Given $f : [0,\infty) \to [0,\infty)$ is increasing and satisfy

- $f(0) = 0$
- $f(x) > 0 \quad \forall\ x > 0$

If f also satisfies $f(x+y) \leq f(x) + f(y)\ \forall\ x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric.

Question: Show that each of the following conditions is suff. to ensure that $f(x+y) \leq f(x) + f(y)$.

(a) f has a second derivative satisfying $f'' \leq 0$

(b) f has a decreasing first derivative

(c) $f(x)/x$ is decreasing for $x > 0$.

Solution: We first show that if condition (c) is satisfied then $f(x+y) \leq f(x) + f(y)$, $\forall\ x, y \geq 0$.

Note that if either x or y is 0, then the inequality holds. (In the case $x = y = 0$, use the fact that $f(0) = 0$.)

Assume $x, y > 0$.

Case 1. Suppose $x \leq y$. Then $x \leq y < x+y$.

Since $\frac{f(x)}{x}$ is decreasing, $\frac{f(x+y)}{x+y} \leq \frac{f(y)}{y} \Rightarrow f(x+y) \leq \frac{f(y)}{y} \cdot x + f(y)$

Also, $\frac{f(y)}{y} \leq \frac{f(x)}{x} \Rightarrow \frac{f(y)}{y} \cdot x \leq f(x)$. Hence, $\frac{f(y)}{y} \cdot x + f(y) \leq f(x) + f(y)$.

Therefore, $f(x+y) \leq f(x) + f(y)$.

Case 2. Suppose $y < x$. Then, $y < x < x+y$. Interchanging the role of x and y in Case 1 yields $f(x+y) \leq f(x) + f(y)$.

Therefore, Condition (c) $\Rightarrow f(x+y) \leq f(x) + f(y) \quad \forall\ x, y \geq 0$.

Claim: (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Pf. of claim: Given $f''(x) \leq 0 \quad \forall\ x \geq 0$. For $x_1 < x_2$, we need to show $f'(x_2) \leq f'(x_1)$.

Consider $[x_1, x_2]$ and $f' : [x_1, x_2] \to \mathbb{R}$. Since $f''(x)$ exists $\forall\ x \geq 0$, f' is differentiable in $[x_1, x_2]$. Therefore, by Mean Value Thm,

$$f'(x_2) - f'(x_1) = f''(c)(x_2 - x_1) \text{ for some } c \in (x_1, x_2).$$

Since $f''(x) \leq 0 \ \forall x \geq 0$, $f''(c) \leq 0$, hence $f''(c)(x_2 - x_1) \leq 0$.

Therefore $f'(x_2) \leq f'(x_1)$.

Since x_1 and x_2 were arbitrary pts. chosen on $[0,\infty)$, f' is decreasing.

This proves (a) $\Rightarrow$ (b).

WTS: (b) $\Rightarrow$ (c).

Pf: In order to show $\frac{f(x)}{x}$ for $x > 0$ is a decreasing function,

we need to show $\left(\frac{f(x)}{x}\right)' \leq 0$, $\forall x > 0$.

Note that $\left(\frac{f(x)}{x}\right)' = \frac{x f'(x) - f(x)}{x^2}$. It suffices to show $x f'(x) - f(x) \leq 0$ as $\frac{1}{x^2} > 0$.

That is, it suffices to show $f'(x) \leq \frac{f(x)}{x}$, $\forall x > 0$.

For $x > 0$, consider the interval $[0,x]$ and $f: [0,x] \to \mathbb{R}$.

Since f has a derivative on $[0,\infty)$, f is cts. on $[0,x]$ and f is diff. on $(0,x)$.

By Mean Value Thm,

$$\frac{f(x) - f(0)}{x} = f'(c) \text{ for some } c \in (0,x).$$

Since $f(0) = 0$, $\frac{f(x)}{x} = f'(c)$.

Moreover as f' is decreasing, for $c < x$, $f'(x) \leq f'(c)$

$$\Rightarrow f'(x) \leq \frac{f(x)}{x}.$$

Since x was chosen arbitrarily, one has

$$f'(x) \leq \frac{f(x)}{x} \quad \forall x > 0.$$

This completes the proof of the claim.