MSO201A: Probability & Statistics
Endsem Examination: Full Marks 100

[1] (a) The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{2x^2+1}{10}, & 0 \le x < 1 \\ \frac{4}{5}, & 1 \le x < 2 \\ \frac{(x-2)^4+16}{20}, & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

Find the values of $\alpha, F_d(x)$ and $F_c(x)$ such that $F(x) = \alpha\, F_d(x) + (1-\alpha)\, F_c(x)$; where $F_d(x)$ is distribution function of a discrete random variable, $F_c(x)$ is distribution function of an absolutely continuous random variable and $0 < \alpha < 1$.

(b) Two independent components of a system are connected in parallel; the system functions if at least one of the 2 components is functioning. Let X and Y denote random variables denoting the lifetimes (in years) of the above 2 independent components, with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2}\, e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } g_Y(y) = \begin{cases} \frac{1}{4}\, e^{-\frac{y}{4}}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability that the system will function for at least 2 years.

10 (6+4) marks

[2] (a) Let X be a random variable following a standard normal, $N(0,1)$, distribution. Prove or disprove "$Correlation(-X, |X|) = -1$".

(b) Let X and Y be independent random variables with $X \sim B(1, 0.5)$ and $Y \sim P(2)$. Find $P(XY = 0)$.

10 (5+5) marks

[3] (a) The p.d.f. of a continuous random variable X is given by

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution function of $Y = max(X, 1-X)$.

(b) Let $X_1, \dots, X_n$ be i.i.d. random sample with p.d.f.

$$f_\theta(x) = \begin{cases} \theta\, x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$\theta > 0$. Find the p.d.f. of $Y_i = -2\,\theta\, log X_i$, for $i = 1, \dots, n$; and the m.g.f. of $T = -2\,\theta\, \sum_{i=1}^{n} log X_i$.

[12 (6+6) marks]

[4] (a) Let the conditional p.d.f. of X given $Y = y\ (y > 0)$ be given by

$$f_{X|Y}(x|y) = \begin{cases} e^{y-x}, & 0 < y < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

and let Y have the p.d.f. $g_Y(y) = \begin{cases} 3e^{-3y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$.

Find the conditional p.d.f. of Y given $X = x\ (x > 0)$.

(b) Let $\{X_n\}_{n\ge1}$ be a sequence of i.i.d. random variables with p.d.f. $f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$.

Suppose $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the random variable Y such that $\sqrt{n}\left(e^{-\bar{X}_n} - e^{-1}\right) + \bar{X}_n \xrightarrow{\mathcal{L}} Y$, as $n \to \infty$.

[12 (6+6) marks]

[5] Let $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T \sim N_4(\boldsymbol{0}, \boldsymbol{\Sigma})$; $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{pmatrix}$.

(a) Find the joint p.d.f. of $U = X_1 + X_3 - X_4$ and $V = X_1 + X_3 + X_4$.
(b) Prove or disprove "$(-X_1 + X_3 - X_4)$ and $(-X_2 + X_3 + X_4)$ are independently distributed".
(c) Find the distribution of $W = \left(\frac{X_1}{X_3}\right)^2$.

11 (4+4+3) marks

[6] Let X and Y be i.i.d. random variables with p.d.f. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ and suppose $U = X\,Y$.
(a) Find $Correlation(X, U)$.
(b) Find p.d.f. of U.
(c) Find $E[X|U = 0.25]$.

12 (4+5+3) marks

[7] (a) Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables with $P(X_i = 0) = P(X_i = 1) = 0.5$ for all $i \geq 0$. Define $Y_n = 2^n \prod_{j=0}^{n-1} X_j$ for $n \geq 1$. Verify whether there exists a random variable Z such that $Y_n \xrightarrow{p} Z$, as $n \to \infty$.
(b) Let $X_1, \dots, X_n$ be i.i.d. random sample from a distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \dfrac{6\,x^5}{\theta}\; e^{-\frac{x^6}{\theta}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$\theta > 0$. Find a function of minimal sufficient statistics that is a consistent estimator for θ^2.

