Remark : Impulse response function (IRF)

Covariance stationary VAR(p) $\rightarrow$ VMA(∞)

Covariance stationary VARMA(p, q) $\rightarrow$ VMA(∞)

$$\underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t + \Psi_1 \underset{\sim}{\epsilon}_{t-1} + \Psi_2 \underset{\sim}{\epsilon}_{t-2} + \cdots$$

$$\begin{pmatrix} X_{1t} \\ \vdots \\ X_{Kt} \end{pmatrix} = \begin{pmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{Kt} \end{pmatrix} + \begin{pmatrix} \psi_{11}^{(1)} & \cdots & \psi_{1K}^{(1)} \\ \cdots & \cdots & \cdots \\ \psi_{K1}^{(1)} & \cdots & \psi_{KK}^{(1)} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1} \\ \vdots \\ \epsilon_{K,t-1} \end{pmatrix} + \cdots$$

$$\cdots + \begin{pmatrix} \psi_{11}^{(s)} & \cdots & \psi_{1K}^{(s)} \\ \cdots & \cdots & \cdots \\ \psi_{K1}^{(s)} & \cdots & \psi_{KK}^{(s)} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-s} \\ \vdots \\ \epsilon_{K,t-s} \end{pmatrix} + \cdots$$

$$X_{it} = \epsilon_{it} + \left(\psi_{i1}^{(1)} \epsilon_{1,t-1} + \cdots + \psi_{iK}^{(1)} \epsilon_{K,t-1} \right) + \cdots$$

$$\cdots + \left(\psi_{i1}^{(s)} \epsilon_{1,t-s} + \cdots + \psi_{iK}^{(s)} \epsilon_{K,t-s} \right) + \cdots$$

$i = 1(1)K$

$$\frac{\partial X_{i,t+s}}{\partial \epsilon_{j,t}} = \psi_{ij}^{(s)}$$

$$\frac{\partial \underset{\sim}{X}_{t+s}}{\partial \underset{\sim}{\epsilon}_t} = \Psi_s$$

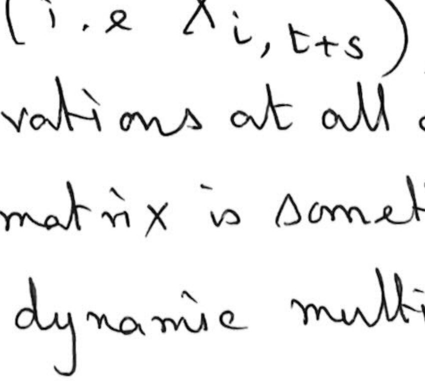
Def^n: Plot of $\dfrac{\partial X_{i,t+s}}{\partial \epsilon_{j,t}}$ as a f^n of s is called the impulse response function (IRF) plot, as it describes the response of X_i at time point $(t+s)$ to a one time impulse in X_j at time point t

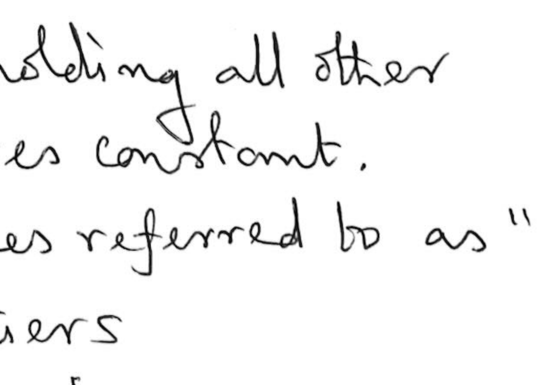
Note: IRF can be interpreted as response of X_i at future time point at lead periods $1, 2, \ldots$; w.r.t. a shock in the jth variable X_j at a time point t.

Note: $\psi_{ij}^{(s)}$ indicates the consequence of one unit increase in the jth variable's innovation at time point t $(\epsilon_{j,t})$ on the value of the ith variable at time $t+s$ (i.e $X_{i,t+s}$), holding all other innovations at all dates constant.

Note: Ψ_s matrix is sometimes referred to as "matrix of dynamic multipliers

Example : IRF





Auto Regressive Integrated Moving Average (ARIMA) model

Realise that many time series processes which are non-stationary but an appropriate order differenced process, derived from the non-stationary process, can be a stationary ARMA.

e.g. (i) $X_t = X_{t-1} + \epsilon_t$; $\epsilon_t \sim WN(0, \sigma^2)$ is a non-stationary process, but

$$Y_t = \nabla X_t = X_t - X_{t-1} = \epsilon_t \text{ is stationary}$$

$Y_t \sim ARMA(0, 0) \equiv WN$

(ii) $X_t = m_t + Y_t$

Y_t : Covariance stationary

$m_t = \sum_{j=0}^{K} \beta_j t^j$ is the time trend

$$\nabla^K X_t = \nabla^K m_t + \nabla^K Y_t$$

$Z_t = \nabla^K X_t = K!\, \beta_K + \nabla^K Y_t$ is stationary and can be modeled using ARMA

Def^n: Integrated process

A time series $\{X_t\}$ is said to be integrated of order d ($X_t \sim I_d$) if d is the smallest integer $\ni$ $\nabla^d X_t$ is a stationary process.

