Midsem Solution

(1)

$$X_t = (-1)^t A + \epsilon_t$$

(a) $\left.\begin{array}{l} A \sim N(0,1) \\ \epsilon_t \overset{i.i.d.}{\sim} N(0,1) \end{array}\right\rangle$ indep $\qquad \gamma_\epsilon(h) = \begin{cases} 1, & h=0 \\ 0, & o/w \end{cases}$

$E X_t = 0 \quad \forall t$

$$\mathrm{Cov}(X_{t+h}, X_t) = E\left((-1)^{t+h} A + \epsilon_{t+h}\right)\left((-1)^t A + \epsilon_t\right)$$

i.e. $\gamma_X(h) = \underbrace{(-1)^{|h|} + \gamma_\epsilon(h)}_{f^n \text{ of } h \text{ only}}$ (2)

$\Rightarrow \{X_t\}$ is covariance stationary — (1)

$\mathrm{Cov}(X_{14}, X_{11}) = \gamma_X(3) = (-1)^3 + \gamma_\epsilon(3) = -1$ (1)

(b) $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}(-1)^i A + \frac{1}{n}\sum_{i=1}^{n}\epsilon_i$

$$= \begin{cases} \frac{1}{n}\sum_{i=1}^{n}\epsilon_i, & \text{If } n \text{ is even} \\ -\frac{A}{n} + \frac{1}{n}\sum_{i=1}^{n}\epsilon_i, & \text{If } n \text{ is odd} \end{cases}$$

$E Y_n = 0 \quad \forall n$

$$V(Y_n) = \begin{cases} \frac{V(\epsilon_1)}{n}, & \text{If } n \text{ is even} \\ \frac{1}{n^2}V(A) + \frac{V(\epsilon_1)}{n}, & \text{If } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{1}{n}, & n \text{ even} \\ \frac{1}{n^2} + \frac{1}{n}, & n \text{ odd} \end{cases}$$

(5)

$\Rightarrow \{Y_n\}$ is not covariance stationary.

(c) $P_t = \epsilon_t + \epsilon_{4t+3}$; $\epsilon_t \overset{i.i.d}{\sim} N(0,1)$

$P_1 = \epsilon_1 + \epsilon_7$

$P_7 = \epsilon_7 + \epsilon_{31}$ (4)

$\text{cov}(P_1, P_7) \neq 0 \Rightarrow \{P_t\}$ is not WN

Note that proving $V(P_t)$ is not const does not imply it is non-white

(d) $Q_t = \epsilon_{2t} + \epsilon_{2t+1}$

$E\, Q_t = 0 \quad \forall t$

$$\text{cov}(Q_{t+h}, Q_t) = \text{cov}(\epsilon_{2(t+h)} + \epsilon_{2(t+h)+1}, \epsilon_{2t} + \epsilon_{2t+1})$$

$$= \gamma_\epsilon(2h) + \gamma_\epsilon(2h-1) + \gamma_\epsilon(2h+1) + \gamma_\epsilon(2h)$$

$\Rightarrow \forall h = \pm 1, \pm 2, \ldots$

$\text{cov}(Q_{t+h}, Q_t) = 0$ (4)

$\Rightarrow \{Q_t\}$ is WN

(2)
(a)
$$f(h) = \begin{cases} 1, & h = 0 \\ 1, & h = \pm 1 \\ 0, & o/w \end{cases}$$

By characterization of ACVF, $f(.)$ is ACVF iff it is even and n.n.d.

