Recall: (M,d): A is dense in M if $\bar{A} = M$.

Note that $\bar{A} = M \Rightarrow \bar{A}^{\circ} = M$.

Def^n: (M,d) metric space and $A \subset M$.
A is said to be nowhere dense if $\bar{A}^{\circ} = \phi$.

(HW) A is nowhere dense iff $(\bar{A})^c$ is dense in M.

Q. Can one write a metric space (M,d) as a "countable" union of nowhere dense sets?

(M,d): $A = \{x\}$ may or may not be a nowhere dense set. (Why?)

→ $\{A_1, .., A_n\}$ each A_i is nowhere dense then $\bigcup_{j=1}^{n} A_j$ is nowhere dense.

Q. $\bigcup_{j=1}^{\infty} A_j$ nowhere dense?

→ $(\mathbb{Q}, |\cdot|)$ $\{r_j\}$ is nowhere dense and $\mathbb{Q} = \bigcup_{j=1}^{\infty} \{r_j\}$

→ $(\mathbb{R}, |\cdot|)$ Is it a countable union of nowhere dense sets?

Thm: (The Baire's Theorem) (M,d) complete metric space
Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of nowhere dense sets.

Then $M \setminus \bigcup_{1}^{\infty} A_n \neq \phi$.

(In other words, M cannot be written as the countable union of nowhere dense sets.)

Pf: (*) Since M is open and A_1 is nowhere dense, there is an open ball B_1 of radius less than 1 s.t. $B_1 \cap A_1 = \phi$.
Let F_1 be the closed ball whose radius is one-half of B_1 and consider $\mathrm{int}(F_1)$.
Then $\exists$ open ball B_2 of radius less than $\frac{1}{2}$ s.t. $B_2 \cap A_2 = \phi$.
with $B_2 \subset \mathrm{int}(F_1)$ and $F_1 \subset B_1$

Let F_2 be the closed ball whose radius is one-half of B_2 and consider $\text{int}(F_2)$,
i.e., $F_2 \subset B_2$ s.t. radius of F_2 is one-half of B_2.

Then $\exists$ B_3 of radius less than $\frac{1}{2^2}$ s.t. $B_3 \cap A_3 = \phi$ where $B_3 \subset \text{int}(F_2)$

Continuing this way, one obtains a $(F_n)\downarrow$ non-empty closed sets with $\text{diam}(F_n) \to 0$.

Since M is complete, $\exists$ $x \in M$ s.t. $x \in F_n$ $\forall n \geq 1$. Note that $x \in B_n$ but $x \notin A_n$ $\forall n \geq 1$.

Hence $M \setminus \bigcup_{n=1}^{\infty} A_n \neq \phi$.

(*) (HW): A is nowhere dense iff each non-empty open set contains an open ball disjoint from A.

EQUIVALENTLY:

(The Baire's Thm.)

(M,d) complete metric space

$\{A_n\} \subset M$ arbitrary sets s.t. $M = \bigcup_{n=1}^{\infty} A_n$ then $\exists$ $n \in \mathbb{N}$ s.t. $\overset{\circ}{\overline{A_n}} \neq \phi$.

"Category":

Def^n: $A \subset M$ is said to be of first category in M if A can be written as a countable union of nowhere dense sets.

Def^n: $A \subset M$ is said to be of second category if whenever $A = \bigcup_1^{\infty} A_n$ then $\overset{\circ}{\overline{A_m}} \neq \phi$ for some $m \in \mathbb{N}$.

HW → The Baire "Category" Thm. for $(\mathbb{R}, |\cdot|)$: $\mathbb{R}$ is of the second category.

Equivalently, If (G_n) is a seq. of dense open sets in $\mathbb{R}$, then $\bigcap G_n \neq \phi$. Moreover, $\bigcap G_n$ is dense in $\mathbb{R}$.

- $\mathbb{Q}$ cannot be written as the countable intersection of open subsets of $\mathbb{R}$.
- If $\mathbb{R} = \bigcup_1^{\infty} E_n$ where E_n is closed, then $\exists$ $m \in \mathbb{N}$ s.t. E_m contains an open interval.
- If $\mathbb{R} \setminus \mathbb{Q} = \bigcup_1^{\infty} E_n$ then $\exists$ $m \in \mathbb{N}$ s.t. $\overline{E_m}$ contains an interval.

HW: Show that $\mathbb{R}$ is an uncountable set.

$\longrightarrow$ The Baire Category Thm. for (M,d):

A complete metric space is of the second category.

Equivalently, if (G_n) is a sequence of dense open sets in M, then $\bigcap G_n \neq \phi$. Moreover, $\bigcap G_n$ is dense in M.

Consequence:

In a complete metric space, the complement of every first category set is dense.

