\underset{\sim}{X})\right) = \frac{\frac{1}{2}\sqrt{n} + E(\sum X_i)}{n + \sqrt{n}} = \frac{\frac{1}{2}\sqrt{n} + n\theta}{n + \sqrt{n}} \neq \theta$$

$\Rightarrow$ T($\underset{\sim}{X}$) is not u. e. of θ.

$$\lim_{n\to\infty} E(T(\underset{\sim}{X})) = \lim_{n\to\infty} \frac{\frac{1}{2} + n\theta}{n + \sqrt{n}} = \theta$$

$\Rightarrow T(\underset{\sim}{X})$ is unbiased in the limit for θ

(7) $X_1, \dots, X_n$ r. s. from (μ, σ^2)

$$\Rightarrow \left.\begin{matrix} \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \\ Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}{}^2 \end{matrix}\right\rangle indep.$$

If Z$\sim \chi_m{}^2, then$

$$E\left(\frac{1}{z}\right) = \frac{1}{2^{\frac{m}{2}}\sqrt{m/2}} \int_0^\infty z^{-1} e^{-\frac{z}{2}} z^{\frac{m}{2}-1}\, dz$$

$$= \frac{1}{2^{\frac{m}{2}}\sqrt{\frac{m}{2}}} \int_0^\infty e^{-\frac{z}{2}} z^{\frac{m}{2}-1-1}\, dz$$

$$= \frac{\sqrt{\frac{m}{2} - 1}\ 2^{\frac{m}{2} - 1}}{2^{\frac{m}{2}}\sqrt{\frac{m}{2}}} = \frac{1}{m-2}$$

$$\&\ E\left(\frac{1}{\sqrt{z}}\right) = \frac{1}{2^{m/2}\sqrt{\frac{m}{2}}} \int_0^\infty e^{-\frac{z}{2}} \quad Z^{\frac{m}{2}-\frac{1}{2}-1}\, dz$$

$$= \frac{\sqrt{\frac{m-1}{2}}\, 2^{\frac{\cancel{m}}{2}-\frac{1}{2}}}{2^{\frac{\cancel{m}}{2}}\sqrt{\frac{m}{2}}} = \frac{\sqrt{\frac{m-1}{2}}}{\sqrt{2}\sqrt{\frac{m}{2}}}$$

$$\Longrightarrow E\left(\frac{1}{Y}\right) = E\left(\frac{\sigma^2}{(n-1)s^2}\right) = \frac{1}{(n-1)-2} = \frac{1}{n-3}$$

$$\Longrightarrow E\left(\frac{1}{s^2}\right) = \frac{n-1}{n-3}.\frac{1}{\sigma^2}.$$

& $E\left(\frac{1}{\sqrt{Y}}\right) = E\left(\frac{\sigma}{\sqrt{n-1}\,S}\right) = \frac{\sqrt{\frac{n-2}{2}}}{\sqrt{2}\,\Gamma\frac{n-1}{2}}$

$$\Longrightarrow E\left(\frac{1}{S}\right) = \frac{\sqrt{n-1}\,\Gamma\frac{n-1}{2}}{\sqrt{2}\;\Gamma\frac{n-1}{2}} = \frac{1}{\sigma}$$

$since$ $\overline{X}$ & s^2 are indep.

$$E\left(\frac{\overline{X}}{s^2}\right) = E(\bar{X}).E\left(\frac{1}{s^2}\right)$$

$$= \mu.\frac{n-1}{n-3}.\frac{1}{\sigma^2}.$$

$\Longrightarrow E\left(\frac{n-3}{n-1}.\frac{\bar{X}}{s^2}\right) = \frac{\mu}{\sigma^2}$

$\Longrightarrow \frac{n-3}{n-1}.\frac{\bar{X}}{s^2}$ is an unbiased estimator of $\frac{\mu}{\sigma^2}$.

Further

$$E\left(\frac{\overline{X}}{s}\right) = E(\bar{X}).E\left(\frac{1}{S}\right)$$

$$= \mu.\frac{\sqrt{n-1}\,\Gamma\frac{n-1}{2}}{\sqrt{2}\;\Gamma\frac{n-1}{2}}.\frac{1}{\sigma}$$

$$\Longrightarrow E\left(\sqrt{\frac{2}{n-1}}\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n-1}{2}}\frac{\overline{X}}{s}\right) = \frac{\mu}{\sigma}$$

$\Longrightarrow \sqrt{\frac{2}{n-1}}.\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n-1}{2}}.\frac{\overline{X}}{s}$ is an unbiased estimator of $\frac{\mu}{\sigma}$.

(8) $X_1, \ldots, X_n$ are i. i. d. B(1, θ)

$$g(\theta) = \theta^2(1-\theta)$$

$Define$ $(\underline{X})= \begin{cases} 1 & if\ X_1 = 1, X_2 = 1, X_3 = 0 \\ 0 & o/w \end{cases}$

$$E_\theta\ \delta(\underline{X}) = P(X_1 = 1, X_2 = 1, X_3 = 0)$$

$$= P(X_1 = 1)P(X_2 = 1)P(X_3 = 0)$$

$$= \theta^2(1-\theta)$$

$\Longrightarrow$ $(\underline{X})$ is an u. e. of g(θ)= $\theta^2(1-\theta)$.

(9)

(a) f(x | α)= $\frac{1}{\alpha}\ e^{-\frac{x}{\alpha}};\quad x>0$

jt p. d. f. $f(\underline{x}|\alpha)=\frac{1}{\alpha^n}\quad e^{-\frac{1}{\alpha}\sum x_i}\ X_1,\dots,X_n>0$

$$=\left(\frac{1}{\alpha^n}\quad e^{-\frac{1}{\alpha}\sum x_i}\right).1$$

$$i.e.\ f(\underline{x}|\alpha)=g\left(\alpha,\sum_1^n x_i\right).h(\underline{x})(h(\underline{x})=1).$$

By NFFT, T (X) = $\sum_1^n x_i$ is suff for α

(9)(b) $f(x|\beta)=\ e^{-(x-\beta)}\quad x>\beta$

$$f(\underline{x}|\beta)=\begin{cases}e^{-\sum(x_i-\beta)}, & X_1,\dots,X_n>\beta\\ 0 & o/w\end{cases}$$

$$i.e. f(\underset{\sim}{x}|\beta)=\begin{cases}e^{-\sum x_i+n\beta}, & x_{(1)}>\beta\\ 0 & o/w\end{cases}$$

$$i.e.\ f(\underset{\sim}{x}|\beta)=e^{n\beta-\sum x_i}\ I_{(\beta,x_{(1)})}\left[I_{(a,b)}=\begin{cases}1 & a<1\\ 0 & o/w\end{cases}\right]$$

$$=\left(e^{-\sum x_i}\right)\left(e^{n\beta}I_{(\beta,x_{(1)})}\right)$$

$$=h(\underline{x})g\left(\beta,x_{(1)}\right)$$

By NFFT, T(X)= $X_{(1)}$ is a suff statistic.

(9) (c) $f(\underset{\sim}{x}|\alpha,\beta)=\frac{1}{\alpha^n}\exp\left(\frac{-\sum x_i}{\alpha}+\frac{n\beta}{\alpha}\right)I_{(\beta,x_{(1)})}$

$$=\left(\frac{1}{\alpha^n}\exp\left(\frac{-\sum x_i}{\alpha}+\frac{n\beta}{\alpha}\right).I_{(\beta,x_{(1)})}\right).\underset{\downarrow}{1}$$

$$=g\left((\alpha,\beta);\sum x_i\ \ x_{(1)}\right).\ h(\underline{x})$$

By, NFFT, T(X)= $\left(\sum x_i\ \ x_{(1)}\right)$ is jointly sufficient for (α, β).

(9)(d) $f(\underset{\sim}{x}|\mu,\sigma)=\ \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n\left[\prod_{i=1}^n\left(\frac{1}{x_i}\right)\right].\exp\left(-\frac{1}{2\sigma^2}\sum(logx_i-\mu)^2\right)$

$$=\left(\frac{1}{\sigma^n}\exp\left(-\frac{n\mu^2}{2\sigma^2}-\frac{1}{2\sigma^2}\sum(logx_i)^2+\frac{\mu}{\sigma^2}\sum logx_i\right)\right)\times\left(\left(\frac{1}{\sqrt{2\pi}}\right)^n\prod_{i=1}^n{x_i}^{-1}\right)$$

$$= g(\mu,\sigma); \left(\sum logx_i , \sum (logx_i)^2\right). h(\underset{\sim}{x}) \left(\left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} {x_i}^{-1}\right)$$

By NFFT ($\sum logx_i$,$\sum(logx_i)^2$) is jointly sufficient for (μ, σ)

(9)(e)f($\underset{\sim}{x}$|θ)= $\begin{cases} \frac{1}{\theta^n}; & -\frac{\theta}{2} < x_1 \dots x_n < \theta/2 \\ 0 & o/w \end{cases}$

$$= \begin{cases} \frac{1}{\theta^n}; & |x_i| < \frac{\theta}{2}; i = 1,2,\dots n \\ 0 & o/w \end{cases}$$

i.e. f($\underset{\sim}{x}$|θ)= $\begin{cases} \frac{1}{\theta^n}; & \text{Max}_i |x_i| < \frac{\theta}{2}; i = 1,2,\dots n \\ 0 & o/w \end{cases}$

$\Longrightarrow$ f ($\underset{\sim}{x}$|θ)= $\frac{1}{\theta^n} I\left(\text{Max}_i |x_i| , \frac{\theta}{2}\right)$

$\Longrightarrow$By NFFT T($\underline{X}$)= $\underset{i}{\text{Max}} |x_i|$ is suff for θ.

(10) Joint p. d. f. of X_1 & X_2

$$f(x_1,x_2) = \theta e^{-\theta x_1}.2\theta\, e^{-2\theta x_2}$$
$$f(x_1,x_2) = \left(\theta^2 e^{-\theta(x_1+2x_2)}\right)$$
$$f(x_1,x_2) = \left(\theta^2 e^{-\theta(x_1+2x_2)}\right)(2)$$

BY NFFT T$(X_1 , X_2) = X_1 + 2X_2$ is suff for θ.

(11) jt p. d. f.

f ($\underset{\sim}{x}$|θ)= $\begin{cases} \prod_{i=1}^{n} \exp(i\theta - x_i) & if \frac{x_1}{1}, \frac{x_2}{2}, \dots, \frac{x_n}{n} \geq \theta \\ 0 & o/w \end{cases}$

$$= \begin{cases} e^{-\sum_1^n x_i}\, e^{\theta\frac{n(n+1)}{2}} & if \frac{x_i}{i} \geq \theta \\ 0 & o/w \end{cases}$$

$$i.e. \text{f}\,(\underset{\sim}{x}|\theta) = \left(e^{\theta\frac{n(n+1)}{2}} \text{I}\left(\theta, \underset{i}{\text{Min}} \frac{x_i}{i}\right)\right)\left(e^{-\sum_{i=1}^{n} x_i}\right)$$

$$\swarrow \qquad\qquad\qquad \swarrow$$

$$= \text{g}\left(\theta, \underset{i}{\text{Min}} \frac{x_i}{i}\right) \qquad \times \qquad \text{h}(\underset{\sim}{x})$$

$\Longrightarrow$ T ($\underset{\sim}{X}$) = $\text{Min}_i \frac{X_i}{i}$ is suff.

(12)$X_1, \dots, X_n$ $i.i.d$ $Beta(\alpha,\beta)$

(a) β is known – α is the unknown parameter

f ($\underset{\sim}{x}$|α) = $\begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\alpha}\right)^n (\pi x_i)^{\alpha-1}\right]\left[\underbrace{\left(\pi(1-x_i)\right)^{\beta-1} \left(\frac{1}{\Gamma\beta}\right)^n}_{h(\underset{\sim}{x})}\right] & 0 < x_1, \dots, x_n < 1 \\ 0 & o/w \end{cases}$

By NFFT $(\prod_{i=1}^{n} X_i)$ is suff for α.