10 (5+5) marks

[8] Let $X_1, \dots, X_n$ be i.i.d. random sample with p.d.f. $f_\theta(x) = \frac{1}{2\,\theta}\, e^{-\frac{|x|}{\theta}}, x \in (-\infty, \infty),\ \theta > 0$.
(a) Find UMVUE of $g(\theta) = \theta$.
(b) Find CRLB for $g(\theta) = \theta^2$.

11 (6+5) marks

[9] (a) Let $X_1, \dots, X_n$ be i.i.d. random sample with p.d.f. $f_\theta(x) = \begin{cases} \theta\, 3^\theta x^{-(\theta+1)}, & x > 3 \\ 0 & \text{otherwise} \end{cases}; \theta > 0$. Prove or disprove "minimal sufficient statistic is NOT complete".
(b) Let $X_1, \dots, X_n$ be i.i.d. random sample with p.d.f. $f_\theta(x) = \begin{cases} \frac{3}{2\theta}, & -\frac{\theta}{3} \leq x \leq \frac{\theta}{3} \\ 0 & \text{otherwise} \end{cases}; \theta > 0$. Find MLE of θ.

12 (6+6) marks

USEFUL INFORMATION

- If $\boldsymbol{X} = (X_1, \dots, X_p)^T \sim N_p(\mu, \Sigma), \mu \in \mathcal{R}^p, \Sigma > 0$; then p.d.f. of $\boldsymbol{X}$ is $f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}; \boldsymbol{x} \in \mathcal{R}^n$; and m.g.f. of $\boldsymbol{X}$ is $M_{\boldsymbol{X}}(\boldsymbol{t}) = e^{t^T\mu + \frac{1}{2}t^T\Sigma t}$
- If $X \sim B(1, \theta), 0 < \theta < 1$; then p.m.f. of X is $f(x) = P(X = x) = \theta^x(1-\theta)^{1-x}, x = 0,1$; and m.g.f. of X is $M_X(t) = (1 - \theta + \theta\, e^t)$
- If $X \sim P(\theta), \theta > 0$; then p.m.f. of X is $f(x) = P(X = x) = \frac{e^{-\theta}\theta^x}{x!}, x = 0,1, \dots \dots$; and m.g.f. of X is $M_X(t) = e^{\theta(e^t - 1)}$
- If $X \sim \chi^2_n$; then p.d.f. of X is $f(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}, x > 0$; and m.g.f of X is $M_X(t) = (1-2t)^{-\frac{n}{2}}, t < \frac{1}{2}$
- If $X \sim N(\mu, \sigma^2), \mu \in \mathcal{R}, \sigma > 0$, then the p.d.f. of X is $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, x \in \mathcal{R}$; and m.g.f. of X is $M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$

1.

(a) $P(X=0) = \frac{1}{10}$; $P(X=1) = \frac{5}{10}$, $P(X=3) = \frac{3}{20}$

$\alpha = \frac{3}{4}$ — (1)

$$F_d(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{15}, & 0 \le x < 1 \\ \frac{4}{5}, & 1 \le x < 3 \\ 1, & x \geq 3 \end{cases}$$

— (2)

Deduct (1) mark if "≤", "<" are wrongly put.

$1-\alpha = \frac{1}{4}$

$$F_c(x) = \begin{cases} 0, & x < 0 \\ \frac{4}{5}x^2, & 0 \le x < 1 \\ \frac{4}{5}, & 1 \le x < 2 \\ \frac{(x-2)^4 + 4}{5}, & 2 \le x < \\ 1, & x \geq 3 \end{cases}$$

— (3)

(b) Reqd prob

$= P(X > 2 \text{ or } Y > 2)$

$= P(X>2) + P(Y>2) - P(X>2)\,P(Y>2)$ — (1)

$P(X>2) = e^{-1}$
$P(Y>2) = e^{-1/2}$ — (1)

$\Rightarrow$ reqd prob $= e^{-1} + e^{-1/2} - e^{-1}e^{-1/2}$ — (2)

