(1)

(a) $E X_1 = 0 = E X_2$; $V(X_1) = 1 = V(X_2)$

$E Y_t = 0 \quad - \forall t$

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t+h}) &= E\, Y_t Y_{t+h} \\
&= E\left(X_1 \cos \tfrac{\pi}{2} t - X_2 \sin \tfrac{\pi}{2} t\right)\left(X_1 \cos \tfrac{\pi}{2}(t+h) - X_2 \sin \tfrac{\pi}{2}(t+h)\right) \\
&= \cos \tfrac{\pi}{2} h
\end{aligned}$$

$\Rightarrow$ $\{Y_t\}$ is covariance stationary

$Y_1 = -X_2 \nsim N_1 \Rightarrow \{Y_t\}$ is not a Gaussian process

$\mathrm{Cov}(Y_{10}, Y_{20}) = \cos 5\pi = -1$

(b)
$$\begin{aligned}
E|X_{t+h} - X_t|^2 &= E(X_{t+h} - X_t)^*(X_{t+h} - X_t) \\
&= E(X_{t+h}^* X_{t+h}) + E(X_t^* X_t) - E(X_t^* X_{t+h}) - E(X_{t+h}^* X_t) \\
&= 2\gamma_X(0) - \gamma_X(h) - \gamma_X(-h) \quad — \quad (*)
\end{aligned}$$

Now, $\gamma_X(h) = (\gamma_X(-h))^* \quad \forall h$

$$\left.\begin{aligned}
\Rightarrow \mathrm{Re}(\gamma_X(h)) &= \mathrm{Re}(\gamma_X(-h)) \\
\mathrm{Im}(\gamma_X(h)) &= -\mathrm{Im}(\gamma_X(-h))
\end{aligned}\right\} \Rightarrow \gamma_X(h) + \gamma_X(-h) = 2\,\mathrm{Re}(\gamma_X(-h))$$

$$\begin{aligned}
\Rightarrow (*) &= 2\gamma_X(0) - 2\,\mathrm{Re}(\gamma_X(-h)) \\
&= 2\left(\gamma_X(0) - \mathrm{Re}(\gamma_X(-h))\right)
\end{aligned}$$

(2)

(a)

$$f(1) = \frac{1}{9}\sum_{1}^{9} \epsilon_t \epsilon_{t+1}$$
$$= \frac{1}{9}\left(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \cdots + \epsilon_9\epsilon_{10}\right)$$
$$E\, f(1) = 0 \; ; \; V(f(1)) = E\, f(1)^2$$
i.e. $$V(f(1)) = \frac{1}{81}E\left(\epsilon_1\epsilon_2 + \cdots + \epsilon_9\epsilon_{10}\right)^2$$
$$= \frac{1}{81}\, 9\sigma^4 = \frac{\sigma^4}{9}$$

$$\mathrm{Cov}(f(1), f(2)) = \mathrm{Cov}\left(\frac{1}{9}(\epsilon_1\epsilon_2 + \cdots + \epsilon_9\epsilon_{10}), \frac{1}{8}(\epsilon_1\epsilon_3 + \cdots + \epsilon_8\epsilon_{10})\right)$$
$$= 0$$

(b)
$$\gamma_X(h) = \begin{cases} 0.15(1+\theta_1^2+\theta_2^2), & h=0 \\ 0.15(\theta_1+\theta_1\theta_2), & h=\pm 1 \\ 0.15\,\theta_2, & h=\pm 2 \\ 0, & \text{o/w} \end{cases}$$

$$\gamma_Y(h) = \begin{cases} 0.15(1+\gamma_1^2+\gamma_2^2), & h=0 \\ 0.15(\gamma_1+\gamma_1\gamma_2), & h=\pm 1 \\ 0.15\,\gamma_2, & h=\pm 2 \\ 0, & \text{o/w} \end{cases}$$

$$g_Z(x) \Rightarrow \gamma_Z(h) = \begin{cases} 1, & h=0 \\ 0.3, & h=\pm 1 \\ 0, & \text{o/w} \end{cases}$$

$$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$$

$$\gamma_Z(\pm 2) = 0 \Rightarrow \theta_2 = -\gamma_2$$

$\gamma_z(\pm 1) = 0.3 \Rightarrow$

$$0.15\,(\theta_1 + \theta_1\theta_2 + \gamma_1 + \gamma_1\gamma_2) = 0.3$$

i.e. $\theta_1 + \gamma_1 + \theta_2(\theta_1 - \gamma_1) = 2$

$\theta_1 = \gamma_1 = 1$ satisfies the above

$\gamma_z(0) = 1 \Rightarrow$

$$0.15\,(2 + \theta_1^2 + \gamma_1^2 + 2\theta_2^2) = 1$$

using $\theta_1 = \gamma_1 = 1$; $0.3\,(2 + \theta_2^2) = 1$

$$2 + \theta_2^2 = \frac{10}{3} \Rightarrow \theta_2 = \pm \frac{2}{\sqrt{3}}$$

$\Rightarrow$ a soln is $\theta_1 = \gamma_1 = 1$; $\theta_2 = \frac{2}{\sqrt{3}}$, $\gamma_2 = -\frac{2}{\sqrt{3}}$

