Tarun Goyal

Complex Analysis

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After Logarithm Notes

Logarithm of general functions

X is a metric space :-

Let $X \xrightarrow{f} \mathbb{C}/\{0\}$. Then

we say:

$g : X \longrightarrow \mathbb{C}$ is a log of f if $\forall x \in X$, $f(x) = e^{g(x)}$

$\boxed{\text{$g(x)$ is continuous log if f and g are continuous}}$

Special Case

If $X \underset{\text{open}}{\subseteq} \mathbb{C}$, f is holomorphic

If g is holo $\Rightarrow$ g is holomorphic log of f.

eg: $X = \mathbb{C} \setminus \overline{R_\alpha}$. $f(z) = z \Rightarrow g(z) = \log_\alpha z$

$g(z)$ is holo log of f.

we assume $f(x)$ is continuous

Continuous Arguement

★ $\theta : X \longrightarrow \mathbb{R}$, cont^n arg of f

if $\forall x \in X$, $f(x) = |f(x)| e^{i\theta(x)}$

θ = continuous function

Theorem:

Continuous Arguement $\Leftrightarrow$ Continuous logarithm

Proof:-

$$g(x) = u(x) + i v(x)$$

$$f = e^{g} = e^{u} \cdot e^{iv}$$

$|f| = e^{u} = |f(x)|$ | define $g = \log|f(x)| + i v(x)$

$u = \log|f|$

i.e

$$f(x) = |f(x)| \cdot e^{i \operatorname{Im}(g(x))}$$

ie if $v(x)$ is continuous, g is continuous

Illy $\Leftarrow$

$$f(x) = e^{\log|f(x)|} \cdot e^{i\theta(x)}$$

Assumption: $\log|f(x)| + i\theta(x)$ is contn. ✓✓

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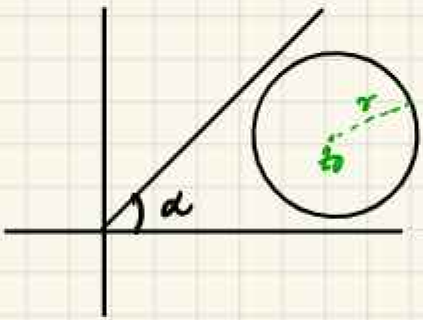
★ if a holo f^n has a cont^n log, then is the log holo?

Ans: $f : V \xrightarrow{\text{holo}} \mathbb{C}$, Assume $\exists\, \alpha \in \mathbb{R}$

$\ni \quad f(V) \cap \overline{R}_\alpha = \phi$

$\Rightarrow \quad \log_\alpha f$ is holomorphic



we can Always, get a ray avoiding $D(z_0, r)$ if $0 \notin D(z_0, r)$

Then,

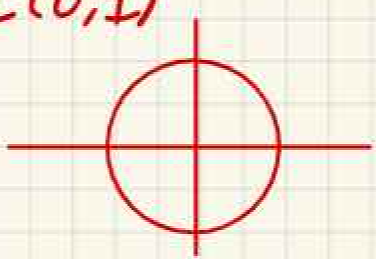
$$\exists\, \alpha \in \mathbb{R} \ni \overline{R}_\alpha \cap D(z_0; r) = \phi$$

$$\therefore \forall f : U \xrightarrow{\text{holo}} D(z_0; r),$$

$\exists$ a holomorphic function.

$f(t) = \dfrac{\theta(e^{it}) + \theta(e^{-it})}{2\pi}$ is a choice for $A(1)$ hence, $f(t)$ is constant

Q $C(0;1)$



$f(z) = z$.

Here it doesn't have a continuous arguement

$\downarrow \alpha$: ... on S^1 because if $\theta(z_1)=\theta(z_2)$ then $z_1 = z_2 \Rightarrow \theta(z)-\theta(-z) \neq 0$

Check:- $f = \dfrac{\theta(z)-\theta(-z)}{|\theta(z)-\theta(-z)|}$

see, $f(-z) = -f(z)$

Uniqueness of log

$$e^{g(x)} = e^{g_1(x)}$$

$$g(x) = u + iv \qquad g_1 = u_1 + iv_1$$

$$* \; u = u_1 = \log|f|$$

and $v(x) - v_1(x) \in 2\pi\mathbb{Z}$

Also,

Assume f, g, g_1 are continuous & X is Connected

then, $v - v_1$ is a constant

because $g - g_1$ is continuous function $\Rightarrow g - g_1 = 2\pi i k$ is cont^n $(k \in \mathbb{Z})$ i.e. k is constant

Lemma: $\gamma : [a,b] \xrightarrow{\text{cont}^n} \mathbb{C}\setminus\{0\}$

(I) Then γ has a cont^n arg.

we'll prove general version:-

(II) $f : [a,b] \times [c,d] \xrightarrow{\text{cont}^n} \mathbb{C}\setminus\{0\}$

then f has a cont^n arg

Claim II $\implies$ I

$\boxed{\text{proof}}$ $\tilde{\gamma} : [a,b] \times [0,1] \longrightarrow \mathbb{C} \setminus \{0\}$

$$\tilde{\gamma}(t,s) \longrightarrow \gamma(t)$$

$\implies \tilde{\gamma}$ is continuous (γ is continuous)

Also,
$$\tilde{\gamma}(t,s) = |\tilde{\gamma}(t,s)|\, e^{\tilde{\theta}(t,s)}$$

$\tilde{\gamma}$ is cont^n $\Rightarrow$ $\tilde{\theta}(t,s)$ is cont^n

Put $s=0$ {Assumption}

$$\gamma(t) = |\gamma(t)|\, e^{i\theta(t)}$$

where $\theta(t) = \tilde{\theta}(t,0)$ continuous

$$|\gamma(t)| = |\tilde{\gamma}(t,0)|$$

Hence, $\theta(t)$ is continuous $\Rightarrow$ γ has a continuous arguement

① $(\theta(b) - \theta(a))/2\pi$ doesn't depend on $\bar{\theta}$ if γ is closed & cont^n

Let θ be a cts arg and γ is closed

i.e $\gamma(a) = \gamma(b)$

i.e $\gamma(a) = \underbrace{|\gamma(a)|}\, \underline{e^{i\theta(a)}} = \underbrace{|\gamma(b)|}\, e^{i\theta(b)}$

i.e $\dfrac{\theta(b)-\theta(a)}{2\pi} \in \mathbb{Z}$

Let θ_1 be another cont^n arg, $\exists k_{\theta\theta_1} \ni$ $\theta_1 = \theta + 2\pi k_{\theta\theta_1}$

$$\Rightarrow \frac{\theta_1(b)-\theta_1(a)}{2\pi} = \frac{\theta(b)-\theta(a)}{2\pi}$$

i.e It doesn't depend on choice of θ

2. $\gamma : [a,b] \xrightarrow[\text{closed}]{\text{cont}^n} \mathbb{C}$ & $z_0 \notin \gamma^*$ (Range of γ)

Then, $\gamma_1(t) = \gamma(t) - z_0$ is a closed curve not passing through 0. Let θ be a cts arg of γ_1 then $\dfrac{\theta(b)-\theta(a)}{2\pi} \in \mathbb{Z}$ independent on θ.

$$\underbrace{\frac{\theta(b)-\theta(a)}{2\pi}}$$

= no. of revolutions completed around z_0 by γ = Net change in angle/arg

= Index of z_0 w.r.t γ

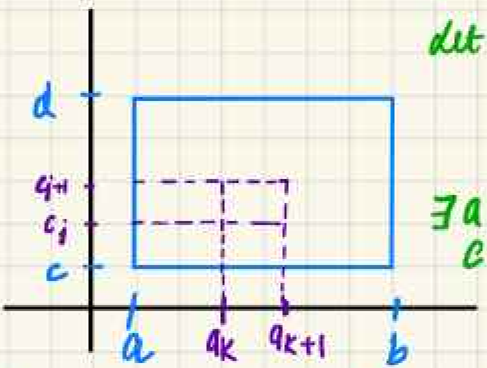
= winding number of γ w.r.t z_0

$$\boxed{\text{Notation:-}\quad \frac{\theta(b)-\theta(a)}{2\pi} = \operatorname{Ind}_\gamma(z_0)}$$

θ is cts arg of $\gamma_1(t) = \gamma(t) - z_0$

proof of (II)

As $[a,b]\times[c,d]$ is compact, f is uniformly continuous



Let $\varepsilon = \min\{|f(x,y)| : (x,y)\in[a,b]\times[c,d]\}$

$\exists\ a=a_0<a_1\cdots<b=a_n$

$c=c_0<c_1\cdots<d=c_m$

$R_{k,j} : [a_k,a_{k+1}]\times[c_j,c_{j+1}]$

$\ni\ \forall (x,y)\in R_{k,j}$

$|f(x,y)-f(a_k,c_j)|<\varepsilon$

[uniformly cts $\Rightarrow \exists\,\delta \ni \|(x,y)-(h,k)\|<\delta \Rightarrow |f(x,y)-f(h,k)|<\varepsilon$]

$\Rightarrow\ f(x,y)\in D(f(a_k,c_j);\varepsilon)$ $\forall (x,y)\in R_{k,j}$

Also,

① — $0\notin D(f(a_k,c_j);\varepsilon)$

[as if $0\in$ ✓ $|f(a_k,c_j)|<\varepsilon$]

Now, ① $\Rightarrow$ $f|_{R_{k,j}} : R_{k,j}\longrightarrow D(f(a_k,c_j);\varepsilon)$ must have a cts arg.

Summary: we sliced domain so f admits a cts arg in each slice.

Now, $R_{0j}\cap R_{1j} = \{a_1\}\times[c_j,c_{j+1}]$

$\theta_{0j}-\theta_{1j} = 2\pi l$ for some $l\in\mathbb{Z}$

[Because, they have a common domain
So, they have to give same value on this domain]

$\Rightarrow$ ✓.

we replace, $\theta_{ij} \longrightarrow \theta_{ij} + 2\pi l$ { no diff, still cts + $\in A(R_{ij})$ }

now we have a cts
arguement over $R_{0j} \cup R_{ij}$

Extend this arguement, finitely many times

<u>Hence Proved.</u>

Riemann Integration

$f : [a,b] \longrightarrow \mathbb{C}$

$f = u + iv$ then, $\int f = \int u + iv$

$$= \boxed{\int_a^b u + i\int_a^b v}$$

Basic Properties

i) Linearity, $f, g \in R$ then,

$$f+g \in R$$

ii) Fundamental theorem I

$$F(x) = \int_a^x f(x)\,dx \quad , \text{if } f \in R$$

- $F(x)$ is uniformly cts
- If $f(x)$ is cts at 'c' then F is diff at 'c' and $F'(c) = f(c)$

iii) **Fundamental Theorem - II**

If f is differentiable and f' is R-I then

$$\boxed{\int_a^b f' = f(b) - f(a)}$$

iv) <u>Integration by Parts</u>

v) <u>Change of Variables</u>

$$\int_c^d (f \circ h)(s)\, h'(s)\, ds = \int_{h(c)}^{h(d)} f(t)\, dt$$

vi) <u>Triangle Inequality</u>

$f \in \mathcal{R} \implies |f| \in \mathcal{R}$ &

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Path

$\gamma : [a,b] \xrightarrow{\text{cont}} \mathbb{C}$

$\gamma^* : \gamma([a,b]) \subseteq U$ (we say γ is a curve in U)

If γ is piecewise C' i.e $\exists$ a partition

$a = a_0 < a_1 \cdots < a_n = b \ni$

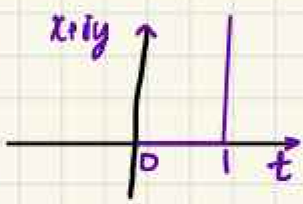
$\gamma|_{[a_j, a_{j+1}]}$ is differentiable and its derivative is cts.

then we call it a path

eg:- $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$

= C^1 curve,

$$\gamma(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 + i(t-1) & 1 \leq t \leq 2 \end{cases}$$



joining 2 curves

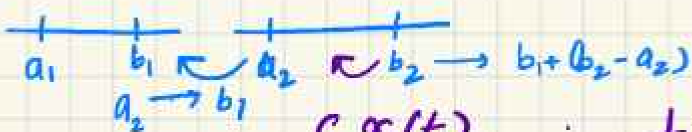
$\gamma_1 : [a_1, b_1] \xrightarrow{\text{cont}} \mathbb{C}$

$\gamma_2 : [a_2, b_2] \xrightarrow{\text{cont}} \mathbb{C}$ $\quad : \gamma_1(b_1) = \gamma_2(a_2)$

we need a cts domain

↓

shift γ_2



$$\underbrace{\gamma(t)}_{\text{cts}} = \begin{cases} \gamma_1(t) & : \quad t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2) & : \quad t \in [b_1, b_1 + (b_2 - a_2)] \end{cases}$$

→ If γ_1 & γ_2 are path then

$(\gamma_1 + \gamma_2) = \gamma$ is also path.

★ Its the mapping and not the image that matters.

Reparameterization

$\gamma : [a,b] \longrightarrow \mathbb{C}$ path,

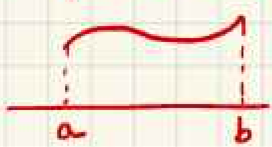
$[c,d] \xrightarrow{\phi} [a,b]$, ϕ is one-one & onto, differentiable map.

(ϕ^{-1} is also diff)

we say, $\gamma \circ \phi$ is reparameterization of γ.

eg: $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a path

$\tilde{\gamma} : \gamma(a+b-t)$; $t \in [a,b]$



opposite path

Remember Notation

$$\int_\gamma f = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$\int f(z)\,dz$ via path by $\gamma(t)$

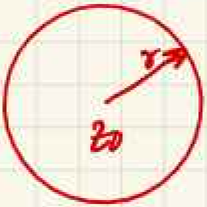
$z \longrightarrow \gamma(t)$

$\Downarrow$ change of variable

$\int_a^b f(\gamma)\,\gamma'(t)\,dt$

Examples :- Circle i.e $\gamma(t) = z_0 + r e^{it}$, $t \in [0, 2\pi)$

$= C(z_0, r)$



$$\int_{C(z_0,r)} f(z)\,dz = \int_0^{2\pi} f(z_0 + re^{it})(ire^{it})\,dt$$

$f(z) = z^n, \ n \geq 1$

$$\int_{\gamma_1} f(z)\,dz = \int_0^{2\pi} (e^{it})^n \cdot (ie^{it})\,dt$$

$$= i\int_0^{2\pi} e^{it(n+1)}\,dt$$

$$= \frac{1}{n+1}\Big[\qquad\Big]_0^{2\pi}$$

$$= \underline{\underline{0}}$$

Q what if

-1

$n < 0, \ n \neq -1$

Fundamental Theorem

$$\xrightarrow[\text{holo}]{f} \mathbb{C}$$

$$\gamma : [a,b] \longrightarrow \mathbb{C}$$

If $\exists\, F : U \longrightarrow \mathbb{C} \ni F' = f$ then

* $$\int_\gamma f = F(b) - F(a)$$

* In particular, if γ = closed, $\int_\gamma f = 0$

$$\boxed{\text{FTC Chain Rule Corollary}}$$

$$\underbrace{(F\circ\gamma)'}_{F} = \underbrace{F'(\gamma(t))\,\gamma'(t)}_{f}$$

$$\int_a^b f(\gamma(t))\,\gamma'(t) = F(\gamma(b)) - F(\gamma(a)) \quad \left\{\begin{matrix} F' = f \\ \text{where} \end{matrix}\right\}$$

Note:- $z^{\overbrace{n}^{F'=f}}$, $n \neq -1$

Always admits primitive $\longrightarrow$ $\overbrace{\dfrac{z^{n+1}}{n+1}}^{F}$

for any path $\gamma : [a,b] \longrightarrow \mathbb{C}$

$$\int_\gamma z^n\,dz = \frac{(\gamma(b))^{n+1} - (\gamma(a))^{n+1}}{n+1}$$

Integration over Line Segments

$$[z, w] = (1-t)z + tw \ , \quad t \in [0,1]$$

order important (direction ke liye)

$$\int_{[w,z]} f = \int_0^1 f((1-t)z + tw)(w-z)\,dt$$

$$= (w-z)\int_0^1 f((1-t)z + tw)\,dt$$

Integration over Re-parameterization

$$[c,d] \xrightarrow{\phi} [a,b]$$

$$[a,b] \xrightarrow{\gamma} \mathbb{C} \quad \text{path}$$

$$f : \gamma^* \xrightarrow{\text{cont}} \mathbb{C}$$

$$\int_{\gamma\circ\phi} f = \int_c^d f(\gamma\circ\phi)\,(\gamma\circ\phi)'(t)\,dt$$

$$= \int_c^d f(\gamma\circ\phi)\,\gamma'(\phi(t))\,\phi'(t)\,dt \quad \left[\text{chain Rule}\right]$$

$\boxed{\mathbb{R} \to \mathbb{R}}$

Change of Variables

$$[c,d] \xrightarrow[\text{diff}]{\phi} [a,b] \xrightarrow[\text{cont}]{g} \mathbb{C}$$

Assume $\phi' \in \mathcal{R}$

$$\int_c^d g(\phi(x))\,\phi'(x)\,dx = \int_{\phi(c)}^{\phi(d)} g(y)\,dy$$

Orientation

$\gamma(\phi)$

- **preserving** (ϕ gives $a \to b$)

$$\int_{\gamma \circ \phi} f = \int_\gamma f$$

- **reversing** (ϕ gives $b \to c$)

$$\int_{\gamma \circ \phi} f = -\int_\gamma f$$

How reparametrization affects integral value

In particular

$$\int_\gamma f = -\int_{\tilde{\gamma}} f$$

- $\tilde{\gamma}$ is opposite path of γ, same mapping

Ex: Prove/check $\displaystyle\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

ML inequality

$$\left|\int_\gamma f\right| \leq M\underbrace{\int_a^b |\gamma'(t)|dt}_{L_\gamma}$$

Suppose $\forall t \in [a,b]$,

$|f(\gamma(t))| \leq M$

$\int_a^b |\gamma'(t)|\, dt = L_\gamma$

as γ is compact

f is continuous

$\Rightarrow \gamma^*$ has a maxima and a minima

Index of a Curve

eg:- $f(z) = \frac{1}{z}$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-it} \cdot e^{it} \cdot i \, dt = 2\pi i$$

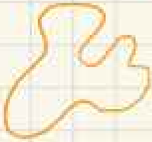
$$\Rightarrow \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{z}\, dz = 1 \xrightarrow[\text{the}]{\text{which is}} \text{Index }(0) \text{ of curve } C(0;1)$$

$\left[\frac{2\pi - 0}{2\pi} = 1\right]$

Theorem: Let $\gamma : [a,b] \xrightarrow[\text{path}]{\text{closed}} \mathbb{C}$. Then

$\forall z \in \mathbb{C} \setminus \gamma^*$, $\quad Ind_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z}$

Proof: Let $z \notin \gamma^*$, consider $d(z, \gamma^*) = \varepsilon > 0$



z

$\overline{\text{a closed} \subseteq \text{of compact}}$

$= \text{compact}$

[to find intervals in $a \to b$]

Claim: $\exists\ a = a_0 < a_1 \cdots < a_n = b$

$\forall j = 0, 1, \cdots, n-1$

$t \in [a_j, a_{j+1}]\ |\gamma(t) - \gamma(a_j)| < \varepsilon$ —— ①

Proof: γ is uniformly cts $\therefore \forall \varepsilon > 0\ \exists \delta > 0 \ni$

$|a_{j+1} - a_j| < \delta \Rightarrow |\gamma(a_{j+1}) - \gamma(a_j)| < \varepsilon$

Hence ✓.

In other words,

$$\gamma([a_j, a_{j+1}]) \in D(\gamma(a_j); \varepsilon)$$

Also, by the choice of ε,

$$z \notin D(\gamma(a_j); \varepsilon)$$

Now,
Define $f: w \longrightarrow w - z$ (holo as polynomial)

$$f(D(\gamma(a_j); \varepsilon)) = D(\gamma(a_j) - z; \varepsilon)$$

[Find a domain over which f has holo log]

$0 \notin D(\gamma(a_j) - z, \varepsilon)$ as $z \notin \gamma^*$

$\Rightarrow f$ has a holo logarithm say g_j

on $D(\gamma(a_j); \varepsilon)$ as $0 \notin \operatorname{Img}(f)$ on D

ie $f(w) = e^{g_j(w)} \quad \forall\, w \in D(\gamma(a_j); \epsilon)$

$$\Rightarrow f'(w) = g_j'(w)\, e^{g_j(w)}$$

$$\Rightarrow \frac{1}{w-z} = \frac{f'(w)}{f(w)} = g_j'(w)$$

Denote $\gamma\big|_{[a_j, a_{j+1}]}$ by γ_j

Now, $$\int_{\gamma_j} \frac{dw}{w-z} = g_j(\gamma_j(a_{j+1})) - g_j(\gamma_j(a_j))$$

[FTC]

define: $\gamma = \sum_{i=1}^{n-1} \gamma_i$.

then using prev. results $$\int_{\gamma} \frac{dw}{w-z} = \sum_{j=0}^{n-1} \int_{\gamma_j} \frac{dw}{w-z}$$

$$= \sum_{j=0}^{n-1} g_j(\gamma_j(a_{j+1})) - g_j(\gamma_j(a_j))$$

[The reason we sliced is cuz we don't have a holo log entire interval.]

$$\sum \operatorname{Re}(g_j(\gamma_j(a_{j+1})) - g_j(\gamma_j(a_j))) + i \sum \operatorname{Im}(\quad)$$

① ②

let $\theta_j = \operatorname{Im}(g_j)$

θ_j is cont^n on all partitions $(\theta_j = \arg)$

(cont log $\Leftrightarrow$ cont arguement)

①

$$= \sum \left(\log |\gamma(a_{j+1}) - z| - \log |\gamma(a_j) - z|\right)$$

$\downarrow$

Telescopic Sum

$$(1) = \log|\gamma(b) - z| - \log|\gamma(a) - z|$$

$$= 0 \qquad \text{[closed path]}$$

② $$\sum_{j=0}^{n-1} \theta_j(\gamma_j(a_{j+1})) - \theta_j(\gamma_j(a_j))$$

$\theta\big|_{[a_j, a_{j+1}]}$ is also cts on $[a_j, a_{j+1}]$ where θ is $\arg(\gamma(t))$.

$\Downarrow$

$$\theta(a_{j+1}) - \theta(a_j) = \theta_j(\gamma_j(a_{j+1})) - \theta_j(\gamma_j(a_j))$$

$\Downarrow$

$$(\text{II}) = 2\pi i \; \theta(b) - \theta(a)$$

$\Rightarrow$ Hence Proved.

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Some Points :-

① $z \longrightarrow \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$ is a cts f^n.

$\mathbb{C}\setminus\gamma^* \longrightarrow \mathbb{Z}$ = $Ind_\gamma(z)$ is a loc const function

In particular,

If S is connected subset of $\mathbb{C}\setminus\gamma^*$ then Ind_γ f^n is cts on S.

② furthermore,

S: unbounded then, $\forall R>0, \exists z \in S \ni \forall w \in \gamma^*$,

$|w-z| > R$

If not then

$\forall z \in S, \exists w_z \in \gamma^* \ni |z-a_z| \leq R$

$\Rightarrow |z| \leq |z-w_z| + |w_z|$

$\leq R + \sup|\gamma(t)|$ i.e bounded $\Rightarrow\Leftarrow$

Now,

$|Ind_\gamma(z)| \leq \frac{1}{2\pi i}\int \left|\frac{1}{|w-z|}\right| dw \leq \frac{1}{2\pi R} l_\gamma$

$= \boxed{\frac{l_\gamma}{2\pi R}}$

$\Rightarrow Ind_\gamma = 0$ on S. {since continuous}

eg:-

★ $C(z_0;r) : t \longrightarrow z_0 + re^{it}$

$$\frac{1}{2\pi i}\int_{C(w_0,z)} \frac{dw}{w-z} = \begin{cases} 1 & : |z-z_0| < r \\ 0 & : |z-z_0| > r \end{cases}$$

* Zeroes of Polynomial inside a Disc

$$P(z) \in \mathbb{C}[z]$$

$$P(z) = \prod_{i=1}^{k} (z-\alpha_i)^{m_i}$$

$$P(z) = (z-\alpha_1)^{m_1} Q(z)$$

$$\frac{P'(z)}{P(z)} = \frac{m_1 (z-\alpha_1)^{m_1-1} Q(z) + Q'(z)(z-\alpha_1)^{m_1}}{Q(z)(z-\alpha_1)^{m_1}}$$

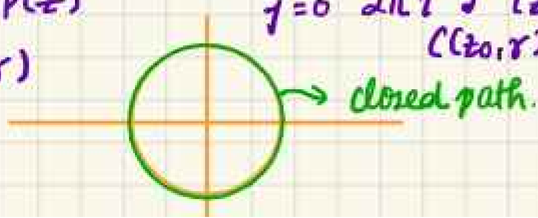
$$= \frac{m_1}{(z-\alpha_1)} + \underbrace{\frac{Q'(z)}{Q(z)}}_{\text{open \& further differentiate}}$$

$$\frac{P'(z)}{P(z)} = \frac{m_1}{(z-\alpha_1)} + \frac{m_2}{(z-\alpha_2)} + \dots + \frac{m_K}{(z-\alpha_K)}$$

$\boxed{\text{Analysis}}$:- Let $D(z_0, r)$ be a disc

If $P(z) \neq 0$ on boundary i.e $|z-z_0| = r$.

$$\frac{1}{2\pi i} \int_{C(z_0,r)} \frac{P'(z)}{P(z)}\, dz = \sum_{j=0}^{k} \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{m_j}{(z-\alpha_j)}\, dz$$



$$\int_{C(z_0,r)} \frac{m_j}{(z-\alpha_j)}\,dz = 1 \text{ or } 0$$ depending if α_j is inside or not.

