Example : VAR(1) with 2 variables

$$\underset{\sim}{X}_{t} = \Phi \underset{\sim}{X}_{t-1} + \underset{\sim}{\epsilon}_{t} \quad ; \quad \underset{\sim}{\epsilon}_{t} \sim VWN(\underset{\sim}{0}, \Sigma)$$
2×1

i.e $$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

i.e. $$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} \phi_{11} X_{1,t-1} + \phi_{12} X_{2,t-1} + \epsilon_{1,t} \\ \phi_{21} X_{1,t-1} + \phi_{22} X_{2,t-1} + \epsilon_{2,t} \end{pmatrix}$$

Also
$$(I_2 - \Phi B) \underset{\sim}{X}_{t} = \underset{\sim}{\epsilon}_{t}$$

i.e $$\Phi(B) \underset{\sim}{X}_{t} = \underset{\sim}{\epsilon}_{t}$$

$$\Phi(B) = \begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix}$$ VAR matrix polynomial

Condition for stationarity of VAR(p)

A K-variate VAR(p) process is covariance stationary if all values of z satisfying $|\Phi(z)| = 0$ ($|A|$ is determinant of A) all lie outside the unit circle

i.e. all z satisfying

$$\left| I_K - \Phi_1 z - \Phi_2 z^2 - \cdots - \Phi_p z^p \right| = 0$$

lie outside the unit circle.

i.e. all y satisfying

$$\left| I_K y^p - \Phi_1 y^{p-1} - \cdots - \Phi_p \right| = 0$$

lie inside the unit circle

<u>Example</u> VAR(1) with K=2

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & -.6 \\ .5 & -.7 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

$$\Phi(B) = I_2 - \begin{pmatrix} 1 & -.6 \\ .5 & -.7 \end{pmatrix} B$$

$$|\underline{\Phi}(z)| = \begin{vmatrix} 1-z & .6z \\ -.5z & 1+.7z \end{vmatrix} = (1-z)(1+.7z)+.3z^2$$

$$= (1+.7z - z - .7z^2) + .3z^2$$

$$= 1 - .3z - .4z^2$$

$$= (1-.8z)(1+.5z)$$

Roots of $|\underline{\Phi}(z)| = 0$ are $\frac{1}{.8}, -\frac{1}{.5}$

$\Rightarrow$ all z satisfying $|\underline{\Phi}(z)| = 0$ lie outside the unit circle

$\Rightarrow$ the VAR(1) process is covariance stationary.

<u>Remark</u>: w.l.o.g. we can take mean vector of a covariance stationary VAR(1) as null vector, i.e. we take w.l.o.g. a VAR(1) (covariance stationary) without a const vector in the model,

o/w $\underset{\sim}{Y}_{t\,(K\times 1)} = \underset{\sim}{\delta} + \underline{\Phi}_1 \underset{\sim}{Y}_{t-1} + \cdots + \underline{\Phi}_p \underset{\sim}{Y}_{t-p} + \underset{\sim}{\epsilon}_t$ $\quad \underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \underline{\Sigma})$

$$E(\underset{\sim}{Y}_t) = E(\underset{\sim}{\delta} + \underline{\Phi}_1 \underset{\sim}{Y}_{t-1} + \cdots + \underline{\Phi}_p \underset{\sim}{Y}_{t-p} + \underset{\sim}{\epsilon}_t)$$

i.e. $\underset{\sim}{\mu} = \underset{\sim}{\delta} + \underline{\Phi}_1 \underset{\sim}{\mu} + \cdots + \underline{\Phi}_p \underset{\sim}{\mu}$

$$\Rightarrow (I_K - \Phi_1 - \cdots - \Phi_p)\,\underset{\sim}{\mu} = \underset{\sim}{\delta}$$

$$\Rightarrow \underset{\sim}{\mu} = (I_K - \Phi_1 - \cdots - \Phi_p)^{-1}\,\underset{\sim}{\delta}$$

i.e. $\underset{\sim}{\mu} = (\underline{\Phi}(1))^{-1}\,\underset{\sim}{\delta}$

(Note that $\underline{\Phi}(z) = I_K - \underline{\Phi}_1 z - \cdots - \Phi_p z^p$

$\underline{\Phi}(1) = I_K - \underline{\Phi}_1 - \cdots - \underline{\Phi}_p$

& $\underline{\Phi}(1)$ is non singular

as $|\underline{\Phi}(1)| \neq 0$ for stationary process.)

$$\underset{\sim}{\delta} = \underline{\Phi}(1)\,\underset{\sim}{\mu}$$

$$\Rightarrow \underset{\sim}{Y}_t - \underset{\sim}{\mu} = \underline{\Phi}_1(\underset{\sim}{Y}_{t-1} - \underset{\sim}{\mu}) + \cdots + \underline{\Phi}_p(\underset{\sim}{Y}_{t-p} - \underset{\sim}{\mu}) + \underset{\sim}{\epsilon}_t$$

let $\underset{\sim}{X}_t = \underline{Y}_t - \underset{\sim}{\mu}$

$$\Rightarrow \underset{\sim}{X}_t = \underline{\Phi}_1(\underset{\sim}{X}_{t-1}) + \cdots + \underline{\Phi}_p \underset{\sim}{X}_{t-p} + \underline{\epsilon}_t$$

equivalent VAR(p) with same VWN and

$\underline{\Phi}_1, \ldots, \underline{\Phi}_p$ and without const vector

A. <u>Mean vector of covariance stationary VAR(p)</u>

$$\underset{\sim}{X}_t = \underline{\Phi}_1 \underset{\sim}{X}_{t-1} + \cdots + \underline{\Phi}_p \underset{\sim}{X}_{t-1} + \underset{\sim}{\epsilon}_t$$

