# Point Estimation

$X_1, \ldots X_n$ random sample from $P_\theta$ with p.d.f. (or p.m.f.) $f_\theta$

$\theta \in \Theta$

$g(\theta)$ : parametric $f^n$ of ~~the~~ interest, called estimand

$\delta(X_1, \ldots, X_n)$ : Estimator (a $f^n$ of r.v.s $X_1, \ldots, X_n$)

**Def$^n$: Unbiased estimator**

$\delta(\underset{\sim}{X})$ is an unbiased estimator for $g(\theta)$ if

$$E\,\delta(\underset{\sim}{X}) = g(\theta) \quad \forall\ \theta \in \Theta$$

**Examples**

(i) $N(\theta, 1)$ pop$^n$ $\quad \theta \in \mathbb{R}$ ; $g(\theta) = \theta$

$X_1, \ldots, X_n$ random sample

$$\left.\begin{aligned} \delta_1(\underset{\sim}{X}) &= \bar{X} \\ \delta_2(\underset{\sim}{X}) &= X_1 \\ \delta_3(\underset{\sim}{X}) &= \frac{X_1 + X_2}{2} \\ \delta_4(\underset{\sim}{X}) &= \sum_{i=1}^{n} a_i X_i \ni \sum a_i = 1 \end{aligned}\right\}$$

$E\,\delta_i(\underset{\sim}{X}) = \theta$

$\delta_1, \delta_2, \delta_3, \delta_4$ are all unbiased estimator of $\theta$

(ii) $U(0, \theta)$ pop$^n$ $\quad \theta > 0 \quad g(\theta) = \theta$

$X_1, \ldots X_n$ random sample

$$\left.\begin{aligned} \delta_1(\underset{\sim}{X}) &= 2X_1 \\ \delta_2(\underset{\sim}{X}) &= X_1 + X_2 \\ \delta_3(\underset{\sim}{X}) &= 2\bar{X} \\ \delta_4(\underset{\sim}{X}) &= \frac{n+1}{n} X_{(n)} \end{aligned}\right\}$$

all are unbiased estimator for $\theta$

(iii) $B(1, \theta)$ pop$^n$ $\quad 0 < \theta < 1$.

$g(\theta) = \theta \qquad \delta_1(\underset{\sim}{X}) = X_i \quad i = 1(1)n$

$\delta_2(\underset{\sim}{X}) = \frac{X_1 + X_2}{2}$ ; $\delta_3(\underset{\sim}{X}) = \sum_1^n X_i / n$

all are u.e. for $\theta$

$g(\theta) = \theta(1-\theta)$

$$\delta(\underset{\sim}{X}) = \begin{cases} 1, & X_1 = 1, X_2 = 0 \\ 0 & \text{o/w} \end{cases}$$

$\delta(\underset{\sim}{X})$ is u.e. of $\theta(1-\theta)$

# Sufficient statistic

Data dimension reduction without loss of information

$X_1, \ldots, X_n$ random sample from a dist$^n$ with p.d.f. or p.m.f. $f_\theta(x)$

$T(\underset{\sim}{X})$ : statistic

$T(\underset{\sim}{X})$ is sufficient for $\theta$ if $T(\underline{X})$ contains all information about $\theta$, that is contained in the entire sample $(X_1, \ldots, X_n)$; i.e. given $T(\underline{X})$, $(X_1, \ldots X_n)$ does not contain any information about $\theta$.

Def$^n$: A statistic $T(\underline{X})$ is said to be sufficient for $\theta$ if the conditional dist$^n$ of $(X_1, \ldots, X_n)$ given $T=t$ is independent of $\theta$.

Example:

(i) $X_1, \ldots, X_n$ i.i.d. random sample from $B(1, \theta)$

Claim: $T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is sufficient for $\theta$

$$\sum_{i=1}^{n} X_i \sim B(n, \theta)$$

$$P(X_1=x_1, \ldots, X_n = x_n \mid T=t)$$

$$= \frac{P(X_1=x_1, \ldots, X_n=x_n ; T=t)}{P(T=t)} = 0 \text{ if } \sum_{i=1}^{n} x_i \neq t$$

indep of $\theta$

otherwise if $\sum_{i=1}^{n} x_i = t$

$$= \frac{P(X_1=x_1, \ldots, X_n=x_n)}{P(T=t)}$$

$$= \frac{P(X_1=x_1) \cdots P(X_n=x_n)}{P(T=t)} \quad \text{(independence)}$$

$$= \frac{\left(\theta^{x_1}(1-\theta)^{1-x_1}\right) \cdots \left(\theta^{x_n}(1-\theta)^{1-x_n}\right)}{\binom{n}{t} \theta^t (1-\theta)^{n-t}}$$

$$= \frac{1}{\binom{n}{t}} \leftarrow \text{indep of } \theta$$

$\Rightarrow T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is sufficient for $\theta$

Example

(ii) $X_1, \ldots, X_n$ i.i.d. random sample from $P(\theta)$; $\theta > 0$

Claim: $T(\underline{X}) = \sum_{i=1}^{n} X_i$ is sufficient for $\theta$

$$P(X_1=x_1, \ldots, X_n = x_n \mid T=t) \qquad T \sim P(n\theta)$$

$$= \frac{P(X_1=x_1, \ldots, X_n=x_n ; T=t)}{P(T=t)} = 0 \text{ if } \sum_{i=1}^{n} x_i \neq t$$

