(1)(a) $f(x) = \alpha\, e^{-x^2-\beta x}$; $-\infty < x < \infty$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \alpha \int_{-\infty}^{\infty} e^{-\left(x^2+2.x.\frac{\beta}{2}+\frac{\beta^2}{4}\right)} e^{\beta^2/4}\, dx = 1$$

$$\Rightarrow \alpha\, e^{\beta^2/4} \int_{-\infty}^{\infty} e^{-\frac{1}{2\left(\frac{1}{2}\right)}\left(x-\left(-\beta/2\right)\right)^2} dx = 1$$

$$\Rightarrow \alpha\, e^{\beta^2/4} \sqrt{2\pi}\sqrt{\tfrac{1}{2}} = 1 \quad \text{— (i)} \quad \text{— (1)}$$

Also $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = -\frac{3}{2}$

i.e. $\alpha\, e^{\beta^2/4} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2\left(\frac{1}{2}\right)}\left(x-\left(-\beta/2\right)\right)^2} dx = -\frac{3}{2}$

$$\Rightarrow \alpha\, e^{\beta^2/4} \sqrt{2\pi}\sqrt{\tfrac{1}{2}}\left(-\frac{\beta}{2}\right) = -\frac{3}{2} \quad \text{— (ii)} \quad \text{— (1)}$$

using (i) & (ii), $\beta = 3$

$$\Rightarrow \alpha\, e^{9/4}\sqrt{\pi} = 1 \Rightarrow \alpha = \frac{e^{-9/4}}{\sqrt{\pi}} \quad \text{— (2)}$$

(b) $f_X(x) = \begin{cases} 1/\theta, & 0 < x < \theta \\ 0, & o/w \end{cases}$ $\quad Y = -\log\left(X/\theta\right)^2 \Rightarrow Y = -2\log\left(X/\theta\right)$

$\Rightarrow X = \theta e^{-Y/2}$ — (1)

$$|J| = \left|\frac{dx}{dy}\right| = \left|\theta e^{-y/2}\left(-\tfrac{1}{2}\right)\right| = \frac{\theta}{2} e^{-y/2} \quad \text{— (1)}$$

$$f_Y(y) = \begin{cases} \frac{1}{\theta}\cdot\frac{\theta}{2} e^{-y/2}, & 0 < y < \infty \\ 0, & o/w \end{cases} \quad \text{— (2)}$$

$$\text{i.e. } f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2}, & 0 < y < \infty \\ 0, & o/w. \end{cases}$$

i.e. $Y \sim \chi^2_2$

1(c) $X \sim N(\theta, 1)$

$$E\left(e^{X^2/2}\int_X^\infty e^{-t^2/2}\,dt\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt\right)e^{-\frac{1}{2}(x-\theta)^2}\,dx. \quad \text{—(1)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_x^\infty \cancel{e^{x^2/2}}\, e^{-t^2/2}\, e^{-\frac{1}{2}(\cancel{x^2}+\theta^2 - 2x\theta)}\,dt\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_x^\infty e^{-t^2/2}\, e^{-\theta^2/2}\, e^{x\theta}\,dt\,dx$$

$-\infty < x < t < \infty$

Interchanging the order of integration

$$= \frac{1}{\sqrt{2\pi}}e^{-\theta^2/2}\int_{-\infty}^{\infty} e^{-t^2/2}\int_{-\infty}^{t} e^{x\theta}\,dx\,dt \quad \text{—(2)}$$

$$= \frac{e^{-\theta^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\frac{1}{\theta}\,e^{\theta t}\,dt$$

$$= \frac{1}{\theta}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2+\theta^2-2\theta t)}\,dt$$

$$= \frac{1}{\theta}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t-\theta)^2}\,dt = \frac{1}{\theta}. \quad \text{—(2)}$$

2 (a) $$f(x) = \begin{cases} 6\alpha^6 x^{-7}, & 0 < \alpha < x < \infty \\ 0, & \text{o/w} \end{cases}$$

$$P(\alpha+2 \le X < \alpha+3 \mid X \ge \alpha+1)$$
$$= \frac{P(\alpha+2 \le X < \alpha+3)}{P(X \ge \alpha+1)}$$
$$= \frac{F_X(\alpha+3) - F_X(\alpha+2)}{1 - F_X(\alpha+1)} \quad - (*) \quad — (2)$$

