MTH 442: Time Series Analysis
Problem Set # 3

[1] Let $\{\varepsilon_t\}$ be a sequence of i.i.d. random variables with mean zero and finite variance σ^2. Define a complex valued time series $Z_t = \varepsilon_t + i\,Y_t$ with $Y_t = \begin{cases} t\,\varepsilon_t, & \text{if } t \text{ is odd,} \\ -t\,\varepsilon_t, & \text{if } t \text{ is even.} \end{cases}$

Find $Cov(Z_{t+h}, Z_t)$ for $h \in \{0, \pm 1, \pm 2, \ldots\}$ and verify whether $\{Z_t\}$ is covariance stationary.

[2] Let $X_t = U_t + i V_t$ be a complex valued stationary process with $\{U_t\}$ and $\{V_t\}$ real valued stationary processes. Prove or disprove " $\gamma_X^*(h) = \gamma_X(-h); \forall h$, where '$*$' denotes the complex conjugate".

[3] Let $Z_1, \ldots.., Z_n$ be n random variables from $\{Z_t\}$ that is $WN(\mu, \sigma^2)$. Show that $\bar{Z}_n \xrightarrow{p} \mu.$

[4] Let $Z_1, \ldots.., Z_n$ be n random variables from a stationary $\{Z_t\}$ with mean μ and ACVF $\gamma_Z(.)$.
Suppose $\gamma_Z(h)$ is estimated by

$$\hat{\gamma}_Z^*(h) = \frac{1}{n-h} \sum_{t=1}^{n-h} (Z_t - \bar{Z}_n)(Z_{t+h} - \bar{Z}_n).$$

Show that if we assume that $\sum_{t=1}^{n-h} (Z_t - \bar{Z}_n) \cong \sum_{t=1}^{n-h} (Z_{t+h} - \bar{Z}_n) \cong \sum_{t=1}^{n} (Z_t - \bar{Z}_n)$, then the bias of $\hat{\gamma}_Z^*(h)$ for estimating $\gamma_Z(h)$ is $-V(\bar{Z}_n)$.

[5] Let $\{X_t\}$ be given by $X_t = \phi\, X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$is $WN(0,1)$.

(a) Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\phi = 0.8$.

(b) Define a new process $Y_t = \sum_{i=1}^{t} X_i$ and verify whether $\{Y_t\}$ is covariance stationary?

[6] Let $\{Z_t\}$be i.i.d. N(0,1) variable and define

$$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even} \\ \left(Z_{t-1}^2 - 1\right)/\sqrt{2}, & \text{if } t \text{ is odd} \end{cases}$$

Show that $\{X_t\}$is WN(0,1).

[7] Consider the following $MA(\infty)$ process

$$X_t = \varepsilon_t + C\left(\varepsilon_{t-1} + \varepsilon_{t-2} + \ldots.\right)$$

where, $\{\varepsilon_t\}$is $WN(0, \sigma^2)$ and $C < \infty$ is a constant.

(a) Is $\{X_t\}$covariance stationary?

(b) Is the first difference series covariance stationary?

[8] Suppose $\{X_t\}$ is an MA(1) process $X_t = \varepsilon_t + 0.5\,\varepsilon_{t-1}$. Verify whether $Y_t = X_t - X_{t-1}$ is covariance stationary and has any standard model.

[9] Let $\{X_t\}$, $\{Y_t\}$ and $\{Z_t\}$ be 3 independent mean zero covariance stationary processes; $\{X_t\}$ having an MA(1) process $X_t = \varepsilon_t + \varepsilon_{t-1}$, $\varepsilon_t \sim WN(0,1)$, $\{Y_t\}$ and $\{Z_t\}$ are $WN(0,1)$ processes. Define $U_t = (1 - Z_t)X_t + Y_t$.

(a) Is $\{U_t\}$covariance stationary?

(b) Does $\{U_t\}$ follow a white noise process?

[10] Prove that sum of two independent white noise processes is also a white noise process. Give an example to show that sum of two stationary independent non-white noise series can also be stationary white.

[11] Let $\{X_t\}$ be a time series given by $X_t = \mu + \varepsilon_t + \varepsilon_{t-1} + \phi\,\varepsilon_{t-2}$; $\{\varepsilon_t\}$is a sequence of i.i.d. $N\left(0,\sigma^2\right)$.

Consider $\delta_1 = \dfrac{2X_1 + X_3}{3}$ and $\delta_2 = \dfrac{X_3 + X_4 + X_5}{3}$ as two estimators of μ.

