Note: If $\Sigma > 0$, then p.d.f. of $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$ is

$$f_{\underset{\sim}{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underset{\sim}{x}-\underline{\mu})} \qquad \underset{\sim}{x} \in \mathbb{R}^p.$$

$p = 2$; $\underline{N_2(\underline{\mu}, \Sigma)}$ $\qquad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$; $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$.

or $\underline{BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)}$ $\qquad \sigma_{ii} = V(X_i) = \sigma_i^2$ (say)

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad \sigma_{12} = \sigma_{21} = Cov(X_1, X_2)$$

$$\rho = \frac{\sigma_{12}}{[\sigma_{11} \cdot \sigma_{22}]^{1/2}}.$$

$$f_X(\underline{x}) = \frac{1}{2\pi \, \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right\}\right]$$

$$\underset{\sim}{x} \in \mathbb{R}^2$$

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

Remark:

Note that if X_1 & X_2 are uncorrelated, i.e. $\rho = 0$, then

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) \, f_{X_2}(x_2)$$

$\Rightarrow X_1$ & X_2 are indep.

(such a conclusion is not true in general; i.e. uncorrelated $\not\Rightarrow$ independence in general).

For $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$

$$\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} \underset{\sim}{X}^{(1)} \\ \hline \underset{\sim}{X}^{(2)} \end{pmatrix} \qquad \underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}^{(1)} \\ \hline \underset{\sim}{\mu}^{(2)} \end{pmatrix}.$$

$$\Sigma = \left(\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right). \qquad \Sigma_{ii} = \text{Cov}(\underset{\sim}{X}^{(i)}).$$

$\underset{\sim}{X}^{(1)}$ & $\underset{\sim}{X}^{(2)}$ are indep iff $\Sigma_{12} = 0$.

<u>Conditional distⁿ</u>

$$\underset{\sim}{X}^{(1)} \mid \underset{\sim}{X}^{(2)} \sim N_q\left(\underset{\sim}{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}\left(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)}\right),\ \Sigma_{11.2} \right)$$

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Sly

$$\underset{\sim}{X}^{(2)} \mid \underset{\sim}{X}^{(1)} \sim N_{p-q}\left(\underset{\sim}{\mu}^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}\left(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}\right),\ \Sigma_{22.1} \right)$$

$$\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

For $p = 2$, i.e. $N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$X_1 \mid X_2 \sim N_1\left(\mu_1 + (\rho\sigma_1\sigma_2)(\sigma_2^2)^{-1}(x_2 - \mu_2),\ \sigma_1^2 - \frac{(\rho\sigma_1\sigma_2)^2}{\sigma_2^2} \right)$$

$$\Sigma = \left(\begin{array}{c|c} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \hline \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array}\right)$$

i.e. $X_1 \mid X_2 \sim N_1\left(\mu_1 + \rho\dfrac{\sigma_1}{\sigma_2}(x_2 - \mu_2),\ \sigma_1^2(1 - \rho^2) \right)$

Sly

$$X_2 \mid X_1 \sim N_1\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1),\ \sigma_2^2(1 - \rho^2) \right).$$

<u>Note</u>: If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$; $\Sigma > 0$

then $\quad (\underline{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu}) \sim \chi^2_p$

Proof of $\underset{\sim}{X}^{(1)}$ & $\underset{\sim}{X}^{(2)}$ indep iff $\Sigma_{12}=0$

If $\underset{\sim}{X}^{(1)}$ & $\underset{\sim}{X}^{(2)}$ are indep then for any X_i in $\underset{\sim}{X}^{(1)}$ and X_j in $\underset{\sim}{X}^{(2)}$,

X_i & X_j are indep $\Rightarrow \operatorname{cov}(X_i, X_j)=0 \Rightarrow \Sigma_{12}=0$

Alternately suppose $\Sigma_{12}=0$, then

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \; ; \; \Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}$$

$$\& \; |\Sigma| = |\Sigma_{11}|\,|\Sigma_{22}|$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}^{(1)}, \underset{\sim}{x}^{(2)}) = \frac{1}{(2\pi)^{p/2}|\Sigma_{11}|^{1/2}|\Sigma_{22}|^{1/2}} \exp\left(-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})'\begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}(\underset{\sim}{x}-\underset{\sim}{\mu})\right)$$

$$= \left(\frac{1}{(2\pi)^{q/2}|\Sigma_{11}|^{1/2}} \exp\left(-\frac{1}{2}(\underset{\sim}{x}^{(1)}-\underset{\sim}{\mu}^{(1)})'\Sigma_{11}^{-1}(\underset{\sim}{x}^{(1)}-\underset{\sim}{\mu}^{(1)})\right)\right)$$

$$\times\left(\frac{1}{(2\pi)^{\frac{p-q}{2}}|\Sigma_{22}|^{1/2}} \exp\left(-\frac{1}{2}(\underset{\sim}{x}^{(2)}-\underset{\sim}{\mu}^{(2)})'\Sigma_{22}^{-1}(\underset{\sim}{x}^{(2)}-\underset{\sim}{\mu}^{(2)})\right)\right)$$

$$= f_{\underset{\sim}{X}^{(1)}}(\underset{\sim}{x}^{(1)})\, f_{\underset{\sim}{X}^{(2)}}(\underset{\sim}{x}^{(2)})$$

$\underset{\sim}{X}^{(1)} \sim N_q(\underset{\sim}{\mu}^{(1)}, \Sigma_{11})$; $\underset{\sim}{X}^{(2)} \sim N_{p-q}(\underset{\sim}{\mu}^{(2)}, \Sigma_{22})$

$\Rightarrow \underset{\sim}{X}^{(1)}$ & $\underset{\sim}{X}^{(2)}$ are indep.

$$\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma) \; ; \; \Sigma > 0$$

Derivation of conditional dist^n

Let

$$\underset{\sim}{Z} = \underbrace{\begin{bmatrix} I_q & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-q} \end{bmatrix}}_{A} \begin{pmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}) \\ \underset{\sim}{X}^{(2)} - \underline{\mu}^{(2)} \end{pmatrix} \sim N_p(\underline{0}, A\Sigma A')$$

$$A\Sigma A' = \begin{bmatrix} I_q & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-q} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I_{p-q} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I_{p-q} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

$\Sigma_{11.2}$

$\Rightarrow \left(\underset{\sim}{X}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}) \right)$ & $(\underline{X}^{(2)} - \underset{\sim}{\mu}^{(2)})$ are indep

$\Rightarrow (\underset{\sim}{X}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}) \mid \underline{X}^{(2)}$ and unconditional dist^n are identical

$\Rightarrow \underset{\sim}{X}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}) \mid \underline{X}^{(2)} \sim N_q(\underline{0}, \Sigma_{11.2}$

$\Rightarrow \underset{\sim}{X}^{(1)} \mid \underset{\sim}{X}^{(2)} \sim N_q\left(\underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}), \Sigma_{11.2}\right)$

joint moment generating f^n

$\underset{\sim}{X} = (X_1, \ldots, X_p)'$

joint m.g.f.

$$M_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{X}}\right) = E\left(e^{t_1 X_1 + \cdots + t_p X_p}\right).$$

$\underset{\sim}{t} \to (t_1, \ldots, t_p)$

provided the expectation exists in some nbd of $\underset{\sim}{0}_{p\times 1}$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\sum_1^p t_i x_i} f_X(\underline{x})\, dx_1 \ldots dx_p.$$

for continuous case

$$= \sum_{x_1} \cdots \sum_{x_p} e^{\sum t_i x_i} P(\underline{X} = \underset{\sim}{x})$$

for discrete case