Department of Mathematics & Statistics

MTH305a

Quiz-I

Marks: 10 Time: 15 minutes

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1. Let $a > b > 0$ and $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x,y,z) := \left(\sqrt{x^2+y^2}-a\right)^2 + z^2 - b^2$. Let $S := \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}$. Given a point $p = (x,y,z)$ in the torus S, find two linearly independent vectors v_1 and v_2 of the tangent space T_pS and express them in terms of the coordinates (x,y,z) of the point p. [5 marks]

For the function $f(x,y,z) = \left(\sqrt{x^2+y^2} - a\right)^2 + z^2 - b^2$ on $\mathbb{R}^3$, the gradient at a point $p = (x,y,z)$ is

$$\nabla f(p) = 2\left(\frac{x\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}, \frac{y\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}, z\right) -$$

For $p \in S$, $\nabla f(p) \neq 0$ and it is normal to the surface S. $\longrightarrow$ 1 mark

If $p = (x,y,z) \in S$ and $z = 0$, then $e_3 = (0,0,1)$ is orthogonal to $\nabla f(p) = 2\left(\frac{x\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}, \frac{y\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}, 0\right)$.

Therefore, $(-y, x, 0)$ and $(0,0,1)$ are in T_pS and they are linearly independent. $\longrightarrow$ 2 marks

If $z \neq 0$, then

$\left(-z, 0, \frac{x\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}\right)$ and $\left(0, -z, \frac{y\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}}\right)$ are two linearly independent vectors in T_pS $\longrightarrow$ 2 marks

2. Let $S := \{(\cosh v \cos u, \cosh v \sin u, v) : v \in \mathbb{R} \text{ and } u \in (-\pi, \pi)\}$ and let $\varphi\colon (-\pi,\pi) \times \mathbb{R} \to S$ be a parametrization defined by the map $\varphi(u,v) := (\cosh v \cos u, \cosh v \sin u, v)$. Show that at every point on the surface the tangent vectors φ_u and φ_v are eigenvectors of the Weingarten map dN. Show that the Gauss curvature at every point on the surface S is negative by finding the eigenvalues of dN. [5 marks]

Since $\varphi(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$, we have

$$\varphi_u = (-\cosh v \sin u, \cosh v \cos u, 0) \quad \text{and}$$

$$\varphi_v = (\sinh v \cos u, \sinh v \sin u, 1) \longrightarrow 1 \text{ mark}$$

Hence

$$\varphi_u \times \varphi_v = (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v) \longrightarrow 1 \text{ mark}$$

$$\|\varphi_u \times \varphi_v\| = \cosh^2 v \quad \text{and}$$

$$N(u,v) = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\tanh v\right) \longrightarrow 1 \text{ mark}$$

Therefore $N_u = \left(\frac{-\sin u}{\cosh v}, \frac{\cos u}{\cosh v}, 0\right) = \frac{1}{\cosh^2 v}\varphi_u = \lambda_1 \varphi_u$

and $N_v = \left(-\frac{\sinh v \cos u}{\cosh^2 v}, \frac{-\sinh v \cos u}{\cosh^2 v}, \frac{-1}{\cosh^2 v}\right) = \frac{-1}{\cosh^2 v}\varphi_v = \lambda_2 \varphi_v$ $\longrightarrow 1$ mark

This proves that $K(u,v) = \lambda_1 \lambda_2$

$$= \frac{-1}{\cosh^4 v} \,(<0) \longrightarrow 1 \text{ mark}$$