<u>Remark</u> : It is always possible to write a stationary VAR(p) process as VMA(∞) process with the VWN process having elements that are mutually uncorrelated.

Consider the VMA(∞) representation of VAR(p).

$$\underset{K\times 1}{\underset{\sim}{X}_t} = \underset{\sim}{\epsilon}_t + \Psi_1 \underset{\sim}{\epsilon}_{t-1} + \Psi_2 \underset{\sim}{\epsilon}_{t-2} + \cdots$$

$$\underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \Sigma), \ \Sigma > 0$$

Note that $\exists$ a non-singular matrix $H \ni$

$$H \Sigma H' = D_\lambda \text{ (diagonal matrix)}$$

Using the above non-singular matrix H, we can write

$$\underset{\sim}{X}_t = H^{-1} H \underset{\sim}{\epsilon}_t + \Psi_1 H^{-1} H \underset{\sim}{\epsilon}_{t-1} + \Psi_2 H^{-1} H \underset{\sim}{\epsilon}_{t-2} + \cdots$$

i.e. $\underset{\sim}{X}_t = \Psi_0^* \underset{\sim}{\eta}_t + \Psi_1^* \underset{\sim}{\eta}_{t-1} + \Psi_2^* \underset{\sim}{\eta}_{t-2} + \cdots$

$$\Psi_i^* = \Psi_i H^{-1} \ ; \ i = 0, 1, \ldots$$

$$\Psi_0 = I_K$$

$$\underset{\sim}{\eta}_t = H \underset{\sim}{\epsilon}_t \text{ is } \ni E(\underset{\sim}{\eta}_t) = H\, E(\underset{\sim}{\epsilon}_t) = \underset{\sim}{0}$$

$$Cov(\underset{\sim}{\eta}_t, \underset{\sim}{\eta}_s) = E(\underset{\sim}{\eta}_t \underset{\sim}{\eta}_s')$$

i.e. $Cov(\underset{\sim}{\eta}_t, \underset{\sim}{\eta}_s) = E(H\underset{\sim}{\epsilon}_t)(H\underset{\sim}{\epsilon}_s)'$

$$= H\, E(\underset{\sim}{\epsilon}_t \underset{\sim}{\epsilon}_s')\, H'$$

$$= \begin{cases} H\Sigma H' = D_\lambda, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases}$$

$\Rightarrow \underset{\sim}{\eta}_t \sim VWN(\underset{\sim}{0}, D_\lambda)$

$\Rightarrow$ elements of $\underset{\sim}{\eta}_t$ are uncorrelated

$\Rightarrow \{\underset{\sim}{X}_t\}$ is expressed as VMA(∞) with VWN process with uncorrelated components.

Vector ARMA(p,q)

$\underset{\sim}{X}_t \sim VARMA(p,q)$ if

$$\underset{K\times 1}{\underset{\sim}{X}_t} = \Phi_1 \underset{\sim}{X}_{t-1} + \cdots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\epsilon}_t + \Theta_1 \underset{\sim}{\epsilon}_{t-1} + \cdots + \Theta_q \underset{\sim}{\epsilon}_{t-q}$$

$$\Phi_p \neq 0, \quad \Theta_q \neq 0, \quad \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

$$Cov(\underset{\sim}{\epsilon}_t, \underset{\sim}{X}_{t-j}) = 0 \quad \forall j > 0$$

Conditions for stationarity

$\underset{\sim}{X}_t$ is covariance stationary if all values of z satisfying $|I_K - \Phi_1 z - \cdots - \Phi_p z^p| = 0$ lie outside the unit circle

i.e. all z satisfying $|\Phi(z)| = 0$ lie outside the unit circle (with model as $\Phi(B)\underset{\sim}{X}_t = \Theta(B)\underset{\sim}{\epsilon}_t$)

Every covariance stationary vector ARMA(p,q) has a causal representation through,

$$\underset{\sim}{X}_t = \Phi(B)^{-1}\Theta(B)\underset{\sim}{\epsilon}_t = \Psi(B)\underset{\sim}{\epsilon}_t, \text{ say}$$

i.e. $\Phi(B)^{-1}\Theta(B) = \Psi(B)$

$$\Rightarrow \Theta(B) = \Phi(B)\Psi(B)$$

i.e. $(I_K + \Theta_1 B + \cdots + \Theta_q B^q) = (I_K - \Phi_1 B - \cdots - \Phi_p B^p)(\Psi_0 + \Psi_1 B + \cdots)$

i.e.

$$(I_K + \Theta_1 B + \cdots + \Theta_q B^q)$$
$$= \Psi_0 + (\Psi_1 - \Phi_1 \Psi_0) B + (\Psi_2 - \Phi_1 \Psi_1 - \Phi_2 \Psi_0) B^2 +$$
$$(\Psi_3 - \Phi_1 \Psi_2 - \Phi_2 \Psi_1 - \Phi_3 \Psi_0) + \cdots$$

comparing coefficients,

B^0 : $\Psi_0 = I_K$

B^1 : $\Psi_1 = \Theta_1 + \Phi_1$

B^2 : $\Psi_2 = \Phi_1 \Psi_1 + \Phi_2 \Psi_0 + \Theta_2$

B^3 : $\Psi_3 = \Phi_1 \Psi_2 + \Phi_2 \Psi_1 + \Phi_3 \Psi_0 + \Theta_3$

$\vdots$

In general,

$$\Psi_s = \Phi_1 \Psi_{s-1} + \Phi_2 \Psi_{s-2} + \cdots + \Phi_p \Psi_{s-p} + \Theta_s \quad \forall\, s \le q$$

$$\& \quad \Psi_s = \Phi_1 \Psi_{s-1} + \Phi_2 \Psi_{s-2} + \cdots + \Phi_p \Psi_{s-p} \quad \forall\, s > q$$

with $\Psi_r = 0 \;\; \forall\, r < 0 \;\&\; \Psi_r = I_K$ if $r = 0$

Def: $\underset{\sim}{X}_t$ is said to be invertible if all values of z satisfying $|\Theta(z)| = 0$ lie outside the unit circle. In this case,

$$\underset{\sim}{\epsilon}_t = \Theta(B)^{-1} \Phi(B) \underset{\sim}{X}_t = \Psi(B) \underset{\sim}{X}_t$$

