ASSIGNMENT 9

MTH 301, 2018

(1) Use $f_n(x) = x^n$ on $[0,1]$ to show that $B = \{f \in C[0,1] : \|f\| \leq 1\}$ is not compact.

(2) Show that $\{f \in C[0,1] : f(x) > 0 \; \forall x \in [0,1]\}$ is open.

(3) What is the interior of $\{f \in C(R) : f(x) > 0 \; x \in \mathbb{R}$ and bounded$\}$?

(4) Prove that the family $\{\sin(nx) : n \geq 1\}$ is not an equicontinuous subset of $C[0,\pi]$.

(5) (a) Show that $\mathcal{F} = \{F(x) = \int_0^x f(t)dt : f \in C[0,1], \|f\|_\infty \leq 1\}$ is a bounded and equicontinuous subset of $C[0,1]$.
 (b) Why is $\mathcal{F}$ not closed?
 (c) Show that the closure of $\mathcal{F}$ is all functions f with Lipschitz constant 1 such that $f(0) = 0$.

(6) (a) Let $\mathcal{F}$ be a subset of $C[0,1]$ that is closed, bounded, and equicontinuous. Prove that there is a function $g\mathcal{F}$ such that

$$\int_0^1 g(x)dx \geq \int_0^1 f(x)dx, \; \forall f \in \mathcal{F}.$$

 (b) Construct a closed bounded subset $\mathcal{F}$ of $C[0,1]$ for which the conclusion of the previous problem is false.

(7) Let $\mathcal{F}$ be an equicontinuous family of functions in $C(X)$, where X is a compact metric space. Prove that if for each $x \in X$, $\sup\{f(x) : f \in \mathcal{F}\} = M_x < \infty$, then $\mathcal{F}$ is bounded.

(8) Let $\mathcal{F}$ be a family of continuous functions defined on $\mathbb{R}$ that is (i) equicontinuous and satisfies (ii) $\sup\{f(x) : f \in \mathcal{F}\} = M_x < \infty$ for every x. Show that every sequence $\{f_n\}$ has a subsequence that converges uniformly on $[-k,k]$ for every $k > 0$.

(9) Let $K(x,t)$ be a continuous function on the square $[a,b] \times [a,b]$. Given $f \in C[a,b]$, define $g(x) = \int_a^b f(t)K(x,t)dt$. Also, consider the operator $T : C[a,b] \to C[a,b]$ defined by $Tf = g$.
 (a) Show that $g \in C[a,b]$.
 (b) Show that T maps bounded set to equicontinuous set.