Assignment 6.

1. Given $A \subset S$. Define $\chi_A : S \to \mathbb{R}$, the characteristic function of A, by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Consider $\chi_A : \mathbb{R} \to \mathbb{R}$. What are the pts. where χ_A is cts.? What are the pts. where it is discontinuous?

2. Let $f : (M, d) \to (N, \rho)$ be continuous, and let A be a separable subset of M. Prove that $f(A)$ is also separable.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz function if $\exists\, K < \infty$ s.t.

$$|f(x) - f(y)| \le K\,|x - y|, \quad \forall\, x, y \in \mathbb{R}.$$

Prove that a Lipschitz function is continuous.

4. Fix $k \geq 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x_k$. Is f continuous? Explain.

5. Define $g : \ell_2 \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} x_n/n$. Is g continuous? Explain.

6. Fix $y \in \ell_\infty$ and define $h : \ell_1 \to \ell_1$ by $h(x) = (x_n y_n)_{n=1}^{\infty}$. Show that h is continuous.

7. Suppose that we are given a point x and a seq. (x_n) in a metric space M, and suppose that $f(x_n) \to f(x)$, $\forall$ continuous, real-valued function f on M. Does it follow that $x_n \to x$ in M? Explain.

8. Given disjoint nonempty closed sets E, F, define $f : M \to \mathbb{R}$ by

$$f(x) = \frac{d(x, E)}{d(x, E) + d(x, F)}.$$

Show that f is a cts. function on M.

9. Let C be a closed set in $\mathbb{R}$ and let $f: C \to \mathbb{R}$ be cts. Show that there is a continuous extension $g: \mathbb{R} \to \mathbb{R}$ with $g(x) = f(x)$, $\forall\, x \in C$.
If we drop the assumption of closedness on C, does every cts. function on C has a continuous extension?

10. Suppose that $f: \mathbb{Q} \to \mathbb{R}$ is Lipschitz. Show that f extends to a continuous function $h: \mathbb{R} \to \mathbb{R}$. Is h unique? Explain.

11. Prove that $\mathbb{N}$ (with usual metric) is homeomorphic to $\{\frac{1}{n} \mid n \geq 1\}$ (with usual metric).

12. Show that every metric space is homeomorphic to one of finite diameter.

13. Prove that $\mathbb{R}$ is homeomorphic to $(0,1)$ and $(0,1)$ is homeomorphic to $(0,\infty)$.
Is $\mathbb{R}$ isometric to $(0,1)$?

14. Let $f: (M,d) \to (N,\rho)$ be a homeomorphism. Prove that M is separable iff N is separable.

15. Let $f: (M,d) \to (N,\rho)$.

(i) (Def^n:) f is an open map if $f(\mathcal{U})$ is open whenever $\mathcal{U}$ is open in M.
Give examples of a continuous map that is not open and an open map that is not continuous

(ii) (Def^n:) f is a closed map if $f(A)$ is closed whenever A is closed in M.
Give examples of a continuous map which is not closed and a closed map which is not continuous.