Frequency Domain Analysis

Aim: To study the frequency (corresponding to periodic component) properties of time series and identify dominant frequencies that drive the time series

Tool: Spectral density function

Def^n: Spectral density

Suppose that $\{X_t\}$ is a stationary zero mean time series with autocovariance f^n $\gamma(\cdot)$ satisfying $\sum_h |\gamma(h)| < \infty$. The spectral density of $\{X_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) ; \quad -\infty < \lambda < \infty$$

Important properties

(1) Alternate form

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \left(\cos \lambda h - i \sin \lambda h \right)$$

$$= \frac{1}{2\pi} \gamma(0) \left(\cos 0 - i \sin 0 \right) + \frac{1}{2\pi} \left(\sum_{h=1}^{\infty} \gamma(h) \left(\cos \lambda h + \cos(-\lambda h) - i \cancel{\sin(\lambda h)} - i \cancel{\sin(-\lambda h)} \right) \right)$$

i.e. $$f(\lambda) = \frac{1}{2\pi} \left(\gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos \lambda h \right)$$

(2) $f(\cdot)$ is even

$$f(\lambda) = \frac{1}{2\pi} \left(\gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos \lambda h \right)$$

$$= \frac{1}{2\pi} \left(\gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos(-\lambda h) \right)$$

$$= f(-\lambda) \quad \forall \lambda$$

(3) $f(\lambda)$ is periodic with period 2π

$$f(\lambda) = \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos\lambda h\right)$$

Note that $\cos((\lambda+2\pi k)h) = \cos\lambda h$ for any integer k & h

$$f(\lambda+2\pi k) = \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos\left((\lambda+2\pi k)h\right)\right)$$

$$= \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos\lambda h\right)$$

$$= f(\lambda) \quad \text{for any integer } k$$

Hence, it suffices to confine attention to the values of f on the interval $-\pi$ to π as if we know the values of $f(\lambda)$ $\forall \lambda \in [-\pi, \pi]$, we can infer the values of $f(\lambda)$ for any λ.

(4) $f(\lambda) \geq 0 \quad \forall \lambda$

For any positive integer N, define

$$f_N(\lambda) = \frac{1}{2\pi N} E\left(\left|\sum_{r=1}^{N} X_r e^{-ir\lambda}\right|^2\right)$$

Note that $f_N(\lambda) \geq 0 \quad \forall N$

$$f_N(\lambda) = \frac{1}{2\pi N} E\left(\left|\sum_{r=1}^{N} X_r e^{-ir\lambda}\right|^2\right)$$

$$= \frac{1}{2\pi N} E\left(\sum_{r=1}^{N} X_r e^{-ir\lambda}\right)\left(\sum_{s=1}^{N} X_s e^{is\lambda}\right)$$

$$= \frac{1}{2\pi N} E\left(X_1 e^{-i\lambda} + X_2 e^{-2i\lambda} + \cdots + X_N e^{-Ni\lambda}\right)$$

$$\left(X_1 e^{i\lambda} + X_2 e^{2i\lambda} + \cdots + X_N e^{Ni\lambda}\right)$$

$$f_N(\lambda) = \frac{1}{2\pi N} \sum_{|h|<N} (N-|h|)\, e^{-ih\lambda}\, \gamma(h)$$

[You may recall a result from Lecture #

$$N V(\bar{X}_N) = \frac{1}{N} \sum_{|h|<N} (N-|h|)\, \gamma(h) \quad \text{with } \sum|\gamma(h)| < \infty$$

$$\longrightarrow \sum_{h=-\infty}^{\infty} \gamma(h) \quad \text{as } N \to \infty$$]

Proceeding exactly along the same lines of the proof of this result, it follows that

$$f_N(\lambda) \longrightarrow \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = f(\lambda)$$

as $N \to \infty$

Thus, $f_N(\lambda) \geq 0 \quad \forall N \quad \forall \lambda$

and $f_N(\lambda) \to f(\lambda)$ as $N \to \infty$

$$\Rightarrow f(\lambda) \geq 0 \quad \forall \lambda$$

(5) Inversion formula

Let $\{\gamma(h)\}$ be an absolutely summable sequence of auto covariances associated with a covariance stationary process and let the spectral density of the process $\{X_t\}$ be

