Assignment - 3 discussion.

1. $(\ell_\infty, \|\cdot\|_\infty)$ $\quad x = (1,1,1,1, \ldots\ldots) \in \ell_\infty$

$$x^{(k)} = (1,1,1,1,\ldots 1, 0)$$

$$\|x - x^{(k)}\|_\infty = 1, \ \forall k$$

$$\not\downarrow 0$$

$\rightarrow (M, \|\cdot\|)$ $\quad d(x,y) := \|x-y\|.$ $\quad$ "$d \sim \rho$"

$\|\cdot\|'$ $\quad d'(x,y) := \|x-y\|'.$

strongly equivalent: $\|\cdot\| \sim_s \|\cdot\|'$ if $\exists$ c_1 and c_2 s.t. $\quad c_1\|\cdot\|' \le \|\cdot\| \le c_2\|\cdot\|'$

$$\Downarrow$$

$$c_1 d'(x,y) \le d(x,y) \le c_2 d'(x,y).$$

$$\Downarrow$$

$$d(x_n,x) \to 0 \text{ iff } d'(x_n,x) \to 0.$$

$$d \sim d'$$

if $d \sim_s d'$ then $d \sim d'$

$$\Leftarrow$$

$$\times$$

Eg: $M = [0,1]$ $\quad d(x,y) = |x-y|$

$$d'(x,y) := \sqrt{|x-y|}$$

$d \sim d'$ but $d \not\sim_s d'$ Suppose $d \sim_s d'$ $\exists$ $c_1 > 0, c_2 > 0$ s.t. $c_1 d'(x,y) \le d(x,y) \le c_2 d'(x,y)$

$$c_1 \sqrt{|x-y|} \le |x-y|, \ \forall x,y \in [0,1]$$

$y = 0$, $\quad c_1\sqrt{x} \le x$ not possible $\forall x$

$$c_1 < \frac{x}{\sqrt{x}} \to \infty$$

4. $X = \{x_1, x_2, \ldots, x_N\}$. $\quad (X,d)$.

$\{x_j\}$ is open in X.

$r := \min\{d(x_j, x_i) \mid 1 \le i \le N, i \ne j\}$

$$B(x_j, r) = \{y \in X \mid d(x_j, y) < r\} = \{x_j\} = \{x_j\}$$

$\rightarrow (y_n) \in X$ s.t. $y_n \to y$ $\quad d \sim d_0$.

$\mathcal{U}$ is open

$\forall x \in \mathcal{U}$ if $\exists$ $r > 0$ st. $B(x,r) \subset \mathcal{U}$.

$B(x,r) = \{y \mid d(x,y) < r\}$

$(\mathbb{Q}, |\cdot|)$ $\left(\frac{1}{n}\right) \in \mathbb{Q}$

$\swarrow_{|\cdot|}$ $\not\searrow_{d_0}$

0 $\quad$ 0

$(\|\cdot\|_\infty, \|\cdot\|_2)$ $\quad \|\cdot\|_\infty \sim_s \|\cdot\|_2$ (HW)

$(\|\cdot\|_\infty, \|\cdot\|_1)$ $\quad \|\cdot\|_\infty \sim_s \|\cdot\|_1$ (HW)

$\|\cdot\|_1$ $\quad X^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$

$\forall\, 1 \le j \le n, \quad |x_j^{(k)} - x_j^{(m)}| \le \|X^{(k)} - X^{(m)}\|_1 \to 0$ $\quad \downarrow_{\|\cdot\|_1}$

$X = (x_1, \dots, x_n)$

$(\ell_1, \|\cdot\|_\infty) \subset (\ell_\infty, \|\cdot\|_\infty)$

$X = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots)$

$X^{(k)} = (1, \frac{1}{2}, \dots, \frac{1}{k}, \quad) \in \ell_1$

$\|X^{(k)} - X^{(m)}\|_\infty = \sup\left\{\frac{1}{m+1}, \frac{1}{m+2}, \dots\right\} = \frac{1}{m+1} \to 0$

$k > m$.

Suppose $X^{(k)} \to X' \in \ell_1$. But also, $X^{(k)} \to X$ in $\|\cdot\|_\infty$

$\cap \ell_\infty$ $\quad \cap \ell_\infty$

$\Rightarrow X' \in \ell_\infty$. $\quad \Rightarrow X' = X$

$\cancel{d(X)}$