(X,d) metric space

- Convergence of sequences in (X,d)
- Notion of Cauchy seq.
- (X,d) is a complete m.s. if every Cauchy seq. cgs. in it
- $\mathbb{R}^d$ is complete
- l_p, $d(x,y) = \|x-y\|_p = \left(\sum_{k=1}^{\infty} |x(k)-y(k)|^p\right)^{1/p}$ $\quad 1 \le p < \infty$

$$\sup_k |x(k)-y(k)|$$

is a m.s.

(Minkowski Inequality)

- $x \in l_p$, $y \in l_{p'}$ $\quad \frac{1}{p} + \frac{1}{p'} = 1$

then, $xy \in l_1$ (Holder's Inequality)

THM: l_p is a complete m.s.

Pf:, Let $\{x_n\}$ be a Cauchy seq. in l_p.

$x_n = (x(1), x(2), \ldots) = \{x_n(k)\}$

i.e. $\varepsilon > 0$ $\exists$ N_0 st.

(1) $d(x_n, x_m) = \|x_n - x_m\|_p = \left(\sum |x_n(k) - x_m(k)|^p\right)^{1/p} < \epsilon$ $\forall$ $n,m \ge N$

Fix n.

$|x_n(k) - x_m(k)| \cancel{\le \|x_n - x_m\|}$

$= \left(|x_n(k) - x_m(k)|^p\right)^{1/p} \le \left(\sum_k |x_n(k) - x_m(k)|^p\right)^{1/p}$

$= \|x_n - x_m\| < \epsilon$ $\forall$ $n,m \ge N$

For each k, $\{x_n(k)\}$ is Cauchy in $\mathbb{R}$

$x_n(k) \longrightarrow x_k\cancel{(k)}$ as $n \longrightarrow \infty$

$\epsilon > 0$, $\exists$ N_k st.

$|x_n(k) - x_k| < \epsilon$ $\forall$ $n \ge N_k$

Date __/__/____

Define $x = \{x(k)\}$; $x(k) = x_k$

To show, $x \in l_p$, $\|x_n - x\|_p \to 0$ as $n \to \infty$.

$$\sum_{k=1}^{N} |x(k)|^p = \sum_{k=1}^{N} \lim_{n\to\infty} |x_n(k)|^p = \lim_{n\to\infty} \sum_{k=1}^{N} |x_n(k)|^p$$

$$\leq \lim_{n\to\infty} \sum_{k=1}^{\infty} |x_n(k)|^p$$

$$= \lim_{n\to\infty} \|x_n\|_p^p < \infty$$

$\Rightarrow x \in l_p$

Now, $\cancel{\sum_{k=1}^{N} |x_n(k) - x(k)|^p < \epsilon}$ ~~want N_1 s.t $n > N_1$~~

$$\sum_{k=1}^{N} |x_{N_0}(k) - x(k)|^p$$

$$= \sum_{k=1}^{N} \lim_{m} |x_{N_0}(k) - x_m(k)|^p = \lim_{m\to\infty} \sum_{k=1}^{N} |x_{N_0}(k) - x_m(k)|$$

$$\leq \lim_{m\to\infty} \|x_{N_0} - x_m\|_p^p < \epsilon$$

From prev', $\|x_{N_0} - x_m\|^p < \epsilon \quad \forall \; m \geq N_0$

We can extend this for $\forall \; n \geq N_0$

$$\sum_{k=1}^{N} |x_n(k) - x(k)|^p < \epsilon \quad \forall \; n \geq N_0$$

$$\Rightarrow \lim_{N\to\infty} \sum_{k=1}^{N} |x_n(k) - x(k)|^p < \epsilon \quad \forall \; n \geq N_0$$

$$\Rightarrow \|x_n - x\|_p < \epsilon$$

Ex: l_∞ is a complete m.s.

$C[0,1]$ is a m.s. ~~dp~~ d_p metric

$$d_p(f,g) = \begin{cases} \left(\int_0^1 |f(x)-g(x)|^p \, dx\right)^{1/p} & 1 \le p < \infty \\ \sup |f(x)-g(x)| & p = \infty \end{cases}$$

$f_n(x) = x^n$ $\quad \{f_n\}$ $\quad 0 \le x \le 1$

$n > m$ $\quad n-m > 0$

$$\int_0^1 |x^n - x^m| \, dx = \int_0^1 x^m |x^{n-m} - 1| \, dx \le \int_0^1 \frac{x^{m+1}}{m+1}\Big|_0^1 = \frac{1}{m+1} \to 0$$

$$f_n(x) \to f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

But, f isn't cts.

$$\int_0^1 |f_n(x) - f(x)| \, dx = \int_0^1 |f_n(x)| \, dx \to 0$$

d_∞ $\quad \|f_n - f_m\|_\infty \to 0$

$\sup |f_n(x) - f_m(x)|$

$$\sup_x |f_n(x) - f(x)| = 1 \not\to 0$$

Ex $\|f_n - f\|_1 \to 0$. Does this $\Rightarrow$ $f_n(x) \to f(x)$??

Neighbourhood:

$f: \mathbb{R} \to \mathbb{R}$ cts.
x if $\epsilon > 0$ $\exists \delta > 0$
s.t. $y \in (x-\delta, x+\delta) \Rightarrow |f(x)-f(y)| < \epsilon$

A set $U \subseteq \mathbb{R}$ is s.t. be open if $x \in U$ $\exists$ $\epsilon > 0$ s.t.
$(x - \epsilon, x + \epsilon) \subset U$

Ex
(i) $\mathbb{R}$ is open
(ii) (a,b), $a, b \in \mathbb{R}$
(iii) $U = \bigcup_{n=1}^{\infty} \left(n - \frac{1}{n}, n + \frac{1}{n}\right)$ $\quad$ Take $x \in U \Rightarrow$ $x \in \left(n_0 - \frac{1}{n_0}, n_0 + \frac{1}{n_0}\right)$
(iv) $U = \bigcup^{\infty} (a_n, b_n)$

(v) $U = (0,2) \cap (1,3)$

$(a,b) \cap (c,d)$ is open
if empty then true
Suppose $x \in (a,b) \cap (c,d)$
$\exists\ \delta_1$ s.t. $(x-\delta_1, x+\delta_1) \subset (a,b)$
$\exists\ \delta_2$ s.t. $(x-\delta_2, x+\delta_2) \subset (c,d)$
$\min(\delta_1, \delta_2) = \delta$

$\{0\} \rightarrow$ Not open

Let $I_n = \left(0-\frac{1}{n}, \frac{1}{n}\right)$
$\bigcap_{n=1}^{\infty} I_n = \{0\}$ not open

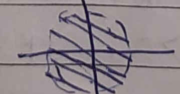
THM: 1. Arbitrary union of open sets is open
2. Finite intersection of open sets is open

04/02/22

U is open in $\mathbb{R}$ if $\forall\ x \in U\ \exists\ \epsilon > 0$ s.t.
$(x-\epsilon, x+\epsilon) \subset U$

(X,d), $x \in X$: open ball of radius $\epsilon > 0$ and centred at x
$B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$

$\mathbb{R}^2$ $d(x,y)$ usual metric i.e. $d(x,y) = \left(\sum_{i=1}^{n} |x_i -\right.$

$B_1(0)$ 

$(C[0,1], \|\cdot\|_\infty)$ $B_\epsilon(f) = \{g \in C[0,1] : \sup_x$

Claim:- For $r > 0$ $B_r(z)$ is open

Let $y \in B_r(x)$, we must find $\epsilon > 0$ s.t $B_\epsilon(y) \subset B_r(x)$

$y \in (x-r, x+r)$

Clearly, $d(x,y) < r$

So, $\exists$ a $\epsilon' > 0$ s.t $\epsilon' + d(x,y) < r$

Consider, $z \in B_{\epsilon'}(y)$

$$d(z,x) \leq d(z,y) + d(y,x) < \epsilon' + d(y,x) < r$$

$\therefore B_{\epsilon'}(y) \subset B_r(x)$

THM. Let (X,d) be a m.s. and $\{U_i\}$ be a family of open sets, then $\bigcup_{i \in I} U_i$ is open and finite intersecⁿ is open

Pf: Let $x \in \bigcup_i U_i$

$\Rightarrow x \in U_{i_0}$ for some i_0

$\Rightarrow \exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset U_{i_0}$ (U_{i_0} is open)

$\Rightarrow B_\epsilon(x) \subset \bigcup_i U_i$

Let $U_1, U_2, \ldots, U_n$ be open.

Either, $\bigcap_{i=1}^{n} U_i$ is empty or non-empty.

$\Downarrow$ open

Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow x \in U_i \ \forall i$

$\exists \epsilon_i > 0$ s.t $B_{\epsilon_i}(x) \subset U_i$

Take $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$

$\Rightarrow B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subset U_i \ \forall i$

$\Rightarrow B_\epsilon(x) \subset \bigcap_{i=1}^{n} U_i$

Date __/__/__

Closed Set :

$A \subseteq X$ is closed if A^c is open.

Ex : (i) $\mathbb{R}, \phi$

(ii) $\mathbb{Z} \subseteq \mathbb{R}$ (usual metric)

$Z^c = (0,1) \cup (1,2) \cup \ldots$

or $\bigcup_{n \in Z} (n, n+1)$ $\Rightarrow$ Z is closed

(iii) $Q \subseteq R$ (usual metric)

$Q^c = R/Q$

Q^c is not open $\Rightarrow$ Q isn't closed

Q is not open as well

(iv) $[0,1)$ not closed

$A = (-\infty, 0) \cup [1, \infty)$

$1 \in A$, $(1-\epsilon, 1+\epsilon) \not\subset A$

(v) $[a,b]$ closed

(vi) $[a,b] \cap [c,d]$ closed

(vii) In discrete metric $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

Open ball in discrete metric

$B_{\frac{1}{2}}(x) = \{ y : d(x,y) < \frac{1}{2} \} = \{x\}$

Every set in a m.s. is open as well as closed.

(viii) In $\mathbb{R}$ with usual metric $\{x\}$ is closed.

THM. 1. Arbitrary intersec^n of closed sets is closed

2. Finite union of closed sets is closed

Date __/__/__

Pf:

Let $\{F_i\}$ closed set.