(b) α is known – β is the unknown parameter

$$f(\underset{\sim}{x}|\alpha) = \begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\beta}\right)^n \left(\pi(1-x_i)\right)^{\beta-1}\right]\left[\underbrace{(\pi x_i)^{\alpha-1}\left(\frac{1}{\Gamma\alpha}\right)^n}_{h(\underset{\sim}{x})}\right] & 0 < x_1, \dots, x_n < 1 \\ 0 & o/w \end{cases}$$

By NFFT $\prod_{i=1}^{n}(1-X_i)$ is suff for β.

(c) α, β both unknown

$$f(\underset{\sim}{x}|\alpha,\beta) = \begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\alpha\;\Gamma\beta}\right)^n (\pi x_i)^{\alpha-1}\left(\pi(1-x_i)\right)^{\beta-1}\right].1 & 0 < x_1, \dots, x_n < 1 \\ 0 & o/w \end{cases}$$

By NFFT $(\pi x_i, \pi(1-x_i)\,)$ is jointly sufficient for (α, β).

(13) T is suff for $\theta \in \Theta$ & T= $\Psi(T^*)$

By NFFT T is suff for θ iff.

$f(\underset{\sim}{x}|\theta) = g(\theta, t(\underset{\sim}{x}))\quad h(\underset{\sim}{x})$

i.e. $f(\underset{\sim}{x}|\theta) = g(\theta, \Psi(t^*(\underset{\sim}{x}))\,.\,h(\underset{\sim}{x})$

$f(\underset{\sim}{x}|\theta) = g'(\theta, t^*(\underset{\sim}{x}))\,.\,h(\underset{\sim}{x})$

$\Longrightarrow T^*(\underset{\sim}{X})$ is suff for θ.

(14) $$f(\underset{\sim}{x}|\theta) = \begin{cases} 1, & \theta - \frac{1}{2} < x_{(1)}, \dots, < x_{(n)} < \theta + \frac{1}{2} \\ 0 & o/w \end{cases}$$

i.e. $f(\underset{\sim}{x}|\theta) = I_{\left(\theta-\frac{1}{2}, x_{(1)}\right)} I_{\left(x_{(n)}, \theta+\frac{1}{2}\right)}$

$$= g\left(\theta, \left(x_{(1)}, x_{(n)}\right)\right) \quad h(\underset{\sim}{x})$$

By NFFT, T($\underset{\sim}{X}$)= $\left(X_{(1)}, X_{(n)}\right)$ is jointly suff for θ.

(15) $f(\underset{\sim}{x}|\theta) = \left(\theta - e^{-\theta x_1}\right)\left(2\theta - e^{-2\theta x_2}\right)\dots\left(n\theta - e^{-n\theta x_n}\right)$

i.e. $f(\underset{\sim}{x}|\theta) = \theta^n (\prod_{i=1}^{n} i) e^{-\theta\, \sum_{i=1}^{n} i x_i}$

$$= \left(\prod_{i=1}^{n} i\right)\left(\theta^n\, e^{-\theta\, \sum_{i=1}^{n} i x_i}\right)$$

$$= h(\underline{x})\, g\left(\theta, \sum_{i=1}^{n} i x_i\right)$$

By NFFT, T($\underline{X}$)= $\sum_{i=1}^{n} iX_i$ is sufficient for θ.

Problem Set-11

[1] Let $X_1, \ldots, X_n$ be a random sample from P(θ), $\theta \in (0, \infty)$. Show that T= $\sum_{i=1}^{n} X_i$ is complete sufficient statistic. Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of the following parametric functions: (a) g(θ) $= \theta$, (b) g(θ) $= e^{-\theta}$ (c) g(θ) $= e^{-\theta}(1+\theta)$.

[2] Suppose $X_1, \ldots, X_n$ be a random sample from B(1, θ), $\theta \in (0, 1)$. Show that T= $\sum_{i=1}^{n} X_i$ is complete sufficient statistic and hence find the UMVUE for each of the following parametric functions : (a) g(θ) $= \theta$, (b) g(θ)= θ^4 and (c) g(θ) $= \theta(1-\theta)^2$.

[3] Let $X_1, \ldots, X_n$ be a random sample from Exp (θ, 1), i. e.

$$f(x \mid \theta) = \begin{cases} e^{-(x-\theta)} & if\ x > \theta \\ 0 & otherwise \end{cases}$$

Show that T= $X_{(1)} = \min\{X_1, \ldots, X_n\}$ is a complete sufficient statistic and hence find the UMVUE of g (θ) $=\theta^2$.

[4] $X_1, \ldots, X_n$ is a random sample from U(0, θ), $\theta > 0$. Show that T=$X_{(n)} = \max\{X_1, \ldots, X_n\}$ is a complete sufficient statistic and find the UMVUE of g(θ) $=\theta^k; k > -n$.

[5] $X_1, \ldots, X_n$ is a random sample from Gamma (2, θ), θ> 0, i.e.

$$f(x \mid \theta) = \begin{cases} \frac{1}{\Gamma 2\theta^2}\, e^{-x/\theta} x & if\ x > 0 \\ 0 & otherwise \end{cases}$$

Show that T= $\sum_{i=1}^{n} X_i$ is complete sufficient statistic and find the UMVUE of θ.

[6] Let $X_1, \ldots, X_n$ be a random sample from U(θ – ½ , θ + ½). Show that sufficient statistic is not complete.

[7] Suppose the statistic T is UMVUE of θ such that V(T) $\neq 0$. Show that T^2 cannot be UMVUE of θ^2.

[8] Let $X_1, \ldots, X_n$ be a random sample from N(0, θ). Find the UMVUE ofθ^2.

[9] Let $X_1, \ldots, X_n$ be a random sample from N(μ, θ). Find the UMVUE of (a) θ when μ is known and (b) θ when μ is not known.

[10] Let $X_1, \ldots, X_n$ be a random sample from N (μ, σ^2). Find the UMVUE of (a) σ^r when μ is known, (b) σ^r when μ is not known and (c) δ, where δ is given by P(X$\geq \delta$)= p for a given p.

[11] $X_1, \dots, X_n$ is a random sample from U $(0, \theta)$, $\theta > 0$. Consider the following 3 estimators for θ;

$$T_1(\underset{\sim}{X}) = \frac{n+1}{n} X_{(n)}, T_2(\underset{\sim}{X}) = 2\bar{X} \text{ and } T_3(\underset{\sim}{X}) = X_{(1)} + X_{(n)}.$$

Show that all the estimators are unbiased for θ. Among the three estimators, which are would you prefer and why?

[12] $X_1, \dots, X_n$ be a random sample from N (μ, σ^2), $\mu \in \Re$, $\sigma \in \Re^+$. Assuming completeness of the associated sufficient statistic find the UMVUE of μ^2 and $\mu +$.

[13] Let $X_1, \dots, X_n$ be a random sample from Exp (a, b). Assuming completeness of the associated sufficient statistic, find the (a) UMVUE of a when b is known and (b) UMVUE of b when a is known.

Solution Key

(1) (a) g(θ)= θ

E(T) = nθ $\Longrightarrow$ E(T/n) = θ

T/n u. e. based on CSS.

$\Longrightarrow \frac{T}{n}$ is UMVUE of θ

(b) g(θ) = $e^{-\theta}$

Consider $\delta_0(X) = \begin{cases} 1, & if\ X_1 = 0 \\ 0, & o/w \end{cases}$

E$\delta_0(X) = P(X_1 = 0) = e^{-\theta} \Longrightarrow \delta_0(X) is\ u.e. of e^{-\theta}$

Rao-blackwellization of $\delta_0(X)$

(t) = E$(\delta_0(X) \,|\, T = t)$

= P$(X_1 = 0 \,|\, T = t) = \frac{P(X_1=0, T=t)}{P(T=t)} = \frac{P(X_1=0, \sum_{i=2}^{n} X_i = t)}{P(T=t)}$

$$= \frac{P(X_1 = 0) P(\sum_{i=2}^{n} X_i = t)}{P(T = t)} = \frac{\left(e^{-\theta}\right)\left(\frac{e^{-(n-1)\theta}\left((n-1)\theta\right)^t}{t!}\right)}{\frac{e^{-n\theta}(n\theta)^t}{t!}}$$

$$= \left(\frac{n-1}{n}\right)^t$$

(T)= $\left(\frac{n-1}{n}\right)^T$ is u. e. based on CSS

$\Longrightarrow \left(\frac{n-1}{n}\right)^T$ is UMVUE of $e^{-\theta}$.

(c)$g(\theta) = e^{-\theta}(1+\theta)$

$$\delta_0(X) = \begin{cases} 1, & if\ X_1 = 0 \\ 0, & o/w \end{cases}$$

$E\delta_0(X) = P(X_1 \leq 1) = P(X_1 = 0) + P(X_1 = 1)$

$= e^{-\theta} + \theta e^{-\theta} = e^{-\theta}(\theta + 1)$

$\Longrightarrow \delta_0(X)$ is u.e. of $e^{-\theta}(\theta + 1)$.

Rao-blackwellization of $\delta_0(X)$

(t) $= E(\delta_0(X) \,\big|\, T = t)$

$= P(X_1 \leq 1 \,\big|\, T = t)$

$$= \frac{P(X_1 \leq 1 | T = t)}{P(T = t)}$$

$$= \frac{P(X_1 = 0\ \cup X_1 = 1)}{P(T = t)}$$

$$= \frac{P(X_1 = 0, T = t) + P(\ X_1 = 1, T = t)}{P(T = t)}$$

$$= \frac{P(X_1 = 0, \sum_2^n X_i = t) + P(X_1 = 1, \sum_2^n X_i = t - 1)}{P(T = t)}$$

$$= \frac{P(X_1 = 0)P(\sum_2^n X_i = t) + P(X_1 = 1)P(\sum_2^n X_i = t - 1)}{P(T = t)}$$

$$= \frac{e^{-\theta}\left(\frac{e^{-(n-1)\theta}\left((n-1)\theta\right)^t}{t!}\right) + \theta e^{-\theta}\left(\frac{e^{-(n-1)\theta}\left((n-1)\theta\right)^{t-1}}{(t-1)!}\right)}{\frac{e^{-n\theta}(n\theta)^t}{t!}}$$

$$= \left(\frac{n-1}{n}\right)^t \left(1 + \frac{t}{n-1}\right)$$

(T)$= \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$ is u.e. of $g(\theta)$ based on C.S.S T

$\Longrightarrow \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$ is UMVUE of $g(\theta) = e^{-\theta}(1+\theta)$.

(2) $X_1, \dots, X_n$ r. s. from $B(1, \theta)$, $\theta \in (0, 1) = \Theta$

By NFFT T= $\sum_{i=1}^{n} X_i$ is sufficient

T ~B(n, θ)

Note that p. m. f. of $X_i \sim B(1,\theta)$ is

f(x)= $\theta^x(1-\theta)^{1-x}$

$= \left(\frac{\theta}{1-\theta}\right)^x (1-\theta)$

$=$ exp(x log $\left(\frac{\theta}{1-\theta}\right) + \log(1-\theta)$)

With h(x) =1, (θ)= log $\left(\frac{\theta}{1-\theta}\right)$, T(x)= x & $\beta(\theta)$= - log (1-θ)

This is 1-parameter exponential family

Natural parameter space {: $\eta\in(0,\infty)$} ({$\eta(\theta)$: $\theta\in$ (0, 1)}).

Natural parameter space contains open intervals.

$\Longrightarrow$ The 1-parameter expo family is of full rank (complete)

$\Longrightarrow$T($\underset{\sim}{X}$) = $\sum_{i=1}^{n} X_i$ is C.S.S.

Alternate proof using convergent power series argument done class.

(a) g(θ)= θ

E(T) = n$\theta \Longrightarrow$ E $\left(\frac{T}{n}\right) = \theta$

$\frac{T}{n}$u. e. based on CSS T

$\Longrightarrow \frac{T}{n}$ is UMVUE of θ.

(b) g(θ)= θ^4

$$\delta_0(\underset{\sim}{X}) = \begin{cases} 1, & if\ X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1 \\ 0, & o/w \end{cases}$$

E$\delta_0(\underset{\sim}{X}) = P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1) = \prod_{i=1}^{4} P(X_i = 1) = \theta^4$

$\Longrightarrow \delta_0(\underset{\sim}{X})$ is u.e. of θ^4.

