Assignment - 12 (Solution/hints)

1. $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$ is not equicontinuous on $[0,1]$.

Suppose $\{f_n\}$ is equicontinuous. Then for $0 < \varepsilon < 1$, $\exists\ \delta_\varepsilon > 0$ s.t.

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{for } |x-y| < \delta_\varepsilon.$$

Take n large enough s.t. $\frac{1}{n} < \delta_\varepsilon$. Then $|f_n(\underbrace{\tfrac{1}{n}}_{x}) - f_n(\underbrace{0}_{y})| = 1 > \varepsilon$

So, $\{f_n\}$ is not equicontinuous on $[0,1]$.

2. $f_n(x) = \dfrac{e^{-nx}}{n^2}$ not equicts. on $\mathbb{R}$.

Suppose it is equicts. Then $|f_n(x) - f_n(y)| < \varepsilon$ whenever $|x-y| < \delta$.
Take $y=0$ and $x = -\frac{\delta}{2}$. Then show that $\dfrac{|e^{n\delta/2} - 1|}{n^2} \to \infty$ as $n \to \infty$

contradiction to $\dfrac{|e^{n\delta/2} - 1|}{n^2} < \varepsilon\ \forall n$

3. (f_n) ptwise bdd. on E a countable set. Let $E = \{x_1, x_2, x_3, \ldots\}$.

(why?) Then, for x_1, $\exists$ a subseq. $f_{11}, f_{12}, f_{13}, \ldots$ s.t.

$$f_{11}(x_1),\ f_{12}(x_1), \ldots \longrightarrow \text{cvgs. to say } f(x_1)$$

define $f: E \to \mathbb{R}$ as $f(x_1) = \lim_{k\to\infty} f_{1k}(x_1)$

Consider the seq. $(f_{11}, f_{12}, \ldots)$

For x_2, $\exists$ a subseq. of $(f_{11}, f_{12}, \ldots)$ say:

$$f_{21}(x_2),\ f_{22}(x_2),\ f_{23}(x_2), \ldots \longrightarrow \text{cvgs. to } f(x_2) \text{ (say)}.$$

Continuing this way, one obtains

$$\begin{array}{lllll} f_{11} & f_{12} & f_{13} & f_{14} & \cdots \\ f_{21} & f_{22} & f_{23} & f_{24} & \cdots \\ f_{31} & f_{32} & f_{33} & f_{34} & \cdots \\ \vdots & & & & \end{array}$$

Choose the diagonal seq. (f_{nn}) which is a subseq. of (f_n) and note that for each $x_k \in E$, $f_{nn}(x_k) \to f(x_k)$ as $n \to \infty$.

4. K: compact set and $f_n : K \to \mathbb{R}$ cts.

Suppose $f_n \to f$ unif.

<u>claim:</u> (f_n) equicts. on K.

HW: (f_n) is unif. Cauchy. Hence, for $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n, m \geq N_\varepsilon$.

$\forall x \in K$, $|f_n(x) - f_m(x)| < \varepsilon$.

Target: Want $|f_n(x) - f_n(y)| < \varepsilon$, $\forall x, y \in K$, $\forall n \geq 1$, whenever $d(x,y) < \delta_\varepsilon$

For N_ε: f_{N_ε} is uniformly cts., so $\exists \delta_{N_\varepsilon}$ s.t. $d(x,y) < \delta_{N_\varepsilon}$, $|f_{N_\varepsilon}(x) - f_{N_\varepsilon}(y)| < \varepsilon$.

For all $n \geq N_\varepsilon$,

(HW): $|f_n(x) - f_n(y)| < 3\varepsilon$.

We are now left with $n = 1, 2, \ldots, N_\varepsilon - 1$. Use uniform continuity of $\{f_1, \ldots, f_{N_\varepsilon - 1}\}$

Then choose $\delta_\varepsilon := \min\{\delta_{1,\varepsilon}, \ldots, \delta_{N_\varepsilon - 1}, \delta_{N_\varepsilon}\}$. Complete the solution.

5. Note that $|f_n'(x)| \leq M$ $\forall x \in [a,b]$, $\forall n \geq 1$.

By MVT on $[x,y]$: $|f_n(x) - f_n(y)| \leq M|x-y|$ $\forall n \geq 1$.

Hence (f_n) is equicts. Since $[a,b]$ is compact & $\{f_n\}$ unif. bounded, by Arzela-Ascoli thm, $\{f_n\}$ has a unif. cvgt. subseqn.

6. Note that $|f_n(x) - f_n(y)| \leq |x-y|$ $\forall n \geq 1$.

Hence (f_n) equi. cts. (why?)

$\to$ $f_n \to 0$ ptwise (HW)

Suppose (f_n) has a unif. cvgt. subseq. $\{f_{n_k}\}$.

Then $f_{n_k} \to 0$ unif.

(HW). Show that $f_n \not\to 0$ unif.

7. Note that $\int_0^1 f(x) P(x) = 0$ for all polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

HW: Use Weierstrass thm. to conclude that $\int_0^1 f^2 dx = 0$. Then conclude that $f = 0$.

8. Suppose (f_n) equicts. on a compact set K.

$f_n \longrightarrow f$ ptwise on K.

<u>claim</u> $f_n \to f$ unif. on K.

pf: For $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f_n(x) - f_n(y)| < \varepsilon/3$ if $d(x,y) < \delta$, $\forall n$, $\forall x, y \in K$.

Consider $\bigcup_{x \in K} B(x, \delta)$ that covers K.

Compactness of K implies that $\exists \{x_1, x_2, .., x_k\}$ s.t. $\bigcup_{i=1}^{k} B(x_i; \delta) \supset K$.

HW: Show that $\exists\ N \in \mathbb{N}$ s.t. $|f_n(x_i) - f_m(x_i)| < \varepsilon/3$ $\forall\ m, n \geq N$, $\forall\ 1 \leq i \leq k$.

Let $x \in K$. and $m, n \geq N$.

Then $x \in B(x_j, \delta)$ for some $1 \leq j \leq k$.

HW: $|f_n(x) - f_m(x)| < \varepsilon$ $\forall\ n, m \geq N$ and $\forall\ x \in K$.