MSO201A (End-Semester Examination)

Name: Time: Three hours

Roll Number: Maximum Marks = 45

Group A

This group consists of fifteen questions, and each of them carries one mark. Each question has only one correct answer.

(i) If you tick (✓) the correct answer, you will get one (i.e., 1) for that question.

(ii) If you don't tick (✓) any answer, you will get zero (i.e., 0) for that question.

(iii) If you tick (✓) a wrong answer, you will get negative one (i.e., -1) for that question.

1. Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with probability density function $f_X(x;\theta) = \frac{1}{\theta}1_{(0<x<\theta)}$, where $\theta > 0$. Here the random variable X has the same distribution as X_1, and $1_{(A)} = 1$ if A is true and $1_{(A)} = 0$ if A is false. The maximum likelihood estimator of θ is

(a) $\max\{X_1, \ldots, X_n\}$ (b) $\min\{X_1, \ldots, X_n\}$ (c) $\frac{1}{n}\sum\limits_{i=1}^{n} X_i$ (d) none of (a), (b) and (c).

2. Let $X_1, \ldots, X_{11}$ be i.i.d. random variables with uniform distribution over $(0, 1)$. Then $E\left[\frac{X_2+X_9+X_{11}}{\sum\limits_{i=1}^{11} X_i}\right] =$

(a) $\frac{11}{3}$ (b) $\frac{9}{11}$ (c) $\frac{3}{11}$ (d) 1.

3. Let X be a continuous random variable with probability density function f is such that $f(x) = f(-x)$ for all x. Suppose that $E[X^k] < \infty$ for any integer k. Then $E[X^{1003}] =$

(a) 1003 (b) 0 (c) $1003f(0)$ (d) $f(0)$.

4. Let X be a random variable with probability mass function $P[X = x] = \binom{n}{x}p^x(1-p)^{n-x}$, where $x = 0, \ldots, n$ and $0 < p < 1$. Suppose that Y is a random variable obtained from the random variable X when $n \to \infty$, $p \to 0$ and $np \to \lambda$, where λ (> 0) is a constant. Then for any integer y, $P[Y = y] =$

(a) $\frac{\exp(-\lambda)\lambda^y}{y!}$ (b) $\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})$ (c) $\binom{n}{y}p^y(1-p)^{n-y}$ (d) 0.

5. Consider a random sample $X_1, \ldots, X_n$ associated with the probability density function $f(x;\theta) = \exp\{-(x - 7\theta)\}$, where $x \geq 7\theta$, and suppose that $X_{(i)}$ is the i-th order statistic, where $i = 1, \ldots, n$. The maximum likelihood estimator of θ is

(a) $X_{(n)}$ (b) $\frac{X_{(n)}}{7}$ (c) $\frac{X_{(1)}}{7}$ (d) $X_{(1)}$.

6. Let X be a Poisson random variable with mean $= \frac{1}{2}$. Then $E[(X + 1)!]$ equals

(a) $2e^{-\frac{1}{2}}$ (b) $4e^{-\frac{1}{2}}$ (c) $4e^{-1}$ (d) $2e^{-1}$

7. Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with normal distribution mean = 5 and variance = 1. Then $\lim\limits_{n\to\infty} P\left[\left|\frac{1}{n}\sum\limits_{i=1}^{n} X_i - 7\right| < 10^{-101}\right] =$

(a) 10^{-101} (b) 1 (c) 0 (d) $\frac{1}{2}$

8. Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with uniform distribution over $(0, \theta)$, where $\theta > 0$, and $X_{(i)}$ $(i = 1, \ldots, n)$ denotes the i-th order statistic. A consistent estimator of θ is

(a) $2X_1$ (b) $2X_{(1)}$ (c) $X_{(n-7)}$ (d) $X_1 + X_2$

9. Suppose that X is a continuous, non-negative random variable with distribution function $F(x)$. Then $E(X) =$

(a) $\int\limits_0^\infty F(x)dx$ (b) $\int\limits_0^\infty \{1 - F(x)\}dx$ (c) $\int\limits_0^1 \{1 - F(x)\}dx$ (d) $\int\limits_0^1 F(x)dx$

10. Let (X, Y) follows bivariate normal distribution with means $= (0, 0)$, variances $= (1, 1)$ and the correlation coefficient $= \rho(\neq 0)$. Then $P[X > 0, Y > 0] =$

