Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex Analysis (MTH 403A)

End semester Examination (2023-24-I)

Solutions

Date: 18 Nov. 2023

Time: 3 hours

Total marks: 45

Instructions

- All questions are compulsory.
- Questions can be answered in any serial order. You, however, **must** ensure that the serial numbers given of the answers are correct and distinct.
- All parts of a question must be answered **together**. Otherwise that question will not be graded.
- The answer to each of these questions must be furnished with all the necessary and relevant details. The rationale behind a step has to be explained in its right place. Insufficient explanations, inarticulate answers may result in deduction of marks.

Questions

1. (a) Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

Solution. As the restriction of f to $\overline{\mathbb{D}}$ is continuous, it follows from 1.10 of Exercise Sheet 8 that either f is either constant, or there exists $|\lambda| = 1$ such that,

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right)^{k_j}, \forall z \in \overline{\mathbb{D}},$$

where $a_1, \ldots, a_n$ are precisely all distinct zeros of f in $\mathbb{D}$ with order $m_1, \ldots, m_n$. It now follows that no a_j can be nonzero, otherwise $\lim\limits_{z \to a_j} |f(z)| = \infty$, which is not possible as f is entire. Hence all such functions are of the following form λz^n, where $|\lambda| = 1$ and $n = 0, 1, 2, \ldots$

(b) Let $A \stackrel{\text{def}}{=} \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Find all bounded holomorphic functions $f : \mathbb{C} \setminus A \longrightarrow \mathbb{C}$.

Solution. For each $n \in \mathbb{N}$, choose $\delta_n > 0$ such that $\frac{1}{m} \notin D(\frac{1}{n}, \delta_n)$, for all $m \neq n$. So each f has a removable singularity at every $1/n$ since f is bounded, consequent we obtain a bounded holomorphic function on $\mathbb{C} \setminus \{0\}$ which is bounded. So we extend this function analytically to $\mathbb{C}$ and the extension also remains bounded. Now from Liouville's theorem, we get f is constant.

Note. It is not correct at all to say at first that f has an isolated singularity at 0.

[2+3=5]

2. (a) Find all harmonic functions $u : \mathbb{R}^2 \longrightarrow [0, \infty)$.

Solution. Since $\mathbb{R}^2$ is simply connected, there exists an entire function f such that u is the real part of f. It now follows from an exercise of the sheet 6 that if the real part of an entire function is nonegative then it has to be constant.

(b) Suppose that $f : \mathbb{H} \cup (0, 1) \longrightarrow \mathbb{C}$ is a continuous function such that f is holomorphic on $\mathbb{H}$ and $\forall x \in (0, 1),\ f(x) = x^4 - 2x^2$. Find the value of $f(i)$.

Solution. *Step 1*. Extend f holomorphically to $U \overset{\text{def}}{=} U^+ \cup I \cup U^-$ by reflection principle, where $U^+ = \mathbb{H}$. Denote the extension by f as well. *Step 2*. Consider the function $g(z) = z^4 - 2z^2$, for all $z \in U$. Using identity theorem, conclude that $f \equiv g$ on U. *Step 3*. Now calculate $f(i)$. The answer is 3.

Note. $(0, 1)$ is not an open subset $\mathbb{C}$, hence $\mathbb{H} \cup (0, 1)$ is not an open subset of $\mathbb{C}$. So it is mathematically wrong to apply identity theorem to f and g on $\mathbb{H} \cup (0, 1)$. Furthermore, unless it is established correctly that $f(z) = z^4 - 2z^2$, for all $z \in \mathbb{H} \cup (0, 1)$, putting i in $x^4 - 2x^2$ does not make any sense.

[2+3=5]

3. (a) For each $n \in \mathbb{N}$, let $f_n(z) \overset{\text{def}}{=} 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$, for all $z \in \mathbb{C} \setminus \{0\}$. Show that, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, all zeros of f_n lie inside $D(0; \varepsilon)$.

Solution. For each $n \in \mathbb{N}$, consider the polynomial $g_n(z) \overset{\text{def}}{=} 1 + z + \dfrac{z^2}{2!} + \cdots + \dfrac{z^n}{n!}$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, g_n has all zeros outside the closed disc $\overline{D(0; \frac{1}{\varepsilon})}$. Since $f_n(z) = g_n\left(\frac{1}{z}\right)$, all zeros of f_n lie inside $D(0; \varepsilon)$, whenever $n \geq N$.

