Problem Set-5
MTH-204, 204A
Abstract Algebra

1. Let G be a group and $a \in G$, then $C_G(a)$, the centralizer or normalizer of a in G is the set $C_G(a) = \{x \in G : xa = ax\}$. Prove that $C_G(a)$ is a subgroup of G. Prove that $C_G(Ta) = T(C_G(a))$, where T is an automorphism of G.

2. Let H be a subgroup of G, then the centralizer $C(H)$ of H is defined by the set $\{x \in G : xh = hx \text{ for all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

3. Let G be a group and T an automorphism of G. If N is a normal subgroup of G such that $T(N) \subset N$, show how you could use T to define an automorphism of G/N.

4. Prove that the multiplicative group $G = \mathbb{C} \setminus \{0\}$ is isomorphic to the group G' of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where not both a and b are 0, under matrix multiplication.

5. Prove that the quotient group $\frac{\mathbb{R}}{\mathbb{Z}}$ is isomorphic to the group of all complex numbers of absolute value 1 under multiplication.

6. Let G be a finite group, T an automorphism of G with the property that $Tx = x$ for $x \in G$ if and only if $x = e$. Prove that every $g \in G$ can be represented as $g = x^{-1}T(x)$ for some $x \in G$.

7. Let G be a finite group, T an automorphism of G with the property that $Tx = x$ if and only if $x = e$. Suppose further that $T^2 = I$. Prove that G must be abelian.

8. Prove that every finite group having more than two elements has a nontrivial automorphism.

9. Let G be a group of order $2n$. Suppose that half of the elements of G are of order 2, and the other half form a subgroup H of order n. Prove that H is of odd order and is an abelian subgroup of G.