ACGF of a filtered process

Let $\{X_t\}$ be a covariance stationary process with ACVF $\gamma(\cdot)$ and ACGF

$$g_X(z) = \sum_{j=-\infty}^{\infty} z^j \gamma(j)$$

Consider a linear filtered process $\{Y_t\}$ $\ni$

$$Y_t = \sum_{i=0}^{q} \theta_i X_{t-i} = \theta(B) X_t$$

$$\theta(B) = \theta_0 + \theta_1 B + \cdots + \theta_q B^q$$

$$\gamma_Y(h) = Cov(Y_{t+h}, Y_t)$$

$$= Cov\left(\sum_{i=0}^{q} \theta_i X_{t+h-i}, \sum_{j=0}^{q} \theta_j X_{t-j}\right)$$

$$\gamma_y(h) = \sum_{i=0}^{q} \sum_{j=0}^{q} \theta_i \theta_j \gamma_X(h-i+j)$$

$$g_y(z) = \sum_{h=-\infty}^{\infty} z^h \gamma_y(h)$$

$$= \sum_{h=-\infty}^{\infty} z^h \left(\sum_{i=0}^{q} \sum_{j=0}^{q} \theta_i \theta_j \gamma_X(h-i+j) \right)$$

$$= \sum_{i=0}^{q} \theta_i \sum_{j=0}^{q} \theta_j \sum_{h=-\infty}^{\infty} z^h \gamma_X(h-i+j)$$

$$= \sum_{i=0}^{q} \theta_i z^i \sum_{j=0}^{q} \theta_j z^{-j} \sum_{h=-\infty}^{\infty} z^{h-i+j} \gamma_X(h-i+j)$$

$$h' = h-i+j$$

$$= \sum_{i=0}^{q} \theta_i z^i \sum_{j=0}^{q} \theta_j z^{-j} \sum_{h'=-\infty}^{\infty} z^{h'} \gamma_X(h')$$

i.e. $\underline{g_y(z)} = \theta(z)\,\theta(z^{-1})\, g_X(z)$

ACGF of the filtered process

Multivariate time series processes

In many situations, it is desirable to study behaviour of related time series processes under a multivariate setup rather than studying these processes in isolation as univariate processes. Under the multivariate setup it is possible to exploit the interrelationship between these processes for identifying the dynamics of these processes.

Let $\{X_{1_t}\}, \{X_{2_t}\}, \ldots \{X_{m_t}\}$ be m time series processes $\ni E X_{i_t}^2 < \infty \; \forall i, \forall t$ and define m-dimensional time series

$$\underset{\sim}{X}_t = (X_{1_t}, \ldots, X_{m_t})'$$

$$E(\underset{\sim}{X}_t) = \begin{pmatrix} E X_{1_t} \\ \vdots \\ E X_{m_t} \end{pmatrix} = \underset{\sim}{\mu}_t$$

$$E(\underset{\sim}{X}_t \underset{\sim}{X}'_{t+h}) = \begin{pmatrix} E(X_{1_t} X_{1_{t+h}}) & \cdots & E(X_{1_t} X_{m_{t+h}}) \\ & E(X_{2_t} X_{2_{t+h}}) \cdots & E(X_{2_t} X_{m_{t+h}}) \\ & \ddots & \vdots \\ & & E(X_{m_t} X_{m_{t+h}}) \end{pmatrix},$$

$$\begin{aligned}\operatorname{cov}(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h}) &= E(\underset{\sim}{X}_t - \underset{\sim}{\mu}_t)(\underset{\sim}{X}_{t+h} - \underset{\sim}{\mu}_{t+h})' \\ &= E(\underset{\sim}{X}_t \underset{\sim}{X}'_{t+h}) - \underset{\sim}{\mu}_t \underset{\sim}{\mu}'_{t+h} \\ &= \begin{pmatrix} \gamma_{11}(t,t+h) & \cdots & \gamma_{1m}(t,t+h) \\ \cdots & \cdots & \cdots \\ \gamma_{m1}(t,t+h) & \cdots & \gamma_{mm}(t,t+h) \end{pmatrix} \\ &= \Gamma_X(t, t+h) = \Gamma_t(h)\end{aligned}$$

where $\gamma_{ij}(t,t+h) = \operatorname{cov}(X_{i_t}, X_{j_{t+h}})$

<u>Def^n</u>: An m-variate process $\{\underset{\sim}{X}_t\}$ is covariance stationary if

