The Real Numbers ($\mathbb{R}$)

Axioms on $\mathbb{R}$:

Axiom 1. $\longrightarrow$ **Algebraic Properties of $\mathbb{R}$:** Binary operations, namely, addition & multiplication,

Additive properties:

- $a+b = b+a$ for all $a,b \in \mathbb{R}$
- $(a+b)+c = a+(b+c)$ for all $a,b,c \in \mathbb{R}$
- there exists an element $0 \in \mathbb{R}$ such that $0+a = a$ and $a+0 = a$, for all $a \in \mathbb{R}$.
- for each $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ such that
$$a+(-a) = 0 = (-a)+a$$

Multiplicative properties:

- $a\cdot b = b\cdot a$ for all $a,b \in \mathbb{R}$
- $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all $a,b,c \in \mathbb{R}$
- there exists an element $1 \in \mathbb{R}$ such that $1\cdot a = a = a\cdot 1$ for all $a \in \mathbb{R}$.
- for each $a \neq 0$ in $\mathbb{R}$, there exists an element $1/a \in \mathbb{R}$ such that
$$a\cdot(1/a) = 1 = (1/a)\cdot a.$$

Compatibility of addition and multiplication.

- $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$ and $(b+c)\cdot a = (b\cdot a)+(c\cdot a)$

Using the above axioms on $\mathbb{R}$, one can deduce the following:

HW: (i) If $z, a \in \mathbb{R}$ with $z+a = a$, then $z = 0$.

(ii) If $u \in \mathbb{R}$, $b \neq 0$ with $u\cdot b = b$, then $u = 1$.

(iii) If $a \in \mathbb{R}$, then $a\cdot 0 = 0$.

(iv) If $a \neq 0$, $b \in \mathbb{R}$ such that $a\cdot b = 1$, then $b = 1/a$

(v) If $a\cdot b = 0$ then either $a = 0$ or $b = 0$.

Axiom 2. $\longrightarrow$ **Order properties of $\mathbb{R}$:** These properties refer to the positivity and inequalities between real nos. There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real nos., that satisfy:

(i) If $a,b \in \mathbb{P}$, then $a+b \in \mathbb{P}$ and $a\cdot b \in \mathbb{P}$.

(ii) If $a \in \mathbb{R}$, then <u>exactly</u> one of the following holds:

(Trichotomy property) $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$. $\boxed{\mathbb{R} = \mathbb{P} \cup \{0\} \cup -\mathbb{P}}$

Def^n:- $a \in \mathbb{R}$ is called a positive real no. if $a \in \mathbb{P}$

$a \in \mathbb{R}$ is called a negative real no. if $-a \in \mathbb{P}$

$a \in \mathbb{P} \cup \{0\}$ then a is called a nonnegative real no.

$-a \in \mathbb{P} \cup \{0\}$ then a is called a nonpositive real no.

Def^n: Define an "order" on $\mathbb{R}$ as:

Let $a, b \in \mathbb{R}$.

- If $a - b \in \mathbb{P}$, we write "$a > b$" or $b < a$.
- If $a - b \in \mathbb{P} \cup \{0\}$ then we write $a \geq b$ or $b \leq a$.

Via the Trichotomy Property, exactly one of these hold:

$a > b$, $a = b$, $a < b$.

HW: Let $a, b, c \in \mathbb{R}$.

(i) If $a > b$ and $b > c$, then $a > c$.

(ii) If $a > b$ then $a + c > b + c$

(iii) If $a > b$ and $c > 0$, then $ca > cb$;
$c < 0$, then $ca < cb$.

Question: Is there a way to describe the elements of $\mathbb{R}$ using the above axioms ???

Recall $1 \in \mathbb{R}$, so does $1 + 1 = 2 \in \mathbb{R}$, ... and so on.

$\mathbb{N}$: denote the set of natural nos. n obtained by adding 1 n-times.

Similarly adding $-1 \in \mathbb{R}$ n-times yield $-n$

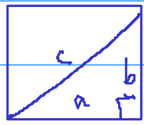
$\mathbb{Z} := \{-n \mid n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$

Elements of $\mathbb{R}$ that can be written as a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$ are called rational nos.

$\mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$ Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Question: Is $\mathbb{R} = \mathbb{Q}$?

Discovered by the ancient Greek society of Pythagoreans in the 6^{th} century B.C.



square with unit length.

Length of the diagonal of a unit square is not a rational no. !!!

Via the Pythagorean thm. for right-angled triangles: $a^2+b^2=c^2$

Here $a=b=1$, so $\boxed{c^2=2}$.

