1 $X_t = (-1)^t A$

$Y_t = A \cos(\pi/2\, t) + B \sin \pi/2\, t$

A & B are indep r.v.s with 0 mean and var 1.

$E X_t = E Y_t = 0$

$\gamma_X(h) = (-1)^{|h|}$; $\gamma_Y(h) = \cos(\pi/2\, h)$

$\Rightarrow \{X_t\}$ & $\{Y_t\}$ are covariance stationary

$\text{Cov}(X_t, Y_{t+h}) = E X_t Y_{t+h}$

$= (-1)^t \cos(\pi/2 (t+h))$ ← depends on t and hence not a f^n of h only

$\Rightarrow \underset{\sim}{Z}_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$ is not covariance stationary

2 $E\, \underset{\sim}{Y}_t = \begin{pmatrix} E X_t \\ E X_{\alpha+\beta t} \end{pmatrix} = \begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad \forall \alpha, \beta.$

$$\text{cov}(\underset{\sim}{Y}_t, \underset{\sim}{Y}_{t+h}) = \begin{pmatrix} \underset{①}{\text{Cov}(X_t, X_{t+h})} & \underset{②}{\text{Cov}(X_t, X_{\alpha+\beta(t+h)})} \\ \underset{③}{\text{Cov}(X_{\alpha+\beta t}, X_{t+h})} & \underset{④}{\text{Cov}(X_{\alpha+\beta(t+h)}, X_{\alpha+\beta t})} \end{pmatrix}$$

① $= \gamma_X(h)$

② $= \gamma_X(\alpha + \beta t + \beta h - t) = \gamma_X(\alpha + t(\beta-1) + \beta h)$
indep of t if $\beta = 1$

③ $= \gamma_X(t + h - \alpha - \beta t) = \gamma_X(h - \alpha - t(\beta-1))$
indep of t if $\beta = 1$

④ $= \gamma_X(\cancel{\alpha} + \cancel{\beta t} - \cancel{\alpha} - \cancel{\beta t} - \beta h)$
$= \gamma_X(\beta h)$

$\Rightarrow \underset{\sim}{Y}_t$ is covariance stationary $\beta = 1$ and $\forall$ integer α

3 $X_t = \phi X_{t-1} + u_t \quad |\phi|<1$

$Y_t = \phi X_{t-2} + \delta_t$

$X_t = \sum_{j=0}^{\infty} \phi^j u_{t-j}$

indep of $\{u_t\}$ & $\{\delta_t\} \Rightarrow \{X_t\}$ & $\{\delta_t\}$ are indep.

$\gamma_{xy}(k) = \mathrm{Cov}(X_t, Y_{t+k}) = E(X_t Y_{t+k})$

$= E(X_t(\phi X_{t-2+k} + \delta_{t+k}))$

$\gamma_{xy}(k) = \phi\, \gamma_x(k-2)$

$\gamma_{xy}(0) = \phi\, \phi^2 \frac{\sigma^2}{1-\phi^2}$; $\gamma_{xy}(1) = \phi\, \phi \frac{\sigma^2}{1-\phi^2}$.

$\forall k \geq 2 \quad \gamma_{xy}(k) = \phi\, \phi^{k-2} \frac{\sigma^2}{1-\phi^2}$.

$\rho_{xy}(k) = \frac{\gamma_{xy}(k)}{(\gamma_x(0)\gamma_y(0))^{1/2}}$; $\gamma_x(0) = \frac{\sigma^2}{1-\phi^2}$

$Y_t = \phi X_{t-2} + \delta_t$

$\gamma_y(0) = \phi^2 \frac{\sigma^2}{1-\phi^2} + \gamma_\delta(0) = \phi^2 \frac{\sigma^2}{1-\phi^2} + \sigma^2$

