Department of Mathematics & Statistics

MTH305a

End-Semester Examination

Marks: 50 Time: 180 minutes

Instructions:

1. If you are using result(s)
   (a) from one variable differentiable calculus,
   (b) from linear algebra

   you need to state the relevant result(s) clearly.
2. You may assume that $\langle\ ,\ \rangle$ is the dot product.

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1. **True or False** (Just write the answer. The rough work for your answer should be in the answer book)

   (a) For a unit speed curve $c : I \to \mathbb{R}^3$ lying on a sphere of radius R, the curvature $\kappa(t) \geq \frac{1}{R}$ for $t \in I$. [2 marks]

   True

   (b) Let $a, b > 0$. For a circular helix $c(s) := \Big(a\cos(\frac{s}{\sqrt{a^2+b^2}}), a\sin(\frac{s}{\sqrt{a^2+b^2}}), \frac{bs}{\sqrt{a^2+b^2}}\Big)$, the torsion $\tau(s)$ is constant. [2 marks]

   True

   (c) Let S be a regular surface and let (U, V, φ) be a local chart around a point p in S. If $N(u, v) := \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}$ is the choice of unit normal on $U \cap S$, then $\langle dN(\varphi_u), \varphi_u\rangle = -\frac{\det(\varphi_u, \varphi_v, \varphi_{uu})}{\|\varphi_u \times \varphi_v\|}$ for every point on $U \cap S$. [2 marks]

   True

   (d) Let $S := \{(\sinh v \cos u, \sinh v \sin u, u) \in \mathbb{R}^3 : (u, v) \in \mathbb{R}^2\}$ be the Helicoid. Then S is a minimal surface. [2 Marks]

   True

   (e) Let S be a regular surface and p be a point in S such that the mean curvature $H(p) = 0$. Then the Gauss curvature $K_S(p) < 0$. [2 Marks]

False

2. **Fill in the blanks** (Write a very brief outline of reasons not exceeding 5 sentences.)

   (a) Let S be a connected regular surface and $a \in \mathbb{R}^3$ is a unit vector with the following property: *For all points p in S and for all vectors $v \in T_pS$, we have $\langle a, v\rangle = 0$.* Then S is contained in a .... [3 Marks]

   Plane [1 Mark]

   Let $f : S \to \mathbb{R}$ be the function defined by $f(p) := \langle (a, b, c), p\rangle$. Then for every point p in S and $w \in T_pS$, we have $df_p(w) = \langle (a, b, c), w\rangle = 0$, by hypothesis. [1 mark]

   Hence f is constant, say d. This proves that S is contained in a plane $\{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d\}$. [1 Mark]

   (b) Let $a > b > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2\}$. Let $C = \{(x, y, z) \in S : x^2 + y^2 = a^2\}$. Then for every point p in C, the Gauss Curvature $K(p) =$ .... [3 Marks]

   0 (zero) [1 Mark]

   A unit normal $N(x, y, z) = \frac{1}{b}(\frac{x(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}}, \frac{x(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}}, z)$ at a point (x, y, z) on the torus S becomes $N(x, y, z) = \frac{1}{b}(0, 0, z) = \pm(0, 0, 1)$ for points $p = (x, y, z)$ on C.

   Hence for any point p on C, the tangent space T_pS is XY-plane. If $p = (a \cos v, a \sin v, \pm b)$ then the curve $\gamma(v) := (a \cos v, a \sin v, \pm b)$ is on C and the unit normal along γ is $N(\gamma(v)) = \pm(0, 0, 1)$. [1 mark]

   Thus 0 is an eigenvalue of the weingartent map dN_p for points p on C and the curvature is zero. [1 mark]

