Continuous Functions:

Recall: $f:(\mathbb{R},|\cdot|)\to(\mathbb{R},|\cdot|)$ is cts. at $x_0\in\mathbb{R}$ if $\forall\ \varepsilon>0$, $\exists\ \delta_\varepsilon(x_0)>0$ s.t.
whenever $|x-x_0|<\delta_\varepsilon \Rightarrow |f(x)-f(x_0)|<\varepsilon$.
$x\in(x_0-\delta_\varepsilon, x_0+\delta_\varepsilon) \Rightarrow f(x)\in(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$.

(HW) Equivalently, $f:(\mathbb{R},|\cdot|)\to(\mathbb{R},|\cdot|)$ is cts. at x_0, if
$\forall\ \underset{\text{open set}}{\mathcal{U}}\ni f(x_0)$, $\exists\ V_{x_0}$ open set s.t $x_0\in V_{x_0}$ and $f(V_{x_0})\subset\mathcal{U}$.

$\to$ f is cts. on $\mathbb{R}$ if $f^{-1}(\mathcal{U})$ is an open set for every open set $\mathcal{U}$ in $\mathbb{R}$.

Def^n: $f:(M,d)\to(N,\rho)$ is cts. at $x_0\in M$ if $\forall\ \varepsilon>0$, $\exists\ \delta_\varepsilon>0$ s.t.
$$f(B_d(x_0,\delta_\varepsilon))\subset B_\rho(f(x_0),\varepsilon)$$

Using the sequential characterizations of open sets and closed sets, one obtains:

Thm: (HW) For $f:(M,d)\to(N,\rho)$, TFAE:
(i) f is cts. on M.
(ii) For any $x\in M$, if $x_n\xrightarrow{d}x$ in M, then $f(x_n)\xrightarrow{\rho}f(x)$ in N.
(iii) If E is a closed set in N, then $f^{-1}(E)$ is closed in M.
(iv) If V is open in N, $f^{-1}(V)$ is open in M.

Examples: (i) Consider $(\mathbb{N},|\cdot|)$ relative metric space induced from $|\cdot|$ on $\mathbb{R}$.
Any function $f:(\mathbb{N},|\cdot|)\to(\mathbb{R},|\cdot|)$ is cts.
Note that every singleton pt. set, $\{n\}$ is open and closed in $(\mathbb{N},|\cdot|)$.
So, $f^{-1}(\mathcal{U})$ is open in $\mathbb{N}$ for every open set $\mathcal{U}$ in $(\mathbb{R},|\cdot|)$.
Q. Is any function $f:(\mathbb{R},|\cdot|)\to(\mathbb{N},|\cdot|)$ cts ?

HW: Example 5.2 (a) – (f).

Recall: $|d(x,z) - d(y,z)| \leq d(x,y)$. $\forall\ x, y, z \in (M,d)$.

$F_z : (M,d) \to \mathbb{R}$ as $F_z(x) := d(x,z)$.

$|d(x_n, z) - d(x,z)| \leq d(x_n, x)$

In particular, if $x_n \xrightarrow{d} x$, i.e., $d(x_n, x) \to 0$ then $d(x_n, z) \to d(x,z)$ as $n \to \infty$.

Similarly $F_y(x) := d(y,x)$ is cts. on (M,d). That is, if $x_n \xrightarrow{d} x$, then $d(y, x_n) \to d(y,x)$ as $n \to \infty$.

So, the metric d as a function on $M \times M$ is cts. separately in each variable, hence, d is also jointly cts. That is, whenever $d(x_n, x) \to 0$, $d(y_n, y) \to 0$ Then $d(x_n, y_n) \to d(x,y)$.

Consider $A \subset M$ and $x \in M$.

Define the distance of x from the set A as

$$d(x, A) := \inf \{ d(x,a) \mid a \in A \}.$$

Claim: The map $F_A : (M,d) \to \mathbb{R}$ defined by $F_A(x) := d(x, A)$ is cts.

Pf: To show if $d(x_n, x) \to 0$ then $d(x_n, A) \to d(x, A)$ as $n \to \infty$.

HW: $|d(x_n, A) - d(x, A)| \leq d(x_n, x)$. Complete the proof!

Note: If $f : (M,d) \to \mathbb{R}$ is cts., then $f^{-1}\{0\}$ is a closed set in M.