$\leftarrow$ think as winding powers of α_j

$C(z_0,r)$
$\downarrow$
$z_0 + re^{it}$

$$= \sum_{0 \leq j \leq k} m_j\,(0 \text{ or } 1)$$

= no. of zeros of $P(z)$ inside the Disc, containing multiplicities

A very Important Result

Lemma $\gamma : [a,b] \longrightarrow \mathbb{C}$ be closed path. $z_0 \notin \gamma^*$. Then

$$\operatorname{Ind}_\gamma(z_0) \in \mathbb{Z}$$

Proof:- $g(t) = \int_a^t \underbrace{\frac{\gamma'(s)}{\gamma(s)-z_0}}_{I}\,ds$. Then if the I is cts at t then,

$$g'(t) = \frac{\gamma'(t)}{\gamma(t)-s}$$

Hence,
$$\frac{d}{dt}\left[e^{-g(t)}(\gamma(t)-z_0)\right] = 0$$

whenever $g'(t)$ exists.

$\Rightarrow$ $e^{-g(t)}(\gamma(t)-s)$ is piecewise constant on $[a,b]$

But $e^{-g(t)}(\gamma(t)-s)$ is cts on $[a,b]$

Hence, its constant.

$$\Rightarrow \quad e^{-g(a)}(\gamma(a)-z_0) = e^{-g(b)}(\gamma(b)-z_0)$$

using $\gamma(a) = \gamma(b)$

we get $e^{-g(a)} = e^{-g(b)}$, as $g(a)=0$, $\boxed{e^{-g(b)} = 1}$

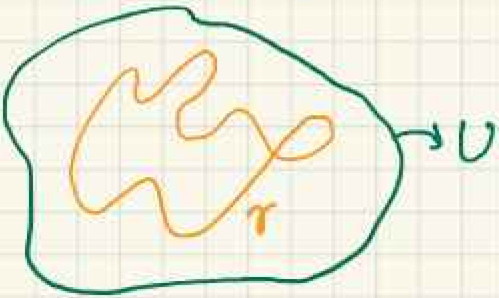
Hence, $g(b) = 2\pi i n$

Check the roots series + Polynomial series $\longrightarrow$ convergence.

Cauchy Theory

$U \underset{\text{open}}{\subseteq} \mathbb{C}$

$\gamma : [a,b] \xrightarrow[\text{closed}]{\text{path}} U$

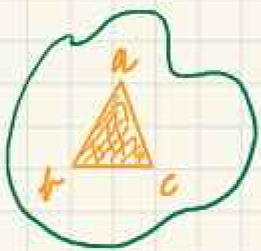


If γ + U satisfy the toprlogical conditions below, $\forall f : U \xrightarrow{\text{holo}} \mathbb{C}$

$$\boxed{\int_\gamma f = 0}$$

Cauchy Theorem for Δs

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, $\Delta \subseteq \mathbb{C}$, $p \in U \to$ Any point



triangle

$\{ t_1 a + t_2 b + t_3 c : t_1, t_2, t_3 \geq 0$ and $\sum_{1 \leq i \leq 3} t_i = 1 \}$

All interior and boundary points included

$f : U \xrightarrow{\text{cont}} \mathbb{C}$ and f is holo on $U \setminus \{p\}$

$$\Rightarrow \int_{\partial\Delta} f = 0$$

Theorem: Continuing, f is forced to be holo at p as well!

Note:- f does not necessarily admit a primitive,

If f has a primitive, $\int_\gamma f = 0$ $\forall$ closed paths γ.

For $[z, w]$, $\gamma(t) = (1-t)z + tw$ (line segment)

$$\int_{[z,w]} f = \int_0^1 f((1-t)z + tw)(w-z)\,dt$$

$$\longrightarrow \underbrace{[z_0, z_1] * [z_1, z_2] * \cdots * [z_{n-1}, z_n]}_{\text{Joining of curves.}}$$

$$\int_{[z_0, z_1, \ldots z_n]} f = \sum_{j=0}^{n-1} \int_{[z_j, z_{j+1}]} f$$

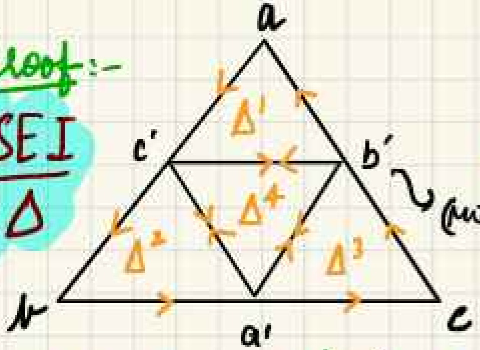
$$\Rightarrow \Delta(a,b,c) \Rightarrow \int_{\partial\Delta(a,b,c)} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f$$

Note:- $\partial\Delta(a,b,c) = \partial\Delta(b,c,a) = \partial\Delta(c,a,b) = -\partial\Delta(a,c,b)$

Here, $\int_{\partial\Delta} f = 0$, so orientation doesn't really matter.

Proof :-

CASE I
$p \notin \Delta$



(mid points of the sides)

$$\int_{\partial\Delta} f = \sum_{j=1}^{4} \int_{\partial\Delta^{(j)}} f$$

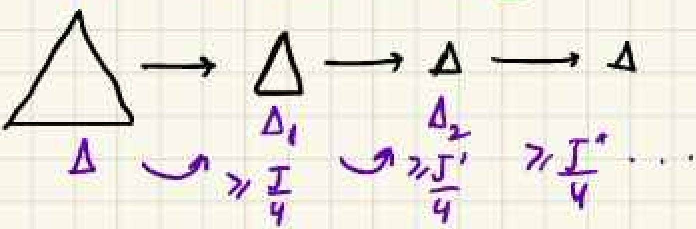
$$\underbrace{\left|\int_{\partial\Delta} f\right|}_{\text{Let } = J} \le \left|\sum_{j=1}^{4}\int_{\partial\Delta^{(j)}} f\right|$$

[exists else $\Rightarrow\Leftarrow$]

$\exists\, j \in \{1,2,3,4\} \ni \left|\int_{\partial\Delta^{(j)}} f\right| \geq \frac{J}{4}$

choose the least $|\int f|$ in this.

Let this $\Delta^{(j)} = \Delta_1$



Keep dividing by 4.

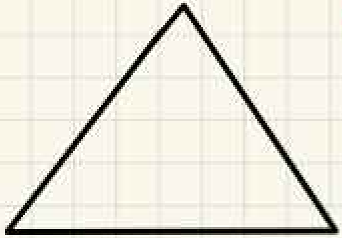
$$\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$$

$$4^2 \left|\int_{\partial\Delta_2} f\right| \geq 4 \left|\int_{\partial\Delta_1} f\right| \geq \underbrace{\left|\int_{\partial\Delta} f\right|}_{= J}$$

Let length of $\partial\Delta$ (perimeter) $= L_\Delta$

$$L_{\Delta_1} = \frac{L_\Delta}{2}$$

$L_{\Delta_2} = \frac{L_\Delta}{4}$ and so on, $\boxed{L_{\Delta_n} = \frac{L_\Delta}{2^n}}$ $\therefore\ 4^n \left|\int_{\partial\Delta_n} f\right| \geq J$

$\Rightarrow$ 

For any 2 points in Δ_n,

$d(x,y) < \text{diam}(\Delta_n)$

[property of diameter]

$\therefore \quad \text{diam}(\Delta_n) \leq L_{\Delta_n} = \dfrac{L_\Delta}{2^n} \Rightarrow \text{diam} \to 0$

$\Rightarrow \exists z_0 \in \Delta \ni \bigcap_{i=1}^{\infty} \Delta_i = \{z_0\}$

[compact sets + diam $\to 0 \Rightarrow$ non ϕ intersection]

as $z_0 \in \Delta \Rightarrow z_0 \neq p \Rightarrow f$ is holo at z_0.

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni |z - z_0| < \delta,$

$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0|$

we are saying after N, every point of Δ_n lies within δ of z_0

Let $N \in \mathbb{N}$ be $\ni \dfrac{L_\Delta}{2^N} < \delta$.

now, $z \in \Delta_N \Rightarrow |z - z_0| \leq \text{diam}\, \Delta_N \leq \dfrac{L_\Delta}{2^N} < \delta$

Hence, $\forall n \geq N, \quad \Delta_n \subseteq D(z_0, \delta)$

$$\left| \int_{\partial \Delta_n} \{ f(z) - f(z_0) - f'(z_0)(z - z_0) \} \, dz \right| \leq \varepsilon L_{\Delta_n}^2$$

① ② ③

[ML inequality]

Also,

② is constant and $\int ③ = 0$

$\int ② = k \int (\partial \Delta_n)' \, dt$

$= 0$

Hence, $\left| \int_{\partial \Delta_n} f(z)\, dz \right| \leq \frac{\varepsilon L_\Delta^2}{4^n}$

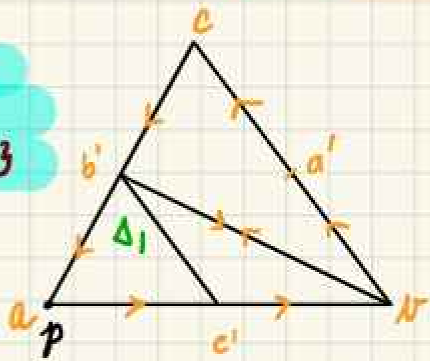
$$4^n \left| \int_{\partial \Delta_n} f(z)\, dz \right| \leq \varepsilon L_\Delta^2$$

Hence, $J \leq \varepsilon L_\Delta^2$

as ε is arbitrary, $\boxed{J = 0}$

CASE II

$p \in \{a, b, c\}$

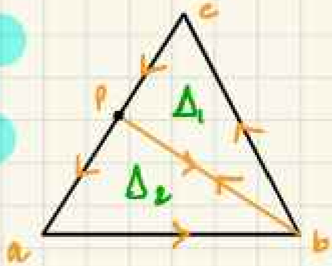


$\int_{\partial \Delta} f = \int_{\partial \Delta_1} f$ (as rest Δ nrto every where → Case-I Hence Done)

We keep reducing the Δs further in $\Delta_1 \to \Delta_2 \ldots$ containing p.

$\int_{\partial \Delta} f = 0$ as diameter $\longrightarrow 0$

CASE III
$p \in$ edge

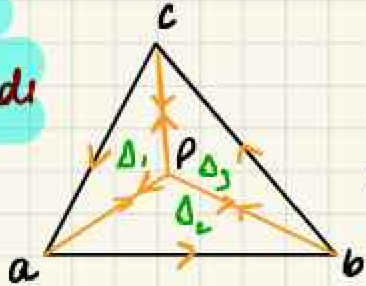


from Case II,

$$\int_{\partial\Delta_1} f = 0 = \int_{\partial\Delta_2} f = 0$$

Hence Proved.

Case IV
p lies inside



$$\int_{\partial\Delta} f = \int_{\partial\Delta_1} f + \int_{\partial\Delta_2} f + \int_{\partial\Delta_3} f$$
$$= 0$$

Cauchy Theorem for Open Convex Region

⊛ $U \underset{\text{open}}{\subseteq} \mathbb{C}$, assume U is convex

i.e. $[z_1, z_2] = \{(1-t)z_1 + tz_2 : 0 \le t \le 1\} \subseteq U$

Conditions

1. $p \in U \subseteq \mathbb{C}$, U is convex and open.
2. $f : U \xrightarrow{\text{cts}} \mathbb{C}$, holo at every point of $U \setminus \{p\}$
3. $\gamma : [a,b] \longrightarrow U$ (closed path)

then $\boxed{\exists F \in H(U) \ni F' = f} \Rightarrow$

f admits primitive and

$$\int_\gamma f = 0$$

③ Region = open connected set.

Proof:- $z_0 \in U$, $F(z) = \int_{[z_0,z]} f$

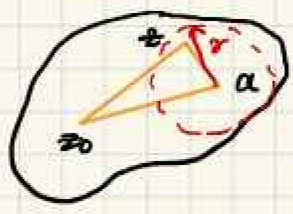
Claim: $F(z)$ is primitive for f

to prove:- F is diff & $F' = f$.

Let $a \in U$, $\exists r > 0 \ni D(a,r) \subseteq U$. $\forall z \in D(a;r)$

$$(1) := \frac{F(z) - F(a)}{z-a} - f(a)$$

$$= \frac{\int_{z_0,z} f - \int_{z_0,a} f}{z-a} - f(a)$$



using Cauchy's theorem for a Δ

$$\int_{[a,z]} f + \int_{[z,z_0]} f + \int_{[z_0,a]} f = 0$$

Tarun Goyal

$$\Rightarrow \quad (1) := \frac{\int_{[a,z]} f}{z-a} - f(a)$$

we can write $f(a) = \dfrac{\int_{[a,z]} f(a)\,dz}{z-a}$

$$(1) := \frac{\int_{[a,z]} (f(z) - f(a))\,dz}{z-a}$$

Now, $f(z)$ is cts at a
$\Rightarrow \exists\ \delta \ni |z-a| < \delta$
$\Rightarrow |f(z) - f(a)| < \frac{\varepsilon}{2}$

very important

$$|(1)| = \frac{1}{|z-a|} \left| \int_{[a,z]} f(a) - f(z) \right|$$

use of ML inequality

$\boxed{\text{By ML inequality}}$

$$|(1)| < \frac{1}{|z-a|} |z-a| \cdot \frac{\varepsilon}{2} < \varepsilon$$

Hence, F is differentiable and $F'(z) = f(z)$

<u>use of convexity:-</u>

If z_0 can be joined to any other point of U
$\Rightarrow$ line joining them $\subseteq U$.

Def:- U is star-shaped if $\exists\ z_0 \in U \ni$
$\forall z \in U, \quad [z_0, z] \subseteq U$

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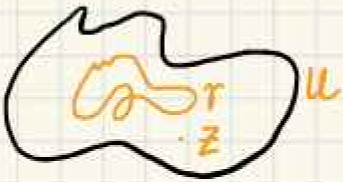
CAUCHY INTEGRAL FORMULA

① $U \underset{\text{open}}{\subseteq} \mathbb{C}$, convex ② $f \in H(U)$, $\gamma : [a,b] \xrightarrow[\text{path}]{\text{closed}} U$

let $z \in U / \gamma^*$.



$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & : w \neq z \\ f'(w) & : w = z \end{cases}$$

clearly, g is holo at every $w \neq z$, g is cts at $w = z$. $\lim_{w \to z} g(w) = g(z)$. We can apply cauchy's theorem for open convex Regions.

$$\int_\gamma g = 0$$

$$\Rightarrow \int_\gamma \frac{f(w) - f(z)}{w - z} = 0 \quad \{z \notin \gamma^*\}$$

$$= \frac{1}{2\pi i} \int_\gamma (\quad) = 0 \Rightarrow \frac{1}{2\pi i} \int \frac{f(w)dw}{w - z} = \frac{1}{2\pi i} \int \frac{\overset{\text{constant}}{f(z)}\, dw}{w - z}$$

Hence,

$$\boxed{\operatorname{Ind}_\gamma(z)\cdot f(z) = \frac{1}{2\pi i}\int \frac{f(\omega)\,d\omega}{\omega - z}}$$

uses:-

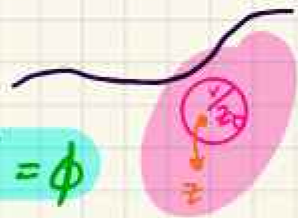
1. Index $f(z)$ express as an Integral
2. Helps in directly calculating integrals of this form.

$\boxed{RHS}$

$$\mathbb{C}/\gamma^* \longrightarrow \mathbb{C}$$

$$z \longrightarrow \frac{1}{2\pi i}\int_a^b \frac{f(\gamma(t))\,\gamma'(t)}{\gamma(t) - z}\,dt$$

take $z_0 \in U/\gamma^*$; $D(z_0; r) \cap \gamma^* = \phi$



Then, $\forall\ z \in D(z_0; r)$

$$\frac{1}{2\pi i}\int_a^b \frac{f(\gamma(t)) \times \gamma'(t)}{\gamma(t) - z}\,dt = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_a^b \frac{f(\gamma(t))\gamma'(t)\,dt}{(\gamma(t) - z_0)^{n+1}}(z - z_0)^n$$

[write $\dfrac{1}{\gamma(t) - z} = \displaystyle\sum_{n=0}^{\infty}\frac{1}{(\gamma(t) - z_0)^{n+1}}(z - z_0)^n$, G.P because norm < 1]

$$= \boxed{\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)\,d\omega}{(\omega - z_0)^{n+1}}\right)(z - z_0)^n}$$

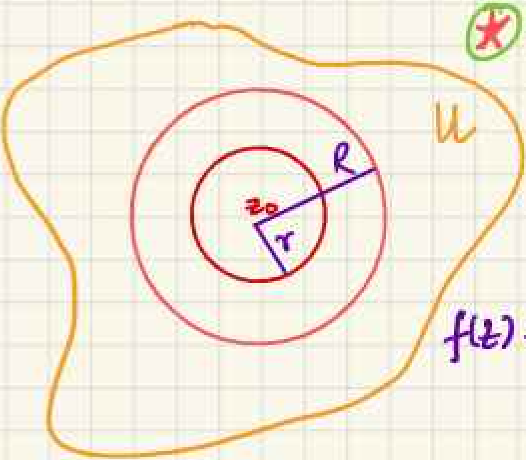
just need a holo f^n and a z $\ni$ z doesn't belong to curve $\Rightarrow$ Expand around z_0 for $f(z)$. find z_0 using z

Some extra analysis

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, $f \in H(U)$, $z_0 \in U \Rightarrow \exists\ R > 0 \ni D(z_0, R) \subseteq U$.

U_{new} which is convex $\Rightarrow$ Cauchy's ~~Integral formula~~ in it

Let $r \in (0, R)$ [opposite Analysis]



fix $z \in D(z_0, r)$ {as Index = 1 ⇒ $z \notin \gamma^*$ }

if $\gamma = C(z_0; r) \Rightarrow D(z_0; r) \cap \gamma^* = \phi$

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0; r)} \frac{f(w)\,dw}{w - z} \quad \left[\begin{array}{l}\text{Index} = 1 \\ \forall z \in D(\,)\end{array}\right]$$

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C(z_0; r)} \frac{f(w)\,dw}{(w - z_0)^{n+1}} \right) (z - z_0)^n$$

Hence, Inside this Disc, $D(z_0; r)$, f can be expressed as a power series, centered at z_0 (S∑XY!)

$\therefore$ f is analytic

f is holo $\Leftrightarrow$ f is analytic

(locally speaking)

Corollary:- If f is holo, then f', f'' ... all are holo

i.e. f^n is ∞ differentiable

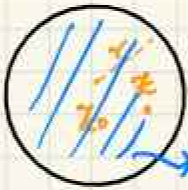
Eg:- $f(x) = \begin{cases} e^{-1/x^2} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

f is ∞ differentiable

$\Rightarrow$

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})(\cancel{r}ie^{it})}{(\cancel{z_0} + r\cancel{e^{it}}) - \cancel{z_0}}\,dt$$

general



curve

Area of Interest

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

= Mean Value property

i.e Value of function at centre of circle,

= average of function along boundary of circle

Continuing the Analytic representation

$$\boxed{\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f(w)}{(w - z_0)^{n+1}}\,dw} \quad (*)$$

n^{th} — coefficient of power series about

Remark: (*) holds $\forall\ z \in D(z_0;r)$

on circle

Cauchy's estimate

Assume, $\exists\, M > 0 \ni \forall\, |w - z_0| = r$, $|f(w)| \le M$.
Then from ML inequality,

$$|f^{(n)}(z_0)| \le n! \frac{1}{2\pi} \frac{M}{r^{n+1}} (2\pi r)$$

i.e $\boxed{|f^{(n)}(z_0)| \le n! \frac{M}{r^n}}$ ← very powerful

Liouville's Theorem

$\underbrace{f \in H(\mathbb{C})}_{:=\ f \text{ is entire}}$ & f is bounded $\Rightarrow$ f is constant

Observe

(*) doesn't change with r.

i.e $\forall\, r \in (0, R)$, $z \in D(z_0; r)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

basically $\forall\, z \in$ disc around z_0.

$$a_n = \frac{f^n(z_0)}{n!}$$

Morera's Theorem

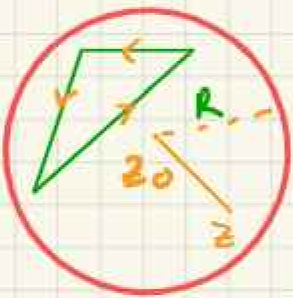
Let, $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $f : U \xrightarrow{\text{cts}} \mathbb{C}$

Assume $\forall \Delta \subseteq U$, $\displaystyle\int_{\partial\Delta} f = 0$. Then $f \in H(U)$.

proof :-

Let $R > 0$ be $\ni$ $D(z_0; R) \subseteq U$.

Note :- $D(z_0; R)$ is open and convex.



$$F(z) = \int_{[z_0, z]} f \quad , \forall z \in D(z_0; R)$$

F is primitive of f.

[see earlier theorem and use a similar proof]

Hence, F is holomorphic

$\Downarrow$

[complex differentiable is holomorphic]

F' is holomorphic

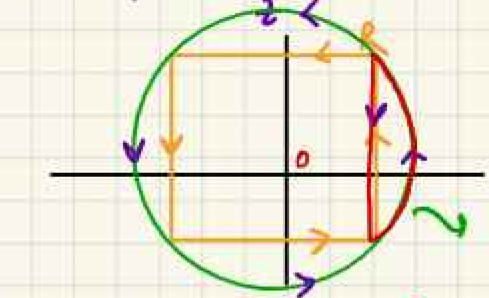
i.e $F' = f$ = holomorphic

$\Rightarrow f \in H(U)$

Hence proved

Applications

1. $f(z) = \frac{1}{z}$, $z \in \mathbb{C}\setminus\{0\}$



Q: $\int_{\partial R} f$

consider circumcircle

$$\int_{D} f = 0$$ (By Cauchy's Theorem as you get a disc that contains this and avoids origin.)

$$\Rightarrow \int_{\downarrow} f = \int_{\uparrow} f$$ and so on.

$$\Rightarrow \int_{\partial R} f = \int_{C(0;r)} f = 2\pi i$$

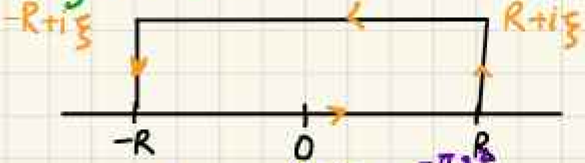
we shifted the contour of the funct.

2. $\forall\ \xi \in \mathbb{R}$, $e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} \, dx$ — (**)

$\Rightarrow$ (i) $\xi = 0$, ** becomes,

$$1 = \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2} e^{2\pi i \pi \xi}}_{\text{use } \Gamma(1/2) = \sqrt{\pi}} dx$$

(ii) $\xi > 0$,



$t \in [0, \xi]$

$f(z) = e^{-\pi z^2}$

$$0 = \int_{-R}^{R} e^{-\pi x^2}\,dx + \int_{0}^{\xi} e^{-\pi(R^2 + 2iRt - t^2)}\, i\,dt$$

$$+ \int_{R}^{-R} e^{-\pi(x+i\xi)^2}\,dx$$

$$+ \int \cdots$$

Cauchy's Theorem

as $R \to \infty$ is RHS (almost)

2nd term:-
+ 4th term:-

$$\left| \int_{0}^{\xi} e^{-\pi(R^2 - t^2 + 2itR)}\, i\,dt \right| \leq \underbrace{\frac{e^{\pi t^2}}{e^{\pi R^2}}}_{\longrightarrow 0} \xi i$$

as $R \to 0$

3rd term:-

$$\int_{-R}^{R} e^{-\pi(x^2 - \xi^2 + 2xi\xi)}\,dx$$

$$= e^{\pi\xi^2} \underbrace{\int_{-R}^{R} e^{-\pi x^2} \cdot e^{-2\pi i x\xi}\,dx}$$

as $R \to 0$,

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} \cdots$$

Illy for $\xi < 0$

Hence Proved

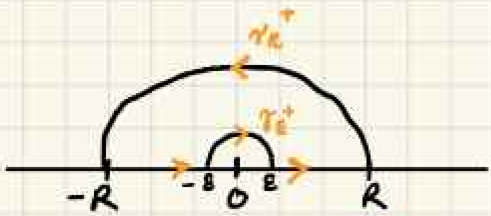
3. $$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi$$

Because symmetric function about 0.

