Name: ____________________

Roll Number: ______________

Final Exam Solutions

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.

2. This exam contains **6 + 1** questions and is worth **60%** of your grade.

3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There exists a countable $X \subseteq \omega_1$ such that $\sup(X) = \omega_1$.

(ii) There exists a bijection $f : \mathbb{R}^7 \to \mathbb{R}^9$ satisfying: For every x, y in $\mathbb{R}^7$, $f(x-y) = f(x) - f(y)$.

(iii) If $f : \omega \to \omega$ is a strictly increasing computable function, then range(f) is computable.

(iv) The set of all subsets of ω that are definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ is countable.

(v) TA is ω-categorical.

Solution

(i) False. Since the union of a countable family of countable ordinals is countable.

(ii) True.

(iii) True. To check if $m \in \text{range}(f)$, we just need to check if $m \in \{f(0), f(1), \ldots, f(m)\}$.

(iv) True. Since each definable set is uniquely determined by an $\mathcal{L}_{PA}$-formula and there are only countably many $\mathcal{L}_{PA}$-formulas.

(v) False. TA has countable non-standard models.

Question 2. [10 Points]

(a) [**5 Points**] Let $\mathcal{F}$ be the set of all **continuous** functions $f : \mathbb{R} \to \mathbb{R}$. Show that $|\mathcal{F}| = \mathfrak{c}$.

(b) [**5 Points**] Let $\mathcal{E}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Show that $|\mathcal{E}| > \mathfrak{c}$.

Solution

(a) See Homework 19(b). $\square$

(b) First observe that $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}| = \mathfrak{c}$. Next note that the function $H : \mathcal{P}(\mathbb{R}) \to \mathcal{E}$ defined by $H(X) = 1_X$ (where $1_X : \mathbb{R} \to \{0, 1\}$ is the characteristic function of X) is an injection. So by Cantor's theorem, $|\mathcal{E}| \geq |\mathcal{P}(\mathbb{R})| > |\mathbb{R}| = \mathfrak{c}$. $\square$

Question 3. [10 Points]

Using transfinite recursion, construct a function $f : \mathbb{R} \to \mathbb{R}$ such that for every interval $(a, b) \subseteq \mathbb{R}$ and $y \in \mathbb{R}$, there exists an **irrational** $x \in (a, b)$ such that $f(x) = y$.

Solution. Let $\mathcal{F}$ be the set of all pairs (J, y) where J is an open interval in $\mathbb{R}$ and $y \in \mathbb{R}$. It is easy to see that $|\mathcal{F}| = \mathfrak{c}$. Let $\langle (J_\alpha, y_\alpha) : \alpha < \mathfrak{c} \rangle$ be an injective sequence with range $\mathcal{F}$. Using transfinite recursion define $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ such that the following hold.

(a) Each f_α is a function and $\mathrm{dom}(f_\alpha) \cup \mathrm{range}(f_\alpha) \subseteq \mathbb{R}$.

(b) For every $\alpha < \mathfrak{c}$, $|f_\alpha| \leq |\alpha + \omega| < \mathfrak{c}$.

(c) $f_0 = \emptyset$ and if $\alpha < \mathfrak{c}$ is a limit ordinal, then $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$.

(d) If $\alpha < \beta < \mathfrak{c}$, then $f_\alpha \subseteq f_\beta$.

(e) For every $\alpha < \mathfrak{c}$, there exists an irrational $x \in J_\alpha \cap \mathrm{dom}(f_{\alpha+1})$ such that $f_{\alpha+1}(x) = y_\alpha$.

Suppose $\alpha < \mathfrak{c}$ and $\langle f_\beta : \beta < \alpha \rangle$ has been defined. We would like to define f_α such that Clauses (a)-(e) are preserved. If $\alpha = 0$ or a limit ordinal, we define f_α using Clause (c). It is easy to check that Clauses (a)-(e) are preserved.

