$F(t) \geq F(t_0)$

OR.

$F(t) \leq F(t_0)$. $\forall\, t \in B(t_0)$

Date 07/09/22

Differential Geometry

Outline

1) Def^n. of Parametrized Curves
2) Examples
3) Tangent vectors and tangent lines, speed, etc.
4) characterization of E straight lines in $\mathbb{R}^n$.

Goal: To study differential geometry of curves in $\mathbb{R}^n$, $n \geq 2$.

$n=2$, plane curves

$n=3$, space curves.

Two aspects

1) classical differential geometry of curves (——):
   Study of local properties of curves.
   Eg:- Tangent, normal, torsion, curvature, etc.

2) Global differential geometry of curves.
   Study of entire curves.
   Eg:- Isoperimetric inequality.

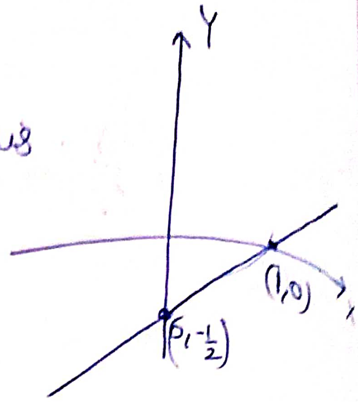
Tool/Method used: Differential (Integral) calculus.

<u>Study</u>: 1-dimensional subsets of $\mathbb{R}^n$ — objects.
(in some sense)

defined in terms of differentiable functions.

<u>Examples</u>. 1) $x-2y=1$.

$(1,0)$ and $(0,-\frac{1}{2})$ satisfy this

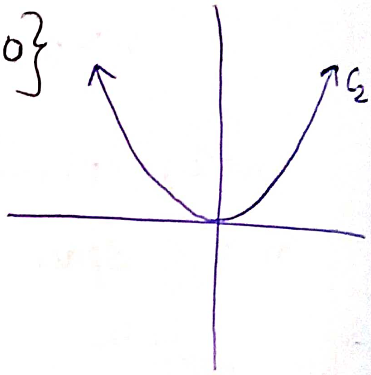


$f_1(x,y) = x-2y-1$

$C_1 = \{(x,y) \in \mathbb{R}^2 \mid f_1(x,y)=0\}$.

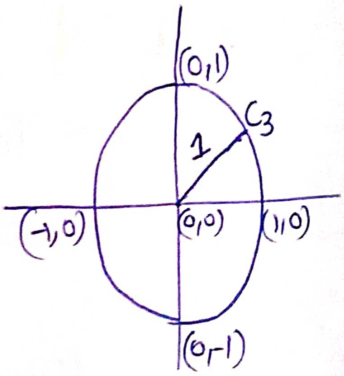
2) $f_2(x,y) = x^2-y$

$C_2 = \{(x,y) \in \mathbb{R}^2 \mid f_2(x,y)=0\}$



3) $f_3(x,y) = x^2+y^2-1$

$C_3 = \{(x,y) \in \mathbb{R}^2 \mid f_3(x,y)=0\}$



Sometimes, Curves will be intersection of two surfaces.

Level Curve. $C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = l\}$.

C is called level curve - the set of points (x,y) in $\mathbb{R}^2$. at which $f(x,y)$ reaches the level l.

Defn: (Parametrized Curves)

A ~~function~~ map $\alpha : I \to \mathbb{R}^n$, where $I = (a,b)$ (where $-\infty \le a < b \le +\infty$) is called a parametrized curve.
$\downarrow$
(a continuous function) = map.

$\alpha = (\alpha_1, \ldots, \alpha_n)$

$\forall\, t \in I, \quad \alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$

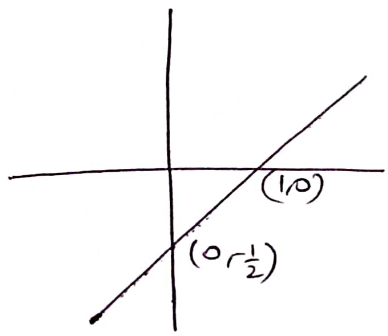
$\alpha_i : I \to \mathbb{R}$.

Defn:- α is smooth if $\alpha_i : I \to \mathbb{R}$, $i = 1, \ldots, n$ are smooth.

α_i is smooth (C^∞) if $\forall\, n \in \mathbb{N}$, $\dfrac{d^n \alpha_i}{dx^n}$ exist on I.