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$$

Relate with a contour

convert in complex numbers



, $R > 0$, $\varepsilon > 0$

$$f(z) = \frac{1 - e^{iz}}{z^2}, \quad z \in \mathbb{C}\setminus\{0\}$$

move along the curve

Remarks

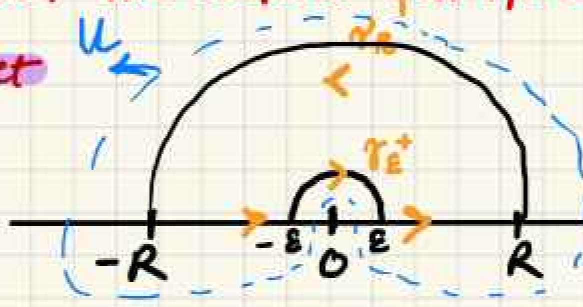
★ to apply Cauchy's Theorem for open convex sets-

→ f = cts everywhere, holo except at most 1 point

✓ → closed curve

✓ → star-like set

for this we exclude 0.



Therefore, $\displaystyle\int \frac{1-e^{ix}}{x^2}\,dx = 0$

$$= \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0 \quad \text{①}$$

→ Now we want to use ML inequality

pre computation

$z = x+iy,\ y \geq 0$

$|e^{iz}| = |e^{-y+ix}| = e^{-y} \leq 1$

$$\Rightarrow \frac{|1-e^{iz}|}{|z^2|} \overset{\Delta\text{ ineq.}}{\leq} \frac{2}{|z|^2}$$

$$\left| \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz \right| \leq \frac{2}{R^2}(\pi R) \longrightarrow 0 \quad (R \to \infty)$$

expansion

for $\gamma_\varepsilon^+ :\to \quad 1-e^{iz} = -iz - \frac{(iz)^2}{2!} - \frac{(iz)^3}{3!} \cdots$

$$\frac{1-e^{iz}}{z^2} = \frac{-i}{z} - \left[\frac{i^2}{2!} + \frac{i^3 z}{3!} + \cdots\right]$$

Let $F(z) = \frac{i}{2!} + \frac{i^3 z}{3!}$ $\quad \forall z \in \mathbb{C}$

$\forall\ |z| \leq 1,\quad |E(z)| \leq \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \cdots < 1$

as $\quad \leq \frac{1}{2^2} + \frac{1}{2^2} + \cdots \longrightarrow 1$

when $z \neq 0$, $\dfrac{1-e^{iz}}{z^2} = \dfrac{-i}{z} + E(z)$

$$\int_{\gamma_\varepsilon^+} (\quad) = -i \int_{\gamma_\varepsilon^+} \frac{dz}{z} + \int_{\gamma_\varepsilon^+} E(z)\,dz$$

$\left|\int\right| \leq 1 \cdot \pi\varepsilon \longrightarrow 0$, $\forall \varepsilon \in (0,1)$

$$\int_{\gamma_\varepsilon^+} \frac{dz}{z} = \int_0^\pi \frac{i\varepsilon \cdot i e^{i(\pi-t)}}{\varepsilon e^{i(\pi-t)}}\,dt$$

$= \pi i$, $\forall \varepsilon > 0$

Hence, $\lim\limits_{\varepsilon \to 0}$ RHS $= \pi$

in ①, $\lim\limits_{\varepsilon \to 0} \displaystyle\int_{|x| \geq \varepsilon} = -\int_{\gamma_\varepsilon^+} (\quad)$

$= \pi$

i.e $\displaystyle\int_{-\infty}^{\infty} \frac{1-e^{it}}{t^2}\,dt = \pi$, taking Real part,

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi$$

Hence Proved

Equicontinuity

$\mathcal{F}$ - family of functions, $f : X \longrightarrow \mathbb{C}$. fix $x_0 \in X$.
We say $\mathcal{F}$ is equicontinuous at $x_0 \in X$, if $\forall \epsilon > 0$ $\exists \delta > 0 \ni d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$, $\forall f \in \mathcal{F}$

Theorem Let $\underset{\text{open}}{U \subseteq \mathbb{C}}$, $\mathcal{F} \subseteq H(U)$. Assume $\mathcal{F}$ is uniformly bounded on each $\underset{\text{compact}}{K \subseteq U}$ (i.e, $\forall \underset{\text{compact}}{K \subseteq U}$, $\exists M > 0 \ni \forall z \in K, f \in \mathcal{F}, |f(z)| \leq M$)

Then, $\mathcal{F}$ is equicontinuous.

Proof:

$D(z_0; R) \subseteq U$, $0 < r < R$

$\Rightarrow \overline{D(z_0; R)} \subseteq U$.

Let $f \in \mathcal{F}$, using Cauchy's Integral formula,

① — $f(z) - f(z_0) \overset{\text{CIF}}{=} \frac{1}{2\pi i} \int_{C(z_0; r)} f(w) \frac{z - z_0}{(w - z)(w - z_0)} \, dw$

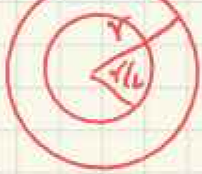
Let $M > 0 \ni \forall f \in \mathcal{F}$ & $|w - z_0| = r$, $|f(w)| \leq M$ (uniformly bounded) use CIF for these

①
$\Rightarrow |f(z) - f(z_0)| \leq \frac{1}{2\pi} M \cdot |z - z_0| \cdots$

Now, $\left| \frac{1}{(w - z)(w - z_0)} \right| = \frac{1}{|w - z| r}$

to have a bound for $|w-z|$, z can't be very close to w.

$\Rightarrow$



If $z \in D(z_0, r/2)$. Then $\forall f \in \mathcal{F}$

$$|f(z)-f(z_0)| \le \frac{M}{2\pi}|z-z_0| \cdot \frac{2}{r} \cdot \frac{1}{r}$$

$$= \underbrace{\frac{M}{\pi r^2}|z-z_0|}_{\text{independent of } z}$$

given $\varepsilon > 0$, clearly, we can choose $\delta < \frac{r}{2}$ and $\delta < \frac{\pi r^2 \varepsilon}{M}$ and then

$$|f(z)-f(z_0)| < \varepsilon$$

$\forall f \in \mathcal{F}$

<u>Hence equicontinuity proved</u>

Remark:- This was an application of Cauchy's Integral Formula

Applications of Morera's Theorem

★ Cauchy's Theorem for a Δ : $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $p \in U$, $f : U \xrightarrow{\text{cts}} \mathbb{C}$, $f \in H(U \setminus \{p\})$

$$\Rightarrow \underbrace{\int_{\partial\Delta} f = 0}$$

$\Rightarrow$ Morera's Theorem $\Rightarrow f \in H(U)$
In particular f is holo at p.

1. $U \underset{\text{open}}{\subseteq} \mathbb{C}$. $\forall n \in \mathbb{N}$, $f_n \in H(U)$. Let $f_n \xrightarrow[n\to\infty]{} f$ pt.wise where $f : U \longrightarrow \mathbb{C}$.

Assume $f_n \longrightarrow f$ uniformly on each $K \underset{\text{compact}}{\subseteq} U$
[i.e almost uniform convergence]

Then $f \in H(U)$. Also, $f_n^{(k)} \xrightarrow{n\to\infty} f^{(k)}$, almost uniformly

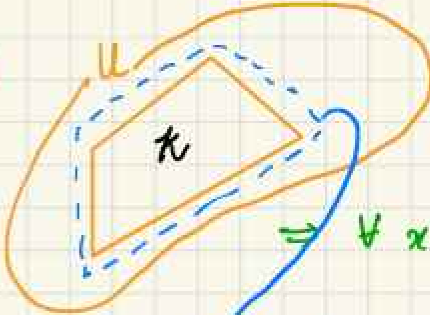
$\Rightarrow$ (i) Let $\Delta \subseteq U$, As $\partial\Delta$ is cpt, $f_n \xrightarrow[n\to\infty]{} f$ unif

Hence $\displaystyle\int_\gamma f_n \xrightarrow[n\to\infty]{} \int_\gamma f$

$\Rightarrow \displaystyle\int_{\partial\Delta} f = 0 \Rightarrow$ using Morera's theorem ✓.

[exercise:-
$f_n : \gamma^* \xrightarrow{\text{cts}} \mathbb{C}$
$f_n \longrightarrow f$ uniformly
$\Rightarrow f$ is cts &
$\underbrace{\displaystyle\int_\gamma f_n \xrightarrow[n\to\infty]{} \int_\gamma f}$
true even if f_n is not cts.]

(ii)



$\varepsilon := d(K, \mathbb{C}\setminus U) > 0$

$\quad = \inf\{|k-w| : k\in K,\ w\in \mathbb{C}\setminus U\}$

$\Rightarrow \forall\, x\in K,\ D(x;\varepsilon)\subseteq U$ (o/w $y\notin U \ni y\in D(x;\varepsilon)$ $\Rightarrow\Leftarrow$)

pre requisite

Now, $0<r<\varepsilon$, $\forall\, x\in K$, $\overline{D}(x;r)\subseteq D(x;\varepsilon)\subseteq U$

$C := \{z\in\mathbb{C} : d(z,K)\le r\}$

C is compact
- $K\subseteq C$
- C is closed, as d is cts
- C is bounded

because

$\exists\, M>0 \ni \forall\, x\in K,\ |x|\le M$.

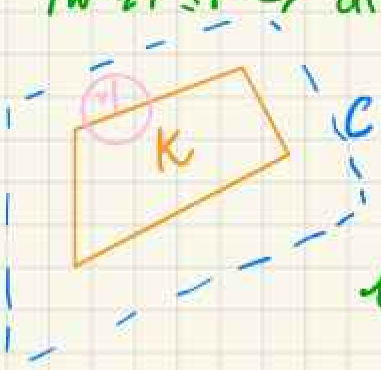
Let $z\in C$, Since K is compact $\exists\, k\in K \ni d(z,K) = |z-k|$

$|z|\le |z-k|+|k| \le d(z,K)+|k|$
$\le r+M\le \varepsilon+M$

This C is called thickening of K

Let $z\in K$. Clearly, $\overline{D(z;r)}\subseteq C$ because if

$|w-z|\le r \Rightarrow d(w,K)\le |w-z|\le r$



Since, C is cpt,

$f_n \longrightarrow f$ uniformly on C. (Assumptions)

Use Cauchy's estimate on $C(z;r)$

we get, $|f_n'(z) - f'(z)| = |(f_n - f)'(z)|$

$\leq \frac{1}{r} \sup_C |f_n - f|$, $\forall z \in K$ [Bounded because C is bounded]

as $f_n \longrightarrow f$ uniformly,

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n \geq N, w \in C$,

$$|f_n(\omega) - f(\omega)| \leq \frac{\varepsilon}{2r}$$

$\Rightarrow \forall n \geq N$, $\sup_C |f_n - f| \leq \frac{r\varepsilon}{2} < r\varepsilon$

$\Rightarrow \forall n \geq N$, $|f_n'(z) - f'(z)| < \varepsilon$, $\forall z \in K$

Hence, converges uniformly almost.

Extend this finitely to K.

$\boxed{\text{Crux}}$

1. Cauchy estimate applicable :- $f \in H(U)$, U is convex and Bounded over a circle

for this we used thickening of compact set

The convergence of power series and their derivatives is a special case of this.

2. $f(z) = \int_\gamma \frac{f(t)}{\gamma(t) - z}\, dt$, $z \in \mathbb{C} \setminus \gamma^*$ is holo.

Leibniz Rule

General version:-

2 variable, 1 variable $\int$

$$f(z) = \int_a^b \underbrace{f(z,t)}_{\text{cts}}\, dt, \ \forall z \in U.$$

* $f : U \times [a,b] \xrightarrow{\text{cts}} \mathbb{C}$, $U \underset{\text{open}}{\subseteq} \mathbb{C}$
* f is holo in z ($\forall$ fixed t, $z \longrightarrow f(z,t)$ is holo.)
* $U \times [a,b]$ has sup metric i.e. $d((z_1,t_1),(z_2,t_2)) = \max\{|z_1 - z_2|, |t_1 - t_2|\}$

Then, f is holo and holo at z, fix t

$\forall z \in U$,

$$\boxed{f'(z) = \int_a^b \frac{\partial f}{\partial z}(z,t)\, dt}$$

uses:- consider $f(z)$, holo, γ be a path, $z_0 \in \mathbb{C} \setminus \gamma^*$

$$\int_a^b \frac{f(\gamma(t))\, \gamma'(t)}{\gamma(t) - z_0} \equiv \int_a^b \frac{h(t)}{\gamma(t) - z_0}$$ is holo

This is called

Liebniz Rule of diff under Integation

take $f(z,t) = \frac{f(t)}{\gamma(t) - z}$

clearly, $z \in \mathbb{C} \setminus \gamma^*$

$\Rightarrow$ holo

$\Rightarrow$ use Liebniz Rule

Proof:- $\Delta \subseteq U$ $\forall t \in [a,b]$,

Cauchy theorem for $\Delta \Rightarrow \int_{\partial\Delta} f(z,t)\,dz = 0$

$$\Rightarrow \int_a^b \int_{\partial\Delta} f(z,t)\,dz\,dt = 0 \xrightarrow[\text{order}]{\text{change of}} \int_{\partial\Delta} \int_a^b f(z,t)\,dt\,dz = 0$$

①

Hence, $\int_{\partial\Delta} f(z)\,dz = 0$ $\Rightarrow$ f is holo using Morera's theorem.

Another Proof

Recall :- $\varphi : [a,b] \xrightarrow{\text{cts}} \mathbb{R}$,

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n-1} \varphi\left(a + r\frac{b-a}{n}\right) = \int_a^b \varphi$$

$$f_n(z) = \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(z, a + r\frac{b-a}{n}\right), \quad \forall z \in U$$

Clearly, $\forall z \in U$, $f_n(z) \xrightarrow[n\to\infty]{} f(z)$

To show, $f_n(z) \longrightarrow f(z)$ a.u.

Let $K \underset{\text{cpt}}{\subseteq} U$. Since f is cts on $K\times[a,b]$, f is uniformly cts on it.

Let $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$, $\forall n \geq N$ & $j = 0, 1, \dots, n-1$,

as then $|z-s|<\delta$ $\Big|$ $z, s \in \left[a + j\frac{b-a}{n}, a + (j+1)\frac{b-a}{n}\right]$

f is uniformly cts $\Rightarrow$ $|f(z,t) - f(z,s)| < \varepsilon/2$

as N avoids δ. Suppose $z \in K$

$$|F_n(z) - F(z)| = \left| \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(z, r\cdot\frac{b-a}{n}\right) - \int_a^b f(z,t)\,dt \right|$$

$$\leq \left| \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(z, r\frac{b-a}{n}\right) - \sum_{r=0}^{n-1} \int_{a+r\frac{b-a}{n}}^{a+(r+1)\frac{b-a}{n}} f(z,t)\,dt \right|$$

$$= \left| \sum_{r=0}^{n-1} \left(\int_{a+r\frac{b-a}{n}}^{a+(r+1)\frac{b-a}{n}} f\left(z, r\cdot\frac{b-a}{n}\right) dt - \sum{}'' \right) \right|$$

$$= \left| \sum_{r=0}^{n-1} \int_{a+r\frac{(b-a)}{n}}^{a+(r+1)\frac{b-a}{n}} \left(f\left(z, a+r\cdot\frac{b-a}{n}\right) - f(z,t)\right) dt \right|$$

outside then

$$\leq \sum_{r=0}^{n-1} \varepsilon \frac{(b-a)}{n} = \boxed{\varepsilon(b-a)}$$

[$r\frac{b-a}{n}$ and t are in same interval for]

Hence $f(z) \longrightarrow f_n(z)$ a.u

Hence $f(z)$ is holomorphic

Also, seq. of derivatives also converges.

$$f_n'(z) = \frac{b-a}{n} \sum_{r=0}^{n-1} \frac{\partial f}{\partial z}\left(z, a + r\cdot\frac{b-a}{n}\right)$$

$$\xrightarrow{n \to \infty} \int_a^b \frac{\partial f}{\partial z}(z,t)\,dt$$

Liouville's Theorem :- $f \in H(\mathbb{C})$ & bdd $\Rightarrow f$ is constant

Fundamental theorem of Algebra

$P(z) \in \mathbb{C}[z]$, $\deg P(z) \neq 0 \Rightarrow P(z)$ has a zero.

Proof:- Assume contrary,

$$f(z) = \frac{1}{P(z)}, \ z \in \mathbb{C} \text{ is entire}$$

We show $\exists\, M \geq 0 \ni |P(z)| \geq M$. W.L.G assume P is monic, i.e $P(z) = z^d + a_{d-1}z^{d-1} \cdots + a_0$

$$\Rightarrow |P(z)| \geq |z|^d \left(1 - \left|\frac{a_{d-1}}{z} + \cdots + \frac{a_0}{z^d}\right|\right)$$

$$\left|\frac{a_{d-1}}{z}+\cdots+\frac{a_0}{z^d}\right| \leq \left|\frac{a_{d-1}}{z}\right|+\cdots+\left|\frac{a_0}{z^d}\right| \quad — (*)$$

$$\leq \left(|a_{d-1}|+\cdots+|a_0|\right)\cdot\frac{1}{|z|}, \text{ whenever } |z|\geq 1$$

$\exists\, N\in\mathbb{N} \ni$, $\left(|a_{d-1}|+\cdots+|a_0|\right)\frac{1}{N} < \frac{1}{2}$

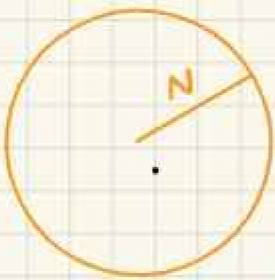
$\Rightarrow$ $|z|\geq N$,

$$\left(|a_{d-1}| \cdots\right)\frac{1}{|z|} \leq \frac{1}{2}$$

$\Rightarrow$ From $(*)$,

$$\left|\qquad\qquad\right| \leq \frac{1}{2}$$

$\Rightarrow$ $\forall\, |z|\geq N$, $|P(z)| > \frac{|z|^d}{2} > \frac{N^d}{2}$

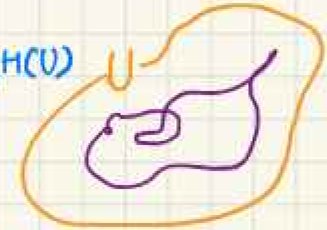


$|P(z)| > \frac{N^d}{2}$. Now as $\overline{D(0,N)}$ is compact, $\exists\, M_1>0 \ni$ $|P(z)|\geq M_1$, $\forall\, z\in\overline{D(0,N)}$

Let $M := \min\left\{\frac{N}{2}, M_1\right\}$. Then $\forall\, z\in\mathbb{C}$, $|P(z)|\geq M$.

GLOBAL CAUCHY THEOREM

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, γ - closed path, $\gamma^* \subseteq U$, $f \in H(U)$

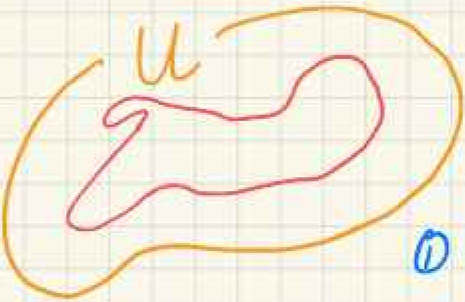


- U is convex $\Rightarrow \int = 0$
- $\Delta \Rightarrow \int = 0$

Condition on γ or U was enough to get Result

Conditions are stringent $\Rightarrow$ Extent of application is less.

New Conditions [Both γ and U]



GLOBAL CAUCHY THEOREM

① $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $\gamma : [a,b] \longrightarrow U$ closed path

② $\forall z \in \mathbb{C} \setminus U$, $\text{Ind}_\gamma(z) = 0$

$\hookrightarrow \gamma \underset{U}{\sim} 0$

Then, at least one (and hence both) of following Equivalent statement holds:-

(C1) $\forall f \in H(U)$, $\int_\gamma f = 0$

(C.2) $\forall f \in H(U)$, $z \in U \setminus \gamma^*$

$$\text{Ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw$$

Proof of Equivalence of $\boxed{C.1}$ and $\boxed{C.2}$

$C.1 \Rightarrow C.2$

$$g(w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & : w \neq z \\ f'(z) & : w = z \end{cases}$$

$\Rightarrow g \in H(U \setminus \{z\})$ + cts on U.

- Let $D(z;r) \subseteq U$. As g is cts on disc & holo on $D(z,r)\setminus\{z\}$. g is holo at z as well.

- from C.1, $\displaystyle\int_\gamma g(w)\,dw = 0 \Rightarrow \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw = 0$ $[z \notin U]$

$\boxed{\text{Hence Done}}$

$C.2 \Rightarrow C.1$

Let $f \in H(U)$. Consider $g(w) = (w-z)f(w)$, $\forall w \in U$.
$[\text{Fix } z \in U \setminus \gamma^*]$

$$\Rightarrow \underset{0}{\underset{\parallel}{g(z)}}\, \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_\gamma f$$

Hence proved.

Tarun Goyal

Proof of C.1

$\Rightarrow$ $g : U \times U \longrightarrow \mathbb{C}$

$$g(w,z) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & : w \neq z \\ f'(z) & : w = z \end{cases}$$

we show that $\forall z \in U \setminus \gamma^*$, $\int_\gamma g(w,z)\,dz = 0$

Step 1 :- $z \longmapsto \int_\gamma g(w,z)\,dw$, $\forall z \in U$ is holo

1. Claim :- g is cts {use sup norm}

Proof :- Observe $w \neq z$, $g(w,z) = \dfrac{f(w)-f(z)}{w-z}$

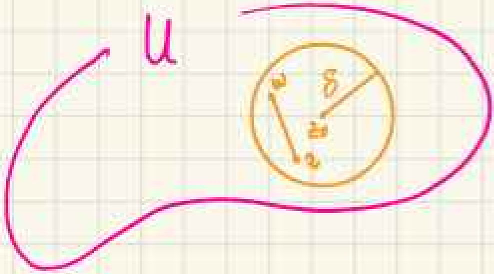
$$[\text{FTC}] \quad = \frac{1}{w-z}\int_{[z,w]} f' = \int_0^1 f'((1-t)z + tw)\,dt$$

$$w = z, \quad g(w,z) = f'(z) = \int_0^1 f'((1-t)z + tw)\,dt$$

Let $(w_0, z_0) \in U \times U$. If $w_0 \neq z_0$, g is continuous at (w_0, z_0)

So Assume, $w_0 = z_0$. Since f' is cts at z_0.

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ni \forall u \in D(z_0, \delta),\ |f'(u) - f'(z_0)| < \frac{\varepsilon}{2}$$



Let, $w, z \in D(z_0;\delta)$.
Then
$|w-w_0|, |z-z_0| < \delta$, i.e.,
$d((w,z),(w_0,z_0)) < \delta$
$\Rightarrow \forall t \in [0,1]$,
$(1-t)z + tw \in D(z_0;\delta) \Rightarrow$

$$|g(w,z) - g(w_0,z_0)| = \left| \int_0^1 f'((1-t)z+tw) - \int f'(1-t\ldots) \right|$$
$$\leq \left| \int_0^1 f'((1-t)z+tw) - f'((1-t)z_0+tz_0) \right|$$
$$\leq \int_0^1 \left| \qquad \right| dt$$
$$\leq \frac{\varepsilon}{2} < \underline{\underline{\varepsilon}}.$$

Hence its continuous

② $\forall z \in U$, $w \longmapsto g(w,z)$ is holo. Similarly
$\forall w \in U$, $z \longmapsto g(w,z)$ " " .