$F = \bigcap_i F_i$, $F^c = \bigcup_i F_i^c$ open

$\Rightarrow F$ is closed

$F = \bigcup_{i=1}^{n} F_i \Rightarrow F^c = \bigcap_{i=1}^{n} F_i^c$ open $\Rightarrow F$ is closed

$[0,1) \rightarrow$ Neither open nor closed

Does $\exists A \subsetneq \mathbb{R}$ s.t. A is both open and closed.

$(A \neq \{\phi\})$

Interior Pt.:

$x \in A$ $(A \subseteq \mathbb{R})$ is said to be an int. pt. if $\exists \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \subseteq A$

Exterior Pt.:

~~$x \in A$~~ $x \in \mathbb{R}$ is said to be an ext. pt. of A if $\exists \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \subset A^c$

Boundary Pt.:

$x \in \mathbb{R}$ is s.t. be bound. pt. of A if for every $\epsilon > 0$

$(x-\epsilon, x+\epsilon) \cap A \neq \phi$ and $(x-\epsilon, x+\epsilon) \cap A^c \neq \phi$

(Neither interior nor exterior)

~~For~~ A^o = set of all int. pts. of A

A_{ext} = " " ext. "

∂A = " " bd. "

$\mathbb{Q} \rightarrow \mathbb{Q}^o = \phi$

$\mathbb{Z} = \phi$

Let A be an open subset of $\mathbb{R}$. $A \subseteq A^o$,

$A^o \subseteq A \Rightarrow A = A^o$

Page No.

Date __/__/__

Propos^n A is open subset of $\mathbb{R} \iff A = A^o$

Pf:

($\Leftarrow$) $A = A^o$

$x \in A = A^o$ $\exists\ \epsilon > 0$

$(x-\epsilon, x+\epsilon) \subseteq A$

A open $\iff A = A^o$

Clearly, $A \supseteq A^o$

Does $\exists$ U open s.t. $A \supseteq U \supset A^o$

Propos^n ~~Pf~~ A^o is largest open set contained in A

Pf: $x \in A^o$

$U \subset A$ is open

$x \in U \Rightarrow B_\epsilon(x) \subset U$

$\Rightarrow x \in A^o \Rightarrow U \subseteq A^o$

DEF^N $\bar{A}$ to be the smallest closed set containing A.

$x \in \bar{A} \iff B_\epsilon(x) \cap A \neq \phi \quad \forall\ \epsilon > 0$

Claim A^o is open or $x \in A^o \Rightarrow B_\epsilon(x) \subseteq A^o$

$x \in A^o \Rightarrow B_\epsilon(x) \subseteq A^o$

If $y \in B_\epsilon(x)$ $\exists$ $\delta > 0$ $B_\delta(y) \subset A \Rightarrow y \in A^o$

$\delta = \min(y-(x-\epsilon), (x+\epsilon)-y)$

~~$(y-\delta)$~~ $(y-\delta, y+\delta) \subseteq (x-\epsilon, x+\epsilon) \subseteq A$

$\Rightarrow y \in A^o$

THM. Let (X, d) be a m.s. $F \subset X$. Then, following are eqvt.

i) F is closed

ii) If $B_\epsilon(x) \cap F \neq \phi$ for every $\epsilon > 0$ then $x \in F$

iii) If a seq. $\{x_n\}$ in F cgs. to some $x \in X$, then $x \in F$

Pf: $(i) \Rightarrow (ii)$

F^c is open

(ii) ~~$\Rightarrow (iii)$~~

Let x be a pt. st. $B_\epsilon(x) \cap F \neq \phi$

To show $x \in F$. If not, ~~$x \notin$~~ $x \in F^c$

$\Rightarrow \exists\ \epsilon > 0$ st. $B_\epsilon(x) \subset F^c \Rightarrow \Leftarrow$

$(ii) \Rightarrow (i)$ Want to show F^c is open

Let $x \in F^c$, want to show $\exists \epsilon > 0$ st $B_\epsilon(x) \subset F^c$

Suppose not true.

Then, $\forall\ \epsilon > 0$, $B_\epsilon(x) \cap F \neq \phi \Rightarrow x \in F \Rightarrow \Leftarrow$

$(iii) \Rightarrow (ii)$

Let $x_n \to x$, For every $\epsilon > 0$, $\exists\ N_0$ st.

$d(x_n, x)$ ~~$|x_n - x|$~~ $< \epsilon\ \forall\ n \geq N_0$

$\Rightarrow B_\epsilon(x) \cap F \neq \phi \Rightarrow x \in F$

$(ii) \Rightarrow (iii)$

Given $B_\epsilon(x) \cap F \neq \phi$ For every $\epsilon > 0$

$B_{\frac{1}{n}}(x) \cap F \neq \phi$. Take $x_n \in B_{\frac{1}{n}}(x) \cap F$

$d(x_n, x) < \frac{1}{n} \Rightarrow x_n \to x$

Pt. of closure:

$x \in$ ~~X~~ is s.t. be pt. of closure if $\forall\ \epsilon > 0$, $B_\epsilon(x) \cap F \neq \phi$

Date ___/___/___

Limit Pt.

$x \in X$ is s.t. be limit pt. of F if $\exists$ seq. $\{x_n\} \subseteq F$ s.t. $x_n \longrightarrow x$

DEF^N $\bar{A}$ is the smallest closed subset containing A

THM. $x \in \bar{A} \Leftrightarrow x$ is a pt. of closure of

Pf: Ex

Ex - Cantor Set:

$I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ $I_2^1 = \left(\frac{1}{9}, \frac{2}{9}\right)$ $I_2^2 = \left(\frac{7}{9}, \frac{8}{9}\right)$

$I_3^1 = \left(\frac{1}{27}, \frac{2}{27}\right)$, $I_3^2 = \left(\frac{7}{27}, \frac{8}{27}\right)$ $I_3^4 = \left(\frac{25}{27}, \frac{26}{27}\right)$

$C = [0,1] \setminus \left(\bigcup_{n,i} I_n^i\right)$ $F \setminus A = F \cap A^c$

THM. i) Arbitrary intersec^n of closed sets is closed
ii) Finite union of closed sets is closed

$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0,1] \rightarrow$ Not closed

THM. Let $A \subseteq \mathbb{R}$. If A is both closed and open then $A = \mathbb{R}$ or φ.

Pf: Suppose $A \subsetneq \mathbb{R}$, $a \in A$ and $b \in A^c$
Assume, $a < b$

$S = \{x : [a, x] \subseteq A\}$
S is bdd. as $x < b$ $\forall x \in S$
Let α be the sup S
If $\alpha \in A$, $\alpha < \alpha + \epsilon' \in A$ (A is open) $\epsilon' > 0$
$[a, \alpha] \subseteq A \Rightarrow [a, \alpha + \epsilon'] \subset A$
$\Rightarrow\Leftarrow$

Date __/__/__

$\alpha \in A^c$

If $\alpha \in A^c$, A^c is also open

$\exists \epsilon > 0$ s.t. $(\alpha - \epsilon, \alpha + \epsilon) \subseteq A^c$

$\Rightarrow\Leftarrow$ ($\because \alpha - \epsilon \in A^c$, $\alpha \in A$)

So, $\alpha \notin A^c$ and $\alpha \notin A$

$\Rightarrow$ If A is closed and open, then A is $\mathbb{R}$ or ϕ

$\bar{\mathbb{Q}} = \mathbb{R}$

DEF^N $A \subseteq X$ is said to be dense in X if $\bar{A} = X$

Ex: $\mathbb{R} \setminus \mathbb{Q}$

$\mathbb{Q} \subseteq Y \subseteq \mathbb{R}$ $\qquad A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

$\mathbb{R} = \bar{\mathbb{Q}} \subseteq \bar{Y} \subseteq \bar{\mathbb{R}} = \mathbb{R}$

$\Rightarrow \bar{Y} = \mathbb{R} \Rightarrow Y$ is dense.

$Y = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$ is dense in $\mathbb{R}$

$Y_0 = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$?

Prop: A set A is dense in $\mathbb{R}$ iff. $A \cap I \neq \phi$ for any open interval I of $\mathbb{R}$

Pf: A is dense $\Rightarrow \bar{A} = \mathbb{R}$

$x \in \mathbb{R} \Rightarrow B_\epsilon(x) \cap A \neq \phi \quad \forall \epsilon > 0$

$\Rightarrow A \cap I \neq \phi$

$(\Leftarrow)$ $x \in \mathbb{R}$, $B_\epsilon(x) \cap A \neq \phi$ for every $\epsilon > 0$

$\Rightarrow x \in \bar{A}$

$\mathbb{R} \subseteq \bar{A}$ and $\bar{A} \subseteq \mathbb{R}$.

$\Rightarrow \mathbb{R} = \bar{A}$

Enough to show $Y_0 = \{m + n\sqrt{2}\}$ is dense in $(0,1)$

i.e. take any $I \subseteq (0,1)$ if $I \cap Y_0 \neq \phi$, then we are done.

Date __/__/____

~~Let~~ $n \in \mathbb{Z}$ $\quad n\sqrt{2} - [n\sqrt{2}]_{GIF} \in (0,1)$

if $n_1 \neq n_2$

$$n_1\sqrt{2} - [n_1\sqrt{2}] \neq n_2\sqrt{2} - [n_2\sqrt{2}]_{GIF}$$

$S = \{ n\sqrt{2} - [n\sqrt{2}] : n \in \mathbb{Z} \} \subseteq (0,1)$ and ∞

Take any interval $I \subset (0,1)$. Let $l(I) = \epsilon$

ϵ Divide $(0,1)$ into finitely many subintervals (of the $(a,b]$) of length $< \epsilon$

$(0,\epsilon] \cup (\epsilon, 2\epsilon] \cup (2\epsilon, 3\epsilon] \cup \dots \cup (N\epsilon, 1)$ ~~$\delta<\epsilon$~~

Atleast One of these subinterval contains more than 1 pt. of S

Suppose, if we show $(0,\epsilon) \cap S \neq \phi$

$ks > \epsilon \quad (s \in (0,\epsilon))$

$k_0 = \min\{k : ks > \epsilon\}$

$(k_0 - 1)s < \epsilon \Rightarrow k_0 s < \epsilon + s < 2\epsilon$

$\Rightarrow \epsilon < k_0 s < 2\epsilon$

$k_0 s = k_0(m + n\sqrt{2}) = k_0 m + k_0 n\sqrt{2} \in S$

~~All you need +~~

s_2, s_1 s.t $s_2 > s_1$

$s_1 = m_1 + n_1\sqrt{2}, \quad s_2 = m_2 + n_2\sqrt{2}$

$0 < s_2 - s_1 = (m_2 - m_1) + (n_2 - n_1)\sqrt{2} \in S$

$0 < s_2 - s_1 < \epsilon$

Date 08/02/22

$A \subseteq \mathbb{R} \longrightarrow$ Open — $\bigcup A_i$ Open, $\bigcap_{i=1}^{n} A_i$ open

Closed $\rightarrow A^c$ is open

$\Leftrightarrow \epsilon > 0 \quad B_\epsilon(x) \cap A \neq \varphi$

$\Leftrightarrow$ A contains all limit pts.