(a) $\frac{1}{4}$ (b) 0 (c) $\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho$ (d) $\frac{1}{4} + \frac{1}{2\pi}\cos^{-1}\rho$

11. Let Φ and ϕ be the CDF and the PDF of the standard normal distribution, respectively. Then $\int\limits_{-\infty}^{\infty} \Phi(x)\phi(x)dx =$

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{4}$ (d) None of (a), (b) and (c)

12. Let X be a random variable with p.m.f. $P[X = n] = \frac{1}{10}$, where $n = 1, 2, \ldots, 10$. Then $E[\max\{X, 5\}] =$

(a) 5 (b) 6.5 (c) 1 (d) 10

13. Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with p.d.f. $f(x) = \frac{1}{4}e^{-|x-4|} + \frac{1}{4}e^{-|x-6|}$, $x \in \mathbb{R}$. Then as $n \to \infty$, $\frac{1}{n}\sum\limits_{i=1}^{n} X_i$ converges in probability to

(a) 5 (b) 4 (c) 6 (d) 0.

14. Let E, F and G be three events such that $P(E \cap F \cap G) = 0.1$, $P(G|F) = 0.3$ and $P(E|F \cap G) = P(E|F)$. Then $P(G|E \cap F) =$

(a) 1 (b) 0.5 (c) 0 (d) 0.3

15. Suppose that X_1 and X_2 are two independent random variables (identical with X) with probability mass function $f(x|\theta)$, where $x = 0$ and 1. We now want to test $H_0 : \theta = 0$ against $H_1 : \theta = 1$. The form of $f(x|\theta)$ is as follows. At $\theta = 0$, $P[X = 0] = 0.3$ and $P[X = 1] = 0.7$, whereas at $\theta = 1$, $P[X = 0] = 0.5$ and $P[X = 1] = 0.5$. Suppose, we reject H_0 when $X_1 + X_2 < 2$. Here, the probability of Type-I error is

(a) 0.25 (b) 0.75 (c) 0.51 (d) None of (a), (b) and (c) are true.

Group B

This group consists of fifteen questions, and each of them carries two marks. Each question may have more than one correct answer.

(i) If you tick (✓) all correct answers, you will get two (i.e., 2) for that question.

(ii) If you don't tick (✓) all correct answers but tick (✓) at least one correct answer (without any wrong answer), you will get one (i.e., 1) for that question.

(iiI) If you don't tick (✓) any answer, you will get zero (i.e., 0) for that question.

(iv) If you tick (✓) any wrong answer, you will get negative one (i.e., -1) for that question.

1. Let X and Y have the joint p.d.f. $f_{(X,Y)}(x,y) = e^{-y}$ if $0 < x < y < \infty$, and $= 0$, otherwise. Then the correlation coefficient between X and Y equals

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2}{\sqrt{3}}$

2. Let $X_1, \ldots, X_n$ be a sequence i.i.d. random variables from uniform distribution over $(\theta, \theta+1)$, where $\theta \in \mathbb{R}$. Let $U_n = \max\{X_1, \ldots, X_n\}$ and $V_n = \min\{X_1, \ldots, X_n\}$. Then

(a) U_n is consistent for θ
(b) V_n is consistent for θ
(c) $2U_n - V_n - 2$ is a consistent estimator of θ
(d) $2V_n - U_n + 1$ is a consistent estimator for θ

3. Let A_1, A_2 and A_3 be three events such that $P(A_i) = \frac{1}{3}$, $i = 1, 2, 3$; $P(A_i \cap A_j) = \frac{1}{6}$, $1 \leq i \neq j \leq 3$ and $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$. Then the probability that none of the events A_1, A_2 and A_3 occur equals

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 0

4. Let $X_1, \ldots, X_n$ be a sequence i.i.d. random variables with normal distribution having mean $= \mu$ and variance $= 1$. Then which of the following statement(s) is (are) true?

(a) X_1 is an unbiased estimator of μ
(b) $\frac{X_1+X_2+X_3}{3}$ is an unbiased estimator of μ
(c) $\frac{1}{n-3} \sum\limits_{i=4}^{n} X_i$ is a consistent estimator of μ
(d) X_1 is a consistent estimator of μ.

5. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with unknown parameter p, where $0 < p < 1$. Let $T_{1,n} = X_1$ and $T_{2,n} = \frac{1}{n} \sum\limits_{i=1}^{n} X_i$. Which of the following statement(s) is (are) true?

