iii) Auto Regressive (AR) processes

AR(1) or Markov process

$$X_t = \phi X_{t-1} + \epsilon_t \; ; \; \epsilon_t \sim WN(0, \sigma^2)$$

$$\mathrm{cov}(\epsilon_t, X_{t-j}) = 0 \quad \forall j > 0$$

unlike MA, finite order, processes, AR processes are not always stationary

Note that for AR(1) process with $\phi = 1$ is a random walk and as we have already discussed it is non-stationary

Further, for $|\phi|>1$ the series explodes and hence can't be stationary

Note that
$$
\begin{aligned}
X_t &= \phi X_{t-1} + \epsilon_t \\
&= \phi(\phi X_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= \phi^2 X_{t-2} + \phi\,\epsilon_{t-1} + \epsilon_t \\
&= \phi^2(\phi X_{t-3} + \epsilon_{t-2}) + \phi\,\epsilon_{t-1} + \epsilon_t \\
&= \phi^3 X_{t-3} + \phi^2 \epsilon_{t-2} + \phi\epsilon_{t-1} + \epsilon_t \\
&\ \ \vdots \\
&= \phi^t X_0 + \phi^{t-1}\epsilon_1 + \cdots + \epsilon_t
\end{aligned}
$$

Starting from any arbitrary X_0, series would explode for $|\phi|>1$

If $\{X_t\}$ is stationary, then

$$V(X_t) = V(\phi X_{t-1} + \epsilon_t) = \phi^2 V(X_{t-1}) + V(\epsilon_t)$$

i.e $\sigma_X^2 = \phi^2 \sigma_X^2 + \sigma^2$

$$\Rightarrow \sigma_X^2 = \gamma_X(0) = \frac{\sigma^2}{1-\phi^2} \leftarrow |\phi|<1 \text{ is a valid region for this}$$

For an AR(1) process $|\phi|<1$ is the region for stationarity

Alternate formulation for stationarity:

$$\phi(B) X_t = \epsilon_t$$
$$\phi(B) = 1 - \phi B$$

Consider root of $\phi(z)=0$; i.e. $1-\phi z = 0 \Rightarrow z = \frac{1}{\phi}$

$|\phi|<1 \Leftrightarrow$ roots of $\phi(z)=0$ lie outside unit circle

Note: Condition for stationarity of AR is usually in terms of the above

Consider a covariance stationary AR(1)

$$X_t = \phi X_{t-1} + \epsilon_t$$

$$E(X_t) = E(\phi X_{t-1} + \epsilon_t)$$

$$\mu_X = \phi \mu_X$$

i.e $\mu_X(1-\phi) = 0$

$1-\phi \neq 0$ (condition of stationarity)

$\Rightarrow \mu_X = 0$

$$\gamma_X(1) = Cov(X_{t+1}, X_t) = E(X_{t+1} X_t)$$

$$= E(\phi X_t + \epsilon_{t+1}) X_t$$

$$= \phi \sigma_X^2 + 0 \quad (Cov(\epsilon_t, X_{t-j}) = 0 \ \forall j > 0)$$

i.e. $\gamma_X(1) = \phi \dfrac{\sigma^2}{1-\phi^2} = \gamma_X(-1)$

$$\gamma_X(2) = Cov(X_{t+2}, X_t) = E(X_{t+2} X_t)$$

$$= E(\phi X_{t+1} + \epsilon_{t+2}) X_t$$

$$= \phi \gamma_X(1) = \phi^2 \frac{\sigma^2}{1-\phi^2}.$$

<u>$\forall h > 0$</u>

$$\gamma_X(h) = E(X_{t+h} X_t)$$

Now $X_{t+h} = \phi X_{t+h-1} + \epsilon_{t+h}$

$$= \phi(\phi X_{t+h-2} + \epsilon_{t+h-1}) + \epsilon_{t+h}$$

$$= \phi^2 X_{t+h-2} + \phi \epsilon_{t+h-1} + \epsilon_{t+h}$$

$\vdots$

$$= \phi^h X_{t+h-h} + \phi^{h-1} \epsilon_{t+1} + \cdots + \epsilon_{t+h}$$

$$\Rightarrow \gamma_X(h) = E(X_{t+h} X_t)$$
$$= \phi^h \gamma_X(0) = \phi^h \frac{\sigma^2}{1-\phi^2} \quad \left(\begin{array}{c} \mathrm{Cov}(\epsilon_t, X_{t-j}) = 0 \\ \forall j > 0 \end{array} \right).$$

$$\gamma_X(h) = \gamma_X(-h) = \frac{\sigma^2}{1-\phi^2} \phi^{|h|} \; ; \quad h = 0, \pm 1, \pm 2, \ldots$$

ACF $\rho_X(h) = \begin{cases} 1, & h = 0 \\ \phi^{|h|} & \text{If } h \neq 0 \end{cases}$

Note : unlike MA(q), AR process's ACVF/ACF does not cut off (to zero) beyond the lag order of the model

Note : $\forall\, h > 0$

$$\gamma_X(h) = E(X_{t+h} X_t)$$
$$= E(\phi X_{t+h-1} + \epsilon_{t+h}) X_t$$
$$= \phi\, \gamma_X(h-1)$$

$$\text{i.e. } \gamma_X(h) = \phi\, \gamma_X(h-1) \quad - (*)$$

i.e. the ACVF satisfies relationship similar to the data eq^n

The above eq^n (*) is called the Yule-Walker eq^n.

