Assignment-8

1. Define $d(m,n) := \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Show that d is equivalent to the usual metric on $\mathbb{N}$ but $(\mathbb{N}, d)$ is not complete.

2. Show that $\mathbb{R}^n$ is complete under $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ norms.

3. Given metric spaces M and N, show that $M \times N$ is complete iff both M and N are complete.

4. Prove that the Hilbert cube H^∞ is complete.

5. Is it essential that the sets F_n in the Nested Set Thm. be both closed & bdd? Justify. Is the condition really necessary?

6. Prove that a normed linear space X is complete iff its closed unit ball $B = \{x \in X \mid \|x\| \leq 1\}$ is complete.

7. Let $E = \{x \in \mathbb{Q} \mid 2 < x^2 < 3\}$ considered as a subset of $\mathbb{Q}$ wrt. the usual metric. Show that E is closed and bdd. but not compact.

8. If A is compact in M, prove that $\operatorname{diam}(A)$ is finite.

9. Prove or disprove: M is compact iff every closed ball in M is compact.

10. If $A \subset M$ compact and $B \subset N$ compact, show that $A \times B$ is compact in $M \times N$.

Evaluation

11. Prove that the set $\{x \in \mathbb{R}^n \mid \|x\|_1 = 1\}$ is compact in $\mathbb{R}^n$ under the Euclidean norm.

12. Show that the Heine-Borel thm on $\mathbb{R}$ implies the Bolzano-Weierstrass thm. Conclude that the Heine-Borel thm. is equivalent to the completeness of $\mathbb{R}$.

13. Show that $A = \{ x \in \ell_2 \mid |x_n| \le \frac{1}{n}, n = 1, 2 \dots \}$ is compact in $(\ell_2, \|\cdot\|_2)$.

14. If M is compact, then M is separable.

15. Suppose M is compact and $f: M \to N$ is cts., one-one and onto. Prove that f is a homeomorphism.

16. Given $f: [a,b] \to \mathbb{R}$. Define $G: [a,b] \to \mathbb{R}^2$ by $G(x) = (x, f(x))$. Prove that TFAE:

(i) f is cts.

(ii) G is cts.

(iii) the graph of f is a compact subset of $\mathbb{R}^2$.

17. Show that TFAE:

(i) Every decreasing seq. of nonempty closed sets in M has nonempty intersection.

(ii) Every countable open cover of M admits a finite subcover.