$\Leftrightarrow A = \bar{A}$

Let $x, y \in \mathbb{R}$ s.t. $x \neq y$

Let $|x-y| = \epsilon$

then $(x-\frac{\epsilon}{4}, x+\frac{\epsilon}{4}) \cap (y-\frac{\epsilon}{4}, y+\frac{\epsilon}{4}) = \varphi$

$d(x,y) = \epsilon$. Consider $B_{\epsilon/4}(x)$ and $B_{\epsilon/4}(y)$

They are disjoint

$(\mathbb{R}, d)$

$\mathbb{R} \setminus \{x\}$ is open, dense

$\bigcap \mathbb{R} \setminus \{x\} \neq \varphi$

$(\mathbb{Q}, d)$

$\mathbb{Q} \setminus \{x\}$ is open, dense

Take an enumer^n of $\mathbb{Q}$

say $\{r_n\}$

$\bigcap_{n=1}^{\infty} \mathbb{Q} / \{r_n\} = \varphi$

<u>THM</u>: Let $\{U_i\}_{i=1}^{\infty}$ be a collec^n of dense, open subsets of $\mathbb{R}$. Then, $\bigcap_{i=1}^{\infty} U_i$ is dense in $\mathbb{R}$

Pf: Let I be an open interval in $\mathbb{R}$

$I \cap U_1 \neq \varphi$, $I \cap U_1$ is open

Let $J_1 = [a_1, b_1] \subset I \cap U_1$

($\because U_2$ is dense) $J_1 \cap U_2$ is non-empty and contain an interval

$J_2 = [a_2, b_2] \subset J_1 \cap U_2$

$J_2 \subset J_1$

We get $J_1 \supset J_2 \supset J_3 \ldots$

Date __/__/__ (Nested interval)

By Cantor's Intersecⁿ Thm:

$\cap J_i \neq \varphi$

Let $x \in \cap J_i \ \forall i$

$x \in \cap U_i$, $x \in I$

$I \cap (\bigcap_{i=1}^{\infty} U_i) \neq \varphi$. So, $\bigcap_{i=1}^{\infty} U_i$ is dense.

If $R = \bigcup_{i=1}^{\infty} F_i$, F_i's are closed.

Then, atleast one F_i has non-empty interior

Suppose $F_i^{o} = \varphi \ \forall i$

$\Rightarrow$ there is no open int. I s.t. $I \subseteq F_i$

$\Rightarrow$ If I is open, then $I \cap F_i^c \neq \varphi$

$\Rightarrow F_i^c$ is dense.

F_i^c open and dense.

$\varphi \neq \bigcap_{i=1}^{\infty} F_i^c = R^c = \varphi \quad \Rightarrow \Leftarrow$

F closed and "empty int."

We say that $A \subseteq R$ is no-where dense if $\bar{A}^{o} = \varphi$

$A = N$

$A = C$ a Cantor set.

$U_n \to$ open

$\bigcap_{n=1}^{\infty} U_n$

G_δ set: $A \subseteq R$ is G_δ if $A = \bigcap_{n=1}^{\infty} U_n$, U_n's op

Ex: If $F \subseteq R$ is closed, then is it true that F is G_δ

F_σ - $A = \bigcup_{n=1}^{\infty} F_n$, F_n closed

Date __/__/__

Ex i) If O open, Does this imply $O \in F_\sigma$
ii) Can we say that if $A \subseteq \mathbb{R}$ then either A is G_δ or A is F_σ.

BWP: A subset $A \subseteq \mathbb{R}$ is said to have BWP if every seq. in A has a cgt. subseq. which cgs. in A

i.e. $\{x_n\} \subseteq A$ $\exists$ $\{x_{n_k}\}$ s.t. $x_{n_k} \to x$, $x \in A$.

THM: ~~If~~ A has BWP, ~~then~~ iff. A is closed and bdd.

Pf: ($\Leftarrow$) A closed, bdd.

$\{x_n\} \subseteq A$, $\{x_n\}$ bdd. $\Rightarrow \{x_{n_k}\} \to x \in A$

$\Rightarrow$ A has BWP

($\Rightarrow$) A has BWP

(i) A is bdd.

If not, $\forall M \in \mathbb{N}$, $\exists$ x_m s.t. $|x_m| > M$

$\{x_n\} \subseteq A$. $\exists$ a cgt. subs. $\{x_{m_k}\}$

$|x_{m_k}| \le M_0$ $\forall m_k$

Choose $M > M_0$

$M < m_k \le |x_{m_k}| \le M_0 < M$ $\Rightarrow \Leftarrow$

(ii) $x \in \bar{A}$ (Want to show, $x \in A$)

$\Downarrow$

$\exists \{x_n\}$ in A s.t. $x_n \longrightarrow x$

$\exists \{x_{n_k}\}$ cgs. in A.

$x_{n_k} \xrightarrow{} x \in A$

So, A is closed.

Date 09/02/22

Compact Set: A subset $K \subseteq \mathbb{R}$ is s.t. be compact if K has BWP

In $\mathbb{R}^d$, K is compact iff K is closed & bd.

$\{x_n\} \subseteq \mathbb{R}^d \qquad x_n = (x_1^n, \ldots, x_d^n)$

$d(x,y)$

Let $A \subseteq \mathbb{R}$

$d(x, A) := \inf\{d(x,y) : y \in A\}$

If $x \in A$, then $d(x, A) = 0$

If $x \notin A$, then $d(x, 0) \geq 0$

$A = (0,1)$, $x = 0$ $d(x, A) = 0$

Let A be a closed set.

If $d(x, A) = 0$ then $x \in A$

or if $x \notin A$, then $d(x, a) > 0$

$d(x, A) = \inf\{d(x,y) : y \in A\} = 0$

$\Rightarrow \epsilon > 0, \exists y \in A$ s.t. $d(x,y) < \epsilon$ $(\because A$

$\Updownarrow$

$B_\epsilon(x) \cap A \neq \varphi \Rightarrow x \in \bar{A} =$

$\therefore d(x, A) = 0 \Rightarrow x \in A$

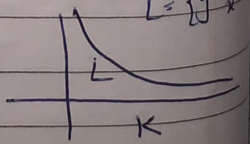
$K, L \subseteq X$

$d(K, L) := \inf\{d(x,y) : x \in K, y \in L\}$

If $K \cap L = \varphi$, not necessarily $d(K,L) = 0$

If Both K, L closed : $K \cap L = \varphi$

$K \cap L = \varphi$, $d(K, L) = 0$



K, L closed

Date __/__/__

$K = \{n : n \in \mathbb{N}\}$, $L = \{n + \frac{1}{2n} : n \in \mathbb{N}\}$

Both closed

$d(K,L) = 0$, $K \cap L = \phi$

If K compact and L closed, ~~then~~ $K \cap L = \phi$,
then $d(K,L) > 0$.

If not, $d(K,L) = 0$

$\exists \{x_n\} \subseteq K$ and $\{y_n\} \subseteq L$ s.t. $d(x_n, y_n) \to 0$

$\Downarrow$

$\exists \{x_{n_k}\} \subseteq K$ s.t. $x_{n_k} \to x \in K$ (K is compact)

$$d(x, y_{n_k}) \leq \underbrace{d(x, x_{n_k})}_{\downarrow \, 0} + \underbrace{d(x_{n_k}, y_{n_k})}_{\downarrow \, 0}$$

$y_{n_k} \to x \Rightarrow x \in L$ (L is closed)

$\not\Rightarrow$ $x \in K \cap L$

So, $d(K,L) > 0$

DEF^N: Let $\{U_i\}_{i \in I}$ be a family of open subsets of $\mathbb{R}$.
We say that $\{U_i\}$ is an open cover of $A \subseteq \mathbb{R}$
if $A \subseteq \bigcup_{i \in I} U_i$

$A = (0,1) \to U = A$ or $U = \{(0, \frac{1}{2}), (-\frac{1}{4}, 1)\}$

or $U = \left\{ \left(\frac{1}{n+2}, \frac{1}{n}\right), \left(\frac{1}{n+1}, \frac{1}{n+3}\right) \right\}$

$(0,1) = \bigcup_{n} \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$

$[0,1] = \bigcup_{n} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) \cup U_1 \cup U_2$

$0 \in U_1$, $\exists N, \frac{1}{n} \in U_1 \; \forall n \geq N$

$(-\epsilon, \epsilon) \subset U_1$ $\frac{1}{n} < \epsilon$

$[0,1] \subseteq U_1 \cup \left(\bigcup_{n=N}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) \right) \cup U_2$

$1 \in U_2$

$(1-\delta, 1+\delta) \subset U_2$

$\exists M \quad 1-\delta < 1-\frac{1}{n} \quad \forall n > M$

$$[0,1] \subseteq U_1 \cup \left(\bigcup_{n=N}^{M} \left(\frac{1}{n}, 1-\frac{1}{n}\right)\right) \cup U_2$$

DEF^N :

~~Let $U = \{U_i\}$ be an open cover for A~~
~~A set $A \subseteq \mathbb{R}$ is so~~
~~$U = \{U_i\}$ an open cover for A if $\exists$ finitely many $U_i \in U$ s.t. $A \subseteq \bigcup_{i \in F} U_i$~~

~~then we say that $\{U_i\}_{i \in F}$ is finite subcover for A~~

THM. K is compact iff every open cover for K has finite subcover

DEF^N $U = \{U_i\}$ an open cover for A.
We say U has a finite subcover if $\exists$ finitely many $U_i \in U$ s.t. $A \subseteq \bigcup_{i \in F} U_i$

Pf :

$A \subseteq \bigcup_{i \in F}$

($\Leftarrow$) Consider $U_x = \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$ → open

$K \subseteq \bigcup_{x \in K} U_x$ $\qquad \{U_x\}_{x \in K}$ is an open cover
K

$\exists x_1, \dots, x_n \in K$ st $K \subseteq \bigcup_{i=1}^{n} U_{x_i}$

$x \in K \Rightarrow x \in U_{x_{i_0}} \qquad i_0 \in \{1, \dots, n\}$

$|x| \le |x - x_{i_0}| + |x_{i_0}| \le 1 + M$, $M = \max\{|x_1|, \dots\}$

So, K is bdd.

