Assignment-3

1. Given a fixed element $x \in \ell_1$, show that the seq. $x^{(k)} = (x_1, x_2, \dots, x_k, 0, 0 \dots)$ converges to x in ℓ_1-norm. Show that $x^{(k)}$ also cvgs. to x in ℓ_2-norm.
Give an example to show that $x^{(k)} \not\to x$ in ℓ_∞-norm.

2. Given $x, y \in \ell_2$, define $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$. Show that if $x^{(k)} \to x$ & $y^{(k)} \to y$ in ℓ_2 then $\langle x^{(k)}, y^{(k)} \rangle \to \langle x, y \rangle$.

3. Two metrics d and ρ on a set M are said to be equivalent if
$$d(x_n, x) \to 0 \text{ iff } \rho(x_n, x) \to 0 .$$
For (M,d) a metric space, show that
(i) $\rho(x,y) := \sqrt{d(x,y)}$
(ii) $\sigma(x,y) := d(x,y)/1+d(x,y)$
(iii) $\tau(x,y) := \min\{d(x,y), 1\}$ all these metrics (i)–(iii) are equivalent to d.

4. (a) Show that the usual metric on $\mathbb{N}$, i.e., $(\mathbb{N}, |\cdot|)$ is equivalent to the discrete metric on $\mathbb{N}$. Show that any metric on a finite set is equivalent to the discrete metric.
(b) If M is a countable set and d is a metric on M, then is d equivalent to the discrete metric on M?

5. Show that the metrics induced by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are equivalent.

6. (The product metric) Given two metric spaces (M,d) and (N,ρ), define the following metrics on $M \times N = \{(x,y) \mid x \in M, y \in N\}$. as

(i) $d_1\big((a,x), (b,y)\big) := d(a,b) + \rho(x,y)$
(ii) $d_2\big((a,x), (b,y)\big) := \left[d(a,b)^2 + \rho(x,y)^2\right]^{1/2}$
(iii) $d_\infty\big((a,x), (b,y)\big) := \max\{d(a,b), \rho(a,b)\}$

Show that all three metrics on $M \times N$ are equivalent.

(Note that a particular example of this product metric space is given in Question 5.)

evaluation

7. Show that every Cauchy seq. in $(\mathbb{R}^n, \|\cdot\|_2)$ cvg. in $\mathbb{R}^n$ for $n \geq 2$.
Does every Cauchy seq. in $(\mathbb{R}^n, \|\cdot\|_\infty)$ and in $(\mathbb{R}^n, \|\cdot\|_1)$ also cvg. in $\mathbb{R}^n$ w.r.t these norms?
Does every Cauchy seq. in $(\ell^1, \|\cdot\|_\infty)$ cvg. in ℓ^1? Is the $\|\cdot\|_1$-norm equivalent to $\|\cdot\|_\infty$-norm?

(We say $\|\cdot\|$ and $\|\cdot\|'$ on M are equivalent if the metrics induced by these norms, namely, $d_{\|\cdot\|}$ and $d_{\|\cdot\|'}$, are equivalent.)