MTH 442: Time Series Analysis
Problem Set # 4

[1] Let $\{X_t\}$ be an MA(1) process $X_t = \varepsilon_t + \theta\,\varepsilon_{t-1}$, where $|\theta| < 1$ and $\varepsilon_t \sim WN\left(0, \sigma^2\right)$.

Find $\lim_{k\to\infty} E\left(X_t + \sum_{i=1}^{k} (-\theta)^i X_{t-i} - \varepsilon_t \right)^2$ for $|\theta| < 1$.

[2] Let $\{X_t\}$ be a stationary $AR(1)$ process $X_t = \phi X_{t-1} + \varepsilon_t;\ \varepsilon_t \sim WN\left(0, \sigma^2\right), |\phi| < 1$. Find $\lim_{N\to\infty} E\left(X_t - \sum_{i=0}^{N} \phi^i\, \varepsilon_{t-i} \right)^2$.

[3] Let $\{X_t\}$ be a stationary AR(1) process and suppose that it is possible to observe every second value only, i.e. $Y_t = X_{2t}$ is observable. Express $\{Y_t\}$ as $Y_t = \sum_{i=0}^{\infty} \psi_i\, Z_{t-i}$, with appropriate white noise process $\{Z_t\}$

[4] Consider a sequence $\{Z_t\}$ of i.i.d. random variables with mean 0 and variance σ^2. Define a time series $\{X_t\}$ as $X_t = Z_t Z_{t-1} + Z_{t-1} Z_{t-2} + \frac{1}{4} Z_{t-2} Z_{t-3}$. Verify whether $\{X_t\}$ is an invertible MA process.

[5] Suppose $\{X_t\}$ be AR(1) process $X_t = 0.25\,X_{t-1} + \varepsilon_t,\ \varepsilon_t \sim WN(0,1)$ and let $\{Y_t\}$ be an MA(1) process $Y_t = \delta_t - \delta_{t-1},\ \delta_t \sim WN(0,1)$. $\{\delta_t\}$ and $\{\varepsilon_t\}$ are assumed to be independently distributed. Verify whether $Z_t = (1 - X_t)(1 + Y_t)$ is a white noise process.

[6] Consider the ARMA(1,1) process $\{X_t\}$

$$X_t = \frac{1}{2} X_{t-1} + \varepsilon_t + \frac{2}{5} \varepsilon_{t-1}$$

(a) Check stationarity, causality and invertibility of $\{X_t\}$.
(b) If the process is causal, obtain the corresponding $MA(\infty)$ form of $\{X_t\}$.
(c) If the process is invertible, obtain the corresponding $AR(\infty)$ form

[7] Let $\{X_t\}$ be a stationary $MA(q)$ process with Auto Covariance function (ACVF) $\gamma_X(h)$. Define $Y_t = \sum_{j=0}^{\infty} a_j\, X_{t-j} + \varepsilon_t$; where, $\sum_{j=0}^{\infty} |a_j| < \infty$, $\{\varepsilon_t\}$ is $WN(0,1)$ and $\{\varepsilon_t\}$ and $\{X_t\}$ are independently distributed.

(a) Express the Auto Covariance Generating Function (ACGF) of $\{Y_t\}$ in terms of ACGF of $\{X_t\}$, ACGF of $\{\varepsilon_t\}$ and $g(z) = \sum_{j=0}^{\infty} a_j\, z^j$.

(b) Suppose $q = 2$ and $\{a_j\}_{j=0}^{\infty}$ is given by $a_0 = 1,\ a_1 = 2$, and $a_j = 0\ \forall\, j > 1$. Using the ACGF of $\{Y_t\}$ obtained in (a), find $\gamma_Y(2)$.

[8] Let $\{X_t\}$ be an AR(1) process $X_t = \frac{1}{2}X_{t-1} + \varepsilon_t$, $\varepsilon_t \sim WN(0,\sigma^2)$ and $\{Y_t\}$ be MA(2) process $Y_t = \delta_t - \delta_{t-1} - \delta_{t-2}$, $\delta_t \sim WN(0,\sigma^2)$. Furthermore, $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independent. Express the ACGF of $Z_t = X_t - X_{t-1} + Y_t$ in terms of ACGFs of $\{X_t\}$ and $\{Y_t\}$.

[9] Let $\{X_t\}$ be a *MA(1)* process $X_t = \varepsilon_t + \theta\, \varepsilon_{t-1}$; $\varepsilon_t \sim WN(0,\sigma^2)$, $|\theta| > 1$. Define a new process $\{Y_t\}$ as $Y_t = \sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j}$. Verify whether $\{Y_t\}$ is stationary and/or white.