Joint m.g.f.

$\underset{\sim}{X} = (X_1, \ldots, X_p)'$

joint m.g.f.

$$M_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{X}}\right) = E\left(e^{t_1X_1 + \cdots + t_pX_p}\right)$$

$(t_1, \ldots t_p)$

provided the expectation exists in some nbd of $\underset{\sim}{0}_{p\times 1}$

$$M_X(\underset{\sim}{t}) = \int \cdots \int e^{\sum_1^p t_i x_i} f_{\underset{\sim}{X}}(\underset{\sim}{x})\, dx_1 \cdots dx_p \quad \text{for continuous case}$$

$$= \sum_{x_1} \cdots \sum_{x_p} e^{\sum_1^p t_i x_i} P(\underset{\sim}{X} = \underset{\sim}{x}) \quad \text{for discrete case}$$

$$\mu'_{k_1, \ldots, k_p} = E\left(X_1^{k_1} \cdots X_p^{k_p}\right) = \left.\frac{\partial^{k_1 + \cdots + k_p} M_{\underset{\sim}{X}}(\underset{\sim}{t})}{\partial t_1^{k_1} \cdots \partial t_p^{k_p}}\right|_{\underset{\sim}{t} = \underset{\sim}{0}}$$

joint moment of order $k_1 + \cdots + k_p$

k_is are non-negative integers

Note that we can get marginal distn m.g.fs from the joint m.g.f.

$$M_{\underset{\sim}{X}}(0, \ldots 0, t_i, 0 \ldots 0) = E\left(e^{t_i X_i}\right) = M_{X_i}(t_i)$$

i^{th} position

Sly marginal joint m.g.f. for any subset can be obtained from $M_X(\underset{\sim}{t})$.

<u>Note</u>: m.g.f. of $X_1 + \cdots + X_k$ for $k \le p$ can be obtained from $M_{\underset{\sim}{X}}(\underset{\sim}{t})$

<u>Remark</u>: $X_1, \ldots, X_p$ are independent iff

$$M_{\underset{\sim}{X}}(\underset{\sim}{t}) = \prod_{i=1}^p M_{X_i}(t_i)$$

M.g.f. of $N_p(\underset{\sim}{\mu}, \Sigma)$

$$M_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{X}}\right)$$

$$= M_{\underset{\sim}{t}'\underset{\sim}{X}}(1)$$

$$= e^{\underset{\sim}{t}'\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}'\Sigma\underset{\sim}{t}}$$

$\underset{\sim}{t}'\underset{\sim}{X} \sim N_1(\underset{\sim}{t}'\underset{\sim}{\mu}, \underset{\sim}{t}'\Sigma\underset{\sim}{t})$

m.g.f.

$$M_{\underset{\sim}{t}'\underset{\sim}{X}}(z) = E\left(e^{z(\underset{\sim}{t}'\underset{\sim}{X})}\right)$$

$$= e^{z(\underset{\sim}{t}'\underset{\sim}{\mu}) + \frac{z^2}{2}(\underset{\sim}{t}'\Sigma\underset{\sim}{t})}$$

- Marginal m.g.f.s can be derived using the above.

M.g.f. of trinomial (n, θ_1, θ_2)

$\underset{\sim}{X} = (X_1, X_2) \sim \text{trinomial}(n, \theta_1, \theta_2)$

$$M_{\underset{\sim}{X}}(t_1, t_2) = E\left(e^{t_1X_1 + t_2X_2}\right)$$

$$= \sum_{x_1=0}^{n}\sum_{x_2=0}^{n-x_1} e^{t_1x_1 + t_2x_2} \frac{n!}{x_1!\,x_2!\,(n-x_1-x_2)!}\theta_1^{x_1}\theta_2^{x_2}(1-\theta_1-\theta_2)^{n-x_1-x_2}$$

$$= \sum_{x_1=0}^{n} \frac{n!}{x_1!\,(n-x_1)!}(\theta_1e^{t_1})^{x_1} \sum_{x_2=0}^{n-x_1} \frac{(n-x_1)!}{x_2!\,(n-x_1-x_2)!}(\theta_2e^{t_2})^{x_2}(1-\theta_1-\theta_2)^{n-x_1-x_2}$$

