Case 2 : Non-Gaussian linear process

We only have asymptotic distⁿ result for such non-Gaussian linear processes.

Let $\{X_t\}$ be a covariance stationary linear time series

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j} \ ; \ \epsilon_t \sim WN(0, \sigma^2)$$

where $\sum_j |\psi_j| < \infty$ & $\sum_j \psi_j \neq 0$, then

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{L} N\left(0, \sum_{h=-\infty}^{\infty} \gamma_h\right)$$

i.e $\sqrt{n}(\bar{X}_n - \mu) \overset{asym}{\sim} N\left(0, \left(\sum \psi_j\right)^2 \sigma^2\right)$.

Example : $\{X_t\}$ is non Gaussian stationary AR(1)

$$X_t = \delta + \phi X_{t-1} + \epsilon_t \ ; \ \epsilon_t \sim WN(0, \sigma^2)$$

$$|\phi| < 1$$

$$\delta = \mu(1-\phi) \ ; \ \mu = \frac{\delta}{1-\phi}$$

$$(1-\phi B) X_t = \delta + \epsilon_t$$

$$X_t = (1-\phi B)^{-1} \delta + (1-\phi B)^{-1} \epsilon_t$$

i.e $$X_t = \left(\frac{\delta}{1-\phi}\right) + \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$$

i.e. $X_t = \mu + \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$

For AR(1), $\sum_{-\infty}^{\infty} \gamma_h = g_X(1) = \frac{\sigma^2}{(1-\phi)^2}$.

Using the asymptotic result, we have

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{\mathcal{L}} N\left(0, \frac{\sigma^2}{(1-\phi)^2}\right)$$

<u>Estimation of $\gamma_h / \rho_h$</u>

$$\hat{\gamma}_h = \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \bar{X}_n)(X_{t+h} - \bar{X}_n)$$

i.e. $\hat{\gamma}_h = \frac{1}{n} \sum_{t=h+1}^{n} (X_t - \bar{X}_n)(X_{t-h} - \bar{X}_n)$

$$\hat{\rho}_h = \hat{\gamma}_h / \hat{\gamma}_0$$

<u>Asymptotic Result for distn</u>

Suppose $\{X_t\}$ is covariance stationary linear process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j}$$

$\epsilon_t \sim WN(0, \sigma^2)$, $\sum_j |\psi_j| < \infty$ & $E(\epsilon_t^4) < \infty$

Then for each $h$, we have

$$\hat{\underset{\sim}{\rho}}(h) \overset{asym}{\sim} N_h\left(\underset{\sim}{\rho}(h), \frac{1}{n}W\right)$$

i.e. $$\sqrt{n}\left(\hat{\underset{\sim}{\rho}}(h) - \underset{\sim}{\rho}(h)\right) \xrightarrow{d} N_h(\underset{\sim}{0}, W)$$

where $$\hat{\underset{\sim}{\rho}}(h) = (\hat{\rho}_1, \dots, \hat{\rho}_h)'$$

$$\underset{\sim}{\rho}(h) = (\rho_1, \dots, \rho_h)'$$

$$W = ((w_{ij}))$$

$$w_{ij} = \sum_{k=-\infty}^{\infty}\Big(\rho(k+i)\rho(k+j) + \rho(k-i)\rho(k+j) + 2\rho(i)\rho(j)\rho^2(k) - 2\rho(i)\rho(k)\rho(k+j) - 2\rho(j)\rho(k)\rho(k+i)\Big)$$

Alt form

$$= \sum_{k=1}^{\infty}\Big(\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\Big)\Big(\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\Big)$$

The above is called the "Bartlett's formula".

## Application of the above result

Example 1: Let $\{X_t\}$ be an i.i.d. $\underset{WN}{(0, \sigma^2)}$ sequence

$$\rho_h = 0 \quad \forall\ |h| > 0$$

Applying Bartlett's formula

$$w_{ii} = \sum_{k=1}^{\infty}\Big(\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\Big)^2 = \rho(0)^2 = 1$$

$$W_{ij} = \sum_{k=1}^{\infty} \left( \rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k) \right) \left( \rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k) \right)$$

$$= 0 \quad \forall\, i \neq j$$

i.e. $W_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{o/w} \end{cases}$

$$\sqrt{n}\left( \hat{\underset{\sim}{\rho}}(h) - \underset{\sim}{\rho}(h) \right) \xrightarrow{L} N_h(\underset{\sim}{0}, I_h)$$

i.e. $\hat{\underset{\sim}{\rho}}(h) \overset{asym}{\sim} N_h\left( \underset{\sim}{\rho}(h), \frac{1}{n} I_h \right)$

So, for large $n$, $\hat{\rho}_1, \ldots, \hat{\rho}_h$ are approximately independent and identically distributed univariate normal r.v.s with mean 0 and variance $\frac{1}{n}$.

