MTH 442: Time Series Analysis
Problem Set#5

[1] Let $\{X_t\}$ and $\{Y_t\}$ be two covariance stationary processes. Define a bivariate process $\underset{\sim}{Z}_t = (X_t, Y_t)'$. Prove or give a counter example "$\underset{\sim}{Z}_t$ is covariance stationary".

[2] Let $\{X_t\}$ be a covariance stationary process with mean μ and ACVF $\gamma_X(h)$. Define $\{\underset{\sim}{Y}_t\}$ as $\underset{\sim}{Y}_t = (X_t, X_{\alpha+\beta t})'$. Find the values of α and β for which $\{\underset{\sim}{Y}_t\}$ is stationary.

[3] Let $X_t = \phi X_{t-1} + \mu_t$ and $Y_t = \phi X_{t-2} + \delta_t$, where $|\phi| < 1$, $\{\mu_t\}$ and $\{\delta_t\}$ are independent $WN(0, \sigma^2)$ sequences with $Cov(\mu_t, X_{t-j}) = 0$ for all $j > 0$. Find the cross correlation $\rho_{XY}(k); k = 0,1,2,...$

[4] Let $\{X_t\}$ be a stationary time series with mean zero and ACVF $\gamma_X(h) = \left(\frac{1}{2}\right)^{|h|}$ and $\{Y_t\}$ be a $WN(0, \sigma_Y^2)$ independent of $\{X_t\}$. Define a bivariate time series

$$\underset{\sim}{Z}_t = \begin{pmatrix} X_t(Y_t + Y_{t-1}) \\ X_t Y_{t-2} \end{pmatrix}$$

Show that $\underset{\sim}{Z}_t$ is a covariance stationary vector process. Further, find k such that $Cov(\underset{\sim}{Z}_t, \underset{\sim}{Z}_{t+h})$ is a null matrix for all $|h| > k$.

[5] A and B are constants and $\theta \sim U(0, 2\pi)$, verify whether $\underset{\sim}{X}_t = (A\cos(t+\theta), B\cos(2t+\theta))'$ is covariance stationary bivariate process.

[6] Let $\{X_t\}$ be a sequence of independent and identically distributed $N(0,1)$ random variables. Define $Y_t = (-1)^t + X_t + X_{t-1} + X_{t-2}$ and $Z_t = (-1)^t X_t$. Verify whether $\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$ is covariance stationary bivariate process.

[7] Suppose $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$ are two independent $WN(0, \sigma_\varepsilon^2)$ processes. Define

$$X_t = 0.5 X_{t-1} + \varepsilon_{1,t}$$
$$Y_t = 0.5 X_{t-2} + \varepsilon_{2,t}$$

Find the cross-covariance between X_t and Y_t at lags 2 and 3.

[8] Let $\{X_t\}$ be zero mean stationary process with absolutely summable ACVF $\gamma_X(h)$. Define a vector process $\underset{\sim}{Z}_t$ as

$$\underset{\sim}{Z}_t = \begin{pmatrix} \sum_{h=-\infty}^{\infty} \psi_{1,h} B^h X_t \\ \sum_{h=-\infty}^{\infty} \psi_{2,h} B^h X_{a+bt} \end{pmatrix}$$

where, $\sum_{h=-\infty}^{\infty} |\psi_{i,h}| < \infty; i = 1, 2$. Find the values of a and b such that the vector process $\underset{\sim}{Z}_t$ is covariance stationary. Further, consider $\underset{\sim}{Z}_t$ with $a = 1, b = 1$ and $X_t \sim WN\left(0, \sigma^2\right)$. Prove or disprove "$\underset{\sim}{Z}_t$ is a vector white noise process".

[9] Suppose $\underline{\epsilon_t} \sim VWN(0, \Sigma), \Sigma > 0$ and let $\underline{Y_t} = \begin{pmatrix} \underline{\epsilon_t} \\ 2\underline{\epsilon_{2t+3}} \end{pmatrix}$. Verify whether or not $\underline{Y_t} \sim VWN$.

[10] Suppose $\epsilon_t \sim WN(0,1)$; $X_t = 2\epsilon_t + \epsilon_{t-1} + \epsilon_{t+1}$ and $Y_t = 2 + \epsilon_t - \epsilon_{t-1}$.

Define $\underline{Z_t} = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$.

(a) Verify whether or not $\underline{Z_t}$ is covariance stationary.

(b) Find the smallest integer k such that $Cov\left(\underline{Z_t}, \underline{Z_{t+h}}\right)$ is a null matrix for $|h| \geq k$.

[11] Let $\{e_t\}$ be a sequence of i.i.d. $N(0,1)$ random variables. Define

$$X_t = e_1 + e_2 \cos t + e_3 \sin t$$
$$Y_t = t + e_1 \cos t + e_2 \sin t$$
$$Z_t = (-1)^t e_4 + e_5 \cos t + e_6 \sin t$$

Verify whether the bivariate processes $\underset{\sim}{P}_t = \begin{pmatrix} X_t \\ Y_t - t \end{pmatrix}$ and $\underset{\sim}{Q}_t = \begin{pmatrix} X_t \\ Z_t \end{pmatrix}$ are covariance stationary.

[12] Consider the 3-variate stationary *VAR(1)*, $\underset{\sim}{X}_t = \Phi \underset{\sim}{X}_{t-1} + \underset{\sim}{\varepsilon}_t$; where, $\{\underset{\sim}{\varepsilon}_t\} \sim VWN\left(\underset{\sim}{0}, \Sigma\right)$.

