Assignment-5

(2) Is sum of two compact subsets of $\mathbb{R}$ is compact? Is sum of a closed set and compact set is closed?

sum of a closed set and compact set is closed (can be proved using sequences)

Since closedness can be charaterized by sequence in $\mathbb{R}^n$,if $(x_n) \in A + B$ we need to show limit of the convergence sequence still lies in it.assume A is compact B is closed.

Since $x_n = a_n + b_n \to x$,compactness implies sequential compactness hence $a_{n_k} \to a \in A$ for some subsequence. now $x_{n_k} \to x$ which means subsequence $b_{n_k} \to x - a$ converge,since B is closed,$x - a \in B$,hence $x = a + b \in A + B$,which means the sum is closed.

For the first part we have shown that it'll be closed, and boundedness is easy.. hence the sum is compact

(3) Show that if Y is a subset of a complete metric space X, then Y is compact if and only if it is closed and totally bounded.

Proposition A.0.5 (Compactness in complete metric spaces) *In a complete metric space* (X, d) *a subset* F *is compact if and only if it is closed and totally bounded.*

Proof The $\Rightarrow$ part is self-evident: F is compact (also sequentially) and it contains all of its limit points.

Conversely, take a sequence $\{x_n\}_{n\geq 1}$ in F. By assumption F is totally bounded, so for every $\epsilon = 1/m$, $m \geq 1$, F can be covered by finitely many balls $B(x_n, 1/m) = \{y \in X : d(x_n, y) \leq 1/m\}$, $n = 1, \ldots, k$.

Choosing $\epsilon = 1$, let $\{x_{1_1}, \ldots, x_{k_1}\}$ denote the subset of X such that the closed balls $B(x_{i_1}, 1)$ of radius 1 finitely cover F. Among these there is at least one that contains infinitely many terms of $\{x_n\}_{n\geq 1} \subseteq F$, call it $B_1 = B(x_{n_1}, 1)$. Then define $N_1 = \{n : x_n \in B_1\}$, an infinite set.

Pick $n_1 \in N_1$ and $\epsilon = 1/2$. Let $\{x_{1_2}, \ldots, x_{k_2}\}$ be the finite subset of X for which the closed balls $B(x_{i_2}, 1/2)$ cover F. One of these, say $B_2 = B(x_{n_2}, 1/2)$, contains infinitely many terms of $\{x_n\}_{n\geq 1} \subseteq F$. Define $N_2 = \{n > n_1 : x_n \in B_1 \cap B_2\}$ and so forth.

Eventually we obtain a subsequence $\{x_{n_i}\}_{i\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that, for every $n_j \geq n_i$, the terms x_{n_j} belong to $B_i = B(x_{n_i}, 1/i)$, and therefore $\{x_{n_i}\}_{i\geq 1}$ is Cauchy in $B_i \subseteq F$.

By completeness $\{x_n\}_{n\geq 1}$ converges in X, and as a matter of fact in F, since F is closed.

Thus we proved $\{x_n\}_{n\geq 1} \subseteq F$ has a convergent subsequence, and F is compact.

Also proved in notes

(4) Show that a closed subset of a compact metric space is compact.

Proof: (a) Let $\mathcal{M}$ be a closed subspace of the compact space $\mathcal{X}$, and let $\{\mathcal{U}_\alpha\}$ be an open cover of $\mathcal{M}$, since $\mathcal{M}$ is closed, the collection $\{\mathcal{U}_\alpha\} \cup \{\mathcal{M}^c\}$ is an open cover of $\mathcal{X}$ which contains a finite subcover since $\mathcal{X}$ is compact. Therefore, there is a collection $\{\alpha_1, \ldots, \alpha_N\}$ such that $\{\mathcal{U}_{\alpha_n}\}_{n=1}^N \cup \{\mathcal{M}^c\}$ covers $\mathcal{X}$. Hence $\{\mathcal{U}_{\alpha_n}\}_{n=1}^N$ must cover $\mathcal{M}$. Therefore, $\mathcal{M}$ is compact.

(5) Show that every compact metric space has a countable dense subset.

