Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Midsem solution (MTH305A)

Semester: 2022-2023, I

Full Marks. 50 Time. 120 Minutes

Note. To answer the questions, if you use any theorem please state it properly.

(1) Let $f(x,y) = \frac{x^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(a) Show that $D_1 f, D_2 f$ are bounded functions in $\mathbb{R}^2$ and hence f is continuous.

(b) Show that directional derivative at $(0,0)$ exists in all directions.

(c) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be differentiable map with $\gamma(0) = (0,0)$ and $\|\gamma'(0)\| \neq 0$. Let $g(t) = f(\gamma(t))$. Show that g is differentiable for all $t \in \mathbb{R}$. But f is not differentiable at $(0,0)$.

[4+2+4=10 points]

Solution.

a)

- Boundedness of $D_1 f(x,y)$:

$$D_1 f(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^3/t^2}{t} = 1.$$

For $(x,y) \neq (0,0)$,

$$D_1 f(x,y) = \frac{3x^2(x^2+y^2) - x^3(2x)}{(x^2+y^2)^2} = \frac{x^2}{x^2+y^2} + \frac{2x^2y^2}{(x^2+y^2)^2} \leq 2.$$

- Boundedness of $D_2 f(x,y)$:

$$D_2 f(0,0) = 0$$

$$|D_2 f(x,y)| = \left| -\frac{2x^3y}{(x^2+y^2)^2} \right| = \left| \frac{x^2}{x^2+y^2} \right| \times \left| \frac{2xy}{x^2+y^2} \right| \leq 1.$$

b)Existence of directional derivatives:

For $(u,v) \neq (0,0) \in \mathbb{R}^2$, the directional derivative of f at the point $(0,0)$ along the direction (u,v) is

$$\begin{aligned} D_{(u,v)} f(0,0) &= \lim_{t\to 0} \frac{f(tu,tv) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{t^3u^3}{t(t^2u^2+t^2v^2)} \\ &= \frac{u^3}{u^2+v^2}. \end{aligned}$$

c) Differentiability of g:

- To show show that g is differentiable on $\mathbb{R}$, it suffices to show that it is differentiable at 0.

$$\begin{aligned}\lim_{t\to 0}\frac{g(t)-g(0)}{t} &= \lim_{t\to 0}\frac{f\circ\gamma(t)}{t}\\ &= \lim_{t\to 0}\frac{\gamma_1(t)^3}{t(\gamma_1(t)^2+\gamma_2(t)^2)}\\ &= \lim_{t\to 0}\frac{\left(\frac{\gamma_1(t)}{t}\right)^3}{\left(\frac{\gamma_1(t)}{t}\right)^2+\left(\frac{\gamma_2(t)}{t}\right)^2}\\ &= \frac{(\gamma_1'(0))^3}{(\gamma_1'(0))^2+(\gamma_2'(0))^2}\end{aligned}$$

- Finally, we show that f is not differentiable. If f is differentiable at $(0,0)$, then $T=f'(0,0):\mathbb{R}^2\to\mathbb{R}$ is defined by

$$T(h,k)=D_1f(0,0)h+D_2f(0,0)k=h.$$

$$\begin{aligned}\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-h|}{\|(h,k)\|} &= \lim_{(h,k)\to(0,0)}\frac{|hk^2|}{(h^2+k^2)^{\frac{3}{2}}}\\ &= \lim_{h\to 0}\frac{h^3m^2}{(h^2+m^2h^2)^{\frac{3}{2}}}\text{ along } k=mh\\ &= \frac{m^2}{(1+m^2)^{\frac{3}{2}}}.\end{aligned}$$

Now, it follows that the function f is not differentiable at $(0,0)$.

(2) (a) Suppose $f:\mathbb{R}^2\to\mathbb{R}$ is a function such that the partial derivatives D_xf, D_yf exist and are continuous in $\mathbb{R}^2$. Show that the function f is differentiable on its domain $\mathbb{R}^2$.

Solution.

- Let $(a,b)\in\mathbb{R}^2$.

$$\begin{aligned}f(a+h,b+k)-f(a,b) &= f(a+h,b+k)-f(a+h,b)+f(a+h,b)-f(a,b)\\ &= kD_2f(a+h,b+k')+hD_1f(a+h',b),\end{aligned}$$

where $(a+h,b+k')\in L[(a+h,b+k),(a+h,b)]$ and $(a+h',b)\in L[(a+h,b),(a,b)]$.