2 (a)

$$\operatorname{Cov}(-X, |X|) = E(-X|X|) \quad — (1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -x|x| \, e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x^2 e^{-x^2/2} dx + \int_{0}^{\infty} -x^2 e^{-x^2/2} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} x^2 e^{-x^2/2} dx - \int_{0}^{\infty} x^2 e^{-x^2/2} dx \right)$$

$$= 0 \quad — (3)$$

$$\Rightarrow \quad \operatorname{Corr}^n(-X, |X|) = 0 \quad — (1)$$

(b) $X \sim B(1, \frac{1}{2}) \qquad Y \sim P(2)$

$$P(XY=0) = P(X=0, Y=0) + P(X=0)\,P(Y \neq 0) + P(X \neq 0)\,P(Y=0)$$

$$= P(X=0)\,P(Y=0) + P(X=0)\,P(Y \neq 0) + P(X \neq 0)\,P(Y=0) \quad — (2)$$

$$= \frac{1}{2} e^{-2} + \frac{1}{2}(1 - e^{-2}) + \frac{1}{2} e^{-2}$$

$$= \frac{1}{2}(1 + e^{-2}) \quad — (3)$$

3. p.d.f. of X

(a) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & o/w \end{cases}$

$Y = \max(X, 1-X) \in (\frac{1}{2}, 2)$

d.f. of Y

$F_Y(y) = P(Y \le y) = P(\max(X, 1-X) \le y)$

$= P(X \le y, 1-X \le y)$

$= P(X \le y, X \ge 1-y)$ — (1)

$= \begin{cases} 0, & y \le \frac{1}{2} \\ \int_{1-y}^{y} x\,dx, & \frac{1}{2} < y \le 1 \text{ — line 2} \\ \int_{0}^{1} x\,dx + \int_{1}^{y} (2-x)\,dx, & 1 < y \le 2 \text{ — line 3} \\ 1, & y > 2 \end{cases}$ — (2)

Give partial mark if any of line 2/3 is correct

i.e. $F_Y(y) = \begin{cases} 0, & y \le \frac{1}{2} \\ (y - \frac{1}{2}), & \frac{1}{2} < y \le 1 \text{ — line 2*} \\ 2y - y^2/2 - 1, & 1 < y \le 2 \text{ — line 3*} \\ 1, & y > 2 \end{cases}$ — (3)

Give partial mark if any of the line 2*/3* is correct

Alt solⁿ: Finding p.d.f and then d.f. using p.d.f.

3(b) p.d.f of X_i

$$f(x) = \begin{cases} \theta x^{\theta-1}, & 0<x<1 \\ 0, & o/w \end{cases}$$

$$Y_i = -2\theta \log X_i$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{2}\frac{x}{\theta} \quad ; \quad x^\theta = e^{-y/2}$$

p.d.f. of Y_i

$$f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2}, & y>0 \\ 0, & o/w \end{cases} \quad \text{i.e. } Y_i \sim \chi^2_2$$

$(2\frac{1}{2})$

$Y_1, \ldots, Y_n$ are i.i.d.

$$T = -2\theta \sum_{i=1}^n \log X_i$$

$$M_T(t) = E\left(e^{t(-2\theta\sum \log X_i)}\right)$$

$$= E\left(e^{\sum_1^n t(-2\theta \log X_i)}\right)$$

$$= \prod_{i=1}^n E\left(e^{-t\, 2\theta \log X_i}\right) = \prod_{i=1}^n M_{(-2\theta \log X_i)}(t)$$

(1)

Now m.g.f of Y_i is

$$E(e^{tY_i}) = \frac{1}{2}\int_0^\infty e^{ty} e^{-y/2} dy = \frac{1}{2}\int_0^\infty e^{-y(\frac{1}{2}-t)} dy$$

$$= \frac{1}{1-2t} \quad \text{for } t<\frac{1}{2}$$

$$\Rightarrow M_T(t) = \frac{1}{(1-2t)^n} \quad \text{for } t<\frac{1}{2}$$

$(2\frac{1}{2})$

3(b) Alt solution

$$Y_i = -2\theta \log X_i \sim \chi^2_2 \quad i = 1, \ldots n \text{ \& indep}$$

$$T = \sum_{i=1}^{n} Y_i \sim \chi^2_{2n} \text{ by additive prop of indep } \chi^2$$

$$\text{m.g.f. } M_T(t) = \frac{1}{(1-2t)^{2n/2}} = \frac{1}{(1-2t)^n} \text{ for } t < \frac{1}{2}$$

