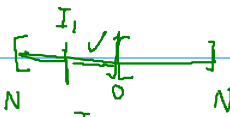
Assignment - 7 - Solution/Hints

2. $(\mathbb{R}, |\cdot|)$ A bdd set $\Rightarrow$ totally bdd.

A: $(a_n) \in A$ $\quad \{a_n \mid n \geq 1\}$ — finite ✓ / infinite

(a_{n_k}) constant

$\Rightarrow A \subseteq [-N, N]$ for some $N > 0$.
I_0



$\{a_n \mid n \geq 1\}$.

Choose $a_{n_1} \in \{a_n \mid n \geq 1\}$ s.t. $a_{n_1} \in I_1$. $\ell(I_1) = \frac{\ell(I_0)}{2}$

Choose $a_{n_2} \neq a_{n_1}$ s.t. $a_{n_2} \in I_2$, $\ell(I_2) = \frac{\ell(I_0)}{2^2}$

I_n : closed & bdd.

$(a_{n_k})_{k=1}^{\infty}$ $\quad a_{n_k} \in I_k$ s.t. $\ell(I_k) = \frac{\ell(I_0)}{2^k} \to 0$ as $k \to \infty$.

$I_1 \supseteq I_2 \supseteq \cdots\cdots$

HW: (a_{n_k}) is Cauchy seq. $\quad \varepsilon > 0$ choose m: $\frac{\ell(I_0)}{2^m} < \varepsilon$.

$N_\varepsilon := n_m$ $\quad n_i, n_j \cdots\cdots$ $\quad |a_{n_i} - a_{n_j}| \underset{?}{\leq} \ell(I_{n_m}) < \varepsilon$

$n > 1$. $(\mathbb{R}^n, \|\cdot\|_2)$. We know: $(\mathbb{R}, |\cdot|)$ is tot. bdd.

(*) (X, d) and (Y, ρ) $\quad (X \times Y, d \times \rho)$

(M, d), (M, ρ)

$d \sim_s \rho$ $\quad \exists\, C_1, C_2 > 0$

$C_1 \rho(\,) \leq d(\,) \leq C_2 \rho$

$A \subseteq X$ and $B \subseteq Y$ which are totally bdd in X and Y respectively.
Then $A \times B$ is also totally bdd. w.r.t. $d_2((x,y),(x',y')) := \left[d^2(x,x') + \rho^2(y,y')\right]^{1/2}$

• $d_\infty((x,y),(x',y')) := \max\{d(x,x'), \rho(y,y')\}$.

Pf: (HW): $B_{d_\infty}((x,y), r) = B_d(x,r) \times B_\rho(y,r)$

<u>Claim:</u> $A \times B$ is totally bdd.

Pf: For $\varepsilon > 0$. $\exists$ finite sets $F \subset A$, $E \subset B$ s.t. $A \subset \bigcup_{x \in F} B_d(x, \varepsilon)$

$F = \{a_1, \dots, a_n\}$, $E = \{b_1, \dots, b_m\}$

$B \subset \bigcup_{y \in E} B_\rho(y, \varepsilon)$

Consider $A \times B \subset \bigcup_{\substack{x \in F \\ y \in E}} B_d(x, \varepsilon) \times B_\rho(y, \varepsilon) \overset{(HW)}{=} \bigcup_{(x,y) \in F \times E} B_{d_\infty}((x,y), \varepsilon)$

$F \times E$. finite subset of $X \times Y$.

$(\mathbb{R}^n, \|\cdot\|_2)$ $\quad X$ bdd set in $\mathbb{R}^n$. $\quad X \in [-N, N]^n$ is totally bdd.

$[-N,N] \times \dots \times [-N,N]$
t.b. $\qquad$ t.b.

$\mathbb{R}^2$

$[-N,N] \times [-N,N]$ is totally bdd (by result proved $\circledast$)

3. $\left(\left\{\frac{1}{n} \mid n \geq 1\right\}, |\cdot|\right)$, $(\mathbb{N}, |\cdot|)$

(HW) totally bdd. $\qquad$ not totally bdd.

