Sequence behavior depends on the "choice" of the metric!

(M,d) metric space

- Not every Cauchy seq. converge.
- Not every bdd. seq. has a convergent subsequence

Recall: $(\mathbb{R}, |\cdot|)$:

- Every Cauchy seq. converges
- Every bdd. seq. has a cvgt. subseq.

Example. $M = (0,1)$ and $d(x,y) := |x-y|$.

Consider $(\frac{1}{n})$. Note that $(\frac{1}{n})$ is Cauchy which does not have a limit in $(0,1)$.

($(0,1) \subset \mathbb{R}$
$\frac{1}{n} \to x = 0$

Example. (a) $M = \mathbb{R}$ and d_0 (discrete metric)

Consider (n). $d_0(n,m) = 1 \quad \forall\, n,m$ s.t. $n \neq m$.

So, (n) is a bdd. seq., but it has no cvgt. subsequence!

(HW) $\to$ A seq. (x_n) in (M, d_0) is cvgt. iff (x_n) is eventually a constant seq.

Every Cauchy seq. converges in (M, d_0).

(b) $M = \mathbb{R}$ with $d_{|\cdot|}(x,y) := |x-y|$, the seq. (n) is <u>not</u> a bdd. seq.
(usual metric)

(c) The seq. $(\frac{1}{n})$ is Cauchy in $(\mathbb{R}, d_{|\cdot|})$, but $(\frac{1}{n})$ is <u>not</u> Cauchy in $(\mathbb{R}, d_0)$.

Example: For $n > 1$, $1 \le p < \infty$, $(\mathbb{R}^n, \|\cdot\|_p)$, $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

For $p = \infty$: $(\mathbb{R}^n, \|\cdot\|_\infty)$, $\|x\|_\infty := \max_{1 \le i \le n} \{|x_i|\}$.

$\to$ For $p \neq q$, $1 \le p, q < \infty$,

- A seq. (x_m) cvgs. (Cauchy) in $(\mathbb{R}^n, \|\cdot\|_p)$ <u>iff</u> (x_m) cvgs. (Cauchy) in $(\mathbb{R}^n, \|\cdot\|_q)$.
- A seq. (x_m) cvgs. (Cauchy) in $(\mathbb{R}^n, \|\cdot\|_p)$ iff (x_m) cvgs. (Cauchy) in $(\mathbb{R}^n, \|\cdot\|_\infty)$

Note: For $1 \le i \le n$, $|x_i| \le \sum_{j=1}^{n} |x_j|^p$. Let $x_m = (x_1^{(m)}, x_2^{(m)}, \dots, x_n^{(m)})$.

If $x_m \xrightarrow{\|\cdot\|_p} x$ where $x = (x_1, x_2, \dots, x_n)$, then $x_i^{(m)} \to x_i$ for each $1 \le i \le n$.

(HW) If $x_i^{(m)} \to x_i$ for $1 \le i \le n$, then $x_m \xrightarrow{\|\cdot\|_p} x$.

(HW) $x_m \xrightarrow{\|\cdot\|_p} x \iff x_m \xrightarrow{\|\cdot\|_q} x$.

(HW) $x_m \xrightarrow{\|\cdot\|_p} x \iff x_m \xrightarrow{\|\cdot\|_\infty} x$.

Example. $M = \ell_1$ and consider $(\ell_1, \|\cdot\|_1)$ and $(\ell_1, \|\cdot\|_\infty)$

In ℓ_1, for $x \in \ell_1$, i.e., $\sum_{i=1}^{\infty} |x_i| < \infty$, consider

$$x^{(n)} := (x_1, x_2, \cdots, x_n, 0, 0, 0, \cdots)$$

Then $\|x^{(n)} - x\|_1 \to 0$ as $n \to \infty$. for every $x \in \ell_1$.

Question: If $x^{(n)} \xrightarrow{\|\cdot\|_1} x$, then does $x^{(n)} \xrightarrow{\|\cdot\|_\infty} x$ in ℓ_1 ?

Ans. $\|x^{(n)} - x\|_\infty = \sup_{i \geq 1} \{ |x_i^{(n)} - x_i| \}$

Since $\|x^{(n)} - x\|_1 \to 0$ so for $\varepsilon > 0$ $\exists N_\varepsilon$, s.t. $\forall n \geq N_\varepsilon$

$$\|x^{(n)} - x\|_1 < \varepsilon$$

$$\text{i.e., } \sum_{i=1}^{\infty} |x_i^{(n)} - x_i| < \varepsilon \quad \forall n \geq N_\varepsilon$$

But also, for each i, $|x_i^{(n)} - x_i| \leq \sum_{1}^{\infty} |x_i^{(n)} - x_i| < \varepsilon \quad \forall n \geq N_\varepsilon$

$\Rightarrow$ for each i, $|x_i^{(n)} - x_i| < \varepsilon$, $\forall n \geq N_\varepsilon$

$$\Rightarrow \sup_{i \geq 1} \{ |x_i^{(n)} - x_i| \} < \varepsilon, \ \forall n \geq N_\varepsilon$$

$$\Rightarrow \|x^{(n)} - x\|_\infty \to 0 \quad \text{as } n \to \infty$$

(HW) Question: If $x^{(n)} \xrightarrow{\|\cdot\|_\infty} x$, then does $x^{(n)} \xrightarrow{\|\cdot\|_1} x$? (NO!)

$\rightarrow$ For $1 \leq p < \infty$, $(\ell_p, \|\cdot\|_p)$, $(\ell_p, \|\cdot\|_\infty)$, and $p = \infty$: $(\ell_\infty, \|\cdot\|_\infty)$:

$$\text{Norm-convergence} \Longrightarrow \text{coordinatewise convergence}$$
$$\not\Longleftarrow$$

Example: Consider $e_n := (0, \cdots, 0, \underset{n\text{-th}}{1}, 0, 0 \cdots)$ in $(\ell_\infty, \|\cdot\|_\infty)$

Then, coordinatewise. each coordinate seq. converge to 0. But $e_n \not\to 0$ in $\|\cdot\|_\infty$ and $e_n \not\to 0$ in $\|\cdot\|_p$.

$\longrightarrow$ For $1 \le p \le \infty$, $(\mathbb{R}^n, \|\cdot\|_p)$ every Cauchy seq. converge.

(HW) Question. For $1 \le p \le \infty$: $(\ell_p, \|\cdot\|_p)$ does every Cauchy seq. converge? (YES!)

(HW) Question. For $1 \le p < \infty$, $(\ell_p, \|\cdot\|_\infty)$ does every Cauchy seq. converge? (NO!)

Hint. Consider $p = 1$. Note that $(\ell_1, \|\cdot\|_\infty) \subset (\ell_\infty, \|\cdot\|_\infty)$

Construct a seq. $(x^{(n)})$ s.t. $x^{(n)} \in \ell_1$, $(x^{(n)})$ Cauchy but $(x^{(n)}) \not\to x$ in $(\ell_1, \|\cdot\|_\infty)$.