Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Quiz -1 (MTH305A)

Semester: 2022-2023, I

Full Marks–20 Time - 45 Minutes

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(1) Are the following statements TRUE/FALSE? Answer with justification.

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.
The set $f(A) \subset \mathbb{R}^m$ is compact implies $A \subset \mathbb{R}^n$ is compact.

[2 points]

Answer. FALSE
Example. Consider $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x, y) = x$ and $A = \{0\} \times \mathbb{R}$. Then $f(A) = \{0\}$ is compact but A is not compact.

(b) Suppose $f : \Omega \to \mathbb{R}^m$ is a differentiable function, where $\Omega \subset \mathbb{R}^n$ is open. If $df_a = 0$ for all $a \in \Omega$, then f is a constant function. [2 points]

Answer. FALSE
Example. Consider $f : (0, 1) \cup (2, 3) \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) \text{ and} \\ 1 & \text{if } x \in (2,3) \end{cases}.$$

The function is not constant but $f'(x) = 0$, for all x.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}^n, n \in \mathbb{N}$ be a continuously differentiable function with $\det(f'(a)) \neq 0$ for all $a \in \mathbb{R}^n$. Then the function $f : \mathbb{R}^n \to f(\mathbb{R}^n)$ is a diffeomorphism.

[2 points]

Answer. FALSE
Example. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (e^x \cos y, e^x \sin y)$. Then

$$\det(f'(x,y)) = \det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x} \neq 0.$$

But f is in injective ($f(x, 0) = f(x, 2\pi)$) and hence $f : \mathbb{R}^2 \to f(\mathbb{R}^2)$ is not invertible.

(2) Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \mathbb{R}^n$ be compact with $K \subset \Omega$. Show that there exists a compact set $D \subset \mathbb{R}^n$ satisfying

$$K \subset \text{Interior}(D) \subset D \subset \Omega.$$

[5 points]

Answer.

- Let $x \in K \subset \Omega$. There exists $r_x > 0$ such that $B_{2r_x}(x) \subset \Omega$.
- Consider $\mathcal{O} = \{B_{r_x}(x) \mid x \in K\}$ an open cover of K.
- We have a finite sub-cover $\{B_{r_{x_i}}(x_i) \mid i = 1, \dots m\}$.
- Define $D = \bigcup_{i=1}^{m} \bar{B}_{r_{x_i}}(x_i)$ which has the required properties.

(3) Let $\Omega \subset \mathbb{R}^2$ be open and $f : \Omega \to \mathbb{R}^2$ be a differentiable map. Write $f(x, y) = (u(x, y), v(x, y))$. Then we know that $f'(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map of $\mathbb{R}$-vector spaces. Considering $\mathbb{R}^2$ as $\mathbb{C}$, prove that f' is a $\mathbb{C}$-linear map if and only if (u, v) satisfies the Cauchy Riemann equations, i.e., $u_x = v_y,\ u_y = -v_x$.

Answer.

$$f'(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = T(\text{say}).$$

If $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ linear then there exists $c = a + ib \in \mathbb{C}$ such that $T(z = x + iy) = cz = (a+ib)(x+iy) = ax - by + i(bx + ay)$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \equiv x + iy$, the matrix representation of T as $T : \mathbb{R}^2 \to \mathbb{R}^2$ and comparing, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

which implies that $u_x = v_y$ and $u_y = -v_x$.

[5 points]

(4) Let $R = (a, b) \times (c, d) \subset \mathbb{R}^2$ be an open rectangle and $f : R \to \mathbb{R}$ be a function such that its partial derivatives exists and bounded on R. Show that f is continuous.

[5 points]

Answer.

Let $\left|\frac{\partial f}{\partial x}(x, y)\right| \leq M$ and $\left|\frac{\partial f}{\partial y}(x, y)\right| \leq M$ for all $(x, y) \in R$.

We have

$$\begin{aligned} & f(x+h, y+k) - f(x, y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x, y) \\ \Longrightarrow\ & |f(x+h, y+k) - f(x, y)| \leq |f(x+h, y+k) - f(x+h, y)| + |f(x+h, y) - f(x, y)| \\ & \qquad = |k| \left|\frac{\partial f}{\partial y}(x+h, y+\theta_2 k)\right| + |h| \left|\frac{\partial f}{\partial x}(x+\theta_1 h, y)\right| \text{(applying MVT)}. \\ \Longrightarrow\ & |f(x+h, y+k) - f(x, y)| \leq M[|h| + |k|] \leq 2M\|(h, k)\|. \end{aligned}$$

Now, it follows that f is continuous.