① Given cond^n : $\{X_t\}$ covariance stationary

& $\gamma(h) \to 0$ as $h \to \infty$

$$\mathrm{Cov}(\bar{X}_n, X_n) = \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} X_i, X_n\right)$$

$$= \frac{1}{n}\mathrm{Cov}\left(\sum_{1}^{n} X_i, X_n\right)$$

$$= \frac{1}{n}\left(\gamma(n-1) + \gamma(n-2) + \cdots + \gamma(0)\right)$$

$$= \frac{1}{n}\sum_{h=0}^{n-1}\gamma(h)$$

$$\left|\frac{1}{n}\sum_{0}^{n-1}\gamma(h)\right| \le \frac{1}{n}\sum_{0}^{n-1}|\gamma(h)| = \frac{1}{n}\sum_{0}^{K-1}|\gamma(h)| + \frac{1}{n}\sum_{K}^{n-1}|\gamma(h)|$$

Since $\gamma(h) \to 0$ as $h \to \infty$; for any given $\epsilon > 0$ $\exists$ an n_0 $\ni$ $\forall\, n \ge n_0$ $\exists$ a K $\ni$

$$|\gamma(h)| < \frac{\epsilon}{2} \quad \forall\, h \ge K$$

Also for this K, $\sum_{0}^{K-1}|\gamma(h)|$ is a fixed qty so for a given $\epsilon > 0$ $\exists$ n_1 $\ni$ $\forall\, n \ge n_1$

$$\frac{1}{n}\sum_{0}^{K-1}|\gamma(h)| < \frac{\epsilon}{2}$$

Let $n^* = \max(n_0, n_1)$, then $\forall\, n \ge n^*$

$$\left|\frac{1}{n}\sum_{0}^{n-1}\gamma(h)\right| \le \frac{1}{n}\sum_{0}^{K-1}|\gamma(h)| + \frac{1}{n}\sum_{K}^{n-1}|\gamma(h)|$$

$$< \frac{\epsilon}{2} + \frac{(n-K)}{n}\frac{\epsilon}{2} < \epsilon$$

$$\Rightarrow \frac{1}{n}\sum_{0}^{n-1}\gamma(h) \to 0 \quad \text{as } n \to \infty$$

(2)

$$V(\bar{X}_n) = \frac{1}{n}\sum_{|h|<n}\left(1-\frac{|h|}{n}\right)\gamma_h$$

$$= \left|\frac{1}{n}\sum_{|h|<n}\left(1-\frac{|h|}{n}\right)\gamma_h\right|$$

$$\leq \frac{1}{n}\sum_{-(n-1)}^{(n-1)}\left(1-\frac{|h|}{n}\right)|\gamma_h|$$

$$\leq \frac{1}{n}\sum_{-(n-1)}^{(n-1)}|\gamma_h|$$

$$= \frac{2}{n}\sum_{0}^{n-1}|\gamma_h| - \frac{1}{n}|\gamma_0|$$

$$< \frac{2}{n}\sum_{0}^{n-1}|\gamma_h| \to 0 \text{ as } n\to\infty \quad \text{as } \gamma_h \to 0 \text{ as } h\to\infty$$

(using the previous problem).

<u>Counter example</u>: Converse is NOT true

$X_t = (-1)^t Z$; Z is a r.v. with mean 0, Var 1.

$$\bar{X}_n = \begin{cases} 0, & n \text{ is even} \\ -\frac{Z}{n}, & n \text{ is odd} \end{cases}$$

$$V(\bar{X}_n) = \begin{cases} 0, & n \text{ is even} \\ \frac{1}{n^2}, & n \text{ is odd} \end{cases}$$

$\longrightarrow 0$ as $n\to\infty$

$\gamma_h = \text{Cov}(X_t, X_{t+h}) = (-1)^h \not\to 0$ as $h\to\infty$.