4

(a) $f_{X|Y} = e^{y-x}, \quad 0 < y < x < \infty$

$0, \quad o/w$

$g_Y(y) = 3e^{-3y}, \quad y > 0$

$0, \quad o/w$

Jt p.d.f. of X & Y

$f_{X,Y}(x,y) = f_{X|Y}\, g_Y$

$$= \begin{cases} 3\, e^{y-x}\, e^{-3y}, & \underline{0 < y < x < \infty} \\ 0, & o/w \end{cases}$$ (1)

Marginal p.d.f. of X:

$$f_X(x) = 3\, e^{-x} \int_0^x e^{-2y}\, dy \qquad 0 < x < \infty$$

$$= \frac{3}{2} e^{-x}\left(1 - e^{-2x}\right) \qquad 0 < x < \infty \quad \left(2\tfrac{1}{2}\right)$$

Conditional p.d.f. of Y given X = x

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{2\, e^{-2y}}{1 - e^{-2x}}, & 0 < \underline{\underline{y < x}} < \infty \\ 0, & o/w \end{cases}$$ $\left(2\tfrac{1}{2}\right)$

4 (b) $X_1, \ldots$ seq of i.i.d p.d.f. $f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & o/w \end{cases}$

$EX_i = 1$; $V(X_i) = 1$, $i = 1, 2, \ldots$

By CLT $\quad \sqrt{n}(\bar{X}_n - 1) \xrightarrow{\mathcal{L}} N(0,1)$ as $n \to \infty$ (1)

$g(x) = e^{-x}$; $g'(x) = -e^{-x}$; $(g'(1))^2 = e^{-2}$

By Δ-rule

$$\sqrt{n}(g(\bar{X}_n) - g(1)) \xrightarrow{\mathcal{L}} N(0, (g'(1))^2) \text{ as } n \to \infty$$

i.e. $\sqrt{n}(e^{-\bar{X}_n} - e^{-1}) \xrightarrow{\mathcal{L}} N(0, e^{-2})$ as $n \to \infty$ (2)

By WLLN $\quad \bar{X}_n \xrightarrow{p} 1$ as $n \to \infty$ (1)

$\Rightarrow \sqrt{n}(e^{-\bar{X}_n} - e^{-1}) + \bar{X}_n \longrightarrow N(1, e^{-2})$ as $n \to \infty$

(2)

(5) $\underset{\sim}{X} \sim N(\underline{0}, \Sigma)$

$$\Sigma = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ & 1 & 1/2 & 0 \\ & & 1 & -1/2 \\ & & & 1 \end{pmatrix}$$

(a)

$$\underset{\sim}{y} = \begin{pmatrix} X_1 + X_3 - X_4 \\ X_1 + X_3 + X_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \underset{\sim}{X} = A\underset{\sim}{X}$$

$\forall \underline{\alpha} \in \mathbb{R}^2 \quad \underline{\alpha}'\underset{\sim}{y} = \underline{\alpha}' A \underset{\sim}{X} = \underline{\beta}' \underset{\sim}{X} \sim N_1 \quad (\underline{\beta} \in \mathbb{R}^4)$

$\Rightarrow \underline{\underset{\sim}{y} = A\underset{\sim}{X} \sim N_2(\underline{0}, A\Sigma A')}$ (2)

$$A\Sigma A' = \Sigma^* = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

If $A\Sigma A'$ is not calculated explicitly don't deduct marks

p.d.f. of $\underset{\sim}{y}$

$$\underline{f_y(\underset{\sim}{y}) = (2\pi)^{-1} |\Sigma^*|^{-1/2} e^{-\frac{1}{2}\underset{\sim}{y}'\Sigma^{*-1}\underset{\sim}{y}}} \; ; \; y \in \mathbb{R}^2$$ (2)

(b)

$$\begin{pmatrix} -X_1 + X_3 - X_4 \\ -X_2 + X_3 + X_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \underset{\sim}{X} = B\underset{\sim}{X}$$

$\forall \underline{\alpha} \in \mathbb{R}^2 \quad \underline{\alpha}' B \underset{\sim}{X} \sim N_2$

$$\Rightarrow \begin{pmatrix} -X_1 + X_3 - X_4 \\ -X_2 + X_3 + X_4 \end{pmatrix} \sim \underline{N_2(\underline{0}, B\Sigma B')}$$ $\left(1\frac{1}{2}\right)$