Examples. — 1) $x - 2y = 1$

$x = 1 + 2y$

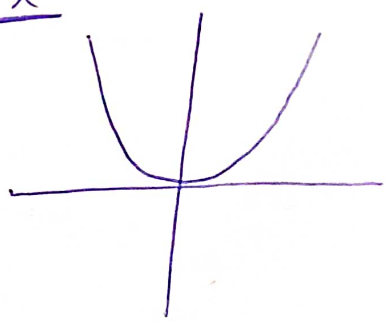


Parametrized curve.

$\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = (1+2t, t)$

$y = x^2$



2) $\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = (t, t^2)$

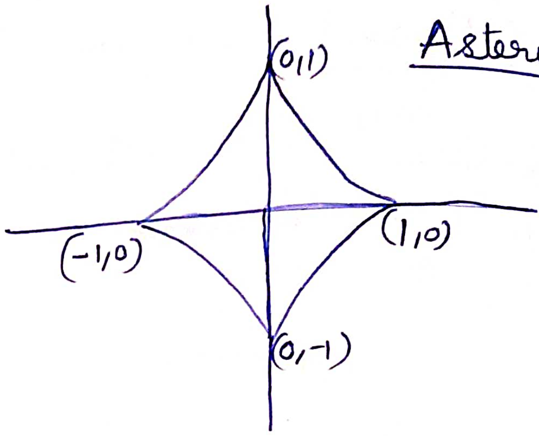
$\alpha^*(t) = (t^3, t^6)$

$\alpha^{**}(t) = (2t, 4t^2)$

• Parametrization is not unique, there exist infinitely many Parametrized for a given curve.

3) $\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = (\cos^3 t, \sin^3 t)$; $f(x,y) = x^{2/3} + y^{2/3} - 1$



Asteroid

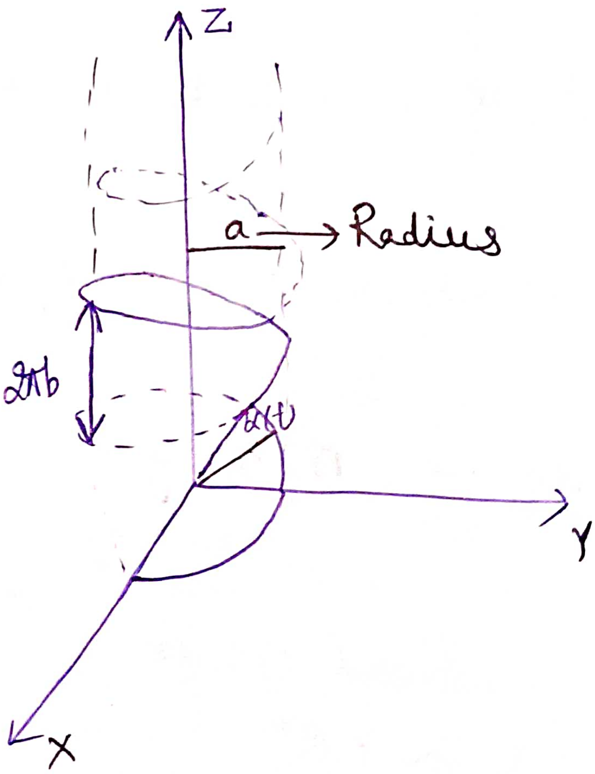
4) Circular Helix.

$\alpha : \mathbb{R} \to \mathbb{R}^3$

$\alpha(t) = (a \cos t, a \sin t, bt)$, $a, b \neq 0$.

5.) $\alpha(t) = (t, |t|)$ Pitch - $2\pi b$

Continuous Curve but not Smooth.



Differentiability of Curves.

Suppose $\alpha : I \longrightarrow \mathbb{R}^n$

$$t \longmapsto \alpha(t) = (\alpha_1(t), \dots, \alpha_n(t)) \text{ is smooth.}$$

$$\alpha'(t) = (\alpha_1'(t), \dots, \alpha_n'(t))$$

Defn – (1) Trace α is defined by

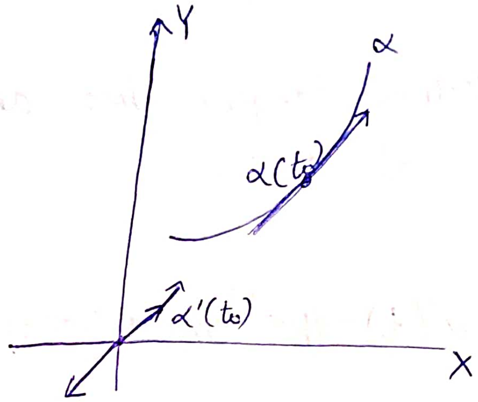
$$Im(\alpha) = Tr(\alpha) = \{\alpha(t) \mid t \in I\}.$$

2) Tangent Vector.

