Assignment #4

1) Let $S \subseteq \mathbb{R}^2$ be open and $f : S \to \mathbb{R}$ be diff'ble at each pt of S. let $x, y \in S$ s.t. $L(x,y) \subseteq S$. Then show that $\exists$ a pt z in the interior of $L(x,y)$ s.t. $f(y) - f(x) = f'(z)(y-x)$.

Sol^n In the MVT, consider $m = 1$ and $a = 1 \in \mathbb{R}^m$. It follows that

$$f(y) - f(x) = f'(z)(y-x).$$

2) Find all first and 2nd partial derivative of z w.r.t. x and y if $xy + yz + xz = 1$.

Sol^n

x, y - indep
z depends on x and y

$$\frac{\partial x}{\partial x} y + x\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x} z + y\frac{\partial z}{\partial x} + \frac{\partial x}{\partial x} z + x\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0$$

$$(x+y)\frac{\partial z}{\partial x} = -(y+z) \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}}$$

- $$\frac{\partial^2 z}{\partial x^2} = +\frac{(y+z)}{(x+y)^2} - \frac{1}{x+y}\cdot\frac{\partial z}{\partial x} = \frac{y+z}{(x+y)^2} + \frac{y+z}{(x+y)^2} = \frac{2(y+z)}{(x+y)^2}$$

$\frac{\partial z}{\partial y} = ?$

$$x + z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}}.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x+z}{(x+y)^2} - \frac{1}{x+y}\frac{\partial z}{\partial y} = 2\frac{(x+z)}{(x+y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\left(1+\frac{\partial z}{\partial x}\right)(x+y) - (x+z)\cdot 1}{(x+y)^2} = \frac{x+y+\frac{-(y+z)}{\cancel{x+y}}\cancel{(x+y)} - x - z}{(x+y)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\cancel{y} - \cancel{y} - z - z}{(x+y)^2} = \frac{-2z}{(x+y)^2} = \frac{\partial^2 z}{\partial y \partial x}.$$

3) Prove that $f(x,y) = \sin(y - ax) + \ln(y + ax)$ is a sol^n to the wave equation $D_{1,1}f = a^2 D_{2,2}f$.

$$D_1 f = -a\cos(y-ax) + \frac{a}{(y+ax)} = a\left[\frac{1}{y+ax} - \cos(y-ax)\right]$$

$$D_{1,1}f = a\left[-\frac{1\cdot a}{(y+ax)^2} + \sin(y-ax)\cdot(-a)\right]$$

$$= -a^2\left[\frac{1}{(y+ax)^2} + \sin(y-ax)\right]$$

$$D_2 f = \cos(y - ax) + \frac{1}{y + ax}$$

$$a^2 D_{2,2} f = -a^2\left[\sin(y - ax) + \frac{1}{(y+ax)^2}\right]$$

Hence $\quad D_{1,1} f = a^2 D_{2,2} f$

4) Let α and k be constants. Prove that the function $f(x,t) = e^{-\alpha^2 k^2 t} \sin(kx)$ is a sol^n to the heat eq^n

$$D_2 f = \alpha^2 D_{1,1} f.$$

$$D_2 f = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$$

$$D_1 f = k \, e^{-\alpha^2 k^2 t} \cos kx$$

$$D_{1,1} f = -k^2 e^{-\alpha^2 k^2 t} \sin kt$$

$$\Rightarrow \alpha^2 D_{1,1} f = D_2 f. \qquad \square$$

5) Find all local maxima and local minima for the following functions.

(i) $f_1(x,y) = x^2 + y^2$

(ii) $f_2(x,y) = x^2 - y^2$

(iii) $f_3(x,y) = x^4 + y^4$

(iv) $f_4(x,y) = x^3 + y^3$.

Theorem. $f: S \to \mathbb{R}$, $S \subseteq \mathbb{R}^2$ with cts 2nd-ord partial derivatives at a stationary pt $a \in \mathbb{R}^2$. Let

$$\begin{pmatrix} D_{11} f(a) & D_{12} f(a) \\ D_{12} f(a) & D_{22} f(a) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$A = D_{11} f(a)$

$B = D_{12} f(a)$

$C = D_{22} f(a)$

$\Delta = AC - B^2$

$\Delta_0 = 1, \quad \Delta_1 = A, \quad \Delta_2 = \Delta$

a) $\Delta > 0$, $A > 0$ ($A = \Delta_1$) $\Rightarrow$ At a f has a local min

b) $\Delta > 0$, $A < 0$ " " " " max

c) If $\Delta < 0$, then f has a saddle pt. at a.