REMARK:- use Moreras Theorem

$$\int_\gamma g(w,z)\,dz = \int_a^b g(\gamma(t),z)\,\gamma'(t)\,dt$$

Consider $[a,b] \times U \longrightarrow \mathbb{C}$
$(t,z) \mapsto g(\gamma(t),z)\gamma'(t)$
[using ②, $\downarrow$ is holo if 't' is fixed]

2. If γ is C^1 then cts [if piecewise take partitions of $[a,b]$]

$$\Rightarrow \quad f(z) = \int_a^b g(\gamma(t), b)\, \gamma'(t)\, dt$$

is holo [Leibnitz theorem]

$\int_a^b$ = sum of finitely many holo $\Rightarrow$ holo.

step 2 – extending $f(t)$ from $U \to \mathbb{C}$

$$V = \{ z \in \mathbb{C} \setminus \gamma^* : \mathrm{Ind}_\gamma(z) = 0 \}$$

Clearly $\mathbb{C} \setminus U \subseteq V \Rightarrow$

$$U \cup V = \mathbb{C}$$

Let $z \in U \cap V$

$$\int_\gamma g(w,z)\,dw = \int_\gamma \frac{f(w) - f(z)}{w - z}\,dw$$

$$= \int_\gamma \frac{f(w)}{w-z} - \underbrace{2\pi i\, f(z)\, \mathrm{Ind}_\gamma(z)}_{\text{as } z \in V \Rightarrow = 0}$$

Define $h : \mathbb{C} \to \mathbb{C}$

$$h(z) = \begin{cases} \displaystyle\int_\gamma g(w,z)\,dw, & z \in U \\[2ex] \displaystyle\int_\gamma \frac{f(w)}{w-z}\,dw, & z \in V \end{cases}$$

Since, $h(z)$ is holo at U, V and $U \cap V \Rightarrow$

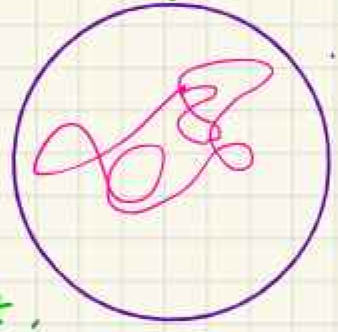
$\boxed{h(z) \text{ is entire}}$

Step 3: h is bounded, hence it is constant by Liouville's theorem

γ^* is compact $\Rightarrow \exists R>0 \ni \gamma^* \subseteq D(0,R)$

$\Rightarrow \mathbb{C}\setminus\gamma^* = C_1 \cup C_2 \cup \cdots$

↑ connected components



$\mathbb{C}\setminus D(0,R)$ is connected open set $\Rightarrow$ $\exists$ connected component, say $\mathcal{C}$ of $\mathbb{C}\setminus\gamma^*$ $\Rightarrow$ $\mathcal{C}$ is unbounded. As

$\text{Ind}_\gamma \equiv 0$ on any unbounded component of $\mathbb{C}\setminus\gamma^*$, we get

$$\mathcal{C} \subseteq V.$$

$\mathbb{C}\setminus\overline{D(0,R)} \subseteq \mathcal{C} \subseteq V$. Hence $\mathbb{C}\setminus V \subseteq \overline{D(0,R)}$

Thus $\mathbb{C}\setminus V$ is bounded.

Since h is cts, h is bounded $\overline{\mathbb{C}\setminus V}$ [since it is compact]

$\Rightarrow$ h is bounded on $\mathbb{C}\setminus V$.

* To show h is bounded on V.

$M > R$ $\quad V_M' = \{z \in V : d(z,\gamma^*) < M\}$

$V_M'' = \{z \in V : d(z,\gamma^*) \geq M\}$

clearly $V = V_M' \sqcup V_M''$

① **Obs** : V_M' is bounded

Let $z \in V_M'$. $\exists\, w \in \gamma^* \ni |w-z| < M$

$$|z| \leq |z-w| + |w| < M + R$$

$\Rightarrow h$ is bdd on V_M'

② Let $z \in V_M''$.

$$|h(z)| = \left| \int_\gamma \frac{f(w)}{w-z}\, dw \right| \leq \sup_{\gamma^*} |f| \cdot \frac{L_\gamma}{M}$$

Hence, h is bounded V_M''.

$\Rightarrow$ h is bounded everywhere

$\Rightarrow$ h is constant

Step 4 :- $C = 0$ $[\, h(z) = c\,]$

$\forall\ M \geq R$, V_M', V_M'' defined before.

Observe that : $V_M'' \neq \phi$. otherwise $V = \underset{\text{bounded}}{\underset{\downarrow}{V_M'}}$ and V can't be bounded ($\mathbb{C} \subseteq V$)

Hence, $\underset{= |c|}{|h(z)|} \leq \dfrac{L_\gamma |f|_{\sup_{\gamma^*}}}{M}$, $\forall M > R$

M = arbitrary $\Rightarrow$ $h(z) = 0$.

If γ satisfies $\forall z \in \mathbb{C} \setminus U$ $\text{Ind}_\gamma(z) = 0$

Then γ is U- homologous to 0

Defn

Given 2 paths γ & η.

If $\text{Ind}_\gamma(z) = \text{Ind}_\eta(z)$ $\forall z \in \mathbb{C} \setminus U$

Then γ and η are "homologous in U"

denoted by $\gamma \underset{U}{\sim} \eta$

Theorem: $\forall f \in H(U)$, $\int_\gamma f = \int_\eta f$ is $\gamma \underset{U}{\sim} \eta$

proof later

Chains

$\gamma_1, \ldots, \gamma_n$ curves in U

$k_1, \ldots, k_n \in \mathbb{Z}$.

$$\gamma = \sum_1^n k_i \cdot \gamma_i$$

If γ_i is closed $\forall i$ then γ is said to be a Cycle

Now, 1) $\gamma^* = \gamma_1^* \cup \gamma_2^* \cup \cdots \gamma_n^*$

2) $f : \gamma^* \longrightarrow \mathbb{C}$

$$\int_\gamma f = k_1 \int_{\gamma_1} f + \cdots + k_n \int_{\gamma_n} f$$

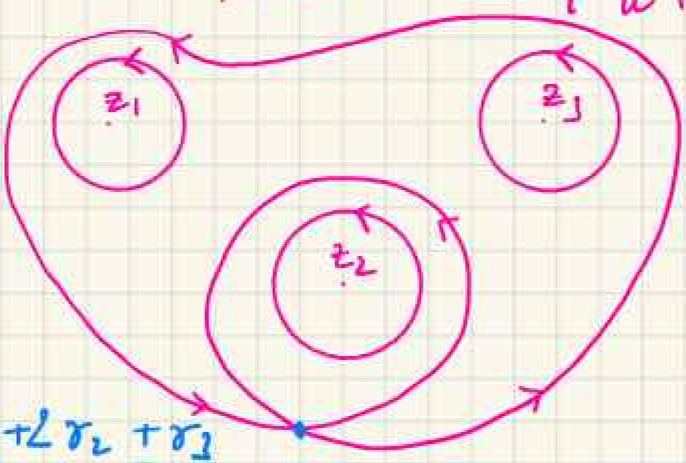
3) γ is a cycle, $\forall z \in \mathbb{C}\setminus\gamma^*$

$$\gamma \underset{U\setminus\{z_1,\ldots,z_n\}}{\sim} \sum k_i\gamma_i \Leftarrow \left\{ \operatorname{Ind}_\gamma(z) = k_1 \operatorname{Ind}_{\gamma_1}(z) + \cdots + k_n \operatorname{Ind}_{\gamma_n}(z) \right.$$

$$\left[\int_\gamma h(w)\,dw = \sum k_i \int_{\gamma_i} h(w)\,dw \quad \forall h \in H(U\setminus\{z_1,\ldots,z_n\}) \right]$$

Examples of homotopy

$\gamma \underset{U}{\sim} \gamma_1 + 2\gamma_2 + \gamma_3$



$\gamma \underset{U}{\sim} \gamma_1 + 2\gamma_2 + \gamma_3$

$\gamma \underset{U}{\sim} \gamma_1$

Theorem:-

Let γ be a cycle $z_1, \ldots, z_n \in U$ & $D_i = D(z_i, r_i)$ $\ni D_i \cap D_j = \phi$ and $D_i \subseteq U$, $\gamma_i = C(z_i; r_i)$

Then, $\gamma \underset{U}{\sim} \sum k_i \gamma_i$

where $k_i = \operatorname{Ind}_\gamma(z_i)$

* Clearly

$U \subseteq \mathbb{C}$, γ is a cycle in U : $\{\gamma = \sum_{i}^{n} k_i \gamma_i\}$

$$\forall z \in \mathbb{C} \setminus U, \quad \mathrm{Ind}_\gamma(z) = 0$$

i.e $\gamma \underset{U}{\sim} 0$

$$\Rightarrow \forall z \in U \setminus \gamma^*$$

$$\mathrm{Ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

and hence $\int_{\gamma} f = 0$, $\gamma \to$ closed.

* Corollary is the above mentioned theorem

Proof:-

$$g : U \times U \longrightarrow \mathbb{C}$$

$$g(w, z) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & : w \neq z \\ f'(w) & : w = z \end{cases}$$

Step 1 :- $\forall z \in U$, $z \longmapsto \int_\gamma g(w, z)\, dw$ is holo

Step 2 :- $V = \{z \in \mathbb{C} \setminus \gamma^* : \mathrm{Ind}_\gamma(z) = 0\}$

$$\Rightarrow \mathbb{C} \setminus U \subseteq V \Rightarrow U \cup V = \mathbb{C}, \quad z \in U \cap V$$

$$\int g(w, z)\, dz = \int_\gamma \frac{f(w)}{w - z}\, dw$$

$$h : \mathbb{C} \longrightarrow \mathbb{C}$$

$$h(z) = \begin{cases} \int_\gamma g(w, z)\, dz & : z \in U \\ \int_\gamma \frac{f(w)}{w - z} & : z \in V \end{cases}$$

unbounded

Step 3 :- $V_M' = \{ — : d(z, \gamma_j^*) \leq M\} \rightarrow$ bad ✓.

$V_M'' : \{ — : d(z, \gamma_j^*) > M\}$

$\downarrow$

$z \in V_M''$, $h(z) = \sum_{j=1}^{n} k_j \int_{\gamma_j} \frac{f(w)}{w-z}$

$\Rightarrow |h(z)| \leqslant \sum_{j=1}^{n} |k_j| \frac{\sup |f(w)|}{M} \cdot l\gamma_j$

$\leqslant K'$

$\Rightarrow$ $h(z)$ is constant + since $V_M'' \neq \phi$

$\boxed{h(z) = 0}$

Homotopy

- $U \underset{\text{open}}{\subseteq} \mathbb{C}$. We say γ_0 & γ_1 are homotopic in U if there is a continuous map

$$H : [a,b] \times [0,1] \xrightarrow{\text{cts}} U \ni$$

(i) $H(t,0) = \gamma_0(t) \quad \forall\, t \in [a,b]$

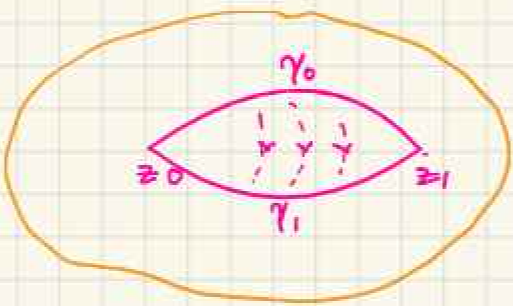
(ii) $H(t,1) = \gamma_1(t) \quad \forall\, t \in [a,b]$

(iii) $\forall\, s \in [0,1]$

$H(a,s) = z_0$, $H(b,s) = z_1$] $\Rightarrow$ endpoints are fixed.

For any $s \in [0,1]$, $\underbrace{\gamma_s(t)}_{\text{curve at "time" } s} = H(t,s)$, $t \in [a,b]$

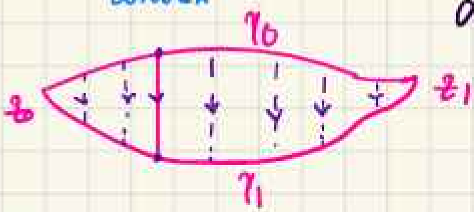
$\approx$ deformation



- We deform $\gamma_0 \to \gamma_1$ in time 1
- Curve doesn't leave U.
- γ_s need not be a path.

If $U \underset{\substack{\text{open} \\ \text{convex}}}{\subseteq} \mathbb{C} \implies$ Any 2 curves will be homotopic

Linear Homotopy



Line segment joining any 2 points $\in U$.
Push γ_0 along that line

Here map will be :-

$$H(t,s) = (1-s)\gamma_0(t) + s\gamma_1(t)$$

Same holds if U is star-like Region

Example 2 :- $U: \mathbb{C}\setminus R_\alpha$, $\gamma_0, \gamma_1 \to$ curves. Here also any 2 curves will be homotopic to each oth.

$$\gamma_0(t) = |\gamma_0(t)| e^{i \arg_\alpha(\gamma_0(t))}$$

$\hookrightarrow$ $\arg_\alpha$ & $\log_\alpha$ cts in $\mathbb{C}\setminus R_\alpha$

$$\gamma_1(t) = |\gamma_1(t)| e^{i\arg_\alpha(\gamma_1(t))}$$

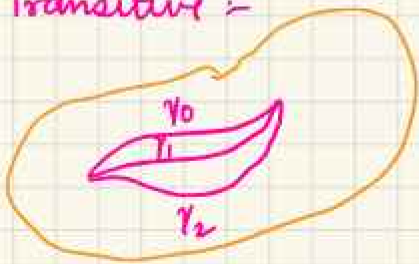
$$H(s,t) = \Big((1-s)|\gamma_0(t)| + s|\gamma_1(t)|\Big) \cdot e^{i(1-s)\arg_\alpha(\gamma_0(t)) + s\arg_\alpha(\gamma_1(t))}$$

✓. Verify

Some Algebra

★ Homotopy is equivalence relation.

- Reflexive :- ✓
- Symmetric :- $\tilde{H}(t,s) = H(t,1-s)$
- Transitive :-



$H(t,2s)$: $0 \le s \le \frac{1}{2}$

$\gamma_0 \longrightarrow \gamma_1$ $(\frac{1}{2})$ and

$\gamma_1 \longrightarrow \gamma_2$ (second half)

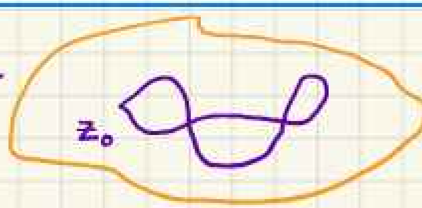
$$H(t,s) = \begin{cases} H_1(t,2s) & : \quad 0 \le s \le \frac{1}{2} \\ H_2(t,2s-1) & : \quad \text{o/w} \end{cases}$$

Path-Homotopy

Suppose γ_0 & γ_1 are paths.

Convention:- If γ_0 & γ_1 are paths & homotopic in U then γ_s is also a path $\forall s \in [0,1]$

For closed Path :-



$\gamma_0(a) = \gamma_0(b) = z_0$

$\gamma_1(t) = z_0$

★ Every closed path in a Convex set is homotopic to its end point ($\gamma(a) = z_0$). We also say γ is Null homotopic in U.

★ Let $U \subseteq \mathbb{C}$ open connected. If every closed path is null homotopic then U is simply connected.

Examples :-

① Open convex subsets of $\mathbb{C}$

② $\mathbb{C} \setminus R_\alpha \ \forall \alpha \in \mathbb{R}$

★ $U = \mathbb{C} \setminus \{0\}$ Not simply connected

$\gamma_0(t) = e^{it}, \ 0 \leq t < 2\pi$ X

Consider $z \in \mathbb{C} \setminus U$ where γ_0 & γ_1 are closed path in U, homotopic. Let $(t, s) \longmapsto H(s,t) - z \ : \neq 0$

$\Rightarrow \theta$ (its arg exists)

Fix $s \in [0,1]$, $\theta_s(t) = \theta(t,s) \longrightarrow$ always as

$$\gamma_s(t) - z = |\gamma_s(t) - z| \, e^{i\theta_s(t)}$$

$$\operatorname{Ind}_{\gamma_s}(z) = \frac{\theta_s(b) - \theta_s(a)}{2\pi} = \frac{\theta(b,s) - \theta(a,s)}{2\pi} \in \mathbb{Z}$$

$= f_n$ of s

given z = fixed,

$s \longrightarrow \mathrm{Ind}_{\gamma_s}(z)$ is a continuous function

$$\Rightarrow \quad \boxed{\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z)}$$

Same holds for closed curves :-

$$\forall z \in \mathbb{C} \setminus U, \quad \mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z)$$

$\Rightarrow$ γ_0 and γ_1 are homologous

$$\Rightarrow \forall f \in H(U), \quad \int_{\gamma_0} f = \int_{\gamma_1} f$$

γ_0 & γ_1 are homotopic $\Rightarrow$ homologous

Tarun Goyal

Homotopy Version of Cauchys Theorem

$U \underset{open}{\subseteq} \mathbb{C}$, γ is closed path which is null homotopic in U. $\forall f \in H(U)$,

$$\int_{\gamma} f = \int_{\substack{const\\ path\ (z_0)}} f = 0$$

$\Rightarrow \gamma \underset{U}{\simeq} 0$

Because const path

i.e if U is simply connected in U, then $\forall \gamma \to$ closed path,

$\forall f \in H(U)$, $\int_{\gamma} f = 0$ as all $\gamma \longrightarrow$ null homotopic

Conclusions

1) $\forall f \in H(U)$ (simply connected U) $\exists F \in H(U) \ni F' = f$. Infact

for $z_0 \in U$, $F(z) = \int_{z_0}^{z} f \quad \forall z \in U$

(Assignment)

2) Let $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ (connected U) analytic function

$\Rightarrow \dfrac{f'}{f} \in H(U)$ (Algebra of holo)

First fix $z_0 \in U$

$$h(z) = \int_{z_0}^{z} \frac{f'}{f}\,dz \quad \forall z \in U \quad \begin{bmatrix} \text{using} \\ ① \end{bmatrix}$$

$\Rightarrow \forall z \in U,\ h'(z) = \frac{f'(z)}{f(z)} \quad \Rightarrow \quad \phi = e^{-h} f,\ f \in H(U)$

$$\phi' = -h' e^{-h} f + e^{-h} f' = 0$$

$\Rightarrow \phi$ is constant (U is connected)

$$\Rightarrow \phi(z) = f(z_0)$$

hence, (Let $w_0 \in \mathbb{C} \ni f(z_0) = e^{w_0}$)

Then,

$$e^{-h(z)} f(z) = f(z_0) \Rightarrow f(z) = e^{w_0 + h(z)}$$

an analytic logarithm namely,

$$z \longmapsto w_0 + \int_{z_0}^{z} \frac{f'}{f}$$

<u>Note</u> :- The above depends on z_0 & w_0. But any other choice will differ by 2π

<u>Special Case</u>

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, simply connected, $0 \notin U$.

$f(z) = z \ , \forall z \in U$

$\Rightarrow f$ vanishes everywhere

$\Rightarrow$ Fix $z_0 \in U$, $z \longmapsto w_0 + \int_{z_0}^{z} \frac{dw}{w}$ is analytic log of

$f(z) = z$. Then $\exists$ a branch of log on U

$$\left\{ \log x = \log x_0 + \int_{x_0}^{x} \frac{dt}{t} \right\}$$

Corollary :-

$\forall f \in H(U)$ f is 0 free. $\forall n \in \mathbb{N}$ f has an analytic n^{th} root

i.e. $\exists F \in H(U) \ni F^n = f$

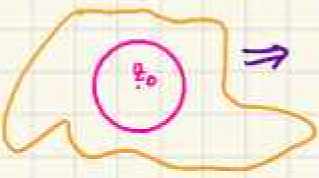
Tf?

Zeros of Analytic functions

$U \subseteq \mathbb{C}$ open connected, $f \in H(U)$

$$Z(f) = \{z \in U : f(z) = 0\} \subseteq U$$

Let $z_0 \in Z(f) \subseteq U$ & $\exists R > 0 \ni D(z_0; R) \subseteq U$



$\Rightarrow \forall z \in D(z_0; R)$ $f(z) = \sum_0^\infty a_n (z - z_0)^n$

Assume that all $a_m \neq 0$. Let $m \geq 0$ be the least Index

$$\ni a_m \neq 0$$

$$f(z) = \sum_{i=m}^{\infty} a_i (z-z_0)^i$$

Let

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & : z \neq z_0 \\ a_m & : z = z_0 \end{cases}$$

$\Rightarrow$ g is holo $U \setminus \{z_0\}$ $\Rightarrow g \in H(U)$ (Morera's Theorem)

Theorem:- Let $U \subseteq \mathbb{C}$ connected open, $f : U \xrightarrow{\text{holo}} \mathbb{C}$. Suppose $Z(f)$ has a clusterpoint in U. Then f vanishes i.e $f = 0$ $\forall$.

proof:- let z_0 be a c.p of $Z(f)$. Clearly, f being cts at $z_0 \Rightarrow$

$$f(z_0) = 0 \Rightarrow z_0 \in Z(f).$$

Lets expand f into powerseries around z_0. (Analytic to yes!)

$$f(z) = \sum a_n (z-z_0)^n, \forall z \in D(z_0, r) \subseteq U$$

Claim:- $a_n = 0 \;\forall n \geq 0$.

If not let $m =$ least $\mathbb{Z}^+ \ni a_m \neq 0$. Then

$$f(z) = (z-z_0)^m g(z)$$
$$g(z_0) \neq 0$$

[Before Analysis]

Since g is cts,

$$\exists\, \varepsilon < r \ni g(z) \neq 0 \;\forall\; D(z_0, \varepsilon).$$

$$\Rightarrow \nexists \text{ a seq}^n \ni z_n \longrightarrow z_0 \;\&\; f(z_n) \longrightarrow f(z_0)$$

i.e z_0 is not a cluster point $\Rightarrow\Leftarrow$

Let $A = \{ t \in U : z = \text{cluster point } (Z(f)) \}$. Since $z_0 \in A$, $A \neq \phi$. If $z \in A$, by above arguement,

$$\forall\, z \in D(z, \varepsilon),\ f(z) = 0 \Rightarrow B(z, \varepsilon) \subseteq A$$

① Hence, A is open.

② Let $z =$ l.p of A. $\exists$ seq^n z_n in $A \ni z_n \to \underset{\in U}{z}$.
If $z_n = z$ for some n.
$z \in A$. Else if $z_n \neq z$ $\forall n, \forall z \in A$

then, z is a cluster point of $\{z_n : n \in \mathbb{N}\}$
$\Rightarrow z$ is c.p of $Z(f) \Rightarrow z \in A$

Hence, A is closed

Since, U is connected $\Rightarrow A = U$ or $A = \phi$.

Identity Theorem

$U \subseteq \mathbb{C}$, connected, $f, g : U \xrightarrow{\text{holo}} \mathbb{C}$, $S = \{ z \in U : f(z) = g(z) \}$
$= Z(f - g)$

If S has a limit point then $f \equiv g$

★ $U \subseteq \mathbb{C} \longrightarrow$ open & connected $\Leftrightarrow$ $U =$ Region

$\longrightarrow$ $U = \mathbb{C} \setminus \{0\}$, $f(z) = \sin \frac{1}{z} : z \neq 0$

$$\left\{ \frac{1}{n\pi} : n \in \mathbb{Z} \right\} \subseteq Z(f)$$

★ (limit points should be considered inside U)

Theorem:- If $f \not\equiv 0$ in U, $f : U \to \mathbb{C}$. then the set $Z(f)$ is countable proof. $U \subseteq \mathbb{C}$, $f \in H(U)$. $Z(f)$ doesn't have a limit pt point in U.

$U = \bigcup_{n=1}^{\infty} K_n$. $\underset{\text{compact}}{K_n \subseteq U}$ and

$K_n \subseteq K_{n+1} \; \forall n$.

If $Z(f) \cap K_n = \infty$, being ∞ subset of a compact set, it has a limit point in K_n.
$\Rightarrow$ l.p in U

That limit point is in $K_n \subseteq U \Rightarrow\Leftarrow$
$\Rightarrow$
$\forall n \geq N$, $Z(f) \cap K_n$ is finite

$$Z(f) = \bigcup_{n=1}^{\infty} (Z(f) \cap K_n)$$

$\Rightarrow$ $Z(f)$ is countable

Assume $Z(f)$ doesn't have a l.p in U. Let $z_0 \in Z(f)$

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^n} & : z \neq z_0 \\ a_m & : z = z_0 \end{cases}$$

$$f(z) = (z-z_0)^m g(z), \; \forall z \in U. \; g(z_0) \neq 0$$

$\downarrow$ polynomial

Exercise:- Show, $m \in \mathbb{N}$, g are !, take $m, g, \cdots$

Maximum Modulus Principle

$U \underset{rg}{\subseteq} \mathbb{C}$, $f \in H(U)$. Let $r > 0 \ni \overline{D(z_0, r)} \subseteq U$.

Assume $|f(z_0 + re^{it})| \leq |f(z_0)| \ \forall t \in [0, 2\pi]$

$$\frac{1}{2\pi}\int |f(z_0 + re^{it})|^2 = \sum_0^\infty |a_n|^2 r^{2n}$$ [Proved in assignments] Parseval's Identity

$$\Rightarrow |f(z_0)|^2 \geq \sum_0^\infty |a_n|^2 r^{2n}$$

$$\Rightarrow |a_0|^2 \geq \sum_0^\infty |a_n|^2 r^{2n} \geq |a_0|^2$$

$$\Rightarrow a_1 = a_2 = \ldots = 0 \ \forall n \geq 1$$

f is constant on $D(z_0; r) \subseteq \overline{D(z_0; r)} \subseteq U$

$\downarrow$ has a l.p in U

from Id theorem, f is a constant

Hence, Value of f at centre dominates value at boundary
$\Rightarrow$ f is a constant.

- ★ $|f(z_0)| \leq \max\limits_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})|$ and $\underbrace{\text{equality when}}_{= \text{Max. Modulus Principle}}$

Theorem:- $U \underset{\text{region, bounded}}{\subseteq} \mathbb{C}$. Let $f \in H(U)$.

