$$\gamma_X(h) = Cov(X_{t+h}, X_t) = Cov(X_t, X_{t-h})$$
$$= E(X_t X_{t-h})$$
$$= E(\phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}) X_{t-h}$$

$$\gamma_X(h) = \phi_1 \gamma_X(h-1) + \cdots + \phi_p \gamma_X(h-p)$$
$$+ E(\epsilon_t X_{t-h}) + \theta_1 E(\epsilon_{t-1} X_{t-h}) + \cdots + E(\theta_q \epsilon_{t-q} X_{t-h}) \quad \longleftarrow (*)$$

Note that

$$\gamma_X(h) = \phi_1 \gamma_X(h-1) + \cdots + \phi_p \gamma_X(h-p)$$
$$\forall h > q$$

otherwise, r.h.s. of $(*)$ will have additional terms involving θ_js and ϕ_js & σ^2

<u>e.g.</u> suppose $q > 1$ and take $h = 1$

$$\gamma_X(1) = \phi_1 \gamma_X(0) + \phi_2 \gamma_X(1) + \cdots + \phi_p \gamma_X(p-1)$$
$$+ \sum_{i=1}^{q} \theta_i E(\epsilon_{t-i} X_{t-1})$$

$$E(\epsilon_{t-1} X_{t-1}) = E(\epsilon_{t-1}(\phi_1 X_{t-2} + \cdots + \phi_p X_{t-p-1} + \epsilon_{t-1}$$
$$+ \theta_1 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q-1}))$$
$$= \sigma^2 \quad \text{(always the case when time indices of } \epsilon \text{ \& } X \text{ are same)}$$

$$E(\epsilon_{t-2} X_{t-1}) = E\,\epsilon_{t-2}(\phi_1 X_{t-2} + \cdots + \phi_p X_{t-p-1} + \epsilon_{t-1}$$
$$+ \theta_1 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q-1})$$
$$= \sigma^2 \phi_1 + \sigma^2 \theta_1$$

Invertibility of stationary processes

Invertibility of AR(1) : AR(1) to MA(∞) representation

Let $\{X_t\}$ be a covariance stationary AR(1)

$$X_t = \phi X_{t-1} + \epsilon_t \ ; \quad \epsilon_t \sim WN(0, \sigma^2) ; |\phi| < 1$$

$$(1-\phi B) X_t = \epsilon_t \quad \text{i.e.} \quad \phi(B) X_t = \epsilon_t$$

Define $\phi_j^*(B) = 1 + \phi B + \phi^2 B^2 + \cdots + \phi^j B^j$

We have by multiplying both sides of AR(1) model equation with $\phi_j^*(B)$

$$\phi_j^*(B)\, \phi(B) X_t = \phi_j^*(B)\, \epsilon_t$$

Note that $\phi_j^*(B)\, \phi(B)$

$$= (1 + \phi B + \cdots + \phi^j B^j)(1 - \phi B)$$

$$= (1 + \phi B + \cdots + \phi^j B^j)$$

$$- (\phi B + \phi^2 B^2 + \cdots + \phi^{j+1} B^{j+1})$$

$$= 1 - \phi^{j+1} B^{j+1}$$

$$\Rightarrow \phi_j^*(B)\, \phi(B) X_t = X_t - \phi^{j+1} B^{j+1} X_t$$

$$= X_t - \phi^{j+1} X_{t-j-1}$$

$$= \phi_j^*(B)\, \epsilon_t$$

$$\Rightarrow X_t - \phi^{j+1} X_{t-j-1} = \epsilon_t + \phi\epsilon_{t-1} + \cdots + \phi^j \epsilon_{t-j}$$

Realize that

$$\phi_j^*(B)\phi(B)X_t - X_t = -\phi^{j+1} X_{t-j-1}$$

hence

$$E\left(\phi_j^*(B)\phi(B)X_t - X_t\right)^2 = \phi^{2(j+1)} E\left(X_{t-j-1}^2\right)$$

$$\lim_{j\to\infty} E\left(\phi_j^*(B)\phi(B)X_t - X_t\right)^2 = \lim_{j\to\infty} \phi^{2(j+1)} E\left(X_{t-j-1}^2\right).$$

Since $V(X_t) = E(X_t^2) < \infty \ \forall t$ (as the process is covariance stationary)

& $|\phi| < 1$

$$\lim_{j\to\infty} \phi^{2(j+1)} E\left(X_{t-j-1}^2\right) = 0$$

$$\Rightarrow \lim_{j\to\infty} E\left(\phi_j^*(B)\phi(B)X_t - X_t\right)^2 = 0$$

$$\text{i.e. } \lim_{j\to\infty} E\left(\phi_j^*(B)\epsilon_t - X_t\right)^2 = 0$$

$$\left(\phi_j^*(B)\phi(B)X_t = \phi_j^*(B)\epsilon_t\right)$$

$$\text{i.e. } \lim_{j\to\infty} E\left(\sum_{i=0}^{j} \phi^i \epsilon_{t-i} - X_t\right)^2 = 0$$

$$\text{i.e. } \sum_{i=0}^{j} \phi^i \epsilon_{t-i} \xrightarrow{m.s.} X_t \text{ as } j \to \infty$$

($\xrightarrow{m.s.}$: means convergence in mean square sense)

$$\lim_{j \to \infty} \sum_{i=0}^{j} \phi^i \epsilon_{t-i} = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}$$

Thus

$$X_t \overset{m.s.}{=\!=} \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \leftarrow MA(\infty)$$

i.e. $X_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}$ (usually we just say this) in m.s. sense.

with $\lim_{j \to \infty} (1 + \phi B + \cdots + \phi^j B^j)$ acting as $(1-\phi B)^{-1}$

so that $$(1-\phi B) X_t = \epsilon_t$$

$$\Rightarrow (1-\phi B)^{-1}(1-\phi B) X_t = (1-\phi B)^{-1} \epsilon_t$$

i.e. $$X_t = \sum_{i=0}^{\infty} \phi^i B^i \epsilon_t$$

i.e. $$X_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}$$

MA(∞) representation of stationary ~~AR~~ AR(1).