$\exists\ \varepsilon > 0$ s.t.

$\circledast$ A is not totally bdd. $\{x_n\} \subset A$ s.t. $d(x_n, x_m) \geq \varepsilon$. $x_n \neq x_m$.

$B := \{x_n \mid n \geq 1\}$. consider B w.r.t. relative metric induced by M.

Consider $B(x_n, \varepsilon) \cap B$ open ε-ball in B.

$= \{x_n\}$ since $d(x_n, x_m) \geq \varepsilon \quad \forall\ n \neq m$.

$\Rightarrow \{x_n\}$ is both open & closed.

$F: B \longrightarrow \mathbb{N}$

$x_n \longrightarrow n$

$\leftarrow$ (M,d) totally bdd.

<u>claim:</u> M is separable

Pf $\varepsilon_n := \frac{1}{n}$ $\exists$ $A_1^{(n)}, \dots, A_{k_n}^{(n)}$ s.t. $M = \bigcup_{i=1}^{k_n} A_i^{(n)}$, $\operatorname{diam}(A_i^{(n)}) < \frac{1}{n}$.

$a_1^{(n)}, \quad \dots, a_{k_n}^{(n)}$

$S := \bigcup_{n=1}^{\infty} \{a_1^{(n)}, \dots, a_{k_n}^{(n)}\}$ WTS: $\overline{S} = M$.

$\overline{S} \subset M$.

$x \in M$ & $\varepsilon > 0$ $\exists$ N s.t. $\frac{1}{N} < \varepsilon$.

$x \in M = \bigcup_{i=1}^{k_N} A_i^{(N)} \Rightarrow x \in A_{i_0}^{(N)}$ $\exists$ $a_{i_0}^{(N)} \in A_{i_0}^{(N)}$ s.t.

$d(x, a_{i_0}^{(N)}) < \frac{1}{N} < \varepsilon$.

$B(x,\varepsilon) \cap S \neq \phi$.

$\Rightarrow \overline{S} = M$.

$\rightarrow$ H^∞ is totally bdd.

$H^\infty := \{(x_n) \mid |x_n| \le 1\} = [-1,1]^{\mathbb{N}}$ $\qquad d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$.

IGNORE! DONE ON THE NEXT PAGE.

$x = (x_1, x_2, \dots \mid \dots)$

For $\varepsilon > 0$ <u>choose</u> $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \frac{\varepsilon}{2}$.

$x \in H^\infty \rightsquigarrow x^{(n)} := (\underbrace{x_1, \dots, x_n}, \quad 0 \quad)$.

$A_n := \{x^{(n)} \mid x \in H^\infty\}$ Note that $d(x, x^{(n)}) \underset{(\text{HW})}{\le} \dots < \frac{\varepsilon}{2}$.

$x^{(n)} \in A_n \rightsquigarrow \{(x_1, \dots, x_n)\} \in \mathbb{R}^n$

$\{x_1 \in [-1,1], x_2 \in [-1,1], \dots, x_n \in [-1,1]$

$\bigcup_{j \in \text{finite}} B^{(1)}(a_j^{(1)}, \varepsilon/2)$

$a = (a_1^{(1)}, a_2^{(2)}, \dots, a_n^{(n)}, \quad 0 \quad)$

$\to$ H^∞

$\to$ • $d_\infty(x,y) := \sup_{n \ge 1} \{\frac{1}{2^n} |x_n - y_n|\}$

<u>Show:</u> $d_\infty \sim_s d$

After some thought, I decided to stick to the def^n of metric on H^∞ mentioned in Carothers. Nevertheless, it will be good for you to know about d_∞ defined on the previous page. I will give the proof with $d(x,y) := \sum_1^\infty \frac{1}{2^n}|x_n-y_n|$. It's going to be cumbersome. You can pick the idea here and then try to prove totally bdd. with d_∞.