3. Let $I \subseteq \mathbb{R}$ be an open interval and $(f(u), 0, g(u))$ be a unit speed curve with $f(u) > 0$ for $u \in I$. Let $S := \{(f(u) \cos v, f(u) \sin v, g(u)) : u \in I \text{ and } v \in (0, 2\pi)\}$ be the surface of revolution obtained by rotating the curve about Z-axis. Show that, at every point $\varphi(u, v) := (f(u) \cos v, f(u) \sin v, g(u))$ on the surface S, φ_u and φ_v are eigenvectors with eigenvalues $\lambda_1 = f''(u)g'(u) - f'(u)g''(u)$ and $\lambda_2 = \frac{-g'(u)}{f(u)}$ respectively of the Weingarten map dN. [5 Marks]

   For the parametrization $\varphi(u, v) := (f(u) \cos v, f(u) \sin v, g(u))$, the coordinate tangent vectors are $\varphi_u = (f'(u) \cos v, f'(u) \sin v, g'(u))$ and $\varphi_v = (-f(u) \sin v, f(u) \cos v, 0)$. Since $(f'(u))^2 + (g'(u))^2 = 1$ for $u \in I$, it follows that $N(u, v) = (-g'(u) \cos v, -g'(u) \sin v, f'(u))$ is a unit normal for S. [1 mark]

   Observe that $dN(\varphi_v) = \frac{\partial N}{\partial v} = (g'(u) \sin v, -g'(u) \cos v, 0)$.

Since $f(u) > 0$ for $u \in I$, we write $(g'(u)\sin v, -g'(u)\cos v, 0)$ as $-\frac{g'(u)}{f(u)}\varphi_v$ to conclude that φ_v is an eigenvector of the Weingarten map dN with eigenvalue $-\frac{g'(u)}{f(u)}$. [1 mark]

Now $dN(\varphi_u) = \frac{\partial N}{\partial u} = (-g''(u)\cos v, -g''(u)\sin v, f''(u))$.

Since $(f'(u))^2 + (g'(u))^2 = 1$, a differentiation of this equation results in $f'(u)f''(u) + g'(u)g''(u) = 0$.

Hence, for every $u \in I$ there exists $\lambda(u)$ in $\mathbb{R}$ such that $f''(u) = \lambda(u)g'(u)$ and $g''(u) = -\lambda(u)f'(u)$ and using this we conclude that

$$\begin{aligned} dN(\varphi_u) &= (-g''(u)\cos v, -g''(u)\sin v, f''(u)) \\ &= \lambda(u)(f'(u)\cos v, f'(u)\sin v, g'(u)) \\ &= \lambda(u)\varphi_u. \end{aligned}$$ [1 mark]

We need to determine $\lambda(u)$ now. Observe that by multiplying $f''(u) = \lambda(u)g'(u)$ by $g'(u)$ and $-g''(u) = \lambda(u)f'(u)$ by $f'(u)$ and adding we get $f''(u)g'(u) - g''(u)f'(u) = \lambda(u)[f'(u)^2 + g'(u)^2] = \lambda(u)$. [2 marks]

This proves that φ_u and φ_v are eigenvectors of the Weingarten map with eigenvalues $\lambda_1 = g'(v)f''(v) - f'(v)g''(v)$ and $\lambda_2 = -\frac{g'(v)}{f(v)}$ respectively.

4. (a) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the differentiable map defined by $f(x,y,z) := (xy, yz, zx)$. Determine the set $A := \{p \in \mathbb{R}^3 : \|p\| = 1 \text{ and } df_p \text{ is singular}\}$. [3 marks]

For the differentiable map $f(x,y,z) = (xy, yz, zx)$, we compute the derivative at every point $p = (x,y,z)$.

Observe that $df_p(e_1) = (y, 0, z)$, $df_p(e_2) = (x, z, 0)$ and $df_p(e_3) = (0, y, x)$. [1 mark]

For points p in A, the derivative is singular which means that the determinant $\det\begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix} = xyz = 0$. This shows that either $x = 0$ or $y = 0$ or $z = 0$. [1 mark]

Since $\|p\|^2 = x^2 + y^2 + z^2 = 1$, it follows that either $x^2 + y^2 = 1$ or $y^2 + z^2 = 1$ or $z^2 + x^2 = 1$.