$f(h)$ clearly is even

For n.n.d.ness we need

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i f(t_i - t_j) a_j \geq 0 \quad \forall n \ \forall (t_1, \ldots, t_n) \in \mathbb{Z}^n$$
$$\forall (a_1, \ldots, a_n) \in \mathbb{R}^n$$

Take $n = 3$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$

$f(t_1 - t_2) = f(t_2 - t_3) = 1 = f(t_1 - t_1) = f(t_2 - t_2) = f(t_3 - t_3)$

$f(t_1 - t_3) = 0 = f(t_3 - t_1)$

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Consider $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_i f(t_i - t_j) a_j$$

$$= a_1^2 f(0) + a_1 a_2 f(1) + a_1 a_3 f(2)$$
$$+ a_2 a_1 f(1) + a_2^2 f(0) + a_2 a_3 f(1)$$
$$+ a_3 a_1 f(2) + a_3 a_2 f(1) + a_3^2 f(0)$$

$$= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_2a_3$$

$$= 1 + 1 + 1 - 2 - 2 = -1 < 0$$

$\Rightarrow f(\cdot)$ is not n.n.d.

$\Rightarrow f(\cdot)$ can not be ACVF (7)

(b) $Y_t = (a + bt) S_t + \epsilon_t$

(i) $Z_1 = Y_1$

$Z_2 = \alpha Y_2 + (1-\alpha) Y_1$

$Z_3 = \alpha Y_3 + (1-\alpha)(\alpha Y_2 + (1-\alpha) Y_1)$

i.e. $Z_3 = \alpha Y_3 + (1-\alpha)\alpha Y_2 + (1-\alpha)^2 Y_1$

$Z_4 = \alpha Y_4 + (1-\alpha)(\alpha Y_3 + \alpha(1-\alpha) Y_2 + (1-\alpha)^2 Y_1)$

i.e. $Z_4 = \alpha Y_4 + \alpha(1-\alpha) Y_3 + \alpha(1-\alpha)^2 Y_2 + (1-\alpha)^3 Y_1$

$\forall\, n$ & $\forall\, (t_1, \ldots, t_n)$

$\underset{\sim t}{Z} = \begin{pmatrix} Z_{t_1} \\ \vdots \\ Z_{t_n} \end{pmatrix} \sim N_n$ as $\underset{\sim}{\alpha}' \underset{\sim}{Z}$ = lin comb of indep $N(0,1)$ r.v.s ($\{Y_t\}$ seq)

+ a non-random const

$\sim N_1 \quad \forall\, \underline{\alpha} \in \mathbb{R}^n$

$$\Rightarrow \underset{\sim}{Z}^* = \begin{pmatrix} Z_1 \\ Z_3 \\ Z_4 \end{pmatrix} \sim N_3\left(E\, Z^*,\ \mathrm{Cov}\, Z^*\right) \qquad -(2)$$

Note that $\underset{\sim}{Z}^* = A(\alpha)\,\underset{\sim}{Y}$; $A(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-\alpha)^2 & \alpha(1-\alpha) & \alpha & 0 \\ (1-\alpha)^3 & \alpha(1-\alpha)^2 & \alpha(1-\alpha) & \alpha \end{pmatrix}$

$$\Rightarrow E(\underline{Z}^*) = A(\alpha)\ E(\underline{Y})$$

$$\mathrm{Cov}(\underline{Z}^*) = A(\alpha)\ \mathrm{Cov}(\underline{Y})\ A(\alpha)'$$

$$E(\underset{\sim}{Y}) = \begin{pmatrix} (a+b)S_1 \\ (a+2b)S_2 \\ (a+3b)S_3 \\ (a+4b)S_4 \end{pmatrix} \ \& \ \mathrm{Cov}(\underline{Y}) = I_4$$

$$\Rightarrow \underset{\sim}{Z}^* \sim N_3\left(A(\alpha)E(\underline{Y}),\ A(\alpha)\,A'(\alpha)\right).$$

(ii) $E\, Z_1 = E\, Y_1 = (a+b)\, s_1$

$$E\, Z_2 = \frac{2}{3} E\, Y_2 + \frac{1}{3} E\, Y_1$$

$$= \frac{2}{3}(a+2b)\, s_2 + \frac{1}{3}(a+b)\, s_1 \neq E\, Z_1$$

$\Rightarrow \{Z_t\}$ is not covariance stationary (2)