Now, $F_X(x) = 0$ if $x \le \alpha$

& $\forall\, x > \alpha$, $F_X(x) = \int_\alpha^x 6\alpha^6 t^{-7}\,dt = 6\alpha^6\left(\frac{1}{6}(\alpha^{-6} - x^{-6})\right)$
$$= 1 - \left(\frac{\alpha}{x}\right)^6 \quad — (1)$$

i.e. $$F_X(x) = \begin{cases} 0, & x \le \alpha \\ 1 - \left(\frac{\alpha}{x}\right)^6, & x > \alpha \end{cases}$$

$$\Rightarrow (*) = \frac{\left(1 - \left(\frac{\alpha}{\alpha+3}\right)^6\right) - \left(1 - \left(\frac{\alpha}{\alpha+2}\right)^6\right)}{\left(\frac{\alpha}{\alpha+1}\right)^6} \quad — (2)$$

$$= \frac{\left(\frac{\alpha}{\alpha+2}\right)^6 - \left(\frac{\alpha}{\alpha+3}\right)^6}{\left(\frac{\alpha}{\alpha+1}\right)^6} = \left(\frac{\alpha+1}{\alpha+2}\right)^6 - \left(\frac{\alpha+1}{\alpha+3}\right)^6.$$

(b) $Y = \log X$

m.g.f. $M_Y(t) = E\left(e^{t\log X}\right) = E\left(e^{\log X^t}\right) = E(X^t)$ — (2)

$$\Rightarrow M_Y(t) = E(X^t) = 6\alpha^6 \int_\alpha^\infty x^t x^{-7}\,dx \quad — (1)$$
$$= 6\alpha^6 \int_\alpha^\infty x^{(t-6)-1}\,dx$$
$$= 6\alpha^6 \times \left(\frac{\alpha^{t-6}}{6-t}\right) \quad t < 6$$

i.e. $$M_Y(t) = \frac{6\alpha^t}{6-t} \quad \text{for } t < 6. \quad — (2)$$

Note: One can also obtain the m.g.f. of Y by first finding the density of Y and then obtain the m.g.f.

(3). jt p.d.f. of X_1, X_2

(a) $f_{X_1,X_2}(x_1,x_2) = \begin{cases} e^{-x_1} e^{-x_2}; & 0<x_1<\infty,\ 0<x_2<\infty \\ 0, & o/w. \end{cases}$

Transformation $\left.\begin{array}{l} U = X_1 + X_2 \\ V = X_1 - X_2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} X_1 = \frac{U+V}{2} \\ X_2 = \frac{U-V}{2} \end{array}\right\}.$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

$|J| = \frac{1}{2}$ — (2)

Note that unconditionally; $0<u<\infty$, $-\infty<v<\infty$ — (1/2)

Also note that $0<x_1<\infty \Rightarrow 0 < \frac{u+v}{2} < \infty$

i.e. $-v<u<\infty$ & $-u<v<\infty$ — (i).

and $0<x_2<\infty \Rightarrow 0 < \frac{u-v}{2} < \infty$

i.e. $v<u<\infty$ & $-\infty<v<u$ — (ii)

Combining (i) & (ii), we have

$\max(v,-v) < u < \infty$ & $-u<v<u$. — (2)

$\Rightarrow$ if $-\infty<v<0$; then $-v<u<\infty$

& if $0<v<\infty$; then $v<u<\infty$

$\Rightarrow$ the jt p.d.f. of U & V is

$$f_{U,V} = \begin{cases} \frac{1}{2} e^{-u}, & \text{if } -\infty<v<0, -v<u<\infty \ \text{ or } \ 0<v<\infty,\ v<u<\infty \\ 0, & o/w. \end{cases}$$

(1/2) (1)

Marks to be awarded only if range is written properly.

Note: One can also obtain this range graphically.

(b) Marginal p.d.f. of U

$$f_U(u) = \int_{-u}^{u} \frac{1}{2} e^{-u}\, dv = u e^{-u}$$

$$\Rightarrow f_U(u) = \begin{cases} u e^{-u}, & u > 0 \\ 0, & o/w \end{cases} \quad \text{— (2)}$$

Marginal p.d.f. of V

If $-\infty < v < 0$;

$$f_V(v) = \int_{-v}^{\infty} \frac{1}{2} e^{-u}\, du = \frac{1}{2} e^{v} \quad \text{— (2)}$$

