Date. October 27, 2022

Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Quiz -3 (MTH305A)

Semester: 2022-2023, I

Full Marks. 20 Time. 45 Minutes

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(1) Are the following statements TRUE/FALSE? **Justify your answer.**

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and $r \in \mathbb{R}$ such that $f^{-1}(r)$ is a regular surface. Then $r \in f(\mathbb{R}^3)$ is a regular value of f.
Solution.
FALSE
Consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2$ for all $(x, y, z) \in \mathbb{R}^2$. Note that $f^{-1}(0) = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ is a regular surface but 0 is a critical value of f.
[3 points]

(b) The set $C = \{(\sin\theta, \cos\theta, t) \mid \theta \in \mathbb{R} \text{ and } -1 < t < 1\} \subset \mathbb{R}^3$ is a regular surface.
Solution.
TRUE
Consider $U = \{(x, y) \in \mathbb{R}^2 \mid 1 < \|(x, y)\| < 3\}$. Define $\phi : U \to C$ by

$$\phi(x, y) = \left(\frac{x}{\|(x, y)\|}, \frac{y}{\|(x, y)\|}, \|(x, y)\| - 2\right).$$

Then $\{(U, \phi)\}$ is an atlas showing that C is a regular surface.
[3 points]

(c) The set $H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \text{ and } \}$ is a regular surface.
Solution.
FALSE
If H is a regular surface, then there exists a local co-ordinate chart $\phi : \mathbb{R}^2 \to V$, where V is an open set in H containing $(0, 0, 0)$. Let $\phi(q) = (0, 0, 0)$. Note that ϕ is a homeomorphism and hence $\phi : \mathbb{R}^2 \setminus \{q\} \to V \setminus \{(0, 0, 0)\}$ is a homeomorphism, which cannot exists as $\mathbb{R}^2 \setminus \{q\}$ is connected but $V \setminus \{(0, 0, 0)\}$ is disconnected.
[2 points]

(2) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = xyz + x^3 + y^3 + z^3$ for all $(x, y, z) \in \mathbb{R}^3$. For each $\alpha \in \mathbb{R}$, we denote $F_\alpha = f^{-1}(\alpha)$.

(a) Determine the set of (i) critical points, (ii) critical values (iii) regular points and (iv) regular values of the function f.
Solution.

$Df(x,y,z) = 0 \implies (yz + 3x^2, xz + 3y^2, xy + 3z^2) = (0,0,0)$.
(i) Critical points: $(0,0,0) \in \mathbb{R}^3$.
(ii) Critical values: $0 = f(0,0,0) \in \mathbb{R}$.
(iii) Regular points: $\mathbb{R}^3 \setminus \{(0,0,0)\}$.
(iv) Regular values: $\mathbb{R} \setminus \{0\}$.

(b) Show that $F_0 \setminus \{(0,0,0)\}$ is a regular surface.
Solution.
Let $U = \mathbb{R}^3 \setminus \{(0,0,0)\} \subset_{\text{open}} \mathbb{R}^3$ and $g : U \to \mathbb{R}$ be defined by $g(x,y,z) = xyz + x^3 + y^3 + z^3$. Then $g^{-1}(0) = F_0 \setminus \{(0,0,0)\}$ is a regular surface as 0 is a regular value of the function g.

[3+3=6 points]

(3) (a) Let $f : [a,b] \to \mathbb{R}$ be a function. Suppose $F, \tilde{F} : I \to \mathbb{R}^2$, where I is an open set in $\mathbb{R}$ containing $[a,b]$, are two smooth functions such that $F|_{[a,b]} = f$ and $\tilde{F}|_{[a,b]} = f$. Show that $F^{(k)}(a) = \tilde{F}^{(k)}(b)$ for all $k \in \mathbb{N}$.
Solution.

- Note that f is smooth on (a,b) and for all $t \in (a,b)$, we have $F^{(k)}(t) = \tilde{F}^{(k)}(t) = f^{(k)}(t)$ for all $k \in \mathbb{N}$.
- $F^{(k)}(a) = \lim_{t \to a-} F^{(k)}(t) = \lim_{t \to a+} F^{(k)}(t) = \lim_{t \to a+} f^{(k)}(t)$ and $\tilde{F}^{(k)}(a) = \lim_{t \to a-} \tilde{F}^{(k)}(t) = \lim_{t \to a+} \tilde{F}^{(k)}(t) = \lim_{t \to a+} f^{(k)}(t)$.
- Hence, we conclude that $F^{(k)}(a) = \tilde{F}^{(k)}(a)$.

(b) Consider $\alpha : [-2.0, 2.0] \to \mathbb{R}^2$ defined by $\alpha(t) = (t^3 - 4t, t^2 - 4)$. Determine whether α is (i) Closed, (ii) Simple, (iii) Smooth closed curve.
Solution.

- $\alpha(-2.0) = (0,0) = \alpha(+2.0)$ implies that α is closed.
- Note that α has self intersection at $(0,0)$, i.e., for $t = \pm 2$. Hence, α is injective on $[-2,2)$ implies that α simple.
- $\alpha'(2) = (8,4)$ and $\alpha'(-2) = (8,-4)$ and $\alpha'(2) = (8,4)$. Therefore, α is not a smooth closed curve.

[3+3=6 points]