Claim: K is closed

If not true, $\exists\ x \in \overline{K}/K$

$\exists$ a seq. $\{x_n\} \subseteq K$ s.t. $x_n \longrightarrow x$

$x \notin K \Rightarrow \epsilon_y = d(x,y) > 0 \quad \forall\ y \in K$

$y \in K$, define $O_y = (y - \frac{\epsilon_y}{3}, y + \frac{\epsilon_y}{3})$

$\Rightarrow K \subseteq \bigcup_{y \in K} O_y \quad \Rightarrow K \subseteq \bigcup_{i=1}^{n} O_{y_i}$

Take $\epsilon = \min\{\epsilon_{y_1}, \dots, \epsilon_{y_n}\}$ $\quad x \notin O_y$

$(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \cap O_{y_i} = \phi$

$\Rightarrow (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \cap K = \phi$

$\Rightarrow x \notin \overline{K} \quad \Rightarrow\Leftarrow$

Hence, $\overline{K} = K \Rightarrow K$ is closed.

$(\Rightarrow)$ K compact $\Rightarrow$ every open cover has finite subcover $(*)$

$K \subseteq [a,b]$

Let $[a,b] \subseteq \bigcup U_i$

$S = \{ \cancel{[a,x]}\ x : [a,x]$ ~~can be expressed~~ has finite subcover ~~property~~ $\}$ i.e. $[a,x] \subseteq \bigcup_{i \in F} U_i$

S is bdd.

Let $\alpha = \sup S$, $\alpha \leq b$

If $\alpha < b$

Then, $\alpha \in K \subseteq \bigcup_i U_i \Rightarrow \alpha \in U_{i_0}$ for some i_0

So, $(\alpha - \epsilon, \alpha + \epsilon) \subset K \cap U_{i_0}$

$[a, \alpha + \epsilon]$ has finite subcover $\Rightarrow\Leftarrow$

Hence, $\alpha = b$

$K \subseteq [a,b]$ $\quad (a,b) \setminus K$ is non-empty open

Take any open cover of K.

finite subcover $\leftarrow$ $[a,b] \rightarrow \{(a-\epsilon, a+\epsilon), (a,b)\setminus K, (b-\epsilon, b+\epsilon)\} \cup U_i\}$

Date __/__/__

$\overset{K}{A}$ $\overset{K}{A}$ $\overset{K}{A}$ $\overset{K}{A}$ $\overset{K}{A}$

$K \subseteq [a, b] \subseteq (a,b) \setminus K \cup (a-\epsilon, a+\epsilon) \cup (b-\epsilon, b+\epsilon)$

$K \subseteq \bigcup_{i \in F} U_i$

Cts. Func^n :

f is cts. at x_0 iff. given $\epsilon > 0$, $\exists \delta$ s.t. $(x_0 - \delta, x_0 + \delta) \subset f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$

$$f^{-1}(A) = \{x : f(x) \in A\}$$

or for every seq. $x_n \neq x_0$, $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

If $A \subseteq X$, f is cts. on A if f is cts. at every pt. in A

f is cts on A
$\Leftrightarrow$ U open in $\mathbb{R}$ $\Rightarrow$ $f^{-1}(U)$ is open A
$\Leftrightarrow$ S closed in $\mathbb{R}$ $\Rightarrow$ $f^{-1}(S)$ is closed.

Ex :

Saathi

U open $\Rightarrow f(U)$ open $\times$ (Take const. func^n)

A closed $\Rightarrow f(A)$ closed ✓

1. $f: \mathbb{R} \to \mathbb{R}$ cts. everywhere

Polynomials in $\mathbb{R}^d$: $x \in \mathbb{R}^d$, $x = (x_1, \dots, x_d)$

$\alpha \in \mathbb{N}^d \cup \{0\}$, $\alpha = (\alpha_1, \dots, \alpha_d)$

$$P(x) = \sum_{|\alpha| \le N} a_\alpha x^\alpha \qquad |\alpha| = \sum_{i=1}^{d} \alpha_i$$

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$

deg. of P is N

$(x_1, x_2)^{(2,2)} = x_1^2 x_2^2$

if the highest $N \in \mathbb{N}$ s.t.

$a_\alpha \neq 0$ atleast for some α with $|\alpha| = N$

$|\alpha| = 2$, (x, y)

$a_1 x^2 + a_2 xy + a_3 y^2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^x = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{x^n}{n!}$$

15/02/22

1. Cts except at 1 pt.
2. " " " finitely many pt.
3. " " " $\mathbb{Z}$
4. " " " $\mathbb{Q}$

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ q > 0,\ (p, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $x \in \mathbb{Q}$, then f isn't cts. at x

$\exists$ $\mathbb{R} \setminus \mathbb{Q} \ni x_n \to x$ $\quad f(x_n) = 0$, $f(x) = \frac{1}{q} \neq 0$

Date __/__/__

If $x \in \mathbb{R} \setminus \mathbb{Q}$

Claim: f is cts. at x

$\exists\, \epsilon > 0$ s.t. $|f(y) - f(x)| < \epsilon$

$\Rightarrow |f(y)| < \epsilon$

if $y \in \mathbb{Q}$, $y \in (x-\delta, x+\delta)$

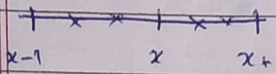
$f(y) = \frac{1}{q}$

We want $\frac{1}{q} < \epsilon$

Assume, $\frac{1}{q} \geq \epsilon \Rightarrow q \leq \frac{1}{\epsilon}$ so finitely many q's.

Consider $(x-1, x+1)$

$\hookrightarrow \frac{p}{q}$ s.t. $\frac{1}{q} \geq \epsilon$



$\frac{p}{q} < x+1 \Rightarrow p < (x+1)q$

$x-1 < \frac{p}{q} \Rightarrow p > (x-1)q$

For each q, there are finitely many

Now, we have finitely many q's

So, $\#\left\{\frac{p}{q} : r = \frac{p}{q} \in (x-1, x+1) \text{ and } \frac{1}{q} \geq \epsilon\right\} < \infty$

Say, $r_1, \dots, r_m$

$\delta = \min_{1 \leq i \leq m} |x - r_i| > 0$

If ~~interval~~ $(x-\delta, x+\delta)$ $\ni r \in \mathbb{Q}$, then $r = \frac{p}{q}$ with

$\frac{1}{q} < \epsilon$

Hence, f is cts. only at x.

Does $\exists$ a func^n s.t. it is cts. only at rationals ?

Denote $C_f = \{x : f \text{ is cts. at } x\}$
$D_f = \{x : f \text{ is discts. at } x\}$

THM. If f is monotonic, then D_f is countable. Moreover, the discont. is jump discontinuity.

Pf: Assume f is mono. $\uparrow$ing.

$$f(x^+) = \lim_{y \to x^+} f(y) \qquad f(x^-) = \lim_{y \to x^-} f(y)$$

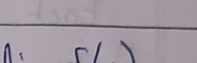
If f is discts. at x, $f(x^+) > f(x^-) \Rightarrow f(x^+) - f(x^-) > 0$

$(f(x^-), f(x^+))$ is non-empty.

$$A_n = \{x : f(x^+) - f(x^-) > 1/n\}$$

$$D_f = \bigcup_n A_n$$

Observe, if $x, y \in D_f$, $x \neq y$



$\{x_n\} \downarrow x$, $x_n < \frac{y-x}{2}$

$\{y_n\} \uparrow x$, $y_n > \frac{y-x}{2}$

$f(x^+) \leq f(y^-)$
$(f(x^-), f(x^+)) \cap (f(y^-), f(y^+)) = \varphi$

$x_1, \ldots, x_m \in A_n$

$$\frac{m}{n} \leq f(x_1^+) - f(x_1^-) + f(x_2^+) - f(x_2^-) + \ldots + f(x_m^+) - f(x_m^-) \leq f(b) - f(a)$$

$\# A_n$ is finite

Hence, D_f is countable.

Date __/__/__

Q Given a countable set S, can we find an func^n f s.t. f is cts. everywhere ex S.

S countable. Let $\{x_1, x_2, \ldots,\}$ be an enum of S.

For each n, $f_n(x) := \begin{cases} -1/2^n & \text{if } x < x_n \\ 0 & \text{if } x = x_n \\ 1/2^n & \text{if } x > x_n \end{cases}$

Each f_n ~~cts.~~ cts. except at x_n.

$f(x) = \sum_n f_n(x)$ $\qquad S_n = \sum_{n=1}^{N} f_n(x)$, S_N corollary cts

OK

If $x \in \mathbb{R} \setminus S$, f cts(!) $\qquad S_N(x) \to f(x)$

If $x \in S \Rightarrow x = x_m$

$f(x) = \underset{\text{not cts}}{f_m(x)} + \boxed{\sum_{n \neq m} f_n(x)}$ cts at x

f isn't cts. at x_m.

THM. For any f, C_f is G_δ

Pf: $n \in \mathbb{N}$, $A_n = \{x : \exists \delta > 0$ s.t. whenever $s, t \in (x-\delta, x+\delta)$, $|f(s) - f(t)| < 1/n\}$

$C_f = \bigcap_n A_n$

$x \in C_f$, $\exists \delta > 0$ s.t. $y \in (x-\delta, x+\delta)$
$\Rightarrow |f(y) - f(x)| < \frac{1}{2n}$

$|f(y) - f(t)| \leq |f(y) - f(x)| + |f(x) - f(t)| <$

$x \in A_n$ $\forall n$ δ

$C_f \subseteq \bigcap A_n$ 16/02/22

~~$y \in (x-\delta, x+\delta) \Rightarrow |f(x) - f(y)| < \frac{1}{n}$~~

$x \in \bigcap_n A_n$

$\epsilon > 0$ $\exists$ n_0 s.t. $\frac{1}{n_0} < \epsilon$

Date __/__/__

$x \in A_{n_0} \Rightarrow \exists\ \delta > 0$ s.t. $y \in (x-\delta, x+\delta)$
$\Rightarrow |f(y) - f(x)| < \frac{1}{n_0} < \epsilon$

$C_f \supseteq \cap A_n$

$\Rightarrow C_f = \bigcap_n A_n$

A_n open. $x \in A_n$, $(x-\delta, x+\delta) \subset A_n$
Easy.

