(b) $$F(x+) = \lim_{h \downarrow 0} F(x+h)$$

$$= \lim_{n \to \infty} F\left(x + \frac{1}{n}\right)$$

$$= \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right)$$

Realize that $A_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, \ldots$ is $\ni A_n \downarrow$

and $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$

$$\Rightarrow \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right) = P_X\left(\lim_{n \to \infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} \left(-\infty, x + \frac{1}{n}\right]\right)$$

$$= P_X\left((-\infty, x]\right) = F(x)$$

$\Rightarrow F(x+) = F(x)$; i.e. $F(.)$ is right continuous

(c) $$F(-\infty) = \lim_{n \to \infty} F(-n)$$

$$= \lim_{n \to \infty} P_X\left((-\infty, -n]\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} (-\infty, -n]\right) \quad \left((-\infty, -n] \downarrow\right)$$

$$= P_X(\phi) = 0 \qquad \bigcap_{1}^{\infty} (-\infty, -n] = \phi$$

$$F(+\infty) = \lim_{n \to \infty} F(n) \qquad \left((-\infty, n] \uparrow\right)$$

$$= \lim_{n \to \infty} P_X\left((-\infty, n]\right)$$

$$= P_X\left(\lim_{n \to \infty} (-\infty, n]\right) = P_X\left(\bigcup_{1}^{\infty} (-\infty, n]\right)$$

$= P_X(\mathbb{R}) \qquad \left(\bigcup_{n=1}^{\infty} (-\infty, n] = \mathbb{R} \right)$

$= 1$

Remark: Converse of the prev result is true, i.e. If $F(.)$ be a function $F : \mathbb{R} \to [0,1] \ni$

(i) $F(.)$ is non-decreasing

(ii) $F(.)$ is right continuous

(iii) $F(-\infty) = 0$ and $F(\infty) = 1$

then $F(.)$ is d.f. of some appropriate random variable

Remark: For a d.f. $F(.)$, both $F(x+)$ and $F(x-)$ exist $\forall x \in \mathbb{R}$ as $F(.)$ is monotone, bdd below and bdd above.

Remark: A d.f. is continuous at $a \in \mathbb{R}$ iff $F(a) = F(a-)$

Remark: For any $a \in \mathbb{R}$

$$P(X = a) = P(X \le a) - P(X < a) = F(a) - F(a-)$$

If d.f. is continuous at a, then $P(X = a) = 0$

Remark: For $-\infty < a < b < \infty$

$$P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$$

$$P(a < X < b) = P(X < b) - P(X \le a) = F(b-) - F(a)$$

$$P(a \le X < b) = P(X < b) - P(X < a) = F(b-) - F(a-)$$

$$P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a-)$$

If $F(.)$ is continuous at a and b, then all the above are equal and equal to $F(b) - F(a)$

Remark: $F(.)$ can only have points of jump discontinuities and can have atmost countable number of such jump discontinuities

Discrete random variable

$(\Omega, \mathcal{F}, P)$: prob space

$X : \Omega \to \mathbb{R}$ be a r.v.

$(\mathbb{R}, \mathcal{B}, P_X)$: induced prob space (induced by X)

$F(.)$: d.f. of X

Defn: Random variable X is said to be a discrete r.v.

If $\exists$ a countable set $D \subset \mathbb{R}$ $\ni$

$$P(X = x) = F(x) - F(x-) > 0 \quad \forall x \in D$$

and $P(X \in D) = 1$

$\therefore$ D is called the support of the r.v. X

D is the set of all discontinuity pts of $F(.)$

Defn: Let $D = \{x_1, x_2, \ldots\}$ ← countable (finite or infinite)

$$P(X = x_i) = p_i \text{, say}, \quad p_i > 0 \;\forall i$$

$$P(X \in D) = \sum_i p_i = 1$$

The collection $\{p_1, p_2, \ldots\}$ is called the probability mass f^n of r.v. X.

i.e. $f_X(x) = P(X = x)$ $\quad x \in S$ is the p.m.f. of X

$= F(x) - F(x-)$; $f(x) > 0 \;\forall x \in D$

$\sum_{x \in D} f(x) = 1$

Remark: (i) d.f. of a discrete r.v. increases only by jumps

(ii) number of jump discontinuities are atmost countable

(iii) d.f. determines the p.m.f uniquely and vice-versa

Example :

(1) X : r.v.

$F(.)$: d.f. of X

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{3}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases}$$

pts of jump discontinuities $\{0, 1, 2, 3\} = D$ (finite collection)

$P(X \in D) = 1$;

X is a discrete r.v. with support D

p.m.f.

x	$P(X = x)$
0	$F(0) - F(0-) = \frac{1}{4}$
1	$F(1) - F(1-) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$
2	$F(2) - F(2-) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$
3	$F(3) - F(3-) = \frac{1}{4}$

i.e. p.m.f is

$$f(x) = \begin{cases} \frac{1}{4}, & \text{If } x = 0, 3 \\ \frac{1}{12}, & \text{If } x = 1 \\ \frac{5}{12}, & \text{If } x = 2 \\ 0, & \text{o/w.} \end{cases}$$

Example :

(2) Random exp : tossing coin until head appears

$\Omega = \{H, TH, TTH, \ldots\}$

X : r.v. which counts number of tosses reqd to get 1^{st} H

i.e. $X(\omega) = \text{no. of } T \text{ in } \omega + 1$

possible values of X : $1, 2, 3, \ldots$

$P(X = i) = \frac{1}{2^i}$; $i = 1, 2, \ldots$

d.f. of X

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ \vdots & \end{cases}$$

magnitude of jump at $i = \frac{1}{2^i}$ for $i = 1, 2, \ldots$

$D = \{1, 2, 3, \ldots\}$

countably infinite

Support

p.m.f. $f(x) = \frac{1}{2^x}$; $x = 1, 2, \ldots$

Continuous random variable

<u>Def^n</u>: A random variable X is said to be a continuous r.v. if $\exists$ a non-negative, integrable function $f : \mathbb{R} \to [0, \infty)$ such that for any $x \in \mathbb{R}$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$f(.)$ is called the probability density function (p.d.f.) of X.

