Assignment 6: Several variables calculus & differential geometry (MTH305A)

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(1) Calculate the torsion of the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\alpha(t) = \left(\frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, \frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t \right).$$

(2) Consider the space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$, defined by

$$\alpha(t) = (3t - t^3, 3t^2, 3t + t^3).$$

(a) Compute curvature and torsion of the curve α

(b) Show that there exists a unit vector $A \in \mathbb{R}^3$, such that the tangent vectors $T(t)$ make a constant angle with A. Find such a vector and compute the fixed angle for A

(3) Let α be a unit-speed curve with non-vanishing curvature $\kappa(s)$.

(a) If the tangent vector $T(s)$ to the curve α make a constant angle with a fixed unit vector, show that $\frac{\tau(s)}{\kappa(s)}$ is a constant, where $\tau(s)$ is the torsion of the curve α.

(b) If $\frac{\tau(s)}{\kappa(s)}$ is constant, where $\tau(s)$ is the torsion of α, then show that the tangent vectors $T(s)$ to the curve α make a constant angle with a fixed unit vector.

(4) Compute curvature of $\gamma(t) = (\cos^3 t, \sin^3 t)$.

(5) Consider the circular helix

$$\gamma(\theta) = (a \cos\theta, a \sin\theta, b\theta),$$

where a, b are constants not both zero. Compute curvature of γ.

(6) Show that $\gamma(s) = \left(x_0 + R\cos\frac{s}{R}, y_0 + R\sin\frac{s}{R}\right)$ is unit-speed curve. Compute the curvature of γ.

(7) Let $\gamma(t) = ((1 + a\cos t)\cos t, (1 + a\cos t)\sin t)$, where a is a constant. Show that

(a) γ is a simple closed curve if $|a| < 1$.

(b) For $|a| > 1$, γ is not a simple closed curve.

(c) What if, $|a| = 1$?

(8) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a simple closed curve and $M : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Show that

(a) $\text{lenght}(\gamma) = \text{length}(M \circ \gamma)$ and

(b) $\text{area}(\gamma) = \text{area}(M \circ \gamma)$.