Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex Analysis (MTH 403A)

Solutions for Mid-semester Examination

2023-24-I

Date: 19 Sept. 2023

Time: 2 hours

Total marks: 35

Instructions

- All questions are compulsory.
- Questions can be answered in any serial order. You, however, **must** ensure that the serial numbers given of the answers are correct and distinct.
- All parts of a question must be answered **together**. Otherwise that question will not be graded.
- The answer to each of these questions must be furnished with all the necessary and relevant details. Insufficient explanations, inarticulate answers may result in deduction of marks.

Questions

1. (a) Show that i^i is **always** a real number.

Solution. Let $\alpha \in \mathbb{R}$. Choose $k \in \mathbb{Z}$ such that

$$\frac{(\alpha - \frac{\pi}{2})}{2\pi} \leq k < \frac{(\alpha - \frac{\pi}{2})}{2\pi} + 1,$$

i.e., equivalently, $\alpha \leq \frac{\pi}{2} + 2k\pi < \alpha + 2\pi$. Considering the branch $\log_\alpha$, we obtain that

$$i^i = e^{i \log_\alpha i} = e^{i \log_\alpha \left(e^{\frac{i\pi}{2}}\right)} = e^{i \cdot i \left(\frac{\pi}{2} + 2k\pi\right)} = e^{-(4k+1)\frac{\pi}{2}} \in \mathbb{R}.$$

(b) Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ be a bounded holomorphic function. Will f necessarily be constant? Justify your claim.

Solution. From hypothesis, we have f is bounded on $\mathbb{D} \setminus \{0\}$, hence f can be redefined at 0 so that the extension, say g is holomorphic. Clearly g is bounded as f is bounded, hence Liouville's theorem yields that g is constant. Therefore f is also constant.

Note: Here we have used Exercise 1.17 of Exercise Sheet 5.

(c) Let $U \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $f(z) \stackrel{\text{def}}{=} e^z$, for all $z \in \mathbb{C}$. Clearly $|f|$ is not bounded on U but $|f| \equiv 1$ on ∂U. Does this contradict the corollary of 'Maximum modulus principle' that yields $\sup_{\overline{U}} |f| = \sup_{\partial U} |f|$? Justify your answer.

Solution. This does not contradict the above-mentioned corollary at all, since the corollary assumes that U is bounded.

Note: This rather shows that the boundedness assumption on U cannot be dropped in order to obtain $\sup_{\overline{U}} |f| = \sup_{\partial U} |f|$.

[3+2+2=7]

2. Let $U \subseteq \mathbb{C}$ be a region and $f : U \longrightarrow \mathbb{C}$. Suppose that $\{P_n\}_{n=1}^{\infty}$ is sequence of polynomial with complex coefficients which converges to f uniformly on each compact subset of U. Assume that there exists $d \in \mathbb{N}$ such that $\deg P_n \leq d$ for all $n \in \mathbb{N}$. Show that f is also a polynomial with degree at most d.

Solution. Since $P_n \xrightarrow[n\to\infty]{} f$ uniformly on each compact subset of U, it follows that, for all $k \geq 0$, $P_n^{(k)} \xrightarrow[n\to\infty]{} f^{(k)}$ uniformly on each compact subset of U. Since each P_n has degree $\leq d$, $P_n^{(k)} \equiv 0$, for all $k \geq d+1$. This implies that $f^{(k)} \equiv 0$ whenever $k \geq d+1$. Let $z_0 \in U$. Choose $R > 0$ such that $D(z_0; R) \subseteq U$. Then, for all $z \in D(z_0; R)$, $f(z) = \sum_{n=0}^{d} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$. Since the polynomial $\sum_{n=0}^{d} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ agrees with f on $D(z_0; R)$, which has a limit point in U, namely z_0, it follows from the 'Identity theorem' that for all $z \in U$, $f(z) = \sum_{n=0}^{d} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$.

Note: Note the use of the 'Identity theorem' here. In order to show f is a polynomial, it is not enough to show that, for all $z_0 \in U$, it agrees with the polynomial $\sum_{n=0}^{d} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ in an open disc centered at z_0, as you can see the polynomial apparently varies with z_0. Hence some reason needs to be provided why a single polynomial will work for every z_0.

