Midsem solution and marking scheme

(1)

(a) $F(-\infty) = 0 \Rightarrow \beta = 0$ — (1).

$F(.)$ is right cont $\Rightarrow F(3) = F(3+)$

i.e. $\dfrac{4\alpha^2 - 9\alpha + 6}{4} = 1 \Rightarrow 4\alpha^2 - 9\alpha + 2 = 0$

$(4\alpha - 1)(\alpha - 2) = 0$

$\Rightarrow \alpha = 2$ or $1/4$ — (1)

$\alpha = 2$ is not possible as $F(.) \geq 0$

$\Rightarrow \underline{\alpha = \frac{1}{4}}$ (4)

(b) P(at least 2 of A, B, C)

$= P(ABC^c) + P(AB^cC) + P(A^cBC) + P(ABC)$

$= P(AB) + P(AC) + P(BC) - 2P(ABC)$

$= \underline{\frac{1}{3} + \frac{1}{3} + \frac{1}{4} - \frac{2}{6}}$ — (4).

$= \frac{7}{12}$

(c) Suppose $F_{X,Y}(x, y)$ is d.f. of (X, Y)

We must have

$$P(a_1 < X_1 \leq b_1,\ a_2 < X_2 \leq b_2) = F_{X,Y}(b_1, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(b_1, a_2) + F_{X,Y}(a_1, a_2) \geq 0 \quad \forall (a_1, b_1), (a_2, b_2)$$

$a_1 < b_1 \quad a_2 < b_2$

But for given $F_{X,Y}(.,.)$

$$P(0 < X_1 \leq 2, -1 < X_2 \leq 1) = F_{X,Y}(2,1) - F_{X,Y}(0,1) - F_{X,Y}(2,-1) + F_{X,Y}(0,-1) = -1$$

$\Rightarrow F_{X,Y}(.,.)$ is not jt d.f. (4)

(*) Note: Other rectangles can be considered.

(2)

(a).

$$P(X \le x \mid X < 1) = \begin{cases} 0, & x \le 0 \\ \dfrac{P(X \le x)}{P(X < 1)}, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

$$F_X(1) = \lambda^2 \int_0^1 x e^{-\lambda x}\, dx = \underline{(1 - e^{-\lambda} - \lambda e^{-\lambda})}$$ Deduct $1\frac{1}{2}$ mark If this is wrong

$$E(X \mid X < 1) = \frac{1}{F_X(1)} \lambda^2 \int_0^1 x^2 e^{-\lambda x}\, dx \quad — (2)$$

$$= \frac{1}{F_X(1)} \left(\frac{2}{\lambda}(1 - e^{-\lambda} - \lambda e^{-\lambda}) - \lambda e^{-\lambda} \right)$$

$$= \left(\frac{2}{\lambda} - \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda} - \lambda e^{-\lambda}} \right) = \left(\frac{2}{\lambda} - \frac{\lambda}{e^{\lambda} - (1+\lambda)} \right)$$

(4)

(b) $X \sim U(-4, 4)$

$Y = X^3 - X$

$$F_Y(0) = P(X^3 - X \le 0) = P(X(X^2 - 1) \le 0)$$

$$= P(X \le 0, X^2 - 1 \ge 0) + P(X \ge 0, X^2 - 1 \le 0)$$

$$= P(X \le 0, X^2 \ge 1) + P(X \ge 0, X^2 \le 1) \quad — (2)$$

$$= P(X \le -1) + P(0 \le X \le 1) \quad — \quad (1)$$

$$= \frac{3}{8} + F_X(1) - F_X(0)$$

$$= \frac{4}{8} = \frac{1}{2} \quad \longrightarrow (3)$$

(3) $Y = [X]$ $\qquad F_X(x) = 1 - e^{-x}$; $0 < x < \infty$

$\mathcal{Y} = \{0, 1, 2, \ldots\}$

$$P(Y=k) = P(k \leq X < k+1)$$

$$= F_X(k+1) - F_X(k)$$

$$= \left(1 - e^{-(k+1)}\right) - \left(1 - e^{-k}\right)$$

$$= e^{-k} - e^{-(k+1)} \quad ; \; k = 0, 1, 2, \ldots$$

(a) $$P(6 \leq Y < 9 \mid Y < 10) = \frac{P(6 \leq Y < 9)}{P(Y < 10)}$$

$$= \frac{P(Y=6) + P(Y=7) + P(Y=8)}{\sum_{y=0}^{9} P(Y=y)} \quad \text{— (1)}$$

$$= \frac{(e^{-6} - \cancel{e^{-7}}) + (\cancel{e^{-7}} - \cancel{e^{-8}}) + (\cancel{e^{-8}} - e^{-9})}{1 - e^{-10}}$$

$$= \frac{e^{-6} - e^{-9}}{1 - e^{-10}} \quad \text{— (2)}$$

(b) d.f.

