Metric Spaces

M: any set

A function $d: M \times M \to \mathbb{R}$ is called a metric on M if

(i) $0 \le d(x,y) < \infty \quad \forall\ x,y \in M$

(ii) $d(x,y) = 0$ iff $x = y$

(iii) $d(x,y) = d(y,x) \quad \forall\ x,y \in M$

(iv) $d(x,y) \le d(x,z) + d(z,y)$; $\forall\ x,y,z \in M$.

(M,d) is called a metric space.

Que. Given a set M, does there always exists a metric d on M?

Define $d: M \times M \to \mathbb{R}$ as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$ (d is called the discrete metric on M)

(M,d) is called a discrete metric space.

Example: $(\mathbb{R}, |\cdot|)$, where $d(x,y) := |x-y|$, is a metric space.

$|\cdot|$ is called the usual metric on $\mathbb{R}$.

Example (Important) Vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Using the norm function on V, a metric is defined on V as follows:

Def^n: A norm on a vector space V is $\|\cdot\| : V \to [0,\infty)$ satisfying

(i) $0 \le \|x\| < \infty \quad \forall\ x \in V$

(ii) $\|x\| = 0$ iff $x = 0$

(iii) $\|\alpha x\| = |\alpha|\,\|x\| \quad \forall\ x \in V,\ \alpha \in \mathbb{R}$ (or $\mathbb{C}$)

(iv) $\|x+y\| \le \|x\| + \|y\|$

$(V, \|\cdot\|)$ is called a normed linear space.

$d: V \times V \to \mathbb{R}$ as $d(x,y) := \|x-y\|$. (HW: Show that d is a metric on V.)

Examples of normed linear spaces:

(a) Consider $(\mathbb{R}^n, \|\cdot\|_2)$ where $\|x\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ the Euclidean-norm for $n \geq 1$.

(b) For $1 \leq p < \infty$, $x \in \mathbb{R}^n$, define $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ where $x = (x_1, x_2, \dots, x_n)$

$\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ (proved later).

(c) $C[a,b] := \{ f : [a,b] \to \mathbb{R} \text{ cts.} \}$ Define $\|f\|_1 := \int_a^b |f|\,dt$ and $\|f\|_\infty := \max_{a \leq t \leq b} \{|f(t)|\}$

HW: Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $C[a,b]$.

(d) (Sequence space) For $1 \leq p < \infty$,

$$\ell_p := \left\{ (x_n)_{n=1}^\infty \;\middle|\; \sum_{n=1}^\infty |x_n|^p < \infty \right\}$$

For $p = \infty$, $\ell_\infty := \{ (x_n)_{n=1}^\infty \mid (x_n) \text{ is a bounded seq.} \}$.

For $x \in \ell_p$, $\|x\|_p := \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p}$ is a norm on ℓ_p. (proved later)

For $x \in \ell_\infty$, $\|x\|_\infty := \sup_n \{|x_n|\}$ is a norm on ℓ_∞. (HW)

→ For $1 < p < \infty$,

- ℓ_p is a vector space.

  Claim: $x + y \in \ell_p$ if $x, y \in \ell_p$.

  Pf: Let $a, b \geq 0$.

  Then $(a+b)^p \leq (2 \max\{a,b\})^p = 2^p \max\{a^p, b^p\} \leq 2^p(a^p + b^p)$

  For $x, y \in \ell_p$, $\sum_{n=1}^\infty |x_n + y_n|^p \leq 2^p \sum_1^\infty |x_n|^p + 2^p \sum_1^\infty |y_n|^p < \infty$

- ℓ_p is a normed linear space

  claim: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ for $x, y \in \ell_p$.

$$\|x+y\|_p^p = \sum_{i=1}^\infty |x_i + y_i|^p = \sum |x_i + y_i| \, |x_i + y_i|^{p-1} \leq \sum \left(|x_i| \, |x_i + y_i|^{p-1} + |y_i| \, |x_i + y_i|^{p-1} \right)$$

$$(\text{why}) \overset{?}{=} \sum_1^\infty |x_i| \, |x_i + y_i|^{p-1} + \sum_1^\infty |y_i| \, |x_i + y_i|^{p-1}$$