Define $\underset{\sim}{Y}_t = \begin{pmatrix} \underset{\sim}{X}_t \\ \alpha \underset{\sim}{X}_{t-1} \end{pmatrix}, \alpha \in \Re$. Prove or disprove "$\underset{\sim}{Y}_t$ is covariance stationary *VAR* $\forall\ \alpha \in \Re$".

[13] Consider a n-dimensional stationary VAR(p) process

$$\underset{\sim}{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underset{\sim}{X}_{t-2} + \dots + \Phi_p \underset{\sim}{X}_{t-p} + \underset{\sim}{\varepsilon}_t; \underset{\sim}{\varepsilon}_t \sim VWN\left(\underset{\sim}{0}, \Sigma\right).$$

Define a new np-dimensional vector process $\underset{\sim}{Z}_t = \left(\underset{\sim}{X}_t' \underset{\sim}{X}_{t-1}' \dots \underset{\sim}{X}_{t-p+1}'\right)'$. Let us define

$$\underset{np\times np}{F} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_{p-1} & \Phi_p \\ I_n & 0 & 0 & \dots & & 0 \\ 0 & I_n & 0 & \dots & & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & I_n & 0 \end{bmatrix} \text{ and } \underset{np\times 1}{\underset{\sim}{\eta_t}} = \begin{pmatrix} \underset{\sim}{\varepsilon}_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using F and $\underset{\sim}{\eta}_t$, express $\underset{\sim}{Z}_t$ as a VAR process. Further, if $E(\underset{\sim}{Z}_t \underset{\sim}{Z}_t') = \Omega$, prove that $\Omega = F\Omega F' + Q$; where $E(\underset{\sim}{\eta}_t \underset{\sim}{\eta}_t') = Q$.

[14] Consider the following 2-variate $VAR(2)$ process;

$$\underset{\sim}{X}_t = \underset{\sim}{\varepsilon}_t - \underset{\sim}{X}_{t-1} - \frac{1}{4} \underset{\sim}{X}_{t-2};$$

where, $\{\underset{\sim}{\varepsilon}_t\} \sim VWN(\underset{\sim}{0}, \Sigma), \Sigma > 0$. Is the given $VAR(2)$ process covariance stationary? In case the process is causal, obtain its $VMA(\infty)$ representation.

[15] Consider a k-dimensional stationary VAR(2) process

$\underset{\sim}{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underset{\sim}{X}_{t-2} + \underset{\sim}{\varepsilon}_t; \underset{\sim}{\varepsilon}_t \sim VWN(\underset{\sim}{0}, \Sigma)$. Prove or disprove " $\underset{\sim}{Z}_t = \begin{pmatrix} \underset{\sim}{X}_t \\ \underset{\sim}{X}_{t-1} \end{pmatrix}$ is a stationary VAR(1) process".

[16] Let $\{\underset{\sim}{X}_t\}$ be a bivariate vector *ARMA(1,1)* process:

$$\underset{\sim}{X}_t = \Phi \underset{\sim}{X}_{t-1} + \underset{\sim}{\varepsilon}_t + \Theta \underset{\sim}{\varepsilon}_{t-1}; \{\underset{\sim}{\varepsilon}_t\} \sim VWN(\underset{\sim}{0}, I_2), \text{ where, } \Phi = \Theta = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}.$$

(a) Find the value(s) of the constant a such that $\{\underset{\sim}{X}_t\}$ is causal and invertible.

(b) Using an appropriate value of a (obtained in (1)), find the $VMA(\infty)$ and $VAR(\infty)$ representation of $\{\underset{\sim}{X}_t\}$.

(c) Verify whether the sequence of matrices, say $\{\Psi_j\}_{j=0}^{\infty}$, associated with the $VMA(\infty)$ representation is absolutely summable or not.

(d) Find the impulse response of the two component variables of $\{\underset{\sim}{X}_t\}$ with respect to shocks in the other variable.

[17] Let $\underline{X}_t$ be a 2-variate VAR(1) process $\underline{X}_t = \Phi \underline{X}_{t-1} +, \underline{\epsilon}_t \sim VWN(0, \Sigma);\ \Sigma > 0,\ \Phi = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}$. Which of the following statements is (are) CORRECT?

(a) For $a = 0.5$ and $b = 0.5$, $\underline{X}_t$ is covariance stationary

(b) For $a = 0.5$ and $b = 1$, $\underline{X}_t$ is covariance stationary

(c) For all $a, b \in \mathbb{R}$, $\underline{X}_t$ is covariance stationary

[18] Suppose the $k \times 1$ random vector $\underline{\epsilon}_t \sim VWN(0, \Sigma);\ \Sigma > 0$ and $\underline{X}_t = \underline{\epsilon}_t + \Theta \underline{\epsilon}_{t-1}$, Θ is a $k \times k$ matrix of constants. Define $\underline{Z}_t = \begin{pmatrix} \underline{X}_t \\ \underline{\epsilon}_{t-1} \end{pmatrix}$. Prove or disprove the following statements.

(a) $\underline{Z}_t$ is covariance stationary for all Θ

(b) $\underline{Z}_t \sim VWN$

(c) $\underline{Z}_t$ has a $VMA(1)$ representation