So it suffices to define $f_{\alpha+1}$ assuming $\langle f_\beta : \beta \leq \alpha \rangle$ has already been defined. But this is easy: Since $|\mathrm{dom}(f_\alpha)| = |f_\alpha| < \mathfrak{c}$, $|\mathbb{Q}| = \omega < \mathfrak{c}$ and $|J_\alpha| = \mathfrak{c}$, we can choose an $x \in (J_\alpha \setminus (\mathrm{dom}(f_\alpha) \cup \mathbb{Q}))$ and define

$$f_{\alpha+1} = f_\alpha \cup \{(x, y_\alpha)\}$$

Having constructed $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$, define $g = \bigcup_{\alpha < \mathfrak{c}} f_\alpha$. Note that $\mathrm{dom}(g) \subseteq \mathbb{R}$ and for every open interval J and $y \in \mathbb{R}$, there exists an irrational $x \in J \cap \mathrm{dom}(g)$ such that $g(x) = y$. Extend g to a function $f : \mathbb{R} \to \mathbb{R}$ by defining f to be identically zero outside $\mathrm{dom}(g)$. It is clear that f is as required. $\square$

Question 4. [10 Points]

Recall that DLO is the theory of dense linear orderings without end-points.

(a) [**2 Points**] Show that $(\mathbb{Z}, <)$ is not an elementary submodel of $(\mathbb{Q}, <)$. Here $\mathbb{Z}$ is the set of all integers and $\mathbb{Q}$ is the set of all rationals.

(b) [**8 Points**] Let $M \subseteq \mathbb{R}$ be countable. Assume that $(M, <) \models DLO$. Show that $(M, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Solution

(a) The sentence $(\forall x)(\forall y)(\exists z)(x < y \implies ((x < z) \wedge (z < y)))$ is true in $(\mathbb{Q}, <)$ but false in $(\mathbb{Z}, <)$. So $Th(\mathbb{Q}, <) \neq Th(\mathbb{Z}, <)$ which implies the claim. $\square$

(b) Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol. Suppose $\psi(x, y_1, \ldots, y_n)$ is an $\mathcal{L}$-formula whose free variables are among $x, y_1, \ldots, y_n$ and $a_1, \ldots, a_n$ are in M. Assume that there exists $a \in \mathbb{R}$ such that $(\mathbb{R}, <) \models \psi(a, a_1, \ldots, a_n)$. We will show that there exists $b \in M$ such that $(\mathbb{R}, <) \models \psi(b, a_1, \ldots, a_n)$. By the Tarski-Vaught criterion, it will follow that $(M, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Using the Lemma on Slide 157, choose a countable $N \subseteq \mathbb{R}$ such that $M \subseteq N$ and $(N, <)$ is an elementary submodel of $(\mathbb{R}, <)$. Since $(N, <)$ is elementary submodel of $(\mathbb{R}, <)$ and $(\mathbb{R}, <) \models (\exists x)(\psi(x, a_1, \ldots, a_n))$ we can find $c \in N$ such that $(N, <) \models \psi(c, a_1, \ldots, a_n)$.

By an argument similar to Practice Final problem 4(a), we can find an isomorphism $f : (N, <) \to (N, <)$ such that $f(a_k) = a_k$ for every $1 \leq k \leq n$ and $f(c) \in M$. Put $f(c) = b$.

We claim that $(\mathbb{R}, <) \models \psi(b, a_1, \ldots, a_n)$ and therefore $b \in M$ is as required. Since f is an isomorphism, by the Lemma on Slide 148, we get $(N, <) \models \psi(c, a_1, \ldots, a_n)$ iff $(N, <) \models \psi(f(c), f(a_1), \ldots, f(a_n))$ iff $(N, <) \models \psi(b, a_1, \ldots, a_n)$. So $(N, <) \models \psi(b, a_1, \ldots, a_n)$. As $(N, <)$ is an elementary submodel of $(\mathbb{R}, <)$, it follows that $(\mathbb{R}, <) \models \psi(b, a_1, \ldots, a_n)$ as claimed. $\square$

Question 5. [10 Points]

(a) [**5 Points**] Let $W \subseteq \omega$ be an infinite c.e. set. Show that there is an infinite $X \subseteq W$ such that X is computable.

(b) [**5 Points**] Show that $\omega \setminus True_{\mathcal{N}}$ (defined on Slide 199) is not c.e.

