Remark: Dist^n of r^{th} order statistic : $X_{(r)}$

$X_{(r)} = r^{th}$ smallest of $\{X_1, \ldots, X_n\}$ $\quad r = 1, \ldots, n$

p.d.f. of $X_{(r)}$:

$$\frac{n!}{(r-1)!\,(n-r)!} (F(x))^{r-1} (1 - F(x))^{n-r} f(x) \qquad x \in \mathbb{R}$$

jt p.d.f. of $X_{(r)}$ & $X_{(s)}$ $\qquad 1 \leq r < s \leq n$

$$f_{X_{(r)}, X_{(s)}}(x, y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} (F(x))^{r-1} (F(y) - F(x))^{s-r-1} \times$$

$$(1 - F(y))^{n-s} f(x) f(y)$$

$$-\infty < y < x < \infty$$

e.g. jt dist^n of $(X_{(1)}, X_{(n)})$

$$f_{X_{(1)}, X_{(n)}}(x, y) = \frac{n!}{(n-2)!} (F(y) - F(x))^{n-2} f(x) f(y)$$

can be used to obtain p.d.f of range statistic $\quad R = X_{(n)} - X_{(1)}$

(iii) p.m.f. based approach for discrete setup

Suppose $(X_1, \ldots, X_n)$ have joint p.m.f.

$$P(\underset{\sim}{X} = \underset{\sim}{x}) = P(X_1 = x_1, \ldots, X_n = x_n) \qquad \underset{\sim}{x} \in \mathcal{X}$$

$(X_1, \ldots, X_n) \rightarrow Y = u(X_1, \ldots, X_n)$ with $\mathcal{Y}$ as possible values of Y

$$P(Y = y) = \sum_{\substack{\underset{\sim}{x} \in \mathcal{X} \\ \ni u(\underset{\sim}{x}) = y}} P(\underset{\sim}{X} = \underset{\sim}{x})$$

In case we have $X_1, \ldots, X_n$ a random sample (implying independence) with a common p.m.f (identical dist^n) then $P(\underset{\sim}{X} = \underset{\sim}{x})$ factors into n components with identical marginal p.m.f.s.

__Example__: Let X_1 and X_2 are indep with

$$X_i \sim P(\lambda_i) \qquad i = 1, 2$$

$$Y = X_1 + X_2 \qquad \mathcal{Y} = \{0, 1, 2, \ldots\}$$

$$P(Y = y) = \sum_{x=0}^{y} P(X_1 = x, X_2 = y - x)$$

$$= \sum_{x=0}^{y} P(X_1 = x) \, P(X_2 = y - x)$$

$$= \sum_{x=0}^{y} \frac{e^{-\lambda_1} \lambda_1^x}{x!} \frac{e^{-\lambda_2} \lambda_2^{y-x}}{(y-x)!}$$

$$= e^{-(\lambda_1 + \lambda_2)} \frac{1}{y!} \sum_{x=0}^{y} \binom{y}{x} \lambda_1^x \lambda_2^{y-x}$$

$$= e^{-(\lambda_1 + \lambda_2)} \frac{1}{y!} (\lambda_1 + \lambda_2)^y$$

i.e. $Y = X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$

Remark : If $X_1, \ldots, X_n$ are indep with $X_i \sim P(\lambda_i); i=1,\ldots,n$

then $$Y = \sum_{i=1}^{n} X_i \sim P\left(\sum_{i=1}^{n} \lambda_i\right)$$

$\overline{V}$ <u>Jacobian based approach</u> (cont case only)

$\underset{\sim}{X} = (X_1, \ldots, X_n)'$

jt p.d.f. of $X_1, \ldots, X_n$: $f_{X_1,\ldots X_n}(x_1, \ldots, x_n)$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) > 0 \text{ for } \underset{\sim}{x} \in \mathcal{X} \subset \mathbb{R}^n$$

suppose
$$y_1 = h_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$y_n = h_n(x_1, \ldots, x_n)$$

be 1-1 transformation $\mathcal{X} \to \mathcal{Y}$ with inverse as

$$x_i^{-1} = h_i^{-1}(y_1, \ldots, y_n) \; ; \; i=1,\ldots,n$$

$$\mathcal{Y} = \underset{\sim}{h}(\mathcal{X}) = \{\underset{\sim}{h}(\underset{\sim}{x}) \in \mathbb{R}^n : \underset{\sim}{x} \in \mathcal{X}\}$$

Suppose further that

(i) $\dfrac{\partial h_i^{-1}(\underset{\sim}{y})}{\partial y_j}$ exists $\forall i, j$ and are continuous

& (ii) the Jacobian determinant

$$J = \begin{vmatrix} \dfrac{\partial h_1^{-1}(\underset{\sim}{y})}{\partial y_1} & \cdots & \dfrac{\partial h_1^{-1}(\underset{\sim}{y})}{\partial y_n} \\ - & - & - \\ \dfrac{\partial h_n^{-1}(\underset{\sim}{y})}{\partial y_1} & \cdots & \dfrac{\partial h_n^{-1}(\underset{\sim}{y})}{\partial y_n} \end{vmatrix} \neq 0.$$

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \to \begin{pmatrix} Y_1 = h_1(\underset{\sim}{X}) \\ \vdots \\ Y_n = h_n(\underset{\sim}{X}) \end{pmatrix}$$

