Uniform cvgs. "preserve" Riemann integration:

2 $f_n : [a,b] \to \mathbb{R}$ cts. and $f_n \to f$ unif.

Then $\int_a^b f_n \to \int_a^b f$. That is, $\int_a^b f = \int_a^b \lim f_n = \lim \int_a^b f_n$.

Idea: Since $\|f_n - f\|_\infty \to 0$, $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq N_\varepsilon$ .....

Then $\left| \int_a^b f_n - f \right| \leq \|f_n - f\|_\infty (b-a)$

Uniform convergence "almost" preserve differentiation (i.e, the limit function is differentiable

3 $f_n : [a,b] \to \mathbb{R}$ diff.

Suppose $\exists$ $x_0 \in [a,b]$ s.t. $(f_n(x_0))$ cvgs.

If $f_n' \to g$ uniformly on $[a,b]$,

then, $f_n \to f$ unif. for some f,

and $f_n' \to f'$.

Pf. (Rudin: Thm. 7.17)

- Use Cauchy Criterion for uniform cvgs. (f_n) is unif. Cauchy, if for $\varepsilon > 0$, $\exists$ $N_\varepsilon(?) \in \mathbb{N}$ s.t. $\forall x \in [a,b]$, $|f_n(x) - f_m(x)| < \varepsilon$.

- $(f_n(x_0))$ Cauchy: for that $\varepsilon > 0$ (first bullet),

  $\exists$ $N_\varepsilon^{(1)}$ s.t. $\forall$ $n, m \geq N_\varepsilon^{(1)}$, $|f_n(x_0) - f_m(x_0)| < \varepsilon$.

- Since (f_n') cvgs. uniformly, (f_n') is uniformly Cauchy,

  $\exists$ $N_\varepsilon^{(2)}$ s.t. $\forall$ $n, m \geq N_\varepsilon^{(2)}$, $\forall x \in [a,b]$, $|f_n'(x) - f_m'(x)| < \varepsilon' := \frac{\varepsilon}{(b-a)}$.

Choose $N_\varepsilon := \min\{N_\varepsilon^{(1)}, N_\varepsilon^{(2)}\}$.

- For $\varepsilon > 0$ & $n,m \geq N_\varepsilon$, for $x \in [a,b]$, consider

$$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)|$$
$$\leq |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |f_n(x_0) - f_m(x_0)|$$

Apply MVT for $f_n - f_m$ on $[a,b]$:

$$|(f_n - f_m)(x) - (f_n - f_m)(x_0)| = |(f_n' - f_m')(t)|(x - x_0)$$
$$< \frac{\varepsilon}{(b-a)} \cdot (x - x_0) = \varepsilon \frac{(x - x_0)}{(b-a)} \leq \varepsilon$$

Hence, $|(f_n - f_m)(x) - (f_n - f_m)(x_0)| < \varepsilon$

Therefore, $|f_n(x) - f_m(x)| < 2\varepsilon \quad \forall x \in [a,b], \forall n,m \geq N_\varepsilon$.

$\Rightarrow (f_n)$ uniformly Cauchy $\Rightarrow (f_n)$ cvgs. uniformly.

Let $f(x) := \lim_{n\to\infty} f_n(x)$ for $x \in [a,b]$.

- WTS: f is differentiable on $[a,b]$ and $f_n' \to f'$ uniformly.

Let $x \in [a,b]$ (fix). WTS: $f'(x)$ exists. That is,

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

Since f_n is diff. $\underbrace{\lim_{t\to x} \frac{f_n(t) - f_n(x)}{t - x} \text{ exists}}$, for each $n \geq 1$.

$$\lim_{t\to x} \frac{f_n(t) - f_n(x)}{t - x} = f_n'(x)$$

Define $\left.\begin{aligned} \phi(t) &:= \frac{f(t)-f(x)}{t-x} \\ \phi_n(t) &:= \frac{f_n(t)-f_n(x)}{t-x} \end{aligned}\right\}$ for $t \neq x$.

