ASSIGNMENT 7

MTH 301, 2018

(1) Consider the set $\mathbb{Q}$ of rational numbers with the metric $d(x, y) = |x-y|;\ x, y \in \mathbb{Q}$. Show that the Baire category theorem does not hold in this metric space.

(2) Using Baire Ctegory Theorem prove the following
 (a) $[0, 1]$ contains uncountably many elements.
 (b) The linear space of all polynomials in one variable is not a Banach space in any norm.
 (c) Let (X, d) be a complete metric space with no isolated points. Then (X, d) is uncountable.
 (d) There exists a continuous function $f : [0, 1] \to \mathbb{R}$ that is not monotone on any interval of positive length.

(3) Let $(X, \|\ \|)$ be a normed space, $Y \subsetneq X$, i.e. $Y \neq X$ a proper linear subspace. Prove that Y contains no ball (that is, its interior is empty).

(4) Prove that any finite dimensional linear subspace of $(X, \|\ \|)$ is closed.

(5) Let $C^\infty(\mathbb{R})$ i.e. infintely differentiable function and suppose that for all $x \in \mathbb{R}$ there exists $n_x \in \mathbb{Z}_+$ such that $f^{(n_x)}(x) = 0$. Show that there exists a nonempty open interval $(a, b) \subset \mathbb{R}$ such that the restriction of f to (a, b) is a polynomial.

(6) Consider $X = C[0, 1]$ with sup metric. Define $E_m = \{f \in X : \exists x \in [0, 1]\ with\ |f(x + h) - f(x)| \leq m|h|\ for\ all\ x + h \in [0, 1]\}$. It is clear from the definition of differentiability that all functions differentiable at some point in $(0, 1)$ lie in one of these sets E_m.
 (a) Show that E_m is closed for each m.
 (b) Show that set of piecewise linear functions P_L is dense in X. ($p \in P_L$ if there exits a partition $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $p|_{[a_1, a_{i+1}]}$ is linear for each $i = 0, \ldots, k-1$.
 (c) If E_m contains some open ball $B_\epsilon(f)$ then there exists a $p \in P_L$ such that $p \in B_\epsilon(f)$.
 (d) Observe that for a piecewise linear function the slopes are bounded.
 (e) For given M consider the partition $P = \{a_j\}$ where $a_j = \frac{j}{M}, j = 0, \ldots, M$. Define a continuous function $g : [0, 1] \to \mathbb{R}$ by $g(x) = \begin{cases} 1 & if\ x = a_j,\ j \in 2\mathbb{Z} \\ -1 & if\ x = a_j,\ j \in 2\mathbb{Z} + 1 \\ \text{linear} & otherwise \end{cases}$.
 Consider $h(x) = p(x) + \frac{\epsilon}{2} g(x)$. Clearly, $h \in B_\epsilon(p)$. By taking M large conclude that E_m cannot contain an open ball.
 (f) From above conclude that there exists a continuous function which is not differentiable at any point.

(7) Show that the set of nowhere differentiable functions is residual (hence dense) in $C[0, 1]$.