Random experiment

A random experiment is an experiment such that

(i) all possible outcome of the exp are known in advance

(ii) outcome of a particular trial is not known or can not be predicted in advance

(iii) the exp can be repeated under identical conditions

Sample space : Set of all possible outcomes of a random experiment (~~usually~~ usually denoted by Ω)

Event : Suppose Ω is the sample space of a random experiment. If the outcome of the random experiment is a member of a set E, we say that the event E has happened; $E \subset \Omega$, thus is a collection of possible outcomes.

Examples

(1) Random experiment : tossing a coin until a head is observed

Sample space : $\Omega = \{H, TH, TTH, \ldots\}$

Event : number of tails reqd is odd

$E = \{TH, TTTH, \ldots\} \subset \Omega$

(2) 3 white, 4 red balls are numbered 1, 2, 3, 4, 5, 6, 7

Random exp : drawing a ball

Sample space : $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$

Event : drawing a white ball

$$E = \{1, 2, 3\} \subset \Omega$$

Mutually exclusive event : 2 event A and B are mutually exclusive if occurrence of one signifies non-occurrence of the others, i.e. the 2 events can not occur simultaneously.
(i.e. $A_i \cap A_j = \phi \ \forall i \neq j$ for n events)

Exhaustive events : $A_1, \ldots A_n$ events are said to be exhaustive if one of them must necessarily occur.

i.e. $\bigcup_{i=1}^{n} A_i = \Omega$

Classical definition of probability

Setup : random experiment ~~results~~ has finite number of equally likely possible outcomes

$\Omega = \{w_1, w_2, \ldots, w_n\}$ say.

An outcome $w \in \Omega$ is said to be favorable to event E if $w \in E$.

$$P(E) = \frac{\text{no. of outcomes favorable to } E}{\text{total no. of outcomes}} = \frac{\text{no. of elements in } E}{\text{no. of elements in } \Omega}$$

Note: under the above def^n

(i) $\forall$ event $E \subset \Omega$; $P(E) \geq 0$

(ii) $P(\Omega) = 1$

(iii) If $E_1, \ldots E_n$ are mutually exclusive ($E_i \cap E_j = \phi$ $\forall i \neq j$)

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \frac{\text{no. of elements in } \bigcup E_i}{\text{no of elements in } \Omega}$$

$$= \frac{\sum_{i=1}^{n} (\text{no of elements in } E_i)}{\text{no. of elements in } \Omega} = \sum_{i=1}^{n} P(E_i)$$

Note: The defn depends on 2 crucial restrictive assumptions; (i) finite number of possible outcomes & (ii) equally likely outcomes.

Relative frequency defn of probability

Ω : sample space

E : $E \subset \Omega$ is an event

Suppose the random experiment is repeated N times

$f_N(E)$: number of times event E occurs out of N

Relative frequency of event $E = \frac{f_N(E)}{N}$

Relative freq defn of prob

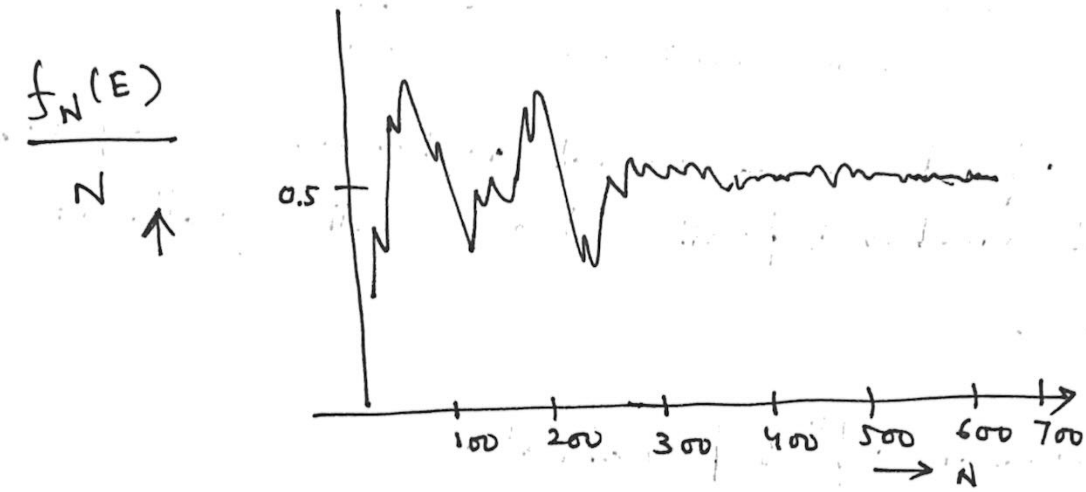
$$p_E = \lim_{N \to \infty} \frac{f_N(E)}{N}$$

prob of E

Note: p_E depends on "large" number of replications of the random exp which may not be feasible.

