Some important dist^n obtained through transformations

(I) $X_1, \ldots, X_n$ i.i.d $N(0,1)$

$X_i^2 \sim \chi_1^2$ and are i.i.d.

$\sum_{i=1}^{n} X_i^2 \sim \chi_n^2 \leftarrow$ chi-square on n degrees of freedom

(II)

$U \sim N(0,1)$

$V \sim \chi_r^2$

U & V are indep

$$T = \frac{U}{\sqrt{V/r}} \sim \text{Student's t-dist}^n \text{ on } r \text{ degrees of freedom}$$

$T \sim t_r$

(III)

$U \sim \chi_r^2$

$V \sim \chi_s^2$

U & V are indep

$$F = \frac{U/r}{V/s} \sim F \text{ dist}^n \text{ with } (r,s) \text{ degrees of freedom}$$

$F \sim F_{r,s}$

Remark: $F \sim F_{r,s} \Leftrightarrow \frac{1}{F} \sim F_{s,r}$

$F_{1,s} \stackrel{d}{=} t_s^2$

An important result: Sampling from $N(\mu, \sigma^2)$

Suppose $X_1, \ldots, X_n$ be a random sample (i.e $X_1, \ldots X_n$ are independent and identically distributed) from $N(\mu, \sigma^2)$ distribution; $\mu \in \mathbb{R}$, $\sigma > 0$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$: sample mean random variable

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$: sample variance random variable.

i.e. $\bar{X} = f_1(X_1, \ldots, X_n)$

$S^2 = f_2(X_1, \ldots, X_n)$

Then

(i) $\bar{X} \sim N(\mu, \sigma^2/n)$

(ii) $\frac{(n-1) S^2}{\sigma^2} \sim \chi^2_{n-1}$

& (iii) $\bar{X}$ & S^2 are independent

Remark:

$$\left.\begin{aligned} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} &\sim N(0,1) \\ \frac{(n-1) S^2}{\sigma^2} &\sim \chi^2_{n-1} \end{aligned}\right\} \text{independent}$$

$$\Rightarrow \quad \frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2} \Big/ (n-1)}} = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Proof of the results (i)-(iii)

jt p.d.f. of $(X_1, \ldots, X_n)$

$$f_{X_1,\ldots X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

i.e. $f_{\underset{\sim}{X}}(\underset{\sim}{x}) = K \exp\left(-\frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2\right)$

K does not depend on $(x_1, \ldots, x_n)$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = K \exp\left(-\frac{1}{2\sigma^2}\sum_1^n (x_i - \bar{x} + \bar{x} - \mu)^2\right)$$

$$= K \exp\left(-\frac{1}{2\sigma^2}\sum (x_i-\bar{x})^2 - \frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right)$$

Make the following transformation

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \longrightarrow \begin{pmatrix} Y_1 = \frac{1}{n}(X_1 + \cdots + X_n) \\ Y_2 = X_2 - \bar{X} \\ \vdots \\ Y_n = X_n - \bar{X} \end{pmatrix} \qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Inverse transformation:

$$X_1 = Y_1 - Y_2 - \cdots - Y_n$$

$$X_2 = Y_1 + Y_2$$

$$X_3 = Y_1 + Y_3$$

$$\vdots$$

$$X_n = Y_1 + Y_n$$

Jacobian determinant

$$J = \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 \cdots & 0 \\ \cdot & - & - & - & \cdot \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} n & 0 & 0 \cdots & 0 \\ 1 & & & \\ \vdots & & \left(I_n \right) & \\ 1 & & & \end{vmatrix} = n$$

From the inverse transformation note that

$$(x_i - \bar{x}) = y_i \quad \text{for } i = 2, 3, \ldots, n$$

and $x_1 - \bar{x} = (y_1 - y_2 - \cdots - y_n) - y_1 = \left(-\sum_{i=2}^{n} y_i\right)$

Thus the jt p.d.f. of the random variables $Y_1, \ldots, Y_n$ is

$$f_{\underset{\sim}{y}}(\underline{y}) = K' \exp\left(-\frac{1}{2\sigma^2}(-y_2 - y_3 - \cdots - y_n)^2 - \frac{1}{2\sigma^2}\sum_{i=2}^{n} y_i^2 - \frac{n}{2\sigma^2}(y_1 - \mu)^2\right)$$

$$= f_{Y_2, \ldots, Y_n}(y_2, \ldots, y_n)\, f_{Y_1}(y_1) \qquad \underset{\sim}{y} \in \mathbb{R}^n$$