This proves that $A = \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \cup \{(0,y,z) \in \mathbb{R}^3 : y^2 + z^2 = 1\} \cup \{(x,0,z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}$. [1 mark]

(b) Let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1\}$. Find the equation and a basis of the tangent space T_pS to S at the point $p = (4, -8, -9)$ on the surface S. [3 marks]

The gradient of the function $f(x,y,z) = x^2 + y^2 - z^2$ at point $p = (x,y,z)$ is $\nabla f(p) = 2(x, y, -z)$. [1 mark]

Observe that for point $p \in S$, the gradient $\nabla f(p) \neq (0,0,0)$. Hence the tangent space T_pS at any point $p = (x,y,z)$ in S is defined by $T_pS = \{(u,v,w) \in \mathbb{R}^3 : xu + yv - zw = 0\}$.

For the point $p = (4, -8, -9)$ on S, the tangent space $T_pS = \{(u, v, w) \in \mathbb{R}^3 : 4u - 8v + 9w = 0\}$. [1 mark]

The vectors $(9, 0, 4)$ and $(0, 9, 8)$ are in T_pS and these two vectors are linearly independent. Hence they form a basis of T_pS. [1 mark]

(c) Let S be a regular surface. Let $\{w_1, w_2\}$ be a basis of the tangent space T_pS at some point p in S such that $\langle dN_p(w_1), w_1\rangle < 0 < \langle dN_p(w_2), w_2\rangle$; here $dN_p : T_pS \to T_pS$ is the Weingarten map. Show that the Gauss curvature $K(p) \leq 0$. [3 marks]

Since the set of vectors $\{w_1, w_2\}$ forms a basis of T_pS, it follows that the Gauss curvature

$$K(p) = \frac{\det\begin{pmatrix}\langle dN_p(w_1), w_1\rangle & \langle dN_p(w_1), w_2\rangle \\ \langle dN_p(w_1), w_2\rangle & \langle dN_p(w_2), w_2\rangle\end{pmatrix}}{\det\begin{pmatrix}\langle w_1, w_1\rangle & \langle w_1, w_2\rangle \\ \langle w_1, w_2\rangle & \langle w_2, w_2\rangle\end{pmatrix}}. \qquad \text{[1 mark]}$$

By hypothesis the product $\langle dN_p(w_1), w_1\rangle\,\langle dN_p(w_1), w_2\rangle$ is negative and $\langle dN_p(w_1), w_2\rangle^2 \geq 0$. [1 mark]

Hence $K(p) = \frac{\langle dN_p(w_1),w_1\rangle\langle dN_p(w_1),w_2\rangle - \langle dN_p(w_1),w_2\rangle^2}{\langle w_1,w_1\rangle\langle w_2,w_2\rangle - \langle w_1,w_2\rangle^2} \leq 0$. [1 mark]

5. (a) Let $S^2 := \{p \in \mathbb{R}^3 : \|p\| = 1\}$ be the unit sphere and $\varphi : \mathbb{R}^2 \to S^2 \setminus \{(0, 0, 1)\}$ be the coordinate system defined by the inverse of the stereographic projection. Find $E = \langle \varphi_u, \varphi_u\rangle$, $F = \langle \varphi_u, \varphi_v\rangle$ and $G = \langle \varphi_v, \varphi_v\rangle$ with respect to this coordinate system $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in \mathbb{R}^2$. [6 marks]

For $(u, v, 0) \in \mathbb{R}^3$, the line segment joining $(u, v, 0)$ and $(0, 0, 1)$ is given by $\gamma(t) = (1-t)(u, v, 0) + t(0, 0, 1)$ for $t \in [0, 1]$. Then the map $\varphi(u, v)$ is the point of intersection of $\gamma(t)$ and the unit sphere S^2 that is not equal to $(0, 0, 1)$.

