Recall: The Bolzano-Weierstrass Thm (for seq.):

A bounded seq. has a cvgt. subseq. in $(\mathbb{R}, |\cdot|)$.

(A bounded seq. has a cvgt. subseq. in $(\mathbb{R}^n, d_{\|\cdot\|_2})$)

Consider $(M, d) = (\ell_\infty, d_{\|\cdot\|_\infty})$

Take $(X_n)_{n=1}^{\infty}$ in ℓ_∞ where $X_n := (0, 0, \dots, \underset{n\text{-th}}{1}, 0, \dots)$

$(X_n)_{n=1}^{\infty}$ is a bdd. seq. but does not have any cvgt. subseq.

Note that if a set A is bounded, then $\exists\ x \in M$ s.t. $A \subset B(x, r)$ for "some $r > 0$".

"loose a bit" "tighten"

$$\forall\ r > 0,\ \exists\ x_1, \dots, x_n \in M \text{ s.t. } A \subset \bigcup_{j=1}^{n} B(x_j, r)$$

Def^n: (M, d): metric space

$A \subset M$ is said to be **totally bdd.** if, given $\varepsilon > 0$ $\exists\ x_1, \dots, x_n \in M$ s.t. $A \subset \bigcup_{j=1}^{n} B(x_j, \varepsilon)$.

- Every totally bdd. set is a bdd. set. ($\not\Leftarrow$)

e.g. Consider $X_n \in \ell_\infty$. Then $\|X_n - X_m\|_\infty = 1$. $\forall\ n, m$.

Take $\varepsilon := \frac{1}{2}$. For any finite subset $\{x_1, \dots, x_n\} \subset M$, Consider $\bigcup_{j=1}^{n} B(x_j, \frac{1}{2})$.

Then $A = \{X_n\} \not\subset \bigcup_{j=1}^{n} B(x_j, \frac{1}{2})$ (why?)

(HW) $\star$ A is totally bdd iff $\forall$ $\varepsilon > 0$, there are finitely many sets $A_1, \dots, A_n \subset A$, with $\operatorname{diam}(A_j) < \varepsilon$ for $1 \le j \le n$, s.t. $A \subset \bigcup_{j=1}^{n} A_j$.

For (x_n) a seq. in (M, d), let $A := \{x_n \mid n \geq 1\}$.

$\star$ $\{x_n \mid n \geq 1\}$ is totally bdd. iff $\forall$ $\varepsilon > 0$, $\exists$ finitely many sets $A_1, \dots, A_n \subset A$ s.t. $\operatorname{diam}(A_j) < \varepsilon$ and $A \subset \bigcup_{j=1}^{n} A_j$. (Smell Cauchy ???)

Recall: $(\mathbb{R}, |\cdot|)$ Consider (x_n) : Cauchy seq.

Then, $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon$ s.t. $\forall\, n, m \geq N_\varepsilon$, $|x_n - x_m| < \varepsilon$

$\forall\, n \geq N_\varepsilon$, $|x_n| \leq |x_n - x_{N_\varepsilon}| + |x_{N_\varepsilon}| < \varepsilon + |x_{N_\varepsilon}|$.

Let $C := \max\{|x_1|, |x_2|, \dots, |x_{N_\varepsilon - 1}|, \varepsilon + |x_{N_\varepsilon}|\}$

Then $|x_n| \leq C \ \forall\, n \geq 1$. Hence $\{x_n \mid n \geq 1\}$ is a bdd set.

(x_n): Cauchy seq. For $\varepsilon > 0$, let $A_\varepsilon := \{x_n \mid n \geq N_\varepsilon\}$. Then $\mathrm{diam}(A_\varepsilon) < \varepsilon$.

Consider $B(x_1, \varepsilon), \dots, B(x_{N_\varepsilon - 1}, \varepsilon)$, A_ε

$A_j := B(x_j, \varepsilon) \cap \{x_n \mid n \geq 1\} = \{x_j\}$, $\forall\, 1 \leq j \leq N_\varepsilon - 1$

Note that $A = \{x_n \mid n \geq 1\} = \bigcup_{j=1}^{N_\varepsilon - 1} A_j \cup A_\varepsilon$. where $\mathrm{diam}(A_\varepsilon) < \varepsilon$ and $\mathrm{diam}(A_j) = 0 < \varepsilon$

→ If (x_n) Cauchy, then $\{x_n \mid n \geq 1\}$ is totally bdd. ($(\mathbb{R}, |\cdot|)$: totally bdd $\Leftrightarrow$ bdd.)

→ If $\{x_n \mid n \geq 1\}$ is a bdd set, then $\exists$ a cvgt. subsequence (BW-Thm)
(Completeness property of $\mathbb{R}$)

Upshot: $(\mathbb{R}, |\cdot|)$: (x_n) Cauchy $\Longrightarrow$ $\{x_n \mid n \geq 1\}$ is a bdd set. (equivalently, totally bdd.)
$\exists$ cvgt. subseq. $\Longleftarrow$

- In general metric spaces, not every bdd seq. has a cvgt. subseq.
- In a metric space, totally bdd set is a bdd set, but converse is not true.