$$F_Y(y) = \begin{cases} 0, & x < 0 \\ 1 - e^{-1}, & 0 \leq x < 1 \\ 1 - e^{-2}, & 1 \leq x < 2 \\ \vdots & \\ 1 - e^{-k} & k-1 \leq x < k \\ 1 - e^{-(k+1)} & k \leq x < k+1 \\ \vdots & \end{cases}$$

jump of magnitude $e^{-k} - e^{-(k+1)}$ at k for $k = 0, 1, 2, \ldots$

(5)

Deduct marks for wrong range

(c) $E(y) = \sum_{0}^{\infty} k\left(e^{-k} - e^{-(k+1)}\right)$

$$= \sum_{k=1}^{\infty} e^{-k} = \left(\frac{1}{e-1}\right) \quad — (1)$$

$$E(y^2) = \sum_{0}^{\infty} k^2\left(e^{-k} - e^{-(k+1)}\right)$$

$$= 1e^{-1} + 3e^{-2} + 5e^{-3}$$

$$= \sum_{1}^{\infty} (2k-1)\, e^{-k} = 2\sum_{1}^{\infty} k e^{-k} - \sum_{1}^{\infty} e^{-k}$$

$$= \frac{2e}{(e-1)^2} - \frac{1}{e-1} = \frac{e+1}{(e-1)^2} \quad — (2)$$

$$V(y) = \frac{e+1}{(e-1)^2} - \frac{1}{(e-1)^2} = \frac{e}{(e-1)^2} \quad — ([illegible]) - (2).$$

$$\Rightarrow \quad V(y) = e\left(E(y)\right)^2$$

Given expression is thus wrong - disprove

(d) $Z = X - [X]$

$$F_Z(c) = P(X - [X] \le c)$$

If $c < 0$, then $F_Z(c) = 0$

If $0 < c < 1$; $F_Z(c) = P(X - [X] \le c)$

$$= \sum_{k=0}^{\infty} P(k \le X \le k + c)$$

$$= \sum_{k=0}^{\infty} \left(F_X(k+c) - F_X(k)\right)$$

$$= \sum_{k=0}^{\infty} \left(e^{-k} - e^{-(k+c)}\right)$$

$$= \sum_{k=0}^{\infty} e^{-k}\left(1 - e^{-c}\right) = \frac{1-e^{-c}}{1-e^{-1}}.$$

i.e. $F_Z(c) = \begin{cases} 0, & c < 0 \\ \dfrac{1-e^{-c}}{1-e^{-1}}, & 0 \le c \le 1 \\ 1, & c > 1 \end{cases}$ — (5).

(Z is a cont r.v.; $F_Z(.)$ does not have any point of jump discontinuity.)

p.d.f. $f_Z(z) = \begin{cases} \dfrac{e^{-z}}{1-e^{-1}}, & 0 < z < 1 \\ 0, & o/w \end{cases}$

(e) Median of Z be m

$$F_Z(m) = \frac{1}{2}$$

i.e. $\dfrac{1-e^{-m}}{1-e^{-1}} = \dfrac{1}{2}$ — (1)

$$2 - 2e^{-m} = 1 - e^{-1}$$

$$1 + e^{-1} = 2e^{-m}$$

$$e^{-m} = \tfrac{1}{2}(1+e^{-1})$$

median is : $m = -\log\left(\tfrac{1}{2}(1+e^{-1})\right)$ — (2).

4

(a) $f_X(k+1) = \frac{3}{k+1} f_X(k) \qquad k = 0, 1, 2, \ldots$

$k=0 \qquad f_X(1) = \frac{3}{1} f_X(0)$

$k=1 \qquad f_X(2) = \frac{3}{2} f_X(1) = \frac{3^2}{2\times 1} f_X(0)$

$k=2 \qquad f_X(3) = \frac{3}{3} f_X(2) = \frac{3^3}{3\times 2\times 1} f_X(0)$

$\vdots$

$k=j-1 \qquad f_X(j) = \frac{3^j}{j!} f_X(0)$

$\vdots$

$$\sum_{j=0}^{\infty} f_X(j) = 1$$

$= 0$

$\Rightarrow \quad f_X(0) + f_X(1) + \ldots$

$f_X(0) + \frac{3}{1} f_X(0) + \frac{3^2}{2!} f_X(0) + \frac{3^3}{3!} f_X(0) + \ldots = 0$

