Assignment-8

1. Define $d(m,n) := \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Show that d is equivalent to the usual metric on $\mathbb{N}$ but $(\mathbb{N}, d)$ is not complete.

$(\mathbb{N}, |\cdot|)$ is a discrete metric space. A $(x_n) \in \mathbb{N}$ cvgs. w.r.t $|\cdot|$ iff (x_n) is eventually constant. If $|x_n - x| \to 0$ then $x_n = x \ \forall n \geq N$. Hence $d(x_n, x) = \left|\frac{1}{x_n} - \frac{1}{x}\right| = 0 \ \forall n \geq N$.
Conversely, if $d(x_n, x) \to 0$, then $\left|\frac{1}{x_n} - \frac{1}{x}\right| \to 0$ as $n \to \infty$ ...... (x_n) must be eventually const. (why?)
Completeness: (n) is Cauchy wrt. d, but not w.r.t. $|\cdot|$.

2. Show that $\mathbb{R}^n$ is complete under $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ norms.

Hint: Show that these norms are strongly equivalent.

3. Given metric spaces M and N, show that $M \times N$ is complete iff both M and N are complete.

$M \times N$ $\quad d_2(X, Y)$ or $d_\infty(X, Y)$ where $X = (x, y)$, $Y = (a, b)$

4. Prove that the Hilbert cube H^∞ is complete.

$\sum \frac{1}{2^k} |x_k^{(n)} - x_k^{(m)}| < \varepsilon \ \forall m, n \geq N_\varepsilon \Rightarrow |x_k^{(n)} - x_k^{(m)}| < \varepsilon \ \forall n, m \geq N_\varepsilon$ & $\forall k \geq 1$.

$\to$ let $x_k^{(n)} \to x_k$ where $|x_k^{(n)} - x_k| < \varepsilon$, $\forall n \geq N_\varepsilon$ and $\forall k \geq 1$.

(HW) $x = (x_k) \in H^\infty$

$\to$ for each fixed $m \geq N_\varepsilon$, consider $\sum_{k=1}^{\ell} \frac{1}{2^k} |x_k^{(n)} - x_k^{(m)}| < \varepsilon$ (why?)

$x_k^{(n)} \to x_k$

why? $\Rightarrow \sum_{k=1}^{\ell} \frac{1}{2^k} |x_k - x_k^{(m)}| < \varepsilon$. Therefore, $\sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - x_k^{(m)}| < \varepsilon \ \forall m \geq N_\varepsilon$.

$\qquad\qquad\qquad\qquad \| $

$d(x_k, x) \to 0$ as $k \to \infty$.

5. Is it essential that the sets F_n in the Nested Set Thm. be both closed & bdd? Justify. Is the condition really necessary? yes!

$M = \mathbb{R}$ $\quad F_n := (1, 1 + \frac{1}{n})$ nested but $\cap F_n = \phi$.
$\qquad\qquad F_n := [n, \infty)$ $\quad \cap F_n = ?$

6. Prove that a normed linear space X is complete iff its closed unit ball $B = \{x \in X \mid \|x\| \leq 1\}$ is complete.

(HW): (X, d): metric space. X complete $\Leftrightarrow \forall r > 0$, $\overline{B(x, r)} := \{y \in X \mid d(x, y) \leq r\}$ is complete.

X: complete $\Rightarrow$ $B = \{x \mid \|x\| \le 1\}$ complete. $\quad\|$ (M,d) complete. ACM. (A,d) closed iff complete.

Suppose not. $\exists$ A infinite. totally bdd. set which has no limit pt. in B.

Since tot. bdd., every seq. in A has a Cauchy subseq. - - - (HW)

$\{x : \|x\| \le 1\}$ complete $\Rightarrow$ X: complete.

Suppose not. $\exists$ $(x_n) \in X$ Cauchy s.t. (x_n) does not cvg. in X.

$\Downarrow$

$\|x_n\| \le M$ for some $M > 0$.

$\Rightarrow$ $\|\frac{1}{M} \cdot x_n\| \le 1$. $\Rightarrow$ $\frac{1}{M} \cdot x_n \in B$.

(HW) $\left(\frac{1}{M} \cdot x_n\right)$ Cauchy in B. Here $\left(\frac{1}{M} \cdot x_n\right)$ cvgs. in B. . . - - - - - . . finish the proof.

