Result: Suppose $\{X_t\}$ be a zero mean stationary process with absolutely summable ACVF $\gamma_X(\cdot)$ and $\{a_j\}_{j=-\infty}^{\infty}$ be an absolutely summable seq, then the spectral density f^n of the filtered process

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}$$ is given by

$$f_Y(\lambda) = (2\pi)^2 f_X(\lambda)\, f_a(\lambda)\, f_a^*(\lambda)$$

where, $f_X(\cdot)$: spectral density of input process $\{X_t\}$

$$f_a(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} a_j e^{-i\lambda j}$$

$f_a^*(\lambda)$: conjugate of $f_a(\lambda)$

Pf: $E\, Y_t = 0$

$$\begin{aligned}
\gamma_Y(h) &= E\, Y_t Y_{t+h} \\
&= E\left(\sum_{j=-\infty}^{\infty} a_j X_{t-j}\right)\left(\sum_{k=-\infty}^{\infty} a_k X_{t+h-k}\right) \\
&= E\left(\sum_j \sum_k a_j a_k X_{t-j} X_{t+h-k}\right) \\
&= \sum_j \sum_k a_j a_k E\left(X_{t-j} X_{t+h-k}\right)
\end{aligned}$$

i.e. $\gamma_y(h) = \sum_j \sum_k a_j a_k \gamma_x(h-k+j)$

$$f_y(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_y(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \left(\sum_j \sum_k a_j a_k \gamma_x(h-k+j) \right)$$

$$= \frac{1}{2\pi} \sum_j \sum_k a_j a_k \sum_h e^{-ih\lambda} \gamma_x(h-k+j)$$

$$= \frac{1}{2\pi} \sum_j \sum_k a_j a_k e^{-i\lambda k} e^{ij\lambda} \sum_h e^{-i\lambda(h-k+j)} \gamma_x(h-k+j)$$

$$= \frac{1}{2\pi} \sum_j \sum_k a_j a_k e^{-ik\lambda} e^{ij\lambda} \sum_{h'=-\infty}^{\infty} e^{-i\lambda h'} \gamma_x(h')$$

$h' = h-k+j)$

$$= \sum_j a_j e^{ij\lambda} \sum_k a_k e^{-ik\lambda} \frac{1}{2\pi} \sum_{h'=-\infty}^{\infty} e^{-ih'\lambda} \gamma_x(h')$$

i.e. $f_y(\lambda) = (2\pi)^2 f_a(\lambda) f_a^*(\lambda) f_x(\lambda)$

<u>Remark</u>: The above result can be used to desired spectral densities of AR, MA, ARMA processes.

<u>Remark</u>: $\sum_k a_k e^{-ik\lambda}$ is called transfer f^n of the filter with $\{a_k\}$ as filter coeffs

$\left|\sum_k a_k e^{-ik\lambda}\right|^2$ is called power transfer f^n of the filter.

Spectral density of MA(q) thro filtering result

$$X_t = \sum_{j=0}^{q} \theta_j \epsilon_{t-j} \quad ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$\{\epsilon_t\}$ is the input seq & $\{X_t\}$ is the output process after linear filtering.

$$\Rightarrow \quad f_X(\lambda) = (2\pi)^2 f_\theta(\lambda) f_\theta^*(\lambda) f_\epsilon(\lambda)$$

$$= \left(\sum_{j=0}^{q} \theta_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{q} \theta_j e^{ij\lambda}\right) \frac{\sigma^2}{2\pi}$$

i.e. $f_X(\lambda) = \dfrac{\sigma^2}{2\pi} \theta(e^{-i\lambda}) \theta(e^{i\lambda})$

as obtained earlier.

MA(∞) :
$$X_t = \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j}$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left(\sum_{0}^{\infty} \theta_j e^{-ij\lambda}\right)\left(\sum_{0}^{\infty} \theta_j e^{ij\lambda}\right)$$

$$= \frac{\sigma^2}{2\pi} \theta(e^{-i\lambda}) \theta(e^{i\lambda})$$

Spectral density of AR(p) thro filtering result

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t \quad ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\phi(B) X_t = \epsilon_t \quad ; \quad \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\epsilon_t = \sum_{j=0}^{p} \tilde{\phi}_j X_{t-j} \; ;$$

with $\tilde{\phi}_0 = 1, \quad \tilde{\phi}_j = -\phi_j \quad \forall j \geq 1$

using the filtering result

$$f_\epsilon(\lambda) = f_X(\lambda) \left(\sum_{0}^{p} \tilde{\phi}_j e^{ij\lambda}\right)\left(\sum_{0}^{p} \tilde{\phi}_j e^{-ij\lambda}\right)$$

i.e. $$\frac{\sigma^2}{2\pi} = f_X(\lambda)\left(\sum_0^p \tilde{\phi}_j e^{ij\lambda}\right)\left(\sum_0^p \tilde{\phi}_j e^{-ij\lambda}\right)$$

$$\Rightarrow \quad f_X(\lambda) = \frac{\sigma^2}{2\pi}\left[\left(\sum_0^p \tilde{\phi}_j e^{i\lambda j}\right)\left(\sum_0^p \tilde{\phi}_j e^{-ij\lambda}\right)\right]^{-1}$$

i.e. $$f_X(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{\phi(e^{i\lambda})\,\phi(e^{-i\lambda})}$$

