Compact Metric Spaces

Def^n: A metric space (M,d) is said to be compact if for every collection of open sets $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ in M s.t. $M = \bigcup_{\alpha\in I} \mathcal{U}_\alpha$, $\exists$ a finite subcollection $\{\mathcal{U}_1, \dots, \mathcal{U}_n\}$ s.t. $M = \bigcup_{j=1}^{n} \mathcal{U}_j$.

In other words, (M,d) is compact if every open cover of M has a finite subcover.

(HW): (M,d) is compact iff for any collection of closed sets $\mathcal{F}$ in M s.t. $\bigcap_{j=1}^{n} F_j \neq \phi$ for all choices of finitely many sets $F_1, \dots, F_n \in \mathcal{F}$ then $\bigcap_{F\in\mathcal{F}} F \neq \phi$. (Finite Intersection property).

$\to$ (M,d) If every open covering of M has a finite subcover, then (M,d) is totally bdd.

Indeed, for $\varepsilon > 0$, consider $\mathcal{A} = \{ B(x,\varepsilon) \mid x \in M\}$. Then $\mathcal{A}$ forms an open covering of M. Hence, $\exists$ $x_1, x_2, \dots, x_n \in M$ s.t. $M = \bigcup_{j=1}^{n} B(x_j, \varepsilon)$.

So (M,d) is totally bdd.

$\to$ If (M,d) has the finite intersection property, then M is complete.

Indeed, we will show that the Nested Set Thm. holds in M.

Let $\{F_n\}$ be a decreasing seq. of nonempty closed sets in M with $\operatorname{diam}(F_n) \to 0$.

Then $\bigcap_{j=1}^{n} F_j = F_n$ (as F_j's are nested). Hence $\bigcap_{j=1}^{n} F_j \neq \phi$, $\forall$ $n \geq 1$.

Hence by the finite intersection property, $\bigcap_{j=1}^{\infty} F_j \neq \phi$. Therefore M is complete.

(HW). Q. If M is totally bdd, then is M compact (Def^n)?

Q. If M is complete, then is M compact?

Open cover property $\Rightarrow$ totally bdd.

$\Updownarrow$

finite intersection property $\Rightarrow$ complete.

(HW) $\to$ (M,d) is compact $\Longleftrightarrow$ totally bdd and complete.

Remark: Recall that M is complete iff Nested Set Thm.

- M is compact iff for a nested seq. of nonempty closed sets $\{F_n\}$, $\bigcap_1^\infty F_n \neq \phi$. (Condition $\operatorname{diam}(F_n) \to 0$ not required)

- M is compact iff every "countable" open cover has a finite subcover.

$\to$ An analogue of the Heine-Borel Thm:

Recall: (Heine-Borel Thm.) $(\mathbb{R}^n, \|\cdot\|_2)$ $n \geq 1$.
A is compact iff A is closed and bdd.

$\to$ (M,d) complete metric space and $A \subset M$.
A is compact iff A is closed and totally bdd.

Def^n: (M,d) is sequentially compact if for every seq. (x_n) in M, $\exists$ a subseq. that converges in M.

(HW) $\to$ (M,d) is compact iff M is sequentially compact.

Consequence: A compact $\Rightarrow$ A is closed in M

$\not\Leftarrow$ (HW)

A compact. $\Leftarrow$ M compact + A closed in M

$\ast$. $f : (M,d) \to (N,\rho)$ cts. map.

K: compact in M

Then $f(K)$ is compact in N.

Consequences ?

- (M,d) compact If $f: M \to \mathbb{R}$ is cts., then f is bdd (i.e., $\{f(x) \mid x \in M\}$ is a bdd. set in $\mathbb{R}$)

Recall: $f: [a,b] \to \mathbb{R}$ cts. then f is bdd.

- (M,d) compact $f: (M,d) \to \mathbb{R}$ cts.
  Then f attains its max. and min. values.

Recall: $f: [a,b] \to \mathbb{R}$ cts. then f attains its max and min.

- $f: [a,b] \to \mathbb{R}$ cts.

  Then $f[a,b]$ is compact & connected subset of $\mathbb{R}$.

  If f is constant, then $f[a,b] = \{c\}$.

  If f is nonconstant, then $f[a,b]$ connected implies it is an interval.

  Moreover, $f[a,b]$ is compact implies $f[a,b]$ is closed & bdd.

  Hence $f[a,b] = [c,d]$.

- (M,d) compact.

$C(M) := \{ f : (M,d) \to \mathbb{R} \text{ cts.} \}$

Then, $\|f\|_\infty := \sup_{x \in M} \{ |f(x)| \}$ defines a norm on $C(M)$.

Recall: $(C[a,b], \|\cdot\|_\infty)$ is a normed linear space.

$\ast$ $C(M)$ is a complete metric space

Q. Is $(C(M), \|\cdot\|_\infty)$ totally bdd ?