Department of Mathematics

Calculus of Several Variables and Differential Geometry

Assignment-IV

1. Let $a > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : e^{x^2} + e^{y^2} + e^{z^2} = a\}$. Find the condition on a so that S is a regular surface in $\mathbb{R}^3$.

2. Let $a > b > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2\}$. Show that S is a regular surface in $\mathbb{R}^3$.

3. Let S be a compact surface. Show that, for every vector $a \in \mathbb{R}^3 \setminus \{0\}$, there is a point p in S such that the vector a is normal to S at the point p.

4. Let S be a regular connected surface in $\mathbb{R}^3$ and p_0 in $\mathbb{R}^3$ be such that all the normal lines to S passes through the point p_0. Show that S is contained in a sphere.

5. Let S be a regular connected surface in $\mathbb{R}^3$ and ℓ in $\mathbb{R}^3$ be such that all the normal lines to S meets the line ℓ orthogonally. Show that S is contained in a cylinder.

6. Let S be a connected surface and $a \in \mathbb{R}^3$ be a unit vector such that for all points p in S, the vector a is orthogonal to T_pS. Show that S is contained in a plane.

7. Find the image of the Gauss map for the following surfaces. Further compute the derivative of the Gauss map in each of the cases.

   (a) Let $S := \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d\}$ where (a, b, c) is a non-zero vector in $\mathbb{R}^3$ and d be a real number.

   (b) Let $R > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}$.

   (c) Let $R > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\}$.

   (d) Let $S := \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$.

   (e) Let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \text{ and } z > 0\}$.

   (f) Let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$.

   (g) Let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1\}$.

   (h) Let $a > b > 0$ and $S := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2\}$.

8. Let I be an open interval and $c : I \to \mathbb{R}^3$ be the unit speed curve lying in the plane $Y = 0$. If $c(u) := (f(u), 0, g(u))$, let $S := \{(f(u)\cos v, f(u)\sin v, g(u)) : u \in I \text{ and } v \in (0, 2\pi)\}$. Show that S is a regular surface. Find the Gauss map and find its derivative.

9. Write down a parametrization for each of surfaces in Exericses 7 and 8 and find $E = \langle \varphi_u, \varphi_u \rangle$, $F = \langle \varphi_u, \varphi_v \rangle$ and $G = \langle \varphi_u, \varphi_u \rangle$ for the parametrization you find.

10. Find the Gauss curvature and mean curvature of the surfaces in Exercises 7 and 8.

11. Let S be the unit sphere in $\mathbb{R}^3$. Let $\phi \colon \mathbb{R}^2 \to S^2 \setminus \{(0, 0, 1)\}$ be the inverse of the stereographic projection map. Find E, F and G for this parametrization.

12. Let S be the unit sphere in $\mathbb{R}^3$. Let $\ell := \{tv : t \in \mathbb{R}\}$ for a unit vector v be a straight line passing through origin in $\mathbb{R}^2$ and $\gamma(t) = \varphi(tv)$ where φ is as in Exercise 11. For $a < b \in \mathbb{R}$, find the length $\int_a^b \|\gamma'(t)\| dt$ of the curve γ in $[a, b]$. Further find $\lim_{b\to\infty} \int_0^b \|\gamma'(t)\| dt$.

13. For any line $\ell_v := \{tv : t \in \mathbb{R}\}$ in $\mathbb{R}^2$, find the length of the image curve $\gamma(t) := \varphi(tv)$ on any finite interval, for the parametrization $\varphi(u, v) := (u, v, u^2 + v^2)$ of the elliptic paraboloid $S := \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$.

14. For any line $\ell_v := \{tv : t \in \mathbb{R}\}$ in $\mathbb{R}^2$, find the length of the image curve $\gamma(t) := \varphi(tv)$ on any finite interval, for the parametrization $\varphi(u, v) := (u, v, u^2 - v^2)$ of the elliptic paraboloid $S := \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$.