Remark:

$\alpha F_1(x) + (1-\alpha) F_2(x)$ will be d.f. of mixed type

$\forall \alpha \ni 0 < \alpha < 1$

if $\alpha = 0$; X is continuous r.v.

If $\alpha = 1$. X is discrete r.v.

Remark: Any dist^n f^n $F(.)$ can be expressed as.

$$F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$$

d.f. of discrete r.v.   d.f. of cont r.v.

Example: Let X be r.v. with d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ x/4, & 0 \le x < 1 \\ x/3, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

jump discontinuities at $x = 1, 2$

$D = \{1, 2\}$; $D \neq \phi$

$\Rightarrow X$ is not cont r.v.

$$P(X \in D) = \sum_{x \in D} P(X = x)$$

$$= \sum_{x \in D} \left(F(x) - F(x-) \right)$$

$$= \sum_{x = 1,2} \left(F(x) - F(x-) \right)$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(1 - \frac{2}{3} \right) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

$P(X \in D) = \frac{5}{12} < 1$

$\Rightarrow X$ is not discrete r.v.

$\Rightarrow X$ is neither discrete or cont

Discrete part of $F(.)$:

$$F_1(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{12}, & 1 \le x < 2 \\ \frac{5}{12} & x \geq 2 \end{cases}$$

Take $\alpha = \frac{5}{12}$; $\alpha F_d(x) = F_1(x)$

$$F_d(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{5}, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

$F_d(x)$ is d.f. of a discrete r.v. $\rightarrow$ p.m.f. $\left\{ \begin{array}{l} P(X_d = 1) = \frac{1}{5} \\ P(X_d = 2) = \frac{4}{5} \end{array} \right\}$.

$F_2(x)$: Continuous part of $F(.)$

$F_2(x) = F(x) - F_1(x)$

$$F_2(x) = \begin{cases} 0, & x < 0 \\ x/4, & 0 \le x < 1 \\ x/3 - \frac{1}{12}, & 1 \le x < 2 \\ \frac{7}{12} & x \geq 2 \end{cases}$$

$F_2(x) = (1-\alpha) F_c(x)$; $1 - \alpha = \frac{7}{12}$

$$F_c(x) = \begin{cases} 0, & x < 0 \\ 3x/7, & 0 \le x < 1 \\ \frac{12}{7}\left(\frac{x}{3} - \frac{1}{12}\right) = \frac{4}{7}x - \frac{1}{7}, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

$F_c(x)$ is continuous everywhere

$F_c(x)$ is the d.f. of a cont r.v.

p.d.f. of X_c

$$f_{X_c}(x) = \begin{cases} 3/7, & 0 \le x < 1 \\ 4/7, & 1 \le x < 2 \\ 0, & o/w \end{cases}$$

$$f_{X_c}(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f_{X_c}(x)\, dx = \int_0^1 \frac{3}{7}\, dx + \int_1^2 \frac{4}{7}\, dx = 1$$

$$F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$$

Mathematical Expectation

$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X)$

$g : \mathbb{R} \xrightarrow{g} \mathbb{R}$

Expected value of $g(X)$: mathematical expectation of $g(X)$

$E(g(X))$ exists if $E|g(X)| < \infty$

Suppose, X is a discrete r.v. with p.m.f.

$X : x_1, x_2, \ldots$

$P(X=x) : p_1, p_2, \ldots$

$E\,g(X)$ is said to exist and equals $\sum_{i=1}^{\infty} g(x_i)\,p_i$ provided $\sum_i |g(x_i)|\,p_i < \infty$

If X is continuous with p.d.f. $f_X(x)$, then $E(g(X))$ exists and equals $\int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$ provided $\int_{-\infty}^{\infty} |g(x)|\, f_X(x)\,dx < \infty$

Special cases

(i) $g(X) = X$

$E\,g(X) = E\,X = \mu_1'$: mean of the distⁿ of X

(ii) $g(X) = X^n$: n is a positive integer

$E\,g(X) = E\,X^n = \mu_n'$

n^{th} moment about origin of r.v. X

(iii) $g(X) = (X-a)^n$

$E\,g(X) = E(X-a)^n$: nth moment of X about the pt a

If $a = E(X)$, then

$E(X - E(X))^n = \mu_n$: nth order central moment of X

$n = 2$; $\mu_2 = E(X - E(X))^2 \rightarrow$ variance of X

$= \sigma^2$

$\mu_2^{1/2} = \sigma$: standard deviation of X