Random sampling from stationary time series

Let $X_1, \ldots, X_n$ be a sample of size n from a stationary time series with

(i) $E X_t = \mu \quad \forall t$

(ii) $\gamma_h = Cov(X_t, X_{t+h}) = E(X_t - \mu)(X_{t+h} - \mu) \quad \forall t$

& (iii) $\sum_h |\gamma_h| < \infty$

Estimation of μ

$\bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$ is an unbiased estimator for μ

$E \bar{X}_n = \mu$

$V \bar{X}_n = \frac{1}{n} \sum_{|h| \leq n} \left(1 - \frac{|h|}{n}\right) \gamma_h$

Some important asymptotic results:

Result 1: $E(\bar{X}_n - \mu)^2 \longrightarrow 0$ as $n \to \infty$

i.e. $\bar{X}_n \xrightarrow{m.s.} \mu$

Pf: $n E(\bar{X}_n - \mu)^2 = n V(\bar{X}_n) = \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_h$

i.e. $n E(\bar{X}_n - \mu)^2 = \left| \gamma_0 + \left(1 - \frac{1}{n}\right) 2\gamma_1 + \left(1 - \frac{2}{n}\right) 2\gamma_2 + \cdots \right.$

$\left. \cdots + \left(\frac{n-(n-1)}{n}\right) 2\gamma_{n-1} \right|$

$\leq |\gamma_0| + \left(\frac{n-1}{n}\right) 2|\gamma_1| + \left(\frac{n-2}{n}\right) 2|\gamma_2| + \cdots$

$\cdots + \frac{1}{n} 2|\gamma_{n-1}|$

$< |\gamma_0| + 2|\gamma_1| + \cdots + 2|\gamma_{n-1}|$

$\leq 2 \sum_{h=0}^{n-1} |\gamma_h| \to 2 \sum_{0}^{\infty} |\gamma_h| < \infty$

$$\Rightarrow E(\bar{X}_n - \mu)^2 = O\left(\tfrac{1}{n}\right)$$

$$\Rightarrow E(\bar{X}_n - \mu)^2 \to 0 \text{ as } n \to \infty$$

i.e. $\bar{X}_n \xrightarrow{m.s.} \mu$

<u>Remark</u> : as $\bar{X}_n \xrightarrow{m.s.} \mu$; we also have $\bar{X}_n \xrightarrow{p} \mu$

<u>Result 2</u> : $\displaystyle\lim_{n\to\infty} n\, V(\bar{X}_n) = \sum_{h=-\infty}^{\infty} \gamma_h$

<u>Pf</u> :
$$n\, V(\bar{X}_n) = n\, E(\bar{X}_n - \mu)^2$$
$$= \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_h$$
$$= \sum_{h=-n}^{n} \gamma_h - \sum_{h=-n}^{n} \frac{|h|}{n} \gamma_h$$

$$\lim_{n\to\infty} \sum_{h=-n}^{n} \gamma_h = \sum_{-\infty}^{\infty} \gamma_h$$

Also, $\displaystyle\sum_{h=-n}^{n} \frac{|h|}{n} \gamma_h = 2 \sum_{h=0}^{n} \frac{h\gamma_h}{n}$

Now $\displaystyle\sum_{h=0}^{n} \frac{h\gamma_h}{n} = \sum_{h=0}^{N} \frac{h\gamma_h}{n} + \sum_{N+1}^{n} \frac{h}{n} \gamma_h$

$$\lim_{n\to\infty} \frac{1}{n} \underbrace{\sum_{h=0}^{N} h\gamma_h}_{\text{finite qty}} = 0$$

& $\displaystyle\left| \sum_{N+1}^{n} \frac{h}{n} \gamma_h \right| \le \sum_{N+1}^{n} \left| \frac{h}{n} \gamma_h \right| \le \sum_{N+1}^{n} |\gamma_h|$

Since $\sum_{0}^{\infty} |\gamma_h| < \infty$, $\forall\ \epsilon > 0$ $\exists$ an N_0 (for large n)

$\ni \forall\ N \geq N_0$ $\sum_{N+1}^{\infty} |\gamma_h| < \epsilon$

$\Rightarrow \left| \sum_{N+1}^{n} \frac{h}{n} \gamma_h \right| < \epsilon$ for large n

$\Rightarrow \lim_{n \to \infty} \sum_{h=-n}^{n} \frac{|h|}{n} \gamma_h = 0$

Hence $\lim_{n \to \infty} n V(\bar{X}_n) = \sum_{h=-\infty}^{\infty} \gamma_h$

Example 1

AR(1) $\quad X_t = \phi X_{t-1} + \epsilon_t$; $|\phi| < 1$, $\epsilon_t \sim WN(0, \sigma^2)$

$$\gamma_h = \frac{\sigma^2}{1-\phi^2} \phi^{|h|}$$

$\Rightarrow \sum |\gamma_h| < \infty$ (as $|\phi| < 1$)

$$n V(\bar{X}_n) = \sum_{h=-\infty}^{\infty} \frac{\sigma^2}{1-\phi^2} \cdot \phi^{|h|} \quad \text{for large } n$$

$$= \frac{\sigma^2}{1-\phi^2} \left(1 + 2\phi + 2\phi^2 + \cdots \right)$$

$$= \frac{\sigma^2}{1-\phi^2} \left(1 + \frac{2\phi}{1-\phi} \right) = \frac{\sigma^2}{1-\phi^2} \cdot \frac{1+\phi}{1-\phi}$$

$$= \frac{\sigma^2}{(1-\phi)^2} ; \quad V(\bar{X}_n) \approx \frac{1}{n} \frac{\sigma^2}{(1-\phi)^2} \text{ for large } n$$

