(1) $$Z_t = \begin{cases} \epsilon_t + it\,\epsilon_t, & t \text{ odd} \\ \epsilon_t - it\,\epsilon_t, & t \text{ even} \end{cases}$$ $E(Z_t) = 0 \ \forall t$

$\epsilon_t \overset{i.i.d}{\sim} (0, \sigma^2)$

$Cov(Z_{t+h}, Z_t)$

Case 1 : t & h even

$$Cov(Z_{t+h}, Z_t) = E(\epsilon_{t+h} - i(t+h)\,\epsilon_{t+h})^* (\epsilon_t - it\,\epsilon_t)$$
$$= E(\epsilon_{t+h} + i(t+h)\,\epsilon_{t+h})(\epsilon_t - it\,\epsilon_t)$$
$$= 0 \ \forall\ h \neq 0$$

Sly for other cases $Cov(Z_{t+h}, Z_t) = 0 \ \forall\ h \neq 0$

t even

$$V(Y_t) = E(\epsilon_t - it\epsilon_t)^* (\epsilon_t - it\epsilon_t)$$
$$= E(\epsilon_t + it\,\epsilon_t)(\epsilon_t - it\,\epsilon_t)$$
$$= \sigma^2(1+t^2)$$

t odd

$$V(Y_t) = E(\epsilon_t + it\,\epsilon_t)^* (\epsilon_t - it\,\epsilon_t)$$
$$= \sigma^2(1+t^2)$$

$\{Z_t\}$ is not covariance stationary

(2) $\gamma_X(h) = E(X_{t+h}-\mu)^*(X_t-\mu)$ $\left(X_t \text{ is covariance stationary}\right)$

$= E(X_{t+h-h}-\mu)^*(X_{t-h}-\mu)$

$= E(X_t-\mu)^*(X_{t-h}-\mu)$

$= \left(E(X_{t-h}-\mu)^*(X_t-\mu)\right)^*$

$= \left(\gamma_X(-h)\right)^*$

$\gamma_X(h)^* = \gamma_X(-h) \ \forall h$

(3) By Weak law of large numbers

$$\frac{1}{n}\sum_{t=1}^{n} Z_t \xrightarrow{p} EZ_1 = \mu$$

i.e. $\bar{Z}_n \xrightarrow{p} \mu$.

(4) $\displaystyle\sum_{t=1}^{n-h} (Z_t-\mu)(Z_{t+h}-\mu)$

$= \displaystyle\sum_{t=1}^{n-h} (Z_t-\bar{Z}_n+\bar{Z}_n-\mu)(Z_{t+h}-\bar{Z}_n+\bar{Z}_n-\mu)$

$= \displaystyle\sum_{t=1}^{n-h} (Z_t-\bar{Z}_n)(Z_{t+h}-\bar{Z}_n) + (n-h)(\bar{Z}_n-\mu)^2$

$+ (\bar{Z}_n-\mu)\displaystyle\sum_{t=1}^{n-h}(Z_{t+h}-\bar{Z}_n)$

$+ (\bar{Z}_n-\mu)\displaystyle\sum_{t=1}^{n-h}(Z_t-\bar{Z}_n)$

Using the given approximation

$$\sum_{1}^{n-h}\left(z_t-\bar{z}_n\right) \approx \sum_{t=1}^{n-h}\left(z_{t+h}-\bar{z}_n\right) \approx \sum_{t=1}^{n}\left(z_t-\bar{z}_n\right)=0$$

the last 2 terms are ≈ 0

Thus

$$\sum_{t=1}^{n-h}\left(z_t-\mu\right)\left(z_{t+h}-\mu\right)$$

$$\approx \sum_{1}^{n-h}\left(z_t-\bar{z}_n\right)\left(z_{t+h}-\bar{z}_n\right)+(n-h)\left(\bar{z}_n-\mu\right)^2$$

$$\Rightarrow E\left(\sum_{1}^{n-h}\left(z_t-\mu\right)\left(z_{t+h}-\mu\right)\right) \approx \sum_{1}^{n-h} E\left(z_t-\bar{z}_n\right)\left(z_{t+h}-\bar{z}_n\right) + (n-h)V\left(\bar{z}_n\right)$$

i.e. $(n-h)\,\gamma_z(h) \approx (n-h)\,E\left(\hat{\gamma}_z^*(h)\right)+(n-h)V\left(\bar{z}_n\right)$

$$\left(\hat{\gamma}_z^*(h)=\frac{1}{n-h}\sum_{1}^{n-h}\left(z_t-\bar{z}_n\right)\left(z_{t+h}-\bar{z}_n\right).\right).$$

bias $E\left(\hat{\gamma}_z^*(h)\right)-\gamma_z(h) \approx -V\left(\bar{z}_n\right).$

