Compact subsets of $C(X)$ where X is a compact metric space.

Recall that $(C(X), \|\cdot\|_\infty)$ is a complete metric space.

And that,

(M,d) is a compact metric space $\Leftrightarrow$ (M,d) complete and totally bdd.

$C(X)$ is not a compact metric space.

What about compact subsets of $C(X)$?

For A a closed set in $C(X)$, A is compact $\Leftrightarrow$ A is totally bdd.

Familiar model: $(\mathbb{R}, |\cdot|)$ Let $A \subset \mathbb{R}$ closed.

A is compact $\Leftrightarrow$ A is bdd.

Recall: Totally bdd $\Rightarrow$ bdd.

$\not\Leftarrow$ (in general)

Target: Totally bdd. $\Leftrightarrow$ bdd. + ?

(suffices to figure out for any set A, b/c. A is totally bdd $\Leftrightarrow$ $\overline{A}$ is totally bdd.)

In $C(X)$, we are able to come up with a condition which fills up the (?) above.

Recall: A cts. function on a compact metric space is unif. cts.

That is, $\forall\, \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s.t. $d(x,y) < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

Going to "strengthen" this unif. cts. property of each cts function to a family of cts. fnc.

Def^n: For $A \subset C(X)$, A is said to be equicontinuous if

for each $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s.t. $\forall\, f \in A$, $d(x,y) < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

In general: If A is equicontinuous, then A may not be totally bdd.

(HW): Example: For $0 < K < \infty$ and $0 < \alpha \leq 1$,

$$A = \{ f: [0,1] \to \mathbb{R} \text{ cts. s.t. } |f(x) - f(y)| \leq K |x-y|^\alpha \}$$

Show that A is equicts. but not totally bdd.

Q. If $A \subset C(X)$ is uniformly bdd, then is A totally bdd.?

A. NO! Let (f_n) be unif. bdd. To check whether (f_n) is totally bdd, need to check if $\exists$ a Cauchy subseq. of (f_n). That is, if $\exists (f_{n_k})$ s.t. (f_{n_k}) uniformly cvgs. (why?).

Example: (A seq. (f_n) which is unif. bdd. but does not have any unif. cvgt. subseq.)

Define $f_n : [0,1] \to \mathbb{R}$ as $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$, $n = 1, 2, 3, \ldots$

Note that $|f_n(x)| \leq 1 \quad \forall x \in [0,1], \forall n \geq 1$.

Moreover $f_n(x) \to 0$ ptwise.

Since $f_n(\frac{1}{n}) = 1 \quad \forall n$, there is no subseq. of (f_n) that cvgs. unif.

Remark: One cannot generalize the Bolzano-Weierstrass thm. on $\mathbb{R}$ to arbitrary metric spaces (even if its complete).

Upshot: Uniformly bdd. set $A \not\Rightarrow$ totally bdd.

Equicontinuous set $A \not\Rightarrow$ totally bdd.

Lemma 11.16 (Carothers). Let A be a totally bdd. set in $C(X)$. Then A is unif. bdd. and equicontinuous. | Self-read.

Q. Is the converse true?

A. YES! (Arzela-Ascoli Thm)

→ Arzela-Ascoli Thm: X : compact metric space and $A \subset C(X)$.
Then A is compact iff A is closed, unif. bdd, and equicontinuous.

Idea: $\Rightarrow$: Previous lemma.

$\Leftarrow$: Enough to show that every seq. (f_n) has a uniformly cvgt. subsequence.

Idea is to construct a uniformly Cauchy subsequence.

Since (f_n) is equicts., for $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. $\forall$ $n \geq 1$,

$$d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon. \quad \text{—— } (1)$$

Since X is compact, X is totally bdd, hence $\exists$ $\{x_1, \dots, x_k\} \subset X$ s.t.

$$X = \bigcup_{i=1}^{k} B(x_i, \delta).$$

Since (f_n) unif. bdd, $(f_n(x_i))$ is bdd. in $\mathbb{R}$, $\forall$ $1 \leq i \leq k$.

x_1 : $f_{1,1}(x_1), f_{1,2}(x_1), \dots\dots$ cvgs.

Consider $(f_{1,k})$ as the seq. and one has $(f_{1,k}(x_2))$ also bdd.

$f_{2,1}(x_2), f_{2,2}(x_2), \dots\dots$ cvgs.

$\dots\dots$ $f_{3,1}(x_3), f_{3,3}(x_3), \dots\dots$ cvgs.

$\vdots$

Pick the diagonal seq: $(f_{1,1}, f_{2,2}, f_{3,3}, \dots)$

Then, for each $1 \leq i \leq k$, $(f_{n,n}(x_i))$ cvgs.

Since there are finitely many i's, $1 \leq i \leq k$, for which $(f_{n,n})$ cvgs.

So, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ m, n \geq N$ and $1 \leq i \leq k$,

$$\left|f_{n,n}(x_i) - f_{m,m}(x_i)\right| < \varepsilon. \quad — (2)$$

Given $x \in X$, $\exists\ 1 \leq i \leq k$, s.t. $x \in B(x_i, \delta)$. and $\forall\ m, n \geq N$,

$$\left|f_{n,n}(x) - f_{m,m}(x)\right| \leq \underbrace{\left|f_{n,n}(x) - f_{n,n}(x_i)\right|}_{\substack{<\varepsilon \\ \text{via } (1)}} + \underbrace{\left|f_{n,n}(x_i) - f_{m,m}(x_i)\right|}_{\substack{<\varepsilon \\ \text{via } (2)}} + \underbrace{\left|f_{m,m}(x_i) - f_{m,m}(x)\right|}_{\substack{<\varepsilon \\ \text{via } (1)}}$$

Hence the subseq. $(f_{n,n})$ of (f_n) is unif. Cauchy. $\blacksquare$.