→ $\mathbb{Q}$ does not have the completeness property.

Recall: $x^2 = 2$ does not have a solution in $\mathbb{Q}$.
Note that $a^2 = 2$ if and only if $(-a)^2 = 2$.
So without loss of generality, one can say $x^2 = 2$ does not have a solution in the set of all positive rational numbers.

Que.? Does there exists a solution of the equation $x^2 = 2$ in $\mathbb{R}$?

Ans. Yes! Thanks to Axiom 3. Consider $E = \{x > 0 \mid x^2 < 2\}$.

- More generally, for $a > 0$ and $n \in \mathbb{N}$, there is a (unique) solution $x \in \mathbb{R}$ such that $x^n = a$. (see Rudin: Thm 1.37 for the proof)

Back to the question on the completeness of $\mathbb{Q}$:

Idea! Consider $S := \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 < 2\}$.
S has the supremum in $\mathbb{R}$ (Why?). Let $t := \sup S$.
HW → t satisfies $t^2 = 2$. And, we have just seen that $t \notin \mathbb{Q}$.
Therefore, $\sup S \notin \mathbb{Q}$. Hence $\mathbb{Q}$ does not have the completeness prop.

→ $\mathbb{Q}$ has gaps in $\mathbb{R}$!!!

→ $\mathbb{Q}$ is "dense" in $\mathbb{R}$. That is,

Theorem: Given $x, y \in \mathbb{R}$ such that $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Pf: WLOG assume $x > 0$ (why?). Since $y - x > 0$, $\exists$ (there exists) $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y - x$. Hence $nx + 1 < ny$.

Note that $nx > 0$, so $\exists\, m \in \mathbb{N}$ s.t. $m - 1 \le nx < m$.
(Consequences of the Archimedean property)

Therefore, $m \le nx + 1 < ny \Rightarrow nx < m < ny$
$\Rightarrow x < \frac{m}{n} < y$ ∎

HW: If $x, y \in \mathbb{R}$ such that $x < y$, then $\exists$ an irrational number z such that $x < z < y$.

Upshot: Rationals are dense in $\mathbb{R}$ and Irrationals are also dense in $\mathbb{R}$.

Sequences and Sets of Real Numbers.

HW: Every real number is the limit of a (monotone) sequence of rational numbers.
(Every real number is the limit of a (monotone) sequence of (irrational numbers.))

$\longrightarrow$ A monotone bounded sequence of real nos. converges.

Pf. Consider (x_n) monotone and bounded.
Suppose (x_n) is increasing. Since (x_n) bounded, by the least upper bound property (Axiom 3) of $\mathbb{R}$, $\sup\{x_n\} < \infty$. Let $x := \sup\{x_n\}$.

Claim: $\lim x_n = x$

Pf of claim: For $\varepsilon > 0$, $\exists$ $N_\varepsilon \in \mathbb{N}$ such that $x - \varepsilon < x_{N_\varepsilon}$.

Since x_n's increasing, $x - \varepsilon < x_n$ $\forall$ (for all) $n \geq N_\varepsilon$ as $x_{N_\varepsilon} \leq x_{N_\varepsilon+1} \leq \cdots\cdots$

But also, $x_n \leq x$ because $x = \sup\{x_n\}$.

So, $\forall$ $n \geq N_\varepsilon$, $x - \varepsilon < x_n \leq x < x + \varepsilon$

$$\therefore \; |x_n - x| < \varepsilon.$$

Suppose (x_n) is decreasing. Then $(-x_n)$ is increasing.

HW: Complete the proof!

The Nested "Interval" Thm.

If (I_n) is a sequence of closed, bounded, and nonempty intervals in $\mathbb{R}$ with $I_1 \supseteq I_2 \supseteq \cdots\cdots$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \phi .$$

Moreover, if length $(I_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} I_n$ contains only one pt.

Idea of the proof:

$$I_n = [a_n, b_n]$$

- $a_1 \leq a_2 \leq a_3 \leq \cdots\cdots \leq b_1$ so by the monotone seq. convergence result,

$$a := \lim_{n\to\infty} a_n = \sup\{a_n\} < \infty$$

- $b_1 \geq b_2 \geq b_3 \geq \cdots\cdots \geq a_1$,

$$b := \lim_{n\to\infty} b_n = \inf\{b_n\} < \infty$$

- $a_{n+1} < b_{n+1}$ for all n, and $\lim a_n$, $\lim b_n$ exists.

so $a \leq b$.

claim: $\bigcap_n [a_n, b_n] = [a,b]$.

pf: If $x \in [a_n, b_n]$, $\forall n$ then $a_n \leq x \leq b_n$.

$$\Rightarrow a \leq x \leq b.$$

$\therefore \bigcap [a_n, b_n] \subseteq [a,b]$.

