Recall: A monotone bounded seq. converges in $\mathbb{R}$.

Question: What if "monotone" condition is dropped?

Ans. NO! Plenty of examples of bounded sequences that do not converge.

Question: Can we say something about bounded sequences of real nos.?

Introduce a new notion!

(a_n): bounded seq. in $\mathbb{R}$

Define $t_n := \inf\{a_n, a_{n+1}, \cdots\}$ for each $n \geq 1$.

$T_n := \sup\{a_n, a_{n+1}, \cdots\}$ for each $n \geq 1$.

$t_n \uparrow$ $T_n \downarrow$

Note that t_n $\cancel{\downarrow}$ and T_n $\cancel{\uparrow}$ sequences.

Since (a_n) is bdd., $\inf\limits_{k\geq 1}\{a_k\} \leq t_n \leq T_n \leq \sup\limits_{k \geq 1}\{a_k\}$.

Therefore, (t_n) converges and (T_n) converges to $\underset{\text{sup}}{\cancel{\inf}}\{t_n\}$ and $\underset{\text{inf}}{\cancel{\sup}}\{T_n\}$, respectively.

That is, $\lim\limits_{n\to\infty} t_n$ exists and $\lim\limits_{n\to\infty} T_n$ exists.

- $\lim\limits_{n\to\infty} t_n = \inf\limits_{n\geq 1} \sup\{a_n, a_{n+1}, \cdots\} = \inf\limits_{n \geq 1} \sup\limits_{k \geq n}\{a_k\}$
- $\lim\limits_{n\to\infty} T_n = \sup\limits_{n\geq 1} \inf\{a_n, a_{n+1}, \cdots\} = \sup\limits_{n \geq 1} \inf\limits_{k \geq n}\{a_k\}$.

Question. Can we extend this idea to "any" sequence?

Ans. Yes! need to consider the extended real number system $\mathbb{R} \cup \{\pm\infty\}$.

Given any seq. (a_n) of real numbers, define

(limit inferior) $\liminf\limits_{n\to\infty} a_n := \sup\limits_{n\geq 1} \left(\inf \{a_n, a_{n+1}, \dots\} \right)$

(limit superior) $\limsup\limits_{n\to\infty} a_n := \inf\limits_{n\geq 1} \left(\sup \{a_n, a_{n+1}, \dots\} \right)$

→ If (a_n) is convergent, then $\liminf a_n = \limsup a_n = \lim a_n$

→ If (a_n) is a bounded seq., then

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup \{a_k \mid k \geq n\} \right) \neq \pm\infty$$

→ Characterization of $\limsup\limits_{n\to\infty} a_n$, if (a_n) is a bdd. seq.

Let $M := \limsup\limits_{n\to\infty} \{a_n\}$.

(HW) " $\forall\, \varepsilon > 0$, $a_n < M + \varepsilon$ for all but finitely many n, and

$M - \varepsilon < a_n$ for infinitely many n.

Theorem: The Bolzano–Weierstrass Thm (for sequences).

Every bounded sequence of real nos. has a convergent subsequence.

Pf: Let (a_n) be a bounded seq. and let $M := \limsup\limits_{n\to\infty} \{a_n\} < \infty$.

For $\varepsilon_k := \frac{1}{k}$, $\exists\, N_k \in \mathbb{N}$ such that $\forall\, n \geq N_k$,

$a_n < M + \frac{1}{k}$, and

$M - \frac{1}{k} < a_n$ for infinitely many n.

Choose $n_k > N_k$ such that $M - \frac{1}{k} < a_{n_k}$. Also, $a_{n_k} < M + \frac{1}{k}$.

$\therefore$ $M - \frac{1}{k} < a_{n_k} < M + \frac{1}{k}$. Hence $|a_{n_k} - M| < \frac{1}{k}$, $\forall\, k \geq 1$.

Therefore, $a_{n_k} \to M$ as $k \to \infty$. ∎

Corollary: Every bdd. seq. of real numbers has a Cauchy subsequence. (HW)

$\rightarrow$ Every Cauchy seq. of real numbers converges.

pf: Let (a_n) be a Cauchy seq.

For $\varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ st. $\forall\, n, m \geq N_\varepsilon$,

$$|a_n - a_m| < \varepsilon.$$

$$\Rightarrow a_{N_\varepsilon} - \varepsilon < a_n < a_{N_\varepsilon} + \varepsilon \quad \forall\, n \geq N_\varepsilon.$$

$$\Rightarrow (a_n) \text{ is a bdd. seq.}$$

Moreover, $a_{N_\varepsilon} - \varepsilon < a_n \quad \forall\, n \geq N_\varepsilon$

$$\Rightarrow a_{N_\varepsilon} - \varepsilon \leq \liminf a_n$$

and $a_n < a_{N_\varepsilon} + \varepsilon \quad \forall\, n \geq N_\varepsilon$

$$\Rightarrow \limsup a_n \leq a_{N_\varepsilon} + \varepsilon$$

Since (a_n) is bdd., $\liminf a_n \neq \pm\infty$ and $\limsup a_n \neq \pm\infty$.

Also, $\limsup a_n - \liminf a_n \geq 0$.

(why?) $\limsup a_n - \liminf a_n < 2\varepsilon$. Since this inequality is true for every $\varepsilon > 0$,

so, $\limsup a_n = \liminf a_n$. $(0 \leq a < \varepsilon,\ \forall \varepsilon > 0 \Rightarrow a = 0)$

Therefore, $\lim\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n \left(= \liminf\limits_{n\to\infty} a_n\right)$. ∎