BLP eq^{n}s :

$$\frac{\partial S(\underset{\sim}{a})}{\partial a_j} = 0 \qquad j = 0, 1, \ldots, n$$

$\frac{\partial S}{\partial a_0} = 0$ gives

$$E\left(X_{n+h} - a_0 - \sum_{i=1}^{n} a_i X_{n+1-i}\right) = 0 \qquad - (i)$$

$\frac{\partial S}{\partial a_j} = 0$ gives

$j = 1, \ldots, n \qquad E\left(X_{n+h} - a_0 - \sum_{i=1}^{n} a_i X_{n+1-i}\right) X_{n+1-j} = 0 \qquad - (ii)$

$$(i) \Rightarrow \quad \mu - a_0 - \sum_{i=1}^{n} a_i \mu = 0$$

$$\text{i.e. } a_0 = \mu\left(1 - \sum_{i=1}^{n} a_i\right)$$

$$(ii) \Rightarrow E\left(X_{n+h} X_{n+1-j}\right) - a_0 \mu - \sum_{1}^{n} a_i E\left(X_{n+1-i} X_{n+1-j}\right) = 0$$
$$j = 1(1)n$$

$$\text{i.e. } E\left(X_{n+h} X_{n+1-j}\right) - \mu^2\left(1 - \sum_{1}^{n} a_i\right) - \sum_{1}^{n} a_i E\left(X_{n+1-i} X_{n+1-j}\right) = 0$$
$$j = 1(1)n$$

$$\text{i.e. } \left(E\left(X_{n+h} X_{n+1-j}\right) - \mu^2\right) - \sum_{1}^{n} a_i \left(E\left(X_{n+1-i} X_{n+1-j}\right) - \mu^2\right) = 0$$
$$j = 1(1)n$$

$$\text{i.e. } \gamma_{h+j-1} = \sum_{i=1}^{n} a_i \gamma_{i-j}; \qquad j = 1(1)n$$

Thus the BLP prediction eqns are

$$a_0 = \mu\left(1 - \sum_{1}^{n} a_i\right) \qquad - (iii)$$

$$\gamma_{h+j-1} = \sum_{i=1}^{n} a_i \gamma_{i-j}; \quad j = 1(1)n \qquad - (iv)$$

Let $\underset{\sim}{\gamma}_n(h) = (\gamma_h, \gamma_{h+1}, \ldots, \gamma_{h+n-1})'$

$\underset{\sim}{a}_n = (a_1, \ldots, a_n)'$

$\Gamma_n = ((\gamma_{i-j}))$

(iv) is $\underset{\sim}{\gamma}_n(h) = \Gamma_n \underset{\sim}{a}_n$

BLP is $P_n X_{n+h} = \mu(1 - \sum a_i) + \sum_{i=1}^{n} a_i X_{n+1-i}$

$= \mu + \sum_{i=1}^{n} a_i (X_{n+1-i} - \mu)$

$\underset{\sim}{a}_n \ni \underset{\sim}{\gamma}_n(h) = \Gamma_n \underset{\sim}{a}_n$

Note (i) $E(X_{n+h} - P_n X_{n+h}) = 0$

(ii) $E(X_{n+h} - P_n X_{n+h})^2$

$= E(X_{n+h} - \mu - \underset{\sim}{a}_n' \underset{\sim}{Y}_n)^2$

$(\underset{\sim}{Y}_n = (X_n - \mu, \ldots, X_1 - \mu)')$

$= E((X_{n+h} - \mu) - \underset{\sim}{a}_n' \underset{\sim}{Y}_n)((X_{n+h} - \mu) - \underset{\sim}{a}_n' \underset{\sim}{Y}_n)'$

$= E(X_{n+h} - \mu)^2 + \underset{\sim}{a}_n' E \underset{\sim}{Y}_n \underset{\sim}{Y}_n' \underset{\sim}{a}_n - 2E(X_{n+h} - \mu)\underset{\sim}{a}_n' \underset{\sim}{Y}_n$

$= \gamma_0 + \underset{\sim}{a}_n' \Gamma_n \underset{\sim}{a}_n - 2 \underset{\sim}{a}_n' \underset{\sim}{\gamma}_n(h)$

$= \gamma_0 + \underset{\sim}{a}_n' \underset{\sim}{\gamma}_n(h) - 2 \underset{\sim}{a}_n' \underset{\sim}{\gamma}_n(h)$ $\left(\underset{\sim}{a}_n \text{ is } \ni \; \underset{\sim}{\gamma}_n(h) = \Gamma_n \underset{\sim}{a}_n\right)$

$= \gamma_0 - \underset{\sim}{a}_n' \underset{\sim}{\gamma}_n(h)$

This is the mean square prediction error corresponding to BLP.