$\Rightarrow C_f$ is G_δ.

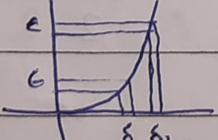
Corollary: $\nexists\ f$ s.t. $C_f = Q$

Pf: Q is not G_δ.
If so, $Q = \bigcap_{n=1}^{\infty} A_n$, A_n's open
$Q \subseteq A_n \Rightarrow A_n$'s are open and dense.

~~Define~~ Let $\{r_1, r_2, \dots\}$ be enumer^n of Q.
Define $V_n = A_n \setminus \{r_n\}$.
V_n open, V_n dense.
~~$\Rightarrow \cap V_n$ is dense.~~ $\cap V_n$ is dense
$\cap V_n = \cap (A_n \setminus \{r_n\})$
$= (\bigcap_n A_n) \cap (\cap \{r_n\}^c) = Q \cap (R \setminus Q) = \varphi$

So, Q isn't G_δ.
$\Rightarrow Q$ cannot be equal to C_f for any f.

16/02/22



$f(x) = x^2$ — depends on ϵ, x
$\epsilon > 0\ \exists\ \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
$|x^2 - y^2| < \epsilon$
$(x+y)|x-y| < \epsilon$
$|x-y| < \frac{\epsilon}{|x+y|}$

No uniform δ.

Date __/__/__

- f cts. on X, then if $x_n \longrightarrow x \Rightarrow f(x_n)$

$X = (0, \infty)$, d = usual metric

Define $f(x) = \frac{1}{x}$ cts.

$\epsilon > 0$ $\exists N_0$ s.t.

$x_n = \frac{1}{n}$, $\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} <$

$\{x_n\}$ is Cauchy.

$f(x_n) = n \leftarrow$ unbdd.

<u>Uniform Continuity</u>:

$f : X \longrightarrow R$, f is s.t. be uniformly c on X if for $\epsilon > 0$, $\exists \delta > 0$ (depending s.t. ~~wh~~ $d(f(x), f(y)) < \epsilon$ whenever $d(x,y)$

Ex: Consider $f : [0,2] \longrightarrow R$ $f(x) = x^2$

$\epsilon > 0$. $|x-y| < \frac{\epsilon}{2x}$ $\frac{\epsilon}{x+y} < \frac{\epsilon}{2}$ (∵

$x^2 - y^2$

$f : [0,1] \longrightarrow R$ $f(x) = x^2$

$\epsilon > 0$

$|x^2 - y^2| = |(x+y)(x-y)| \leq 2|x-y|$

If $\delta = \frac{\epsilon}{4}$ $< \frac{\epsilon}{2} < \epsilon$

THM: Let K be a compact set. $f : K \rightarrow R$ is cts., then f is unif. c

Pf: Suppose not.

Then, $\exists \epsilon > 0$ s.t. for each $\delta > 0$,

$|x-y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon$

Date __/__/__

For each $n \in \mathbb{N}$ $\quad |x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$

$\{x_n\}, \{y_n\} \subseteq K$ $\quad \exists$ subseq $x_{n_k} \to x_0 \in K$

$$|y_{n_k} - x_0| \leq |x_{n_k} - y_{n_k}| + |x_{n_k} - x_0|$$
$$< \frac{1}{n_k} + |x_{n_k} - x_0| \to 0 \text{ as } k \to \infty$$

$y_{n_k} \to x_0$ $\qquad f(x_{n_k}) \to f(x_0)$

$\qquad\qquad f(y_{n_k}) \to f(x_0)$

$$\lim_{k \to \infty} |f(x_{n_k}) - f(y_{n_k})| = 0$$

$\Rightarrow \Leftarrow$

Remark Let f be unif. cts.

If $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

$\{x_n\}$ Cauchy

Want N_0 s.t. $|f(x_n) - f(x_m)| < \epsilon \ \forall\, n, m \geq N_0$

f is unif cts.

$\Rightarrow \epsilon > 0 \ \exists\, \delta > 0$ s.t. $d(f(x), f(y)) < \epsilon$ when $d(x,y) < \delta$

$\exists N_0$ s.t. $d(x_n, x_m) < \delta \quad \forall\, n, m \geq N_0$

$\Rightarrow |f(x_n) - f(x_m)| < \epsilon \quad \forall\, n, m \geq N_0$

$\Rightarrow \{f(x_n)\}$ is Cauchy.

Ex $\quad f : (0,1) \to \mathbb{R}$. $\quad f(x) = \frac{1}{x}$ cts. but not bdd.

If f is unif. cts.

Take $\epsilon = 1$, $\exists\, \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$

$0 \ \delta \ 2\delta \ 3\delta \ldots \ N\delta$

Divide $(0,1)$ into finitely many disjoint intervals with length δ.

$|f(x)| = |f(x) - f(n\delta)| + |f(n\delta)|$ $\qquad M = \max\{|f(n\delta)| : n = 1, \ldots N\}$

$< 1 + M$

Date __/__/__

$a \in \mathbb{R}$, $\delta > 0$

$$\omega_f(a,\delta) = \sup\{|f(x) - f(a)| : |x-a| < \delta\}$$

↑ Modulus of continuity

$$\omega(a) = \inf_{\delta} \omega_f(a,\delta) \leftarrow \text{Oscill}^n \text{ of } f$$

THM: (i) f cts at $a \iff \omega(a) = 0$

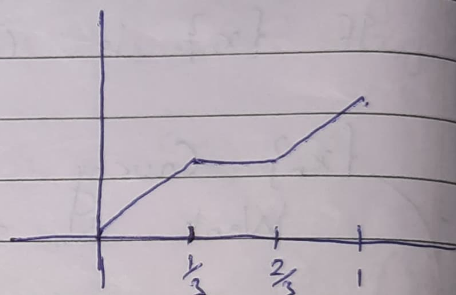
(ii) f is unif. cts. on $X \iff \lim_{\delta \to 0^+} \sup_{a \in X} \omega_f(a,\delta) = 0$

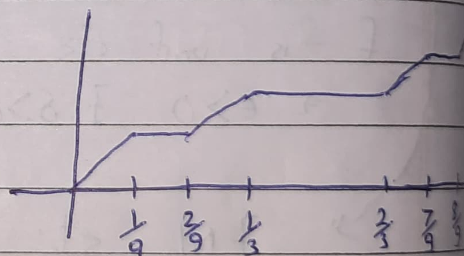
Ques^n in Midsem

Pf: Ex!!

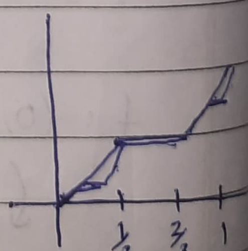
Devil's Staircase

Cantor–Lebesgue Func^n





$$f_1(x) = \begin{cases} \frac{3}{2}x & : 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & : \frac{1}{3} < x \le \frac{2}{3} \\ \frac{3x}{2} - \frac{1}{2} & : \frac{2}{3} < x < 1 \end{cases}$$



$$f_2(x) = \begin{cases} \frac{f_1(3x)}{2} & 0 \le x < \frac{1}{9} \\ & \cancel{\frac{1}{9} < x < \frac{2}{9}} \\ f_1(x) & \frac{1}{3} \le x < \frac{2}{3} \end{cases}$$

Date __/__/__

$$f_2(x) = \begin{cases} \frac{1}{2} f_1(3x) & 0 \leq x \leq \frac{1}{3} \\ f_1(x) & \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{1}{2} f_1(3x-2) + \frac{1}{2} & \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_3(x) = \begin{cases} \frac{1}{2} f_2(3x) & \text{''} \\ f_2(x) & \text{''} \\ \frac{1}{2} f_2(3x-2) + \frac{1}{2} & \text{''} \end{cases}$$

$\vdots$

Seq. of cts. func^n

$$f_n(x) \geq f_{n+1}(x)$$

Look at Cantor Set $\quad I_1^1 = (\frac{1}{3}, \frac{2}{3})$

$I_2^1 \ (\frac{1}{9}, \frac{2}{9}) \qquad I_2^2 = (\frac{7}{9}, \frac{8}{9})$

~~$|f(f_n(x) - f_{n+1}(x)|$~~ $\quad |f_n(x) - f_{n+1}(x)| \leq \frac{1}{2^n}$

$$f_N(x) - f_1(x) = \sum_{n=1}^{N-1} \left(f_{n+1}(x) - f_n(x)\right)$$

$$\leq \sum_{n=1}^{N-1} \frac{1}{2^n} \quad \text{cgs.}$$

$\Rightarrow \{f_n(x)\}$ cgs. uniformly.

$\hookrightarrow f(x)$ cts. $\leftarrow$ Cantor-Lebesgue Func^n

Date 02/03/22

$K \subseteq X$ is s.t. be compact if every bdd. seq. $\{x_n\}$ in K has cgt. subseq. $\{x_{n_k}\}$ which cgs. in it.

$K \subseteq \mathbb{R}$ compact $\Leftrightarrow$ K is closed and bdd.
$\Leftrightarrow$ Every open cover for K has a finite subseq.

$f : [a,b] \xrightarrow{cts.} \mathbb{R}$, then f attains max., min.

Pf : f bdd.
Suppose not.
Then, for every $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ s.t.
$|f(x_n)| > n$

$\{x_n\} \subseteq [a,b]$
By BWP, $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x$
$\Rightarrow f(x_{n_k}) \to f(x)$

$\{f(x_{n_k})\}$ is bdd. $\Rightarrow\Leftarrow$

If we change $[a,b]$ with compact set K, pf. remains same.

