Note: We can also define a autocorrelation matrix function for a covariance stationary process as

$$R_X(h) = \begin{pmatrix} \rho_{11}(h) & \rho_{12}(h) & \cdots & \rho_{1m}(h) \\ & \ddots & & \\ & & \ddots & \\ & & & \rho_{mm}(h) \end{pmatrix}$$

$$\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{(\gamma_{ii}(0)\, \gamma_{jj}(0))^{1/2}}$$

$$R_X(h) = D_0^{-1/2}\, \Gamma_X(h)\, D_0^{-1/2}$$

$$D_0 = \text{diag}(\gamma_{11}(0), \ldots, \gamma_{mm}(0))$$

Standard multivariate processes

(I) Vector White noise (VWN)

$$\underset{\sim}{Z}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

($\underset{\sim}{Z}_t$ is a VWN with mean vector $\underset{\sim}{0}$ and covariance matrix f^n Σ)

$E\, \underset{\sim}{Z}_t = \underset{\sim}{0}$ and covariance matrix f^n as

$$\Gamma(h) = \begin{cases} \Sigma, & h = 0 \\ 0, & o/w \end{cases} \quad \text{i.e. } E(\underset{\sim}{Z}_t \underset{\sim}{Z}_s') = \begin{cases} \Sigma, & t = s \\ 0, & t \neq s \end{cases}$$

Note : Note that for VWN process the vector process is uncorrelated; however, It is not necessary that the components are uncorrelated.

II : Vector Moving Average (VMA)

VMA(1): $\underset{K\times 1}{\underset{\sim}{X}_t} = \underset{K\times K}{\Theta}\, \underset{\sim}{\epsilon}_{t-1} + \underset{\sim}{\epsilon}_t$

$\underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$

Θ : $K\times K$ matrix of constants

$$\underset{\sim}{X}_t = (I_K + \Theta B)\, \underset{\sim}{\epsilon}_t$$

i.e. $\underset{\sim}{X}_t = \Theta(B)\, \underset{\sim}{\epsilon}_t$

$\Theta(B) = I_K + \Theta B$: MA matrix polynomial

$E\, \underset{\sim}{X}_t = \underset{\sim}{0} \quad \forall\, t$

$$\begin{aligned}
Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h}) &= E(\underset{\sim}{X}_t \underset{\sim}{X}_{t+h}') \\
&= E\left(\underset{\sim}{\epsilon}_t + \Theta\, \underset{\sim}{\epsilon}_{t-1}\right)\left(\underset{\sim}{\epsilon}_{t+h} + \Theta\, \underset{\sim}{\epsilon}_{t+h-1}\right)' \\
&= E\left(\underset{\sim}{\epsilon}_t \underset{\sim}{\epsilon}_{t+h}'\right) + E\left(\underset{\sim}{\epsilon}_t \underset{\sim}{\epsilon}_{t+h-1}' \Theta'\right) \\
&\quad + E\left(\Theta\, \underset{\sim}{\epsilon}_{t-1} \underset{\sim}{\epsilon}_{t+h}'\right) \\
&\quad + E\left(\Theta\, \underset{\sim}{\epsilon}_{t-1} \underset{\sim}{\epsilon}_{t+h-1}' \Theta'\right)
\end{aligned}$$

$$\text{i.e. } \Gamma_X(h) = \Sigma\, I_0(h) + \Sigma\, \Theta'\, I_1(h) + \Theta\, \Sigma\, I_{-1}(h) + \Theta\, \Sigma\, \Theta'\, I_0(h)$$

i.e. $$\Gamma_X(h) = \begin{cases} \Sigma + \Theta \Sigma \Theta', & h = 0 \\ \Sigma \Theta', & h = 1 \\ \Theta \Sigma, & h = -1 \\ 0, & |h| > 1 \end{cases}$$

$\underset{\sim}{X}_t$ is always covariance stationary $\forall$ $K \times K$ Θ.

