# Department of Mathematics

**Calculus of Several Variables and Differential Geometry**

Assignment-III

1. Find the unit speed parametrization of the following curves.
   (a) For $a > 0$, let $\gamma(t) := (a\cos t, a\sin t)$ for $t \in \mathbb{R}$.
   (b) For $a, b > 0$, let $\gamma(t) := (ae^{bt}\cos t, ae^{bt}\sin t)$ for $t \in \mathbb{R}$.
   (c) For $a, b > 0$, let $\gamma(t) := (a\cos t, a\sin t, bt)$ for $t \in \mathbb{R}$.

2. Find the curvature of the curves
   (a) $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = e^t(\cos\, t, \sin\, t)$.
   (b) $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t, t^2)$.
   (c) $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (\cosh t, \sinh t)$.
   (d) $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (a\cos t, b\sin t)$ where $a, b > 0$ are fixed.
   (e) $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma(t) = (t, \sin t)$.

   In each of the problem sketch the trace of the curves.

3. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve and $\psi\colon [c, d] \to [a, b]$ be a smooth bijection. Let $\sigma\colon [c, d] \to \mathbb{R}^2$ defined by $\sigma(s) := \gamma(\psi(s))$ be a reparametrization of the curve $\gamma$. Show that length of the curves $\gamma$ and $\sigma$ are equal.

4. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve and $A$ be a $2 \times 2$ orthogonal matrix. Let $\gamma_A\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve defined by $\gamma_A(t) := A(\gamma(t))$. Find the relation between the curvature $\kappa_\gamma(t)$ and $\kappa_{\gamma_A}(t)$ for $t \in [a, b]$.

5. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve parametrized by arc length. For $s \in [a, b]$, we denote the tangent vector $\gamma'(s) = (x'(s), y'(s))$ by $t(s)$ and the unit postive normal $(-y'(s), x'(s))$ by $N(s)$. By definition $t'(s) = \kappa(s)N(s)$. Prove that $N'(s) = -\kappa(s)t(s)$ for $s \in [a, b]$.

6. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve. Assume that there exists $t_0$ in $[a, b]$ such that $\|\gamma(t\| \leq \|\gamma(t_0)\|$. Show that $\kappa(t_0) \geq \frac{1}{\|\gamma(t_0)\|}$. Can you infer some information about the curvature if $\|\gamma(t\| \geq \|\gamma(t_0)\|$?

7. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular smooth curve with constant positive curvature $\kappa$. Show that the trace of $\gamma$ is contained in a circle of radius $1/\kappa$. What can you say when the curvature $\kappa$ is a non-positive constant.

8. Can you find a regular smooth curve $\gamma\colon [a, b] \to \mathbb{R}^2$ such that $\|\gamma'(t)\| = 3$ and $\kappa(t) = 4$ for $t \in [a, b]$.

9. Can you find a regular smooth curve $\gamma\colon [a, b] \to \mathbb{R}^2$ such that $\|\gamma'(t)\| = 3$ and $\kappa(t) = -4$ for $t \in [a, b]$.

10. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a regular curve such that all the normal lines to the curve pass through a point $a \in \mathbb{R}^2$. Show that the trace of the curve $\gamma$ is contained in a circle.

11. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve such that, for all $t \in \mathbb{R}$, $\|\gamma(t)\| = R$ for some $R > 0$. Show that the curvature $\kappa(t) \geq 1/R$.

12. Find the curvature and torsion of the curve $\gamma(t) := (a\cos t, a\sin t, bt)$ for $t \in \mathbb{R}$ and $a, b > 0$ are fixed.

13. Find the curvature and torsion of the curve $\gamma(t) := (e^t \cos t, e^t \sin t, e^t)$ for $t \in \mathbb{R}$.

Apart from the exercises above you may solve the problems in the book *Curves and Surfaces* by Sebastian Montiel and Antonio Ros. The exercise with $\uparrow$ mark provided with hints in the book to help you.