1. Suppose $E \not\subset A$ and $E \not\subset B$. Then $E \cap A \neq \phi$ and $E \cap B \neq \phi$.

$E \cap A$ is open set in E, $E \cap B$ also open in E.

$\underset{\text{open in } M.}{A}$

$E = E \cap A \,\dot{\cup}\, E \cap B. \Rightarrow E$ is disconnected.

3. Suppose M is disconnected. $\exists\, A, B \neq \phi$ open s.t. $M = A \,\dot{\cup}\, B$.

$A \neq \phi$, $\exists\, a \in A$

$B \neq \phi$, $\exists\, b \in B$. Consider $E := \{a, b\}$.

By hypothesis, E is connected in M.

$E \subset A \,\dot{\cup}\, B$

$\cdots \Rightarrow$ either $E \subset A$ or $E \subset B$

$\cdots$

4. Suppose $\bar{E} \cap \bar{F} = \phi$. Then $\bar{E}^c \cup \bar{F}^c = M$.

$E \subset \bar{E} \subset \bar{F}^c$ (open)

$F \subset \bar{F} \subset \bar{E}^c$ (open)

$E \cup F \subset \bar{F}^c \cup \bar{E}^c$

$(E \cup F) \cap \bar{F}^c$

$(E \cup F) \cap \bar{E}^c$.

$E \cup F$ is connected

5. M : connected (any metric space.) Let say $a, b \in M$.

claim M is uncountable.

? $f(M)$ is conneted subset of $\mathbb{R}$

$\Rightarrow$ why M is uncountable.

$f : M \to \mathbb{R}$ as $f(x) = \dfrac{d(x,a)}{d(a,b)}$ is cts.

& $f(a) = 0$, $f(b) = 1$ f is a non-constant cts function.

$M = (\mathbb{R}, |\cdot|)$

$x \mapsto \dfrac{x-a}{b-a} \to \dfrac{|x-a|}{|b-a|}$

$d(x,a)$

6. $f: \mathbb{R} \to \mathbb{R}$ cts. and open.

claim: f is strictly monotone.

$a < c < b$

$f(a) \leq f(c) \leq f(b)$

$f: [a,b] \to \mathbb{R}$ is also cts.

$\exists\ x_0 \in [a,b]$ s.t. $f(x_0) = \max_{x \in [a,b]} f(x)$.

$\exists\ [a,b]$ s.t $\exists\ c \in (a,b)$

s.t. $a < c < b$ and

$\underbrace{f(a) \leq f(c)}\ \underbrace{\geq f(b)}$

$f(b) \leq f(c)$

$x_0 = a$ or $x_0 = b$

$a < x_0 < b$

$f(a,b)$

$(m, f(x_0)]$

$[\quad]$

$x_0 = a$ $f(a) = f(x_0)$

$\to f(a) \geq f(c)$

$\to f(a) = f(c) \Rightarrow$ c is also a pt. where max. is attained.

Consider $f(a,b)$

$[m, f(x_0)]$ not open in $\mathbb{R}$

$(m, f(x_0)]$ not open in $\mathbb{R}$.

Darbouxs' Thm:

$f'(a) < k < f'(b)$ $\exists\ c \in (a,b)$ s.t. $f'(c) = k$.

$f: [a,b]$ $\Rightarrow$ f' may not cts. as a function.

$f': [a,b] \to \mathbb{R}$ is not cts!

Complex analysis

$f: \mathbb{C} \to \mathbb{C}$

$f'(z)$ analytic fnctn.

7. Suppose $\exists$ $f: \mathbb{R} \to \mathbb{R}$ cts. s.t. $f(\mathbb{Q}) \subset \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}$.

Note that $f(\mathbb{Q})$ is at most a countable set. Since $f(\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{Q}$ is also a countable set, $f(\mathbb{R}) = f(\mathbb{Q} \cup \mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{Q}$. Moreover, since f is cts. and $\mathbb{R}$ is connected, $f(\mathbb{R})$ is a connected subset of $\mathbb{R}$. Since $f(\mathbb{Q}) \subset \mathbb{R}\setminus\mathbb{Q}$ and $f(\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{Q}$ so $f(\mathbb{R})$ has at least two pts. Hence, $f(\mathbb{R})$ is a connected subset of $\mathbb{R}$ that consists of at least two pts. Therefore $f(\mathbb{R})$ must be an interval which is an uncountable set. which is a contradiction to $f(\mathbb{R}) \subset \mathbb{Q}$.
Therefore there is no such cts. funtion $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(\mathbb{Q}) \subset \mathbb{R}\setminus\mathbb{Q}$ and $f(\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{Q}$.

8. Given: A, B : closed subsets of M
$A \cap B$ and $A \cup B$ are connected sets.

Claim: A and B are connected.

Pf: We first prove that A is connected.

Suppose A is not connected. Hence $\exists$ a cts. function $g: A \to \{0,1\}$ which is onto.
Since $A \cap B \subset A$, $A \cap B$ is connected and g is a cts. function, so has $g(A \cap B)$ is connected in $\{0,1\}$. Let $g(A \cap B) = 0$.
Since $A = (A \cap B) \cup (A \cap B^c)$ and g is onto, so $g(A \cap B^c) = 1$.
Continuity of g and $\{1\}$ closed in $\{0,1\}$ implies that $g^{-1}(1) = A \cap B^c$ is closed in A.

Define $f: A \cup B \to \{0,1\}$ as

$$f(x) := \begin{cases} g(x), & x \in A \\ 0, & x \in B \end{cases}$$

f is cts. Indeed, $f^{-1}(1) = g^{-1}(1) = A \cap B^c$
$f^{-1}(0) = B \cup (A \cap B)$.

Since A and B are closed, $A \cap B$ is closed and so $B \cup (A \cap B)$ is also closed.
Since $A \cap B^c \subset A \subset A \cup B$ and $A \cap B^c$ closed in A and A closed in $A \cup B$, $A \cap B^c$ is closed in $A \cup B$. (Recall: In general, $A \subset B \subset C$ with A closed in B, B closed in C, then A closed in C).

Therefore, f is a cts. nonconstant function from $A \cup B$ onto $\{0,1\}$.
Hence $A \cup B$ is disconnected which is a contradiction to the connectedness hypothesis on $A \cup B$.

10. Hints: Suppose $f'(a) < k < f'(b)$.

Define $g : [a,b] \to \mathbb{R}$ as $g(x) := kx - f(x)$.

Since g is cts., g attains its maximum on $[a,b]$.

(HW): $g'(a) > 0$ and $g'(b) < 0$ so g cannot attain its max. at a and b.

Therefore g attains its max at $c \in (a,b)$.

(HW) Hence, $g'(c) = 0$. This implies that $g'(c) = k - f'(c) = 0$. Hence $f'(c) = k$.