Name:
Roll No:

MTH442A: Time Series Analysis
Quiz #2; Full Marks-20

[1] Let $\{\epsilon_t\}$ and $\{\delta_t\}$ be mutually independent sequences of i.i.d. $N(0,1)$ random variables. Define

$$P_t = \epsilon_t + \epsilon_{t-1} + \delta_t; \quad Q_t = \left(\cos\left(\frac{\pi}{4}t\right)\right)\epsilon_1 + \left(\sin\left(\frac{\pi}{4}t\right)\right)\epsilon_2;$$
$$R_t = \left(\cos\left(\frac{\pi}{2}t\right)\right)\epsilon_{10} + \left(\sin\left(\frac{\pi}{2}t\right)\right)\epsilon_{20} \quad \text{and} \quad S_t = \delta_{t-1} + \delta_{t-3} + \epsilon_{t+4}.$$

Prove or disprove the following statements:

(a) $\underline{\eta}_t = (\epsilon_t, \delta_t, \epsilon_{3t+7})^T \sim VWN$ — 2

(b) $\underline{\gamma}_t = (\delta_{2t}, \epsilon_{t-1}, \delta_{2t-1})^T \sim VWN$ — 2

(c) $\begin{pmatrix} P_t \\ Q_t \end{pmatrix}$ is covariance stationary — 3

(d) $\begin{pmatrix} Q_t \\ R_t \end{pmatrix}$ is covariance stationary — 3

(e) If $\underline{Z}_t = \begin{pmatrix} P_t \\ S_t \end{pmatrix}$, then $Cov(\underline{Z}_t, \underline{Z}_{t+h}) \overset{=0}{}$ for all $|h| > 3$ — 3

[2] Let $\underline{X}_t$ be a 2-variate vector $ARMA(2,1)$ process $\underline{X}_t = \underline{X}_{t-1} - \frac{1}{4}\underline{X}_{t-2} + \underline{\epsilon}_t + 3\underline{\epsilon}_{t-1}$; $\underline{\epsilon}_t \sim VWN(\underline{0}, \Sigma), \Sigma > 0$.

Prove or disprove the following statements:

(a) $\underline{X}_t$ is stationary and causal — 2

(b) $\underline{X}_t$ is invertible — 2

(c) $\underline{X}_t$ can be represented as $\underline{X}_t = \sum_{j=0}^{\infty} \Psi_j \, \underline{\epsilon}_{t-j}$ with $\Psi_2 = \frac{13}{4} I_2$ — 3

1) (a) $\underset{\sim}{\eta}_t = (\epsilon_t, \delta_t, \epsilon_{3t+7})'$

$Cov(\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_{10}) \neq 0 \Rightarrow \underset{\sim}{\eta}_t \nsim VWN$ — (2)

(b) $\underline{\gamma}_t = (\delta_{2t}, \epsilon_{t-1}, \delta_{2t-1})'$;

$Cov(\underset{\sim}{\gamma}_t, \underset{\sim}{\gamma}_s) = \begin{cases} 0, & t \neq s \\ I_3, & t = s \end{cases} \Rightarrow \underset{\sim}{\gamma}_t \sim VWN(\underset{\sim}{0}, I_3)$ — (2)

(c) $Cov(P_t, Q_{t+h})$ depends on t.

e.g. for $h=0$; $Cov(P_1, Q_1) = \cos \pi/4$; $Cov(P_4, Q_4) = 0$

$\Rightarrow \begin{pmatrix} P_t \\ Q_t \end{pmatrix}$ is not covariance stationary — (3)

(d) $\underset{\sim}{A}_t = \begin{pmatrix} Q_t \\ R_t \end{pmatrix}$ $\quad E\,\underline{A}_t = \underset{\sim}{0}$

$$Cov(\underset{\sim}{A}_t, \underline{A}_{t+h}) = \begin{pmatrix} Cov(Q_t, Q_{t+h}) & Cov(Q_t, R_{t+h}) \\ Cov(R_t, Q_{t+h}) & Cov(R_t, R_{t+h}) \end{pmatrix}$$

$\text{Cov}(Q_t, R_{t+h}) = \text{Cov}(R_t, Q_{t+h}) = 0 \quad \forall h \ \forall t$

$\text{Cov}(Q_t, Q_{t+h}) = \text{Cos}(\frac{\pi}{4} h) \quad \forall t$

$\text{Cov}(R_t, R_{t+h}) = \text{Cos}(\pi/2 \, h) \quad \forall t$

$\Rightarrow$ $\text{Cov}(\underline{A}_t, \underline{A}_{t+h})$ is a f^n of h only $\Rightarrow$ $\underline{A}_t$ is covariance stationary — (3)

(e) $\text{Cov}(P_t, S_{t-4}) = 1$

$\Rightarrow \text{Cov}(\underline{Z}_t, \underline{Z}_{t+h}) \neq 0 \quad h = -4$

$\Rightarrow \text{Cov}(\underline{Z}_t, \underline{Z}_{t+h}) \neq 0$ for all $|h| > 3$ — (3)

(2) $(I_2 - I_2 B + \frac{1}{4} I_2 B^2) \underline{X}_t = (I_2 + 3 I_2 B) \underline{\epsilon}_t$

$\Phi(B) \underline{X}_t = \Theta(B) \underline{\epsilon}_t$

(a) $|\Phi(z)| = \begin{vmatrix} 1 - z + \frac{1}{4} z^2 & 0 \\ 0 & 1 - z + \frac{1}{4} z^2 \end{vmatrix}$

$\Rightarrow |\Phi(z)| = 0 \Rightarrow z = 2$

$\Rightarrow \underline{X}_t$ is causal & stationary — (2)

(b) $|\Theta(z)| = \begin{vmatrix} 1+3z & 0 \\ 0 & 1+3z \end{vmatrix} \Rightarrow |\Theta(z)| = 0 \Rightarrow z = -\frac{1}{3}$

$\Rightarrow \underline{X}_t$ is not invertible — (2)

(c) $\underline{X}_t$ is causal $\Rightarrow$ $\underline{X}_t = \Phi(B)^{-1} \Theta(B) \underline{\epsilon}_t = \Psi(B) \underline{\epsilon}_t = \sum_{j=0}^{4} \Psi_j \underline{\epsilon}_{t-j}$

$\Theta(B) = \Phi(B) \Psi(B)$

i.e. $I_2 + 3 I_2 B = (I_2 - I_2 B + \frac{1}{4} I_2 B^2)(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \dots)$

Comparing coeffs of B^j:

$j = 0; \quad \Psi_0 = I_2$

$j = 1; \quad 3 I_2 = \Psi_1 - \Psi_0 \Rightarrow \Psi_1 = 4 I_2$

$j = 2; \quad 0 = \Psi_2 - \Psi_1 + \frac{1}{4} \Psi_0$

$\Rightarrow \Psi_2 = \Psi_1 - \frac{1}{4} \Psi_0$

$= 4 I_2 - \frac{1}{4} I_2 = \frac{15}{4} I_2$ — (3)