(3)

(a)

$$\underline{y}_t = \begin{pmatrix} \underline{x}_t \\ 2\,\underset{\sim}{x}_{t-1} \\ 3\,\underline{x}_{t-2} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi\,\underset{\sim}{x}_{t-1} + \underline{\epsilon}_t \\ 2\,\underset{\sim}{x}_{t-1} \\ 3\,\underline{x}_{t-2} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi & 0 & 0 \\ 2I_2 & 0 & 0 \\ 0 & \frac{3}{2}I_2 & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_{t-1} \\ 2\,\underset{\sim}{x}_{t-2} \\ 3\,\underset{\sim}{x}_{t-3} \end{pmatrix} + \begin{pmatrix} \underline{\epsilon}_t \\ \underline{0} \\ \underline{0} \end{pmatrix}$$

i.e. $$\underset{6\times 1}{\underline{y}_t} = \underset{6\times 6}{\underline{\Phi}^*}\,\underset{\sim}{y}_{t-1} + \underline{\eta}_t$$

$$\underline{\eta}_t \sim VWN\Big(\underline{0},\ \Sigma^* = \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Big)$$

singular

$\Rightarrow$ $\underset{\sim}{y}_t \sim VAR(1)$

stationarity of $\underline{y}_t \sim VAR(1)$

$\underline{y}_t$ is stat if all z satisfying $|I_6 - \underline{\Phi}^* z| = 0$ lie outside unit circle

$$|I_6 - \underline{\Phi}^* z| = \begin{vmatrix} I_2 - \underline{\Phi} z & 0 & 0 \\ -2z & I_2 & 0 \\ 0 & -\frac{3}{2}z\,I_2 & I_2 \end{vmatrix} = |I_2 - \underline{\Phi}\,z|$$

As $\{\underline{X}_t\}$ is cov stat all z satisfying $|I_2 - \Phi z| = 0$ lie outside unit circle

$\Rightarrow \underset{\sim}{Y}_t$ is also cov stat VAR(1)

(b) $$\Gamma(0) = E\,\underline{X}_t\,\underline{X}_t'$$
$$= E(\Phi \underset{\sim}{X}_{t-1} + \underline{\epsilon}_t)\,\underline{X}_t'$$
$$= \Phi(\Gamma(-1))' + \Sigma = \Phi\,\Gamma(1) + \Sigma$$

$$\Gamma(1) = E\,\underline{X}_t\,\underline{X}_{t+1}' = E\,\underline{X}_t(\Phi\,\underline{X}_t + \underline{\epsilon}_{t+1})'$$

i.e. $\Gamma(1) = \Gamma(0)\,\Phi'$

$$\Rightarrow \Gamma(0) = \Phi\,\Gamma(0)\,\Phi' + \Sigma$$

(c) $$(I_2 - \Phi B)\,\underline{X}_t = \underline{\epsilon}_t \quad \text{cov stat}$$

Causal rep : $$\underline{X}_t = (I_2 - \Phi B)^{-1}\,\underline{\epsilon}_t$$
$$\underline{X}_t = \sum_{j=0}^{\infty} \Phi^j\,\epsilon_{t-j}$$

$$\Phi = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{pmatrix}, \quad \Phi^2 = \begin{pmatrix} (0.5)^2 & 2(0.5)^2 \\ & (0.5)^2 \end{pmatrix} \cdots$$

$$\therefore \Phi^s = \begin{pmatrix} (0.5)^s & s(0.5)^s \\ 0 & (0.5)^s \end{pmatrix}$$

Impulse response :

$$\frac{\partial X_{1,t+s}}{\partial \epsilon_{2,t}} = \psi_{12}^{(s)} = s\,(0.5)^s$$

(4)

