$\bar{A}$

$A \subseteq X$

W relatively open if $W = U \cap A$, $U \overset{open}{\subseteq} X$

A disconnected if $\exists$ non-empty W_1, W_2 s.t. $W_1 \cap W_2 = \phi$, $A = W_1 \cup W_2$

A Connected if A isn't disconnected.

- $\mathbb{R}$ is connected
- $A \subseteq \mathbb{R}$ connected $\Leftrightarrow$ A is an interval
- $A \subseteq X$ " , then $f(A)$ is connected
- A connected, then $\bar{A}$ is also connected

Let $X = U \cup V$, U and V sepᵃʳⁿ $\left(\begin{smallmatrix}\text{open} \\ \text{and}\end{smallmatrix}\text{ disjoint}\right)$
$A \subseteq X$ s.t. A is connected.
Then, either $A \subseteq U$ or $A \subseteq V$.

Suppose not, then $\exists\, x \in A \setminus V$ $\overset{\Rightarrow x \in U}{}$ and $y \in A \setminus U$
$\Rightarrow \exists\, \epsilon > 0$ s.t. $B_{\epsilon_x}(x) \subseteq U$, $B_{\epsilon_y}(y) \subseteq V$

$$A \subseteq \bigcup_{x \in A} B_{\epsilon_x}(x)$$

Take $W_1 = \bigcup_{x \in A \setminus V} B_{\epsilon_x}(x) \cap A$, $W_2 = \bigcup_{x \in A \setminus U} B_{\epsilon_x}(x) \cap A$

$$W_1 \cap W_2 = \phi, \quad A = W_1 \cup W_2$$

THM. If X is connected, then only "clopen" sets are X or ϕ.

Let $A \overset{\text{non-empty}}{\subsetneq} X$ "clopen"
$\Rightarrow X = A \cup A^c$
A, A^c are sepᵃʳⁿ $\Rightarrow X$ is disconnected
So, A doesn't exist.

Date __/__/__

THM: Let A and B be connected subset of X. If $A \cap B \neq \phi$, then $A \cup B$ is connected.

Pf: Suppose not.

Then, $\exists$ "separ" of $A \cup B$ say ~~W,~~ W_1, W_2.

$A \cup B = W_1 \cup W_2$

$\Rightarrow A \subseteq W_1$ or $A \subseteq W_2$

If $A \subseteq W_1$, then $B \subseteq W_1$

Then, W_2 is empty

$\Rightarrow \Leftarrow$

- $B \subseteq A \subseteq X$

If ~~A~~B is connected in $A \Rightarrow B$ is connected in X

$W_1 = U \cap A$, $W_2 = V \cap A$

$Y_1 = U_1 \cap B$, $Y_2 = U_2 \cap B$

$= (U \cap A) \cap B = U \cap B$

Path:

$x, y \in X$

$y : [0,1] \xrightarrow{cts.} X$

s.t. $y(0) = x$, $y(1) = y$

If $\exists$ a path between any two $x, y \in X$, then we say that X is path connected

THM. ~~X~~ path connected $\Rightarrow$ Connected

Pf: If not, ~~X~~ $= U \cup V = A$

Take $x \in A \setminus U$, $y \in A \setminus V$

$\exists\ y [0,1] \xrightarrow{cts.} A$

s.t. $y(0) = x$, $y(1) = y$

Date ___/___/___

$\gamma[0,1]$ is connected

So, either $\gamma[0,1] \subseteq U$ or V $\Rightarrow \Leftarrow$

as $\gamma(1) \in V$, $\gamma(0) \in U$

$\rightarrow$ $C[0,1]$. Is it connected

Take $f, g \in C[0,1]$

To show $\gamma : [0,1] \xrightarrow{cts.} C[0,1]$

s.t. $\gamma(0) = f$, $\gamma(1) = g$

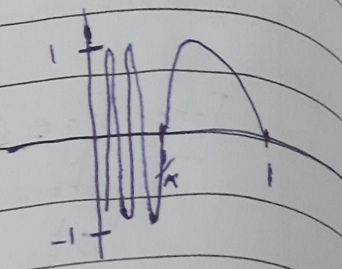
$$\gamma(t) = tg + (1-t)f$$

So, $C[0,1]$ is path connected $\Rightarrow$ connected.

Path connected $\Rightarrow$ Connected
Converse may not be true.

Topologist Sine Curve

$A = \{(x, \sin\frac{1}{x}) : x \in (0,1]\}$



A connected
$\bar{A}$ connected

$\bar{A} = A \cup \{(0,x) : x \in [-1,1]\}$

Suppose $\bar{A}$ is path connected
$\exists$ a cts. ~~map~~ $y : [0,1] \to \bar{A}$
s.t. $y(0) = (0,1)$, $y(1) = (1, \sin 1)$

Take $\epsilon = \frac{1}{2}$
$\exists \delta > 0$ s.t. $\forall x \in (0,\delta)$ then $|y(x) - y(0)| < \frac{1}{2}$
$\Rightarrow |y(0)| - |y(x)| < \frac{1}{2}$
$|y(x)| > |y(0)| - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$
$\Rightarrow \Leftarrow$ as for $x \in (0,\delta)$ $y(x) \in [-1,1]$
So, $\bar{A}$ isn't path connected

$\mathbb{R}^n \to$ Complete

$1 \leq p \leq \infty$ l_p -
$C[0,1]$

$C_{0,0} = \{x \in l_\infty : x(j) = 0$ except for finitely many

$C_0 = \{x \in l_\infty : x(j) \to 0$ as $j \to \infty\}$

Date

$a_1 = (1, 0, \dots)$

$a_2 = (1, \frac{1}{2}, 0, \dots)$

$\vdots$

$a_n = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, \dots)$

$\vdots$

$a = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

$d(a, a_n) = \|a - a_n\|_\infty$

$= \sup_j |a(j) - a_n(j)| = \sup_{j \geq n+1} |a(j)| = \frac{1}{n+1} \to 0$

So, $a_n \to a$

But $a \notin \cancel{a}\ C_{0,0}$.

