Example 2 : $\{X_t\}$ is $\ni$

$$X_t = A \cos \omega t + B \sin \omega t$$

A & B are uncorrelated r.v.s with

mean zero and variance σ^2

$\omega \in (-\pi, \pi)$ and is fixed

$$E X_t = 0 \quad \forall t \quad \text{---(i)}$$

$$\begin{aligned} \operatorname{Cov}(X_t, X_{t+h}) &= \operatorname{Cov}\Big(A \cos \omega t + B \sin \omega t,\ A \cos \omega (t+h) \\ &\qquad + B \sin \omega (t+h)\Big) \\ &= \sigma^2 \Big(\cos \omega t \cos \omega (t+h) + \sin \omega t \sin \omega (t+h)\Big) \\ &= \underline{\sigma^2 \cos \omega h} = \gamma_X(h) \quad \text{---(ii)} \end{aligned}$$

f^n of h only and is indep of t

(i) & (ii) $\Rightarrow$ $\{X_t\}$ is covariance stationary

(*) What happens If I make A & B to have identical but non zero mean ??

Example 3

$\{X_t\}$ is $\ni$

$$X_t = \sum_{j=1}^{K} \left(A_j \cos(j\omega t) + B_j \sin(j\omega t) \right)$$

$\{A_j\}_{j=1}^{K}$ seq of independent $N(0, \sigma^2)$

$\{B_j\}_{j=1}^{K}$ seq of indep $N(0, \sigma^2)$

Further $\{A_j\}$ & $\{B_j\}$ sequences are mutually indep

$$E X_t = 0 \ \forall t \quad (i)$$

$$\operatorname{cov}(X_t, X_{t+h}) = E\, X_t X_{t+h}$$

$$= E\left(\sum_{j=1}^{K} \{A_j \cos(j\omega t) + B_j \sin(j\omega t)\} \right) \left(\sum_{j=1}^{K} \{A_j \cos(j\omega(t+h)) + B_j \sin(j\omega(t+h))\} \right)$$

$$= \sum_{j=1}^{K} \Big\{ E(A_j^2) \cos(j\omega t) \cos(j\omega(t+h)) + E(B_j^2) \sin(j\omega t) \sin(j\omega(t+h)) \Big\}$$

(remaining terms are zero due to serial & mutual independence of $\{A_j\}$ & $\{B_j\}$ seqs)

$$= \sigma^2 \sum_{j=1}^{K} \cos(j\omega h) \leftarrow \text{indep of } t;\ f^n \text{ of } h \text{ only} \quad \text{—(ii)}$$

(i) & (ii) $\Rightarrow \{X_t\}$ is covariance stationary

Example 3 (contd)

Consider the random ~~vec~~ vector $\underset{\sim}{Z} = \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}$ for any n and any adm $t_1, \ldots t_n$

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} \left(A_j \cos(j\omega t_1) + B_j \sin(j\omega t_1) \right) \\ \vdots \\ \sum_{j=1}^{K} \left(A_j \cos(j\omega t_n) + B_j \sin(j\omega t_n) \right) \end{pmatrix}$$

$\forall \underset{\sim}{\alpha} \in \mathbb{R}^n$; $\underset{\sim}{\alpha}' \underset{\sim}{Z} = \alpha_1 \left(\sum_{j=1}^{K} \left(A_j \cos(j\omega t_1) + B_j \sin(j\omega t_1) \right) \right)$

$$+ \cdots$$

$$\vdots$$

$$+ \alpha_n \left(\sum_{j=1}^{K} \left(A_j \cos(j\omega t_1) + B_j \sin(j\omega t_1) \right) \right)$$

$$= \beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_K A_K$$

$$+ \gamma_1 B_1 + \cdots + \gamma_K B_K \quad - (*)$$

$(\beta_1, \ldots, \beta_K$ and $\gamma_1, \ldots \gamma_K$ are consts$)$.

$(*)$ is a linear combination of indep. N_1 random variables

$\Rightarrow$ $\underset{\sim}{\alpha}' \underset{\sim}{Z}$ is a linear combination of indep N_1 random variables

$\Rightarrow$ $\underset{\sim}{\alpha}' \underset{\sim}{Z} \sim N_1 \quad \forall \underset{\sim}{\alpha} \in \mathbb{R}^n \; (\underset{\sim}{\alpha} \neq \underset{\sim}{0})$

$$\Rightarrow \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} \sim N_n(\underset{\sim}{0}, \Sigma) \quad — (*^2)$$

$$\Sigma = \begin{pmatrix} K\sigma^2 & \sigma^2 \sum_{j=1}^{K} \cos(j\omega(t_2-t_1)) & \cdots & \sigma^2 \sum_{j=1}^{K} \cos(j\omega(t_n-t_1)) \\ & K\sigma^2 & & \sigma^2 \sum_{j=1}^{K} \cos(j\omega(t_n-t_2)) \\ & & \ddots & \\ & & & K\sigma^2 \end{pmatrix}$$

$(*^2) \Rightarrow \{X_t\}$ is a Gaussian process.

Further, we have already proved that $\{X_t\}$ is Covariance stationary

$\Rightarrow \{X_t\}$ is strict stationary also.

Note that
$$\begin{pmatrix} X_{t_1+K} \\ \vdots \\ X_{t_n+K} \end{pmatrix} \sim N_n(\underset{\sim}{0}, \Sigma) \quad \forall \text{ int } K$$

Example 4

$X_t = Z_t + \theta Z_{t-1}$; $\{Z_t\}$ is a seq of i.i.d. zero mean finite variance σ^2 process.

$$E X_t = 0 \quad \forall t \qquad \text{— (i)}$$

$$\begin{aligned}
Cov(X_t, X_{t+h}) &= Cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\
&= Cov(Z_t, Z_{t+h}) + \theta\, Cov(Z_t, Z_{t+h-1}) \\
&\quad + \theta\, Cov(Z_{t-1}, Z_{t+h}) + \theta^2 Cov(Z_{t-1}, Z_{t+h-1}) \\
&= \sigma^2 I_{(0)}(h) + \theta \sigma^2 I_{(1)}(h) \\
&\quad + \theta \sigma^2 I_{(-1)}(h) + \theta^2 \sigma^2 I_{(0)}(h) \\
&= \begin{cases} \sigma^2(1+\theta^2), & \text{if } h = 0 \\ \theta\sigma^2, & \text{if } h = \pm 1 \\ 0, & \text{o/w} \end{cases} \qquad \text{— (ii)}
\end{aligned}$$

indep of t ; f^n of h only

(i) & (ii) $\Rightarrow \{X_t\}$ is covariance stationary