ASSIGNMENT 5

MTH 301, 2021-22

(1) Suppose U is a dense subset of $\mathbb{R}^n$.
   (a) Show that if U is open in $\mathbb{R}^n$ then $A \cap U$ is dense in U.
   (b) Show by example that this may fail for sets that are not open.
(2) Is sum of two compact subsets of $\mathbb{R}$ is compact? Is sum of a closed set and compact set is closed?
(3) Show that if Y is a subset of a complete metric space X, then Y is compact if and only if it is closed and totally bounded.
(4) Show that a closed subset of a compact metric space is compact.
(5) Show that every compact metric space has a countable dense subset.
(6) Prove that every open subset U of $\mathbb{R}^n$ is the countable union of compact subsets. Show that every open cover of an open subset of $\mathbb{R}^n$ has a countable subcover.
(7) **Cantor's Intersection Theorem:** A decreasing sequence of nonempty compact subsets $A_1, A_2, \dots$ of a metric space (X, d) has nonempty intersection.
(8) Show that a continuous function from a compact metric space (X, d) into a metric space (Y, ρ) is uniformly continuous.
(9) Let S_n for $n \geq 1$ be a finite union of disjoint closed balls in R^k of radius at most 2^{-n} such that $S_{n+1} \subset S_n$ and S_{n+1} has at least two balls inside each ball of S_n. Prove that $C = \bigcap_{n=1}^{\infty} S_n$ is a perfect, nowhere dense compact subset of $\mathbb{R}^k$.
(10) If f is a continuous one-to-one function of a compact metric space X onto Y, show that f^{-1} is continuous. Show that the above statement is false if X is not compact.
(11) We say that (X, d) is a second countable metric space if there is a countable collection $\mathcal{U}$ of open balls in X such that for every $x \in X$ and $r > 0$, there is a ball $U \in \mathcal{U}$ with $x \in U \subseteq B_r(x)$. Prove that (X, d) is second countable if and only if it is separable (having a countable dense set).
(12) Let $A_1, A_2 \subset \mathbb{R}^n$ such that $A_1 \cap A_2 = \Phi$. If $x \in A_i$ $i = 1, 2$ then $d(x, A_j) > 0$ for $i \neq j$ then show that there exists a continuous function f on $\mathbb{R}^n$ such that $f\big|_{A_1} = 1$ and $f|_{A_2} = 0$. Is this condition necessary? In particular if A_i's are compact then it holds.
(13) If A is a non-compact subset of $\mathbb{R}^n$, show that there is a bounded continuous real valued function on A that does not attain its maximum.
(14) Let $S \subset \mathbb{R}^n$ is compact. Show that $f : S \to \mathbb{R}^m$ is continuous if and only if its graph $G_f = \{(x, f(x) : x \in S\}$ is compact. Give a function defined on $[0, 1]$ that has a closed graph but it is not continuous.
(15) Let $A \subset \mathbb{R}^n$ is compact. Show that for any point $x \in \mathbb{R}^n$ there is a closest point $a \in A$ to x i.e. $a \in A$ satisfies $\|x - a\| \leq \|x - b\|, \forall b \in A$.