WTS: (ϕ_n) cvgs. uniformly (Use Cauchy Criterion)

$$|\phi_n(t)-\phi_m(t)| = \frac{|f_n(t)-f_n(x)-f_m(t)+f_m(x)|}{|t-x|}$$

Apply MVT to $f_n - f_m$ on $[x,t]$:

$$\frac{|f_n(t)-f_n(x)-f_m(t)+f_m(x)|}{|t-x|} = \frac{|(f_n'-f_m')(c)|\,(t-x)}{(t-x)}$$

$$= |(f_n'-f_m')(c)|$$

$$< \frac{\varepsilon}{b-a}.$$

So, $|\phi_n(t)-\phi_m(t)| < \frac{\varepsilon}{b-a} \quad \forall\ t \neq x.$

Hence (ϕ_n) unif. converges on $[a,b] \setminus \{x\}$.

Moreover, since $f_n(t) \to f(t)$ ptwise for each $t \in [a,b]$,

$$\phi_n(t) \to \frac{f(t)-f(x)}{t-x} \ (= \phi(t)) \text{ as } n \to \infty.$$

In fact, $\phi_n \to \phi$ uniformly on $[a,b] \setminus \{x\}$ as $f_n \to f$ unif. on $[a,b]$.

Using Remark(ii) from Lecture-20, one has

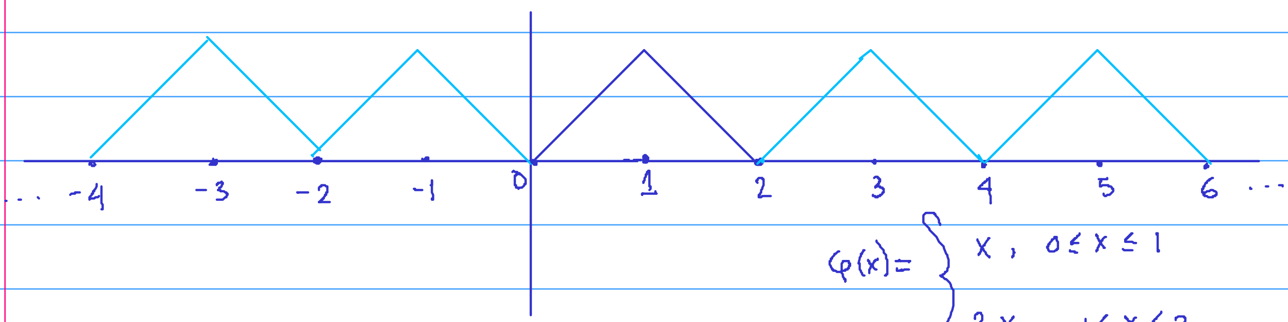
$$\lim_{t\to x} \phi(t) \overset{\phi_n\to\phi}{\underset{\text{ptwise.}}{=}} \lim_{t\to x}\lim_{n\to\infty} \phi_n(t) \overset{\phi_n\to\phi}{\underset{\text{unif.}}{=}} \lim_{n\to\infty}\lim_{t\to x} \phi_n(t)$$

$$= \lim_{n\to\infty} f_n'(x) = g(x) \text{ which exists.}$$

Therefore, $f'(x) := \lim_{t\to x} \phi(t)$ exists. HW: Why $f_n' \to f'$ uniformly?

HW: If in (3) one further assumes that the derivative of f_n as a function on $[a,b]$ is cts., i.e., $f_n' : [a,b] \to \mathbb{R}$ is cts., then one obtains an easier proof to the conclusion in (3): Read Carothers Thm. 10.7.

→ Example of a cts. function which is nowhere differentiable:



$$\varphi(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$

Extend φ to $\mathbb{R}$ as $\varphi(x+2) = \varphi(x)$.

→ $\varphi : \mathbb{R} \to \mathbb{R}$ is cts.

- $0 \le \varphi(x) \le 1, \ \forall\, x \in \mathbb{R}$
- $\varphi(2n) = 0, \ n \in \mathbb{Z}$
- $\varphi(2n+1) = 1, \ n \in \mathbb{Z}$
- For $0 \le x \le 1$, φ is increasing and $1 \le x \le 2$ φ is decreasing

Consider "magic function" $f(x) := \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$.

HW: Show that $\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \varphi(4^n x)$ cvgs. uniformly.