(a) BLP of X_{t+1} based on Y_t, Y_{t-1}

BLP eq^{n}s:

$$\begin{cases} E\left(X_{t+1} - \alpha - \beta Y_t - \gamma Y_{t-1}\right) = 0 & - \\ E\left(X_{t+1} - \alpha - \beta Y_t - \gamma Y_{t-1}\right) Y_t = 0 & - \\ E\left(X_{t+1} - \alpha - \beta Y_t - \gamma Y_{t-1}\right) Y_{t-1} = 0 & - \end{cases}$$

i.e. $\alpha = E X_{t+1} - \beta E Y_t - \gamma E Y_{t-1}$

$E X_t = 1 \ \forall t \quad E Y_t = 2 \ \forall t$

$\Rightarrow \alpha = 1 - 2(\beta + \gamma)$

$$\left.\begin{array}{l} Cov(X_{t+1}, Y_t) = \beta \gamma_y(0) + \gamma \gamma_y(1) \\ Cov(X_{t+1}, Y_{t-1}) = \beta \gamma_y(1) + \gamma \gamma_y(0) \end{array}\right\}$$

$$\gamma_y(h) = \begin{cases} 17, & h = 0 \\ -4, & h = \pm 3 \\ 0, & o/w \end{cases}$$

$Cov(X_{t+1}, Y_t) = Cov\left(\frac{1}{2} + \frac{1}{2} X_t + \epsilon_{t+1}, 2 + \epsilon_t - 4\epsilon_{t-3}\right)$

$= \frac{1}{2} Cov(X_t, \epsilon_t) - 2 Cov(X_t, \epsilon_{t-3})$

$Cov(X_t, \epsilon_t) = 1$; $Cov(X_t, \epsilon_{t-3}) = Cov\left(\frac{1}{2} + \frac{1}{2} X_{t-1} + \epsilon_t, \epsilon_{t-3}\right)$

X_t

$= \frac{1}{2} + \frac{1}{2} X_{t-1} + \epsilon_t = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} X_{t-2} + \epsilon_{t-1}\right) + \epsilon_t$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} X_{t-3} + \epsilon_{t-2}\right) + \frac{1}{2}\epsilon_{t-1} + \epsilon_t$

$\Rightarrow Cov(X_t, \epsilon_{t-3}) = \frac{1}{8}$

$\Rightarrow Cov(X_{t+1}, Y_t) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$$\text{Cov}(X_{t+1}, Y_{t-1}) = \text{Cov}\left(\tfrac{1}{2}+\tfrac{1}{2}X_t + \epsilon_{t+1},\ 2+\epsilon_{t-1} - 4\epsilon_{t-4}\right)$$

$$= \tfrac{1}{2}\text{Cov}(X_t, \epsilon_{t-1}) - 2\,\text{Cov}(X_t, \epsilon_{t-4})$$

$$\text{Cov}(X_t, \epsilon_{t-1}) = \text{Cov}\left(\tfrac{1}{2}+\tfrac{1}{2}X_{t-1}+\epsilon_t,\ \epsilon_{t-1}\right)$$

$$= \frac{1}{2}$$

$$\text{Cov}(X_t, \epsilon_{t-4})$$

$$X_t = \underbrace{\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}}+\tfrac{1}{8}X_{t-3}+\underbrace{\tfrac{1}{4}\epsilon_{t-2}+\tfrac{1}{2}\epsilon_{t-1}+\epsilon_t}$$

$$= (\quad) + \tfrac{1}{8}\left(\tfrac{1}{2}+\tfrac{1}{2}X_{t-4}+\epsilon_{t-3}\right) + (\quad)$$

$$\Rightarrow \text{Cov}(X_t, \epsilon_{t-4}) = \frac{1}{16}$$

$$\Rightarrow \text{Cov}(X_{t+1}, Y_{t-1}) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

BLP eq^{n}s

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \beta \\ \gamma \end{pmatrix}_{BLP} = \begin{pmatrix} \frac{1}{68} \\ \frac{1}{136} \end{pmatrix}.$$

$$\alpha = 1 - 2 \times \frac{1}{17} \times \frac{3}{8} = \frac{65}{68}$$

$$\Rightarrow \text{BLP is : } \frac{65}{68} + \frac{1}{68}Y_t + \frac{1}{136}Y_{t-1}$$

(b) PACF at lag 4

$\alpha(4) = \text{Corr}^n\left(Y_1 - P_{Y_2,Y_3,Y_4} Y_1 ,\ Y_5 - P_{Y_2,Y_3,Y_4} Y_5 \right)$