Then i) f = constant OR

ii) f doesn't have a local maxima in U

Tarun Goyal

Minimum Modulus Principle

* Let $U \subseteq \mathbb{C}$ (region), $f \in H(U)$. Let $z_0 \subseteq U$ & $r > 0$ be such that $\overline{D(z_0;r)} \subseteq U$ and f vanishes nowhere in $\overline{D(z_0;r)}$ then, $|f(z_0)| \geq \min_{t \in [0,2\pi]} |f(z_0 + re^{it})|$

* Let $\overline{U} \subseteq \mathbb{C}$ (region), $f \in H(U)$. $f \neq 0 \ \forall z \in \overline{U}$, then

$$\min_{\overline{U}} f = \min_{\partial U} f$$

* If a local minima exists, then $f = 0$ at that point, else f = constant

Corollary :- Maximum Modulus Principle

★ $U \underset{\text{reg}}{\subseteq} \mathbb{C}$. $f \in H(U)$. Then if f has a local max.

$\Rightarrow$ f is constant

$|f(z_0)| \geq |f(z)| \quad \forall z \in D$

$\Rightarrow$ ✓.

★ Let U be bdd. $f : \overline{U} \xrightarrow{\text{cts}} \mathbb{C}$. Assume $f \in H(U)$
Then,

$$\max_{\overline{U}} |f| = \max_{\partial U} |f|$$

proof: If f is constant ✓.
else, $|f|_{max}$ can't lie inside

Open Mapping Theorem

Theorem :- $f : U \xrightarrow{\text{holo}} \mathbb{C}$ such that $f'(z) \neq 0$ for some $z \in U$. Then $\exists\, V \underset{\text{open}}{\subseteq} U \ni z \in V$ and f is 1-1, onto on V, f^{-1} is holo on $f(V)$.

proof :- Consider f as a function in $\mathbb{R}^2$

$$f' = u_x + i u_y \quad \Rightarrow \det(\nabla f) = |f'(z)|$$

$$= v_x - i v_y \qquad f'(z) \neq 0 \Rightarrow \det(\nabla f) \neq 0$$

i.e ∇f is non singular. Using IMT

$\exists\, V \underset{\text{open}}{\subseteq} U$ and $W \underset{\text{open}}{\subseteq} \mathbb{C}$ $\ni$

i) f is one-one on V
ii) $f^{-1} : W \longrightarrow V$ is one-one one

Now observe $g : W \longrightarrow W$

$$g = f(f^{-1}(z)) = z$$

Since, $f \in H(V)$.
f^{-1} is cts
and $f'(f^{-1}(z)) \neq 0$ (By INT again)

hence

$f^{-1} \in H(W)$

Hence Proved.

Theorem :- Let $U \underset{\text{region}}{\subseteq} \mathbb{C}$, $f \neq$ constant, if $f'(z_0) = 0$ then $\exists\, V \underset{\text{open}}{\subseteq} U$, $z_0 \in V$, $\exists\, \rho > 0$, $m > 1$, $m \in \mathbb{N}$ and an onto holomorphic function $\psi : V \longrightarrow D(0, \rho)$ such that-

i) $\forall\, z \in V$, $f(z) = f(z_0) + (\psi(z))^m$

ii) $\psi(z_0) = 0$ iii) ψ' vanishes nowhere in V

iv) ψ is m to 1 on $V \setminus \{z_0\}$

Analysis

Clearly,
If $f'(z_0) = 0$, f is open mapping from an open neighbourhood of z_0. Else,
$D(f(z_0); \rho^m) = f(V) \subseteq f(U)$, if U is connected and f is non-constant.

Hence, $\forall$ non-constant $f \in H(U)$, $f(z_0)$ is an interior point of $f(U)$. $\Rightarrow f(U) =$ open

Tarun Goyal

Theorem :- Let U be a region, $f \in H(U)$. Then, either $f(U)$ = open or a point.

Observe, Max Modulus principle is a special Case of OMT.

Some takeaways from Identity Theorem

1. $f \equiv 0$ or all zeros are isolated points

2. $Z(f) \cap K$ = finite $\forall\ K \subseteq U$ compact

3. In particular, $\overline{D(z_0;r)} \subseteq U \Rightarrow f$ has finitely many zeros in $\overline{D(z_0;r)}$

* If $\overline{D(z_0;r)} \subseteq U \Rightarrow \exists\ R > r \ni D(z_0;R) \subseteq U$.

Then $Z(f) \cap D(z_0;r)$ = finite

i.e. Assumption of U-connected can be dropped.

* $\underset{\text{open}}{U} \subseteq \mathbb{C}$, $\{f_n\}_{n=1}^{\infty} \subseteq H(U)$, $f_n \xrightarrow{a.u.} f$, Suppose each f_n has exactly k zeros in $\overline{D(z_0;r)}$.

Q Does f have k zeros in $\overline{D(z_0;r)}$

Ans NO!,

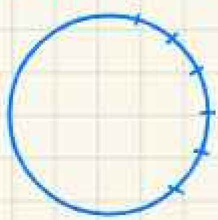
eg:- $f_n(z) = \frac{z}{n}$, $\mathbb{D}$

boundary removed

Q Does f has AT LEAST! k zeros in $D(z_0;r)$?

Ans NO!

$\mathbb{D}$, $k \in \mathbb{N}$



$1 \le r < k$, $\beta \in S^1$, $l = k - r$

$\alpha_i \neq \alpha_j$, $i \neq j$

Choose a seq^n $\{\beta_n\}_1^{\infty} \subseteq \mathbb{D}$

$\ni \beta_n \longrightarrow \beta$.

Consider $f_n(z) = (z-\beta_n)^l (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)$

Check, $f_n(z) \longrightarrow f(z)$, uniformly

But no. of zeros will drop down!

— change assumptions —

proposition:- Let $f \in H(U)$, $U \subseteq \mathbb{C}$. Then $\int_{C(z_0;r)} \frac{f'}{f} = \sum m_i$ if f is zero-free on the open circle and $\overline{D(z_0;r)} \subseteq U$.

proof:- Let, $\alpha_1, \dots, \alpha_k$ are precisely all distinct zeros of f in $D(z_0;r)$

[we can get $R > 0 \ni \overline{D(z_0;R)} \in U$ & f has exactly k zeros $\alpha_1, \dots, \alpha_k$ in $D(z_0;R)$

restrict our discussion to $\overline{D(z_0;R)}$

$$f(z) = (z-\alpha_1)^{m_1} g_1(z) = (z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2} g_2(z) \cdots$$

$$f(z) = (z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2} \cdots (z-\alpha_k)^{m_k} g(z) \;:\; g(z) \in H(U),\ g(z) \neq 0 \ \forall z \in \{\alpha_1, \dots, \alpha_k\}$$

Observe g is zero-free in $\overline{D(z_0;R)}$

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z-\alpha_1} + \cdots + \frac{m_k}{z-\alpha_k} + \frac{g'(z)}{g(z)}$$

$$\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f'(z)}{f(z)}\,dz = \sum m_i + \frac{1}{2\pi i}\int_{C(z_0;r)} \underbrace{\frac{g'}{g}}_{\in H(D(z_0;r))}$$

$$= \sum m_i$$

[$\in H(D(z_0;r)) \Rightarrow$ CIF $\Rightarrow \int = 0$]

Hurwitz Theorem

Theorem :- $U \subseteq \mathbb{C}$ $\{f_n\}_1^\infty \in H(U)$, $f_n \xrightarrow{u.c} f$. Assume f doesn't have any roots on $|z-z_0|=r$ $\Rightarrow \exists N \in \mathbb{N}$ $\forall n \geq N$

then

f_n and f have same no. zeroes in $D(z_0;r)$

proof:-

$S = \{z \in U \mid |z-z_0| = r\}$. As f doesn't vanish anywhere on S

$\hookrightarrow$ compact

$\Rightarrow \exists M > 0 \ni \forall w \in S, \quad |f(w)| \geq M$

$\Rightarrow \exists N \in \mathbb{N} \ni \forall n \geq N, \quad |f_n(w) - f(w)| \leq \frac{M}{2} \quad \forall w \in S$ — ①

$\Rightarrow f_n(w) \neq 0 \;\forall n \geq N$

Now, <u>To show</u> :-

$$\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f_n'}{f_n} \longrightarrow \frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f'}{f}$$

[Idea:- sequence of $\mathbb{Z}$ converging to another integer

$$\left| \int \frac{f_n'}{f_n} - \int \frac{f'}{f} \right|$$

$$= \left| \int_{C(z_0;r)} \left(\frac{f_n'}{f_n} - \frac{f'}{f} \right) \right|$$

Observe - $\left| \frac{f_n'}{f_n} - \frac{f'}{f} \right| = \left| \frac{f_n' f - f' f_n}{f_n f} \right|$

$\leq \frac{2}{M^2} |f_n' f - f' f_n|$ [using ①]

$\leq \frac{2}{M^2}(|f_n' f - f' f| + |f' f_n - f' f|)$

$= \frac{2}{M^2}(|f||f_n' - f'| + |f'||f_n - f|)$

$M_1 = \sup |f|$, $M_2 = \sup |f'|$

$\leq \frac{2}{M^2}(M_1 + M_2)\varepsilon \quad \forall n \geq N$ (f' also converges uniformly)

$\leq \frac{4\pi r}{M^2}(M_1 + M_2)\varepsilon$

Let $\varepsilon' = \frac{\varepsilon M^2}{(M_1+M_2) 4\pi r^2}$

$\Rightarrow \int \frac{f_n'}{f_n}$ converges uniformly

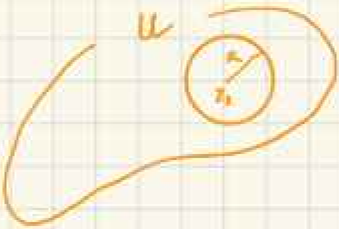
$\Rightarrow$ a seq^n of $\mathbb{Z}$ converges

$\Rightarrow$ <u>Constant</u> <u>Hence Proved</u>

Corollary:- U-region, $\{f_n\}_1^\infty \in H(U)$, $f_n \xrightarrow{a.u} f$. Assume that, for ∞-many n, f_n doesn't have a zero.

Then $f \equiv 0$ or f doesn't have a zero.

proof:- Assume $f \not\equiv 0$, let $f(z_0) = 0$, $z_0 \in U$.
$\exists R > 0 \ni D(z_0; R) \subseteq U$ & $f(z) \neq 0 \; \forall z \in D(z_0; R) \setminus \{z_0\}$



Consider $0 < r < R$,
$\overline{D(z_0; r)} \subseteq U$,
f_n, $\forall n \geq N$, no zeros in Disk

using Hourwitz theorem, f has no zeros

$\not\Longleftarrow$

Hence, $f \equiv 0$ as $\nexists R > 0 \ni \ldots$

Cor 2:- Let U, $\{f_n\}_1^\infty \subseteq H(U)$. $f_n \xrightarrow{a.u} f$. Assume all f_n are injective beyond a stage.

Then, $f \equiv$ constant, or f is one-one.

proof:- Assume f is not constant,
Pick $z_0 \in U$.

Consider, $g_n = f_n - f_n(z_0)$, $g = f - f(z_0)$

Clearly, $g_n \xrightarrow{a.u} g$ on U and hence on $\boxed{U \setminus \{z_0\}}$

Consider $U \setminus \{z_0\}$. connected ✓.
Note, g_n doesn't have a zero on $U \setminus \{z_0\}$

[To use Hurwitz theorem on $U \setminus \{z_0\}$, as analyzing zeros become easy]

$\Rightarrow$ $g_n \equiv 0$ or g_n has no zero $\forall n \geq N$

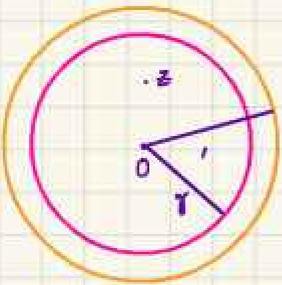
Hence proved

$\mathbb{D} = D(0;1)$, $f \in H(\mathbb{D})$: $f : \mathbb{D} \longrightarrow \mathbb{D}$, $f(0) = 0$.

Consider, $g(z) = \begin{cases} \dfrac{f(z)}{z} & : z \neq 0 \\ f'(0) & : z = 0 \end{cases}$

$\lim\limits_{z \to 0} g(z) = f'(0) \Rightarrow g$ is cts $\Rightarrow g \in H(\mathbb{D})$.

Now, let $0 < r < 1$, $|z| < r$. —



Using Maximum Modulus principle,

$$|g(z)| \leq \max_{|w| = r} \frac{|f(w)|}{r} \leq \frac{1}{r}$$

Let $\lim\limits_{r} r \to 1$

$$\Rightarrow |g(z)| \leq 1$$

Hence, i) $\boxed{|f(z)| \leq |z| \ , \ \forall z \in \mathbb{D}}$

ii) $\boxed{|g(0)| = |f'(0)| \leq 1}$

iii) If equality holds in i) or ii) for some $z \neq 0$,
Then, $\exists \lambda \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $f(z) = \lambda z \ \forall z \in \mathbb{D}$

SCHWARZ LEMMA

$f \in H(\mathbb{D})$. that's it
+ $f(0) = 0$ $\rightarrow$ if not take $\underbrace{\varphi_{f(0)}}_{\substack{0 \to 0 \\ \mathbb{D} \to \mathbb{D}}} \circ f$

Aut($\mathbb{D}$)

examples:-

1. $f_\lambda(z) = \lambda z$, where $|\lambda| = 1$

2. Fix $w \in \mathbb{D}$, Define, $\varphi_w(z) = \dfrac{w - z}{1 - \bar{w} z}$, $\forall z \in \mathbb{D}$

- self-inverse
- holo everywhere in $\mathbb{C} \setminus \{\frac{1}{\bar{w}}\}$
- $\varphi_w(\partial \mathbb{D}) = \partial \mathbb{D}$ and $\varphi_w(\mathbb{D}) = \mathbb{D}$

PROBLEM :- $f : \mathbb{D} \longrightarrow \mathbb{D}$, $\alpha, \beta \in \mathbb{D}$, $f \in H(\mathbb{D})$

How large can $|f'(\alpha)|$ be, given $f(\alpha) = \beta$?

$\Rightarrow$ $\underbrace{(\varphi_\beta \circ f \circ \varphi_\alpha)}_{= g}(0) = 0$, using Schwarz lemma,

$|g'(0)| \leq 1 \Rightarrow$

$$|\varphi_\beta'(\beta) f'(\alpha) \varphi_\alpha'(0)| \leq 1$$

$$\left[\varphi_w'(z) = \frac{(1 - \bar{w} z)(-1) + (w - z)\bar{w}}{(1 - \bar{w} z)^2} \Rightarrow |\varphi_w'(0)| = |w|^2 - 1, \quad |\varphi_w'(w)| = \frac{1}{|w|^2 - 1}\right]$$

Hence, $\boxed{|f'(\alpha)| \leq \dfrac{|\beta|^2 - 1}{|\alpha|^2 - 1}}$

Remark :- ① Maximizers are all Rational functions
② " " 1-1, onto mapping

★ **Theorem :-** All Analytic Automorphisms are of form

$\varphi_\alpha f() \varphi_B$

proof :-

Automorphism ⟶ bijective holo & inverse also holo

$\Rightarrow$ $f \in Aut(\mathbb{D})$

$\Rightarrow \exists!\ w \in \mathbb{D} \ni f(w) = 0$

Let $g = f^{-1}$

$\forall z \in \mathbb{D},\ g(f(z)) = z$

$g'(f(z))f'(z) = 1 \Rightarrow g'(f(w))f'(w) = 1$

hence, $g'(0)f'(w) = 1 \Rightarrow |g'(0)f'(w)| = 1$ — ①

Also, $f(w) = 0 \Rightarrow$ using Schwarz lemma, $|f'(w)| \leq \frac{1}{1-|w|^2}$ — ②

$g: \mathbb{D} \longrightarrow \mathbb{D}$, holo $g(0) = w$, $g(w) = 0$

using Schwarz lemma

$|g'(0)| \leq 1-|w|^2$

inserting in ①, $|f'(w)| \geq \frac{1}{1-|w|^2}$

using ②, equality holds

$\Rightarrow g'(0) = 1-|w|^2$ and $|f'(w)| = \frac{1}{1-|w|^2}$

Hence, f is a maximizer $\Rightarrow \exists\, \lambda,\ |\lambda| = 1 \ni$

$$f(z) = \varphi_0(\lambda \varphi_w(z)) \quad \text{———} \ \star$$

using $\varphi_0(z) = -z \implies f(z) = -\lambda \varphi_w(z)$, $\lambda \to -\lambda$

$$\Rightarrow \boxed{f(z) = \lambda \varphi_w(z)} \qquad w = f^{-1}(0)$$

Hence proved

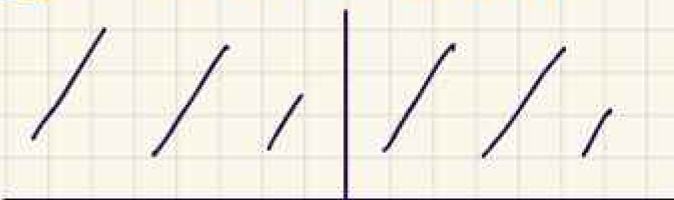
Corollary:- $f \in \text{Aut}(\mathbb{D})$, $f(0) = 0 \Rightarrow \exists\ |\lambda| = 1 \ni$

$$\boxed{f(z) = \lambda z}$$

origin fixing Automorphisms are all ROTATIONS!

Structural Description

Theorem:- $\text{Aut}(\mathbb{D}) \cong \text{Aut}(\mathbb{H})$



$F : \mathbb{H} \longrightarrow \mathbb{D}$ $\qquad$ $G : \mathbb{D} \longrightarrow \mathbb{H}$

$$F(z) = \frac{i - z}{i + z} \qquad G(z) = i \cdot \frac{1 - w}{1 + w}$$

Clearly, $F(i) = 0$, $G(0) = i$

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] & \mathbb{H} \arrow[d, "F"] \\ \mathbb{D} \arrow[u, "G"] & \mathbb{D} \arrow[l, "F \circ f \circ F^{-1}"] \end{tikzcd}$$

$f \in \text{Aut}(\mathbb{H})$

$\Rightarrow F \circ f \circ F^{-1} \cong \text{Aut}(\mathbb{D})$

Hence, $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{Aut}(\mathbb{D})$

Definition : Mobius Transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * z = \frac{az+b}{cz+d}$$

$$\equiv M$$

$$\boxed{M_z = f_M(z)}$$

Observe, this is a group action from $SL_2(\mathbb{R})$ on $\mathbb{H}$

Hence,

$$\underbrace{f_{M_1 M_2} = f_{M_1} \cdot f_{M_2}} \Rightarrow (f_M)^{-1} = f_{M^{-1}}$$

★ each (2×2) matrix gives rise to an automorphism of upper half plane

Theorem :- Every Auto in $\mathbb{H}$ is of form M_z

proof :- $f\left(\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{= M_\theta} z\right) = f\left(\frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}\right)$

$$= \frac{i - \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}}{i + \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}} = \frac{-z(\cos\theta - i\sin\theta) + (i\cos\theta + \sin\theta)}{z(\cos\theta + i\sin\theta) + (i\cos\theta - \sin\theta)}$$

$$= \frac{-z e^{-i\theta} + i e^{-i\theta}}{z e^{i\theta} + i e^{i\theta}} = \frac{e^{-i\theta}(i-z)}{e^{i\theta}(i+z)} = e^{-2i\theta} F(z)$$

hence, $f \circ M_\theta = e^{-2i\theta} f(z)$

hence, $\boxed{f \circ M_\theta \circ f^{-1}(z) = e^{-2i\theta} z}$

Now, $f \in Aut(\mathbb{H})$, $f(w) = i \in \mathbb{H}$. From transitivity [of group action]

$\exists$ a matrix $N \in SL_2(\mathbb{R})$ such that $f_N(i) = w$

$(f \circ f_N)(i) = f(w) = i$

$\Rightarrow f \circ f_N \in Aut(\mathbb{H})$

Hence, $f \circ (f \circ f_N) \circ f^{-1} \in Aut(\mathbb{D})$

$f \circ (f \circ f_N) \circ f^{-1}(0) = 0$ $\quad [f^{-1}(0) = i]$

$\exists\ \psi \in \mathbb{R} \ni f \circ (f \circ f_N) \circ f^{-1}(z) = e^{i\psi} z$ (Aut $(\mathbb{D})$ fixing 0)

choose $\theta = -\frac{\psi}{2}$, we get $f \circ (f \circ f_N) \circ f^{-1}(z) = e^{-2i\theta} z$

i.e $f \circ (f \circ f_N) \circ f^{-1} = f \circ M_\theta \circ f^{-1}$

hence, $f \circ f_N = f_{M_\theta}$

i.e

$f = f_{M_\theta} \circ f_{N^{-1}} = f_{M_\theta N^{-1}}$

Hence proved

$SL_2(\mathbb{R}) \longrightarrow Aut(\mathbb{H})$

$M \longrightarrow f_M$

$\hookrightarrow$ group homomorphisms

From first Isomorphism theorem, we have

$$\frac{SL_2(\mathbb{R})}{\text{Kernel}} \cong Aut(\mathbb{H}) \cong Aut(\mathbb{D})$$

Once we get kernel, we are done!

Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker$

$f_M(z) = z \quad \forall z \in \mathbb{H}$, Since M fixes $z \quad \forall z \in \mathbb{H}$,

M fixes $i \Rightarrow M$ is orthogonal $\Rightarrow M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

for some $\theta \in \mathbb{R}$

Then we have, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}(2i) = 2i$

$\Rightarrow \sin\theta = 0$, $\cos\theta = \pm 1$, $M = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ if $M \in$

$= \pm I_2$

Kernel $= \pm I_2$.

$$Aut(\mathbb{D}) \cong Aut(\mathbb{H}) \cong \frac{SL_2(\mathbb{R})}{\{\pm I_2\}}$$

$$\frac{SL_2(\mathbb{R})}{\{\pm I_2\}} = \underline{PSL_2(\mathbb{R})}$$

Tarun Goyal

PROBLEM 7.5

Consider $g(z) = \overline{f(\bar{z})}$

Clearly, $g(z) \in H(U)$.

$g \in H(U)$.
1. use Cauchy Riemann
or
2. basic def^n
$\lim_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0}$

Let $S = \{z : g(z) = f(z), z \in U\}$

$J \subseteq S$. and J has a l.p
hence S has a l.p hence
& U is connected
$f(z) = g(z) \quad \forall z \in U$

hence, $f(z) = \overline{f(\bar{z})} \Rightarrow \boxed{\overline{f(z)} = f(\bar{z})}$

Extension of this

$U^+ \underset{\text{open}}{\subseteq} \mathbb{H}$ and $I \subseteq \partial U^+$

$U^- = \{\bar{z} : z \in \bar{U}\}$, $U = U^+ \cup U^- \cup I$

Consider $f : U^+ \cup I \longrightarrow \mathbb{C}$, continuous

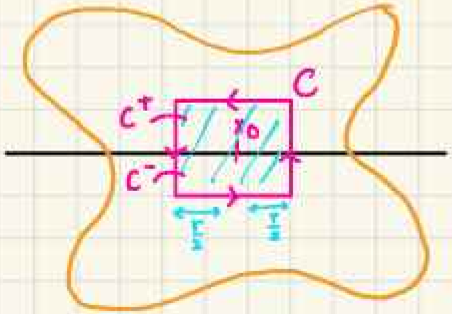
and f is holo on U^+. Assume $f(I) \subset \mathbb{R}$

Let $F(z) = \begin{cases} f(z) & : z \in U^+ \cup I \\ \overline{f(\bar{z})} & : z \in U^- \end{cases}$

★ Claim :- F is holo on U^+ and U^-. $F \in H(U)$.