$\underline{\Phi}_p \neq 0$, $\epsilon_t \sim VWN(\underline{0}, \Sigma)$

$Cov(\underset{\sim}{\epsilon}_t, \underset{\sim}{X}_{t-j}) = 0 \;\; \forall\, j > 0$

$$\underset{\sim}{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t$$

$$\Rightarrow \underset{\sim}{\mu} = E(\underset{\sim}{X}_t) = \Phi_1 E(\underset{\sim}{X}_{t-1}) + \cdots + \Phi_p E(\underset{\sim}{X}_{t-p}) + \underset{\sim}{0}$$

i.e. $\underset{\sim}{\mu} = \Phi_1 \underset{\sim}{\mu} + \cdots + \Phi_p \underset{\sim}{\mu}$

i.e. $(I_K - \Phi_1 - \cdots - \Phi_p)\underset{\sim}{\mu} = \underset{\sim}{0}$

i.e. $\Phi(1)\underset{\sim}{\mu} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{\mu} = \underset{\sim}{0} \quad (|\Phi(1)| \neq 0).$

Auto covariance matrix function

$$\Gamma_0 = Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_t) = E(\underset{\sim}{X}_t \underset{\sim}{X}_t')$$

$$= E\left(\underset{\sim}{X}_t (\Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t)'\right)$$

$$= E(\underset{\sim}{X}_t \underset{\sim}{X}_{t-1}')\Phi_1' + E(\underset{\sim}{X}_t \underset{\sim}{X}_{t-2}')\Phi_2' + \cdots$$

$$+ E(\underset{\sim}{X}_t \underset{\sim}{X}_{t-p}')\Phi_p' + E(\underset{\sim}{X}_t \underset{\sim}{\epsilon}_t')$$

Note that

$$E(\underset{\sim}{X}_t \underset{\sim}{\epsilon}_t') = E\left(\left(\Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p}) + \epsilon_t\right)\underset{\sim}{\epsilon}_t'$$

$$= \Phi_1 Cov(\underset{\sim}{X}_{t-1}, \underset{\sim}{\epsilon}_t) + \cdots + \Phi_p Cov(\underset{\sim}{X}_{t-p}, \underset{\sim}{\epsilon}_t)$$

$$+ Cov(\underset{\sim}{\epsilon}_t, \underset{\sim}{\epsilon}_t)$$

$$= \Sigma \quad \left(Cov(\underset{\sim}{X}_{t-j}, \underset{\sim}{\epsilon}_t) = 0 \ \forall j > 0\right)$$

$$\Rightarrow \Gamma_0 = \Gamma(-1)\Phi_1' + \Gamma(-2)\Phi_2' + \cdots + \Gamma(-p)\Phi_p' + \Sigma \quad \text{--- } (*')$$

Note further that

$\Gamma_0 = E(\underline{X}_t \underline{X}_t')$ can be derived through

$$= E\left(\Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t\right)\underline{X}_t'$$

$$= \Phi_1 E\left(\underset{\sim}{X}_{t-1}\underset{\sim}{X}_t'\right) + \cdots + \Phi_p E\left(\underset{\sim}{X}_{t-p}\underline{X}_t'\right) + E\left(\underline{\epsilon}_t \underline{X}_t'\right)$$

$$= \Phi_1 \left(E\left(\underline{X}_t \underset{\sim}{X}_{t-1}'\right)\right)' + \cdots + \Phi_p \left(E\left(\underline{X}_t \underset{\sim}{X}_{t-p}'\right)\right)' + \Sigma$$

i.e $\Gamma_0 = \Phi_1 (\Gamma(-1))' + \cdots + \Phi_p (\Gamma(-p))' + \Sigma$

i.e $\Gamma_0 = \Phi_1 \Gamma(1) + \cdots + \Phi_p \Gamma(p) + \Sigma \quad \text{--- } (*^2)$

$\left((\Gamma(-h))' = \Gamma(h)\right)$

$(*') = (*^2)$ – Γ_0 is anyway symmetric matrix

$$\mathrm{Cov}(\underset{\sim}{X}_t, X_{t+h}) = E(\underset{\sim}{X}_t \underset{\sim}{X}_{t+h}')$$

$$= E(\underset{\sim}{X}_t (\Phi_1 \underset{\sim}{X}_{t+h-1} + \cdots + \Phi_p \underset{\sim}{X}_{t+h-p} + \underset{\sim}{\epsilon}_{t+h})')$$

$\forall h > 0$

$$\Gamma_h = \mathrm{Cov}(\underset{\sim}{X}_t, X_{t+h})$$

$$= E(\underset{\sim}{X}_t \underset{\sim}{X}_{t+h-1}')\Phi_1' + \cdots + E(\underset{\sim}{X}_t X_{t+h-p}')\Phi_p'$$

$$\left(E(\underset{\sim}{X}_t \underset{\sim}{\epsilon}_{t+h}') = \mathrm{Cov}(\underset{\sim}{X}_t, \epsilon_{t+h}) = 0 \text{ as } h > 0 \right)$$

$$\Rightarrow \Gamma(h) = \Gamma(h-1)\Phi_1' + \cdots + \Gamma(h-p)\Phi_p'$$

matrix Yule-Walker eq^n.

