ASSIGNMENT 8

MTH 301, 2018

(1) A set $E \subset \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $\{x_n\} \to x$ with $\{x_n\}$ contained in one of A or B, and x an element of the other.

(2) Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty. Show that a set $E \subset \mathbb{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

(3) A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.
  (a) Show that $\mathbb{Q}$ is totally disconnected.
  (b) Is the set of irrational number totally disconnected?
  (c) Is Cantor set C is totally disconnected?

(4) Let $\mathcal{F}$ be a collection of connected subsets of a metric space X such that the intersection $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$. Show that $\bigcup_{A \in \mathcal{F}} A$ is connected.

(5) From the above exercise we see the following: If $x \in X$ then $\bigcup A$ where A is a connected subset containing x is connected. Call this maximal connected set as the *component* of X containing x.

Show that every point of a metric space X belongs to a uniquely determined component of X. i.e. The component of X form a collection of disjoint sets whose union is X.

(6) In $\mathbb{R}^n$ we have seen that if a set is connected it may not necessarily be path connected. However, show that every open connected set in $\mathbb{R}^n$ is connected.

(7) Show that every open set U in $\mathbb{R}^n$ can be expressed as countable union of disjoint open connected sets.

(8) Prove that a metric space X is connected if and only if every non-empty proper subset of X has a non-empty boundary.

(9) Let $U \subset \mathbb{R}^n$ open connected. Let T be a component of $\mathbb{R}^n \setminus U$. Show that $\mathbb{R}^n \setminus T$ is connected.

(10) Let (X, d) be a metric space which is not bounded. Show that for every $a \in X$ and every $r > 0$ the set $S = \{x : d(x, a) = r\}$ is non-empty.

(11) Prove that no pair of the following subspaces of $\mathbb{R}$ are *homeomorphic*: $(0, 1), [0, 1), [0, 1]$.

(12) Let $S = \{(1, 0) \bigcup_{n \in \mathbb{N} L_n}$ be a subset of $\mathbb{R}^2$ where L_n is the closed line segment connecting the origin $(0, 0)$ to the point $(1, \frac{1}{n})$. Show that S is connected but not path connected.