Certain Sets in Point-Set-Topology.

Two classes of subsets: Open sets and Closed sets

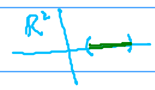
Def^n: (M,d) metric space Let $U \subset M$. Then U is said to be an open set if
$\forall\, x \in U,\ \exists\ r_x > 0$ s.t. $B(x, r_x) \subset U$.

For $F \subset M$, F is said to be a closed set if $M \setminus F$ is an open set.
Note: The empty set, denoted as ϕ, is always an open set (by def^n.), so is also a closed set!

Examples: • $(\mathbb{R}, |\cdot|)$ Every open interval is an open set.
$(0,1]$ or $[0,1)$ or $[0,1]$ is not an open set.

• $(\mathbb{R}, d_0)$ where d_0 is the discrete metric.
Question. What are the open sets?

• $(\mathbb{R}^2, \|\cdot\|_2)$ Question: Is the open interval of $\mathbb{R}$ in $\mathbb{R}^2$ an open set in $\mathbb{R}^2$?
$x \mapsto (x,0) \subset \mathbb{R}^2$



HW. Every ball $B(x,r)$ in any metric space (M,d) is always an open set.
HW. Every open set $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B(x, r_x)$.

→ For $\{U_\alpha\}$ a collection of open sets in (M,d):
• $\bigcup_\alpha U_\alpha$ is again an open set
• $\bigcap_{k=1}^{n} U_k$ is an open set (Why $\bigcap_1^\infty U_n$ is not an open set?)

→ Sequential Characterization of open sets:

Recall: $\mathcal{U} \subset M$ is an open set if $\forall\, x \in \mathcal{U},\ \exists\ r_x > 0$ s.t. $B(x, r_x) \subset \mathcal{U}$

Thm: $\mathcal{U}$ is open in (M,d) iff whenever a sequence $x_n \to x \in \mathcal{U}$, (x_n) is eventually in $\mathcal{U}$.

$\Rightarrow$): Note that $x_n \to x$ in (M,d) iff (x_n) is eventually in $B(x, \varepsilon)$, for any $\varepsilon > 0$.

Pf: ($\Leftarrow$:) Suppose $\mathcal{U}$ is not open. Then $\exists\ x \in \mathcal{U}$ s.t. $\forall\ \varepsilon > 0$, $B(x,\varepsilon) \not\subset \mathcal{U}$.

In particular, for $\varepsilon_n := \frac{1}{n}$, $\exists\ x_n \in B(x, \frac{1}{n})$ s.t. $x_n \notin \mathcal{U}$.

Claim: $x_n \to x$ in (M,d).

Pf of claim: For a given $\varepsilon > 0$, $\exists\ N_\varepsilon > 0$ st. $\forall\ n \geq N_\varepsilon$, $\frac{1}{n} < \varepsilon$.

(HW) Note that $B(x, \frac{1}{n}) \underset{?}{\subset} B(x,\varepsilon)$ for $n \geq N_\varepsilon$.

$$x_n \in B(x, \tfrac{1}{n}) \Rightarrow x_n \in B(x,\varepsilon) \text{ for } n \geq N_\varepsilon.$$

So, $x_n \in B(x,\varepsilon)$ eventually. Hence $x_n \to x$.

But, $x_n \notin \mathcal{U}$, $\forall\ n \geq N_\varepsilon$, i.e., $(x_n) \notin \mathcal{U}$ eventually.

- Recall that $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B(x, r_x)$.

Question: Is it possible to rewrite $\mathcal{U}$ as a union of pairwise disjoint open balls?

Ans. Not always!

Open sets in $\mathbb{R}$: $(\mathbb{R}, |\cdot|)$

- Union of a countable collection of disjoint open intervals is an open set.

Thm: Every nonempty open subset of $\mathbb{R}$ can be written as "the" countable union of "open intervals."

Pf: For a set $S \subset \mathbb{R}$ s.t. $I \subset S$ where I is an interval, the interval I is said to be maximal, if there is no other interval $I \subset J \subset S$ properly containing I.

Step 1. For $x \in \mathcal{U}$, $\exists$ a maximal interval I_x s.t. $x \in I_x \subset \mathcal{U}$.

$$\text{Define } A := \{ a \in \mathbb{R} \mid (a,x] \subset \mathcal{U} \} \neq \phi \text{ (why?)}$$
$$B := \{ b \in \mathbb{R} \mid [x,b) \subset \mathcal{U} \} \neq \phi \text{ (why?)}$$

Let $a_x := \inf A$ and $b_x := \sup A$. (Note a_x could be $-\infty$, b_x could be $+\infty$)

Then, $x \in (a_x, b_x) \subset \mathcal{U}$. (if $a_x = -\infty$ and $b_x = +\infty$
$(-\infty, \infty) = \mathbb{R}$. $\mathcal{U} = \mathbb{R}$.

