Example 5

Let $\{X_t\}$ be a seq of i.i.d. random variables with

$$E X_t = 0 \quad \text{and} \quad V X_t = \sigma^2 < \infty$$

Define
$$S_t = \sum_{i=1}^{t} X_i$$

$$E S_t = 0 \quad \forall t$$

$\Rightarrow \{S_t\}$ is a mean stationary process

$$V S_t = t \sigma^2$$

$$\operatorname{Cov}(S_{t+h}, S_t) = \begin{cases} t\sigma^2, & \text{if } h > 0 \\ (t+h)\sigma^2, & \text{if } h \le 0 \end{cases}$$

$\Rightarrow \{S_t\}$ is not covariance stationary

Note that $\quad S_t = S_{t-1} + X_t \leftarrow$ Random-Walk model

Further, although $\{S_t\}$ is not covariance stationary

$\nabla S_t = S_t - S_{t-1} = X_t$ is covariance stationary

Remark: If $\{X_t\}$ is covariance stationary, then

(i) $Y_t = \alpha X_t \; ; \forall \alpha \in \mathbb{R}$, is covariance stationary

(ii) $Y_t = X_t \pm \alpha \; ; \alpha \in \mathbb{R}$ is covariance stationary

(iii) $$Y_t = \begin{cases} X_t, & \text{If } t \text{ is odd} \\ X_t + \alpha, & \text{If } t \text{ is even} \end{cases}$$

$$E\,Y_t = \begin{cases} E X_t, & t \text{ is odd} \\ \alpha + E X_t, & t \text{ is even} \end{cases}$$

$\Rightarrow$ $\{Y_t\}$ is not covariance stationary (not even mean stationary)

Sly, If $Y_t = \begin{cases} X_t, & t \text{ odd} \\ \alpha X_t, & t \text{ even} \end{cases}$

$\{Y_t\}$ is not covariance stationary (as $V Y_t = \begin{cases} V X_t, & \text{odd} \\ \alpha^2 V X_t, & \text{even} \end{cases}$)

$\{Y_t\}$ is not mean stationary if $E X_t \neq 0$

<u>Remark</u>: If $\{X_t\}$ and $\{Y_t\}$ are covariance stationary processes and $\{X_t\}$ and $\{Y_t\}$ are independent, then

$Z_t = X_t + Y_t$ is $\Rightarrow$

$E Z_t = \mu_x + \mu_y \quad \forall t$ (indep of t)

$$\operatorname{Cov}(Z_{t+h}, Z_t) = \underbrace{\operatorname{Cov}(X_{t+h}, X_t)}_{f^n \text{ of } h \text{ only}} + \underbrace{\operatorname{Cov}(Y_{t+h}, Y_t)}_{f^n \text{ of } h \text{ only}}$$

$\Rightarrow$ $\operatorname{Cov}(Z_{t+h}, Z_t)$ is a f^n of h only

$\Rightarrow$ $\{Z_t\}$ is also covariance stationary

Note that if $\{X_t\}$ & $\{Y_t\}$ are ~~to~~ uncorrelated covariance stationary processes, then also $\{Z_t\}$ is covariance stationary

<u>Remark</u>: If $Z_t = X_t + Y_t$ is covariance stationary, then it is not necessary that $\{X_t\}$ & $\{Y_t\}$ are covariance stationary

Counter example !!

Remark We can also define a complex valued time series process in the following way

Let $\{X_t\}$ & $\{Y_t\}$ be 2 real valued time series processes. Define

$$U_t = X_t + iY_t \; ; \quad i = \sqrt{-1}$$

$\{U_t\}$ is a complex valued time series with the properties:

(i) $E U_t = E X_t + i\, E Y_t$

(ii) $Cov(U_{t+h}, U_t) = E\left(U_{t+h} - E(U_{t+h})\right)^{*}\left(U_t - E(U_t)\right)$

$\{U_t\}$ is said to be covariance stationary if

(a) $E(U_t) = \mu$ indep of t

& (b) $Cov(U_{t+h}, U_t)$ is a f^n of h only

Example: $Y_t = A e^{i\omega t}$

A is random variable $\ni E(A) = 0 \; ; V(A) = \sigma^2 < \infty$

$$Y_t = A \cos\omega t + i A \sin\omega t$$

i.e $Y_t = U_t + iV_t \; ; \; U_t = A\cos\omega t$

$V_t = A \sin\omega t$

$\{U_t\}$ & $\{V_t\}$ are real valued time series.

$$E Y_t = E U_t + i E V_t = 0 \quad \forall t \quad -(i)$$

$$\begin{aligned} Cov(Y_{t+h}, Y_t) &= E(Y_{t+h}^* Y_t) \\ &= E\left(A e^{-i\omega(t+h)} A e^{i\omega t}\right) \\ &= E\left(A^2 e^{-i\omega h}\right) \\ &= \underline{\sigma^2 e^{-i\omega h}} \quad -(ii) \end{aligned}$$

indep of t; f^n of h only

(i) & (ii) $\Rightarrow$ $\{Y_t\}$ is Covariance stationary complex valued time series.

<u>Remark</u>: Suppose

$$X_t = U_t + i V_t$$

For stationarity of $\{X_t\}$, is it necessary that $\{U_t\}$ & $\{V_t\}$ need to be covariance stationary? Think about it.