Spectral density of ARMA(p,q) thro filtering argument

Let the ARMA(p,q) model be
$$\sum_0^p \phi_j X_{t-j} = \sum_{j=0}^q \theta_j \epsilon_{t-j} \; ; \; \epsilon_t \sim WN(0,\sigma^2)$$

$$\phi(B)X_t = \theta(B)\epsilon_t$$

Spectral density of l.h.s
$$f_X(\lambda)\left(\sum_0^p \phi_j e^{ij\lambda}\right)\left(\sum_0^p \phi_j e^{-ij\lambda}\right)$$

spectral density of r.h.s.
$$f_\epsilon(\lambda)\left(\sum_0^q \theta_j e^{ij\lambda}\right)\left(\sum_0^q \theta_j e^{-ij\lambda}\right)$$

$$\Rightarrow \quad f_X(\lambda) = \frac{\sigma^2}{2\pi}\,\frac{\left(\sum_0^q \theta_j e^{ij\lambda}\right)\left(\sum_0^q \theta_j e^{-ij\lambda}\right)}{\left(\sum_0^p \phi_j e^{ij\lambda}\right)\left(\sum_0^p \phi_j e^{-ij\lambda}\right)}$$

i.e. $$f_X(\lambda) = \frac{\sigma^2}{2\pi}\,\frac{\theta(e^{i\lambda})\,\theta(e^{-i\lambda})}{\phi(e^{i\lambda})\,\phi(e^{-i\lambda})}$$

Estimation of spectral density

Non-parametric approach:

$$\hat{f}_X(\lambda) = \frac{1}{2\pi} \sum_{h=-(n-1)}^{n-1} e^{-ih\lambda} \hat{\gamma}(h)$$

where, $$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \bar{X}_n)(X_{t+h} - \bar{X}_n)$$

Parametric approach:

Suppose $\{X_t\}$ is a Gaussian, stationary and invertible ARMA(p, q)

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left(\sum_{j=0}^{q} \theta_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{q} \theta_j e^{ij\lambda}\right)}{\left(\sum_{j=0}^{p} \phi_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{p} \phi_j e^{ij\lambda}\right)}.$$

Obtain parameter estimates (say MLE under Gaussian) as $\hat{\phi}_1, \ldots, \hat{\phi}_p, \hat{\theta}_1, \ldots, \hat{\theta}_q$ and $\hat{\sigma}^2$ and hence

$$\hat{f}_X(\lambda) = \frac{\hat{\sigma}^2}{2\pi} \frac{\left(\sum_{j=0}^{q} \hat{\theta}_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{q} \hat{\theta}_j e^{ij\lambda}\right)}{\left(\sum_{j=0}^{p} \hat{\phi}_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{p} \hat{\phi}_j e^{ij\lambda}\right)}$$

Spectral distribution function

Spectral representation theorem: A function $\gamma(\cdot)$ defined on the set of integers is the ACVF of a stationary process iff there exists a f^n $F(\cdot)$ which is right continuous, non decreasing, bounded on $[-\pi,\pi]$ with $F(-\pi)=0$ $\ni$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda) \quad \forall \text{ integer } h$$

Note: Spectral representation theorem ensures that there exists such an F for every stationary process. $F(\cdot)$ is called the spectral distribution function.

Note: F defined through the spectral representation theorem is a generalized distribution function on $[-\pi,\pi]$ in the sense that

$$G(\lambda) = \frac{F(\lambda)}{F(\pi)}$$ is a proper distribution function on $[-\pi,\pi]$.

Note: Note that $F(\pi) = \gamma(0) = V(X_1)$; from

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda)$$

we have

$$\frac{\gamma(h)}{F(\pi)} = \frac{\gamma(h)}{\gamma(0)} = \rho(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, d\left(\frac{F(\lambda)}{F(\pi)}\right)$$

i.e. $$\rho(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dG(\lambda)$$

The above is the spectral representation of $\rho(h)$.

