Metric Spaces

Parasar Mohanty, 2021

0.1 Metric Spaces

As we have seen in $\mathbb{R}$ we have a notion of distance. i.e. distance between two real numbers x and y is given by $|x-y|$. One would like to ask "What is distance?" Here we have a map $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d(x, y) = |x-y|$. Do we have any such map in arbitrary set X? What property does this map d has.

Definition 0.1. *We say that (X, d) is a metric space if $d : X \times X \to \mathbb{R}$ satisfying*

1. $d(x, y) = 0 \Leftrightarrow x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y), \quad \forall x, y, z \in X$.

One can see easily that if all the above properties are satisfied by d then $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$. So $d(x, y) \geq 0$ for all $x, y \in X$.

Example 0.2. 1. *In $\mathbb{R}$*

   (a) **Usual metric** *or standard metric $d(x, y) = |x-y|$.*

   (b) **Discrete Metric** $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

2. *In $\mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.*

   (a) $d(x, y) = \sum_{k=1}^{n} |x_k - y_k|$.

   (b) $d(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$.

   (c) $d(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^2 \right)^2$.

   In order to show this as a metric we need following important inequality.

   Theorem 0.3 (Cauchy-Schwartz Inequality)**.** *Let $x, y \in \mathbb{R}^n$. Denote $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$.. Then,*

$$|\langle x, y \rangle| \leq \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |y_k|^2 \right)^{\frac{1}{2}}.$$

   Equality occurs if and only if x and y are co-linear.

Proof. We can assume that both x and y are non-zero elements in $\mathbb{R}^n$. Consider $x - ty$. Then

$$\begin{aligned} 0 \leq \langle x - ty, x - ty \rangle &= \|x\|^2 + t^2\|y\|^2 - \langle x, ty \rangle - t\langle y, x \rangle \\ &= \|x\|^2 + t^2\|y\|^2 - 2t\langle x, y \rangle. \end{aligned}$$

Put $t = \frac{\langle x,y \rangle}{\|y\|^2}$ then we have $0 \leq \|x\|^2 - \frac{\langle x,y \rangle^2}{\|y\|^2}$.
For equality $\|x - ty\| = 0$. So we have the conclusion. $\square$

As a consequence of the above theorem we have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \left| \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle \right| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Thus, $d(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - x\| = d(x, z) + d(z, y)$.

(d) *Let* $1 < p \neq 2 < \infty$ *Define* $d(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$. *Will it be metric? For triangle inequality we need extra effort. We need another inequality which is called Minkowski inequality.*

(e) **Discrete Metric** $d(x, y) == \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

3. *Consider* $C[0, 1] = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$. *For* $f, g \in C[0, 1]$

   (a) $d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$.

   (b) $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.

   (c) *For* $1 < p < \infty$ $d_p(f, g) = \left(\int_0^1 |f(x) - g(x)|^p \right)^{\frac{1}{p}}$. *(requires extra work!)*

4. *Sequence Spaces*

   (a) $l_1(\mathbb{N}) = \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n| < \infty\}$. *Define* $d_1(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$.

   (b) $l_\infty(\mathbb{N}) = \{x = \{x_n\} : \sup_{1 \leq n \infty} |x_n| < \infty\}$. *Define* $d_\infty(x, y) = \sup_{1 \leq n < \infty} |x_n - y_n|$.

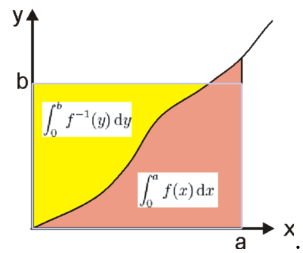
   (c) *For* $1 < p < \infty$, $l_p(\mathbb{N}) = \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. *Define* $d_p(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$.