Your basic idea is just fine, but you're making it too complicated; that's part of why you're having trouble expressing it clearly. As others have suggested, you don't need to try to relate the centres of one 'level' to those of any other 'level'. Here's a relatively efficient version of the idea:

> For each $n \in \mathbb{Z}^+$ let
>
> $$\mathscr{U}_n = \left\{ B\left(x, \frac{1}{n}\right) : x \in X \right\} ;$$
>
> $\mathscr{U}_n$ is an open cover of the compact space X, so there is a finite $F_n \subseteq X$ such that
>
> $$\left\{ B\left(x, \frac{1}{n}\right) : x \in F_n \right\}$$
>
> covers X. Let $D = \bigcup_{n \in \mathbb{Z}^+} F_n$; D is a countable union of finite sets, so D is countable. To see that D is dense in X let $y \in X$ and $\epsilon > 0$ be arbitrary. There is an $n \in \mathbb{Z}^+$ such that $\frac{1}{n} \le \epsilon$, and there is then an $x \in F_n$ such that $y \in B\left(x, \frac{1}{n}\right)$. But then $d(x, y) < \frac{1}{n} \le \epsilon$, so $y \in D \cap B(x, \epsilon)$, and D is indeed dense in X. $\dashv$

You might like to note, by the way, that we could have got the same result had each of the sets F_n been countable: we did not actually need them to be finite. Thus, the same argument shows that every Lindelöf metric space is separable. And this actually is a stronger result, since $\mathbb{R}$ with its usual metric is Lindelöf but not compact.

(6) Prove that every open subset U of $\mathbb{R}^n$ is the countable union of compact subsets. Show that every open cover of an open subset of $\mathbb{R}^n$ has a countable subcover.

(7) **Cantor's Intersection Theorem:** A decreasing sequence of nonempty compact subsets $A_1, A_2, \ldots$ of a metric space (X, d) has nonempty intersection.

Theorem. *Let S be a topological space. A decreasing nested sequence of non-empty compact, closed subsets of S has a non-empty intersection. In other words, supposing $(C_k)_{k\geq 0}$ is a sequence of non-empty compact, closed subsets of S satisfying*

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots,$$

it follows that

$$\bigcap_{k=0}^{\infty} C_k \neq \emptyset.$$

The closedness condition may be omitted in situations where every compact subset of S is closed, for example when S is Hausdorff.

Proof. Assume, by way of contradiction, that $\bigcap_{k=0}^{\infty} C_k = \emptyset$. For each k, let $U_k = C_0 \setminus C_k$. Since $\bigcup_{k=0}^{\infty} U_k = C_0 \setminus \bigcap_{k=0}^{\infty} C_k$ and $\bigcap_{k=0}^{\infty} C_k = \emptyset$, we have $\bigcup_{k=0}^{\infty} U_k = C_0$. Since the C_k are closed relative to S and therefore, also closed relative to C_0, the U_k, their set complements in C_0, are open relative to C_0. Since $C_0 \subset S$ is compact and $\{U_k | k \geq 0\}$ is an open cover (on C_0) of C_0, a finite cover $\{U_{k_1}, U_{k_2}, \ldots, U_{k_m}\}$ can be extracted. Let $M = \max_{1 \leq i \leq m} k_i$. Then $\bigcup_{i=1}^{m} U_{k_i} = U_M$ because $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \cdots$, by the nesting hypothesis for the collection $(C_k)_{k\geq 0}$. Consequently, $C_0 = \bigcup_{i=1}^{m} U_{k_i} = U_M$. But then $C_M = C_0 \setminus U_M = \emptyset$, a contradiction. ■

The Normal Nowhere Dense Statement:

Let X be a metric space. A subset $A \subseteq X$ is called nowhere dense in X if the interior of the closure of A is empty, i.e. $(\overline{A})^{\circ} = \emptyset$. Otherwise put, A is nowhere dense iff it is contained in a closed set with empty interior.

Alternate Formulation:

"Passing to complements, we can say equivalently that A is nowhere dense iff its complement contains a dense open set."