<u>VMA(q)</u>

$$\underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t + \Theta_1 \underset{\sim}{\epsilon}_{t-1} + \cdots + \Theta_q \underset{\sim}{\epsilon}_{t-q}$$

$$\Theta_q \neq 0 \ ; \ \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

$$\underset{\sim}{X}_t = (I_K + \Theta_1 B + \cdots + \Theta_q B^q) \underset{\sim}{\epsilon}_t$$

i.e. $\underset{\sim}{X}_t = \Theta(B) \underset{\sim}{\epsilon}_t$

$\Theta(B) = I_K + \Theta_1 B + \cdots + \Theta_q B^q$ is the MA ~~matrix~~ polynomial

$$E \underset{\sim}{X}_t = \underset{\sim}{0}$$

$$Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h}) = E(\underset{\sim}{X}_t \underset{\sim}{X}'_{t+h})$$

$(\Theta_0 = I_K) \longrightarrow$ $$= E(\Theta_0 \underset{\sim}{\epsilon}_t + \Theta_1 \underset{\sim}{\epsilon}_{t-1} + \cdots + \Theta_q \underset{\sim}{\epsilon}_{t-q})(\Theta_0 \underset{\sim}{\epsilon}_{t+h} + \Theta_1 \epsilon_{t+h-1} + \cdots + \Theta_q \epsilon_{t+h-q})'$$

$$\Gamma_0 = Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_t) = \sum_{j=0}^{q} \Theta_j \Sigma \Theta_j'$$

$$\Gamma(1) = Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+1}) = \sum_{j=0}^{q-1} \Theta_j \Sigma \Theta_{j+1}'$$

$$\Gamma(-1) = \Gamma(1)'$$

$$\Gamma(2) = \text{Cov}(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+2}) = \sum_{j=0}^{q-2} \Theta_j \Sigma \Theta_{j+2}'$$

$$\Gamma(-2) = \Gamma(2)'$$

$\forall\ h \leq q$

$$\Gamma(h) = \sum_{j=0}^{q-h} \Theta_j \Sigma \Theta_{j+h}'$$

$$\Gamma(-h) = \Gamma(h)' = \sum_{j=0}^{q-h} \Theta_{j+h} \Sigma \Theta_j'$$

e.g. $\Gamma(-1) = E\left(\Theta_0 \underset{\sim}{\epsilon}_t + \Theta_1 \underset{\sim}{\epsilon}_{t-1} + \cdots + \Theta_q \underset{\sim}{\epsilon}_{t-q}\right)$

$$\left(\underset{\sim}{\epsilon}_{t-1}' \Theta_0' + \underset{\sim}{\epsilon}_{t-2}' \Theta_1' + \cdots + \underset{\sim}{\epsilon}_{t-q-1}' \Theta_q'\right)$$

$$= \Theta_1 \Sigma \Theta_0' + \cdots + \Theta_q \Sigma \Theta_{q-1}'$$

i.e. $\Gamma(-1) = \sum_{j=0}^{q-1} \Theta_{j+1} \Sigma \Theta_j' = \Gamma(1)'$

Also $\Gamma(h) = \text{Cov}(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h})$

$$= 0 \quad \forall\ |h| > q$$

VMA(q) is always Covariance stationary $\forall\ \Theta_1, \ldots, \Theta_q$, matrices of consts
(a property like the univariate MA processes)

VMA(∞)

$$\underset{\sim}{X}_t = \sum_{j=0}^{\infty} \Theta_j \, \underset{\sim}{\epsilon}_{t-j} \; ; \quad \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

Def^n: A seq of $n \times m$ matrices $\{H_s\}_{s=0}^{\infty}$ is said to be absolutely summable if each of its elements forms an absolutely summable scalar sequence
i.e. $\sum_{s=0}^{\infty} |H_{ij(s)}| < \infty \quad \forall i,j; \; i = 1(1)n, \; j = 1(1)m$

Suppose that for the VMA(∞) process the sequence of matrices $\{\Theta_j\}_{j=0}^{\infty}$ is absolutely summable,

i.e $\sum_{s=0}^{\infty} |\Theta_{ij(s)}| < \infty \quad \forall i,j = 1(1)K$

($\Theta_{ij(s)}$: $(i,j)^{th}$ element of Θ_s)

then VMA(∞) is covariance stationary

Note: If $\{\Theta_j\}$ is square summable, then also VMA(∞) would be covariance stationary.

