Name: **Date: April 05, 2014**

Roll No:

MSO201A/ESO 209: PROBABILITY & STATISTICS

Quiz # 2 Full Marks 20

[1] $X_1,....,X_n$ is a random sample from $U(0,\theta), \theta>0$.

(a) Find $c(\theta)$ such that $e^{\bar{X}_n^2} \xrightarrow{p} c(\theta)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(b) Find the distribution function of the random variable X such that $n\left(\theta - X_{(n)}\right) \xrightarrow{\mathcal{L}} X$, where $X_{(n)} = \max\left(X_1,\ldots,X_n\right)$.

[2] $X_1,....,X_n$ is a random sample from the exponential distribution having p.d.f.

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & \text{if } x>\theta \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify whether $\sum_{i=1}^{n} X_i$ is a sufficient statistic or not.

(b) Verify whether $X_{(1)} = \min\left(X_1,\ldots,X_n\right)$ is a sufficient statistic or not.

(c) Find an unbiased estimator of θ, if any, based on $\sum_{i=1}^{n} X_i$.

(d) Find an unbiased estimator of θ, if any, based on $X_{(1)}$.

(1) $X_1, \ldots X_n$ r.s. from $U(0,\theta)$, $\theta>0$

(a) $X_1, \ldots X_n$ i.i.d with $EX_i = \frac{\theta}{2}$ $\forall i$

By Khintchin's WLLN, $\frac{1}{n}\sum_1^n X_i \xrightarrow{p} EX_1 = \frac{\theta}{2}$ — (2)

i.e. $\bar{X}_n \xrightarrow{p} \theta/2$

$\Rightarrow e^{\bar{X}_n^2} \xrightarrow{p} e^{\theta^2/4}$ — (2)

(b) $Z_n = n(\theta - X_{(n)})$

d.f. of Z_n: $F_Z(x) = P[n(\theta - X_{(n)}) \le x] = P\left(X_{(n)} \ge \theta - \frac{x}{n}\right)$

$= 1 - P\left(X_{(n)} < \theta - \frac{x}{n}\right)$ — (2)

d.f. of $X_{(n)}$

$$F_{X_{(n)}}(x) = \begin{cases} 0, & x \le 0 \\ (F_X(x))^n, & 0<x<\theta \\ 1, & x \ge \theta \end{cases}$$

$$= \begin{cases} 0, & x \le 0 \\ 1-\left(\frac{\theta - x/n}{\theta}\right)^n, & 0<x<n\theta \\ 1, & x \ge n\theta \end{cases}$$ — (2)

$$\longrightarrow F_X(x) = \begin{cases} 0, & x \le 0 \\ 1-e^{-x/\theta}, & x>0 \end{cases}$$ (2)

(2) $X_1, \ldots X_n$ r.s. from $f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & o/w \end{cases}$

(a) & (b) joint p.d.f. of $X_1, \ldots X_n$

$$f_\theta(\underset{\sim}{x}) = \begin{cases} e^{-\sum_1^n (x_i - \theta)}, & x_1, \ldots x_n > \theta \\ 0 & o/w \end{cases}$$

$$= \begin{cases} e^{n\theta} e^{-\sum_1^n x_i}, & x_{(1)} > \theta \\ 0, & o/w. \end{cases}$$

i.e. $f_\theta(\underset{\sim}{x}) = e^{n\theta} e^{-\sum_1^n x_i} I_{(\theta, x_{(1)})}$

$$= \underbrace{\left(e^{-\sum_1^n x_i}\right)}_{h(\underset{\sim}{x})} \underbrace{\left(e^{n\theta} I_{(\theta, x_{(1)})}\right)}_{g(\theta, x_{(1)})} \quad \text{— (2)}$$

$\Rightarrow$ ΣX_i is NOT suff for θ (by NFFT) — (1)

& $X_{(1)}$ is suff for θ — (1)

(c) $E X_i = \int_\theta^\infty x e^{-(x-\theta)} dx = \int_0^\infty (y+\theta) e^{-y} dy = 1 + \theta$ — (1)

$\Rightarrow$ $E(\Sigma X_i) = n(1+\theta)$

$\Rightarrow$ $E(\bar{X}) = 1+\theta) \Rightarrow E(\bar{X} - 1) = \theta$

$\Rightarrow (\bar{X}-1)$ is u.e. of θ — (2)

(d) $f_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f_X(x)$

$$= \begin{cases} n e^{-n(x-\theta)}, & x > \theta \\ 0, & o/w \end{cases} \quad \text{— (1)}$$

$E X_{(1)} = n\int_\theta^\infty x e^{-n(x-\theta)} dx = n\int_0^\infty (y+\theta) e^{-ny} dy = \theta + \frac{1}{n}$

$E(X_{(1)}) = \theta + \frac{1}{n}$; i.e $E\left(X_{(1)} - \frac{1}{n}\right) = \theta$

$\Rightarrow X_{(1)} - \frac{1}{n}$ is u.e. of θ — (2)