Remark: Converse of the previous result (strict stationarity + existence of moments upto order 2 $\Rightarrow$ weak stationarity) is NOT true (in general)

A counter example for weak stationarity $\not\Rightarrow$ strict stationarity

Let the time series $\{X_t\}$ be a seq of indep r.v.s $\ni$

$$X_t \sim \begin{cases} \exp(1), & \text{if } t \text{ is odd} \\ N(1,1), & \text{if } t \text{ is even} \end{cases}$$

$$EX_t = 1 \ \forall t$$

$$\mathrm{Cov}(X_t, X_{t+h}) = \begin{cases} 1, & \text{if } h=0 \\ 0 & \text{o/w} \end{cases} \leftarrow \text{indep of } t, \ f^n \text{ of } h \text{ only}$$

$\Rightarrow \{X_t\}$ is Covariance stationary

Realize that distr^n of X_1 & X_2 are different and hence $\{X_t\}$ can not be strict stationary.

Remark: Can you give an example of a time series which is strict stationary but is not Covariance stationary?

Example is obvious!! Isn't it?

Remark: There is a special (particular) type of time series for which covariance stationarity $\Rightarrow$ strict stationarity.

Def^n: Gaussian process

A time series $\{X_t\}$ is said to be Gaussian if for any n and any admissible $t_1, \ldots, t_n$, the jt dist^n of $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normal; i.e. $(X_{t_1}, \ldots, X_{t_n})' \sim N_n$

[Def^n of multivariate normal

A multivariate random vector $\underset{\sim}{X}_{p\times 1}$ with mean vector $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$ and covariance matrix $E(\underset{\sim}{X} - \underline{\mu})(\underset{\sim}{X} - \underline{\mu})' = \Sigma = Cov(\underset{\sim}{X})$ is said to follow a multivariate normal iff $\forall \underset{\sim}{\alpha} \in \mathbb{R}^p$ $(\underset{\sim}{\alpha} \neq \underset{\sim}{0})$

$\underline{\alpha}' \underset{\sim}{X} \sim N_1$, i.e. $\underset{\sim}{X} \sim N_p$ iff $\underset{\sim}{\alpha}' \underset{\sim}{x} \sim$ univariate normal $\forall \underset{\sim}{\alpha} \in \mathbb{R}^p$ $(\underset{\sim}{\alpha} \neq \underset{\sim}{0})$]

We write $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$

If $\Sigma > 0$, then the jt p.d.f. of $\underset{\sim}{X}$ is

$$f_{\underset{\sim}{X}}(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\underset{\sim}{x} - \underline{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underline{\mu})\right)$$

Note that N_p dist^n is completely specified by $\underset{\sim}{\mu}$ & Σ

If $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$

Covariance stationary Gaussian process

Suppose $\{X_t\}$ is a Gaussian process and further that the process $\{X_t\}$ is covariance stationary

i.e. $EX_t = \mu \ \forall t$

$\text{Cov}(X_t, X_{t+h}) = f^n$ of h only $\forall t$; $VX_t = \sigma^2 \ \forall t$ (finite)

Since $\{X_t\}$ is Gaussian, the jt distr^n of

$$\underset{\sim}{Z} = (X_{t_1}, \ldots, X_{t_n})' \text{ is } N_n(\underset{\sim}{\mu}, \Sigma)$$

where
$$\underset{\sim}{\mu} = E(\underset{\sim}{Z}) = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}$$

$\cancel{\Sigma = \text{Cov}(\underset{\sim}{Z})} =$

$$\Sigma = \text{Cov}(\underset{\sim}{Z}) = \begin{pmatrix} V(X_{t_1}) & \text{Cov}(X_{t_1}, X_{t_2}) & \cdots & \text{Cov}(X_{t_1}, X_{t_n}) \\ & V(X_{t_2}) & \cdots & \text{Cov}(X_{t_2}, X_{t_n}) \\ & & \ddots & \vdots \\ & & & V(X_{t_n}) \end{pmatrix}$$

$$\text{i.e. } \Sigma = \begin{pmatrix} \sigma^2 & f^n \text{ of } (t_2 - t_1) & \cdots & f^n \text{ of } (t_n - t_1) \\ & \sigma^2 & \cdots & f^n \text{ of } (t_n - t_2) \\ & & \ddots & \vdots \\ & & & \sigma^2 \end{pmatrix}$$

Consider now shifted set of r.vs

$\underset{\sim}{Y} = (X_{t_1+k}, \ldots, X_{t_n+k})$ for any int k

$\underset{\sim}{Y} \sim N_n$ with $E(\underset{\sim}{Y}) = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}$.

$$Cov(\underset{\sim}{Y}) = \begin{pmatrix} V(X_{t_1+k}) & Cov(X_{t_1+k}, X_{t_2+k}) & \cdots & Cov(X_{t_1+k}, X_{t_n+k}) \\ & V(X_{t_2+k}) & & Cov(X_{t_2+k}, X_{t_n+k}) \\ & & \ddots & \\ & & & V(X_{t_n+k}) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2 & f^n \text{ of } (t_2-t_1) \text{ only} & \cdots & f^n \text{ of } (t_n-t_1) \text{ only} \\ & \sigma^2 & & f^n \text{ of } (t_n-t_2) \text{ only} \\ & & \ddots & \\ & & & \sigma^2 \end{pmatrix}$$

$$= \Sigma$$

$$\Rightarrow (X_{t_1}, \ldots, X_{t_n}) \overset{d}{=} (X_{t_1+k}, \ldots, X_{t_n+k})$$

identical in dist^n

$\Rightarrow \{X_t\}$ is strict stationary

Thus, if $\{X_t\}$ is Gaussian, then if $\{X_t\}$ is covariance stationary $\Rightarrow \{X_t\}$ is strict stationary

<u>Note</u>: This is a sp case when cov stat $\Rightarrow$ strict stationary

Some examples

Example 1 : $\{X_t\}$ is a seq of i.i.d. random variables

$\{X_t\}$ is strict stationary

Is $\{X_t\}$ covariance stationary

Alternately, suppose $\{X_t\}$ is a seq of i.i.d. random variables with finite variance σ^2. Then $\{X_t\}$ is clearly covariance stationary and strict stationary.

Example 2 :

$$Y_t = \alpha + \beta t + \epsilon_t$$

$\{\epsilon_t\}$ is a seq of i.i.d. random variables

$$\Rightarrow E(\epsilon_t) = 0 \ \forall t \ \& \ Cov(\epsilon_t, \epsilon_s) = \begin{cases} \sigma^2, & t = s \\ 0, & o/w \end{cases}$$

Seq $\{Y_t\}$ is thus an independent random variable

with $E Y_t = \alpha + \beta t \leftarrow f^n$ of t

$\& \ V Y_t = \sigma^2$

As $E Y_t$ is a f^n of t, $\{Y_t\}$ is not even mean stationary and hence is not covariance stationary.