Name: ______________________

Roll Number: ________________

Practice Final Exam Solutions

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **6 + 1** questions and is worth **60%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) If $(L, \prec)$ is a linear ordering and L is uncountable, then there exists an infinite $X \subseteq L$ such that X is well-ordered by $\prec$.

(ii) There exists a bijection $f : \mathbb{R}^7 \to \mathbb{R}^9$ satisfying: For every x, y in $\mathbb{R}^7$, $f(x + y) = f(x) + f(y)$.

(iii) The set of all non-computable functions $f : \omega \to \omega$ has the same cardinality as the set of all real numbers.

(iv) There exists a finite $F \subseteq TA$ such that $PA \cup F$ is a complete $\mathcal{L}_{PA}$-theory.

(v) The theory DLO (dense linear orderings without end points) is decidable (as defined on Slide 188).

Solution

(i) False. Take $L = \omega_1$ and for $\alpha, \beta \in L$, define $\alpha \prec \beta$ iff $\beta < \alpha$.

(ii) True. Both $(\mathbb{R}^7, +)$ and $(\mathbb{R}^9, +)$ are models of TFDAG of the same uncountable cardinality $\mathfrak{c}$.

(iii) True. Since the set of computable functions is countable.

(iv) False. Otherwise TA would be computably axiomatizable.

(v) True. It is both complete and computably axiomatizable.

Question 2. [10 Points]

(a) [**5 Points**] Let $\mathcal{F}$ be the set of all strictly increasing functions $f : \omega \to \omega$. Show that $|\mathcal{F}| = \mathfrak{c}$.

(b) [**5 Points**] Let $\mathcal{E}$ be the set of all countable subsets of ω_1. Show that $|\mathcal{E}| = \mathfrak{c}$.

Solution

(a) First note that $|\mathcal{F}| \le |\omega^\omega| \le |\mathbb{R}^\omega| = \mathfrak{c}$.

Next, let W be the set of all infinite subsets of ω. Since the set of all finite subsets of ω is countable, $|W| = \mathfrak{c}$. Define $H : \mathcal{F} \to W$ by $H(f) = \mathsf{range}(f)$. We claim that $\mathsf{range}(H) = W$. This is because for each $S = \{n_0 < n_1 < \dots\} \in W$ the function $f(k) = n_k$ is strictly increasing and has range S. It follows that $|\mathcal{F}| \ge |W| = \mathfrak{c}$.

It follows that $|\mathcal{F}| = \mathfrak{c}$. □

(b) As $\mathcal{P}(\omega) \subseteq \mathcal{E}$, we get $\mathfrak{c} = |\mathcal{P}(\omega)| \le |\mathcal{E}|$.

Let $\mathcal{W}$ be the set of all nonempty countable subsets of $\mathbb{R}$. Note that for each $S \in \mathcal{W}$, there exists $f : \omega \to \mathbb{R}$ such that $\mathsf{range}(f) = S$. So there is a surjection from $\mathbb{R}^\omega$ to $\mathcal{W}$. Hence $|\mathcal{W}| \le |\mathbb{R}^\omega| = \mathfrak{c}$. Since $|\mathbb{R}| \ge \omega_1$, we get $|\mathcal{E}| \le |\mathcal{W}| \le \mathfrak{c}$.

It follows now that $|\mathcal{E}| = \mathfrak{c}$. □

Question 3. [10 Points]

Using transfinite recursion, construct a function $f : \mathbb{R} \to \mathbb{R}$ such that for every interval $(a, b) \subseteq \mathbb{R}$,

$$\mathsf{range}(f \restriction (a, b)) = \mathbb{R}$$

Solution. Let $\mathcal{F}$ be the set of all pairs (J, y) where J is an open interval in $\mathbb{R}$ and $y \in \mathbb{R}$. It is easy to see that $|\mathcal{F}| = |\mathfrak{c}$. Let $\langle (J_\alpha, y_\alpha) : \alpha < \mathfrak{c} \rangle$ be an injective sequence with range $\mathcal{F}$. Using transfinite recursion define $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ such that the following hold.

(a) Each f_α is a function and $\mathrm{dom}(f_\alpha) \cup \mathrm{range}(f_\alpha) \subseteq \mathbb{R}$.

(b) For every $\alpha < \mathfrak{c}$, $|f_\alpha| \leq |\alpha + \omega| < \mathfrak{c}$.

(c) $f_0 = \emptyset$ and if $\alpha < \mathfrak{c}$ is a limit ordinal, then $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$.

(d) If $\alpha < \beta < \mathfrak{c}$, then $f_\alpha \subseteq f_\beta$.

(e) For every $\alpha < \mathfrak{c}$, there exists $x \in J_\alpha \cap \mathrm{dom}(f_{\alpha+1})$ such that $f_{\alpha+1}(x) = y_\alpha$.

Suppose $\alpha < \mathfrak{c}$ and $\langle f_\beta : \beta < \alpha \rangle$ has been defined. We would like to define f_α such that Clauses (a)-(e) are preserved. If $\alpha = 0$ or a limit ordinal, we define f_α using Clause (c). It is easy to check that Clauses (a)-(e) are preserved.

