Assignment 4a.

1. $X \subset M$ $\quad x \in X$ $\quad B(x, \frac{1}{n})$ $\quad \bigcap_{n=1}^{\infty} \bigcup_{x \in X} B(x, \frac{1}{n}) = X$

2. H^∞ $\quad [0,1]^{\mathbb{N}}$ $\quad A = B(y_1, \varepsilon_1) \times B(y_2, \varepsilon_2) \times \cdots \times B(y_k, \varepsilon_k) \times [0,1] \times \cdots$

$y = (y_n)_{n=1}^{\infty}$ $\quad$ claim A is open.

N

$$A = \bigcap_{j=1}^{k} \Big([0,1] \times \cdots \times \underbrace{B(y_j, \varepsilon_j)}_{j^{th}} \times [0,1] \times \cdots \Big)$$

open

3. $A = \{ e^{(k)} \}_{k=1}^{\infty}$ $\quad$ A is closed (claim)

$x \in \overline{A} \Rightarrow x \in A.$

$\Downarrow$

$\exists\ e_n^{(k)} \in A$ s.t. $\| e_n^{(k)} - x \|_1 \to 0.$

$\| e_n^{(k)} - e_m^{(k)} \|_1 \to 0$

$\Updownarrow$

$\left(e_n^{(k)} \right)_{n=1}^{\infty}$ is eventually a constant seq.

N_0 $\quad e_n^{(k)} = e_{N_0}^{(k)} \quad \forall\ n \geq N_0$

$\qquad\qquad\quad \| \qquad \in A.$

$\qquad\qquad\quad x$

4. ℓ^∞ $\quad F =: c_{00}$

$x = (1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots)$ $\quad x^{(n)} := (1, \frac{1}{2}, \ldots, \frac{1}{n}, \ \ \ ) \in c_{00}.$

$\downarrow \|\cdot\|_\infty$

x

$\to$ F is not closed.

$\to$ $0 \in \ell^\infty$ $\quad \forall\ \varepsilon > 0,$ $\quad B(0, \varepsilon) \not\subset F.$ $\quad 0 \in F.$

$y_n := (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots\ldots) \xrightarrow{\|\cdot\|_\infty} 0$

Recall: $x_n \to x \in \mathcal{U}$ $\qquad B(0, \varepsilon)$

$\frac{1}{n_\varepsilon} < \varepsilon$ $\quad y_{n_\varepsilon} \in B(0, \varepsilon)$ $\quad B(0,\varepsilon) \not\subset F \ \ldots$ F is not open.

$\|y_n\|_\infty = \frac{1}{n_\varepsilon} < \varepsilon$

5. c_0 closed in ℓ_∞. $x \in \overline{c_0}^{\|\cdot\|_\infty}$ $x^{(n)} \in c_0$.

For $j \geq 1$, $|x_j^{(n)} - x_j| \leq \|x^{(n)} - x\|_\infty \to 0$ $\quad \overset{\shortparallel}{(x_j)_{j=1}^\infty}$ $\quad x^{(n)} = (x_j^{(n)})_{j=1}^\infty$

$\downarrow$

x_j

..... $x \in c_0$

$|x_j| \leq |x_j - x_j^{(N)}| + |x_j^{(N)}| \ldots\ldots < \varepsilon$

$\forall j \geq j_0$.

6. $B = \{x \in \ell_2 \mid |x_n| < \frac{1}{n},\ n = 1, \ldots\ldots\}$ is not open.

$B(0, \varepsilon) \not\subset B$.

For $\varepsilon > 0$, $\exists N$ st. $\forall n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{2}$

$x := (0 \ldots\ldots 0, \underset{N^{th}}{\varepsilon/2}, \ldots)$ $\quad x \in B(0, \varepsilon)$

$\|x\|_2 < \varepsilon$

$x \notin B$ $\quad n = N$

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8. $\|\cdot\| : V \to \mathbb{R}$ is cts function! (HW)

$\to 0$ $\quad \underset{\underset{a,b \in A}{\sup\{d(a,b)\}}}{\overset{\shortparallel}{\mathrm{diam}(A)}} \overset{\leq}{=} \mathrm{diam}(\overline{A})$. $\quad A \subset \overline{A}$ $\quad$ $a \leq b$

$\mathrm{diam}(\overline{A}) \leq \mathrm{diam}(A)$ $\quad$ $a < b + \varepsilon \quad \forall \varepsilon > 0$

$\Downarrow$

$a \leq b$

$\mathrm{diam}(\overline{A}) < \mathrm{diam}(A) + \varepsilon$

$d(a,b)$ $\quad d(x,a)$ $\quad d(x,y)$

$x \in \overline{A} \setminus A$ $\quad x \in \overline{A} \setminus A$

$a \in A$ $\quad y \in \overline{A} \setminus A$

$x \in \overline{A}$ $\exists\ a_n \in A$ s.t. $d(a_n, x) \to 0$.

~~$x \in$~~ ......

$d(x,a) \leq d(x, a_n) + d(a_n, a)$ $\quad$ ... (HW) ---- $\quad a_n \to x$, $b_n \to y$

$< \varepsilon + d(a_n, a)$

$\leq \mathrm{diam}(A)$

11. $A = \mathbb{Q}$ $\quad B = \mathbb{R} \setminus \mathbb{Q}$. $\quad \overline{A \cap B} = \phi$ $\quad \phi \subsetneq \mathbb{R}$

$(\mathbb{R}, |\cdot|)$ $\quad \overline{A} \cap \overline{B} = \mathbb{R}$

$A = \mathbb{Q}$ $\quad B = \mathbb{Q}^c \cup \mathbb{N}$ $\quad \overline{A \cap B} = \overline{\mathbb{N}} = \mathbb{N}$ $\quad \overline{A} \cap \overline{B} = \mathbb{R}$ $\quad \mathbb{N} \subsetneq \mathbb{R}$

$\overline{A} \overset{\subset}{\underset{\supset}{=}} (\mathrm{int}\, A^c)^c$

$\mathrm{int}\, A^c \subset A^c$

$A \subset (\mathrm{int}\, A^c)^c$

$\Rightarrow \overline{A} \subset (\mathrm{int}\, A^c)^c$

$x \in (\mathrm{int}\, A^c)^c \Rightarrow x \notin \mathrm{int}(A^c) \Rightarrow \forall \frac{1}{n} \ \exists a_n \in A$ s.t $B(x, \frac{1}{n}) \cap A \neq \phi$

$\left(\text{o/w: } \exists \varepsilon > 0, \ B(x,\varepsilon) \subset A^c \right)$

$\Downarrow$

$x \in \mathrm{int}\, A^c$

$\underset{\in A}{a_n} \longrightarrow x \Rightarrow x \in \overline{A}$.

$\mathcal{U} = \bigcup_{n=1}^{\infty} (a_n, b_n)$ $\quad \overline{\mathcal{U}} \ni \{a_n, b_n\}$ $\quad \mathcal{U} \neq \mathbb{R} = (-\infty, \infty)$

$\mathcal{U} \subsetneq \overline{\mathcal{U}}$