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Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Quiz -2 Solution (MTH305A)

Semester: 2022-2023, I

Full Marks. 20 Time. 45 Minutes

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(1) Are the following statements TRUE/FALSE? **Justify your answer.**

(a) If $B \subset \mathbb{R}^n$ is closed, $x \in \mathbb{R}^m$ and $\mathcal{O}$ is an open cover of $\{x\} \times B \subset \mathbb{R}^m \times \mathbb{R}^n$, then there is an open set $U_x \subset \mathbb{R}^m$ containing x such that $U \times B$ is covered by a finite number of sets in $\mathcal{O}$.

[3 points]

Solution.
FALSE.
Counter-example. Consider $m = 1 = n, B = [1, \infty) \subset \mathbb{R}, x = 1 \in \mathbb{R}$ and

$$N = \left\{ (x, y) \in \mathbb{R}^2 \mid y > \frac{1}{2}, |x - 1| < \frac{1}{y} \right\}.$$

Then $\mathcal{O} = \{N\}$ is open cover of $\{x\} \times B$. But for every $\epsilon > 0$,

$$(1 - \epsilon, 1 + \epsilon) \times B \not\subset N.$$

Draw a figure for better understanding.

(b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

Solution. TRUE
If f is differentiable at $(0, 0)$, then the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(t, t)$ is also differentiable at 0. But $g(t) = f(t, t) = \sqrt{t^2} = |t|$ is not differentiable at 0.

[2 points]

(2) Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. Consider the function $f : B^n \to \mathbb{R}^n$ defined by

$$f(x) = \frac{x}{\sqrt{1 - \|x\|^2}}.$$

(a) f is continuously differentiable.
(b) f is invertible.
(c) $f^{-1} : \mathbb{R}^n \to B^n$ is also continuously differentiable.
(d) Find a diffeomorphism $f : B^n_r(a) \to \mathbb{R}^n$, where $r > 0$ and
$B^n_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$.

Solution.

If $f(x) = y$, then we have

$$\begin{aligned} \frac{x}{\sqrt{1-\|x\|^2}} &= y \\ \implies \frac{\|x\|}{\sqrt{1-\|x\|^2}} &= \|y\| \\ \implies \frac{\|x\|^2}{1-\|x\|^2} + 1 &= \|y\|^2 + 1 \\ \implies \sqrt{1-\|x\|^2} &= \frac{1}{\sqrt{1+\|y\|^2}}. \end{aligned}$$

Now, $\frac{x}{\sqrt{1-\|x\|^2}} = y \implies x = y\sqrt{1-\|x\|^2} = \frac{y}{\sqrt{1+\|y\|^2}}$. Define

$$g(y) = \frac{y}{\sqrt{1+\|y\|^2}}, \text{ for all } y \in \mathbb{R}^n.$$

We note that $g : \mathbb{R}^n \to B^n$ is inverse of $f : B^n \to \mathbb{R}^n$.

(a) The function $h : B^n \to \mathbb{R}$ defined by $h(x) = \sqrt{1-\|x\|^2}$ is C^∞ and no-where zero. The function $k_i : B^n \to \mathbb{R}$ defined by $k_i(x) = x_i$ is C^∞ (note $k_i = \pi_i|_{B^n}$), $i = 1, \ldots, n$. Therefore $f_i(x) = \frac{k_i(x)}{h(x)}$ is C^∞. Now, it follows that $f = (f_1, \ldots, f_n)$ if C^∞.

(b) g is inverse of f, hence f is invertible.

(c) A similar argument as in part (a) shows that $f^{-1} = g$ is also C^∞.

(d) The function $d : B^n_r(a) \to B^n$ defined by $d(x) = \frac{x-a}{r}$ is a diffeomorphism. Therefore, $F : B^n_r(a) \to \mathbb{R}^n$ defined by $F(x) = f \circ d(x) = \frac{x-a}{\sqrt{a^2-\|x-a\|^2}}$ is also a diffeomorphism as a composition of two diffeomorphism.

[8 points]

(3) Apply Lagrange's multiplier method to find the point on the line of intersection of the two planes $x_1 + x_2 + x_3 + 2 = 0$ and $x_1 - x_2 - x_3 - 2 = 0$ (in $\mathbb{R}^3$) which is nearest to the origin.

[7 points]

- Objective function: $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$
- Constraint equations:

$$\begin{aligned} g_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 + 2 = 0 \text{ and} \\ g_1(x_1, x_2, x_3) &= x_1 - x_2 - x_3 - 2 = 0. \end{aligned}$$

- Lagrangian function:

$$\begin{aligned} \phi(x_1, x_2, x_3) &= f(x_1, x_2, x_3) + \lambda_1 g_1(x_1, x_2, x_3) + \lambda_2 g_2(x_1, x_2, x_3) \\ &= x_1^2 + x_2^2 + x_3^2 + \lambda_1(x_1 + x_2 + x_3 + 2) + \lambda_2(x_1 - x_2 - x_3 - 2) \\ &= x_1^2 + x_2^2 + x_3^2 + (\lambda_1 + \lambda_2)x_1 + (\lambda_1 - \lambda_2)x_2 + (\lambda_1 - \lambda_2)x_3 + 2(\lambda_1 - \lambda_2). \end{aligned}$$

- The system of equations are: $\frac{\partial \phi}{\partial x_i} = 0, i = 1, 2, 3$, give

$$2x_1 + \lambda_1 + \lambda_2 = 0$$
$$2x_2 + \lambda_1 - \lambda_2 = 0$$
$$2x_3 + \lambda_1 - \lambda_2 = 0$$

- Solving $\frac{\partial \phi}{\partial x_i} = 0, i = 1, 2, 3$ and the constraint equations we have $(x_1, x_2, x_3) = (0, -1, -1)$ and $\lambda_1 = 1, \lambda_2 = -1$.