Rao-blackwellization of θ^4

(t) = E($\delta_0(\underset{\sim}{X}) \,\big|\, T = t$)

$$= X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1 \,\big|\, T = t$$

$$= \frac{P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, \sum_{i=5}^{n} X_t = t-4)}{P(T = t)}$$

$$= \frac{P(X_1 = 1)P(\,X_2 = 1)P(\,X_3 = 1)P(\,X_4 = 1)P(\sum_{i=5}^{n} X_t = t-4)}{P(T=t)}$$

$$= \frac{\theta.\theta.\theta.\theta.\left(\binom{n-4}{t-4}\theta^{t-4}(1-\theta)^{n-t}\right)}{\binom{n}{t}\theta^{t}(1-\theta)^{n-t}}$$

$$= \frac{\binom{n-4}{t-4}}{\binom{n}{t}} = \frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)}$$

(T)= $\frac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)}$ is u.e. based on C.S.S.

$\Longrightarrow$(T) is UMVUE of θ^4.

(c) $g(\theta) = \theta(1-\theta)^2$

$$\delta_0(\underset{\sim}{X}) = \begin{cases} 1, & if\ X_1 = 1, X_2 = 0, X_3 = 0 \\ 0, & o/w \end{cases}$$

$E\delta_0(\underset{\sim}{X}) = P(X_1 = 1, X_2 = 0, X_3 = 0) = \theta(1-\theta)^2$

$\Longrightarrow \delta_0(\underset{\sim}{X})$ is u.e. of $\theta(1-\theta)^2$

Rao-blackwellization of $\delta_0(\underset{\sim}{X})$

(t) = $E(\delta_0(\underset{\sim}{X}) \,|\, T = t)$

$$= P(X_1 = 1, X_2 = 0, X_3 = 0 | T = t)$$

$$= \frac{P(X_1 = 1, X_2 = 0, X_3 = 0, \sum_4^n X_i = t-1\,)}{P(T=t)}$$

$$= \frac{P(X_1 = 1)P(\,X_2 = 0)P(\,X_3 = 0)P(\sum_4^n X_i = t-1)}{P(T=t)}$$

$$= \frac{\theta.(1-\theta).(1-\theta)\binom{n-3}{t-1}\theta^{t-1}\,(1-\theta)^{n-3-t+1}}{\binom{n}{t}\theta^{t}(1-\theta)^{n-t}}$$

$$= \frac{\binom{n-3}{t-1}}{\binom{n}{t}} = \frac{\frac{(n-3)!}{(t-1)!\,(n-t-2)!}}{\frac{n!}{t!\,(n-t)!}}$$

$$= \frac{t.(n-t).(n-t-1)}{n(n-1)(n-2)}$$

(T)= $\frac{T.(n-T).(n-T-1)}{n(n-1)(n-2)}$ is u.e. based on C.S.S. T

$\Longrightarrow$(T) is UMVUE of $\theta(1-\theta)^2$.

(3) $X_1, \ldots, X_n$ r.s. from f(x)= $\begin{cases} e^{-(x-\theta)} & if\ x > \theta \\ 0 & otherwise \end{cases}$

By NFFT T= $X_{(1)}$ is sufficient.

$$f_T(t) = \begin{matrix} ne^{-n(t-\theta)} & if\ t > \theta \\ 0 & o/w \end{matrix}$$

Now E g(T)= 0 $\forall\, \theta \in \Theta$

$\Longrightarrow \int_\theta^\infty g(t)\, ne^{-n(t-\theta)}\, dt = 0 \quad \forall\, \theta \in \Theta$

i.e. $\int_\theta^\infty g(t)\, e^{-nt}\, dt = 0 \quad \forall\, \theta \in \Theta$

Differentiating w.r.t. θ, we get

g(θ) $e^{-n\theta} = 0 \quad \forall\, \theta \in \Theta$

$\Longrightarrow$ g(θ)= 0 $\forall$ 0<θ <t ; 0< t < ∞

$\Longrightarrow$ g(t) = 0 a.e. (θ<t <∞)

$\leftrightarrow$

$\downarrow$

Range of T=$X_{(1)}$

$\Longrightarrow$T= $X_{(1)}$ is complete.

g(θ)= θ^2

E $X_{(1)}$=$\int_\theta^\infty t\, ne^{-n(t-\theta)} dt$

$$= n \int_0^\infty (y+\theta)\, e^{-ny}\, dy = n\left[\frac{\Gamma 2}{n^2} + \theta.\frac{1}{n}\right] = \theta + \frac{1}{n}$$

E($X_{(1)} - \frac{1}{n}$)= θ

Sly E ${X_{(1)}}^2 = n \int_\theta^\infty t^2\, e^{-n(t-\theta)}\, dt$

y= t- θ

= n$\int_0^\infty (y+\theta)^2\, e^{-ny}\, dy$

$$= n \int_0^\infty (y^2 + \theta^2 + 2\theta y\, e^{-ny}\, dy$$

$$= n\left[\frac{\Gamma 3}{n^3} + \theta^2 \frac{1}{n} + 2\theta \frac{\Gamma 2}{n^2}\right]$$

$$= \frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}$$

$\Longrightarrow$ E ${X_{(1)}}^2 = \frac{2}{n^2} + \theta^2 + \frac{2}{n}\, E\left(X_{(1)} - \frac{1}{n}\right)$

$\Longrightarrow E\left({X_{(1)}}^2 - \frac{2}{n^2} - \frac{2}{n}\left(X_{(1)} - \frac{1}{n}\right)\right) = \theta^2$

$$i.e. E({X_{(1)}}^2 - \frac{\cancel{2}}{n^2} - \frac{2}{n} X_{(1)} + \frac{\cancel{2}}{n^2} = \theta^2$$

$\Longrightarrow$ E$\left({X_{(1)}}^2 - \frac{2}{n} X_{(1)}\right) = \theta^2$

${X_{(1)}}^2 - \frac{2}{n} X_{(1)}$ is u.e. of θ^2 based on C.S.S. $X_{(1)}$

$\Longrightarrow {X_{(1)}}^2 - \frac{2}{n} X_{(1)}$ is UMVUE of θ^2

(4) $X_1, \dots, X_n$ r.s. from U(0, θ)

T= $X_{(n)}$ is C.S.S. (proved in class)

g(θ) $= \theta^k$

p.d.f. of T; $f_T(t)$ $\begin{cases} \frac{n}{\theta^n}\ t^{n-1}, & 0 < t < \theta \\ 0\ , & o/w \end{cases}$

E$\left({X_{(n)}}^k\right) = \frac{n}{\theta^n} \int_0^\theta t^k\, t^{n-1}\, dt = \frac{n}{n+k} \theta^k.$

$\Longrightarrow E\left(\frac{n+k}{n} {X_{(n)}}^k\right) = \theta^k.$

$\frac{n+k}{n}\ {X_{(n)}}^k$ is u.e. based of C.S.S. $X_{(n)}$

$\Longrightarrow \frac{n+k}{n}\ {X_{(n)}}^k$ is UMVUE of θ^k.

(5) $X_1, \dots, X_n$ r.s. from G(2, θ); θ> 0

p.d.f.

f(x)= $\begin{cases} \frac{1}{\Gamma 2 \theta^2}\ e^{-x/\theta} x^{\ \ 2-1}\ \ if\ x > 0 \\ \qquad 0 \quad o/w \end{cases}$

Note that the p.d.f. can be written as

f(x)= x exp $(-\frac{x}{\theta} -\ 2\ log\theta)$

with h(x)= x ; $\eta(\theta)$= $-\frac{1}{\theta}$ $\ T(x) = x\ ;\ \ \beta(\theta) = 2\ log\theta$

the above is 1-parameter expo family distn.

Natural parameter space

$\{:\eta < 0\}$ $(\{(\theta) : \theta \in (0, \infty)\})$.

The above contains open intervals

$\Longrightarrow$ 1-parameter expo family is full rank (complete)

$\Longrightarrow$ $T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is C.S.S.

$E(T) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E\, X_i = 2n\theta$

$$\left[E\, X_i = \frac{1}{\theta^2}\int_0^\infty x^2 e^{-\frac{x}{\theta}}\, dx = \frac{\Gamma 3\, \theta^3}{\theta^2} = 2\theta\right]$$

$\Longrightarrow E\left(\frac{T}{2n}\right) = \theta$

$\frac{T}{2n}$ is u.e. based on C.S.S. $T = \sum_{i=1}^{n} X_i$

$\Longrightarrow \frac{T}{2n}$ is UMVUE of θ.

(6) $X_1, \ldots, X_n$ be a r.s. from $U(\theta - ½, \theta + ½)$.

jt. P.d.f $f(\underset{\sim}{x}) = \begin{cases} 1, & \theta - \frac{1}{2} < x_{(1)} < \cdots < x_{(n)} < \theta + \frac{1}{2} \\ 0 & o/w \end{cases}$

i.e. $f(\underset{\sim}{x}) = 1.\, I_{\left(\theta - \frac{1}{2}, x_{(1)}\right)}\ I_{\left(x_{(n)}, \theta + \frac{1}{2}\right)}$

$$\left(I_{(a,b)} = \begin{cases} 1, & a < b \\ 0, & o/w \end{cases}\right)$$

By NFFT, $T(\underset{\sim}{X}) = \left(X_{(1)}, X_{(n)}\right)$ is jointly suff for θ

$$f_{X_{(1)}}(x) = n\big(1 - F_X(x)\big)^{n-1} f_X(x);\ \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$$

$$= \begin{cases} n\left(\theta - x + \frac{1}{2}\right)^{n-1}, & \theta - \frac{1}{2} < x < \theta + \frac{1}{2} \\ 0 & o/w \end{cases}$$

$$E\, X_{(1)} = n\int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} x\left(\theta - x + \frac{1}{2}\right)^{n-1} dx = \theta + \frac{1}{2} - \frac{n}{n+1}$$

$$f_{X_{(n)}}(x) = n\big(F_X(x)\big)^{n-1} f_X(x);\ \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$$

$$= \begin{cases} n\left(x-\theta+\frac{1}{2}\right)^{n-1}, & \theta-\frac{1}{2}<x<\theta+\frac{1}{2} \\ 0 \quad o/w \end{cases}$$

$$E\ X_{(n)} = n\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\left(x-\theta+\frac{1}{2}\right)^{n-1} dx = \frac{n}{n+1} - \frac{1}{2} + \theta$$

$\Longrightarrow E\left(X_{(n)} - X_{(1)} - \frac{n-1}{n+1}\right) = 0 \quad \forall\ \theta$

$\nRightarrow X_{(n)} - X_{(1)} = \frac{n-1}{n+1}$ a.e.

i.e. with g(T)= $X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$

E g($\underset{\sim}{T}$)= 0 $\forall\ \theta \nRightarrow$ g(t)= 0 a.e.

$\Longrightarrow \underset{\sim}{T}= \left(X_{(n)} - X_{(1)}\right)$ is not complete.

(7) T is UMVUE of θ

$\Longrightarrow$ E(T)= θ

Suppose T^2 is UMVUE of $\theta^2 \Longrightarrow T^2$ is u.e. of θ^2

$\Longrightarrow E(T^2) = \theta^2$

$\Longrightarrow$ V(T)= $E(T^2) - E^2(T)$

i.e. V(T)= $\theta^2 - \theta^2 = 0$

whis is a cantvadiction

$\Longrightarrow T^2$ cannot be u.e. of θ^2

$\Longrightarrow T^2$ cannot be UMVUE of θ^2

(8) $X_1, \dots, X_n$ r.s. from N(0, θ)

T= $\sum_{i=1}^{n} {X_i}^2\ \ is\ C.S.S.$

$$\frac{T}{\theta} \sim {\chi_n}^2;\ \ E\left(\frac{T}{\theta}\right) = n; V\left(\frac{T}{\theta}\right) = 2n$$

$$E(T) = n\theta; V(T) = 2n\theta^2$$

$$ET^2 = V(T) + \left(E(T)\right)^2 = 2n\theta^2 + n^2\theta^2 = n(n+2)\theta^2$$

$\Longrightarrow E\left(\frac{T^2}{n(n+2)}\right) = \theta^2$

$\frac{T^2}{n(n+2)}$ is u.e. based on C.S.S. $\sum X_i{}^2 = T$

$\Rightarrow \frac{T^2}{n(n+2)}$ is UMVUE of θ^2.

