(iii) $g(X) = (X-a)^n$

$E\,g(X) = E(X-a)^n$: n^{th} moment of X about the pt a

If $a = E(X)$, then

$E(X - E(X))^n = \mu_n$: n^{th} order central moment of X.

$n=2$; $\mu_2 = E(X - E(X))^2 \rightarrow$ variance of X

$= \sigma^2$

$\mu_2^{1/2} = \sigma$: standard deviation of X

<u>Remark</u>: Measure of skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \left(\frac{\mu_3^2}{\mu_2^3}\right)^{1/2} = \frac{\mu_3}{(\sigma^2)^{3/2}} = \frac{\mu_3}{\sigma^3}$$

$= 0$ for $dist^n$ sym about $E(X)$

> 0 for positively skewed $dist^n$

< 0 for negatively skewed $dist^n$

<u>Remark</u>: Measure of peakedness.

Kurtosis measure $\gamma_2 = \frac{\mu_4}{\sigma^4}$

<u>Remark</u>: If $g_1, g_2, \ldots g_r$ be r real valued f^ns on $\mathbb{R}$ and let X be a r.v. $\ni E\,g_i(X)$ exists for $i = 1, \ldots r$, then $E\left(\sum_{i=1}^{r} g_i(X)\right)$ exists and is equal to $\sum_{i=1}^{r} E\,g_i(X)$

<u>e.g</u>: $\mu_n = E(X - E(X))^n$

$$= E\left(X^n + \binom{n}{1} X^{n-1}(-E(X)) + \binom{n}{2} X^{n-2}(-E(X))^2 + \cdots \cdots + (-1)^n E X^n\right)$$

i.e. $\mu_n = \mu_n' - \binom{n}{1}\mu_{n-1}'\mu_1' + \binom{n}{2}\mu_{n-2}'(\mu_1')^2 - \cdots + (-1)^n(\mu_1')^n$

$n=2$

$$\sigma^2 = \mu_2 = E(X - E(X))^2$$
$$= \mu_2' - 2(\mu_1')^2 + (\mu_1')^2$$
$$= \mu_2' - (\mu_1')^2$$
$$= EX^2 - (E(X))^2$$

Remark: Suppose EX^m exists for a positive int m then $\forall$ positive int $K \ni K \leq m$, EX^K exists

$$\int_{-\infty}^{\infty} |x|^K f(x)\,dx = \int_{|x|\leq 1} |x|^K f(x)\,dx + \int_{|x|\geq 1} |x|^K f(x)\,dx$$

(for cont case)

$$\leq \int_{|x|\leq 1} f(x)\,dx + \int_{|x|>1} |x|^K f(x)\,dx$$

$$\leq \int_{|x|\leq 1} f(x)\,dx + \int_{|x|>1} |x|^m f(x)\,dx$$

$$\leq \int_{-\infty}^{\infty} f(x)\,dx + \int_{-\infty}^{\infty} |x|^m f(x)\,dx$$

$$= 1 + E|X|^m < \infty$$

$\Rightarrow$ EX^K exists.

Moment Generating function (m.g.f)

Def^n: Let X be r.v. The function

$M_X(t) = E(e^{tX})$ is known as the m.g.f of the r.v. X if the expectation exists in some neighborhood of origin

Note: If m.g.f. exists, it determines the d.f. uniquely.

Note: Suppose all derivates of $M_X(t)$ exists at $t=0$ and can be obtained by differentiating under the expectation, then

$$\frac{\partial^k}{\partial t^k} M_X(t)\Big|_{t=0} = \frac{\partial}{\partial t^k} E(e^{tX})\Big|_{t=0}$$

$$= E\left(\frac{\partial^k}{\partial t^k} e^{tX}\Big|_{t=0}\right)$$

$$= E\left(X^k e^{tX}\Big|_{t=0}\right) = \mu_k'$$

i.e. $M_X(t)$ generates the moments of X.

Note: Taylor series expansion of $M_X(t)$ about 0 gives

$$M_X(t) = M_X(0) + \frac{M_X'(0)}{1!} t + \frac{M_X''(0)}{2!} t^2 + - - - -$$

$$\text{Coeff of } \frac{t^k}{k!} = \mu_k' = E X^k$$

Note: Although m.g.f. exists for most of the common distributions, there are cases when it does not exist

e.g. a Cauchy distⁿ

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \, ; \quad x \in \mathbb{R}$$

Some standard inequalities

(1) Chebyshev's inequality

X is a r.v. with $E(X) = \mu$ and $V(X) = \mu_2 = \sigma^2$

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \forall \, \epsilon > 0$$

(2) Generalization of Chebyshev's inequality

$h : \mathbb{R} \to \mathbb{R}$ be non-negative f^n $E\,h(X)$ is finite.

then $\forall \, c > 0$

$$P(h(X) \geq c) \leq \frac{E\,h(X)}{c} \, ;$$

$h(X) = |X|^r$ gives Markov's inequality

(3) Let g be a non-negative and strictly increasing function $g : [0, \infty) \to \mathbb{R}$ such that $E\,g(X)$ is finite, then for any $c > 0 \ni$ $g(c) > 0$

$$P(|X| \geq c) \leq \frac{E(g(|X|))}{g(c)}$$

(4) Jensen's inequality

Let $\Psi : (a, b) \to \mathbb{R}$ be a convex function and let X be r.v. with d.f. F having support $S \subseteq (a, b)$.

Then

$$E\,\Psi(X) \geq \Psi(E(X))$$

provided the expectations exist.

<u>Note</u> : From Jensen's inequality it follows that, for any r.v. X

$$E X^2 \geq (E X)^2$$

$$E|X| \geq |E(X)|$$

$$E e^X \geq e^{EX}$$

. For r.v. $X \ni P(X>0)=1$

$$E(\ln X) \leq \ln(EX)$$

<u>Quantile & percentile</u>

<u>Def^n</u>: Let $0<p<1$. The quantile of order p of the dist^n of a r.v. X, say ξ_p, is a point $\ni$

$$P(X<\xi_p) \leq p \quad \text{and} \quad P(X \leq \xi_p) \geq p$$

ξ_p is also called the $(100 \times p)$th percentile

<u>Note</u> : ξ_p is $\ni$

$$P(X \leq \xi_p) \geq p \quad \text{and} \quad P(X \geq \xi_p) \geq 1-p$$

$$\left(\begin{array}{l} 1-P(X<\xi_p) \geq 1-p \\ \text{i.e. } P(X<\xi_p) \leq p \end{array} \right).$$

Further

i.e. $P(X \leq \xi_p) - P(X=\xi_p) \leq p$

$\cancel{P(X \leq \xi_p) \geq p}$

$$p \leq P(X \leq \xi_p) \leq p + P(X=\xi_p)$$

<u>Note</u>: For a continuous dist^n ξ_p is thus solution of

$$F(\xi_p) = p$$

If $F(.)$ is strictly increasing then sol^n is unique, o/w there can be many solutions.