Proof -

First, you should know that, for any $B \subseteq X$, $X \setminus \overline{B} = (X \setminus B)^\circ$ and that $X \setminus B^\circ = \overline{X \setminus B}$. Now

$$\begin{aligned} A \text{ nowhere dense} &\iff \left(\overline{A}\right)^\circ = \varnothing \\ &\iff X \setminus (\overline{A})^\circ = X \\ &\iff \overline{X \setminus \overline{A}} = X \\ &\iff \overline{(X \setminus A)^\circ} = X \\ &\iff (X \setminus A)^\circ \text{ is dense in } X \\ &\iff (X \setminus A) \text{ contains a dense open subset.} \end{aligned}$$

The last equivalence may not be so obvious if you're not very used to metric spaces. See below, if necessary:

> If $(X \setminus A)^\circ$ is dense in X, then $(X \setminus A)^\circ$ is a dense open subset of $X \setminus A$.
>
> Conversely, if $(X \setminus A)$ contains a dense open subset D, then $D \subseteq (X \setminus A)^\circ$, so $(X \setminus A)^\circ$ is dense as well.

Assignment - 7

(1) Consider the set $\mathbb{Q}$ of rational numbers with the metric $d(x, y) = |x - y|$; that the Baire category theorem does not hold in this metric space.

Remark. The completeness assumption is necessary in this theorem as the following example illustrates. Let $X = \mathbb{Q}$. Write $\mathbb{Q} = \{r_n \mid n \in \mathbb{N}\}$ and let $G_n = \mathbb{Q} - \{r_n\}$ for each $n \in \mathbb{N}$. Then G_n is open and dense in $\mathbb{Q}$ for each n, but $\bigcap_{n=1}^{\infty} G_n = \varnothing$.

2. prove using baire category theorem

(a) $[0, 1]$ contains uncountably many elements.

https://math.stackexchange.com/questions/165696/your-favourite-application-of-the-baire-category-theorem

Proof: Assume that it contains countably many. Then $[0, 1] = \bigcup_{x \in (0,1)} \{x\}$ and since $\{x\}$ are nowhere dense sets, X is a countable union of nowhere dense sets. But $[0, 1]$ is complete, so we have a contradiction. Hence X has to be uncountable.

(b) The linear space of all polynomials in one variable is not a Banach space in any norm.

Proof: "The subspace of polynomials of degree $\leq n$ is closed in any norm because it is finite-dimensional. Hence the space of all polynomials can be written as countable union of closed nowhere dense sets. If there were a complete norm this would contradict the Baire Category Theorem."

(c) Let (X, d) be a complete metric space with no isolated points. Then (X, d) is uncountable.

https://math.stackexchange.com/questions/2523793/show-that-a-complete-metric-space-without-isolated-points-is-uncountable

Say (X, d) is a countable complete metric space without isolated points.
Then, we can write $X = \{x_n : n \geq 1\}$(as we assume it is countable).
Now, consider the Sets $U_n = X \backslash \{x_n\}$. Since $\{x_n\}$ is closed, U_n is open. Also because $\{x_n\}$ is not isolated(as given), for all $\varepsilon > 0$, $B(x_n, \varepsilon) \cap U_n \neq \phi$,
Hence U_n is dense in X(Any other point not equal to X already in U_n).
Now, by Baire's Category Theorem $\bigcap_{i=1}^{\infty} U_n$ is dense in X. But since $\bigcap_{i=1}^{\infty} U_n \neq \phi$ is a contradiction so our assumption that X is countable is not true hence, X must by uncountable.

Stefan Meskan also explains up to this point in his answer. Only point need to reconsider is why $\bigcap_{i=1}^{\infty} U_n \neq \phi$ is contradiction.
Let's consider a general case for X
Since $\mathbb{X}$ is countable we can write $\mathbb{X} = \{x_1, x_2, x_3 \ldots, x_{n-1} x_n\} : n \in \mathbb{N}$ & as we defined

$$U_1 = \{ \quad x_2, x_3, x_4 \ldots, x_{n-1}, x_n\}$$

$$U_2 = \{x_1, \quad x_3, x_4 \ldots, x_{n-1}, x_n\}$$

$$U_3 = \{x_1, x_2, \quad , x_4 \ldots, x_{n-1}, x_n\}$$

$$\vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$U_n = \{x_1, x_2, x_3, x_4 \ldots x_{n-1}, \quad \}$$

So, cleary $\bigcap_{i=1}^{\infty} U_n = \phi$. (As $\forall x_n \in X \; \exists \; U_n$ and $\bigcup_{i=1}^{\infty} U_n = X$)

(d) There exists a continuous function $f : [0,1] \to \mathbb{R}$ that is not monotone on any interval of positive length.

