Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex analysis (MTH 403)

Quiz 1 solutions

Date: 12 August 2023

Time: 30 minutes | Total marks: 08

Answer to any question must be written within the space provided below the same.

1. Let $U \neq \emptyset$ be an open and connected subset of $\mathbb{C}$. Find all holomorphic functions $f : U \longrightarrow \mathbb{C}$ such that $f(U)$ is contained in either a line or a circle. [4]

   To the students: When do you say a set A is contained in a set B? It means every element of A is conatined in B. Isn't it?

   Solution:

Case 1. Assume $f(U)$ is contained in a line. Then there is some line whose equation must be satisfied by every $f(z)$, i.e., there exists $a \neq 0$ and $b \in \mathbb{R}$ such that $\mathrm{Re}(af(z)) = b$. As $\mathrm{Re}\, af$ is constant and U is connected, it follows that af is a constant function. Since $a \neq 0$, so f must be a constant function.

Case 2. Assume $f(U)$ is contained in a circle. Then there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that, $|f(z) - z_0| = r$, for all $z \in U$. Since $|f - z_0|$ is constant and U is connected, it follows that $f - z_0$ is a constant function. Hence f must be a constant function.

2. Let $P(z)$ be polynomial with complex coefficients. Assume that $P(n) \neq 0$, for any $n \in \mathbb{N} \cup \{0\}$. Find the radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} P(n) z^n.$$

[4]

Solution: Let $P(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, where $a_0, \ldots, a_k \in \mathbb{C}$ and $a_k \neq 0$. Then, for all $n \in \mathbb{N} \cup \{0\}$, one has

$$\begin{aligned}\left|\frac{P(n)}{P(n+1)}\right| &= \left|\frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0}{a_k (n+1)^k + a_{k-1}(n+1)^{k-1} + \cdots + a_1(n+1) + a_0}\right| \\ &= \left|\frac{\frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0}{n^k}}{\frac{a_k (n+1)^k + a_{k-1}(n+1)^{k-1} + \cdots + a_1(n+1) + a_0}{n^k}}\right| \\ &= \left|\frac{a_k + \frac{a_{k-1}}{n} + \cdots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}}{a_k(1+\frac{1}{n})^k + \frac{a_{k-1}}{n}(1+\frac{1}{n})^{k-1} + \ldots \frac{a_1}{n^{k-1}}(1+\frac{1}{n}) + \frac{a_0}{n^k}}\right| \end{aligned} \tag{2.1}$$

From (2.1), it is clear that $\lim\limits_{n\to\infty} \left|\frac{P(n)}{P(n+1)}\right| = \frac{|a_k|}{|a_k|} = 1$. Hence the radius of convergence of the power series is equal to $\lim\limits_{n\to\infty} \left|\frac{P(n)}{P(n+1)}\right| = 1$.

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