4 ACVF of $\{X_t\}$: $\gamma_X(h) = \left(\frac{1}{2}\right)^{|h|}$

$$Y_t \sim WN(0, \sigma_Y^2)$$

$$\underline{Z}_t = \begin{pmatrix} X_t(Y_t + Y_{t-1}) = U_t \\ X_t Y_{t-2} = V_t \end{pmatrix}; \quad E\, \underline{Z}_t = \underline{0}.$$

$$\mathrm{Cov}(\underline{Z}_t, \underline{Z}_{t+h}) = \begin{pmatrix} \underset{①}{\mathrm{Cov}(U_t, U_{t+h})} & \underset{②}{\mathrm{Cov}(U_t, V_{t+h})} \\ \underset{③}{\mathrm{Cov}(V_t, U_{t+h})} & \underset{④}{\mathrm{Cov}(V_t, V_{t+h})} \end{pmatrix}.$$

$$① = \gamma_X(h)\left(2\gamma_Y(h) + \gamma_Y(h-1) + \gamma_Y(h+1)\right)$$

$$= \gamma_X(h)\, \sigma_Y^2 \left(2I_0 + I_1 + I_{-1}\right)$$

$$④ = \gamma_X(h)\, \gamma_Y(h)$$

$$② = \sigma_Y^2\, \gamma_X(h)\left(I_2 + I_1\right)$$

$$③ = \sigma_Y^2\, \gamma_X(h)\left(I_{-2} + I_{-1}\right)$$

$\Rightarrow \underset{\sim}{Z}_t$ is covariance stationary

$\forall\ |h| > 2 \quad \Gamma_Z(h)$ is a null matrix

5 $$\underline{Z}_t = \begin{pmatrix} X_t = A\cos(t+\theta) \\ Y_t = A\cos(2t+\theta) \end{pmatrix}$$

$$E X_t = A\ E\cos(t+\theta) = A \cdot \frac{1}{2\pi}\int_0^{2\pi} \cos(t+\theta)\, d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi} (\cos t \cos\theta - \sin t \cdot \sin\theta)\, d\theta = 0.$$

$$\mathrm{Cov}(X_t, Y_{t+h}) = E\, X_t Y_{t+h}$$

$$= AB \frac{1}{2\pi} \int_0^{2\pi} \cos(t+\theta)\cos(2(t+h)+\theta)\, d\theta$$

$$= \frac{AB}{4\pi} \int_0^{2\pi} \left[\begin{array}{l} \cos(t+\theta-2t-2h-\theta) \\ + \cos(t+\theta+2t+2h+\theta) \end{array} \right] d\theta$$

$$= \frac{AB}{4\pi} \int_0^{2\pi} \{ \cos(t+2h) + \cos(3t+2h+2\theta)) \, d\theta$$

$$= \frac{AB}{4\pi} \cdot \cos(t+2h)\, 2\pi \leftarrow f^n \text{ of } t$$

$\Rightarrow \underline{Z}_t$ is not covariance stationary

⑥ $Y_t = (-1)^t + X_t + X_{t-1} + X_{t-2}$

$E Y_t = (-1)^t \leftarrow f^n$ of time

$\Rightarrow \begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$ can not be covariance stationary

⑦ $X_t = \phi X_{t-1} + \epsilon_{1t}$; ϵ_{1t} & ϵ_{2t} are indep $WN(0, \sigma_\epsilon^2)$

$Y_t = \phi X_{t-2} + \epsilon_{2t}$; $\phi = \frac{1}{2}$

$X_t = \sum_0^\infty \phi^j \epsilon_{1,t-j}$; $E X_t = 0$

$\gamma_{xy}(2) = \mathrm{Cov}(X_t, Y_{t+2}) = E\, X_t Y_{t+2}$

$$= E X_t (\phi X_t + \epsilon_{2,t+2})$$

$$= \phi \frac{\sigma_\epsilon^2}{1-\phi^2} \quad \text{Sly } \gamma_{xy}(3).$$

8 $$\underset{\sim}{Z}_t = \begin{pmatrix} A_t = \sum_{-\infty}^{\infty} \psi_{1,h} X_{t-h} \\ B_t = \sum_{-\infty}^{\infty} \psi_{2,h} X_{a+bt-h} \end{pmatrix} \qquad E \underline{Z}_t = \underline{0}$$

$$\mathrm{Cov}(\underline{Z}_t, Z_{t+k}) = \begin{pmatrix} \mathrm{Cov}(A_t, A_{t+k}) & \mathrm{cov}(A_t, B_{t+k}) \\ \mathrm{Cov}(B_t, A_{t+k}) & \mathrm{cov}(B_t, B_{t+k}) \end{pmatrix}$$

$$\mathrm{Cov}(A_t, A_{t+k}) = \mathrm{Cov}\Big(\sum_h \psi_{1,h} X_{t-h},\ \sum_{h'} \psi_{1,h'} X_{t+k-h'}\Big)$$
$$= \sum_h \sum_{h'} \psi_{1,h} \psi_{1,h'} \gamma_X(k-h'+h)$$
$\longrightarrow$ f^n of k only

