AR(p) to MA(∞)

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

$$\phi(B) X_t = \epsilon_t$$

$$X_t = \phi(B)^{-1} \epsilon_t$$

Let $\theta_1, \ldots, \theta_p$ be the roots of $\phi(z) = 0$

For $\{X_t\}$ covariance stationary $|\theta_i| > 1 \quad \forall\, i = 1(1)p$

$$\phi(B) = (1 - \theta_1^{-1} B)(1 - \theta_2^{-1} B) \cdots (1 - \theta_p^{-1} B)$$

$$\text{or} = (1 - \lambda_1 B)(1 - \lambda_2 B) \cdots (1 - \lambda_p B)$$

$$|\lambda_i| < 1 \quad \forall\, i = 1(1)p.$$

$\Rightarrow (1 - \lambda_i B)^{-1}$ exists and

$$(1 - \lambda_i B)^{-1} = \sum_{j=0}^{\infty} \lambda_i^j B^j$$

Now, using partial fraction approach

$$\phi(B)^{-1} = \frac{1}{\phi(B)} = \frac{c_1}{1 - \lambda_1 B} + \cdots + \frac{c_p}{1 - \lambda_p B}$$ for some suitable consts $c_1, \ldots, c_p$

$$= c_1 \sum_{j=0}^{\infty} \lambda_1^j B^j + \cdots + c_p \sum_{j=0}^{\infty} \lambda_p^j B^j$$

$$= \sum_{i=1}^{p} c_i \sum_{j=0}^{\infty} \lambda_i^j B^j$$

$$\Rightarrow X_t = \phi(B)^{-1} \epsilon_t = \left(\sum_{i=1}^{p} c_i \sum_{j=0}^{\infty} \lambda_i^j B^j \right) \epsilon_t$$

$$= \sum_{j=0}^{\infty} \underbrace{\left(\sum_{i=1}^{p} c_i \lambda_i^j \right)}_{\psi_j} \epsilon_{t-j} = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \leftarrow \text{MA}(\infty)$$

with $\psi_j = \sum_{i=1}^{p} c_i \lambda_i^j = \sum_{i=1}^{p} c_i \theta_i^{-j}$

Note: Method of comparing coefficients can be used to find the seq $\psi_0, \psi_1, \psi_2, \dots$ and hence the MA(∞) representation

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

Remark

Causal AR process: An AR(p) process

$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \epsilon_t$ is said to be causal if it can be expressed in terms of a white noise sequence in MA(∞) form $X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ for appropriate constants.

Thus an AR(p) is causal if roots of $\phi(z) = 0$ all lie outside unit circle (i.e. if is covariance stationary).

Invertible representations are thus causal representations of stationary AR processes.

Invertibility of MA processes

MA(1) $\quad X_t = \epsilon_t + \theta \epsilon_{t-1}; \quad \epsilon_t \sim WN(0, \sigma^2)$

always covariance stationary $\forall \theta$

$$X_t = \theta(B)\,\epsilon_t$$

$$\theta(B) = 1 + \theta B$$

Suppose now that $|\theta| < 1$, then $(1+\theta B)^{-1}$ exists and is given by

$$(1+\theta B)^{-1} = (1 - (-\theta)B)^{-1}$$
$$= 1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots$$

$\Rightarrow \quad X_t = \theta(B)\,\epsilon_t$ can be expressed as

$$\epsilon_t = \theta(B)^{-1} X_t$$

i.e. $\quad \epsilon_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \cdots$

i.e. $\quad \epsilon_t = \sum_{i=0}^{\infty} (-\theta)^i X_{t-i} \quad \leftarrow AR(\infty)$ form

i.e. $\quad X_t = -\sum_{i=1}^{\infty} (-\theta)^i X_{t-i} + \epsilon_t$

Note that the mean square sense convergence interpretation is also valid for this MA setup.

$$X_t = \epsilon_t + \theta \epsilon_{t-1}$$
$$= \epsilon_t + \theta(X_{t-1} - \theta \epsilon_{t-2})$$
$$= \epsilon_t + \theta X_{t-1} - \theta^2 \epsilon_{t-2}$$

i.e. $X_t = \theta X_{t-1} - \theta^2 \epsilon_{t-2} + \epsilon_t$

$$= \theta X_{t-1} - \theta^2 (X_{t-2} - \theta \epsilon_{t-3}) + \epsilon_t$$

i.e. $X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 \epsilon_{t-3} + \epsilon_t$

$\vdots$ continuing K times substitution

$$X_t = -\sum_{i=1}^{K} (-\theta)^i X_{t-i} - (-\theta)^{K+1} \epsilon_{t-(K+1)} + \epsilon_t$$

$$\Rightarrow E\left(X_t - \epsilon_t + \sum_{i=1}^{K} (-\theta)^i X_{t-i}\right)^2 = E\left((-\theta)^{K+1} \epsilon_{t-K-1}\right)^2$$

Since $E(\epsilon_t^2) < \infty \ \forall t$ & $|\theta| < 1$

$$\lim_{K \to \infty} E\left(X_t - \epsilon_t + \sum_{i=1}^{K} (-\theta)^i X_{t-i}\right)^2 = 0$$

$$\Rightarrow \quad X_t \overset{m.s.}{=\!=} \epsilon_t - \sum_{i=1}^{\infty} (-\theta)^i X_{t-i}$$

Note: (unlike AR(1)) covariance stationary MA(1) is not necessarily invertible.

MA(q)

$$X_t = (1+\theta_1 B + \cdots + \theta_q B^q)\,\epsilon_t$$

$$X_t = \theta(B)\,\epsilon_t$$

Let $\theta(B) = (1-\lambda_1 B)\cdots(1-\lambda_q B)$

If $|\lambda_i| < 1 \ \forall i$ then roots of $\theta(z)=0$ all lie outside the unit circle and each of $(1-\lambda_i B)$ is invertible and $\theta(B)^{-1}$ exists and

$$\theta(B)^{-1} = (1-\lambda_1 B)^{-1}\cdots(1-\lambda_q B)^{-1}$$

with $(1-\lambda_i B)^{-1} = \sum_{j=0}^{\infty} \lambda_i^j B^j \quad \forall i = 1(1)q$

We can either use partial fraction approach or method of comparing coefficients to find the AR(∞) representation of the invertible MA(q).

e.g.

$$X_t = \theta(B)\,\epsilon_t$$

$$\Rightarrow \theta(B)^{-1} X_t = \epsilon_t$$

i.e. $\epsilon_t = \psi(B)\, X_t$ say

$$\psi(B) = \psi_0 + \psi_1 B + \cdots$$

$$\Rightarrow \psi(B) = \theta(B)^{-1}$$

or $\theta(B)\psi(B) = 1$

i.e. $(1+\theta_1 B + \cdots + \theta_q B^q)(\psi_0 + \psi_1 B + \cdots) = 1$

Comparing coeffs of B^j from both the sides we can express ψ_j s in terms of θ_j s.