$\uparrow$ indep of $\theta$

otherwise if $\sum_{i=1}^{n} x_i = t$

$$= \frac{P(X_1=x_1, \ldots, X_n=x_n)}{P(T=t)}$$

$$= \frac{P(X_1=x_1) \cdots P(X_n=x_n)}{P(T=t)} \quad \text{(independence)}$$

$$= \frac{\cancel{P(X_1=x_1)} - \cancel{\ }}{\ }$$

$$= \frac{\left(e^{-\theta}\,\theta^{x_1}/x_1!\right)\cdots\left(e^{-\theta}\,\theta^{x_n}/x_n!\right)}{e^{-n\theta}\,(n\theta)^t/t!}$$

$$= \frac{t!}{\prod_{i=1}^{n} x_i!}\cdot\frac{1}{n^t} \rightarrow \text{indep of } \theta$$

$\Rightarrow T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is sufficient for $\theta$

<u>Remark</u>: The above def^n of sufficient statistic is not a constructive definition.

<u>Neyman-Fisher Factorization Theorem</u>

$X_1, \ldots, X_n$ be a random sample with p.d.f. or p.m.f. $f_\theta(x)$ $\theta \in \Theta$. A statistic $T(\underset{\sim}{X})$ is sufficient for $\theta$ iff $f_\theta(\underset{\sim}{x})$ can be factored as

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\; g_\theta(t(\underset{\sim}{x}))$$

where, $h(\underset{\sim}{x}) > 0$ is a f^n of $(x_1, \ldots, x_n)$ only and indep of $\theta$

and $g_\theta(t(\underset{\sim}{x}))$: f^n of $\theta$ and depends on $(x_1, \ldots, x_n)$ only through $t(x_1, \ldots, x_n)$.

<u>Remark</u>: Every 1-1 f^n of a sufficient statistic $T(\underset{\sim}{X})$ is also ~~of~~ a sufficient statistic.

<u>Remark</u>: $T$ & $T^*$ be 2 statistic $\ni T = \psi(T^*)$

$T$ is suff $\Rightarrow T^*$ is also sufficient statistic

Examples

(1) $X_1, \ldots X_n$ r.s. from $N(\theta, 1)$ $\theta \in \mathbb{R}$

$$f_\theta(\underset{\sim}{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_1^n (x_i-\theta)^2} \qquad -\infty < x_1, \ldots, x_n < \infty$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\left(\Sigma x_i^2 + n\theta^2 - 2\theta\Sigma x_i\right)\right)$$

$$= \underbrace{\left(\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\Sigma x_i^2\right)\right)}_{h(\underset{\sim}{x})} \underbrace{\left(\exp\left(-\frac{n}{2}\theta^2 + \theta\sum_1^n x_i\right)\right)}_{g_\theta(\Sigma x_i)}$$

By NFFT, $T(\underset{\sim}{X}) = \sum_{i=1}^n X_i$ is sufficient for $\theta$

$\bar{X}$ is also sufficient for $\theta$

$(X_1, \ldots, X_n)$ is suff for $\theta$ (it's the trivial suff stat)

$(X_1, \sum_{i=2}^n X_i)$ is also suff for $\theta$

(2) $X_1, \ldots, X_n$ r.s. from $P(\theta)$

$$f_\theta(\underset{\sim}{x}) = \prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\,\theta^{\sum_1^n x_i}}{\prod_{i=1}^n x_i!}$$

i.e. $$f_\theta(\underset{\sim}{x}) = \left(\frac{1}{\prod_1^n x_i!}\right)\left(e^{-n\theta}\,\theta^{\sum_1^n x_i}\right)$$

$$= h(\underset{\sim}{x})\; g_\theta\left(\sum_1^n x_i\right)$$

By NFFT, $T(\underset{\sim}{X}) = \sum_1^n X_i$ is sufficient for $\theta$

(3) $X_1, \ldots, X_n$ random sample from $U(0,\theta)$, $\theta > 0$

st p.d.f.

$$f_\theta(\underline{x}) = \begin{cases} \frac{1}{\theta^n}, & 0 < x_1, \ldots, x_n < \theta \\ 0, & o/w \end{cases}$$

i.e. $$f_\theta(\underline{x}) = \begin{cases} \frac{1}{\theta^n}, & 0 < x_{(1)} < \cdots < x_{(n)} < \theta \\ 0, & o/w \end{cases}$$

i.e. $$f_\theta(\underline{x}) = \frac{1}{\theta^n} I(0, x_{(1)})\, I(x_{(n)}, \theta)$$

where $I(a,b) = \begin{cases} 1, & a < b \\ 0, & o/w \end{cases}$

$$f_\theta(\underline{x}) = \underbrace{\left(I(0, x_{(1)})\right)}_{h(\underline{x})} \left(g_\theta(x_{(n)})\right)$$

$$g_\theta(x_{(n)}) \downarrow \frac{1}{\theta^n} I(x_{(n)}, \theta)$$

By NFFT, $T(\underline{X}) = X_{(n)}$ is suff for $\theta$