Note: If $g(.)$ is not monotone, the above cannot be applied.

e.g. $Y = X^2$ $\quad \mathcal{X} = (-\infty, \infty)$; $\mathcal{Y} = (0, \infty)$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

p.d.f. of Y:

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y)$$

$$= \frac{\partial}{\partial y}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}$$

Example: $X \sim N(0,1)$

$$Y = X^2 \qquad \mathcal{X} = (-\infty, \infty) \; ; \; \mathcal{Y} = (0, \infty)$$

$$F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \qquad y > 0$$

$$= \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y}))$$

$$= 2\,\Phi(\sqrt{y}) - 1$$

p.d.f. of Y: $f_Y(y) = \cancel{2}\,\phi(\sqrt{y}) \frac{1}{\cancel{2}\sqrt{y}} = \frac{\phi(\sqrt{y})}{\sqrt{y}}$

i.e. $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} y^{-1/2} \qquad y > 0$

Realize that the above is density of a χ^2 r.v. with $n = 1$

<u>Example</u>: $X \sim U(-1, 2)$

$$f_X(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & o/w \end{cases}$$

$$\mathcal{X} = (-1, 2)$$

$$Y = X^2 \qquad \mathcal{Y} = (0, 4)$$

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

If $0 \le y < 1$, then $F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3} = \frac{2\sqrt{y}}{3}$

if $1 \le y < 4$, then

$$F_Y(y) = \int_{-1}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{1}{3}(1 + \sqrt{y})$$

If $y \geq 4 \qquad F_Y(y) = 1$

& if $y < 0 \qquad F_Y(y) = 0$

i.e. $$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{2\sqrt{y}}{3}, & 0 \le y < 1 \\ \frac{1}{3}(1+\sqrt{y}), & 1 \le y < 4 \\ 1, & y \geq 4 \end{cases}$$

p.d.f. of Y:

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{6\sqrt{y}}, & 1 < y < 4 \\ 0, & o/w \end{cases}$$

Jacobian based method

Note that from d.f. based method we have for strictly monotone $g(.)$ $\left(y = g(x) \,;\, x = g^{-1}(y) \right)$

$$F_Y(y) = \begin{cases} F_X(g^{-1}(y)), & \text{if } g \text{ is increasing} \\ 1 - F_X(g^{-1}(y)), & \text{if } g \text{ is decreasing} \end{cases}$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \dfrac{\partial}{\partial y} g^{-1}(y), & \text{if } g \text{ is increasing} \\ - f_X(g^{-1}(y)) \dfrac{\partial}{\partial y} g^{-1}(y), & \text{if } g \text{ is decreasing.} \end{cases}$$

i.e.
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial}{\partial y} g^{-1}(y) \right| \quad \forall\, y \in \mathcal{Y}$$
$$= 0 \qquad \text{o/w}$$

$\dfrac{\partial}{\partial y} g^{-1}(y)$ is called the Jacobian of transformation.

Example: $X \sim U(0,1)$

$$Y = -2 \ln X$$

$$y = g(x) = -2 \ln x$$

$$x = e^{-y/2} = g^{-1}(y)$$

$$J = \frac{dx}{dy} = e^{-y/2}\left(-\tfrac{1}{2}\right) = \frac{d\, g^{-1}(y)}{dy} \quad \leftarrow \text{Jacobian}$$

$$f_Y(y) = f_X(g^{-1}(y)) \, |J| \qquad 0 < y < \infty$$
$$= 0, \qquad \text{o/w}$$

i.e. $f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2}, & y > 0 \\ 0, & o/w \end{cases}$

Note that $Y \sim \chi^2_2$

Example : $X \sim \text{Gamma}(n, \beta)$

$$f_X(x) = \frac{1}{\Gamma n \ \beta^n} e^{-x/\beta} x^{n-1} , \quad 0 < x < \infty$$