(5)

$$X_t = \phi X_{t-1} + \epsilon_t \qquad \epsilon_t \sim WN(0,1)$$

(a) $V\left(\frac{1}{4}(X_1+X_2+X_3+X_4)\right)$

$$= \frac{1}{16}\left(4\gamma_0 + 2\sum_{i<j} Cov(X_i, X_j)\right)$$

$$= \frac{1}{16}\left(4\gamma_0 + 6\gamma_1 + 4\gamma_2 + 2\gamma_3\right)$$

$$\gamma_X(h) = \frac{\sigma^2}{1-\phi^2}\cdot \phi^{|h|}; \quad \phi = 0.8$$

(b) $Y_t = \sum_{i=1}^{t} X_i$

$$Y_1 = X_1 \qquad Y_2 = X_1 + X_2$$

$$V(Y_1) = \gamma_0 \neq V(Y_2) = 2\gamma_0 + 2\gamma_1 \quad ; \ \phi = 0.8$$

(6) $X_t = \begin{cases} Z_t, & t \text{ even} \\ (Z_{t-1}^2 - 1)/\sqrt{2}, & t \text{ odd} \end{cases} \qquad Z_t \overset{i.i.d}{\sim} N(0,1)$

$$E X_t = 0 \ \forall t \left(= \begin{cases} E Z_t, & t \text{ even} \\ \frac{1}{\sqrt{2}}(E Z_{t-1}^2 - 1), & t \text{ odd} \end{cases} \right).$$

<u>t even</u> $V(X_t) = E Z_t^2 = 1$

t odd $V(X_t) = \frac{1}{2}\left(E Z_{t-1}^4 + 1 - 2 E Z_{t-1}^2\right)$

$$= \frac{1}{2}(3 + 1 - 2) = 1$$

t even & h even

$$Cov(X_{t+h}, X_t) = E(Z_{t+h} Z_t) = 0 \quad \forall h \neq 0$$

t even, h odd

$$Cov(X_{t+h}, X_t) = \frac{1}{\sqrt{2}} E(Z^2_{t-1+h} - 1) Z_t$$

$$= 0 \quad \forall h \neq 0 \quad \left(\begin{array}{l} h=1 \text{ odd order moment} \\ h \neq 1 \text{ indep} \end{array} \right).$$

Sly other cases.

$\Rightarrow \{X_t\} \sim WN(0,1)$

(7)

(a) $V(X_t) = \sigma^2 + c^2 \sigma^2 \sum_{1}^{\infty} 1 \not< \infty$

$\Rightarrow \{X_t\}$ is not covariance stationary

(b) $X_t = \epsilon_t + c\,\epsilon_{t-1} + c\,\epsilon_{t-2} + \cdots$

$X_{t-1} = \epsilon_{t-1} + c\,\epsilon_{t-2} + c\,\epsilon_{t-3} + \cdots$

$\nabla X_t = X_t - X_{t-1}$

$Z_t = \nabla X_t = \epsilon_t + (c-1)\,\epsilon_{t-1} \leftarrow MA(1)$

$\Rightarrow \{Z_t\}$ is covariance stationary

(8)

$$Y_t = \nabla X_t = \left(\epsilon_t + \tfrac{1}{2}\epsilon_{t-1}\right) - \left(\epsilon_{t-1} + \tfrac{1}{2}\epsilon_{t-2}\right)$$

MA(2)

(9) $$U_t = (1 - Z_t) X_t + Y_t$$

$$X_t = \epsilon_t + \epsilon_{t-1} \qquad \left.\begin{array}{l} \epsilon_t \sim WN(0,1) \\ Y_t \sim WN(0,1) \\ Z_t \sim WN(0,1) \end{array}\right\} \text{mutually indep.}$$

