# MTH 442: Time Series Analysis
## Problem Set # 7

**[1]** The spectral density of a real valued time series $\{X_t\}$ is defined on $[0,\pi]$ as

$$f(\lambda)=\begin{cases}100, & \text{if } \pi/6-0.01\le\lambda\le\pi/6+0.01\\ 0, & \text{otherwise.}\end{cases}$$

and on $[-\pi,0]$ by $f(\lambda)=f(-\lambda)$. Obtain $\gamma_X(0)$ and $\gamma_X(1)$.

**[2]** A stationary time series $\{X_t\}$has a spectral density $f(\lambda)=\lambda^2, \lambda\in[-\pi,\pi]$;find the auto covariance sequence.

**[3]** $\{X_t\}$and $\{Y_t\}$are two uncorrelated stationary time series processes with absolutely summable auto covariances $\gamma_X(.)$and $\gamma_Y(.)$ respectively. Obtain the spectral density function of $Z_t=X_t+Y_t$ in terms of the spectral densities of $\{X_t\}$and $\{Y_t\}$.

**[4]** Let $\{X_t\}$ be a causal AR(1) process. Derive the spectral density function of the filtered process

$$Y_t=\frac{1}{3}\left(X_{t-1}+X_t+X_{t+1}\right).$$

**[5]** Suppose $\{Z_t\}$ be $WN(0,1)$ process. Define a new process $\{X_t\}$

$$X_t=\sum_{j=1}^{4}\alpha_j\, Z_{t-j+1}\, Z_{t-j};\ \alpha_1=\frac{1}{8},\alpha_2=\frac{3}{4},\alpha_3=\frac{3}{2},\alpha_4=1.$$

Find the spectral density function of $\{X_t\}$.

**[6]** Obtain autocovariance sequence of an *MA(q)* process using its spectral density.

**[7]** Using the characterization of ACVF through the spectral density, check whether or not the following functions are auto covariance functions

$$\text{(a)}\ \gamma(h)=\begin{cases}1 & h=0\\ -0.5 & h=\pm2\\ -0.25 & h=\pm3\\ 0 & \text{otherwise.}\end{cases}\qquad \text{(b)}\ \gamma(h)=\begin{cases}1 & |h|\le1\\ 0 & |h|\ge2.\end{cases}$$

**[8]** Find the value of $a$ for which $\gamma(h)=\begin{cases}1, & h=0\\ a, & |h|=1\\ 0, & \text{otherwise.}\end{cases}$ is ACVF.

**[9]** Let $\{Z_t\}$ be a stationary time series with spectral density

$$f_Z(\lambda)=1,\ -\pi\le\lambda\le\pi.$$

$\{X_t\}$ is a time series obtained from $\{Z_t\}$ by applying a linear filter $g(B)=\sum_{j=-\infty}^{\infty}g_j B^j$. The ACVF of the filtered process $\{X_t\}$ is $\gamma_X(h)=e^{-|h|}$. Obtain the spectral density of $\{Y_t\}$, the filtered process obtained by applying the same filter $g(B)$ on $\{X_t\}$.

**[10]** Let $\{X_t\}$ and $\{Y_t\}$ be 2 zero mean uncorrelated time series; $\{X_t\}$ having an invertible MA(1) model $X_t=\varepsilon_t+\theta\varepsilon_{t-1},\ \varepsilon_t\sim WN(0,1)$ and $Y_t\sim WN(0,1)$.

Define $Z_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} + \sum_{j=-\infty}^{\infty} \theta_j Y_{t-j}$; with

$$\psi_j = \begin{cases} 0.5, & \text{if } j = \pm 1 \\ 0, & \text{otherwise.} \end{cases} \qquad \theta_j = \begin{cases} 1, & \text{if } j = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the spectral density function of $\{Z_t\}$ and it's value at $\pi$.

**[11]** Let $\{X_t\}$ be a time series defined by

$$X_t = A\cos\left(\frac{\pi}{4}t\right) + B\sin\left(\frac{\pi}{4}t\right) + Y_t.$$

$A$ and $B$ are uncorrelated random variables with mean 0 and variance $\sigma^2$; $Y_t = 0.5\,\varepsilon_t$, $\varepsilon_t \sim WN\left(0, \sigma_\varepsilon^2\right)$. Further, $\varepsilon_t$ is uncorrelated with $A$ and $B$ for every *t*. Find the spectral distribution function of $\{X_t\}$.

**[12]** Let $X_t = \alpha_1 \cos(t) + \alpha_2 \sin(t) + Y_t$; where, $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)^T \sim N_2\left(\underset{\sim}{0}, diag(3, 3)\right)$, $Y_t = \varepsilon_t - \varepsilon_{t-2}$, $\varepsilon_t \sim WN(0,3)$ and is independent of $\underset{\sim}{\alpha}$. Find the spectral distribution function of $\{X_t\}$. Using the spectral distribution function of $\{X_t\}$, derive the value of $\gamma_X(0)$.