- Now, we have

$$\begin{aligned}&\lim_{(h,k)\to(0,0)}\frac{|f(a+h,b+k)-f(a,b)-[D_1f(a,b)h+D_2f(a,b)k]|}{\|(h,k)\|}\\ &\le \lim_{(h,k)\to(0,0)}|D_1f(a+h',b)-D_1(a,b)|\frac{|h|}{\|h,k\|}+\lim_{(h,k)\to(0,0)}|D_2f(a+h,b+k')-D_2(a,b)|\frac{|k|}{\|h,k\|}\\ &\le \lim_{(h,k)\to(0,0)}|D_1f(a+h',b)-D_1(a,b)|+\lim_{(h,k)\to(0,0)}|D_2f(a+h,b+k')-D_2(a,b)|=0,\end{aligned}$$

as D_1f and D_2f both are continuous at (a,b).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$, but its partial derivatives $D_i f, i = 1,2$, are not continuous at $(0,0)$.

[5+5=10 points]

Solution.

- For $(x,y) \in \mathbb{R}^2 \setminus (0,0)$, we have

$$D_1 f(x,y) = 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$
$$D_2 f(x,y) = 2y \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

- For $(x,y) = (0,0)$, we have

$$D_1 f(x,y) = 0$$
$$D_2 f(x,y) = 0.$$

- If the function $D_1 f(x,y)$ is continuous at $(0,0)$, then $\lim\limits_{(x,y)\to(0,0)} D_1 f(x,y)$ must exists and equal to 0. But, along the line $y = mx$, we have

$$\lim_{x\to 0} 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 \text{ and}$$
$$\lim_{x\to 0+} \frac{x}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$
$$= \lim_{x\to 0+} \frac{1}{\sqrt{1+m^2}} \cos\left(\frac{1}{x\sqrt{1+m^2}}\right) \text{ does not exists}$$

Therefore, $D_1(x,y)$ is not continuous at $(0,0)$. A similar argument shows that $D_2 f(x,y)$ is also not continuous at $(0,0)$.

- To show that f is differentiable, it suffices to show that it is differentiable at $(0,0)$.

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - 0(h,k)|}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{|\|(h,k)\|^2 \sin\left(\frac{1}{\|h,k\|}\right)|}{\|(h,k)\|}$$
$$= \lim_{(h,k)\to(0,0)} \|h,k\| \times \left|\sin\left(\frac{1}{\|h,k\|}\right)\right|$$
$$= 0.$$

(3) Consider the equation $x^2 + y^2 - 25 = 0$. Can we write y as a unique differentiable function of x near $(3,4)$.

[3 points]

Solution. We apply implicit function theorem to answer the question affirmative.

Theorem 1 (Implicit function theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be C^1 on an open set containing $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f(a, b) = 0$. Let*

$$M = \left(D_{n+j} f_i(a, b)\right)_{m\times m}.$$

If $\det(M) \neq 0$, then there exist open sets $a \in A \subset \mathbb{R}^n$ and $b \in B \subset \mathbb{R}^m$ with the following properties: For each $x \in A$, there exists a unique $g(x) \in B$ such that

$$f(x, g(x)) = 0.$$

and g is differentiable.

(4) Show that vector product is not associative.

Solution.

Consider $u = (1, 0, 0), v = (1, 1, 0)$ and $w = (1, 1, 1)$. Then $u \times v = (0, 0, 1)$ and $v \times w = (1, -1, 0)$.

$$\begin{aligned}(u \times v) \times w &= (0, 0, 1) \times (1, 1, 1) = (-1, 1, 0)\\ u \times (v \times w) &= (1, 0, 0) \times (1, -1, 0) = (0, 0, -1).\end{aligned}$$

Hence we conclude that $(u \times v) \times w \neq u \times (v \times w)$.

[3 points]

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x^2 - y^2, 2xy).$$

(a) Show that the function f is a local diffeomorphism except possibly at the origin.

Solution.

$$f'(x, y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \implies \det(f'(x, y)) = 4(x^2 + y^2).$$

Therefore, $\det(f'(x, y)) = 0 \iff (x, y) = (0, 0)$. Inverse function theorem implies that f is a local diffeomorphism except at $(0, 0)$.

(b) Let $U \subset \mathbb{R}^2$ be an open set containing $(1, 1)$ chosen so that the restricted function $f|_U : U \to f(U)$ is a diffeomorphism. Let $g = (g_1, g_2) : f(U) \to U$ be the inverse of $f|_U$. Calculate the partial derivatives

$$\frac{\partial g_1}{\partial x}(0, 2), \frac{\partial g_1}{\partial y}(0, 2), \frac{\partial g_2}{\partial x}(0, 2), \text{ and } \frac{\partial g_2}{\partial y}(0, 2).$$

Solution.

$$f'(x, y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \implies f'(1, 1) = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \text{ and } \left(f'(1, 1)\right)^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Now, applying chain rule we have

$$g \circ f(x,y) = (x,y) \implies g'(f(1,1))f'(1,1) = I_{2\times 2} \implies g'(0,2) = (f'(1,1))^{-1}.$$

Thus we have

$$g'(0,2) = \begin{bmatrix} \frac{\partial g_1}{\partial x}(0,2) & \frac{\partial g_1}{\partial y}(0,2) \\ \frac{\partial g_2}{\partial x}(0,2) & \frac{\partial g_2}{\partial y}(0,2) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

[2+3=5 points]

(6) Find a unit-speed re-parameterisation of $\alpha(t) = (e^t \cos t, e^t \sin t), t \in \mathbb{R}$.
Solution.