$\underline{\operatorname{cov}(-X_1 + X_3 - X_4, -X_2 + X_3 + X_4) = 0}$ $\left(1\frac{1}{2}\right)$

$\Rightarrow \underline{-X_1 + X_3 - X_4 \;\&\; -X_2 + X_3 + X_4 \text{ are indep}}$ (1)

(c) $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N_2\left(\underline{0}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

<u>Alt soln</u>: Deriving p.d.f of $\frac{X_1^2}{X_3^2}$

$\Rightarrow X_1 \;\&\; X_3$ are indep $N(0,1)$

$\Rightarrow X_1^2 \;\&\; X_3^2$ are indep χ^2_1

$\Rightarrow X_1^2/X_3^2 \sim F_{1,1}$ —— (3)

(6) $X \sim U(0,1) \Rightarrow E(X) = \frac{1}{2}; \; EX^2 = \frac{1}{3}; \; VX = \frac{1}{12}$

(a) $\text{Cov}(X, U) = \text{Cov}(X, XY)$

$$= E(X^2Y) - E(X)E(XY)$$

$$= E(X^2)E(Y) - (E(X))^2 E(Y)$$

$$= \frac{1}{3}\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{24} \quad \left(1\tfrac{1}{2}\right)$$

$$V(U) = V(XY) = E(X^2Y^2) - (E(XY))^2$$

$$= E(X^2)EY^2 - (E(X)E(Y))^2$$

$$= \frac{1}{9} - \frac{1}{16} = \frac{7}{144} \quad \left(1\tfrac{1}{2}\right)$$

$$\Rightarrow \text{Corrl}^n(X, U) = \frac{1/24}{\left(\frac{1}{12}\times\frac{7}{144}\right)^{1/2}} = \left(\frac{3}{7}\right)^{1/2} \quad (1)$$

(b) $\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} U = XY \\ V = X \end{pmatrix}$ Inverse: $X = V$, $Y = \frac{U}{V}$ (there can be other dummy in transformation.)

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$$

unconditional range: $0 < u < 1$; $0 < v < 1$

conditional range: $0 < \frac{u}{v} < 1$, i.e. $0 < u < v < 1$

jt p.d.f of (U,V)

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{v}, & 0 < u < v < 1 \\ 0, & o/w \end{cases} \quad (2)$$

Marginal p.d.f. of U

$$f_U(u) = \int_u^1 \frac{1}{v}\,dv = \begin{cases} -\log u, & 0 < u < 1 \\ 0, & o/w \end{cases} \quad (3)$$

6.(c) Conditional p.d.f. of $X|U$

$$f_{X|U}(=f_{V|U}) = \frac{f_{UV}}{f_U} = \begin{cases} -\dfrac{1}{x\log u}, & 0<u<x<1 \\ 0, & \text{o/w} \end{cases} \quad \left(1\tfrac{1}{2}\right)$$

$$E(X|U) = -\frac{1}{\log u}\int_u^1 x\,\frac{1}{x}\,dx$$

$$= -\frac{(1-u)}{\log u}$$

$$\Rightarrow E\left(X|U=\tfrac{1}{4}\right) = -\frac{3/4}{\log\frac{1}{4}} = \frac{3/4}{\log 4} \quad \left(1\tfrac{1}{2}\right)$$

7(a) $Y_n = 2^n X_0 X_1 \cdots X_{n-1}$

$$\Rightarrow Y_n = \begin{cases} 2^n & \text{w.p. } \left(\frac{1}{2}\right)^n \\ 0, & \text{w.p. } 1-\left(\frac{1}{2}\right)^n \end{cases} \quad (1)$$

$$\Rightarrow P(|Y_n|>\epsilon) = \left(\tfrac{1}{2}\right)^n \quad \text{if } 0<\epsilon<2^n$$
$$= 0 \quad \text{if } \epsilon \geq 2^n \quad (2)$$

$$\Rightarrow P(|Y_n|>\epsilon) \to 0 \text{ as } n\to\infty \quad \forall \epsilon>0$$
$$\Rightarrow Y_n \xrightarrow{p} 0 \text{ as } n\to\infty \quad (2)$$

(b) $$f_\theta(x) = \begin{cases} \dfrac{6x^5}{\theta}\, e^{-x^6/\theta}, & x>0 \\ 0, & \text{o/w} \end{cases}$$