If VMA(∞) is covariance stationary, then it's auto covariance matrix function is

$$\Gamma(h) = \sum_{j=0}^{\infty} \Theta_j \, \Sigma \, \Theta_{j+h}' \; ; \quad h = 0, 1, 2, \ldots$$

$$\Gamma(-h) = \Gamma(h)'$$

VAR models

Consider a VAR(p) process (say K-variate)

$$\underset{K\times 1}{\underset{\sim}{X}_t} = \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underset{\sim}{X}_{t-2} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t ;$$

$$\Phi_p \neq 0, \quad \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

$$Cov(\underset{\sim}{\epsilon}_t, \underset{\sim}{X}_{t-j}) = 0 \quad \forall j > 0$$

i.e. $E(\underset{\sim}{\epsilon}_t \underset{\sim}{X}'_{t-j}) = 0 \quad \forall j > 0$

$\left[\begin{array}{l} \underset{\sim}{X}_{p\times 1}, \ \underset{\sim}{Y}_{q\times 1} \text{ random vectors} \\ Cov(\underset{\sim}{X}, \underset{\sim}{Y}) = E(\underset{\sim}{X} - E(\underset{\sim}{X}))(\underset{\sim}{Y} - E(\underset{\sim}{Y}))' \end{array}\right]$

$\Phi_1, \Phi_2, \ldots, \Phi_p$ are $K\times K$ matrices of constants – VAR matrices of parameters

Model can be written in terms of AR operator matrix polynomial

$$\Phi(B) \underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t$$

where $\Phi(B) = I_K - \Phi_1 B - \cdots - \Phi_p B^p$.

VAR matrix polynomial

Let us see what the VAR(p) gives us. The l^{th} row of the VAR(p) system is (at time point t)

$$X_{l,t} = \left(\Phi_{l1}^{(1)} X_{1,t-1} + \Phi_{l2}^{(1)} X_{2,t-1} + \cdots \cdots + \Phi_{lK}^{(1)} X_{K,t-1}\right)$$

$$+ \left(\Phi_{l2}^{(2)} X_{1,t-2} + \Phi_{l2}^{(2)} X_{2,t-2} + \cdots \cdots + \Phi_{lK}^{(2)} X_{K,t-2}\right)$$

$$\vdots$$

$$+ \left(\Phi_{l1}^{(p)} X_{1,t-p} + \Phi_{l2}^{(p)} X_{2,t-p} + \cdots \cdots + \Phi_{lK}^{(p)} X_{K,t-p}\right)$$

$$+ \epsilon_{lt}$$

$l = 1(1)K \;;\; t = 1(1)n$

i.e.

$$X_{l,t} = \left(\Phi_{l1}^{(1)} X_{1,t-1} + \Phi_{l1}^{(2)} X_{1,t-2} + \cdots \cdots + \Phi_{l1}^{(p)} X_{1,t-p}\right)$$

$$+ \left(\Phi_{l2}^{(1)} X_{2,t-1} + \Phi_{l2}^{(2)} X_{2,t-2} + \cdots \cdots + \Phi_{l2}^{(p)} X_{2,t-p}\right)$$

$$\vdots$$

$$+ \left(\Phi_{lK}^{(1)} X_{K,t-1} + \Phi_{lK}^{(2)} X_{K,t-2} + \cdots \cdots + \Phi_{lK}^{(p)} X_{K,t-p}\right)$$

$$+ \epsilon_{lt}$$

Thus the model eq^n for the l^{th} variable is expressed in terms of p lags of the l^{th} variable (as it would have been for AR(p)) + p lags of all the remaining $K-1$ variables present inside the VAR(p) system

+ instantaneous noise variable for the l^{th} variable system

Thus the VAR(p) system takes care of using inputs from related variables for a predictive model.