MID-SEMESTER EXAMINATION
MTH-204, MTH-204A
ABSTRACT ALGEBRA
Spring-2023

Time Allowed: 2 hrs Max. Marks: 30

1. Give **complete** and **precise** definitions for the following. [5]

a. Group b. Quotient Group c. Isomorphism of Groups d. Direct Product of Groups e. Group action

2. Give an example of each of the following. [4]

a. A non-abelian group of order 38.

And: $D_{19} = \{1, x, x^2, \cdots, x^{18}, y, xy, x^2y, \cdots, x^{18}y\}$, $x^{19} = y^2 = 1$ and $yx = x^{-1}y$, the dihedral group of order 38.

b. A group G and two elements x, y in G such that $o(x)$ and $o(y)$ are finite but $o(xy)$ is infinite.

Ans: Take $G = GL_2(\mathbb{R})$, and two elements A and B such that

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A and B each have order 2, but there product AB is of infinite order:

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

To see this, note that

$$(AB)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

which is the identity matrix only for $n = 0$.

c. A subgroup of index 6 in S_4.

Ans: $\{1, (12)(34), (13)(24), (14)(23)\}$.

d. A non-identity automorphism of Q_8.

Ans: $i \mapsto j$, $j \mapsto i$ and $k \mapsto -k$ is a non-identity automorphism.

3. State the following theorems and prove any one of them. [10]

a. Lagrange's theorem.

b. First isomorphism theorem for groups.

c. Sylow's theorems.

d. Cayley's theorem.

e. Burnside's lemma.

4. Find all the subgroups of D_4 and determine which are normal. [4]

Ans: The subgroups of $D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ are:
$D_4, \{e\}, \{1, y\}, \{1, xy\}, \{1, x^2y\}, \{1, x^3y\}, \{1, x^2\}, \{1, x, x^2, x^3\}, \{1, x^2, y, x^2y\}, \{1, x^2, xy, x^3y\}$.

The normal subgroups are $D_4, \{e\}, \{1, x^2\}, \{1, x, x^2, x^3\}, \{1, x^2, y, x^2y\}, \{1, x^2, xy, x^3y\}$.

$Z(D_4) = \{1, x^2\}$ and hence it is normal whereas $\{1, x, x^2, x^3\}, \{1, x^2, y, x^2y\}, \{1, x^2, xy, x^3y\}$ are of index 2, hence normal.

5. Prove that in a group of odd order a non identity element is not conjugate to its inverse. [4]

Ans: Suppose $x \neq 1$ is conjugate to x^{-1}. If $x = x^{-1}$, then $x^2 = 1$. Hence $|G|$ is divisible by 2. This is a contradiction. Hence $x \neq x^{-1}$. Since $|C_x|$ is odd, C_x contains an element y which is neither x nor x^{-1}. Since x is conjugate to y, x^{-1} is conjugate to y^{-1}. Hence $y^{-1} \in C_x$. Since $y \neq y^{-1}$, we must conclude that $|C_x|$ is even. This is a contradiction.

6. Let S_3 act on the set $A = \{(i, j) : 1 \leq i, j \leq 3\}$ by $\sigma.(i, j) = (\sigma(i), \sigma(j))$ for $\sigma \in S_3$ and $(i, j) \in A$. [4]
Compute the orbits and stabilizers of this action.

Ans: $O_{(1,1)} = O_{(2,2)} = O_{(3,3)} = \{(1, 1), (2, 2), (3, 3)\}$.

If $i \neq j$ the orbit $O_{(i,j)} = \{(k, l) : 1 \leq k, l \leq 3 \text{ and } k \neq l\}$.

The stabilizer of $(1, 1)$ is the set $\{id, (23)\}$ where id is the identity element in S_3.

The stabilizer of $(2, 2)$ is the set $\{id, (13)\}$.

The stabilizer of $(3, 3)$ is the set $\{id, (12)\}$.

If $i \neq j$ then the stabilizer of (i, j) contains only the identity element.