Vector ARMA$(p, q) \longrightarrow$ VAR(∞) use comparing coeffs.

VMA(q) to VAR(∞)

VMA(q) model

VMA(q) is invertible if all z satisfying $|\Theta(z)| = 0$ lie outside the unit circle.

$$\underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t + \Theta_1 \underset{\sim}{\epsilon}_{t-1} + \cdots + \Theta_q \underset{\sim}{\epsilon}_{t-q}$$

$$\Theta_q \neq 0, \quad \underset{\sim}{\epsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$$

$$\underset{\sim}{X}_t = \Theta(B) \underset{\sim}{\epsilon}_t$$

$$\Theta(B) = I_K + \Theta_1 B + \cdots + \Theta_q B^q$$

Let $\Theta(B)^{-1}$ denote the inverse operator of $\Theta(B)$, then

$$\underset{\sim}{\epsilon}_t = \Theta(B)^{-1} \underset{\sim}{X}_t = \underline{\Psi}(B) \underset{\sim}{X}_t = \sum_{j=0}^{\infty} \Psi_j \underset{\sim}{X}_{t-j}$$

$$\underline{\Psi}(B) = \Theta(B)^{-1}$$

i.e. $I_K = \Theta(B) \underline{\Psi}(B)$

$$I_K = (I_K + \Theta_1 B + \cdots + \Theta_q B^q)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$$

i.e. $I_K = \Psi_0 + (\Theta_1 \Psi_0 + \Psi_1) B + (\Psi_2 + \Theta_1 \Psi_1 + \Theta_2 \Psi_0) B^2 + \cdots$

Comparing coeff of B^j:

B^0 : $\Psi_0 = I_K$

B^1 : $\Psi_1 = -\Theta_1 \Psi_0 = -\Theta_1$

B^2 : $\Psi_2 = -\Theta_1 \Psi_1 - \Theta_2 \Psi_0$

$\Psi_2 = \Theta_1^2 - \Theta_2$

$\rightarrow \Psi_s = -(\Theta_1 \Psi_{s-1} + \Theta_2 \Psi_{s-2} + \cdots + \Theta_q \Psi_{s-q})$

$\Psi_0 = I$, $\Psi_l = 0 \ \forall l < 0$

B^3 : $\Psi_3 = -\Theta_1 \Psi_2 - \Theta_2 \Psi_1 - \Theta_3 \Psi_0$

i.e. $\Psi_3 = -\Theta_1(\Theta_1^2 - \Theta_2) - \Theta_2(-\Theta_1) - \Theta_3$

$$-\Theta_1^3 + \Theta_1 \Theta_2 + \Theta_2 \Theta_1 - \Theta_3$$

Remark : Impulse response function (IRF)

Covariance stationary VAR(p) $\rightarrow$ VMA(∞)

Covariance stationary VARMA(p, q) $\rightarrow$ VMA(∞)

$$X_t = \underset{\sim}{\epsilon}_t + \Psi_1 \underset{\sim}{\epsilon}_{t-1} + \Psi_2 \underset{\sim}{\epsilon}_{t-2} + \cdots$$

$$\begin{pmatrix} X_{1t} \\ \vdots \\ X_{Kt} \end{pmatrix} = \begin{pmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{Kt} \end{pmatrix} + \begin{pmatrix} \psi_{11}^{(1)} & \cdots & \psi_{1K}^{(1)} \\ \cdots & \cdots & \cdots \\ \psi_{K1}^{(1)} & \cdots & \psi_{KK}^{(1)} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-1} \\ \vdots \\ \epsilon_{K,t-1} \end{pmatrix} + \cdots$$

$$\cdots + \begin{pmatrix} \psi_{11}^{(s)} & \cdots & \psi_{1K}^{(s)} \\ \cdots & \cdots & \cdots \\ \psi_{K1}^{(s)} & \cdots & \psi_{KK}^{(s)} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-s} \\ \vdots \\ \epsilon_{K,t-s} \end{pmatrix} + \cdots$$

$$X_{it} = \epsilon_{it} + \left(\psi_{i1}^{(1)} \epsilon_{1,t-1} + \cdots + \psi_{iK}^{(1)} \epsilon_{K,t-1} \right) + \cdots$$

$$\cdots + \left(\psi_{i1}^{(s)} \epsilon_{1,t-s} + \cdots + \psi_{iK}^{(s)} \epsilon_{K,t-s} \right) + \cdots$$

$i = 1(1)K$

$$\frac{\partial X_{i,t+s}}{\partial \epsilon_{j,t}} = \psi_{ij}^{(s)}$$

$$\frac{\partial \underset{\sim}{X}_{t+s}}{\partial \underset{\sim}{\epsilon}_t} = \Psi_s$$