<u>Note</u>: $\forall |\phi| < 1$; we will take

$\sum_{i=0}^{\infty} \phi^i B^i = (1-\phi B)^{-1}$ the inverse operator of $(1-\phi B)$ operator

AR(2) to MA(∞) representation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\phi(B) X_t = \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 = (1-\lambda_1 B)(1-\lambda_2 B) \text{ say}$$

$\Rightarrow$ Roots of $\phi(z) = 0$ are $\frac{1}{\lambda_1}$ & $\frac{1}{\lambda_2}$

For $\{X_t\}$, a covariance stationary process,

$|\lambda_i| < 1 \quad i = 1, 2$

Further if $|\lambda_i| < 1 \ \forall i$ then $(1-\lambda_i B)^{-1}$

exists $\forall i$ and is given by

$$(1-\lambda_i B)^{-1} = \sum_{k=0}^{\infty} \lambda_i^k B^k$$

Let us consider a partial fraction approach to obtain MA(∞) representation of AR(2).

$$X_t = \frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} \epsilon_t$$

Let
$$\frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} = \frac{a}{1-\lambda_1 B} + \frac{b}{1-\lambda_2 B}$$

$$= \frac{a(1-\lambda_2 B) + b(1-\lambda_1 B)}{(1-\lambda_1 B)(1-\lambda_2 B)}$$

$$= \frac{(a+b) - B(a\lambda_2 + b\lambda_1)}{(1-\lambda_1 B)(1-\lambda_2 B)}$$

$\Rightarrow a + b = 1$ & $a\lambda_2 + b\lambda_1 = 0$

$$(1-b)\lambda_2 + b\lambda_1 = 0 \Rightarrow b = \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

& $a = \dfrac{\lambda_1}{\lambda_1 - \lambda_2}$

$$\Rightarrow \frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} = (1-\lambda_1 B)^{-1}(1-\lambda_2 B)^{-1}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2}(1-\lambda_1 B)^{-1} + \frac{\lambda_2}{\lambda_2 - \lambda_1}(1-\lambda_2 B)^{-1}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2}\sum_{j=0}^{\infty}\lambda_1^j B^j + \frac{\lambda_2}{\lambda_2 - \lambda_1}\sum_{j=0}^{\infty}\lambda_2^j B^j$$

$\Rightarrow (1-\lambda_1 B)(1-\lambda_2 B) X_t = \epsilon_t$ can be expressed as

$$X_t = (1-\lambda_1 B)^{-1}(1-\lambda_2 B)^{-1}\epsilon_t$$

i.e. $$X_t = \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\sum_{j=0}^{\infty}\lambda_1^j B^j + \frac{\lambda_2}{\lambda_2 - \lambda_1}\sum_{j=0}^{\infty}\lambda_2^j B^j\right)\epsilon_t$$

i.e. $$X_t = \frac{\lambda_1}{\lambda_1 - \lambda_2}\sum_{j=0}^{\infty}\lambda_1^j \epsilon_{t-j} + \frac{\lambda_2}{\lambda_2 - \lambda_1}\sum_{j=0}^{\infty}\lambda_2^j \epsilon_{t-j}$$

i.e. $$X_t = \sum_{j=0}^{\infty}\left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\lambda_1^j + \frac{\lambda_2}{\lambda_2 - \lambda_1}\lambda_2^j\right)\epsilon_{t-j}$$

MA(∞) representation of AR(2)

<u>Note</u>: The MA(∞) representation of AR(2) can also be obtained using the "method of comparing coefficients".

Illustration of "method of comparing coefficients".

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\phi(B) X_t = \epsilon_t$$

$$X_t = \phi(B)^{-1} \epsilon_t \quad \left(\text{as } \{X_t\} \text{ is covariance stationary}\right)$$

$$X_t = \Psi(B)\epsilon_t, \text{ say, with}$$

$$\Psi(B) = \Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots.$$

i.e. $\phi(B)^{-1} = \Psi(B)$

i.e. $1 = \phi(B)\Psi(B)$

i.e. $1 = (1 - \phi_1 B - \phi_2 B^2)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \cdots)$

i.e. $1 = \Psi_0 + B(\Psi_1 - \Psi_0\phi_1) + B^2(\Psi_2 - \phi_1\Psi_1 - \phi_2\Psi_0) + \cdots$

Comparing coeffs of B^j from both sides we can solve for $\Psi_0, \Psi_1, \Psi_2, \ldots$ and hence the $MA(\infty)$ rep $X_t = \sum_{j=0}^{\infty} \Psi_j \epsilon_{t-j}$

e.g.

Comparing coeff

of B^0 : $\Psi_0 = 1$

of B^1 : $\Psi_1 = \Psi_0\phi_1 = \phi_1$

B^2 : $\Psi_2 = \phi_1\Psi_1 + \phi_2\Psi_0 = \phi_1^2 + \phi_2$

$\vdots$