$\alpha'(t_0)$ is called tangent vector at time $t_0 \in I$.

↓

we have unique tangent.



3) Tangent line to α at time t_0. $\quad T : t \to \alpha(t_0) + t\alpha'(t_0) \quad t \in \mathbb{R}$.

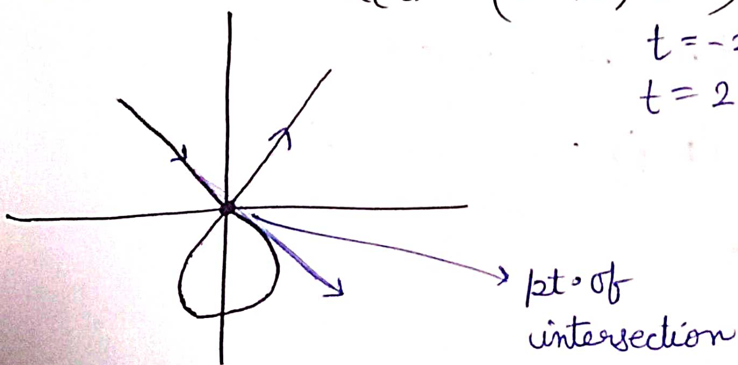
4) $\|\alpha'(t_0)\|$ – is called speed of α at time t_0.

Example – (1) $\alpha : \mathbb{R} \to \mathbb{R}^2$.

$$\alpha(t) = (t^3 - 4t, \; t^2 - 4)$$

$t = -2, \quad \alpha(-2) = (0,0)$

$t = 2, \quad \alpha(2) = (0,0)$



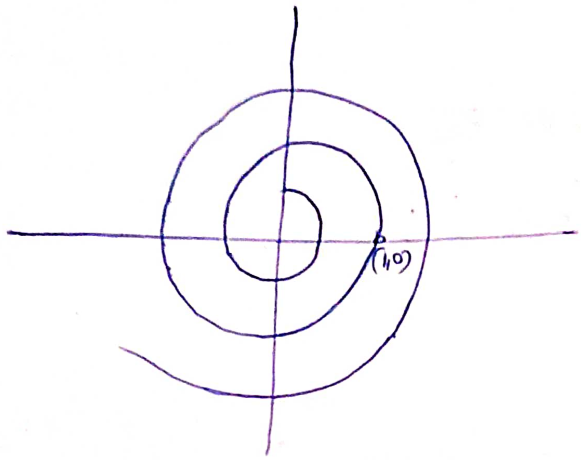
$\alpha'(t) = (3t^2 - 4, \; 2t)$ — tangent vector at time t.

$\alpha'(-2) = (8, -4) \qquad \alpha'(2) = (8, 4)$

$T_{(t=-2)}(t) = t(8,-4)$.

Example-2: $\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = (e^{bt}\cos t, e^{bt}\sin t)$ — logarithmic spiral.



$\alpha(0) = (1,0)$

$b \in \mathbb{R}$.

$\downarrow$

whether it goes anticlockwise or clockwise, it is totally determined by the sign of b.

Ex. Compute Tangent vector, tangent line and speed of log. spiral at $t=0$.

Tangent vector — $\alpha'(t) = (b e^{bt}\cos t + (-\sin t)\cdot e^{bt},\ b\cdot e^{bt}\sin t + e^{bt}\cos t)$

$$\alpha'(0) = (b, 1)$$

Speed. $\|\alpha'(0)\| = \sqrt{1+b^2}$

Tangent line of α at time 0.

$\alpha(t_0) + t\alpha'(t_0)$, $t \in \mathbb{R}$.

$(1,0) + t(b,1)$

$\Rightarrow (1+tb, t)$

Proposition 1:- If the tangent vector of a parametrized curve is constant, then the Image / trace of the curve is a part of straight lines.

Proof: Suppose $\alpha'(t) = a \quad \forall\ t \in I$, and $a \in \mathbb{R}^n$ - Constant vector.

$= (a_1, \ldots, a_n)$

$$\Rightarrow \alpha(t) = \cancel{\int \alpha'(u)\,du} = \int \alpha'(t)\cdot dt = at + b.$$

$\Rightarrow b$ is another constant vector.

$\alpha(t) = at + b$ if $a = 0$, $\alpha(t) = b$

(Part of straight line)

if $a \neq 0$

still straight line direction of a through b.

Curve:-

1. $x^{2/3} + y^{2/3} = 1 \rightarrow$ Asteroid.
2. $(a\cos t, a\sin t, bt) \longrightarrow$ Helix.
3. $(e^{bt}\cos t, e^{bt}\sin t) \rightarrow$ logarithmic spiral.