General proposition :- Let $f : U \longrightarrow \mathbb{C}$. If f is holo on U^+, U^-, f is cts on U then $f \in H(U)$.

proof:-



Let $x_0 \in I$ $\exists\, r>0 \ni D(x_0, r) \subseteq U$

Define

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$

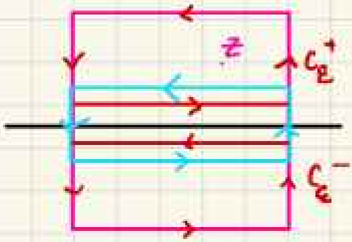
$\forall z \in \left(x_0 - \frac{r}{2}, x_0 + \frac{r}{2}\right) \times \left(-\frac{r}{2}, \frac{r}{2}\right)$

g is holo (Because f is continuous)
& use Leibniz theorem

Break $C \longrightarrow 2$ parts, C^+ and C^-

If $z \notin \left[x_0 - \frac{r}{2}, x_0 + \frac{r}{2}\right]$ $\Rightarrow$ $g(z) = \frac{1}{2\pi i}\int_{C^+}\frac{f(w)\,dw}{w-z} + \int_{C^-}\frac{f(w)\,dw}{w-z}$

$\Rightarrow z \in C^+ \cup C^-$



take $\varepsilon < |z| \ni$ we can form 2 curves C_ε^+, C_ε^- since $f \in H(U^+)$ & $H(U^-)$

we can use CIF $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_{C_\varepsilon^+}\frac{f(w)}{w-z}\,dw + \underbrace{\frac{1}{2\pi i}\int_{C_\varepsilon^-}\frac{f(w)}{w-z}\,dw}_{\text{will be '0'.}}$$

$$\int_{C_\varepsilon^+}\frac{f(w)}{w-z}\,dw = \underbrace{\int_{x_0-\frac{r}{2}}^{x_0+\frac{r}{2}}\frac{f(x+i\varepsilon)\,dx}{x+i\varepsilon-z}}_{1} + \underbrace{\int_{\varepsilon}^{r/2}\frac{f(x_0+r/2+ix)\,dx}{x_0+\frac{r}{2}+ix-z}}_{2}$$

$$- \underbrace{\int_{x_0-r/2}^{x_0+\frac{r}{2}}\frac{f(ir/2+x)\,dx}{\frac{ir}{2}+x-z}}_{3} - \underbrace{\int_{\varepsilon}^{r/2}\frac{f(x_0-r/2+ix)\,dx}{x_0-\frac{r}{2}+ix-z}}_{4}$$

$\lim_{\varepsilon \to 0}$ ① :-

$$\left| \int_{x_0 - r/2}^{x_0 + r/2} \frac{f(x+i\varepsilon)\,dx}{x+i\varepsilon - z} - \int_{x_0 - \frac{r}{2}}^{x_0 + \frac{r}{2}} \frac{f(x)\,dx}{x-z} \right| \leq \int_{x_0 - \frac{r}{2}}^{x_0 + \frac{r}{2}} \left| \frac{f(x+i\varepsilon)}{x+i\varepsilon - z} - \frac{f(x)}{x-z} \right| \quad \text{——— } (*)$$

f will be continuous on upper red rectangle (closed) as $z \notin \mathbb{R}$

from uniform continuity, given $\theta > 0$, $\exists\, \delta > 0$

if $|z - z_0| < \delta$, $|h(z) - h(z_0)| < \theta$

where,

$h(z_0) = \dfrac{h(z_0)}{z_0 - z}$. Now if $\varepsilon < \delta$ we have

$(*) < \theta \cdot r$

hence,

$$\int_{C_\varepsilon^+} \frac{f(\omega)}{\omega - z}\,d\omega \xrightarrow{\varepsilon \to 0} \int_{C^+} \frac{f(\omega)\,d\omega}{\omega - z}$$

Illy

$$\underbrace{\int_{C_\varepsilon^-} \frac{f(\omega)}{\omega - z}\,d\omega}_{=0} \xrightarrow{\varepsilon \to 0} \int_{C^-} \frac{f(\omega)}{\omega - z}\,d\omega$$

hence $= 0$

Let $\varepsilon \longrightarrow 0$

$$\therefore \quad f(z) = \frac{1}{2\pi i}\left(\int_{C^+} + \int_{C^-} \right)$$

$f(z) = g(z)$ $\forall\, z \in$ square except line $\left(z_0 - \frac{r}{2},\ z_0 + \frac{r}{2}\right)$

from continuity of f, $\exists$ a seq^n $z_n \to$ pt of line

f is cts $\Rightarrow$ holo at that point also
$\Rightarrow$
$f(z) = g(z)$ everywhere in square

★ hence, f is a special case of this proposition
i.e. $f \in H(U)$

CONFORMALLY EQUIVALENT

$U, V \underset{\text{open}}{\subseteq} \mathbb{C}$

we say U & V are "conformally equi"

if $\exists$ $f: U \xrightarrow[\text{onto}]{1-1} V$, holo

examples :-

1. $\mathbb{D} \cong \mathbb{H}$
2. $\mathbb{R} \times (\alpha, \alpha+2\pi) \cong \mathbb{C} \setminus \overline{\mathbb{R}}_\alpha$ [$z \longrightarrow e^z$ map]

Check :-

Non-examples

Theory

Let $\mathcal{F}$ be family of functions $U \xrightarrow{} \mathbb{C}$, $U \overset{\text{open}\ \mathbb{C}}{\subseteq}$.

$\mathcal{F}$ is equicts at $z_0 \in U$ if

$$\forall \varepsilon > 0\ \exists \delta > 0 \ni |f(z) - f(z_0)| < \varepsilon \quad \forall f \in \mathcal{F},\ |z - z_0| < \delta.$$

proposition :- Let $\{f_n\}_1^\infty \subseteq C(U) \ni \{f_n\}_1^\infty$ converges pt wise

Denote $f(z) = \lim_{n\to\infty} f_n(z)$, $\forall z \in U$. Assume $\{f_n : n \in \mathbb{N}\}$ is equicontinuous then:-

a) f is cts
b) f_n converges a.u

proof:-

Let $K \underset{\text{cpt}}{\subseteq} U$, $\forall w \in K\ \exists \delta_w > 0 \ni \forall z \in D(w, \delta_w)$

$$|f_n - f_n(z_0)| < \varepsilon$$

$$\Rightarrow |f_n(z) - f(z_0)| \leq \varepsilon \quad \text{(point wise convergence)}$$

Now, $K \underset{\text{cpt}}{\subseteq} U \Rightarrow \exists\ w_1, \ldots, w_s \in K \ni K \subseteq \bigcup_{j=1}^{s} D(w_j, \delta_{w_j})$

Let $j = 1, \ldots, s$. $\exists N_j \in \mathbb{N} \ni \forall n \geq N_j$

$$|f_n(w_j) - f(w_j)| < \varepsilon$$

Let $N = \max_{1 \leq j \leq s} N_j$, $\forall n \geq N$ $|f_n(w_j) - f(w_j)| < \varepsilon$ $\forall j = 1, \ldots, s$

pick $z \in K$, choose $j \ni z \in D(w_j, \delta_{w_j})$

$$\Rightarrow \quad |f_n(z) - f(z)| \leq |f_n(z) - f_n(w_j)| + |f_n(w_j) - f(w_j)| + |f(w_j) - f(z)|$$

$$\leq 3\varepsilon$$

Lets choose a Metric

$$d(f,g) = \underbrace{\sum_{1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}}_{\text{* use all } K_n \text{ and bdd Metric}}$$

[can't use sup as U is not compact]

Ex :- ① d is a metric on $C(U)$

② $\{f_n\}$ converges to f in $d \Leftrightarrow f_n \xrightarrow{\text{a.u}} f$

* $H(U) \subseteq C(U)$

↓

closed subset

Theorem :- Montel's Theorem

Let $\mathcal{F} \subseteq H(U)$ be u. bdd on each $K \underset{}{\overset{cpt}{\subseteq}} U$, i.e

$\forall K \underset{cpt}{\subseteq} U$, $\exists M_K > 0 \ni \forall z \in K$, $|f(z)| < M_K \; \forall f \in \mathcal{F}$

Then $\overline{\mathcal{F}}$ is compact.

Defining the Metric

Tarun Goyal

Theorem:- $U \overset{\text{open}}{\subseteq} \mathbb{C}$,

$\exists \{K_n\} \underset{\text{compact}}{\subseteq} U \ni K_n \subseteq K_{n+1}^{\circ}$ and

$U = \bigcup^{\infty} K_n$

proof:- let $K_n = \{z : |z| \leq n\} \cap \{z : d(z, \mathbb{C}\setminus U) \geq \frac{1}{n}\}$

bdd

compact as intersection of 2 compact sets

done in A:1

AGAIN

Also, $K_n \subseteq \{z : |z| < n+1\} \cap \{z : d(\mathbb{C}\setminus U, z) > \frac{1}{n+1}\} \underset{\text{open}}{\subseteq} K_{n+1}$

hence $K_n \subseteq K_{n+1}^{\circ}$.

hence, $U = \bigcup K_n$

$U = \bigcup K_n^{\circ} \equiv \boxed{\text{open cover of } U} \Rightarrow K \subset G \Rightarrow K \subset K_n$ for some $n \in \mathbb{N}$

* using same notations

define $\rho_n(f,g) = \sup\{d(f,g) : z \in K_n\}$

use it

$$\rho(f,g) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

* converges because $\frac{t}{1+t} < 1 \;\forall t \geq 0$ and $\frac{1}{2^n} \xrightarrow{\text{decr. seq}^n} 0$

hence, converges

(using 3.42, Rudin)

Lemma: Let (S,d) be a metric space, then, $\mu(a,b) = \frac{d(a,b)}{1+d(a,b)}$ is a metric on S. (i) $X \underset{\text{open}}{\subseteq} (S,d) \Leftrightarrow X \underset{\text{open}}{\subseteq} (S,\mu)$

(ii) $\{a_n\}$ Cauchy in $(S,d) \Leftrightarrow \{a_n\}$ Cauchy in (S,μ)

Theorem:- $(C(U), \rho)$ is a metric space.

Lemma:- Given $\varepsilon > 0$, $\exists\, \delta > 0$ and $K \underset{\text{compact}}{\subseteq} U \ni$ for $f,g \in C(U)$

$\sup\{d(f,g) : z \in K\} < \delta$

$\Rightarrow \rho(f,g) < \varepsilon$

* Converse is Also true!

proof:- fix $p \in \mathbb{N} \ni \sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{2}$. Let $K = K_p$, defined above

Now, $\sup_K d(f,g) < \delta \Rightarrow \sum_1^{p} \frac{1}{2^n}\left(\frac{\delta}{\delta+1}\right) < \frac{\delta}{\delta+1}$

hence choose $\delta \ni \forall\, 0 \le t < \delta$, $\frac{t}{t+1} < \frac{\varepsilon}{2}$

Now,

$$\rho(f,g) = \underbrace{\sum_1^p \rho_n}_{\le \frac{\varepsilon}{2}} + \underbrace{\sum_{p+1}^{\infty} \rho_n}_{\le \frac{\varepsilon}{2}} < \varepsilon$$

Theorem:- a) $\mathcal{O} \subseteq C(U)$ is open $\Leftrightarrow \forall f \in \mathcal{O}\ \exists \underbrace{K \subseteq U}_{compact}$, $\delta > 0 \ni$

$$\mathcal{O} \supset \{g : d(f,g) < \delta : z \in K\}$$

proof:- $\Rightarrow$ $\mathcal{O}$ = open, $f \in \mathcal{O} \Rightarrow \exists\, \varepsilon > 0 \ni \mathcal{O} \supset \{g : \rho(f,g) < \varepsilon\}$. Using above lemma, $\exists\, \delta > 0$, $K \subseteq U \ni \sup_K d(f,g) < \delta \Rightarrow \rho(f,g) < \varepsilon \Rightarrow$ All $g \ni \sup_K d(f,g) < \delta$ $\subset \mathcal{O}$

$\Leftarrow$ Let $f \in \mathcal{O}$, using converse of above lemma

$\exists\, \varepsilon > 0 \ni \rho(f,g) < \varepsilon$ then $\sup_K d(f,g) < \delta$

$\Rightarrow \{g : \rho(f,g) < \varepsilon\} \subseteq \mathcal{O} \Rightarrow \mathcal{O}$ = open.

Lemma:- $\{f_n\}$ in $C(U,\rho)$ converges to $f \Leftrightarrow \{f_n\} \xrightarrow{a.u} f$.

proof: $\Rightarrow$ $\exists\, N \in \mathbb{N} \ni \rho(f_n, f) < \varepsilon\ \forall\, n \geq N$.

Let $K \subseteq U$, $\delta > 0$ be given,

To prove: $\sup_K d(f, f_n) < \delta\ \forall\, n \geq N$

using prev. lemma, $\exists\, \varepsilon > 0 \ni \rho(f, f_n) < \varepsilon \Rightarrow$

hence proved.

$\Leftarrow$ Let $f_n \xrightarrow{a.u} f$

$\Rightarrow \forall\, K \subseteq U$, $f_n \xrightarrow{u} f$ i.e $\forall\, z \in K$, $d(f_n, f) < \varepsilon\ \forall\, n \geq N$ — ①

now, let $\varepsilon > 0$. we know $\exists\, \delta > 0$, $K \subseteq U \ni \sup_K d(f, f_n) < \delta$

using ①, $\exists\, N \geq 1 \ni \forall\, n \geq N$, $\sup_K d(f_n, f) < \delta$ $\Rightarrow \rho(f, f_n) < \varepsilon$

hence proved.

Theorem :- $\mathcal{F} \subseteq H(U)$, uniformly bounded on all $\underset{\text{compact}}{K \subseteq U}$.
then,

i) $\overline{\mathcal{F}}$ is equicontinuous on all $\underset{\text{compact}}{K \subseteq U}$

ii) $\overline{\mathcal{F}}$ is compact

Montel's Theorem

(i) is by CIF.
as follows:-

proof :- (i)

Montels Theorem proof (ii)

① $\mathcal{F} \subseteq H(U)$ such that it is uniformly bounded on each $K \underset{\text{compact}}{\subseteq} U$. Then $\overline{\mathcal{F}}$ is compact

② $\mathcal{F} \subseteq H(U)$.

every sequence in $\mathcal{F}$ has a l.p in $H(U)$ $\Leftrightarrow$ $\mathcal{F}$ is locally bounded

[converse of above statement, proved in section "Compactness Criterion"]

Statement ①

proof:- $\{f_n\}_1^\infty \subseteq \mathcal{F}$. To show: it has a conv. subse.

that is, on some Dense Subset.

Consider $S = \{z_1, \ldots\}$ a dense subset of U.

we show $\exists$ $\{f_{n_k}\}_1 \ni \{f_{n_k}\}_1^\infty$ converges pt. wise on S.

$S = \{z_1, z_2 \ldots\}$.

$f_1(z_1), f_2(z_1), \ldots,$ bdd seq. $\Rightarrow$ convergent subsequence

Let it be $\{f_{1_j}(z)\}_{j=1}^\infty$, convergent on z_1

Now write, $f_{11}(z_2), f_{12}(z_2), \ldots$

again bounded $\Rightarrow$ $\{f_{2_j}(z)\}$ convergent on z_1, z_2

So on, Inductively Doable.

create a sequence, $f_{11}(z), f_{22}(z), \ldots$

Clearly, convergent $\forall\ z \in S$.

Compactness Criterion

$\mathcal{F} \in H(U)$ is compact $\Leftrightarrow$ $\mathcal{F}$ is closed and uniformly bounded on each $K \underset{cpt}{\subseteq} U$

$\Rightarrow$

proof:- Assume $\mathcal{F}$ is compact.
$\Rightarrow \mathcal{F}$ is closed.

Let $K \underset{cpt}{\subset} U$.

Consider, $\mathcal{F} \longrightarrow \mathbb{R}_{\geq 0}$:- $f \longrightarrow \|f\|_K$ $= \left\{ \sup_{z \in K} |f(z)| \right.$

Note:- $X \rightarrow$ compact Metric Space
$C(X) \longrightarrow \mathbb{R}_{\geq 0}$
$f \longrightarrow \|f\|_X$ is a metric. Consider a
(Fix g) set, S such that all $f \in S$ be $\ni \|f-g\|_K < r$.
Clearly on X, this set is bounded. (uniformly)

This is just to use that on K, f is bounded

$\mathcal{F}$ being compact $\Rightarrow \exists$ finite no. of such sets covering $\mathcal{F}$ up, hence, uniformly bounded on X.

Hence proved.

$\Leftarrow$ The converse,

$\mathcal{F}$ is uniformly bounded $\Rightarrow \overline{\mathcal{F}}$ = compact. (Montel Theorem)

$\mathcal{F}$ = closed $\Rightarrow \mathcal{F} = \overline{\mathcal{F}}$.

hence proved.

Remark:- 1. It doesn't work on $C(U)$.

2. It proves converse of Montel's Theorem.

Theorem:- $\mathcal{F} \underset{compact}{\subseteq} H(U)$ & $z_0 \in U$. $\Rightarrow \exists\, g \in \mathcal{F} \ni \forall f \in \mathcal{F}$

$|g'(z_0)| \geq |f'(z_0)|$

proof:- $\mathcal{F}_\epsilon \longrightarrow \mathbb{R}_{\geq 0}$

$f \longrightarrow \|f'(z_0)\|$ is continuous

Assume $f_n \xrightarrow{a.u} f \Rightarrow f_n'(z_0) \longrightarrow f'(z_0)$

Hence, maxima exists

Corollary (Compactness criterion)

$U \to$ Region. $z_0 \in U$, $\epsilon > 0$. Consider $\mathcal{F} = \{f : U \to \mathbb{D} : f$ is 1-1, holo, $|f'(z_0)| \geq \epsilon\}$

Then, $\mathcal{F}$ is compact

proof:- Enough to show $\mathcal{F}$ is closed, [$\|f\| \leq 1$, $\Rightarrow$ ubdd]

Consider, $\{f_n\}_1^\infty \subseteq \mathcal{F}$

$f_n \xrightarrow[n\to\infty]{a.u} f$, $f \in C(U)$

[because equicontinuous family because u-bdd each compact set] Also, $f_n' \in H(U) \Rightarrow f \in H(U)$

$\Rightarrow f_n' \xrightarrow{a.u} f' \Rightarrow |f_n'(z_0)| \longrightarrow |f'(z_0)|$

as $|f_n'(z_0)| \geq \epsilon \Rightarrow |f_n'(z_0)| \geq \epsilon$. Now, only thing left is f is one-one proof.

Using previous corollaries of Hurwitz Theorem

$\Rightarrow f \equiv$ constant or 1-1

$\downarrow$

X because $|f'(z_0)| > \epsilon$

hence f is one-one. $\Rightarrow f \in \mathcal{F}_\epsilon \Rightarrow \mathcal{F}_\epsilon$ is closed

Hence Proved

Remember:- uniformly bdd on each $K \underset{\text{compact}}{\subseteq} U$

$\Rightarrow$ equicontinuity

$\Rightarrow$ (convergence $\equiv$ a.u convergent)

$\Rightarrow$ holo $\longrightarrow f =$ holo

Theorem :- $\underset{\text{region}}{U \subseteq} \mathbb{C}$, $U \neq \mathbb{C} \ni \forall f \in H(U) (f \neq 0 \ \forall z \in U)$ has an analytic square root. Then $\exists f_0 \in H(U)$ $f_0 : U \xrightarrow{1-1} \mathbb{D}$

proof :- Let $a \in \mathbb{C} \setminus U$. Consider $z \longrightarrow z - a$: $z \in U$.
$\Rightarrow \exists h \in H(U) \ni \forall z \in U$, $h(z) = z - a$.
Observe, h is zero-free, one-one.
Also, from Open Mapping Theorem, $h(U) \underset{\text{open}}{\subseteq} \mathbb{C}$.
Define $-h(U) = \{-h(z) : z \in U\}$ is also open as : $\varphi(h(U)) = -h(U)$ homeomorphism

claim :- $h(U) \cap (-h(U)) = \phi$
proof :-

Let $w_0 \in -h(U)$
$\Rightarrow \exists r > 0 \ni D(w_0; r) \subseteq -h(U)$
$\Rightarrow D(w_0; r) \cap h(U) = \phi \Rightarrow$
$\forall z \in U$, $|h(z) - w_0| > r$. Consider $\varphi(z) = \frac{1}{h(z) - w_0}$, $z \in U$.

Clearly, $\varphi \in H(U)$ & $|\varphi(z)| < \frac{1}{r}$ $\forall z \in U$.

φ is 1-1.
Let, $f_0 = r\varphi$. Hence proved.

Consider $\mathcal{F} = \{ f : U \xrightarrow[\text{holo}]{1-1} \mathbb{D} : |f'(z_0)| \geq |f_0'(z_0)| \}$
where $z_0 \in U$ is fixed. Note $f_0'(z_0) \neq 0$

Since $\mathcal{F}$ is compact (proved above),
$\exists g \in \mathcal{F} \ni \forall f \in \mathcal{F}$, $|g'(z_0)| \geq |f'(z_0)|$.
We claim,
$g(U) = \mathbb{D}$

Assume Contrary, $g(U) \subset \mathbb{D}$. Then, $\exists\, \alpha \in \mathbb{D} \setminus g(U)$ Consider $\psi = \varphi_\alpha \circ g$.

Clearly, ψ is analytic, zero free $\Rightarrow$ admits an analytic square root, say h i.e $h^2 = \varphi_\alpha \circ g$.

Clearly, h = zero-free, one-one. Put $b = h(z_0)$.

Define, $f = \varphi_b \circ h$. Then $f(z_0) = \varphi_b(b) = 0$

Now, $g = \varphi_\alpha \circ h^2 = \varphi_\alpha \circ (\varphi_b \circ f)^2 = \varphi_\alpha \circ (\varphi_b^2 \circ f)$

$= (\varphi_\alpha \circ \varphi_b^2) \circ f$

hence, $|g'(z_0)| = |(\varphi_\alpha \circ \varphi_b^2)'(0)|\,|f'(z_0)|$ — $*$

as φ_b is 1-1 $\Rightarrow$ $\varphi_\alpha \circ \varphi_b^2$ is not one-one.

Hence, $|(\varphi_\alpha \circ \varphi_b^2)'(0)| < (1 - |\varphi_\alpha \circ \varphi_b^2(0)|^2)$

using $*$, $|g'(z_0)| < |f'(z_0)|$. But f should be injective and also,

$|f'(z_0)| > |g'(z_0)| \geq |f_0'(z_0)|$. This implies $f \in \mathcal{F}$

$\Rightarrow\Leftarrow$

<u>Remark</u>

① If g is a maximizer of $\{|f'(z_0)| : f \in \mathcal{F}\}$ then $g(z_0) = 0$. Otherwise consider $\varphi_\alpha \circ g$, $\alpha = g(z_0)$. Then,
$|(\varphi_\alpha \circ g)'(z_0)| = |\varphi_\alpha'(\alpha)\, g'(z_0)| = \frac{1}{1-|\alpha|^2}\, g'(z_0) > |g'(z_0)| \geq |f'(z_0)|$

$\Rightarrow \varphi_\alpha \circ g \in \mathcal{F}$ $\Rightarrow\Leftarrow$

② Let g be as above. Suppose $f : U \xrightarrow{\text{holo}} \mathbb{D}$, $f(z_0) = 0$. Then $f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ fixes origin. From Schwarz lemma $\forall z \in \mathbb{D}$,

$|f \circ g^{-1}| \leq |z| \Rightarrow |f(w)| \leq |g(w)| \;\; \forall w \in U$

and

$|(f \circ g^{-1})'(0)| = \left| f'(z_0) \cdot \frac{1}{g'(z_0)} \right| \leq 1 \Rightarrow |f'(z_0)| \leq |g'(z_0)|$

Futhermore, If $|f(\omega)| = |g(\omega)|$ for some $\omega \neq z_0$,
$\Leftrightarrow$ $f = \lambda g$ for some $|\lambda| = 1$.

Uniqueness :-

Suppose $g_1, g_2 : U \longrightarrow \mathbb{D}$ are bijective, holo. $g_1(z_0) = g_2(z_0) = 0$. Then. $g_1 \circ g_2^{-1} \in \mathrm{Aut}(\mathbb{D})$, fixes origin. hence,

$$g_1 = \lambda g_2 \quad , \text{ for some } |\lambda| = 1$$

* Let U be as before. Then $\exists !$ bijective holo function $g : U \to \mathbb{D}$ satisfying $g(z_0) = 0$, $g'(z_0) > 0$ *

If $g_1 = \lambda g_2 \ni$ ↗, then, $g_1' = \lambda g_2' \Rightarrow \lambda > 0 \Rightarrow \lambda = 1$

* We proved that, $U \subseteq \mathbb{C}$. open. connected, simply connected then if every zero-free analytic function admits an analytic square root, then $\exists f : U \xrightarrow[\substack{\text{onto} \\ \text{holo}}]{1\text{-}1} \mathbb{D}$

Goal is to reduce this square root assumption to simply connectedness only.

Theorem :- T.F.A.E, $\underset{\text{open, connected}}{U \subseteq \mathbb{C}}$

1. $\forall z \in \mathbb{C} \setminus U$ & closed path γ in U, $\mathrm{Ind}_\gamma(z) = 0$
2. $\forall f \in H(U)$, " " " " " $\int_\gamma f = 0$
3. $\forall f \in H(U)$, f admits a primitive
4. Every zero-free $f \in H(U)$, admits a analytic logarithm
5. " " " " , " a analytic n^{th} root $\forall n = 1, 2, \ldots$
6. " " " , " " " root for infinitely many n.
7. " " " admits an analytic square root.

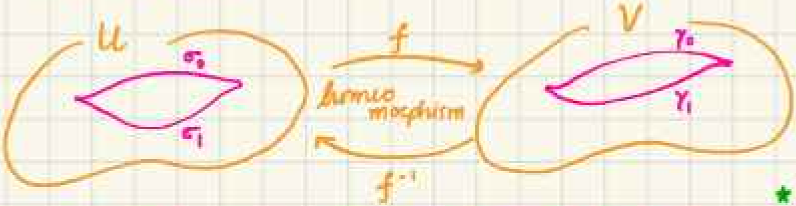
8. U is conformally equivalent to $\mathbb{D}$ provided $U \subsetneq \mathbb{C}$

9. U is homeomorphic to $\mathbb{D}$. $\left(\mathbb{C} \longrightarrow \mathbb{D},\ z \longrightarrow \frac{z}{1+|z|} \right)$
"a homeomorphism"

10. Every closed curve in U is null-homotopic.
[path replaced by curve]

11. Every closed path in U is null-homotopic.
i.e U is simply connected

$1. \longrightarrow 9.$ is done.