- f is cts. on $\mathbb{R}$.

Claim: f is not diff. anywhere on $\mathbb{R}$.

Pf. (Rudin - Thm. 5.19): If f is differentiable at x, then for $\forall\, \alpha_m, \beta_m$ s.t. $\alpha_m < x < \beta_m$ with $\alpha_m \to x$, $\beta_m \to x$, one has

$$\frac{f(\beta_m) - f(\alpha_m)}{\beta_m - \alpha_m} \to f'(x)$$

To show that f is not diff. at x, suffices to construct (α_m), (β_m) s.t.

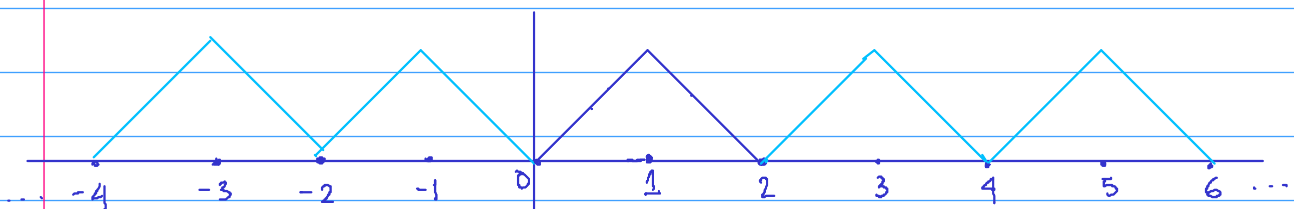
$\alpha_m < x < \beta_m$ with $\alpha_m \to x$, $\beta_m \to x$, BUT

$$\frac{f(\beta_m) - f(\alpha_m)}{\beta_m - \alpha_m} \to \infty.$$

Fix $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Since $\bigcup_{k \in \mathbb{Z}} [k, k+1] = \mathbb{R}$, $\exists$ $\underset{\substack{\uparrow \\ k(x,m)}}{k \in \mathbb{Z}}$ s.t.

$$k \le 4^m x \le k+1$$

$$\Rightarrow \quad 4^{-m} k \le x \le 4^{-m}(k+1)$$



$$\varphi(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$

Recall $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$.

Hence, $\quad 4^{n-m} k \le 4^n x \le 4^{n-m}(k+1)$

- $n > m$, $\quad \varphi(4^{n-m} k) = 0 = \varphi(4^{n-m}(k+1)) \Rightarrow \left|\varphi(4^{n-m}(k+1)) - \varphi(4^{n-m} k)\right| = 0$
- $n = m$, $\quad \left|\varphi(k+1) - \varphi(k)\right| = \underline{1}$
- $n < m$, $\quad 4^{n-m} k,\ 4^{n-m}(k+1) \notin \mathbb{Z}$. Since $\varphi(x) = \varphi(x+2)$ where $\varphi(x) = x$ or $\varphi(x) = 2-x$

so, $\left|\varphi(4^{n-m}(k+1)) - \varphi(4^{n-m} k)\right| = 4^{n-m}$.

$$\left|\varphi(4^{n-m}(k+1)) - \varphi(4^{n-m} k)\right| = \begin{cases} 0, & n > m \\ 4^{n-m}, & n \le m. \end{cases}$$

Let $\alpha_m := 4^{-m} k$ and $\beta_m := 4^{-m}(k+1)$. Note that $\alpha_m \leq x \leq \beta_m$ and $\beta_m - \alpha_m \to 0$ as $m \to \infty$.

$$f(\beta_m) - f(\alpha_m) = \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \left[\varphi(4^n \beta_m) - \varphi(4^n \alpha_m)\right]$$

$$= \left(\frac{3}{4}\right)^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n 4^{n-m}.$$

$$|f(\beta_m) - f(\alpha_m)| \overset{?}{\underset{HW}{>}} \frac{1}{2}\left(\frac{3}{4}\right)^m$$

Hence, $$\frac{|f(\beta_m) - f(\alpha_m)|}{(\beta_m - \alpha_m)} > \frac{1}{2} \cdot 3^m \longrightarrow \infty \quad \text{as } m \to \infty.$$