Note (iii) $E\left(X_{n+h} - P_n X_{n+h}\right) X_j = 0 \quad \forall j = 1(1)n$

Note (iv) If $\mu = 0$, then $a_0 = 0$, so we can start with prediction eq^n without a constant

Remark : sp case - one step ahead prediction for zero mean

$$P_n X_{n+1} = \sum_{i=1}^{n} \beta_i X_{n+1-i}$$

$\underset{\sim}{\beta}_n = (\beta_1, \ldots, \beta_n)'$ is $\ni$

$$\Gamma_n \underset{\sim}{\beta}_n = \underset{\sim}{\gamma}_n(1)$$

$$\underset{\sim}{\gamma}_n(1) = (\gamma_1, \ldots, \gamma_n)' \qquad \Gamma_n = ((\gamma_{i-j}))$$

$$\underset{\sim}{\beta}_n^{(BLP)} = \Gamma_n^{-1} \underset{\sim}{\gamma}_n(1)$$

use estimates $\hat{\gamma}_i$ to get

$$\hat{\underset{\sim}{\beta}}_n^{BLP} = \hat{\Gamma}_n^{-1} \hat{\underset{\sim}{\gamma}}_n(1)$$

Predicted X_{n+1} : $\hat{X}_{n+1}^{BLP} = \left(\hat{\Gamma}_n^{-1} \hat{\underset{\sim}{\gamma}}_n(1)\right)' \underset{\sim}{X}_n$

Example : Prediction for AR(2)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

Prediction of X_2 using X_1

$$P_{X_1} X_2 = \beta_1 X_1$$

β_1 is $\ni$ $E(X_2 - \beta X_1)^2$ is minimized w.r.t. β

$$g(\beta) = E(X_2 - \beta X_1)^2$$

$$\frac{\partial g}{\partial \beta} = 0 \Rightarrow E(X_2 - \beta X_1) X_1 = 0$$

i.e. $\gamma_1 = \beta \gamma_0$

$$\Rightarrow \beta = \frac{\gamma_1}{\gamma_0} = \rho_1$$

$$\frac{\partial^2 g}{\partial \beta^2} = +\gamma_0 \qquad P_{X_1} X_2 = \rho_1 X_1$$

Prediction of X_3 using X_2 & X_1

$$P_{(X_2, X_1)} X_3 = \beta_1^* X_2 + \beta_2^* X_1$$

β_1^*, β_2^* are $\ni$ $g(\beta_1, \beta_2) = E(X_3 - \beta_1 X_2 - \beta_2 X_1)^2$ is minimized w.r.t. β_1 & β_2

BLP eq^{n}s : $\dfrac{\partial g}{\partial \beta_1} = 0 \Rightarrow E(X_3 - \beta_1 X_2 - \beta_2 X_1) X_2 = 0$

i.e. $\gamma_1 = \beta_1 \gamma_0 + \beta_2 \gamma_1$

$$\frac{\partial g}{\partial \beta_2} = 0 \Rightarrow E(X_3 - \beta_1 X_2 - \beta_2 X_1) X_1 = 0$$

i.e. $\gamma_2 = \beta_1 \gamma_1 + \beta_2 \gamma_0$

$$\begin{pmatrix} \beta_1^* \\ \beta_2^* \end{pmatrix}_{BLP} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \leftarrow \text{as expected}$$

Notice the similarity of BLP eq^{n}s with Yule-Walker eq^{n}s.

Further,

$$\forall n \geq 2 ; \quad P_{(X_n, \ldots, X_1)} X_{n+1} = \phi_1 X_n + \phi_2 X_{n-1}$$

i.e. $\beta_1^* = \phi_1$, $\beta_2^* = \phi_2$; $\beta_3^* = \cdots = \beta_n^* = 0$