Assignment 1: Several variables calculus & differential geometry (MTH305A)

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(1) Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a continuously differentiable injective function such that $\det(f'(x)) \neq 0, \forall x \in \Omega$.

  (a) Show that $f(\Omega)$ is open set in $\mathbb{R}^n$

  (b) Show that $f^{-1} : f(\Omega) \to \Omega$ is differentiable.

  (c) Show that $f(B)$ is open in $\mathbb{R}^n$ for any open set $B \subset \Omega$.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is NOT injective.

(3) (a) If $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$f'(a) \neq 0, \text{for all } a \in \mathbb{R},$$

  show that f is injective.

  (b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

  Show that $\det(f'(x, y)$ for all $(x, y) \in \mathbb{R}^2$ but f is not injective.

(4) Consider the following system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0. \end{aligned}$$

Show that this system of equations can be solved for

  (a) x, y, u in terms of z;

  (b) x, z, u in terms of y and

  (c) y, z, u in terms of x.

Does there exist a solution of x, y, z in terms of u?

(5) Let us consider a function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_1(x, y) = x^2 - y^2 \text{ and}$$
$$f_2(x, y) = 2xy.$$

(a) What is the range of f?

(b) Show that $\det(f'(a)) \neq 0$ for all $a \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(c) Show that every point $a \in \mathbb{R}^2 \setminus \{(0, 0)\}$ has a neighbourhood in which f is injective, but f is not globally injective.

(d) consider $a = (3, 5)$ and $b = f(a)$. Let g be a continuous inverse of f defined in an open neighbourhood of b such that $g(b) = a$. Find an explicit formula for g and verify

$$g'(b) = [f'(a)]^{-1}.$$

(6) Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$J(p_1, p_2) = (-p_2, p_1), \text{ for all } (p_1, p_2) \in \mathbb{R}^2.$$

Show that

(a) $J^2 = J \circ J = -Id$.

(b) J is inner product preserving and hence it is norm preserving.

(c) For every $p = (p_1, p_2) \in \mathbb{R}^2$, the vectors $J(p)$ and p are perpendicular to each other.

(d) Let $p, q \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Show that there exists a unique number θ satisfying

$$\cos\theta = \frac{\langle p, q\rangle}{\|p\|\|q\|} \text{ and } \sin\theta = \frac{\langle p, J(q)\rangle}{\|p\|\|q\|}, \quad 0 \leq \theta < 2\pi.$$

The oriented angle from q to p is θ.