<u>Remark</u>: Ex 1, 2, 3 are direct approaches to prove convergence in law/distribution.

Convergence in law can also be proved using m.g.f. convergence.

(4) $X_n \sim Bin(n, \theta)$

Suppose $n \to \infty \ni n\theta = \lambda$ is fixed i.e. $\theta = \frac{\lambda}{n}$

$$M_{X_n}(t) = \left((1-\theta) + \theta e^t\right)^n$$

$$= \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n$$

$$\longrightarrow e^{\lambda\cancel{\theta}(e^t-1)} \quad \text{as } n \to \infty$$

$\Rightarrow X_n \xrightarrow{L} X$; where $X \sim P(\lambda)$

(5) $X_1, \dots$ i.i.d. $N(0,1)$

$\bar{X}_n \sim N(0, \frac{1}{n})$

$M_{\bar{X}_n}(t) = e^{\frac{t^2}{2n}} \to 1$ as $n \to \infty$

$\Rightarrow \bar{X}_n \xrightarrow{L} X$; where X is degenerate at 0

(6) $X_n \sim \chi^2_n$; $E X_n = n$; $V X_n = 2n$

$M_{X_n}(t) = (1-2t)^{-n/2}$ $\quad t < \frac{1}{2}$

$$Y_n = \frac{X_n - n}{\sqrt{2n}}$$

$$M_{Y_n}(t) = E\left(e^{t\left(\frac{X_n - n}{\sqrt{2n}}\right)}\right)$$

$$= e^{-\frac{tn}{\sqrt{2n}}} E\left(e^{\frac{tX_n}{\sqrt{2n}}}\right)$$

$$= e^{-\frac{tn}{\sqrt{2n}}} \left(1 - \frac{2t}{\sqrt{2n}}\right)^{-n/2} \qquad t < \sqrt{\frac{n}{2}}$$

$$= \left(e^{t\sqrt{\frac{2}{n}}}\right)^{-n/2} \left(1 - \sqrt{\frac{2}{n}}\, t\right)^{-n/2}$$

$$= \left[\left(1 + t\sqrt{\frac{2}{n}} + \frac{t^2\, 2/n}{2!} + \frac{t^3\, 2^{3/2}/n^{3/2}}{3!} \dots\right) - t\sqrt{\frac{2}{n}}\left(1 + t\sqrt{\frac{2}{n}} + \frac{t^2\, 2/n}{2!} + \dots\right)\right]^{-n/2}$$

$$= \left(1 - \frac{t^2}{n} + \frac{k_1}{n^{3/2}} - \dots\right)^{-n/2}$$

$$= \left(1 - \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^{-n/2}$$

$$\to e^{t^2/2} \leftarrow \text{m.g.f. of } N(0,1)$$

$$\Rightarrow \frac{X_n - n}{\sqrt{2n}} \xrightarrow{L} N(0,1) \text{ r.v.}$$

Some important results

(i) If $X_n \xrightarrow{p} X$ then $X_n \xrightarrow{L} X$

Converse is not true in general.

However, if $X_n \xrightarrow{L} c$ (a const), then

$$X_n \xrightarrow{p} c$$

(ii) Slutsky's Lemma

If $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{p} c$, then

(i) $X_n \pm Y_n \xrightarrow{L} X \pm c$

(ii) $X_n Y_n \xrightarrow{L} cX$

(iii) $\frac{X_n}{Y_n} \xrightarrow{L} X/c$ (for $c \neq 0$)

(iii) Δ-method or Δ-rule

Let $\{X_n\}$ be a seq of random variables $\ni$

$$\sqrt{n}(X_n - \theta) \xrightarrow{L} N(0, \sigma^2)$$

Suppose g be real valued f^n differentiable at θ $\ni$

$g'(\theta) \neq 0$, then

$$\sqrt{n}\left(g(X_n) - g(\theta)\right) \xrightarrow{L} N\left(0, (g'(\theta))^2 \sigma^2\right)$$

Central Limit Theorem (CLT)

In WLLN, we investigated

$$\frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

i.e convergence of $\frac{S_n - a_n}{b_n}$ to a degenerate distⁿ (degenerate at 0)

In CLT, we investigate convergence of $\frac{S_n - a_n}{b_n}$ to a non-degenerate distⁿ.

Defⁿ: Let $X_1, X_2, \ldots$ be a seq of i.i.d. r.v.s with common d.f F. We say that F belongs to the domain of attraction of a distⁿ V if there exists centering constants $\{A_n\}$ and norming constants $\{B_n\}$ $(B_n > 0)$ $\ni$

as $n \to \infty$

$$P\left(\frac{S_n - A_n}{B_n} \leq x\right) \to V(x) \quad \text{at all continuity points of } V$$

$$S_n = \sum_{i=1}^{n} X_i$$

i.e. $\frac{S_n - A_n}{B_n} \xrightarrow{L} X$; where d.f. of X is $V(\cdot)$

Lindeberg - Levy CLT

Let $\{X_n\}$ be a seq of i.i.d. r.v.s with $E(X_i) = \mu$ and $V(X_i) = \sigma^2 < \infty$, then for $S_n = \sum_{i=1}^{n} X_i$

$$\frac{S_n - E S_n}{\sqrt{V(S_n)}} \xrightarrow{L} N(0,1) \text{ r.v.}$$

i.e. $$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{L} X \quad ; \quad X \sim N(0,1)$$

i.e. $$\frac{n\bar{X}_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{L} X \quad ; \quad X \sim N(0,1)$$

i.e. $$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{L} X ; \quad X \sim N(0,1)$$

i.e. $$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{L} Y ; \quad Y \sim N(0,\sigma^2)$$

Applications of CLT

(1) $X_1, \dots X_n$ are i.i.d. $\mathrm{Exp}(\theta)$ (exponential with mean θ)

i.e. $f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0 \\ 0, & o/w \end{cases}$

$E(X_i) = \theta \quad ; \quad V(X_i) = \theta^2$

By CLT

$$\frac{\sqrt{n}(\bar{X}_n - \theta)}{\theta} \xrightarrow{L} N(0,1)$$

i.e. $$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{L} N(0,\theta^2)$$

Further, suppose we are interested in ~~as~~ asymptotic distn ($n \to \infty$) of $\frac{1}{\bar{X}_n}$; we can apply Δ-rule on the CLT result with

$$g(x) = \frac{1}{x} \qquad g'(x) = -\frac{1}{x^2} \qquad g'(\theta) \neq 0$$

Δ-rule $\Rightarrow$ $$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) \xrightarrow{L} N\Big(0, \underbrace{\left(\frac{1}{\theta^4}\right)}_{(g'(\theta))^2} \cdot \theta^2\Big)$$

i.e. $$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) \xrightarrow{L} N\left(0, \frac{1}{\theta^2}\right)$$