Assignment-1

2. Equivalent cond. $u = \sup A$ $\iff$ $\forall\ \varepsilon > 0$, $u-\varepsilon$
- u: upper bound $\exists\ a \in A$ s.t. $u - \varepsilon < a$
- $u \le V$

- $\varepsilon_n \downarrow 0$ $\varepsilon_n := \frac{1}{n}$ $\forall\ n \geq 1$, $\exists\ a_n \in A$ s.t. $u - \frac{1}{n} < a_n \le u$ $\quad n\to\infty \downarrow$

$\lim_{n\to\infty} a_n = u$

3. claim: $\inf_n a_n \overset{?}{\le} \lim_{n\to\infty} a_n \overset{\checkmark}{\le} \sup a_n$.

Since $a_n \to a$ (say). $\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ s.t. $\forall\ n \geq N_\varepsilon$,

$\underbrace{a - \varepsilon < a_n}_{\sup}\ \underbrace{< a + \varepsilon}_{\inf}$

$\inf a_n \le a_n \le \sup a_n \quad \forall n$

since $\lim a_n$ exist.

$\to M < a_n < N$

$a < a_n + \varepsilon \le \sup_{n\geq 1} a_n + \varepsilon$

$a \le \sup_n a_n + \varepsilon \Rightarrow a \le \sup_{n \geq 1} a_n$

$\subset \mathbb{N} \subset \mathbb{Z}$

4. $\mathbb{Z}$ $\quad \overset{\neq\phi}{A} \subset \mathbb{Z} \to A \cap \mathbb{N} \neq \phi$ $\quad S := \{ n \in \mathbb{N} \mid a \le n\ \forall\ a \in A\} \neq \phi$

by the WOP: $\min S = k$.

$\begin{cases} u \ \& \ V \\ u \le V \end{cases}$

$A \cap \mathbb{N} = \phi$ $\quad a \le k \ \ \forall\ a \in A$

V be any other upper bound, then

$0 \in A$ $\quad A \subset -\mathbb{N}$ $\quad V \in S$

$a \le 0$ $\quad$ Consider $-A$ $\quad \Rightarrow k \le V$

$0 \le V$ $\quad -A \subset \mathbb{N}$

u^{ε} (HW)

$-u \in A$

- u: upper bd.

$\exists\ a$ s.t. $u - a = 0$ i.e. $a = u \in A$.

$\to u - a = 0$

$u - a = k \to u = a + k$

$u - a = k+1$

$\exists\ a \in A$ s.t. $u - a = k$ (say)

assume. sup exists in $\mathbb{Z}$.

$\exists\ a$ s.t. $u - a = k+1$ $\quad \to u \in A$ $\quad \to u \notin A$

$(u-1) - a = k$ $\quad u' = u - 1$

$\mathbb{R}$ $\quad n \quad M \quad n+1$

$\mathbb{Z}$ $\quad M \notin \mathbb{Z}$.

$n < M < n+1$

$n < M - \varepsilon < n+1$

8. $\Leftarrow$ $\forall$ (x_{n_k}) $\exists$ $(x_{n_{k_l}}) \to x$.

claim: $x_n \to x$ $\quad \forall \varepsilon > 0, \exists N_\varepsilon$ st. $\forall n \geq N_\varepsilon$, $|x_n - x| < \varepsilon$.

Suppose $x_n \not\to x$ $\quad \exists \varepsilon_0 > 0$ st $\forall k \geq 1$ $\exists n_k \in \mathbb{N}$ st. $|x_{n_k} - x| > \varepsilon_0$

$\exists$ (x_{n_k}) $\quad$ Suppose $x_{n_{k_l}} \to x$ $\quad \exists M_\varepsilon$ $\quad |x_{n_{k_l}} - x| > \varepsilon_0$

for $\varepsilon_0 > 0$ $\exists$ $\quad |x_{n_{k_l}} - x| < \varepsilon_0$

$n_1 < n_2 < n_3 <$

n_{k_1} n_{k_2}

9. $\limsup = \inf_{k \geq 1} \{\sup n a_n\} > 0$ $\quad a_n$ $\quad \sum_1^\infty a_n < \infty$

$\Downarrow$

$2 \quad 2^3 \quad 2^3 \quad 2^4 \quad \dots$ $\quad \sum_1^\infty a_{2^n} + \sum a_k < \infty$

$2^n < m < 2^{n+1}$

define $a_m := \frac{1}{2^m}$ and $a_{2^n} := \frac{1}{2^n}$ $\quad \sup_{n \geq k} \{n a_n\} \geq 1$

$n a_n$ $\quad 2^n \cdot a_{2^n} = 2^n \cdot \frac{1}{2^n} = 1$ $\quad \Rightarrow \inf\{\sup \dots\} \geq 1 > 0$

cvgt seq $\Rightarrow$ bdd. seq

$x_n \to x$ $\quad \not\Leftarrow$ $(1,0,1,0,1,0,\dots)$ $\quad |x_n| \leq M, \forall n$

$1, 1, 1, 1, \longrightarrow 1$

$0, 0, 0, \longrightarrow 0$

glb $<$ $u = lb$, $w \leq u$ $\quad$ A is bdd below $\quad$ A bdd set

$\forall a \in A$ $\quad |a| \leq M$.

lub $\quad$ A bdd above $\quad -M < a \leq M$