Date
8/Sept/22

Theorem : Let $f : S \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}$, $g = (g_1, \dots, g_m) : S \longrightarrow \mathbb{R}^m$ be C^1, $m < n$.

$x_0 \in X_0 = \{x \in S \mid g(x) = 0\}$. Suppose $\exists$ $B(x_0) \subseteq \mathbb{R}^n$, such that $f(x_0) \leq f(x)$ $\forall$ $x \in B(x_0) \cap X_0$.

OR.

such that $f(x_0) \geq f(x)$ $\forall$ $x \in B(x_0) \cap X_0$. Assume $\det \left(D_j g_i(x_0) \right)_{m \times m} \neq 0$.

Then, $\exists$ $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ satisfying the following.

$$D_r f(x_0) + \sum_{i=1}^{m} \lambda_i D_r g_i(x_0) = 0 \quad \forall \; r = 1, \dots, n. \quad (1)$$

Proof Consider, system of linear equations with unknowns $\lambda_1, \dots, \lambda_m$.

$$\sum_{i=1}^{m} \lambda_i D_r g_i(x_0) = D_r f(x_0) \quad , r = 1, \dots, m. \quad (2)$$

— admits a unique solution

$(\lambda_1, \dots, \lambda_m)$ - solution of (2).

Goal : To show $(\lambda_1, \dots, \lambda_m)$ satisfy the last $(n-m)$ equations in system -(1)

Tool : Implicit function theorem,

$g : S \subseteq_{\text{open}} \mathbb{R}^m \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}$.

$x = (\underbrace{x_1, \dots}_{x'}, \underbrace{x_n}_{t})$ $\qquad$ $x = (x', t)$.

$x' = (x_1, \dots, x_m)$

$t = (x_{m+1}, \dots, x_n)$.

$$\left.\begin{array}{l} g(x_0) = 0 \\ g - C^1. \\ \det(D_j g_i(x_0)) \neq 0 \end{array}\right\} \text{The condition of Implicit function theorem are satisfied.}$$

Implicit function theorem $\Rightarrow \exists\, h : T_0 \to \mathbb{R}^m$ C^1

$h(t_0) = x_0'$ and $g(h(t), t) = 0 \;\forall\, t \in T_0$.

$g_1(x_1, \ldots, x_m, x_{m+1}, \ldots x_n) = 0$

figure

$\vdots$

$g_m(x_1, \ldots x_m, x_{m+1}, \ldots x_n) = 0$

$$x_i = h_i(x_{m+1}, \ldots, x_n) = h_i(t),$$

$$t = (x_{m+1}, \ldots x_n) \in T_0.$$

$$G_p(t) = g_p\left(h_1(x_{m+1}, \ldots, x_n), \ldots, h_m(x_{m+1}, \ldots, x_n), x_{m+1}, \ldots, x_n\right) = g_p(h(t), t) \quad t \in T_0.$$

$$= g_0 \circ H(t)$$

$t \in T_0$

$H(t) = (h(t), t)$

$g_p(h(t), t) = 0 \;\forall\, t \in T_0$

$1 \le p \le m$

$$F(t) = f(h_1(t), \ldots, h_m(t), t) = f \circ H(t).$$

$G_p(t) \equiv 0$, so its partial derivative is 0.

$H(t) = (h(t), t) \in \mathbb{R}^n$

- cts.

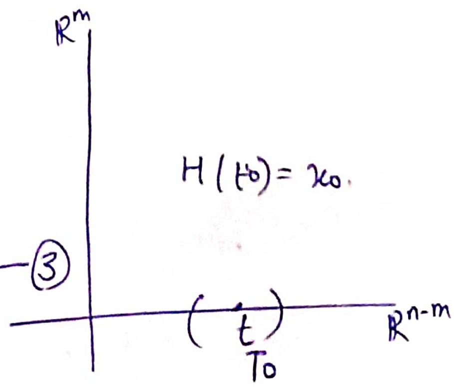
$H_k(\quad)$

$$H_k(t) = \begin{cases} h_k(t) & k \le m \\ x_k & k \ge m+1 \end{cases}$$

- $D_r G_p(t_0) = 0 \;\forall\; 1 \le r \le n-m$

Apply chain rule, we have

$$\Rightarrow \sum_{k=1}^{n} D_r g_p(x_0) \cdot D_r H_k(t_0) = 0.$$



$$\Rightarrow \sum_{k=1}^{m} D_r g_p(x_0)\, D_r h_k(t_0) + D_{m+r}\, g_p(x_0) = 0 \quad \text{— (3)}$$

By the given hyphothesis in theorem, apply on $F(t)$.
and using the continuity of $H(t)$.

