Assignment - 9

1. Show that any Lipschitz map $f : (M,d) \to (N,\rho)$ is uniformly cts.

2. Prove that every map $f : \mathbb{N} \to \mathbb{R}$ is uniformly cts.

3. If $f : (0,1) \to \mathbb{R}$ is uniformly cts., show that $\lim_{x \to 0+} f(x)$ exists.

4. For $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, define
$$F(x) = \frac{f(x) - f(a)}{x - a} \quad \text{for } x \neq a.$$
Prove that f is differentiable at "a" iff F is unif. cts. in some punctured disk around a.

5. Let E be a bdd., noncompact subset of $\mathbb{R}$. Show that there is a cts. function function $f : E \to \mathbb{R}$ that is not uniformly cts.

6. Give an example of a bdd. cts. map $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly cts. Can an unbdd. cts. function $f : \mathbb{R} \to \mathbb{R}$ be uniformly cts.? Explain.

7. A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order α, where $\alpha > 0$ if $\exists$ $0 < K < \infty$ s.t. $|f(x) - f(y)| \leq K |x-y|^{\alpha}$ for all $x, y \in \mathbb{R}$. Prove that such a function is uniformly cts.

8. Show that any function $f : \mathbb{R} \to \mathbb{R}$ having a bdd. derivative is Lipschitz of order 1.

9. Show that a function satisfying Lipschitz condition of order $\alpha > 1$ is constant.

10. Show that x^{α} is unif. cts. on $(0,\infty)$ iff $0 \leq \alpha \leq 1$.

11. Define $f : \ell_2 \to \ell_1$ by $f(x) = (x_n/n)_{n=1}^{\infty}$. Show that f is unif. cts.

Evaluation.

12. (a) If $f, g : \mathbb{R} \to \mathbb{R}$ s.t. f and g are uniformly cts. and bounded, then show that $f \cdot g$ is also uniformly cts. (5 pts.)

(b) Suppose $f : (M, d) \to (N, \rho)$ is cts.

Prove or disprove: f is uniformly cts. $\Longleftrightarrow$ f maps Cauchy seq. to Cauchy seq.

(5 pts.)