Note! In (i) $X_t \sim I_1$ (or $I(1)$)

(ii) $X_t \sim I_K$ (or $I(K)$)

Defⁿ: ARIMA model

$\{X_t\}$ is said to follow an Auto Regressive Integrated Moving Average (ARIMA) model of order (p, d, q) if

$$Z_t = \nabla^d X_t = (1-B)^d X_t \sim ARMA(p, q)$$

i.e. $Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$

i.e. $\phi(B) Z_t = \theta(B) \epsilon_t$ $\quad \left(\begin{array}{l} \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \\ \theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q \end{array} \right)$

i.e. $\phi(B) \nabla^d X_t = \theta(B) \epsilon_t$

i.e. $\underbrace{\phi(B)(1-B)^d X_t = \theta(B) \epsilon_t}_{\text{model in terms of } \{X_t\}}$ — (*)

Note that (i) $ARIMA(p, 0, q) \equiv ARMA(p, q)$

(ii) $ARIMA(p, d, q) \equiv ARMA(p+d, q)$

from (*), the model for $\{X_t\}$ is

$$\phi^*(B) X_t = \theta(B) \epsilon_t.$$

$$\phi^*(B) = \phi(B)(1-B)^d \Rightarrow X_t \sim ARMA(p+d, q)$$

(iii) $X_t \sim ARIMA(p, d, q)$

i.e. $X_t \sim ARMA(p+d, q)$

X_t is always a non-stationary ARMA, ~~irrespective~~ irrespective of covariance stationarity of $Z_t \sim ARMA(p,q)$ $(Z_t = \nabla^d X_t)$

$\phi^*(z) = \phi(z)(1-z)^d$ has d roots on unit circle $\Rightarrow X_t$ is non-stationary.

<u>Example</u>: $X_t \sim ARIMA(1,1,1)$

i.e. $\nabla X_t = Z_t$ is $\ni$

$$Z_t = \phi Z_{t-1} + \epsilon_t + \theta \epsilon_{t-1}$$

$$|\phi| < 1, \quad \epsilon_t \sim WN(0, \sigma^2)$$

i.e $X_t - X_{t-1} = \phi(X_{t-1} - X_{t-2}) + \epsilon_t + \theta \epsilon_{t-1}$

i.e. $X_t = (1+\phi) X_{t-1} - \phi X_{t-2} + \epsilon_t + \theta \epsilon_{t-1}$

i.e. $X_t \sim ARMA(2,1)$

Also $(1-\phi B)(1-B) X_t = (1+\theta B)\epsilon_t$

i.e. $\phi^*(B) X_t = \theta(B) \epsilon_t$

$\phi^*(z)$ has one unit root

$\Rightarrow X_t$ is non-stationary ARMA(2,1)

Seasonal ARMA model

Seasonal MA(Q): $MA(Q)_s$

$$X_t = \epsilon_t + \theta_1^{(s)} \epsilon_{t-s} + \theta_2^{(s)} \epsilon_{t-2s} + \cdots + \theta_Q^{(s)} \epsilon_{t-Qs}$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

s: period of seasonality

Seasonal AR(P): $AR(P)_s$

$$X_t = \phi_1^{(s)} X_{t-s} + \phi_2^{(s)} X_{t-2s} + \cdots + \phi_P^{(s)} X_{t-Ps} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

s: period of seasonality

Seasonal ARMA(P, Q): $ARMA(P,Q)_s$

$$X_t = \phi_1^{(s)} X_{t-s} + \phi_2^{(s)} X_{t-2s} + \cdots + \phi_P^{(s)} X_{t-Ps}$$
$$+ \epsilon_t + \theta_1^{(s)} \epsilon_{t-s} + \theta_2^{(s)} \epsilon_{t-2s} + \cdots + \theta_Q^{(s)} \epsilon_{t-Qs}$$

i.e. $\left(1 - \phi_1^{(s)} B^s - \phi_2^{(s)} B^{2s} - \cdots - \phi_P^{(s)} B^{Ps}\right) X_t$

$$= \left(1 + \theta_1^{(s)} B^s + \theta_2^{(s)} B^{2s} + \cdots + \theta_Q^{(s)} B^{Qs}\right) \epsilon_t$$

$$\underbrace{\Phi^{(s)}(B^s)}_{\text{Seasonal AR polynomial}} X_t = \underbrace{\Theta^{(s)}(B^s)}_{\text{Seasonal MA polynomial}} \epsilon_t$$

Note: $ARMA(P,Q)_s$ is stationary and causal if roots of $0 = \Phi^{(s)}(z)$ all lie outside the unit circle; it is invertible if roots of $\Theta^{(s)}(z) = 0$ all lie outside unit circle