AR(2) process or Yule process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$Cov(\epsilon_t, X_{t-j}) = 0 \quad \forall j > 0$$

$$\phi(B) X_t = \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

AR(2) process is stationary if roots of $\phi(z) = 0$ all lie outside the unit circle

i.e. roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside unit circle

i.e. roots of $y^2 - \phi_1 y - \phi_2 = 0$ lie inside unit circle

Let π_1 & π_2 be the roots of $y^2 - \phi_1 y - \phi_2 = 0$

$|\pi_i| < 1$; $i = 1, 2$, if

$$\left| \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \right| < 1$$

The 2 roots are

$$\pi_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad \& \quad \pi_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

Case 1 : Roots are real

$$|\pi_i| < 1$$

$$\Leftrightarrow \quad \underbrace{-1 < \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}}_{(ii)} < \underbrace{\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1}_{(i)}$$

(i) $\Rightarrow$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\phi_1^2 + 4\phi_2 < 4 + \phi_1^2 - 4\phi_1$$

i.e. $1 - \phi_1 - \phi_2 > 0$ — (1)

(ii) $\Rightarrow$ $-2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2}$

$(2+\phi_1)^2 > \phi_1^2 + 4\phi_2$

i.e. $1 + \phi_1 - \phi_2 > 0$ (2)

Also $\pi_1 + \pi_2 = \phi_1$ & $-\pi_1\pi_2 = \phi_2$

$|\pi_i| < 1 \Rightarrow |\phi_2| < 1$

Case 2 : Roots are complex

$$\pi_1 = a + ib = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

$$\pi_2 = a - ib = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

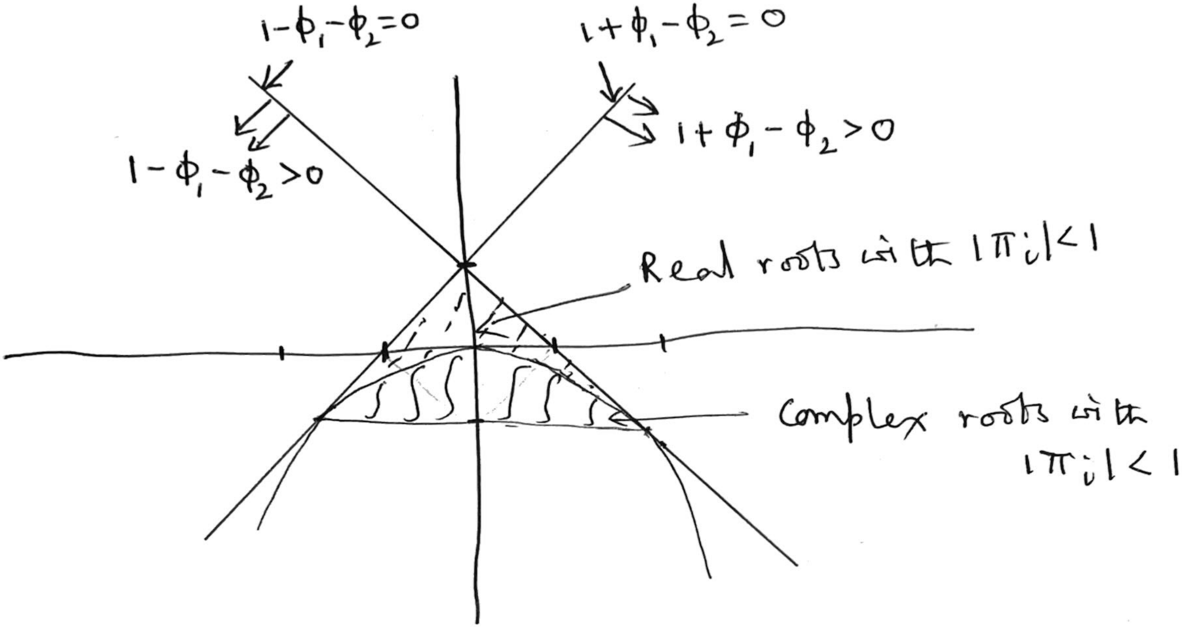
$$a = \phi_1/2 \qquad ib = \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2}$$

$$\Rightarrow b = \frac{-\sqrt{-\phi_1^2 - 4\phi_2}}{2}$$

$$a^2 + b^2 = -\phi_2$$

$|\pi_i| < 1 \Rightarrow 1 + \phi_2 > 0.$

Region of stationarity in $\phi_1 - \phi_2$ plane



Region of stationarity is the triangular region bounded by

$$\left.\begin{aligned} 1-\phi_1-\phi_2 &> 0 \\ 1+\phi_1-\phi_2 &> 0 \\ |\phi_2| &< 1 \end{aligned}\right\}.$$