ASSIGNMENT 4

MTH 301, 2021-22

(1) Let $x \in \mathbb{R}^n$. For $1 \leq p < \infty$ define $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. For $r > 0$ denote $B_r(x) = \{y : \|x - y\|_p < r\}$. Sketch $B_1(0)$ for $p = 1, \frac{4}{3}, 2$ and ∞ in $\mathbb{R}^2$. Also show that for any pair of $p_1, p_2,\ 1 \leq p_1, p_2 \leq \infty$ there exists constants C_1, C_2 such that

$$C_1 \|x\|_{p_1} \leq \|x\|_{p_2} \leq C_2 \|x\|_{p_1}, \ \ \forall x \in \mathbb{R}^n.$$

(2) Show that $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ is a metric for $x, y \in (0, \infty)$.

(3) Let d be a metric on $\mathbb{R}^n$. Show that ρ as defined below are metrics.

(a) $\rho(x, y) = \sqrt{d(x, y)}$.

(b) $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$.

(c) $\rho(x, y) = \min\{d(x, y), 1\}$.

(4) For a $n \times m$ real matrix $A = (a_{ij})$ define $\|A\| = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}}$. Identify $\mathbb{R}^{n \times m}$ as set of all $n \times m$ real matrices. Show that $d(A, B) = \|A - B\|$ is a metric on $\mathbb{R}^{n \times m}$.

(5) Denote $\mathbb{R}^\infty$ to be the collection of all real sequences $x = \{x_n\}$. Show that

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on $\mathbb{R}^\infty$.

(6) For $x = (x_1.x_2, \ldots, x_n) \in \mathbb{R}^n$ denote $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$. For $x, y \in \mathbb{R}^n$ denote

$$< x, y >= \sum_{i=1}^{n} x_i y_i.$$

(a) If $< x, y >= 0$ i.e. they are orthogonal then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

(b) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in \mathbb{R}^n$.

(c) Let $\|x\| = \|y\| = 1$ and $\|\frac{x+y}{2}\| = 1$. Then show that $x = y$.

(d) Suppose U is a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^m$. U is called an **isometry** if $\|Ux\| = \|x\|$, **self-adjoint** if $U = U^*$, **unitary** if $UU^* = U^*U = I$.

(i) Let U be an isometry.

(A) Show that $\langle Ux, Uy \rangle = \langle x, y \rangle$.

(B) If $\{v_1, \ldots, v_k\}$ is an orthonormal set in $\mathbb{R}^n$ then show that $\{Uv_1, \ldots, Uv_k\}$ is also orthonormal.

(C) Will U be unitary? Will U be self adjoint?

(ii) If U satisfies $UU^* = U^*U$ then will it necessarily be unitary?

(e) Let M be a subspace of $\mathbb{R}^n$ with an orthonormal basis $\{v_1, \ldots, v_k\}$. Define a linear transformation on $\mathbb{R}^n$ by $Px = \sum_{i=1}^{k} < x, v_i > v_i$.

(i) Show that Px belongs to M, and $Py = y$ for all $y \in M$. Hence show that $P^2 = P$.

(ii) Show that $< Px, x - Px >= 0$.

(iii) Hence show that $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$.

(iv) If y belongs to M, show that $\|x-y\|^2 = \|y - Px\|^2 + \|x - Px\|^2$.

(v) Hence show that $\|Px\|$ is the closest point in M to x.

(7) If $a_i < b_i \in \mathbb{R}$ for $i = 1, \ldots, n$ then the subset $(a_1, b_1) \times \cdots \times (a_n, b_n)$ of $\mathbb{R}^n$ is called an *open rectangle* and the subset $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is called a *closed rectangle.*

(a) Show that an open rectangle is an open set and a closed rectangle is a closed set.

(b) Show that a sub set U of $\mathbb{R}^n$ is an open set if and only if for every $x_0 \in U$ there exists an open rectangle R such that $x_0 \in R \subset U$.

(8) Show that every open set in $\mathbb{R}$ can be expressed as countable union of disjoint open intervals.

(9) Show that every set in $\mathbb{R}^n$ can be expressed as intersection of open sets.

(10)

Definition 0.1. *Let (X, d) be a metric space and $A \subseteq X$.*

A point $x \in A$ is said to be an **interior point** *of A if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset A$. Denote A° to be the set of all interior points of A.*

A point $x \in X$ is said to be an **exterior point** *of A if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset A^c$. Denote A_{ext} to be the set of all interior points of A.*

A point $x \in X$ is said to be a **boundary point** *of A if for every $\epsilon > 0$ the ball $B_\epsilon(x)$ contains points of both A and A^c. Denote ∂A the set of boundary points of A .(is also called boundary of A)*

(a) Find $\partial \mathbb{Q}$.

(b) Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Find ∂A.

(c) Let $x \in A$. Show that either $x \in \partial A$ or $x \in A^o$.

(d) Show that if A is closed then $\partial A \subseteq A$ and if A is open then $\partial A \cup A = \emptyset$.

(e) Show that A° is the largest open set contained in A.

(f) Let $\bar{A}$ be the smallest closed set containing A. Show that $\bar{A} \setminus A^0 = \partial A$.

(11) Let A, B be two closed subsets of $\mathbb{R}^n$. Define

$$d(A, B) = \inf\{\|a - b\| = d(a, b) : a \in A \text{ and } b \in B\}.$$

(a) If $A = \{a\}$ is a singleton set then show that $d(A, B) > 0$.

(b) Give example of two disjoint closed sets A and B such that $d(A, B) = 0$.

(c) If A is compact and $A \cap B = \emptyset$ then show that $d(A, B) > 0$.