Countable and Uncountable Sets

Def^n: Two sets A and B are said to be equivalent if there is a function $f: A \longrightarrow B$ such that f is one-to-one and onto.

Notation: $A \sim B$.

HW: Show that $A \sim B$ in the above sense is an equivalence relation on the collection of all sets.

Def^n: If $A \sim B$ then we say that A and B have the same number of elements. That is, A and B have the same cardinality.

Def^n: A set X is called a finite set if either $A = \phi$ or $A \sim \{1, 2, .., n\}$ for some $n \in \mathbb{N}$.

Otherwise, X is called an infinite set.

Def^n: The number of elements in a set X is called its cardinal number.

e.g. If X is a finite set, then the cardinal number of X is n (if $X \sim \{1, \dots, n\}$).

- Finite sets have finite cardinal numbers.

Infinite sets and infinite Cardinal numbers

Def^n: Any set X which is equivalent to $\mathbb{N}$ is said to be a countably infinite set.

A set X is countable if either X is a finite set or X is countably infinite.

Def^n: A set which is not countable is called an uncountable set.

In other words, if X is a countably infinite set, then one can enumerate the elements X, i.e., $X = \{x_1, x_2, \dots\}$.

For example, although $X = \{2, 4, 6, \dots\} \subset \mathbb{N} = \{1, 2, 3, \dots\}$,

$\mathbb{N} \sim X$ (moreover, $\mathbb{N}$ "numerically equiv." X)

via the bijection $f: \mathbb{N} \to X$ defined as $f(n) := 2n$.

Example: $\mathbb{Z} \sim \mathbb{N}$ $\quad f: \mathbb{Z} \to \mathbb{N}$ as $f(n) = \begin{cases} 2n, & n \geq 1 \\ -2n+1, & n \leq 0 \end{cases}$

- $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ Use the fact that every $k \in \mathbb{N}$ is uniquely written as $k = 2^{m-1}(2n-1)$ for some $m, n \in \mathbb{N}$.

Question: What can we say about infinite subsets of $\mathbb{N}$?

Thm: An infinite subset of $\mathbb{N}$ is countably infinite.

Pf. Make use of the well ordering property of $\mathbb{N}$. Let A be an infinite subset of $\mathbb{N}$. Since $A \neq \phi$, there is a smallest element in A, say, a_1.
Then consider $A \setminus \{a_1\}$. Again $A \setminus \{a_1\}$ has a smallest element, say, a_2.
Continuing this way, we obtain a collection $\{a_1, a_2, a_3, \cdots\}$ such that

$$a_n = \min A \setminus \{a_1, \ldots, a_{n-1}\} \quad \text{for} \quad n \geq 1.$$

Claim: $A = \{a_1, a_2, \cdots\}$.

Pf of claim: Suppose $A \setminus \{a_1, a_2, \cdots\} \neq \phi$. Let $a \in A \setminus \{a_1, a_2, \cdots\}$.
Consider $\{k : a_k > a\}$.
If $\{k : a_k > a\} = \phi$, then $a < a_1$ (why?)
This implies that $a = \min A$ contradiction to $a_1 = \min A$.
So, $\{k : a_k > a\} \neq \phi$. Hence $\{k : a_k > a\}$ has a smallest element, say, n. That is, $a_{n-1} < a < a_n$. Note also that $a_1 < a_2 \cdots < a_{n-1}$.

note that $a_{n-1} \neq a$ b/c. $a \notin \{a_1, \ldots\}$

This contradicts $a_n = \min A \setminus \{a_1, \ldots, a_{n-1}\}$. Therefore, $A = \{a_1, a_2, \cdots\}$.

HW: Show that $\{a_1, a_2, \cdots\}$ is equivalent to $\mathbb{N}$ (implying that A is a countable set).

Thm: Every sequence of real nos. has a monotone subsequence.

