Name: ______________________

Roll Number: ________________

Final Exam

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

<u>INSTRUCTIONS</u>

1. Write your **Name** and **Roll number** above.
2. This exam contains **6 + 1** questions and is worth **60%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There exists a countable $X \subseteq \omega_1$ such that $\sup(X) = \omega_1$.

(ii) There exists a bijection $f : \mathbb{R}^7 \to \mathbb{R}^9$ satisfying: For every x, y in $\mathbb{R}^7$, $f(x - y) = f(x) - f(y)$.

(iii) If $f : \omega \to \omega$ is a strictly increasing computable function, then $\text{range}(f)$ is computable.

(iv) The set of all subsets of ω that are definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ is countable.

(v) TA is ω-categorical.

Question 2. [10 Points]

(a) [**5 Points**] Let $\mathcal{F}$ be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Show that $|\mathcal{F}| = \mathfrak{c}$.

(b) [**5 Points**] Let $\mathcal{E}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Show that $|\mathcal{E}| > \mathfrak{c}$.

Question 3. [10 Points]

Using transfinite recursion, construct a function $f : \mathbb{R} \to \mathbb{R}$ such that for every interval $(a, b) \subseteq \mathbb{R}$ and $y \in \mathbb{R}$, there exists an **irrational** $x \in (a, b)$ such that $f(x) = y$.

Question 4. [10 Points]

Recall that DLO is the theory of dense linear orderings without end-points.

(a) [**2 Points**] Show that $(\mathbb{Z}, <)$ is not an elementary submodel of $(\mathbb{Q}, <)$. Here $\mathbb{Z}$ is the set of all integers and $\mathbb{Q}$ is the set of all rationals.

(b) [**8 Points**] Let $M \subseteq \mathbb{R}$ be countable. Assume that $(M, <) \models DLO$. Show that $(M, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Question 5. [10 Points]

(a) [**5 Points**] Let $W \subseteq \omega$ be an infinite c.e. set. Show that there is an infinite $X \subseteq W$ such that X is computable.

(b) [**5 Points**] Show that $\omega \setminus True_{\mathcal{N}}$ (defined on Slide 199) is not c.e.

Question 6. [10 Points]

Let T be a computable $\mathcal{L}_{PA}$-theory such that $PA \subseteq T \subseteq TA$. For $f : \omega \to \omega$, we say that f is **numeralwise representable in** T iff there is an $\mathcal{L}_{PA}$-formula $\psi(y, x)$ such that for every $(m, n) \in \omega^2$,

(i) If $f(m) = n$, then $T \vdash \psi(\overline{n}, \overline{m})$.

(ii) If $f(m) \neq n$, then $T \vdash \neg\psi(\overline{n}, \overline{m})$.

(a) [**4 Points**] Let $f : \omega \to \omega$. Show that f is numeralwise representable in T iff f is computable.

(b) [**6 Points**] Show that T is undecidable.

Bonus Question [5 Points]

Let $\langle X_n : n < \omega \rangle$ be a sequence of **uncountable** sets. Show that there exists $\langle Y_n : n < \omega \rangle$ such that

(a) For every $n < \omega$, Y_n is uncountable and $Y_n \subseteq X_n$.

(b) For every $m < n < \omega$, $Y_n \cap Y_m = \emptyset$.

End of Exam