BLP : $P_{Y_2,Y_3,Y_4} Y_1$

<u>BLP eqns</u> (i) $E(Y_1 - \alpha - \beta Y_4 - \gamma Y_3 - \delta Y_2) = 0$

$\mu_Y (1 - \beta - \gamma - \delta) = \alpha$

(ii) $E(Y_1 - \alpha - \beta Y_4 - \gamma Y_3 - \delta Y_2) Y_4 = 0$

$\Rightarrow \gamma_3 = \beta \gamma_0 \Rightarrow \beta = \frac{\gamma_3}{\gamma_0} = -\frac{4}{17}$

(iii) $E(Y_1 - \alpha - \beta Y_4 - \gamma Y_3 - \delta Y_2) Y_3 = 0$

$\Rightarrow \gamma = 0$

(iv) $E(Y_1 - \alpha - \beta Y_4 - \gamma Y_3 - \delta Y_2) Y_2 = 0$

$\Rightarrow \delta = 0$

$\Rightarrow \alpha = 2\left(1 + \frac{4}{17}\right) = \frac{42}{17} \Rightarrow P_{Y_2,Y_3,Y_4} Y_1 = \frac{42}{17} - \frac{4}{17} Y_4$

BLP : $P_{Y_2,Y_3,Y_4} Y_5 = \frac{42}{17} - \frac{4}{17} Y_2$

$\text{Cov}\left(Y_1 - P_{Y_2,Y_3,Y_4} Y_1 ,\ Y_5 - P_{Y_2,Y_3,Y_4} Y_5 \right)$

$= \text{Cov}\left(Y_1 + \frac{4}{17} Y_4 ,\ Y_5 + \frac{4}{17} Y_2 \right) = 0$

$\Rightarrow \alpha(4) = 0$

(5)

(a) $$X_1 \sim N\left(\frac{\delta}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right)$$

$$X_2 | X_1 \sim N(\delta + \phi x_1, \sigma^2)$$

$$X_3 | X_2, X_1 \sim N(\delta + \phi x_2, \sigma^2)$$

Likelihood f^n

$$L(\underline{\theta}) = f_{X_1} \; f_{X_2|X_1} \; f_{X_3|X_2,X_1}$$

i.e. $$L(\underline{\theta}) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{1-\phi^2}{\sigma^2}}\exp\left(-\frac{1-\phi^2}{2\sigma^2}\left(x_1-\frac{\delta}{1-\phi}\right)^2\right)\right)$$
$$\times\left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_2-\delta-\phi x_1)^2\right)\right)$$
$$\times\left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_3-\delta-\phi x_2)^2\right)\right)$$

(b) $\log f(x_1, x_2, x_3; \delta, \phi, \sigma^2)$

$$= -\frac{3}{2}\log 2\pi - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi^2}\right) - \frac{1}{2}\log\sigma^2 - \frac{1}{2}\log\sigma^2$$
$$- \frac{1-\phi^2}{2\sigma^2}\left(x_1 - \frac{\delta}{1-\phi}\right)^2 - \frac{1}{2\sigma^2}(x_2-\delta-\phi x_1)^2$$
$$- \frac{1}{2\sigma^2}(x_3-\delta-\phi x_1)^2$$

$$\frac{\partial \log f(X_1, X_2, X_3; \delta, \phi, \sigma^2)}{\partial\sigma^2}$$
$$= -\frac{1}{2}\frac{1}{\sigma^2}\times 3 + \frac{1-\phi^2}{2}\left(X_1-\frac{\delta}{1-\phi}\right)^2\frac{1}{\sigma^4} + \frac{1}{2\sigma^4}(X_2-\delta-\phi X_1)^2$$
$$+ \frac{1}{2\sigma^4}(X_3-\delta-\phi X_1)^2$$

$$\frac{\partial^2 \log}{\partial(\sigma^2)^2} = \frac{3}{2}\frac{1}{\sigma^4} + \frac{1-\phi^2}{2}\left(X_1-\frac{\delta}{1-\phi}\right)^2\frac{(-2)}{\sigma^6} + \frac{(X_2-\delta-\phi X_1)^2}{2}\frac{(-2)}{\sigma^6}$$
$$+ \frac{(X_3-\delta-\phi X_1)^2}{2}\frac{(-2)}{\sigma^6}$$

$$E\left(\frac{\partial^2 \log f}{\partial(\sigma^2)^2}\right) = \frac{3}{2}\frac{1}{\sigma^4} - \frac{1}{\sigma^4} - \frac{1}{\sigma^4} - \frac{1}{\sigma^4}$$

$$= -\frac{3}{2}\frac{1}{\sigma^4}$$

6 (a)

$$X_t + \frac{3}{2}X_{t-1} + \frac{3}{4}X_{t-2} + \frac{1}{8}X_{t-3} = \epsilon_t + 6\epsilon_{t-1} + 12\epsilon_{t-2} + 8\epsilon_{t-3}$$

