**Name:** ____________________

**Roll Number:** _______________

# Practice Midterm Solutions

## MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

**INSTRUCTIONS**

1. Write your **Name** and **Roll number** above.
2. This exam contains **4 + 1** questions and is worth **40%** of your grade.
3. Answer **ALL** questions.

**Question 1. [5 × 2 Points]**

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) For every infinite limit ordinal $\alpha < \omega_1$, there is an ordinal $\beta$ such that $\alpha = \beta + \omega$.

(ii) The set of irrationals numbers has the same cardinality as the set of real numbers.

(iii) For every function $f : \mathbb{R} \to \mathbb{R}$, there are injective functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g - h$.

(iv) Let $\mathcal{L}$ be a first order language, $T$ be an $\mathcal{L}$-theory and $\phi, \psi$ be $\mathcal{L}$-sentences. Assume $T \vdash (\phi \vee \psi)$. Then either $T \vdash \phi$ or $T \vdash \psi$.

(v) Every consistent first order theory has an infinite model.

**Solution**.

(i) False. For example, take $\alpha = \omega \cdot \omega$. Then

$$(\forall \gamma < \alpha)(\exists \delta)(\gamma < \delta < \alpha \text{ and } |\{\xi : \gamma < \xi < \delta\}| = \omega)$$

On the other hand,

$$(\forall \delta)(\beta < \delta < \beta + \omega \implies |\{\xi : \beta < \xi < \delta\}| < \omega)$$

So $\alpha \neq \beta + \omega$ for any $\beta$.

(ii) True. Since $|\mathbb{R}| = \mathfrak{c} > \omega = |\mathbb{Q}|$.

(iii) True. By Homework Problem (15), there are injective functions $g, h' : \mathbb{R} \to \mathbb{R}$ such that $f = g + h'$. Take $h = -h'$. Then $h$ is also injective and $f = g - h$.

(iv) False. Take $T$ to be the empty theory, $\phi \equiv (\exists x)(\exists y)(x \neq y)$ and $\psi \equiv \neg\phi$.

(v) False. Let $T = \{(\forall x)(\forall y)(x = y)\}$.

**Question 2. [10 Points]**

Let $\mathcal{F}$ be the set of all bijections from $\omega$ to $\omega$.

(a) [**4 Points**] Define $H : \mathcal{F} \to \mathcal{P}(\omega)$ by $H(f) = \{n < \omega : f(n) = n\}$. Show that

$$\mathsf{range}(H) = \mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\}.$$

(b) [**4 Points**] Show that $|\mathcal{F}| \leq |\mathcal{P}(\omega)|$.

(c) [**2 Points**] Show that $|\mathcal{F}| = \mathfrak{c}$.

**Solution**.

(a) We first show that $\mathsf{range}(H) \subseteq \mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\}$. Let $f : \omega \to \omega$ be a bijection. Then $H(f) \subseteq \omega$ and so $H(f) \in \mathcal{P}(\omega)$. Towards a contradiction, suppose $H(f) = \omega \setminus \{n\}$ for some $n$. Since $f$ is a bijection on $\omega$, $f(n) \notin \mathsf{range}(f \restriction (\omega \setminus \{n\})) = \omega \setminus \{n\}$. So $f(n) = n$. But then $H(f) = \omega$ which is a contradiction.

Next fix $A \subseteq \omega$ such that $|\omega \setminus A| \neq 1$. We will find $f \in \mathcal{F}$ such that $H(f) = A$. If $A = \omega$, take $f$ to be the identity function. So assume $A \neq \omega$. We consider two cases.

**Case 1**. $\omega \setminus A$ is finite. Let $\{n_1 < n_2 < \cdots < n_k\}$ list the members of $\omega \setminus A$ in increasing order. Note that $k \geq 2$. Define

$$f(n) = \begin{cases} n & \text{if } n \in A \\ n_{j+1} & \text{if } n = n_j \text{ and } 1 \leq j < k \\ n_1 & \text{if } n = n_k \end{cases}$$

Then $f$ is a bijection on $\omega$ and $H(f) = A$.