$\star$ H^∞ is totally bdd. w.r.t. $d(x,y) = \sum_1^\infty \frac{1}{2^n}|x_n-y_n|$ for $x_n, y_n \in [-1,1]$.

Since total boundedness deals with ε-balls, first let's look at what are ε-balls in H^∞.

For $x \in H^\infty$ and $r>0$,

$B(x,r) = \{y \in H^\infty \mid d(x,y) < r\}$. That is, $\sum_1^\infty \frac{|x_n-y_n|}{2^n} < r$.

For $r>0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\frac{2^{n-1} r}{2} > 1$

and $\sum_{n=N}^\infty \frac{|x_n-y_n|}{2^n} < \frac{r}{2}$. $\forall n \geq N$ and $x_n, y_n \in [-1,1]$.

So we need to only look at the first $(N-1)$-coordinates for y.

(HW): Show that $B(x,r) = B\left(x_1, \frac{2r}{2^N}\right) \times B\left(x_2, \frac{2^2 r}{2^N}\right) \times \cdots \times B\left(x_{N-1}, \frac{2^{N-1} r}{2^N}\right) \times [-1,1] \times \cdots$

Now for $\varepsilon>0$, $\exists$ N s.t. $\sum_N^\infty \frac{|x_n-y_n|}{2^n} < \frac{\varepsilon}{2}$. $\forall n \geq N$ and $\frac{2^{n-1}\varepsilon}{2} > 1$. $\forall n \geq N$.

For $x \in H^\infty$, consider $x^{(N-1)} \doteq (x_1, x_2, \ldots, x_{N-1}, \quad)$

Let $A := \{x^{(N-1)} \mid x \in H^\infty\}$. Consider $\tilde{A} := \{(x_1, \ldots, x_{N-1}) \mid x^{(N-1)} \in A\}$.

Then $\tilde{A} \subset [-1,1] \times \cdots \times [-1,1] \subset \mathbb{R}^{N-1}$.

Since $\underbrace{[-1,1] \times \cdots \times [-1,1]}_{\text{w.r.t relative metric where } [-1,1] \text{ is relative metric space induced by } (\mathbb{R}, |\cdot|)}$ is totally bdd. in $\mathbb{R}^{N-1}$, $\tilde{A}$ is also totally bdd. in $\mathbb{R}^{N-1}$.

$$[-1,1] = \bigcup_{j=1}^{m_1} B\left(a_j^{(1)}, \frac{\varepsilon}{2}\right), \ldots, \quad [-1,1] = \bigcup_{j=1}^{m_{N-1}} B\left(a_j^{(N-1)}, \frac{\varepsilon}{2}\right)$$

$$F_1 := \{a_j^{(1)} \mid 1 \le j \le m_1\}, \ldots, F_{N-1} := \{a_j^{(N-1)} \mid 1 \le j \le m_{N-1}\}.$$

Let $\underset{\uparrow}{F} := F_1 \times F_2 \times \cdots \times F_{N-1} \times 0 \times \cdots \in H^\infty.$
(finite set)

So, $$A \subset \bigcup_{(a_j^{(1)}, \ldots, a_j^{(N-1)}, 0) \in F} \left(B\left(a_j^{(1)}, \frac{\varepsilon}{2}\right) \times \cdots \times B\left(a_j^{(N-1)}, \frac{\varepsilon}{2}\right) \times [-1,1] \times \cdots \right)$$

Hence, $$H^\infty = \bigcup_{(a_j^{(1)}, \ldots, a_j^{(N-1)}, 0) \in F} \left(B\left(a_j^{(1)}, \varepsilon\right) \times \cdots \times B\left(a_j^{(N-1)}, \varepsilon\right) \times [-1,1] \times \cdots \right) \quad \text{---} (*)$$

Note that $d(x, x^N) = \sum_{N+1}^{\infty} \frac{1}{2^n} |x_n - y_n| < \frac{\varepsilon}{2}$. Then use triangle inequality to conclude $(*)$