Assignment - 3

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(1) (a) Consider any $y \in f(\Omega)$

then $\exists x: \quad y = f(x)$.

By Inverse Function Theorem there exist open sets

$U: \quad x \in U$

$V: \quad y \in V$

$f(U) = V$

Now since $V = f(U) \subseteq f(\Omega)$

we have an open set about y in $f(\Omega)$.

$\therefore$ we can find $r: \quad B_r(y) \subseteq f(\Omega)$

$\therefore \quad f(\Omega)$ is open.

(b) By IFT, f^{-1} is $C^1(f(\Omega))$, therefore diff.

(c) Since f^{-1} is continuous for any open set $B \subset \Omega$,

$(f^{-1})^{-1}(B)$ is open

$\Rightarrow \quad f(B)$ is open in $\mathbb{R}^n$

Towards a contradiction assume f is injective.

Q2. Take any 3 distinct non-collinear points in $\mathbb{R}^2$ say a, b & c.

WLOG, let $f(a) < f(b) < f(c)$

if any two are equal then f is not injective and we are done.

Now, define $\xi: L(a,c) \to \mathbb{R}$

$\xi(t) = f((1-t)a + tc)$

So, $\xi(0) = f(a)$

$\xi(1) = f(c)$

& ξ is a compos^n of continuous fns $\therefore$ continuous.

$\therefore$ By IVT $\exists\, t_0$:

$\xi(t_0) = f(b)$

$\Rightarrow$ i.e. $f((1-t_0)a + t_0 c) = f(b)$

Since a, b, c are non-collinear

$b \neq (1-t_0)a + t_0 c \quad \forall t_0 \in [0,1]$

$\therefore \quad f$ is not-injective.

Q3. (a) For any $x, y \in \mathbb{R}: \quad x<y$

$\exists\, c \in (x,y)$:

$\dfrac{f(y) - f(x)}{y - x} = f'(c) \neq 0$

$f(y) \neq f(x)$

So, injective

(b)

$$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$\det f'(x,y) = e^{2x}(\cos^2 y + \sin^2 y)$

$= e^{2x} \neq 0 \quad \forall x \in \mathbb{R}$

However it is not injective $\because$

$f(x, y+2\pi) = f(x,y)$ trivially

$\forall x, y \in \mathbb{R}$

Let (x_0, y_0, z_0, u_0) be a sol^n to the system.

(4) Consider the implicit functions:

(1) $f_1: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$

$f_1((x,y,u),z) = (3x + y - z + u^2,\ x - 2y + 2z + u,\ 2x + 2y - 3z + 2u)$

Now,

$$f_1'(x,y,u,z) = \begin{pmatrix} 3 & 1 & 2u & -1 \\ 1 & -2 & 1 & 2 \\ 2 & 2 & 2 & -3 \end{pmatrix}$$

$$\det(f'_{1\,3\times3}) = \begin{vmatrix} 3 & 1 & 2u \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{vmatrix}$$

$= 3(-6) - 1(-2) + 2u(6)$

$= 12u - 20$

So whenever $12u - 20 \neq 0$

$u \neq \frac{20}{12} = \frac{5}{3}$

By Implicit Function Theorem,

$\exists\, \xi: \mathbb{R} \to \mathbb{R}^3$ and open sets $U \subseteq \mathbb{R}^3 \times \mathbb{R}$

$W \subseteq \mathbb{R}$:

$f_1(\xi(z), z) = 0 \quad \forall z \in W$

i.e. $(x,y,u) = \xi(z) \quad \forall z \in W$

||^{ly}, we can show (b) & (c).

But not for u $\because$ the matrix for that case has det. 0.

(5) With $z = x + iy$ observe

$f(z) = z^2$

So by MTH403A,

(a) $\mathbb{C}$ $\because$ square root of each complex no. exists

(b)

$$f'(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$\det f'(x,y) = 4(x^2 + y^2) > 0$ whenever $(x,y) \neq 0$

(c) By (b) above, $\det f'(x,y) \neq 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{0,0\}$

$\therefore$ By the inverse function theorem for each $x \in \mathbb{R}^2 \setminus \{0,0\}$ $\exists$ open sets U & V

$x \in U$

$f(x) \in V$

f is bijective from $U \to V$

& $V = f(U)$

& f, f^{-1} are C^1

So, f is injective on this U.

Globally it is not injective $\because \exists$ 2 square roots for each complex no. $z \neq 0$

$\sqrt{z}$ & $-\sqrt{z}$

& $f(\sqrt{z}) = f(-\sqrt{z}) = z$

$$\xi(x,y) = \left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}},\ \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \right) \quad (x,y) \in B_1(3,4)$$

is a cts. choice for sqrt $\because$ sgn$(y) > 0$ on this ball

$$\xi'(x,y) = \begin{pmatrix} \frac{1}{2\sqrt{2}} \dfrac{1 + \frac{x}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}} & \frac{1}{2\sqrt{2}} \dfrac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}+x}} \\ \frac{1}{2\sqrt{2}} \dfrac{\frac{x}{\sqrt{x^2+y^2}} - 1}{\sqrt{\sqrt{x^2+y^2}-x}} & \frac{1}{2\sqrt{2}} \dfrac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{\sqrt{x^2+y^2}-x}} \end{pmatrix}$$

$b = (16, 30) \qquad \sqrt{x^2+y^2} = 34$

(6)

(a)

$J^2(p_1,p_2) = (J \circ J)(p_1,p_2)$

$= J(-p_2, p_1)$

$= (-p_1, -p_2)$

$= -id(p_1,p_2)$

So, $J^2 = -id$

(b) $\langle J(u_1,u_2), J(v_1,v_2) \rangle$

$= \langle (-u_2,u_1), (-v_2,v_1) \rangle$

$= u_2 v_2 + u_1 v_1$

$= \langle (u_1,u_2), (v_1,v_2) \rangle$

So, IP preserving

$\Rightarrow$ Norm preserving

$\because \quad \|p\| = (\langle p,p \rangle)^{\frac{1}{2}}$

(c) $\langle J(p), p \rangle$

$= \langle (-p_2, p_1), (p_1, p_2) \rangle$

$= -p_1 p_2 + p_1 p_2$

$= 0$

(d) We show that,

$(\langle p,q \rangle)^2 + (\langle p, J(q) \rangle)^2 = \|p\|^2 \|q\|^2$

$\Rightarrow \quad (p_1 q_1 + p_2 q_2)^2 + (-p_1 q_2 + p_2 q_1)^2$

$= p_1^2 q_1^2 + p_2^2 q_2^2 + \cancel{2 p_1 p_2 q_1 q_2}$

$+ p_1^2 q_2^2 + p_2^2 q_1^2 - \cancel{2 p_1 p_2 q_1 q_2}$

$= (p_1^2 + p_2^2)(q_1^2 + q_2^2)$

$= \|p\|^2 \|q\|^2$

So, setting $\theta = \text{atan2}(\langle p,q \rangle, \langle p, J(q) \rangle)$

$$\text{where } \text{atan2}(x,y) = \begin{cases} \tan^{-1} \frac{y}{x} ; & x>0 \quad y>0 \\ \pi - \tan^{-1}\left(\frac{-y}{x}\right) ; & x<0 \quad y>0 \\ \pi + \tan^{-1}\left(\frac{-y}{-x}\right) ; & x<0, y<0 \\ \pi/2 ; & x=0 \quad y>0 \\ 3\pi/2 ; & x=0 \quad y<0 \\ 2\pi - \tan^{-1}\left(\frac{-y}{x}\right) ; & x>0 \quad y<0 \end{cases}$$

works.