**Case 2**. $\omega \setminus A$ is infinite. Let $\{n_0 < n_1 < n_2 < \dots\}$ list the members of $\omega \setminus A$ in increasing order and define

$$f(n) = \begin{cases} n & \text{if } n \in A \\ n_{2k+1} & \text{if } n = n_{2k} \\ n_{2k} & \text{if } n = n_{2k+1} \end{cases}$$

Then $f$ is a bijection on $\omega$ and $H(f) = A$. It follows that $\mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\} \subseteq \mathsf{range}(H)$. Hence $\mathsf{range}(H) = \mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\}$.

(b) Since $\mathcal{F} \subseteq \mathcal{P}(\omega \times \omega)$, we get $|\mathcal{F}| \leq |\mathcal{P}(\omega \times \omega)|$. Also, $|\omega \times \omega| = \omega$. So $|\mathcal{P}(\omega \times \omega)| = |\mathcal{P}(\omega)|$. It follows that $|\mathcal{F}| \leq |\mathcal{P}(\omega)|$.

(c) By part (a), $\mathcal{F} \geq |\mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\}|$. Since $|\{\omega \setminus \{n\} : n < \omega\}| = \omega < |\mathcal{P}(\omega)| = \mathfrak{c}$, it follows that $|\mathcal{P}(\omega) \setminus \{\omega \setminus \{n\} : n < \omega\}| = \mathfrak{c}$. So $|\mathcal{F}| \geq \mathfrak{c}$. By part (b), $|\mathcal{F}| \leq |\mathcal{P}(\omega)| = \mathfrak{c}$. Hence $|\mathcal{F}| = \mathfrak{c}$.

**Question 3. [10 Points]**

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$,

$$f(x+y) = f(x) + f(y) + f(x)f(y)$$

(a) [**2 Points**] Define $h = 1 + f$. Show that $h(x+y) = h(x)h(y)$.

(b) [**8 Points**] Suppose $f$ is continuous and not identically equal to $-1$. Show that $f(x) = a^x - 1$ for some $a > 0$.

**Solution**.

(a) $h(x+y) = 1 + f(x+y) = 1 + f(x) + f(y) + f(x)f(y) = (1+f(x))(1+f(y)) = h(x)h(y)$.

(b) Since $f$ is continuous, $h = 1 + f$ is also continuous. Since $f$ is not identically $-1$, $h$ is not identically 0. By Homework Problem 16, it follows that $h(x) = a^x$ for some $a > 0$. Hence $f(x) = h(x) - 1 = a^x - 1$.

**Question 4. [10 Points]**

Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol.

(a) [**4 Points**] Show that there is an $\mathcal{L}$-theory $T$ such that for every $\mathcal{L}$-structure $\mathcal{M} = (M, \prec^{\mathcal{M}})$,

$$\mathcal{M} \models T \text{ iff "}(M, \prec^{\mathcal{M}}) \text{ is a linear ordering".}$$

(b) [**6 Points**] Show that there is no $\mathcal{L}$-theory $T$ such that for every $\mathcal{L}$-structure $\mathcal{M} = (M, \prec^{\mathcal{M}})$,

$$\mathcal{M} \models T \text{ iff "}(M, \prec^{\mathcal{M}}) \text{ is a well-ordering".}$$

**Solution**.

(a) Let $T$ consist of the following axioms.

(1) $(\forall x)(\neg(x \prec x))$

(2) $(\forall x)(\forall y)(\forall z)((x \prec y \wedge y \prec z) \implies (x \prec z))$

