Name: ______________________

Roll Number: ________________

Midsemester Exam Solutions

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2021/22, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.

2. This exam contains **4 + 1** questions and is worth **40%** of your grade.

3. Answer **ALL** questions.

Question 1. [5 × 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There exists a sequence $\langle X_n : n < \omega \rangle$ such that for each $n < \omega$, $|X_{n+1}| < |X_n|$.

(ii) The set of all injective functions from ω to ω is countable.

(iii) For every uncountable $X \subseteq \mathbb{R}$, there exists an uncountable $Y \subseteq X$ such that for every distinct a, b, c in Y, we have $a + c \neq 2b$.

(iv) Let $\mathcal{L}$ be a first order language, T be an $\mathcal{L}$-theory and ϕ, ψ be $\mathcal{L}$-sentences. Assume $T \vdash (\phi \wedge \psi)$. Then $T \vdash \phi$ and $T \vdash \psi$.

(v) Let T be a first order theory which has no uncountable model. Then T is inconsistent.

Solution 1.

(i) False. Otherwise, $\{|X_n| : n < \omega\}$ would be a set of ordinals with no least member.

(ii) False. See Question 2 in practice midsem.

(iii) True. This follows from the Rado's theorem on lecture slide 85.

(iv) True. Since $(\phi \wedge \psi) \implies \phi$ is a propositional tautology, $T \vdash (\phi \wedge \psi) \implies \phi$. By Modus Ponens, $T \vdash \phi$. Similarly, $T \vdash \psi$.

(v) False. Take $T = \{(\forall x)(\forall y)(x = y)\}$.

Question 2. [10 Points]

(a) [**2 Points**] State the continuum hypothesis.

(b) [**3 Points**] Assume the continuum hypothesis. Show that there exists a well-order $\prec$ on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, $\mathsf{pred}(\mathbb{R}, \prec, x)$ is countable.

(c) [**5 Points**] Now assume that there exists a **linear order** $\prec$ on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, $\mathsf{pred}(\mathbb{R}, \prec, x)$ is countable. Show that the continuum hypothesis holds.

Solution 2.

(a) $|\mathbb{R}| = \omega_1$. ☕

(b) Assume $|\mathbb{R}| = \omega_1$. Then there is a bijection $f : \mathbb{R} \to \omega_1$. For $x, y \in \mathbb{R}$, define $x \prec y$ iff $f(x) < f(y)$. Then f is an isomorphism from $(\mathbb{R}, \prec)$ to $(\omega_1, <)$. So $\prec$ is a well-order on $\mathbb{R}$. Since ω_1 is the least uncountable ordinal, for every $\alpha \in \omega_1$, $\mathsf{pred}(\omega_1, <, \alpha) = \alpha$ is countable. So $\mathsf{pred}(\mathbb{R}, \prec, x)$ is also countable for every $x \in \mathbb{R}$. ☕

(c) **Proof 1**: Fix a linear order $\prec$ on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, $\mathsf{pred}(\mathbb{R}, \prec, x)$ is countable. Towards a contradiction, suppose $|\mathbb{R}| \neq \omega_1$. Then $|\mathbb{R}| \geq \omega_2$.

Using transfinite recursion, define $\langle x_\alpha : \alpha < \omega_1 \rangle$ as follows. $x_0 \in \mathbb{R}$ is arbitrary. Having defined $\langle x_\beta : \beta < \alpha \rangle$, put $P = \bigcup\{\mathsf{pred}(\mathbb{R}, \prec, x_\beta) : \beta < \alpha\}$. Note that P is a countable union of countable sets and so P is countable. Choose $x_\alpha \in \mathbb{R} \setminus W$. This completes the definition of $\langle x_\alpha : \alpha < \omega_1 \rangle$.

