Construction of $\mathbb{R}$

MTH 301

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0.1 Real Numbers

Definition 0.1. *A sequence is a map $f : \mathbb{N} \to \mathbb{Q}$. If $f(n) = x_n$ then we denote $\{x_n\}$ as a sequence.*

Example 0.2. 1. $x_n = 1,\ x_n = n,\ x_n = n^2,\ x_n = (-1)^n$.

2. *In definition of sequence we need only $x_n \in \mathbb{Q}$ for every n. We may not write always x_n is a compact form.*

3. *Consider $x_1 = 1$ and $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.*

Definition 0.3. 1. **Bounded Sequence***: A sequence $\{x_n\}$ is said to be bounded if there exists $M \in \mathbb{N}$ such that $|x_n| \leq M,\ \forall n \in \mathbb{N}$.*

2. **Cauchy Sequence***: A sequence $\{x_n\}$ is said to be Cauchy if for every $\epsilon > 0$ there exists a $N_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon,\ \forall n, m > N_0$.*

3. **Convergence***: A sequence $\{x_n\}$ is said to converge to some $x \in \mathbb{Q}$ if for every $\epsilon > 0$ there exists a $N_0 \in \mathbb{N}$ such that $|x_n - x| < \epsilon,\ \forall n > N_0$.*

Proposition 0.1. *Every Cauchy sequence is bounded.*

Proof. Let $\epsilon = 1$. Then there exists N_0 such that $|x_n - x_m| < 1$ for all $n \geq N_0$. Clearly, $|x_n| \leq \max\{|x_1|, \ldots, |x_{N_0-1}|, 1 + |x_{N_0}|\},\ \ \forall n \in \mathbb{N}$. □

Proposition 0.2. *Every convergent sequence is Cauchy.*

Proof. Let $x_n \to x$. For $\epsilon > 0$ there exists a $N_0 \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2},\ \forall n > N_0$. Let $n, m > N_0$ then

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x| + |x_m - x| \\ &< \epsilon. \end{aligned}$$

□

However every bounded sequence may not be Cauchy. Consider $x_n = (-1)^n$. Take $\epsilon = 1$ then for every N we have $|x_{2N} - x_{2N+1}| = 2$! Now we would like to show that every Cauchy sequence may not converge. Consider the sequence $x_1 = 1$ and $x_{n+1} = \frac{2(1+x_n)}{2+x_n}$. Now, $x_n \geq 0$ (by induction!) So,

$$\begin{aligned} |x_{n_1} - x_n| &= \frac{2(1 + x_n)}{2 + x_n} - \frac{2(1 + x_{n-1})}{2 + x_{n-1}} \\ &= \frac{2}{(2 + x_n)(2 + x_{n-1})}\left[(1 + x_n)(2 + x_{n-1}) - (1 + x_{n-1})(2 + x_n)\right] \\ &= \frac{2}{(2 + x_n)(2 + x_{n-1})}(x_n - x_{n-1}). \end{aligned}$$

Thus, $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$. Hence, $\{x_n\}$ is Cauchy. This does not converge in $\mathbb{Q}$!!

By using triangle inequality and above proposition we have the following.

Lemma 0.4. *If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are also Cauchy sequences.*

Definition 0.5 (Equivalent)**.** *We say that a Cauchy sequence is equivalent to 0 if $\lim_{n\to\infty} |x_n| = 0$.*

Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if $\{x_n - y_n\}$ is equivalent to 0.

Denote $\mathbb{R}$ **to be the set of equivalence classes of Cauchy sequences***.*

Clearly $\mathbb{Q} \subset \mathbb{R}$. (if $x \in \mathbb{Q}$ then consider $x = [\{x_n\}]$ where $x_n = x$ for each n.)