$$E\,U_t = 0$$

$$\begin{aligned} Cov(U_t, U_{t+h}) &= E\Big(\big((1-Z_t)X_t + Y_t\big)\big((1-Z_{t+h})X_{t+h} + Y_{t+h}\big)\Big) \\ &= E\big((1-Z_t)(1-Z_{t+h})\big)\, E(X_t X_{t+h}) \\ &\quad + E\big(1-Z_t\big)\, \cancel{E X_t\, E(Y_{t+h})}^{\,0} \\ &\quad + E Y_t\, \cancel{E(1-Z_{t+h})\, E X_{t+h}}^{\,0} \\ &\quad + E(Y_t Y_{t+h}) \\ &= (1 + \gamma_z(h))\,\gamma_x(h) + \gamma_y(h) \end{aligned}$$

$$\begin{aligned} \gamma_U(1) &= (1 + \gamma_z(1))\,\gamma_x(1) + \gamma_y(1) \\ &= \gamma_x(1) \neq 0 \end{aligned}$$

$$\Rightarrow U_t \not\sim WN$$

(10)

$$\left.\begin{array}{l} X_t \sim WN(0, \sigma_x^2) \\ Y_t \sim WN(0, \sigma_y^2) \end{array}\right\rangle \text{indep}$$

$Z_t = X_t + Y_t$

$E\, Z_t = 0 \;\forall t$

$$\begin{aligned} Cov(Z_t, Z_s) &= E\, Z_t Z_s \\ &= E(X_t + Y_t)(X_s + Y_s) \\ &= E(X_t X_s) + E(Y_t Y_s) \\ &= \begin{cases} \sigma_x^2 + \sigma_y^2, & t = s \\ 0, & o/w \end{cases} \end{aligned}$$

$\Rightarrow Z_t \sim WN(0, \sigma_x^2 + \sigma_y^2)$

Let $X_t = \epsilon_t + \epsilon_{t-1}$ $\quad \epsilon_t \sim WN(0,1)$

$Y_t = \delta_t - \delta_{t-1}$ $\quad \delta_t \sim WN(0,1)$

$\{\epsilon_t\}$ & $\{\delta_t\}$ are indep.

$$\gamma_X(h) = \begin{cases} 2, & h = 0 \\ 1, & h = \pm 1 \\ 0, & o/w \end{cases}$$

$$\gamma_Y(h) = \begin{cases} 2, & h = 0 \\ -1, & h = \pm 1 \\ 0, & o/w \end{cases}$$

$Z_t = X_t + Y_t$; $\{X_t\}$ & $\{Y_t\}$ are indep.

$$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$$
$$= \begin{cases} 4, & h=0 \\ 0, & o/w \end{cases}$$

$\Rightarrow \quad Z_t \sim WN(0,4)$

(11) $\quad X_t = \mu + \epsilon_t + \epsilon_{t-1} + \phi\,\epsilon_{t-2}\,; \quad \epsilon_t \overset{i.i.d}{\sim} N(0,\sigma^2)$

(a) $\quad \delta_1 = \dfrac{2X_1 + X_3}{3} \qquad \delta_2 = \dfrac{X_3 + X_4 + X_5}{3}$

$E X_t = \mu \quad \forall t$

$E(\delta_1) = \mu \qquad E\,\delta_2 = \mu$

both u.e.

(b) $\quad Cov(X_t, X_{t+h}) = \begin{cases} (2+\phi^2)\sigma^2, & h=0 \\ (1+\phi)\sigma^2, & h=\pm 1 \\ \phi\sigma^2, & h=\pm 2 \\ 0, & o/w \end{cases}$

$$V(\delta_1) = \frac{1}{9}\left(4V(X_1) + V(X_3) + 4\,Cov(X_1, X_3)\right)$$
$$= \frac{1}{9}\left(4\gamma_0 + \gamma_0 + 4\gamma_2\right) = \frac{1}{9}\left(5\gamma_0 + 4\gamma_2\right)$$

$$V\,\delta_2 = \frac{1}{9}\left(3\gamma_0 + 4\gamma_1 + 2\gamma_2\right)$$

$$V\,\delta_1 - V\,\delta_2 = \frac{1}{9}\left(2\gamma_0 - 4\gamma_1 + 2\gamma_2\right)$$

i.e.