If $0 < v < \infty$;

$$f_V(v) = \int_{v}^{\infty} \frac{1}{2} e^{-u}\, du = \frac{1}{2} e^{-v} \quad \text{— (2)}$$

$$\Rightarrow f_V(v) = \begin{cases} \frac{1}{2} e^{v}, & v < 0 \\ \frac{1}{2} e^{-v}, & v > 0 \end{cases} \quad \text{— } *^1$$

i.e. $f_V(v) = \frac{1}{2} e^{-|v|}$; $-\infty < v < \infty$. — $*^2$

Deduct (2) Mark If one of the 2 final forms of f_V ($*^1 / *^2$) is not written

4

(a). $f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & o/w \end{cases}$ $\quad F_X(x) = 1 - e^{-(x-\theta)}$ $\quad x \geq \theta.$

$X_{(1)} = \min(X_1, \ldots X_n)$

p.d.f. of $X_{(1)}$ $\quad f_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x) \quad x \geq \theta$

i.e. $f_X(x) = \begin{cases} n\, e^{-n(x-\theta)}, & x \geq \theta \\ 0, & o/w. \end{cases}$ — (1)

Note that

$$P\left[\,|X_{(1)} - \theta| \geq \epsilon\right] \leq \frac{E(X_{(1)} - \theta)^2}{\epsilon^2} = \frac{EX_{(1)}^2 + \theta^2 - 2\theta E X_{(1)}}{\epsilon^2} \quad \text{— (1)}$$

— (*)

Now, $E\,X_{(1)} = n \int_\theta^\infty x\, e^{-n(x-\theta)}\, dx$

$y = (x-\theta)$

$$= n \int_\theta^\infty (y+\theta)\, e^{-ny}\, dy$$

$$= n\left[\int_\theta^\infty y\, e^{-ny}\, dy + \theta \int_\theta^\infty e^{-ny}\, dy\right]$$

i.e. $E\,X_{(1)} = \frac{1}{n} + \theta.$

Also $E\,X_{(1)}^2 = n \int_\theta^\infty x^2\, e^{-n(x-\theta)}\, dx$

$y = x - \theta$

$$= n \int_\theta^\infty (y^2 + \theta^2 + 2\theta y)\, e^{-ny}\, dy$$

$$= n\left[\int_\theta^\infty y^2 e^{-ny}\, dy + \theta^2 \int_\theta^\infty e^{-ny}\, dy + 2\theta \int_\theta^\infty y\, e^{-ny}\, dy\right]$$

i.e. $EX_{(1)}^2 = n\left[\frac{2}{n^3} + \frac{\theta^2}{n} + 2\theta\, \frac{1}{n^2}\right]$

i.e. $E\,X_{(1)}^2 = \frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}$

$$\Rightarrow E X_{(1)}^2 + \theta^2 - 2\theta E X_{(1)}$$

$$= \left(\frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}\right) + \theta^2 - 2\theta\left(\theta + \frac{1}{n}\right)$$

$$= \frac{2}{n^2} + \cancel{\theta^2} + \cancel{\frac{2\theta}{n}} + \cancel{\theta^2} - 2\cancel{\theta^2} - \cancel{\frac{2\theta}{n}} = \frac{2}{n^2} \quad -(1)$$

$$\Rightarrow P\left[|X_{(1)} - \theta| \geq \epsilon\right] \leq \frac{2}{n^2 \epsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow X_{(1)} \xrightarrow{p} \theta \quad -(1)$$

$$\Rightarrow \sqrt{X_{(1)}} \xrightarrow{p} \sqrt{\theta}. \quad -(1)$$

(b) d.f. of $X_{(1)}$.

$$F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - (1 - F_X(x))^n \quad x \geq \theta$$
$$= 0 \quad \text{o/w.}$$

i.e $F_{X_{(1)}}(x) = \begin{cases} 0, & x < \theta \\ 1 - (1 - F_X(x))^n, & x \geq \theta \end{cases} \quad -(1)$

i.e $F_{X_{(1)}}(x) = \begin{cases} 0, & x < \theta \\ 1 - e^{-n(x-\theta)}, & x \geq \theta \end{cases} \quad -(2)$

$$\xrightarrow[\text{as } n \to \infty]{} = \begin{cases} 0, & x < \theta \\ 1, & x \geq \theta. \end{cases}$$

i.e. $F_X(x) = \begin{cases} 0, & x < \theta \\ 1, & x \geq \theta \end{cases} \quad -(2)$

$\Rightarrow X_{(n)} \xrightarrow{L} X$; X is degenerate r.v., degenerate at θ.

<u>Note</u>: One can also argue that since

$X_{(n)} \xrightarrow{p} \theta \Rightarrow X_{(1)} \xrightarrow{L} \theta \leftarrow$ degenerate r.v.