(3) $(x_1, \ldots, x_{100})$ from AR(1) with unknown mean μ

$\bar{x}_{100} = 0.157$, $\hat{\phi} = 0.6$, $\hat{\sigma}^2 = 2$

$H_0: \mu = 0$ ag $H_A: \mu \neq 0$

Using asymptotic result

$$\sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{L} N\left(0, \sum_{-\infty}^{\infty} \gamma_h = \frac{\sigma^2}{(1-\phi)^2}\right)$$

Also
$$Z = \frac{\sqrt{n}\,(\bar{X}_n - 0)}{\sqrt{\frac{\hat{\sigma}^2}{(1-\hat{\phi})^2}}} \overset{asym}{\sim} N(0,1) \text{ under } H_0$$

Reject H_0 at 5% level of significance if

obsd $|Z| > \tau_{0.05/2}$

obsd $$|Z| = \left| \frac{\sqrt{100}\,(0.157)}{\sqrt{\frac{4}{(1-.6)^2}}} \right| \simeq .31 \not> 1.96 (\tau_{0.05/2})$$

$\Rightarrow$ Accept H_0.

$100(1-\alpha)\%$ asymptotic confidence interval

$$\bar{x}_n \mp \tau_{\alpha/2} \sqrt{\frac{\hat{\sigma}^2}{(1-\hat{\phi})^2}}$$

④

$$X_t = \theta + \epsilon_t + \frac{1}{2}\epsilon_{t-1} + \frac{1}{2}\epsilon_{t-2} \; ; \; \epsilon_t \overset{i.i.d}{\sim} N(0,1)$$

$$\gamma_h = \begin{cases} 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}, & h = 0 \\ \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, & h = \pm 1 \\ \frac{1}{2}, & h = \pm 2 \\ 0, & o/w \end{cases}$$

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{L} N\left(0, \sum_{-\infty}^{\infty} \gamma_h = 4\right)$$

CI : $\bar{X}_n \mp \tau_{\alpha/2}\sqrt{\frac{4}{n}}$

⑤ $$\lim_{n\to\infty} n V(\bar{X}_n) = \sum_{-\infty}^{\infty} \gamma_h$$

$$= \sum_{-\infty}^{\infty} \left(.6^{|h|} + 2(0.3)^{|h|} + (0.1)^{|h|} \right) - (*)$$

$$\left(\sum_{-\infty}^{\infty} a^{|h|} = 2\sum_0^{\infty} a^h - 1 = \frac{2}{1-a} - 1 = \frac{1+a}{1-a} \right)$$
$$|a| < 1$$

$$(*) = \frac{1.6}{.4} + 2\frac{1.3}{.7} + \frac{1.1}{.9} \approx 8.93$$

$$\sqrt{n}(\bar{X}_n - \mu) \overset{asym}{\sim} N(0, 8.93) \; - (*')$$

$$P(\bar{X}_n - 0.49 \le \mu \le \bar{X}_n + 0.49)$$
$$= P(|\bar{X}_n - \mu| \le 0.49) \ge 0.95 \quad - (i)$$

Using $(*')$ $$P\left(\frac{|\bar{X}_n - \mu|}{\sqrt{\frac{8.93}{n}}} \le \underset{\tau_{0.05/2}}{1.96} \right) = 0.95$$

i.e. $P\left(|\bar{X}_n - \mu| \leq \sqrt{\frac{8.93}{n}} \times 1.96\right) = 0.95$ —(ii)

Comparing (i) & (ii)

$$0.49 \geq \sqrt{\frac{8.93}{n}}\ 1.96$$

$$\Rightarrow n \geq 16 \times 8.93 = 142.88$$

(6) $$X_t = \mu + \frac{1}{2} X_{t-1} + \epsilon_t$$

$$\Rightarrow \mu_X = 2\mu \ ; \ \gamma_X(h) = \frac{\sigma^2}{1-\frac{1}{4}} \cdot \left(\frac{1}{2}\right)^{|h|} = \frac{4}{3}\sigma^2 \left(\frac{1}{2}\right)^{|h|}$$

