Note: If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

Let $Z = \frac{X-\mu}{\sigma}$

d.f. of Z: $P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right)$

$$= P(X \leq \mu + \sigma z)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$\frac{x-\mu}{\sigma} = y$; $$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy = \Phi(z)$$

i.e. $Z \sim N(0,1)$

Alt: use m.g.f.

Note: Normal approximation to Binomial

$X \sim B(n, p)$ $EX = np$; $V(X) = np(1-p)$

To compute $P(\alpha \leq X \leq \beta) = \sum_{x=\alpha}^{\beta} \binom{n}{x} p^x (1-p)^{n-x}$

$\alpha, \beta \to$ points in support of X for large n $\approx \Phi\left(\frac{\beta + \frac{1}{2} - np}{\sqrt{npq}}\right) - \Phi\left(\frac{\alpha - \frac{1}{2} - np}{\sqrt{npq}}\right)$

Note that the above approximation is based on

- For large n, $\frac{X - np}{\sqrt{npq}}$ has a limiting dist^n $N(0,1)$
- $P(\alpha \leq X \leq \beta) = P(\alpha - \frac{1}{2} < X < \beta + \frac{1}{2})$

Continuity correction

(iii) Gamma dist^n

$X \sim \text{Gamma}(\alpha, \beta) \qquad \alpha > 0,\ \beta > 0$

p.d.f. $f(x) = \begin{cases} \dfrac{1}{\Gamma\alpha\, \beta^\alpha} \cdot e^{-x/\beta}\, x^{\alpha-1}, & x > 0 \\ 0, & o/w. \end{cases}$

m.g.f.

$$M_X(t) = E(e^{tX}) = \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{tx} e^{-x/\beta} x^{\alpha-1}\, dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{-x\left(\frac{1}{\beta} - t\right)} x^{\alpha-1}\, dx$$

$$= \frac{\Gamma\alpha}{\left(\frac{1}{\beta} - t\right)^\alpha} \cdot \frac{1}{\beta^\alpha \Gamma\alpha} = \frac{1}{(1-\beta t)^\alpha}$$

Note that $M_X(t)$ exists if $t < \frac{1}{\beta}$

$$E X^k = \mu_k' = \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty x^k e^{-x/\beta} x^{\alpha-1}\, dx$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \cdot \beta^k$$

$$\left.\begin{aligned} E(X) &= \frac{\Gamma(\alpha+1)}{\Gamma\alpha} \beta = \alpha\beta \\ E X^2 &= \frac{\Gamma(\alpha+2)}{\Gamma\alpha} \beta^2 = \alpha(\alpha+1)\beta^2 \end{aligned}\right\} \Rightarrow V(X) = \alpha\beta^2$$

Sp. case: $\alpha = 1 \rightarrow$ exponential dist^n with scale parameter β

$X \sim \exp(\beta)$ or $\exp(0, \beta)$

$\exp(\underbrace{0}_{\text{location}}, \underbrace{\beta}_{\text{scale parameter}})$

$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & o/w. \end{cases}$

· m.g.f. $M_X(t) = \dfrac{1}{1-\beta t}$ exists for $t < \dfrac{1}{\beta}$

$$E(X) = \beta, \quad V(X) = \beta^2$$

<u>Note</u>: Lack of memory property of $\exp(\beta)$

Note that

$$\underset{x>0}{P(X \geq x)} = \frac{1}{\beta} \int_x^\infty e^{-t/\beta}\, dt = \frac{1}{\beta} \cdot \left. \frac{e^{-t/\beta}}{-\frac{1}{\beta}} \right|_x^\infty$$

$$= e^{-x/\beta}$$

$\Rightarrow$

$$\underset{r,s>0}{P(X \geq r+s \mid X \geq r)} = \frac{P(X \geq r+s,\ X \geq r)}{P(X \geq r)}$$

$$= \frac{P(X \geq r+s)}{P(X \geq r)} = \frac{e^{-(r+s)/\beta}}{e^{-r/\beta}} = e^{-s/\beta}$$

$$= P(X \geq s)$$

<u>Note</u>: Alternate def^n of Gamma(α, β)

p.d.f. $f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1}, & x > 0 \\ 0, & o/w \end{cases}$

<u>Note:</u> Chi-square dist^n as a special case of Gamma dist^n

Consider $G(\alpha, \beta)$ $\quad \alpha = p/2, \ \beta = 2$

i.e. $G(p/2, 2)$ $\quad p = 1, 2, \ldots$

$$f(x) = \begin{cases} \dfrac{1}{2^{p/2}\,\Gamma p/2}\, e^{-x/2}\, x^{p/2 - 1}, & x > 0 \\ 0, & o/w \end{cases}$$

$E(X) = p \ ; \ V(X) = 2p$

$\underline{\text{IV}}$ Exponential distⁿ

As the sp case of Gamma (α, β) with $\alpha = 1$, we have $\exp(\beta)$

$\uparrow$ scale

In general, 2-parameter exp distⁿ.

$X \sim exp(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x-\alpha}{\beta}}, & x > \alpha \\ 0, & o/w \end{cases}$$

$\hat{V}$ Beta distⁿ

$X \sim Beta(\alpha, \beta)$

p.d.f. $$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & o/w \end{cases}$$

$\alpha, \beta > 0$; $B(\alpha, \beta) = \frac{\Gamma\alpha \, \Gamma\beta}{\Gamma\alpha+\beta}$.

i.e. $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$

Sp. case : $\alpha = 1, \beta = 1$

i.e. $X \sim B(1,1) \leftarrow U[0,1]$, a cont uniform dist^n

Moments

$$E(X^K) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^K x^{\alpha-1} (1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+K, \beta)}{B(\alpha, \beta)} = \frac{\Gamma\alpha+K \, \Gamma\beta / \Gamma\alpha+\beta+K}{\Gamma\alpha \, \Gamma\beta / \Gamma\alpha+\beta}$$

$$= \frac{\Gamma\alpha+K \, \Gamma\alpha+\beta}{\Gamma\alpha+\beta+K \, \Gamma\alpha}$$

$E(X) = \frac{\alpha}{\alpha+\beta}$ $\quad E X^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

$V(X) = E X^2 - (E(X))^2 = - -.$

m.g.f.

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\Gamma(\alpha+j)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+j)\,\Gamma(\alpha)}$$

$\hat{\underline{VI}}$ Double exponential or a Laplace distⁿ

$$f(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad x \in \mathbb{R}$$

$\mu \in \mathbb{R}, \sigma > 0$

$X \sim DE(\mu, \sigma)$ m.g.f : $\dfrac{e^{t\mu}}{1-(\sigma t)^2}$; $|t| < \dfrac{1}{\sigma}$

$EX = \mu$; $V(X) = 2\sigma^2$

$\hat{\underline{VII}}$ Cauchy distⁿ

$X \sim C(a, b)$

$$f(x) = \frac{1}{\pi b} \frac{1}{1+\left(\frac{x-a}{b}\right)^2}, \quad x \in \mathbb{R}$$

$a \in \mathbb{R}, b > 0$

None of the moments exist

m.g.f does not exist