(iii) MA filter of window length 6

output $\rightarrow$ $$\hat{m}_t = \frac{1}{6}\left(\frac{1}{2} Y_{t-3} + Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2} + \frac{1}{2} Y_{t+3}\right)$$

i.e. $$\hat{m}_t = \frac{1}{6}\left(\frac{1}{2}(a+b(t-3))\, s_{t-3} + (a+b(t-2))\, s_{t-2} + \cdots + \frac{1}{2}(a+b(t+3))\, s_{t+3} + \left\{\frac{1}{2}\epsilon_{t-3} + \epsilon_{t-2} + \cdots + \frac{1}{2}\epsilon_{t+3}\right\}\right)$$

$a\, s_{t-i}$ terms would cancel out

3/2/1 ~~const~~ $b\, s_{t-i}$ terms would not cancel out

$$\left(\frac{3}{2} b\, s_{t+3} + 2b\, s_{t+2} + b\, s_{t+1} \cdots \neq 0\right)$$

(2)

$\Rightarrow$ $\hat{m}_t$ would contain s_t terms.

$\Rightarrow$ seasonal component will not be eliminated.

(3) $Y_t = (\epsilon_t + i\,\delta_t)(A \cos \omega t + i\, B \sin \omega t)$

(a) $E\, Y_t = 0 \quad \forall\, t$ $\qquad\qquad \omega = \pi/2$

$\mathrm{Cov}(Y_{t+h}, Y_t) = E\Big[(\epsilon_{t+h} - i\,\delta_{t+h})(A \cos \omega(t+h) - i\, B \sin \omega(t+h))\Big]$

$= E(Y_{t+h}^{*} Y_t)$ $\qquad \Big[(\epsilon_t + i\,\delta_t)(A \cos \omega t + i\, B \sin \omega t)\Big]$

$= (\gamma_\epsilon(h) + \gamma_\delta(h))(\cos \omega t \cos \omega(t+h) + \sin \omega t \sin \omega(t+h))$

$\gamma_y(h) = (\gamma_\epsilon(h) + \gamma_\delta(h)) \cos \omega h$

$\nearrow f^n$ of h only $\quad$ — (3)

$\Rightarrow \{Y_t\}$ is covariance stationary

$Z_t = (\epsilon_t + i\,\delta_t) + (A \cos \omega t + i\, B \sin \omega t)$

$E\, Z_t = 0 \quad \forall\, t$

$\mathrm{Cov}(Z_{t+h}, Z_t) = E(Z_{t+h}^{*} Z_t)$

$= E\Big[(\epsilon_{t+h} \cancel{+} - i\,\delta_{t+h}) + (A \cos \omega(\cancel{t+h}) - i\, B \sin \omega(t+h))\Big]$

$\Big[(\epsilon_t + i\,\delta_t) + (A \cos \omega t + i\, B \sin \omega t)\Big]$

$= (\gamma_\epsilon(h) + \gamma_\delta(h)) + \cos \omega h$ $\leftarrow f^n$ of h only

$\Rightarrow \{Z_t\}$ is covariance stationary — (3)

~~(b)~~ (c) $\gamma_\epsilon(h) = \gamma_\delta(h) = \begin{cases} 1, & h = 0 \\ 0, & \text{o/w} \end{cases}$

$\Rightarrow \quad \gamma_y(h) = 0 \quad \forall\, h \neq 0$ $\qquad$ (2)

(d) $\gamma_Z(h) = \left(\gamma_\epsilon(h) + \gamma_\delta(h)\right) + \cos\omega h$

$h \doteq 4$

$$\gamma_Z(4) = 0 + \cos 2\pi = +1 \neq 0$$

$$\Rightarrow \gamma_Z(h) \neq 0 \quad \forall\ |h| \geq 3$$

(2)

