In such cases, we can use a symmetric padding or end point padding to get rough estimates of trend.

<u>Note</u> : For even order window length moving average, a simple mean of adjacent trend values is computed so as to have trend value correspond to time points. e.g. a 4pt ma

y_1

y_2 $\longrightarrow (y_1+y_2+y_3+y_4)/4$

y_3 $\longrightarrow \hat{m}_3 = \frac{1}{2}\left\{\frac{y_1+y_2+y_3+y_4}{4} + \frac{y_2+y_3+y_4+y_5}{4}\right\}.$

y_4 $\longrightarrow (y_2+y_3+y_4+y_5)/4$

y_5

y_6

$\vdots$

In general, for even order window length

$$\hat{m}_t = \frac{1}{2q}\left(\frac{1}{2}y_{t-q} + y_{t-q+1} + \cdots + y_t + \cdots + y_{t+q-1} + \frac{1}{2}y_{t+q}\right)$$

$$q+1 \le t \le n-q$$

Note: Moving averages are called "low-pass filters" as it filters out the rapidly fluctuating component and passes the low-frequency content (the less volatile smooth part) of the data.

$$y_t \longrightarrow \boxed{} \longrightarrow \hat{m}_t = \sum_{-\infty}^{\infty} a_j \, y_{t+j} \qquad \text{e.g. } a_j = \begin{cases} \frac{1}{2q+1}, & |j| \le q \\ 0, & \text{o/w} \end{cases}$$

filter with coeffs $\{a_j\}$ — a linear filter — equal wts ma filter

Note: $\hat{m}_t$ defined earlier is a 2-sided moving average; one can also define a one-sided moving average and trend estimate at the last pt in the window is considered.

Exponentially Weighted Moving Average (EWMA)

EWMA is an example of one-sided moving average filtering with weights decreasing exponentially inside MA window as one moves further and further away from the time pt, at which the trend is estimated.

For a fixed $\alpha \in (\frac{1}{2}, 1)$, one sided EWMA is defined as

$$\hat{m}_t = \alpha \, Y_t + (1-\alpha)\, \hat{m}_{t-1}; \quad t = 2(1)n$$

$$\hat{m}_1 = Y_1$$

Realize that, we have

$$\hat{m}_1 = Y_1$$

$$\hat{m}_2 = \alpha Y_2 + (1-\alpha)\hat{m}_1$$

i.e. $\hat{m}_2 = \alpha Y_2 + (1-\alpha) Y_1$

$$\hat{m}_3 = \alpha Y_3 + (1-\alpha)\hat{m}_2$$

i.e. $\hat{m}_3 = \alpha Y_3 + (1-\alpha)(\alpha Y_2 + (1-\alpha) Y_1)$

i.e. $\hat{m}_3 = \alpha Y_3 + \alpha(1-\alpha) Y_2 + (1-\alpha)(1-\alpha) Y_1$

$$(\alpha > \alpha(1-\alpha) > (1-\alpha)^2).$$

In general $\forall t \geq 2$

$$\hat{m}_t = \alpha Y_t + (1-\alpha)\hat{m}_{t-1}$$

i.e. $\hat{m}_t = \alpha Y_t + (1-\alpha)(\alpha Y_{t-1} + (1-\alpha)\hat{m}_{t-2})$

i.e $\hat{m}_t = \alpha Y_t + \alpha(1-\alpha) Y_{t-1} + (1-\alpha)^2 \hat{m}_{t-2}$

i.e. $\hat{m}_t = \alpha Y_t + \alpha(1-\alpha) Y_{t-1} + (1-\alpha)^2 (\alpha Y_{t-2} + (1-\alpha)\hat{m}_{t-3})$

i.e. $\hat{m}_t = \alpha Y_t + \alpha(1-\alpha) Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + (1-\alpha)^3 \hat{m}_{t-3}$

$\vdots \qquad \vdots \qquad \vdots$

$\alpha(1-\alpha)^{t-2} Y_2 + (1-\alpha)^{t-1} \hat{m}_1$

Y_1

$$\hat{m}_t = \sum_{j=0}^{t-2} \alpha(1-\alpha)^j Y_{t-j} + (1-\alpha)^{t-1} Y_1$$

EWMA estimate of trend at time pt t

Method 3 : Trend removal/elimination by differencing

This is a method of trend elimination without estimating the trend component.

Define,

lag operator : B ; $BY_t = Y_{t-1}$; $B^j Y_t = Y_{t-j}$

First difference operator : ∇

$$\nabla Y_t = Y_t - Y_{t-1} = (1-B) Y_t$$

$$\nabla^2 Y_t = \nabla (\nabla Y_t)$$

$$\text{i.e. } \nabla^2 Y_t = \nabla (Y_t - Y_{t-1})$$

$$\text{i.e. } \nabla^2 Y_t = \nabla Y_t - \nabla Y_{t-1}$$

$$\text{i.e. } \nabla^2 Y_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})$$

$$\text{i.e } \nabla^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$$

$$= (1 - 2B + B^2) Y_t$$

$$= (1-B)^2 Y_t$$

$$\nabla^j Y_t = \nabla(\nabla^{j-1} Y_t) = (1-B)^j Y_t$$

Suppose, $m_t = a + bt$ - linear time trend

then $\nabla m_t = b$

and suppose that we have the model as

$$Y_t = m_t + e_t \; ; \; E(e_t) = 0$$

$$V(e_t) = \sigma^2 < \infty$$

then $\nabla Y_t = \nabla m_t + \nabla e_t$

i.e $\nabla Y_t = b + (e_t - e_{t-1})$.

∇Y_t series is free from time trend

Sly if $m_t = a + bt + ct^2$

$$\nabla^2 m_t = 2c.$$

& $Y_t = (a + bt + ct^2) + e_t$

$$\nabla^2 Y_t = 2c + \nabla^2 e_t$$

$\nabla^2 Y_t$ would be series free from trend.

In general if m_t is a trend of degree k, then $\nabla^k y_t$ will be a series free from trend as

with $m_t = \sum_{j=0}^{k} a_j t^j$

$$\nabla^k m_t = k!\, a_k$$

and if $y_t = \sum_{j=0}^{k} a_j t^j + e_t$

$$\nabla^k y_t = (k!\, a_k) + \nabla^k e_t$$

$\{\nabla^k y_t\}$ will be a time series with mean $k!\, a_k$ and no time trend component

* <u>Remark</u>: For a given time series, to eliminate trend by differencing we look at the least number of differencing required to reduce it to a series which is free from trend.

(why ??)