Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Mid Semester (MTH305A)

Semester: 2020-2021, I

Full Marks–50 Time - 120 Minutes

Section: A
Are the folowing statements TRUE/FALSE?

4 points

(1) Consider the following system of equations

$$u = ax + by,$$
$$v = cx + dy.$$

Then x, y can be expressed in terms of u and v if and only if $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Answer: TRUE

4 points

(2) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth at $x = q \in \mathbb{R}^n$. If $\det(F'(q)) = 0$, then the function F must not be injective and therefore inverse function does not exist.
Answer: FALSE. Consider $f(x) = x^3, q = 0$.

4 points

(3) For an equation $x^2 + y^2 - 25 = 0$, we can write y as a differentiable function of x near $(3, 4)$.
Answer: TRUE

4 points

(4) Given a function $F(x)$. suppose all conditions in the inverse function theorem are satisfied. Then a global inverse F^{-1} exists.
Answer: FALSE

4 points

(5) Let $A : V \to V$ be an invertible linear transformation. Suppose $\mathcal{B}_0 = \{v_1, \ldots, v_n\}$ is an ordered basis of V and $A(\mathcal{B}_0) = \{A(v_1), \ldots, A(v_n)\}$ is the standard ordered basis of V.
If $\mathcal{B}_0$ and $A(\mathcal{B}_0)$ are equivalently orientated, then for every ordered basis $\mathcal{B}$ of V,

the order bases $\mathcal{B}$ and $A(\mathcal{B})$ are equivalently orientated.
Answer: TRUE

Section: B

6 points

(1) Which of the following statement(s) is(are) correct.

(a) If $A_k \subset \mathbb{R}^n$, for $k \in \mathbb{N}$, are compact subsets of the Euclidean space $\mathbb{R}^n$, then

$$A = \bigcup_{k \in \mathbb{N}} A_k$$

is also compact.
FALSE statement ↑

(b) Let $A_i \subset \mathbb{R}^n$, for $i = 1, \ldots, k$, be compact sets, then

$$A = \bigcup_{i=1}^{k} A_i$$

is also compact.
TRUE statement ↑

(c) Let $A_i \subset \mathbb{R}^n$, for $i = 1, \ldots, k$, be compact sets, then

$$A = \bigcap_{i=1}^{k} A_i$$

is also compact.
TRUE statement ↑

(d) Let X be a closed subset of $\mathbb{R}^n$ and U be open in $\mathbb{R}^n$ such that $X \subset U$. Then there exists $\epsilon > 0$ such that $N(X, \epsilon) \subset U$, where

$$N(X, \epsilon) = \{a \in \mathbb{R}^n \mid \|x - a\| < \epsilon, \text{ for some } x \in X\}.$$

FALSE statement ↑

6 points

(2) Let us consider $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Which of the following statement(s) is(are) correct.

(a) For every $u \in \mathbb{R}^2$, with $u \neq 0$, the directional derivative $f'(0; u)$ of the function f at the point $0 = (0, 0)$ along the direction u exists.
FALSE statement ↑

(b) The partial derivatives $D_1 f$ and $D_2 f$ exist at $(0, 0)$.
TRUE statement ↑

(c) f is differentiable at $(0,0)$.
FALSE statement ↑

(d) f is continuous at $(0,0)$.
FALSE statement ↑

6 points

(3) Let $\alpha : (0,\infty) \to \mathbb{R}^3$ be defined by $\alpha(t) = (\sqrt{t}, 1, t^4)$, for $t \in (0,\infty)$. Let $p = (1,1,1)$.

(a) The curve α is a parameterized regular smooth curve in $\mathbb{R}^3$.
TRUE statement ↑

(b) The tangent vector to α at p is $\left(\frac{1}{2}, 0, 4\right)$.
TRUE statement ↑

(c) The tangent line $T_p : \mathbb{R} \to \mathbb{R}^3$ to α at p is given by

$$T_p(t) = \left(\frac{t}{2} + 1, 1, 1 + 4t\right), \ \forall t \in \mathbb{R}.$$

TRUE statement ↑

(d) The tangent line $T_p : \mathbb{R} \to \mathbb{R}^3$ to α at p is given by

$$T_p(t) = \left(-\frac{t}{2} + 1, 1, 1 - 4t\right), \ \forall t \in \mathbb{R}.$$

TRUE statement ↑

6 points

(4) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be defined by $\alpha(t) = (e^t, e^t \sin t, e^t \cos t)$, for $t \in \mathbb{R}$ and $p = (1,0,1), q = (e^{2\pi}, 0, e^{2\pi}) \in \mathbb{R}^3$.

(a) The curve $\tilde{\alpha} : \mathbb{R} \to \mathbb{R}^3$ defined by $\tilde{\alpha}(t) = (e^{t^3+t}, e^{t^3+t}\sin(t^3+t), e^{t^3+t}\cos(t^3+t))$ is a reparameterization of α.
TRUE statement ↑

(b) The curve α is a regular and hence admits unit-speed reparameterization.
TRUE statement ↑

(c) Length of α between the points p and q is $\sqrt{3}(e^{2\pi} - 1)$.
TRUE statement ↑

(d) Let $\beta(t) = (e^{2t}, e^{2t}\sin 2t, e^{2t}\cos 2t)$, then the length of β between the points p and q is $2\sqrt{3}(e^{2\pi} - 1)$.
FALSE statement ↑

6 points

(5) Let $\alpha(t) = (a\cos t, a\sin t, bt)$, for $t \in \mathbb{R}$ and $a, b \neq 0$.

(a) $\tilde{\alpha}_1(t) = (a\cos(t^3), a\sin(t^3), bt^3)$, for $t \in \mathbb{R}$, is a reparameterization of α.
FALSE statement $\uparrow$

(b) $\tilde{\alpha}_2(t) = (a\cos\left(\frac{t}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{t}{\sqrt{a^2+b^2}}\right), \frac{bt}{\sqrt{a^2+b^2}})$, for $t \in \mathbb{R}$, is a unit-speed reparameterization of α.
TRUE statement $\uparrow$

(c) $\tilde{\alpha}_3(t) = (a\cos\left(\frac{t}{\sqrt{a^2+b^2}}\right), -a\sin\left(\frac{t}{\sqrt{a^2+b^2}}\right), -\frac{bt}{\sqrt{a^2+b^2}})$, for $t \in \mathbb{R}$ is a unit-speed reparameterization of α.
TRUE statement $\uparrow$

(d) $\tilde{\alpha}_3$ is a unit-speed reparameterization of $\tilde{\alpha}_2$.
TRUE statement $\uparrow$

(e) The curves $\alpha(t)$ and $\tilde{\alpha}_1(t)$ have equal trace.
TRUE statement $\uparrow$