Note: If $F(\lambda)$ is $\ni$ it can be expressed as.

$$F(\lambda) = \int_{-\pi}^{\lambda} f(y)\,dy \quad \forall\, \lambda \in [-\pi, \pi] \;:\; f(.) \geq 0$$

i.e. $F(.)$ is a generalised continuous distⁿ fⁿ

(in other words $G(.)$ is distⁿ fⁿ corresponding to a continuous random variable)

$f(.)$ in the above representation is the spectral density function and the associated time series is said to have a continuous spectrum

Note: If $F(.)$ is a generalized discrete distribution function (i.e. $G(.)$ is a proper ~~dist~~ discrete distribution function), increasing only by jumps, then the associated time series is said to have a discrete spectrum

Note: If $F(.)$ is a generalized mixed distribution function ($G(.)$ is a proper mixed distribution function), then we have a mixed spectrum; i.e. time series will have a continuous spectrum part and a discrete spectrum part.

Example 1 : Discrete spectrum

$$X_t = A\cos\omega t + B\sin\omega t$$

A & B are uncorrelated random variables with mean 0 and variance 1; $\omega \in (0,\pi)$ is fixed

$\gamma_X(h) = \cos(\omega h)$ is not absolutely summable and hence we can't talk about spectral density

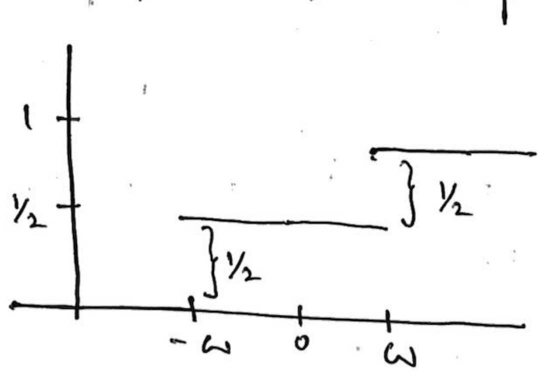
By spectral representation theorem,

$$\gamma(h) = \cos(\omega h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda)$$

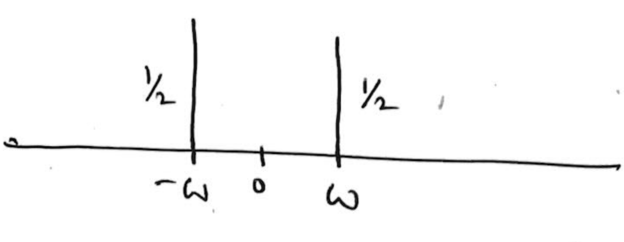
$$= \frac{1}{2} e^{-ih\omega} + \frac{1}{2} e^{ih\omega}$$

$$\Rightarrow F(\lambda) = \begin{cases} 0, & \lambda < -\omega \\ \frac{1}{2}, & -\omega \le \lambda < \omega \\ 1, & \lambda \ge \omega \end{cases} \quad \leftarrow \text{spectral dist}^n \text{ f}^n \text{ of } \{X_t\}$$

$F(\pi) = 1$; $F(\cdot)$ is a proper dist^n f^n



Discrete spectrum

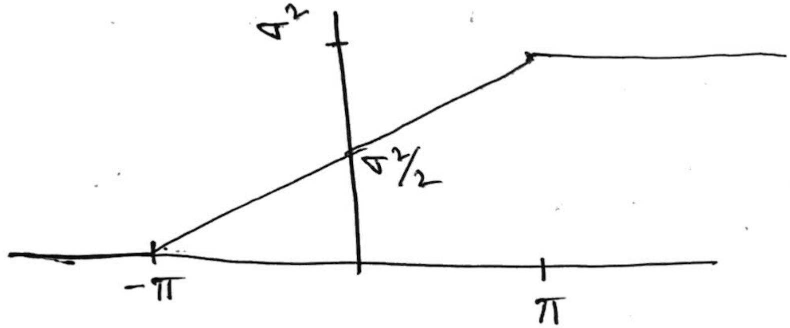


Example 2 Continuous spectrum

$X_t \sim WN(0, \sigma^2)$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \quad \forall \lambda \in [-\pi, \pi]$$

$$F_X(\lambda) = \int_{-\pi}^{\lambda} \frac{\sigma^2}{2\pi} \, d\lambda = \frac{\sigma^2}{2\pi}(\lambda + \pi)$$



$F_X(\lambda)$ is the spectral distn fn of $\{X_t\}$.

Note: If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary processes with ACVF $\gamma_X(\cdot)$ and $\gamma_Y(\cdot)$ and spectral distn functions $F_X(\cdot)$ and $F_Y(\cdot)$ then spectral distn function of $Z_t = X_t + Y_t$ is $F_Z(\lambda) = F_X(\lambda) + F_Y(\lambda)$.