So, $C_{0,0}$ isn't complete.

THM: Let (X, d) be a complete m.s.
$A \subseteq X$. (A, d) is complete $\Leftrightarrow$ A is closed

Pf: $\Rightarrow$

(A, d) complete.

If $x_n \in A$ s.t. $x_n \to x$

$d(x_n, x_m) \to 0$ as $n, m \to \infty$.

$\{x_n\}$ Cauchy in A $\Rightarrow x \in A$.

$\Leftarrow$

$\{x_n\}$ Cauchy in A

$\Rightarrow \{x_n\}$ Cauchy in X

$\Rightarrow x_n \to x \in X$

But A is close, $\{x_n\} \subset A \Rightarrow x \in A$

$\Rightarrow (A, d)$ is Complete.

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Claim : $\overline{C_{0,0}} = C_0$ ~~In sup~~ ~~me~~

$a = (a(1), a(2), \dots) \in C_0$

$\epsilon > 0 \quad \exists N_0$ s.t.
$|a(j)| < \epsilon \quad \forall j \geq N_0$

Define $a_{N_0} = (a(1), a(2), \dots, a(N_0-1), 0, 0, \dots)$
$C_{0,0} \ni$

$d(a, a_{N_0}) = \sup_j |a(j) - a_{N_0}(j)|$

$= \sup_{j \geq N_0} |a(j)| < \epsilon$

$C_{0,0}$ is dense in C_0

Ex : $\overline{C_{0,0}}^{\,\ell_p}$,,

$a = \ell_1$

$\sum_{j=1}^{\infty} |a(j)| < \infty$

$\Rightarrow \exists N_0$ s.t. $\sum_{j=N_0}^{\infty} |a(j)| < \epsilon$

$a_{N_0} = (a(1), \dots, a(N_0-1), 0, 0, \dots)$

$d(a, a_{N_0}) = \sum_{j=N_0}^{\infty} |a(j)| < \epsilon$

DEF^N Let X be a V.S. over $\mathbb{R}$ or $\mathbb{C}$

$$\|\cdot\| : X \to \mathbb{R}^+ \cup \{0\}$$

if (i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in F$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

$(X, \|\cdot\|)$ is called "normed linear space"

nlsp.

Define, $d(x,y) = \|x-y\|$

Consider l_p, $0 \leq p < 1$

$$\{\{x_n\} : \sum |x_n|^p < \infty\}.$$

$$d(x,y) = \sum |x_n - y_n|^p$$

- $p < 1$ $\quad (a+b)^p \leq a^p + b^p$
  $p \geq 1$ $\quad (a+b)^p \geq a^p + b^p$

$$\|x\| = d(x,0) = \sum |x_n|^p$$
$$\|\lambda x\| = |\lambda|^p \|x\|$$

Not a normed linear space.

→ A complete "nlsp" is called Banach Space.

i.e.

$$\|x_n - x_m\| \to 0 \Rightarrow \exists x \in X \text{ s.t. } \|x_n - x\| \to 0.$$

THM. X is Banach $\Leftrightarrow$ Every abs. cgt. series cgs.

i.e. $\sum_{n=1}^{\infty} x_n$, $S_N = \sum_{n=1}^{N} x_n$

$\exists S$ s.t. $\|S_N - S\| \to 0$

if $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then $\sum_{n=1}^{\infty} x_n$ cgs.

Date __/__/__

Pf: X Banach

Given $\sum_{n=1}^{\infty} \|x_n\|$ cgs.

$$S_N = \sum_{n=1}^{N} x_n$$

$$\|S_N - S_M\| = \left\|\sum_{n=M+1}^{N} x_n\right\| \le \sum_{M+1}^{N} \|x_n\|$$

$\epsilon > 0$, $\exists N_0$ s.t. $\forall N, M \ge N_0$

~~$\|S_N - S$~~ $\sum_{M+1}^{N} \|x_n\| < \epsilon$

$$\Rightarrow \|S_N - S_M\| < \epsilon$$

$\{S_n\}$ Cauchy $\Rightarrow$ S_n cgs.

Converse

$(\Leftarrow)$

Let $\{x_n\}$ be Cauchy seq. in X.

$\exists \{x_{n_k}\}$ s.t. $\|x_{n_{k+1}} - x_{n_k}\| \le \frac{1}{2^k}$

$$x_{n_{k+1}} = \sum_{l=1}^{k} (x_{n_{l+1}} - x_{n_l}) + x_{n_1}$$

$\sum_{l=1}^{\infty} (x_{n_{l+1}} - x_{n_l})$ abs. cgt.