(9) (a) $X_1, \dots, X_n$ r.s. from N(μ, θ); μ is known

T= $\sum_{i=1}^{n}(X_i - \mu)^2$ is CSS (μ is known)

$$E\left(\frac{T}{\theta}\right) = n \quad \left(\frac{\sum(X_i - \mu)^2}{\theta} \sim \chi_n{}^2\right)$$

$\Rightarrow \frac{\sum(X_i-\mu)^2}{n}$ is UMVUE for θ when μ is known.

(9) (b) $X_1, \dots, X_n$ r.s. from N(μ, θ); μ, θ both known

$\left(\sum X_i, \sum_{i=1}^{n} X_i{}^2\right) \Leftrightarrow \left(\bar{X}, s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2\right)$ *is C.S.S.* (2-parameter full rank expo family)

$$\frac{(n-1)s^2}{\theta} = \frac{\sum(X_i - \bar{X})^2}{\theta} \sim \chi_{n-1}{}^2$$

$$\Rightarrow E\left(\frac{\sum(X_i - \bar{X})^2}{n-1}\right) = \theta$$

$\frac{\sum(X_i-\bar{X})^2}{n-1}$ is u.e. of θ based on C.S.S.

$\Rightarrow \frac{\sum(X_i-\bar{X})^2}{n-1}$ is UMVUE of θ.

(10) (a) $X_1, \dots, X_n$ r.s. from N(μ, σ^2) μ is known

T= $\sum(X_i - \mu)^2$ is C.S.S.

g(σ)= σ^r

Y= $\frac{T}{\sigma^2} \sim \chi_n{}^2$

$$E\left(Y^{\frac{r}{2}}\right) = \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \int_0^{\infty} y^{\frac{r}{2}} e^{-\frac{y}{2}} y^{-\frac{n}{2}-1} \, dy$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \int_0^{\infty} e^{-\frac{y}{2}} y^{-\frac{n+r}{2}-1} dy$$

$$= \frac{2^{\frac{r}{2}} \Gamma\frac{n+r}{2}}{\Gamma\frac{n}{2}}$$

$$\Longrightarrow E\left(Y^{\frac{r}{2}}\right) = E\left(\frac{T^{\frac{r}{2}}}{\sigma^r}\right) = \frac{2^{\frac{r}{2}}\ \Gamma\frac{n+r}{2}}{\Gamma\frac{n}{2}}$$

$$\Longrightarrow E\left(\frac{\Gamma\frac{n}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n+r}{2}}\right) = \sigma^r$$

$\frac{\Gamma\frac{n}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n+r}{2}} T^{\frac{r}{2}}$ is u.e. of σ^r based on C.S.S.

$\Longrightarrow \frac{\Gamma\frac{n}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n+r}{2}} (\sum(X_i-\mu)^2)^{\frac{r}{2}}$ is UMVUE of σ^r (when μ is known).

(b) $X_1, \dots, X_n$ r.s. from N(μ, σ^2) ; μ & σ both unknown

$(\bar{X}, \sum(X_i-\mu)^2)$ is C.S.S.

$= (\bar{X}, {S_X}^2)$

Y$= \frac{{S_X}^2}{\sigma^2} \sim {\chi_{n-1}}^2$

$E\left(Y^{\frac{r}{2}}\right) = E\ \left(\frac{{S_X}^r}{\sigma^r}\right) = \frac{2^{\frac{r}{2}}\ \Gamma\frac{n-1+r}{2}}{\Gamma\frac{n-1}{2}}$ (following the derivation in part (a))

$\Longrightarrow E\left(\frac{\Gamma\frac{n-1}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n-1+r}{2}} {S_X}^r\right) = \sigma^r$

$\frac{\Gamma\frac{n-1}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n-1+r}{2}} {S_X}^r$ is u.e. of σ^r based on C.S.S.

$\Longrightarrow \frac{\Gamma\frac{n-1}{2}}{2^{\frac{r}{2}}\ \Gamma\frac{n-1+r}{2}} {S_X}^r$ is UMVUE of σ^r.

(c)p= P(X $\le \delta$) = P$\left(\frac{X-\mu}{\sigma} \le \frac{\delta-\mu}{\sigma}\right) = \ \Phi\left(\frac{\delta-\mu}{\sigma}\right)$

↑

Given value ; To find the UMVUE of δ

p$= \Phi\left(\frac{\delta-\mu}{\sigma}\right) \Longrightarrow \frac{\delta-\mu}{\sigma} = \ \Phi^{-1}(p)$

i.e. $\delta = \mu + \sigma\ \Phi^{-1}(p) = g(\theta)$

↓

Known $\underset{\sim}{\theta} = (\mu, \sigma)'$.

Note that E($\bar{X}$)= μ & E $\left(\frac{\Gamma\frac{n-1}{2}}{2^{\frac{1}{2}}\ \Gamma\frac{n}{2}} S_X\right) = \ \sigma$ ↖(from part (b) with r= 1)

$\Longrightarrow$ [$\bar{X} + \frac{\Gamma\frac{n-1}{2}}{2^{\frac{1}{2}}\ \Gamma\frac{n}{2}} S_X\ \Phi^{-1}(p)$] is u.e. of $\mu + \sigma\,\Phi^{-1}(p)$ based on C.S.S.

$\Longrightarrow\quad \bar{X} + \frac{\Gamma\frac{n-1}{2}}{2^{\frac{1}{2}}\ \Gamma\frac{n}{2}} S_X\ \Phi^{-1}(p)$ is UMVUE of $\delta = \ \mu + \ \Phi^{-1}(p)$.

(11) $X_1, \dots, X_n$ r.s. from U(0, θ)

It's easy to check that

$$EX_i = \frac{\theta}{2}\ \ \forall\ i$$

$$EX_{(n)} = \frac{n}{n+1}\theta \ \&\ EX_{(1)} = \frac{\theta}{n+1}$$

$$\Longrightarrow E\big(T_1(X)\big) = \frac{n+1}{n}\ EX_{(n)} = \ \theta$$

$$E\big(T_2(X)\big) = \ \theta \ \&\ E\big(T_3(X)\big) = \frac{\theta}{n+1} + \frac{n}{n+1}\theta = \theta$$

T_1, T_2, T_3 are all u.e. of

Among the 3 estimators T_1 is UMVUE and hence is the preferred estimator.

(12) $X_1, \dots, X_n$ r.s. from N(μ, σ^2)

$$\left(\bar{X}, S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2\right)\ is\ C.S.S$$

$\bar{X} \sim$ N($\mu, \sigma^2/n$) $\qquad >$ indep.

$\frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}{}^2$

$$E(\bar{X}^2) = V(\bar{X}) + \big(E(\bar{X})\big)^2$$

i.e. $E(\bar{X}^2) = \frac{\sigma^2}{n} + \ \mu^2$

$$E(\bar{X}^2) = \frac{1}{n}E(S^2) + \mu^2$$

i.e. $E\left(\bar{X}^2 - \frac{1}{n}S^2\right) = \mu^2$

$\Longrightarrow \bar{X}^2 - \frac{1}{n}S^2$ is u.e. of μ^2 based only on the C. S. S.

$\Longrightarrow \bar{X}^2 - \frac{1}{n}S^2$ is UMVUE of μ^2.

If $\underline{g(\mu, \sigma) = \mu + \sigma}$

$$E(\bar{X}) = \mu$$

And $E\left(\frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\ \Gamma\frac{n}{2}}\left((\sum(X_i - \bar{X})^2)^{\frac{1}{2}}\right)\right) = \sigma$ (from problem # 10 (c)/(b))

$\Longrightarrow E\left(\bar{X} + \frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\ \Gamma\frac{n}{2}}\left((\sum(X_i - \bar{X})^2)^{\frac{1}{2}}\right)\right) = \mu + \sigma$

$\Longrightarrow \bar{X} + \frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\ \Gamma\frac{n}{2}}(\sum(X_i - \bar{X})^2)^{\frac{1}{2}}$ is u.e. of $\mu + \sigma$ based on C.S.S.

$\Longrightarrow \bar{X} + \frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\ \Gamma\frac{n}{2}}(\sum(X_i - \bar{X})^2)^{\frac{1}{2}}$ is UMVUE of $\mu + \sigma$.

(13) $X_1, \ldots, X_n$ r.s. from

$$f(x) = \begin{cases} \frac{1}{b}\, e^{-\left(\frac{x-a}{b}\right)}, & x > a \\ 0 & o/w \end{cases}$$

(This is exponential distn with location parameter 'a' and scale parameter 'b')

(a) If <u>b is known</u> then $X_{(1)}$ is C.S.S., in such a situation

$$EX_{(1)} = a + \frac{b}{n}$$

i.e. $E\left(X_{(1)} - \frac{b}{n}\right) = a$

$X_{(1)} - \frac{b}{n}$ is u.e. based on C.S.S.

$\Longrightarrow X_{(1)} - \frac{b}{n}$ is UMVUE of a, when b is known

(b) If <u>a is known</u>, then $\sum X_i$ is C.S.S.

$$E\left(\sum X_i\right) = \sum EX_i = n(a+b)$$

$$\Longrightarrow E\left(\frac{\sum X_i}{n} - a\right) = b$$

$\Longrightarrow \bar{X}$ - a is u.e. based on C.S.S. and hence is UMVUE of b.

Remark:

[when both a & b are unknown, then

T($\underset{\sim}{X}$)= $(X_{(1)}, \sum X_i)$ or $(X_{(1)}, \sum(X_i - X_{(1)}))$ is C.S.S

We may note that under such a setup

$$X_{(1)} \sim Exp\left(a, \frac{b}{n}\right) \& \frac{2}{b}\sum\left(X_i - X_{(1)}\right) \sim \chi_{2(n-1)}{}^2$$

And the 2 are independent.]

i.e. $X_{(1)}$ & $\frac{2}{b}\sum\left(X_i - X_{(1)}\right)$ are indep.

Problem Set-12

[1] $X_1, \dots, X_n$ be a random sample from N(μ, σ^2) distribution. Find the Cramer-Rao Lower Bounds (CRLB) on the variances of unbiased estimators of μ andσ^2. Can you find unbiased estimators μ and σ^2 whose variance attains the respective CRLB?

[2] $X_1, \dots, X_n$ is a random sample from Gamma(α, β)

f(x|α, β) $= \begin{cases} \frac{1}{\Gamma\alpha\, \beta^\alpha}\, e^{-x/\beta}\, x^{\alpha-1} & if\ x > 0 \\ 0 & otherwise. \end{cases}$

$\alpha$is assumed to be known. Find the Fisher Information I(β) and CRLB on the variances of unbiased estimators of β.

[3] $X_1, \dots, X_n$ be a random sample from P(θ), $\theta \in (0, \infty)$. Find the CRLB on the variances of unbiased estimators of the following estimands: (a) g(θ)= θ , (b) g(θ)= θ^2 and (c) g(θ)= $e^{-\theta}$.

[4] Suppose $X_1, \dots, X_n$ be a random sample from B(1, θ), $\theta \in$ (0, 1). Find the CRLB on the variances of unbiased estimators of the following estimands: (a) g(θ)= θ^4 (b) g(θ)= θ(1- θ).

[5] $X_1, \dots, X_n$ be a random sample from U(0, θ), $\theta > 0$. Show that (a) $\frac{n}{n+1} X_{(n)}$ is a consistent estimator of θ and (b) $e^{X_{(n)}}$ is consistent for e^θ, where $X_{(n)} = \max(X_1, \dots, X_n)$.

[6] $X_1, \dots, X_n$ be a random sample from U(θ- ½ , θ + ½), $\theta \in \Re$. Show that

$X_{(1)} + \frac{1}{2}, X_{(n)} - \frac{1}{2}\ and \frac{(X_{(1)} + X_{(n)})}{2}$ are all consistent estimators of θ, $X_{(n)} = \max(X_1, \dots, X_n)$ and $X_{(1)} = \min(X_1, \dots, X_n)$.

