Assignment 4a (No evaluation on this)

1. Prove that every subset of a metric space M can be written as the intersection of open sets.

2. Given $y = (y_n) \in H^\infty$ (Hilbert cube), $N \in \mathbb{N}$ and $\varepsilon > 0$.
Show that $\{ x = (x_n) \in H^\infty \mid |x_k - y_k| < \varepsilon,\ k = 1, \dots, N \}$ is open in H^∞.

3. Let $e^{(k)} = (0, \dots, 0, \underset{k^{\text{th}}\text{ position}}{1}, 0 \cdots 0 \cdots)$. Show that $\{ e^{(k)} \mid k \geq 1 \}$ is closed as a subset of ℓ_1.

4. Let F be the set of all $x \in \ell_\infty$ s.t. $x_n = 0$ for all but finitely many n.
Is F closed? open? neither? Explain.

5. Show that c_0 is a closed subset of ℓ_∞.

6. Show that $A = \{ x \in \ell_2 \mid |x_n| \leq \frac{1}{n},\ n = 1, 2, \dots \}$ is a closed set in ℓ_2, but that $B = \{ x \in \ell_2 \mid |x_n| < \frac{1}{n},\ n = 1, 2, \dots \}$ is not an open set in ℓ_2.

7. The set $A = \{ y \in M \mid d(x,y) \leq r \}$ is called the closed ball about x of radius r.
Show that A is a closed set, but give an example of a set A which need not be the closure of the open ball $B(x,r)$.

8. If $(V, \|\cdot\|)$ is any normed linear space, prove that the closed ball $\{ x \in V \mid \|x\| \leq 1 \}$ is always the closure of the open ball $\{ x \in V \mid \|x\| \leq 1 \}$.

9. Show that A is open iff $A^\circ = A$ and that A is closed iff $\bar{A} = A$.

10. Show that $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$.

11. If A, B any sets in M, then show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.
Give an example of A and B where $\overline{A \cap B} \subsetneq \bar{A} \cap \bar{B}$.

$\boxed{\mathrm{int}(A) = A^{\circ}}$

12. Show that $\overline{A} = \left(\mathrm{int}\, A^c\right)^c$ and that $A^{\circ} = \left(\mathrm{cl}(A^c)\right)^c$.

13. A set that is both open and closed is said to be a "clopen" set. Show that $(\mathbb{R}, |\cdot|)$ has no nontrivial clopen sets.