Connectedness

Recall that every (nonempty) open set in $(\mathbb{R}, |\cdot|)$ is the countable union of pairwise disjoint (maximal) open intervals.

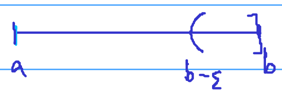
Q. Can we "decompose" any interval into disjoint union of open intervals?

A. An interval in $(\mathbb{R}, |\cdot|)$ cannot be written as a disjoint union of nonempty open sets.

For example, if $[a,b] = A \cup B$ where A, B are nonempty open sets in $[a,b]$ and $A \cap B = \phi$.

Say $b \in B$. Since B is open in $[a,b]$, $\exists\ \varepsilon > 0$ s.t. $(b-\varepsilon, b] \subset B$.

Since $A \subset [a,b]$, let $c := \sup A$.



Then, $a \le c \le b$

Note that if $c = a$, then $A = \{a\}$. But A is open in $[a,b]$, so $\exists\ \delta > 0$ s.t. $[a, a+\delta) \subset A$. Therefore, $c \neq a$.

If $c = b$, then for the given $\varepsilon > 0$ above, $\exists\ a \in A$ s.t. $b - \varepsilon < a$ as $b = c = \sup A$.

This implies that $A \cap B \neq \phi$. contradicting $A \cap B = \phi$. Hence $c \neq b$.

Therefore, $a < c < b$.

(HW.) $\quad c \in \bar{A} \cap \bar{B} \underset{\text{why?}}{=} A \cap B = \phi$, not possible.

(HW). Show that any interval in $\mathbb{R}$ cannot be written as a disjoint union of nonempty open sets in $(\mathbb{R}, |\cdot|)$.

There are special subsets in $(\mathbb{R}, |\cdot|)$ which cannot be decomposed into disjoint union of open sets!

Def^n: For (M, d), we say M is connected if M cannot be written as a disjoint union of nonempty open sets.

In other words, M is disconnected if $\exists\ A, B$ nonempty open sets s.t.

$$A \cap B = \phi \quad \text{and} \quad M = A \cup B.$$

Equivalently, if M is disconnected, then $\exists\ A, B \neq \phi$, open, $A \cap B = \phi$ s.t $A \cup B = M$

Then, $F := A^c$ and $E := B^c$ nonempty closed sets s.t. $M = E \cup F$.

Conversely, if $M = X \cup Y$ s.t. $X, Y \neq \phi$, closed sets then $A := X^c$ and $B := Y^c$ and $M = A \cup B$.

→ (Criterion) M is connected iff there is no nontrivial clopen sets.

(A set which is both open and closed.)

Examples:

- $\mathbb{R}$ is connected.
- Any discrete metric space with two or more pts. is disconnected.

Def^n: A subset $E \subset M$ is <u>disconnected in E</u> if $\exists\ \mathcal{U}, V$ nonempty open sets "in E" s.t.

$$E = \mathcal{U} \cup V. \quad \text{with} \quad \mathcal{U} \cap V = \phi.$$

In other words, if $\exists\ A, B$ open in M s.t. $\mathcal{U} = A \cap E$, $V = B \cap E$ where $\mathcal{U} \neq \phi, V \neq \phi$ and $E = (A \cap E) \cup (B \cap E)$.

Note that if $A \cap B = \phi$, then it suffices to have $E \subset A \cup B$ where $A \neq \phi, B = \phi$, A, B open in M with $A \cap B = \phi$.

(HW) → (M, d) Let $E \subset M$. If $\mathcal{U}$ and V are disjoint open sets in E, then $\exists$ disjoint open sets in M s.t. $\mathcal{U} = A \cap E$ and $V = B \cap E$.

Example: The Cantor set $\mathcal{C}$ is disconnected. For $x, y \in \mathcal{C}$ with $x < y$, then $\exists\ z \notin \mathcal{C}$ s.t. $x < z < y$. Note that $\mathcal{C} \subset [0,1]$. Take the relative metric on $[0,1]$.

Then $[0,z)$ and $(z,1]$ are open sets in $\underset{E}{\underset{\parallel}{[0,1]}}$. So, $\mathcal{C} = \underbrace{([0,z) \cap \mathcal{C})}_{A \cap E\ \cap\ \mathcal{C}} \cup \underbrace{((z,1] \cap \mathcal{C})}_{B \cap E\ \cap\ \mathcal{C}}$.

$$\mathcal{C} = \bigcap_{k=1}^{\infty} C_k \qquad C_k := \bigcup_{n=1}^{2^k} I_n \quad \text{with} \quad \ell(I_n) = \frac{1}{3^k}.$$

$x, y \in C_k \Rightarrow x \in I_{n_1}$ and $y \in I_{n_2}$ where $I_{n_1} \cap I_{n_2} = \phi$. So, $|x - y| > \frac{1}{3^k}$.

Hence, $\exists\ z \notin \mathcal{C}$ s.t. $x < z < y$.

→ Characterization of connected subsets of $\mathbb{R}$ w.r.t the usual metric.

The connected subsets of $\mathbb{R}$ containing more than one pt are intervals.

Pf:

Recall that $E \subset \mathbb{R}$ containing more than two pts. is an interval iff $\forall\ x, y \in E$ s.t. $x < y$, $[x,y] \subset E$.

Suppose E is connected. If E is not an interval, then $\exists\ x, y \in E$ s.t. $x < y$ & $[x,y] \not\subset E$. That is, $\exists\ z \in (x,y)$ s.t. $z \notin E$.

Consider $(-\infty, z)$ and (z, ∞) open in $\mathbb{R}$. Then $E = E \cap (-\infty, z) \ \dot\cup\ E \cap (z, \infty)$

contradicting connectedness assumption on E.

Conversely, intervals are connected (proved on the first page).

Recall that every nonempty open set in $(\mathbb{R}, |\cdot|)$ is uniquely decomposed into pairwise disjoint (maximal) intervals. That is,

$$\mathcal{U} = \dot\bigcup_{n=1}^{\infty} I_n \quad \text{where } I_n : \text{maximal open interval contained in } \mathcal{U}$$

→ $\not\exists\ J$ s.t. $I_n \subsetneq J \subsetneq \mathcal{U}$

→ I_n: connected

SKIP! (The discussion below is optional, mentioned for the sake of completeness in this context.)

Q. Is there an analog of this characterization of open sets in any (M,d)?

A. (M,d) A set V is said to be a maximal connected set if there is no other connected set properly containing V. Such a V is called a connected component of M.

Def^n: Given $E \subset M$, the maximal connected subsets of E are called the connected components of E.

Every nonempty subset of (M,d) can be written uniquely as the disjoint union of its connected components.