$f_X(0) \sum_{j=0}^{\infty} \frac{3^j}{j!} = 1 \quad \Rightarrow f_X(0) = e^{-3} \quad — (2)$

$\Rightarrow f_X(j) = \frac{e^{-3}\, 3^j}{j!} \; ; \; j = 1, 2, \ldots \quad — (2)$

$\Rightarrow$ p.m.f is $P(X = x) = \frac{e^{-3}\, 3^x}{x!} \; ; \; x = 0, 1, \ldots$

$X \sim P(3)$

$E(e^X) = M_X(1) = e^{-3} \sum_{x=0}^{\infty} \frac{(3e)^x}{x!} = e^{-3} e^{3e} = e^{3(e-1)}$

(2)

(4)

(b) $f_X(x) = \begin{cases} 1/3, & -1 < x < 0 \text{ or } 0 < x < 2 \\ 0, & \text{o/w} \end{cases}$

$Y = \dfrac{1}{|X|^{1/2}}$. $\mathcal{Y} = \left(\dfrac{1}{\sqrt{2}}, \infty\right)$

$\mathcal{X}_1 = (-1, 0)$	$\mathcal{X}_2 = (0, 1)$	$\mathcal{X}_3 = (1, 2)$
$\mathcal{Y} = (1, \infty)$	$\mathcal{Y} = (1, \infty)$	$\mathcal{Y} = \left(\frac{1}{\sqrt{2}}, 1\right)$
$y^2 = -\frac{1}{x}$	$y^2 = \frac{1}{x}$	$y^2 = \frac{1}{x}$
$x = -\frac{1}{y^2}$	$x = \frac{1}{y^2}$	$x = \frac{1}{y^2}$
$\frac{dx}{dy} = \frac{2}{y^3}$	$\frac{dx}{dy} = -\frac{2}{y^3}$	$\frac{dx}{dy} = -\frac{2}{y^3}$

If $y \in (1, \infty)$,

p.d.f. is $f_Y(y) = \dfrac{1}{3}\,\dfrac{2}{y^3} + \dfrac{1}{3}\,\dfrac{2}{y^3} = \dfrac{4}{3} y^{-3}$ (3)

If $y \in \left(\frac{1}{\sqrt{2}}, 1\right)$,

p.d.f. is $f_Y(y) = \dfrac{1}{3}\,\dfrac{2}{y^3} = \dfrac{2}{3} y^{-3}$ (3)

i.e. $f_Y(y) = \begin{cases} \frac{2}{3} y^{-3}, & \frac{1}{\sqrt{2}} < y < 1 \\ \frac{4}{3} y^{-3}, & 1 < y < 2 \\ 0, & \text{o/w.} \end{cases}$

(5) $\mathscr{X} = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}$

w.p. $\frac{1}{4}$ for each of the 4 p.ts.

(a) Marginal jt p.m.f. of X & Z is

$$P(X=1, Z=1) = \frac{1}{4}$$
$$P(X=1, Z=-1) = \frac{1}{4}$$
$$P(X=-1, Z=-1) = \frac{1}{4}$$
$$P(X=-1, Z=1) = \frac{1}{4}$$

Marginal of X is

$$P(X=1) = \frac{1}{2} = P(X=-1)$$

Sly marginal of Z is

$$P(Z=1) = \frac{1}{2} = P(Z=-1)$$

(3)

$\Rightarrow P(X=x, Z=z) = P(X=x)\, P(Z=z)$; $x, z = \pm 1$ i.e. $\forall (x,z)$

$\Rightarrow$ X & Z are indep

(b) $P(X=1, Y=1, Z=1) = \frac{1}{4}$

$P(X=1) = \frac{1}{2} = P(Z=1)$ as in (a)

$P(Y=1) = \frac{1}{2}$ also

$\Rightarrow P(X=1)\,P(Z=1)\,P(Y=1) = \frac{1}{8} \neq P(X=1, Y=1, Z=1)$.

$\Rightarrow$ X, Y, Z are not indep. (3)

(c). $P(X=1, \cancel{Y} Z=1 \mid Y=1) = \dfrac{1/4}{1/2} = \dfrac{1}{2}$

$P(X=1 \mid Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{1}{2}$

Sly $P(Z=1 \mid Y=1) = \frac{1}{2}$

$\Rightarrow P(X=1, Z=1 \mid Y=1) \neq P(X=1 \mid Y=1)\, P(Z=1 \mid Y=1)$ (4)

i.e. $P(X = x, Z = z \mid Y = y) = P(X = x \mid Y = y)\, P(Z = z \mid Y = y)$.

does not hold $\forall\, \underset{\sim}{x} \in \mathcal{X}$.