Why $(a_x, b_x) \subset \mathcal{U}$?

Hint. Let $y \in (a_x, b_x)$. Then $a_x < y < b_x$.

$\exists\, a \in A$ s.t. $a_x < a < y$. where $(a, x] \subset \mathcal{U}$.

Case 1. $y \leq x$ ----

Case 2. $x < y$. Also $y < b_x$ so $\exists\, b \in B$ s.t. $x < y < b < b_x$ ----

(Note: Implicitly we are using a nice characterization of intervals on $\mathbb{R}$:
"A set $S \subset \mathbb{R}$ is an interval iff for any $x, y \in S$ s.t. $x < y$, $[x, y] \subset S$.")
must have at least two points

Note that (a_x, b_x) is a maximal interval inside $\mathcal{U}$ containing x.
That is, $\nexists$ J : interval s.t. $(a_x, b_x) \subsetneq J \subset \mathcal{U}$. ?

Claim: (a_x, b_x) is "the" maximal interval inside $\mathcal{U}$ containing x.

Pf. of claim: Suppose I is another maximal interval inside $\mathcal{U}$ containing x.

Then $x \in I$ and $x \in (a_x, b_x)$. Note that $x \in I \cap (a_x, b_x) \subset \mathcal{U}$.

HW: Show that $I \cup (a_x, b_x)$ is again an interval which properly contains (a_x, b_x) and $I \cup (a_x, b_x) \subset \mathcal{U}$.

This contradicts the maximality of (a_x, b_x) containing x in $\mathcal{U}$.
Therefore, (a_x, b_x) is the (one and only one) maximal interval containing x in $\mathcal{U}$.

Step 2: For $x \neq y$ in $\mathcal{U}$, either $I_x \cap I_y = \phi$ or $I_x = I_y$ (HW).

So, $\mathcal{U} = \bigcup_{x \in \mathcal{U}} I_x$ where I_x is the maximal interval containing x.

$\{I_x\}$ is a pairwise disjoint collection of intervals.

Step 3: $\{I_x\}_{x \in \mathcal{U}}$ is a countable collection.

Let $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$ the countable collection of rational nos.

For each $x \in \mathcal{U}$, I_x consists of infinitely many rational nos. r_j.
Let n be the smallest index such that $r_n \in I_x$.

Define $F : \{I_x\} \to \mathbb{N}$ as $F(I_x) := n$.

F is injective b/c. $F(I_x) = F(I_y) \Rightarrow r_n \in I_x \cap I_y$

$\Rightarrow I_x = I_y$.

Hence, $\mathcal{U}$ is the countable union of pairwise disjoint "intervals". (maximal intervals).

Remark: (i) Every point in $\mathcal{U} \subset \mathbb{R}$ is contained in exactly one maximal interval.
(ii) The above representation of $\mathcal{U}$ is unique. That is, if $\mathcal{U}$ is a countable union of disjoint intervals, then these intervals must be maximal.

Upshot: Every nonempty open set in $\mathbb{R}$ has a unique representation as the countable union of maximal intervals. (Note that the totally ordered structure and the least upper property on $\mathbb{R}$ plays an important role in this representation.)

Intervals in $\mathbb{R}$ are special subsets that cannot split into disjoint union of open sets.

$\Downarrow$ Generalization!

"Connected sets" in (M,d) any metric space: To be introduced later.

(Interval $\rightsquigarrow$ open set "maximal" interval $\rightsquigarrow$ maximal open set)

Food for thought!

Q. $(\mathbb{R}^2, \|\cdot\|_\infty)$ $\|(x,y)\|_\infty := \max\{|x|, |y|\}$. Show that an open disk (open set in $\mathbb{R}^2$) cannot be written as the disjoint union of open rectangles.
(Note that an open ball in $\|\cdot\|_\infty$ is an open rectangle.)

Q. $(\mathbb{R}^2, \|\cdot\|_2)$ Consider the open rectangle (which is an open set in $\mathbb{R}^2$).
Show that the open rectangle cannot be written as union of disjoint open disks.
(Note that open balls in $(\mathbb{R}^2, \|\cdot\|_2)$ are open disks.)

Q. Can we have such a characterization / representation of open sets in any metric space (M,d)?
(To be addressed later.)