Department of Mathematics & Statistics

MTH403a

Quiz-I

Name: .... Roll No: ....

Marks: 10 Time: 25 minutes

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1. Let $n \geq 2$ and $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear. Show that the function $f\colon \mathbb{R}6n \to \mathbb{R}$ defined by $f(x) := \langle Ax, x \rangle$ is differentiable and find the jacobian of f at every point $x = (x_1, x_2, \ldots, x_n)$ with respect to the standard bases. [5 marks]

   Let $x \in \mathbb{R}^n$. For $h \in \mathbb{R}^n$, we have $f(x+h) - f(x) = \langle Ax, h \rangle + \langle Ah, x \rangle + \langle Ah, h \rangle$. Observe that the map $T(h) := \langle Ax, h \rangle + \langle Ah, x \rangle + \langle Ah, h \rangle$ is linear in $h \in \mathbb{R}^n$.
   Further by Cauchy-Schwartz inequality, $|f(x+h) - f(x) - T(h)| = |\langle Ah, h \rangle| \leq \|Ah\| \|h\|$. [1 mark]
   Since A is linear there exists a constant $C > 0$ such that $\|Ah\| \leq C\|h\|$ for $h \in \mathbb{R}^n$. Therefore $\frac{|f(x+h)-f(x)-T(h)|}{\|h\|} \leq C\|h\| \to 0$ as $\|h\| \to 0$. [1 mark]
   Hence f is differentiable on $\mathbb{R}^n$ and for $x \in \mathbb{R}^n$, the derivative $df_x(h) = \langle Ax, h \rangle + \langle Ah, x \rangle$.
   Observe that the Jacobian matrix of the given differentiable function f at x is $J(f)(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$ and for $1 \leq i \leq n$, $\frac{\partial f}{\partial x_i} = df_x(e_i) = \langle Ax, e_i \rangle + \langle Ae_i, x \rangle$, where e_i's are the standard basis vectors of $\mathbb{R}^n$. [1 mark]
   Let $x = \sum_{k=1}^n x_k e_k$ and $Ae_i = \sum_{k=1}^n a_{ik} e_k$.
   Then $\langle Ax, e_i \rangle = \sum_k x_k \langle Ae_k, e_i \rangle = \sum_k a_{ki} x_k$ and $\langle Ae_i, x \rangle = \sum_k a_{ik} \langle e_k, x \rangle = \sum_k a_{ik} x_k$. [1 mark]
   Therefore $\langle Ax, e_i \rangle + \langle Ae_i, x \rangle = \sum_k (a_{ik} + a_{ki}) x_k$ and the Jacobian $J(f)(x) = \big(\sum_k (a_{1k} + a_{k1}) x_k, \ldots, \sum_k (a_{nk} + a_{kn}) x_k \big)$. [1 mark]

2. Find and classify the critical points of the smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2) e^{x^2 - y^2}$ for $(x, y) \in \mathbb{R}^2$. [5 marks]

   For the function $f(x, y) = (x^2 + y^2) e^{x^2 - y^2}$, we have $\frac{\partial f}{\partial x} = 2x(1 + x^2 + y^2) e^{x^2 - y^2}$ and $\frac{\partial f}{\partial y} = 2y(1 - x^2 - y^2) e^{x^2 - y^2}$. Therefore $\nabla f(x, y) = (0, 0)$ iff $2x(1 + x^2 + y^2) = 0$ and

$2y(1-x^2-y^2)=0$. Solving these two equations we find that $(x,y)=(0,0)$ and $\pm(0,1)$ are the three critical points of the function f. [1 mark]

Observe that

$$\frac{\partial^2 f}{\partial x^2} = \left[2(1+x^2+y^2)+4x^2+4x^2(1+x^2+y^2)\right]e^{x^2-y^2}$$
$$= \begin{cases} 2 & \text{if } x=0=y \\ 4/e & \text{if } x=0, y=\pm 1 \end{cases}$$

$$\frac{\partial^2 f}{\partial y^2} = \left[2(1-x^2-y^2)-4y^2-4y^2(1-x^2-y^2)\right]e^{x^2-y^2}$$
$$= \begin{cases} 2 & \text{if } x=0=y \\ -4/e & \text{if } x=0, y=\pm 1 \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\left[4xy(x^2+y^2)\right]e^{x^2-y^2}$$
$$= \begin{cases} 0 & \text{if } x=0=y \\ 0 & \text{if } x=0, y=\pm 1 \end{cases}$$ [2 marks]

`(2 marks if all three calculations are correct. Otherwise 1 mark)`

Therefore at the point $(0,0)$, the hessian matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Hence for all non-zero vectors $v \in \mathbb{R}^2$, $\langle Av, v\rangle = 2\|v\|^2 > 0$. This proves that $(0,0)$ is a local minima for the function. [1 mark]

At the points $(0,\pm 1)$, the Hessian matrix is $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 4/e & 0 \\ 0 & -4/e \end{pmatrix}$.

This proves that the points $(0,\pm 1)$ are saddle points for the function f. [1 mark]