Problem Set - 1
MTH-204, MTH-204A
Abstract Algebra

1. Prove that the set of all rational numbers with odd denominators is a group with respect to addition.

2. Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(a.b)^n = a^n.b^n$.

3. If G is a group such that $(a.b)^2 = a^2.b^2$ for all $a, b \in G$, show that G must be abelian.

4. In S_3 give an example of two elements x, y such that $(x.y)^2 \neq x^2.y^2$.

5. In S_3 show that there are four elements satisfying $x^2 = e$ and three elements satisfying $y^3 = e$.

6. If G is a finite group, show that there exists a positive integer n such that $a^n = e$ for all $a \in G$.

7. Show that if every element of the group G has its own inverse, then G is abelian.

8. If G is a group of even order, prove that it has an element $a \neq e$ satisfying $a^2 = e$.

9. Let p be a prime and let $GL_n(\mathbb{Z}_p)$ be the set of all $n \times n$ invertible matrices whose entries are from the set $\mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$.

(a) Prove that $GL_n(\mathbb{Z}_p)$ is a group with respect to matrix multiplication.

(b) What is the order of $GL_n(\mathbb{Z}_p)$?