$\Rightarrow \{x_{n_{k+1}}\}$ cgs. in X. $\Rightarrow \{x_n\}$ cgs. in X

(As $\{x_n\}$ is Cauchy)

$f: [a,b] \longrightarrow [a,b]$ cts.

$\exists x \in [a,b]$ st. $f(x) = x$.

Define. $g(x) = x - f(x)$

$g(a) = a - f(a) \le 0$ (as $f(a) \ge a$)

$g(b) = b - f(b) \ge 0$ (as $f(b) \le b$)

If one of them is 0, then we are done

Otherwise, we use IVP.

Date __/__/__

Ex: $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x+1$

Doesn't have fixed pt.

Ex: $X = \{0, 1\}$.

$f : X \to X$ as $f(0) = 1$, $f(1) = 0$

No fixed pt.

Date 01/04/22 Tutorial (Assgn 5)

Example of f

(i) f cts. but not open

(ii) " " " closed

(iii) f closed " " open

(iv) " open " " closed

X compact

$f : X \xrightarrow[\text{bijec}^n]{\text{cts.}} Y$ f is closed

"homeomorphism"

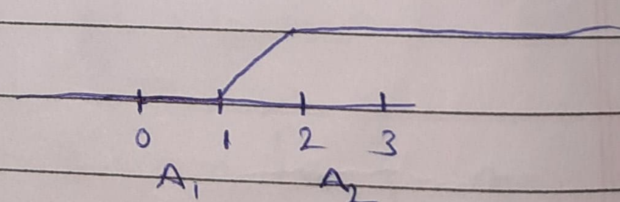
(X,d) is separable if $\exists$ a countable dense set.

Ex: l_∞ isn't separable.

(11) (X,d) is 2nd countable if $\exists$ a countable collec^n $\mathcal{U}$ of open balls in X s.t. $\forall x \in X, r > 0, \exists U \in \mathcal{U}$ s.t. $x \in U \subseteq B_r(x)$

$\rightarrow$ (X,d) is second countable iff it is separable.

(12) $$f(x) = \frac{d(x, A_1)}{d(x, A_1) + d(x, A_2)}$$



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$A \subset X$. A is complete $\Leftrightarrow$ A is closed

Prop:

DEF^N The map $f: X \longrightarrow X$ is said to be a contrac^n if $\exists c < 1$ s.t. for every $x, y \in X$.

$$d(f(x), f(y)) \leq c\, d(x, y)$$

THM: (Banach Contrac^n Principle)

Let (X, d) be a complete m.s.
If $f: X \longrightarrow X$ is a contrac^n. Then,
$\exists!$ fixed pt. i.e. $\exists!$ $x_0 \in X$ s.t. $f(x_0) = x_0$
↳ there exists a unit

-set.

Pf:

Let $x_0 \in X$. Define $x_1 := f(x_0)$, $x_2 := f \circ f(x_0)$
..... $x_n := f^n(x_0)$

If $x_n \longrightarrow x$, $\Rightarrow$ $f(x_n) \longrightarrow f(x)$

$$f(x_n) = f \circ f^n(x_0) = f^{n+1}(x_0) = x_{n+1} \longrightarrow x$$

Claim: $\{x_n\}'$ is Cauchy.

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq c\, d(x_n, x_{n-1})$$

$c < 1$

x_n's are Cauchy.

Hence, $x_n \longrightarrow x$ for some $x \in X$
and $f(x) = x$

Remark (i) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 1$
$d(f(x), f(y)) = d(x, y)$ (Isometry)

(ii) Suppose $d(f(x), f(y)) < d(x, y)$ $\forall x, y$

Date __/__/__

$X = [1, \infty)$ ← Complete as closed

$f(x) = x + \frac{1}{x}$

$d(f(x), f(y)) = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = |x-y| \left| 1 - \frac{1}{xy} \right|$
$< |x-y| = d(x,y)$

Doesn't have a fixed pt.
Contracⁿ condⁿ is necessary

Ex: $f(x) = ax + b$, $a, b \in \mathbb{R}$, $x \in \mathbb{R}$

If $|a| < 1$, then f is contracⁿ.

$f(x) = Ax + b$, $b, x \in \mathbb{R}^n$, $A \in M_n(\mathbb{R}^n)$
If $Ax + b = x$??
or $(A - I)x + b = 0$ $\qquad A = (a_{ij})$

$d(f(x), f(y)) = \| \cancel{\sup}\, f(x) - f(y) \|_\infty = \| Ax - Ay \|_\infty$
$= \max \left(\left| \sum_j a_{ij}(x_j - y_j) \right|, \left| \sum_{j=1}^n a_{ij} \ldots \right| \right)$
$\leq \max_i \left(\sum_{j=1}^n |a_{ij}| \right) \| x - y \|_\infty$

If $\cancel{\max_i} \sum_{j=1}^n |a_{ij}| < c < 1$ $\forall i$, then contracⁿ

Ex: Try in $p = 1, 2$ metric.