(a) Both $T_{1,n}$ and $T_{2,n}$ are unbiased estimators of p.
(b) $T_{1,n}$ is more efficient than $T_{2,n}$ when $n \geq 2$.
(c) Variance $(T_{1,n})$ = Variance $(T_{2,n})$ for all n.
(d) $T_{2,n}$ is the UMVUE of p.

6. Let P be a probability function that assigns the same weight to each of the points of the sample space $\Omega = \{1, 2, 3, 4\}$. Consider the events $E = \{1, 2\}$, $F = \{1, 3\}$ and $G = \{3, 4\}$. Then

(a) E and F are independent.
(b) E and G are independent
(c) F and G are independent
(d) E, F and G are independent.

7. Let X be a random variable, whose moment generating function is $M_X(t) = e^{\frac{t^2}{2}}$, and let $\mathbb{Q}$ denotes the set of rational number. Then $P[X \in \mathbb{Q}] =$

(a) 0 (b) 1 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$.

8. Let X and Y be two independent standard normal random variables. Then the pdf of $Z = \frac{|X|}{|Y|}$ is

(a) $f_Z(z) = e^{-z}$, $z > 0$ (b) $f_Z(z) = \frac{1}{\pi}\frac{1}{(1+z^2)}$, $z > 0$
(c) $f_Z(z) = \frac{2}{\pi}\frac{1}{(1+z^2)}$, $z > 0$ (d) $f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$, $z > 0$.

9. Let $X_1, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta) = c(\theta)e^{-(x-\theta)}$ if $x \geq 2\theta$, and $= 0$, otherwise, where $\theta \in \mathbb{R}$ is the unknown parameter. Then

(a) The maximum likelihood estimator of θ is $\frac{\min\{X_1,\ldots,X_n\}}{2}$ (b) $c(\theta) = 1$ for all $\theta \in \mathbb{R}$.
(c) The maximum likelihood estimator of θ is $\min\{X_1, \ldots, X_n\}$ (d) $c(\theta) = \theta$ for all $\theta \in \mathbb{R}$.

10. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with uniform distribution over $(0, 1)$, and as $n \to \infty$, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} c$. Then

(a) $c = \frac{1}{2}$ (b) $c = 1$ (c) $c = \frac{1}{3}$ (d) $c = \frac{1}{12}$

11. Let (X, Y) has the joint p.d.f. $f(x, y) = 2$ if $0 \leq x \leq y \leq 1$, and $= 0$, otherwise.
Let $a = E(Y|X = \frac{1}{2})$ and $b = Var(Y|X = \frac{1}{2})$. Then $(a, b) =$

(a) $(\frac{3}{4}, \frac{7}{12})$ (b) $(\frac{1}{4}, \frac{1}{48})$ (c) $(\frac{1}{4}, \frac{7}{12})$ (d) $(\frac{3}{4}, \frac{1}{48})$

12. Let $X_1, \ldots, X_n$ be a random sample from normal distribution with mean $= \mu$ and variance $= \sigma^2 > 0$ and define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Then

(a) $\bar{X}_n$ and S_n^2 are independent of each other (b) $E(\bar{X}_n) = \mu$ and $Var(\bar{X}_n) = \frac{\sigma^2}{n}$
(c) $Var\left(\frac{(n-1)S_n^2}{\sigma^2}\right) = 2(n-1)$ (d) $E\left(\frac{(n-1)S_n^2}{\sigma^2}\right) = (n-1)$

13. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$, and $\rho(X, Y) = \frac{1}{3}$, where ρ denotes the correlation coefficient. Then $\rho\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) =$

(a) $\frac{34}{38}$ (b) $\frac{38}{27}$ (c) $\frac{2}{3}$ (d) 1

14. The cumulative distribution function of a random variable X defined over $\mathbb{R}$ is $F_X(x) = 1 - e^{-\beta x^2}$, where $\beta > 0$. Then

(a) $E(X) = \frac{\sqrt{\pi}}{2\sqrt{\beta}}$ (b) $Variance(X) = \frac{1}{\beta} - \frac{\pi}{4\beta}$ (c) $E(X) = \frac{1}{\beta}$ (d) $E(X^2) = \frac{1}{\beta}$

15. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with mean $= \mu$ and finite variance. Then as $n \to \infty$,

(a) $\frac{1}{n-3}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$ (b) $\frac{2}{n(n+1)}\sum_{i=1}^{n} iX_i \xrightarrow{p} \mu$ (c) $\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2X_i \xrightarrow{p} \mu$ (d) $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$