Assignment-10 discussion

1. G: open and dense in $\mathbb{R}$ and $x \in G$.

claim: $G \setminus \{x\}$ is open and dense in $\mathbb{R}$.

Open: $y \in G \setminus \{x\}$. $y \in G \Rightarrow \exists\ r_1 > 0$ $(y-r_1, y+r_1) \subset G$.

Since $y \neq x$, $\exists\ r_2 > 0$ s.t. $x \notin (y-r_2, y+r_2) \subset G \setminus \{x\}$

dense: Let I be an open interval in $\mathbb{R}$. If $x \notin I$, then $I \cap G \setminus \{x\}$ as G is dense in $\mathbb{R}$. Suppose $x \in I$. $\exists$ J open in I s.t. $x \notin J$.

Then $J \cap G \setminus \{x\} \neq \phi$ as $I \cap G \neq \phi$ and open.

Hint: Consider $M = \{\frac{1}{n} \mid n \geq 1\} \cup \{0\}$. Take any isolated pt. in M, say, $\frac{1}{2}$.

$G = \{\frac{1}{n} \mid n \geq 1\}$ is dense in M but $G \setminus \{\frac{1}{2}\}$ is not dense in M.

2. $\mathbb{N}$ first category in $\mathbb{R}$: Each $n \in \mathbb{N}$ is a nowhere dense in $\mathbb{R}$.

$$\mathbb{N} = \bigcup_{n \geq 1} \{n\}$$

$\mathbb{N}$ is second category in $\mathbb{N}$: Suppose $\mathbb{N} = \bigcup_{n=1}^{\infty} E_n$ then $\exists$ m st. $\overline{E_m}^{\circ} \neq \phi$

because any nonempty subset of $\mathbb{N}$ is both open and closed in $\mathbb{N}$. ------ finish.

3. M: complete metric space

$\mathcal{U}$: nonempty open set in M.

claim: $\mathcal{U}$ is of second category.

pf: Let $\mathcal{U} = \bigcup_{1}^{\infty} E_n$ where $E_n \subset M$.

WTS: $\exists\ m \in \mathbb{N}$ s.t. $\overline{E_m}^{\circ} \neq \phi$.

Suppose not. Then $\mathcal{U} = \bigcup_{1}^{\infty} E_n$ s.t. E_n is nowhere dense.

Then $\mathcal{U}$ is of first category in M.

Since M is complete, M is of second category. Therefore $\mathcal{U}^c \neq \phi$ and $\mathcal{U}^c$ is dense in M. Since $\mathcal{U}^c$ is dense in M, every open set in M must intersect $\mathcal{U}^c$. Take $\mathcal{U}$ in particular. Then $\mathcal{U} \cap \mathcal{U}^c \neq \phi$ but $\mathcal{U} \cap \mathcal{U}^c = \phi$. Therefore, $\mathcal{U}$ is of second category.

Here we used Corollary 9.12: If A is of first category, then $\overline{A^c} = M$.
Pf. of Corollary 9.2: Suppose not. One has $M = A \cup A^c$.

$A = \bigcup_{n=1}^{\infty} E_n$ with $\overline{E_n}^{\circ} = \phi$ as A is of first category.

Since $E_n \subset \overline{E_n}$, $\overline{E_n}^c \subset E_n^c$. Moreover, E_n is nowhere dense implies that $\overline{E_n}^c$ is dense in M.

Since M is complete, by the Baire's Category thm, $\bigcap \overline{E_n}^c$ is dense in M.

$$\bigcap \overline{E_n}^c = \left(\bigcup \overline{E_n}\right)^c \text{ and as } E_n \subset \overline{E_n} \text{ so } \left(\bigcup \overline{E_n}\right)^c \subset \left(\bigcup E_n\right)^c$$

$$\bigcap_1^{\infty} \overline{E_n}^c \subset \left(\bigcup E_n\right)^c$$

Since $\bigcap_1^{\infty} \overline{E_n}^c$ is dense in M, $\left(\bigcup E_n\right)^c$ is dense in M. That is, A^c is dense in M. ∎

4. WTS: If $\mathbb{N} = \bigcup_{n=1}^{\infty} E_n$ then $\exists\, n \in \mathbb{N}$ s.t. $\overline{E_n}^{\circ} \neq \phi$. Hint $B_d(k, \frac{1}{k}) = \left\{ m \,\middle|\, \frac{|m-k|}{mk} < \frac{1}{k} \right\}$
$(\mathbb{N}, d)$ $d(m,n) := \left|\frac{1}{m} - \frac{1}{n}\right|$. $(\mathbb{N}, d)$ HW: $(\mathbb{N}, d)$ homeo $(\mathbb{N}, |\cdot|)$
not complete.

5. $N \overset{\text{homeo}}{\sim} M$ (complete).

Let $\{G_n\} \subset N$ s.t. G_n: dense & open in N.

Hint Note that homeomorphism is an open map and preserve denseness property.

$\mathring{L} = \mathring{L} = \phi$

6 Show that $\bar{L}^{\circ} = \phi$ where L is a line in $\mathbb{R}^2$. Then use completeness of $\mathbb{R}^2$ & Baire's thm.

7. Show that $\overline{W_n}^{\circ} = \phi$. Take the hint from Q.6. and to show $\overline{W_n}^{\circ} = \phi$ assume that $\overline{W_n}^{\circ} \neq \phi$ to get $\underset{\text{infi.}}{V} \subset \underset{\text{finite.}}{W_n}$.

W_n : closed subspace of V.

$W_n^{\circ} = \phi$