as
$$X_t - \mu = \delta + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} - \mu + \epsilon_t$$

i.e. $X_t - \mu = \mu(1-\phi_1 - \cdots - \phi_p) + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} - \mu + \epsilon_t$

i.e. $(X_t - \mu) = \phi_1 (X_{t-1} - \mu) + \cdots + \phi_p (X_{t-p} - \mu) + \epsilon_t$

Define, $Y_t = X_t - \mu$

model is $Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \epsilon_t$

$\{X_t\}$ Covariance stationary $\Rightarrow \{Y_t\}$ is also covariance stationary

with identical ACVF & ACF as $\{X_t\}$

Note: A special case AR process

Random Walk: $X_t = X_{t-1} + \epsilon_t$; ϵ_t i.i.d (μ, σ^2) (or $\epsilon_t \sim WN(0,\sigma^2)$)

Let $X_0 = 0$ initialization

$X_1 = \epsilon_1$

$X_2 = X_1 + \epsilon_2 = \epsilon_1 + \epsilon_2$

$\vdots$

$X_t = \sum_{i=1}^{t} \epsilon_i$; $E(X_t) = t\mu$ (or 0 if $\epsilon_t \sim WN(0,\sigma^2)$)

$V(X_t) = \underline{\underline{t\sigma^2}} \Rightarrow \{X_t\}$ is non-stationary

Note that for such a process
$Y_t = \nabla X_t = X_t - X_{t-1}$ is always stationary

IV : Auto Regressive Moving Average (ARMA) process

$\epsilon_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is an ARMA(p,q) process if

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p}$$
$$+ \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

$$\phi_p \neq 0;\ \theta_q \neq 0;\ Cov(\epsilon_t, X_{t-j}) = 0\ \forall j > 0$$

$\phi_1, \ldots, \phi_p$: AR parameters of AR part of ARMA(p,q)

$\theta_1, \ldots, \theta_q$; MA parameters of MA part of ARMA(p,q)

Time domain properties of standard models

I : White noise $X_t \sim WN(0, \sigma^2)$

$$\gamma_X(h) = \begin{cases} \sigma^2, & h = 0 \\ 0, & o/w \end{cases} \qquad \rho_X(h) = \begin{cases} 1, & h = 0 \\ 0, & o/w \end{cases}$$

$\{X_t\}$ is always covariance stationary

II : MA models

MA(1) $\quad X_t = \epsilon_t + \theta \epsilon_{t-1}$; $\epsilon_t \sim WN(0, \sigma^2)$

$E X_t = 0 \quad \forall t$

$V X_t = (1+\theta^2)\sigma^2 \quad \forall t$

$$\gamma_X(h) = Cov(X_{t+h}, X_t) = E(\epsilon_{t+h} + \theta\epsilon_{t+h-1})(\epsilon_t + \theta\epsilon_{t-1})$$

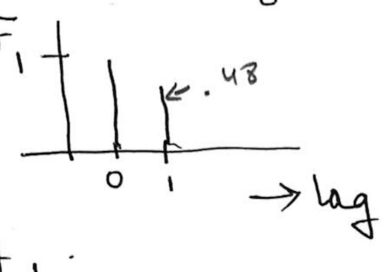
i.e. $$\gamma_X(h) = \begin{cases} (1+\theta^2)\sigma^2, & \text{if } h=0 \\ \theta\sigma^2, & \text{if } h=\pm 1 \\ 0, & o/w \end{cases}$$

ACF $$\rho_X(h) = \begin{cases} 1, & h=0 \\ \theta/1+\theta^2, & h=\pm 1 \\ 0, & o/w \end{cases}$$

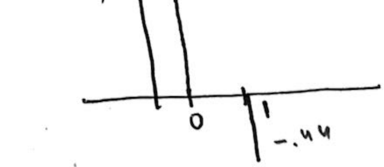
$\{X_t\}$ is covariance stationary $\forall\, \theta$

Note: Shape of ACF depends on the value of θ

$$X_t = \epsilon_t + 0.8\,\epsilon_{t-1}$$



$$X_t = \epsilon_t - 0.6\,\epsilon_{t-1}$$



Note: $\underset{\theta}{\text{Max}}\ \rho_X(1) = \frac{1}{2}$ attained at $\theta = 1$

$\underset{\theta}{\text{Min}}\ \rho_X(1) = -\frac{1}{2}$ attained at $\theta = -1$

Note: No unique representation

Note that

If $X_t = \epsilon_t + \theta\,\epsilon_{t-1}$ then $\rho_X(1) = \dfrac{\theta}{1+\theta^2}$

also if $X_t = \epsilon_t + \frac{1}{\theta}\,\epsilon_{t-1}$ then $\rho_X(1) = \dfrac{\theta}{1+\theta^2}$

$\Rightarrow \forall \rho \in [-\frac{1}{2}, \frac{1}{2}]$ $\exists$ 2 different MA(1) models that gives the same ACF at lag 1