Note. Here we have used Exercise 1.1 of Exercise Sheet 3. Furthermore, although the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $e^{1/z}$ on every compact subset of $\mathbb{C} \setminus \{0\}$, Hurwitz's theorem does not apply to this case on any $D(0; \varepsilon)$ as $e^{1/z}$ has an essential singularity at 0.

(b) Let $\lambda > 1$. Locate the solutions of the equation $ze^{\lambda - z} = 1$ in the unit disc $\mathbb{D}$. Be as precise as possible.

Solution. Since $|(ze^{\lambda - z} - 1) - ze^{\lambda - z}| = |1| < e^{\lambda - \mathrm{Re}(z)} = |ze^{\lambda - z}|$ on $|z| = 1$, it follows from Rouche's theorem that $ze^{\lambda - z} - 1$ has exactly same number of zeros with $ze^{\lambda - z}$ counting orders in $\mathbb{D}$. Hence $ze^{\lambda - z} - 1$ has exactly one zero, which is simple, in $\mathbb{D}$. Now using the Intermediate value theorem, we obtain that zero must be real.

Note. The conclusion that the equation has a unique solution in the open unit disc is not precise enough.

[2+3=5]

4. (a) Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ be holomorphic. Determine whether or not the following statements are equivalent for f:

(i) f has an analytic n-th root on U for some positive integer $n > 1$.

(ii) f has an analytic logarithm on U.

Solution. Consider the function $f(z) = z^2$, for $z \in \mathbb{D} \setminus \{0\}$. Clearly $g(z) = z$ is an analytic square root of that, however f does not have an analytic logarithm as $f'(z)/f(z) = 2/z$ does not have a primitive on $\mathbb{D} \setminus \{0\}$.

Note. We have done this in Exercise Sheet 6. See 4.2 and 4.3.

(b) Let T be a nonidentity Möbius transformation on $\hat{\mathbb{C}}$. Show that a Möbius transformation S on $\hat{\mathbb{C}}$ commutes with T, i.e., $S \circ T = T \circ S$, if T and S have the same fixed points.

Solution. Let $z \neq w \in \hat{\mathbb{C}}$ be the fixed points of T and S. Choose a Möbius transformation P on $\hat{\mathbb{C}}$ such that $P(z) = 0$ and $P(w) = \infty$. Then both PTP^{-1} and PSP^{-1} fix 0 and ∞, consequently they are scalar multiplications so that they commute. Since PTP^{-1} and PSP^{-1} commute, it follows that S and T also commute.

Note.

(a) This is nothing but Exercise 1.4 of Exercise Sheet 10.

(b) It can also be done in a different way. Let $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, where $j = 1, 2$, be the corresponding matrices. If one $c_j \neq 0$ then ∞ cannot be a fixed point, so that both fixed points are in $\mathbb{C}$. Then by comparing the two quadratic equations obtained from the fixed points one can show S and T. Similarly one HAS to deal with the case that ∞ is a fixed point separately.

[2+3=5]

5. Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and $f \in H(\mathbb{D})$. Assume that $|f(z)| \leq |e^z|$, whenever $|z| = 1$. Maximize $|f(\log 2)|$ subject to the condition $f(-\log 2) = 0$.

Solution. This is Exercise 1.3 of Exercise Sheet 8.

3. Let $\omega = -\log 2$. Then $f \circ \varphi_\omega(0) = 0$. As for all $z \in \mathbb{D}$, $|f(\varphi_\omega(z))| \leq |e^{\varphi_\omega(z)}|$, we have $|e^{-\varphi_\omega(z)} \cdot f(\varphi_\omega(z))| \leq |z|$, $\forall z \in \mathbb{D}$.

$\Rightarrow |f(\varphi_\omega(z))| \leq |z| |e^{\varphi_\omega(z)}|$, $\forall z \in \mathbb{D}$.

Now, observe that $\varphi_\omega(z) = -\omega \Leftrightarrow \frac{\omega - z}{1 - \omega z} = -\omega \Leftrightarrow z = \frac{2\omega}{1+\omega^2}$

Hence, one obtains that, $|f(\log 2)| \leq \frac{2\log 2}{1+(\log 2)^2} \cdot e^{\log 2} = \frac{4 \log 2}{1+(\log 2)^2}$

Furthermore, equality occurs iff $\exists\, |\lambda| = 1$ s.t. $f(z) = \lambda e^z \varphi_\omega(z)$, $\forall z \in \mathbb{D}$.