[7] $X_1, \dots, X_n$ be a random sample from

f(x)= $\begin{cases} \frac{1}{2}(1 + \theta x) - 1 < x, 1 \\ 0 \quad otherwise. \end{cases}$

Where, $\theta \in (-1, 1)$. Fin a consistent estimator for θ.

[8] $X_1, \dots, X_n$ be a random sample from P(θ). Find a consistent estimator of $\theta^3(3\sqrt{\theta} + \theta + 12)$.

[9] Let $X_1, \dots, X_n$ be a random sample from Gamma (α, β) with density

$$f(x) = \begin{cases} \frac{1}{\Gamma\alpha\beta^\alpha} \ e^{-x/\beta} x^{\alpha-1} & x > 0 \\ 0 \quad otherwise. \end{cases}$$

Where, α is a known constant and β is an unknown parameter, show that $\frac{\sum_{i=1}^{n} X_i}{n\alpha}$ is a consistent estimator of β.

[10] Let $X_1, \dots, X_n$ be a random sample from each of the following distributions having the following density or mass functions. Find the maximum likelihood estimator (MLE) of θ in each case.

(a) $f(x; \theta) = \begin{cases} \frac{e^{-\theta}\theta^x}{x!} & x = 0, 1, 2, \dots \\ 0 \ otherwise. \end{cases}$

(b) $f(x; \theta) = \begin{cases} \theta \ x^{\theta-1} & 0 < x < 1 \\ 0 \quad otherwise. \end{cases}$

(c) $f(x; \theta) = \begin{cases} \frac{1}{\theta} \ e^{-x/\theta} & x > 0 \\ 0 \quad otherwise. \end{cases}$

(d) $f(x; \theta) = \begin{cases} \frac{1}{2} \ e^{-|x-\theta|} & -\infty < x, \infty \\ 0 \quad otherwise. \end{cases}$

(e) $X \sim U\left(-\frac{\theta}{2}, \frac{\theta}{2}\right)$.

[11] Let $X_1, \dots, X_n$ be a random sample from the distribution having p.d.f.

$$f(x; \theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2} & x \geq \theta_1 \\ 0 \quad otherwise. \end{cases}$$

Find the MLEs of θ_1 *and* θ_2.

[12] Let $X_1, \dots, X_n$ be a random sample from the distribution having p.d.f.

$$f(x; \alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1} & x \geq 0 \\ 0 \quad otherwise. \end{cases}$$

Find the MLEs of α and λ.

[13] Let $X_1, \dots, X_n$ be a random sample from the function having p.d.f.

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{2\sqrt{3}\sigma} & \mu - \sqrt{3}\,\sigma, x < \mu + \sqrt{3}\,\sigma \\ 0 \quad otherwise. \end{cases}$$

Find the MLEs of μ and σ.

[14] Let $X_1, \dots, X_n$ be a random sample from U (θ- ½ , θ + ½), $\theta \in \Re$. Show that any statistic u($X_1, \dots, X_n$) such that it satisfies

$$X_{(n)} - \frac{1}{2} \leq u(X_1, \dots, X_n) \leq X_{(1)} + \frac{1}{2}$$

Is a maximum likelihood estimator of θ. in particular $\frac{(X_{(1)} + X_{(n)})}{2}$ $and \frac{3}{4}\left(X_{(1)} + \frac{1}{2}\right) + \frac{1}{4}\left(X_{(n)} - \frac{1}{2}\right)$ are MLEs of θ.

[15] The lifetimes of a component are assumed to be exponential with parameter λ. Ten of these components were placed on a test independently. The only data recorded were the number of components that had failed (out of 10 put to test) in less than 100 hours, which was recorded to be 3. Find the maximum likelihood estimate of λ.

[16] A salesman of used cars is willing to assume that the number of sales he makes per day is a Poisson random variable with parameter μ. Over the past 30 days he made no sales on 20 days and one or mare sales on each of the remaining 10 days. Find the maximum likelihood estimate of μ.

[17] Let $X_1, \dots, X_n$ be a random sample from each of the following distributions. Find the method of moments estimator (MOME) of the corresponding unknown parameters in each of the situations.

(a) X~P (θ); (b) X~ (-$\theta/2$, $\theta/2$);

(c) X~ Exp (0, θ); (d) X~ Exp (α, β);

(e) X~ G(α, β) with p.d.f. f(x ; α, β)= $\begin{cases} \frac{1}{\beta^\alpha \, \Gamma\alpha} \; e^{-x/\beta} \; x^{\alpha-1} & if \; x \geq 0 \\ 0 \quad otherwise. \end{cases}$

Solution Key

(1) $X_1, \dots, X_n$ i.i.d. N(μ, σ^2)

f(x| μ, σ^2)= $\frac{1}{\sqrt{2\pi \, \sigma^2}} \; e^{-\frac{1}{2\,\sigma^2}(x-\mu)^2}$

$$\log f = k - \frac{1}{2}\log \sigma^2 - \frac{1}{2\,\sigma^2}(x-\mu)^2 \; \frac{\partial \log f}{\partial \mu} = \frac{(x-\mu)}{\sigma^2}; \frac{\partial^2 \log f}{\partial \mu^2} = -\frac{1}{\sigma^2}.$$

$$-E\left(\frac{\partial^2 \log f}{\partial \mu^2}\right) = \frac{1}{\sigma^2} = I(\mu)$$

CRLB for an u.e. for μ= $\frac{\sigma^2}{n}$.

Since V($\bar{X}$)= $\frac{\sigma^2}{n}$; $\bar{X}$ attains CRLB.

$\frac{\partial \log f}{\partial \sigma^2} = -\frac{1}{2\,\sigma^2} + \frac{1}{2\,\sigma^4}(x-\mu)^2$

$$\frac{\partial^2 \log f}{\partial (\sigma^2)^2} = \frac{1}{2\,\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}.$$

$$I(\sigma^2) = -E\left(\frac{\partial^2 \log f}{\partial (\sigma^2)^2}\right) = -\frac{1}{2\,\sigma^4} + \frac{1}{\sigma^4} = \frac{1}{2\,\sigma^4}$$

CRLB for an u.e. for$\sigma^2 = \frac{2\sigma^4}{n}$.

Now $S^2 = \frac{1}{n-1}\sum_1^n (X_i - \bar{X})^2$ is UMVUE for σ^2 with

$\mathrm{V}(S^2) = \frac{2\sigma^4}{n-1} > \mathrm{CRLB}$

Since UMVUE is the unbiased with lowest variance in the class of all unbiased estimators, CRLB can't be attained by any unbiased estimator of σ^2.

(2) $\mathrm{f}(\mathrm{x}|\alpha,\beta) = \frac{1}{\Gamma\alpha\,\beta^\alpha}\, e^{-\frac{x}{\beta}}\, x^{\alpha-1} \quad if\ x > 0$

$$\log f = -log\,\Gamma\alpha - \alpha\log\beta - \frac{x}{\beta} + (\alpha-1)\log x$$

$$\frac{\partial \log f}{\partial \beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\frac{\partial^2 \log f}{\partial \beta^2} = \frac{\alpha}{\beta^2} - 2\frac{x}{\beta^3}.$$

$$I(\beta) = -E\left(\frac{\partial^2 \log f}{\partial \beta^2}\right) = -\frac{\alpha}{\beta^2} + 2\frac{\alpha\beta}{\beta^3} = \frac{\alpha}{\beta^2}.$$

$\Longrightarrow$ CRLB for u.e. of $\beta : \frac{1}{n.\frac{\alpha}{\beta^2}} = \frac{\beta^2}{n\alpha}$.

(3) $X_1, \ldots, X_n$ i.i.d. P (θ)

$\mathrm{f}(\mathrm{x}|\theta) = \frac{e^{-\theta}\theta^x}{x!}$

$$\log \mathrm{f}(\,\mathrm{x} \mid \theta\,) = -\theta + x log\theta - \log x!$$

$$\frac{\partial \log f}{\partial \theta} = -1 + \frac{x}{\theta};\frac{\partial^2 \log f}{\partial \theta^2} = -\frac{x}{\theta^2}.$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f}{\partial \theta^2}\right) = \frac{1}{\theta}$$

CRLB for any u.e. of $\theta = \frac{1}{n.\frac{1}{\theta}} = \frac{\theta}{n}$

CRLB for any u.e. of g $(\theta) = \theta^2$: $\frac{(2\theta)^2}{\frac{n}{\theta}} = \frac{4\theta^3}{n}$

CRLB for any u.e. of g $(\theta) = e^{-\theta}$: $\frac{\left(-e^{-\theta}\right)^2}{\frac{n}{\theta}} = \frac{\theta\, e^{-2\theta}}{n}$.

(4) $X_1, \dots, X_n$ i.i.d. B $(1, \theta)$

f(x|θ)= $\theta^x\ (1-\theta)^{1-x}$

$$\log f(\,x \mid \theta\,) = x\log\theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f}{\partial\theta} = \frac{x}{\theta} + \frac{(1-x)}{1-\theta}(-1) = \frac{x}{\theta(1-\theta)} - \frac{1}{1-\theta}.$$

$$E(\theta) = E\left(\frac{\partial \log f}{\partial\theta}\right)^2 = V\left(\frac{\partial \log f}{\partial\theta}\right) = \frac{\theta(1-\theta)}{\left(\theta(1-\theta)\right)^2} = \frac{1}{\theta(1-\theta)}.$$

CRLB for u.e. of θ^4: $\frac{\left(4\theta^3\right)^2}{n.\frac{1}{\theta(1-\theta)}} = \frac{16\theta^7(1-\theta)}{n}$

CRLB for u.e. of θ(1-θ): $\frac{(1-2\theta)^2}{n.\frac{1}{\theta(1-\theta)}} = \frac{(1-2\theta)^2\theta(1-\theta)}{n}$.

(5) $X_1, \dots, X_n$ i.i.d. U$(0, \theta)$

P[$|X_{(n)} - \theta| \geq \epsilon$] $\leq \frac{E\left(X_{(n)}-\theta\right)^2}{\epsilon^2} = \frac{EX_{(n)}{}^2+\theta^2-2\theta\, EX_{(n)}}{\epsilon^2}$

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1}, & 0 < x < \theta \\ 0, o/w \end{cases}$$

$$EX_{(n)} = \frac{n}{\theta^n}\int_0^\theta x^n\, dx = \frac{n}{n+1}.\theta$$

$$EX_{(n)}{}^2 = \frac{n}{\theta^n}\int_0^\theta x^{n+1}\, dx = \frac{n}{n+2}\theta^2$$

$$\Longrightarrow \text{P}[\mid X_{(n)} - \theta \mid \geq \epsilon] \leq \frac{1}{\epsilon^2}\left[\frac{n}{n+2}\theta^2 + \theta^2 - 2\theta\frac{n}{n+1}.\theta\right]$$

$\to 0$ as n$\to\infty$

$\Longrightarrow X_{(n)} \to \theta$.

$\Longrightarrow \frac{n}{n+1} X_{(n)} \rightarrow \theta$

$\Longrightarrow \frac{n}{n+1} X_{(n)}$ is a consistent estimator for θ

Further since $X_{(n)} \rightarrow \theta$

$e^{X_{(n)}} = g\left(X_{(n)}\right) \rightarrow g(\theta) = e^{\theta}$.

$\Longrightarrow e^{X_{(n)}}$ is a consistent estimator for e^{θ}.