Endsem

If f is contracⁿ $\overset{?}{\Rightarrow}$ $\sum_{j=1}^n |a_{ij}| < c < 1$

$\varphi : \mathbb{Q} \to \mathbb{R}$, $\varphi(x) = x$, $d(\varphi(x), \varphi(y)) = d(x, y)$
$\mathbb{Q} \subseteq \mathbb{R}$,
$\overline{\mathbb{Q}} = \mathbb{R}$ complete
↳ dense

Date __/__/__

DEF^N Let (X,d) be a m.s.
We say that (X^*,d^*) is a completion of (X,d) if
(i) (X^*,d^*) is complete
(ii) $\exists$ an isometry, $\varphi : X \longrightarrow X^*$ s.t. $\varphi(X)$ is dense in X^*.

Lemma: Let A be a dense subset of X. If every Cauchy seq. in A cgs in X, then X is complete.

Pf: $\{x_n\}$ Cauchy in X.

~~For $\epsilon>0$,~~

~~For $\epsilon>0$,~~ $\exists y_n \in A$ st $d(x_n,y_n) < \cancel{\frac{1}{2^n}}\ \frac{1}{2^n}$

$$d(y_n,y_m) \leq d(y_n,x_n) + d(x_n,x_m) + d(x_m,y_m)$$
$$\leq \frac{1}{2^n} + \frac{1}{2^m} + \epsilon$$

$y_n \longrightarrow x$

$$d(x_n,x) \leq d(x_n,y_n) + d(y_n,x)$$
$$\Rightarrow x_n \longrightarrow x$$

(X,d)

Let $\{x_n\}, \{y_n\}$ be Cauchy in X.

Define $\{x_n\} \sim \{y_n\}$ if $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$

↳ equivalence rel^n

$$X^* = \{[\{x_n\}] : \{x_n\} \text{ Cauchy in } X\}$$

Define $d^*([\{x_n\}],[\{y_n\}]) = \lim\limits_{n\to\infty} d(x_n,y_n)$

Well defined:

Let $\{x_n\} \sim \{x_n'\}$, $\{y_n\} \sim \{y_n'\}$

Date __/__/__

$$d(x_n, y_n) \leq d(x_n, x_n') + d(x_n', y_n') + d(y_n', y_n)$$

$$d(x_n', y_n') \leq d(x_n', x_n) + d(x_n, y_n) + d(y_n, y_n')$$

$\Rightarrow$ For $n \to \infty$, $d(x_n, y_n) = d(x_n', y_n')$

So, d^* is well-defined.

(X, d)

$X^* = \{ [\{x_n\}] : \{x_n\} \text{ Cauchy in } X \}$

$d^*([\{x_n\}], [\{y_n\}]) = \lim_{n \to \infty} d(x_n, y_n)$

d^* is a metric

(i) $d^*([\{x_n\}], [\{y_n\}]) = 0 \iff [\{x_n\}] = [\{y_n\}]$

(ii) $d^*(\quad " \quad) =$

(iii) $d^*([\{x_n\}], [\{y_n\}]) \leq d^*([\{x_n\}], [\{z_n\}]) + d^*([\{z_n\}], [\{y_n\}])$

Define $\varphi : X \longrightarrow X^*$

$\varphi(x) = \hat{x}$, $\quad \hat{x} = [(x, x, \ldots)]$

$\varphi(x) \subseteq X^*$

$x^* \in X^*$

$\shortparallel$

$[\{x_n\}]$ $\qquad$ Need to show $\epsilon > 0$ $\exists z \in X$ s.t. $d^*(x^*, \hat{z}) < \epsilon$

For every $\epsilon > 0$, $\exists N$ s.t. $d(x_n, x_m) < \epsilon$ $\forall n, m \geq N$

$\hat{x}_N$

$d^*(x^*, \hat{x}_N) = \lim_{n \to \infty} d(x_n, x_N) < \epsilon$

$d^*(\varphi(x), \varphi(y)) = d(x, y)$

Take a Cauchy seq. in $\varphi(X)$

$\{x_k^*\}$ a Cauchy seq. in X^*

$x_k^* = [\{x_n^k\}]$

Each fixed k, $\{x_n^k\}$ is Cauchy in X.

Want to show $\{x_k^*\}$ cgs. in X^*

Date __/__/__

$$x_k^* = [(x_k, x_k, \ldots)]$$

$$d^*(x_k^*, x_l^*) < \epsilon \quad \forall\, k, l \geq N_0$$

$$= \lim_{n\to\infty} d(x_n^k, x_n^l)$$

$$= d(x_k^0, x_l^0)$$

$$x_k^{**} = (x_k, \ldots)$$

$$[\{x_k\}]$$

(X, d)

$$((X_1^*, d_1^*), \varphi_1) \quad , \quad ((X_2^*, d_2^*), \varphi_2)$$

$$\exists\, \phi : X_1^* \xrightarrow{\text{isometry}} X_2^*$$

$$\varphi_1 : X_0 \longrightarrow X_1^* \quad , \quad \varphi_1 : X_0 \longrightarrow \varphi_1(X) \ (\text{Restricted to } \varphi(X))$$