This is got by solving for t such that $\|\gamma(t)\| = 1$. Observe that $\|\gamma(t)\| = 1$ iff $t^2(u^2 + v^2 + 1) - 2t(u^2 + v^2) + (u^2 + v^2 - 1) = 0$. Solving this equation, we get $t = \frac{u^2+v^2-1}{u^2+v^2+1}$. [1 mark]

Hence $\varphi(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ for $(u, v) \in \mathbb{R}^2$ is the map from $\mathbb{R}^2 \to S^2$. [1 mark]

Note that each component of the map φ is differentiable. Hence the map φ is differentiable on $\mathbb{R}^2$.

The partial derivative

$\varphi_u = d\varphi_{(u,v)}(e_1) = \frac{\left(2(1-u^2+v^2), -4uv, 4u\right)}{(1+u^2+v^2)^2}$ [1 mark]

and the partial derivative

$\varphi_v = d\varphi_{(u,v)}(e_2) = \frac{\left(-4uv, 2(1+u^2-v^2), 4v\right)}{(1+u^2+v^2)^2}$. [1 mark]

Therefore

$$
\begin{aligned}
E = \langle \varphi_u, \varphi_u \rangle &= 4\frac{(1-u^2+v^2)^2 + 4u^2v^2 + 4u^2}{(1+u^2+v^2)4} \\
&= 4\frac{(1+u^2+v^2)^2}{(1+u^2+v^2)^4} \\
&= \frac{4}{(1+u^2+v^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
G = \langle \varphi_v, \varphi_v \rangle &= 4\frac{4u^2v^2 + (1-u^2+v^2)^2 + 4u^2}{(1+u^2+v^2)4} \\
&= 4\frac{(1+u^2+v^2)^2}{(1+u^2+v^2)^4} \\
&= \frac{4}{(1+u^2+v^2)^2} \qquad \text{[1mark]}
\end{aligned}
$$

and

$$
\begin{aligned}
F = \langle \varphi_u, \varphi_u \rangle &= \frac{-8uv(1-u^2+v^2) - 8uv(1+u^2-v^2) + 16uv}{(1+u^2+v^2)4} \\
&= 0. \qquad \text{[1mark]}
\end{aligned}
$$

(b) Let (u,v) be a unit vector in $\mathbb{R}^2$ and $\ell_{(u,v)} = \{t(u,v) : t \in \mathbb{R}\}$ be a line in $\mathbb{R}^2$ and let $\gamma(t) = \varphi(t(u,v))$ for $t \in \mathbb{R}$. For $a < b$ in $\mathbb{R}$, find the length $\int_a^b \|\gamma'(t)\| dt$ of the curve $\gamma|_{[a,b]}$. Find $\lim_{b\to\infty} = \int_0^b \|\gamma'(t)\| dt$. [4 marks]

Given a unit vector $(u,v) \in \mathbb{R}^2$, the curve

$$
\begin{aligned}
\gamma(t) &:= \varphi(t(u,v)) \\
&= \left(\frac{2tu}{1+t^2(u^2+v^2)}, \frac{2tv}{1+t^2(u^2+v^2)}, \frac{t^2(u^2+v^2)-1}{1+t^2(u^2+v^2)} \right) \\
&= \left(\frac{2t}{1+t^2}u, \frac{2t}{1+t^2}v, \frac{t^2-1}{1+t^2}, \right). \qquad \text{[1mark]}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\gamma'(t) &= \frac{1}{(1+t^2)^2}\big((2(1+t^2)-4t^2)u, (2(1+t^2)-4t^2)v, 2t(t^2+1)-2t(t^2-1)\big) \\
&= \frac{1}{(1+t^2)^2}\big(2(1-t^2)u, 2(1-t^2)v, 4t)\big)
\end{aligned}
$$

and

$$\begin{aligned}\|\gamma'(t)\|^2 &= \frac{1}{(1+t^2)^4}\left(4(1-t^2)^2+16t^2\right)\\ &= \frac{4(1+t^2)^2}{(1+t^2)^4}\\ &= \frac{4}{(1+t^2)^2}.\end{aligned}$$ [1mark]