(6) $X_1, \dots, X_n$ i.i.d. U(θ- ½ , θ + ½)

$$F_X(x) = \int_{\theta-\frac{1}{2}}^{x} dx = \left(x - \theta + \frac{1}{2}\right)$$

$$f_{X_{(1)}}(x) = n\left(1 - F_X(x)\right)^{n-1} f(x)$$

$$i.e. f_{X_{(1)}}(x) = \begin{cases} n\left(\theta - x + \frac{1}{2}\right)^{n-1} & , \theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2} \\ 0 & , o/w \end{cases}$$

$$EX_{(1)} = n \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\left(\theta - x + \frac{1}{2}\right)^{n-1} dx = \theta + \frac{1}{2} - \frac{n}{n+1}.$$

$$EX_{(1)}{}^2 = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x^2\left(\theta - x + \frac{1}{2}\right)^{n-1} dx$$

$$= \left(\theta + \frac{1}{2}\right)^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)$$

$$P\left[\left|X_{(n)} - \left(\theta - \frac{1}{2}\right)\right| \geq \epsilon\right] \leq \frac{E\left(X_{(1)} - \left(\theta - \frac{1}{2}\right)\right)^2}{\epsilon^2}$$

r.h.s. $= \frac{1}{\epsilon^2}\left[E\left(X_{(1)}{}^2\right) + \left(\theta - \frac{1}{2}\right)^2 - 2\left(\theta - \frac{1}{2}\right)E\left(X_{(1)}\right)\right]$

$$= \frac{1}{\epsilon^2}\left[\left\{\left(\theta + \frac{1}{2}\right)^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)\right\} + \left(\theta - \frac{1}{2}\right)^2 - 2\left(\theta - \frac{1}{2}\right)\left(\theta + \frac{1}{2} - \frac{n}{n+1}\right)\right]$$

$$\rightarrow \frac{1}{\epsilon^2}\left[\left\{\left(\theta + \frac{1}{2}\right)^2 + 1 - (2\theta + 1)\right\} + \left(\theta - \frac{1}{2}\right)^2 - 2\left(\theta - \frac{1}{2}\right)\left(\theta - \frac{1}{2}\right)\right] as\ n \rightarrow \infty$$

$$= 0$$

$\Longrightarrow P\left[\left|X_{(n)} - \left(\theta - \frac{1}{2}\right)\right| \geq \epsilon\right] \longrightarrow 0 \; as \; n \to \infty$

$\Longrightarrow X_{(1)} \longrightarrow \theta - \frac{1}{2}$. __________(1)

We can similarly prove that

$X_{(n)} \longrightarrow \theta + \frac{1}{2}$. ________________(2)

Combining (1) & (2), we get.

$$\frac{X_{(1)} + X_{(n)}}{2} \longrightarrow \theta$$

$\Longrightarrow \frac{X_{(1)}+X_{(n)}}{2}$ is a consistent estimator for θ

So,

$X_{(1)} + \frac{1}{2}$ is a consistent estimator for θ (from (1))

& $X_{(n)} - \frac{1}{2}$ is a consistent estimator for θ (from (2)).

(7) $X_1, \ldots, X_n$ i. i. d. $f_X(x) = \begin{cases} \frac{1}{2}(1+\theta x) - 1 < x, 1 \\ 0 \quad otherwise. \end{cases}$

$$E(X) = \frac{1}{2}\int_{-1}^{1}(1+\theta x)\,dx = \frac{\theta}{3}$$

$\Longrightarrow X_1, \ldots, X_n$ are i. i. d. with $E(X_1) = \frac{\theta}{3}$

By khintchime's WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \longrightarrow E(X_1)$$

$$i.e. \bar{X} \longrightarrow \frac{\theta}{3} \Longrightarrow 3\bar{X} \longrightarrow \theta$$

$\Longrightarrow 3\bar{X} \longrightarrow \theta$ is a consistent estimator for θ.

(8) $X_1, \ldots, X_n$ i. i. d. P (θ)

$$E(X_i) = \theta \quad \forall \; i = 1(1)n$$

By WLLN $\overline{X_n} \longrightarrow \theta$

$\Longrightarrow$ g$(\overline{X_n})\longrightarrow$g$(\theta)$

$\Longrightarrow \overline{X_n}^3\left(3\sqrt{\overline{X_n}}+\overline{X_n}+12\right) \longrightarrow \theta^3(3\sqrt{\theta}+\ \theta+12)$

$\Longrightarrow \overline{X_n}^3\left(3\sqrt{\overline{X_n}}+\overline{X_n}+12\right)$ is a consistent estimator for $\theta^3(3\sqrt{\theta}+\ \theta+12)$.

(9) $X_1, \dots, X_n$ i. i. d. Gamma (α, β)

α is known

$$E(X) = \ \alpha\beta \ for\ X \sim Gamma(\alpha, \beta)$$

By WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \longrightarrow E(X_1)$$

$$i.e. \frac{1}{n}\sum_{1}^{n} X_i \longrightarrow \alpha\beta$$

$\Longrightarrow \frac{1}{n\alpha}\sum_1^n X_i \longrightarrow \beta.$

$\Longrightarrow\ \frac{1}{n\alpha}\sum_1^n X_i$ is a consistent estimator for β.

[Note: T= $\sum_1^n X_i \sim Gamma(n\alpha, \beta)$can be proved using m. g. f. approach]

(10) (a) $X_1, \dots, X_n$ i.i.d. P (θ)

Likelihood function L($\theta|\underset{\sim}{x}$)= $\theta^{\sum x_i} e^{-n\theta} \left(\prod x_i!\right)^{-1}$

$$l(\theta|\underset{\sim}{x}) = \log \mathrm{L}(\theta|\underset{\sim}{x}) = \sum x_i\ \ log\theta - n\theta - \log\left(\prod x_i!\right)$$

$$\frac{\partial l}{\partial\theta} = \frac{\sum x_i}{\theta} - n;\ \frac{\partial l}{\partial\theta} = 0 \Longrightarrow \theta\hat{} = \bar{x}$$

$$\frac{\partial^2 l}{\partial\theta^2}|\theta = \theta\hat{} = \ -\frac{\sum x_i}{\theta\hat{}^2} < 0$$

$\Longrightarrow \theta\hat{}_{MLE} = \bar{X}$

(10) (b) $X_1, \dots, X_n$ i.i.d. with p. d. f.

$$f_X(x) = \begin{cases} \theta\ x^{\theta-1} & 0 < x < 1 \\ 0 & otherwise. \end{cases}$$

$$\mathrm{L}(\theta|\underset{\sim}{x}) = \ \theta^n\left(\prod_{1}^{n} x_i\right)^{\theta-1}$$

$$l(\theta|\underset{\sim}{x}) = \log L(\theta|\underset{\sim}{x}) = n \log\theta + (\theta - 1)\sum_{1}^{n} \log x_i$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum logx_i$$

$$\frac{\partial l}{\partial \theta} = 0 \Longrightarrow \theta\hat{} = -\frac{n}{\sum_1^n \log x_i}$$

$$\frac{\partial^2 l}{\partial \theta^2}|\theta = \theta\hat{} = -\frac{n}{\theta\hat{}^2} < 0 \Longrightarrow \theta\hat{}_{MLE} = -\frac{n}{\sum_1^n \log x_i}.$$

(c) $X_1, \dots, X_n$ i.i.d. with p. d. f.

$$f_X(x) = \begin{cases} \frac{1}{\theta}\ e^{-x/\theta} & x > 0 \\ 0 \quad otherwise. \end{cases}$$

$$L(\theta|\underset{\sim}{x}) = \frac{1}{\theta^n} e^{-\sum x_i/\theta}$$

$$l(\theta|\underset{\sim}{x}) = \log L(\theta|\underset{\sim}{x}) = -n \log\theta + \frac{1}{\theta}\sum x_i$$

$$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i$$

$$\frac{\partial l}{\partial \theta} = 0 \Longrightarrow \theta\hat{} = \bar{x}$$

$$\frac{\partial^2 l}{\partial \theta^2}|\theta = \theta\hat{} = \left(\frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}\right)|\ \theta = \theta\hat{} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = -\frac{n}{\bar{x}^2} < 0$$

$\Longrightarrow \theta\hat{}_{MLE} = \bar{X}$.

(a) (10) (d) $X_1, \dots, X_n$ i.i.d. with p. d. f.

$$f_X(x) = \begin{cases} \frac{1}{2}\ e^{-|x-\theta|} & -\infty < x, \infty \\ \quad 0 \quad otherwise. \end{cases}$$

$L(\theta|\underset{\sim}{x})$ is maximized if $\sum |\ x_i - \theta\ |$ is minimized

Realize that $\sum |\ x_i - \theta\ |$ is minimized w.r.t. θ at

$$\theta\hat{} = median(x_1, \dots, x_n)$$

$\Longrightarrow \theta\hat{}_{MLE} = median\ (X_1, \dots, X_n)$

(10) (e) $X_1, \dots, X_n$ i.i.d. with $U\left(-\frac{\theta}{2}, \frac{\theta}{2}\right)$

Likelihood function $L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & -\frac{\theta}{2} \le x_1, \dots, x_n \le \frac{\theta}{2} \\ 0, & \text{otherwise} \end{cases}$

$$L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & \text{if } |x_i| \leq \frac{\theta}{2}, \text{i} = 1(1)\text{n} \\ 0, & \text{otherwise} \end{cases}$$

i.e.

$$L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & \text{if} \max_{\text{i}} |x_i| \leq \frac{\theta}{2} \\ 0, & \text{otherwise} \end{cases}$$

$L(\theta|\underset{\sim}{x})$ is maximized at minimum value of θ given $\underset{\sim}{x}$

$\Longrightarrow \theta\hat{}_{MLE} = 2 \max_{\text{i}} |x_i|$

(11) $X_1, \ldots, X_n$ i.i.d. $\text{Exp}(\theta_1, \theta_2)$

$$\theta_1\hat{}_{MLE} = X_{(1)}$$

$$\theta_2\hat{}_{MLE} = \frac{1}{n}\sum_1^n \left(X_i - X_{(1)}\right)$$

Done in class.

(12) $X_1, \ldots, X_n$ i.i.d. $f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1} & x \geq 0 \\ 0 & otherwise. \end{cases}$

$$L(\theta|\underset{\sim}{x}) = \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} e^{-\lambda \sum_1^n x_i} \prod_1^n x_i^{\alpha-1}$$

$$l(\theta|\underset{\sim}{x}) = \log L(\theta|\underset{\sim}{x}) = \text{n}\alpha \log \lambda - n \log \Gamma\alpha + (\alpha - 1) \sum logx_i - \lambda \sum_1^n x_i$$

Likelihood equations

$$\frac{\partial\ logl}{\partial\lambda} = \frac{n\alpha}{\lambda} - \sum x_i = 0\_\_\_\_\_\_\_\_(1)$$

$$\frac{\partial\ logl}{\partial\alpha} = nlog\lambda - n\frac{\Gamma\alpha'}{\Gamma\alpha} + \sum logx_i = 0\ \_\_\_\_\_\_\_(2)$$

(1) $\Longrightarrow \lambda = \frac{\alpha}{\bar{x}}$

From (2), we get

N $\log\left(\frac{\alpha}{\bar{x}}\right) - n\frac{\Gamma\alpha'}{\Gamma\alpha} + \sum logx_i = 0$ (*)

Solving (*) by numerical method gives $\alpha\hat{}_{MLE}$

& $\lambda\hat{}_{MLE} = \frac{\alpha\hat{}_{MLE}}{\bar{X}}$.

(13) $X_1, \dots, X_n$ i.i.d. with p.d.f.

$$f(x\,;\mu,\sigma)=\begin{cases}\frac{1}{2\sqrt{3}\sigma}, & \mu-\sqrt{3}\,\sigma\,, x<\mu+\sqrt{3}\,\sigma \\ 0, & otherwise.\end{cases}$$

Likelihood function

$$L((\mu,\theta)|\underset{\sim}{x}) = \begin{cases}\left(\frac{1}{2\sqrt{3}\sigma}\right)^n, \quad \underset{(*)}{\underline{\mu-\sqrt{3}\,\sigma \le x_{(1)} \le \cdots \le x_{(n)} \le \mu+\sqrt{3}\,\sigma}} \\ 0\,, \quad otherwise.\end{cases}$$

Using condition (*),

$\mu-\sqrt{3}\,\sigma \le x_{(1)}$ & $x_{(n)} \le \mu+\sqrt{3}\,\sigma$

$\Rightarrow \mu \le x_{(1)}+\sqrt{3}\,\sigma$ & $x_{(n)}-\sqrt{3}\,\sigma \le \mu$

$\Rightarrow x_{(n)}-\sqrt{3}\,\sigma \le \mu \le x_{(1)}+\sqrt{3}\,\sigma$

For a given σ, $L((\mu,\theta)|\underset{\sim}{x})$ is maximized w.r.t. μ if

$\mu \in \left(x_{(n)}-\sqrt{3}\,\sigma, x_{(1)}+\sqrt{3}\,\sigma\right)$ (o/w $L((\mu,\theta)|\underset{\sim}{x}) = 0$)

$\Rightarrow$ Any value of μ in the above internal is an MLE of μ

In particular

$$\frac{\left(X_{(n)}-\sqrt{3}\,\sigma+X_{(1)}+\sqrt{3}\,\sigma\right)}{2} = \frac{X_{(n)}+X_{(1)}}{2} = \mu\hat{}(\sigma)_{MLE}$$

Since the above MLE is indep of σ, it is MLE of $\mu\ \forall\ \sigma$

$\Rightarrow \mu\hat{}_{MLE} = \frac{X_{(n)}+X_{(1)}}{2}$

Further, $L(\mu\hat{}, \sigma)$ is maximized w.r.t. σ if σ is minimum.