Define $\Phi = \varphi_2 \circ \varphi_1^{-1}\Big|_{\varphi_1(X)}$

$C[0,1]$, $\|f\|_1 = \int_0^1 |f(x)|\, dx$

$d_1(f,g) = \|f-g\|_1$

Denote complete of $(C[0,1], \|\cdot\|_1)$ as $L^1[0,1]$ ↳ Lebesgue space

$f \in L^1[0,1] \Rightarrow \exists \{f_n\} \subseteq C[0,1]$ s.t. $\|f - f_n\| \to 0$ as

$$\int_{[0,1]} f = \lim_{n\to\infty} \int_0^1 f_n$$

$$\Phi : C[0,1] \longrightarrow \mathbb{C}$$

$$\Phi(f) = \int_0^1 f(t)\, dt$$

$$|\Phi(f)| \leq \|f\|_1$$

$f \in L^1[0,1]$

$\exists f_n \in C[0,1]$ s.t. $f_n \to f$ in $\|\cdot\|_1$

$\{\varphi(f_n)\}$

$$|\varphi(f_n) - \varphi(f_m)| = |\varphi(f_n - f_m)| \leq \|f_n - f_m\|_1$$

$$\varphi(f_n) \longrightarrow \alpha \in \mathbb{C}$$

Define $\int_{[0,1]} f = \alpha$

Saathi

Date ___/___/___

$A \subseteq [0,1]$ χ_A

$\int |f_n(x) - \chi_A(x)|\, dx \longrightarrow 0$, then A is measure.

THM: (Baire Category

Let X be a complete m.s. Then, a countable intersecⁿ of open dense subsets of X is dense

~~DEFⁿ~~ Pf:

Let $V_1, V_2, \ldots$ be open, dense subsets of X

Denote $V = \bigcap_{n=1}^{\infty} V_n$

Let U be an open subset of X.

$U \cap V_1 \neq \phi$

$\hookrightarrow$ Non-empty, open

$\Rightarrow x_1 \in U \cap V_1$, $\exists r_1 > 0$ s.t. $\overline{B_{r_1}(x_1)} \subseteq U \cap$

$r_1 < 1$

$\overline{B_{r_1}(x_1)} \cap V_2 \neq \phi$

$\Rightarrow \exists r_2 < \frac{1}{2}$ s.t. $\overline{B_{r_2}} \subseteq B_{r_1}(x_1) \cap V_2$

$\overline{B_{r_2}} \cap V_3 \neq \phi$

$\Rightarrow \exists r_3 < \frac{1}{3}$ s.t. "

$\Rightarrow$ We have $\overline{B_{r_1}} \supseteq \overline{B_{r_2}} \supseteq \overline{B_{r_3}} \ldots$

$diam(\overline{B_{r_n}}) \leq \frac{2}{n} \longrightarrow 0$

$x \in \bigcap_i \overline{B_{r_i}} \neq \phi \Rightarrow x \in U$ and $x \in V_n \ \forall n$

DEFⁿ We say $A \subseteq X$ is of "1st Category" (or meagre) if is expressed as a countable union of nowhere dense subsets. If a set is not of first category then it is called as a set of second category

Complement of a "meager" set is called residual

Ex (i) X complete m.s.
$A \subseteq X$ is of first category. Then $\mathring{A} = \phi$

$A = \bigcup_i V_i$

Ex (ii) Complement of a "meager set" is dense

(iii) X cannot be written as countable union of nowhere dense sets.

(iv) $X = \bigcup_n F_n$, F_n closed, $\exists\, n_0$ s.t. $\mathring{F}_{n_0} \neq \phi$

THM: An infinite dimensional Banach space cannot have a countable basis.

Pf: Suppose $\{e_1, e_2, \dots\}$ be a basis of X.
Define $H_n = \overline{\text{Span}}\{e_1, \dots, e_n\}$
H_n's closed subspace of X.

$X = \bigcup_n H_n$ $\quad (x \in X,\ x = \sum_{n \in F} \alpha_n e_n \subseteq H_{N_0})$

$\Rightarrow$ Atleast one H_n has non-empty interior. (By BCT)
$\exists\, r > 0$ s.t. $B_r(x) \subseteq H_{N_0}$

$0 \neq y \in X$, $z = \dfrac{y}{\|y\|} \in B_1(0)$

$rz \in B_r(0)$

$x + B_r(0) = B_r(x) \subseteq H_{N_0}$

$x + rz \in H_{N_0}$

$\Rightarrow z \in H_{N_0} \Rightarrow y \in H_{N_0}$

$\Rightarrow X = H_{N_0}$ (f.d.) $\Rightarrow \Leftarrow$

Date 12/04/22

$X \to$ Compact m.s.

$C(X) = \{f : X \to \mathbb{R}$ or $\mathbb{C}$: f is cts.$\}$

$$\|f\|_\infty = \sup_{x \in X} |f(x)| < \infty$$

$$d(f,g) = \|f-g\|_\infty$$

$C(X)$ is complete

We say $\{f_n\}$ is Cauchy if $\|f_n - f_m\|_\infty \to 0$ as $n,m \to \infty$

$f_n \to f$, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$

$$\mathcal{F} = \{Tf : Tf(x) = \int_0^x f(t)\,dt : f \in C[0,1], \|f\|_\infty \leq 1$$

Compact subset in $C(X)$

$A = \{f_n\}$ s.t. $f_n \xrightarrow{\|\cdot\|_\infty} f$, $\bar{A}$ is seq. subset

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)|$$

$\exists N_0, n \geq N_0$ $\leq \|f_n - f\|_\infty + |f(x) - f(y)| + \|f - f_n\|_\infty$

$$< \frac{\epsilon}{3} + |f(x) - f(y)| + \frac{\epsilon}{3} < \epsilon$$

f cts. $\Rightarrow \exists\ \delta > 0$ $\exists \epsilon$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y))$

$\epsilon > 0$, $\exists \delta_n > 0$, st. $d(x,y) < \delta_n \Rightarrow |f_n(x) - f_n(y)| <$

$1 \leq n \leq N_0$

$d(x,y) < \delta^2$ $\quad \delta^1 = \min\{\delta_1, \ldots, \delta_{N_0 - 1}\}$

$\delta^2 = \min\{\delta^1, \delta\}$

$\Rightarrow |f_n(x) - f_n(y)| < \epsilon \ \forall n$

Date __/__/__

DEF^N (Equicts.)

