Note: For $p = \frac{1}{2}$, we get median of a distⁿ

i.e. If $z_{1/2} = \text{Med}$, then

$$P(X < \text{Med}) \leq \frac{1}{2} \text{ and } P(X \leq \text{Med}) \geq \frac{1}{2}$$

Like mean, $E(X)$, Med is a measure of central tendency.

Some Standard discrete distⁿs

(I) 1-point / degenerate distⁿ

$P(X=c) = 1$

d.f. $F(x) = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$

$E(X) = c \quad V(X) = 0$

m.g.f. $M_X(t) = E(e^{tX}) = e^{tc}$

(II) 2-pt distⁿ

$X=x$	x_1	x_2	$(x_1 < x_2$, say$)$
$P(X=x)$	p	$1-p$	

d.f. $F(x) = \begin{cases} 0, & x < x_1 \\ p, & x_1 \leq x < x_2 \\ 1, & x_2 \leq x \end{cases}$

m.g.f. $M_X(t) = e^{tx_1} p + e^{tx_2}(1-p)$

$E(X) = x_1 p + (1-p)x_2$

$EX^2 = x_1^2 p + x_2^2 (1-p)$

$VX = EX^2 - (E(X))^2 = \ldots$

Sp. case : Bernoulli r.v. $\quad x_1 = 1, \; x_2 = 0$

success/ failure

occurrence/non-occurrence

$X = \begin{cases} 1, & p \\ 0, & 1-p \end{cases}$

$M_X(t) = pe^t + (1-p)$

$EX = p \; ; \; V(X) = p(1-p) < EX$

(III) Binomial: Suppose we perform n indep Bernoulli trials (outcome 0 or 1) with prob of 1 (success) in each trial p

X : r.v. counting the number of successes in n trials

Possible values of X : 0, 1, . . . n

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \dots n \\ 0, & o/w \end{cases}$$

$X \sim Bin(n, p) \qquad 0 < p < 1$

m.g.f : $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \left((1-p) + pe^t\right)^n$$

$$EX = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = np$$

$$EX^2 = \frac{\partial^2 M_X(t)}{\partial t^2}\Big|_{t=0} = \frac{\partial}{\partial t}\left(n(q+pe^t)^{n-1} pe^t\right)\Big|_{t=0}$$

$$= npq + n^2p^2 \qquad (q = 1-p)$$

$$VX = EX^2 - (EX)^2 = npq$$

IV) Negative Binomial

Repeat independent Bernoulli trials until r successes

$$\Omega = \{ \underbrace{S \dots S}_{r\,S}, \underbrace{\dots\dots S}_{1F,\, rS}, \underbrace{\dots\dots S}_{2F,\, rS} \dots$$

X : number of failures preceeding the r^{th} success

$$P(X=x) = \begin{cases} \binom{x+r-1}{x} q^x p^{r-1} \cdot p, & x = 0, 1, 2, \dots \\ 0, & o/w \end{cases}$$

$q = 1-p$

$X \sim NB(r, p)$

Note : $\binom{x+r-1}{x} = (-1)^x \binom{-r}{x}$ → hence the name "negative binomial".

Note: Negative Binomial series

$$(x+a)^{-n} = \sum_{K=0}^{\infty} \binom{-n}{K} x^K a^{-n-K}$$

Note that

$$\binom{n+K-1}{K} = \frac{(n+K-1)!}{K!\,(n-1)!}$$

$$= \frac{(n+K-1)(n+K-2)\cdots n}{K!}$$

$$= (-1)^K \frac{\left((-n)(-n-1)\cdots\cdots(-n-K+1)\right)}{K!}$$

$$= (-1)^K \binom{-n}{K}$$

i.e.

$$(x+a)^{-n} = \sum_{K=0}^{\infty} \binom{-n}{K} x^K a^{-n-K}$$

$$= \sum_{K=0}^{\infty} (-1)^K \binom{n+K-1}{K} x^K a^{-n-K}$$

$$\sum_0^\infty P(X=x) = \sum_0^\infty \binom{x+r-1}{x} q^x p^r$$

$$= p^r \sum_0^\infty \binom{-r}{x} (-q)^x$$

$$= p^r (1-q)^{-r} = 1$$

m.g.f. : $M_X(t) = E(e^{tX}) = \sum_0^\infty e^{tx} \binom{x+r-1}{x} q^x p^r$

$$= p^r \sum_0^\infty \binom{-r}{x} (-qe^t)^x$$

$$= p^r (1-qe^t)^{-r}$$

$$E\,X = \left.\frac{\partial M_X(t)}{\partial t}\right|_{t=0} = \left. p^r r (1-qe^t)^{-r-1} qe^t \right|_{t=0} = \frac{rq}{p}$$

$$V\,X = \frac{rq}{p^2}$$

Note: Sp. case of $NB(r, p)$ - $r=1$, Geometric distn

p.m.f. $P(X=x) = q^x p$; $x = 0, 1, 2, \ldots$

Note: $P(X \geq m) = \sum_m^\infty p q^x = p q^m \frac{1}{1-q} = q^m$

For $m, n \geq 0$

$$P(X \geq m+n \mid X \geq m) = \frac{P(X \geq m+n \text{ and } X \geq m)}{P(X \geq m)}$$

$$= \frac{P(X \geq m+n)}{P(X \geq m)} = \frac{q^{m+n}}{q^m} = q^n = P(X \geq n)$$

also $P(X = m+n \mid X \geq m) = \frac{q^{m+n} p}{q^m} = q^n p = P(X=n)$

The above is called "lack of memory property"

(v) Discrete uniform

uniform distⁿ on n pts

$$P(X = x_i) = \frac{1}{n} \; ; \quad i = 1, 2, \dots n$$

$$M_X(t) = \frac{1}{n} \sum_{1}^{n} e^{t x_i}$$

$$E X = \frac{1}{n} \sum_{1}^{n} x_i$$

$$V X = \frac{1}{n} \sum_{1}^{n} (x_i - \bar{x})^2$$

(vi) Hypergeometric

Urn containing M white & $N-M$ balls of some other color

n balls drawn at a time or one after another without replacement

X : number of white balls in the sample

minimum value of X : $\max(0, n-(N-M))$

maximum value of X : $\min(n, M)$

$$P(X = x) = \begin{cases} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, & x = \max(0, n-(N-M)), \dots, \min(n, M) \\ 0, & \text{o/w.} \end{cases}$$