$$\mathrm{Cov}(B_t, B_{t+k}) = \sum_h \sum_{h'} \psi_{2,h} \psi_{2,h'} \gamma_X(bk-h'+h)$$
$\longrightarrow$ f^n of k only

$$\mathrm{Cov}(A_t, B_{t+k}) = \mathrm{Cov}\Big(\sum_h \psi_{1,h} X_{t-h},\ \sum_{h'} \psi_{2,h'} X_{a+b(t+k)-h'}\Big)$$
$$= \sum_h \sum_{h'} \psi_{1,h} \psi_{2,h'} \gamma_X(\underbrace{a+bt+bk-h'-t+h})$$
$$= \gamma_X(a+t(b-1)+bk-h'+h)$$

Sly

$$\mathrm{Cov}(B_t, A_{t+k}) = \sum_h \sum_{h'} \psi_{1,h} \psi_{2,h'} \gamma_X\big(a+t(b-1)-k-h'+h\big)$$

If $b=1$ and any $a \in \mathbb{Z}$, $\underline{Z}_t$ is covariance stationary.

For $a=1$, $b=1$

$$\mathrm{Cov}(A_t, A_{t+k}) = \sum_h \sum_{h'} \psi_{1,h} \psi_{1,h'} \gamma_X(k-h'+h) \neq 0 \quad \forall k \neq 0 \quad (k = h'-h)$$

$\Rightarrow \underline{Z}_t \not\sim$ VWN

9 $\underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \Sigma)$, $\Sigma > 0$

$$\underset{\sim}{Y}_t = \begin{pmatrix} \underset{\sim}{\epsilon}_t \\ 2\underset{\sim}{\epsilon}_{2t+3} \end{pmatrix}$$

$$\underset{\sim}{Y}_1 = \begin{pmatrix} \underset{\sim}{\epsilon}_1 \\ 2\underset{\sim}{\epsilon}_5 \end{pmatrix}, \quad \underset{\sim}{Y}_5 = \begin{pmatrix} \underset{\sim}{\epsilon}_5 \\ 2\underset{\sim}{\epsilon}_{13} \end{pmatrix}$$

$$Cov(\underline{Y}_1, \underline{Y}_5) = E(\underline{Y}_1 \underline{Y}_5') = \begin{pmatrix} 0 & 0 \\ 2I & 0 \end{pmatrix} \neq 0$$

$\Rightarrow \underset{\sim}{Y}_t \nsim VWN$

10 $\epsilon_t \sim WN(0,1)$

$$X_t = 2\epsilon_t + \epsilon_{t-1} + \epsilon_{t+1}$$

$$Y_t = 2 + \epsilon_t - \epsilon_{t-1}$$

$$\underset{\sim}{Z}_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}; \quad E\,\underset{\sim}{Z}_t = \underline{0} \ \forall t$$

$$Cov(\underline{Z}_t, \underline{Z}_{t+h}) = \begin{pmatrix} Cov(X_t, X_{t+h}) & Cov(X_t, Y_{t+h}) \\ Cov(Y_t, X_{t+h}) & Cov(Y_t, Y_{t+h}) \end{pmatrix}.$$

$$Cov(X_t, X_{t+h}) = Cov(2\epsilon_t + \epsilon_{t-1} + \epsilon_{t+1}, 2\epsilon_{t+h} + \epsilon_{t+h-1} + \epsilon_{t+h+1})$$

$$= 4I_0(h) + 2I_1(h) + 2I_{-1}(h)$$
$$+ 2I_{-1}(h) + I_0(h) + I_{-2}(h)$$
$$+ 2I_1(h) + I_2(h) + I_0(h) \quad - (1)$$

$$\operatorname{Cov}(X_t, Y_{t+h}) = \operatorname{Cov}(2\epsilon_t + \epsilon_{t-1} + \epsilon_{t+1}, 2 + \epsilon_{t+h} - \epsilon_{t+h-1})$$

$$= 2 I_0(h) - 2 I_1(h)$$
$$+ I_{-1}(h) - I_0(h)$$
$$+ I_1(h) - I_2(h) \quad - (2)$$

$$\operatorname{Cov}(Y_t, X_{t+h}) = \operatorname{Cov}(2 + \epsilon_t - \epsilon_{t-1}, 2\epsilon_{t+h} + \epsilon_{t+h-1} + \epsilon_{t+h+1})$$
$$= 2 I_0(h) + I_1(h) + I_{-1}(h)$$
$$- 2 I_{-1}(h) - I_0(h) - I_{-2}(h) \quad - (3)$$

$$\operatorname{Cov}(Y_t, Y_{t+h}) = \operatorname{Cov}(2 + \epsilon_t - \epsilon_{t-1}, 2 + \epsilon_{t+h} - \epsilon_{t+h-1})$$
$$= I_0(h) - I_1(h)$$
$$- I_{-1}(h) + I_0(h) \quad - (4)$$

