Midsem solution:

1. $a_n \geq 0$, $\sum a_n < \infty$ then $\liminf \{na_n\} = 0$

Pf.

$$\liminf \{na_n\} = \sup_{k\geq 1} \left\{ \inf_{n\geq k} \{na_n\} \right\}$$

(1) If $\exists \{n_k a_{n_k}\}$ s.t. $n_k a_{n_k} \to 0$ then for each $k \geq 1$, $\inf_{n\geq k}\{na_n\} = 0 \Rightarrow \liminf\{na_n\} = 0$.

Suffices to prove: If $a_{n_k} \downarrow 0$ and $\sum_{k=1}^{\infty} a_{n_k} < \infty$, then $n_k a_{n_k} \to 0$.

To keep the computation notationally simpler, we will first prove the following for sequences:

(2) Let $a_n \downarrow 0$ and $\sum a_n < \infty$. Then $\lim(na_n) = 0$

By the Cauchy condensation test, $\sum a_n < \infty$ iff $\sum 2^n a_{2^n} < \infty$

Since $\sum a_n < \infty$, $\sum 2^n a_{2^n} < \infty$. Then $2^n a_{2^n} \to 0$ as $n \to \infty$.

For $k \in \mathbb{N}$, $\exists$ $n \in \mathbb{N}$ s.t. $2^n \leq k < 2^{n+1}$.

Using $a_n \downarrow 0$, $\underbrace{\frac{1}{2}\left(2^{n+1} a_{2^{n+1}}\right)}_{\downarrow\, 0} \leq k a_k < \underbrace{2\left(2^n a_{2^n}\right)}_{\downarrow\, 0}$ So by Squeeze Thm, $ka_k \to 0$.

Back to our question: $a_n \geq 0$ & $\sum a_n < \infty$. Then $a_n \to 0 \Rightarrow \exists (n_k)_{k=1}^{\infty}$ s.t. $a_{n_k} \downarrow 0$.

And as $\sum a_n < \infty$, $\sum_{k=1}^{\infty} a_{n_k} < \infty$

So, now we fit in (2) which implies $n_k a_{n_k} \to 0$ which combined with (1) completes the proof.

Converse not true: Consider $a_n = \begin{cases} \frac{1}{n}, & n: \text{odd} \\ 0, & n: \text{even} \end{cases}$. Then $\inf\{na_n\} = 0$. hence $\liminf\{na_n\} = 0$

But, $\sum_{1}^{\infty} a_n = \sum_{n:\text{odd}} \frac{1}{n} = \infty$.

2. $A \subset \mathbb{R}$ perfect set (nonempty)

Claim: A is uncountable.

Pf: Suppose A is countable. Let $A = \{x_1, x_2, \dots\}$.

Consider x_1 and an interval $[a_1, b_1]$ s.t. $x_1 \in (a_1, b_1)$.

Since x_1 is a limit pt., $(a_1, b_1) \setminus \{x_1\} \cap A \neq \phi$ (in fact, intersects at infinitely many pts)

Choose $x_{n_1} \in A$ s.t. $x_{n_1} \neq x_1$ and $x_{n_1} \neq x_2$ and consider $(a_2, b_2) \ni x_{n_1}$

s.t. $(a_2, b_2) \subset (a_1, b_1)$ and $b_2 - a_2 \leq \frac{1}{2}(b_1 - a_1)$ and $x_1, x_2 \notin [a_2, b_2]$.

Again $(a_2, b_2) \setminus \{x_{n_1}\} \cap A$ has infinitely many pts. Choose x_{n_2} s.t. $x_{n_2} \neq x_1, x_2, x_3$.

Consider $(a_3, b_3) \ni x_{n_2}$ s.t. $(a_3, b_3) \subset (a_2, b_2)$ and $b_3 - a_3 \leq \frac{1}{2}(b_2 - a_2) \leq \frac{1}{2^2}(b_1 - a_1)$.

s.t. $x_1, x_2, x_3 \notin [a_3, b_3]$.

Continuing this way, one obtains a nested sequence of closed & bdd intervals

$\{[a_k, b_k]\}$ s.t. $b_k - a_k \leq \frac{1}{2^{k-1}}(b_1 - a_1)$ and $x_{n_{k-1}} \neq x_1, x_2, \dots, x_k$ with $x_{n_{k-1}} \in (a_k, b_k)$, $k \geq 2$

s.t. $x_1, x_2, \dots, x_k \notin [a_k, b_k]$.

By the Nested Interval Property, $\bigcap_{k=1}^{\infty} [a_k, b_k] \neq \phi$, say, $x \in \bigcap_{k=1}^{\infty} [a_k, b_k]$

Observe that $x \neq x_n \ \forall n \geq 1$. Indeed, suppose $x = x_N$ for some $N \geq 1$.