Now, f is bdd. in $\mathbb{R}$
$\alpha = \sup\{ |f(x)| : x \in K\}$

$n \in \mathbb{N}$, $\exists x_n \in K$ s.t. $\alpha - \frac{1}{n} < |f(x_n)|$

$\exists x_{n_k}$ s.t. $x_{n_k} \to x \in K$
$f(x_{n_k}) \to f(x)$

$$\alpha - \frac{1}{n_k} < |f(x_{n_k})| < \alpha + \frac{1}{n_k}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\alpha \qquad\quad |f(x)| \qquad\quad \alpha$$

$\Rightarrow |f(x)| = \alpha$

l_1

$e_n = (0, 0, \dots, 0, \underset{\text{n-th pos}^n}{1}, 0, \dots)$

$d(e_n, e_m) = \|e_n - e_m\|_1 = \sqrt{2}$

$K = \{e_n\}$, $\|e_n\| \leq 1$

K is closed and bdd. but not compact

What is reqr. so that this is true for any arbitrary m.s.

Total Bdd.ness :

$A \subseteq X$ is said to be totally bdd. if for every $\epsilon > 0$. $\exists$ finitely many A_i s.t.

$A_1, \dots, A_N$ s.t.

(i) $d(A_i) < \epsilon$

(ii) $A \subseteq \bigcup_{i=1}^{N} A_i$

A is totally bdd., ~~then~~ iff. for $\epsilon > 0$, $\exists$ $x_1, \dots, x_N \in A$ s.t. $A \subseteq \bigcup_{i=1}^{N} B_\epsilon(x_i)$

Let $\{x_n\} \subseteq A \leftarrow$ Totally bdd.

Let $\epsilon = 1$

$x_1, \dots, x_N \in A$

s.t. $A \subseteq \bigcup_{i=1}^{N} B_1(x_i)$

So, atleast one of $B_1(x_i)$ contains ∞-ly many $\{x_n\}$.

Call that set A_1.

A_1 is also TB $A_1 \subseteq \bigcup_{i=1}^{N_2} B_{\frac{1}{2}}(x_i)$

Pick the one which has ∞ly many say A_2.

$A_1 \supseteq A_2 \supseteq \dots$

$d(A_k) \leq \frac{1}{2^{k-1}}$

Date __/__/__

Choose $\{x_{n_k}\}$ s.t. $x_{n_k} \in A_k$

$\Rightarrow \{x_{n_k}\}$ is Cauchy.

(X,d) if X is complete and T.B.

X is compact.

~~X com~~

04/03/22

Let $\{x_n\}$ be Cauchy in X.

X complete $\Rightarrow$ ~~st~~ $\exists$ $\{x_{n_k}\}$ cgs. in it.

$\Rightarrow \{x_n\}$ cgs.

Suppose not T.B., $\exists\ \epsilon$ s.t. finitely many balls of radius ϵ will not cover X.

$x_1 \in X$, $\exists\ x_2 \in X \setminus B_\epsilon(x_1)$,

$x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\epsilon(x_i)$

$\exists$ a subseq. $\{x_{n_k}\}$ which cgs. to $x \in X$.

$x_{n_k} \longrightarrow x$

$\epsilon > 0$, $\exists\ N_0$ s.t. $d(x_{n_k}, x) < \frac{\epsilon}{2}\ \forall n_k \geq N_0$

$n_k > N$ $\quad d(x_{n_0}, x_N) < \frac{1}{N}$, $\frac{1}{N} < \frac{\epsilon}{2}$

$\Rightarrow x_{n_1} \in B_{\frac{1}{N}}(x_N)$ $\Rightarrow\Leftarrow$

Cantor Intersecⁿ:

(X,d) m.s.

$A_1 \supseteq A_2 \cdots \supseteq A_i$'s closed

$diam(A_n) \longrightarrow 0$. Then, $\bigcap_{n=1}^{\infty} A_n$ is singleton

$\epsilon > 0$, $\exists N$ $d(A_n) < \epsilon\ \forall n \geq N$

Date __/__/____

$m, n \geq N$

$x_m \in A_m \subseteq A_N$

$x_n \in A_n \subseteq A_N$

$d(x_m, x_n) < \epsilon \Rightarrow \{x_n\}$ is Cauchy

$\Rightarrow x_n \longrightarrow x$

Fix m, $x_n \in A_m \;\; \forall n \geq m$

$\Rightarrow x \in \overline{A_m} = A_m$

$\Rightarrow x \in \bigcap_{m=1}^{\infty} A_m$, $y \in \bigcap_{n=1}^{\infty} A_m$

$d(x, y) < \epsilon \;\; \forall \epsilon$, $\Rightarrow x = y$.

X compact $\Rightarrow$ Every open cover has a finite subcover

Let $X = \bigcup_{i \in I} U_i$, U_i's open.

(*) If $\exists r > 0$, s.t. for $x \in X$.

$B_r(x) \subset U_i$ for some i.

X totally bdd. $\Rightarrow \exists x_1, \ldots, x_N$ s.t. $X \subseteq \bigcup_{j=1}^{N} B_r(x_j)$

$\subseteq \bigcup_{i=1}^{N} U_{i_{x_j}}$

Suppose (*) doesn't hold

For any $r > 0$, $\exists x \in X$: $B_r(x) \not\subseteq U_i \;\; \forall i$

$n \in \mathbb{N}$, $\exists x_n \in X$ s.t. $B_{1/n}(x_n) \not\subseteq U_i \;\; \forall i$

$\{x_n\} \subseteq X$

$\Rightarrow \exists \{x_{n_k}\}$ cgs. to $x \in X$.

$x \in U_{i_0}$ for some i_0

As U_{i_0} open, $\exists \epsilon > 0$ st. $B_\epsilon(x) \subseteq U_{i_0}$

Date __/__/__

$d(x, x_{n_k}) < \epsilon/2 \quad \forall n \geq N$

Choose N large s.t. $\frac{1}{N} < \epsilon/2$

$y \in B_{1/N}(x_N) \quad d(x,y) \leq d(x, x_N) + d(x_N, y)$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow B_{1/N}(x_N) \subset B_\epsilon(x) \subset U_{i_0} \Rightarrow\Leftarrow$

Suppose (X,d) a compact m.s.
$A_1 \supseteq A_2 \supseteq \cdots$ closed subset of X.
Then, $\bigcap_{n=1}^{\infty} A_n \neq \phi$.

Suppose $\bigcap_{n=1}^{\infty} A_n = \phi \Rightarrow \bigcup_{n=1}^{\infty} A_n^c = X$

$\exists$ finite ~~subcover~~ set $F \subseteq N$ s.t.
$\bigcup_{n \in F} A_n^c = X \Rightarrow \bigcap_{n \in F} A_n = \phi \Rightarrow \Leftarrow$

$\rightarrow$ $\{A_n\}$ a seq. of closed sets s.t. every finite intersection of subcollec non-empty.

07/03/22

X compact $\Leftrightarrow$ X is complete, T.B.

X compact $\Rightarrow$ Every open cover has finite su

Let $\{U_i\}$ be open cover for X.
Suppose, it does not have a finite subcover
$F \subset N$ finite

$\bigcup_{i \in F} U_i \neq X, \quad \bigcap_{i \in F} U_i^c \neq \phi$

$\Rightarrow \{U_i^c\}$ has finite intersec

Date __/__/__

(X, d)

Every open covers has a finite subcovers iff. $\{F_i\}$ collec^n of closed sets with FIP has non-empty intersec^n.

Let $\{x_n\}$ be a seq. in X.

$F_n = \{x_k : k > n\}$

↳ closed as it contains all limits pts.

$\{F_n\}$ has F.I.P.

$F_1 \supseteq F_2 \supseteq \dots$

$\exists\, x \in \bigcap_n F_n$.

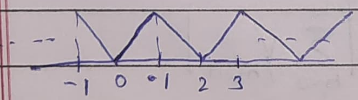
We can get a subseq. $\{x_{n_k}\}$ s.t. $(n_k > n_{k-1})$

$d(x, x_{n_k}) < \frac{1}{k}$

X compact $\iff$ Every collec^n of closed sets with FIP has non-empty intersec^n

~~FI~~

—— x ——



$\varphi(x) = |x|$ if $x \in [-1, 1]$

$\varphi(x+2) = \varphi(x)$

$\varphi_n(x) = \left(\frac{3}{4}\right)^n \varphi(4^n(x))$

$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$

f cts.

$\frac{f(x+h_n) - f(x)}{h_n}$, $h_n = \pm \frac{1}{2} 4^{-n}$ Choose sign s.t.

there is no integer (strictly) between $4^n x, 4^n(x+h_n)$

Page No.

Date __/__/__

$$4^n\left(x+\frac{1}{2}4^{-n}\right) - 4^n\left(x-\frac{1}{2}4^{-n}\right) = 1$$

Int can be here or here

$4^n(x+h_n)$ $4^n x$ $4^n(x+h_n)$

$$Y_{m,n} = \frac{1}{h_n}\left(\frac{3}{4}\right)^n\left[\varphi\left(4^n(x+h_m)\right) - \varphi(4^n x)\right]$$

$$= \pm 2\, 3^n 4^{n-m}\left[\varphi\left(4^n x + \frac{1}{2}4^{n-m}\right) - \varphi(4^n x)\right]$$

Suppose $n > m$

$$\frac{1}{2}4^{n-m} \in 2\mathbb{Z} \Rightarrow Y_{m,n} = 0$$

If $n = m$ $\quad (4^n x, 4^m(x+h_m)) \cap \mathbb{Z} = \varphi$

$$|Y_{m,n}| = 2\cdot 3^m \cdot \frac{1}{2} = 3^m$$

If $n < m$

$$|\varphi(y) - \varphi(x)| \leq |x-y|$$

$$|Y_{m,n}| \leq 2\cdot 3^n \cdot 4^{m-n}\, \frac{1}{2}4^{n-m} = 3^n$$

$$\left|\frac{1}{h_n}\left[f(x+h_n) - f(x)\right]\right| = \left|\sum_{n=0}^{m} Y_{m,n}\right|$$

$$\geq Y_{m,n} - \sum_{n=0}^{m-1}|Y_{m,n}|$$

$$\geq 3^n - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^n+1)$$

$$D[a,b] = \{ f : [a,b] \to \mathbb{R} : f \text{ diff.} \}$$

$$d(f,g) = \|f-g\|_\infty$$

$\{f_n\} \subseteq D[a,b]$

$f_n \to f \qquad f \in C[a,b]$

Does $f \in D[a,b]$??