[4]

3. Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\left\{b_n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ is unbounded. Prove or disprove the following: there exists a holomorphic function f, defined on an open subset of $\mathbb{C}$ containing 0, such that $|f^{(n)}(0)| \geq n! b_n$ for all $n \geq 0$.

Solution. Suppose that such a holomorphic function, defined on an open subset of $\mathbb{C}$ containing 0, say U, exists. Then there exists $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \ \forall z \in D(0; r).$$

This shows that the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ has radius of convergence $\geq r > 0$. On the other hand, we have

$$\left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}} \geq b_n^{\frac{1}{n}} \ \forall n \geq 0.$$

It follows from this that $\displaystyle\limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}} = \infty$, as $\left\{b_n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ is unbounded. Consequently, we obtain that the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ is $\displaystyle\frac{1}{\limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}}} = 0$, which is a contradiction. Hence no such function exists.

[4]

4. Let $z_0 \in \mathbb{C}$, $R > 0$ and $f \in H(D(z_0; R))$. Denote $\operatorname{Re} f$ by u. Show that, for any $r \in (0, R)$, $f'(z_0) = \dfrac{1}{\pi r} \displaystyle\int_0^{2\pi} u(z_0 + re^{it}) e^{-it}\, dt$.

Solution. Denote $\operatorname{Im} f$ by v. From Cauchy's integral formula for derivatives, we get

$$\begin{aligned}
f'(z_0) &= \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{(w - z_0)^2}\, dw \\
&= \frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{it}) e^{-it}\, dt \\
&= \frac{1}{2\pi r} \left(\int_0^{2\pi} u(z_0 + re^{it}) e^{-it}\, dt + i \int_0^{2\pi} v(z_0 + re^{it}) e^{-it}\, dt \right)
\end{aligned}$$

From Cauchy's theorem for open convex sets, one obtains that

$$\begin{aligned}
\int_{C(z_0;r)} f = 0 &\implies \int_0^{2\pi} f(z_0 + re^{it}) e^{it}\, dt = 0 \\
&\implies \int_0^{2\pi} \overline{f(z_0 + re^{it}) e^{it}}\, dt = 0 \\
&\implies \int_0^{2\pi} u(z_0 + re^{it}) e^{-it}\, dt = i \int_0^{2\pi} v(z_0 + re^{it}) e^{-it}\, dt
\end{aligned}$$

In view of the above, is now clear that

$$f'(z_0) = \frac{1}{\pi r} \int_0^{2\pi} u(z_0 + re^{it}) e^{-it}\, dt.$$

[5]

5. Find all entire function f with the following property: there exists $A > 0$ and $k \in \mathbb{N}$ such that, for all $r > 0$, $\displaystyle\int_0^{2\pi} \left|f(re^{it})\right|^2\, dt \leq A r^{2k}$.

Solution. Since f is entire, so it is representable by a power series, say $\sum_{n=0}^{\infty} a_n z^n$, on entire $\mathbb{C}$. Parseval's identity yields that, for all $r > 0$,

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{it})\right|^2 \, dt \leq \frac{A}{2\pi} r^{2k}$$

Let $n > k$. Then from the above, we have

$$|a_n| \leq \left(\frac{A}{2\pi}\right)^{\frac{1}{2}} \frac{1}{r^{n-k}}, \forall r > 0.$$

Now let $r \to \infty$ to conclude that $a_n = 0$. When $n < k$, we again get $a_n = 0$ by letting $r \to 0$. Thus there exists $\alpha \in \mathbb{C}$ such that $f(z) = \alpha z^k$, for all $z \in \mathbb{C}$. Conversely if f is of the form αz^k, where $\alpha \in \mathbb{C}$, it is clear that $\int_0^{2\pi} \left|f(re^{it})\right|^2 dt = \int_0^{2\pi} \alpha^2 r^{2k} \, dt = \alpha^2 \cdot 2\pi i \cdot r^{2k}$. Hence αz^k, for $\alpha \in \mathbb{C}$, are precisely all the enture functions satisfying the above-mentioned condition.

Note.