(1), (2), (3), (4)

$\Rightarrow \operatorname{Cov}(\underset{\sim}{Z}_t, \underset{\sim}{Z}_{t+h})$ is a f^n of h only, indep of t

$\Rightarrow \underset{\sim}{Z}_t$ is covariance stationary

Further, (1), (2), (3) & (4) $\Rightarrow$

$$\operatorname{Cov}(\underset{\sim}{Z}_t, \underset{\sim}{Z}_{t+h}) = 0 \quad \forall\ |h| \geq 3$$

3 is the smallest integer $\ni \Gamma_Z(h) = 0$.

11. $X_t = e_1 + e_2 \cos t + e_3 \sin t$

$Y_t = t + e_1 \cos t + e_2 \sin t$

$Z_t = (-1)^t e_4 + e_5 \cos t + e_6 \sin t$

$$\underset{\sim}{P}_t = \begin{pmatrix} X_t \\ R_t = Y_t - t = e_1 \cos t + e_2 \sin t \end{pmatrix}.$$

$\mathrm{Cov}(X_t, R_t) = \cos t + \cos t \sin t \leftarrow f^n$ of t

$\mathrm{Cov}(X_t, R_{t+h}) = \cos(t+h) + \cos t \sin(t+h)$.

$\Rightarrow \underset{\sim}{P}_t$ is not covariance stationary

$\underline{Q}_t = \begin{pmatrix} X_t \\ Z_t \end{pmatrix}$ $\{X_t\}$ & $\{Z_t\}$ are clearly covariance stationary

$$\mathrm{Cov}(X_t, Z_{t+h}) = \mathrm{Cov}\big(e_1 + e_2 \cos t + e_3 \sin t,\ (-1)^{t+h} e_4 + e_5 \cos(t+h) + e_6 \sin(t+h)\big).$$

$= 0 \quad \forall h$

Sly $\mathrm{Cov}(Z_t, X_{t+h}) = 0 \ \forall h$

$$\Rightarrow \mathrm{Cov}(\underline{Q}_t, \underline{Q}_{t+h}) = \begin{pmatrix} \gamma_X(h) & 0 \\ 0 & \gamma_Z(h) \end{pmatrix}$$

$\Rightarrow \underline{Q}_t$ is covariance stationary. $\forall h$

(12)

$$\underline{Y}_t = \begin{pmatrix} \underset{\sim}{X}_t \\ \alpha \underset{\sim}{X}_{t-1} \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ \alpha I_3 & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{X}_{t-1} \\ \alpha \underline{X}_{t-2} \end{pmatrix} + \begin{pmatrix} \underline{\epsilon}_t \\ \underline{0} \end{pmatrix}.$$

i.e. $\underline{Y}_t = \Phi^* \underset{\sim}{Y}_{t-1} + \underset{\sim}{\eta}_t$

$\underset{\sim}{\eta}_t$ is $\ni E\, \underline{\eta}_t = \underline{0}$

$$\mathrm{Cov}(\underline{\eta}_t, \underline{\eta}_s) = E(\underline{\eta}_t \underline{\eta}_s') = \begin{cases} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} = \Sigma^*, & \text{if } t = s \\ 0, & \text{o/w.} \end{cases}$$

$$\underset{j>0}{\mathrm{Cov}(\underline{\eta}_t, \underset{\sim}{Y}_{t-j})} = E\left(\underline{\eta}_t \begin{pmatrix} \underset{\sim}{X}_{t-j} \\ \alpha \underset{\sim}{X}_{t-j-1} \end{pmatrix}'\right)$$

$$= 0.$$

$\Rightarrow \{\underline{Y}_t\}$ is VAR(1)

