Likelihood f^n

$$L(\underset{\sim}{\theta}) = f_{X_1}(x_1;\underset{\sim}{\theta}) \prod_{t=2}^{n} f_{X_t|X_{t-1}}(x_t;\underset{\sim}{\theta}|x_{t-1}).$$

Explicit form of log likelihood f^n

$$l(\underset{\sim}{\theta}) = \log f_{X_1}(x_1;\underset{\sim}{\theta}) + \sum_{t=2}^{n} \log f_{X_t|X_{t-1}}(x_t;\underset{\sim}{\theta}|x_{t-1})$$

$$l(\underset{\sim}{\theta}) = \left(-\frac{1}{2}\log 2\pi - \frac{1}{2}\log\frac{\sigma^2}{1-\phi^2} - \frac{1}{2}\frac{(x_1 - c/1-\phi)^2}{\sigma^2/1-\phi^2}\right)$$

$$\left(-\frac{n-1}{2}\log 2\pi - \frac{n-1}{2}\log\sigma^2 - \sum_{t=2}^{n}\frac{(x_t - c - \phi x_{t-1})^2}{2\sigma^2}\right) \quad (*')$$

$$\hat{\underset{\sim}{\theta}}_{EMLE} = \underset{\underset{\sim}{\theta}}{\arg\max}\ l(\underset{\sim}{\theta})$$

Note that the above EMLE does not have a closed form solution. Iterative search procedures are used for obtaining the values of EMLE (e.g. Newton-Raphson, Levenburg-Marquardt, Downhill simplex)

<u>Remark</u> : Alternate multivariate approach to derive the likelihood f^n.

Consider $(X_1, \ldots, X_n)$ as a random vector from an n-dimensional Gaussian ($\{X_t\}$ is a Gaussian process).

$$E(\underset{\sim}{X}) = \mu \underset{\sim}{1}_n \; ; \qquad \mu = \frac{c}{1-\phi}$$

$$Cov(\underset{\sim}{X}) = \Sigma = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ & \ddots & & \vdots \\ & & \ddots & \gamma_1 \\ & & & \gamma_0 \end{pmatrix}$$

i.e. $$\Sigma = \frac{\sigma^2}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-1} \\ & 1 & \phi & \cdots & \phi^{n-2} \\ & & \ddots & & \vdots \\ & & & \ddots & \phi \\ & & & & 1 \end{pmatrix}$$

$$\underset{\sim}{X} \sim N_n(\mu \underset{\sim}{1}_n, \Sigma)$$

Likelihood f^n

$$L(\underset{\sim}{\theta}) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\underset{\sim}{x} - \mu \underset{\sim}{1}_n)' \Sigma^{-1} (\underset{\sim}{x} - \mu \underset{\sim}{1}_n)\right)$$

log likelihood f^n

$$\ell(\underset{\sim}{\theta}) = -\frac{n}{2}\log(2\pi) + \frac{1}{2}\log(|\Sigma^{-1}|) - \frac{1}{2}(\underset{\sim}{x} - \mu \underset{\sim}{1}_n)' \Sigma^{-1} (\underset{\sim}{x} - \mu \underset{\sim}{1}_n)$$

Approach iii : Conditional MLE approach

Regard the value of X_1 (observed as x_1) as deterministic and maximize the likelihood conditional on the first observation

i.e. $L_c(\underset{\sim}{\theta}) = f_{X_n,\dots X_2|X_1}(x_n,\dots x_2|x_1)$

Realize that

$$f_{X_n,\dots X_2|X_1} = f_{X_n|X_{n-1},\dots X_1}\, f_{X_{n-1},\dots X_2|X_1}$$

i.e. $f_{X_n,\dots X_2|X_1} = \left(f_{X_n|X_{n-1}}\right)\left(f_{X_{n-1}|X_{n-2},\dots X_1}\, f_{X_{n-2},\dots X_2|X_1}\right)$

$$\vdots$$

$$f_{X_n,\dots X_2|X_1} = f_{X_n|X_{n-1}}\, f_{X_{n-1}|X_{n-2}} \cdots f_{X_2|X_1}$$

$$f_{X_n,\dots X_2|X_1} = \prod_{t=2}^{n} f_{X_t|X_{t-1}}(x_t;\underset{\sim}{\theta}|x_{t-1})$$

i.e $L_c(\underset{\sim}{\theta}) = \prod_{t=2}^{n} f_{X_t|X_{t-1}}(x_t;\underset{\sim}{\theta}|x_{t-1})$

Now $X_t|X_{t-1} \sim N_1(c+\phi x_{t-1}, \sigma^2)$

$\forall\, t \geq 2$

Conditional log likelihood

$$l_c(\underset{\sim}{\theta}) = -\frac{n-1}{2}\log 2\pi - \frac{n-1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_2^n (x_t - c - \phi x_{t-1})^2$$

$$\underset{\sim}{\hat{\theta}}_{CMLE} = \underset{\underset{\sim}{\theta}}{\arg\max}\; l_c(\underset{\sim}{\theta})$$

Realize that maximization of $l_c(\underset{\sim}{\theta})$ w.r.t. c & ϕ is equivalent to minimization of

$$\sum_{t=2}^n (x_t - c - \phi x_{t-1})^2$$

i.e. $\Rightarrow$ CMLEs of c & ϕ are the ordinary LSE (that was described in Approach I).