$\Rightarrow (Y_2, \ldots, Y_n)$ and Y_1 are independent

$\Rightarrow Y_1$ & (any function of $Y_2, \dots, Y_n$) are independent

Note further that $Y_1 = \bar{X}$

& $$s^2 = \frac{1}{n-1}\left[(X_1-\bar{X})^2 + (X_2-\bar{X})^2 + \dots + (X_n-\bar{X})^2\right]$$

i.e. $$s^2 = \frac{1}{n-1}\left[(-Y_2 \dots - Y_n) + Y_2^2 + \dots + Y_n^2\right]$$

f^n of $(Y_2, \dots, Y_n)$

$\Rightarrow \bar{X}$ & s^2 are independent

Further $$f_{Y_1}(y_1) = K_1 \exp\left(-\frac{n}{2\sigma^2}(y_1-\mu)^2\right) \qquad -\infty < y_1 < \infty$$

$$\Rightarrow Y_1 = \bar{X} \sim N(\mu, \sigma^2/n)$$

Note that

$$X_i \sim N(\mu, \sigma^2) \quad \text{i.i.d.}$$

$$\frac{X_i-\mu}{\sigma} \sim N(0,1) \quad \text{i.i.d.}$$

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi_1^2 \quad \text{i.i.d.}$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \chi_n^2$$

and $$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

$$= \frac{(n-1)s^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2} \quad \text{— } (*)$$

Note that m.g.f. of l.h.s. of (*) is $(1-2t)^{-n/2}$

(m.g.f. of χ^2_n)

i.e. $$(1-2t)^{-n/2} = E\left(e^{t\left(\frac{1}{\sigma^2}\sum_1^n (x_i-\mu)^2\right)}\right)$$

$$= E\left(e^{t\left\{\frac{(n-1)s^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}\right\}}\right)$$

$$= E\left(e^{t\frac{(n-1)s^2}{\sigma^2}}\right) E\left(e^{t\frac{n(\bar{x}-\mu)^2}{\sigma^2}}\right)$$

(using independence)

$$= E\left(e^{t\frac{(n-1)s^2}{\sigma^2}}\right) (1-2t)^{-1/2}$$ m.g.f. of χ^2_1

$$\Rightarrow E\left(e^{t\frac{(n-1)s^2}{\sigma^2}}\right) = (1-2t)^{-\frac{(n-1)}{2}} = M_{\frac{(n-1)s^2}{\sigma^2}}(t)$$

By uniqueness of m.g.f. it follows that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

—x—

Convergence of sequence of random variables :

Modes of convergence :

- Convergence in probability - to be covered in this course
- Convergence in distribution - to be covered in this course
- Convergence almost surely
- Convergence in r^{th} mean

Convergence in probability

Let $\{X_n\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. $\{X_n\}$ is said to converge in probability to a random variable X (we write $X_n \xrightarrow{p} X$ as $n \to \infty$) if

$$P(|X_n - X| > \epsilon) \to 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0$$

Some important results

(i) If $X_n \xrightarrow{p} X$ and 'a' is a constant, then

$$aX_n \xrightarrow{p} aX$$

(ii) If $X_n \xrightarrow{p} X$ and $g(.)$ is any continuous function,

then $\quad g(X_n) \xrightarrow{p} g(X)$

(iii) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then

$$X_n \pm Y_n \xrightarrow{p} X + Y$$

$$X_n Y_n \xrightarrow{p} XY$$

$$\frac{X_n}{Y_n} \xrightarrow{p} \frac{X}{Y} \quad (\text{provided } P(Y=0) = 0)$$

Remark: Approaches to verify convergence in prob

(i) Direct approach (by calculating limiting prob)

(ii) Using Chebyshev's inequality (provided 2^{nd} order moment exists)

Examples

(1) $X_1, \ldots, X_n$ are i.i.d. Bernoulli $(1, \theta)$; $0 < \theta < 1$

Let $Z_n = \sum_{i=1}^{n} X_i \sim B(n, \theta)$

Consider the r.v. $Y_n = \frac{Z_n}{n}$

$$P(|Y_n - \theta| > \epsilon) \le \frac{E(Y_n - \theta)^2}{\epsilon^2}$$

$$= \frac{E\left(\frac{Z_n}{n} - \theta\right)^2}{\epsilon^2} = \frac{E(Z_n - n\theta)^2}{n^2 \cdot \epsilon^2} = \frac{V(Z_n)}{n^2 \epsilon^2}$$

$$= \frac{n\theta(1-\theta)}{n^2 \epsilon^2} \to 0 \text{ as } n \to \infty \quad \forall \epsilon > 0$$

$\Rightarrow Y_n \xrightarrow{p} \theta$

i.e. $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \theta \; (= E(X_i))$