$$V\delta_1 - V\delta_2 = \frac{2\sigma^2}{9}\left((2+\phi^2) - 2(1+\phi) + \phi\right)$$

$$= \frac{2\sigma^2}{9}\left(\phi^2 - \phi\right)$$

Let $g(\phi) = \phi^2 - \phi$

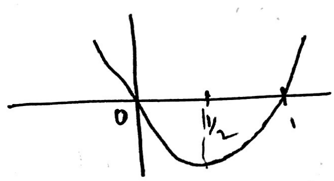
$g''(\phi) = 2 > 0 \quad \forall \phi$

$g'(\phi) = 0 \Rightarrow \phi = \frac{1}{2}$

$g(\frac{1}{2}) = -\frac{1}{4} < 0$

$g(\phi) = 0 \Rightarrow \phi = 0, 1$

g is cnvx with min at $\phi = \frac{1}{2}$ & $g(\phi) = 0$ at 0 & 1



$\Rightarrow g(\phi) > 0 \quad \forall \phi > 1$ & $\phi < 0$

i.e. $V(\delta_1) > V(\delta_2)$ if $\phi > 1$ & $\phi < 0$

(c) $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$

$\underset{\sim}{\alpha} \in \mathbb{R}^n \ (\underset{\sim}{\alpha} \neq \underset{\sim}{0})$

$\underset{\sim}{\alpha}' \underset{\sim}{X}$ = lin comb of indep $N_1 \sim N_1 \ \forall \underset{\sim}{\alpha} \in \mathbb{R}^n \ (\underset{\sim}{\alpha} \neq \underset{\sim}{0})$

$\Rightarrow \underset{\sim}{X} \sim N_n$

$$E(\underset{\sim}{X}) = \mu \underset{\sim}{1}_n$$

$$\mathrm{Cov}(\underset{\sim}{X}) = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & & & 0 \\ \gamma_1 & \ddots & \ddots & \ddots & & \\ \gamma_2 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \gamma_2 \\ & & \ddots & \ddots & \ddots & \gamma_1 \\ 0 & & & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} = \Sigma$$

$$\underset{\sim}{X} \sim N_n\left(\mu \underset{\sim}{1}_n , \Sigma\right)$$

$\{X_t\}$ is MA(2) – cov stationary

$\{X_t\}$ is Gaussian.

Hence $\{X_t\}$ is strict stationary

(12). $X_t = \phi X_{t-1} + \epsilon_t$; $|\phi|<1$ $\epsilon_t \sim WN(0,\sigma^2)$

$$Y_t = X_t - \frac{1}{\phi} X_{t-1}$$

$$EY_t = 0 \quad \forall t$$

$$\mathrm{Cov}(Y_{t+h}, Y_t) = E\left(X_{t+h} - \frac{1}{\phi}X_{t+h-1}\right)\left(X_t - \frac{1}{\phi}X_{t-1}\right)$$

$$= \gamma_X(h) - \frac{1}{\phi}\gamma_X(h+1) - \frac{1}{\phi}\gamma_X(h-1) + \frac{1}{\phi^2}\gamma_X(h)$$

For $h \geq 1$ $\quad = \frac{\sigma^2}{1-\phi^2}\left(\phi^h - \phi^h - \phi^{h-2} + \phi^{h-2}\right) = 0$

Sly $\forall h \leq -1 \quad = 0$

i.e. $\gamma_y(h) = 0 \quad \forall\ h \neq 0$

$$\gamma_y(0) = \gamma_x(0)\left(1+\frac{1}{\phi^2}\right) - \frac{2}{\phi}\gamma_x(1)$$

$$Y_t \sim WN\left(0, \gamma_y(0)\right)$$

(13)

$T_1 = \frac{X_4+X_5}{2}$ $\quad V(T_1) = \frac{1}{4}\left(2\gamma_0 + 2\gamma_1\right) = \frac{1}{2}\left(1+\phi^2+\phi\right)$

$$V(T_2) = \frac{1}{9}\left(3\gamma_0 + 4\gamma_1\right)$$

$T_2 = \frac{X_3+X_4+X_5}{3}$ $\quad = \frac{1}{9}\left(3(1+\phi^2)+4\phi\right)$

$$V(T_1) - V(T_2) = \frac{9(1+\phi^2+\phi) - 2(3+3\phi^2+4\phi)}{18}$$

$$= \frac{1}{18}\left(3+3\phi^2+\phi\right)$$

$$g(\phi) = 3\phi^2+\phi+3$$

$$g''(\phi) = 6 > 0 \quad \forall\ \phi$$

$$g'(\phi) = 0 \quad \Rightarrow \phi = -\frac{1}{6}$$

$$g\left(-\frac{1}{6}\right) = 3\,\frac{1}{6^2} - \frac{1}{6} + 3 > 0$$

$$g(\phi) > 0 \quad \forall\ \phi$$

$$\Rightarrow V(T_1) > V(T_2) \quad \forall\ \phi$$

(14) $Y_t = \sum_{j=0}^{\infty} \left(-\frac{1}{\theta}\right)^j X_{t-j}$; $|\theta| > 1$