- $\|\alpha'(t)\| = \sqrt{2}e^t$.
- $s(t) = \int_0^t \|\alpha'(u)\| du = \sqrt{2}(e^t - 1)$. Note that $s : \mathbb{R} \to (-\sqrt{2}, \infty)$ is a diffeomorphism.
- $s^{-1} = h : (-\sqrt{2}, \infty) \to \mathbb{R}$ is given by $h(s) = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$
- $\beta(s) = \alpha \circ h(s) = \frac{s+\sqrt{2}}{\sqrt{2}}\left(\cos\left(\ln \frac{s+\sqrt{2}}{\sqrt{2}}\right), \sin\left(\ln \frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)$ is a unit speed re-parameterisation of α

[4 points]

(7) Let $\alpha : (a,b) \to \mathbb{R}^3$ be a regular curve. Show that there exists a reparameterisation $\beta : (0,1) \to \mathbb{R}^3$ of α such that $\|\beta'(u)\| = m$ for all $u \in (0,1)$, where $m \in \mathbb{R}$ is a constant.

- By the fact that every regular curve admits a unit-speed re-parameterisation, consider $\tilde{\alpha} : (c,d) \to \mathbb{R}^3$ a unit-speed re-parameterisation of α.
- Now define $h : (0,1) \to (c,d)$ by $h(t) = c + t(d-c)$. Then h is a diffeomorphism.
- Then $\beta : (0,1) \to \mathbb{R}^3$ defined by $\beta(u) = \tilde{\alpha} \circ h(u)$ is a constant speed re-parameterisation of the original regular curve α.

[5 points]

(8) (a) Let $\alpha : (a,b) \to \mathbb{R}^2$, $(\alpha(t) = (\alpha_1(t), \alpha_2(t))$ be a regular curve and $J : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $J(x,y) = (-y,x)$. Then show that the unsigned curvature of α is

$$\kappa(t) = \frac{|\langle \alpha'(t), J(\alpha''(t)) \rangle|}{\|\alpha'(t)\|^3}.$$

Solution.

- We write

$$\begin{aligned} \alpha(t) &= (\alpha_1(t), \alpha_2(t), 0) \\ \implies \alpha'(t) &= (\alpha_1'(t), \alpha_2'(t), 0) \\ \implies \alpha''(t) &= (\alpha_1''(t), \alpha_2''(t), 0). \end{aligned}$$

- $\|\alpha'(t) \times \alpha''(t)\| = |\langle \alpha'(t), J(\alpha''(t)) \rangle|$
- $\kappa(t) = \frac{\|\alpha'(t)\times\alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{|\langle \alpha'(t), J(\alpha''(t))\rangle|}{\|\alpha'(t)\|^3}$.

(b) Find a unit-speed parameterised plane curve whose signed curvature is

$$K(s) = -\frac{1}{1+s^2}.$$

Solution.

- Define $\phi(u) := \int\limits_0^u \kappa(s)ds = \int\limits_0^u -\frac{1}{1+s^2} ds = -\tan^{-1}(u)$.
- $\alpha(s) = \left(\int\limits_0^s \cos\phi(u)du, \int\limits_0^s \sin\phi(u)du\right) = \left(\int\limits_0^s \cos\left(\tan^{-1}(u)\right)du, -\int\limits_0^s \sin\left(\tan^{-1}(u)\right)du\right)$
- Then we have,

$$\begin{aligned} \cos\left(\tan^{-1}(u)\right) &= \frac{1}{\sqrt{1+u^2}} \\ \sin\left(\tan^{-1}(u)\right) &= \frac{u}{\sqrt{1+u^2}} \end{aligned}$$

- $$\begin{aligned} \int\limits_0^s \cos\left(\tan^{-1}(u)\right)du &= \int\limits_0^s \frac{1}{\sqrt{1+u^2}}du = \operatorname{arcsinh}(s) \\ \int\limits_0^s \sin\left(\tan^{-1}(u)\right)du &= \int\limits_0^s \frac{u}{\sqrt{1+u^2}}du = \sqrt{1+s^2}. \end{aligned}$$
- $\alpha(s) = (\operatorname{arcsinh}(s), -\sqrt{1+s^2})$ is the required curve

[5+5=10 points]