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h).$$

Then

$$\gamma(k) = \int_{-\pi}^{\pi} f(\lambda) e^{i\lambda k} \, d\lambda$$

Consider,

$$\int_{-\pi}^{\pi} e^{i\lambda k} f(\lambda) \, d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\lambda h} \right) e^{i\lambda k} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{i\lambda(k-h)} \, d\lambda$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{i\lambda(k-h)} \, d\lambda \quad - (*)$$

Note that

$$\int_{-\pi}^{\pi} e^{i\lambda(k-h)} \, d\lambda$$

$$= \int_{-\pi}^{\pi} \left(\cos(\lambda(k-h)) + i \sin(\lambda(k-h)) \right) d\lambda$$

$$= \begin{cases} 2\pi, & \text{if } h = k \\ 0, & \text{o/w (i.e. } h \neq k) \end{cases}$$

$$\Rightarrow (*) = \gamma(k)$$

Characterization of spectral density function

A real valued function $f(\cdot)$ defined on $[-\pi, \pi]$ is the spectral density of a stationary process with absolutely summable auto covariance function iff

(i) $f(\lambda) = f(-\lambda) \quad \forall \lambda$

(ii) $f(\lambda) \geq 0 \quad \forall \lambda$

& (iii) $\int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$

Alternate characterization of ACVF

An absolutely summable f^n $\gamma(\cdot)$ defined on the set of integers is the ACVF of a stationary process iff it is

(i) even and

(ii) $$f(\lambda) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0 \quad \forall \lambda \in [-\pi, \pi]$$

in which case $f(\lambda)$ is the corresponding spectral density function.

Spectral density of standard models

(I) White noise process

Let $X_t \sim WN(0, \sigma^2)$

$$\gamma(h) = \begin{cases} \sigma^2, & h=0 \\ 0, & o/w \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\lambda h} = \frac{1}{2\pi} \gamma(0) = \frac{\sigma^2}{2\pi} \quad \forall \lambda$$

Alternatively, given $f(\lambda) = \frac{\sigma^2}{2\pi}$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} e^{ih\lambda}\, d\lambda$$

$$= \frac{\sigma^2}{2\pi} \begin{cases} 2\pi, & \text{if } h=0 \\ 0, & o/w \end{cases}$$

i.e. $\gamma(h) = \begin{cases} \sigma^2, & h=0 \\ 0, & o/w. \end{cases}$

II AR(1) process

$$X_t = \phi X_{t-1} + \epsilon_t; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\gamma(h) = \frac{\sigma^2}{1-\phi^2} \phi^{|h|}$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \left(\frac{\sigma^2}{1-\phi^2} \phi^{|h|} \right)$$

$$= \frac{1}{2\pi} \frac{\sigma^2}{1-\phi^2} \left(1 + \sum_{h=1}^{\infty} \phi^h e^{-ih\lambda} + \sum_{h=1}^{\infty} \phi^h e^{ih\lambda} \right)$$

$$= \frac{\sigma^2}{2\pi(1-\phi^2)} \left(1 + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}} \right)$$

i.e. $f(\lambda) = \frac{\sigma^2}{2\pi(1-\phi^2)}\left(\frac{1+\cancel{\phi^2}-\cancel{\phi e^{-i\lambda}}-\cancel{\phi e^{i\lambda}}+\cancel{\phi e^{-i\lambda}}-\cancel{\phi^2}+\cancel{\phi e^{i\lambda}}-\phi^2}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})}\right)$

$$= \frac{\sigma^2}{2\pi\cancel{(1-\phi^2)}} \cdot \frac{\cancel{(1-\phi^2)}}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})}$$

i.e. $f(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})} = \frac{\sigma^2}{2\pi} \frac{1}{(1-2\phi\cos\lambda+\phi^2)}$.