Proving the existence of a non-monotone continuous function defined on $[0,1]$

Let $(I_n)_{n \in \mathbb N}$ be the sequence of intervals of $[0,1]$ with rational endpoints, and for every $n \in \mathbb N~$ let $E_n=\{f \in C[0,1] : f \:\text{is monotone in}\: I_n\}$. Prove t...

https://math.stackexchange.com/questions/546578/proving-the-existence-of-a-non-monotone-continuous-function-defined-on-0-1

(3) Let $(X, \| \ \|)$ be a normed space, $Y \subsetneq X$, i.e. $Y \neq X$ a proper linear subspace. Prove that Y contains no ball (that is, its interior is empty).

https://math.stackexchange.com/questions/148850/every-proper-subspace-of-a-normed-vector-space-has-empty-interior

Suppose S has a nonempty interior. Then it contains some ball $B(x,r) = \{y : \|y - x\| < r\}$. Now the idea is that every point of V can be translated and rescaled to put it inside the ball $B(x,r)$. Namely, if $z \in V$, then set $y = x + \frac{r}{2\|z\|} z$, so that $y \in B(x,r) \subset S$. Since S is a subspace, we have $z = \frac{2\|z\|}{r}(y - x) \in S$. So $S = V$.

A nice consequence of this is that any *closed* proper subspace is necessarily nowhere dense. So if V is a Banach space, the Baire category theorem implies that V cannot be a countable union of closed proper subspaces. In particular, an infinite dimensional Banach space cannot be a countable union of finite dimensional subspaces. This means, for example, that a vector space of countable dimension (e.g. the space of polynomials) cannot be equipped with a complete norm.

(4) Prove that any finite dimensional linear subspace of $(X, \| \ \|)$ is closed.

Theorem 1 *Any finite dimensional subspace of a normed vector space is closed.*

Proof. Let $(V, \| \cdot \|)$ be such a normed vector space, and $S \subset V$ a finite dimensional vector subspace.

Let $x \in V$, and let $(s_n)_n$ be a sequence in S which converges to x. We want to prove that $x \in S$. Because S has finite dimension, we have a basis $\{x_1, \ldots, x_k\}$ of S. Also, $x \in \mathrm{span}(x_1, \ldots, x_k, x)$. But, as proved in the case when V is finite dimensional (see this parent (http://planetmath.org/EverySubspaceOfANormedSpaceOfFiniteDimensionIsClosed)), we have that S is closed in $\mathrm{span}(x_1, \ldots, x_k, x)$ (taken with the norm induced by $(V, \| \cdot \|)$) with $s_n \to x$, and then $x \in S$. QED.

(5) Let $C^{\infty}(\mathbb{R})$ i.e. infintely differentiable function and suppose that for all $x \in \mathbb{R}$ there exists $n_x \in \mathbb{Z}_+$ such that $f^{(n_x)}(x) = 0$. Show that there exists a nonempty open interval $(a,b) \subset \mathbb{R}$ such that the restriction of f to (a,b) is a polynomial.

If f is infinitely differentiable then f coincides with a polynomial

Let f be an infinitely differentiable function on $[0,1]$ and suppose that for each $x \in [0,1]$ there is an integer $n \in \mathbb{N}$ such that $f^{(n)}(x)=0$. Then does f coincide on $[0,1]...

https://mathoverflow.net/questions/34059/if-f-is-infinitely-differentiable-then-f-coincides-with-a-polynomial

(6) Consider $X = C[0,1]$ with sup metric. Define $E_m = \{f \in X : \exists x \in [0,1] \; with \; |f(x+h) - f(x)| \leq m|h| \; for \; all \; x+h \in [0,1]\}$. It is clear from the definition of differentiability that all functions differentiable at some point in $(0,1)$ lie in one of these sets E_m.

(a) Show that E_m is closed for each m.

(b) Show that set of piecewise linear functions P_L is dense in X. ($p \in P_L$ if there exits a partition $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $p|_{[a_1, a_{i+1}]}$ is linear for each $i = 0, \ldots, k-1$.

(c) If E_m contains some open ball $B_\epsilon(f)$ then there exists a $p \in P_L$ such that $p \in B_\epsilon(f)$.

(d) Observe that for a piecewise linear function the slopes are bounded.

