<u>Remark</u>: We can show that here $\underset{\sim}{T} = (T_1, T_2)$; $T_1 = \Sigma X_i$
& $T_2 = \Sigma X_i^2$

is not complete

$$E\, T_2 = n\, E X_1^2 = n\, 2\theta$$

$$E\, T_1^2 = V(T_1) + (E\, T_1)^2 = n\theta^2 + (n\theta)^2$$

$$= \theta^2 n(n+1)$$

$$\left.\begin{aligned} &\Rightarrow E\left(\frac{T_1^2}{n(n+1)} - \frac{T_2}{2n}\right) = \theta^2 - \theta^2 = 0 \quad \forall\, \theta \in \Theta \\ &\underline{\underline{\not\Rightarrow}} \quad \frac{T_1^2}{n(n+1)} = \frac{T_2}{2n} \quad \text{w.p. } 1 \text{ (a.e.)} \end{aligned}\right\} - (*)$$

In fact $P\left(\frac{T_1^2}{n(n+1)} = \frac{T_2}{2n}\right) = 0$!!

(note that $\frac{T_1^2}{n(n+1)} - \frac{T_2}{2n}$ is a cont r.v.)

(*) $\Rightarrow$ $T(\underline{X}) = \left(\sum_{i=1}^{n} X_i , \sum_{i=1}^{n} X_i^2 \right)$ is NOT complete although

It is minimal suff.

Remark: Approaches to find UMVUE when minimal suff stat is complete

Step I : Find complete suff statistic

Step II : Find a function of complete suff stat which is unbiased for the estimand - this will be the UMVUE

Step II calculations

- For simple estimands it is easy to find u.e. based on c.s.s.
- For complicated estimands use Rao-Blackwellization or solve for the eq^n

$$E(\delta(T)) = g(\theta) \quad \forall \theta \in \Theta$$

- Find UMVUE thro Cramér-Rao Lower Bound (If the bound is attainable)

Cramer–Rao Lower Bound (CRLB)

CRLB provides lower bound for the variance of any unbiased estimator of $g(\theta)$.

$X_1, \dots, X_n$ be i.i.d. random sample from $f_\theta(x)$

$$\theta \in \Theta$$

$g(\theta)$: estimand

$g(\theta)$ is $\ni$ $\exists$ unbiased estimator of $g(\theta)$

Suppose the following regularity conditions hold

(i) support of the r.v.s does not depend on θ

(ii) $g(\theta)$ is differentiable

(iii) derivate of $\frac{\partial}{\partial\theta} f_\theta(x)$ exists and is finite

(iv) derivate of $\int f_\theta(x)\,dx$, w.r.t. θ, can be obtained by differentiating under the integral.

Let $\delta(\underline{X})$ be any unbiased estimator of $g(\theta)$

then

$$V(\delta(\underline{X})) \geq \left(\frac{(g'(\theta))^2}{n\,E\left(\frac{\partial}{\partial\theta} \log f_\theta(X)\right)^2} \right) \leftarrow \underline{\underline{\text{CRLB}}}$$

~~Alternate~~

Alternate form of CRLB

$$V(\delta(\underline{X})) \geq \frac{(g'(\theta))^2}{-n\,E\left(\frac{\partial^2}{\partial\theta^2} \log f_\theta(X)\right)}$$

provided that 2^{nd} derivate conditions (existence and interchange of differentiation and integration) holds

Remark: $E\left(\frac{\partial}{\partial\theta}\log f_\theta(x)\right)^2 = I(\theta)$ is called the Fisher information

$\nwarrow V\left(\frac{\partial}{\partial\theta}\log f_\theta(x)\right)$ as $E\left(\frac{\partial}{\partial\theta}\log f_\theta(x)\right) = 0$

Alt form: $I(\theta) = -E\left(\frac{\partial^2 \log f_\theta(x)}{\partial\theta^2}\right)$

Remark: If $\exists$ an unbiased estimator whose variance equals CRLB, then it is UMVUE.

Remark: There can be situations wherein UMVUE has variance higher than CRLB. In such cases, CRLB is not achievable.