If $x \in [a,b]$, then $\sup a_n = a \leq x \leq b = \inf b_n$

$$\Downarrow \qquad\qquad \Downarrow$$

$$a_n \leq a \qquad\qquad b \leq b_n$$

$$\Rightarrow a_n \leq x \leq b_n, \forall n.$$

$\therefore [a,b] \subseteq \bigcap [a_n, b_n]$

Moreover, if length $[a_n, b_n] = b_n - a_n \to 0$ as $n \to \infty$, then

$$a_n = a_n - b_n + b_n$$
$$\Rightarrow \lim a_n = \lim (a_n - b_n) + \lim b_n$$
$$\Rightarrow \quad a = b \quad \text{as} \quad \lim (b_n - a_n) = 0. \qquad \blacksquare$$

Remark: Note that I_n's in the Nested Interval Thm are required to be closed & bdd. Construct examples where if one or more assumptions are dropped, then the conclusion fails!

Least upper bound axiom $\Rightarrow$ Monotone bounded seq. converges

$\Downarrow$

? $\Nwarrow$ Nested Interval thm.

claim: Nested Interval thm $\Rightarrow$ Least upper bound axiom!

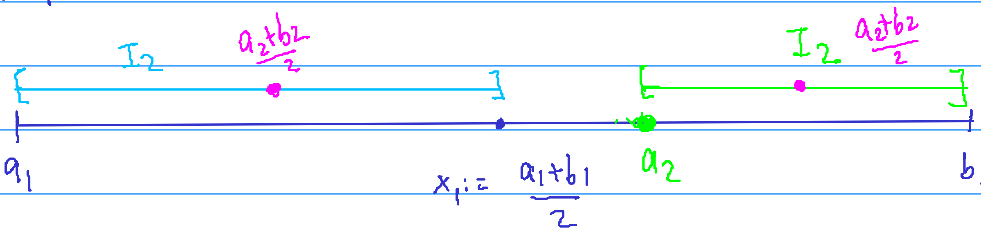
Pf: Idea: Consider $A \subseteq \mathbb{R}$ such that A is bounded above.

Want to show: A has the least upper bound in $\mathbb{R}$.

Case 1: A is a finite set in $\mathbb{R}$ (HW) Then we are done!

Case 2: A has infinitely many pts.

Let b_1 be an upper bound of A. And let $a_1 \in A$ such that $a_1 \neq b_1$.



Let $I_1 := [a_1, b_1]$

How to construct I_2?

- If x_1 is an upper bound for A • take $b_2 = x_1$
  $I_2 = [a_2, b_2]$ with $a_2 = a_1$, $b_2 = x_1$.
- If not, then $\exists\, a_2 \in A$ st. $x_1 < a_2$.
  • $I_2 = [a_2, b_2]$ where $b_2 = b_1$

Note that $a_1 \leq a_2$ and $b_2 \leq b_1$.

Consider I_2 (in blue or green depending on the case)

Also note that length$(I_1) = b_1 - a_1$

length$(I_2) = b_2 - a_2 \leq$ length$(I_1)/2$

length$(I_3) \leq$ length$(I_1)/2^2$ and so on!

These I_n's are nonempty, closed, and bounded, nested seq. with length decreasing to 0. By Nested Interval thm,

$\bigcap_n I_n = \{s\}$. Also, $a_n \to s$ as $n \to \infty$ where $a_n \in A$,

(Recall that in the proof of Nested Interval thm $\lim a_n = a$.)

Since $a_n \to s$ as $n \to \infty$, s is the least upper bound of A.

Recall (HW): For $A \subseteq \mathbb{R}$ such that A is bounded above, the following are equivalent

(i) $s = \sup A$

(ii) $\exists\, a_n \in A$ such that $a_n \to s$ as $n \to \infty$.

Upshot: Least upper bound axiom $\Leftrightarrow$ Monotone bdd. seq. convergence

$\Updownarrow$

Nested Interval thm.