Further $\Gamma(-h) = (\Gamma(h))'$

Causal VAR process

Def^n: A VAR(p) process is said to be Causal if it can be expressed in terms of a VWN sequence as a VMA(∞) form

VMA(∞) representation of VAR

VAR(1) $\quad \underset{\sim}{X}_t = \Phi\, \underset{\sim}{X}_{t-1} + \underset{\sim}{\epsilon}_t \,; \quad \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$

$$\underline{\Phi}(B)\, \underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t$$

$$\underline{\Phi}(B) = I_K - \underline{\Phi}\, B$$

Suppose $\underline{\Phi}(B)^{-1}$ is the inverse of the operator $\underline{\Phi}(B)$

i.e. $\underline{\Phi}(B)^{-1}\, \underline{\Phi}(B) = I_K$

then $\quad \underset{\sim}{X}_t = \underline{\Phi}(B)^{-1} \underset{\sim}{\epsilon}_t = \underline{\Psi}(B)\, \underline{\epsilon}_t$, say

where $\underline{\Psi}(B) = \Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots\cdots$

$$\underline{\Phi}(B)^{-1} = \underline{\Psi}(B)$$

i.e $\quad I_K = \underline{\Phi}(B)\, \underline{\Psi}(B)$

i.e. $I_K = (I_K - \underline{\Phi} B)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$

$$= (\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$$
$$- (\underline{\Phi}\Psi_0 B + \underline{\Phi}\Psi_1 B^2 + \cdots)$$

i.e. $I_K = \Psi_0 + (\Psi_1 - \underline{\Phi}\Psi_0) B + (\Psi_2 - \underline{\Phi}\Psi_1) B^2 + \cdots$

Comparing coefficient of B^j

B^0 : $\Psi_0 = I_K$

B^1 : $\Psi_1 = \Phi$

B^2 : $\Psi_2 = \Phi \Psi_1 = \Phi^2$

$\vdots$

B^j : $\Psi_j = \Phi^j$

$$\Rightarrow \quad \underset{\sim}{X}_t = \sum_{j=0}^{\infty} \Phi^j \underset{\sim}{\epsilon}_{t-j}$$

This is the causal representation a covariance stationary VAR(1)

<u>VAR(p)</u> Suppose $\{X_t\}$ is covariance stationary VAR(p)

$$\underset{\sim}{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t$$

$$\underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

$$\Phi(B) \underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t \; ; \quad \Phi(B) = I_K - \sum_{i=1}^{p} \Phi_i B^i$$

Suppose $\underset{\sim}{X}_t = \Phi(B)^{-1} \underset{\sim}{\epsilon}_t = \Psi(B) \underset{\sim}{\epsilon}_t = \sum_{j=0}^{\infty} \Psi_j \underset{\sim}{\epsilon}_{t-j}$

$\Psi(B)$ is $\ni$ $\Phi(B) \Psi(B) = I_K$

i.e. $\left(I_K - \sum_{i=1}^{p} \Phi_i B^i \right) \left(\sum_{j=0}^{\infty} \Psi_j B^j \right) = I_K$

i.e. $\left(I_K - \Phi_1 B - \Phi_2 B^2 - \cdots - \Phi_p B^p\right)$

$\left(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots\right) = I_K$

i.e. $\Psi_0 B^0 + (\Psi_1 - \Phi_1 \Psi_0) B + (\Psi_2 - \Phi_1 \Psi_1 - \Phi_2 \Psi_0) B^2$
$+ (\Psi_3 - \Phi_1 \Psi_2 - \Phi_2 \Psi_1 - \Phi_3 \Psi_0) B^3 + \cdots$
$= I_K$

Comparing coefficients, we have

$$\Psi_0 = I_K$$
$$\Psi_1 = \Phi_1$$
$$\Psi_2 = \Phi_1 \Psi_1 + \Phi_2 \Psi_0$$
$$\Psi_3 = \Phi_1 \Psi_2 + \Phi_2 \Psi_1 + \Phi_3 \Psi_0$$

In general, $\forall s \geq 2$

$$\Psi_s = \Phi_1 \Psi_{s-1} + \Phi_2 \Psi_{s-2} + \cdots + \Phi_p \Psi_{s-p}$$

with $\Psi_0 = I_K$ & $\Psi_l = 0 \ \forall \ l < 0$

$\Psi_0, \Psi_1, \Psi_2, \ldots$ gives the Causal VMA(∞) representation of VAR(p).