(a) Verify whether the estimators δ_1 and δ_2 are unbiased or not.

(b) Find the values of ϕ, if any, for which $Var\left(\delta_1\right) > Var\left(\delta_2\right)$.

(c) Find the joint distribution of $\left(X_1, X_2, \ldots X_n\right)$ and hence (or otherwise) verify whether or not $\{X_t\}$ is strict stationary.

[12] Let $\{X_t\}$ be an *AR(1)* process $X_t = \phi\, X_{t-1} + \varepsilon_t$; $|\phi| < 1$; $\varepsilon_t \sim WN\left(0,\sigma^2\right)$. Define $Y_t = X_t - \dfrac{1}{\phi} X_{t-1}$.

Verify whether $\{Y_t\}$ is a white noise process.

[13] Let $\{X_t\}$ be a stationary MA(1) process

$$X_t = \varepsilon_t + \phi\,\varepsilon_{t-1};\ \varepsilon_t \sim WN\left(0,1\right).$$

Define $T_1 = \dfrac{X_4 + X_5}{2}$ and $T_2 = \dfrac{X_3 + X_4 + X_5}{3}$.

Does any of the two estimators of mean dominate the other in terms of lower variance (for all values of ϕ) ?

[14] Let $\{X_t\}$ be a *MA(1)* process $X_t = \varepsilon_t + \theta\ \varepsilon_{t-1}$; $\varepsilon_t \sim WN\left(0,\sigma^2\right)$, $|\theta| > 1$. Define a new process $\{Y_t\}$ as $Y_t = \sum_{j=0}^{\infty}\left(-\theta\right)^{-j} X_{t-j}$. Verify whether $\{Y_t\}$ is stationary and/or white.

[15] Consider the AR(2) process $\{Y_t\}$satisfying

$$Y_t - \phi\, Y_{t-1} - \phi^2\ Y_{t-2} = \varepsilon_t; \varepsilon_t \sim WN(0,\sigma^2).$$

Find the value (s) of ϕ for which the above process is stationary.

[16] Show that the AR(2) process $X_t = X_{t-1} + c\ X_{t-2} + \varepsilon_t$ is stationary provided $-1 < c < 0$

[17] Let $\{X_t\}$be a stationary AR(2) process with ACVF $\gamma_X\left(.\right)$. If it is given that $\gamma_X\left(1\right)/\gamma_X\left(0\right) = 1/2$ and $\gamma_X\left(2\right)/\gamma_X\left(1\right) = 1/4$, determine $\gamma_X\left(3\right)/\gamma_X\left(2\right)$.

[18] For a stationary AR(1) process $Y_t = \phi\, Y_{t-1} + \varepsilon_t; \varepsilon_t \sim WN(0,\sigma^2)$. Prove that $\gamma\left(-h\right) = \phi\,\gamma\left(-h+1\right)$, for all $h > 0$.

[19] $\{X_t\}$ and $\{Y_t\}$are two independent covariance stationary ARMA processes given by $\left(1-\phi_1^{(1)}B\right)X_t = \left(1+\theta_1^{(1)}B + \theta_2^{(1)}B^2 + \theta_3^{(1)}B^3\right)\varepsilon_t$ and $\left(1-\phi_1^{(2)}B\right)Y_t = \left(1+\theta_1^{(2)}B + \theta_2^{(2)}B^2\right)\delta_t$; $\left|\phi_1^{(i)}\right| < 1, i = 1,2$; $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independent white noise processes, $\varepsilon_t \sim WN\left(0,\sigma^2\right)$ and $\delta_t \sim WN\left(0,\sigma^2\right)$.

(a) Prove or disprove, " $Z_t = \left(1-\phi_1^{(1)}B\right)\left(1-\phi_1^{(2)}B\right)X_t$" is a stationary MA process.

(b) Let $U_t = \left(1-\phi_1^{(1)}B\right)\left(1-\phi_1^{(2)}B\right)\left(X_t + Y_t\right)$. Find the smallest *k*, if any, such that $Cov\left(U_t, U_{t+h}\right) = 0,\ \forall\, h \geq k.$

[20] Prove that an $MA(\infty)$ process $X_t = \sum_{j=0}^{\infty} \psi_j \, \varepsilon_{t-j}$ with absolutely summable coefficients $\left\{\psi_j\right\}_{j=0}^{\infty}$ has absolutely summable autocovariance sequence $\left\{\gamma_j\right\}_{j=0}^{\infty}$.