$\to$ $A \subset M$ is closed in M iff A is the zero set of some real-valued cts. function.

Pf: $\Rightarrow$: (HW). $d(x, A) = 0$ iff $x \in \bar{A}$.

Consider $F_A : M \to \mathbb{R}$ given by $F_A(x) = d(x, A)$.

$Z(F_A) = \{ x \in M \mid F_A(x) = 0 \}$. If $x \in A$, then $d(x, A) = 0$.

So, $A \subset Z(F_A)$.

Let $x \in Z(F_A)$. Then $d(x,A)=0$. This implies that $x \in \overline{A}$.
But A is closed, so $A = \overline{A}$. Hence $x \in A$. Therefore, $Z(F_A) \subset A$, hence equality.

Conversely, since zero set of a cts. function is a closed set, so A is closed. ∎

Remark. $\{\text{Closed sets in } M\} \leftrightarrow \{f: M \to \mathbb{R} \text{ cts.}\}$.

Equivalently, $\{\text{Open sets in } M\} \leftrightarrow \{f: M \to \mathbb{R} \text{ cts.}\}$.

(A set $\mathcal{U}$ is open in M iff $\exists$ a cts. $f: M \to \mathbb{R}$ s.t. $f^{-1}(V) = \mathcal{U}$ for some open set $V \subset \mathbb{R}$.)

Recall: Sequential characterization of an open set: $\mathcal{U}$ is open in M iff for $x_n \xrightarrow{d} x \in \mathcal{U}$, then $x_n \in \mathcal{U}$ eventually.

$\{\text{cvgt. subsequences}\} \leftrightarrow \{\text{open sets}\} \leftrightarrow \{f: (M,d) \to \mathbb{R} \text{ cts.}\}$.

One can say the same thing for closed sets as well.

$\rightarrow$ M with two metrics d and ρ.

Q. Can we "compare" (M,d) and (M,ρ) in some sense?

A. Via the notion of equivalent metrics?

Recall $d \sim \rho$ if $d(x_n,x) \to 0$ iff $\rho(x_n,x) \to 0$.

equivalently, $\mathcal{U}$ open in (M,d) iff $\mathcal{U}$ open in (M,ρ)

equivalently, A closed in (M,d) iff A closed in (M,ρ)

$$\begin{tikzcd} (M,d) \arrow[rr, leftrightarrow, dashed, "\mathrm{Id}"] \arrow[d, "f \text{ cts.}"'] & & (M,\rho) \arrow[d, "g \text{ cts.}"] \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$

HW: $d \sim \rho$ iff $\mathrm{Id}: (M,d) \to (M,\rho)$ is cts. and $(\mathrm{Id})^{-1}: (M,\rho) \to (M,d)$ is cts. on M.

- Note that Id is bijective.

? If we replace (M,ρ) with (N,ρ), any metric space, can we compare them?

$$(M,d) \xrightarrow{?} (N,\rho)$$

<u>Homeomorphism:</u> (M,d) and (N,ρ) are homeomorphic if $\exists$

$F:(M,d)\to(N,\rho)$ s.t.

- F is bijective
- F is cts. and F^{-1} is cts.

Example. Consider $(M,d):=(\mathbb{N},|\cdot|)$ and $(N,\rho):=(\mathbb{Z},|\cdot|)$.

Define $F:\mathbb{Z}\to\mathbb{N}$ as

$$F(n)=\begin{cases}2n, & n\geq 1\\ -2n+1, & n<1\end{cases}$$

F is a bijection. Note that every subset of $\mathbb{Z}$ and $\mathbb{N}$ are open. (Use the fact that every singleton set is open!)
So, F is cts. and F^{-1} also cts.

Therefore, $(\mathbb{N},|\cdot|)$ is homeomorphic to $(\mathbb{Z},|\cdot|)$.

Q. Is $(\mathbb{Q},|\cdot|)$ homeomorphic to $(\mathbb{N},|\cdot|)$ or $(\mathbb{Z},|\cdot|)$?

$\to$ "homeomorphism" is an equivalence relation.

<u>Remark.</u> If $(M,d)\overset{\text{homeo}}{\sim}(M,\rho)$, then d may not be equivalent to ρ!

e.g., $M=\left\{\{0\}\cup\{1/n, n\geq 1\}\right\}$.