Note that for every $\alpha < \beta < \omega_1$, we have $x_\alpha \prec x_\beta$. Put $L = \bigcup\{\mathsf{pred}(\mathbb{R}, \prec, x_\beta) : \beta < \omega_1\}$. Then $|L| \leq \omega_1$ since L is a union of ω_1 sets each of which is countable. Since $|\mathbb{R}| \geq \omega_2$, we can choose $x \in \mathbb{R} \setminus L$. Now observe that every member of the uncountable set $\{x_\alpha : \alpha < \omega_1\}$ is $\prec$-below x. So $\mathsf{pred}(\mathbb{R}, \prec, x)$ is uncountable. A contradiction. ☕

Proof 2: Fix a linear order $\prec$ on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, $\mathsf{pred}(\mathbb{R}, \prec, x)$ is countable. Define $A = \{(x, y) \in \mathbb{R}^2 : y \prec x\}$. Then for any $x \in \mathbb{R}$, the vertical section of A at x is countable because

$$A_x = \{y : (x, y) \in A\} = \{y \in \mathbb{R} : y \prec x\} = \mathsf{pred}(\mathbb{R}, \prec, x)$$

For any $y \in \mathbb{R}$, the vertical section of $\mathbb{R}^2 \setminus A$ at y is also countable since

$$(\mathbb{R}^2 \setminus A)^y = \{x : (x, y) \in \mathbb{R}^2 \setminus A\} = \{x \in \mathbb{R} : x \preceq y\} = \mathsf{pred}(\mathbb{R}, \prec, y) \cup \{y\}$$

By Sierpinski's theorem on lecture slide 90, it the continuum hypothesis follows. ☕

Question 3. [10 Points]

(a) [**3 Points**] Show that there is no subset of plane that meets every circle at exactly 2 points.

(b) [**7 Points**] Using transfinite recursion, show that there is a subset of plane that meets every circle at exactly 3 points.

Solution 3.

(a) Suppose not and let $S \subseteq \mathbb{R}^2$ be such a set. Let C_1 be any circle in plane. Put $S \cap C_1 = \{a, b\}$. Choose a circle C_2 such that line passing though a, b does not intersect C_2. Let $S \cap C_2 = \{c, d\}$. Since a, b, c are not collinear, there exists a unique circle C_3 passing thorough a, b, c. But now $\{a, b, c\} \subseteq S \cap C_3$. A contradiction. ☕

(b) We try to modify the 2-point set construction. Let $\mathcal{E}$ be the set of all circles in $\mathbb{R}^2$. Since each circle is uniquely determined by its center and radius, $|\mathcal{E}| \leq |\mathbb{R}^2 \times \mathbb{R}^+| = |\mathfrak{c} \times \mathfrak{c}| = \mathfrak{c}$. Also there are at least $\mathfrak{c}$ distinct circles so $|\mathcal{E}| = \mathfrak{c}$. Let $\langle C_\alpha : \alpha < \mathfrak{c} \rangle$ be an injective sequence with range $\mathcal{E}$. Using transfinite recursion, construct an increasing sequence $\langle S_\alpha : \alpha < \mathfrak{c} \rangle$ of subsets of $\mathbb{R}^2$ such that the following hold.

1. $S_0 = 0$ and if γ is limit, then $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$.
2. $|S_\alpha| \leq |\alpha + \omega| < \mathfrak{c}$.
3. No 4 points in S_α are concyclic.
4. $|S_{\alpha+1} \cap C_\alpha| = 3$.

First observe that at any limit stage $\alpha < \mathfrak{c}$, defining $S_\alpha = \bigcup\{S_\beta : \beta < \alpha\}$ does not violate Clause 3. For suppose S_α does contain 4 concylic points. Then all of these 4 points must appear at some stage $\alpha' < \alpha$ which is impossible.

Having constructed S_α, $S_{\alpha+1}$ is obtained as follows. Let $\mathcal{T}$ be the set of circles that pass through 3 points in S_α. Then $|\mathcal{T}| \leq |S_\alpha \times S_\alpha \times S_\alpha| \leq |\alpha + \omega| < \mathfrak{c}$. Let B be the set of points of intersection of C_α with the circles in $\mathcal{T}$. Note that $|B| \leq |\alpha + \omega| < \mathfrak{c}$. By Clause 3, $|S_\alpha \cap C_\alpha| = n \leq 3$ so we can add $3 - n$ points from $C_\alpha \setminus B$ to S_α to get $S_{\alpha+1}$.