1) $f_1(x,y) = x^2+y^2$

$D_1 f(x,y) = 2x = 0 \Leftrightarrow x=0$

$D_2 f(x,y) = 2y = 0 \Leftrightarrow y=0.$

$a=(0,0)$ - is the only stationary point

$A = D_{11} f(a) = 2$

$B = D_{1,2} f(a) = 0$ $\qquad \Delta = AC - B^2 = AC = 4 > 0.$

$C = D_{2,2} f(a) = 2$

$\Delta_0 \quad \Delta_1 = A = 2 \quad \Delta_2 = \Delta = 4 > 0$

At $a=(0,0)$, the f^n has a local minimum.

2) $f(x,y) = x^2 - y^2$ $\qquad A = D_{11} f(a) = 2$

$a=(0,0)$ - stationary pt $\qquad B = D_{12} f(a) = 0$

$C = D_{22} f(a) = -2$

$\Delta = -4 < 0 \Rightarrow f$ has a saddle point at a.

3) $f(x,y) = x^4+y^4$ $\qquad$ Thm is not applicable

$(0,0)$ stationary pt. $\qquad$ but f has local min. at $(0,0)$

4) $f(x,y) = x^3+y^3$. $\qquad (0,0)$ - is the only stationary point

Thm is not applicable. But one can show

that $(0,0)$ is saddle point of f.

6) The plane $x+y-z=1$ intersects the cylinder $x^2+y^2=1$ in an ellipse. Find the point on the ellipse closest to ~~the~~ and farthest from the origin applying Lagrange's method.

$f(x,y,z) = x^2+y^2+z^2$

$g_1(x,y,z) = x+y-z-1$

$g_2(x,y,z) = x^2+y^2-1$

Lagrangian f^n $\quad \varphi(x,y,z) = x^2+y^2+z^2 + \lambda_1(x+y-z-1) + \lambda_2(x^2+y^2-1)$

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$\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda_1 + 2x\lambda_2 = 0$

$\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda_1 + 2y\lambda_2 = 0$

$\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda_1 = 0$

$\left.\begin{array}{l} 2x^2 + \lambda_1 x + 2x^2\lambda_2 = 0 \\ 2y^2 + \lambda_1 y + 2y^2\lambda_2 = 0 \end{array}\right\} \Rightarrow 2\underline{(x^2+y^2)} + \lambda_1(x+y) + 2\underline{(x^2+y^2)}\lambda_2 = 0$

$\Rightarrow \underline{2 + \lambda_1(1+z) + 2\lambda_2 = 0}$

$2(x+y-z) + 3\lambda_1 + 2(x+y)\lambda_2 = 0$

$2 + 3\lambda_1 + 2(1+z)\lambda_2 = 0$

$\Rightarrow 2 + \lambda_1\left(1+\frac{\lambda_1}{2}\right) + 2\lambda_2 = 0$

$2 + 3\lambda_1 + 2\left(1+\frac{\lambda_1}{2}\right)\lambda_2 = 0$

$\Rightarrow \lambda_2 = -1 - \frac{\lambda_1}{2}\left(1+\frac{\lambda_1}{2}\right)$

$2 + 3\lambda_1 + 2\left(1+\frac{\lambda_1}{2}\right)\left(1+\frac{\lambda_1}{2}\left(1+\frac{\lambda_1}{2}\right)\right) = 0$

$\Rightarrow 2 + 3\lambda_1 - \cancel{2}\left[\left(2+\frac{\lambda_1}{\cancel{2}}\right)\left(1+\frac{\lambda_1}{2}+\frac{\lambda_1^2}{4}\right)\right] = 0$

$\Rightarrow \cancel{2} + \underline{3\lambda_1} - \cancel{2} - \underline{\lambda_1} - \frac{\lambda_1^2}{2} - \underline{\lambda_1} - \frac{\lambda_1^2}{2} - \frac{\lambda^3}{4} = 0$

$\Rightarrow \cancel{2\lambda_1}\ \lambda_1 - \lambda_1^2 - \frac{\lambda^3}{4} = 0$

