

- (1) Show that there is no set V such that every set is a member of V .
- (2) Show that $(x, y) = (a, b)$ iff $x = a$ and $y = b$.
- (3) Suppose R is an equivalence relation on A . For each $a \in A$, define the R -equivalence class of a by $[a] = \{b \in A : aRb\}$. Show that $\{[a] : a \in A\}$ is a partition of A .
Furthermore, show that for every partition \mathcal{F} of A , there is an equivalence relation S on A such that \mathcal{F} is the set of all S -equivalence classes.
- (4) Let (L, \prec) be a linear ordering. Prove the following.
 - (a) (L, \prec) is a well-ordering iff there is no sequence $\langle x_n : n < \omega \rangle$ in L such that $(\forall n < \omega)(x_{n+1} \prec x_n)$.
 - (b) (L, \prec) is a well-ordering iff for every $A \subseteq L$, (A, \prec) is isomorphic to an initial segment of (L, \prec) .
- (5) Suppose (X, \prec_1) and (Y, \prec_2) are well-orderings. Then exactly one of the following holds.
 - (a) $(X, \prec_1) \cong (Y, \prec_2)$.
 - (b) For some $x \in X$, $(\text{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.
 - (c) For some $y \in Y$, $(\text{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

- (6) Let $f : \mathcal{P}(\omega) \setminus \{\emptyset\} \rightarrow \omega$ be defined by $f(X) = \min(X)$. Call a well-orderings (A, \prec) f -directed iff $A \subseteq \omega$ and for every $x \in A$,

$$f(\omega \setminus \text{pred}(A, \prec, x)) = x$$

Describe all f -directed well-orderings.

- (7) Show that if $\alpha < \beta$ are ordinals, then there is a unique ordinal γ such that $\alpha + \gamma = \beta$.
(**Hint:** $\gamma = \text{type}(\beta \setminus \alpha, \in)$).
- (8) Suppose α, β, γ are ordinals and $\alpha + \beta = \alpha + \gamma$. Show that $\beta = \gamma$.
- (9) Suppose $\alpha \cdot \alpha = \beta \cdot \beta$. Show that $\alpha = \beta$.
- (10) Show that there is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. (**Hint:** Identify ω with the set of rationals \mathbb{Q} and for each real number x , consider $\{r \in \mathbb{Q} : r \leq x\}$).
- (11) Call an ordinal α good iff there exists $X \subseteq \mathbb{R}$ such that $(X, <)$ is order isomorphic to α . Show that α is good iff $\alpha < \omega_1$.

- (12) Let (P, \preceq_1) be a partial ordering. Show that there exists \preceq_2 such that (P, \preceq_2) is a linear ordering and \preceq_2 extends \preceq_1 which means the following:

$$(\forall a, b \in P)(a \preceq_1 b \implies a \preceq_2 b)$$

- (13) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is additive and $a = f(1)$.

- (a) Show that $f(0) = 0$.
- (b) Show that for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.
- (c) Show that for every $x \in \mathbb{Q}$, $f(x) = ax$.

- (14) Let $H \subseteq \mathbb{R}$ be a Hamel basis.

- (a) Show that every nonzero $x \in \mathbb{R}$ can be uniquely written as

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $x_1 < x_2 < \cdots < x_n$ are in H and a_1, a_2, \dots, a_n are nonzero rational numbers. Uniqueness means the following: Suppose

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1y_1 + b_2y_2 + \cdots + a_my_m$$

where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ are in H and $a_1, \dots, a_n, b_1, \dots, b_m$ are nonzero rationals. Show that $m = n$ and for every $1 \leq k \leq n$, $x_k = y_k$ and $a_k = b_k$.

- (b) Let $f : H \rightarrow \mathbb{R}$. Show that there is a unique additive function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \subseteq g$.

- (15) Show that for every $f : \mathbb{R} \rightarrow \mathbb{R}$ there are injective functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $f = g + h$.

- (16) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

- (a) Show that either f is identically zero or $\text{range}(f) \subseteq \mathbb{R}^+$.
- (b) Suppose f is continuous and not identically zero. Show that $f(x) = a^x$ for some $a > 0$.

- (17) Show that there is a discontinuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

- (18) Prove the following.

- (a) For every ordinal α , $|\alpha| \leq \alpha$.
- (b) If κ is a cardinal and $\alpha < \kappa$, then $|\alpha| < \kappa$.
- (c) There is an injection from X to Y iff $|X| \leq |Y|$.
- (d) There is a surjection from X to Y iff $|Y| \leq |X|$.
- (e) There is a bijection from X to Y iff $|X| = |Y|$.