Having completed the construction, put $S = \bigcup_{\alpha<\mathfrak{c}} S_\alpha$. It is clear that S meets every circle at exactly 3 points. ☕

Question 4. [10 Points]

Let $\mathcal{L} = \{<\}$ where $<$ is a binary relation symbol. Let $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ be the $\mathcal{L}$-structures where $(\mathbb{R}, <)$ is the usual ordering on the set of all real numbers and $(\mathbb{Q}, <)$ is the usual ordering on the set of all rational numbers.

(a) [**3 Points**] Show that for every $\mathcal{L}$-sentence ϕ,

$$(\mathbb{Q}, <) \models \phi \text{ iff } (\mathbb{R}, <) \models \phi$$

(b) [**5 Points**] Show that $(\mathbb{R}, <)$ is not isomorphic to $(\mathbb{R} \setminus \{0\}, <)$.

(c) [**2 Points**] Recall that DLO is the theory of dense linear orderings without end-points. Show that DLO is not $\mathfrak{c}$-categorical. Here $\mathfrak{c} = |\mathbb{R}|$ is the continuum.

Solution 4.

(a) Let ϕ be any $\mathcal{L}$-sentence. By lecture slide 166, DLO is a complete $\mathcal{L}$-theory. So either $DLO \vdash \phi$ or $DLO \vdash \neg\phi$. Suppose $DLO \vdash \phi$. Since both $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ are models of DLO, it follows that $(\mathbb{R}, <) \models \phi$ and $(\mathbb{Q}, <) \models \phi$. Similarly if $DLO \vdash \neg\phi$, then $(\mathbb{R}, <) \models \neg\phi$ and $(\mathbb{Q}, <) \models \neg\phi$. In follows that $(\mathbb{R}, <) \models \phi$ iff $(\mathbb{Q}, <) \models \phi$. ☕

(b) Suppose not and let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an order preserving bijection. Put $L = \{f(x) : x < 0\}$ and $R = \{f(x) : x > 0\}$. Since $L \subseteq \mathbb{R}$ is bounded from above, we can define $a = \mathsf{sup}(L)$ (supremum of L). Clearly $a \notin L$ since L does not have a largest member. Similarly $a \notin R$ since R does not have a least member. Since range$(f) = L \cup R$ and $a \notin L \cup R$, it follows that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is not surjective. A contradiction. ☕

(c) First note that both $(\mathbb{R}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$ are models of DLO. By part (b), they are not isomorphic. Also $|\mathbb{R}| = |\mathbb{R} \setminus \{0\}| = \mathfrak{c}$. So DLO has two non-isomorphic models of cardinality $\mathfrak{c}$. It follows that DLO is not $\mathfrak{c}$-categorical. ☕

Bonus Question [5 Points]

Let $2^{\mathbb{R}}$ denote the set of all functions from $\mathbb{R}$ to $\{0,1\}$. Show that there exists a **countable** $\mathcal{F} \subseteq 2^{\mathbb{R}}$ such that for every $f \in 2^{\mathbb{R}}$ and for every finite $W \subseteq \mathbb{R}$, there exists $g \in \mathcal{F}$ such that $g \restriction W = f \restriction W$.

Solution. Let $\mathcal{I}$ be the set of all intervals (a,b) where $a < b$ are both rationals. Let $\mathcal{B}$ consist of all sets S such that S is the union of finitely many intervals in $\mathcal{I}$. Note that $\mathcal{B}$ is countable. For each $S \in \mathcal{B}$, define $f_S : \mathbb{R} \to \{0,1\}$ by "$f_S(x) = 1$ iff $x \in S$". Define $\mathcal{F} = \{f_S : S \in \mathcal{B}\}$. Clearly, $\mathcal{F}$ is countable. We claim that $\mathcal{F}$ is as required.

Suppose $f \in 2^{\mathbb{R}}$ and $W \subseteq \mathbb{R}$ is finite. Let $W_0 = \{x \in W : f(x) = 0\}$ and $W_1 = \{x \in W : f(x) = 1\}$. Choose $S \in \mathcal{B}$ such that $W_1 \subseteq S$ and $W_0 \cap S = \emptyset$. Such a set $S \in \mathcal{B}$ exists because rationals are dense in $\mathbb{R}$. Put $g = f_S$. Then $g \in \mathcal{F}$ and $g \restriction W = f \restriction W$. ☕

End of Exam