If $f:[a,b] \xrightarrow{cts} \mathbb{R}$ or $K \subseteq X$ then f attains supremum and infimum.

Proof: ① claim: f is bdd.

Suppose not then for every $n \in \mathbb{N}$ $\exists$ $x_n \in [a,b]$ s.t

$|f(x_n)| > n$

Now since $\{x_n\} \subseteq [a,b]$

by BWT $\exists$ $\{x_{n_k}\}$ s.t.

$x_{n_k} \to x \Rightarrow f(x_{n_k}) \to f(x)$

$\therefore$ $\{f(x_{n_k})\}$ is bdd.

but $f(x_{n_k}) > n_k$

$\Rightarrow \Leftarrow$

② $\alpha = \sup\{|f(x)| : x \in K\}$

$n \in \mathbb{N}$ $\exists$ $x_n \in K$ s.t.

$\alpha - \frac{1}{n} < |f(x_n)|$

$\{x_n\}$ $\exists$ $\{x_{n_k}\}$ s.t.

$x_{n_k} \to x \in K$

$f(x_{n_k}) \to f(x)$

$\alpha - \frac{1}{n_k} < |f(x_{n_k})| < \alpha + \frac{1}{n_k}$

$|f(x_{n_k})| \to \alpha$

example: take metric space l_1

with $e_n = (0, 0, \ldots 0, \underset{\uparrow}{\overset{n^{th}}{1}}, 0, \ldots)$

$d(e_n, e_m) = \|e_n - e_m\| = \sqrt{2}$

$E = \langle e_n \rangle$, $\quad \|e_n\| \le 1 \quad$ Not compact

$A \subseteq X$ is said to be $\boxed{\text{totally bounded}}$ if for every $\epsilon > 0$ $\exists$ finitely many A_i i.e. $A_1, \ldots A_N$ s.t.

(i) $d(A_i) < \epsilon$

(ii) $A \subseteq \bigcup_{i=1}^{N} A_i$

A is totally bounded $\Leftrightarrow$ for $\epsilon > 0$ $\exists$ $x_1, \ldots x_N \in A$

s.t. $A \subseteq \bigcup_{i=1}^{N} B_\epsilon(x_i)$

Let $\langle x_n \rangle \subseteq A$ T.B.

Let $\epsilon = 1$

$\exists$ $x_1, \ldots x_n \in A$

s.t. $A \subseteq \bigcup_{i=1}^{N} B_1(x_i)$

→ so, one of $B_1(x_i)$ will contain infinitely many $\langle x_n \rangle$

say that set is A_1.

$A_1 \subseteq \bigcup_{i=1}^{N_2} B_{1/2}(x_i)$

Similarly take A_2 (say it has infinitely many)

$A_1 \supseteq A_2$

$\therefore$ we get $A_1 \supseteq A_2 \supseteq \cdots$

$d(A_k) < \frac{1}{2^k}$

Choose

$\{x_{n_k}\}$ s.t. $x_{n_k} \in A_k$

$\therefore$ we can get $d(x_{n_k}, x_{n_m}) < \epsilon$

$\therefore$ $\{x_{n_k}\}$ is Cauchy.

Remark:
For any arbitratry metric space (X,d) if X complete & totaly bdd then X is compact.

Thm: X is compact $\Rightarrow$ X complete & totally bdd.

Pf: ① X is compact.

Let $\{x_n\}$ be a Cauchy seq in X

then X is compact $\Rightarrow \exists \{x_{n_k}\}$ cgs in it.

$\Rightarrow$ $\{x_n\}$ cgs. i.e, X is complete

② Supp. not T.B. then $\exists$ ϵ s.t. finitely many balls of radius ϵ will not cover X.

$x_1 \in X$ $\exists$ $x_2 \in X \setminus B_\epsilon(x_1)$, $x_3 \in X \setminus \bigcup_{i=1}^{2} B_\epsilon(x_i)$

$\therefore$ $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\epsilon(x_i)$

Consider $\{x_n\}$.

$\exists$ a subseq. $\{x_{n_k}\}$ which cgs to $x \in X$.

$\epsilon > 0$, $\exists N_0$ s.t. $d(x_{n_k}, x) < \epsilon/2$ $\forall$ $n_k \geq N_0$.