$\beta > 0$, n is a positive integer

$\mathcal{X} = (0, \infty)$

$X \to Y = \frac{1}{X}$ $\qquad \mathcal{Y} = (0, \infty)$

$y = g(x) = \frac{1}{x}$; $\quad x = \frac{1}{y} = g^{-1}(y)$

$$J = \frac{\partial x}{\partial y} = \frac{\partial g^{-1}(y)}{\partial y} = -\frac{1}{y^2} \leftarrow \text{Jacobian}$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \, |J|$$

$$= \left(\frac{1}{\Gamma n \ \beta^n} e^{-\frac{1}{y\beta}} \left(\frac{1}{y}\right)^{n-1} \right) \left(\cdot \frac{1}{y^2} \right)$$

$$= \frac{1}{\Gamma n \ \beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}} \qquad y > 0$$

Note: The above works if $g(.)$ is strictly monotone in $\mathcal{X}$.

The following extension to non-monotone setup is useful:

Suppose $\exists$ a partition $A_0, A_1, A_2, \ldots A_K$ of $\mathcal{X}$ such that

$P(X \in A_0) = 0$

$f_X(x)$ is continuous on each A_i

Suppose further that there exist functions $g_1(x), \ldots g_k(x)$ defined on $A_1, \ldots, A_K$ satisfying

(a) $g(x) = g_i(x)$ for $x \in A_i$

(b) $g_i(x)$ is strictly monotone on A_i

(c) $\mathcal{Y} = \{y : y = g_i(x) \text{ for } x \in A_i\}$ is same for each $i = 1(1)K$

(d) $g_i^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$ $\forall i$

Then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{\partial g_i^{-1}(y)}{\partial y} \right|, \quad y \in \mathcal{Y}$$

$$0, \quad \text{o/w.}$$

Example: $X \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < +\infty$$

$$X \to Y = X^2$$

Note that $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$

$$\mathcal{Y} = (0, \infty)$$

Apply the above result with

$A_0 = \{0\}$; $A_1 = (-\infty, 0)$; $A_2 = (0, \infty)$

For the region $(-\infty, 0) = A_1$

$$g_1(x) = x^2 = y \; ; \; x = -\sqrt{y} = g_1^{-1}(y)$$

& for $A_2 = (0, \infty)$; $g_2(x) = x^2 = y$; $x = \sqrt{y}$

$$g_2^{-1}(y) = \sqrt{y}$$

$\therefore$ p.d.f. of $Y = X^2$

$$f_Y(y) = \underbrace{f_X(g_1^{-1}(y)) \left| \frac{\partial g_1^{-1}(y)}{\partial y} \right|}_{(-\infty,0)} + \underbrace{f_X(g_2^{-1}(y)) \left| \frac{\partial g_2^{-1}(y)}{\partial y} \right|}_{(0,\infty)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{y})^2} \left| -\frac{1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} \left| \frac{1}{2\sqrt{y}} \right|$$

i.e. $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} y^{-1/2}$

i.e. $Y \sim \chi_1^2$

<u>Note</u>: For the example of $X \sim U(-1, 2)$; $X \to Y = X^2$

for $y \in (0, 1)$. $f_Y(y) = \underbrace{f_X(g_1^{-1}(y)) \left| \frac{\partial g_1^{-1}(y)}{\partial y} \right|}_{(-1,0) \in \mathcal{X}} + \underbrace{f_X(g_2^{-1}(y)) \left| \frac{\partial g_2^{-1}(y)}{\partial y} \right|}_{(0,1) \in \mathcal{X}}$

$g(x) = x^2$ is monotone strictly in $(-1, 0)$ and $(0, 1)$ $= \frac{1}{3} \frac{1}{2\sqrt{y}} + \frac{1}{3} \frac{1}{2\sqrt{y}}$

$(-1, 0) \to (0, 1)$

& $(0, 1) \to (0, 1)$ $= \frac{1}{3\sqrt{y}}$

for $y \in (1, 4)$

$g(x) = x^2$ is strictly monotone

$(1, 2) \to (1, 4)$ $\quad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right|$

$$= \frac{1}{3} \frac{1}{2\sqrt{y}} = \frac{1}{6\sqrt{y}}$$