Pf: Let (a_n) be a seq. of real nos.

$$\text{Consider } S = \{ n \mid a_m > a_n \ \forall\, m > n \}$$

Suppose S is an infinite set. Then, by previous Thm, $S = \{n_1, n_2, n_3, \cdots\}$ with $n_1 < n_2 < n_3 < \cdots\cdots$. This implies that

$$a_{n_1} < a_{n_2} < a_{n_3} < \cdots \quad (\text{monotone } \uparrow \text{ subseq.})$$

Suppose S is a finite set.

Then $\mathbb{N} \setminus S \neq \phi$. Hence there is a smallest element $n_1 \in \mathbb{N}\setminus S$ s.t. $\forall\, n > n_1$, $n \notin S$.

Since $n_1 \notin S$, $\exists\, n_2 > n_1$ such that $a_{n_2} \leq a_{n_1}$.

Since $n_2 \notin S$, so $\exists\, n_3 > n_2$ such that $a_{n_3} \leq a_{n_2}$. Continuing this way, one obtains

$$\cdots\cdots \leq a_{n_k} \leq a_{n_{k-1}} \leq \cdots \leq a_{n_2} \leq a_{n_1} \quad (\text{monotone } \downarrow \text{ subseq.})$$

Corollary: Prove the Bolzano-Weierstrass thm for sequences:

Let (x_n) be a bdd. seq. By the above thm, $\exists$ a monotone $\uparrow$ or $\downarrow$ subseq. $(x_{n_k})_{k=1}^{\infty}$. Then by Monotone bounded convergent thm, conclusion holds.

HW: Fill in the details.

Corollary: Every Cauchy sequence of real nos. converges.

Pf: Hints.
- (x_n): Cauchy seq. $\Rightarrow$ (x_n) is bounded.
- $\exists\, (x_{n_k})_{k=1}^{\infty}$ monotone subseq. which is convergent.
- Combine these two facts to conclude the full seq. (x_n) converges!

The next goal is to show $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Thm: (suff. conditions to show equivalence of countable sets)

TFAE:

(i) X is a countable set. $\quad$ (i) $\Rightarrow$ (ii) HW.

(ii) $\exists$ a surjection $f : \mathbb{N} \longrightarrow X$.

(iii) $\exists$ an injective map $g : X \longrightarrow \mathbb{N}$.

(ii) $\Rightarrow$ (iii): Define $g : X \to \mathbb{N}$ as $g(x) := \min\{ n \in \mathbb{N} \mid f(n) = x \}$

HW: Show that g is one-to-one!

(iii) $\Rightarrow$ (i):
- If $g(X)$ is a finite set, then ?
- If $g(X)$ infinite set, then use the fact that infinite subsets of $\mathbb{N}$ are countable.

For each $i \in \mathbb{N}$, if A_i is countable, then $\bigcup_{i=1}^{\infty} A_i$ is also countable.

Pf. $A_1 := \{ a_1^{(1)}, a_2^{(1)}, \dots\dots \}$

$\vdots$

$A_k := \{ a_1^{(k)}, a_2^{(k)}, \dots\dots \}$

$\vdots$

$f_1 : A_1 \longrightarrow \mathbb{N} \times \mathbb{N}$ defined as $f_1(a_k^{(1)}) := (1, k)$ injective

$\vdots$

$f_i : A_i \rightarrow \mathbb{N} \times \mathbb{N}$ defined as $f_i(a_k^{(i)}) := (i, k)$ injective

$\vdots$

Define $F : \bigcup_{i=1}^{\infty} A_i \rightarrow \mathbb{N} \times \mathbb{N}$ as $F(a_m^{(n)}) := f_{k_n}(a_m^{(k_n)})$ where $k_n = \min\{ j \mid a_i^{(j)} = a_m^{(n)} \}$

Then F is an injective map. Hence $\bigcup_{i=1}^{\infty} A_i$ is countable because $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

$\mathbb{Q}$ is a countable set.

Pf. Note that $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$.

Since $\mathbb{Q}^+ = \{ \frac{m}{n} \mid m, n \in \mathbb{N} \}$, suffices to define a surjective map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{Q}^+$ (why?)