$$Y_t = \mu + \frac{1}{3} Y_{t-1} + \delta_t$$

$$\mu_Y = \frac{3}{2}\mu \ ; \ \gamma_Y(h) = \frac{9}{8}\sigma^2 \left(\frac{1}{3}\right)^{|h|}$$

$$Z_t = X_t + Y_t \ ; \ \mu_Z = 2\mu + \frac{3\mu}{2} = \frac{7\mu}{2}$$

$$\sqrt{n}\left(\bar{Z}_n - \frac{7\mu}{2}\right) \xrightarrow{L} N\left(0, \sum_{-\infty}^{\infty} \gamma_Z(h)\right)$$

$$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$$

$$\sum_{-\infty}^{\infty} \gamma_Z(h) = \sum_{-\infty}^{\infty} \gamma_X(h) + \sum_{-\infty}^{\infty} \gamma_Y(h)$$

$$= \frac{\sigma^2}{(1-\frac{1}{2})^2} + \frac{\sigma^2}{(1-\frac{1}{3})^2} = 4\sigma^2 + \frac{9}{4}\sigma^2 = \frac{25}{4}\sigma^2$$

i.e. $$\sqrt{n}\left(\bar{Z}_n - \frac{7\mu}{2}\right) \xrightarrow{L} N\left(0, \frac{25}{4}\sigma^2\right)$$

$$P\left(\left|\bar{Z}_n - \frac{7\mu}{2}\right| \leq \frac{5}{2} \frac{1.96}{\sqrt{n}}\right) = 0.95$$

given condition

$$P\left(\left|\bar{Z}_n - \frac{7\mu}{2}\right| \le 0.098\right) \geq 0.95$$

$$\Rightarrow 0.098 \geq \frac{5}{2} \frac{1.96}{\sqrt{n}}$$

i.e. $\sqrt{n} \geq 50$

i.e. $n \geq 2500$

(7) $X_t = \mu + \phi X_{t-1} + \epsilon_t \qquad |\phi| < 1$

$Y_t = \delta + \eta_t + \theta \eta_{t-1} \qquad |\theta| < 1$

$\{\epsilon_t\}$ & $\{\delta_t\}$ are indep.

$Z_t = X_t + Y_t$

$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$

$$\gamma_X(h) = \frac{\sigma^2}{1-\phi^2}\phi^{|h|}; \quad \gamma_Y(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \theta\sigma^2, & h = \pm 1 \\ 0, & o/w \end{cases}$$

$$\sum_0^\infty |\gamma_Z(h)| = \sum_0^\infty |\gamma_X(h) + \gamma_Y(h)|$$

$$\leq \sum |\gamma_X(h)| + \sum |\gamma_Y(h)|$$

$$= \sum \left| \frac{\sigma^2}{1-\phi^2}\phi^{|h|} \right| + \left((1+\theta^2)\sigma^2 + 2\theta\sigma^2\right)$$

$$< \infty$$

$\Rightarrow \bar{Z}_n \xrightarrow{m.s.} \mu_Z = E Z_1$ (using the asymptotic result proved in Lec 23) $\sum|\gamma_h| < \infty \Rightarrow \bar{X}_n \xrightarrow{m.s.} \mu$.

⑧ $P_{(X_1,\dots X_n)} Z = a_1 X_1 + \dots + a_n X_n$

$V(Z - P_{(\ )} Z) = V(Z) + V(P_{(\ )} Z) - 2\,Cov(Z, P_{(\ )} Z)$

$a_1, \dots, a_n \ni$

$E(Z - P_{(\ )} Z)^2$ is min w.r.t $\underset{\sim}{a}$

$\Rightarrow E(Z - P_{(\ )} Z) X_i = 0 \quad \forall i$

$\Rightarrow Cov(Z - P_{(\ )} Z, X_i) = 0 \quad \forall i$

$Cov(Z - P_{(\ )} Z, \Sigma a_i X_i) = 0$

i.e. $Cov(Z - P_{(\ )} Z, P_{(\ )} Z) = 0$

i.e. $Cov(Z, P_{(\ )} Z) = V(P_{(\ )} Z)$

$\Rightarrow V[Z - P_{(\ )} Z] = V(Z) - V(P_{(\ )} Z)$

⑨ $X_t = \begin{cases} Z_1, & t \text{ is even} \\ Z_2, & t \text{ is odd} \end{cases}$