As $x \in [a_{N+1}, b_{N+1}]$, $x_N \in [a_{N+1}, b_{N+1}]$. But by the construction above,

$x_N \notin [a_{N+1}, b_{N+1}]$.

So, $\exists\ x \in \bigcap_{k=1}^{\infty} [a_k, b_k]$ s.t. $x \in A$ and $x \neq x_n \ \forall n \geq 1$. This is not possible

as $A = \{x_n\}_{n=1}^{\infty}$.

Therefore, A is uncountable.

(ii) Consider $A = [0,1] \cup \{2\}$. Then A is nonempty uncountable closed set which is not perfect, because $2 \in A$ is not a limit pt. of A.

(iii) NO! Consider $(M,d) = (\mathbb{Q}, |\cdot|)$. Take $A \doteq \mathbb{Q}^+ \cup \{0\}$.

Then, A is a closed set b/c. $A = \mathbb{Q} \cap [0,\infty)$ closed in the relative metric space $(\mathbb{Q}, |\cdot|)$.

Every pt. in A is a limit pt. Indeed, for $a \in A$ and $\mathcal{U} \cap \mathbb{Q}$ open in $(\mathbb{Q}, |\cdot|)$, s.t. $a \in \mathcal{U} \cap \mathbb{Q}$, $\therefore \mathcal{U} \cap \mathbb{Q} \setminus \{a\} \cap A \neq \phi$ (every $\mathcal{U}$ open in $\mathbb{R}$ s.t. $a \in \mathcal{U}$ contains an interval which intersects infinitely many positive rational nos.)

3. (i) NO! Consider the homeomorphism $f : (\mathbb{N}, |\cdot|) \to (\{\frac{1}{n} \mid n \geq 1\}, |\cdot|)$.
$n \mapsto \frac{1}{n}$.

The sequence $(\frac{1}{n})$ is Cauchy in $(\{\frac{1}{n} \mid n \geq 1\}, |\cdot|)$, but $f^{-1}(\frac{1}{n}) = (n)$ is not Cauchy.

(ii) NO! $(\ell^2, \|\cdot\|_2)$ is a separable space and $(\ell^\infty, \|\cdot\|_\infty)$ is a nonseparable space.
A homeomorphism preserves separability.

4. $B[0,1] := \{ f : [0,1] \to \mathbb{R} \mid f \text{ is a bounded function} \}$.

$(B[0,1], \|\cdot\|_\infty)$ is a nonseparable normed linear space.

Consider $A = \{ \chi_{\{t\}} \in B[0,1] \mid t \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \}$.

A is an uncountable set and $\|\chi_{\{t\}} - \chi_{\{s\}}\|_\infty = 1 \quad \forall\, t \neq s$.

Then explain why $(B[0,1], \|\cdot\|_\infty)$ is nonseparable by using the ideas involved in proving the nonseparability of $(\ell_\infty, \|\cdot\|_\infty)$.

(ii) $\|f\|_1 := \int_0^1 |f(t)|\,dt$ is not a norm on $B[0,1]$. because $\|f\|_1 = 0 \not\Rightarrow f = 0$.

For example, take $f(t) = \begin{cases} 1, & t = 0 \\ 0, & 0 < t \leq 1 \end{cases}$. Then by the Lebesgue Integrability Criteria, f is Riemann Integrable and one can show that $\int_0^1 f = 0$, but $f \not\equiv 0$.

5. YES! Consider $\chi_{\mathbb{Q}} : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$.

$\chi_{\mathbb{Q}}$ is discontinuous at every pt in $\mathbb{R}$. Indeed, for $x \in \mathbb{R}$,
$\exists \; x_n \to x$ s.t. $x_n \in \mathbb{Q}$ and $\exists \; y_n \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $y_n \to x$.

But $\chi_{\mathbb{Q}}(x_n) = 1 \; \forall n \Rightarrow x = 1$ and $\chi_{\mathbb{Q}}(y_n) = 0 \Rightarrow x = 0$ (not possible)

Note that $\chi_{\mathbb{Q}} : (\mathbb{Q}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is the constant function $\chi_{\mathbb{Q}}(\mathbb{Q}) = 1$.
which is continuous.

6. Recall (discussed in one of the tutorial sessions):

(M,d), (M,ρ) metric spaces. We say $d \sim_s \rho$ (strongly equivalent)
if $\exists \; C_1 > 0$ and $C_2 > 0$ s.t. $C_1 \cdot \rho \leq d \leq C_2 \cdot \rho$.

Since $\max\{d, \rho\} \leq d + \rho \leq 2 \max\{d, \rho\}$, $d_1 \sim_s d_2$.