<u>Note</u>: Lag operator representation

$$X_t = \epsilon_t + \theta \epsilon_{t-1} = \epsilon_t + \theta B \epsilon_t$$

i.e. $X_t = \theta(B)\epsilon_t$

$\theta(B) = 1 + \theta B \leftarrow$ MA polynomial

<u>MA(2) process</u>

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}; \; \epsilon_t \sim WN(0, \sigma^2)$$

$$X_t = (1 + \theta_1 B + \theta_2 B^2)\epsilon_t$$

$$X_t = \theta(B)\epsilon_t$$

$$E X_t = 0 \;\; \forall t; \;\; V(X_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2) \;\; \forall t$$

$$\gamma_X(1) = Cov(X_{t+1}, X_t) = Cov(\epsilon_{t+1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1}, \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2})$$

$$= \sigma^2(\theta_1 + \theta_1\theta_2) = \gamma_X(-1)$$

$$\gamma_X(2) = Cov(X_{t+2}, X_t) = Cov(\epsilon_{t+2} + \theta_1 \epsilon_{t+1} + \theta_2 \epsilon_t, \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2})$$

$$= \sigma^2 \theta_2 = \gamma_X(-2)$$

$$\gamma_X(\pm 3) = \gamma_X(\pm 4) = \ldots = 0$$

$$\rho_X(\pm 3) = \rho_X(\pm 4) = \ldots = 0$$

$$\rho_X(1) = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \rho_X(-1); \;\; \rho_X(\pm 2) = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$\{X_t\}$ is covariance stationary $\forall \theta_1, \theta_2$

MA(q)

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} ; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$X_t = \theta(B) \epsilon_t$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

$$E X_t = 0 \quad \forall t$$

$$V X_t = \sigma^2 \left(1 + \sum_{j=1}^{q} \theta_j^2\right).$$

$$Cov(X_{t+h}, X_t) = Cov\big(\epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \cdots + \theta_q \epsilon_{t+h-q},$$
$$\epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}\big)$$

$= 0 \quad \forall |h| > q$ (as there are no common terms)

$$\gamma_X(1) = \sigma^2 (\theta_1 \theta_0 + \theta_2 \theta_1 + \cdots + \theta_q \theta_{q-1}) \qquad (\theta_0 = 1)$$

$$\gamma_X(2) = \sigma^2 (\theta_2 \theta_0 + \theta_3 \theta_1 + \cdots + \theta_q \theta_{q-2})$$

$$\forall\, 0 \le h \le q ; \quad \gamma_X(h) = \sigma^2 (\theta_h \theta_0 + \theta_{h+1} \theta_1 + \cdots + \theta_q \theta_{q-h})$$

i.e. $\gamma_X(h) = \sigma^2 \sum_{j=h}^{q} \theta_j \theta_{j-h}$ $\qquad j' = j - h, \; j = j' + h.$

i.e. $\gamma_X(h) = \sigma^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}$

Further $\gamma_X(-h) = \gamma_X(h)$

$$\Rightarrow \gamma_X(h) = \begin{cases} \sigma^2 \left(1 + \sum_1^q \theta_j^2\right), & \text{if } h = 0 \\ \left(\sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}\right) \sigma^2, & \text{if } |h| \le q \\ 0, & \text{o/w} \end{cases}$$

ACF
$$\rho_X(h) = \begin{cases} 1, & \text{if } h=0 \\ \dfrac{\sum_{0}^{q-|h|} \theta_j \theta_{j+|h|}}{\left(1+\sum_{1}^{q} \theta_j^2\right)}, & \text{if } |h| \le q \\ 0, & \text{o/w} \end{cases}$$

MA(q) is stationary $\forall\ \theta_1, \ldots, \theta_q$

Remark : All finite order MA processes are always covariance stationary, irrespective of the values of MA parameters.

Remark : ACF value is always equal to zero beyond the lag which equals the order of the finite order MA model. This gives a way to identify MA processes from sample ACF plot.

MA(∞) process $\quad X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$; $\epsilon_t \sim WN(0, \sigma^2)$

$EX_t = 0\ \forall t$; $VX_t = \sigma^2\left(\sum_j \psi_j^2\right)$
$< \infty$ if $\sum \psi_j^2 < \infty$
or if $\sum |\psi_j| < \infty$

MA(∞) is not always covariance stationary

For covariance stationary process

$$\gamma_h = \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h} \right) \sigma^2$$

Note: Although square summability, $\sum \psi_j^2 < \infty$ is enough to ensure that MA(∞) is covariance stationary, absolute summability $\sum |\psi_j| < \infty$ is assumed for MA(∞).

$\sum_j |\psi_j| < \infty$ ensures that $\sum_h |\gamma_h| < \infty$ which is required for further theoretical results for MA(∞) process.