Remark: under the same setup

i.e. $X_1, X_2, \ldots$ uncorrelated with $E X_i = \mu_i$

and $V(X_i) = \sigma_i^2$

suppose $\sum_1^n \sigma_i^2 \to \infty$, then we can take

$$a_n = \sum_{i=1}^n \mu_i \quad \& \quad b_n = \sum_1^n \sigma_i^2$$

with
$$\frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

Remark: If $\sigma_i^2 = \sigma^2 \ \forall i$, then the condition

$\frac{1}{n^2} \sum \sigma_i^2 \to 0$ as $n \to \infty$ is automatically

satisfies and WLLN holds

Remark: If $X_1, \ldots$ is a seq of i.i.d. r.v.s with

mean μ and variance $\sigma^2 < \infty$, then

WLLN holds for $\{X_n\}$ &

$$\bar{X}_n \xrightarrow{p} \mu = E X_1$$

Note that finiteness of variance is not reqd by

'Khintchine's WLLN.

Convergence in distribution (or law)

Def^n: We say that a seq of r.v.s $\{X_n\}$ converges in distribution to X ($X_n \xrightarrow{L} X$ as $n \to \infty$)

if $F_{X_n}(x) \longrightarrow F_X(x)$ $\forall x$ at which the limiting dist^n f^n is continuous.

$F_{X_n}(\cdot)$: d.f. of X_n

$F_X(\cdot)$: d.f. of X

Examples

(1) $X_1, \ldots, X_n$ i.i.d. $N(0,1)$

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$$

$$Y_n \sim N\left(0, \frac{1}{n}\right) \quad ; \quad Y_n \xrightarrow{P} 0.$$

$$F_{Y_n}(y) = P(Y_n \leq y) = P(\sqrt{n}\, Y_n \leq \sqrt{n}\, y)$$

$$= \Phi(\sqrt{n}\, y) \longrightarrow \begin{cases} 0, & y < 0 \\ \frac{1}{2}, & y = 0 \\ 1, & y > 0 \end{cases}$$

Consider a d.f $F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y \geq 0 \end{cases}$

i.e. $P(Y = 0) = 1$

then

$$F_{Y_n}(y) \to F_Y(y). \quad \forall\ y \text{ at which } F_Y(\cdot) \text{ is continuous}$$

$$\Rightarrow Y_n \xrightarrow{L} Y \ (\leftarrow \text{a degenerate r.v.})$$

i.e. $Y_n \xrightarrow{L} 0$

(2) $X_1, X_2 \ldots$ i.i.d. $U(0, \theta)$; $\theta > 0$

$$Y_n = X_{(n)} = \max\{X_1, \ldots, X_n\} \ ; \ Y_n \xrightarrow{P} \theta$$

$$F_{Y_n}(y) = \begin{cases} 0, & y < 0 \\ \left(\frac{y}{\theta}\right)^n, & 0 \le y < \theta \\ 1, & y \ge \theta \end{cases} \longrightarrow \begin{cases} 0, & y < \theta \\ 1, & y \ge \theta \end{cases}$$

$\Rightarrow Y_n \xrightarrow{L} Y$, a degenerate r.v.; degenerate at θ

i.e. $Y_n \xrightarrow{L} \theta$

Consider
$$Z_n = n(\theta - X_{(n)}) = n(\theta - Y_n)$$

$$\begin{aligned} F_{Z_n}(x) &= P(Z_n \le x) \\ &= P(n(\theta - X_{(n)}) \le x) \\ &= P\left(X_{(n)} \ge \theta - \frac{x}{n}\right) = 1 - F_{X_{(n)}}\left(\theta - \frac{x}{n}\right) \\ &= \begin{cases} 0, & x < 0 \\ 1 - \left(\frac{\theta - x/n}{\theta}\right)^n & 0 \le x < n\theta \\ 1, & x \ge n\theta \end{cases} \end{aligned}$$

$$F_{Z_n}(x) \to \begin{cases} 0, & x < 0 \\ 1 - e^{-x/\theta}, & x \geq 0 \end{cases}$$

i.e. $Z_n \xrightarrow{L} Z$; where $f_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{1}{\theta} e^{-z/\theta}, & z \geq 0 \end{cases}$

i.e. $Z_n \xrightarrow{L} Z \sim \exp(0, \theta)$

scale θ.

(3) $\{X_n\}$ - seq of discrete r.v.s.

$$P(X_n = x) = \begin{cases} 1, & \text{if } x = 2 + \frac{1}{n} \\ 0, & \text{o/w} \end{cases}$$

$$F_{X_n}(x) = \begin{cases} 0, & x < 2 + \frac{1}{n} \\ 1, & x \geq 2 + \frac{1}{n} \end{cases}$$

$$\to F_X(x) = \begin{cases} 0, & x < 2 \\ 1, & x \geq 2 \end{cases}$$

$$X_n \xrightarrow{L} 2$$

<u>Remark</u>: Ex 1, 2, 3 are direct approaches to prove convergence in law/distribution.

Convergence in law can also be proved using m.g.f. convergence.

(4) $X_n \sim Bin(n, \theta)$

Suppose $n \to \infty$, $\ni np = \lambda$ is fixed i.e. $\theta = \frac{\lambda}{n}$

$$M_{X_n}(t) = ((1-\theta) + \theta e^t)^n$$

$$= \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n$$

$$\to e^{\lambda \cancel{\theta}(e^t - 1)} \quad \text{as } n \to \infty$$

$\Rightarrow X_n \xrightarrow{L} X$; where $X \sim P(\lambda)$

(5) $X_1, \ldots$ i.i.d. $N(0,1)$

$\bar{X}_n \sim N(0, \frac{1}{n})$

$M_{\bar{X}_n}(t) = e^{\frac{t^2}{2n}} \rightarrow 1$ as $n \rightarrow \infty$

$\Rightarrow \bar{X}_n \xrightarrow{L} X$; where X is degenerate at 0