Lets get back to the blued result

generalization: (M, d): metric space and (x_n) a Cauchy seq.
Then $\{x_n \mid n \geq 1\}$ is totally bdd.

Q: What about the converse? That is, if $\{x_n \mid n \geq 1\}$ is totally bdd, then is (x_n): a Cauchy seq. ???

Doodle: $A = \{x_n \mid n \geq 1\}$ totally bdd. Then $\forall\, \varepsilon > 0$, $\exists\, A_1, \dots, A_{N_\varepsilon} \subset A$ s.t. $A \subset \bigcup_{j=1}^{N_\varepsilon} A_j$ and $\operatorname{diam}(A_j) < \varepsilon$. Q: How to guarantee after a certain N onwards, $d(x_n, x_m) < \varepsilon$???

Analyze "what it means to say $\{x_n \mid n \geq 1\}$ is a totally bdd. set ?"

- If A is a finite set, then $\exists\, (n_k)_{k=1}^{\infty}$ s.t. $x_{n_k} = x_j$ for some $1 \leq j \leq N$ and $\forall\, k \geq 1$. (say, $A = \{x_1, \dots, x_N\}$)

Then, $(x_{n_k})_{k=1}^{\infty}$ is a cvgt. subseq.

- Suppose A is an infinite set which is totally bdd.

For $\varepsilon = 1$, $\exists$ finitely many sets $A_1, A_2, \dots, A_{N(1)}$ s.t. $\operatorname{diam}(A_j) < 1$ and
$$A \subset \bigcup_{j=1}^{N(1)} A_j .$$

Since A has infinitely many pts; $\exists$ a set in $\{A_1, \dots, A_{N(1)}\}$ which contain infinitely many pts. of A. Let say that set is A_1.

Since A is totally bdd., A_1 is totally bdd. (HW). So for $\varepsilon := \frac{1}{2}$ $\exists$ a finite collection of subsets of A_1 s.t. diameter of these sets are less than $\frac{1}{2}$.
Continuing this way, one obtains
$$A \supset A_1 \supset A_2 \supset \cdots \quad \text{s.t. } A_k \text{ has infinitely many pts. of } A \text{ and } \operatorname{diam}(A_k) < \frac{1}{k}.$$

Choose a subseq. (x_{n_k}) s.t. $x_{n_k} \in A_k$ for $k \geq 1$.

Hence, for $\varepsilon > 0$, $\exists\, K_\varepsilon$ s.t. $\forall\, k \geq K_\varepsilon$, $\frac{1}{k} < \varepsilon$.

Moreover, $d(x_{n_k}, x_{n_m}) < \frac{1}{k}$, $\forall\, k, m \geq K_\varepsilon$. Therefore, (x_n) has a Cauchy subseq.

If $A = \{x_n \mid n \geq 1\}$ is totally bdd, then $\exists$ a Cauchy subsequence of $(x_n)_{n=1}^{\infty}$.

Upshot: (M, d)

$$\{x_n \mid n \geq 1\} \text{ totally bdd.} \implies \exists\, (x_{n_k})_{k=1}^{\infty} \text{ a Cauchy subseq.}$$

$$\{x_n \mid n \geq 1\} \text{ totally bdd.} \impliedby (x_n)_{n=1}^{\infty} \text{ a Cauchy seq.}$$

(Sequential Characterization of totally bdd. sets)

Thm: A is totally bdd. iff every seq. in A has a Cauchy subseq.

Idea: $\Leftarrow$: Suppose A is not totally bdd. (Hence A is an infinite set.)
Then, $\exists\ \varepsilon > 0$ s.t. A cannot be covered by finitely many sets in A with diameter less than ε.

In particular, for $a_1 \in A$ consider $B(a_1, \varepsilon)$. Since $A \not\subset B(a_1, \varepsilon)$, one has $a_2 \in A$ s.t. $a_2 \notin B(a_1, \varepsilon)$ i.e., $d(a_1, a_2) \geq \varepsilon$.

Consider $B(a_1, \varepsilon) \cup B(a_2, \varepsilon)$. Then $A \not\subset B(a_1, \varepsilon) \cup B(a_2, \varepsilon)$. Hence $\exists\ a_3 \in A$ s.t. $d(a_1, a_3) \geq \varepsilon$, $d(a_2, a_3) \geq \varepsilon$.

Continuing this way, one obtains a seq. (a_n) in A s.t. $d(a_n, a_m) \geq \varepsilon$, $\forall\ n \neq m$. which does not have any Cauchy subseq. contradicting the hypothesis that every seq. in A has a Cauchy subseq.

$\blacksquare$

Corollary: (HW) (Bolzano–Weierstrass Thm) Every bdd. infinite subset of $\mathbb{R}$ has a limit pt. in $\mathbb{R}$