Auto Covariance function (ACVF) of a stationary process

Def^n: Let $\{X_t : t \in T\}$ be a covariance stationary time series process. The ACVF of $\{X_t\}$ at lag h is given by

$$\gamma_X(h) = Cov(X_{t+h}, X_t) = Cov(X_t, X_{t+h})$$

i.e. $\gamma_X(h) = E(X_{t+h} - \mu)(X_t - \mu)$

$$h = 0, \pm 1, \pm 2, \ldots$$

$$(\mu = EX_t)$$

Properties of ACVF

Property 1: $\gamma_X(0) \geq 0$ - trivial

$\overset{\shortparallel}{V(X_t)}$

Property 2 $|\gamma_X(h)| \leq \gamma_X(0) \quad \forall h$

Note that by Cauchy-Schwarz inequality, we have

$$|Cov(X_{t+h}, X_t)| = |E(X_{t+h} - \mu)(X_t - \mu)|$$

$$\leq \left(E(X_{t+h} - \mu)^2\right)^{1/2} \left(E(X_t - \mu)^2\right)^{1/2}$$

$$= (V(X_{t+h}))^{1/2} (V(X_t))^{1/2}$$

i.e. $|Cov(X_{t+h}, X_t)| \leq (V(X_{t+h}))^{1/2} (V(X_t))^{1/2} \quad \forall h$

i.e. $|\gamma_X(h)| \leq \gamma_X(0) \quad \forall h$

Property 3 : $\gamma_X(\cdot)$ is even f^n

$$\gamma_X(h) = E(X_{t+h}-\mu)(X_t-\mu)$$
$$= E(X_{t+h-h}-\mu)(X_{t-h}-\mu) \quad [\because \{X_t\} \text{ is cov stat}]$$
$$= E(X_t-\mu)(X_{t-h}-\mu)$$

i.e. $\gamma_X(h) = \gamma_X(-h) \quad \forall h$

Property 4 : $\gamma_X(\cdot)$ is non negative definite

[A real valued f^n on integers ($f: Z \to R$) is said to be non-negative definite iff

$$\sum_{i,j=1}^{n} a_i f(t_i - t_j) a_j \geq 0$$

$\forall$ positive int n

$\forall\ \underset{\sim}{a} \in R^n \quad \forall\ \underset{\sim}{t} = (t_1, \ldots t_n)' \in Z^n$]

Proof of property 4 : Let $\{X_t\}$ be a covariance stationary process

$\underset{\sim}{a} = (a_1, \ldots, a_n)' \in R^n$

$\underset{\sim}{t} = (t_1, \ldots, t_n)' \in Z^n$; $Z = \{0, \pm 1, \pm 2, \ldots\}$

Define $U_{t_i} = X_{t_i} - \mu$; $i = 1(1)n$; $\mu = E(X_{t_i}) \ \forall i$

$\underset{\sim}{U}_t = (U_{t_1}, \ldots, U_{t_n})'$

Note that $\quad 0 \leq V(\underset{\sim}{a}' \underset{\sim}{U}_t)$

$$V(\underset{\sim}{a}'\underset{\sim}{U}_t) = E(\underset{\sim}{a}'\underset{\sim}{U}_t - E(\underset{\sim}{a}'\underset{\sim}{U}_t))(\underset{\sim}{a}'\underset{\sim}{U}_t - E(\underset{\sim}{a}'\underset{\sim}{U}_t))'$$
$$= E(\underset{\sim}{a}'\underset{\sim}{U}_t)(\underset{\sim}{a}'\underset{\sim}{U}_t)'$$
$$= \underset{\sim}{a}' E(\underset{\sim}{U}_t \underset{\sim}{U}_t') \underset{\sim}{a}$$

$$= \underset{\sim}{a}' \Gamma_n \underset{\sim}{a}$$

Where, Γ_n = Covariance matrix of $(X_{t_1}, \ldots, X_{t_n})$

$$= \begin{pmatrix} Cov(X_{t_1}, X_{t_1}) & Cov(X_{t_1}, X_{t_2}) & \cdots & Cov(X_{t_1}, X_{t_n}) \\ & Cov(X_{t_2}, X_{t_2}) & \cdots & Cov(X_{t_2}, X_{t_n}) \\ & & \ddots & \vdots \\ & & & Cov(X_{t_n}, X_{t_n}) \end{pmatrix}$$

$\underset{\text{stationarity}}{\overset{\text{using}}{=}}$

$$\begin{pmatrix} \gamma_X(0) & \gamma_X(t_1-t_2) & \cdots & \gamma_X(t_1-t_n) \\ & \gamma_X(0) & \cdots & \gamma_X(t_2-t_n) \\ & & \ddots & \vdots \\ & & & \gamma_X(t_n-t_n) = \gamma_X(0) \end{pmatrix}$$

Thus $\underset{\sim}{a}' \Gamma_n \underset{\sim}{a} = V(\underset{\sim}{a}' \underset{\sim}{U}_t) \geq 0 \quad \forall \underset{\sim}{a} \quad \forall \underset{\sim}{t}$

$$\Rightarrow \sum_{i,j=1}^{n} a_i \gamma_X(t_i - t_j) a_j \geq 0 \quad \forall \underset{\sim}{a} \quad \forall \underset{\sim}{t}$$

$\Rightarrow \gamma_X(\cdot)$ is n.n.d.

<u>Remark</u>: Converse of Property 3 & Property 4 taken together is also true.

i.e. a real valued f^n defined on the set of integers which is even and n.n.d is a Covariance function of a covariance stationary time series.

Remark: In light of the previous remark and the properties (3 & 4) of ACVF, we have the following characterization of ACVF:

"A real valued function defined on integers is the ACVF of a covariance stationary time series iff it is even and n.n.d."

Rem: $\{X_t\}$ & $\{Y_t\}$ u.c. cov stat

Auto Correlation Function (ACF)

ACF of a stationary time series is given by

$$\rho_X(h) = \text{Corr}(X_{t+h}, X_t) = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

$$h = 0, \pm 1, \pm 2, \ldots$$

Using the properties of ACVF, $\gamma_X(\cdot)$, we can easily prove the following properties of ACF

(i) $\rho_X(0) = 1$

(ii) $|\rho_X(h)| \leq 1 \quad \forall h$

$Z_t = X_t + Y_t$

(iii) $\rho_X(h) = \rho_X(-h)$

(iv) $\rho_X(\cdot)$ is n.n.d.

(v) If X_{t+h} & X_t are indep then $\rho_X(h) = 0$.

Remark : ACVF & ACF for complex valued time series.

Let $\{X_t\}$ be a complex valued covariance stationary time series

$$X_t = U_t + i V_t$$

ACVF of $\{X_t\}$: $\gamma_X(h) = E(X_{t+h} - \mu)^*(X_t - \mu)$

$$\mu = E X_t$$

Note that $\gamma_X(h)$ is complex valued $(\forall h \neq 0)$, $\gamma_X(0)$ is real valued.

ACF of $\{X_t\}$: $\rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)}$.

Estimation of μ and $\gamma_X(h)$ for covariance stationary process

Let $X_1, \ldots, X_n$ be a sample of size n from a covariance stationary $\{X_t\}$ with $E X_t = \mu$ (unknown) and ACVF $\gamma_X(h) = E(X_{t+h} - \mu)(X_t - \mu)$ $\forall h$ (unknown)

Estimation of μ: μ is estimated by sample mean,

$$\bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$$

Realize that $E(\bar{X}_n) = \frac{1}{n} \sum_{t=1}^{n} E X_t = \mu$,

thus $\bar{X}_n$ is an unbiased estimator of μ