Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex analysis (MTH 403)

Quiz 2 solutions

Date: 09 Sept. 2023

Time: 30 minutes | Total marks: 08

1. Prove or disprove the following: Let γ be a path in $\mathbb{C}$ and $f : \gamma^* \longrightarrow \mathbb{C}$ be a continuous function. Then
$$\left|\int_\gamma f\right| \le \int_\gamma |f|.$$
[3]

   To the students: Given a path σ in $\mathbb{C}$ and a continuous function $g : \sigma^* \longrightarrow \mathbb{C}$, the integral $\int_\sigma g$, by definition, contains a factor σ' in the integrand. Since $\sigma'(t)$ is always a complex number, even if g is real valued, $\int_\sigma g$ may be a nonreal complex number.

   Solution. The statement is FALSE. Let f be the constant function 1 and $\gamma(t) \stackrel{\text{def}}{=} it$, where $0 \le t \le 1$. Then one has
$$\int_\gamma |f| = \int_0^1 1 \cdot i = i,$$
   whereas $\left|\int_\gamma f\right|$ is always a nonnegative real number. Hence there is no comparison between them.

2. Let $f(z) \stackrel{\text{def}}{=} z^2$, for all $z \in \mathbb{C}$. Cauchy's theorem for an open convex set asserts that
$$\int_{C(2;1)} f = 0.$$
   But when you calculate $\displaystyle\int_0^{2\pi} f(2 + e^{it})\, dt$ you can see it is $8\pi \neq 0$. Are the two statements made above contradictory? Substantiate your claim. [2]

   Solution. By definition, $\displaystyle\int_{C(2;1)} f$ is equal to $\displaystyle\int_0^{2\pi} f(2 + e^{it}) \cdot i \cdot e^{it}\, dt$, which is not same as $\displaystyle\int_0^{2\pi} f(2+e^{it})\, dt$. Hence, despite the fact that both the statements under consideration are true, they are not contradicting with each other.

   To the students: Neither you have been asked to show $\int_{C(2;1)} f = 0$, nor $\int_0^{2\pi} f(2 + e^{it})\, dt = 8\pi$. Hence there is no point proving any of these.

3. Evaluate $\displaystyle\int_{C(0;1)} \frac{\sin z}{z^4}\, dz.$ [3]

Solution 1 *(Without using Cauchy theory)*. Since $\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$, for all $z \in \mathbb{C}$, we have

$$\frac{\sin z}{z^4} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-3}}{(2n+1)!}, \text{ whenever } |z| = 1.$$

Now, for all $n \geq 0$ and $|z| = 1$, $\left|(-1)^n \frac{z^{2n+1}}{(2n+1)!}\right| \leq \frac{1}{(2n+1)!}$. Since $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ converges, as for all $N \in \mathbb{N}$, $\sum_{n=0}^{N} \frac{1}{(2n+1)!} \leq \sum_{k=0}^{2N+1} \frac{1}{(2k+1)!} < e$, it follows from Weirstrass M-test that, $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-3}}{(2n+1)!}$ converges uniformly on the unit circle $|z| = 1$. Therefore,

$$\begin{aligned}
\int_{C(0;1)} \frac{\sin z}{z^4}\, dz &= \int_{C(0;1)} \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-3}}{(2n+1)!}\, dz \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_{C(0;1)} z^{2n-3}\, dz \\
&= -\frac{1}{6} \cdot 2\pi i \\
&= -\frac{i\pi}{3}.
\end{aligned}$$

To the students:

1. The uniform convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-3}}{(2n+1)!}$ on the unit circle $|z| = 1$ must be established beforehand in order to interchange the sum and integral.
2. You may provide a different but mathematically correct argument for the above mentioned uniform convergence.

Solution 2 *(Using Cauchy theory)*. Let $f(z) \overset{\text{def}}{=} \sin z$, for all $z \in \mathbb{C}$. Since f is holomorphic everywhere, from Cauchy's integral formula for derivatives, we obtain that

$$\frac{f^{(3)}(0)}{3!} = \frac{1}{2\pi i} \int_{C(0;1)} \frac{f(z)}{z^4}\, dz.$$

Since, for all $w \in \mathbb{C}$, $f'(w) = \cos w$, $f''(w) = -\sin w$ and $f'''(w) = -\cos w$, it now follows that

$$\int_{C(0;1)} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!} \cdot (-1) = -\frac{i\pi}{3}.$$