Observe that

$$\sqrt{3}\,\sigma \ge \mu - x_{(1)} \ \&\sqrt{3}\,\sigma \ge x_{(n)} - \mu$$

At the MLE of μ;

$$\sqrt{3}\,\sigma \ge \frac{x_{(n)}-x_{(1)}}{2}$$

$\Longrightarrow \hat{\sigma}_{MLE} = \frac{X_{(n)}-X_{(1)}}{2\sqrt{3}}$

(14) $X_1, \dots, X_n$ i.i.d. U(θ- ½ , θ + ½), $\theta \in \Re$

$$L(\theta|\underset{\sim}{x}) = \begin{cases} 1, & \theta - \frac{1}{2} \le x_{(1)} \le \cdots \le x_{(n)} \le \theta + \frac{1}{2} \\ 0, & \text{o/w} \end{cases}$$

L is maximized w.r.t. θ of

$\theta - \frac{1}{2} \le x_{(1)}$ & $x_{(n)} \le \theta + \frac{1}{2}$ $(\text{Max}_\theta L = 1)$

$$i.e.\, x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}.$$

$\Longrightarrow$Any statistic U($\underset{\sim}{X}$)$\ni$

$x_{(n)} - \frac{1}{2} \le u(x_1, \dots, x_n) \le x_{(1)} + \frac{1}{2}$ is an MLE of θ

In particular, $\frac{X_{(1)}+X_{(n)}}{2}$ is an MLE of θ

In general, $\alpha(X_{(1)} + \frac{1}{2})$ +(1- α)($X_{(n)} - \frac{1}{2}$); $\forall$ $0 < \alpha < 1$ is an MLE of θ

With $\alpha = \frac{3}{4}$, we have the above estimator as

$\frac{3}{4}\left(X_{(1)} + \frac{1}{2}\right) + \frac{1}{4}\left(X_{(n)} - \frac{1}{2}\right)$ is an MLE of θ.

(15)X : r.v. denoting lifetime of the component

X~ Exp distn with mean λ

$$f_X(x) = \begin{cases} \frac{1}{\lambda}\, e^{-x/\lambda} & x > 0 \\ 0 & otherwise. \end{cases}$$

$$Define\ Y_i = \begin{cases} 1, if\ ith\ component\ has\ life < 100hrs \\ 0,\ o/w \end{cases}$$

$$P(Y_i = 1) = P(X < 100) = \frac{1}{\lambda}\int_0^{100} e^{-x/\lambda}\, dx = \left(1 - e^{-\frac{100}{\lambda}}\right)$$

$Y_1 \dots Y_n$ i.i.d. $B\left(1, \left(1 - e^{-\frac{100}{\lambda}}\right)\right) \equiv B(1, \theta)$

(n= 10) with $\theta = 1 - e^{-\frac{100}{\lambda}}$.

$\Longrightarrow \hat{\theta}_{MLE} = \bar{Y}$(done in class)

Further, $\lambda = -\frac{100}{\log(1-\theta)} = g(\theta)$

$\Longrightarrow$ MLE of $g(\theta)$ is $g(\hat{\theta}_{MLE})$.

$\Longrightarrow \hat{\lambda}_{MLE} = -\frac{100}{\log(1-\hat{\theta}_{MLE})} = -\frac{100}{\log(1-\bar{X})}$

From the given data $\bar{X} = \frac{3}{10}$

$\Longrightarrow$ The maximum likelihood estimate of λ computed

From the given data is $\left(-\frac{100}{\log\left(\frac{7}{10}\right)}\right)$

(16)X: r.v. denoting the no. of sales per day

X~ P(μ) (from the assumptions)

Define $Y_i = \begin{cases} 1, & if\ 0\ sales\ on\ day\ i \\ & 0, o/w \end{cases}$

P$(Y_i = 1) = P(X = 0) = e^{-\mu}$;

$Y_1 \dots Y_{30}$ i.i.d. B(1, $e^{-\mu}$)≡B(1, θ)(θ= $e^{-\mu}$)

$$\hat{\theta}_{MLE} = \bar{Y}$$

Further, μ= -log θ

$\Longrightarrow \hat{\mu}_{MLE} = -\log \hat{\theta}_{MLE}$

$\Longrightarrow$ MLE estimate of μ from the given data is

Given by -log$\left(\frac{20}{30}\right)$.

(17)(a)

$X_1, \dots, X_n$ i.i.d. P(θ)

$$\mu_1^1 = E(X) = \theta$$

MOME of θ, $\hat{\theta}_{MOME} = m_i = \bar{X}$

(b) $X_1, \dots, X_n$ i.i.d. $U\left(-\frac{\theta}{2}, \frac{\theta}{2}\right)$

done in class.

(c) $X_1, \dots, X_n$ i.i.d. Exp (0, θ)

$\mu_i = E(X) = \frac{1}{\theta}\int_0^{\infty} x\, e^{-\frac{x}{\theta}}\, dx = \theta$

$\Longrightarrow \theta\hat{}_{MOME} = m_i = \bar{X}$

(d) $X_1, \dots, X_n$ i.i.d. Exp(α, β)

Done in class.

(e) $X_1, \dots, X_n$ i.i.d. G(α, β) with p.d.f.

$$f(x;\ \alpha, \beta) = \begin{cases} \frac{1}{\beta^{\alpha}\ \Gamma\alpha}\ e^{-x/\beta}\ x^{\alpha-1} & if\ x \geq 0 \\ 0 & otherwise. \end{cases}$$

$$\mu_1{}^1 = E(X) = \frac{1}{\beta^{\alpha}\ \Gamma\alpha}\int_0^{\infty} x^{\alpha+1-1}\, e^{-x/\beta}\, dx = \frac{\Gamma\alpha+1\ \beta^{\alpha+1}}{\Gamma\alpha\beta^{\alpha}} = \alpha\beta$$

$$\mu_2{}^1 = E(X^2) = \frac{1}{\beta^{\alpha}\ \Gamma\alpha}\int_0^{\infty} x^{\alpha+2-1}\, e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{\Gamma\alpha+2\ \beta^{\alpha+2}}{\Gamma\alpha\beta^{\alpha}} = (\alpha+1)\alpha\beta^2$$

$$Equate \left.\begin{matrix} \bar{X} = m_i = \alpha\beta \\ \frac{1}{n}\sum X_i{}^2 = m_2{}^1 = \alpha(\alpha+1)\beta^2 \end{matrix}\right\}$$

$$\Longrightarrow \frac{m_2{}^1}{m_1{}^1} = \alpha\beta + \beta = m_i + \beta$$

$$\Longrightarrow \beta\hat{}_{MOME} = \frac{m_2{}^1 - (m_1{}^1)^2}{m_i} = \frac{\frac{1}{n}\sum X_i{}^2 - \bar{X}^2}{\bar{X}}$$

$$i.e.\ \beta\hat{}_{MOME} = \frac{\frac{1}{n}\sum(X_i - \bar{X})^2}{\bar{X}}$$

$$\&\ \alpha\hat{}_{MOME} = \frac{\bar{X}}{\left(\frac{\frac{1}{n}\sum(X_i - \bar{X})^2}{\bar{X}}\right)} = \frac{\bar{X}^2}{\frac{1}{n}\sum(X_i - \bar{X})^2}.$$

Assignment-13

[1] The observed value of the mean of a random sample of size 20 from N (μ, 80) be 81.2. Find the equal tail 95% and the equal tail 99% confidence interval of μ. Which one is shorter?

[2] Let $\bar{X}$ be the mean of a random sample of size n from N (μ, 9). Find n such that, approximately, P ($\bar{X}$ -1 <μ < $\bar{X}$ +1) = 0.90

[3] Let a random sample of size 25 from a normal distribution N (μ, σ^2) yield $\bar{x}$ = 4.7 and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = 5.76$. Determine a 90% confidence interval for μ.

[4] Let $X_1,\ \dots, X_n$ be random sample of size be random sample of size 9 from N (μ, σ^2). Find the expected length of 95% confidence interval for μ when (a) σ is known and (b) σ is unknown.

[5] Let $X_1, \dots, X_n$ be random sample of size n from a distribution with p.d.f.

f(x|θ)= $\begin{cases} \frac{3x^2}{\theta^3} & 0 < x < \theta \\ 0 & otherwise. \end{cases}$

(a) Find the distribution of $\frac{X_{(n)}}{\theta}$, where $X_{(n)} = \max(X_1, \dots, X_n)$.
(b) Show that $\left(X_{(n)}, \alpha^{-\frac{1}{3n}} X_{(n)}\right)$ gives a 100(1- α)% confidence interval for θ.

[6] Let $X_1,\ \dots, X_n$ be random sample of size n from U(0, θ), $\theta \in \Re$. Show that $\left(X_{(n)}, \alpha^{-\frac{1}{n}} X_{(n)}\right)$ and $(\left(1-\alpha\right)^{-\frac{1}{n}} X_{(n)}, \infty\right)$ are both 100(1- α)% confidence intervals for θ.

[7] Let two independent random samples, each of size 5, from two normal distributions N $(\mu_1, {\sigma_1}^2) and\ (\mu_2, {\sigma_2}^2)$ are; 1.5, 2.8 , 3.3 , 3.9 , 7.2 and 2.8 , 1.8 , 3.1 , 6.5 , 6.9 respectively.

(a) If it is known${\sigma_1}^2 = {\sigma_2}^2 = 3.5$, find a 95% confidence interval for $\mu_1 - \mu_2$.

(b) If it is known that $\mu_1 = \mu_2 = 4$, find a 95% confidence interval for ${\sigma_1}^2 / {\sigma_2}^2$.

Solution Key

(1) $X_1, \dots, X_{20}$ r.s. from N (μ, 80)

Confidence int for μ

X~ N (μ,$\frac{\sigma^2}{n}$), i.e. N (μ, 4)

$\Rightarrow$ Y= $\frac{\bar{X}-\mu}{2} \sim N(0,1)$

$$\Rightarrow 1-\alpha = P\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{X}-\mu}{2} \le z_{\frac{\alpha}{2}}\right)\left[z_{\frac{\alpha}{2}}\ is\ \ni for\ z \sim N(0,1)\ \ P\left(z > z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}\right]$$

$$\Rightarrow 1-\alpha = P\left(\bar{X} - 2z_{\frac{\alpha}{2}} \le \mu \le \bar{X} + 2z_{\frac{\alpha}{2}}\right)$$