The fact (which will be proved shortly) that there is no rational number whose square is equal to 2 led to the discovery of irrational numbers.

Def^n: Irrational number: A number in $\mathbb{R}$ that is not in $\mathbb{Q}$.

Thm: There does not exist a rational number r such that $r^2=2$.

Pf: Hint: Suppose there is an r such that $r^2=2$.

$r = p/q$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

One can assume $p, q > 0$ (why?) and no common divisors (why?)

$$p^2 = 2q^2.$$

This implies that p is even. Deduce from this that q is also even!

This leads to a contradictory conclusion about q (find that) and finish the proof.

(Recall that $a \in \mathbb{N}$ is even if $a = 2n$ for some $n \in \mathbb{N}$
$a \in \mathbb{N}$ is odd if $a = 2n-1$ for some $n \in \mathbb{N}$)

Question: Where does $\mathbb{N}$ belong to in the Trichotomy $\mathbb{R} = \mathbb{P} \cup \{0\} \cup -\mathbb{P}$?

$\mathbb{N} \subset \mathbb{P}$ Why ?

Thm: (i) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
(HW) (ii) $1 > 0$

Using (i) and (ii), prove that if $n \in \mathbb{N}$, then $n > 0$. (i.e., $n \in \mathbb{P}$).

$\mathbb{N}$ has the well-ordering principle:

→ There is no smallest positive real number in $\mathbb{R}$!!! (Prove it.)
($\mathbb{P}$)

Hint: Note that if $a > 0$ then $0 < \frac{1}{2}a < a$.

Consequently, If $a \in \mathbb{R}$ such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.
(This result is important in analysis as is frequently used as a method of proof.)

Algebraic and order properties of $\mathbb{R}$ are not "enough" to characterize $\mathbb{R}$.

The fact that there are real nos. that are not rational nos. necessitates the existence of a new property to characterize $\mathbb{R}$.

Def^n's.

- $A \subseteq \mathbb{R}$ is said to be bounded above if there exists some $x \in \mathbb{R}$ such that $a \leq x$ for all $a \in A$.
- Any number x that satisfies above condition is called an upper bound of A.
- For $A \subseteq \mathbb{R}$ a nonempty set, a number $u \in \mathbb{R}$ is said to be "a" supremum (or a least upper bound) of A if it satisfies;
  - (i) u is an upper bound of A,
  - (ii) If v is any upper bound of A, then $u \leq v$.

Note that supremum of a given set (if finite) is a unique number.
So "a" supremum is in fact "the" supremum.

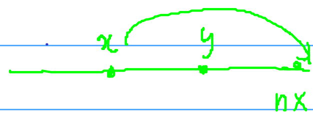
It is not possible to prove using Axioms 1-2 that every nonempty subset of $\mathbb{R}$ that is bounded above in $\mathbb{R}$ has the supremum in $\mathbb{R}$!!!

It is a deep and fundamental property of $\mathbb{R}$.

Axiom 3 $\to$ The Completeness property of $\mathbb{R}$:
Every nonempty set of real numbers that is bounded above has the least upper bound (or the supremum).

An application of Axiom 3:

Archimedean property of $\mathbb{R}$: If x and y are positive real numbers, then there is some natural number $n \in \mathbb{N}$ such that $nx > y$.



Pf: Suppose no such $n \in \mathbb{N}$ exists. That is $nx \leq y$ for all $n \in \mathbb{N}$.
Then the set $A_x := \{nx \mid n \in \mathbb{N}\}$ is bounded above by $y \in \mathbb{R}$.
So, by the completeness property (Axiom 3), $s := \sup A_x$.
Since $s - x < s$, there is some $nx \in A_x$ such that $s - x < nx \leq s$ (why?)
Then, $s < (n+1)x$. But $(n+1)x \in A_x$, contradicting s is an upper bound of A! ∎

HW: If $a, b \in \mathbb{R}$ st. (such that) $a < b$, then there is a $r \in \mathbb{Q}$ with $a < r < b$.

Completeness property of $\mathbb{R}$ $\Longrightarrow$ Archimedean property of $\mathbb{R}$.

Question: (i) Does $\mathbb{Q}$ have the Archimedean property?
yes! (That is, for $x, y \in \mathbb{Q}$ such that x, y positive rational numbers, then there is some $r \in \mathbb{Q}$ st. $rx > y$?)

NO! (ii) Does $\mathbb{Q}$ have the completeness property? (That is, does every subset of $\mathbb{Q}$ that has an upper bound in $\mathbb{Q}$, has the least upper bound?)