Assignment 1: Several variables calculus & differential geometry (MTH305A)

(1) For $x_0 \in \mathbb{R}^n$, define open ball
$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$
A subset $\Omega \subset \mathbb{R}^n$ is called *open* (in $\mathbb{R}^n$) if for any $x_0 \in \Omega$ there exists $r > 0$ such that $B_r(x_0) \subset \Omega$. A subset $\Omega \subset \mathbb{R}^n$ is called *closed* (in $\mathbb{R}^n$) if $\mathbb{R}^n - \Omega$ is open.
(i) Prove that an open ball is an open set.
(ii) Which of the following is open or closed ?
(a) $\{(x, y) \in \mathbb{R}^2 |\ x > y\}$
(b) $\{(x, y) \in \mathbb{R}^2 |\ |x| + |y| < 1\}$
(c) $\{(x, y) \in \mathbb{R}^2 |\ 3x^2 + 2y^2 = 5\}$
(d) $\{(x, y, z) \in \mathbb{R}^3 |\ x > 0, y > 0, z > 0\}$
(e) $\{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1\}$

(2) Let f and g be integrable on $[a, b]$.
(a) Prove that
$$\left| \int_a^b f \cdot g \right| \leq \left(\int_a^b f^2 \right)^{\frac{1}{2}} \cdot \left(\int_a^b g^2 \right)^{\frac{1}{2}}.$$
(b) If equality holds, must $f = \lambda g$ for some $\lambda \in \mathbb{R}$? What if f and g are continuous?

(3) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **norm preserving** if $\|T(x)\| = \|x\|$, for all $x \in \mathbb{R}^n$; and **inner product preserving** if $\langle T(x), T(y)\rangle = \langle x, y\rangle$, for all $x, y \in \mathbb{R}^n$.
(a) Prove that T is norm preserving if and only if T is inner product preserving.
(b) Prove that such a linear transformation is $1-1$ and T^{-1} is of the same sort.

(4) If $x, y \in \mathbb{R}^n$ are nonzero, the angle between x and y, denoted by $\angle(x, y)$, is defined by
$$\angle(x, y) = arc \cos \left(\frac{\langle x, y\rangle}{\|x\|\|y\|} \right)$$
The linear transformation T is called **angle preserving** if
$$\angle(T(x), T(y)) = \angle(x, y), \forall x, y \in \mathbb{R}^n.$$
(a) Prove that if T is norm preserving, then T is angle preserving.
(b) If there is a basis $\{x_i \mid i = 1, \ldots, n\}$ of $\mathbb{R}^n$ and real numbers $\lambda_i, i = 1, \ldots, n$, such that $T(x_i) = \lambda_i x$. If $|\lambda_i|$'s are equal, does it imply that T is angle preserving?

(5) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, show that $\exists\ M \in \mathbb{R}$ such that
$$\|T(h)\| \leq M\|h\|, \forall h \in \mathbb{R}^n.$$

(6) Let $A \subset \mathbb{R}^n$ be a compact set and $B \subset \mathbb{R}^n$ be a closed subset. Show that: if $B \subset A$, then B is also compact.

(7) Let $A \subset \mathbb{R}^n$. Show that A is compact if and only if A is closed and bounded.