(i) $\underset{\sim}{\mu}_t$ is independent of t

(ii) $\Gamma_X(t,t+h)$ is independent of t and is a f^n of h only

In such a situation

$$\underset{\sim}{\mu} = E\,\underset{\sim}{X}_t \quad \forall t$$

$$E(\underset{\sim}{X}_t - \underset{\sim}{\mu})(\underset{\sim}{X}_{t+h} - \underset{\sim}{\mu})' = \Gamma_X(h)$$

$$= \begin{pmatrix} \gamma_{11}(h) & \cdots & \gamma_{1m}(h) \\ & \ddots & \\ & & \gamma_{mm}(h) \end{pmatrix}$$

$\Gamma_X(h)$ is called the Auto covariance matrix function

Remark: If a vector process is stationary then all the components are stationary. Converse is not true, i.e. If the component processes are stationary, the vector process can be non stationary. Can you give a counter example to show that converse is not true?

Basic properties of $\Gamma_X(\cdot)$ for a stationary process

Let $\{\underset{\sim}{X}_t\}$ be a covariance stationary m-variate process. The autocovariance matrix f^n of $\underset{\sim}{X}_t$ is defined as

$$\Gamma_X(h) = Cov(\underset{\sim}{X}_t, \underset{\sim}{X}_{t+h}) = E(\underset{\sim}{X}_t - \underset{\sim}{\mu})(\underset{\sim}{X}_{t+h} - \underset{\sim}{\mu})'$$

$$h = 0, \pm 1, \pm 2, \ldots$$

$$E(\underset{\sim}{X}_t) = \underset{\sim}{\mu} \quad \forall t$$

Properties

(i) $\Gamma_X(h) = \Gamma_X'(-h)$

$$\Gamma_X(h) = E(\underset{\sim}{X}_t - \underset{\sim}{\mu})(\underset{\sim}{X}_{t+h} - \underset{\sim}{\mu})'$$

$$= E(\underset{\sim}{X}_{t-h} - \underset{\sim}{\mu})(\underset{\sim}{X}_t - \underset{\sim}{\mu})' \quad (\because \underset{\sim}{X}_t \text{ is covariance stationary})$$

$$= \left(E(\underset{\sim}{X}_t - \underset{\sim}{\mu})(\underset{\sim}{X}_{t-h} - \underset{\sim}{\mu})'\right)' = \left(\Gamma_X(-h)\right)'$$

(ii) $|\gamma_{ij}(h)| \le [\gamma_{ii}(0)\, \gamma_{jj}(0)]^{1/2}$; $i, j = 1(1)m$

This follows from Cauchy-Schwarz inequality

$\gamma_{ij} = Cov(X_{i_t}, X_{j_{t+h}})$

(iii) $\Gamma_X(h) = ((\gamma_{ij}(h)))$

$\{\gamma_{ii}(h)\}$ is the ACVF seq of $\{X_{i_t}\}$
$i = 1(1)m$

(iv) $\forall\, \underset{\sim}{a}_j \in \mathbb{R}^m$; $j = 1(1)m$, $\sum_{k,j=1}^{n} \underset{\sim}{a}_j' \Gamma_X(k-j)\, \underset{\sim}{a}_k \ge 0$

Note that $\forall\, \underset{\sim}{a}_j \in \mathbb{R}^m$ $j = 1(1)m$, the random variable

$$Y = (\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_n') \begin{pmatrix} (\underset{\sim}{X}_1 - \underset{\sim}{\mu}) \\ \vdots \\ (\underset{\sim}{X}_n - \underset{\sim}{\mu}) \end{pmatrix}$$

$$E Y^2 = E\left(\sum_{j=1}^{n} \underset{\sim}{a}_j' (\underset{\sim}{X}_j - \underset{\sim}{\mu})\right)^2 \ge 0$$

i.e. $E\left(\sum_{j=1}^{n} \underset{\sim}{a}_j' (\underset{\sim}{X}_j - \underset{\sim}{\mu})\right)\left(\sum_{j=1}^{n} \underset{\sim}{a}_j' (\underset{\sim}{X}_j - \underset{\sim}{\mu})\right)' \ge 0$

i.e. $E\left(\sum_j \sum_k \underset{\sim}{a}_j' (\underset{\sim}{X}_j - \underset{\sim}{\mu})(\underset{\sim}{X}_k - \underset{\sim}{\mu})' \underset{\sim}{a}_k\right) \ge 0$

i.e. $\sum_{j,k} \underset{\sim}{a}_j' \left(E(\underset{\sim}{X}_j - \underset{\sim}{\mu})(\underset{\sim}{X}_k - \underset{\sim}{\mu})'\right) \underset{\sim}{a}_k \ge 0$

i.e. $\sum_{j,k} \underset{\sim}{a}_j' \Gamma_X(k-j)\, \underset{\sim}{a}_k \ge 0$