$$= \sum_{x_1=0}^{n} \binom{n}{x_1}(\theta_1e^{t_1})^{x_1}\left(1-\theta_1-\theta_2+\theta_2e^{t_2}\right)^{n-x_1}$$

$$= \left(1-\theta_1-\theta_2+\theta_1e^{t_1}+\theta_2e^{t_2}\right)^n$$

Sly Tf $(X_1, \ldots, X_{p-1}) \sim \text{mult}(n, \theta_1, \theta_2, \ldots, \theta_{p-1})$

$$M_{\underset{\sim}{X}}(t_1, \ldots, t_{p-1}) = \left(1-\theta_1-\theta_2\cdots-\theta_{p-1}+\theta_1e^{t_1}+\cdots+\theta_{p-1}e^{t_{p-1}}\right)^n$$

Marginal m.g.f.s can be derived.

Dist^n of $X_i + X_j$ can be derived thro joint m.g.f.

Conditional expectation

Consider a bivariate setup $(X, Y) \rightarrow$ jt p.d.f $f_{X,Y}(x,y)$

marginal p.d.f.s : $f_X(x)$, $f_Y(y)$

Conditional p.d.f.s : $f_{X|Y}(x)$, $f_{Y|X}(y)$

Note that $\quad f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y)$

$$= f_Y(y)\, f_{X|Y}(x)$$

Let $g(X,Y)$ be some f^n of X & Y

$E(g(X,Y)|X)$: Conditional expectation of $g(X,Y)$ given X

$E(g(X,Y)|Y)$: Conditional expectation of $g(X,Y)$ given Y

$$E(g(X,Y)|X) = E_{Y|X}(g(X,Y)|X)$$

$$= \int_{-\infty}^{\infty} g(x,y)\, f_{Y|X}(y)\, dy \rightarrow f^n \text{ of } X \text{ only}$$

$$= \phi(X) \text{ , say}$$

$$E_X(\phi(X)) = E_X\, E_{Y|X}(g(X,Y)|X)$$

$$= \int_{-\infty}^{\infty} \phi(x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{Y|X}(y)\, f_X(x)\, dy\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dy\, dx$$

$$= E_{X,Y}(g(X,Y))$$

i.e $\quad E_X E_{Y|X}\left(g(X,Y)\,|\,X\right) = E_{X,Y}\left(g(X,Y)\right)$

Sp. Case : $g(X,Y) = Y$ say

$$E\,E\,(Y|X) = E(Y)$$

sly $E\,E\,(X|Y) = E(X)$

Sp. Case : $\quad g(X,Y) = (Y - E(Y))^2$

$$E\,(Y - E(Y))^2 = E_X E_{Y|X}\left((Y - E(Y))^2\,|\,X\right)$$

Note that

$$E\,(Y - E(Y))^2 | X$$

$$= E\,(Y - E(Y|X) + E(Y|X) - E(Y))^2 \,|\, X$$

$$= E\,(Y - E(Y|X))^2 \,|\, X + (E(Y|X) - E(Y))^2$$

$$+ 2\,\cancel{E\left((Y - E(Y|X))|X\right)}\left(E(Y|X) - E(Y)\right)$$

0 (as $E(Y - E(Y|X))|X$
$= E(Y|X) - E(Y|X) = 0$)

$$= V(Y|X) + (E(Y|X) - E(Y))^2$$

$$\Rightarrow E\,E\left[(Y - E(Y))^2 \,|\, X\right] \; (= V(Y)$$

$$= E\,(V(Y|X)) + V(E(Y|X)))$$

i.e. $\quad V(Y) = E(V(Y|X)) + V(E(Y|X))$

<u>Note</u>: Joint m.g.f. can be derived through m.g.f. of conditional distⁿ using conditional expectation

e.g. $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$M_{X_1,X_2}(t_1, t_2) = E\left(e^{t_1 X_1 + t_2 X_2}\right) \quad ; \quad g(x_1, x_2) = e^{t_1 x_1 + t_2 x_2}$$

$$= E\, E\left(e^{t_1 X_1 + t_2 X_2} \middle| X_1\right)$$

$$= E_{X_1}\left(e^{t_1 X_1}\left(E_{X_2|X_1} e^{t_2 X_2} \middle| X_1\right)\right)$$

Now, use the fact that

$$X_2 | X_1 \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1-\rho^2)\right).$$