**Problem Set - 2**
**MTH-204, MTH-204A**
**Abstract Algebra**

1. Let $G$ be a group such that the intersection of all its subgroups which are different from $\{e\}$ is a subgroup different from $\{e\}$. Prove that every element in $G$ has finite order.

2. If $G$ has no nontrivial subgroups, show that $G$ must be finite of prime order.

3. If $H$ is a subgroup of $G$, and $a \in G$, let $aHa^{-1} = \{aha^{-1} : h \in H\}$. Show that $aHa^{-1}$ is a subgroup of $G$. If $H$ is finite, what is $o(aHa^{-1})$ ?

4. Suppose that $H$ is a subgroup of $G$ such that whenever $Ha \neq Hb$ then $aH \neq bH$. Prove that $gHg^{-1} \subseteq H$ for all $g \in G$.

5. For $m, n \in \mathbb{Z}$, compute $m\mathbb{Z} \cap n\mathbb{Z}$.

6. Let $G$ be an abelian group and suppose that $G$ has elements of orders $m$ and $n$, respectively. Prove that $G$ has an element whose order is the least common multiple of $m$ and $n$.

7. Prove that every subgroup of a cyclic group is cyclic.

8. Let $G$ be a cyclic group of order $n$, then prove that for each $d$ dividing $n$, $G$ has a unique subgroup of order $d$.

9. Let $G$ be a cyclic group of order $n$. Prove that $G$ has $\phi(n)$ generators.

10. Let $G$ be a cyclic group of order $n$. If $d$ divides $n$, show that the number of elements of order $d$ in $G$ is $\phi(d)$. It is 0 otherwise.

11. Show that $U_9, U_{17}, U_{18}$ are cyclic groups whereas $U_8, U_{20}$ are not cyclic.

12. If $p$ is a prime, prove that $\phi(p^a) = p^a - p^{a-1}$.

13. If $gcd(m, n) = 1$, prove that $\phi(mn) = \phi(m).\phi(n)$.