Functions of random variables

Discrete r.v. Case

X : discrete r.v. $\quad \mathcal{X}$: range space of X

$Y = g(X)$; $g(\cdot)$ a real valued f^n

$\mathcal{Y}$: range space of Y

$\mathcal{Y} = \{ g(x) : x \in \mathcal{X} \}$

Problem is to find p.m.f. of Y given the p.m.f. of X

Note that $\quad g(x) : \mathcal{X} \to \mathcal{Y}$

Inverse mapping : $g^{-1} \ni$

$$g^{-1}(y) = \{ x \in \mathcal{X} : g(x) = y \}$$

In general, for $A \subset \mathcal{Y}$

$$g^{-1}(A) = \{ x \in \mathcal{X} : g(x) \in A \}$$

* If there is only one x for which $g(x) = y$, then $g^{-1}(y) = x$

p.m.f. of Y :

for $y \in \mathcal{Y}$; $P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x : g(x) = y} P(X = x)$

for $y \notin \mathcal{Y}$: $\quad P(Y = y) = 0$

Example: $\quad X \sim Bin(n, p)$

$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots n$

$$\mathcal{X} = \{0, 1, \ldots n\}$$

$X \to Y = n - X$

$\Rightarrow \mathcal{Y} = \{0, 1, \ldots n\}$

For any $y \in \mathcal{Y}$, $y = g(x) = n - x$ iff $x = n - y$ and $g^{-1}(y)$ is a single pt p.m.f of Y :

$$f_Y(y) = P(Y=y) = \sum_{x:\, g(x)=y} f_X(x) = f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

$$= \binom{n}{y} (1-p)^{y} p^{n-y} \; ; \; y = 0, 1, \ldots n$$

$$= 0 \quad o/w$$

$$\Rightarrow Y \sim B(n, 1-p)$$

Example: $\mathcal{X} = \{-1, 0, 1, 2\}$

p.m.f. of X : $P(X=-1) = 0.2$; $P(X=0) = 0.3$; $P(X=1) = .4$

$P(X=2) = 0.1$

$X \to Y = X^2$; $\mathcal{Y} = \{0, 1, 4\}$

p.m.f. of Y: $P(Y=0) = P(X=0) = 0.3$

$P(Y=1) = P(X=1) + P(X=-1) = 0.6$

$P(Y=4) = P(X=2) = 0.1$

Example: $X \sim P(\lambda)$; $\mathcal{X} = \{0, 1, 2, \ldots\}$

$X \to Y = X^2 + 3$; $\mathcal{Y} = \{3, 4, 7, \ldots\}$

p.m.f. of Y : $P(Y=y) = P(X^2 + 3 = y)$

$$= P(X = \sqrt{y-3})$$

$$= \begin{cases} \dfrac{e^{-\lambda} \lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!}, & y = 3, 4, 7, \ldots \\ 0, & o/w \end{cases}$$

Transformation: Continuous r.v.

Distribution f^n method:

$$X \to Y = g(X) \qquad \mathcal{X} = \{x : f_X(x) > 0\}$$

d.f. of Y: $\qquad \mathcal{Y} = \{y : g(x) = y, \ x \in \mathcal{X}\}$

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

$$= P(\{x : g(x) \leq y\})$$

$$= \int_{x : g(x) \leq y} f_X(x)\, dx$$

Note: The d.f. approach is straightforward if $g(\cdot)$ is strictly monotone (increasing or decreasing). In such a case

$$y = g(x) \Rightarrow x = g^{-1}(y)$$

If $g(x)$ is increasing, then

$$F_Y(y) = \int_{x : x \leq g^{-1}(y)} f_X(x)\, dx = \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx = F_X(g^{-1}(y))$$

If $g(x)$ is decreasing, then

$$F_Y(y) = \int_{x : x \geq g^{-1}(y)} f_X(x)\, dx = \int_{g^{-1}(y)}^{\infty} f_X(x)\, dx = 1 - F_X(g^{-1}(y))$$

Example:

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{o/w} \end{cases} \qquad X \sim U(0,1)$$

$$F(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$Y = -\ln X \quad \downarrow$ in x

$y = -\ln x \Rightarrow x = e^{-y}$ i.e. $g^{-1}(y) = e^{-y}$

$\mathcal{Y} = (0, \infty)$ from $\mathcal{X} = (0,1)$

Thus for $y > 0$

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$
$$= 1 - F_X(e^{-y})$$
$$= 1 - e^{-y}$$

& $F_Y(y) = 0$ for $y \le 0$

p.d.f. of Y: $f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & o/w \end{cases}$

i.e. $Y \sim exp(1)$

<u>Note</u>: If $g(.)$ is not monotone, the above cannot be applied.

e.g. $Y = X^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

p.d.f. of Y:

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y)$$
$$= \frac{\partial}{\partial y}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$
$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}$$