For 100(1- α)Y= 95%, α= 0.05; $\frac{\alpha}{2} = 0.025$

C I $\longrightarrow$ $(\bar{X} -\ 2z_{0.025}, \bar{X} + 2z_{0.025})$

Observed value of $\bar{X}$ is 81.2 & $z_{0.025} = 1.96$

$$(81.2 - 2 \times 1.96, 81.2 + 2 \times 1.96)\_\_\_\_\_\_(1)$$

For 100(1- α)% = 99% , α= $0.01, \frac{\alpha}{2} = 0.005$

C I $\longrightarrow (\bar{X} - \ 2z_{0.025}, \bar{X} + 2z_{0.025})$

x̅= 81.2, $z_{0.025} = 2.575$

$$(81.2 - 2 \times 2.575, 81.2 + 2 \times 2.575)\_\_\_\_\_\_\_(2)$$

(2) $X_1 \dots X_n$ r.s. N (μ, 9)

$$\bar{X} \sim N\left(\mu, \frac{9}{n}\right) \Longrightarrow \frac{(\bar{X} - \mu)}{\left(\frac{3}{\sqrt{n}}\right)} \sim N(0,1)$$

$$P[\bar{X} - 1 < \mu < \bar{X} + 1] = P(-1 \leq \bar{X} - \mu \leq 1)$$

$$= P\left(\frac{-1}{\frac{3}{\sqrt{n}}} \leq \frac{(\bar{X} - \mu)}{\left(\frac{3}{\sqrt{n}}\right)} \leq \frac{1}{\frac{3}{\sqrt{n}}}\right)$$

$$= \Phi\left(\frac{\sqrt{n}}{3}\right) - \Phi\left(-\frac{\sqrt{n}}{3}\right) = 2\Phi\left(\frac{\sqrt{n}}{3}\right) - 1 = 0.90(given\ condition)$$

$$\Longrightarrow \Phi\left(\frac{\sqrt{n}}{3}\right) = 0.95 = \Phi(1.96)$$

$$\Longrightarrow \sqrt{n} = 3 \times 1.96 \ \Longrightarrow n = \cdots$$

(3) $X_1 \dots X_n$ r.s. N (μ, σ^2)

$$\left.\begin{array}{c}\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \\ \frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}{}^2\end{array}\right\rangle indep$$

$$\Longrightarrow \frac{(\bar{X} - \mu)}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

$$P\left[-t_{\frac{\alpha}{2};n-1} \leq \frac{(\bar{X} - \mu)}{\frac{s}{\sqrt{n}}} \leq t_{\frac{\alpha}{2};n-1}\right] = 1 - \alpha$$

$$For\ 1 - \alpha = 0.90;\ \alpha = 0.1, t_{\frac{\alpha}{2};n-1} = t_{0.05,24} = 1.711$$

$$\Longrightarrow P\left[\bar{X} - \frac{s}{\sqrt{n}} t_{0.05,24} \leq \mu \leq \bar{X} + \frac{s}{\sqrt{n}} t_{0.05,24}\right] = 0.90$$

$$100(1-\alpha)\%\ CI\ with\ observed\ \bar{x} = 4.7, s^2 = 5.76$$

$$\left(4.7 - \sqrt{\frac{5.76}{25}} \times 1.711, 4.7 + \sqrt{\frac{5.76}{25}} \times 1.711\right)$$

$$= \cdots$$

(4) $X_1 \dots X_9$ r.s. N (μ, σ^2)

(a) CI for μ when σ^2 is known at 95% level

$P\left[-z_{\frac{\alpha}{2}} \le \frac{\bar{X}-\mu}{2} \le z_{\frac{\alpha}{2}}\right] = 0.95$

$$(\alpha = 0.05)P\left[-z_{0.025} \le \frac{\bar{X}-\mu}{\frac{\sigma}{3}} \le z_{0.025}\right] = 0.95$$

$$(=)\ P\left[\bar{X} - \frac{\sigma}{3} \times 1.96 \le \mu \le \bar{X} + \frac{\sigma}{3} \times 1.96\right] = 0.95[z_{0.025} = 1.96]$$

$$CI\ \left(\bar{X} - \frac{\sigma}{3} \times 1.96, \bar{X} + \frac{\sigma}{3} \times 1.96\right)$$

Length $2\frac{\sigma}{3} \times 1.96 = L$

$$E(L) = 2 \times \frac{\sigma}{3} \times 1.96 = \cdots$$

(b) σ^2 unknown case

$$CI\ based\ on$$

$$P\left[-t_{\frac{\alpha}{2};n-1} \le \frac{(\bar{X}-\mu)}{\frac{s}{\sqrt{n}}} \le t_{\frac{\alpha}{2};n-1}\right] = 0.95$$

$$t_{0.025,8} = 2.306,$$

$$CI\ \left(\bar{X} - \frac{s}{\sqrt{n}} t_{0.025,8}, \bar{X} + \frac{s}{\sqrt{n}} t_{0.025,8}\right)$$

$$Length\ L = 2 \times \frac{s}{\sqrt{n}} \times 2.306$$

$$E(L) = \frac{2 \times 2.306}{\sqrt{n}} \times E(S)$$

$$using\ \frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}{}^2\ \ let\ Y = \frac{(n-1)s^2}{\sigma^2}$$

$$E(\sqrt{Y}) = E\left(\frac{\sqrt{n-1}}{\sigma} s\right) = \frac{1}{2^{\frac{n-1}{2}} \ulcorner \frac{n-1}{2}} \int_0^\infty y^{\frac{1}{2}} e^{-\frac{y}{2}} \quad y^{\frac{n-1}{2}} - 1\, dy$$

$$= \frac{1}{2^{\frac{n-1}{2}} \ulcorner \frac{n-1}{2}} \int_0^\infty e^{-\frac{y}{2}} \quad y^{\frac{n}{2}-1} dy$$

$$= \frac{\ulcorner \frac{n}{2}\ 2^{\frac{n}{2}}}{2^{\frac{n-1}{2}} \ulcorner \frac{n-1}{2}}$$

$$\Rightarrow E(S) = \frac{\sigma}{\sqrt{n-1}} \cdot \frac{\ulcorner \frac{n}{2}\ 2^{\frac{n}{2}}}{2^{\frac{n-1}{2}} \ulcorner \frac{n-1}{2}}$$

$$E(L) = \frac{2 \times 2.306}{\sqrt{n}} \cdot \frac{\sigma}{\sqrt{n-1}} \cdot \frac{\Gamma\frac{n}{2}\, 2^{\frac{n}{2}}}{2^{\frac{n-1}{2}}\, \Gamma\frac{n-1}{2}} = \cdots$$

(5) $X_1 \ldots X_n$ r.s. from f(x|θ)=$\frac{3x^2}{\theta^3}$ $\quad 0 < x < \theta$

(a) $X_{(n)} = Max(X_1 \ldots X_n)$

$$f_{X_{(n)}}(x) = n\big(F_X(x)\big)^{n-1} f_X(x).\ \left[F_X(x) = \int_0^x \frac{3y^2}{\theta^3}dy = \frac{3}{\theta^3}.\frac{x^3}{3} = \left(\frac{x}{\theta}\right)^3\right]$$

$$f_{X_{(n)}}(x) = n\left(\frac{x^3}{\theta^3}\right)^{n-1}.\frac{3x^2}{\theta^3}.\ 0 < x < \theta$$

$$i.e. f_{X_{(n)}}(x) = \frac{3n}{\theta^{3n}} x^{3n-1}\ 0 < x < \theta$$

$$Y = \frac{X_{(n)}}{\theta}\ ;\ f_Y(y) = \frac{3n}{\theta^{3n}}(y\theta)^{3n-1}\theta;\ \ 0 < y < 1$$

$$f_Y(y) = 3n\, y^{3n-1}\ \ 0 < y < 1$$

(b) $P\left(X_{(n)} \le \theta \le \alpha^{-\frac{1}{3n}} X_{(n)}\right) = P\left(\alpha^{\frac{1}{3n}} \le X_{(n)} \le 1\right)$

$$= P\left(\alpha^{\frac{1}{3n}} \le Y \le 1\right)$$

$$= 3n \int_{\alpha^{\frac{1}{3n}}}^{1} y^{3n-1}\ dy = \frac{3n}{3n}(1-\alpha) = 1-\alpha$$

$\Longrightarrow \left(X_{(n)}, \alpha^{-\frac{1}{3n}} X_{(n)}\right)$ provides at 100(1- α) % CI for θ.

(6) $X_1 \ldots X_n$ r.s. U (0, θ)

$$f_{X_{(n)}}(x) = \frac{n}{\theta^n} x^{n-1}\ \ 0 < x < \theta$$

$$Y = \frac{X_{(n)}}{\theta};\ \ f_Y(y) = n\, y^{n-1}; 0 < y < 1$$

$$P\left(X_{(n)} \le \theta \le \alpha^{-\frac{1}{n}} X_{(n)}\right)$$

$$= P\left(1 \le \frac{\theta}{X_{(n)}} \le \alpha^{-\frac{1}{n}}\right)$$

$$= P\left(\alpha^{\frac{1}{n}} \le \frac{X_{(n)}}{\theta} \le 1\right)$$

$$= n \int_{\alpha^{\frac{1}{n}}}^{1} y^{n-1}\ dy = \frac{n}{n}(1-\alpha) = 1-\alpha$$

$$\Longrightarrow \left(X_{(n)}, \alpha^{-\frac{1}{n}} X_{(n)}\right)\ is\ 100(1-\alpha)\%\ CI\ for\ \theta$$

$$Also\ P\left((1-\alpha)^{-\frac{1}{n}} X_{(n)} \le \theta < \infty\right)$$

$$= P\left((1-\alpha)^{-\frac{1}{n}} \le \frac{\theta}{X_{(n)}} < \infty\right)$$

$$= P\left(0 < \frac{X_{(n)}}{\theta} < (1-\alpha)^{-\frac{1}{n}}\right)$$

$\Longrightarrow \left((1-\alpha)^{-\frac{1}{n}} \frac{\theta}{X_{(n)}}, \infty\right)$ is also 100(1- α)% CI for θ.

(7) $X_1, \dots, X_5$ i.i.d. N $((\mu_1, {\sigma_1}^2)$- value(1.5, 2.8 , 3.3 , 3.9 , 7.2)

$Y_1, \dots, Y_5$ i.i.d. N $((\mu_2, {\sigma_2}^2)$- value(2.8 , 1.8 , 3.1 , 6.5 , 6.9)

(a) ${\sigma_1}^2 = {\sigma_2}^2 = 3.5 (= \sigma^2 \; say)$

$$\left.\begin{array}{c} \bar{X} \sim N\left(\mu_1, \frac{\sigma^2}{5}\right) \\ \bar{Y} \sim N\left(\mu_2, \frac{\sigma^2}{5}\right) \end{array}\right> indep.$$

$$\Rightarrow \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{2}{5}}} \sim N(0,1)$$

$$100(1-\alpha)\% \; CI \; for \; ((\mu_1 - \mu_2) \; at \; \alpha = 0.05$$

$$\left((\bar{X} - \bar{Y}) - z_{0.025}\sqrt{3.5}\sqrt{\frac{2}{5}}, (\bar{X} - \bar{Y}) + z_{0.025}\sqrt{3.5}\sqrt{\frac{2}{5}}\right)$$

$use \; z_{0.025} = 1.96 \; \& \; computed \; \bar{x} \& \; \bar{y}$ to get the computed CI

$\mu_1 - \mu_2 = \mu$ (say); μ known

(b) $$\left.\begin{array}{l} \frac{n(\bar{X}-\mu)^2}{{\sigma_1}^2} \sim {\chi_1}^2 \quad (\mu_1 = \mu) \\ \bar{Y} \sim N\left(\mu, \frac{{\sigma_2}^2}{n}\right) \\ \frac{n(\bar{Y}-\mu)^2}{{\sigma_2}^2} \sim {\chi_1}^2 \end{array}\right> indep.$$

$$\Rightarrow \frac{\frac{n(\bar{X} - \mu)^2}{{\sigma_1}^2}/1}{\frac{n(\bar{Y} - \mu)^2}{{\sigma_2}^2}/1}$$

$$= \frac{{\sigma_2}^2}{{\sigma_1}^2} \cdot \left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \sim F_{1,1}$$

$$\Rightarrow P\left(F_{1,1;1-\frac{\alpha}{2}} \leq \frac{{\sigma_2}^2}{{\sigma_1}^2}\left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \leq F_{1,1;\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \frac{1}{F_{1,1;\frac{\alpha}{2}}} \leq \frac{{\sigma_1}^2}{{\sigma_2}^2} \leq \left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \frac{1}{F_{1,1;1-\frac{\alpha}{2}}}\right) = 1 - \alpha$$

$$100(1-\alpha)\% \; CI \; for \; \frac{{\sigma_1}^2}{{\sigma_2}^2} \; is$$

$$\left(\left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \frac{1}{F_{1,1;\frac{\alpha}{2}}}, \left(\frac{\bar{X} - \mu}{\bar{Y} - \mu}\right)^2 \frac{1}{F_{1,1;1-\frac{\alpha}{2}}}\right).$$

Using computed $\bar{x}$, $\bar{y}$ and $F_{1,1;\frac{\alpha}{2}}, F_{1,1;1-\frac{\alpha}{2}}$

At α = 0.05 gives the computed CI.