Take $f_n(x) = |x|^{1+\frac{1}{n}} \in D[-1,1]$

$f_n \to |x| \notin D[-1,1]$

$\|f_n - f\|_\infty \to 0$

$\quad \|$

$\sup_{x\in[-1,1]} |f_n(x) - f(x)|$

$f(x) = |x|$

$S_n(x) = |f_n(x) - f(x)|$

$$S_n(x) = \begin{cases} |x^{1+\frac{1}{n}} - x| & x>0 \\ 0 & x=0 \\ |(-x)^{1+\frac{1}{n}} \cancel{+x} - (-x)| & x<0 \end{cases}$$

$$= \begin{cases} x - x^{1+\frac{1}{n}} & x>0 \\ 0 & x=0 \\ -x-(-x)^{1+\frac{1}{n}} & x<0 \end{cases}$$

For max.

$$S_n'(x) = \begin{cases} 1-\left(1+\frac{1}{n}\right)x^{\frac{1}{n}} & x>0 \\ 0 & x=0 \\ -1+\left(1+\frac{1}{n}\right)(-x)^{\frac{1}{n}} & x<0 \end{cases}$$

$$S_n'(x) = 0 \Rightarrow |x| = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$S_n\left(\frac{1}{(1+\frac{1}{n})^n}\right) = \frac{1}{(1+\frac{1}{n})^n}\left[1 - \frac{1}{1+\frac{1}{n}}\right]$$

$$= \frac{1}{(1+\frac{1}{n})^n}\frac{1}{(n+1)} \to 0 \text{ as } n$$

$f_n \to f$ uniformly

THM.

Let $f_n \in D[a,b]$; $f_n(x_0)$ cgs for some x_0
f_n' cgs uniformly.
Then, $f_n \to f$ uniformly for some
s.t. $f'(x) = \lim_{n\to\infty} f_n'(x)$

Pf: To prove $f_n \xrightarrow{\text{uniformly}} f$

$$|f_n(x) - f_m(x)| \leq |(f_n - f_m)x - (f_n - f_m)x_0| + |f_n(x_0) - f_m(x_0)|$$

$$= |(f_n' - f_m')(\alpha)||x - x_0| + |f_n(x_0) - f_m(x_0)|$$

(MVT)

$\alpha \in (x_0, x)$ or (x, x_0)

$\epsilon > 0$, $\exists N_1$ s.t. $\|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)}$

$\exists N_0$ s.t. $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$

$N_2 = \max(N_0, N_1)$

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t\to x} \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t - x}$$

Proof in next page $\leftarrow$?? $= \lim_{n\to\infty} \lim_{t\to x} \frac{f_n(t) - f_n(x)}{t - x}$

$$= \lim_{n\to\infty} f_n'(x)$$

Date __/__/__

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$$

$$\varphi(t) = \frac{f(t) - f(x)}{t - x}$$

$$\|\varphi_n - \varphi_m\|_\infty \leq \|f_n' - f_m'\|_\infty \quad \text{(MVT)}$$

$\{\varphi_n\}$ cgs. uniformly to φ.

$$\lim_{t \to x} \varphi(t) = f'(x)$$

Let $g(x) = \lim_{n \to \infty} f_n'(x)$

$$|\varphi(t) - g(x)| \leq |\varphi(t) - \varphi_n(t)| + |\varphi_n(t) - f_n'(x)| + |f_n'(x) - g(x)|$$

$$\lim_{t \to x} \varphi_n(t) = f_n'(x)$$

$$f'(x) = \lim_{t \to x} \varphi(t) = g(x) = \lim_{n \to \infty} f_n'(x)$$

$\rightarrow$ $C'[0,1] = \{ f : [0,1] \rightarrow \mathbb{R},\ f \text{ cts., diff.} \}$.

$d(f,g) = \|f - g\|_\infty + \|f' - g'\|_\infty$

Show that this is complete.

THM. Let $\{f_n\} \subseteq C[0,1]$ s.t.

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Then, $D_f = \{x : f \text{ is not cts.}\}$ is countable union of closed, nowhere dense subsets

Corollary $C_{f'}$ is dense in $\mathbb{R}$

$$D_{f'} = \bigcup_n A_n, \quad C_{f'} = \bigcap_n A_n^c$$

Date 04/03/22

THM. $\{f_n\}$ cts. on $[a,b]$.
$f_n(x) \to f(x)$
Then, $D_f = \bigcup_n A_n$, A_n closed, $A_n^\circ = \varphi$
C_f is dense.

Define for $\delta > 0$, $\omega_f(x,\delta) = \sup_{y \in (x-\delta, x+\delta)} |f(y) - f(x)|$

$$\omega_f(x) = \lim_{\delta \to 0^+} \omega_f(x,\delta) = \inf_{\delta > 0} \omega_f(x,\delta)$$

Pf: For $\epsilon > 0$, $D_\epsilon = \{x : \omega_f(x) \geq \epsilon\}$

$$D_f = \bigcup_{n \in \mathbb{N}} D_{1/n}$$

D_ϵ closed: Let $x_n \in D_\epsilon$ s.t. $x_n \to x$
Suppose $x \notin D_\epsilon$, then $\omega_f(x) < \epsilon$
$\omega_f(x) = \epsilon - \delta$ for some $\delta > 0$

$\exists\, r > 0$ s.t. $\omega_f(x, r) < \epsilon - \delta/2$ (by def^n of inf)

Consider x_n s.t. $|x_n - x| < r/2$
$\omega_f(x_n, r/2) < \epsilon$

$\omega_f(x_n) \leq \omega_f(x_n, r/2) < \epsilon \Rightarrow\Leftarrow$

So, $x \in D_\epsilon$
$\Rightarrow D_\epsilon$ is closed.

$D_{1/n}^\circ = \varphi$. Suppose not, then $\exists$ an interval I

$C_N = \{x \in I : |f_m(x) - f_n(x)| < \epsilon/3,\ n,m \geq N\}$

$\bigcup_N C_N = I$

Date __/__/__

$\exists\, N_0$ s.t. C_{N_0} has non-empty interior (Baire-Category)

$J \subseteq C_{N_0} \subseteq I$

$x \in C_{N_0}$

$$|f(x) - f_{N_0}(x)| = \lim_{n\to\infty} |f_n(x) - f_{N_0}(x)| < \epsilon/3$$

Any g, if $|f(y) - g(y)| < \epsilon/3$ on $\forall\, y \in (x-\delta, x+\delta)$, then

$$|\omega_f(x) - \omega_g(x)| < \epsilon/3$$

$s, t \in (x-\delta, x+\delta)$ $\quad f(t) > f(s)$ WLOG.

$$|f(s) - f(t)| = f(t) - f(s)$$

$$\leq g(t) - g(s) + 2\sup_{z\in(x-\delta,x+\delta)} |f(z) - g(z)|$$

$$\leq |g(t) - g(s)| + 2\epsilon/3$$

$$\omega_f(x) \leq \omega_g(x) + \frac{2\epsilon}{3}$$

$|f(x) - f_{N_0}(x)| < \epsilon/3$ on J

$$\Rightarrow \omega_f(x) \leq \omega_{f_{N_0}}(x) + 2\epsilon/3 = 2\epsilon/3 < \epsilon$$

$\Rightarrow\Leftarrow$

$$D^\circ_{1/n} = \varphi$$

Date ___/___/___ Integr^n

Let $f : [a,b] \longrightarrow \mathbb{R}$ and $P = \{a = x_0 < x_1 \dots < x_n$

$$V(f, P) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

Var^n of f on P ~~d~~

$$V_f = \sup_P V_f(f, P)$$

BV

We say that f is a func^n of bdd. Var^n if $V_f < \infty$.

Propos^n: If $f \in BV[0,1]$, then $f \in R[a,b]$

Pf: $P = \{a = x_0 < x_1 \dots < x_n = b\}$

$$x_{j+1} - x_j = \frac{b-a}{n}$$

$$A = U(P,f) - L(P,f) = \frac{b-a}{n} \sum_{j=1}^{n} (M_j - m_j)$$

$c_j, d_j \in [x_{j-1}, x_j]$ s.t. $f(c_j) > M_j - \frac{1}{n}$, $f(d_j) < m_j$

$$A \leq \frac{b-a}{n} \sum_{j=1}^{n} \left(f(c_j) + \frac{1}{n} - f(d_j) + \frac{1}{n}\right)$$

$$= \frac{b-a}{n} \sum_{j=1}^{n} \left[|f(c_j) - f(d_j)| + \frac{2}{n}\right]$$

$$\leq \frac{b-a}{n} \left(V_f + \frac{\Sigma 2}{n}\right)$$

$$\leq \frac{(b-a)}{n}(V_f + 2)$$

$\epsilon > 0$, choose n s.t. $< \epsilon'$

Date 21/03/22

$f \in R[a,b]$

$\epsilon > 0$, $\exists\, g \in C[a,b]$ s.t.

$$\int_a^b |f(x) - g(x)|\, dx < \epsilon$$

$C \to$ Cantor set

$f = \chi_C$ (Characteristic func^n of C)

f is discts. on C.

$\epsilon > 0$, $\exists$ finitely many open interval I_n with

$C \subseteq \bigcup_{n=1}^{N} I_n$ and $\sum_{n=1}^{N} |I_n| < \epsilon$

& $C \subseteq I = \bigcup_{n=1}^{N} I_n$, I^c is compact (if we include endpts)

$x \in I^c$, $\exists\, \delta > 0$ s.t. $(x - \delta, x + \delta) \subseteq I^c$

$I^c \subseteq \bigcup_{x \in I} I_x$, $\exists\, I_{N+1}, \dots, I_{N'}$ s.t. $I^c \subseteq \bigcup_{n \in} I_n$

Consider the partn. consisting of end pts. of $I_1, \dots, I_N, I_{N+1}, \dots, I_{N'}$

$$U(P,f) - L(P,f) = 2 \sum_{j=1}^{N} |I_j| + \sum_{j=N+1}^{N'-1} (f(c_j) - f(c_j'))(x_j - x_{j+1})$$

$$< 2\epsilon$$

Define:

A subset $E \subseteq \mathbb{R}$ is said to have "measure zero" if for every $\epsilon > 0$,

$\exists\, \{I_n\}$ s.t. $E \subseteq \bigcup_n I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$

Ex: (i) Cantor Set has measure zero

(ii) $\{x\}$, $I_n = \left(x - \frac{\epsilon}{2^{n+1}}, x + \frac{\epsilon}{2^{n+1}}\right)$

Date ___/___/___

(ii) Any countable set, $x_1, \ldots$

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\right)$$

Now consider, f s.t. D_F has measure zero

$$\omega_f(x, \delta) = \sup_{y \in (x-\delta, x+\delta)} |f(y) - f(x)|$$

$$\omega_f(x) = \inf_{\delta > 0} \omega_f(x, \delta)$$

$\epsilon > 0$, $D_\epsilon = \{x : \omega_f(x) \geq \epsilon\}$

$D_\epsilon \to$ closed

$D_\epsilon \to$ compact (Prev. lect.)

