Name: ____________________

Roll Number: ________________

Practice Final Exam

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.

2. This exam contains **6 + 1** questions and is worth **60%** of your grade.

3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) If $(L, \prec)$ is a linear ordering and L is uncountable, then there exists an infinite $X \subseteq L$ such that X is well-ordered by $\prec$.

(ii) There exists a bijection $f : \mathbb{R}^7 \to \mathbb{R}^9$ satisfying: For every x, y in $\mathbb{R}^7$, $f(x + y) = f(x) + f(y)$.

(iii) The set of all non-computable functions $f : \omega \to \omega$ has the same cardinality as the set of all real numbers.

(iv) There exists a finite $F \subseteq TA$ such that $PA \cup F$ is a complete $\mathcal{L}_{PA}$-theory.

(v) The theory DLO (dense linear orderings without end points) is decidable (as defined on Slide 188).

Question 2. [10 Points]

(a) [**5 Points**] Let $\mathcal{F}$ be the set of all strictly increasing functions $f : \omega \to \omega$. Show that $|\mathcal{F}| = \mathfrak{c}$.

(b) [**5 Points**] Let $\mathcal{E}$ be the set of all countable subsets of ω_1. Show that $|\mathcal{E}| = \mathfrak{c}$.

Question 3. [10 Points]

Using transfinite recursion, construct a function $f : \mathbb{R} \to \mathbb{R}$ such that for every interval $(a, b) \subseteq \mathbb{R}$,

$$\mathsf{range}(f \upharpoonright (a, b)) = \mathbb{R}$$

Question 4. [10 Points]

(a) [**4 Points**] Suppose $(\mathbb{Q}, <)$ is the usual ordering on rationals and $(M, \prec)$ is a countable dense linear ordering without end points. Suppose $x_1 \prec x_2 \prec \cdots \prec x_n$ are in M and $a_1 < a_2, \cdots < a_n$ are in $\mathbb{Q}$. Show that there is an isomorphism $f : (M, \prec) \to (\mathbb{Q}, <)$ such that $f(x_k) = a_k$ for every $1 \leq k \leq n$.

(b) [**6 Points**] Use Tarski-Vaught criterion to show that $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Question 5. [10 Points]

(a) [**5 Points**] Let $W \subseteq \omega$ be an infinite c.e. set. Show that there is a computable **injective** function $f : \omega \to \omega$ such that $\mathsf{range}(f) = W$.

(b) [**5 Points**] Show that $True_{\mathcal{N}}$ (defined on Slide 199) is not c.e.

Question 6. [10 Points]

Let $\mathcal{N} = (\omega, 0, +, \cdot)$ be the standard model of PA.

(a) [**6 Points**] Define $False_{\mathcal{N}} = \{\ulcorner \psi \urcorner : \mathcal{N} \models \neg\psi\}$. Show that $False_{\mathcal{N}}$ is not definable in $\mathcal{N}$.

(b) [**4 Points**] Show that there are $\mathcal{L}_{PA}$-sentences ϕ and ψ such that PA does not prove either one of the following four sentences.

(i) ϕ

(ii) $\neg\phi$

(iii) $\phi \implies \psi$

(iv) $\phi \implies (\neg\psi)$

Bonus Question [5 Points]

Let X be an uncountable set and suppose $\prec_1$ and $\prec_2$ are two well-orders on X. Show that there is an uncountable $Y \subseteq X$ such that for every $a, b \in Y$,

$$a \prec_1 b \iff a \prec_2 b$$

End of Exam