Mixed seasonal model

$$ARMA(p,q) \times ARMA(P,Q)_s \equiv ARMA(p,q)(P,Q)_s$$

$$\Phi^{(s)}(B^s)\,\phi(B)\,X_t = \Theta^{(s)}(B^s)\,\theta(B)\,\epsilon_t$$

$\epsilon_t \sim WN(0,\sigma^2)$

$\Phi^{(s)}(B^s)$: seasonal AR polynomial (order P)

$\phi(B)$: non-seasonal AR polynomial (order p)

$\Theta^{(s)}(B^s)$: seasonal MA polynomial (order Q)

$\theta(B)$: non-seasonal MA polynomial (order q)

Note that $ARMA(p,q)(P,Q)_s \equiv ARMA(Ps+p, Qs+q)$

Mixed Seasonal ARIMA model

Let $\{X_t\}$ be a non-stationary process and suppose

$Y_t = \nabla^d \nabla_s^D X_t$ is stationary

s: period of seasonality

d: order of non-seasonal differencing

D: order of seasonal differencing

Suppose $Y_t = (1-B)^d(1-B^s)^D X_t$ follows an

ARMA(p,q) for non-seasonal part and ARMA$(P,Q)_s$ for the seasonal part

i.e. $Y_t \sim$ ARMA $(p,q)(P,Q)_s$

i.e. $\Phi^{(s)}(B^s)\,\phi(B)\,Y_t = \Theta^{(s)}(B^s)\,\theta(B)\,\epsilon_t$

$\epsilon_t \sim WN(0,\sigma^2)$

then X_t is said to have a mixed seasonal ARIMA $(p,d,q)(P,D,Q)_s$ (or just S-ARIMA $(p,d,q)(P,D,Q)_s$ model).

Note: $Y_t \sim$ ARMA$(p,q)(P,Q)_s$

$\Rightarrow Y_t \sim$ ARMA$(Ps+p,\ Qs+q)$ with some coeffs 0, i.e. a restricted ARMA$(Ps+p, Qs+q)$

Note: $Y_t = \nabla^d \nabla_s^D X_t$

$Y_t \sim$ ARMA$(p,q)(P,Q)_s$

$$\Phi^{(s)}(B^s)\,\phi(B)\,Y_t = \Theta^{(s)}(B^s)\,\theta(B)\,\epsilon_t$$

i.e. $$\Phi^{(s)}(B^s)\,\phi(B)\,(1-B)^d(1-B^s)^D X_t = \Theta^{(s)}(B^s)\,\theta(B)\,\epsilon_t$$

i.e. $\phi^*(B) X_t = \theta^*(B) \epsilon_t$

$$\phi^*(B) = \Phi^{(s)}(B^s)\, \phi(B)\, (1-B)^d (1-B^s)^D$$

$$\theta^*(B) = \Theta^{(s)}(B^s)\, \theta(B)$$

i.e. $X_t \sim \text{ARMA}(Ps + p + d + Ds,\ Qs + q)$

a restricted, non-stationary ARMA

Parameter estimation in time series models

Parameter estimation for AR models:

AR(p) $\quad X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$

Given an observed sample $(x_1, \ldots, x_n)$ from the above model, the problem is to estimate the model parameter $(c, \phi_1, \ldots, \phi_p)$

Approach I: Least squares estimation

Let $\Psi(c, \phi_1, \ldots, \phi_p) = \sum_{t=p+1}^{n} (x_t - c - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p})^2$

$\hat{c}, \hat{\phi}_1, \ldots, \hat{\phi}_p$ which minimizes $\Psi(c, \phi_1, \ldots, \phi_p)$ is the least squares estimates.

$$\frac{\partial \Psi}{\partial c} = 0 \Rightarrow \sum_{t=p+1}^{n} (x_t - c - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p}) = 0$$

$$\frac{\partial \Psi}{\partial \phi_i} = 0 \Rightarrow \sum_{t=p+1}^{n} (x_t - c - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p}) x_{t-i} = 0.$$

$i = 1(1)p$

This lead to the following linear system of equations

$$\sum_{p+1}^{n} x_t = (n-p)c + \phi_1 \sum_{p+1}^{n} x_{t-1} + \cdots + \phi_p \sum_{p+1}^{n} x_{t-p}$$

$$\sum x_t x_{t-1} = c \sum x_{t-1} + \phi_1 \sum x_{t-1}^2 + \cdots + \phi_p \sum x_{t-1} x_{t-p}$$

$$\sum x_t x_{t-2} = c \sum x_{t-2} + \phi_1 \sum x_{t-1} x_{t-2} + \cdots + \phi_p \sum x_{t-2} x_{t-p}.$$

$$\vdots$$

$$\sum x_t x_{t-p} = c \sum x_{t-p} + \phi_1 \sum x_{t-1} x_{t-p} + \cdots + \phi_p \sum x_{t-p}^2$$

Solving the above we get the least squares estimates of $(c, \phi_1, \ldots, \phi_p)$.

Approach II : Exact Maximum Likelihood estimation

Suppose $\{X_t\}$ is Gaussian AR(1)

$$X_t = c + \phi X_{t-1} + \epsilon_t \, ; \quad \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Model parameters : c, ϕ

Noise parameter : σ^2

Observation set : $(x_1, \ldots, x_n)$

Note that, $E(X_1) = \dfrac{c}{1-\phi} = \mu$ (say)

$$V(X_1) = \gamma_0 \text{ of } AR(1) = \frac{\sigma^2}{1-\phi^2}$$

$$X_1 \sim N\left(\frac{c}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right) ; \quad X_2 \sim N\left(\frac{c}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right)$$

But these are not independent!

$$f_{X_1}(x_1;\theta) = \left(\sqrt{2\pi}\sqrt{\sigma^2/1-\phi^2}\right)^{-1} \exp\left(-\frac{1}{2}\frac{(x_1 - c/1-\phi)^2}{\sigma^2/1-\phi^2}\right)$$

$$X_2 = c + \phi X_1 + \epsilon_2$$

$$\Rightarrow X_2 | X_1 \sim N(c + \phi x_1, \sigma^2)$$

$$f_{X_2|X_1} = \left(\sqrt{2\pi\sigma^2}\right)^{-1} \exp\left(-\frac{1}{2}\frac{(x_2 - c - \phi x_1)^2}{\sigma^2}\right)$$

$$X_3 = c + \phi X_2 + \epsilon_3$$

dist^n of $X_3 | X_2, X_1 \equiv$ dist^n of $X_3 | X_2 \sim N(c + \phi x_2, \sigma^2)$

In general, $\forall\, t \geq 2$

$$X_t | X_{t-1}, \ldots X_1 \equiv X_t | X_{t-1}$$
$$\sim N(c + \phi x_{t-1}, \sigma^2)$$

jt p.d.f. of $X_1, \ldots, X_n$

$$f_{X_1, \ldots X_n} = f_{X_n|X_{n-1}, \ldots, X_1} \; f_{X_{n-1}, \ldots X_1}$$

i.e. $f_{X_1, \ldots X_n} = f_{X_n| X_{n-1}, \ldots X_1} \left(f_{X_{n-1}|X_{n-2} \ldots X_1} f_{X_{n-2}, \ldots X_1} \right)$

split

$\vdots$

$$f_{X_1, \ldots X_n} = f_{X_n|X_{n-1}} f_{X_{n-1}|X_{n-2}} \cdots f_{X_2|X_1} f_{X_1}$$
$$= f_{X_1} \prod_{t=2}^{n} f_{X_t|X_{t-1}}$$