<u>Remark</u>: Support of a continuous r.v is the set

$$S = \{ x \in \mathbb{R} : F(x+h) - F(x-h) > 0, \ \forall h > 0 \}$$

<u>Remark</u>: For a cont r.v. ($F(.)$ is continuous everywhere)

$$P(X = x) = F(x) - F(x-) = 0 \quad \forall x \in \mathbb{R}$$

In general, suppose $A \subset \mathbb{R}$ is any countable subset, then

$$P(X \in A) = \sum_{x \in A} P(X = x) = 0$$

<u>Remark</u>: p.d.f. $f(x)$, then

(i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

and (ii) $\int_{-\infty}^{\infty} f(t)\, dt = 1$

Converse is also true

<u>Remark</u>: F d.f. of r.v. X, if F is differentiable, then

$$f(x) = \frac{d}{dx} F(x)$$

Remark: For a Continuous r.v.

$$P(X < x) = P(X \le x) = F(x) \quad \forall\, x \in \mathbb{R}$$

$$P(X \ge x) = 1 - P(X < x) = 1 - F(x) \quad \forall\, x \in \mathbb{R}$$

$\forall \quad -\infty < a < b < \infty$

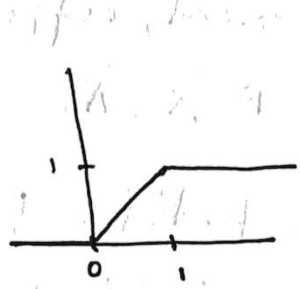
$$P(a < X < b) = P(a \le X < b) = P(a < X \le b)$$
$$= P(a \le X \le b)$$
$$= F(b) - F(a)$$

$$\& \quad F(b) - F(a) = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$
$$= \int_{a}^{b} f(t)\,dt .$$

Remark: p.d.f. determines the d.f. uniquely. Converse is not true

Examples

(1)
$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$



$F(.)$ is cont everywhere

p.d.f. of X is $\quad f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & o/w \end{cases}$

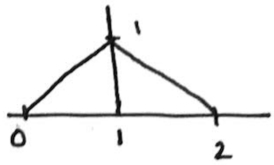
$\frac{d}{dx} F(x)$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

(2)

p.d.f. of a r.v. X is

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & o/w \end{cases}$$



$f(x) \geq 0 \ \forall x$

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{2} + \left(2 - \frac{3}{2}\right) = 1$$

$\Rightarrow f(\cdot)$ is a p.d.f.

dist^n f^n: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x \le 1 \\ \frac{1}{2} + \left(\left(2x - \frac{x^2}{2}\right) - \frac{3}{2}\right), & 1 \le x \le 2 \\ 1, & x \geq 2 \end{cases}$$

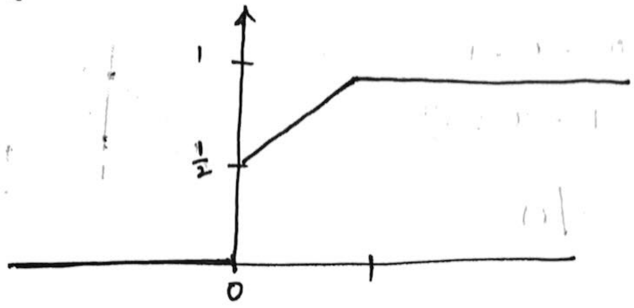
$$P\left(\frac{1}{4} \le X \le \frac{3}{2}\right) = F(3/2) - F(1/4)$$

<u>Remark</u>: There are random variables that are neither discrete nor continuous - random variables of mixed type

<u>Example</u>:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x+1}{2}, & 0 \le x < 1 \\ 1, & x \geq 1 \end{cases}$$

It's easy to check that $F(.)$ is a d.f.



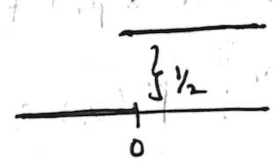
$F(.)$ has jump discontinuity at 0 - jump size $\frac{1}{2}$

$F(.)$ is continuous everywhere, except at 0

$$P(X=0) = F(0) - F(0-) = \frac{1}{2}$$

Discrete part:
(f^n which increases by jump only)

$$F_d = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & x \geq 0 \end{cases}$$



Continuous part:
(increasing continuously part)

$$F_c = \begin{cases} 0, & x<0 \\ x/2, & 0 \leq x < 1 \\ \frac{1}{2}, & x \geq 1 \end{cases}$$

$$F_c = F - F_d$$

Note: F_d & F_c are not d.f.s.

Realise that

$$F(x) = \frac{1}{2} F_1(x) + \frac{1}{2} F_2(x)$$

$$F_1(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases} \qquad F_2(x) = \begin{cases} 0, & x<0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$F_1(x)$ & $F_2(x)$ are proper d.f.s.

d.f. of a discrete r.v.

d.f. of a continuous r.v.