Example 2 :

$$X_t = \epsilon_t + \theta\, \epsilon_{t-1} \;;\; \epsilon_t \sim WN(0, \sigma^2)$$

$$W_{ii} = \sum_{k=1}^{\infty} \left( \rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k) \right)^2$$

$$W_{11} = \sum_{k=1}^{\infty} \left( \rho(k+1) + \rho(k-1) - 2\rho(1)\rho(k) \right)^2$$

$$= \underset{k=1}{\underbrace{\left( \rho(0) - 2\rho(1)\rho(1) \right)^2}} + \underset{k=2}{\underbrace{\left( \rho(1) \right)^2}} + 0 \cdots$$

i.e. $W_{11} = 1 + 4\rho(1)^4 - 4\rho(1)^2 + \rho(1)^2$

$W_{11} = 1 - 3\rho(1)^2 + 4\rho(1)^4$

$$W_{22} = \sum_{K=1}^{\infty} \left( \rho(K+2) + \rho(K-2) - 2\rho(2)\rho(K) \right)^2$$

$$W_{22} = \underset{\uparrow \atop K=1}{\rho(1)^2} + \underset{\uparrow \atop K=2}{\rho(0)^2} + \underset{\uparrow \atop K=3}{\rho(1)^2} = 1 + 2\rho(1)^2$$

$\forall i \geq 2$

$$W_{ii} = \sum_{K=1}^{\infty} \left( \rho(K+i) + \rho(K-i) - 2\rho(i)\rho(K) \right)^2$$

i.e. $W_{ii} = \underbrace{\rho(1)^2}_{K=i-1} + \underbrace{\rho(0)^2}_{K=i} + \underbrace{\rho(1)^2}_{K=i+1}$

$W_{ii} = 1 + 2\rho(1)^2 \quad \forall i \geq 2$

Further $W_{ij} \neq 0$ for $i \neq j$.

$$\sqrt{n}\left(\hat{\rho}_1 - \rho_1\right) \xrightarrow{L} N\left(0, 1 - 3\rho_1^2 + 4\rho_1^4\right)$$

i.e. $\hat{\rho}_1 \overset{asym}{\sim} N\left(\rho_1, \frac{1}{n}\left(1 - 3\rho_1^2 + 4\rho_1^4\right)\right)$

& $\forall i > 1$

$$\sqrt{n}\,\hat{\rho}_i \xrightarrow{L} N\left(0, 1 + 2\rho_1^2\right)$$ (Note that $\rho_i = 0 \ \forall i > 1$)

Forecasting in stationary time series: Best Linear Predictor (BLP)

$\{X_t\}$ - Covariance stationary time series with mean $\mu$ and ACVF $\{\gamma_h\}$

Given information upto time $n$, $\{X_1, \dots, X_n\}$ problem is to predict $X_{n+h}$ for some $h > 0$.

**BLP approach**: Find the linear combination of $X_n, \dots, X_1$ that provides the "best" forecast of $X_{n+h}$

"best" : w.r.t. minimum mean square prediction error

**Def$^n$**: Best linear Predictor (BLP)

BLP of $X_{n+h}$ in terms of $(X_n, X_{n-1}, \dots, X_1)$ denoted by $P_{(X_n, \dots, X_1)} X_{n+h} = P_n X_{n+h}$ is the linear $f^n$

$$a_0^* + a_1^* X_n + \dots + a_n^* X_1 \text{ if}$$

$$E\left(X_{n+h} - P_{(X_n, \dots, X_1)} X_{n+h}\right)^2 \text{ is minimum}$$

among all such linear functions.

**Derivation of BLP**

Let $\underset{\sim}{a} = (a_0, a_1, \dots, a_n)'$ and

$$S(\underset{\sim}{a}) = E\left(X_{n+h} - a_0 - \sum_{i=1}^{n} a_i X_{n+1-i}\right)^2$$

$$\underset{\sim}{a}_{BLP} = \underset{\underset{\sim}{a}}{\arg\min}\ S(\underset{\sim}{a})$$