Therefore, for real numbers $a < b$,
$\int_a^b \|\gamma'(t)\|dt = 2\int_a^b \frac{dt}{1+t^2} = \tan^{-1} b - \tan^{-1} a$ [1 mark]
and
$\lim_{b\to\infty}\int_0^b \|\gamma'(t)\|dt = 2\lim_{b\to\infty}\tan^{-1} b = \pi.$ [1 mark]

6. Let S be a compact regular surface. Show that there exists a point p in S such that the Gauss curvature $K(p) > 0$. [10 marks]

Let $f\colon S \to \mathbb{R}$ be the function defined $f(q) := \langle q, q\rangle$. We know that the function f is differentiable on any surface S in $\mathbb{R}^3$.

Since S is a compact surface, there exists a point p in S such that p is a maxima for the function $f(p) = \|p\|^2$. [1 mark]

Let $v \in T_pS$. Then there exists $\varepsilon > 0$ and a smooth curve $\gamma\colon(-\varepsilon,\varepsilon)\to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Then, since p is a point of maximum for the function f, for $t \in (-\varepsilon,\varepsilon)$, the value of the function $f(\gamma(t)) \le f(p) = f(\gamma(0))$. This shows that $t = 0$ is a maximum for the smooth function $f\circ\gamma$. Hence $\frac{d}{dt}\mid_{t=0} f\circ\gamma(t) = df_{\gamma(0)}(\gamma'(0)) = 2\langle p, v\rangle = 0.$ [1 mark]

Since this is true for all $v \in T_pS$, it follows that the point p is normal to the surface S at p. We may now choose a choice of unit normal around p such that $N(p) = \frac{p}{\|p\|}$. [1 Mark]

Since $t = 0$ is a maximum for the smooth function $f\circ\gamma$, it follows that $\frac{d^2}{dt^2}\langle\gamma(t),\gamma(t)\rangle\mid_{t=0}\le 0$

This shows that

$$\begin{aligned}\frac{d^2}{dt^2}\langle\gamma(t),\gamma(t)\rangle\mid_{t=0} &= 2\left[\left\langle\gamma'(0),\gamma'(0)\right\rangle + \left\langle\gamma(0),\gamma''(0)\right\rangle\right]\\ &= 2\left[\|v\|^2 + \left\langle p,\gamma''(0)\right\rangle\right]\\ &\le 0.\end{aligned}$$

That is $-\langle p,\gamma''(0)\rangle \ge \|v^2\|$ for all tangent vectors $v \in T_pS$. [1 mark]

We will now read $\langle p,\gamma''(0)\rangle$ in terms of the Weingarten map as follows.

$$\begin{aligned}\left\langle p,\gamma''(0)\right\rangle &= \|p\|\left\langle N(p),\gamma''(0)\right\rangle\\ &= \left\langle N(\gamma(t)),\gamma''(t)\right\rangle\mid_{t=0}.\end{aligned}$$ [1 mark]

Observe that for all $t \in (-\varepsilon, \varepsilon)$, $\langle N(\gamma(t)), \gamma'(t)\rangle = 0$. We differentiate this equation to get

$$\begin{aligned} 0 &= \frac{d}{dt}\left\langle N(\gamma(t)), \gamma'(t)\right\rangle \\ &= \left\langle dN_{\gamma(t)}(\gamma'(t)), \gamma'(t)\right\rangle + \left\langle N(\gamma(t)), \gamma''(t)\right\rangle . \end{aligned}$$

Hence $\langle dN_p(v), v\rangle = -\langle N(p), \gamma''(0)\rangle \geq \frac{1}{\|p\|}\|v\|^2$. [2 marks]

Since dN_p is a self-adjoint linear map, it follows that the eigenvalues $\lambda_1(p), \lambda_2(p) \geq \frac{1}{\|p\|}$. [2 marks]

This proves that the Gauss curvature $K(p) \geq \frac{1}{\|p\|^2} > 0$. [1 mark]