$-\frac{\lambda_1}{4}\left(\lambda_1^2 + 4\lambda_1 - 4\right) = 0$

$\lambda_1 = 0$

$(\lambda_1+2)^2 = 8$

$\lambda_1 = -2 \pm 2\sqrt{2}$

$x = \frac{1}{\sqrt{2}}$

$y = \frac{1}{\sqrt{2}}$

$z = -1 \pm \sqrt{2}$

$x^2+y^2+z^2$

$= 1 + (-1 \pm \sqrt{2})^2 = 1 + 1 + 2 + \left(2\cdot(-1)(\pm\sqrt{2})\right)$

$= 4 \mp 2\sqrt{2}$

$\boxed{z = -1 \pm \sqrt{2}}$

$\boxed{\lambda_1 = -2 \pm 2\sqrt{2}}$

$2 + (-2 \pm 2\sqrt{2})(\pm\sqrt{2})$

$+ 2\lambda_2 = 0$

$1 + (-1 \pm \sqrt{2})(\pm\sqrt{2}) + \lambda_2 = 0$

$1 \mp \sqrt{2} + 2 = \lambda_2$

$\Rightarrow \boxed{\lambda_2 = -3 \pm \sqrt{2}}$

$2x(1+\lambda_2) = -\lambda_1$

$x = \frac{-\lambda_1}{2(1+\lambda_2)}$

$y = \frac{-\lambda_1}{2(1+\lambda_2)}$

$x = y \Rightarrow x^2+y^2 = 1$

$2x^2 = 1$

7) Find all points on the surface $xy - z^2 + 1 = 0$ that are closest to the origin.

$f(x,y,z) = x^2 + y^2 + z^2$

$\varphi(x,y,z) = x^2 + y^2 + z^2 + \lambda(xy - z^2 + 1)$

$$\left.\begin{aligned} 2x + \lambda y &= 0 \\ 2y + \lambda x &= 0 \end{aligned}\right\}$$

$2z - 2\lambda z = 0$

$z(1-\lambda) = 0$

$xy + 1 - z^2 = 0$

$\Rightarrow z = \pm 1$

$\underline{\lambda = 1}$. $\Rightarrow$ $\left.\begin{aligned} 2x + y &= 0 \\ 2y + x &= 0 \end{aligned}\right] \Rightarrow x = y = 0$

$(x,y,z) = \underbrace{(0, 0, \pm 1)}$.

pts closest to origin.

$(1, -1, 0)$

$(-1, 1, 0)$

$\underline{z = 0}$

$xy = -1$.

~~$x = y = \pm 1$~~

$2x^2 + \lambda = 0$

$2y^2 + \lambda = 0$

$x = \pm y$ $\quad x = -y$

$\underline{\text{Thm}}$. $f : S \to \mathbb{R}$, $g = (g_1, \dots, g_m) : S \to \mathbb{R}^m$ C^1, $S \subseteq_{\text{open}} \mathbb{R}^n$, $m < n$.

$x_0 \in X_0 = \{ x \in S \mid g(x) = 0 \}$. Assume $B(x_0) \subseteq \mathbb{R}^n$ n-ball s.t. $\forall x \in X_0 \cap B(x_0)$, $f(x) \geq f(x_0)$ ($\leftarrow$ local min)

or $\forall x \in X_0 \cap B(x_0)$ $f(x) \leq f(x_0)$ ($\leftarrow$ local max).

Assume that $\det (D_j g_i(x_0))_{m \times m} \neq 0$.

