Computation of MLE

Examples

(i) $X_1, \dots, X_n$ r.s. from $B(1,\theta)$; $0<\theta<1$

Likelihood f^n: $L(\theta) = \theta^{\sum_1^n x_i} (1-\theta)^{n-\sum_1^n x_i}$

log likelihood f^n: $l(\theta) = L(\theta) = \sum x_i \log\theta + (n-\sum_1^n x_i)\log(1-\theta)$

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{1-\theta}$$

likelihood eq^n: $\frac{\partial l(\theta)}{\partial\theta} = 0$

$$\Rightarrow \hat{\theta} = \sum_1^n x_i / n$$

Further $\frac{\partial^2 l(\theta)}{\partial\theta^2} = -\frac{\sum x_i}{\theta^2} - \frac{(n-\sum x_i)}{(1-\theta)^2}$

$$\Rightarrow \left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\hat{\theta}} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_1^n x_i$$

(ii) $X_1, \dots, X_n$ r.s. from $P(\theta)$; $\theta>0$

Likelihood f^n: $L(\theta) = \frac{e^{-n\theta}\,\theta^{\sum_1^n x_i}}{\prod x_i!}$

Log likelihood f^n: $l(\theta) = -n\theta + \sum x_i \log\theta - \log(\prod x_i!)$

$$\frac{\partial l(\theta)}{\partial\theta} = -n + \frac{\sum x_i}{\theta}$$

$$\frac{\partial l(\theta)}{\partial\theta} = 0 \Rightarrow \hat{\theta} = \frac{\sum x_i}{n}$$

Further $\left.\frac{\partial^2 l(\theta)}{\partial \theta^2}\right|_{\hat{\theta}} = -\left.\frac{\sum x_i}{\theta^2}\right|_{\hat{\theta}} < 0$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_1^n X_i$$

(3) $X_1, \ldots, X_n$ r.s. from a distⁿ (exp) with

p.d.f. $f_\theta(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & o/w \end{cases}$

$$L(\theta) = \theta^n e^{-\theta \sum x_i}$$

$$l(\theta) = n \log \theta - \theta \sum_1^n x_i$$

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{n}{\theta} - \sum x_i$$

$$\frac{\partial l(\theta)}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \frac{n}{\sum_1^n x_i}$$

$$\left.\frac{\partial^2 l(\theta)}{\partial \theta^2}\right|_{\hat{\theta}} = -\left.\frac{n}{\theta^2}\right|_{\hat{\theta}} < 0$$

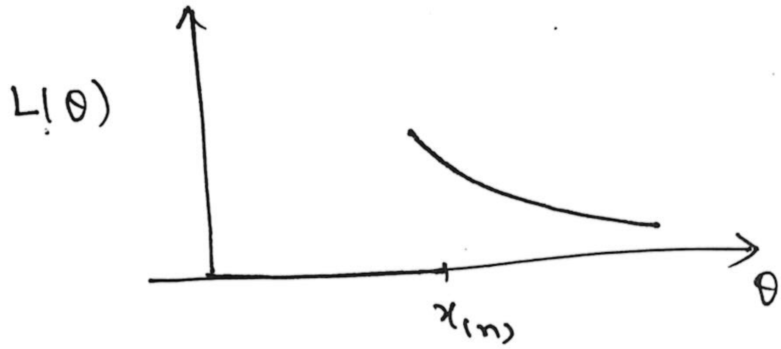
$$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{\sum_1^n X_i}$$

(4) $X_1, \ldots, X_n$ r.s. from $U[0,\theta]$; $\theta > 0$

$$f_\theta(x) = \begin{cases} 1/\theta, & 0 \le x \le \theta \\ 0, & o/w \end{cases}$$

$$L(\theta) = \frac{1}{\theta^n} I(0, x_{(1)})\, I(x_{(n)}, \theta) \; ; \; I(a,b) = \begin{cases} 1, & a \le b \\ 0, & o/w \end{cases}$$

Note that $L(\theta)$ is not differentiable at $x_{(n)}$



i.e. $L(\theta) = 0 \quad \forall\, \theta < x_{(n)}$

$\quad > 0 \;\&\; \downarrow \theta \quad \forall\, \theta \geq x_{(n)}$

$\Rightarrow L(\theta)$ is maximized at $x_{(n)}$

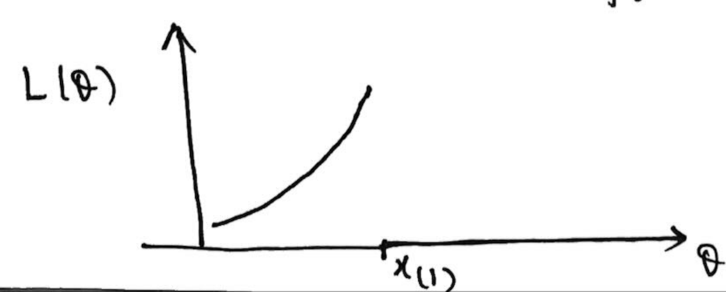
$$\Rightarrow \quad \hat{\theta}_{MLE} = X_{(n)}$$

(5) $X_1, \ldots, X_n$ r.s. from expo with location parameter θ

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & o/w \end{cases}$$

$$L(\theta) = e^{-\Sigma x_i} e^{n\theta} I(\theta, x_{(1)})$$

Note that $L(\theta)$ is not differentiable at $x_{(1)}$



$$L(\theta) = 0 \quad \forall \ \theta > x_{(1)}$$
$$\quad > 0 \ \& \ \uparrow \quad \forall \ \theta \le x_{(1)}$$

$\Rightarrow L(\theta)$ is maximised at $x_{(1)}$

$$\Rightarrow \hat{\theta}_{MLE} = X_{(1)}$$

<u>Remark</u>: As the examples show, MLE need not be unbiased