Closed Sets:

Def^n: A set $F \subset M$ is closed if $M \setminus F$ is open.

Properties:

- For $\{F_\alpha\}$ collection of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
- For $\{F_1, \dots, F_n\}$ closed sets, $\bigcup_{k=1}^{n} F_k$ is closed.
- In (M,d), every finite set of points is closed.

There are examples of sets which are neither open nor closed: e.g. $(0,1]$ in $(\mathbb{R}, |\cdot|)$.

Q. Is there any set $A \subset M$, $A \neq \phi$, $A \neq M$ s.t. A is both open and closed? (Answered later in this lecture)

→ Sequential characterization of closed sets.

Thm: (M,d) TFAE:

(i) F is closed.

(ii) $\forall \varepsilon > 0$, if $B(x,\varepsilon) \cap F \neq \phi$, then $x \in F$.

(iii) If a seq. $(x_n) \subset F$ s.t. $x_n \xrightarrow{d} x$ in M, then $x \in F$.

(See its application in Assignments)

$A \subset M$ any arbitrary subset (M,d)

→ $\exists\ \mathcal{U}$ open s.t. $A \subset \mathcal{U}$. ? smallest/minimal

$\mathcal{U}_a \subseteq A$ $\bigcup_{a \in A} \mathcal{U}_a$ open

Q. Can we construct an open set or a closed set from a given set A in (M,d)?

→ Interior and Closure of a set:

Given $A \subset M$, define interior of $A := \bigcup \{\mathcal{U} \mid \mathcal{U} \text{ open and } \mathcal{U} \supset A\}$

why $\overset{?}{=} \{x \in A \mid B(x,\varepsilon) \subset A \text{ for some } \varepsilon > 0\}$.

Notation: $\text{int}(A)$ or A°. HW: $\text{int}(A)$ is an open set contained in A.

define closure of $A := \bigcap \{F \mid F \text{ closed and } A \subset F\}$

Notation: $cl(A)$ or $\bar{A}$. HW. $cl(A)$ is a closed set containing A.

$$\boxed{\text{int}(A) \subset A \subset cl(A)}$$

Q. Is $int(A)$ always a nonempty set? (HW)

→ Characterization of $cl(A)$ or $\bar{A}$

(HW). Given (M,d) and $A \subset M$, TFAE:

(i) $x \in \bar{A}$

(ii) $\forall\, \varepsilon > 0$, $B(x,\varepsilon) \cap A \neq \phi$

(iii) $\exists\, (x_n) \subset A$ s.t. $x_n \xrightarrow{d} x$.

(See more of it and some new important concepts related to these in Assignments)

There are examples of metric spaces which have nonempty and proper subsets that are both open and closed !!! (Introduce "Relative" Metric Spaces)

→ The Relative Metric

Given (M,d). For $A \subset M$, one can consider (A,d) as a metric space where the metric d is the restriction of d on the set A.

Q. What are open sets, closed sets in (A,d)?

Recall: For (M,d), $B(x,r) := \{ y \in M \mid d(x,y) < r \}$.

An open ball in A is $B_A(a,r) := \{ b \in A \mid d(a,b) < r \} \overset{?}{=} A \cap B(a,r)$

Def^n: A set $V \subset A$ is open in A if for each $x \in V$, $\exists\, B(x,r)$ in M s.t.

$$V \cap B(x,r) \subset V.$$

Proposition: Given (M,d) a metric space. For $A \subset M$, in (A,d) metric space one has:

(i) A set $V \subset A$ is open in A iff $V = A \cap \mathcal{U}$ for some open set $\mathcal{U}$ in M.

(ii) A set $E \subset A$ is closed in A iff $E = A \cap F$ for some closed set F in M.

(iii) For $B \subset A$, $cl_A(B) = A \cap cl(B)$

$$int_A(B) = A \cap int(B)$$

Example of a metric space which has sets that are both open and closed:

(i) Consider $(\mathbb{R}, |\cdot|)$. Take $A = (0,1) \cup \{2\}$. Consider the metric space $(A, |\cdot|)$

Note that $\{2\} = \{2\} \cap (1,3)$, hence open.

$\{2\} = \{2\} \cap [2,3]$, hence closed.

(ii) Consider $(\mathbb{R}, |\cdot|)$. Take $A = \mathbb{Q}$. Define $B := \{ x \in \mathbb{Q} \mid x < \sqrt{2} \}$

HW: Show that B is both open and closed!

(iii) Consider $(\mathbb{R}^2, \|\cdot\|_2)$. Take $A = \mathbb{Q} \times \mathbb{Q}$.

HW: Can you construct a subset of A which is both open and closed?

(iv) Show that if $A \subset M$ is open in M and $V \subset A$ open, then V is open in M.

(Here (A,d) is the relative metric space)

Q. If $A \subset M$ is closed in M and $E \subset A$ is closed in A, then is E closed in M?

(v) $\mathbb{R}$ embeds in $\mathbb{R}^2$ via $x \mapsto (x,0)$.

Consider $(\mathbb{R}^2, \|\cdot\|_2)$. Take $A = \mathbb{R}$. Then $(A, \|\cdot\|_2) = (\mathbb{R}, |\cdot|)$ (why?)

Q. Is a set that is open in $\mathbb{R}$ wrt relative metric, open in $(\mathbb{R}^2, \|\cdot\|_2)$?

Q. Is a set that is closed in $\mathbb{R}$ wrt. relative metric, closed in $(\mathbb{R}^2, \|\cdot\|_2)$?