$\rightarrow$ $f : (0,1) \longrightarrow \mathbb{R}$ is uniformly cts.

Claim: $\lim_{x \to 0^+} f(x)$ exists. $\Leftrightarrow$ $\forall\ x_n \to 0^+ \Rightarrow (f(x_n))$ cvgs.

suffices to show $(f(x_n))$ is Cauchy.

$x_n \to 0^+ \Rightarrow (x_n)$ is Cauchy. Since f is unif. cts., $(f(x_n))$ is Cauchy.

$\Downarrow$

$f(x_n) \to \ell$ (say)

$y_n \to 0^+ \Rightarrow f(y_n) \to \ell'$ (say).

Since $(x_n - y_n) \to 0$ and f is unif. cts., $f(x_n) - f(y_n) \to 0$ as $n \to \infty$.

HW: $\ell = \ell'$.

HW: $f : (a,b) \to \mathbb{R}$ is unif. cts. $\Leftrightarrow$ $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist s.t. $\hat{f} : [a,b] \to \mathbb{R}$ is cts.

$\rightarrow$ $f : \mathbb{R} \to \mathbb{R}$ $\qquad F(x) := \dfrac{f(x) - f(a)}{x - a}$ for $x \neq a$.

claim: f is diff. at a iff F is unif. cts. in $B(a,\delta) \setminus \{a\}$.

Pf: For $\varepsilon > 0$, $\exists\ \delta > 0$ st. $|x - a| < \delta$, $\left| \dfrac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon$.

$\lim_{x \to a} F(x) = f'(a)$. For $x, y \in (a - \delta, a + \delta) \setminus \{a\}$,

$$|F(x) - F(y)| \leq |F(x) - f'(a)| + |f'(a) - F(y)|$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

So accordingly do the scaling: in the def^n. of $f'(a)$ use $\varepsilon/2$.

$\Leftarrow$: claim. $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. That is, $\lim_{x \to a} F(x)$ exists.

$\Leftrightarrow$ $\forall\ (x_n)$ s.t. $x_n \to a$ $\lim F(x_n)$ exists.

Let $x_n \to a$. Then (x_n) is Cauchy. F unif. cts. $\Rightarrow$ $(F(x_n))$ Cauchy in $\mathbb{R}$

$y_n \to a$ $\qquad (F(y_n))$ Cauchy in $\mathbb{R}$

Use unif. cts. criterion in terms of seq. to finish the proof!

→ $f:(0,1)\to\mathbb{R}$ as $f(x) := \frac{1}{x}$.

Claim: f is not unif. cts.

HW: f is not unif. cts. iff. $\exists\ \varepsilon_0>0$ and $(x_n),(y_n)$ s.t. $|x_n-y_n|\to 0$ but $|f(x_n)-f(y_n)|\geq\varepsilon_0\ \forall n\geq 1$.

→ $f(x) = \sin(x^2)$

8 → Use Rolle's Thm. $x,y\in\mathbb{R}$ consider $f:[x,y]\to\mathbb{R}$

→ $|f(x)-f(y)|\leq K|x-y|^{\alpha} \Rightarrow |f(x)-f(y)|\leq K\cdot|x-y|\,|x-y|^{\alpha-1}$.

Suffices to show $f'(x)=0\ \forall\ x\in\mathbb{R}$. (why?)

$$-K\cdot|x-y|^{\alpha-1} < \frac{|f(x)-f(y)|}{|x-y|} \leq K\cdot|x-y|^{\alpha-1} \quad \text{for } x\neq y.$$

$f'(x) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ fix x and let $y\to x$. ……… $f'(x)=0$

11. Use Cauchy-Schwartz's Inequality.

10. Suppose x^{α} is unif. cts. on $[0,\infty)$.

Claim: $0\leq\alpha\leq 1$.

pf: Suppose $\alpha>1$. x^{α} unif. cts. on $[0,\infty) \Rightarrow x^{\alpha}$ unif. cts. on $(1,\infty)$

Note that x^{α} is differentiable on $(1,\infty)$.