$h \to$ ~~odd~~ even

$P_{(X_h, X_{h-1}, \dots X_2)} X_{h+1} = \alpha_1 X_h + \dots + \alpha_{h-1} X_2$

$S(\underset{\sim}{\alpha}) = E(X_{h+1} - \alpha_1 X_h - \dots - \alpha_{h-1} X_2)^2$

BLP eq's

$E(Z_2 - \alpha_1 Z_1 - \dots - \alpha_{h-1} Z_1) Z_1 = 0$

$E(Z_2 \dots) Z_2 = 0.$

$0 = (\alpha_1 + \alpha_3 + \dots + \alpha_{h-1}) \sigma^2$

$\& \ \sigma^2 = (\alpha_2 + \alpha_4 + \dots + \alpha_{h-2}) \sigma^2$

$$\Rightarrow \alpha_1 + \alpha_3 + \cdots + \alpha_{h-1} = 0$$

$$\& \quad \alpha_2 + \alpha_4 + \cdots + \alpha_{h-2} = 1$$

$$P_{(X_h, X_{h-1}, \ldots, X_2)} X_{h+1} = Z_2 (\alpha_2 + \alpha_4 + \cdots + \alpha_{h-2}) = Z_2$$

(10) $$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$P_{(Y_n, \ldots, Y_1)} Y_{n+1} = a_1 Y_n + \cdots + a_n Y_1$$

$a_1, \ldots, a_n \ni$

$E(Y_{n+1} - a_1 Y_n - \cdots - a_n Y_1)^2$ is min w.r.t. $a_1, \ldots, a_n$

$$S(\underset{\sim}{a}) = E(Y_{n+1} - a_1 Y_n - \cdots - a_n Y_1)^2$$

$$= E(\phi_1 Y_n + \phi_2 Y_{n-1} + \cdots + \phi_p Y_{n+1-p} + \epsilon_{n+1} - a_1 Y_n - \cdots - a_n Y_1)^2$$

BLP eq^{n}s $\dfrac{\partial S}{\partial a_j} = 0 \qquad j = 1(1)p$

i.e. $E\left((\phi_1 - a_1) Y_n + \cdots + (\phi_p - a_p) Y_{n+1-p} - a_{p+1} Y_{n-p} - \cdots - a_n Y_1\right) Y_{n+1-j} = 0$

$(\mathrm{cov}(\epsilon_t, X_{t-j}) = 0 \ \forall j > 0) \qquad j = 1(1)p.$

Solution is $a_1 = \phi_1, \ldots, a_p = \phi_p, \ a_{p+1} = \cdots = a_n = 0$

The above can also be derived using Yule-Walker eq^n argument.

(11)

$$Y_t = X_{2t} = \phi X_{2t-1} + \epsilon_{2t} \ ; \ \epsilon_t \sim WN(0,\sigma^2) ; |\phi|<1$$

i.e. $$Y_t = \phi(\phi X_{2t-2} + \epsilon_{2t-1}) + \epsilon_{2t}$$
$$= \phi^2 X_{2t-2} + \phi \epsilon_{2t-1} + \epsilon_{2t}$$

i.e. $$Y_t = \phi^2 Y_{t-1} + \eta_t ;$$

where $$\eta_t = \phi \epsilon_{2t-1} + \epsilon_{2t} \sim WN(0, \sigma^2(1+\phi^2))$$