Example 2 :

X_t is stationary ARMA(p, q)

$$\phi(B) X_t = \theta(B) \epsilon_t$$

$$X_t = \phi(B)^{-1} \theta(B) \epsilon_t$$

i.e. $X_t = \Psi(B) \epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$

If $\sum |\psi_j| < \infty$, then $\sum |\gamma_h| < \infty$

then $n V(\bar{X}_n) \simeq \sum_{h=-\infty}^{\infty} \gamma_h$ for large n

Remark : Note that ACGF of $\{X_t\}$ is

$$g_X(z) = \sum_{h=-\infty}^{\infty} \gamma_h z^h$$

$$\Rightarrow \sum_{-\infty}^{\infty} \gamma_h = \lim_{n \to \infty} n V(\bar{X}_n) = g_X(1)$$

MA(1)

$X_t = \theta(B) \epsilon_t$

$g_X(z) = \sigma^2 \theta(z) \theta(z^{-1})$

$g_X(1) = \sigma^2 (1+\theta)^2$

(i) $X_t \sim AR(1)$; $\phi(B) X_t = \epsilon_t$

ACGF $g_X(z) = \dfrac{\sigma^2}{\phi(z) \phi(z^{-1})} = \dfrac{\sigma^2}{(1-\phi z)(1-\phi z^{-1})}$

$$\Rightarrow \lim_{n \to \infty} n V(\bar{X}_n) = \sum_{h=-\infty}^{\infty} \gamma_h = g_X(1) = \frac{\sigma^2}{(1-\phi)^2}.$$

(ii) $X_t \sim MA(\infty)$ $\quad X_t = \Psi(B) \epsilon_t = \sum_0^{\infty} \psi_j \epsilon_{t-j}$

with $\sum |\psi_j| < \infty$

then $\sum_{-\infty}^{\infty} \gamma_h = g_X(1) = \sigma^2 (\Psi(1))^2 = \sigma^2 (1 + \psi_1 + \psi_2 + \cdots)^2$

Distribution of $\bar{X}_n$

Case 1: Gaussian time series

Suppose $\{X_t\}$ is a Gaussian time series.

i.e. $\forall n$, $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N_n(\mu \underset{\sim}{1}_n, \Gamma_n)$

$$\Gamma_n = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \gamma_2 \\ & & & \ddots & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}.$$

[Note that if $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ then

(i) $\underset{\sim}{X} - \underset{\sim}{\mu} \sim N_p(\underset{\sim}{0}, \Sigma)$

(ii) $\underset{q\times p}{A}\underset{\sim}{X} + \underset{q\times 1}{\underset{\sim}{b}} \sim N_q(A\underset{\sim}{\mu} + \underset{\sim}{b}, A\Sigma A')$

(i) & (ii) follows from the def^n of N_p dist^n.]

$$\bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t = 1'_n \underset{\sim}{X}/n \; ; \; \underset{\sim}{X} = (X_1, \ldots, X_n)'$$

Using (ii) above,

$$\bar{X}_n = \left(\frac{1}{n} 1'_n\right) \underset{\sim}{X}$$

$$\sim N_1\left(\frac{1}{n} 1'_n \cdot \mu \underset{\sim}{1}_n , \frac{1}{n^2} \underset{\sim}{1}'_n \Gamma_n \underset{\sim}{1}_n\right)$$

i.e. $\bar{X}_n \sim N_1\left(\mu, \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma_h\right)$

i.e. $\sqrt{n}\left(\bar{X}_n - \mu\right) \sim N_1\left(0, \sum_{|h|\leq n}\left(1-\frac{|h|}{n}\right)\gamma_h\right)$

Let $v = \sum_{|h|\leq n}\left(1-\frac{|h|}{n}\right)\gamma_h$

$$P\left(\frac{\sqrt{n}\,|\bar{X}_n-\mu|}{v^{1/2}} \leq \tau_{\alpha/2}\right) = 1-\alpha \quad -(*)$$

$\tau_{\alpha/2}$ is a pt $\ni P(Z > \tau_{\alpha/2}) = \alpha/2$; $Z \sim N(0,1)$

$(*) \Rightarrow$ the $100(1-\alpha)\%$ confidence interval for μ in such a case is

$\bar{X}_n \mp \tau_{\alpha/2}\, v^{1/2}$ (provided v is known)

Note that v is usually unknown as $\{\gamma_h\}$ is unknown and we use estimate of v as

$$\hat{v} = \sum_{|h|\leq n}\left(1-\frac{|h|}{n}\right)\hat{\gamma}_h$$

Note

$$\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sqrt{\hat{v}}} \xrightarrow{L} N(0,1)$$

$\xrightarrow{L}$: convergence in law

So, the asymptotic $100(1-\alpha)\%$ confidence interval for μ is $\bar{X}_n \mp \tau_{\alpha/2}\sqrt{\hat{v}}$