# PACF for AR(2)

$\{X_t\}$ is covariance stationary AR(2)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$\alpha(1) = \rho(1)$$

$$\alpha(2) = \text{Corr}\left(X_3 - P_{X_2} X_3 \,,\, X_1 - P_{X_2} X_1\right)$$

$$P_{X_2} X_3 = \rho(1)\, X_2 \qquad P_{X_2} X_1 = \rho(1)\, X_2$$

$$\alpha(2) = \text{Corr}\left(X_3 - \rho(1) X_2 \,,\, X_1 - \rho(1) X_2\right)$$

$$= \frac{\text{Cov}\left(X_3 - \rho(1) X_2 \,,\, X_1 - \rho(1) X_2\right)}{\left[V\left(X_3 - \rho(1) X_2\right) V\left(X_1 - \rho(1) X_2\right)\right]^{1/2}}$$

$$V\left(X_3 - \rho(1) X_2\right) = \gamma_0 + \rho(1)^2 \gamma_0 - 2\rho(1)\gamma_1$$

$$= V\left(X_1 - \rho(1) X_2\right)$$

$$\text{Cov}\left(X_3 - \rho(1) X_2 \,,\, X_1 - \rho(1) X_2\right)$$

$$= \gamma_2 - \rho(1)\gamma_1 - \rho(1)\gamma_1 + \rho(1)^2 \gamma_0 .$$

$$= \gamma_2 - 2\rho(1)\gamma_1 + \rho(1)^2 \gamma_0 \neq 0$$

$$\alpha(2) \neq 0 .$$

$$\alpha(3) = \text{Corr}\left(X_4 - P_{(X_2, X_3)} X_4 \,,\, X_1 - P_{(X_2, X_3)} X_1\right)$$

$$= \text{Corr}\left(X_4 - \phi_1 X_3 - \phi_2 X_2 \,,\, X_1 - \alpha_{1(BLP)} X_2 - \alpha_{2(BLP)} X_3\right)$$

$$= \text{Corr}\left(\epsilon_4 \,,\, X_1 - \alpha_{1(BLP)} X_2 - \alpha_{2(BLP)} X_3\right)$$

$$= 0 .$$

Further $\forall K \geq 3$ ; $\alpha(K) = 0$.

<u>Remark</u> : The above pattern holds true for a stationary AR(p) and for such a model

$$\alpha(K) = 0 \quad \forall\ K > p$$

<u>Remark</u> : For a stationary AR(p)

$\forall K > p$ ; $\hat{\alpha}(K) \overset{asym}{\sim} N\left(0, \frac{1}{n}\right)$ for large $n$

estimator of $\alpha(K)$

<u>PACF for MA(1)</u>

$$X_t = \epsilon_t + \theta\epsilon_{t-1} \ ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\alpha(1) = \rho_1 = \frac{\theta}{1+\theta^2}.$$

$$\alpha(2) = Corr\left(X_3 - P_{X_2}X_3 \,,\, X_1 - P_{X_2}X_1\right)$$

$$P_{X_2}X_3 = \rho_1 X_2 \quad \& \quad P_{X_2}X_1 = \rho_1 X_2$$

$$= \theta^* X_2 \text{, say} \qquad\qquad = \theta^* X_2$$

$$\theta^* = \frac{\theta}{1+\theta^2}.$$

$$\alpha(2) = Corr\left(X_3 - \theta^* X_2 \,,\, X_1 - \theta^* X_2\right)$$

$$= \frac{Cov\left(X_3 - \theta^* X_2 \,,\, X_1 - \theta^* X_2\right)}{\left[V\left(X_3 - \theta^* X_2\right) V\left(X_1 - \theta^* X_2\right)\right]^{1/2}}$$

$$V(X_3 - \theta^* X_2) = \gamma_0 + \theta^{*2}\gamma_0 - 2\theta^*\gamma_1$$

$$= \sigma^2(1+\theta^2) + \left(\frac{\theta}{1+\theta^2}\right)^2 (1+\theta^2)\sigma^2 - 2\left(\frac{\theta}{1+\theta^2}\right)\theta\sigma^2$$

$$= \sigma^2\left((1+\theta^2) - \frac{\theta^2}{1+\theta^2}\right)$$

$$= \sigma^2\left(\frac{1+\theta^2+\theta^4}{1+\theta^2}\right) = V(X_1 - \theta^* X_2)$$

$$\mathrm{Cov}(X_3 - \theta^* X_2, X_1 - \theta^* X_2)$$

$$= \underbrace{0}_{\gamma_2} - \theta^*\gamma_1 - \theta^*\gamma_1 + \theta^{*2}\gamma_0$$

$$= -2\left(\frac{\theta}{1+\theta^2}\right)\theta\sigma^2 + \left(\frac{\theta}{1+\theta^2}\right)^2 \sigma^2(1+\theta^2)$$

$$= -\frac{\theta^2}{1+\theta^2}\cdot\sigma^2$$

$$\Rightarrow \alpha(2) = -\frac{\theta^2}{1+\theta^2+\theta^4}.$$

in general, $\alpha(k) = -\dfrac{(-\theta)^k}{1+\theta^2+\cdots+\theta^{2k}}$.

__Remark__: PACF of MA(1) does not cut off but tails off

__Remark__: MA(q) process has a similar behavior of PACF.

__Remark__: As ARMA(p, q) has MA part, PACF of ARMA process also does not cut off.

Remark: To sum up the behavior of ACF & PACF

| Model | ACF | PACF |
|---|---|---|
| AR(1) | decays exponentially (tails off) | single spike (cuts off) |
| MA(1) | cuts off (after 1 spike) | tails off |
| AR(p) | tails off | cuts off (after p spikes) |
| MA(q) | cuts off (after q spikes) | tails off |
| ARMA(p,q) | tails off | tails off |

The above table is to be used for model identification

# Model order estimation

Using penalized log likelihood criteria or the information theoretic criteria

## Akaike information Criterion (AIC)

General form of AIC:

$$AIC(K) = -2 \log \hat{L} + 2K$$

K: # of parameters in the model

ARMA(p, q) model

$$AIC(p, q) = -2 \log \hat{L} + 2(p+q+1)$$

$$(\hat{p}, \hat{q}) = \underset{\substack{p \in \{0,1,\dots,P\} \\ q \in \{0,1,\dots Q\}}}{\arg\min} AIC(p, q)$$

## Bayesian Information Criterion (BIC)

General form of BIC:

$$BIC(K) = -2 \log \hat{L} + (\log n) K$$

ARMA(p, q) model

$$BIC(p, q) = -2 \log \hat{L} + (\log n)(p+q+1)$$

$$(\hat{p}, \hat{q}) = \underset{\substack{p \in \{0,1,\dots,P\} \\ q \in \{0,1,\dots,Q\}}}{\arg\min} BIC(p, q)$$

# Frequency Domain Analysis

**Aim** : To study the frequency (corresponding to periodic component) properties of time series and identify dominant frequencies that drive the time series

**Tool** : Spectral density function

**Def**$^n$: Spectral density

Suppose that $\{X_t\}$ is a stationary zero mean time series with autocovariance f$^n$ $\gamma(\cdot)$ satisfying $\sum_h |\gamma(h)| < \infty$. The spectral density of $\{X_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \; ; \quad -\infty < \lambda < \infty$$