Assignment 1

MTH 301A, 2022

1. Show that following sets are countable by producing a bijection from $\mathbb{N}$.
   (a) $\{2^n : n \in \mathbb{Z}\}$
   (b) $\{1, \frac{1}{4}, \frac{1}{3}, \frac{1}{16}, \ldots\}$
   (c) $\{0, 4, 0, 8, 0, 12, 0, \ldots\}$
2. (a) Show that $\mathbb{N}$ contains two pairwise disjoint infinite subsets.
   (b) Show that $\mathbb{N}$ contains three pairwise disjoint infinite subsets.
   (c) Show that $\mathbb{N}$ contains 100 pairwise disjoint infinite subsets.
   (d) Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.
3. Show that following sets are countable
   (a) Set of all functions $f : \{0, 1\} \to \mathbb{N}$.
   (b) The set of all points in the plane with rational co-ordinates.
   (c) Set of all words (a finite string of letters in the alphabets).
   (d) Set of all finite subsets of $\mathbb{N}$.
4. If A is an infinite set then it has countably infinite subset.
5. Show that A is infinite if and only if there is a proper subset B of A such that $|A| = |B|$.
6. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

7. A real number $x \in \mathbb{R}$ is called **algebraic** if there exist integers $a_0, a_1, a_2, \ldots a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n11} x^{n11} + \cdots + a_1 x + a_0 = 0.$$

Real numbers that are not algebraic are called **transcendental** numbers.

   (a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

   (b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

   (c) What do you conclude about the set of transcendental numbers?

8. (a) Give an example of a countable collection of disjoint open intervals.
(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

9. Give an explicit one-to-one correspondence between
(a) the points of two open intervals;
(b) the points of two closed intervals;
(c) the points of a closed interval and the points of an open interval;
(d) the points of the closed interval $[0, 1]$ and the set $\mathbb{R}$.

10. On the closed interval $[0, 1] \subset \mathbb{R}$ describe the sets of numbers $x \in [0, 1]$ whose ternary representation $x = 0.\alpha_1\alpha_2\ldots, \alpha_i \in \{0, 1, 2\}$, has the property:
(a) $\alpha_1 \neq 1$
(b) $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$
(c)$\forall i \in \mathbb{N},\ \alpha_i \neq 1$ (the Cantor set).
(d) the Cantor set has the same cardinality as the closed interval $[0, 1]$.