7. $E = \{x \in \mathbb{Q} \mid 2 < x^2 < 3\} \subset \mathbb{Q}$

closed: $x_n \in E$ s.t. $x_n \to x$. Since $2 < x_n^2 < 3$, $2 < x^2 < 3$ as $x_n^2 \to x^2$

bdd: $|x| < \sqrt{3}$ $\forall x \in E$

NOT compact: $\exists$ $r_n \in \mathbb{Q}$ s.t. $\sqrt{2} < r_n < \sqrt{3}$ and $r_n \to \sqrt{2}$.

8. A compact $\Rightarrow$ $\operatorname{diam}(A) < \infty$.

$\Downarrow$

Hint. totally bdd. $\nearrow$

9. M is compact iff every closed ball in M is compact.

$\Rightarrow$: Closed ball is complete b/c closed subsets of complete metric space is complete.

Let $B = \{y \in M \mid d(x,y) \le r\}$ is a closed ball in M.

Suppose B is not totally bdd. Then $\exists$ a seq. $(x_n) \in B$ s.t. (x_n) does not have a Cauchy subseq. - - - - - finish the proof!

Hint. $\Leftarrow$: M not compact $<$ not complete $\rightsquigarrow$ $\exists$ (x_n) Cauchy not cvgt. But (x_n) is bdd. - - - - .
not tot. bdd. $\rightsquigarrow$

10. $A \subset M$ compact $\quad A \times B \subset M \times N$ $\quad M \times N$: complete metric space.

$B \subset N$ compact. (7) suffices to consider any one of d_∞, d_2, or d_1, show that $A \times B$ is closed & totally bdd.

11. Let $A := \{x \in \mathbb{R}^n \mid \|x\|_1 = 1\} = \{x \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| = 1\}$.

Define $f : (\mathbb{R}^n, \|\cdot\|_2) \to \mathbb{R}$ as

$$f(x_1, \ldots, x_n) = |x_1| + \cdots + |x_n|.$$

Idea! Show that f is cts. on $\mathbb{R}^n$. Then take $\{1\} \subset \mathbb{R}$. Since f is cts. so $f^{-1}\{1\}$ is closed in $\mathbb{R}^n$. But $f^{-1}\{1\} = A$. So A is closed in $\mathbb{R}^n$.

$\forall\, x \in A$, $\|x\|_2^2 = |x_1|^2 + \cdots + |x_n|^2 \leq n$ b/c. $|x_1| + \cdots + |x_n| = 1 \Rightarrow |x_j| \leq 1$, $\forall\, 1 \leq j \leq n$.

Since A is closed and bdd. in $\mathbb{R}^n$ and that bdd. sets are totally bdd. and closed sets of complete metric space is complete, so A is complete and totally bdd, hence compact.

Alternatively, since A is closed and bdd. set in $\mathbb{R}^n$, by Heine-Borel thm, A is compact.

12. Heine-Borel Thm $\Rightarrow$ Bolzano-Weierstrass Thm.

(Every infinite (bdd.) set has a limit pt. in $\mathbb{R}$)

$\Updownarrow$

(totally bdd.)

Idea: Let A be an infinite bdd. set in $\mathbb{R}$. Then $\overline{A}$ is a closed and bdd. set in $\mathbb{R}$. (why?)

By Heine-Borel thm, $\overline{A}$ is compact. Since A is an infinite set in $\mathbb{R}$, let (x_n) be a seq. of distinct pts. in A. Then $(x_n) \in \overline{A}$. Compactness of $\overline{A}$ implies that every seq. in $\overline{A}$ has a cvgt. subseq. in $\overline{A} \subset \mathbb{R}$. ---- finish the proof.

Heine-Borel $\Leftrightarrow$ Completeness of $\mathbb{R}$.

$\Downarrow$

Bolzano-Weierstrass

$\nearrow$ Thm. 7.11 (Carothers)

Suppose $\mathbb{R}$ is complete.

claim: A closed & bdd in $\mathbb{R}$ is compact.

Pf: A closed in $\mathbb{R}$ (complete) $\Rightarrow$ A is complete.

A bdd. in $\mathbb{R}$ $\Rightarrow$ A is tot. bdd. in $\mathbb{R}$.

Hence A is compact.

Completeness of $\mathbb{R}$ $\Rightarrow$ Heine-Borel thm.

13. Show that $A = \{x \in \ell_2 \mid |x_n| \le \frac{1}{n}, n = 1, 2 \dots\}$ is compact in $(\ell_2, \|\cdot\|_2)$.

Show that A is closed and totally bdd.