$\Rightarrow \{Y_t\}$ is stationary AR(1)

$$\Rightarrow \hat{Y}_{t+1} = \phi^2 Y_t = P_{(Y_t, Y_{t-1}, \ldots)} Y_{t+1}$$

min mean sq prediction error

$$E(Y_{t+1} - P_{(\cdot)} Y_{t+1})^2 = E(Y_{t+1} - \phi^2 Y_t)^2$$
$$= E(\eta_{t+1}^2) = \sigma^2(1+\phi^2).$$

(12)

$$X_t = \phi_X X_{t-1} + Z_t \qquad Z_t \sim WN(0,\sigma_Z^2)$$
$$Y_t = \phi_Y Y_{t-1} + Z_t + U_t \qquad U_t \sim WN(0,\sigma_U^2)$$

$\Big\}$ indep.

$$|\phi_X|, |\phi_Y| < 1$$

$$P_{Y_t} X_{t+1} = a Y_t$$

$$a = \frac{E(X_{t+1} Y_t)}{E Y_t^2} \quad - (*)$$

Note that

$$Y_t - \phi_Y Y_{t-1} = Z_t + U_t$$

$$(1-\phi_Y B) Y_t = Z_t + U_t$$
$$Y_t = (1-\phi_Y B)^{-1} (Z_t + U_t)$$

$$y_t = \sum_0^\infty \phi_y^j B^j (z_t + u_t) = \sum_{j=0}^\infty \phi_y^j (z_{t-j} + u_{t-j})$$

$$E y_t = 0$$

$$V(y_t) = E y_t^2 = \sum_{j=0}^\infty \phi_y^{2j} (\sigma_u^2 + \sigma_z^2)$$

$$\text{i.e.} \quad E y_t^2 = (\sigma_z^2 + \sigma_u^2)(1 - \phi_y^2)^{-1}$$

$$E(x_{t+1} y_t) = E\left(\sum_0^\infty \phi_x^j z_{t+1-j}\right)\left(\sum_{k=0}^\infty \phi_y^k (z_{t-k} + u_{t-k})\right)$$

$$= E\left(z_{t+1} + \phi_x z_t + \phi_x^2 z_{t-1} + \cdots\right)\left((z_t + u_t) + \phi_y (z_{t-1} + u_{t-1}) + \phi_y^2 (z_{t-2} + u_{t-2}) + \cdots\right)$$

$$= \phi_x \sigma_z^2 + \phi_x^2 \phi_y \sigma_z^2 + \phi_x^3 \phi_y^2 \sigma_z^2$$

$$= \phi_x \sigma_z^2 (1 + \phi_x \phi_y + \phi_x^2 \phi_y^2 + \cdots)$$

$$= \sigma_z^2 \phi_x (1 - \phi_x \phi_y)^{-1}$$

Using (*)

$$a = \frac{\phi_x \sigma_z^2 (1 - \phi_x \phi_y)^{-1}}{(\sigma_z^2 + \sigma_u^2)(1 - \phi_y^2)^{-1}} = \frac{\sigma_z^2}{(\sigma_z^2 + \sigma_u^2)} \cdot \frac{\phi_x (1 - \phi_y^2)}{1 - \phi_x \phi_y}$$

(13)

$$X_t = \frac{1}{2} X_{t-1} + \epsilon_t \ ; \quad \epsilon_t \sim WN(0,1)$$

$$Y_t = X_t + \eta_t \ ; \quad \eta_t \sim WN(0,\sigma^2)$$

indep.

$\{\epsilon_t\}$ & $\{\eta_t\}$ indep $\Rightarrow$ $\{X_t\}$ & $\{\eta_t\}$ are indep.

$\Rightarrow$ Y_t is covariance stationary &

$$\gamma_y(h) = \gamma_X(h) + \gamma_\eta(h) \quad \text{--- (1)}$$

$$\gamma_X(h) = \frac{\sigma^2}{1-\phi^2} \phi^{|h|} \ ; \quad \gamma_\eta(h) = \sigma^2 I(h=0)$$

BLP: $P_{(Y_2,Y_1)} Y_3 = \alpha Y_2 + \beta Y_1$

$\alpha, \beta \ni E\left(Y_3 - P_{(Y_2,Y_1)} Y_3\right)^2$ is min w.r.t. α, β.