(5) $X_1, \dots X_n$ i.i.d. $U(0,1)$

(a) $\left(\prod_{i=1}^{n} X_i\right)^{1/n} = G_n$ (say)

Let $H_n = -\log G_n = \frac{1}{n}\sum_{i=1}^{n}(-\log X_i)$

i.e. $G_n = e^{-H_n}$. — (1)

Note that for $X_i \sim U(0,1)$; $Y_i = -\log X_i$ has the following p.d.f. (using problem # 1(b) with $\theta = 1$ and r/p case)

$Y_i = -\log X_i \Rightarrow X_i = e^{-Y_i}$

$|J| = e^{-y}$.

$$\Rightarrow f_{Y_i}(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & o/w \end{cases}$$ — (1)

$$E Y_i = \int_0^{\infty} y e^{-y} dy = 1 \; ; E Y_i^2 = \int_0^{\infty} y^2 e^{-y} dy = 2$$

$$\Rightarrow V Y_i = 2 - 1 = 1$$

$-\log X_1, -\log X_2, \dots, -\log X_n$ are i.i.d. Exp(1) with mean $(E(-\log X_1) = 1)$. — (1)

$\Rightarrow$ By Khintchin's WLLN

$$\frac{1}{n}\sum_{i=1}^{n}(-\log X_i) \xrightarrow{p} E(-\log X_i) = 1$$

i.e. $H_n \xrightarrow{p} 1$ — (2)

$$\Rightarrow e^{-H_n} \xrightarrow{p} e^{-1}$$ — (1)

i.e. $G_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n} \xrightarrow{p} e^{-1}$

(b) Once again note that

$-\log X_1, -\log X_2, \ldots, -\log X_n$ are i.i.d with

$E(-\log X_1) = 1$ & $V(-\log X_1) = 1$

$\Rightarrow$ By CLT

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(-\log X_i) - 1\right) \xrightarrow{L} N(0,1)$$

i.e. $\sqrt{n}(H_n - 1) \xrightarrow{L} N(0,1)$. — (2)

Apply Δ-rule with $g(x) = e^{-x}$

$g'(x) = -e^{-x}$

$g'(1) = -e^{-1} \neq 0$. — (2)

$\Rightarrow$ By Δ-rule

$$\sqrt{n}\left(g(H_n) - g(1)\right) \xrightarrow{L} N(0, (g'(1))^2)$$

$$\Rightarrow \sqrt{n}\left(e^{-H_n} - e^{-1}\right) \xrightarrow{L} N(0, e^{-2})$$

i.e. $\sqrt{n}(G_n - e^{-1}) \xrightarrow{L} N(0, e^{-2})$ — (2)

i.e. $\sqrt{n}\left(\left(\prod_1^n X_i\right)^{1/n} - e^{-1}\right) \xrightarrow{L} N(0, e^{-2})$.

(6)

(a) $X_1, \ldots X_n$ r.s. from $N(\theta, 2\theta^2)$; $\theta > 0$

jt. p.d.f. of $X_1, \ldots X_n$

$$\prod_{i=1}^{n} f_\theta(x_i) = \left(\frac{1}{\sqrt{2\pi}\sqrt{2\theta^2}}\right)^n \exp\left(-\frac{1}{4\theta^2}\Sigma(x_i-\theta)^2\right)$$

$$= \left(\frac{1}{2\sqrt{\pi}\ \theta}\right)^n \exp\left(-\frac{1}{4\theta^2}\left(\Sigma x_i^2 + n\theta^2 - 2\theta\Sigma x_i\right)\right)$$

$$= \left(\frac{1}{2\sqrt{\pi}\ \theta}\right)^n \exp\left(-\frac{1}{4\theta^2}\Sigma x_i^2 - \frac{n}{4} + \frac{1}{2\theta}\Sigma x_i\right)$$

i.e. $\prod_{i=1}^{n} f_\theta(x_i) = \left(\left(\frac{1}{2\sqrt{\pi}}\right)^n e^{-n/4}\right)\left(\frac{1}{\theta^n}\exp\left(-\frac{1}{4\theta^2}\Sigma x_i^2 + \frac{1}{2\theta}\Sigma x_i\right)\right)$

i.e. $\prod_{i=1}^{n} f_\theta(x_i) = h(\underset{\sim}{x})\ g(\theta; \Sigma x_i, \Sigma x_i^2)$ — (1)

$$h(\underset{\sim}{x}) = \left(\frac{1}{2\sqrt{\pi}}\right)^n e^{-n/4}$$

$$g(\theta; \Sigma x_i, \Sigma x_i^2) = \frac{1}{\theta^n}\exp\left(-\frac{1}{4\theta^2}\Sigma x_i^2 + \frac{1}{2\theta}\Sigma x_i\right).$$

$\Rightarrow$ By NFFT, $T(\underset{\sim}{X}) = \left(\sum_1^n X_i, \sum_1^n X_i^2\right)$ is jtly suff for θ.

