Recall: $f:(M,d)\to(N,\rho)$ is cts. at $x\in M$ if

$$\forall\ \varepsilon>0,\ \exists\ \delta(\varepsilon,x)>0 \text{ s.t. } f(B_d(x,\delta))\subset B_\rho(f(x),\varepsilon).$$

For example, $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}$. Then f is cts. at each pt. and at each pt. x, the $\delta(\varepsilon,x)>0$ is dependent on the choice of x. (HW).

- $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=2x$. Then the $\delta>0$ can be chosen independent of the pt. x and depend only on $\varepsilon>0$.

Def^n: $f:(M,d)\to(N,\rho)$ is said to be uniformly cts. if

$$\forall\ \varepsilon>0,\ \exists\ \delta(\varepsilon)>0 \text{ s.t. } f(B_d(x,\delta))\subset B_\rho(f(x),\varepsilon) \text{ for any } x\in M.$$

(i.e., $\forall\ \varepsilon>0$, $\exists\ \delta(\varepsilon)>0$ s.t. if$\ _\wedge$ $d(x,y)<\delta$ then $\rho(f(x),f(y))<\varepsilon$.)
for any $x,y\in M$ s.t.

→ (HW). (Sequential characterization of uniformly cts. function)

TFAE:

(i) $f:(M,d)\to(N,\rho)$ is uniformly cts.

(ii) for every pair of seq. (x_n) and (y_n) s.t. $d(x_n,y_n)\to0$ then $\rho(f(x_n),f(y_n))\to0$.

Recall:
- Cauchy sequences are NOT preserved under homeomorphism
- Cts. image of a totally bdd. set is NOT necessarily totally bdd.
- If A is dense in (M,d) and (N,ρ) is a metric space, then NOT every cts. function on A extends to a cts. function on M.

- Let $f:(M,d)\to(N,\rho)$ be a bijection. If f and f^{-1} are uniformly cts., then

$$(x_n) \text{ Cauchy} \iff (f(x_n)) \text{ Cauchy}.$$

Hint: Given (x_n) Cauchy. For $\varepsilon>0$, $\exists\ N_\varepsilon$ s.t. $\forall\ n,m\geq N_\varepsilon$, $d(x_n,x_m)<\varepsilon$.
Since f is uniformly cts., for $\varepsilon>0$ choose $\delta_\varepsilon>0$ s.t. $\delta_\varepsilon<\varepsilon$. s.t.

$$d(x_n,x_m)<\delta_\varepsilon \Rightarrow \rho(f(x_n),f(x_m))<\varepsilon\ \forall\ n,m\geq N_\varepsilon.$$

- $f: (M,d) \to (N,\rho)$ uniformly cts.

Since f is unif. cts., for $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.

Let A be a totally bdd. set. Then for $\delta > 0$, $\exists\ \{x_1, \dots, x_n\} \subset M$ s.t.

$$A \subset \bigcup_{i=1}^{n} B(x_i, \delta).$$

$$\text{Then} \quad f(A) \subset \bigcup_{i=1}^{n} f(B(x_i,\delta)) \subset \bigcup_{i=1}^{n} B(f(x_i), \varepsilon).$$

$$\Big(\text{since } f(B(x_i,\delta)) \underset{\substack{\uparrow \\ \text{via uniform cts.}}}{\subset} B(f(x_i), \varepsilon) \Big)$$

- Let A be dense in (M,d) and (N,ρ) complete and let $f: (A,d) \to (N,\rho)$ be uniformly cts.

Then, f extends (uniquely) to a uniformly cts. map $F: (M,d) \to (N,\rho)$ s.t. $F\big|_A = f$.
Moreover, if f is an isometry then F is also an isometry.

We will skip this proof as it requires "Fixed Pt." idea which is excluded from this course.

→ Recall(?) Every cts. map $f: [a,b] \to \mathbb{R}$ is uniformly cts.

- (M,d) compact metric space and $f: (M,d) \to (N,\rho)$ cts., then f is uniformly cts.

Idea: For $\varepsilon > 0$ find a $\delta > 0$ (?) s.t. $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.

- For each $x \in M$, choose $\delta_x > 0$ s.t. $f(B_d(x, \delta_x)) \subset B_\rho(f(x), \varepsilon)$.

No compactness required → Case 1. $\delta := \inf\{\delta_x \mid x \in M\} > 0$: $d(x,y) < \delta \Rightarrow y \in B_d(x, \delta_x) \Rightarrow \rho(f(x), f(y)) < \varepsilon$

Case 2. Suppose $\inf\{\delta_x \mid x \in M\} = 0$.

- $\bigcup_{x \in M} B_d(x, \delta_x)$ is an open cover of M.

Consider $S := \{ B_d(x, \delta_x) \mid \delta_x > 0,\ x \in M \}$. Then S forms an open cover of M.

Since M is compact, $\exists$ $\{x_1, \dots, x_n\}$ and $\{\delta_{x_1}, \dots, \delta_{x_n}\}$ s.t.

$$M = \bigcup_{i=1}^{n} B(x_i, \delta_{x_i}) .$$

Choose $\delta := \min\{\delta_{x_1}, \dots, \delta_{x_n}\}$.

.... finish the proof!

Prove using the sequential characterization of uniform cts. that if M is compact and $f : (M, d) \to (N, \rho)$ cts. then f is unif. cts.