1. We first show that any entire function satisfying the condition mentioned in the question must be of the form αz^k, where $\alpha \in \mathbb{C}$. This does not mean that functions of this form will satisfy the required condition. Hence, it needs to be verified that such functions must obey the condition given by the question. Without the last step, the solution is NOT complete.
2. The idea behind the solution is same as that of Exercise 1.16 of Exercise Sheet 5. In fact, in that tutorial, one student pointed out this. This exercise has been made of that, with an additional change that now there is a square in the integrand and also with r. This paves the way for Parseval's identity.

[5]

6. Let $a \in \mathbb{R}$. Show that the integral $\int_{-\infty}^{\infty} e^{-(x+ia)^2} \, dx$ is independent of a.

Note. This problem is essentially Exercise 1.1 of Exercise Sheet 6.

Solution. Assume first $a > 0$. For any $R > 0$, the integral of e^{-z^2} along the positively oriented boundary of the rectangle with vertices $-R, R, R + ia$ and $-R + ia$ is 0 by Cauchy's theorem. This provides

$$\int_{-R}^{R} e^{-t^2} \, dt + \int_0^a e^{-(R+it)^2} \cdot i \, dt = \int_{-R}^{R} e^{-(t+ia)^2} \, dt + \int_0^a e^{-(-R+it)^2} \cdot i \, dt$$

Using triangle inequality,

$$\left|\int_0^a e^{-(R+it)^2} \cdot i \, dt\right| \leq \int_0^a \left|e^{-R^2+t^2-2itR}\right| dt \leq \frac{\int_0^a e^{t^2} \, dt}{e^{R^2}}.$$

This shows that $\int_0^a e^{-(R+it)^2} \cdot i\, dt \longrightarrow 0$ as $R \to \infty$. Similarly one has $\int_0^a e^{-(-R+it)^2} \cdot i\, dt \xrightarrow[R\to\infty]{} 0$.
Now, since $\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$ in the above equation, the LHS has a limit as $R \to \infty$, hence so does the RHS. From this, it now follows that

$$\int_{-\infty}^{\infty} e^{-(t+ia)^2}\, dt = \lim_{R\to\infty} \int_{-R}^{R} e^{-(t+ia)^2}\, dt = \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi},$$

since other two terms goes to 0 as $R \to \infty$. For the other case, i.e., when $a < 0$, the boundary of the rectangle with vertices $-R, R, R+ia$ and $-R+ia$ will be negatively oriented, however the argument is exactly similar. Thus, for all $a \in \mathbb{R}$, we obtain that $\int_{-\infty}^{\infty} e^{-(x+ia)^2}\, dx = \sqrt{\pi}$. [5]

7. Evaluate $\int_{C(0;3)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz$.

Note. We have discussed Exercise 3.2 of Exercise Sheet 6 in the tutorial.

Solution. let $U \stackrel{\text{def}}{=} D(0;4)$. Clearly the path $C(0;3)$ is homologous to 0 in U. Consider the positively oriented circles with radius $\frac{1}{4}$ centered at $i, -i$ and 2 respectively. Then we get

$$\begin{aligned}\int_{C(0;3)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz &= \int_{C(i;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz \\ &\quad + \int_{C(-i;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz \\ &\quad + \int_{C(2;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz.\end{aligned}$$

We use Cauchy integral formula to evaluate each of the integrals appeared in the RHS above:

$$\int_{C(i;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz = 2\pi i \cdot \frac{\sin \pi i^2 + \cos \pi i^2}{(i+i)(i-2)} = -\frac{\pi}{(i-2)},$$

$$\int_{C(-i;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz = 2\pi i \cdot \frac{\sin \pi i^2 + \cos \pi i^2}{(-2i)(-i-2)} = -\frac{\pi}{(i+2)},$$

and

$$\int_{C(2;1/4)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz = 2\pi i \cdot \frac{\sin 4\pi + \cos 4\pi}{(2^2+1)} = \frac{2\pi i}{5},$$

Hence we obtain

$$\begin{aligned}\int_{C(0;3)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z^2+1)(z-2)}\, dz &= -\pi\left(\frac{1}{i-2} + \frac{1}{i+2}\right) + \frac{2\pi i}{5} \\ &= \frac{2\pi i}{5} + \frac{2\pi i}{5} \\ &= \frac{4\pi i}{5}.\end{aligned}$$

[5]