$n > N$ $d(x_n, x_N) < \frac{1}{N}$ | $\frac{1}{N} < \frac{\epsilon}{2}$

$\therefore$ $x_n \in B_{\frac{1}{N}}(x_N)$

$\Rightarrow \Leftarrow$

$\therefore$ X is T.B.

Centor Intersection

(X,d) metric space

$A_1 \supseteq A_2 \supseteq A_3 \cdots$ A_i's closed.

$diam(A_n) \to 0$

ex:

$\mathbb{R}$

$A_1 = \mathbb{N}$

$A_2 = \{2, 3, \ldots\}$

$A_n = \{n, n+1, \cdots\}$

$\vdots$

~~intersection~~ $\cap A_i = \phi$

$\epsilon > 0$

$\exists N$

$d(A_n) < \epsilon \quad \forall n \geq N$

$m, n > N$

$x_m \in A_m \subseteq A_N$

$x_n \in A_n \subseteq A_N$

$d(x_m, x_n) < \epsilon$

$\Rightarrow \{x_n\}$ is Cauchy.

$\Rightarrow$ ~~$\{x_n\}$~~ $x_n \to x$.

$\Rightarrow$ Fix m

$\Rightarrow x_n \in A_m \quad \forall n \geq m$

$\Rightarrow x \in A_m = A_n$

$\Rightarrow x \in \bigcap_{m=1}^{\infty} A_m$

$y \in \bigcap_{m=1}^{\infty} A_m$

$\Rightarrow d(x, y) < \epsilon \quad \forall \epsilon$

$\Rightarrow x = y$.

\# Let (X,d) be a complete metric space

s.t. $A_1 \supseteq A_2 \supseteq \cdots$, A_i' closed.

$diam(A_n) \to 0$

Then $\bigcap_{n=1}^{\infty} A_n$ is singleton.

Let X be compact $\Rightarrow$ Every open cover has a finite subcover.

Pf: Let $X = \bigcup_{i \in I} U_i$ U_i's open.

(*) If $\exists\, r > 0$ s.t. for $x \in X$

$B_r(x) \subset U_i$ for some i

Now since X is T.B. $\Rightarrow \exists\, x_1, \dots, x_n$ s.t.

$$X \subseteq \bigcup_{j=1}^{N} B_r(x_j) \subseteq \bigcup_{i=1}^{N} U_{i x_j}$$

Suppose (*) does not hold.

$r > 0$, $\exists\, x \in X$, $B_r(x) \not\subseteq U_i$ $\forall i$

$n \in \mathbb{N}$ $\exists\, x_n \in X$ s.t. $B_{\frac{1}{n}}(x_n) \not\subseteq U_i$ $\forall i$

$\{x_n\} \subseteq X$

$\Rightarrow \exists\, \{x_{n_k}\}$ cgs to $x \in X$.

$x \in U_{i_0}$ for some i_0.

As U_{i_0} open $\exists\, \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U_{i_0}$.

Now, $d(x, x_{n_k}) < \epsilon/2$ $\forall n_k \geq N$.

choose N large s.t $\frac{1}{N} < \epsilon/2$

take $y \in B_{\frac{1}{N}}(x_N)$

$$d(x, y) \leq d(x, x_N) + d(x_N, y)$$
$$< \epsilon/2 + \epsilon/2$$

$\Rightarrow B_{\frac{1}{N}}(x_N) \subset B_\epsilon(x) \subset U_{i_0}$

$\Rightarrow \Leftarrow$

~~Every open cover~~

Suppose (X, d) a compact metric space.

$A_1 \supseteq A_2 \supseteq \dots$ closed subsets of X.

Then $\bigcap_{n=1}^{\infty} A_n \neq \phi$

Pf: suppose $\bigcap_{n=1}^{\infty} A_n = \phi$

$\Rightarrow \underbrace{\bigcup_{n=1}^{\infty} A_n^c}_{\text{open cover for } X} = X$ (since A_n^c are open)

$\exists$ a finite set $F \subseteq \mathbb{N}$ s.t. $\bigcup_{n \in F} A_n^c = X$.

$\Downarrow$

$\bigcap_{n \in F} A_n = \phi$

$\Rightarrow \Leftarrow$

$\boxed{\langle A_n \rangle \text{ is a seq of closed sets st. intersection of every finite subcollection is non-empty}}$