BLP eq^{n}s:
$$E(Y_3 - \alpha Y_2 - \beta Y_1) Y_2 = 0$$
$$E(Y_3 - \alpha Y_2 - \beta Y_1) Y_1 = 0$$

i.e.
$$\gamma_1 - \alpha \gamma_0 - \beta \gamma_1 = 0$$
$$\gamma_2 - \alpha \gamma_1 - \beta \gamma_0 = 0$$

i.e.
$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{BLP} \\ \beta_{BLP} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

use (1) to get $\gamma_0, \gamma_1, \gamma_2$ and simplify.

BLP of X_2 based on X_1, X_3, X_4

(14) $P_{(X_1,X_3,X_4)} X_2 = \alpha_1 X_1 + \alpha_2 X_3 + \alpha_3 X_4$

$\alpha_1, \alpha_2, \alpha_3 \ni$

$E(X_2 - \alpha_1 X_1 - \alpha_2 X_3 - \alpha_3 X_4)^2$ is min w.r.t. $\alpha_1, \alpha_3, \alpha_4$

BLP eq^{n}s :

$$E(X_2 - \alpha_1 X_1 - \alpha_2 X_3 - \alpha_3 X_4) X_i = 0 \qquad i = 1, 3, 4$$

i.e. $\gamma_1 - \alpha_1 \gamma_0 - \alpha_2 \gamma_2 - \alpha_3 \gamma_3 = 0$

$\gamma_1 - \alpha_1 \gamma_2 - \alpha_2 \gamma_0 - \alpha_3 \gamma_1 = 0$

$\gamma_2 - \alpha_1 \gamma_3 - \alpha_2 \gamma_1 - \alpha_3 \gamma_0 = 0$

$\gamma(.)$ structure of MA : $\gamma_h = \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \theta\sigma^2, & h = \pm 1 \\ 0, & o/w \end{cases}$

BLP eq^n : $\begin{pmatrix} \gamma_1 \\ \gamma_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & 0 & 0 \\ 0 & \gamma_0 & \gamma_1 \\ 0 & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$$\begin{pmatrix} \alpha_{1(BLP)} \\ \alpha_{2(BLP)} \\ \alpha_{3(BLP)} \end{pmatrix} = \begin{pmatrix} \gamma_0^{-1} & 0 & 0 \\ 0 & \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} & \\ 0 & & \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_1 \\ 0 \end{pmatrix}$$

use ACVF structure of MA(1) to simplify.

(15)

$$X_t = \mu + \phi X_{t-1} + \epsilon_t \quad |\phi|<1$$

$$Y_t = \delta + \eta_t + \theta \eta_{t-1}, \quad |\theta|<1$$

$$\left.\begin{array}{l}\epsilon_t \sim WN(0,\sigma^2) \\ \eta_t \sim WN(0,\sigma^2)\end{array}\right\} \text{indep}$$

$\Rightarrow \{X_t\}$ & $\{Y_t\}$ are indep

BLP: $P_{(Y_1,\ldots,Y_n)} X_{n+1} = a_0 + a_1 Y_1 + \cdots + a_n Y_n$

$a_0, a_1, \ldots a_n \ni E(X_{n+1} - P_{(Y_1,\ldots,Y_n)} X_{n+1})^2$ is

min w.r.t. $a_0, a_1, \ldots, a_n$

BLP eq^{n}s

$$E(X_{n+1} - a_0 - a_1 Y_1 - \cdots - a_n Y_n) = 0 \quad -(1)$$

$$E(X_{n+1} - a_0 - a_1 Y_1 - \cdots - a_n Y_n) Y_j = 0 \quad -(2)$$

$j = 1(1)n$

$(1) \Rightarrow \mu_X - a_0 - a_1 \mu_Y - \cdots - a_n \mu_Y = 0$

$(2) \Rightarrow E(X_{n+1} Y_j) - a_0 E(Y_j) - a_1 E(Y_1 Y_j) - \cdots - a_n E(Y_n Y_j) = 0$