$H: [a,b] \times [0,1] \longrightarrow V$

$H(t,0) = \gamma_0(t)$

$H(t,1) = \gamma_1(t)$

* γ_0 is homotopic to γ_1

Now, take pre-image from f of γs.

$f \circ \sigma_0 = \gamma_0$

$f \circ \sigma_1 = \gamma_1$. taking appropriate composition, we get σ_0 is homotopic to σ_1.

* hence, homotopic is homeomorphically transferable and it is an iff statement

↑

to prove $9. \longrightarrow 10.$, we use and claim that every closed curve is null-homotopic.

$11. \longrightarrow 1.$

is clear, $\int_\gamma f = \int_{\gamma_{z_0}} f = 0$ $\quad$ $Ind_\gamma(z) = 0$

* in these statements, diffeomorphism is a speciality !!!

More on CONFORMAL MAPS :-

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, $f: U \xrightarrow{\text{holo}} \mathbb{C}$, $f'(z) \neq 0$, $\forall z \in U$

Recall :- $V = \mathbb{R}^n$, $u, w \in V$, $\neq 0$

$-1 \leq \frac{\langle v, w \rangle}{\|v\| \|w\|} \leq 1 \qquad \theta = \cos^{-1}(\ \ )$.

preservation of Angles

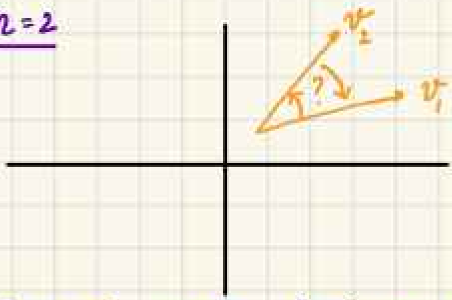
Def^n :- $T: \mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R}^n$ preserves angle if $\frac{\langle v, u \rangle}{\|v\| \|u\|} = \frac{\langle Tv, Tu \rangle}{\|Tv\| \|Tu\|}$ $\forall v, w \neq 0$

* Observe that, $\forall v \neq 0$, $Tv \neq 0$
$\Rightarrow$ Ker(T) = trivial $\Rightarrow$ Invertible

Theorem :- $T: \mathbb{R}^n \to \mathbb{R}^n$ preserves angle iff $\exists \lambda > 0 \ni \lambda T$ is orthogonal.

Now,

$n = 2$



v_1, v_2 l.I

$\Rightarrow$ There comes an idea of orientation. $\{v_1, v_2\}$ or $\{v_2, v_1\}$

Def^n :- Given 2 ordered basis of $\mathbb{R}^n$, $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_n\}$ are of same orientation if the! linear map T that sends v_i to w_i has positive determinant.

Def^n :- $\{v_1, \ldots, v_n\}$ is positively oriented if $\det(v_1 \, v_2 \cdots v_n) > 0$

In $n = 2$, when we say angle preserving, we will imply that it is oriented angles preserving linear maps i.e

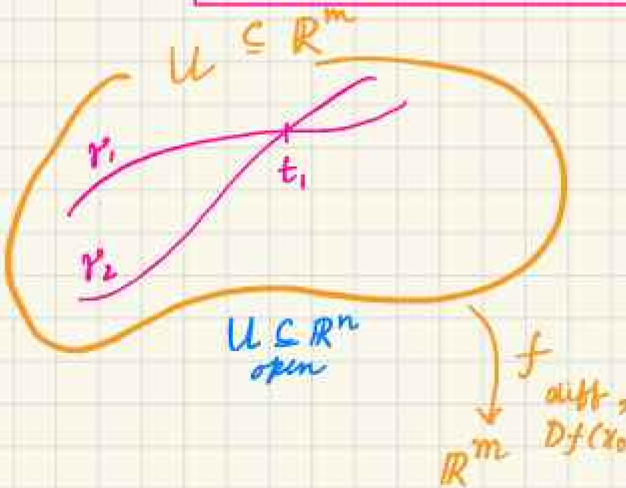
T preserves angles + orientation of angle formed by v_1, v_2.

In $\mathbb{R}^2$, combining with Theorem, we get:
Any linear map preserving angles with orientation must have determinant +ve and $\exists\ \lambda > 0 \ni \lambda T$ is orthogonal

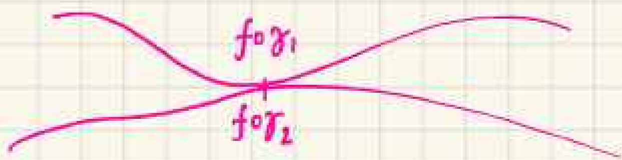
Thus the matrix of λT w.r.t standard basis,

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ [because det > 0, reflection det = −1]

$$\Rightarrow \boxed{T = \lambda^{-1}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$



Consider 2 diff curves γ_1, γ_2
$\gamma_1 : [a,b] \xrightarrow{\text{diff}} U$,
$\gamma_2 : [a,b] \xrightarrow{\text{diff}} U \ni \exists\ t_1, t_2 \in [a_1,b_1], [a_2,b_2]$.
$\gamma_1(t_1) = \gamma_2(t_2) = z_0$
and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$



Theorem:- f preserves angle b/w curves iff $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is angle preserving

Observe for $n = 2$, orientation is also preserved

Analysis

Theorem $\Rightarrow Df(x_0) = \lambda^{-1}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

i.e It satisfies Cauchy-Riemann eq^{n}s $\Rightarrow \boxed{f \text{ is Holo}}$ at z_0

also, $f'(z_0) \neq 0$, $|f'(z_0)|^2 = \det(Df(x_0,y_0))$

★ Holomorphic maps with non-zero derivatives are characterized by angle preserving maps. ★

Remember Mobius Transformation

$$f_M : \mathbb{H} \longrightarrow \mathbb{H}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \qquad f_M(z) = \frac{az+b}{cz+d}$$

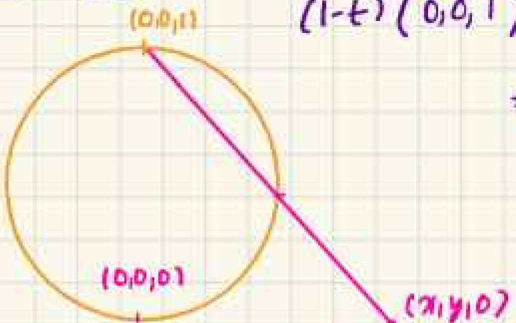
Stereographic Maps

Consider $(x,y) \longleftrightarrow (x,y,0)$

a surface

$$(1-t)(0,0,1) + t(x,y,0) = (tx, ty, 1-t) \in \text{surface}$$

$$\Rightarrow \quad t = \frac{1}{1+x^2+y^2}$$



$$\{(x_1,x_2,x_3) : x_1^2 + x_2^2 + \left(x_3 - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}\}$$

Stereographic Projection

$$\phi : (x,y) \longmapsto \left(\frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{x^2+y^2}{1+x^2+y^2} \right)$$

★ Every point on plane has ! point on sphere when joined with North Pole

Because

$$\psi : (x_1, x_2, x_3) \longrightarrow \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right), \quad \psi : S / \{0,0,1\} \xrightarrow{\text{cts}} \mathbb{R}^2$$

ϕ & ψ are inverse to each other

$\mathbb{R}^2$ is homeomorphic to $S \setminus \{$one point of $y\}$

Define $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$\Phi : \mathbb{C} \cup \{\infty\} \longrightarrow S \subseteq \mathbb{R}^3$ [extend φ for whole S]

$$\Phi(x+iy) = \begin{cases} \phi(x,y) & : (x,y) \in \mathbb{R}^2 \\ (0,0,1) & : (x,y) = \infty \end{cases}$$

By Construction, it is Bijective

Isometric

Consider

$$\underset{\text{set}}{X} \xrightarrow{f} \underset{(Y,d) = \text{metric space}}{Y}$$

Let $x_1, x_2 \in X$. Consider $d(f(x_1), f(x_2))$

$$x_1, x_2 \xrightarrow{d_f} d(f(x_1), f(x_2))$$ helps in defining a metric on X!

Thus $f : (X, d_f) \longrightarrow (Y, d)$
preserves distance $\Rightarrow$ Isometric

Assume f is onto as well! $\qquad$ $\Updownarrow$ f is one-one

$$f^{-1} (Y, d) \longrightarrow (X, d_f)$$

Since f is isometry, so is $f^{-1} \Rightarrow$ continuous
Then X, Y are homeomorphic

We use this Theory to metrize $\hat{\mathbb{C}}$ using Φ

$$\mathbb{C} \subseteq \hat{\mathbb{C}}.$$

Observations.

1. $\hat{\mathbb{C}}$ is homeomorphic to $S \Rightarrow \hat{\mathbb{C}}$ is compact + connected
2. $\hat{\mathbb{C}}$ is isometric to $S \Rightarrow$ complete metric space
   i.e every Cauchy $\Rightarrow$ Convergent
3. $\mathbb{C}$ has 2 metrics.
   (i) one is original.
   (ii)
$$\boxed{\hat{d}(z,w) = \begin{cases} \dfrac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} & : (z,w) \in \mathbb{C}^2 \\ \dfrac{1}{\sqrt{1+|z|^2}} & : z \in \mathbb{C},\ w = \infty \end{cases}}$$

★★ 4. Let $\{z_n\}$ be a seq^n in $\hat{\mathbb{C}}$. Then $\{z_n\}$ converges to ∞ iff $\{|z_n|\}_{n=1}^{\infty}$ diverges to ∞.

General Mobius Transformation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}), \quad ab - cd \neq 0$$
$$a, b, c, d \in \mathbb{C}$$

$$f_g(z) = \begin{cases} \dfrac{az+b}{cz+d} & : z \neq -\dfrac{d}{c} \\ \infty & : z = -\dfrac{d}{c} \\ \dfrac{a}{c} & : z = \infty \end{cases}$$

$w = \dfrac{a}{c}$ $\quad \dfrac{b}{d} = z$

$b = dz$

Clearly,
$$\phi \equiv f_g : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

if $c = 0$, then $f_g(z) = \begin{cases} \dfrac{az+b}{d} & : z \in \mathbb{C} \\ \infty & : z = \infty \end{cases}$

Observe :-

i) $\phi : \mathbb{C} \setminus \{-\frac{d}{c}\} \longrightarrow \mathbb{C}$ is holomorphic

ii) $z \neq -\frac{d}{c} \Rightarrow$ derivative at z is $\dfrac{ad-bc}{(cz+d)^2} \neq 0$

★ Hence, $\phi : \mathbb{C} \setminus \{-\frac{d}{c}\} \longrightarrow \mathbb{C}$ is Conformal map!

iii) $\hat{\mathbb{C}} \xrightarrow[\text{map}]{\text{this}} \hat{\mathbb{C}}$ is its [extend using limit + prev. Theorem]

iv) $\forall g_1, g_2 \in GL_2(\mathbb{C})$

$$f_{g_1 g_2} = f_{g_1} \circ f_{g_2} \ \& \ f_{I_2} = \text{Id map}$$

Thus $\boxed{GL_2(\mathbb{C}) \text{ acts on } \hat{\mathbb{C}}}$

Examples

1) Translations $z \to z+b$ $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

2) Rotation $z \to e^{i\theta} z$ $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$

3) Dilation $z \to az$ $\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$ $a \in \mathbb{R}_{>0}$

4) Inversion $z \to \frac{1}{z}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ All Mobius Transformations can be built from these & Inverse

Observe:- $c=0$ $\frac{az+b}{d} = \underbrace{\left|\frac{a}{d}\right|}_{\text{Dilation}} \cdot \underbrace{\frac{a/d \, z}{|a/d|}}_{\text{Rotation}} + \underbrace{b/d}_{\text{Translation}}$

If $c \neq 0$, $f(z) = \frac{bc-ad}{c^2} \cdot \frac{1}{z + d/c} + \frac{a}{c}$

Analysis Translation $\to$ Inversion $\to$ Scaling $\to$ Translation
$\equiv$ Dilation + Rotation

$f_M(z) = z$ has i) 2 solutions if $M \neq I_2$
ii) ∞ " if $M = I_2$

Lemma :- Given $z_1, z_2, z_3 \in \mathbb{C}$, all distinct, $\exists ! \, M \in GL_2(\mathbb{C}) \ni$
$f_M(z_1) = \infty$, $f_M(z_2) = 1$, $f_M(z_3) = 0$

proof:- existence
Consider $f_M(z) = \frac{(z-z_3)(z_2-z_1)}{(z-z_1)(z_2-z_3)}$. Clearly all properties satisfied. ■

uniqueness
Suppose 2 such Transf.

$f_{M_1} f_{M_2^{-1}}$ fixes 3 pts (z_1, z_2, z_3) $\Rightarrow M_1 M_2^{-1} = I_2 \Rightarrow M_1 = M_2$

Theorem:- $\exists! M \in GL_2(\mathbb{C}) \ni f_M(z_i) = w_i$, $i = 1,2,3 \ni z_i \neq z_j \ \forall i \neq j$, $w_i \neq w_j \ \forall i \neq j$, $z_i, w_i \in \hat{\mathbb{C}} \ \forall i$

proof:- Consider

$$z_i \xrightarrow{M_1} 0,1,\infty \qquad w_i \xrightarrow{M_2} 0,1,\infty$$

Clearly $f_{M_1 M_2^{-1}}$ is the required map.

★ This sends lines + circles to lines + circles

↓

To get image of a map. Knowledge of how it acts on boundary is enough.

proof:- S be a circle in $\mathbb{C}$ $\Rightarrow$ S is invariant to dilations, translations, rotations.

Check for inversions, $|z - z_0| = r$, let $w = \frac{1}{z}$.

then,

$$|z|^2 - 2\mathrm{Re}(\bar{z} z_0) - r^2 = 0 \Rightarrow \frac{1}{|w|^2} - 2\frac{\mathrm{Re}(w z_0)}{|w|^2} + |z_0|^2 - r^2 = 0$$

If $|z_0| \neq r$ then,

$$|w|^2 - \frac{2\mathrm{Re}(w z_0)}{|z_0|^2 - r^2} + \frac{1}{|z_0|^2 - r^2} = 0$$

i.e. $|w|^2 - 2\mathrm{Re}(w\bar{b}) + c = 0$

$=$ circle.

If $|z_0| = r$, then, $2\mathrm{Re}(w z_0) = 1$ $\qquad w = u + iv$

i.e. $(u x_0 - v y_0) = \frac{1}{2}$ = line in $\mathbb{C}$ and ∞ as $w = \frac{1}{z}$ + $z = 0$ at a point

hence a circle as (line $\cup \{\infty\}$)

Let S = line, $2\mathrm{Re}(z\bar{z}_0) = a$, $a \in \mathbb{R}$

$\Rightarrow w = \frac{1}{z} \Rightarrow 2\mathrm{Re}(w z_0) = a|w|^2$. If $a = 0$, line through origin.

else, $w z_0 + \bar{w}\bar{z}_0 - a w\bar{w} = 0$

$$\Rightarrow w\bar{w} - \frac{w z_0}{a} - \frac{\bar{w}\bar{z}_0}{a} + \frac{|z_0|^2}{a^2} - \frac{|z_0|^2}{a^2} = 0$$

= circle in $\mathbb{C}$

SINGULARITIES

Isolated Singularities

Defⁿ:- $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $z_0 \in U$, $f : U \setminus \{z_0\} \longrightarrow \mathbb{C}$

Then we say z_0 is an isolated singularity at z_0.

$\downarrow$ Removable Singularity

If f can be re-defined at z_0 $\ni$ extension is holo at z_0.

Lemma:- f has removable singularity at z_0 iff $\exists\, r > 0$ $\ni$ $D(z_0, r) \setminus \{z_0\} \subseteq U$ and f is bounded on $D(z_0, r) \setminus \{z_0\}$

Types of Singularity

$U \underset{\text{open}}{\subseteq} \mathbb{C}$, $z_0 \in U$, $f : U \setminus \{z_0\} \longrightarrow \mathbb{C}$. Then one of following must be true,

1) $\forall\, r > 0$ $\ni$ $f(D(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$

If not then?

$\forall r>0\ \exists\, w \in \mathbb{C}$, $\delta > 0$ $\ni$

$|f(z) - w| \geq \delta \quad \forall z \in D(z_0, r) \setminus \{z_0\}$

Consider, $g(z) = \dfrac{1}{f(z) - w}$, $z \in D(z_0, r) \setminus \{z_0\}$

$|g(z)| \leq \dfrac{1}{\delta}$

$\Rightarrow$ g has a removable singularity at $z = z_0$.

Extend g, call it g as well!

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Case I:- $g(z_0) \neq 0 \Rightarrow \lim\limits_{z \to z_0} g(z)$ exists $\Rightarrow$

$\lim\limits_{z \to z_0} \dfrac{1}{f(z)-w}$ exists and not ∞

hence, $f(z)$ is bounded $\Rightarrow$ removable Singularity

Case II:- $g(z_0)=0$

Let $m \geq 0$ be the order of the zero z_0.

$\Rightarrow g(z) = (z-z_0)^m g_1(z)$, $g_1 \in H(D(z_0, r))$ & g_1 is zero-free in neighbourhood of z_0, say R

hence,

$$f(z) = w + \frac{1}{(z-z_0)^m} \times \underbrace{\frac{1}{g_1(z)}}$$

analytic because non-zero in $D(z_0, R)$

hence,

$$f(z) = w + \frac{a_0}{(z-z_0)^m} + \cdots + a_m + \left(a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \cdots\right)$$

i.e

$$\boxed{f(z) - \sum_{k=0}^{m} \frac{a_k}{(z-z_0)^{m-k}} = w + \phi(z) \quad \forall z \in D(z_0,R) \setminus \{z_0\}}$$

ϕ defined $\forall z \in D(z_0, R)$

hence, Removable singularity for both sides

$\Rightarrow$

$$\underbrace{f(z) - \sum_{k=0}^{m} \frac{a_k}{(z-z_0)^{m-k}}}_{(k \neq m)}$$ has removable singularity at z_0.

This z_0 is called a "pole"

Principal part of f at z_0.

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Summarise

★ 3 possible scenarios of isolated singularity at z_0 ★

① **Removable** :- $\lim_{z \to z_0} f(z)$ exists and is finite

② **Pole** :- $\lim_{z \to z_0} f(z) = \infty$ or $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

③ **Essential** :- If $\lim_{z \to z_0} f(z)$ D.N.E, or IIIly dense in $\mathbb{C}$ Range.

Analysis of Poles

uniqueness

$\exists r > 0 \ni f(z) = \sum \frac{a_k}{(z-z_0)^{m-k}} + h(z)$ where $h(z) \in H(D(z_0, r))$
$\forall z \in D(z_0, r) \setminus \{z_0\}$

Then, This expansion is unique

proof :- Suppose not, $f(z) = \sum_{k=1}^{m} \frac{b_k}{(z-z_0)^{m-k}} + h_2(z)$ for some $h_2(z) \in H(D(z_0, r))$

$$\underbrace{\sum \frac{b_k - c_k}{(z-z_0)^{m-k}}}_{\substack{\text{non-removable} \\ \text{singularity at } z_0 \text{ if } \exists k \ni c_k \neq b_k}} = h(z) - h_2(z) \in H(D(z_0, r))$$

$\Rightarrow\Leftarrow$.

Observe, m is unique because $h(z)$ is defined at z_0. If we change m, $h(z)$ won't be holo anymore.

MEROMORPHISM

$\underset{\text{open}}{U \subseteq \mathbb{C}}$, $A \subseteq U$.

f is said to be Meromorphic if

1. $f : U \setminus A \xrightarrow{\text{holo}} \mathbb{C}$

2. $\forall z \in A$, z is a pole of f

3. A has no limit point in U.

Remark: ★ $\forall K \subseteq U$, $K \cap A$ is finite
★ A is countable
★ $f \in H(U) \Rightarrow f$ is meromorphic

Residues

$\underset{\text{region}}{U \subseteq \mathbb{C}}$, $f \in H(U)$, assume $f \neq 0 \; \forall z \in U \Rightarrow Z(f)$ has no l.pt in U.

Consider $g = \frac{f'}{f}$. Let $z_0 \in Z(f)$.

Then, $\exists \, m \in \mathbb{N}$ & $g_1 \in H(D(z_0, r))$ $\ni$

$$f(z) = (z - z_0)^m g_1(z) \;\&\; g_1(z_0) \neq 0$$

$$\Rightarrow \frac{f'}{f} = \frac{m}{z - z_0} + \underbrace{\frac{g_1'(z)}{g_1(z)}}_{\in H(D(z_0, r))}$$

★ By uniqueness, $\frac{f'}{f}$ has a pole at z_0 of order 1

Let f be meromorphic, A be set of poles of f.

Define, $z_0 \in A$, $Q(z) = \sum_{k=1}^{m} \frac{c_k}{(z - z_0)^k}$

Then, c_1 is called Residue of f at z_0 denoted by $\operatorname{Res}(f, z_0)$

Observations :-

Let γ be a cycle $\ni z_0 \notin \gamma^*$.

$$\frac{1}{2\pi i}\int_\gamma Q = \frac{c_1}{2\pi i}\int \frac{dz}{z-z_0} + \sum_{k=2}^{m} \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-z_0)^k}$$

have primitive $\Rightarrow \int_\gamma = 0$

Hence, $\frac{1}{2\pi i}\int_\gamma Q(z) = \frac{1}{2\pi i}\int_\gamma f(z) = \text{Res}(f, z_0)\,\text{Ind}_\gamma(z_0)$

where $\gamma \in D(z_0, r)$, $z_0 \notin \gamma^*$

Residue Theorem

Let f be as before, suppose that γ is a cycle $\ni \gamma^* \subseteq U \setminus A$ & assume $\gamma \stackrel{U}{\sim} 0$

Then, $\frac{1}{2\pi i}\int_\gamma f = \sum_{a \in A} \text{Res}(f, a)\,\text{Ind}_\gamma(a)$

Observe:- $\sum_{a \in A} \text{Res}(a, f)\,\text{Ind}_\gamma(a)$ is a finite sum!

Let $B = \{a \in A : \text{Ind}_\gamma(a) \neq 0\}$

i) B is bounded : because, $\mathbb{C} \setminus \gamma^*$ has only one unbounded component and $\forall z$ in it $\text{Ind}_\gamma(z) = 0$

ii) B has no limit points : If it does, say z_0, then $z_0 \notin U$ because $B \subseteq A$ & A has no l.p in U

Observe $\gamma^* \subseteq U \Rightarrow \text{Ind}_\gamma(z_0) = 0$. and because its a l.p $\exists \{z_n\}_1^\infty \subseteq B \ni z_n \longrightarrow z_0 \Rightarrow \underbrace{\text{Ind}_\gamma(z_n)}_{\neq 0, \in \mathbb{Z}^+} \longrightarrow \underbrace{\text{Ind}_\gamma(z_0)}_{=0}$

hence, no limit point

Using Bolzano Wierstrass Theorem, B = finite

Analyses of B & f

Let $B = \{z_1, \ldots, z_n\}$. Let $Q_1, \ldots, Q_n$ be the resp. principal parts.

Consider $f - Q_1 - Q_2 - \cdots - Q_n$, $U \setminus \{z_1, \ldots, z_n\}$

$$= \underbrace{\underbrace{(f - Q_1)}_{\text{Removable sing at } z_1} - \underbrace{(Q_2 + \cdots + Q_n)}_{\text{holo at } z_1}}_{\text{removable sing at } z_1}$$

$\Rightarrow$ $f - \sum_1^n Q_i$ is extendable to a holomorphic function for $U \setminus (A \setminus B) = U_1$

* as we picked only those poles with finite Index

Proposition :- $$\frac{1}{2\pi i} \int_\gamma f - \sum Q_i = 0$$

i.e $$\frac{1}{2\pi i} \int_\gamma f = \sum_1^n \text{Res}(f, z_i)\, \text{Ind}_\gamma(z_i)$$

Zeros & Multiplicity

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Proposition :- Let $\underset{\text{region}}{U} \subseteq \mathbb{C}$, $f \in H(U)$ not identically zero. Let γ be a cycle in $U \ni \gamma \underset{U}{\sim} 0$

Define $U_1 = \{ z \in U : \text{Ind}_\gamma(z) = 1 \} \underset{\text{open}}{\subseteq} U$ [Let no zero in γ^* + $\text{Ind}_\gamma(z) = 0$ or 1 $\forall z \in Z(f)$]

Let N_f = no. of zeros in U_1 counted acc. to multiplicity. Then,

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}$$

proof :- Apply Residue Theorem to

$$\frac{f'}{f} \Rightarrow \frac{1}{2\pi i} \int \frac{f'}{f} = \sum_{a \in Z(f)} \text{Res}\left(\frac{f'}{f}, a\right) \text{Ind}(a)$$

$$= \sum_{a \in Z(f) \cap U_1} \text{Res}\left(\frac{f'}{f}, a\right) \times 1 \text{. Also } \frac{f'}{f} = \frac{m}{z-a} + \text{holo} \Rightarrow \underbrace{\text{Res}\left(\frac{f'}{f}, a\right)}_{= m}$$

hence,

$$\frac{1}{2\pi i} \int \frac{f'}{f} = \sum_{a \in Z(f)} \text{order}(f, a)$$

Hence proved.