Note: Behavior of $\frac{f_N(E)}{N}$ can be quite erratic for small N and would eventually stabilize for "reasonably" large N

Example: To determine prob of head of a coin it is tossed N times; coin is fair



Note: Relative freq def^n of prob also satisfies

(i) $\forall E \subset \Omega \quad P(E) \geq 0$

(ii) $P(\Omega) = 1$

(iii) If $E_1, \ldots E_n$ are mutually exclusive

then $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

Note! relative freq def^n is based on the approximation

$\frac{f_N(E)}{N} \approx p_E$ for large N. Hence is likely to be assigned a different value by different experiment

σ- field

Let Ω be a sample space

Def^n : A σ- field of subsets of Ω, $\mathcal{F}$, is a class of subsets of Ω having the following properties

(i) $\Omega \in \mathcal{F}$

(ii) $\forall A \in \mathcal{F}$, $A^c \in \mathcal{F}$ (i.e closed under complementation)

(iii) If $A_1, A_2, \ldots \in \mathcal{F}$ be a countable collection of sets, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(i.e. closed under countable union)

Note: The above def^n of $\mathcal{F}$ implies

(i) $\phi \in \mathcal{F}$ $(\Omega \in \mathcal{F} \Rightarrow \Omega^c = \phi \in \mathcal{F})$

(ii) $A_1, A_2, \ldots \in \mathcal{F}$

$\Rightarrow A_1^c, A_2^c, \ldots \in \mathcal{F}$

$\Rightarrow \bigcup_1^{\infty} A_i^c \in \mathcal{F}$

$\Rightarrow \left(\bigcup_1^{\infty} A_i^c\right)^c \in \mathcal{F}$

i.e. $\bigcap_1^{\infty} A_i \in \mathcal{F}$

(iii) $A, B \in \mathcal{F}$

$\Rightarrow A - B = A \cap B^c \in \mathcal{F}$

Sly $B - A \in \mathcal{F}$

$\Rightarrow A \Delta B \overset{\text{def}}{=} (A - B) \cup (B - A) \in \mathcal{F}$

(iv) $A_1, \ldots A_n \in \mathcal{F} \Rightarrow \bigcup_1^n A_i \in \mathcal{F}$ & $\bigcap_1^n A_i \in \mathcal{F}$

Take $A_{n+1} = A_{n+2} = \cdots = \phi$

$\Rightarrow \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$ or $A_{n+1} = \cdots = \Omega$ for $\bigcap A_i \in \mathcal{F}$

Remark: Although the power set of Ω, say $P(\Omega)$, is a σ-field of subsets of Ω, in general a σ-field of Ω may not contain all subsets of Ω.

Examples

(i) $\mathcal{F} = \{\phi, \Omega\}$ - a trivial sigma field

(ii) $\forall A \in \Omega$, $\mathcal{F} = \{A, A^c, \phi, \Omega\}$ - σ-field of subsets of Ω

smallest σ-field containing A

(iii) power set of Ω

$\Omega = \{H, T\}$

$\mathcal{F} = \{\{H\}, \{T\}, \{H, T\}, \phi\}$ $\quad 2^2$

$\Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = 2^4$ elements

(iv) Let $\mathcal{C}$ be a class of subsets of Ω and suppose

$\Phi = \{\phi_\alpha : \alpha \in \Delta\}$ be the collection of all σ-fields

that contain $\mathcal{C}$. Then

$$\mathcal{F} = \bigcap_{\alpha \in \Delta} \phi_\alpha$$ is also a σ-field and it is

the σ-field generated by, $\sigma(\mathcal{C})$ ($\sigma(\mathcal{C})$ is the

smallest ~~sigm~~ σ-field that contains $\mathcal{C}$)

If we take $\Omega = \mathbb{R}$ and $\mathcal{C}$ to be the class of all

open intervals in $\mathbb{R}$, then $\sigma(\mathcal{C}) = \mathcal{B}$, say, is called

the Borel σ-field on $\mathbb{R}$.

Axiomatic definition of probability

Ω : sample space

$\mathcal{F}$: σ-field of subsets of Ω

Defⁿ : A probability f^n (or a probability measure) is a real valued set f^n defined on $\mathcal{F}$ which satisfies

(i) $\forall A \in \mathcal{F}$, $P(A) \geq 0$ — non-negativity

(ii) $P(\Omega) = 1$ — normed measure

(iii) If $A_1, A_2, \ldots \in \mathcal{F}$ such that $A_i \cap A_j = \phi \quad \forall i \neq j$

then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ — σ-additivity

$(\Omega, \mathcal{F}, P)$: probability space