vi Hypergeometric

Urn containing M white & $N-M$ balls of some other color

n balls drawn at a time or one after another without replacement

X : number of white balls in the sample

minimum value of X : $\max(0, n-(N-M))$

maximum value of X : $\min(n, M)$

$$P(X=x) = \begin{cases} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, & x = \max(0, n-(N-M)), \ldots, \min(n,M) \\ 0, & o/w. \end{cases}$$

Note: Same setup with replacement sampling – $\text{Bin}\left(n, \frac{M}{N}\right)$

m.g.f. $E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$

(for $n > M$, $n > N-M$

$$E(X) = n\frac{M}{N}$$

$$V(X) = n\frac{M}{N}\left(1-\frac{M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Use $\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$ to derive $E(X)$, $E(X(X-1))$ to get $V(X)$

VII Poisson

$X \sim P(\lambda) \qquad \lambda > 0$

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} \,; & x = 0,1,2,\ldots \\ 0 \,; & \text{o/w} \end{cases}$$

m.g.f : $E(e^{tX}) = \sum_{x=0}^{\infty} \dfrac{e^{tx}\lambda^x}{x!} e^{-\lambda}$

$= e^{\lambda e^t} e^{-\lambda} = e^{-\lambda(1-e^t)}$

$E(X) = V(X) = \lambda$

Note: Poisson dist^n is applicable to model count of events, change of states, failures etc.

Note: Poisson approximation to Binomial

$X \sim Bin(n, p)$

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$= \dfrac{n(n-1)\cdots(n-x+1)}{x!} p^x (1-p)^{n-x}$

$= \dfrac{n}{n} \cdot \dfrac{n-1}{n} \cdots \dfrac{(n-x+1)}{n} \; \dfrac{(np)^x}{x!} (1-p)^n (1-p)^{-x}$

As $n \to \infty$, $p \to 0 \ni np = \lambda$ (fixed)

$P(X=x) \longrightarrow \dfrac{\lambda^x}{x!} e^{-\lambda}$

Some standard continuous distributions

(I) Uniform dist^n

$X \sim U[a, b]$

p.d.f. $f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & o/w. \end{cases}$

d.f. $F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$

m.g.f. $M_X(t) = \frac{1}{b-a} \int_a^b e^{tx}\, dx = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$

II Normal dist^n

$X \sim N(\mu, \sigma^2)$

$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \; ; \; x \in \mathbb{R}$

$\mu \in \mathbb{R}, \sigma > 0$

Dist^n is symmetric around μ $\forall (\mu, \sigma)$

$P(X \le \mu - c) = P(X \ge \mu + c)$

$f(\mu - c) = f(\mu + c) \quad \forall c$

m.g.f: $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$

$$\frac{x-\mu}{\sigma} = z; \quad dx = \sigma\, dz$$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)}\, e^{-z^2/2}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}}\, e^{t^2\sigma^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, dz$$

$$= e^{t\mu + t^2\sigma^2/2} \qquad t \in (-\infty, \infty)$$

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = e^{t\mu + t^2\sigma^2/2}(\mu + t\sigma^2)\Big|_{t=0}$$

$$= \mu$$

$$E X^2 = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = e^{t\mu + t^2\sigma^2/2}(\sigma^2) + (\mu + t\sigma^2)\, e^{t\mu + t^2\sigma^2/2}(\mu + t\sigma^2)\Big|_{t=0}$$

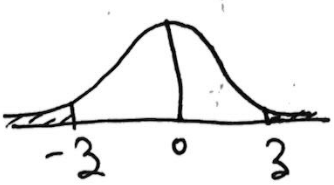
$$= \sigma^2 + \mu^2$$

$$E X^2 = \mu^2 + \sigma^2$$

$$V(X) = \sigma^2$$

Standard normal dist^n $\quad N(0,1)$

$$Z \sim N(0,1) \quad \phi(z) = f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad z \in (-\infty, \infty)$$



$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

(values are tabulated)

$$\Phi(-z) = P(Z \le -z) = P(Z \ge z)$$

$$= 1 - \Phi(z)$$

$$\Phi(-z) + \Phi(z) = 1 \quad ; \quad \Phi(0) = \frac{1}{2}$$

Note: If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

Let $Z = \frac{X-\mu}{\sigma}$

d.f. of Z: $P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right)$

$$= P(X \leq \mu + \sigma z)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$\frac{x-\mu}{\sigma} = y$; $\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy = \Phi(z)$

i.e. $Z \sim N(0,1)$

Alt: use m.g.f.