Assignment 5: Evaluation solution;

(a) Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$.

$$[0,1] = \bigcup_{n=1}^{\infty} [r_n, r_{n+1}]$$

$$\mathbb{R}\setminus\mathbb{Q} \cap [0,1] = \mathbb{R}\setminus\mathbb{Q} \cap \bigcup_{n=1}^{\infty}[r_n, r_{n+1}] = \bigcup_{n=1}^{\infty}\left([r_n, r_{n+1}] \cap \mathbb{R}\setminus\mathbb{Q}\right)$$

$$\mathbb{R}\setminus\mathbb{Q} \cap [0,1] = \bigcup_{n=1}^{\infty} [r_n, r_{n+1}) \cap \mathbb{R}\setminus\mathbb{Q}.$$

Define $f : \mathbb{R}\setminus\mathbb{Q} \cap [0,1] \to \mathbb{Q} \cap [0,1]$ as follows:

For $x \in \mathbb{R}\setminus\mathbb{Q} \cap [0,1] = \bigcup_{n=1}^{\infty} [r_n, r_{n+1}) \cap \mathbb{R}\setminus\mathbb{Q}$, $\exists!$ (unique) $n_0 \in \mathbb{N}$ s.t.

$$x \in [r_{n_0}, r_{n_0+1}) \cap \mathbb{R}\setminus\mathbb{Q}.$$

Define $f(x) = r_{n_0}$.

Claim: f is cts. on $\mathbb{R}\setminus\mathbb{Q} \cap [0,1]$.

pf: For any seq. $(x_n) \in \mathbb{R}\setminus\mathbb{Q} \cap [0,1]$ s.t. $x_n \to x$ in $\mathbb{R}\setminus\mathbb{Q} \cap [0,1]$, one needs to show $f(x_n) \to f(x)$.

Since $x \in \mathbb{R}\setminus\mathbb{Q} \cap [0,1]$, $\exists!$ $m_0 \in \mathbb{N}$ s.t. $x \in [r_{m_0}, r_{m_0+1}) \cap \mathbb{R}\setminus\mathbb{Q}$.

Also note that $x \in (r_{m_0}, r_{m_0+1}) \cap \mathbb{R}\setminus\mathbb{Q}$. Since (r_{m_0}, r_{m_0+1}) is open in $[0,1]$, so $(r_{m_0}, r_{m_0+1}) \cap \mathbb{R}\setminus\mathbb{Q}$ is also open in $\mathbb{R}\setminus\mathbb{Q}$.

(Since we are defining $f : \mathbb{R}\setminus\mathbb{Q} \cap [0,1] \to \mathbb{Q} \cap [0,1]$, $\mathbb{R}\setminus\mathbb{Q} \cap [0,1]$ and $\mathbb{Q} \cap [0,1]$ must be considered as metric spaces by itself which here is the relative metric space induced from the metric space $[0,1]$.)

Moreover, since x is a limit pt. of (x_n) and $(r_{m_0}, r_{m_0+1}) \cap \mathbb{R}\setminus\mathbb{Q} \cap [0,1]$ open set in $\mathbb{R}\setminus\mathbb{Q} \cap [0,1]$, so $\exists N_0$ s.t. $\forall n \geq N_0$,

$$x_n \in (r_{m_0}, r_{m_0+1}) \cap \mathbb{R}\setminus\mathbb{Q} \cap [0,1].$$

By definition of f, $f(x_n) = r_{m_0}$ $\forall n \geq N_0$. Hence $(f(x_n))$ is an eventually constant seq. converging to r_{m_0} which is also equal to $f(x)$.

Therefore, f is cts. on $\mathbb{R}\setminus\mathbb{Q} \cap [0,1]$.

(b) Suppose $\exists$ a cts. function $f: [0,1] \to \mathbb{Q} \cap [0,1]$. which is onto.

For $\delta \in \mathbb{R}\setminus\mathbb{Q} \cap [0,1]$, consider $[0,\delta) \cap \mathbb{Q} \cap [0,1]$ which is open in $\mathbb{Q} \cap [0,1]$ and $(\delta,1] \cap \mathbb{Q} \cap [0,1]$ which is also open in $\mathbb{Q} \cap [0,1]$.

Since f is cts. $f^{-1}([0,\delta) \cap \mathbb{Q} \cap [0,1])$ and $f^{-1}((\delta,1] \cap \mathbb{Q} \cap [0,1])$ are open nonempty disjoint sets in $[0,1]$.

Moreover, since $([0,\delta) \cap \mathbb{Q} \cap [0,1]) \cup ((\delta,1] \cap \mathbb{Q} \cap [0,1]) = \mathbb{Q} \cap [0,1]$

and f is an onto cts. map,

$$[0,1] = f^{-1}([0,\delta) \cap \mathbb{Q} \cap [0,1]) \cup f^{-1}((\delta,1] \cap \mathbb{Q} \cap [0,1])$$

forms a separation of $[0,1]$ implying the disconnectedness of $[0,1]$ which is a contradiction to the connectedness of $[0,1]$.

$\blacksquare$.