Examples:

(i) $X_1, \ldots, X_n$ i.i.d. $B(1,\theta)$

$$f_\theta(x) = \theta^x(1-\theta)^{1-x}$$

$$\log f_\theta(x) = x\log\theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f_\theta(x)}{\partial\theta} = \frac{x}{\theta} + (1-x)\frac{1}{1-\theta}(-1)$$

$$= \frac{x}{\theta} - (1-x)\frac{1}{1-\theta}$$

$$\frac{\partial^2 \log f_\theta(x)}{\partial\theta^2} = -\frac{x}{\theta^2} - (1-x)\frac{1}{(1-\theta)^2}$$

$$E\left(\frac{\partial^2 \log f_\theta(x)}{\partial\theta^2}\right) = -\frac{\theta}{\theta^2} - \frac{1-\theta}{(1-\theta)^2} = -\frac{1}{\theta(1-\theta)}$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f_\theta(x)}{\partial\theta^2}\right) = \frac{1}{\theta(1-\theta)}$$

Estimand: $g(\theta) = \theta$

$$\text{CRLB} = \frac{(g'(\theta))^2}{n\, I(\theta)} = \frac{\theta(1-\theta)}{n}$$

$\delta(\underline{x}) = \frac{\sum X_i}{n}$ u.e. for $g(\theta)$

$$V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}\sum V(X_i)$$

$$= \frac{1}{n^2}\, n\, \theta(1-\theta) = \frac{\theta(1-\theta)}{n} = CRLB$$

$\Rightarrow \overline{X}$ is UMVUE for θ.

Example (ii)

$X_1, \ldots X_n$ r.s $N(\theta, 1)$

$$f_\theta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$$

$$\log f_\theta(x) = c - \frac{1}{2}(x-\theta)^2$$

$$\frac{\partial \log f_\theta(x)}{\partial \theta} = -\frac{1}{\cancel{2}}\cancel{2}(x-\theta)(-1) = x-\theta$$

$$E\left(\frac{\partial \log f_\theta(X)}{\partial \theta}\right)^2 = E(X-\theta)^2 = 1 = I(\theta)$$

or $$\frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = -1 \Rightarrow -E\left(\frac{\partial^2 \log f_\theta(X)}{\partial \theta^2}\right) = I(\theta) = 1$$

$g(\theta) = \theta$ say

$$CRLB = \frac{(g'(\theta))^2}{n\, I(\theta)} = \frac{1}{n}$$

$$V(\bar{X}) = \frac{1}{n} = CRLB$$

$\Rightarrow \bar{X}$ is UMVUE for θ

if $g(\theta) = \theta^2$

$$CRLB = \frac{(g'(\theta))^2}{n\, I(\theta)} = \frac{4\theta^2}{n}$$

Consistent Estimator

A large sample optimal property

Def^n: An estimator $\delta(\underline{X})$ is said to be consistent for $g(\theta)$

If $\delta(\underline{X}) \xrightarrow{p} g(\theta)$ as $n \to \infty$

Remark: Use WLLN to prove consistency or use definition of $\xrightarrow{p}$

EX1: $X_1, \ldots X_n$ r.s. $N(\mu, \sigma^2)$

By WLLN (i) $\frac{1}{n}\sum X_i \xrightarrow{p} \mu$

$\Rightarrow \bar{X}$ is consistent est of μ

(ii) $\frac{1}{n}\sum_1^n (X_i - \bar{X})^2 \xrightarrow{p} \sigma^2$

S_n^2 (say) $\Rightarrow \frac{1}{n}\sum(X_i - \bar{X})^2$ is consistent for σ^2

$$\frac{\bar{X}}{S_n^2} \xrightarrow{p} \frac{\mu}{\sigma^2}$$

$\Rightarrow \frac{\bar{X}}{S_n^2}$ is a consistent estimator for $\frac{\mu}{\sigma^2}$

EX: $X_1, \ldots X_n$ r.s. from $U(0, \theta)$ $\theta > 0$

$X_{(n)} \xrightarrow{p} \theta$ (proved earlier)

$\Rightarrow X_{(n)}$ is consistent for θ