Expression for $V(\bar{X}_n)$ (It is not $\frac{\sigma^2}{n}$!!)

$$V(\bar{X}_n) = V\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right)$$
$$= E\left(\frac{1}{n}\sum_{t} X_t - \mu\right)^2$$
$$= E\left(\frac{1}{n}\sum_{t=1}^{n}(X_t-\mu)\right)^2$$
$$= \frac{1}{n^2}E\Big((X_1-\mu)+\cdots+(X_n-\mu)\Big)\Big((X_1-\mu)+\cdots+(X_n-\mu)\Big)$$
$$= \frac{1}{n^2}\Big[(\gamma_0+\gamma_1+\cdots+\gamma_{n-1}) + (\gamma_1+\gamma_0+\gamma_1+\cdots+\gamma_{n-2})$$
$$- - - - - - - -$$
$$+ (\gamma_{n-1}+\gamma_{n-2}+ - - \cdot + \gamma_0)\Big].$$

$2(n-2)+2 = 2(n-1)$

$$= \frac{1}{n^2}\Big[n\gamma_0 + 2(n-1)\gamma_1 + 2(n-2)\gamma_2 + \cdots \cdots + 2(n-(n-1))\gamma_{n-1}\Big]$$
$$= \frac{1}{n^2}\sum_{i-j=-n}^{n}(n-|i-j|)\gamma_{i-j}$$

i.e. $V(\bar{X}_n) = \frac{1}{n^2}\sum_{h=-n}^{n}(n-|h|)\gamma_h = \frac{1}{n}\sum_{|h|\le n}\left(1-\frac{|h|}{n}\right)\gamma_h$

Estimation of $\gamma(\cdot)$ and $\rho(\cdot)$

Suppose μ is known, an unbiased estimator of $\gamma(h)$ is

$$\hat{\gamma}_{\mu}^{*}(h) = \frac{1}{n-h}\sum_{t=1}^{n-h}(X_t-\mu)(X_{t+h}-\mu)$$

$$h = 0(1)n-1$$

as
$$E\left(\hat{\gamma}_{\mu}^{*}(h)\right) = \frac{1}{n-h}\sum_{t=1}^{n-h}E(X_t-\mu)(X_{t+h}-\mu)$$
$$= \frac{(n-h)\gamma_h}{(n-h)} = \gamma_h$$

An alternate estimator (difference in the divisor only)

$$\hat{\gamma}_{\mu}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_t-\mu)(X_{t+h}-\mu)$$

$$E\left(\hat{\gamma}_{\mu}(h)\right) = \frac{n-h}{n}\gamma_h \neq \gamma_h$$

Bias: $E\left(\hat{\gamma}_{\mu}(h)\right)-\gamma_h = -\frac{h}{n}\gamma_h \to 0$ as $n \to \infty$

$\Rightarrow$ $\hat{\gamma}_{\mu}(h)$ is unbiased in the limit, although It is a biased estimator

Note that

$$\hat{\Gamma}_n = \frac{1}{n} T T'$$

where $n \times 2n$ matrix T is given by

$$T = \begin{pmatrix} 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_n \\ 0 & \cdots & Y_1 & Y_2 & \cdots & Y_n & 0 \\ 0 & \cdots & Y_1 & Y_2 & Y_3 & \cdots & Y_n & 0 & 0 \\ - & - & - & - & - \\ 0 & Y_1 & \cdots & Y_n & 0 & \cdots & 0 \end{pmatrix}$$

$$Y_i = X_i - \bar{X}_n ; \quad i = 1(1)n$$

Thus, $\forall \underset{\sim}{a} \in \mathbb{R}^n \quad \underset{\sim}{a}' \hat{\Gamma}_n \underset{\sim}{a} \geq 0$

$\Rightarrow \hat{\gamma}(h)$ f^n is n.n.d.

<u>Note</u>: $\hat{\gamma}^*(h)$ f^n is not n.n.d.

Standard models of time series

(I) White noise process

$$X_t = \epsilon_t$$

$\{\epsilon_t\}$: sequence of uncorrelated $(0, \sigma^2)$ random variables

$$X_t \sim WN(0, \sigma^2)$$

$$EX_t = 0 \quad \forall t$$

ACVF $\quad \gamma_X(h) = \begin{cases} \sigma^2, & \text{if } h = 0 \\ 0, & \text{o/w} \end{cases}$

ACF $\quad \rho_X(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0, & \text{o/w} \end{cases}$

(II) Moving Average (MA) process

Suppose $\epsilon_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is MA(q) (q is a positive integer) if

$$X_t = \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$$\theta_0 \neq 0, \quad \theta_q \neq 0$$

$\theta_0, \theta_1, \ldots, \theta_q$ are unknown consts – MA parameters

<u>Note</u>: w.l.o.g. θ_0 can be taken as 1

o/w define $\epsilon_t' = \theta_0 \epsilon_t \sim WN(0, \theta_0^2 \sigma^2)$

& $X_t = \epsilon_t' + \left(\frac{\theta_1}{\theta_0}\right) \epsilon_{t-1}' + \cdots + \left(\frac{\theta_q}{\theta_0}\right) \epsilon_{t-q}'$

alternate MA(q) representation

<u>Note</u>: 2-sided MA representation

$$X_t = \sum_{j=-M}^{M} \theta_j \epsilon_{t-j}$$

(iii) <u>Auto Regressive (AR) process</u>

Suppose $\epsilon_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is AR(p) if

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

$$\phi_p \neq 0 \;;\; Cov(\epsilon_t, X_{t-j}) = 0 \;\; \forall j > 0.$$

$\phi_1, \ldots, \phi_p$ are unknown consts; AR parameters.

<u>Note</u>: w.l.o.g. we can take a model without constant term for a stationary process

Suppose we take

$$X_t = \delta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

Since $\{X_t\}$ is stationary

$$\mu = EX_t = \delta + \phi_1 \mu + \cdots + \phi_p \mu$$

$$\Rightarrow \mu(1 - \phi_1 - \cdots - \phi_p) = \delta$$

If $(1 - \phi_1 - \cdots - \phi_p) = 0$ then $\delta = 0$

If o/w, i.e. $(1 - \phi_1 - \cdots - \phi_p) \neq 0$, then we can write

$$X_t = \delta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

as
$$X_t - \mu = \delta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} - \mu + \epsilon_t$$

i.e. $X_t - \mu = \mu(1-\phi_1 - \cdots - \phi_p) + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} - \mu + \epsilon_t$

i.e. $(X_t - \mu) = \phi_1 (X_{t-1} - \mu) + \cdots + \phi_p (X_{t-p} - \mu) + \epsilon_t$

Define, $Y_t = X_t - \mu$

model is $\quad Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \epsilon_t$

$\{X_t\}$ Covariance stationary $\Rightarrow$ $\{Y_t\}$ is also covariance stationary

with identical ACVF & ACF as $\{X_t\}$