Note.

1. In the first step, while showing that

$$\left|e^{-\varphi_w(z)} f(\varphi_w(z))\right| \leq |z|, \ \forall z \in \mathbb{D},$$

one needs to use the Maximum modulus principle. Then only it makes to apply Schwarz lemma in the next step.

2. Showing that $|f(\log 2)| \leq \dfrac{4 \log 2}{1 + (\log 2)^2}$, for any such function f, is not enough in order to conclude that $\dfrac{4 \log 2}{1 + (\log 2)^2}$ is the **maximum possible** value. You must produce at least one f for which the above-mentioned bound is attained.

3. It does not make any sense if one shows for a particular choice of f, $|f(\log 2)| = \dfrac{4 \log 2}{1 + (\log 2)^2}$, before showing that $\dfrac{4 \log 2}{1 + (\log 2)^2}$ is an upper bound. This is because before showing that $\dfrac{4 \log 2}{1 + (\log 2)^2}$ is an upper bound, there is no way one can know whether or not $\dfrac{4 \log 2}{1 + (\log 2)^2}$ is even a candidate.

[4]

6. Let Φ be the family of all analytic functions on $\mathbb{D}$ of the following form:

$$a_1 + a_2 z^2 + a_3 z^3 + \dots,$$

where $|a_n| \leq n$, for all $n \in \mathbb{N}$. Prove or disprove: Φ is compact.

Solution. In view of the compactness criterion, it is enough to show that Φ is closed and uniformly bounded on every compact subset of $\mathbb{D}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in Φ which converges to f almost uniformly. Then f must be holomprhic and, for all $k \geq 0$, $f_n^{(k)} \xrightarrow[n\to]{a.u.} f^{(k)}$. As $|f_n(0)| \leq 1$, $|f_n^{(k)}(0)| \leq k!k$, for all $k = 1, 2, \dots$, and $f_n'(0) = 0$, it follows that $|f(0)| \leq 1$, $|f^{(k)}(0)| \leq k!k$, for all $k = 1, 2, \dots$, and $f'(0) = 0$. Hence Φ is closed. Now take any $r \in (0, 1)$. For any $f \in \Phi$ and $z \in \overline{D(0; r)}$, one has

$$|f(z)| \leq |a_1 + a_2 z^2 + a_3 z^3 + \dots| \leq 1 + \sum_{n=2}^{\infty} n r^n,$$

since the series $\sum_{n=2}^{\infty} n r^n$ converges. [4]

7. Let $U \subseteq \mathbb{C}$ be a region and f be meromorphic on U having only finitely many poles at $z_1, \dots, z_n$ in U with orders $m_1, \dots, m_n$ respectively. Show that there exist $g, h \in H(U)$ with the following three properties:

(i) f and g have exactly same zeros with same orders,

(ii) h has zeros precisely at $z_1, \dots, z_n$ with orders $m_1, \dots, m_n$ respectively, and

(iii) $f(z) = \dfrac{g(z)}{h(z)}$, for all $z \in U \setminus \{z_1, \dots, z_n\}$.

Solution. Consider the following function:

$$g(z) \stackrel{\text{def}}{=} \left(\prod_{j=1}^{n} (z - z_j)^{m_j} \right) f(z)$$

on $U \setminus \{z_1, \ldots, z_n\}$. Since f has a pole at z_j of order m_j, one has $\lim_{z \to z_j} (z - z_j)^{m_j} f(z) \neq 0$. From this it follows that, g has a removable singularity at every z_j and once it is extended analytically to the whole U, it does not have a zero at any z_j. Hence the zeros f and g are precisely same counting orders. Let $h(z) \stackrel{\text{def}}{=} \prod_{j=1}^{n} (z - z_j)^{m_j}$ be the polynomial function. Now all the three conditions (i)-(iii) are immediate. [5]

8. Let f be a holomorphic function defined on $\mathbb{H}$ except possibly at finitely many poles. Suppose that f admits a continuous extension to the real line. Assume that there exist $M, R > 0$ and $a > 1$ such that
$$|f(z)| \leq \frac{M}{|z|^a}, \text{ whenever } |z| > R, z \in \mathbb{H}.$$
Find a formula for $\int_{-\infty}^{\infty} f(t)\, dt$ in terms of residues at the poles. [6]

9. Using residue theory evaluate the following integral:
$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx.$$
[6]