# Assignment 2.

1. If $d$ is a metric on $M$, show that $|d(x,z) - d(y,z)| \le d(x,y)$ $\forall x, y, z \in M$.

2. Consider $\mathbb{R}$. Show that each of the following defines a metric on $\mathbb{R}$:
(i) $\rho(a,b) := \sqrt{|a-b|}$
(ii) $\sigma(a,b) := |a-b| / (1 + |a-b|)$
(iii) $\tau(a,b) := \min\{|a-b|, 1\}$

3. Let $f : [0,\infty) \to [0,\infty)$ be increasing and satisfy $f(0) = 0$, and $f(x) > 0$ $\forall x > 0$. If $f$ also satisfies $f(x+y) \le f(x) + f(y)$, $\forall x, y \geq 0$, then $f \circ d$ is a metric whenever $d$ is a metric. Show that each of the following conditions is sufficient to ensure that $f(x+y) \le f(x) + f(y)$, $\forall x, y \geq 0$:
(i) $f$ has a second derivative satisfying $f'' \le 0$;
(ii) $f$ has a decreasing first derivative;
(iii) $f(x)/x$ is decreasing for $x > 0$.

4. The Hilbert cube $H^\infty$ is the collection of all real seq. $x = (x_n)$ with $|x_n| \le 1$ for $n = 1, 2, \ldots$
(i) Show that $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$ defines a metric on $H^\infty$.

(ii) Given $x, y \in H^\infty$ and $k \in \mathbb{N}$, let $M_k := \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$.
Show that $\frac{1}{2^k} M_k \le d(x,y) \le M_k + \frac{1}{2^k}$

5. Check that $d(f,g) := \max_{a \le t \le b} \{|f(t) - g(t)|\}$ defines a metric on $C[a,b]$, the collection of all cts. real-valued functions on $[a,b]$.

6. A subset $X$ of a metric space $M$ is bounded if $\exists\, x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \le C$ $\forall a \in X$. Show that a finite union of bounded sets is again bounded.

7. Diameter of a nonempty subset $A \subset M$ denoted by $\text{diam}(A)$ is defined as:

$$\text{diam}(A) = \sup \{ d(a,b) \ : a, b \in A \}.$$

Show that $A$ is bounded iff $\text{diam}(A)$ is finite.

8. (Cauchy - Schwarz Inequality) (Self-read from Carothers book)

For $x, y \in \ell_2$, $\sum_1^\infty |x_i y_i| \leq \|x\|_2 \|y\|_2$.

9. Let $V$ be a vector space and let $d$ be a metric on $V$ satisfying $d(x,y) = d(x-y, 0)$ and $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$, $\forall \ x, y \in V$ and scalar $\alpha$.

Show that $\|x\| = d(x,0)$ defines a norm on $V$.

Give an example of a metric on the vector space $\mathbb{R}$ that fails to be associated with a norm in this way.

10. Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$.

Also show that $\|x\|_1 \leq n \|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n} \|x\|_2$.

11. Show that $(\ell_1, \|\cdot\|_1)$ and $(\ell_\infty, \|\cdot\|_\infty)$ are normed linear spaces.

12. Show that $\|x\|_\infty \leq \|x\|_2$ for any $x \in \ell_2$ and

$\|x\|_2 \leq \|x\|_1$ for any $x \in \ell_1$.

13. Let $c_0$ be the collection of all sequences that converge to 0.

Show that $(c_0, \|\cdot\|_\infty)$ is a normed linear space.

Show that one has the following proper set inclusions;

$$\ell_1 \subsetneq \ell_2 \subsetneq c_0 \subsetneq \ell_\infty .$$

14. Show that Hölder's Inequality holds for $p = 1$ and $q = \infty$.

15. Consider $C[0,1]$. For $1 < p < \infty$, define $\|f\|_p := \left( \int_0^1 |f(t)|^p \right)^{1/p}$.

Show that $(C[0,1], \|\cdot\|_p)$ is a normed linear space.

For $p=1$ and $q=\infty$, does the Hölder's and Minkowski's inequality hold?