Define $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}^+$ as $g(m, n) := \frac{m}{n}$.

Also, $\mathbb{Q}^- = \{ \frac{-m}{n} \mid n, m \in \mathbb{N} \}$ is countable. Hence $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ countable.

We next provide an indirect proof of the fact that $\mathbb{R}$ is uncountable.
First we prove that $[0,1]$ is an uncountable set. Then it follows that $\mathbb{R}$ is uncountable (why?)

Claim: $[0,1]$ is an uncountable set.

Pf: Suppose $[0,1]$ is countable. Let $[0,1] = \{ x_1, x_2, \dots x_n, \dots \}$.

Choose $I_1 \subset [0,1]$ s.t. $x_1 \notin I_1$. Choose $I_2 \subset I_1$ s.t. $x_2 \notin I_2$

closed & bdd interval

(such that)

This way we construct nonempty closed and bounded intervals s.t.

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots \text{ s.t. } x_n \notin I_n \ \forall n \geq 1.$$

By the Nested Interval Property of $\mathbb{R}$, $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Let $y \in \bigcap_{n=1}^{\infty} I_n$. Then, $y \neq x_n$, $\forall n \geq 1$. (why?)

Therefore, $[0,1]$ is not countable.

HW: The set of irrational nos. $\mathbb{R} \setminus \mathbb{Q}$ is an uncountable set!

→ For a set A, let $\mathcal{P}(A)$ denote the set of all subsets of A.

(HW) Question: Is $A \sim X$ for some $X \subset \mathcal{P}(A)$?

Question: Is $A \sim \mathcal{P}(A)$?

Ans. NO!

Thm: (Cantor's Thm.) There is no surjective map $F: A \to \mathcal{P}(A)$.

Pf: Let $F: A \to \mathcal{P}(A)$ be any function.

Consider $B \coloneqq \{x \in A \mid x \notin F(x)\} \in \mathcal{P}(A)$.

We will show that B does not have a preimage in A.

Suppose $B = F(b)$ for some $b \in A$.

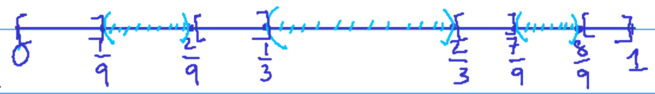
Either $b \in F(b)$ or $b \notin F(b)$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$b \notin F(b)$ $\qquad\qquad$ $b \in B = F(b)$

In either case, we get contradiction. ∎

The Cantor set



$C_0 := [0,1]$

$C_1 := [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 := [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$\vdots$

Define the Cantor set $C := \bigcap_{n=1}^{\infty} C_n$,

where $C_n = \{ x \in [0,1] \mid x = 0.d_1d_2d_3\cdots \text{ has } d_j \in \{0,2\} \text{ for } 1 \le j \le n \}$.

So, $C = \{ x \in [0,1] \mid x \text{ has the ternary expansion containing only } 0\text{'s and } 2\text{'s} \}$.

→ The Cantor set is uncountable.

Pf. Use the Cantor's diagonal argument:

Suppose C is a countable set. Let $C = \{x_1, x_2, x_3, \ldots\}$.

$$x_1 = 0.x_{1,1}\,x_{1,2}\,x_{1,3}\cdots$$
$$x_2 = 0.x_{2,1}\,x_{2,2}\,x_{2,3}\cdots$$
$$x_3 = 0.x_{3,1}\,x_{3,2}\,x_{3,3}\cdots$$

Consider a seq. $(d_1, d_2, d_3, d_4, \ldots)$ such that $d_j = \begin{cases} 0, & \text{if } x_{j,j} = 2 \\ 2, & \text{if } x_{j,j} = 0 \end{cases}$

Then the real no. $x := 0.d_1d_2d_3\cdots \in C$, but $x \neq x_n \;\forall n \ge 1$

(as $d_j \neq x_{j,j} \;\forall j \ge 1$) ∎

Question: Show that every infinite set is equivalent to a proper subset of itself, but a finite set is never equivalent to any proper subset of itself.