(3) $(\forall x)(\forall y)((x = y) \vee (x \prec y) \vee (y \prec x))$

(b) Towards a contradiction suppose $T$ is such a theory. Let $\mathcal{L}'$ be the language obtained by adding the constant symbols $\{c_n : n < \omega\}$ to $\mathcal{L}$. Define an $\mathcal{L}$-theory $S$ as follows

$$S = T \cup \{c_{n+1} \prec c_n : n < \omega\}$$

We claim that every finite subset of $S$ has a model. Suppose $F$ is a finite subset of $T$. Fix $m$ large enough so that for every constant symbol $c_k$ that occurs in some sentence in $F$, we have $k < m$. Let $\mathcal{M} = (\omega, <)$ be the $\mathcal{L}$-structure where $\prec$ is interpreted as the usual ordering $<$ on $\omega$. By assumption, $\mathcal{M} \models T$. Define an $\mathcal{L}'$-structure $\mathcal{N} = (\omega, <, \{c_n^{\mathcal{N}} : n < \omega\})$ as follows: For $k \leq m$, $c_k^{\mathcal{N}} = m - k$ and for $k > m$, $c_k^{\mathcal{N}} = 0$. It is clear that $\mathcal{N} \models F$.

By compactness theorem, it follows that $S$ has a model $\mathcal{A} = (A, \prec^{\mathcal{A}}, \{c_n^{\mathcal{A}} : n < \omega\})$. Put $\mathcal{B} = (A, \prec^{\mathcal{A}})$. Since $T \subseteq S$, we have $\mathcal{B} \models T$ and hence $(A, \prec^{\mathcal{A}})$ must be a well-ordering. But this is impossible since $\{c_n^{\mathcal{N}} : n < \omega\}$ does not have $\prec^{\mathcal{A}}$-least member (as $\mathcal{A} \models c_{n+1} \prec c_n$ for every $n < \omega$).

**Bonus Question [5 Points]**

Let $\mathcal{L}_{PA} = (0, S, +, \cdot)$ be the language of Peano Arithmetic. Let $\mathcal{M} = (\omega, 0, S, +, \cdot)$. Show that there is a countable $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{M}$ is a **proper** elementary submodel of $\mathcal{N}$. This means that $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$.

**Solution**. Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$ where $c$ is a new constant symbol. Define an $\mathcal{L}$-theory

$$T = TA \cup \{c \neq S^n(0) : n < \omega\}$$

Note that every finite subset of $T$ has a model. By compactness theorem, $T$ has a model

$$\mathcal{A} = (A, 0^{\mathcal{A}}, S^{\mathcal{A}}, +^{\mathcal{A}}, \cdot^{\mathcal{A}}, c^{\mathcal{A}})$$

By replacing the interpretations of $S^n(0)$ in $\mathcal{A}$ by $n$, we can assume that $\omega \subseteq A$, $0^{\mathcal{A}} = 0$, $S^{\mathcal{A}} \restriction \omega = S$, $+^{\mathcal{A}} \restriction (\omega \times \omega) = +$ and $\cdot^{\mathcal{A}} \restriction (\omega \times \omega) = \cdot$. Put $\mathcal{N} = (A, 0^{\mathcal{A}}, S^{\mathcal{A}}, +^{\mathcal{A}}, \cdot^{\mathcal{A}})$ and note that $\mathcal{M} = (\omega, 0, S, +, \cdot)$ is a proper substructure of $\mathcal{N}$.

We claim that $\mathcal{M}$ is also an elementary submodel of $\mathcal{N}$. Let $\phi$ be any $\mathcal{L}_{PA}$-formula whose free variables are among $x_1, x_2, \dots, x_n$ and suppose $k_1, k_2, \dots, k_n$ are in $\omega$. Then $\mathcal{M} \models \phi(k_1/x_1, \dots, k_n/x_n)$ iff $\mathcal{M} \models \phi(S^{k_1}(0)/x_1, \dots, S^{k_n}(0)/x_n)$. Since $\phi(S^{k_1}(0)/x_1, \dots, S^{k_n}(0)/x_n) \in TA = Th(\mathcal{M})$, it follows that $\mathcal{M} \models \phi(S^{k_1}(0)/x_1, \dots, S^{k_n}(0)/x_n)$ iff $\mathcal{N} \models \phi(S^{k_1}(0)/x_1, \dots, S^{k_n}(0)/x_n)$ iff $\mathcal{N} \models \phi(k_1/x_1, \dots, k_n/x_n)$. Hence $\mathcal{M} \preceq \mathcal{N}$.

**End of Exam**