(e) For given M consider the partition $P = \{a_j\}$ where $a_j = \frac{j}{M}, j = 0, \ldots, M$. Define a continuous function $g : [0,1] \to \mathbb{R}$ by $g(x) = \begin{cases} 1 & if \; x = a_j, \; j \in 2\mathbb{Z} \\ -1 & if \; x = a_j, \; j \in 2\mathbb{Z}+1 \\ \text{linear} & otherwise \end{cases}$.

Consider $h(x) = p(x) + \frac{\epsilon}{2} g(x)$. Clearly, $h \in B_\epsilon(p)$. By taking M large conclude that E_m cannot contain an open ball.

(f) From above conclude that there exists a continuous function which is not differentiable at any point.

(7) Show that the set of nowhere differentiable functions is residual (hence dense) in $C[0,1]$.

https://math.stackexchange.com/questions/2993112/proving-that-the-set-of-continuous-nowhere-differentiable-functions-is-dense-usi

#imp

Let M be a metric space and let A be a subset of M. Let C be a connected component of A. Prove:

(a) C is closed in A.

Solution. Since C is a maximal connected subset of A and the closure of a connected subset is connected, C must equal its closure in A, hence be closed in A.

(b) If A has only a finite number of connected components, then C is open in A.

Solution. Assume that A has only a finite number of connected components, say $C_1, \ldots, C_n$. They are all closed in A, thus the union of any finite number of them is also closed in A. Now C_1 is the complement in A of $C_2 \cup \cdots \cup C_m$, hence the complement of a closed set, hence open.

(c) Give an example of a metric space M having a connected component that is not open.

Solution. See Exercise 9b.

http://math.fau.edu/schonbek/Analysis/ia1fa14h5s.pdf

Assignment - 8

(1) A set $E \subset \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $\{x_n\} \to x$ with $\{x_n\}$ contained in one of A or B, and x an element of the other.

https://web.math.utk.edu/~swise/classes/spring18/math341/LectureNotes/class16.pdf

Theorem (3.4.6): A set $E \subseteq \mathbb{R}$ is connected iff for all non-empty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof: ($\Rightarrow$) By contrapositive. Suppose that there exist sets A and B such that $E = A \cup B$, but for every convergent sequence $(x_n) \subseteq A$ its limit is never in B and for every convergent sequence $(y_n) \subseteq B$ its limit is never in A. WTS E is disconnected.

Notice that the property just described implies that

$$\bar{A} \cap B = \phi \quad \text{and} \quad A \cap \bar{B} = \phi.$$

Thus E is disconnected. //

($\Leftarrow$) Also by contrapositive. Suppose that E is disconnected. There exist sets A and B such that $E = A \cup B$ and

2

$$\bar{A} \cap B = \phi \quad \text{and} \quad A \cap \bar{B} = \phi$$

let (x_n) be a convergent sequence in A. Then its limit is in $\bar{A}$. By assumption the limit is not in B.

let (y_n) be a convergent sequence in B. Its limit is in $\bar{B}$ and, therefore, not in A. ///

(2) Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty. Show that a set $E \subset \mathbb{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

(3) A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

(b) Is the set of irrational number totally disconnected?

(c) Is Cantor set C is totally disconnected?

(4) Let $\mathcal{F}$ be a collection of connected subsets of a metric space X such that the intersection $\bigcap_{A\in\mathcal{F}} A \neq \emptyset$. Show that $\bigcup_{A\in\mathcal{F}} A$ is connected.

HINT: You're actually about halfway there, though you omitted an important qualification of A and B: if $\bigcup \mathscr{F}$ is not connected, then it can be partitioned into two disjoint, non-empty, **relatively open** subsets A and B. Now fix $x \in \bigcap \mathscr{F}$, and without loss of generality assume that $x \in A$. $B \neq \varnothing$, so pick any $y \in B$. Then there is some $F \in \mathscr{F}$ such that $y \in F$, and of course $x \in F$. Thus, $x \in A \cap F$, and $y \in B \cap F$, so $A \cap F \neq \varnothing \neq B \cap F$. Why is this a contradiction?

