	Recall: A monotone bounded seq. converges in IR.
	THE MET TO THE MET THE
Question:	what if "monotone" condition is dropped?
Äns.	NO! Plenty of examples of bounded sequences that do not converge.
Qupition:	Can we say something about bounded segmences of real nos. ?
	Introduce a new notion!
	(an): bounded seal in R
	Define $t_n := \inf \{a_n, a_{n+1}, \dots \}$ for each $n = 1$.
	To i = sup { 9 mg any, } for each n7/1.
	Note that In D and To D's segmences.
	Since (an) is bdd., inf{ak} = tn = Tn = sup {ak}. kni
	Therefore, (ty) converges and (Ty) converges to institute and sup ETy's, respectively.
	That is, lim to exist and line To exists.
	That is, lim to exists and lim To exists.
۵	$\lim_{h\to\infty} \frac{1}{h} = \inf_{h\to\infty} \sup_{h\to\infty} \left\{ a_h, a_{h+1}, \dots \right\} = \inf_{h\to\infty} \sup_{h\to\infty} \left\{ a_k \right\}$
•	$\lim_{n\to\infty} T_n = \sup_{n\neq 1} \inf_{n} \left\{ a_n, a_{n+1}, \dots \right\} = \sup_{n\neq 1} \inf_{n\neq 1} \left\{ a_n \right\}.$
	N-1 00 W71 K7/N
Question.	Can we extend this idea to "any segmence?
Qn.	Yos! need to consider the extended real number system IRU {±0}.
	· ·

Given any seq. (an) of real numbers, define (limit inferior) liminf an := sub (inf {an, ann, ... }) lin sub an is inf (sub {an, and, ... }) (limit Superior) If (an) is convergent, then liminfan = limsup an = liman If (an) is a bounded segg, then tim sup an = lim (sup \ ak |k>n \) \ \displant \ \pm \omega_n = \lim (sup \ \ ak |k>n \ \) Characterization of lim sup an, if (an) is a bold- seq. Lit Miz lim sup {any. " + E70, an < M+E for all but finitely many n, and (HW) M-E < an for infinitely many n. The Bolzano-Weierstrass Thm (for sequences). Theoremi Every bounded sequence of red row has a convergent subsequence. Let (an) be a bounded seap and let M:= lim sup {and < ov. For Ex := 1 , I NKEN such that + NONE, $a_n < M + \frac{1}{b}$, and M-k < an for infinitely many n. Choose nk > Nk such that M-k < ank. Also, ank < M+k. .. $M - \frac{1}{R} < a_{n_R} < M + \frac{1}{R}$. Here $|a_{n_R} - M| < \frac{1}{R}$, $\forall R > 1$. Therefore, and > M as k+ as.

Corollary:	Every bodd. ceg. of real numbers has a Cauchy subsequence. (HW)
<u></u> ₩	Every Cauchy segs of real numbers converges.
P£:	Lit (an) be a Cauchy seg.
1 12.	For E70, & NE GIN St. H n, m7, NE,
	$ a_n-a_m <\varepsilon$.
	≥> a _n - ε < a _n < a _n + ε + n » N _ε .
	=> (an) is a bdd. seq.
	Moreover, a - E < an + NT/NE
	r.
	$\Rightarrow a_{N_{\varepsilon}} - \varepsilon \leq \lim \inf a_{n}$
	and $a_n < a_{N_{\epsilon}} + \epsilon + N_{\pi}N_{\epsilon}$
	$\Rightarrow \lim_{n \to \infty} a_n \leq a_{N_{\varepsilon}} + \varepsilon$
	Since land is bold., liminfan & too and limisupan & too.
	· Also, lim sup an - lim inf an 70.
	(why 2) him sub an - liminf on < 22. Since this inequality is true for every E>0,
	(why?) himsupan - liminfan < 2 \(\int \). Since this inequality is true for every \(\int \)0, 80, him supan = himinfan. \(0 \le a \le \(\int \) + \(\int \)70 = \(a = 0 \)
	Therefore, lim an = lim sup an (= lim inf an).
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