Consider $(M,|\cdot|)$ and (M,ρ) where

$$\rho\left(\tfrac{1}{n},\tfrac{1}{m}\right)=\left|\tfrac{1}{n}-\tfrac{1}{m}\right|,\quad m,n\geq 2$$

$$\rho\left(\tfrac{1}{n},1\right)=\tfrac{1}{n},\quad n\geq 2$$

$$\rho\left(\tfrac{1}{n},0\right)=1-\tfrac{1}{n},\quad n\geq 2$$

$$\rho(0,1)=1.$$

Thm#. For $f:(M,d)\to(N,\rho)$ a bijection, TFAE:

(i) f is a homeomorphism.

(ii) $d(x_n,x)\to 0$ iff $\rho(f(x_n),f(x))\to 0$.

(iii) $\mathcal{U}$ open in M iff $f(\mathcal{U})$ open in N.

(iv) A closed in M iff $f(A)$ closed in N.

(v) $\tilde{d}(x,y):=\rho(f(x),f(y))$ defines a metric on M which is equivalent to d.

Hint for the above example:

→ Define $F : (M, |\cdot|) \longrightarrow (M, \rho)$ as $F(0) = 1$, $F(1) = 0$

$F\left(\frac{1}{n}\right) = \frac{1}{2n}$ if n: odd

$F\left(\frac{1}{n}\right) = \frac{1}{2n+1}$ if n: even.

- If $x_n \xrightarrow{|\cdot|} x$ in $(M, |\cdot|)$, what can you say about x?

  What seq. are cvgt. in $(M, |\cdot|)$?

  What about $f(x_n)$? (Can you use condition (ii) of the above thm. to show that f is a homeo.?)

- The identity map $Id : (M, |\cdot|) \to (M, \rho)$ is not cts. !!!

  Take $\left(\frac{1}{n}\right)$. Note that $\frac{1}{n} \xrightarrow{|\cdot|} 0$ in M, but $\rho\left(\frac{1}{n}, 0\right) \not\to 0$ as $n \to \infty$.

Example. (HW). $(\mathbb{R}, |\cdot|) \overset{\text{homeo}}{\not\sim} (\mathbb{R}, d_0)$ where d_0: discrete metric.

$(\mathbb{N}, |\cdot|) \overset{\text{homeo}}{\sim} (\mathbb{N}, d_0)$

Def^n: $f : (M, d) \to (N, \rho)$ is an isometry if $d(x,y) = \rho(f(x), f(y))$.

Note that if f is an isometry, then f is injective and $d(x_n, x) \to 0$ iff $\rho(f(x_n), f(x)) \to 0$

Example: $f : [0,1] \longrightarrow [a,b]$ defined as $f(t) = \frac{1}{b-a}\big[(1-t)a + tb\big]$. is an isometry.

HW: Show that f is a homeomorphism.

→ If $f : (M, d) \to (N, \rho)$ is an onto isometry, then f is a homeomorphism.

$\Longleftarrow$ Converse ?

Not True!

Q. Why $(\mathbb{R}, |\cdot|)$ is not isometric onto $(0,1)$?

Q. $[0,1] \overset{\text{homeo}}{\sim} [0, \infty)$? (Ans. NO! see later)

· (HW) Show that $\mathbb{R} \overset{\text{homeo}}{\sim} (0,1) \overset{\text{homeo}}{\sim} (a,b)$.

Q. Is $(0,1) \overset{\text{homeo}}{\sim} (0,1]$???

(HW) If $f:(M,d) \to (N,\rho)$ is a homeomorphism, then $\hat{f}: M\setminus\{x\} \to N\setminus\{f(x)\}$ is again a homeomorphism.

A. Suppose $(0,1) \overset{\text{homeo}}{\sim} (0,1]$. Then $(0,c)\cup(c,1) \overset{\text{homeo}}{\sim} (0,1)$ where $f(c)=1$.
But also, $\mathbb{R} \overset{\text{homeo}}{\sim} (0,1)$. This implies $\mathbb{R} \overset{\text{homeo}}{\sim} (0,c)\cup(c,1)$
Not possible !
HW: Why ?

<u>Remark.</u> For $n \neq m$, $\mathbb{R}^n \overset{\text{homeo}}{\not\sim} \mathbb{R}^m$. And $[0,1] \overset{\text{homeo}}{\not\sim} [0,1]\times[0,1]$. Keep thinking and wait for the right time!

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