→ $\mathbb{R}$ or (M,d) $\quad \tilde{d} := \dfrac{d}{1+d}$ on M. $\quad |\alpha|\tilde{d}(x,y) \neq \tilde{d}(\alpha x, \alpha y)$.

$\tilde{d}$.

Given (M,d)

→ $\left\{ \begin{array}{c} \text{Normed linear space} \\ (M, \|\cdot\|) \end{array} \right\} \underset{\text{proper}}{\subsetneq}$ Metric spaces. (M,d)

$(M,\|\cdot\|) \Downarrow (M, d_{\|\cdot\|})$ $\qquad d \neq d_{\|\cdot\|}$

→ f.d. vector spaces → "infinite-dim. vector spaces"

n.l. V $\qquad \ell^p \qquad V \qquad C[0,1]$

$(V, \|\cdot\|)$ $\qquad \|\cdot\|_p \qquad \exists\ \|\cdot\|, \not\sim \|\cdot\|' \qquad \ell^p(\mathbb{N}, \mu_c)$

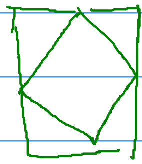
$\|\cdot\|' \quad \|\cdot\| \sim \|\cdot\|'$ $\qquad L^p(X,\mu)$

→ $\|\cdot\| \sim \|\cdot\|'$ if $C_1\|\cdot\|' \leq \|\cdot\| \leq C_2 \|\cdot\|'$

→ $\mathbb{R}^n \quad \|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$

$\mathbb{R}^2 \quad \mathbb{R}^3 \qquad V \leftrightarrow \mathbb{R}^n$

$n = \dim V$



→ $(C[0,1], \|\cdot\|_\infty)$ $\qquad \|f\|_\infty := \sup\limits_{0\leq t\leq 1} \{|f(t)|\}$

n.l.s.

HW: $C[0,1]$ is an inf. dim vector space.

$(C[0,1], \|\cdot\|_1)$ $\qquad \|f\|_1 := \int_0^1 |f(t)|\,dt$

n.l.s.

HW: $(C[0,1], \|\cdot\|_1)$ → There are Cauchy seq. that do not converge in $C[0,1]$ w.r.t. $\|\cdot\|_1$.

→ $\|\cdot\|_\infty \not\sim \|\cdot\|_1$.

→ $1 < p < q < \infty$

HW: $\ell_q \subsetneq \ell_p$ $\qquad \ell_\infty \subsetneq \ell_p \quad \forall\ p \geq 1$

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots\dots) \in \ell^p.$$

$$\|x^{(n)} - x^{(m)}\|_p \to 0 \text{ as } m,n \to \infty$$

$$|x_i^{(n)} - x_i^{(m)}| \le \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p \to 0$$

$$\{(x_i^{(n)})\} \text{ Cauchy seq in } \mathbb{R}$$

$$\downarrow$$

$$x_i$$

$$\cancel{x} = (x_1, x_2, x_3, \dots)$$

$$\text{Goal!} = \cdot\, x \in \ell^p$$

$$\cdot\, \& \;\; \|x^{(n)} - x\|_p \to 0$$

$$[-n, n] \quad \text{in } \mathbb{R}$$

$$\bigcup_{n=1}^{\infty} (0, n)$$

Claim: For $1 \le p < \infty$, every Cauchy seq. in $(\ell^p, \|\cdot\|_p)$ cvgs. in ℓ^p.

Pf: $\quad x^{(n)} := (x_1^{(n)}, x_2^{(n)}, \dots) \in \ell^p$

$$|x_i^{(n)} - x_i^{(m)}| \le \left(\sum_{1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p\right)^{1/p} \to 0 \text{ as } n \to \infty.$$

That is, for $\varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n, m \ge N_\varepsilon$.

$$|x_i^{(n)} - x_i^{(m)}| < \varepsilon, \quad \forall\, i \ge 1.$$

For each i, $\quad x_i^{(n)} \to x_i$ (say). Note that for each i, $|x_i^{(n)} - x_i| < \varepsilon \;\; \forall\, n \ge N_\varepsilon$.

WTS: $\quad x = (x_1, x_2, \dots) \in \ell^p$ & $\|x^{(n)} - x\|_p \to 0$ as $n \to \infty$.

$$\text{Since } \sum_{i=1}^{k} |x_i^{(n)} - x_i^{(m)}|^p \le \sum_{1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \varepsilon^p, \; \forall\, n, m \ge N_\varepsilon,$$

(1) $\longrightarrow$ Fix $m = N_\varepsilon$. Then $\displaystyle\sum_{i=1}^{k} |x_i^{(n)} - x_i^{(N_\varepsilon)}|^p < \varepsilon^p, \; \forall\, n \ge N_\varepsilon.$

Since $\displaystyle\sum_{i=1}^{k} |x_i^{(n)} - x_i^{(N_\varepsilon)}|^p \longrightarrow \sum_{i=1}^{k} |x_i - x_i^{(N_\varepsilon)}|^p, \qquad \sum_{i=1}^{k} |x_i - x_i^{(N_\varepsilon)}|^p < \varepsilon^p$

(2) $\boxed{\Rightarrow \sum_{1}^{\infty} \left| x_i - x_i^{(N_\varepsilon)} \right|^p < \varepsilon^p .}$

$$\Rightarrow y^{(N_\varepsilon)} := \left(x_1 - x_1^{(N_\varepsilon)}, x_2 - x_2^{(N_\varepsilon)}, \dots \right) \in \ell^p .$$

Hence, $x = (x_1, x_2, \dots) = y^{(N_\varepsilon)} + x^{(N_\varepsilon)} \in \ell^p$ as $y^{(N_\varepsilon)}, x^{(N_\varepsilon)} \in \ell^p$.

Note that (1) $\Rightarrow$ (2) for each $m \geq N_\varepsilon$.

Therefore,
$$\sum_{i=1}^{\infty} \left| x_i - x_i^{(n)} \right|^p < \varepsilon^p , \ \forall \ n \geq N_\varepsilon .$$

$$\Rightarrow \left\| x - x^{(n)} \right\|_p < \varepsilon \quad \forall \ n \geq N_\varepsilon .$$

∎