$X_t = \epsilon_t + \theta \epsilon_{t-1}$; $\epsilon_t \sim WN(0, \sigma^2)$

$\rightarrow X_t = \theta(B)\, \epsilon_t$; $\theta(B) = 1 + \theta B$

$\left(1 + \frac{1}{\theta} B\right) Y_t = X_t$

ACGF of l.h.s.

$$\left(1 + \frac{z}{\theta}\right)\left(1 + \frac{z^{-1}}{\theta}\right) g_y(z)$$

ACGF of r.h.s.

$$\sigma^2\, \theta(z)\, \theta(z^{-1})$$

i.e. $\left(1 + \frac{z}{\theta}\right)\left(1 + \frac{z^{-1}}{\theta}\right) g_y(z) = \sigma^2 (1 + \theta z)(1 + \theta z^{-1})$.

$$g_y(z) = \sigma^2 \frac{(1 + \theta z)\left(1 + \theta \frac{1}{z}\right)}{\left(1 + \frac{z}{\theta}\right)\left(1 + \frac{1}{\theta z}\right)}$$

i.e. $g_y(z) = \theta^2 \sigma^2 \leftarrow$ Const ACGF

$\Rightarrow Y_t \sim WN(0, \theta^2 \sigma^2)$

<u>Note</u>: The problem can be solved without using ACGF but that would be involving laborious calculations ....

(15) $\phi(B)\,Y_t = \epsilon_t \quad ; \quad \epsilon_t \sim WN(0,\sigma^2)$

$\phi(B) = 1 - \phi B - \phi^2 B^2$

Roots of $\phi(z)=0 \qquad z_1 = \dfrac{-1-\sqrt{5}}{2\phi} \; ; \; z_2 = \dfrac{-1+\sqrt{5}}{2\phi}$

$|z_1| > 1 \iff |\phi| < \dfrac{\sqrt{5}+1}{2}$

$|z_2| > 1 \iff |\phi| < \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow \{x_t\}$ is stationary for $|\phi| < \dfrac{\sqrt{5}-1}{2}$

(16) Similar to (15); such problem can be solved using triangular stationarity region also

Note that for

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

conditions for stationarity

$$1-\phi_1-\phi_2 > 0, \quad 1-\phi_2+\phi_1 > 0 \;\&\; |\phi_2| < 1. \quad (*)$$

For $\phi_1 = 1$ & $\phi_2 = c$ (*) is

$-c > 0$

$2 > c$

$|c| < 1$

$\Rightarrow -1 < c < 0$ is the region of stationarity

(17) Yule-Walker eq^n

$$\gamma_h = \phi_1 \gamma_{h-1} + \phi_2 \gamma_{h-2} \quad \forall h > 0$$

$h=1$; $\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$

$h=2$; $\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$

$$\Rightarrow \frac{\gamma_1}{\gamma_0} = \phi_1 + \phi_2 \frac{\gamma_1}{\gamma_0}$$

$$\& \quad \frac{\gamma_2}{\gamma_{\cancel{0}1}} = \phi_1 + \phi_2 \frac{\gamma_0}{\gamma_1}$$

$$\Rightarrow \left.\begin{aligned} \frac{1}{2} &= \phi_1 + \frac{1}{2}\phi_2 \\ \frac{1}{4} &= \phi_1 + 2\phi_2 \end{aligned}\right\} \text{sol}^n \quad \begin{aligned} \phi_2 &= -\frac{1}{6} \\ \phi_1 &= \frac{7}{12} \end{aligned}$$

Also for $h=3$, Y-W eq^n is

$$\gamma_3 = \phi_1 \gamma_2 + \phi_2 \gamma_1$$

i.e. $\dfrac{\gamma_3}{\gamma_2} = \phi_1 + \phi_2 \, \gamma_1/\gamma_2$

$$= \frac{7}{12} + \left(-\frac{1}{6}\right) 4 = \ldots$$

(18) Yule-Walker eq^n for AR(1)

$$\gamma_h = \phi \gamma_{h-1}$$

l.h.s $= \gamma_{-h}$ & r.h.s $= \phi \gamma_{-(h-1)} = \phi \gamma_{-h+1}$

i.e. $\gamma_{-h} = \phi \gamma_{-h+1}$. $\longrightarrow$

(19)

$$(1-\phi_1^{(1)}B)X_t = (1+\theta_1^{(1)}B+\theta_2^{(1)}B^2+\theta_3^{(1)}B^3)\epsilon_t$$

