Name:

Roll No:

MSO201A: Probability & Statistics

Optional Quiz: Full Marks 20

[1] Let $\{X_n\}$ be a sequence of i.i.d. random variables with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $T_n = \sum_{i=1}^{n} X_i$, $\bar{X}_n = \frac{T_n}{n}$ and $S_n = \sum_{i=1}^{n} X_i^2$.

(a) Find α such that $\frac{T_n}{\sqrt{n\,S_n}} \xrightarrow{p} \alpha$, as $n \to \infty$.

(b) Find X such that $\sqrt{n}(\bar{X}_n - 1) \xrightarrow{\mathcal{L}} X$, as $n \to \infty$.

(c) Find Y such that $\sqrt{n}(\bar{X}_n^2 - 1) \xrightarrow{\mathcal{L}} Y$, as $n \to \infty$.

10 (4+3+3) marks

[2] Let $X_1, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{3}{\theta}\, x^2\, e^{-\frac{x^3}{\theta}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$\theta > 0$ is unknown.

(a) Find a minimal sufficient statistic for θ.

(b) Find a sufficient statistic for θ, which is **NOT** minimal sufficient.

(c) Find an unbiased estimator of θ^2, which is a function of the minimal sufficient statistic obtained in **(a)**.

10 (3+3+4) marks

$X \sim U(0,2)$; $E(X) = 1$; $V(X) = \frac{1}{3}$; $E X^2 = \frac{4}{3}$

(a) $$\frac{T_n}{\sqrt{n S_n}} = \frac{T_n/n}{\sqrt{n S_n}/n} = \frac{T_n/n}{\sqrt{\frac{S_n}{n}}}.$$

By WLLN; $$\frac{T_n}{n} = \frac{1}{n}\sum_1^n X_i \xrightarrow{p} E X_1 = 1 \quad -(1)$$

$$\frac{S_n}{n} = \frac{1}{n}\sum X_i^2 \xrightarrow{p} E X_1^2 = \frac{4}{3} \Rightarrow \sqrt{\frac{S_n}{n}} \xrightarrow{p} \frac{2}{\sqrt{3}} \quad -\left(1\tfrac{1}{2}\right)$$

$$\Rightarrow \frac{T_n}{\sqrt{n S_n}} \xrightarrow{p} \frac{\sqrt{3}}{2} \quad -\left(1\tfrac{1}{2}\right)$$

(b) $X_1, \ldots, X_n$ i.i.d. $\therefore E X_i = 1$; $V X_i = \frac{1}{3}$

By CLT $$\sqrt{n}(\bar{X}_n - 1) \xrightarrow{L} N\left(0, \frac{1}{3}\right) \quad -(3)$$

Deduct (1) mark if $V(X_i)$ is wrong and answer is otherwise conceptually correct

(c) Take $g(x) = x^2$ $g'(x) = 2x$; $g'(1) = 2$ — (1)

By Δ-method and using (b)

$$\sqrt{n}(\bar{X}_n^2 - 1) \xrightarrow{L} N\left(0, 2^2 \cdot \frac{1}{3}\right) \quad -(2)$$

i.e. $$\sqrt{n}(\bar{X}_n^2 - 1) \xrightarrow{L} N\left(0, \frac{4}{3}\right)$$

Alt sol^n: $$\sqrt{n}(\bar{X}_n^2 - 1) = \underbrace{\sqrt{n}(\bar{X}_n - 1)}_{\xrightarrow{L} N(0, \frac{1}{3})} \underbrace{(\bar{X}_n + 1)}_{\xrightarrow{p} 2} \xrightarrow{L} N\left(0, \frac{4}{3}\right)$$

(2)

(a) $\forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \frac{\left(\frac{3}{\theta}\right)^n (\Pi x_i)^2 \, e^{-\frac{1}{\theta}\Sigma x_i^3}}{\left(\frac{3}{\theta}\right)^n (\Pi y_i)^2 \, e^{-\frac{1}{\theta}\Sigma y_i^3}}$$

$$= \left(\frac{\Pi x_i}{\Pi y_i}\right)^2 e^{-\frac{1}{\theta}\left(\Sigma x_i^3 - \Sigma y_i^3\right)}$$

indep of θ iff $\Sigma x_i^3 = \Sigma y_i^3$

$\Rightarrow T(\underset{\sim}{X}) = \Sigma X_i^3$ is minimal suff stat —— (3)

(b) jt. p.d.f. is

$$f_\theta(\underset{\sim}{x}) = (\Pi x_i)^2 \left(\left(\frac{3}{\theta}\right)^n e^{-\frac{1}{\theta}\Sigma x_i^3}\right)$$

By NFFT, $(X_1, \dots, X_n)$, $(X_1^3, X_2^3, \dots, X_n^3)$, $(X_1^3, \sum_2^n X_i^3)$, there can be other valid non-minimal suff stat all are suff but none are minimal suff (3)

(c) Let $T = \sum_1^n X_i^3$

$$E\,T^2 = E\left(\sum_1^n X_i^3\right)^2$$

$$= E\left(\underline{\sum_1^n X_i^6} + 2\underline{\sum_{i<j} X_i^3 X_j^3}\right)$$

$$= (n(2\theta^2) + n(n-1)\,\theta\cdot\theta)$$

$$= \theta^2(2n + n^2 - n)$$

$$= n(n+1)\theta^2$$

Give partial marks if one is correct $\left(1\frac{1}{2}\right)$

$\Rightarrow \frac{T^2}{n(n+1)}$ is u.e. based on m.s.s. —— (4)