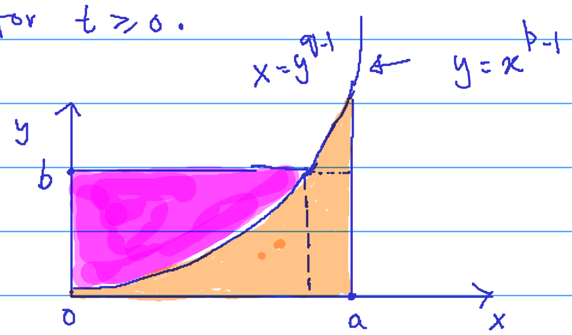
Target: find estimates for product of nonnegative nos. and sums

→ Young's Inequality: For $1 < p < \infty$, let q be s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
Then, for any $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality iff $a^{p-1} = b$.

Idea: Assume $a, b > 0$. Since $\frac{1}{p} + \frac{1}{q} = 1$, $f(t) := t^{p-1}$ and $g(t) := t^{q-1}$ are inverses of each other for $t \geq 0$.



Area of the "pink" portion $= \int_0^b y^{q-1}\,dy = \frac{b^q}{q}$

Area of the "orange" portion $= \int_0^a x^{p-1}\,dx = \frac{a^p}{p}$.

Area of the rectangle determined by a and b is ab.

So, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Equality iff $b = a^{p-1}$.

→ Hölder's Inequality: For $1 < p < \infty$ & q s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
For $x \in \ell_p$, $y \in \ell_q$,

$$\sum_1^\infty |x_i y_i| \leq \|x\|_p \|y\|_q .$$

(Hence, $x \cdot y \in \ell_1$)

pf: Assume $\|x\|_p > 0$ and $\|y\|_q > 0$ (o/w: inequality holds)

Consider $\displaystyle\sum_{i=1}^{\infty} \frac{|x_i|\,|y_i|}{\|x\|_p\,\|y\|_q}$.

For each $i \geq 1$, $\displaystyle\frac{|x_i||y_i|}{\|x\|_p\,\|y\|_q} \underset{\text{Young's Inequality}}{\leq} \frac{1}{p}\cdot\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\,\frac{|y_i|^q}{\|y\|_q^q}$

$$\sum_{1}^{n} \frac{|x_i||y_i|}{\|x\|_p\,\|y\|_q} \leq \frac{1}{p}\sum_{1}^{n}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\sum_{1}^{n}\frac{|y_i|^q}{\|y\|_q^q} \overset{?}{\leq} \frac{1}{p}+\frac{1}{q} = 1.$$

(fill in the gap)

(HW) If $x \in \ell_p$, then $\left(|x_i|^{p-1}\right)_{i=1}^{\infty} \in \ell_q$. Let $z := \left(|x_i|^{p-1}\right)_{i=1}^{\infty}$. Show $\|z\|_q = \|x\|_p^{p-1}$.

Consequently, for $(x_i+y_i)_{i=1}^{\infty} \in \ell_p$, $\left(|x_i+y_i|^{p-1}\right)_{i=1}^{\infty} \in \ell_q$.

Back to proving the triangle inequality: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$. (Minkowski's Inequality)

$$\begin{aligned}
\|x+y\|_p^p &\leq \sum |x_i|\,|x_i+y_i|^{p-1} + \sum |y_i|\,|x_i+y_i|^{p-1} \\
&\underset{\text{Hölder Ineq.}}{\leq} \|x\|_p \,\left\|\left(|x_i+y_i|^{p-1}\right)\right\|_q + \|y\|_p\,\left\|\left(|x_i+y_i|^{p-1}\right)\right\|_q \\
&= \|x+y\|_p^{p-1}\left(\|x\|_p + \|y\|_q\right) \\
&\quad \text{(HW)} \ \vdots
\end{aligned}$$

Hence, $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

Upshot: for $1 \leq p < \infty$, $(\ell_p, \|\cdot\|_p)$ is a normed linear space.

Using this norm, $d_p(x,y) := \|x-y\|_p$ makes (ℓ_p, d_p) a metric space.

Question: If $0 < p < 1$ and $\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$, then is $\|\cdot\|_p$ a norm on ℓ^p?