$\underline{Y}_t$ is stationary VAR(1) if all z satisfying $|I_6 - \Phi^* z| = 0$ lie outside unit circle

i.e. $$\left| \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix} - \begin{pmatrix} \Phi & 0 \\ \alpha I_3 & 0 \end{pmatrix} z \right| = 0$$

i.e. $$\begin{vmatrix} I_3 - \Phi z & 0 \\ -z\alpha I_3 & I_3 \end{vmatrix} = 0$$

i.e. $|I_3 - \Phi z| = 0 \leftarrow$ All z satisfying this are outside unit circle

$\Rightarrow \underline{Y}_t$ is also covariance stationary. ($\because \underline{X}_t$ is cov stationary)

$$\underline{X}_t = \sum_{i=1}^{p} \Phi_i \underset{\sim}{X}_{t-i} + \underline{\epsilon}_t$$

$$\underline{Z}_t = \begin{pmatrix} \underset{\sim}{X}_t \\ \underset{\sim}{X}_{t-1} \\ \vdots \\ \underset{\sim}{X}_{t-(p-1)} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{p} \Phi_i \underset{\sim}{X}_{t-i} + \underset{\sim}{\epsilon}_t \\ \underset{\sim}{X}_{t-1} \\ \vdots \\ \underset{\sim}{X}_{t-(p-1)} \end{pmatrix}.$$

i.e. $$\underline{Z}_t = \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_p \\ I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ 0 & \cdots & & I_n \; 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{X}_{t-1} \\ \vdots \\ \underset{\sim}{X}_{t-p} \end{pmatrix} + \begin{pmatrix} \underset{\sim}{\epsilon}_t \\ \underline{0} \\ \vdots \\ \underline{0} \end{pmatrix}$$

i.e. $$\underline{Z}_t = F \underset{\sim}{Z}_{t-1} + \underset{\sim}{\eta}_t$$

$$\underset{\sim}{\eta}_t \sim VWN(\underline{0}, \underset{\mathcal{Q}}{\underset{\shortparallel}{\Sigma^*}}) \; ; \; \Sigma^* = \begin{pmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \ddots & & \\ & & & 0 \end{pmatrix}.$$

$$E \, \underline{Z}_t \underline{Z}_t' = E \left(F \underset{\sim}{Z}_{t-1} + \eta_t \right) \left(F \underset{\sim}{Z}_{t-1} + \underline{\eta}_t \right)'$$

$$\Omega = F \Omega F' + \underbrace{E\left(F \underset{\sim}{Z}_{t-1} \underline{\eta}_t'\right)}_{0} + \underbrace{E\left(\underline{\eta}_t \underline{Z}_{t-1}' F'\right)}_{0} + \mathcal{Q}$$

i.e. $$\Omega = F \Omega F' + \mathcal{Q}$$

(14) $\underset{\sim}{X}_t + \underset{\sim}{X}_{t-1} + \frac{1}{4}\underset{\sim}{X}_{t-2} = \underset{\sim}{\epsilon}_t$

$$(I_2 + I_2 B + \frac{1}{4} I_2 B^2)\underline{X}_t = \underline{\epsilon}_t$$

$$\underline{\Phi}(B)\underline{X}_t = \underline{\epsilon}_t$$

$$|\underline{\Phi}(z)| = |I_2 + I_2 z + I_2 \tfrac{1}{4} z^2|$$

$$= \begin{vmatrix} 1 + z + \frac{1}{4}z^2 & 0 \\ 0 & 1 + z + \frac{z^2}{4} \end{vmatrix} = \left(1+\frac{z}{2}\right)^4$$

roots of $|\underline{\Phi}(z)| = 0$, $z = -2$, lies outside unit circle

$\Rightarrow$ $\underline{X}_t$ is stationary and causal

$$\underline{X}_t = \underline{\Phi}(B)^{-1}\underline{\epsilon}_t = \underline{\Psi}(B)\underline{\epsilon}_t$$

$$\Rightarrow I_2 = \underline{\Phi}(B)\underline{\Psi}(B)$$

$$= (I_2 + I_2 B + \tfrac{1}{4} I_2 B^2)$$

$$(\underline{\Psi}_0 + \underline{\Psi}_1 B + \underline{\Psi}_2 B^2 + \cdots)$$

$$= \underline{\Psi}_0 + (\underline{\Psi}_1 + \underline{\Psi}_0)B + (\underline{\Psi}_2 + \underline{\Psi}_1 + \tfrac{1}{4}\underline{\Psi}_0)B^2 +$$