$H(t) \in B(x_0)$ $\quad$ $H(t_0) = x_0$

$\forall t \in B(t_0)$ $\quad$ $\exists B(t_0)$ $\quad$ $B(x_0)$.

$F(t)$ has local max./min. at t_0.

$\Rightarrow D_r F(t_0) = 0$

$$= \sum_{k=1}^{m} D_k f(x_0) D_r H_k(t_0) = 0$$

$$= \sum_{k=1}^{m} D_k f(x_0) D_r h_k(t_0) + D_{m+r} f(x_0) = 0. \quad \text{—— } (4)$$

$$\sum_{p=1}^{m} (3) \times \lambda_p + (4) = 0$$

$$\sum_{p=1}^{m} \left(\sum_{k=1}^{m} \lambda_p D_k g_p(x_0) D_r h_k(t_0) + \lambda_p D_{m+r} g_p(x_0) \right).$$
$$+ \sum_{k=1}^{m} D_k f(x_0) D_r h_k(t_0) + D_{m+r} f(x_0) = 0.$$

$$\Rightarrow \sum_{k=1}^{m} \left(\sum_{p=1}^{m} \left[\lambda_p D_k g_p(x_0) D_r h_k(t_0) + D_k f(x_0) \right] D_r h_k(t_0) \right)$$
$$+ \sum_{p=1}^{m} \lambda_p D_{m+r} g_p(x_0) + D_{m+r} f(x_0) = 0.$$

$$\Rightarrow \sum_{p=1}^{m} \lambda_p D_{m+r} g_p(x_0) + \cdots D_{m+r} f(x_0) = 0$$
$$\forall r = 1, \dots, n-m.$$

Correction

Assignment - 4. Q4. $f(x,t) = e^{-\alpha^2 k^2 t} \sin(kx)$.

Date
9/Sept/22.

Goal: To study differential geometry of curves.

Essential: Existence of tangent line at each t.

Today, we study 1) regular and singular curves.

2) Arc length function

3) Re Parametrization

Defn: Singular point and Singular curves

$\alpha : (a,b) \to \mathbb{R}^n$ – smooth

$t_0 \in (a,b)$ is called a singular point if

$\alpha'(t_0) = 0$.

Such a curve is called a singular curve.

Defn: Regular Point and regular curves.

$t_0 \in (a,b)$ is regular if $\alpha'(t_0) \neq 0$.

and α is called a regular curve if

$\alpha'(t) \neq 0 \ \forall \ t \in (a,b)$.

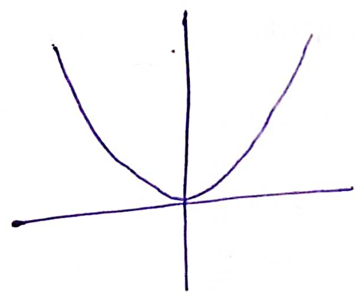
Eg-(1). $\alpha : \mathbb{R} \to \mathbb{R}^2$

(a) $\alpha(t) = (t, t^2)$

$\alpha'(t) = (1, 2t)$

$\|\alpha'(t)\| = \sqrt{1+4t^2} \geq 1$

$\neq 0 \ \forall \ t \in \mathbb{R}$.



$\Rightarrow \alpha'(t) \neq 0$ because norm is zero iff vector is 0.

$\Rightarrow \alpha(t)$ is regular curve.

b) Other parametrization of (1).

$$\alpha^*(t) = (t^3, t^6) \text{ — Singular}$$

$$(\alpha^*)'(0) = 0 \text{ — BAD parametrization}$$

Eg-(2) $\alpha(t) = (e^t \cos t, e^t \sin t)$

$$\alpha'(t) = (e^t(\cos t - \sin t),\ e^t(\sin t + \cos t))$$

$$\|\alpha'(t)\| = \sqrt{e^{2t}\cdot 2} = \sqrt{2}e^t \neq 0 \ \forall\ t \in \mathbb{R}.$$

$\Rightarrow \alpha$ is regular.

Eg-3. $\alpha(t) = (\cos t, \sin t) \longrightarrow \|\alpha(t)\| = 1$

$\alpha'(t) = (-\sin t, \cos t). \qquad \alpha(t) \perp^r \alpha'(t).$

$\neq 0 \ \forall\ t \in \mathbb{R}.$

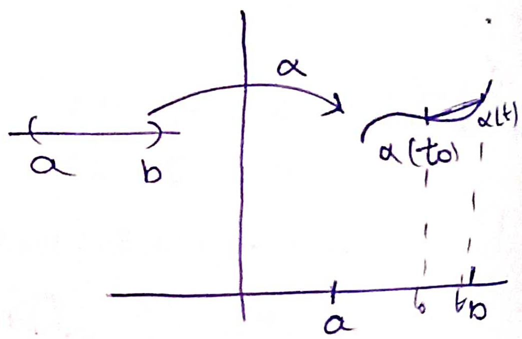
if $\alpha(t) = (k\cos t, k\sin t)$

$\|\alpha(t)\| = k$

Still, $\alpha(t) \perp^r \alpha'(t)$

Exercise. Let $\|\alpha(t)\|$ be constant. Show that $\alpha(t) \perp \alpha'(t)$. $\forall\ t \in I$.

