Limits in Metric Spaces:

Recall: $(\mathbb{R}, |\cdot|)$ $x_n \to x$ as $n \to \infty$:

$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon, |x_n - x| < \varepsilon$.

Reformulation of the def^n towards generalizing the concept!

i.e., $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$, $x - \varepsilon < x_n < x + \varepsilon$

i.e., $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$, $x_n \in (x - \varepsilon, x + \varepsilon)$ "open interval"

Def^n: (M, d): metric space.

- For $x \in M$, $r > 0$, $B(x, r) := \{ y \in M \mid d(x,y) < r \}$ is called the open ball centered at x with radius r.
- Closed ball is the set $\{ y \in M \mid d(x,y) \leq r \}$

In $(\mathbb{R}, |\cdot|)$, for $x \in \mathbb{R}$, $r > 0$ $B(x, r) = (x - r, x + r)$

Take $(\mathbb{R}, d_0)$ where d_0 is the discrete metric. Recall $d_0(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

Let $0 < r < 1$. Then $B(x, r) := \{ y \in \mathbb{R} \mid d_0(x,y) < r < 1 \} = \{x\}$.

For any $r \geq 1$, $B(x, r) = \mathbb{R}$

(This is true for any set M with the discrete metric on it.)

$(V, \|\cdot\|)$ normed linear space $B(0, r) = \{ x \in V \mid \|x\| < r \}$ (Since $d(x,y) \doteq \|x - y\|$ & as $0 \in V$, we can consider $B(0, r)$)

(HW) In $(V, \|\cdot\|)$, $B(0, r) = r B(0, 1)$.

Def^n: A "neighborhood of x" is any set containing an open ball centered at x.

Def^n: (M, d) metric space.

A seq. (x_n) converges to x in M if $d(x_n, x) \to 0$ as $n \to \infty$.

Equivalently, (x_n) converges to x iff $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$, s.t. $\forall n \geq N_\varepsilon$, $d(x_n, x) < \varepsilon$.

iff $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$, $\{ x_n \mid n \geq N_\varepsilon \} \subset B(x, \varepsilon)$

Compare this with the red stuff at the beginning of this lecture to see this "natural" generalization!!!

Def^n: If $\{x_n \mid n \geq N\} \subset A$ for some $N \in \mathbb{N}$, we say that the seq. (x_n) is eventually in A.

- (x_n) converges to x iff $\forall\, \varepsilon > 0$, the seq. (x_n) is eventually in $B(x, \varepsilon)$.
  iff (x_n) is eventually in every neighborhood of x.

Notation: (x_n) converges to x in (M,d) : $x_n \xrightarrow{d} x$ in (M,d).

Def^n: A seq. (x_n) is Cauchy in (M,d) if $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n, m \geq N_\varepsilon$,
$d(x_n, x_m) < \varepsilon$.

Assigment-2 has some exercises to show that there are some limitations to the properties that hold under these generalizations (which is expected!)

Def^n: A subset $X \subset M$ is bounded if $X \subset B(x, r)$ for some $x \in M$, $r > 0$.

(HW). X is bounded iff for any $x \in M$, $\sup\limits_{a \in X} \{d(x,a)\} < \infty$.

Def^n: For $X \subset M$, $\operatorname{diam}(X) := \sup\{d(x,y) \mid x, y \in X\}$.

Exercise problems in the Assignment.