Assignment 7: Several variables calculus & differential geometry (MTH305A)

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(1) Show that $S = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$ is a regular surface.

(2) Let (U, ϕ) be a local coordinate chart of a regular surface and $\sigma : \tilde{U} \to U$ be a diffeomorphism. Then show that $(\tilde{U}, \tilde{\phi})$, where $\tilde{\phi} = \phi \circ \sigma$, is also a local chart of S.

(3) Let S be a regular surface and $p \in S$. Show that there exists a local coordinate chart $(\mathbb{R}^2, \phi)$ such that $p \in \phi(\mathbb{R}^2)$.

(4) Let S be a regular surface in $\mathbb{R}^3$. A subset $\tilde{S}$ of S is called open in S, if for every $p \in \tilde{S}$, there exists $p \in V \subset_{\text{open}} \mathbb{R}^3$ such that $V \cap S \subset \tilde{S}$.

Show that an open subset $\tilde{S}$ of a regular surface is a regular surface.

(5) Show that the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

is a regular surface and find parameterisations whose coordinate neighbourhoods cover the cylinder.

(6) Does there exist a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ and $r \in f(\mathbb{R}^3)$ which is not a regular value of f but $f^{-1}(r)$ is a regular surface.

(7) Let us consider the plane $P = \{(x, y, z) \in \mathbb{R}^3 \mid x = y\}$ in $\mathbb{R}^3$ and $\phi : U \to \mathbb{R}^3$ be given by $\phi(x, y) = (x + y, x + y, xy)$, where $U = \{(x, y) \in \mathbb{R}^2 \mid x > y\}$.

(a) Show that $\phi(U) \subset P$.

(b) Is ϕ a parameterisation of P?