Hints: • $x \in \overline{A} \Rightarrow \exists\, x_k \in A$ s.t. $\|x_k - x\|_2 \to 0$. $\sum_{j=1}^{\infty} |x_j^{(k)} - x_j|^2 < \varepsilon \quad \forall\, k \ge N_\varepsilon$

$x_k = (x_j^{(k)})_{j=1}^{\infty} \in A$ so $|x_j^{(k)}| \le \frac{1}{j} \quad \forall\, k \ge 1$. Then $|x_j - x_j^{(k)}| < \varepsilon \quad \forall\, k \ge N_\varepsilon$ (?)

Take $k := N_\varepsilon$. For each $j \ge 1$, $|x_j| \le \underbrace{|x_j - x_j^{(N_\varepsilon)}|}_{<\varepsilon} + |x_j^{(N_\varepsilon)}| < \varepsilon + \frac{1}{j}$. $\Rightarrow |x_j| \le \frac{1}{j}$

$|x_j| < \varepsilon + \frac{1}{j}$

• For $\varepsilon > 0$, $\exists\, M_\varepsilon \in \mathbb{N}$ s.t. $\sum_{k \ge M_\varepsilon} \frac{1}{k^2} < \varepsilon$. Consider $B = \{x \in A \mid x = (x_1, \dots, x_{M_\varepsilon - 1}, 0)\}$.

For $x \in B$, $|x_j| \le \frac{1}{j} \quad \forall\, 1 \le j \le M_\varepsilon - 1$. Show that B is totally bdd.

Then show that $\overline{B} = A$. in ℓ_2. Hence (?) A is totally bdd.

14. If M is compact, then M is separable.

$\Downarrow$ totally bdd. $\Rightarrow$

15. Suppose M is compact and $f: M \to N$ is cts, one-one and onto.

Prove that f is a homeomorphism. (Self)

E 16. Given $f: [a,b] \to \mathbb{R}$. Define $G: [a,b] \to \mathbb{R}^2$ by $G(x) = (x, f(x))$.

Prove that TFAE:

(i) f is cts.

(ii) G is cts.

(iii) the graph of f is a compact subset of $\mathbb{R}^2$.

17. Show that TFAE:

(i) Every decreasing seq. of nonempty closed sets in M has nonempty intersection.

(ii) Every countable open cover of M admits a finite subcover.

16. (iii) $\Rightarrow$ (i): $G(f)$ compact set in $\mathbb{R}^2$

claim f is cts., i.e., for $x_n \to x$, $f(x_n) \to f(x)$.

Pf: To show $f(x_n) \to f(x)$ it suffices to show that every subseq. of $(f(x_n))$ has a further subseq. that converges to $f(x)$.

Let $(f(x_{n_k}))$ be a subseq. of $(f(x_n))$.

Consider $(x_{n_k}, f(x_{n_k}))$. Note that $(x_{n_k}, f(x_{n_k}))$ is a seq. in $G(f)$.

Since $G(f)$ is compact, hence sequentially compact, $\exists \ (x_{n_{k_m}}, f(x_{n_{k_m}}))_{m=1}^{\infty}$

s.t. $(x_{n_{k_m}}, f(x_{n_{k_m}}))_{m=1}^{\infty}$ cvgs. to $(a, f(a)) \in G(f)$.

Since $x_{n_{k_m}} \xrightarrow{\text{as } m \to \infty} a$ and $x_n \to x$, $x = a$. And $f(x_{n_{k_m}}) \to f(a)$ as $m \to \infty$.

Since $f(x) = f(a)$, so $f(x_{n_{k_m}}) \to f(x)$ as $m \to \infty$.

17. (i) $\Rightarrow$ (ii)

Suppose there is no finite subcover of a given covering $\{G_n\}$ where $\bigcup G_n = M$

$F_1 = G_1^c$, $F_2 = G_1^c \cap G_2^c$, ... Note that $F_n \neq \phi \ \forall n \geq 1$ (why?)

Since $M = \bigcup G_n$, $\bigcap G_n^c = \phi$ but $F_1 \supset F_2 \supset \cdots$ so $\bigcap_{j=1}^{n} F_j \neq \phi$

$\Rightarrow \bigcap G_n^c = \bigcap F_n = \phi$ contradiction ...

(ii) $\Rightarrow$ (i) Suppose $\exists \ \{F_n\} \downarrow$ s.t. $\bigcap F_n = \phi$.

Idea: Then, $F_1^c \subset F_2^c \subset \cdots$ s.t. $(\bigcap F_n)^c = M$, i.e., $\bigcup F_n^c = M$.

(HW). This has no finite subcover.