MIDSEM EXAM
MTH301A-ANALYSIS I
TOTAL SCORE: 50

Please mention all the details to get full credit.

(1) Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $\liminf\{na_n\} = 0$. Is the converse true? Explain. [5+2 points]

(2) (i) Show that any nonempty perfect set of $\mathbb{R}$ with respect to the usual metric is uncountable. [5 points]

(ii) Is every closed uncountable subset of $\mathbb{R}$ with respect to the usual metric a perfect set? Explain. [3 points]

(iii) For (M, d) a metric space, is every nonempty perfect set an uncountable set? Explain. [5 points]

(3) (i) Suppose $f : (M, d) \to (N, \rho)$ is a homeomorphism.
Prove or disprove: (x_n) is Cauchy if and only if $(f(x_n))$ is Cauchy. [5 points]

(ii) Is $(\ell_2, ||\cdot||_2)$ isometrically homeomorphic to $(\ell_\infty, ||\cdot||_\infty)$? Explain. [5 points]

(4) Consider $\mathcal{B}[0, 1] := \{f : [0, 1] \to \mathbb{R} \mid f \text{ is a bounded function}\}$. Define

$$||f||_\infty := \sup_{0\leq t\leq 1}|f(t)|.$$

(i) Is $(\mathcal{B}[0, 1], ||\cdot||_\infty)$ a separable normed linear space? [7 points]

(ii) For $f \in \mathcal{B}[0, 1]$, define $||f||_1 := \int_0^1 |f(t)|dt$. Is this a norm on $\mathcal{B}[0, 1]$? If so, then is $||\cdot||_\infty$ equivalent to $||\cdot||_1$? [3 points]

(5) Is it possible to have a function on a metric space which is discontinuous at every point of the metric space but the restriction of that function on a dense set is continuous? Explain. [5 points]

(6) Given (M, d) and (N, ρ) metric spaces, consider the product metric spaces $(M \times N, d_1)$ and $(M \times N, d_\infty)$ where $d_1((a, x), (b, y)) := d(a, b) + \rho(x, y)$ and $d_\infty((a, x), (b, y)) := \max\{d(a, b), \rho(x, y)\}$.
Is d_1 **strongly equivalent** to d_∞? Explain. [5 points]