i.e. CMLEs of c & ϕ are obtained as solutions of

$$\left.\begin{aligned}\sum_2^n x_t &= c(n-1) + \phi\sum_2^n x_{t-1}\\ \sum_2^n x_t x_{t-1} &= c\sum_2^n x_{t-1} + \phi\sum_2^n x_{t-1}^2\end{aligned}\right\}.$$

$$\begin{pmatrix}\hat{c}\\ \hat{\phi}\end{pmatrix} = \begin{pmatrix} n-1 & \sum_2^n x_t \\ \sum_2^n x_t & \sum x_{t-1}^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_2^n x_t \\ \sum_2^n x_t x_{t-1}\end{pmatrix}$$

Further,

$$\hat{\sigma}^2_{CMLE} = \frac{1}{n-1} \sum_{2}^{n} \left(x_t - \hat{c} - \hat{\phi} x_{t-1} \right)^2$$

Note : Unlike EMLE, we get closed form solution of CMLE

Remark : CMLE & EMLE has the same asymptotic distribution (provided $|\phi|<1$ i.e. stationary

MLE for Gaussian AR(p)

Exact MLE formulation:

$$X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

$$\epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

$$\underset{\sim}{\theta} = (c, \phi_1, \ldots, \phi_p, \sigma^2)'$$

Note first that $\underset{\sim}{X}_p = (X_1, \ldots, X_p)' \sim N_p$

$$E \underset{\sim}{X}_p = \mu \underset{\sim}{1}_p \quad ; \quad \mu = c/(1-\phi_1 - \cdots - \phi_p)$$

Let $\sigma^2 V_p$ denote the covariance matrix of $\underset{\sim}{X}_p$

i.e. $Cov(\underset{\sim}{X}_p) = \sigma^2 V_p = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ & \ddots & & \vdots \\ & & \ddots & \gamma_1 \\ & & & \gamma_0 \end{pmatrix}$

$$\left(\text{for } p=1, \; V_p = (1-\phi^2)^{-1}\right)$$

$$\underset{\sim}{X}_p \sim N_p(\mu \underset{\sim}{1}_p, \sigma^2 V_p)$$

$$f_{X_p, X_{p-1}, \ldots X_1} = (2\pi)^{-p/2} (\sigma^2)^{-p/2} |V_p|^{-1/2} \exp\left(-\frac{1}{2\sigma^2} (\underset{\sim}{x}_p - \mu \underset{\sim}{1}_p)' V_p^{-1} (\underset{\sim}{x}_p - \mu \underset{\sim}{1}_p)\right)$$

For $\forall t > p$; $X_t | X_{t-1}, \ldots X_1 \sim N\left(c + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p}, \sigma^2\right)$

& $X_t | X_{t-1}, \ldots, X_1 \equiv X_t | X_{t-1}, \ldots X_{t-p}$

$\forall t > p$

$$f_{X_t | X_{t-1}, \ldots, X_{t-p}} = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2} \left(x_t - c - \sum_1^p \phi_i x_{t-i}\right)^2\right)$$

jt distⁿ of $X_n, X_{n-1}, \dots X_1$

$$f_{X_n,\dots X_1} = f_{X_n|X_{n-1},\dots,X_1}\, f_{X_{n-1},\dots X_1}$$

$$= f_{X_n|X_{n-1},\dots X_{n-p}}\, f_{X_{n-1}|X_{n-2},\dots X_1}\, f_{X_{n-2},\dots X_1}$$

$$\vdots \qquad\qquad \vdots$$

$$f_{X_n,\dots X_1} = f_{X_n|X_{n-1},\dots X_{n-p}}\, f_{X_{n-1}|X_{n-2}\dots n-1-p} \cdots$$

$$\cdots f_{X_{p+1}|X_p,\dots X_1}\, f_{X_p, X_{p-1},\dots X_1}$$

$$= f_{X_p,\dots X_1} \prod_{t=p+1}^{n} f_{X_t|X_{t-1},\dots X_{t-p}}.$$

Likelihood fⁿ

$$L(\underset{\sim}{\theta}) = f_{\underset{\sim}{X}_p}(\underset{\sim}{x}_p;\underset{\sim}{\theta}) \prod_{t=p+1}^{n} f_{X_t|X_{t-1},\dots,X_{t-p}}(x_t;\underset{\sim}{\theta}|x_{t-1},\dots x_{t-p})$$

log likelihood fⁿ

$$l(\underset{\sim}{\theta}) = \Big(-\frac{p}{2}\log 2\pi - \frac{p}{2}\log\sigma^2 + \frac{1}{2}\log|V_p^{-1}|$$

$$- \frac{1}{2\sigma^2}(\underset{\sim}{x}_p - \mu\underset{\sim}{1}_p)' V_p^{-1}(\underset{\sim}{x}_p - \mu\underset{\sim}{1}_p)\Big)$$

$$+ \Big(-\frac{n-p}{2}\log 2\pi - \frac{n-p}{2}\log\sigma^2$$

$$- \frac{1}{2\sigma^2}\sum_{t=p+1}^{n}\big(x_t - c - \sum_{i=1}^{p}\phi_i x_{t-i}\big)^2\Big)$$

$$\hat{\underset{\sim}{\theta}}_{EMLE} = \underset{\underset{\sim}{\theta}}{\arg\max}\; l(\underset{\sim}{\theta})$$

<u>Note</u>: No closed form solution of $\hat{\underset{\sim}{\theta}}_{EMLE}$

<u>Note</u>: Iterative methods applied to obtain exact maximum likelihood estimates.