- (19) Prove the following.
- (a) $|\mathbb{R}^\omega| = \mathfrak{c}$.
 - (b) $|C(\mathbb{R})| = \mathfrak{c}$ where $C(\mathbb{R})$ is the set of all continuous functions from \mathbb{R} to \mathbb{R} .
 - (c) Let A be the set of all real numbers which are roots of some polynomial equation with rational coefficients. Show that $|A| = \omega$.
- (20) Show that \mathbb{R}^2 cannot be partitioned into circles of positive radii.
- (21) Show that \mathbb{R}^3 can be partitioned into circles of positive radii.
- (22) Suppose $A \subseteq \mathbb{R}^2$ and every vertical section of A is finite. Show that some horizontal section of $\mathbb{R}^2 \setminus A$ is uncountable.
- (23) Let ϕ be a propositional formula in which \neg doesn't occur. Show that ϕ is satisfiable.
- (24) Suppose the set of propositional variables \mathcal{Var} is uncountable. Use Zorn's lemma to show the following: Let S be a set of propositional formulas such that every finite subset of S is satisfiable. Then S is satisfiable. **Hint:** Apply Zorn's lemma to (\mathcal{F}, \subseteq) where \mathcal{F} is the set of all functions h such that $\text{dom}(h) \subseteq \mathcal{Var}$, $\text{range}(h) \subseteq \{0, 1\}$ and for every finite $F \subseteq S$, there exists a valuation $val : \mathcal{Var} \rightarrow \{0, 1\}$ such that $h \subseteq val$ and every formula in F is true under val .
- (25) Let $\mathcal{L}, \mathcal{L}'$ be two first order languages where \mathcal{L}' is obtained from \mathcal{L} by adding a new constant symbol c to \mathcal{L} . Suppose T is an \mathcal{L} -theory, ϕ is an \mathcal{L} -formula with only free variable x , ψ is an \mathcal{L} -sentence and t is an \mathcal{L} -term with no variables. Show that the following hold.
- (UG) If $T \vdash_{\mathcal{L}'} \phi(c/x)$, then $T \vdash_{\mathcal{L}} (\forall x)(\phi)$.
 - (EI) If $T \cup \{\phi(c/x)\} \vdash_{\mathcal{L}'} \psi$, then $T \cup \{(\exists x)(\phi)\} \vdash_{\mathcal{L}} \psi$.
- (26) Suppose $\mathcal{L}, \mathcal{L}'$ are first order languages and \mathcal{L}' extends \mathcal{L} . Let T be an \mathcal{L} -theory and ϕ be an \mathcal{L} -sentence. Then $T \vdash_{\mathcal{L}} \phi$ iff $T \vdash_{\mathcal{L}'} \phi$.
- (27) Suppose T is a maximally consistent \mathcal{L} -theory and ϕ, ψ are \mathcal{L} -sentences. Show the following.
- (a) $T \vdash \phi$ iff $\phi \in T$.
 - (b) $\neg\phi \in T$ iff $\phi \notin T$.
 - (c) $(\phi \wedge \psi) \in T$ iff $\phi \in T$ and $\psi \in T$.
 - (d) $(\phi \vee \psi) \in T$ iff either $\phi \in T$ or $\psi \in T$.
 - (e) $(\phi \implies \psi) \in T$ iff either $\psi \in T$ or $\phi \notin T$.
 - (f) $(\phi \iff \psi) \in T$ iff " $\phi \in T$ iff $\psi \in T$ ".
- (28) Suppose T is a consistent complete \mathcal{L} -theory. Let S be the set all \mathcal{L} -sentences ϕ such that $T \vdash \phi$. Show that S is a maximally consistent \mathcal{L} -theory.

- (29) Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$ where c is a new constant symbol. Let $\mathbf{Primes} = \{2, 3, 5, 7, \dots\}$ be the set of all primes numbers. For each $p \in \mathbf{Primes}$, let “ p divides c ” denote the \mathcal{L} -sentence $(\exists y)(S^p(0) \cdot y = c)$. For each $X \subseteq \mathbf{Primes}$, let T_X be the \mathcal{L} -theory

$$T_X = TA \cup \{(p \text{ divides } c) : p \in X\} \cup \{\neg(p \text{ divides } c) : p \in \mathbf{Primes} \setminus X\}$$

where $TA = Th(\omega, 0, S, +, \cdot)$ denotes true arithmetic.

- (a) Show that T_X is consistent for every $X \subseteq \mathbf{Primes}$.
- (b) Show that TA has continuum many pairwise non-isomorphic countable models.
- (30) Show that every countable linear ordering is isomorphic to a subordering of the rationals $(\mathbb{Q}, <)$.
- (31) Let $W \subseteq \omega$. Show that W is c.e. iff there exists a computable function $f : \omega \rightarrow \omega$ such that $\text{range}(f) = W$.
- (32) Suppose $r_1, r_2, \dots, r_n, d_1, d_2, \dots, d_n$ are natural numbers and for every $1 \leq i \leq n$, $0 \leq r_i < d_i$. Assume that for every $1 \leq i < j \leq n$, d_i and d_j are relatively prime. Show that there exists a positive integer N such that for every $1 \leq i \leq n$, $\text{rem}(N, d_i) = r_i$.
- (33) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable $A \subseteq \omega^2$ such that $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$.
- (34) Suppose $X \subseteq \omega$ is numeralwise representable in PA. Show that X is computable.
- (35) Let $H \subseteq \omega$ be a non-computable c.e. set. Show that H is definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ but not numeralwise representable in PA.
- (36) Do the Exercise on Lecture slide 202.