So it suffices to define $f_{\alpha+1}$ assuming $\langle f_\beta : \beta \leq \alpha \rangle$ has already been defined. But this is easy: Since $|\mathrm{dom}(f_\alpha)| = |f_\alpha| < \mathfrak{c}$ and $|J_\alpha| = \mathfrak{c}$, we can choose $x \in (J_\alpha \setminus \mathrm{dom}(f_\alpha))$ and define

$$f_{\alpha+1} = f_\alpha \cup \{(x, y_\alpha)\}$$

Having constructed $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$, define $g = \bigcup_{\alpha < \mathfrak{c}} f_\alpha$. Note that $\mathrm{dom}(g) \subseteq \mathbb{R}$ and for every open interval J and $y \in \mathbb{R}$, there exists $x \in J \cap \mathrm{dom}(g)$ such that $g(x) = y$. Extend g to a function $f : \mathbb{R} \to \mathbb{R}$ by defining f to be identically zero outside $\mathrm{dom}(g)$. It is clear that f is as required. $\square$

Question 4. [10 Points]

(a) [**4 Points**] Suppose $(\mathbb{Q}, <)$ is the usual ordering on rationals and $(M, \prec)$ is a countable dense linear ordering without end points. Suppose $x_1 \prec x_2 \prec \cdots \prec x_n$ are in M and $a_1 < a_2, \cdots < a_n$ are in $\mathbb{Q}$. Show that there is an isomorphism $f : (M, \prec) \to (\mathbb{Q}, <)$ such that $f(x_k) = a_k$ for every $1 \leq k \leq n$.

(b) [**6 Points**] Use Tarski-Vaught criterion to show that $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Solution

(a) Since any two countable dense linear ordering without end points are isomorphic, we can find functions $f_0, f_1, \dots, f_{n+1}$ such that

(i) f_0 is an order isomorphism from $(\{x \in M : x \prec x_1\}, \prec)$ to $(\{a \in \mathbb{Q} : a < a_1\}, <)$.

(ii) For each $1 \leq k \leq n$, f_k is an order isomorphism from $(\{x \in M : a_k \prec x \prec x_{k+1}\}, \prec)$ to $(\{a \in \mathbb{Q} : a_k \prec a < a_{k+1}\}, <)$.

(iii) f_{n+1} is an order isomorphism from $(\{x \in M : x_n \prec x\}, \prec)$ to $(\{a \in \mathbb{Q} : a_n < a\}, <)$.

Let $g = \bigcup_{0 \leq k \leq n+1} f_k$. Define $f : M \to \mathbb{Q}$ by $g \subseteq f$ and for each $1 \leq k \leq n$, $f(x_k) = a_k$. Then f is as required. $\square$

(b) Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol. Suppose $\psi(x, y_1, \dots, y_n)$ is an $\mathcal{L}$-formula whose free variables are among $x, y_1, \dots, y_n$ and $a_1, \dots, a_n$ are in $\mathbb{Q}$. Assume that there exists $a \in \mathbb{R}$ such that $(\mathbb{R}, <) \models \psi(a, a_1, \dots, a_n)$. We will show that there exists $b \in \mathbb{Q}$ such that $(\mathbb{R}, <) \models \psi(b, a_1, \dots, a_n)$. By the Tarski-Vaught criterion, it will follow that $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Using the Lemma on Slide 157, choose a countable $M \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq M$ and $(M, <)$ is an elementary submodel of $(\mathbb{R}, <)$. Since $(M, <)$ is elementary submodel of $(\mathbb{R}, <)$ and $(\mathbb{R}, <) \models (\exists x)(\psi(x, a_1, \dots, a_n))$ we can find $c \in M$ such that $(M, <) \models \psi(c, a_1, \dots, a_n)$.

By using an argument similar to part (a), we can find an isomorphism $f : (M, <) \to (M, <)$ such that $f(a_k) = a_k$ for every $1 \leq k \leq n$ and $f(c) \in \mathbb{Q}$. Put $f(c) = b$.

We claim that $(\mathbb{R}, <) \models \psi(b, a_1, \dots, a_n)$ and therefore $b \in \mathbb{Q}$ is as required. Since f is an isomorphism, by the Lemma on Slide 148, we get $(M, <) \models \psi(c, a_1, \dots, a_n)$ iff $(M, <) \models \psi(f(c), f(a_1), \dots, f(a_n))$ iff $(M, <) \models \psi(b, a_1, \dots, a_n)$. So $(M, <) \models \psi(b, a_1, \dots, a_n)$. As $(M, <)$ is an elementary submodel of $(\mathbb{R}, <)$, it follows that $(\mathbb{R}, <) \models \psi(b, a_1, \dots, a_n)$ as claimed. $\square$

Question 5. [10 Points]

(a) [**5 Points**] Let $W \subseteq \omega$ be an infinite c.e. set. Show that there is a computable **injective** function $f : \omega \to \omega$ such that $\mathsf{range}(f) = W$.

(b) [**5 Points**] Show that $True_{\mathcal{N}}$ (defined on Slide 199) is not c.e.