$j = 1(1)n$

$(1) \Rightarrow a_0 = \mu_X - \mu_Y \sum_{i=1}^{n} a_i \quad -(3)$

using (3) in (2)

$$E(X_{n+1} Y_j) - (\mu_X \mu_Y - \mu_Y^2 \Sigma a_i)$$
$$- a_1 E(Y_1 Y_j) - \cdots - a_n E(Y_n Y_j) = 0$$

$j = 1(1)n$

i.e. $\quad Cov(X_{n+1}, Y_j) = a_1 \gamma_y(j-1) + \cdots + a_n \gamma_y(j-n)$

$j = 1(1)n$

i.e. $\quad Cov(X_{n+1}, Y_1) = a_1 \gamma_y(0) + a_2 \gamma_y(1) + \cdots + a_n \gamma_y(n-1)$

$\vdots$

$Cov(X_{n+1}, Y_n) = a_1 \gamma_y(n-1) + \cdots + a_n \gamma_y(0)$

$$\begin{pmatrix} Cov(X_{n+1}, Y_1) \\ \vdots \\ Cov(X_{n+1}, Y_1) \end{pmatrix} = \begin{pmatrix} \gamma_y(0) & \gamma_y(1) & \cdots & \gamma_y(n-1) \\ & \ddots & & \\ & & & \gamma_y(0) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

i.e. $\quad \underset{\sim}{0} = \Gamma_y \underset{\sim}{a}_n \quad (\because \{X_t\} \,\&\, \{Y_t\}$ are indep$)$

$\Rightarrow \quad \underset{\sim}{a}_{n(BLP)} = \underset{\sim}{0}$

$\Rightarrow \quad a_{0(BLP)} = \mu_X = \dfrac{\mu}{1-\phi} \quad$ (using (3))

$\Rightarrow \quad P_{(Y_1, \ldots, Y_n)} X_{n+1} = \dfrac{\mu}{1-\phi}$

min mean sq prediction error of BLP

$$E\left(X_{n+1} - \frac{\mu}{1-\phi}\right)^2 = E\left(X_{n+1} - E X_{n+1}\right)^2 = V(X_{n+1})$$

$$= \frac{\sigma^2}{1-\phi^2} \cdot \left(= \gamma_X(0)\right)$$

(1.6) $X_t = \phi X_{t-1} + \epsilon_t$; $\phi = \frac{1}{2}$

$Y_t = X_t + \eta_t$

$\gamma_X(h) = \frac{\sigma_\epsilon^2}{1-\phi^2} \phi^{|h|}$

$= \frac{4}{3} \sigma_\epsilon^2 \left(\frac{1}{2}\right)^{|h|}$

$\gamma_Y(h) = \gamma_X(h) + \gamma_\eta(h)$ — (1)

$\underset{(Y_t, Y_{t-1})}{P} X_{t+2} = \alpha Y_t + \beta Y_{t-1}$

$E\left(X_{t+2} - \alpha Y_t - \beta Y_{t-1}\right)^2$

$E\left(X_{t+2} - \alpha Y_t - \beta Y_{t-1}\right) Y_t = 0$

$E\left(X_{t+2} - \alpha Y_t - \beta Y_{t-1}\right) Y_{t-1} = 0$

$E\left(X_{t+2} Y_t\right) = \alpha \gamma_Y(0) + \beta \gamma_Y(1)$

$E\left(X_{t+2} Y_{t-1}\right) = \alpha \gamma_Y(1) + \beta \gamma_Y(0)$

$E\left(X_{t+2} Y_t\right) = E\left(X_{t+2}(X_t + \eta_t)\right)$

$= \gamma_X(2)$ — (2)

