$(M,d)$ metric space

$C(M) := \{ f : M \to \mathbb{R} \text{ cts.} \}$ carry information about $(M,d)$.

Notation hereon: $M := C(X)$ where $(X,d)$ is a metric space.

Def$^n$: $X$: set Consider $\{f_n\}$ where $f_n : X \to (\mathbb{R}, |\cdot|)$.
Suppose for each $x \in X$, $(f_n(x))$ cvgs. in $\mathbb{R}$, then we say $(f_n)$ cvgs. ptwise.
The function $f : X \to (\mathbb{R}, |\cdot|)$ defined as $f(x) := \lim\limits_{n\to\infty} f_n(x)$ is called the limit function. Notation: $f_n \to f$ ptwise.

Def$^n$: $X$: set Consider $\{f_n\}$ where $f_n : X \to (\mathbb{R}, |\cdot|)$.

$(f_n)$ cvgs. uniformly to $f$ on $X$ if $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq N_\varepsilon$,
$\forall\, x \in X$, $|f_n(x) - f(x)| < \varepsilon$.

Notation: $f_n \to f$ uniformly.

→ $f_n \to f$ uniformly $\Rightarrow$ $f_n \to f$ ptwise.
(HW) $\Leftarrow$ NO!

- Ptwise cvgs. may not preserve continuity:

Example: $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) := x^n$, $n \geq 1$.

Let $f(x) = \begin{cases} 1, & x = 1 \\ 0, & 0 \leq x < 1 \end{cases}$

$f_n \to f$ ptwise. Here $f_n$'s are continuous but $f$ is not cts.

- Infinite sums of cts. functions may not be cts.

Example: $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) := \dfrac{x^2}{(1+x^2)^n}$, $n \geq 1$.

Consider the formal sum $\sum_1^\infty f_n$.

For each $x \in \mathbb{R}$, $\sum_{n=1}^\infty f_n(x) < \infty$ (why?)

Define $f(x) := \sum_{n=1}^\infty f_n(x)$. Note: $f(x) = \begin{cases} 0, & x = 0 \\ 1+x^2, & x \neq 0. \end{cases}$ Hence, $f$ is not cts.

- Suppose $f_n(x) \to f(x)$ ptwise and $f_n, f$ differentiable functions, then $f_n'(x) \not\to f'(x)$ not always.

Example: $f_n(x) := \dfrac{\sin(nx)}{\sqrt{n}}$, $n \geq 1$.

$f_n(x) \to 0$ ptwise.

But $f_n'(x) = \sqrt{n}\cos(nx) \not\to 0$ ptwise. (Why?)

- Suppose $f_n(x) \to f(x)$ ptwise where $f_n, f$ are Riemann Integrable on $[0,1]$.

Then $\int f_n \not\to \int f$ not always.

Consider $f_n(x) := n^2 x (1-x^2)^n$ where $f_n : [0,1] \to \mathbb{R}$.

Note: $f_n(x) \to 0$ ptwise. But $\int f_n \not\to \int f$ (why?)

We look for the convergence that carry over the extra structure and/or properties of the functions $f_n$ to the limit function.

## Uniform cvg. of $(f_n)$

(HW) (Cauchy Criterion for uniform cvg.)

The seq. $(f_n)$ defined on $X$ converges uniformly on $X$

iff

$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall\, x \in X,\ m, n \geq N_\varepsilon,\ |f_m(x) - f_n(x)| \leq \varepsilon.$

- Suppose $f_n \to f$ ptwise. Let $M_n \doteq \operatorname{lub}_{x \in X} \{ |f_n(x) - f(x)| \}$

  Then $f_n \to f$ uniformly iff $M_n \to 0$ as $n \to \infty$.

- For series of functions, one has the Weierstrass M test:

  For $f_n : X \to \mathbb{R}$, suppose $|f_n(x)| \leq M_n$ for $x \in X$ and $n \geq 1$.

  If $\sum_1^\infty M_n$ cvgs., then $\sum_1^m f_n$ cvgs. uniformly on $X$.

  (HW) ✱

Goodies !!!

① $f_n : (X, d) \to (Y, \rho)$ cts. functions.

If $f_n \to f$ uniformly on $X$, then $f$ is cts. on $X$.

(suffices to show $f$ cts. at each $a \in X$)

Pf. W.T.S. For $a \in X$, $\forall\, \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s.t. $d(x,a) < \delta_\varepsilon \Rightarrow \rho\big(f(x), f(a)\big) < \varepsilon$.

unif. cvg. • For $\varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq N_\varepsilon$, $\forall\, x \in X$, $\rho\big(f_n(x), f(x)\big) < \varepsilon$.

cts. of $f_n$. • Fix $n = N_\varepsilon$. Since $f_{N_\varepsilon}$ cts. at $a$, $\exists\, \delta_\varepsilon(a) > 0$ s.t. $d(x,a) < \delta_\varepsilon(a) \Rightarrow \rho\big(f_{N_\varepsilon}(x), f_{N_\varepsilon}(a)\big) < \varepsilon$.

Combine both. • $\rho\big(f(x), f(a)\big) \leq \rho\big(f(x), f_{N_\varepsilon}(x)\big) + \rho\big(f_{N_\varepsilon}(x), f_{N_\varepsilon}(a)\big) + \rho\big(f_{N_\varepsilon}(a), f(a)\big)$

$\forall\, x \in X$ s.t. $d(a,x) < \delta_\varepsilon(a)$,

$\rho\big(f(x), f(a)\big) < \varepsilon + \varepsilon + \varepsilon$ Do appropriate scaling on $\varepsilon > 0$ to get $\rho\big(f(x), f(a)\big) < \varepsilon$.

Remark: (i) $f_n, f \in C(X,Y)$ s.t. $f_n \to f$ ptwise, i.e., $f_n(x) \to f(x)$ for each $x \in X$.

Since $f$ is cts., for $x_m \to x$ in $(X,d)$, $f(x_m) \to f(x)$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n\left(\lim_{m\to\infty} x_m\right) = \lim_{n\to\infty}\lim_{m\to\infty} f_n(x_m)$$

$$\| $$

$$f\left(\lim_{m\to\infty} x_m\right) = \lim_{m\to\infty} f(x_m) = \lim_{m\to\infty}\lim_{n\to\infty} f_n(x_m).$$

If $f_n \to f$ uniformly, then $\boxed{\lim_{n\to\infty}\lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty}\lim_{n\to\infty} f_n(x_m).}$

Q. Converse of (1) does not hold: $f_n \to f$ ptwise but $f_n \not\to f$ uniformly.

Give example.

(Rudin) (Thm. 7.11) (ii) $(X,d)$ metric space and $f_n \to f$ uniformly on $X$.

Let $c$ be a limit pt. of $X$ and that $\lim_{x\to c} f_n(x)$ exists for each $n \geq 1$.

Let $c_n := \lim_{x\to c} f_n(x)$. Then $(c_n)$ cvgs. and

$$\boxed{\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x).}$$

→ A partial converse to (1):

Dini's thm: $X$: compact metric space

$f_n \in C(X)$ s.t. $f_n \uparrow f$ where $f \in C(X)$ and $f_n \to f$ ptwise.

Then, $f_n \to f$ uniformly.

Proof is uploaded at "Resources" on Moodle.