Arc Length Functions



Defn Arc Length functions:

$S: (a,b) \to \mathbb{R}$

$$S(t) = \int_{t_0}^{t} \|\alpha'(u)\|\, du$$

called arc length function of α based at its skating.

1) s - Smooth

2) $s'(t) = \|\alpha'(t)\|$ - Speed of α at t.

3) If α is regular, $s'(t) > 0 \Rightarrow s$ is monotonically increasing

4) for $t < t_0$, $S(t) \begin{cases} \leq 0 & \text{if } t \leq t_0 \\ \geq 0 & \text{if } t \geq t_0 \end{cases}$

5) $S : (a,b) \longrightarrow (c,d)$

Ex. $= S(a,b) = \{ S(t) \mid t \in (a,b) \}$. is a diffeomorphism if α is regular.

Ex. What if, α is not regular.

6) If we have $\tilde{t_0}$ a differentiable base point

$\tilde{s}$ differs by ~~∞~~ $\int_{t_0}^{\tilde{t_0}} \|\alpha'(u)\| \cdot du$.

$$S(t) = \int_{t_0}^{t} \|\alpha'(u)\| \cdot du = \int_{t_0}^{\tilde{t_0}} \|\alpha'(u)\| du + \underbrace{\int_{\tilde{t_0}}^{t} \|\alpha'(u)\| du}_{\tilde{S}(t)}.$$

Example. $\alpha(t) = (e^{Kt} \cos t, e^{Kt} \sin t)$

$\|\alpha'(t)\| = \sqrt{K^2 + 1} \; e^{Kt}$.

$t_0 = 0$.

$$S(t) = \int_0^t \sqrt{K^2+1} \, e^{Ku} \cdot du$$

$$= \frac{\sqrt{K^2+1}}{K} (e^{Kt} - 1).$$

Defn:- $\alpha : (a,b) \to \mathbb{R}^n$ is called a unit speed curve if

$$\|\alpha'(t)\| = 1.$$

3) Examples.

1)

$$\alpha : (0, \|q-p\|) \to \mathbb{R}^n$$
$$t \to p + \frac{t(q-p)}{\|q-p\|}$$

$$\alpha : (0,1) \to \mathbb{R}^n$$
$$t \to (1-t)p + tq$$
$$= p + t(q-p)$$
$$\|\alpha'(t)\| = \|q-p\|$$
constant.

2) $\alpha(t) = (\cos t, \sin t)$.

3) $\alpha(t) = (\cos 2t, \sin 2t)$

Defn. :- (+)ve Reparametrization

$$\alpha : (a,b) \longrightarrow \mathbb{R}^n.$$
$$\beta : (c,d) \to \mathbb{R}^n$$

$$\begin{tikzcd} (a,b) \arrow[r, "\alpha"] & \mathbb{R}^n \\ (c,d) \arrow[u, "h"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

β is a (+)ve representation of α.

if $\exists\, h : (c,d) \to (a,b)$ diffeomorphism

{instead of C^1, we consider smooth}.

Such that

1) $\beta = \alpha \circ h$

2) $h'(t) > 0 \ \forall\ t \in (c,d)$

(–)ve reparametrization defined similarly.

$\downarrow$

$$h'(t) < 0 \ \forall\ t \in (c,d)$$

$\Rightarrow$

$h'(t) \neq 0 \Rightarrow$ reparametrization

Reparametrization

Defn.

1) β is called a reparametrization of α if it is either a (+)ve reparametrization or (-)ve reparametrization of α.

2) if β is reparametrization of α and $\|\beta'(t)\| = 1$. then, β is called a unit speed reparametrization of α.

Ques. Given $\alpha : (a,b) \to \mathbb{R}^n$ smooth curve.
Does, $\exists$ a unit speed reparametrization of α?

Necessary condition. α should be regular.

Chain rule of curve.

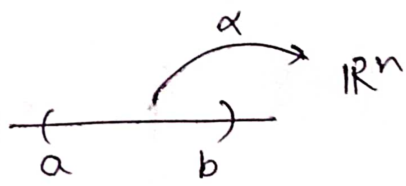
$$\beta = \alpha \circ h .$$

$$\beta'(t) = \alpha'(h(t)) \cdot \underbrace{h'(t)}_{\neq 0}$$

Length of a curve.

