Recall: (Bolzano-Weierstrass Thm)

Every bdd seq. in $(\mathbb{R}, |\cdot|)$ has a cvgt. subsequence.

$\forall$ (M,d) : Every totally bdd. set has a Cauchy subsequence.

Q: Is it possible to get an analog of the Bolzano-Weirstrass thm in (M,d) ?

A: If we assume that every Cauchy seq. in (M,d) cvgs., then "Every totally bdd. set has a cvgt. subsequence."

$\Rightarrow$ Class of metric spaces in which every Cauchy seq. cvgs. : Complete Metric Spaces.

Def^n: A metric space (M,d) is said to be a complete metric space if every Cauchy seq. in M cvgs. to a point in M.

Examples:

- $(\mathbb{R}^n, \|\cdot\|_2)$ complete $\forall\ n \geq 1$.
- for $1 \leq p < \infty$, $(\ell_p, d_{\|\cdot\|_p})$ is complete.
- $(\ell_\infty, d_{\|\cdot\|_\infty})$ is complete.
- $(C[a,b], d_{\|\cdot\|_\infty})$ is complete
- $(0,1)$ w.r.t. the relative metric is NOT complete
- $(\mathcal{P}[a,b], \|\cdot\|_\infty)$ is NOT complete.

Remark: To show a metric space is complete, one needs to show

- the limit pt. x is in (M,d) $\rightarrow$ $(\ell_1, \|\cdot\|_\infty) \subset \ell_\infty$
- $d(x_n, x) \to 0$ as $n \to \infty$ $\rightarrow$ $x_n = (\underbrace{\tfrac{1}{n}, \ldots, \tfrac{1}{n}}_{n\text{-times}}, \ldots) \in (\ell_1, \|\cdot\|_\infty)$

$\downarrow$ 0 $\quad$ $\downarrow$ 0 $\quad$ $x = 0$ "candidate"?

But, $d(x_n, 0) = \|x_n - 0\|_1 = 1 \not\to 0$ as $n \to \infty$.

$\not\vdash$. (M,d) : Complete metric space, $A \subset M$.
(A,d) is complete iff A is closed in M.

Recall: $(\mathbb{R}, |\cdot|)$ Monotone bdd. seq. cvgs. $\Leftrightarrow$ Nested Interval Thm $\Leftrightarrow$ Completeness of $\mathbb{R}$.

(M,d) : metric space.

TFAE:

(i) (M,d) is complete.

(ii) (The Nested set thm) Let $F_1 \supset F_2 \supset \cdots\cdots$ be a decreasing seq. of nonempty closed sets in M with $\mathrm{diam}(F_n) \to 0$. Then $\bigcap F_n \neq \phi$.

(iii) (The Bolzano-Weierstrass Thm.) Every infinite, totally bdd. subset of M has a limit pt. in M.

Pf: (i) $\Rightarrow$ (ii): Given M is complete.

For $F_1 \supset F_2 \supset \cdots\cdots$, choose $x_n \in F_n$ (as $F_n \neq \phi$)

For $n \geq 1$, $\{x_k \mid k \geq n\} \subset F_n$.

Since $\mathrm{diam}(F_n) \to 0$, given $\varepsilon > 0$, $\exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon$, $\mathrm{diam}(F_n) < \varepsilon$.

Since $\{x_k \mid k \geq n\} \subset F_n$, $d(x_n, x_m) \leq \mathrm{diam}(F_n) < \varepsilon \quad \forall n, m \geq N_\varepsilon$.

$\Rightarrow$ (x_n) is a Cauchy seq.

$\Rightarrow$ $x_n \to x$ in M (completeness of M)

Moreover, as F_n is closed for $n \geq 1$ and $\{x_k \mid k \geq n\} \subset F_n$ with $x_n \to x$ as $n \to \infty$,

$x \in F_n$, $\forall n \geq 1$.

$\Rightarrow$ $\bigcap F_n \neq \phi$.

(ii) $\Rightarrow$ (iii): Let A be an infinite totally bdd. set in M.

Then $\exists \{x_n \mid n \geq 1\} \subset A$ s.t. $x_n \neq x_m$, $n \neq m$ & (x_n): Cauchy seq.,

$\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. $\forall n, m \geq N_\varepsilon$, $d(x_n, x_m) < \varepsilon$.

$F_1 := \{x_n \mid n \geq 1\}$, $F_2 := \{x_n \mid n \geq 2\}$, ...

$\overline{F_1} \supset \overline{F_2} \supset \overline{F_3} \supset \ldots\ldots$

Since $x_n \in F_n \subset \overline{F_n}$ and (x_n) is Cauchy, $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$.