$$f(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})}$$

iii) MA(1) process

$$X_t = \epsilon_t + \theta\epsilon_{t-1}\ ;\ \epsilon_t \sim WN(0,\sigma^2)$$

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2), & h=0 \\ \theta\sigma^2, & h=\pm 1 \\ 0, & o/w \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h)$$

$$= \frac{1}{2\pi}\left(\sigma^2(1+\theta^2) + \sigma^2\theta e^{-i\lambda} + \sigma^2\theta e^{i\lambda}\right)$$

$$= \frac{\sigma^2}{2\pi}\left(1+\theta^2+2\theta\cos\lambda\right)$$

$$= \frac{\sigma^2}{2\pi}(1+\theta e^{-i\lambda})(1+\theta e^{i\lambda}) = \frac{\sigma^2}{2\pi}\theta(e^{-i\lambda})\,\theta(e^{i\lambda})$$

Remark : Note that

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}(e^{-i\lambda})^h\gamma(h) = \boxed{\frac{1}{2\pi} g_X(e^{-i\lambda})}$$

where $g_X(\cdot)$ is ACVF of $\{X_t\}$

$f(.)$ of WN, AR(1) and MA(1) are accordingly in term of $g_X(\cdot)$

We can use the above connection between spectral density function and ACGF to obtain spectral densities of standard model for which we have earlier desired ACGF.

<u>Spectral density f^{λ} of AR(p)</u>

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t \; ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\cancel{X_t} \Rightarrow \phi(B) X_t = \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

ACGF $\quad g_X(z) = \dfrac{\sigma^2}{\phi(z)\,\phi(z^{-1})}$

Spectral density

$$f_X(\lambda) = \frac{1}{2\pi}\, g_X(e^{-i\lambda})$$

i.e. $f_X(\lambda) = \dfrac{\sigma^2}{2\pi} \dfrac{1}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})}$

e.g. AR(1) $\quad X_t = \phi X_{t-1} + \epsilon_t$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{(1 - \phi e^{-i\lambda})(1 - \phi e^{i\lambda})}$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{1 - \phi e^{-i\lambda} - \phi e^{i\lambda} + \phi^2}$$

Note that the spectral density f^n of AR(p) can be expressed in terms of roots of AR polynomial.

Suppose $n_1, \ldots, n_p$ are the roots of

$$n^p - \phi_1 n^{p-1} - \cdots - \phi_p = 0$$

then $$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left[\prod_{j=1}^{p}(e^{i\lambda} - n_j)(e^{-i\lambda} - n_j)\right]^{-1}$$

<u>Spectral density of MA (q)</u>

$$Y_t = \sum_{j=0}^{q} \theta_j \epsilon_{t-j} \quad ; \; \epsilon_t \sim WN(0, \sigma^2)$$

<u>ACGF</u>

$$g_Y(z) = \sigma^2 \theta(z)\theta(z^{-1}) \quad ; \; \theta(B) = \theta_0 + \theta_1 B + \cdots + \theta_q B^q$$

$\underline{\theta_0 = 1}$

Spectral density f^n

$$f_Y(\lambda) = \frac{\sigma^2}{2\pi} \theta(e^{-i\lambda})\,\theta(e^{i\lambda})$$

$$= \frac{\sigma^2}{2\pi}\left(\sum_{j=0}^{q}\theta_j e^{-ij\lambda}\right)\left(\sum_{j=0}^{q}\theta_j e^{ij\lambda}\right)$$

if $m_1, m_2, \ldots, m_q$ are the roots of

$$m^q + \theta_1 m^{q-1} + \cdots + \theta_q = 0$$

then $$f_Y(\lambda) = \frac{\sigma^2}{2\pi}\left(\prod_{j=1}^{q}(e^{-i\lambda} - m_j)(e^{i\lambda} - m_j)\right)$$

Spectral density of ARMA(p, q)

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$\phi(B) X_t = \theta(B) \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

ACGF: $g_X(z) = \sigma^2 \dfrac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}$.

Spectral density f^n:

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{\theta(e^{-i\lambda})\,\theta(e^{i\lambda})}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})}.$$

If $n_1, \ldots, n_p$ are the roots of

$$n^p - \phi_1 n^{p-1} - \cdots - \phi_p = 0$$

& $m_1, \ldots, m_q$ are the roots of

$$m^q + \theta_1 m^{q-1} + \cdots + \theta_q = 0$$

then

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left(\prod_{j=1}^{q} (e^{i\lambda} - m_j)(e^{-i\lambda} - m_j)\right)}{\left(\prod_{j=1}^{q} (e^{i\lambda} - n_j)(e^{-i\lambda} - n_j)\right)}.$$