$$\cdots + (\underline{\Psi}_j + \underline{\Psi}_{j-1} + \tfrac{1}{4}\underline{\Psi}_{j-2})B^j + \cdots$$

$$\underline{\Psi}_0 = I$$

$$\underline{\Psi}_1 = -I$$

$$\underline{\Psi}_2 = -\underline{\Psi}_1 - \tfrac{1}{4}\underline{\Psi}_0 \left(= I_2 - \tfrac{1}{4} I_2 = \tfrac{3}{4} I_2\right)$$

$$\underline{\Psi}_j = -\underline{\Psi}_{j-1} - \tfrac{1}{4}\underline{\Psi}_{j-2}$$

(15)

$$\underline{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underline{X}_{t-2} + \underset{\sim}{\epsilon}_t ;$$

$$\underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \Sigma)$$

$$\underline{Z}_t = \begin{pmatrix} \underline{X}_t \\ \underline{X}_{t-1} \end{pmatrix} = \begin{pmatrix} \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underline{X}_{t-2} \\ \underset{\sim}{X}_{t-1} \end{pmatrix} + \begin{pmatrix} \underline{\epsilon}_t \\ \underline{0} \end{pmatrix}$$

$$\underline{Z}_t = \begin{pmatrix} \Phi_1 & \Phi_2 \\ I_K & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{X}_{t-1} \\ \underline{X}_{t-2} \end{pmatrix} + \begin{pmatrix} \underline{\epsilon}_t \\ \underline{0} \end{pmatrix}$$

i.e. $\underline{Z}_t = \underline{\Phi}^* \underset{\sim}{Z}_{t-1} + \underline{\eta}_t$

$$\underline{\eta}_t \sim VWN\left(\underline{0}, \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$(I_{2K} - \underline{\Phi}^* B) \underline{Z}_t = \underline{\eta}_t$$

$$|I_{2K} - \underline{\Phi}^* z| = \begin{vmatrix} I_K - \Phi_1 z & -\Phi_2 z \\ -I_K z & I_K \end{vmatrix}.$$

$$= |I_K - \Phi_1 z - \Phi_2 z^2|$$

$\Rightarrow$ all z satisfying $|I_{2K} - \underline{\Phi}^* z| = 0$ lie outside unit circle (as $\underset{\sim}{X}_t$ is covariance stationary and hence all z satisfying $|I_K - \Phi_1 z - \Phi_2 z^2| = 0$ lie outside unit circle)

16. $\underline{X}_t = \Phi \underset{\sim}{X}_{t-1} + \underline{\epsilon}_t + \Theta \underset{\sim}{\epsilon}_{t-1}$

$$(I_2 - \Phi B)\underline{X}_t = (I_2 + \Theta B)\underline{\epsilon}_t$$

$$|\underline{\Phi}(z)| = \begin{vmatrix} 1-az & -az \\ 0 & 1-az \end{vmatrix} = (1-az)^2$$

$$|\hat{\Phi}(z)| = 0 \iff (1-az)^2 = 0 \Rightarrow z = \frac{1}{a}$$

Condition for stationarity and causal is $|a|<1$

Sly for invertibility

Causal:

$$\underline{X}_t = \hat{\Phi}(B)^{-1}\Theta(B)\underline{\epsilon}_t = \underline{\Psi}(B)\underline{\epsilon}_t$$

$$\Rightarrow \Theta(B) = \hat{\Phi}(B)\hat{\Psi}(B)$$

$$I_2 + \Theta B = (I_2 - \underline{\Phi} B)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$$

$$= \hat{\Psi}_0 + (\Psi_1 - \hat{\Phi}\Psi_0)B + (\Psi_2 - \hat{\Phi}\Psi_1)B^2 + (\hat{\Psi}_3 - \hat{\Phi}\hat{\Psi}_2)B^3 + \cdots$$

$$\Rightarrow \hat{\Psi}_0 = I_2$$

$$\hat{\Psi}_1 = \Theta + \hat{\Phi}$$

$$\hat{\Psi}_2 = \hat{\Phi}(\Theta + \hat{\Phi})$$

$$\hat{\Psi}_3 = \hat{\Phi}^2\Psi_1 = \hat{\Phi}^2(\Theta + \hat{\Phi})$$

$$\vdots$$

$$\hat{\Psi}_j = \hat{\Phi}^{j-1}\Psi_1 = \hat{\Phi}^{j-1}(\Theta + \hat{\Phi})$$