A family $\mathcal{F} \subseteq C(X)$ is said to be equicts. at a pt. $x \in X$ if for $\epsilon > 0$, $\exists \delta > 0$ s.t.

$y \in B_\delta(x) \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall f \in \mathcal{F}$

$\mathcal{F}$ is "equicts." if $\mathcal{F}$; $\epsilon > 0$, $\exists \delta > 0$ s.t.

$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall f \in \mathcal{F}$

Ex: (1) Let $\mathcal{F} = \{ f \in C[0,1]$ s.t. $|f(x) - f(y)| \leq |x-y|^\alpha$, $0 < \alpha \leq 1 \}$

$$X \subseteq \bigcup_{i=1}^{N} B_\delta(x_i)$$

$$x \in B_\delta(x_{i_0})$$

$|f(x) - f(x_{i_0})| < 1$

$\Rightarrow |f(x)| \leq 1 + |f(x_{i_0})|$

$X = \mathbb{R}$, $f(x) = x^2$

PROP: Let X be compact. Then, $\mathcal{F} \subseteq C(X)$ is equicts. iff it is equicts. at each pt. of X.

Pf: $X = \bigcup_{x \in X} B_{\delta_x}(x) \Rightarrow X = \bigcup_{i=1}^{N} B_{\delta_i}(x_i)$

$\delta = \min(\delta_1, \dots, \delta_N)$

$d(x,y) < \delta$, $|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \epsilon$

PROP: Let $\mathcal{F}$ be a totally bdd. subset of $C(X)$. Then, $\mathcal{F}$ is equicts.

Pf: $\epsilon > 0$, $\exists \{x_1, \dots, x_N\}$

$\mathcal{F} \subseteq \bigcup_{i=1}^{N} B_{\epsilon/3}(f_i)$, $f_i \in \mathcal{F}$

$\exists \delta_i > 0$ s.t. $d(x,y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \epsilon/3$

Date ___/___/___

$\delta = \min\{\delta_i\}_{i=1}^{N}$

$y \in B_\delta(x)$, $f \in \mathcal{F}$

$$|f(x)-f(y)| \leq |f(x)-f_{i_0}(x)| + |f_{i_0}(x)-f_{i_0}(y)| + |f_{i_0}(y)-f(y)|$$
$$\leq \|f-f_{i_0}\|_\infty + |f_{i_0}(x)-f_{i_0}(y)| + \|f-f_{i_0}\|_\infty$$
$$< \epsilon$$

$y \in B_{\delta_{i_0}}(x)$

Date 13

Equicontinuity

$\mathcal{F} \subseteq C(X)$ equicts. if

$\epsilon > 0, \exists \delta > 0,\ y \in B_\delta(x) \Rightarrow |f(x) - f(y)| < \epsilon\ \forall f$

THM: (Arzelà - Ascoli Thm.)

Let X be a compact m.s. and $\mathcal{F} \subseteq C(X)$

$\mathcal{F}$ is compact $\Leftrightarrow$ $\mathcal{F}$ closed, uniformly bdd. and equicts.

Pf: $(\Rightarrow)$ done

$(\Leftarrow)$ Let $\{f_n\}$ be a seq. in $\mathcal{F}$

Aim: Find $\{f_{n_k}\}$ s.t. $\|f_{n_k} - f\|_\infty \to 0$ for some $f \in \mathcal{F}$

$\{f_n(x)\} \subseteq \mathbb{R}$ bdd.

$\Rightarrow \exists \{n_k^x\}$ s.t. $\{f_{n_k^x}(x)\}$ cgs.

$\{f_n\}$ equicts. ($\because \mathcal{F}$ is equicts.)

$\Rightarrow \epsilon > 0, \exists \delta > 0,$ s.t. $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $d(x,y) < \delta\ \forall n$

As X compact, $\exists x_1, \dots, x_N \in X$ s.t.

$$X = \bigcup_{i=1}^{N} B_\delta(x_i)$$

Now, $\{f_n(x)\}$ reduces to $\{f_n(x_i)\} \subseteq \mathbb{R}$

$\exists \{n_k^{x_i}\}$ s.t. $\{f_{n_k^{x_i}}(x_i)\}$ cgs.

Let $\Lambda_1 = \{n_k^1\}$

$\{f_{n_k^1}(x_2)\}$ bdd hence,

$\exists \Lambda_2 \subseteq \Lambda_1$ s.t. $\{f_k(x_2)\}_{k \in \Lambda_2}$ cgs.

$\{f_k(x_3)\}_{k \in \Lambda_2}$ bdd, $\exists \Lambda_3 \subseteq \Lambda_2$ s.t. $\{f_k(x_3)\}_{k \in \Lambda_3}$ cgs.