(b) $X_t = Re(Y_t) = \epsilon_t A \cos\omega t - \delta_t B \sin\omega t$

$$E X_t = 0 \quad \forall t$$

$$\operatorname{Cov}(X_{t+h}, X_t) = E\Big(\big(\epsilon_{t+h} A\cos\omega(t+h) - \delta_{t+h} B \sin\omega(t+h)\big)\big(\epsilon_t A\cos\omega t - \delta_t B\sin\omega t\big)\Big)$$

$$= \gamma_\epsilon(h) \cos\omega t \cos\omega(t+h) + \gamma_\delta(h) \sin\omega t \sin\omega(t+h) \quad \text{— } *$$

$h = 0$; $(*) = 1$

$h \neq 0$; $(*) = 0$ as $\gamma_\epsilon(h)$ & $\gamma_\delta(h) = 0$ $\forall h \neq 0$

$$\Rightarrow \gamma_X(h) = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

(4)

$\Rightarrow X_t$ is covariance stationary.

(4)

(a) $X_t = \frac{3}{2} X_{t-1} - \frac{1}{2} X_{t-2} + \epsilon_t + 2\epsilon_{t-1}$

$$\left(1 - \frac{3}{2}B + \frac{1}{2}B^2\right) X_t = (1+2B)\epsilon_t$$

$$\phi(B) X_t = \theta(B)\epsilon_t$$

$$\phi(B) = \left(1 - \frac{1}{2}B\right)(1-B)$$

$\Rightarrow$ roots of $\phi(z) = 0$ are 1 and 2

$\Rightarrow$ $\{X_t\}$ is not covariance stationary — (2)

$$\left(1 - \frac{1}{2}B\right) \underline{(1-B) X_t} = (1+2B)\epsilon_t$$

i.e. $\left(1 - \frac{1}{2}B\right) \nabla X_t = (1+2B)\epsilon_t$

$\Rightarrow$ $\nabla X_t \sim ARMA(1,1)$ with AR polynomial $1 - \frac{1}{2}B$

$\Rightarrow$ ∇X_t is covariance stationary ARMA(1,1) — (5)

(b) $Y_t = \frac{1}{3} Y_{t-1} + \epsilon_t + \epsilon_{t-1}$;

$\phi = \frac{1}{3}$, say $\qquad \epsilon_t \sim WN(0, \sigma^2)$

$E\, Y_t = 0$

$$\gamma_0 = \sigma_y^2 = \phi^2 \sigma_y^2 + \sigma^2 + \sigma^2 + 2\phi\sigma^2$$

$$\Rightarrow \sigma_y^2 (1 - \phi^2) = 2\sigma^2 + 2\sigma^2\phi = 2\sigma^2(1+\phi)$$

$$\Rightarrow \gamma_0 = \sigma_y^2 = 2\sigma^2 \frac{1+\phi}{1-\phi^2} = 2\sigma^2 \frac{1}{1-\phi} = 3\sigma^2 \quad (2)$$

$$\gamma_1 = E(y_{t+1}\, y_t)$$

$$= E(\phi y_t + \epsilon_{t+1} + \epsilon_t)\, y_t$$

$$= \phi \gamma_0 + \sigma^2$$

$$\Rightarrow \gamma_1 = \phi(3\sigma^2) + \sigma^2 = 2\sigma^2$$

$$\gamma_2 = E(y_{t+2}\, y_t)$$

$$= E(\phi y_{t+1} + \epsilon_{t+2} + \epsilon_{t+1})\, y_t$$

$$\gamma_2 = \phi \gamma_1 = \frac{1}{3} 2\sigma^2 = \frac{2}{3}\sigma^2 \quad \text{— }(\;)(2\tfrac{1}{2})$$

$$\forall h > 1 \qquad \gamma_h = \phi \gamma_{h-1}$$

$$\Rightarrow \gamma_6 = \phi \gamma_5 = \phi(\phi \gamma_4)$$

$$= \phi^2(\phi \gamma_3) = \phi^3(\phi \gamma_2)$$

$$= \phi^4 \gamma_2 = \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\sigma^2\right) = \left(\frac{1}{3}\right)^5 2\sigma^2 \quad \text{— }(\;)$$

$$\text{— }(2\tfrac{1}{2})$$