Suppose α is defined on slightly larger interval containing (a,b), then

$$\text{length}(\alpha) = \int_a^b \|\alpha'(t)\| \cdot dt$$

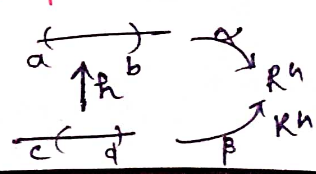


~~$\tilde{\alpha}$ is extention of α.~~

Theorem: If β is reparametrization of α, then length (α) = length β.

$\beta = \alpha \circ h$ $\qquad h'(t) > 0$

$\|\beta'(u)\| = \|\alpha' h(u)\| \; h'(u).$



$$\text{length}(\beta) = \int_c^d \|\beta'(u)\| \, du = \int_c^d \|\alpha'(h(u))\| \cdot h'(u) \cdot du.$$

$t = h(u)$.

$$= \int_a^b \|\alpha'(t)\| \, t \cdot dt = \text{length}(\alpha).$$

12/09/22

Outline

1) Reparametrization and arc length.
2) unit speed reparametrizations
3) Orientation
4) vector product.

Recall, if $\alpha : (a,b) \longrightarrow \mathbb{R}^n$ is defined on a slightly larger interval containing (a,b)

$$l(\alpha) = \int_a^b \|\alpha'(t)\| \cdot dt$$

Theorem : Arc length is invariant under reparametrization i.e., if β is a rep. of α, then length $(\alpha) =$ length(β)

$\beta = \alpha \circ h.$

Proof

$$\beta'(u) = \alpha'(h(u)) \cdot h'(u).$$

$$\|\beta'(u)\| = \|\alpha'(h(u)) \cdot h'(u)\|$$

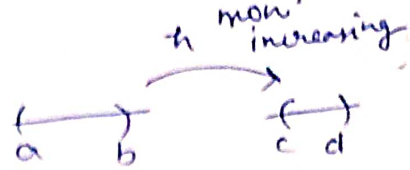
$$= \begin{cases} h'(u) \, \|\alpha'(h(u))\| & +ve \\ -h'(u) \, \|\alpha'(h(u))\| & -ve. \end{cases}$$

Case-1 β is a (+)ve reparametrization of α.

$$\text{length}(\alpha) = \int_a^b \|\alpha'(t)\|\, dt$$

$t = h(u)$

$dt = h'(u)\cdot du$



change accordingly from $(c,d) \to (a,b)$

$$= \int_c^d \|\alpha'(h(u))\|\, h'(u)\cdot du.$$

$\lim_{t \downarrow a} h(t) = c$

$\lim_{t \uparrow b} h(t) = d.$

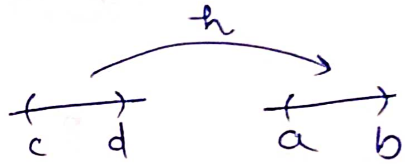
$$= \int_c^d \|\beta'(u)\|\, du = \text{length}(\beta).$$

Case-2. β is a (−)ve reparametrization of α.

$$\text{length}(\alpha) = \int_a^b \|\alpha'(t)\|\cdot dt.$$

$t = h(u)$

$dt = h'(u)\cdot du.$



$$= \int_c^d \|\beta'(u)\|\, du = \text{length}(\beta).$$

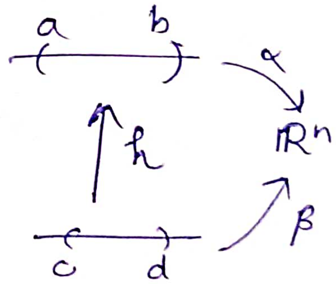
$\lim_{u \downarrow c} h(u) = b$

$\lim_{u \uparrow d} h(u) = a$

Unit-speed reparametrization

$\beta = \alpha \circ h$

1) β is a reparametrization of α.



2) $\|\beta'(u)\| = 1 \;\forall\; u \in (c,d)$.

$\Rightarrow$ β is regular $\Rightarrow$ α is regular.

Remark: If β is reparametrization of α.

Then, α is regular $\Longleftrightarrow$ β is regular.

Theorem: Let $\alpha : (a,b) \to \mathbb{R}^n$ be regular. Then, it admits a unit-speed reparametrization.