Now we define addition and multiplication on $\mathbb{R}$ in the following way

- $[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$
- $[\{x_n\}].[\{y_n\}] = [\{x_n.y_n\}]$

Exercise 0.6. *Show that addition and multiplication are well defined.*

Constant sequences $(0, 0, 0, \dots)$ and $(1, 1, \dots)$ are additive and multiplicative identities. Ever Cauchy sequence has a an additive inverse i.e. $\{-x_n\}$. However, not all have multiplicative inverse! Following lemma is required to show $\mathbb{R}$ is a field.

Lemma 0.7. *Every non-zero equivalence class of Cauchy sequence has a multiplicative inverse.*

Proof. $\{x_n\}$ is a non-zero Cauchy sequence then there exists $\epsilon > 0$ such that $|x_n| > \epsilon$ for $n \geq N$ for some $N \in \mathbb{N}$. If not then for every $\epsilon > 0$ and $n \in \mathbb{N}$ $\exists N_n > n$ such that $|x_{N_n}| < \epsilon$. As $\{x_n\}$ is Cauchy $\exists N_0$ such that $|x_n - x_m| < \epsilon$, $\forall n, m > N_0$. Consider $n > N_0$ then $|x_n| \leq |x_n - x_{n_n}| + |x_{N_n}| < 2\epsilon$ $(N_n > n > N_0)$ i.e. $x_n \to 0$! Consider $y_n = \frac{1}{x_n}$. Now, $|\frac{1}{x_n} - \frac{1}{x_m}| \leq \frac{|x_n - x_m|}{\epsilon}$, $\forall n, m \geq N$. Hence, $\{y_n\}$ is Cauchy. This does the job. $\square$

Corollary 0.8. *$\mathbb{R}$ forms a field.*

Now we extend absolute value to $\mathbb{R}$ by $|[\{x_n\}| = [\{|x_n|\}]$. —bf Order structure on $\mathbb{R}$:

Definition 0.9. *We say that $x = [\{x_n\}] \in \mathbb{R}$ is* **positive** *($x > 0$) if $\exists N \in \mathbb{N}$ such that $x_n > 0$, $\forall n > N$. Given any two $x, y \in \mathbb{R}$ we say that $x > y$ if $x - y > 0$.*
$x \in \mathbb{R}$ is **negative** *if $-x = [\{-x_n\}]$ is positive.*

Proposition 0.3 (Trichotomy)**.** *Let $x \in \mathbb{R}$. Then exactly one of the following three statements is true:*

(i) $x = 0$

(ii) $x > 0$

(iii) $x < 0$

Proposition 0.4. *Let $x, y \in \mathbb{R}$ such that $x > y$. Then $x + t > y + t$, $\forall t \in \mathbb{R}$.*

Proof. Let $x = [\{x_n\}]$, $y = [\{y_n\}]$, and $t = [\{t_n\}]$. As $x > y$ there exists $N \in \mathbb{N}$ such that $x_n > y_n$, $\forall n > N$. Now, $x_n + t_n > y_n + t_n$, $\forall n > N$. Thus $x + t > y + t$. $\square$

Theorem 0.10. *$\mathbb{R}$ is an Archimedian field.*

Proof. Let $x, y > 0$ be two elements of $\mathbb{R}$. We want to show there exists $N \in \mathbb{N}$ such that $N.x > y$. (Here N is considered as the constant sequence of N)

We can assume $x < y$. As $x > 0$ there exist $\epsilon > 0$ and $N_1 \in \mathbb{N}$ such that $x_n > \epsilon > 0,\ \forall n > N_1$. Also, $\{y_n\}$ is Cauchy hence bounded. So there exists $M \in N$ such that $y_n \leq M,\ \forall n \in \mathbb{N}$. Consider ϵ and M there exists $m \in \mathbb{N}$ such that $m\epsilon > M$ (recall all we are doing in $\mathbb{Q}$). Now, $mx_n > m\epsilon > M \geq y_n,\ foralln > N_1$. Hence, $m.x > y$. □