The joint p.d.f. of new r.v.s $(Y_1, \ldots, Y_n)$ is

$$f_{Y_1 \ldots Y_n}(y_1, \ldots, y_n) = f_{\underset{\sim}{X}}\left(h_1^{-1}(\underset{\sim}{y}), \ldots, h_n^{-1}(\underset{\sim}{y})\right) |J| \; \underbrace{I_{(\underset{\sim}{y} \in \underset{\sim}{h}(\underset{\sim}{x}))}}_{\text{inf on the support of } \underset{\sim}{Y}}$$

<u>Remark</u>: From $f_{\underset{\sim}{Y}}(\underset{\sim}{y})$ we can obtain marginal of $Y_i; i=1,\ldots,n$

<u>Remark</u>: Based on $(X_1, \ldots, X_n)$, suppose we are interested in jt distn of $(Y_1, \ldots, Y_K)$ $K < n$, we define $n-K$ dummy transformations (key is to keep the dummies simple!) and integrate out the dummies to get the jt distn of $(Y_1, \ldots, Y_K)$

i.e. $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \to \begin{pmatrix} Y_1 \\ \vdots \\ Y_K \\ \left.\begin{matrix}\vdots \\ Y_n\end{matrix}\right\} \to \text{dummies} \end{pmatrix} \longrightarrow \begin{pmatrix} Y_1 \\ \vdots \\ Y_K \end{pmatrix}$.

<u>Remark</u>: The above approach can be extended to the case where we have 1-1 mapping in mutually disjoint regions of $\mathcal{X}$

$$\mathcal{X} = \bigcup_{i=1}^{K} \mathcal{X}_i \quad \& \quad \mathcal{X}_i \cap \mathcal{X}_j = \phi$$

$\underset{\sim}{h} = (h_1, \ldots, h_n)$ is 1-1 with the inverse

$h_i^{-1}(\underset{\sim}{y}) = (h_{1,i}^{-1}(\underset{\sim}{y}), \ldots, h_{n,i}^{-1}(\underset{\sim}{y}))$ on $\mathcal{X}_i$

$\dfrac{\partial h_{k,i}^{-1}(\underset{\sim}{y})}{\partial y_j}$ exists and are continuous with the Jacobian

determinants

$$J_i = \begin{vmatrix} \dfrac{\partial h_{1,i}^{-1}(\underset{\sim}{y})}{\partial y_1} & \cdots & \dfrac{\partial h_{1,i}^{-1}(\underset{\sim}{y})}{\partial y_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial h_{n,i}^{-1}(\underset{\sim}{y})}{\partial y_1} & \cdots & \dfrac{\partial h_{n,i}^{-1}(\underset{\sim}{y})}{\partial y_n} \end{vmatrix} \neq 0$$

$i = 1, \ldots, K$

then

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \sum_{j=1}^{K} f_{\underset{\sim}{X}}(h_{1,j}^{-1}(\underset{\sim}{y}), \ldots, h_{n,j}^{-1}(\underset{\sim}{y})) |J_j|$$

Example:

X_1, X_2 i.i.d. $\exp(1)$

i.e. $f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & o/w \end{cases}$

$Y = X_1 - X_2$ — interested to know p.d.f. of Y

Define a 1-1 transformation (with a dummy)

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \rightarrow \begin{pmatrix} Y = X_1 - X_2 \\ Z = X_1 + X_2 \end{pmatrix}$$ ← can be different

Inverse transformation

$$X_1 = \frac{Y+Z}{2} = h_1^{-1}(y, z)$$

$$X_2 = \frac{Z-Y}{2} = h_2^{-1}(y, z)$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Support calculation : y

Note that unconditionally $0 < z < \infty$ & $-\infty < y < \infty$

Note further that from inverse transformation we have:

$$\left.\begin{array}{l} 0 < x_1 < \infty; \text{ i.e. } 0 < \frac{y+z}{2} < \infty \\ \text{i.e. } -y < z < \infty \;/\; -z < y < \infty \end{array}\right\} (*^1)$$

$$\& \left.\begin{array}{l} 0 < x_2 < \infty; \text{ i.e. } 0 < \frac{z-y}{2} < \infty \\ \text{i.e. } y < z < \infty \;/\; -\infty < y < z \end{array}\right\} (*^2)$$

Combining $(*^1)$ & $(*^2)$, we get

$$\left.\begin{aligned} &\max(y,-y) < z < \infty \\ &\& \quad -z < y < z \end{aligned}\right\} \begin{aligned} &\text{with} \\ &0 < z < \infty \\ &-\infty < y < \infty \end{aligned}$$

Thus, if $-\infty < y < 0$ then $-y < z < \infty$

& if $0 < y < \infty$ then $y < z < \infty$

$$\Rightarrow f_{Y,Z}(y,z) = \begin{cases} \frac{1}{2} e^{-z}; & (-\infty < y < 0 \text{ and } -y < z < \infty) \\ & \text{or } (0 < y < \infty \text{ and } y < z < \infty) \\ 0, & \text{o/w.} \end{cases}$$

$\Rightarrow$ Marginal p.d.f. of Y (variable of interest)

$$f_Y(y) = \frac{1}{2}\int_{-y}^{\infty} e^{-z}\,dz = \frac{1}{2}e^{y}; \qquad \text{if } -\infty < y < 0$$

$$= \frac{1}{2}\int_{y}^{\infty} e^{-z}\,dz = \frac{1}{2}e^{-y}; \qquad \text{if } 0 < y < \infty$$

i.e. $f_Y(y) = \frac{1}{2} e^{-|y|}$; $\quad y \in (-\infty, \infty)$