# ASSIGNMENT 6

MTH 301, 2021-22

(1) Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.
   (a) If $f'$ exists on an interval and is not constant, then $f'$ must take on some irrational values.
   (b) If $f'$ exists on an open interval and there is some point $c$ where $f'(c) > 0$, then there exists a $\delta$-neighborhood $V_\delta(c)$ around $c$ in which $f'(x) > 0$ for all $x \in V_\delta(c)$..
   (c) If $f$ is differentiable on an interval containing zero and if $\lim_{x \to 0} f'(x) = L$, then it must be that $L = f'(0)$

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(2) Let $f$ and $g$ be decreasing functions defined on $\mathbb{R}$. Is the product $fg$ monotone?
(3) Show that if $f$ is continuous on $[0, 1]$ and one-to-one, then it is monotone.
(4) Let $f$ be Riemann integrable over $[0, 1]$ define $F(x) = \int_0^x f(t)dt$. Give an example of $f$ for which $F$ is not differentiable for all $x$. Show that $F \in \mathcal{BV}[0, 1]$.
(5) Let $A \subset [0, 1]$. Show that $\chi_A$ is Riemann integrable if and only if $\bar{A} \setminus A^\circ$ has zero measure.
(6) Let $f$ be a continuous strictly increasing function on $[a, b]$. Show that $f$ maps $[a, b]$ one-to-one and onto $[f(a), f(b)]$. The inverse function $f^{-1}$ is also continuous and strictly increasing.

   Define $f$ on $S = [0, 1] \cup (2, 3]$ by $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ x - 1 & \text{for } 2 < x \leq 3. \end{cases}$

   (a) Show that $f$ is continuous and strictly increasing on $S$.
   (b) Show that $f$ maps $S$ one-to-one and onto $[0, 2]$.
   (c) Show that $f^{-1}$ is not continuous.
   (d) Why is this not a contradiction to above?
(7) (Extension of Montone functions) Let $\Phi \neq X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be bounded and increasing. Then $f$ can be extended to an increasing function $g : \mathbb{R} \to \mathbb{R}$ as follows. $g(x) = \begin{cases} \sup\{f(t) : t \in X, t \leq x\} & \text{if } X \cap (-\infty, x) \neq \Phi \\ \inf\{f(t) : t \in X\} & \text{otherwise.} \end{cases}$
(8) (a) Let $f : [0, 1] \to \mathbb{R}$ is defined as $f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{otherwise.} \end{cases}$ Show that $f$ is $\mathcal{BV}[a, b]$ on every subinterval $[a, b] \subset (0, 1)$ but $f \notin \mathcal{BV}[0, 1]$.
   (b) Let f be a continuous function defined on [a, b]. The arc length of the curve $y = f(x)$ on the interval [a, b] is defined by $L = \sup S$ where

$$S = \{\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} : a = x_1 < \cdots < x_n = b \text{ a partion of } [a, b]\}$$

   Show that the length of the curve is finite if and only if $f \in \mathcal{BV}[a, b]$.
(9) Give an example of a sequence of $\{f_n\} \subset \mathcal{BV}[a, b]$ such that $f_n(x) \to f(x)$ for all $x \in [a, b]$ but $f \notin \mathcal{BV}[a, b]$. If $V_a^b(f_n) \leq M < \infty \; \forall n$ what can you say about $f$?