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answered Jun 20, 2013 at 0:46
Brian M. Scott
587k ● 51 ■ 699 ▲ 1158

After doing this problem for myself a slightly different way: would the contradiction here be that we have shown that A and B are also disconnected, when they are given to be connected? – Taylor Rendon Oct 15, 2020 at 15:08

2 @TaylorRendon: No, the problem is that A and B split the connected set F. – Brian M. Scott Oct 15, 2020 at 15:12

Gotcha! So that means $F \subset A \cup B$? – Taylor Rendon Oct 15, 2020 at 15:20

@TaylorRendon: Yes, because $F \subseteq \bigcup \mathscr{F} = A \cup B$. – Brian M. Scott Oct 15, 2020 at 15:29

(5) From the above exercise we see the following: If $x \in X$ then $\bigcup A$ where A is a connected subset containing x is connected. Call this maximal connected set as the *component* of X containing x.

Show that every point of a metric space X belongs to a uniquely determined component of X. i.e. The component of X form a collection of disjoint sets whose union is X.

https://gacbe.ac.in/pdf/ematerial/18BMA61C-U1.pdf

Proof :

Since every point $x \in S$ belongs to atleast one connected subset called $\{x\}$, we can say that x belongs to atleast one component of S.

Union of components of S is S.

Components are disjoint.

Suppose $x \in U_1 \cap U_2$ where U_1 and U_2 are two components of S.

$\Rightarrow U_1 \cup U_2$ is a maximal connected set containing X condratracting the fact that U_1 and U_2 are maximal connected sets containing X.

$\Rightarrow U_1 \cap U_2 = \phi$

Hence the proof.

(6) In $\mathbb{R}^n$ we have seen that if a set is connected it may not necessarily be path connected. However, show that every open connected set in $\mathbb{R}^n$ is connected.

https://math.stackexchange.com/questions/1219540/connected-open-subsets-in-mathbbr2-are-path-connected

prolly in notes as well

(7) Show that every open set U in $\mathbb{R}^n$ can be expressed as countable union of disjoint open connected sets.

Open subset of $\mathbb{R}$ can be written as countable union of disjoint open intervals, intuition?

begingroup$ Here is the proof of this proposition from my Real Analysis textbook. Proposition 1.5. Suppose $G \subset \mathbb{R}$ is open. Then G can be written as the countable union of disjoint open intervals. Proof.

https://math.stackexchange.com/questions/1895810/open-subset-of-mathbbr-can-be-written-as-countable-union-of-disjoint-open-i?rq=1

(8) Prove that a metric space X is connected if and only if every non-empty proper subset of X has a non-empty boundary.

https://math.stackexchange.com/questions/118460/x-is-connected-iff-forall-a-subset-x-partial-a-neq-emptyset

(9) Let $U \subset \mathbb{R}^n$ open connected. Let T be a component of $\mathbb{R}^n \setminus U$. Show that $\mathbb{R}^n \setminus T$ is connected.

Let S be an open connected set in $\Bbb R^n$, let T be a component of $\Bbb R^n-S$. Prove that $\Bbb R^n-T$ is connected

begingroup$ $\Bbb R^n-S=T\cup (\bigcup A)$, T and these A are components in $\Bbb R^n-S$. $\Bbb R^n-T=S\cup(\bigcup A)$, $\DeclareMathOperator{\cl}{cl}$ since $\partial S \subseteq \Bbb R^n-S,\partial S\subseteq (\bigcup A)$ I consider $S\cup \partial S \cup (\bigcup A)=\bigcup (\cl S \cup A)$ If S is not an empty set, and if for every A, $\cl S \cup A$ is connected, in that way, $\Bbb R^n-T$ is connected.

https://math.stackexchange.com/questions/3301304/let-s-be-an-open-connected-set-in-bbb-rn-let-t-be-a-component-of-bbb

(11) Prove that no pair of the following subspaces of $\mathbb{R}$ are *homeomorphic*: $(0,1)$, $[0,1)$, $[0,1]$.

Show that no two of the spaces $(0,1), \ (0, 1], \ [0,1]$ are homeomorphic.

I want to show that no two of the spaces $(0,1), \ (0, 1], \ [0,1]$ are homeomorphic, with the hint: what happens if you remove a point from each one of these spaces. No idea where to begin, excep...

https://math.stackexchange.com/questions/504059/show-that-no-two-of-the-spaces-0-1-0-1-0-1-are-homeomorphic