$$\hat{\Phi} = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \quad \hat{\Phi}^2 = \begin{pmatrix} a^2 & 2a^2 \\ 0 & a^2 \end{pmatrix}, \quad \hat{\Phi}^3 = \begin{pmatrix} a^3 & 3a^3 \\ 0 & a^3 \end{pmatrix} \cdots$$

$$\hat{\Phi}^{j-1} = \begin{pmatrix} a^{j-1} & (j-1)a^{j-1} \\ & a^{j-1} \end{pmatrix} \cdots$$

$$\Rightarrow \quad \Psi_j = \begin{pmatrix} a^{j-1} & (j-1)a^{j-1} \\ 0 & a^{j-1} \end{pmatrix} \begin{pmatrix} 2a & 2a \\ 0 & 2a \end{pmatrix}$$

$$\Psi_j = \begin{pmatrix} 2a^j & 2ja^j \\ 0 & 2a^j \end{pmatrix} = \begin{pmatrix} \psi_{11}^{(j)} & \psi_{12}^{(j)} \\ \psi_{21}^{(j)} & \psi_{22}^{(j)} \end{pmatrix} \text{ say}$$

For $\{\Psi_j\}_{j=0}^{\infty}$:

$$\sum_{j=0}^{\infty} |\psi_{11}^{(j)}| = 1 + 2\sum_{j=0}^{\infty} |a|^j < \infty \quad \text{as } |a|<1$$

$$\sum_{j=0}^{\infty} |\psi_{21}^{(j)}| = 0$$

$$\sum_{j=0}^{\infty} |\psi_{22}^{(j)}| = \sum_{j=0}^{\infty} |\psi_{11}^{(j)}| < \infty$$

$$\sum_{j=0}^{\infty} |\psi_{12}^{(j)}| = 2\left(|a| + 2|a|^2 + 3|a|^3 + \cdots\right)$$

Let $S = |a| + 2|a|^2 + 3|a|^3 + \cdots$

$$|a|S = |a|^2 + 2|a|^3 + 3|a|^4 + \cdots$$

$$S(1-|a|) = |a| + |a|^2 + |a|^3 + \cdots$$

$$S(1-|a|) = \frac{|a|}{1-|a|}$$

$$\Rightarrow S = \frac{|a|}{(1-|a|)^2} \Rightarrow \sum_{j=0}^{\infty} |\psi_{12}^{(j)}| < \infty$$

$\Rightarrow \{\Psi_j\}_{j=0}^{\infty}$ is absolutely summable

Impulse Responses:

$$\frac{\partial X_{1,t+s}}{\partial \epsilon_{2,t}} = \psi_{12}^{(s)} = 2 s a^{s}$$

$$\frac{\partial X_{2,t+s}}{\partial \epsilon_{1,t}} = \psi_{21}^{(s)} = 0$$

(17)

$$\underset{\sim}{X}_t = \underline{\Phi}\, \underset{\sim}{X}_{t-1} + \underset{\sim}{\epsilon}_t\ ; \quad \underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \Sigma),\ \Sigma > 0$$

2×1

$$\underline{\Phi} = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}; \quad \underline{\Phi}(B)\, \underset{\sim}{X}_t = \underset{\sim}{\epsilon}_t\ ; \quad \underline{\Phi}(B) = I_2 - \underline{\Phi} B$$

$\underset{\sim}{X}_t$ is Covariance stationary if all z satisfying

$$|\underline{\Phi}(z)| = 0 \quad \text{lie outside the unit circle}$$

$$|\underline{\Phi}(z)| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix} z \right| = \left| \begin{pmatrix} 1 & -bz \\ 0 & 1-az \end{pmatrix} \right|.$$

$$= 1 - az$$

$$|\underline{\Phi}(z)| = 0 \iff 1 - az = 0 \implies z = \frac{1}{a}$$

$\Rightarrow \forall\, a \ni |a| < 1 \cdot \underset{\sim}{X}_t$ is Cov stat

Note that stationarity of $\underset{\sim}{X}_t$ does not depend on the value of b

Hence, $\underset{\sim}{X}_t$ is Cov stat $\forall |a| < 1$ & $\forall\, b \in \mathbb{R}$

$\Rightarrow$ (a) & (b) statements are correct

& (c) is not Correct.