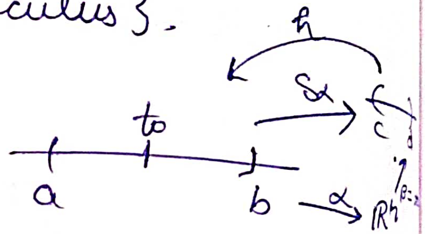
$$\exists \quad \begin{tikzcd} (c,d) \arrow[r, "h"] \arrow[rr, bend right, "\beta = \alpha \circ h"'] & (a,b) \arrow[r, "\alpha"] & \mathbb{R}^n \end{tikzcd}$$

$\beta = \alpha \circ h$ – unit-speed curve.

$S_\alpha : (a,b) \longrightarrow \mathbb{R}$

$$S_\alpha(t) = \int_{t_0}^{t} \|\alpha'(t)\| \cdot du \quad \text{— arc length function}$$

$S_\alpha'(t) = \|\alpha'(t)\|$ { By fundamental theorem of calculus }.



By regularity,

$S_\alpha'(t) = \|\alpha'(t)\| > 0.$

$\Rightarrow$ S_α is monotonically ↑ing
$\Rightarrow$ S_α is 1-1.

IFT ↓

$\Rightarrow$ S_α is locally diffeos at every $t \in (a,b)$

$(c,d) = S_\alpha(a,b) = \{ S_\alpha(t) \mid t \in (a,b) \}$.

$\Rightarrow$ $S_\alpha : (a,b) \longrightarrow (c,d)$ diffeomorphism

{ invertible & locally diffeo $\Rightarrow$ diffeomorphism

$h = S_\alpha^{-1}$.

$\beta = \alpha \circ h$ is a reparametrization of α.

$S_\alpha \circ h = Id$

Claim: $\|\beta'(u)\| = 1 \;\forall\; u \in (c,d)$.

$S_\alpha'(h(u)) \cdot h'(u)$

$\beta'(u) = \alpha'(h(u)) \cdot h'(u).$

$$= \alpha'(h(u)) \cdot \frac{1}{S_\alpha'(h(u))}$$

$$= \alpha'(h(u)) \cdot \frac{1}{\|\alpha'(h(u))\|}.$$

$$\|\beta'(u)\| = \frac{\|\alpha'(h(u))\|}{\|\alpha'(h(u))\|} = 1 \ \forall \ u \in (c,d).$$

$\square$

Orientation

Let V be a finite dimensional vector space over $\mathbb{R}$.

$E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_n\}$ are ordered basis of V.

$\exists \ a_{ij} \in \mathbb{R}$. s.t. $1 \le i, j \le n$.

$$e_1 = a_{11} f_1 + a_{21} f_2 + \cdots + a_{n1} f_n$$

$$e_2 = a_{12} f_1 + a_{22} f_2 + \cdots + a_{n2} f_n.$$

$$\vdots$$

$$e_n = a_{1n} f_1 + a_{2n} f_2 + \cdots + a_{nn} f_n.$$

$$A = (a_{ij})_{n \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$ — change of basis matrix.

Defn: $E \sim F$ if $\det(A) > 0$.

- $\mathcal{B}(V)$ denotes the set of all ordered basis of V.

Lemma:- '$\sim$' an equivalence relation on $\mathcal{B}(V)$.

Proof Exercise.

Equivalence class.

$$\frac{\mathcal{B}(V)}{\sim} = \{O_1, O_2\}.$$

We assign '+' sign to O_1
'−' sign to O_2.

Defn:- 1) We say that $B \in \mathcal{B}(V)$ is (+)vely Oriented if $B \in O_1$

and (−)vely oriented if $B \in O_2$.

2) The standard Ordered basis of $\mathbb{R}^n$ is (+)vely Oriented.

Vector product in $\mathbb{R}^3$

$u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

$u \times v \in \mathbb{R}^n$ characterised by.

$(u \times v) \cdot \omega = \det(u, v, \omega)$.

$$= \det \begin{pmatrix} u_1 & v_1 & \omega_1 \\ u_2 & v_2 & \omega_2 \\ u_3 & v_3 & \omega_3 \end{pmatrix} \quad \text{where } \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3.$$

Ex. Show that.

$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3$$

Properties of Vector product.

1) Anti Commutative

$$u \times v = -v \times u$$

2) for $u, v, w \in \mathbb{R}^3$, $a, b \in \mathbb{R}$,

$$(au + bv) \times w = a(u \times w) + b(v \times w)$$

3) $u \times v = 0 \Leftrightarrow u$ and v are L.D.

4) $(u \times v) \cdot u = 0$

Remark: Let u and v are L.I.,
$P(u, v)$ - plane in $\mathbb{R}^3$ generated by u & v.
$(u \times v)$ is $\perp^r$ to $P(u, v)$.