For each $x,y\in(1,\infty)$, using Rolle's Thm on $[x,y]$:

$$y^{\alpha}-x^{\alpha} = \alpha\cdot c(x,y)^{\alpha-1}(y-x)$$

For $\varepsilon := \frac{\alpha}{2}$, suppose $\exists\ \delta>0$. Then for every x,y s.t. $|y-x|<\delta$ one should get $|y^{\alpha}-x^{\alpha}|<\frac{\alpha}{2}$

But, if we take $y = x+\frac{\delta}{2}$ for a fixed x, then $|y^{\alpha}-x^{\alpha}| = \alpha\cdot c(x,y)^{\alpha-1}\frac{\delta}{2}$

Then, $\alpha \cdot C(x,y)^{\alpha-1}\frac{\delta}{2} < \frac{\alpha}{2} \Rightarrow \delta < \frac{1}{C(x,y)^{\alpha-1}}$. This happens for each $x \in (1,\infty)$ and y s.t. $y-x=\frac{\delta}{2}$.

Now consider $x_n \doteq n$. Then as $n\to\infty$, $C(x,y)\to\infty$

$$\Rightarrow \frac{1}{C^{\alpha-1}(x,y)} \to 0 \text{ as } \alpha-1>0.$$

$$\Rightarrow \delta = 0 \text{ contradicting } \delta>0.$$

$\Leftarrow$: For $0\le\alpha\le 1$, x^α is unif. cts. on $[0,\infty)$. Note: for $\alpha=0$ and $\alpha=1$ (HW).

$[0,\infty) = [0,1]\cup(1,\infty)$ x^α is unif. cts. on $[0,1]$ (why?)

On $(1,\infty)$: x^α is Lipschitz: $|x^\alpha - y^\alpha| = \alpha\, C(x,y)^{\alpha-1}|x-y|$ for some $C(x,y)\in(x,y)$

$C(x,y)>1$ and $\alpha-1<0$ so $C^{\alpha-1}<1$.

HW: x^α unif. cts. on $[0,1]$

x^α unif. cts. on $[1,\infty)$

12. (a) • f, g bdd. so $\exists$ $M>0$ s.t. $|f(x)|\le M$ and $|g(x)|\le M$.

• $\varepsilon>0$, $\exists$ $\delta>0$ s.t. $|x-y|<\delta$, $|f(x)-f(y)|<\varepsilon/2M$ and $|g(x)-g(y)|<\frac{\varepsilon}{2M}$.

$$|f(x)g(x)-f(y)g(y)| \le |f(x)g(x)-f(y)g(x)+f(y)g(x)-f(y)g(y)|$$
$$\vdots$$
$$\le M|f(x)-f(y)| + M|g(x)-g(y)|$$
$$<\varepsilon$$

• $f(x)=x$ and $g(x)=\sin x$ f, g unif. cts. on $\mathbb{R}$

fg not unif. cts. on $\mathbb{R}$.

For $\varepsilon=\frac{1}{2}$ take $x_n = 2n\pi + \frac{1}{2n\pi}$ and $y_n = 2n\pi$. Then $x_n - y_n \to 0$.

$$f(x_n)g(x_n) - f(y_n)g(y_n) > \frac{\sin\left(\frac{1}{2n\pi}\right)}{1/2n\pi}.$$

Since $\frac{\sin x}{x}\to 1$ as $x\to 0$ so given $\delta>0$ $\exists$ $n\in\mathbb{N}$ s.t. $\frac{1}{2n\pi}<\delta$ and

$$\frac{\sin(1/2n\pi)}{1/2n\pi} > \frac{1}{2}.$$

(b) Unif. cts. is necessary for Cauchy seq. to map to Cauchy seq.

Example: Consider $M = \{ \frac{1}{n} \mid n \geq 1 \}$ w.r.t. $|\cdot|$ induced from $(\mathbb{R}, |\cdot|)$.

Define $f : M \longrightarrow \mathbb{R}$

as $f\left(\frac{1}{n}\right) = \begin{cases} 1, & n: \text{odd} \\ -1, & n: \text{even}. \end{cases}$ Then f is cts.

f is not unif. cts. Indeed, $(x_n) = \left(\frac{1}{n}\right)$ is Cauchy in M, but

$\left(f(x_n)\right) = (1, -1, 1, -1, \ldots)$ not Cauchy in $\mathbb{R}$.