Example

$X_1, \ldots, X_n$ random sample from $N(\mu, \sigma^2)$

$\mu \in \mathbb{R}, \sigma > 0 \quad \underset{\sim}{\theta} = (\mu, \sigma)'$

$\forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_{\underset{\sim}{\theta}}(\underset{\sim}{x})}{f_{\underset{\sim}{\theta}}(\underset{\sim}{y})} = \frac{\exp\left(-\frac{1}{2}\left(\frac{\sum x_i^2}{\sigma^2} + \frac{n\mu^2}{\sigma^2} - \frac{2\mu}{\sigma^2}\sum x_i\right)\right)}{\exp\left(-\frac{1}{2}\left(\frac{\sum y_i^2}{\sigma^2} + \frac{n\mu^2}{\sigma^2} - \frac{2\mu}{\sigma^2}\sum y_i\right)\right)}$$

$$= \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\sum x_i^2 - \sum y_i^2\right) - \frac{2\mu}{\sigma^2}\left(\sum x_i - \sum y_i\right)\right)\right)$$

indep of θ iff $\sum x_i = \sum y_i$ & $\sum x_i^2 = \sum y_i^2$

$\Rightarrow T(\underset{\sim}{X}) = \left(\sum_1^n X_i, \sum_1^n X_i^2\right)$ or $\left(\bar{X}, \sum (X_i - \bar{X})^2\right)$ is m.s.s.

(i.e. jointly minimal suff for θ)

Example: $X_1, \ldots, X_n$ random sample from $N(\theta, \theta^2)$

$\theta > 0$

$\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \exp\left(-\frac{1}{2}\left(\frac{1}{\theta^2}\left(\Sigma x_i^2 - \Sigma y_i^2\right) - \frac{2}{\theta}\left(\Sigma x_i - \Sigma y_i\right)\right)\right)$$

indep of θ iff $\Sigma x_i = \Sigma y_i$ & $\Sigma x_i^2 = \Sigma y_i^2$

$\Rightarrow T(\underset{\sim}{X}) = \left(\Sigma X_i, \Sigma X_i^2\right)$ is jointly minimal suff statistic for θ.

Example: $X_1, \ldots, X_n$ random sample from $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$\theta \in \mathbb{R}$

$\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \frac{I_{(\theta-\frac{1}{2},\, x_{(1)})}\, I_{(x_{(n)},\, \theta+\frac{1}{2})}}{I_{(\theta-\frac{1}{2},\, y_{(1)})}\, I_{(y_{(n)},\, \theta+\frac{1}{2})}}$$

indep of θ iff $x_{(1)} = y_{(1)}$ & $x_{(n)} = y_{(n)}$

$\Rightarrow T(\underset{\sim}{X}) = (X_{(1)}, X_{(n)})$ is jointly minimal sufficient statistic for θ.

Improvement of an unbiased estimator using sufficient statistic

Consider the problem of estimation of θ in $N(\theta, 1)$; $\theta \in \mathbb{R}$

$(X_1, \ldots, X_n)$ random sample

$X_i \ (i = 1, \ldots n)$, $\frac{X_1 + X_2}{2}$, $\frac{X_1 + X_n}{2}$, ..., $\bar{X}$ are all unbiased estimator for θ; $\exists$ infinite number of unbiased estimators in this case.

A natural criterion to pickup the "best" unbiased estimator would be to look for unbiased estimator having least variance $\forall \theta$, i.e. find $\delta^*(\underline{X}) \ni$

(i) $E_\theta \, \delta^*(\underline{X}) = g(\theta)$ ← the estimand (θ in $N(\theta, 1)$ example)

& (ii) $V_\theta \, \delta^*(\underline{X}) \leq V_\theta(\delta(\underline{X})) \quad \forall \theta \in \Theta$ and $\forall \delta$ satisfying (i)

Such a $\delta^*(\underline{X})$ is called Uniformly minimum variance unbiased estimator for $g(\theta)$ (or UMVUE for $g(\theta)$)

Note: The following result provides a way to improve upon an unbiased estimator, in terms of lower variance, using information of sufficient statistic.

Rao - Blackwell Theorem

Let $\delta(\underline{X})$ be any unbiased estimator of $g(\theta)$ and $T(\underline{X})$ be a sufficient statistic for θ.

Define $\eta(T) = E(\delta(\underline{X}) \mid T)$

then

(i) $\eta(T)$ is a statistic as T is sufficient

(ii) $E(\eta(T)) = E\, E(\delta(\underline{X})|T)$

$$= E(\delta(\underline{X})) = g(\theta)$$

i.e. $\eta(T)$ is an unbiased estimator of $g(\theta)$

and (iii) $V(\eta(T)) \leq V(\delta(\underline{X}))$ (equality iff $\eta(T) = \delta(\underline{X})$ w.p. 1)

(with probability)

__Remark__: The above th^m leads to a new estimator $\eta(T) = E(\delta(X)|T)$; which is called Rao-Blackwellized version of $\delta(\underline{X})$.