$\vdots$

$\Lambda_N \subseteq \Lambda_{N-1} \subseteq \dots \subseteq \Lambda_1$

$k \in \Lambda_i$

Let n_k be the k-th entry of Λ_k.

$\Lambda = \{n_k : k \in \mathbb{N}\}$, $n_i = \Lambda_i$

Then, atmost i-1 elements of ~~ase~~ Λ are not in

$$f_{n_k}(x_i) \longrightarrow L_i \quad \text{as } k \longrightarrow \infty$$

$$|f_{n_k}(x_i) - f_{n_l}(x_i)| < \epsilon/3 \quad \forall n, l > M \ \forall i$$

$$M = \max(M_i)$$

$$|f_{n_k}(x) - f_{n_l}(x)| \leq |f_{n_k}(x) - f_{n_k}(x_{i_0})| + |f_{n_k}(x_{i_0}) - f_{n_l}(x_{i_0})| + |f_{n_l}(x_{i_0}) - f_{n_l}(x)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \forall n, l > M$$

$\{f_{n_k}\}$ uniformly Cauchy

$f_{n_k} \to f \in \mathcal{F}$

$(X, \| \ \|)$

$B = \{x \in X : \|x\| \leq 1\}$

Suppose B is compact

$$B \subseteq \bigcup_{i=1}^{N} B_{\frac{1}{2}}(x_i) \subseteq Y + \frac{1}{2} B_1(0)$$

$$B_{\frac{1}{2}}(x) = x + B_{\frac{1}{2}}(0) \ \cancel{\subseteq Y + B_{\frac{1}{2}}(0)}$$

Let ~~y~~ $Y = \text{span}\{x_1, \ldots, x_N\}$ (Y is v.s $\Rightarrow Y + \frac{1}{2}Y = Y$)

$$B_1(0) \subseteq B \subseteq Y + \frac{1}{2} B_1(0) \subseteq Y + \frac{1}{2}\left(Y + \frac{1}{2} B_1(0)\right) = Y + \frac{1}{2^2} B_1(0)$$

$$\subseteq Y + \frac{1}{2^N} B_1(0)$$

$x = x_n + y_n$

Date ___/___/___

$(C^1[0,1], \|\cdot\|_\alpha)$

$\|f\|_\alpha = \|f\|_\infty + \|f'_\infty\|$ $\|f\|_\infty \leq 1, \|f'\|_\infty \leq 1$

$|f(x) - f(y)| \leq |f'(x_0)||x-y| \leq |x-y|$

So, equicts.

But not compact

As Arezlà - Ascoli Thm is applicable only when $\mathcal{F} \subseteq C(X)$ i.e. Here $(C^1[0,1], \|\cdot\|_\alpha) = (C(X), \|\cdot\|_\infty)$ for some X

Assgn. 8

(9) $U \underset{\text{open}}{\overset{\text{connected}}{\subseteq}} \mathbb{R}^n$. T a component of $\mathbb{R}^n \setminus U$. Then show $\mathbb{R}^n \setminus T$ is connected

Let S be a component of $\mathbb{R}^n \setminus U$

Suppose, $\partial U \cap S = \varphi$,

$S \subseteq \mathbb{R}^n \setminus U$, $\mathbb{R}^n \setminus \bar{U}$ is open in $\mathbb{R}^n$

$\Rightarrow$ If S is open in $\mathbb{R}^n \setminus \bar{U}$, S is open in $\mathbb{R}^n$

Suppose S is not open

$\exists x \in S$, $\forall \epsilon > 0$ $D_\epsilon = B_\epsilon(x) \cap (\mathbb{R}^n \setminus \bar{U})$ intersec

Contradicts maximality of S. So, S open

Again $\Rightarrow\Leftarrow$

So, $\partial U \cap S \neq \varphi$

$\{C_i\}_{i \in I}$ to be components of $\mathbb{R}^n \setminus U = \bigcup_i C_i$

$U_i' = (\partial U_i \cap C_i) \cup U_i$

$\mathbb{R}^n \setminus T = \bigcup_{\substack{i \\ C_i \neq T}} (C_i \cup U_i')$

$U \subseteq U_i' \subseteq \bar{U}$

connected.

Typo

(10) (X, d) connected m.s. which isn't bdd.

$\forall a \in X$, $\forall r > 0$ $S^a = \{x : d(x,a) = r\}$ is non-empty

Suppose empty.

Then we have $U_1 = \{x : d(x,a) > r\}$ Non-empty

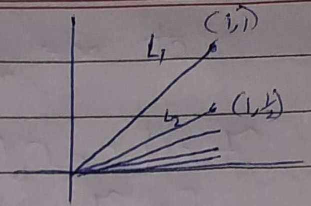
$U_2 = \{x : d(x,a) < r\}$ open

$U_1 \cup U_2 = X$ $\Rightarrow X$ is disconnected $\Rightarrow\Leftarrow$

(12)

Date $\{(1,0)\}$

(12) $S = \cancel{(0,1)} \cup \left(\bigcup_{n\in\mathbb{N}} L_n\right)$



$\left(\bigcup_{n\in\mathbb{N}} L_n\right)$ is connected
(as path connected)

$\left(\overline{\bigcup L_n}\right)$ is connected

$\bigcup L_n \subset S \subseteq \overline{\bigcup L_n}$
So, S is connected.