$E\left(X_{t+2} Y_{t-1}\right) = E\left(X_{t+2}(X_{t-1} + \eta_{t-1})\right)$

$= \gamma_X(3)$ — (3)

$$\begin{pmatrix} \gamma_X(2) \\ \gamma_X(3) \end{pmatrix} = \begin{pmatrix} \gamma_Y(0) & \gamma_Y(1) \\ \gamma_Y(1) & \gamma_Y(0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad — (4)$$

use (1), (2), (3) in (4) to obtain

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{BLP} = \begin{pmatrix} \gamma_y(0) & \gamma_y(1) \\ \gamma_y(1) & \gamma_y(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma_x(2) \\ \gamma_x(3) \end{pmatrix}.$$

(b) PACF at lag 2 of $\{Y_t\}$

$$\alpha(2) = \text{Corr}^n\left(Y_3 - P_{Y_2} Y_3,\ Y_1 - P_{Y_2} Y_1\right). \quad — (*)$$

$$P_{Y_2} Y_3 = \rho_y(1) Y_2 \ ; \quad P_{Y_2} Y_1 = \rho_y(1) Y_2$$

Use eq^n (1) to obtain $\rho_y(1)$

$$\rho_y(1) = \frac{\gamma_y(1)}{\gamma_y(0)}$$

$$\gamma_y(0) = \gamma_x(0) + \gamma_\eta(0) \quad \& \quad \gamma_y(1) = \gamma_x(1)$$

$$\Rightarrow \rho_y(1) = \frac{\gamma_x(1)}{\gamma_x(0) + \gamma_\eta(0)}$$

use $\rho_y(1)$ in $(*)$ to calculate

$\alpha(2)$ as

$$\frac{\text{Cov}\left(Y_3 - \rho_y(1) Y_2,\ Y_1 - \rho_y(1) Y_2\right)}{\left[V\left(Y_3 - \rho_y(1) Y_2\right) V\left(Y_1 - \rho_y(1) Y_2\right)\right]^{1/2}}$$

(17)

$$X_t = \epsilon_t + 2\epsilon_{t-1} - \epsilon_{t-2}; \quad \epsilon_t \sim WN(0, 1)$$

(a) $$E(X_5 - \alpha X_4 - \beta X_3)^2$$

BLP eqn's $$E(X_5 - \alpha X_4 - \beta X_3) X_4 = 0$$

$$\& \ E(X_5 - \alpha X_4 - \beta X_3) X_3 = 0$$

i.e. $$\gamma_1 = \alpha \gamma_0 + \beta \gamma_1$$

$$\gamma_2 = \alpha \gamma_1 + \beta \gamma_0$$

$$\gamma_h = \begin{cases} 6, & h = 0 \\ 0, & h = \pm 1 \\ -1, & h = \pm 2 \\ 0, & o/w \end{cases}$$

$$\Rightarrow \left.\begin{array}{l} 0 = 6\alpha \\ -1 = 6\beta \end{array}\right\} \Rightarrow \alpha = 0; \ \beta = -\frac{1}{6}$$

$$P_{(X_4, X_3)} X_5 = -\frac{1}{6} X_3$$

(b) $$E\left(X_5 + \frac{1}{6} X_3\right)^2$$

$$= E X_5^2 + \frac{1}{36} E X_3^2 + 2 \cdot \frac{1}{6} E X_5 X_3$$

$$= \gamma_0 + \frac{1}{36} \gamma_0 + \frac{1}{3} \gamma_2$$

$$= 6 + \frac{1}{36} 6 + \frac{1}{3}(-1)$$

(c) PACF at lag 2 $= \alpha(2)$

$$= \text{Corrn}\left(X_1 - P_{X_2} X_1, X_3 - P_{X_2} X_3\right) = \text{Corrn}\left(X_1 - \rho(1) X_2, X_3 - \rho(1) X_2\right)$$

$$= \text{Corrn}(X_1, X_3) = \frac{\gamma_2}{\gamma_0} = -\frac{1}{6}$$

(d) $\alpha(2) =$ Coeff of X_3 in BLP of X_5 on X_4, X_3