ACGF

3

$$g_X(z) = \sigma^2 \frac{(1+6z+12z^2+8z^3)(1+6z^{-1}+12z^{-2}+8z^{-3})}{\left(1+\frac{3}{2}z+\frac{3}{4}z^2+\frac{1}{8}z^3\right)\left(1+\frac{3}{2}z^{-1}+\frac{3}{4}z^{-2}+\frac{1}{8}z^{-3}\right)}$$

i.e. $$g_X(z) = \sigma^2 \frac{(1+6z+12z^2+8z^3)(z^3+6z^2+12z+8)/\cancel{z^3}}{(8+12z+6z^2+z^3)(8z^3+12z^2+6z+1)/64\cancel{z^3}}$$

i.e. $$g_X(z) = 64\sigma^2$$

$$\Rightarrow \gamma_X(0) = 64\sigma^2 \; ; \; \gamma_X(q) = 0$$

(b)
$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_X(\lambda)\, d\lambda$$

$$= \int_{-\pi}^{0} e^{i\lambda h} f_X(\lambda)\, d\lambda + \int_{0}^{\pi} e^{i\lambda h} f_X(\lambda)\, d\lambda$$

$$= \int_{0}^{\pi} e^{-i\lambda h} f_X(-\lambda)\, d\lambda + \int_{0}^{\pi} e^{i\lambda h} f_X(\lambda)\, d\lambda$$

$$= \int_{0}^{\pi} (e^{-i\lambda h} + e^{i\lambda h}) f_X(\lambda)\, d\lambda$$

$$= 2\int_{0}^{\pi} \cos\lambda h\, f_X(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos\lambda h\, f_X(\lambda)\, d\lambda$$

7(a): $X_t = (1-B^3)\epsilon_t = \theta(B)\epsilon_t$

$$f_X(\lambda) = \frac{1}{2\pi}\,\theta(e^{-i\lambda})\,\theta(e^{i\lambda})$$

$$= \frac{1}{2\pi}\left(1-e^{-3i\lambda}\right)\left(1-e^{3i\lambda}\right)$$

$$= \frac{1}{2\pi}\left(2-e^{-3i\lambda}-e^{3i\lambda}\right)$$

$$f_X(\lambda) = \frac{1}{2\pi}\left(2-2\cos 3\lambda\right)$$

$$\Rightarrow f_X(\pi) = \frac{4}{2\pi} = \frac{2}{\pi}$$

$$Y_t = X_{t-1} + X_{t+1} = (B+B^{-1})X_t$$

$$\Rightarrow f_Y(\lambda) = \left(e^{-i\lambda}+e^{i\lambda}\right)\left(e^{i\lambda}+e^{-i\lambda}\right)f_X(\lambda)$$

$$= \left(2+e^{2i\lambda}+e^{-2i\lambda}\right)f_X(\lambda)$$

$$f_Y(\lambda) = (2+2\cos 2\lambda)\,f_X(\lambda)$$

$$\Rightarrow f_Y(\pi) = 4 \times \frac{2}{\pi} = \frac{8}{\pi}$$

(b) $\phi(B)X_t = \epsilon_t \; ; \; \phi(B) = 1-\phi B$

$$f_X(\lambda) = \frac{1}{2\pi}\,\frac{1}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})}$$

$$Y_t + 2Y_{t-1} = X_t + X_{t+2}$$

Spectral density of l.h.s $= (1+2e^{-i\lambda})(1+2e^{i\lambda})\,f_Y(\lambda)$

Spectral density of r.h.s. $= (1+e^{-2i\lambda})(1+e^{2i\lambda})\,f_X(\lambda)$

$$\Rightarrow f_Y(\lambda) = f_X(\lambda)\,\frac{(1+e^{-2i\lambda})(1+e^{2i\lambda})}{(1+2e^{-i\lambda})(1+2e^{i\lambda})}$$

$$\Rightarrow f_y(\lambda) = \left(\frac{1}{2\pi} \frac{1}{\left(1-\frac{1}{3}e^{-i\lambda}\right)\left(1-\frac{1}{3}e^{i\lambda}\right)}\right)\left(\frac{(1+e^{-2i\lambda})(1+e^{2i\lambda})}{(1+2e^{i\lambda})(1+2e^{-i\lambda})}\right)$$

$$\Rightarrow f_y(0) = \frac{1}{2\pi} \frac{1}{\left(\frac{4}{9}\right)} \cdot \frac{4}{9} = \frac{1}{2\pi}$$