Proposition 0.5. *$\mathbb{Q}$ is dense in $\mathbb{R}$. i.e. For all $x \in \mathbb{R}$ and $\epsilon \in \mathbb{Q}_{>0}$ there exists a $r \in \mathbb{Q}$ such that $|x - r| < \epsilon$.*

Proof. Let $x = [\{x_n\}]$ and $\epsilon > 0$. As $\{x_n\}$ is Cauchy there exists N_ϵ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N_\epsilon$. Choose $r = \{x_{N_\epsilon + 1}\}$. Then $|x_n - r| < \epsilon,\ \forall N > N_\epsilon$. □

Remark 0.1. *Now we can replace $\epsilon \in \mathbb{Q}_+$ by $\epsilon \in \mathbb{R}_+$.*

0.2 *lub* Property

Now we are ready to prove the most important property of $\mathbb{R}$.

Theorem 0.11. *Let $A \subset \mathbb{R}$. If A has an upper bound then A has least upper bound.*

Proof. First we claim that there exists $p \in \mathbb{Q}$ which is an upper bound for A but $p - 1$ is not an upper bound. Let $a \in A$ then $r = a - 1$ is not an upper bound. If $s < r$ then s is also not an upper bound. Consider the sequence $s_n = s + n$. So there exists $n \in \mathbb{N}$ such that s_n is an upper bound. Denote $N = \{n : s_n \text{ is an upper bound}\}$. Denote $n_0 = \min N$. So s_{n_0} is an upper bound whereas $s_{n_0} - 1 = s_{n_0 - 1}$ is not an upper bound.

Nest we claim that there exists sequences $\{p_n\}$ and $\{q_n\}$ in $\mathbb{Q}$ such that for all $n \in \mathbb{N}$ we have q_n is an upper bound for A but p_n is not, also

$$q_n - p_n = \frac{1}{2^{n-1}}. \tag{1}$$

We will prove it by induction. Let q be an upper bound such that $q - 1$ is not. Define $q_1 = q$ and $p_1 = q - 1$. Then $q_1 - p_1 = 1$. Thus for $n = 1$ claim is true. Suppose qe have p_n and q_n with required property. Now define $s_n = \frac{p_n + q_n}{2}$. If s_n is not an upper bound then define

$$\begin{aligned} p_{n+1} &= s_n \\ q_{n+1} &= q_n. \end{aligned}$$

Clearly, $q_{n+1} - p_{n+1} = \frac{q_n - p_n}{2} = \frac{1}{2^n}$. If s_n is an upper bound then define $p_{n+1} = p_n$ and $q_{n+1} = s_n$. Hence we have the claim.

Our next claim is above chosen sequences $\{p_n\}$ and $\{q_n\}$ are Cauchy and $[\{p_n\}] = [\{q_n\}]$. Observe that $p_n \leq p_{n+1} \leq q_{n+1} \leq q_n$. So by induction $q_n \leq p_{n+k} \leq q_{n+k} \leq q_n,\ \forall k \in \mathbb{N}$. If $m \geq n$ then $|p_n - p_m| = p_m - p_n \leq q_n - p_n = |q_n - p_n| = \frac{1}{2^{n-1}}$. Similarly we can prove $\{q_n\}$ is Cauchy.

Define $[\{p_n\}] = [\{q_n\}] = r$. First we show that r is an upper bound for A. Suppose not then $\exists a \in A$ such that $a > r$. There exists $q \in \mathbb{Q}$ such that $r < q < a$. Then $q - r > 0$ as element of $\mathbb{R}$. So $q - q_n$ eventually a positive sequence in $\mathbb{Q}$. i.e. $\exists N \in \mathbb{N}$ such that $q - q_n > 0,\ \forall n \geq N$. So, $q_n < q < a,\ \forall n \geq N$. A contradiction. So r is an upper bound.