Solution

(a) By Homework problem 31, we can fix a computable $g : \omega \to \omega$ such that $\text{range}(g) = W$. Define $f : \omega \to \omega$ as follows: $f(0) = 0$ and $f(n+1) = g(k)$ where k is the least number such that $g(k) \notin \{f(0), f(1), \ldots, f(n)\}$. Note that f is well-defined because $\text{range}(g) = W$ is infinite. It is clear that f is computable injective and $\text{range}(f) = \text{range}(g) = W$. $\square$

(b) First note that the solution to Homework problem 35 shows that every c.e. subset of ω is definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$. By Tarksi's theorem, $True_{\mathcal{N}}$ is not definable in $\mathcal{N}$. So it cannot be c.e. $\square$

Question 6. [10 Points]

Let $\mathcal{N} = (\omega, 0, +, \cdot)$ be the standard model of PA.

(a) [**6 Points**] Define $False_{\mathcal{N}} = \{\ulcorner \psi \urcorner : \mathcal{N} \models \neg\psi\}$. Show that $False_{\mathcal{N}}$ is not definable in $\mathcal{N}$.

(b) [**4 Points**] Show that there are $\mathcal{L}_{PA}$-sentences ϕ and ψ such that PA does not prove either one of the following four sentences.

(i) ϕ

(ii) $\neg\phi$

(iii) $\phi \implies \psi$

(iv) $\phi \implies (\neg\psi)$

Solution

(a) Define $f : \omega \to \omega$ as follows. If n is the Gödel number of an $\mathcal{L}_{PA}$ formula ϕ, then $f(n)$ is the Gödel number of $\neg\phi$. Otherwise, $f(n) = 0$.

It is clear that f is computable and hence definable in $\mathcal{N}$. so we can fix an $\mathcal{L}_{PA}$-formula $\psi(y, x)$ such that for every $(m, n) \in \omega^2$, $f(m) = n$ iff $\mathcal{N} \models \psi(n, m)$.

Note that $n \in True_{\mathcal{N}}$ iff $f(n) \in False_{\mathcal{N}}$. Towards a contradiction, suppose $False_{\mathcal{N}}$ is definable in $\mathcal{N}$ and fix an $\mathcal{L}_{PA}$-formula $\theta(x)$ such that for every $n < \omega$, $n \in False_{\mathcal{N}}$ iff $\mathcal{N} \models \theta(n)$. Consider the formula $\phi(x) \equiv (\exists y)(\psi(y, x) \wedge \theta(y))$. Observe that for every $n < \omega$, $\mathcal{N} \models \phi(n)$ iff $\mathcal{N} \models \theta(f(n))$ iff $f(n) \in False_{\mathcal{N}}$ iff $n \in True_{\mathcal{N}}$. Hence $True_{\mathcal{N}}$ is definable in $\mathcal{N}$ via the formula $\phi(x)$. But this contradicts Tarksi's theorem. $\square$

(b) Using the incompleteness theorem on Slide 200, choose $\phi \in TA$ such that PA does not prove either one of ϕ and $\neg\phi$. Since $PA \cup \{\phi\}$ is computable subset of TA, we can choose $\psi \in TA$ such that neither one of ψ and $\neg\psi$ is a theorem of $PA \cup \{\phi\}$. By the deduction theorem (Slide 127), it follows that ϕ and ψ are as required. $\square$

Bonus Question [5 Points]

Let X be an uncountable set and suppose $\prec_1$ and $\prec_2$ are two well-orders on X. Show that there is an uncountable $Y \subseteq X$ such that for every $a, b \in Y$,

$$a \prec_1 b \iff a \prec_2 b$$

Solution. Put $\alpha = \mathsf{type}(X, \prec_1)$ and let $f : (X, \prec) \to (\alpha, \in)$ be the unique order isomorphism. Since X is uncountable, $\alpha \geq \omega_1$. Put $Z_1 = \{x \in X : f(x) < \omega_1\}$. Then $\mathsf{type}(Z_1, \prec_1) = \omega_1$. Repeating this argument with $(Z_1, \prec_2)$, we can get $Z \subseteq Z_1$ such that $\mathsf{type}(Z, \prec_2) = \mathsf{type}(Z, \prec_1) = \omega_1$.

Using transfinite recursion, construct $\langle x_\alpha : \alpha < \omega_1 \rangle$ as follows. $x_0 \in Z$ is arbitrary. Having defined x_β for $\beta < \alpha$, choose x_α as follows: Let $W = \{x \in Z : (\exists \beta < \alpha)((x \prec_1 x_\beta) \text{ or } (x \prec_2 x_\beta))\}$. Note that W is countable since $\mathsf{type}(Z, \prec_1) = \mathsf{type}(Z, \prec_2) = \omega_1$. Define x_α to be any member of $Z \setminus W$. This complete the construction of $\langle x_\alpha : \alpha < \omega_1 \rangle$.

Put $Y = \{x_\alpha : \alpha < \omega_1\}$. Then it is easy to see that for every $\alpha < \beta < \omega_1$, we have $x_\alpha \prec_1 x_\beta$ and $x_\alpha \prec_2 x_\beta$. So Y is as required. $\square$

End of Exam