Suppose, D_f has measure zero.

$$D_\epsilon = \bigcup_{n=1}^{N} I_n \quad \text{s.t.} \quad \sum_{n=1}^{N} |I_n| < \epsilon$$

$$I = \bigcup_{n=1}^{N} I_n, \quad I^c \text{ is compact,}$$

$z \in I^c$ $\exists$ I_z s.t.

$$\sup_{x, y \in I_z} |f(x) - f(y)| < \epsilon$$

$$U(P,f) - L(P,f) \leq \underbrace{M \sum_{n=1}^{N} |I_n|}_{\text{Controlled}} + \sum_{j=N+1}^{N'} |f(c_j) - f(c_{j-1})| (x_j - x_{j-1})$$

$< \epsilon(b-$

Is converse true?

Date 2

Length $\leftarrow$ of subset

THM: Le

Pf

Date 22/03/22

Length of subset $\leftarrow m^*(A) = \inf \{ \sum_n |I_n| : A \subseteq \bigcup_n I_n \}$

$m^*(A) = 0$ for set of measure zero

$m^*(D_f) = 0$

$f \in R[a,b]$

THM: Let $f : [a,b] \longrightarrow R$ be bdd., then
$f \in R[a,b] \Leftrightarrow$ Set of discontinuities of f has measure zero.

Pf:

If $D_{1/n}$ has measure zero, then we are done.

$D_f = \bigcup_k D_{1/k}$ $\quad \epsilon > 0$

Want to find $\{I_n\}$ s.t.

$D_f \subseteq \bigcup_n I_n$, $\sum_n |I_n| < \epsilon$

$\exists \{I_j^k\}$ s.t. $D_{1/k} \subseteq \bigcup_j I_j^k$ and $\sum |I_j^k| < \frac{\epsilon}{2^k}$

So, we have $\{I_i^k\}_{i,k}$ s.t. $D_f \subseteq \bigcup_{i,k} I_i^k$

and $\sum_{k,j} |I_j^k| = \sum_k \sum_i |I_i^k| < \sum_k \frac{\epsilon}{2^k} < \epsilon$

Now, t.p. $D_{1/n}$ has measure zero
n-fixed

$\epsilon > 0$, $\exists$ a partitn P s.t. $\{a = x_0 < x_1 < \dots < x_N = b\}$

$U(P,f) - L(P,f) < \frac{\epsilon}{n}$ $\quad$ ($\because f$ is R.I.)

$I_j = (x_{j-1}, x_j)$

Page No.

Date ___/___/___

If $D_{1/n} \cap I_j \neq \phi$, then

$$\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) > \frac{1}{n}$$

$$\frac{1}{n} \sum_{\{j : I_j \cap D_{1/n} \neq \phi\}} |I_j| \leq \sum_{\{j : I_j \cap D_{1/n} \neq \phi\}} \left(\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) \right) (x_j - x_{j-1})$$

$$\leq U(P,f) - L(P,f)$$

$$\leq \epsilon/n$$

So, $D_{1/n}$ has measure zero.

Closed Rectangle :

$$R = \prod_{j=1}^{d} [a_j, b_j] \quad , \quad a_j, b_j \in \mathbb{R}$$

Volume : $|R| = \prod_{j=1}^{d} (b_j - a_j)$

Part^n : $P = (P_1, \ldots, P_d)$ where P_j's are partition of $[a_j, b_j]$.

Subrect. : S_j is a subinterval of P_j.

$$S = \prod_{j=1}^{d} S_j$$

$$f : R \longrightarrow \mathbb{R}$$

Define

$$U(P,f) = \sum_S \left(\sup_{x \in S} f(x) \right) |S|$$

$$L(P,f) = \sum_S \left(\inf_{x \in S} f(x) \right) |S|$$

f is R.I. on R if for every $\epsilon > 0$,

Date __/__/__

$\exists$ a partitⁿ. P s.t. $U(P,f) - L(P,f) < \epsilon$.

THM: Let f be a cts. funcⁿ on $R \subseteq \mathbb{R}^d$
Suppose $R = R_1 \times R_2$, $R_i \subseteq \mathbb{R}^{d_i}$
where $d_1 + d_2 = d$
$X = (x_1, x_2)$, $x_i \in \mathbb{R}^{d_i}$

Then, $F(x_1) = \int_{R_2} f(x_1, x_2)\, dx_2$

$F(x_1)$ is cts. funcⁿ on $\mathbb{R}^{d_1}$

$$\int_{R_1} F(x_1)\, dx_1 = \int_{R_1}\left(\int_{R_2} f(x_1,x_2)\, dx_2\right) dx_1$$

or $$\int_R f(X)\, dX = \int_{R_1}\int_{R_2} \cdots \int_{R_d} f(x_1, x_2, \ldots, x_d)\, dx_d \ldots dx_1\, dx_2$$

$g : A \longrightarrow B$ "cts. diff." , $g^{-1} : B \longrightarrow A$ is diff

$$\int_{g(A)} f(x)\, dx = \int_A f(g(x))\, |\det(Dg)(x)|\, dx$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$f(x+h) - f(x) - hf'(x) \longrightarrow 0$

$\int_{\mathbb{R}^d} f(x)\, dx \rightarrow$ We say f has moderate decay
if $|f(x)| \leqslant \dfrac{A}{1+|x|^{d+1}}$

$f : \mathbb{R}^2 \xrightarrow[\text{mod. decay}]{\text{cts.}} \mathbb{R}$

$F(x_1) = \int f(x_1, x_2)\, dx_2$

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1,x_2)\, dx_1\, dx_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1,x_2)\, dx_2\, dx_1$$

Date 23/03/22

$f : [a,b] \to \mathbb{R}$

I an interval with end pts. a, b
$a = \inf I$, $b = \sup I$

Consider, A, $a = \inf A$, $b = \sup A$

A not interval $\Rightarrow \exists\, x_0 \in (a,b)$ s.t. $x_0 \notin A$.

$\underset{U_1}{(-\infty, x_0)} \cap A \neq \varphi$, $\underset{U_2}{(x_0, \infty)} \cap A \neq \varphi$.

(X, d) m.s.
$\exists$ an open set U_1, U_2 in X s.t.
$A = ((-\infty, x_0) \cap A) \cup (A \cap (x_0, \infty))$ $U_1 \cap U_2 = \varphi$, $U_1 \cap A_1 \neq \varphi$, $U_2 \cap A_1 \neq \varphi$.

DEF^N We say that W is relatively open in A if $\exists$ an open set U in X s.t. $W = U \cap A$

DEF^N $A \subseteq X$ is said to be disconnected if $\exists\, W_1, W_2$ non-empty relatively open subsets of A s.t.
(a) $W_1 \cap W_2 = \varphi$
(b) ~~$W_1 \cap A \neq \varphi$, $W_2 \cap A \neq \varphi$~~ $W_1 \cup W_2 = A$

A connected if A is not disconnected

THM. $A \subseteq \mathbb{R}$
A is connected $\Leftrightarrow$ A is an interval

$\Rightarrow$ (Done above)

$\Leftarrow$ Suppose A is not connected

THM:

Pf:

$\Rightarrow \exists W_1, W_2$ relatively open s.t.
$W_1 \cap A \neq \phi$, $W_2 \cap A \neq \phi$, $W_1 \cap W_2 = \phi$

$A = [a,b]$, $B = \{x \in W_1 : [a,x] \subseteq W_1\}$

Let $\alpha = \sup B$.

If $\alpha \in W_1 = U_1 \cap A$ ($\alpha \neq b$ as then $W_2 \cap A = \phi$)
$\Rightarrow \exists \delta > 0$ s.t. So, $\alpha < b$
$\alpha + \delta \in U_1 \cap A = W_2$
$[a,\alpha) \cup [\alpha, \alpha+\delta)$
$[a, \alpha+\delta)$ $\Rightarrow\Leftarrow$

Similarly, $\alpha \notin W_2$.

(X,d), (Y,ρ) m.s.

THM: Let A be a connected subset of X.
$f : X \longrightarrow Y$ be cts. map.
Then, $f(A)$ is connected.

$\cap A$.

Pf: Suppose not $\exists W_1$ and W_2 (Separa^n)
s.t. $f(A) = W_1 \cup W_2$

$W_1 = U_1 \cap f(A)$
U_1 open in Y
$f^{-1}(W_1) = f^{-1}(U_1) \cap A$
$f^{-1}(U_1)$ open in X
Relatively open in A.
Similarly, $f^{-1}(W_2)$

Suppose $x \in f^{-1}(W_1) \cap f^{-1}(W_2)$
$\Rightarrow f(x) \in W_1 \cap W_2 = \phi$
So, $f^{-1}(W_1) \cap f^{-1}(W_2) = \phi$

Page No.

Date ___/___/___

$A = f^{-1}(W_1) \cup f^{-1}(W_2)$

$\Rightarrow$ A is not connected

$\therefore \Rightarrow \Leftarrow$

So, $f(A)$ is connected.

THM: Let A be connected.
Then $\bar{A}$ is also connected

Pf: Suppose not. Then, $\exists\, W_1, W_2$ separ^n for $\bar{A}$

$\bar{A} = W_1 \cup W_2$, $W_i = U_i \cap \bar{A}$

$A \subseteq \bar{A}$

Claim: Either $A \subseteq W_1$ or W_2

Suppose $A \subseteq W_1$

Let $x \in \bar{A} \setminus A$ and $x \in W_2 = U_2 \cap \bar{A}$

$\epsilon > 0$, $B_\epsilon(x) \subseteq U_2$

$B_\epsilon(x) \cap A = \phi$