Recall :-

1) $U \underset{\text{reg}}{\subseteq} \mathbb{C}$ 2) γ := closed path in U 3) $\gamma \underset{U}{\sim} 0$ 4) $\forall z \in U \setminus \gamma^*$, $\text{Ind}_\gamma(z) \in \{0, 1\}$

Then, $\text{Ind}_{f \circ \gamma}(0) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}$ = Zeros of f in U_1

Rouché's Theorem

Let assumptions be as before, $f, g \in H(U) \ni |f(w) - g(w)| < |f(w)| \; \forall w \in \gamma^*$

Then, no. of zeros in f, g are same in U_1 { U = Region }

i.e. $\text{Ind}_{f \circ \gamma}(0) = \text{Ind}_{g \circ \gamma}(0)$

[Also, they have same no. of zeros]

proof :- Assumptions $\Rightarrow \forall t \in [a, b]$

$$|f(\gamma(t)) - g(\gamma(t))| < |f(\gamma(t))|$$

$\Rightarrow 0 \notin \gamma^*$. Using Ex. sheet 3

Hence proved.

proposition: f and g have same no. of zeros inside γ

proof :-

$$Ind_{f\circ\gamma}(0) = Ind_{g\circ\gamma}(0)$$

$$\Rightarrow \int_{f\circ\gamma} \frac{dz}{z} = \int_{g\circ\gamma} \frac{dz}{z}$$

$$= \int_a^b \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t))} = \int_a^b \frac{g'(\gamma(t))\gamma'(t)\,dt}{g(\gamma(t))} \Rightarrow \int_\gamma \frac{f'}{f} - \frac{g'}{g} = 0$$

$\Rightarrow$ same no. of zeros

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Applications

1) $\int \frac{f'}{f}$ by using an easy f^n g.

Define:- γ be a closed path in $\mathbb{C}$. We say γ has an interior if the Image of

$$Ind_\gamma : \mathbb{C} \setminus \gamma^* \longrightarrow \mathbb{Z} \text{ is precisely } \{0,1\}.$$

And $Int(\gamma) = U_1 = \{ z \in \mathbb{C} \setminus \gamma^* : Ind_\gamma(z) = 1 \}$

Q :- If γ has an interior then, $Int(\gamma) \cup \gamma^*$ is compact

proof :-

Theorem :- Let $U \subseteq \mathbb{C}$ region and γ is a closed path in U with $\gamma \underset{U}{\simeq} 0$. $f \in H(U)$, non constant. Suppose

i) γ and $f\circ\gamma$ both have interiors

ii) $f(\text{Int}\,\gamma) \cap (f\circ\gamma)^* = \phi$

Then $f : \text{Int}\,\gamma \longrightarrow \text{Int}(f\circ\gamma)$

* f is one-one

Also, if $\text{Int}(f\circ\gamma)$ is connected then f is a conformal equivalence b/w Interiors.

Proof :- $z \in \text{Int}\,\gamma$. $\text{Ind}_{f\circ\gamma}(f(z)) = \dfrac{1}{2\pi i}\displaystyle\int_{f\circ\gamma} \dfrac{d\xi}{\xi - f(z)}$

$$= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t)) - f(z)}$$

Claim :- $\text{Ind}_{f\circ\gamma}(f(z)) \geq 1$

Now, Consider $g = f - f(z)$

Ind $= 0$ or 1 but

g has a zero $\Rightarrow \text{Ind}_{g\circ\gamma}(0) \geq 1$

$\qquad \| $

$\text{Ind}_{f\circ\gamma}(f(z)) \geq 1 \Rightarrow \boxed{= 1}$

hence, $f(z) \in \text{Int}(f\circ\gamma)$

$\therefore$ g has only 1 zero, z in the interior (Ind $= 1$)

$\Downarrow$

$\forall\, z, w \in \text{Int}\,\gamma$, $f(z) \neq f(w)$ if $z \neq w$. ■

Now, if $f(\text{Int}\,\gamma)$ is connected.

If we show $f(\text{Int}\,\gamma)$ is closed in $\text{Int}(f\circ\gamma)$ then done because it is open & connected $\Rightarrow$ onto'ness

Let $\{w_n\}_1^\infty$ be conv. in $f(\text{Int}\,\gamma)$ $\longrightarrow w$

$\boxed{\begin{matrix} \| \\ f(z_n) \\ \in \text{Int}\,\gamma \end{matrix}}$ As $\text{Int}\,\gamma \cup \gamma^*$ is compact, $\exists$ conv. subseq. $\{z_{n_k}\}_{k=1}^\infty$ conv. to z_0, say. Observe that $f(z_{n_k}) \longrightarrow f(z_0) = w$

$\in \text{Int}\,\gamma \cup \gamma^*$

To prove $w \in f(\text{Int}\,\gamma)$:- else, $f(z_0) \in (f\circ\gamma)^*$ but $z_0 \in \text{Int}\,\gamma$

Since, $f(\text{Int}\,\gamma) \cap (f\circ\gamma)^* = \phi$

$\rightarrow\!\leftarrow$

Some Analysis :- $U \subseteq \mathbb{C}$, f is meromorphic. Let $P(f)$ = set of all poles

[Recall:- $P(f)$ has no limit point in U]

Arguement Principle

Let f be meromorphic, Let γ_* be closed path or cycle in U.
$\ni \quad \gamma^* \subseteq U \setminus P(f)$ and $\gamma \underset{U}{\simeq} 0$.
If $\forall w \in \gamma^*$. $f(w) \neq 0$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \sum_{z \in P(f) \cup Z(f)} \text{order}(f, z) \cdot \text{Ind}_\gamma(z)$$

proof:- Obs if $\underset{\text{region}}{U \subset \mathbb{C}}$, $A \subseteq U$ doesn't have l.pt $\Rightarrow U \setminus A$ = connected

$\Rightarrow$ $U \setminus P(f)$ is a region. As f vanishes nowhere on γ^* $\Rightarrow$ $Z(f)$ has no lpt in $U \Rightarrow U \setminus (P(f) \cup Z(f))$ is a region

$\Rightarrow \quad \frac{f'}{f} \in H(U \setminus P(f) \cup Z(f))$

$$\sum_{z_0 \in Z(f)} \text{Res}\left(\frac{f'}{f}, z_0\right) \text{Ind}_\gamma(z_0) = \sum_{z_0 \in Z(f)} \text{order}_{z_0}(f) \, \text{Ind}_\gamma(z_0)$$

also, $\displaystyle\sum_{z_0 \in P(f)}$ " $= -\displaystyle\sum_{z_0 \in P(f)} \text{order}_{z_0}(f) \, \text{Ind}_\gamma(z_0)$

Hence proved

Recall:-

Let $R>0$ be radius of conv. of $f(z) = \sum_0^\infty c_n z^n$ then $\sum c_n (z-a)^{-n}$ is convergent for $|z-a| > \frac{1}{R}$.

Cauchy Integral formula for an Annulus

Let $f \in H(D'(z_0; R))$, choose $r_1, r_2 \ni 0<r_1<r_2<R$.

punctured disk

Let $\gamma_j(t) = z_0 + r_j e^{it} : 0 \le t \le 2\pi$. Then $\forall z \ni r_1 < |z-z_0| < r_2$, we have,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z}\, dw$$

proof:-

$$F(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & : w \in D'(z_0; R), w \neq z \\ f'(z) & : w = z \end{cases}$$

Then $F(w) \in H(D'(z_0, r))$.

Now since γ_1 & γ_2 are homotopic in the Annulus,

$$\int_{\gamma_1} F = \int_{\gamma_2} F \text{, hence,}$$

$$\int_{\gamma_2} \frac{f(w)-f(z)}{w-z} - \int_{\gamma_1} \underbrace{\frac{f(w)-f(z)}{w-z}}_{=0} = 0$$

hence proved.

Laurent Expansion

Let $f \in H(A(a;r,R))$ then $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n, z \in A$

where, $a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$ and $\gamma = a + \rho e^{it}$ for some $r < \rho < R$, $0 \leq t < 2\pi$

Also, the series converges absolutely & uniformly on $K \subseteq A$ cpt

proof :- $r < r_1 < |z-a| < r_2 < R$. (choose r_1, r_2)

Then.

$$2\pi i \, f(z) = \int_{\gamma_2} \frac{f(w)}{w-z} dw - \int_{\gamma_1} \frac{f(w)}{w-z} dw = f_2 + f_1 \text{ (say)}$$

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \begin{cases} \dfrac{-1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} & : w \in \gamma_1^* \\ \dfrac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)} & : w \in \gamma_2^* \end{cases}$$

[= geometric series = unif cgt ⇒ swap $\int$ & Σ]

Therefore
$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n$$
$$+ \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n$$

Now, Since $\frac{f(w)}{(w-a)^{n+1}} \in H(A)$ and $\gamma_1 \underset{A}{\sim} \gamma_2$

we are done.

Hence proved.

Remark :- $a_{-1} = \text{Res}(f; a)$

Classifying singularities with Laurent Series

Case I removable sing at z_0.

$\Rightarrow$ $\forall n<0$, $a_n = 0$ as $\frac{f(w)}{(w-z_0)^{n+1}}$ is holo making $\int = 0$

$\Leftarrow$ If $a_n = 0$, $\forall n<0$ then f is power series centred at z_0, then f has a power series centred at $z_0 \Rightarrow f \in H(z_0)$

Hence, $\boxed{\text{Removable sing iff } a_n = 0 \ \forall n<0}$

Case II f has a pole at z_0

$$f(z) - \left(\frac{a_{-1}}{z-z_0} + \cdots + \frac{a_{-k}}{z-z_0}\right) \text{ has a remov. sing}$$

Now it is evident.

Analysis

γ is closed curve in $U \setminus \{z_0\}$, $\gamma \underset{U}{\sim} 0$ and $\mathrm{Ind}_\gamma(z_0) = 1$

$$\Rightarrow \quad C(z_0;r) \underset{U\setminus\{z_0\}}{\sim} \gamma$$

and $w \longmapsto f(w)$, holo on $U \setminus \{z_0\}$

$$\Rightarrow \int_{C(z_0;r)} f(w)\,dw = \int_\gamma f(w)\,dw$$

[homologous version of Cauchy's Theorem]

$$f : U \xrightarrow[\text{holo}]{1\text{-}1} \mathbb{C} \quad , \quad g : U \xrightarrow{\text{holo}} \mathbb{C}$$

Recall, $f' \neq 0$ in U. $\overline{D(z_0;r)} \subseteq U \Rightarrow f(\overline{D(z_0;r)}) \subseteq \mathbb{C}$

Consider, $w \in f(D(z_0;r))$

$$h(z) = \frac{g(z) f'(z)}{f(z) - w} \quad , \quad z \in D(z_0;r) \setminus \{f^{-1}(w)\}$$

Conclusion:-

$$\mathrm{Res}(h, f^{-1}(w)) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{g(z) f'(z)\,dz}{f(z) - w}$$

- $f' \neq 0$ as f is 1-1
- $f^{-1}(w)$ is zero of $z \longmapsto f(z) - w$ (order 1)
- hence a pole of order 1

$\boxed{\star \text{ Poles of order 1 = simple poles}}$

Proposition:- Suppose $g \in H(U)$, f has a simple pole at z_0. Then $\mathrm{Res}(gf, z_0) = g(z_0)\,\mathrm{Res}(f, z_0)$

$$f(z) - \frac{K}{z - z_0} = \varphi(z) \in H(U) \Rightarrow f(z) = \frac{K + \varphi(z)(z - z_0)}{z - z_0} = \frac{\psi(z)}{z - z_0} \quad : z \neq z_0$$

Thus $\psi(z_0) = \mathrm{Res}(f, z_0)$

Conversely, $f(z) = \dfrac{\theta(z)}{z - z_0}$: $\theta \in H(U)$: $\theta(z_0) \neq 0$

Then f has a simple pole at z_0.

Since $f(z) = \dfrac{\phi(z)}{z - z_0}$, $\mathrm{Res}(gf, z_0) = g(z_0)\,\phi(z_0)$

Hence proved

Continuing :- $\text{Res}(h, f^{-1}(w)) = g(f^{-1}(w)) \cdot \text{Res}\left(\frac{f'}{f-w}, f^{-1}(w)\right)$

as $\frac{f'}{f-w}$ has simple pole at $f^{-1}(w)$

$\Rightarrow \quad \text{Res}(h, f^{-1}(w)) = g(f^{-1}(w)) \cdot 1$

$\Rightarrow$

$$g(f^{-1}(w)) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{g(z) f'(z)}{f(z)-w}\, dz \quad \forall w \in f(D(z_0;r))$$

In particular if $g = Id$,

$$\star\star \quad \boxed{f^{-1}(w) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{z f'(z)}{f(z)-w}\, dz} \qquad \forall w \in f(D(z_0;r))$$

Theorem:- f is holo on $\mathbb{C}$ except at finitely many poles, none of which are on real line. Also, the ones on the upper half are $z_1, \dots, z_n$. Assume further that $\exists\, a > 0$ & $R > 0, M > 0 \ni |f(z)| \leq \frac{M}{|z|^{1+a}} : |z| \geq R$

Then,
$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} f = \sum_{k=1}^{n} \text{Res}(f, z_k)$$

proof:-

$M >> 1$ [suff. large]

From Residue theorem, $\frac{1}{2\pi i} \int_{\gamma_r} f = \sum \text{Res}(f, z_k) \quad \forall r >> 1$

$$\Rightarrow \int_{-r}^{r} f(x)\, dx + \int_{\frown} f(z)\, dz$$

$$\left| \underbrace{\quad}_{} \right| \leq \frac{\pi r}{r^{1+a}} M \xrightarrow{r \to \infty} 0$$

hence proved.

Using Residue Theory to Calculate Integrals $\int_{-\infty}^{\infty}$, $\int_{0}^{\infty}$

$$\int_{0}^{\infty} f(x)\, dx = \lim_{R \to \infty} \int_{0}^{R} f(x)\, dx$$

If $\lim_{b\to\infty} \lim_{a\to-\infty} \int_a^b f(x)dx$ exists then we say $\int_{-\infty}^{\infty} f(x)dx$ converges

$$\Downarrow$$

$\exists\, l \ni \forall \varepsilon>0,\ \exists R>0 \ni$ whenever $a<-R$ & $b>R$

$$\left|\int_a^b f - l\right| < \varepsilon$$

Also, $\lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b f \neq \lim_{R\to\infty} \int_{-R}^{R} f$ [In general]

eg:- $\int_{-R}^{R} x\,dx \longrightarrow 0$ but $\int_{-\infty}^{\infty} x\,dx$ is undefined

If we know Integral is convergent, then, $\lim_R \int_{-R}^{R} f$ can be used

Absolute Convergence

If $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, then $\int_{-\infty}^{\infty} f\,dx$ converges

Absolute conv. $\Rightarrow$ conv.

Converse not true eg:- $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

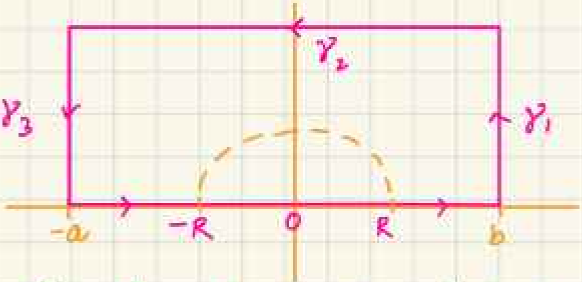
Fourier Transforms

Assume:-

- f is holo on $\mathbb{C}$ except possibly finite points
- None of poles lie on Real line
- Let $\{z_1, \dots, z_n\}$ be all poles in $\mathbb{H}$.
- $f(z) \longrightarrow 0$ as $z \longrightarrow \infty$ i.e $|f(z)| < \varepsilon\ \forall |z| \geq R_\varepsilon$

Then $\int_{-\infty}^{\infty} \overbrace{f(x)e^{iax}}^{\text{Fourier transform}}\,dx = 2\pi i \sum_{K=1}^{n} \operatorname{Res}(f(z)e^{iaz}, z_K)$ $: a>0$

proof:- Given $\varepsilon>0$, choose $R>0 \ni |f(z)| < \varepsilon\ \forall\ |z|>R$ and let $|z_j| < R\ \forall j \leq n$. Also $\frac{t}{e^{at}} \leq 1$, $t \geq R$



Clearly,

$$\int_\gamma \underbrace{f(z)e^{iaz}}_{g}\, dz = 2\pi i \sum_{1}^{n} Res(g; z_k)$$

i.e $\int_{-a}^{b} g + \int_{\gamma_1} g + \int_{\gamma_2} g + \int_{\gamma_3} g$. Now, $\left|\int_{\gamma_1} g\right| \le \varepsilon \int_0^c e^{-at}\,dt = \frac{\varepsilon}{a}\left(1 - \frac{1}{e^{ac}}\right) \le \frac{\varepsilon}{a}$

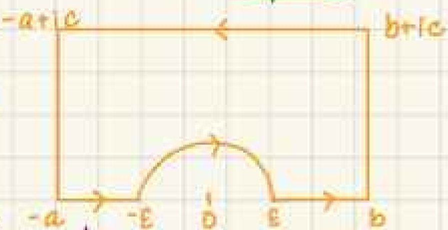
$\left|\int_{\gamma_2} g\right| \le \varepsilon$, $\left|\int_{\gamma_3} g\right| < \frac{\varepsilon}{a}$

hence, $\left|\int_\gamma g - \int_{-a}^{b} g\right| < \varepsilon$

Hence proved ■

Ex :- $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

→ $f(z) = \frac{e^{iz}}{z}$, pole at 0



$0 = \int_\gamma f(z)\, dz$

$= \int_{-a}^{-\varepsilon} f + \int_{\varepsilon}^{b} + \int_{\gamma_1} + \int_{\gamma_2} \dots + \int_{\gamma_3} + \int_{\gamma_\varepsilon} \Rightarrow \int_{-a} f + \int_{\varepsilon} f = -\underbrace{\int_{\gamma_1} \dots}_{\text{made arb. small}} - \int_{\gamma_\varepsilon}$

proposition :- If f has a simple pole at z_0, & suppose $\gamma_k := z_0 + ke^{it}$ $\begin{cases} \alpha \le t \le \beta \\ 0 \le \alpha < \beta \le 2\pi \end{cases}$

Then

$$\lim_{k \to 0^+} \int_{\gamma_k} f = i\, Res(f, z_0)(\beta - \alpha)$$

proof :- $f(z) = \frac{a}{z} + \underbrace{g(z)}_{\in H(U)}$

$\int_{\gamma_k} f(z)\,dz = a \int_{\gamma_k} \frac{dz}{z} + \int_{\gamma_k} g(z)\,dz$. Now $\left|\int_{\gamma_k} g(z)\,dz\right| \le M \cdot 2\pi k$

$\downarrow$

$= ia(\beta - \alpha)$. Hence proved.

continuing!
$$\int_{-a}^{-\varepsilon} f + \int_{\varepsilon}^{b} \longrightarrow \int_{\gamma_\varepsilon} f \quad \text{as } a \to -\infty \quad \varepsilon \to 0^+$$
$$b \to \infty$$
$$\longrightarrow i\pi \quad (\mathrm{Res}(f;0) = 1)$$

Taking Im part of $\int_{-a}^{-\varepsilon} \frac{e^{ix}}{x} + \int_{\varepsilon}^{b} \frac{e^{ix}}{x} \longrightarrow i\pi$

$=:$

Trigonometric Integrals

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta .$$

general method :-

$\sin\theta \longrightarrow \left(z + \frac{1}{z}\right)\frac{1}{2i}$ on unit circle

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{\gamma_1} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{iz}\, dz$$

$\downarrow$ Then Residue Theorem

eg:- $\varphi(x,y) = \frac{1}{a+y}$

$y \to \sin\theta \longrightarrow =:$

[$f(c,s)$ is rational function in 2 variables and makes sense on unit circle (as $\int$ has to be defined)

$\Rightarrow \frac{P_1(x,y)}{P_2(x,y)}$ [when replaced]

finite zeroes $\Rightarrow f$ has finite poles

Q : $\int_0^{2\pi} \sin^{2n}\theta\, d\theta$ $= \frac{\left(z - \frac{1}{z}\right)^{2n}}{2i} \cdot \frac{1}{iz} = \frac{(-1)^n}{4^n} \cdot \frac{-i}{2} \sum_{k=0}^{2n} \binom{2n}{k} z^k \frac{(-1)}{z^{2n-k}}$

only pole is at 0

hence

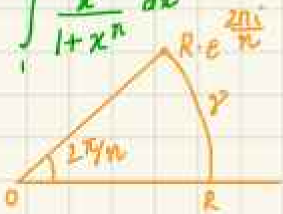
$$\text{Ans} = \pi 4^{-n}\binom{2n}{n}$$

Q $\displaystyle\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n}$: $m<n$, $m,n\in\mathbb{N}$

$$\int_1^\infty \frac{x^{m-1}\,dx}{1+x^n} \le \int_1^\infty \frac{x^n}{1+x^n}\,dx \quad \le \quad 1-\underbrace{\frac{1}{1+x^n}}_{\text{converges}}$$

To prove $\int$ converges

$$\int_1^\infty \frac{x^n}{1+x^n}\,dx$$



All poles except $e^{\frac{i\pi}{n}}$ lie outside.

$$f(z)=\frac{z^{m-1}}{1+z^n}$$

$$\int_\gamma f = \int_0^R \frac{x^{m-1}\,dx}{1+x^m} + \underbrace{\int_{R\frown Re^{i2\pi/n}} f}_{\substack{=0 \text{ as}\\ R\to\infty}} - \underbrace{\int_0^R \frac{t^{m-1}p^{2(m-1)}\cdot p^2\,dt}{1+(tp^2)^n}}_{=p^{2m}\int_0^R \frac{x^{m-1}}{1+x^m}dx} = 2\pi i \operatorname{Res}(f, e^{i\pi/n}) \quad : \; p=e^{i\pi/n}$$

$$f(z)=\frac{z^{m-1}}{(z-p)(z^{n-1}+\dots+p^{n-1})}$$

$$\Rightarrow \operatorname{Res}(f,p)=\frac{p^{m-1}}{n\cdot p^{n-1}}$$

hence $\displaystyle (1-p^{2m})\int_0^R \cdots = \frac{-1}{n}p^m\cdot 2\pi i$

Harmonic functions

A real valued function $u: U \underset{\text{open}}{\subseteq} \mathbb{C} \longrightarrow \mathbb{R}$, is harmonic in U if $u \in C^2(U)$ and

$$\Delta u \equiv 0 \quad \text{i.e} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$$

eg:- If $f \in H(U)$, then $Re(f)$ + $Im(f)$ are harmonic

Q if u is harmonic, Does $\exists f \in H(U) \ni Re(f) = u$.

No, counter :-

$$f: \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{R}, \quad f(z) = \log|z|$$

Analysis :-

Let $f \in H(U)$, $Re f = u$, f is zero free. Suppose $\exists v_1 : U \to \mathbb{R}$ $\ni g = u_1 + i v_1$ is holo. Then, consider $h(z) = \exp(g(z))$ $\forall z \in U$

$$|h(z)| = |e^{g(z)}| = e^{Re(g(z))} = e^{u_1} = e^{\log|f|} \Rightarrow \left|\frac{f}{e^g}\right| = 1$$

hence, if U is connected $\Rightarrow \exists \alpha \in \mathbb{C} \ni f = \alpha e^g = e^w \cdot e^g$

$$\Rightarrow f = e^{(w+g)} \quad \text{i.e } f \text{ has analytic logarithm}$$

Corollary : U = connected, every harmonic f^n has a harmonic conjugate then every 0-free analytic f^n has an analytic log $\Leftrightarrow$ U is simply connected.

Sufficiency

Assume U is simply connected $\Rightarrow$ Every harmonic f^n has a harmonic conjugate

proof:- (To find $f \in H(U)$, $Re(f) = u$). Let $F := u_x - i u_y$. From CR eqns $F \in H(U)$
F is C' as u is C^2.

Let f be primitive of F (simply connected U). then
$f = U + iV$ (say). $f'(z) = U_x + i V_x = U_x - i U_y$
$= u_x - i u_y$

hence, $U = u + c$ (connectedness)

$\Rightarrow$ $f - c$ has real part u + holo in U.

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Theorem :- Every harmonic f^n (in $U \subseteq \mathbb{C}$ region) has a harmonic conjugate iff U is simply connected.