& $$(1-\phi_1^{(2)}B)Y_t = (1+\theta_1^{(2)}B+\theta_2^{(2)}B^2)\delta_t$$

$$\left.\begin{array}{l}\epsilon_t \sim WN(0,\sigma^2)\\ \delta_t \sim WN(0,\sigma^2)\end{array}\right\rangle \text{indep}$$

$\{X_t\}$ & $\{Y_t\}$ are covariance stationary and independent

(a)

$$Z_t = (1-\phi_1^{(1)}B)(1-\phi_1^{(2)}B)X_t$$

$$Z_t = (1-\phi_1^{(2)}B)\underline{(1-\phi_1^{(1)}B)X_t}$$

i.e. $$Z_t = (1-\phi_1^{(2)}B)(1+\theta_1^{(1)}B+\theta_2^{(1)}B^2+\theta_3^{(1)}B^3)\epsilon_t.$$

i.e. $$Z_t = (1+\alpha_1B+\alpha_2B^2+\alpha_3B^3+\alpha_4B^4)\epsilon_t$$

$\Rightarrow$ $Z_t \sim MA(4)$ covariance stationary

(b)

$$U_t = (1-\phi_1^{(1)}B)(1-\phi_1^{(2)}B)(X_t+Y_t)$$

$$U_t = (1-\phi_1^{(1)}B)(1-\phi_1^{(2)}B)X_t + (1-\phi_1^{(1)}B)(1-\phi_1^{(2)}B)Y_t$$

i.e. $$U_t = (1-\phi_1^{(2)}B)\underline{(1-\phi_1^{(1)}B)X_t} + (1-\phi_1^{(1)}B)\underline{(1-\phi_1^{(2)}B)Y_t}$$

i.e. $$U_t = (1-\phi_1^{(2)}B)(1+\theta_1^{(1)}B+\theta_2^{(1)}B^2+\theta_3^{(1)}B^3)\epsilon_t + (1-\phi_1^{(1)}B)(1+\theta_1^{(2)}B+\theta_2^{(2)}B^2)\delta_t.$$

$$= A_t + B_t \text{, say}$$

$$A_t \sim MA(4) \Rightarrow \gamma_A(h) = 0 \quad \forall\ |h| \geq 5$$

& $B_t \sim MA(3) \Rightarrow \gamma_B(h) = 0 \quad \forall\ |h| \geq 4$

Further, $\{A_t\}$ & $\{B_t\}$ are independent as $\{\epsilon_t\}$ & $\{\delta_t\}$ are indep

$$\Rightarrow \gamma_U(h) = \gamma_A(h) + \gamma_B(h)$$

$$\Rightarrow \gamma_U(h) = 0 \quad \forall\ |h| \geq 5$$

(20)

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \ ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\{\psi_j\} \ni \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\gamma_h = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

$$|\gamma_h| \leq \sigma^2 \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$$

& $\displaystyle\sum_{h=0}^{\infty} |\gamma_h| \leq \sigma^2 \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$

Note that

$$\sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$$

$$= \sum_{h=0}^{\infty} \left(|\psi_0| |\psi_h| + |\psi_1| |\psi_{h+1}| + |\psi_2| |\psi_{h+2}| + \cdots \right)$$

$$= |\psi_0| \sum_0^{\infty} |\psi_h| + |\psi_1| \sum_0^{\infty} |\psi_{h+1}| + |\psi_2| \sum_0^{\infty} |\psi_{h+2}| + \cdots$$

$$\leq |\psi_0| \left(\sum_0^{\infty} |\psi_h| \right) + |\psi_1| \left(\sum_0^{\infty} |\psi_h| \right) + |\psi_2| \left(\sum_0^{\infty} |\psi_h| \right) + \cdots$$

$= \left(\sum_0^{\infty} |\psi_h| \right)^2 < \infty$ as $\{\psi_j\}$ is absolutely summable

$\Rightarrow \sum_0^{\infty} |\gamma_h| < \infty \Rightarrow \{\gamma_h\}$ is also absolutely summable