Solution

(a) By Practice Final problem 5(a), we can fix an injective computable function $f : \omega \to \omega$ such that $\text{range}(f) = W$. Define $h : \omega \to \omega$ as follows. $h(0) = f(0)$ and $h(n+1) = f(m)$ where m is least such that $f(m) > h(n)$. Note that h is computable and strictly increasing. Put $\text{range}(h) = X$. Then $X \subseteq W$ is infinite and computable (see Question 1(iii)). $\square$

(b) Suppose not. Then $\omega \setminus True_{\mathcal{N}}$ is definable in $\mathcal{N}$ (by Homework problem (35)). Let $\phi(x)$ be an $\mathcal{L}_{PA}$-formula witnessing this. It follows that $True_{\mathcal{N}}$ is also definable in $\mathcal{N}$ via the formula $\neg\phi(x)$ which is impossible by Tarski's theorem. $\square$

Question 6. [10 Points]

Let T be a computable $\mathcal{L}_{PA}$-theory such that $PA \subseteq T \subseteq TA$. For $f : \omega \to \omega$, we say that f is **numeralwise representable in** T iff there is an $\mathcal{L}_{PA}$-formula $\psi(y, x)$ such that for every $(m, n) \in \omega^2$,

(i) If $f(m) = n$, then $T \vdash \psi(\overline{n}, \overline{m})$.

(ii) If $f(m) \neq n$, then $T \vdash \neg\psi(\overline{n}, \overline{m})$.

(a) [**4 Points**] Let $f : \omega \to \omega$. Show that f is numeralwise representable in T iff f is computable.

(b) [**6 Points**] Show that T is undecidable.

Solution

(a) First assume that f is computable. Then f is numeralwise representable in PA (Slide 201). Since $PA \subseteq T$, f is also numeralwise representable in T via the same formula.

Next suppose f is numeralwise representable in T via $\psi(y, x)$. Since T is computable, the set of theorems of T is c.e. (by Theorem (2) on Slide 186). So we can fix a program P such that for any $\mathcal{L}_{PA}$-sentence ϕ, P halts on input ϕ iff $T \vdash \phi$. Consider the program Q which on input m, starts running P with inputs $\psi(0, \overline{m}), \psi(\overline{1}, \overline{m}), \psi(\overline{2}, \overline{m}), \ldots$ until it finds an n such that P halts on input $\psi(\overline{n}, \overline{m})$ after which it outputs this n. It is clear that Q computes f. $\square$

(b) **(Sketch)** This is exactly like the proof of the fact that PA is undecidable. We just replace PA by T in the proof of the Theorem on Slide 202. The argument goes through because the function f in that proof is numeralwise representable in T (by part (a) above) and $T \subseteq TA$. The latter fact is used in the proof of the Exercise on that slide. $\square$

Bonus Question [5 Points]

Let $\langle X_n : n < \omega\rangle$ be a sequence of **uncountable** sets. Show that there exists $\langle Y_n : n < \omega\rangle$ such that

(a) For every $n < \omega$, Y_n is uncountable and $Y_n \subseteq X_n$.

(b) For every $m < n < \omega$, $Y_n \cap Y_m = \emptyset$.

Solution. Using transfinite recursion, construct $\langle x_{\alpha,n} : n < \omega \text{ and } \alpha < \omega_1\rangle$ as follows.

(a) Each $x_{0,n} \in X_n$ and $x_{0,n}$'s are pairwise distinct.

(b) Suppose $\alpha < \omega_1$ and $x_{\beta,n}$ has been chosen for every $\beta < \alpha$ and $n < \omega$. Put $W = \{x_{\beta,n} : \beta < \alpha \text{ and } n < \omega\}$. Choose $x_{\alpha,n} \in X_n \setminus W$ such that whenever $m \neq n$, $x_{\alpha,n} \neq x_{\alpha,m}$. This can be done because W is countable while each X_n is uncountable.

Put $Y_n = \{x_{\alpha,n} : \alpha < \omega_1\}$. It is clear that Y_n's are pairwise disjoint uncountable subsets of X_n's. $\square$

End of Exam