$\sum_{i=1}^n X_i^6$ is m.s.s. — (1)

$$E X_i^6 = \int_0^\infty x^6\, \frac{6x^5}{\theta}\, e^{-x^6/\theta}\,dx = \sqrt{2}\,\theta \ne \theta$$

$\because X_1, \dots X_n$ i.i.d

$\Rightarrow X_1^6, \dots, X_n^6$ i.i.d. with $E X_i^6 = \theta$

By WLLN $\quad \frac{1}{n} \sum X_i^6 \xrightarrow{p} \theta \quad$ as $n \to \infty$ (2)

(2) $\Rightarrow \left(\frac{1}{n} \sum X_i^6 \right)^2 \xrightarrow{p} \theta^2$ as $n \to \infty$

$\Rightarrow \left(\frac{1}{n} \sum X_i^6 \right)^2$ is a consistent estimator for θ^2

8 (a) $\quad f_\theta(x) = \frac{1}{2\theta} e^{-|x|/\theta}$

1-parameter exp family with full rank

$\Rightarrow \sum_{i=1}^n |X_i|$ is complete suff statistic (~~?~~) (2)

$E|X_i| = \frac{1}{2\theta} \int_{-\infty}^{\infty} |x| e^{-|x|/\theta} dx = \theta$ — $\left(1\frac{1}{2}\right)$

$\Rightarrow \frac{1}{n} \sum_1^n |X_i|$ is u.e. of θ based on c.s.s.

$\Rightarrow \frac{1}{n} \sum_1^n |X_i|$ is UMVUE for θ $\left(2\frac{1}{2}\right)$ Alt soln: using attainment of CRLB

(b) $\quad \log f = -\log 2 - \log \theta - \frac{|x|}{\theta}$

$$\frac{\partial \log f}{\partial \theta} = -\frac{1}{\theta} + \frac{|x|}{\theta^2}$$

$$\frac{\partial^2 \log f}{\partial \theta^2} = \frac{1}{\theta^2} - 2\frac{|x|}{\theta^3} \quad \text{— (1)}$$

$$E\left(\frac{\partial^2 \log f}{\partial \theta^2} \right) = \frac{1}{\theta^2} - \frac{2}{\theta^3} E|X| = -\frac{1}{\theta^2}$$

$\Rightarrow I(\theta) = \frac{1}{\theta^2}$ (2)

$g(\theta) = \theta^2 \Rightarrow (g'(\theta))^2 = (2\theta)^2$

$\Rightarrow$ CRLB for $\theta^2 = \dfrac{(g'(\theta))^2}{n\, I(\theta)} = \dfrac{4\theta^4}{n}$ (2)

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(a) $f_\theta(x) = \theta\, 3^\theta\, x^{-\theta}\, x^{-1}\, I(3, x)$

i.e. $f_\theta(x) = \left(x^{-1} I(3,x)\right) \exp\left(-\theta \log x + \theta \log 3 + \log \theta\right)$

$h(x) = x^{-1} I(3, x)$ ✓ (2)

$\eta_1(\theta) = -\theta \; (=\eta)$

$T(x) = \log x$

$B(\theta) = -(\theta \log 3 + \log \theta)$

The above thus is 1-param exp family with natural parameter space $= \{\eta : \eta < 0\}$ (1)

$\Rightarrow$ the 1-param exp family is of full rank (2)

$\Rightarrow$ the minimal suff statistic $T(\underline{X}) = \sum_{i=1}^{n} \log X_i$ is complete. (1)

9 (b) $L(\theta) = \left(\dfrac{3}{2\theta}\right)^n I\left(3 \max_i |x_i|, \theta\right)$

$L(\theta) = 0 \quad \forall\ \theta < 3 \max_i |x_i|$

$= \left(\dfrac{3}{2\theta}\right)^n \quad \forall\ \theta \geq 3 \max_i |x_i|$ — (2)

i.e. $\downarrow \theta \quad \forall\ \theta \geq 3 \max_i |x_i|$ (2)

$\Rightarrow \hat{\theta}_{MLE} = 3 \max_i |X_i|$ (2)

Alt: Equivalently $3 \max\left(X_{(n)}, -X_{(1)}\right)$ is also MLE