0.2 Inequalities

Theorem 0.4 (Young's Inequality)**.** *Let* $1 \leq p < \infty$ *and* $a, b \geq 0$. *If* $\frac{1}{p} + \frac{1}{q} = 1$ *then*

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Proof. $f(x) = x^{p-1}$



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$$\text{Denote } \|x\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sup_n |x_n| & p = \infty \end{cases}.$$

Theorem 0.5 (Hölder's Inequality)**.** *Let* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \|y\|_q.$$

Proof. We can assume $\|x\|_p \neq 0$, $\|y\|_q \neq 0$. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|x_n|}{\|x\|_p} \frac{|y_n|}{\|y\|_q} &\le \sum_{n=1}^{\infty} \frac{|x_n|^p}{p\|x\|_p^p} \sum_{n=1}^{\infty} \frac{|y_n|^q}{q\|y\|_q^q} \quad \text{(Young's Inq.)} \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

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Theorem 0.6 (Minkowski Inequality)**.** *Let* $1 \le p \le \infty$*. Then*

$$\|x + y\|_p \le \|x\|_p + \|y\|_p.$$

Proof.

$$\begin{aligned} \|x+y\|_p^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \\ &= \sum_{n=1}^{\infty} |x_n + y_n| \; |x_n + y_n|^{p-1} \\ &\le \sum_{n=1}^{\infty} (|x_n| + |y_n|) \; |x_n + y_n|^{p-1} \\ &\le \|x+y\|_p^{\frac{p}{q}} (\|x\|_p + \|y\|_p). \end{aligned}$$

$\square$

Corollary 0.7.

1. *l_p is a metric space.*
2. *$C[0,1], d_p)$ is a metric space.*

Definition 0.8. *Let (X,d) be a metric space.*

1. *A sequence $\{x_n\}$ in X is said to converge to $x \in X$ (i.e. $x_n \to x$ as $n \to \infty$) if for $\epsilon > 0$ $\exists N_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$, $\forall n \geq N_0$.*
2. *A sequence $\{x_n\}$ in X is said to be Cauchy if for $\epsilon > 0$ $\exists N_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, $\forall n, m \geq N_0$.*
3. *We say that (X,d) is a complete metric space if every Cauchy sequence converges to some $x \in X$.*

Theorem 0.9. *$\mathbb{R}^n$ is complete.*

Proof is easy by looking at the each co-ordinates.

Theorem 0.10. *l_p is a complete metric space.*

Sketch of the Proof. Let $\epsilon > 0$. Find N_0 such that

$$\|x_n - x_m\|_p < \epsilon, \quad \forall n, m \geq N_0. \tag{1}$$

For each n we have $x_n = \{x_n(k)\}$. For a fixed k $\{x_n(k)\}$ is Cauchy in $\mathbb{R}$. So $x_n(k) \to x(k)$ as $n \to \infty$ for some $x(k) \in \mathbb{R}$. i.e $\exists$ N_k such that

$$|x_n(k) - x(k)| < \epsilon, \ \forall n \geq N_k. \tag{2}$$

Define $x = \{x(k)\}$.

$$\begin{aligned}
\sum_{k=1}^{N} |x(k)|^p &= \sum_{k=1}^{N} \lim_{n\to\infty} |x_n(k)|^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{N} |x_n(k)|^p \quad (finite\ sum) \\
&\leq \lim_{n\to\infty} \sum_{k=1}^{\infty} |x_n(k)|^p \leq M.
\end{aligned}$$

Last inequality follows as $\{x_n\}$ being Cauchy is bounded. Thus, $x \in l_p$. Now,

$$\begin{aligned}
\sum_{k=1}^{N} |x_{N_0}(k) - x(k)|^p &= \sum_{k=1}^{N} \lim_{n\to\infty} |x_{N_0}(k) - x_n(k)|^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{N} |x_{N_0}(k) - x_n(k)|^p \\
&\leq \lim_{n\to\infty} \|x_{N_0} - x_n\|_p^p = 0.
\end{aligned}$$

This shows $\|x_n - x\|_p \to 0$. $\square$