(18) $\underset{\sim}{\epsilon}_t \sim VWN(\underline{0}, \Sigma), \ \Sigma > 0$

$\underset{\sim}{X}_t = \epsilon_t + \Theta \underset{\sim}{\epsilon}_{t-1}$ — i.e. VMA(1)

$$\underline{Z}_t = \begin{pmatrix} \underline{X}_t \\ \underset{\sim}{\epsilon}_{t-1} \end{pmatrix} = \begin{pmatrix} \underline{\epsilon}_t + \Theta \underset{\sim}{\epsilon}_{t-1} \\ \underset{\sim}{\epsilon}_{t-1} \end{pmatrix}$$

(i)

$E\, \underline{Z}_t = \underset{\sim}{0} \quad \forall t$

$$Cov(\underline{Z}_t, \underline{Z}_{t+h}) = E\, \underline{Z}_t \underline{Z}_{t+h}'$$

$$= \begin{pmatrix} E\, \underline{X}_t \underline{X}_{t+h}' & E\, \underset{\sim}{X}_t \epsilon_{t+h-1}' \\ E\, \underset{\sim}{\epsilon}_{t-1} \underline{X}_{t+h}' & E(\underset{\sim}{\epsilon}_{t-1}) \underset{\sim}{\epsilon}_{t+h-1}' \end{pmatrix}$$

$$E\, \underset{\sim}{X}_t \underline{X}_{t+h}' = E(\underline{\epsilon}_t + \Theta \underset{\sim}{\epsilon}_{t-1})(\underline{\epsilon}_{t+h} + \Theta \epsilon_{t+h-1})'$$

$$= \Sigma I_0(h) + \Sigma \Theta' I_1(h) + \Theta \Sigma I_{-1}(h) + \Theta \Sigma \Theta' I_0(h) \quad -(1)$$

$$E\, \underset{\sim}{X}_t \epsilon_{t+h-1}' = E(\underline{\epsilon}_t + \Theta \underset{\sim}{\epsilon}_{t-1}) \epsilon_{t+h-1}'$$

$$= \Sigma I_1(h) + \Theta \Sigma I_0(h) \quad — (2)$$

$$E\, \underset{\sim}{\epsilon}_{t-1} \underline{X}_{t+h}' = E\, \underset{\sim}{\epsilon}_{t-1} (\underline{\epsilon}_{t+h} + \Theta \underline{\epsilon}_{t+h-1})'$$

$$= \Sigma I_{-1}(h) + \Sigma \Theta' I_0(h) \quad -(3)$$

$$E\, \underset{\sim}{\epsilon}_{t-1} \underset{\sim}{\epsilon}_{t+h-1}' = \Sigma I_0(h) \quad — (4).$$

~~Q~~

(1), (2), (3), (4) along with $E\,\underset{\sim}{Z}_t = \underset{\sim}{0}\ \forall t$

$\Rightarrow \underset{\sim}{Z}_t$ is covariance stationary

(ii)

$$E\,\underset{\sim}{Z}_t Z_{t+1} = \begin{pmatrix} \Sigma \Theta' & \Sigma \\ 0 & 0 \end{pmatrix} \neq \text{null matrix}$$

$\Rightarrow \underset{\sim}{Z}_t \not\sim VWN$

(iii)

$$\underset{2K\times 1}{\underset{\sim}{Z}_t} = \begin{pmatrix} \underset{\sim}{\epsilon}_t + \Theta\, \underset{\sim}{\epsilon}_{t-1} \\ \underset{\sim}{\epsilon}_{t-1} \end{pmatrix}.$$

$$= \begin{pmatrix} \underset{\sim}{\epsilon}_t \\ \underset{\sim}{0} \end{pmatrix} + \begin{pmatrix} \Theta & 0 \\ I_K & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\epsilon}_{t-1} \\ \underset{\sim}{0} \end{pmatrix}$$

$$\underset{\sim}{Z}_t = \underset{\sim}{\eta}_t + \Theta^* \underset{\sim}{\eta}_{t-1}$$

with $\underset{\sim}{\eta}_t = \begin{pmatrix} \underset{\sim}{\epsilon}_t \\ \underset{\sim}{0} \end{pmatrix} \sim VWN\left(\underset{\sim}{0}, \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\right)$

a singular cov matrix

$\Rightarrow \underset{\sim}{Z}_t$ has a VMA(1) representation