then $\exists$ $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

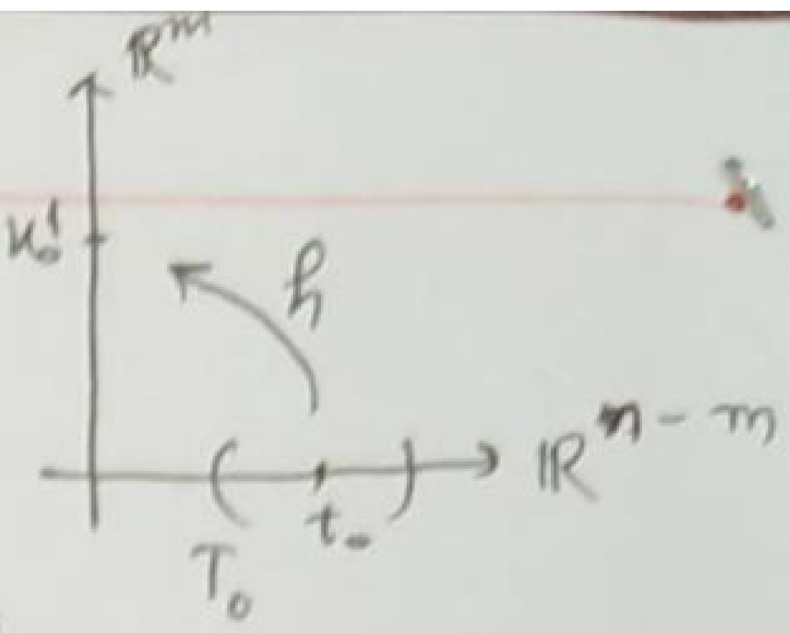
$$D_r f(x_0) + \sum_{i=1}^{m} \lambda_i D_r g_i(x_0) = 0 \quad \forall r = 1, \dots, n \quad \cdots \text{(A)}$$

$\underline{\text{Pf}}$ 1) $\sum_{i=1}^{m} \lambda_i D_r g_i(x_0) = D_r f(x_0)$ --- (1) $r = 1, 2, \dots, m$. sys of m lin. eq$^{\text{n}}$s unknowns λ_i

– has unique sol$^{\text{n}}$ as $\det(D_j g_i(x_0)) \neq 0$.

choose $\lambda_1, \dots, \lambda_m$ sol$^{\text{n}}$ of (1) So first m eq$^{\text{n}}$s in (A) hold true.

Tool Implicit f^n thm. $\quad g: S \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$



$$g(x) = g(x', t)$$

$$x = (x_1, \dots, x_m, x_{m+1}, \dots, x_n)$$
$$= (x', t) \qquad x' = (x_1, \dots, x_m)$$
$$t = (x_{m+1}, \dots, x_n)$$

$$x_0 = (x_0', t_0)$$

Implicit f^n thm $\Rightarrow$

$$\exists \ h : \underset{\substack{\text{open in} \\ \mathbb{R}^{n-m}}}{T_0} \to \mathbb{R}^m \quad C^1 \qquad h(t_0) = x_0'$$

$$\text{s.t.} \quad \underline{g(h(t), t)} = 0 \quad \forall \ t = (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m}.$$
$$\Downarrow$$
$$g_p(h(t), t) = 0$$

$$h = (h_1, \dots, h_m)$$

$$f(x_{m+1}, \dots, x_n) = f(h_1(x_{m+1}, \dots, x_n), \dots, h_m(x_{m+1}, \dots, x_n), x_{m+1}, \dots, x_n)$$
$$= f \circ H(x_{m+1}, \dots, x_n) \qquad H : T_0 \to \mathbb{R}^n; \quad H_k(x_{m+1}, \dots x_n) = \begin{cases} h_k(x) & k \leq m \\ x_k & k \geq m+1 \end{cases}$$

$$G_p(x_{m+1}, \dots, x_n) = g_p(h_1(x_{m+1}, \dots, x_n), \dots, h_m(x_{m+1}, \dots, x_n), x_{m+1}, \dots, x_n)$$
$$= g_p \circ H(x_{1+m}, \dots, x_n)$$

$$D_r G_p(t_0) = 0 \Rightarrow \sum_{k=1}^{n} D_k g_p(x_0) D_r H_k(t_0) = 0 \Rightarrow \sum_{k=1}^{m} D_k g_p(x_0) D_r h_k(x_0) + D_{m+r} g_p(x_0) = 0 \to ②$$

$p = 1, \dots, m$
$r = 1, \dots, n-m.$

f has a local max or min at t_0.

$$\Rightarrow \quad D_r f(t_0) = 0 \qquad r = 1, \dots, n-m$$

$$\Rightarrow \sum_{k=1}^{n} D_k f(x_0) D_r H_k(x_0) = 0 \Rightarrow \sum_{k=1}^{m} D_k f(x_0) D_r h_k(t_0) + D_{m+r} f(x_0) = 0. \to ③$$

$$② \times \lambda_p + ③$$