Now we want to show that r is the *lub*. Suppose $s < r$ and an upper bound. we can find $p \in \mathbb{Q}$ such that $s < p < r$ and $p_n - p$ is eventually positive. So, $p_n - p > 0$ eventually. So $p_n > p > a,\ \forall a \in A$. A contradiction. So we have the result. □

Corollary 0.12. *If $A \subset \mathbb{R}$ is bounded below then it has greatest lower bound.*

Now we are also in a position to prove the following. For $x \in \mathbb{R}$ define $[x] = lub\, \{n \in \mathbb{Z} : x \geq n\}$.

Proposition 0.6. *Every real number has a decimal expansion.*

Proof. Let $x \in \mathbb{R}$. Then we know that $x = a_0 + y_0$ where $0 \leq y_0 < 1$. Denote $a_1 = [10y_0]$. Then

$$x = a_0 + \frac{10y_0}{10} = a_0 + \frac{a_1}{10} + \frac{y_1}{10}, \text{ where } 0 \leq y_1 < 1.$$

Similarly, consider $a_2 = [10y_1]$. Then,

$$x = a_0 + \frac{a_1}{10} + \frac{10y_1}{100} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{y_2}{10^2}, \text{ where } 0 \leq y_2 < 1.$$

Now consider $A = \{a_0, a_0 + \frac{a_1}{10}, a_0 + \frac{a_1}{10} + \frac{a_2}{10^2}, \ldots, a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}, \ldots\}$. Clearly A is bounded above by x. We claim that $x = \sup A$. If not let $\sup A = \alpha < x$.i.e. $x - \alpha = \epsilon > 0$. We can get $m \in \mathbb{N}$ such that $\frac{1}{10^m} < \epsilon$. Then,

$$\begin{aligned} x - \left(a_0 + \frac{a_1}{10} + \cdots + \frac{a_m}{10^m}\right) &= \frac{y_m}{10^m} \\ &< \frac{1}{10^m} \\ &< \epsilon = x - \alpha \\ \Rightarrow \alpha < a_0 + \frac{a_1}{10} + \cdots + \frac{a_m}{10^m}. \end{aligned}$$

This is a contradiction. ∎

0.3 Cardinality of $\mathbb{R}$

Proposition 0.7. *$\mathbb{R}$ is uncountable.*

Proof. Suppose not, then there will be an enumeration of numbers in $(0, 1)$ say $\{y_n\}$, where

$$\begin{aligned} y_1 &= 0.a_{11}a_{12}\ldots a_{1n}\ldots \\ y_2 &= 0.a_{21}a_{22}\ldots a_{2n}\ldots \\ &\vdots \\ y_n &= 0.a_{n1}a_{n2}\ldots a_{nn}\ldots \\ &\vdots \end{aligned}$$

Consider $y = 0.b_1b_2\cdots_n\ldots$, where $b_n = \begin{cases} a_{nn}+1 & \text{if } a_{nn} = 0,1,2,\ldots,8 \\ 0 & \text{if } a_{nn} = 9. \end{cases}$ Clearly, $y \neq y_n$ for any n. Hence, (0,1) is not countable. ∎

As any $x \in (0, 1)$ has a binary representation. We have,

Proposition 0.8. $|\mathcal{P}(\mathbb{N})| = |T(\mathbb{N})| = |\mathbb{R}|$.

Proposition 0.9. *Every infinite set contains a countably infinite subset.*

Proof. First we show that for every $n \in \mathbb{N}$ there exists a subset $A_n \subset A$ such that $|A_n| = n$. As A is non-empty $\exists a_1 \in A$. Let us choose $A_l \subsetneq A$ such that $|A_l| = l$. As A is infinite $\exists x \in A_l^c \cap A$. Define $A_{l+1} = A_l \cup \{x\}$. So by induction we get the desired property. Now for every $n \in \mathbb{N}$ define $B_n = A_{2^n} \cap (\cup_{k=0}^{2^{n-1}} A_k)^c$. Easy to see that $B_n \cap B_m = \phi$ if $n \neq m$. Define $B = \{x_n : x_n \in B_n\}$. (Such a choice is possible by Axiom of Choice!!) $\square$