Q. Does $\exists (?)$ $f : \mathbb{R} \to \mathbb{R}$ which is not cts. only at irrational pts. ???

Q. Can we say something about the set of pts. of continuity of a cts. function ?

Recall: • finite union of closed sets is closed.

• finite intersection of open sets is open.

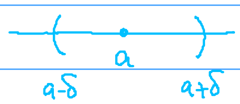
Def^n: F_σ-set : countable union of closed sets

G_δ-set : countable intersection of open sets

Thm: $f : \mathbb{R} \to \mathbb{R}$.

$D_f := \{ a \in \mathbb{R} \mid f \text{ is not cts. at } a \}$ is an F_σ-set.

Pf: For $a \in D_f$, $\exists\ \varepsilon > 0$ s.t. $\forall\ \delta > 0$, $|x-a| < \delta \Rightarrow |f(x) - f(a)| \geq \varepsilon$.



Let I be an open bdd. interval s.t. $a \in I$.

Then, $\exists\ \delta_0 > 0$ s.t. $(a-\delta_0, a+\delta_0) \subset I$. Let $I_0 := (a-\delta_0, a+\delta_0)$.

$$\{ |f(x) - f(a)| \text{ s.t } a, x \in I_0 \} \subset \{ |f(x) - f(y)| \text{ s.t. } x, y \in I \}$$

Since $|f(x) - f(a)| \geq \varepsilon$ for $x, a \in I_0$, $|f(x) - f(y)| \geq \varepsilon\ \forall\ x, y \in I$.

Therefore,

$$\sup_{\substack{I \ni a \\ \text{open bdd. interval}}} \{ |f(x) - f(y)| \text{ s.t. } x, y \in I \} \geq \varepsilon$$

Case 1. f is bdd. in a nghd. of a.

$$\varepsilon \leq \sup_{a \in I} \{ |f(x) - f(y)| \text{ s.t. } x, y \in I \} \leq 2 \sup \{ |f(x)| \}$$

$$w_f : \{ I \text{ open bdd. interval s.t. } a \in I \} \longrightarrow \mathbb{R}$$

$$w_f(I) := \sup_{x, y \in I} \{ |f(x) - f(y)| \}$$

If $J \subset I$, then $w_f(J) \leq w_f(I)$.

So consider I_n s.t. $I_{n+1} \subset I_n$, then $w_f(I_n) \downarrow$, but also $\varepsilon > 0$ is a lower bound for $w_f(I_n)$.

Therefore, $\inf_{I \ni a} \{ w_f(I) \} > 0$. Moreover, $\inf_{I \ni a} \{ w_f(I) \} < \infty$ as f is a bdd. function in a nghd. of a.

$$W_f : D_f \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$W_f(a) := \inf_{I \ni a} \{ w_f(I) \} \quad (\text{if } f \text{ is bdd. in a nghd. of } a)$$

and $W_f(a) := \infty$ (if f is unbdd. in every nghd. of a)

HW: f is cts. at a. $\iff$ $W_f(a) = 0$.

<u>Claim:</u> D_f is an F_σ-set.

Pf: Note that $D_f = \{ a \in \mathbb{R} \mid W_f(a) > 0 \}$

$$= \bigcup_{n=1}^{\infty} \{ a \in \mathbb{R} \mid W_f(a) \geq \tfrac{1}{n} \}$$

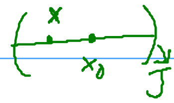
WTS. $\{a \in \mathbb{R} \mid W_f(a) < r\}$ is open for $r > 0$.

Let $x_0 \in \{a \in \mathbb{R} \mid W_f(a) < r\}$. Then $W_f(x_0) < r$.

This implies that $\inf\limits_{I \ni x_0} \{w_f(I)\} < r$

$\Rightarrow \exists\, J$ open bdd. interval s.t. $J \ni x_0$ and $w_f(J) < r$.

For each $x \in J$, $W_f(x) < w_f(J)$



$\|$

$\inf\limits_{\substack{I \ni x \\ \text{open} \\ \text{bdd.}}} \{w_f(I)\}$ so, in particular, $J \ni x$.

Since $w_f(J) < r$, $W_f(x) < r$. That is, for each $x \in J$, $W_f(x) < r$.

Therefore, J is an open interval containing x_0 s.t. $\forall\, x \in J$, $W_f(x) < r$.

Hence, $\{a \in \mathbb{R} \mid W_f(a) < r\}$ for $r > 0$ is an open set in $\mathbb{R}$.

This implies that $\{a \in \mathbb{R} \mid W_f(a) \geq \frac{1}{n}\}$ is closed in $\mathbb{R}$, for each $n \geq 1$.

Therefore, D_f is an F_σ - set.

HW: Q. Why $\mathbb{R} \setminus \mathbb{Q} \neq D_f$ for any $f: \mathbb{R} \to \mathbb{R}$?