Assignment 2

Limits of Sequences, MTH 301A, 2022

1. Let $\{x_n\}$ be a sequence. If $\limsup_{n\to\infty} x_n = l$ and there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = l'$ then $l' \leq l$. Make a similar statement about $\liminf$ and prove it.

2. Let $\{x_n\}$ be a sequence.
   (a) If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = l < 1$ then $\lim_{n\to\infty} x_n = 0$.
   (b) If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = l > 1$ then $\lim_{n\to\infty} x_n = \infty$.
   (c) If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ then we cannot conclude.

3. Find $\limsup x_n$ and $\liminf x_n$ for each sequence.
   (a) $x_n = (-1)^n$
   (b) $x_n = \sin \frac{n\pi}{2}$
   (c) $x_n = \frac{1+(-1)^n}{2}$
   (d) $x_n = n \sin \frac{n\pi}{2}$.
   (e) $x_n = \sin n\pi + \cos n\pi$
   (f) Let $x_0 = -2$
      i. $x_n = 3x_{n-1}$.
      ii. $x_n = x_{n-1}$
      iii. $x_n = \frac{1}{2}x_{n-1}$
      iv. $x_n = \alpha x_{n-1}$ for some $\alpha < 1$.
   (g) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.
      i. Prove that $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.
      ii. Prove that $\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$.
      iii. Find the two counter examples to show that the equalities may not hold in part (a) and part (b).
   (h) Let $\{x_n\}$ be a convergent sequence and $\{y_n\}$ be an arbitrary sequence.
      i. Prove that $\limsup_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.
      ii. Prove that $\liminf_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$.
   (i) Determine whether or not the sequence, x_n defined for each integer n, by the finite series

$$x_n = \arctan 1 + \arctan 2 + \arctan 3 + \cdots + \arctan n,$$

   for $n \geq 1$, converges as $n \to \infty$?

4. Let $\{c_{k,n} : 1 \leq k \leq n, n \geq 1\} \subset \mathbb{R}$ such that

(a) $\lim\limits_{n\to\infty} c_{k,n} = 0$.

(b) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} c_{k,n} = 1$.

(c) There exists $C > 0$ such that $\sum\limits_{k=1}^{n} |c_{k,n}| \leq C, \ \forall n$.

Let $\{a_n\}$ be a sequence define $b_n = \sum\limits_{k=1}^{n} a_k c_{k,n}, \ \forall n \geq 1$. If $\lim\limits_{n\to\infty} a_n = 0$ then show that $b_n \to 0$ as $n \to \infty$. Moreover, if $a_n \to a$ then $b_n \to a$ as $n \to \infty$.

5. Let $\lim\limits_{n\to\infty} a_n = a$. Show that

(a) $\lim\limits_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = a$.

(b) $\lim\limits_{n\to\infty} \frac{na_1+(n-1)a_2+\ldots 1.a_n}{n} = a$.

(c) If $\lim\limits_{n\to\infty} b_n = b$ then $\lim\limits_{n\to\infty} \frac{a_1 b_n+\cdots+a_n b_1}{n} = ab$.

(d) Let $b_n > 0, \ \forall n$ and $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} b_k = +\infty$ with $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \beta$ then $\lim\limits_{n\to\infty} \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} = \beta$ and $\lim\limits_{n\to\infty} \frac{a_1 b_1+\cdots+a_n b_n}{b_1+\cdots+b_n} = a$.

Using this prove that

Theorem 0.1. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that*

i. y_n strictly increases to $+\infty$.

ii. $\lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \beta$.

Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \beta$.

6. Find

(a) $\lim\limits_{n\to\infty} \frac{1}{n^{k+1}} \left(k! + \frac{(k+1)!}{1!} + \cdots + \frac{(k+n)!}{n!} \right), \quad k \in \mathbb{N}$.

(b) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} \right)$.

(c) $\lim\limits_{n\to\infty} \frac{1^k+2^k+\cdots+n^k}{n^{k+1}}, \ k \in \mathbb{N}$.

(d) If $\{a_n\}$ is a sequence such that $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = a$. Then show that $\lim\limits_{n\to\infty} \frac{a_n}{n} = a$.