$(= (T_1, T_2)'$ say$)$ — (2)

(b) $E T_1^2 = V T_1 + (E T_1)^2 = V(\Sigma X_i) + (E\Sigma X_i)^2$

i.e. $E T_1^2 = 2n\theta^2 + n^2\theta^2$

i.e. $E T_1^2 = \theta^2 n(n+2)$

$\Rightarrow E\left(T_1^2 / n(n+2)\right) = \theta^2$ (4).

Alt

$E T_2 = \Sigma E X_i^2 = \Sigma\left(V(X_i) + (E X_i)^2\right)$

i.e. $E T_2 = \Sigma(2\theta^2 + \theta^2) = 3n\theta^2$

$E\left(\frac{T_2}{3n}\right) = \theta^2$

u.e.

Note: There are many other choices!!

6(c) From part (b)

$$E\left(\frac{T_1^2}{n(n+2)} - \frac{T_2}{3n}\right) = 0 \quad \forall\, \theta > 0$$

(4).

$$\not\Rightarrow \quad \frac{T_1^2}{n(n+2)} = \frac{T_2}{3n} \quad a.e.$$

$$\Rightarrow \quad \underset{\sim}{T} = (T_1, T_2) \text{ is not complete}$$

6(d) $X_1, \ldots X_n$ i.i.d. with mean θ & Var $2\theta^2$

$\Rightarrow X_1^2, \ldots X_n^2$ are i.i.d. with mean $EX_1^2 = VX_1 + (EX_1)^2$ $(= 3\theta^2)$ — (*)

By WLLN, $\frac{1}{n}\sum_1^n X_i^2 \xrightarrow{p} EX_1^2 = 3\theta^2$

$$\Rightarrow \frac{1}{3n}\sum_1^n X_i^2 \xrightarrow{p} \theta^2$$

(*)

$\Rightarrow \frac{1}{3n}\sum X_i^2$ is consistent for θ^2 — (2)

Also WLLN $\Rightarrow \frac{1}{n}\sum X_i \xrightarrow{p} EX_1 = \theta$

i.e. $\bar{X}_n \xrightarrow{p} \theta$

$\Rightarrow \bar{X}_n^2 \xrightarrow{p} \theta^2$

$\Rightarrow \bar{X}_n^2$ is a consistent estimator for θ^2.

— (2)

Note 6(c)

(*) If s-parameter exponential family is of full rank then the associated suff statistic is complete. NOT the other way!!

To show that T is not complete, we need to find $g(.) \ni E\,g(T) = 0 \;\forall\, \theta \in \Theta \not\Rightarrow g(t) = 0$ a.e.

(7)

(a) $$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta} \quad x>0$$

$$f_\theta(x) = e^{-x/\theta - \log\theta} \qquad \theta > 0 \Rightarrow \frac{1}{\theta} > 0$$

1-param expo family with full rank

$\Rightarrow T = \sum X_i$ is c.s.s. — (1)

$$E X_1 = \int_0^\infty x \frac{1}{\theta} e^{-x/\theta} dx = \frac{1}{\theta} \Gamma 2 \cdot \theta^2 = \theta.$$

$$\Rightarrow E\left(\frac{T}{n}\right) = \theta \text{ — (1)}$$

T/n is u.e. based on c.s.s. and hence UMVUE

(2)

(b) $$\log f_\theta = -\frac{x}{\theta} - \log\theta$$

$$\frac{\partial \log f_\theta}{\partial\theta} = \frac{x}{\theta^2} - \frac{1}{\theta}$$

$$\frac{\partial^2 \log f_\theta}{\partial\theta^2} = -\frac{2x}{\theta^3} + \frac{1}{\theta^2}$$

$$E\left(\frac{\partial^2 \log f_\theta(x)}{\partial\theta^2}\right) = -\frac{2}{\theta^3} E(X) + \frac{1}{\theta^2} = -\frac{1}{\theta^2}$$

$$CRLB = \frac{(g'(\theta))^2}{-n E\left(\frac{\partial^2 \log f}{\partial\theta^2}\right)} = \frac{\theta^2}{n} \text{ — (3)}$$

$$V(T/n) = \frac{1}{n^2}\sum V(X_i) = \frac{V(X_1)}{n}$$

$$\left[E X^2 = \frac{1}{\theta}\int_0^\infty x^2 e^{-x/\theta} dx = 2\theta^2 \Rightarrow VX = 2\theta^2 - \theta^2 = \theta^2. \right]$$

$$\text{i.e. } V(T/n) = \frac{\theta^2}{n} = CRLB$$

$\Rightarrow$ UMVUE attains the CRLB — (3)