S not path connected.
Suppose $\cancel{\quad}$. $\gamma : [0,1] \to S$ a cts. map st $\gamma(0) = (1,0)$

$A = \{t \in [0,1] : \gamma(t) = (1,0)\}$
A is closed. $(A = \gamma^{-1}((1,0)))$

Take $t_0 \in A$, γ being cts. $\exists\, \delta > 0$ st if $t \in \underbrace{(t_0-\delta, t_0+\delta)}_{I_\delta}$,
$|\gamma(t) - \gamma(t_0)| < \frac{1}{2}$
$\Rightarrow |\gamma(t)| > \frac{1}{2}$
$\Rightarrow$ x coordinate of $\gamma(t)$ is +ve.
$m(x,y) := y/x$

$\varphi : I_\delta \to S$
$\varphi(t) := m \circ \gamma(t)$
Range of $\varphi \subseteq \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}\}$
Range of φ is singleton.

Assg. 9

$\mathcal{F} = \{f \in C[0,1] : |f(x)| > 0 \;\forall x \in [0,1]\}$ is open

$\epsilon > 0$ $\quad B_\epsilon(f) = \{g \in C[0,1] : \|f-g\|_\infty < \epsilon\}$

$|f(x) - g(x)| < \epsilon$ $\quad$ Choose $\epsilon = \alpha/2$
$\Rightarrow |g(x)| > |f(x)| - \epsilon > \alpha - \epsilon$

Date ___/___/___

Let $\alpha = \min_{x\in[0,1]} |f(x)| \Rightarrow \alpha = f(x_0) > 0$

$\mathcal{F} = \{ f \in C_b(\mathbb{R}) : f(x) > 0 \ \forall x \in \mathbb{R} \}$

$f \in \mathcal{F}$ s.t. $\inf_{x\in\mathbb{R}} f(x) = \alpha > 0$

Consider $g \in B_{\alpha/2}(f) \Rightarrow \|g - f\|_\infty < \alpha/2$

$\Rightarrow \|g\|_\infty \leq \|f\|_\infty + \frac{\alpha}{2}$

$|g(x)| > |f(x)| - \frac{\alpha}{2} > \frac{\alpha}{2}$

Suppose $\inf f(x) = 0$

$\epsilon > 0$, $\exists x_0$ s.t. $f(x_0) < \frac{\epsilon}{2}$

$I_{x_0} = (x_0 - \delta, x_0 + \delta)$

$\exists$ a nbd. of x_0 s.t. $f(x) < \frac{\epsilon}{2} \ \forall x \in I_{x_0}$

Let J be interval st $\bar{J} \subsetneq I_{x_0}$

$g(x) = 0$ on J

$\quad f(x)$ on J^c

(4) $\{\sin(nx) : n \geq 1\}$ is not an equicts subset $C[0,1]$.

Take $\epsilon = \frac{1}{2}$

Find a δ st. we can't get $|f(x) - f(y)| < \epsilon$

(5)(a) $\mathcal{F} = \{ F(x) = \int_0^x f(t)\,dt : f \in C[0,1], \|f\|_\infty \leq 1 \}$

$F(x) = \int_0^x f(t)\,dt \leq \int_0^1 f(t)\,dt \leq \int_0^1 1\,dt = 1$

$F \in \mathcal{F}$, $\|F\|_\infty = \sup |F(x)| = \sup_{x\in[0,1]} \left| \int_0^x f(t)\,dt \right|$

$= \sup_x \left| \int_0^x |f(t)|\,dt \right| \leq \|f\|_\infty \sup x$

$\leq \|f\|_\infty$

Date __/__/__

Saathi

$$|F(x)-F(y)| = \left|\int_x^y f(t)\,dt\right| = \int_x^y |f(t)|\,dt$$

$$\cancel{\le (y-x)\|f\|}$$

$$\le |y-x|\,\|f\|_\infty \le |y-x|$$

(b) Choose $F_n(x) = |x|^{\frac{1}{n}+1}$ $\quad F_n(x) \to |x|$ (not diff.)

$|x| \notin \mathcal{F}$ So, $\mathcal{F}$ isn't closed

(c) $F_n \xrightarrow{\|\cdot\|_\infty} F$

$$|F(x)-F(y)| = \lim_n |F_n(x)-F_n(y)|$$

$$\cancel{F_n(x)} = \varphi_n' = n\left(F\left(x+\tfrac{1}{n}\right)-F(x)\right) \to F' = f$$

6) (a) $\mathcal{F} \subset C[0,1]$. Closed, bdd, equicts.

$\exists g \in \mathcal{F}$ st. $\int_0^1 g(x)\,dx \ge \int_0^1 f(x)\,dx,\ \forall f \in \mathcal{F}$.

$T: \mathcal{F} \to \mathbb{R}$

$T(f) = \int_0^1 f(x)\,dx$, $\quad g \in B_\delta(f)\quad \|f-g\|_\infty < \delta$

$$|T(f)-T(g)| \le \int_0^1 |f-g| \le \|f-g\|_\infty < \delta$$

T is cts map.

So, it will attain a bd.

(b) $f_n(x) = x^{\frac{n}{n-1}-1}$