Assignment 4b (Evaluation Q.

1. Define the distance between two (nonempty) sets A and B in M by

$$d(A,B) = \inf \{ d(a,b) \mid a \in A, b \in B \}.$$

Give an example of A, B s.t. A and B are disjoint closed sets in $\mathbb{R}^2$ with $d(A,B)=0$.

2. Let $A \subset M$. A point $x \in M$ is called a limit point of A if every nghd of x contains a point of A that is different from x itself, i.e., if $\forall \varepsilon > 0$, $\exists B(x,\varepsilon)$ s.t.

$$B(x,\varepsilon) \setminus \{x\} \cap A \neq \phi.$$

If x is a limit point of A, show that every nghd of x contains infinitely many points of A.

3. Show that x is a limit point of A iff $\exists$ a seq. (x_n) in A s.t. $x_n \to x$ and $x_n \neq x$ $\forall n \geq 1$.

4. Let A' be the set of limit pts. of a set A. Show that A' is closed and $\bar{A} = A' \cup A$. Show that $A' \subset A$ iff A is closed.

5. Prove the Bolzano-Weierstrass Thm: Every bdd infinite subset of $\mathbb{R}$ has a limit point.

6. A set P is called perfect if it is empty or if it is a closed set and every point of P is a limit point of P. Show that a nonempty perfect subset P of $\mathbb{R}$ is uncountable.

7. If $x \in A$ and x is not a limit pt. of A, then x is called an isolated pt. of A. Show that a point $x \in A$ is an isolated pt. of A iff $B(x,\varepsilon) \setminus \{x\} \cap A = \phi$ for some $\varepsilon > 0$. Prove that a subset of $\mathbb{R}$ can have at most countably many isolated points, thus showing that every uncountable subset of $\mathbb{R}$ has a limit pt.

8. A point $x \in M$ is said to be a boundary pt. of A if each nghd of x hits both A and A^c. That is, $B(x,\varepsilon) \cap A \neq \phi$ and $B(x,\varepsilon) \cap A^c \neq \phi$ $\forall\ \varepsilon > 0$.
Denote $\partial A :=$ the set of boundary pts. of A.
Prove:
(i) $\partial A = \partial(A^c)$
(ii) $cl(A) = \partial A \cup A^o$ (Recall A^o is the interior of A & $cl(A)$ is the closure of A.)
(iii) $M = A^o \cup \partial A \cup (A^c)^o$.

9. Show that ∂A is always a closed set, in fact, $\partial A = cl(A) \setminus A^o$.

10. Show that A is closed iff $\partial A \subset A$.

11. Give examples showing $\partial A = \phi$ and $\partial A = M$ are both possible.

Evaluation 5 pts.

12. (a) Prove that if A is open or A is closed in M, then $(\partial A)^o = \phi$. (4 pts.)
(b) Give an example in which $(\partial A)^o = M$. (1 pt.)

13. A set is said to be dense in M if $cl(A) = M$.
Show that A is dense in M iff any one of them holds;
(a) Every pt. in M is the limit of a seq. from A.
(b) $B(x,\varepsilon) \cap A \neq \phi$ $\forall\ x \in M$ and $\forall\ \varepsilon > 0$.
(c) $\mathcal{U} \cap A \neq \phi$ for every nonempty open set $\mathcal{U}$.
(d) A^c has empty interior, i.e., $(A^c)^o = \phi$.

14. Let G be open and let D be dense in M.
Show that $\overline{G \cap D} = \overline{G}$. Give an example showing that this equality may fail if G is not open. (Recall notation $\overline{A} = cl(A)$.)

15. A metric space is called separable if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}$, $\mathbb{R}^2$, in $\mathbb{R}^n$ for $n>2$.

16. Prove that $(\ell_2, \|\cdot\|_2)$ is separable, but $(\ell^\infty, \|\cdot\|_\infty)$ is not separable.

17. Show that a separable metric space has at most countably many isolated points.

18. If M is separable, show that any collection of disjoint open sets in M is at most countable.

19. A set A is said to be nowhere dense set if $(cl(A))^\circ = \phi$.
Show that $\{x\}$ is a nowhere dense in M iff x is not an isolated pt. in M.

20. If A and B are nowhere dense sets in M, then prove that $A \cup B$ is nowhere dense.
Give an example showing that an infinite union of nowhere dense sets need not be nowhere dense.