Assignment 5: Several variables calculus & differential geometry (MTH305A)

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(1) Let $\alpha : (a, b) \to \mathbb{R}^3$ be a parameterized curve that does not pass through the origin. If $\alpha(t_0)$ is the point on the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, then show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

(2) Let $\alpha : I \to \mathbb{R}^3$ be a parametrised curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t) \perp v$ for all $t \in I$ and that $\alpha(0) \perp v$. Prove that

$$\alpha(t) \perp v, \text{ for all } t \in I.$$

(3) Let $\alpha : I \to \mathbb{R}^3$ be a parameterized curve, with $\alpha'(t) \neq 0$, for all $t \in I$. Show that $\|\alpha(t)\|$ is a non-zero constant if and only if $\alpha(t)$ if orthogonal to $\alpha'(t)$ for all $t \in I$.

(4) Is $\alpha(t) = (t^2, t^4)$ a parameterisation of $y = x^2$?

(5) Find the parametric equation of the level curves:
(a) $y^2 - x^2 = 1$
(b) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

(6) Let $(a_{i,j})$ be a skew-symmetric matrix of order 3×3. Let $v_i, i = 1, 2, 3$, be smooth functions of a parameter s satisfying the system of differential equations

$$\frac{dv_i}{ds} = \sum_{j=1}^{3} a_{i,j} v_j, \text{ for } i = 1, 2, 3.$$

Furthermore, assume that for some initial value s_0, the vectors $v_1(s_0), v_2(s_0)$ and $v_3(s_0)$ are orthonormal. **Show that for all values of s, the vectors $v_1(s), v_2(s)$ and $v_3(s)$ are orthonormal.**

Solution.

- $v_1(s_0), v_2(s_0), v_3(s_0)$ are orthonormal implies

$$\langle v_i(s_0), v_j(s_0)\rangle = \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}.$$

- For $1 \le i, j \le 3$, consider $\alpha_{i,j}(s) = \langle v_i(s), v_j(s) \rangle$.

$$\frac{d\alpha_{i,j}}{ds} = \langle v_i'(s), v_j(s) \rangle + \langle v_i(s), v_j'(s) \rangle$$
$$= \left\langle \sum_{k=1}^{3} a_{i,k} v_k, v_j \right\rangle + \left\langle v_i, \sum_{k=1}^{3} a_{j,k} v_k \right\rangle$$
$$= \sum_{k=1}^{3} [a_{i,k} \langle v_k, v_j \rangle + a_{j,k} \langle v_i, v_k \rangle] = \sum_{k=1}^{3} [a_{i,k} \alpha_{k,j} + a_{j,k} \alpha_{i,k}]$$

- Consider the initial value problem (IVP):
$$\frac{d\alpha_{i,j}}{ds} = \sum_{k=1}^{3} [a_{i,k} \alpha_{k,j} + a_{j,k} \alpha_{i,k}], 1 \le i, j \le 3,$$
with initial condition $\alpha_{i,j}(s_0) = 0$ for $i \ne j$ and $\alpha_{i,j}(s_0) = 1$ for $i = j$.
- As $a_{i,j} = -a_{j,i}$ for all $1 \le i, j \le 3$, we have the function
$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \ne j. \end{cases}$$
is a solution to the IVP.
- By the construction of the IVP, the function $\alpha_{i,j}(s) = \langle v_i(s), v_j(s) \rangle$ is also a solution to the IVP.
- By the uniqueness theorem of IVP, we conclude that
$$\alpha_{i,j}(s) = \delta_{i,j}.$$

(7) Find cartesian equation of
$$\gamma(t) = (e^t, t^2).$$

(8) Calculate the tangent vectors of
$$\gamma(t) = (\cos^2 t, \sin^2 t).$$

(9) Calculate arc-length of the catenary
$$\gamma(t) = (t, \cosh t)$$
starting at a point $(0, 1)$.

(10) Show that the curve

$$\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$$

is unit-speed curve.

(11) Find unit-speed reparameterization of

$$\gamma(t) = (e^t \cos t, e^t \sin t).$$

(12) Determine if the curve $\gamma(t) = (t, \cosh t)$ is regular?

(13) Let γ be a curve in $\mathbb{R}^n$. Let $\tilde{\gamma}$ be a reparameterization of γ with reparameterization map ϕ (so that $\tilde{\gamma}(\tilde{t}) = \gamma \circ \phi(\tilde{t})$). Let $\tilde{t}_0$ be a fixed value of $\tilde{t}$ and $t_0 = \phi(\tilde{t}_0)$. Let S and $\tilde{S}$ be the arc lengths of γ and $\tilde{\gamma}$ starting at the point $\gamma(t_0) = \tilde{\gamma}(\tilde{t}_0)$.
Prove that $\tilde{S} = S$, if $\frac{d\phi}{d\tilde{t}} > 0$ for all $\tilde{t}$ and $\tilde{S} = -S$, if $\frac{d\phi}{d\tilde{t}} < 0$ for all $\tilde{t}$