(c) likelihood f^n

$$L(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} f_\theta(x_i) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i}$$

log likelihood $l(\theta|\underline{x}) = -n\log\theta - \frac{1}{\theta}\Sigma x_i$

$$\frac{\partial l}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\Sigma x_i$$

$$\Rightarrow \frac{\partial l}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} = \frac{1}{\theta^2}\Sigma x_i$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\Sigma x_i = \bar{x} \quad -(1)$$

Further, $\frac{\partial^2 l}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\Sigma x_i$

$$\Rightarrow \left.\frac{\partial^2 l}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3}$$

$$= \frac{n}{\bar{x}^2} - \frac{2n}{\bar{x}^2} = -\frac{n}{\bar{x}^2} < 0 \quad -(1)$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X} \quad -(1)$$

By invariance property of MLE

$$\hat{\theta}^2_{MLE} = \bar{X}^2. \quad -(1)$$

~~6~~7 (d)

$$ET^2 = E\left(\sum X_i\right)^2$$

$$= E\left(\sum_1^n X_i^2 + 2\sum_{i<j} X_i X_j\right)$$

$$= \sum_1^n EX_i^2 + 2\sum_{i<j} EX_i\, EX_j$$

$$= 2n\theta^2 + 2\sum_{i<j} \theta^2$$

$$= 2n\theta^2 + n(n-1)\,\theta^2$$

$$\Rightarrow ET^2 = \theta^2\left(2n + n^2 - n\right) = \theta^2\left(n + n^2\right)$$

i.e. $ET^2 = \theta^2\, n(n+1)$

$$\Rightarrow E\left(\frac{T^2}{n(n+1)}\right) = \theta^2 \qquad -(4).$$

$\frac{T^2}{n(n+1)}$ is u.e. based on css and hence UMVUE

$\left(\frac{n}{n+1}\right) T^2$

8 (a) $X_1, \ldots X_n$ r.s. from $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$f_\theta(x) = \begin{cases} 1, & \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} \\ 0, & o/w \end{cases}$$

likelihood f^n

$$L(\theta | \underset{\sim}{x}) = \prod_{i=1}^{n} f_\theta(x_i) = \begin{cases} 1, & \theta - \frac{1}{2} \le x_{(1)} \le x_{(2)} \ldots \le x_{(n)} \le \theta + \frac{1}{2} \\ 0, & o/w. \end{cases}$$

i.e. $L(\theta | \underset{\sim}{x}) = I_{(\theta - \frac{1}{2}, x_{(1)})} \, I_{(x_{(n)}, \theta + \frac{1}{2})}$ — (1)

Maximum value of the likelihood f^n is 1 if

$$\theta - \frac{1}{2} \le x_{(1)} \quad \& \quad x_{(n)} \le \theta + \frac{1}{2}$$

i.e. $\theta \le x_{(1)} + \frac{1}{2}$ & $x_{(n)} - \frac{1}{2} \le \theta$

i.e. if $x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$ — (2)

(2) Any θ in the above interval would maximise the likelihood f^n and hence is MLE

e.g $X_{(1)} + \frac{1}{2}$ / $X_{(n)} - \frac{1}{2}$ / $\frac{X_{(1)} + X_{(n)}}{2}$ all are MLE

$\alpha \left(X_{(n)} - \frac{1}{2} \right) + (1 - \alpha) \left(X_{(1)} + \frac{1}{2} \right)$ is MLE

$0 < \alpha < 1$

MLE is thus not unique here (1)

Award marks if proper argument is given

8 (b) $E(X) = \mu_1' = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\,dx = \frac{x^2}{2}\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} = \theta$ — (1)

MOME estimator obtained by equating

$m_1' = \frac{1}{n}\sum X_i = \mu_1' = \theta$

$\Rightarrow \hat{\theta}_{MOME} = \frac{1}{n}\sum X_i = \bar{X}$. — (3).