But also, $\operatorname{diam}(F_n) = \operatorname{diam}(\overline{F_n})$, so $\operatorname{diam}(\overline{F_n}) \to 0$ as $n \to \infty$.

Therefore, $\bigcap \overline{F_n} \neq \phi$. Hence $\exists\ x \in \overline{F_n}\ \forall\ n \geq 1$.

Since $x_n \in \overline{F_n}$ and $x \in \overline{F_n}$, $d(x_n, x) \leq \operatorname{diam}(\overline{F_n}) \to 0 \Rightarrow d(x_n, x) \to 0$ as $n \to \infty$.

(Note: Using the Nested Set Thm, we obtained a given Cauchy seq. actually converges.)

(iii) $\Rightarrow$ (i): Take (x_n) Cauchy seq. (Suffices to show $\exists$ subseq. of (x_n) that cvgs.).

Then, $\{x_n \mid n \geq 1\}$ is a totally bdd. set.

finite set | infinite set.

$\Downarrow$ (x_n) has a subsequence which is a constant seq. hence, (x_n) cvgs. to that constant. (HW).

$\Rightarrow$ By the hypothesis, $\{x_n \mid n \geq 1\}$ has a limit pt. say, x in M. Since x is a limit pt. of $\{x_n \mid n \geq 1\}$ $\exists$ a subseq. (x_{n_k}) s.t. $x_{n_k} \to x$ as $k \to \infty$. Hence (x_n) cvgs. to x. (HW).

<u>Corollary:</u> Remark: Consider $(M, d) = (\mathbb{R}, |\cdot|)$.

Since totally bdd. ($\Leftrightarrow$ bdd. so, by the above thm:

$\mathbb{R}$ is complete

$\Updownarrow$

Nested "Interval" Thm. $\Longleftrightarrow$ least upper bound property (Axiom 3)

$\Updownarrow$ $\qquad$ $\Updownarrow$ using extra "order structure" on $\mathbb{R}$.

Bolzano–Weierstrass Thm. $\qquad$ Monotone bounded cvgs. thm.

A metric space with "extra" structure on it provides an "extra" tool to check the completeness property of (M,d).

Normed linear spaces :

Motivation: Recall that on $\mathbb{R}$, $\sum_{n\geq 1}^{\infty} a_n < \infty$ if $\sum_{n=1}^{\infty} |a_n| < \infty$. (HW: Recall the proof)

$$a_n = a_1 + \sum_{k=2}^{n} (a_k - a_{k-1}), \quad \forall\ n \geq 1.$$

In particular, consider $(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ a subseq.

$$a_{n_m} = a_{n_1} + \sum_{k=2}^{m} (a_{n_k} - a_{n_{k-1}})$$

$(\mathbb{R}, |\cdot|)$: Assume: If $\sum |a_n| < \infty$ then $\sum a_n < \infty$. — (1)

Let (x_n) be a Cauchy seq. WTS: (x_n) cvgs.

suffices (*)

$\Downarrow$

$\exists\ (x_{n_k})$ s.t. $|x_{n_k} - x_{n_{k+1}}| < \frac{1}{2^k}$ (**)

$$\Rightarrow \sum_{k=1}^{m} |x_{n_k} - x_{n_{k+1}}| < \sum_{k\geq 1}^{m} \frac{1}{2^k}.$$

$\Rightarrow \sum_{k=1}^{\infty} |x_{n_k} - x_{n_{k+1}}| < \infty$. Hence by hypothesis (1), $\sum_{k=1}^{\infty} (x_{n_k} - x_{n_{k+1}}) < \infty$

Since $x_{n_m} = x_{n_1} + \sum_{k=1}^{m} (x_{n_k} - x_{n_{k-1}})$

$\mathbb{R}$ is complete.

$\rightarrow$ (x_n): Cauchy seq.
To show (x_n) cvgs.
(*) suffices to produce a cvgt. subseq. of (x_n).

$\rightarrow$ (**) (x_n) Cauchy seq.
$\exists\ (x_{n_k})$ s.t.
$d(x_{n_k}, x_{n_{k+1}}) < \frac{1}{2^k}$

Conversely, suppose $\mathbb{R}$ is complete. Then $\sum_{1}^{\infty} |a_n| < \infty$ implies $s_n := \sum_{1}^{n} |a_k|$ is Cauchy.

------ $\sum_{1}^{\infty} a_n < \infty$.

Generalize this to normed linear space :

(HW): (M, d): normed linear space. M is complete iff whenever $\sum_{k=1}^{\infty} \|a_n\| < \infty$ implies $\sum_{1}^{\infty} a_n \in M$.