→ $C[a,b]$ is separable.

Consider $p: [a,b] \to \mathbb{R}$ defined as $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_i \in \mathbb{R}$.

If (P_n) is a seq. of polynomials s.t. $P_n \to f$ uniformly, then f is cts. on $[a,b]$.

Q. Is the converse true? That is, for $f: [a,b] \to \mathbb{R}$ cts.,
does $\exists$ a seq. of polynomials P_n s.t. $P_n \to f$ unif.

A. YES! Weierstrass Approximation Thm. (Intuitive Idea: Carothers Thm. 11.2)

→ Weierstrass Approximation Thm:
$f: [a,b] \to \mathbb{R}$ cts. and $\varepsilon > 0$.
Then $\exists$ a polynomial p with real coefficients s.t. $|f(x) - p(x)| < \varepsilon$ $\forall$ $x \in [a,b]$.

Pf: claim: Suffices to prove the thm. for $[0,1]$.
Pf. of the claim: If $a = b$, take $p(x) := f(a)$. Then $|f(a) - f(a)| = 0 < \varepsilon$.
Assume $a < b$.
Then $\varphi: [0,1] \to [a,b]$ defined as $\varphi(t) = (b-a)t + a$. is cts.

And, $g: [0,1] \to \mathbb{R}$ defined by $g(t) := f(\varphi(t))$ is cts.

Suppose we have the Weierstrass Approx. thm. for $[0,1]$. Then for $\varepsilon > 0$, $\exists$ $p': [0,1] \to \mathbb{R}$
s.t. $|g(t) - p'(t)| < \varepsilon$ $\forall$ $t \in [0,1]$.
Let $x = (b-a)t + a$. Then $t = (x-a)/(b-a)$

Hence, $|f(x) - \underbrace{p'((x-a)/(b-a))}_{= p(x)}| < \varepsilon$. for all $x \in [a,b]$. This proves the claim.

For each n: define the Berstein polynomial

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

- $$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \underset{\substack{\uparrow \\ \text{(Binomial thm.)}}}{=} [x+(1-x)]^n = 1 \quad \text{— (1)}$$

- Differentiate (1) w.r.t. x and then multiply throughout by $x(1-x)$:

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (k-nx) = 0. \quad \text{— (2)}$$

- Differentiate (2) by considering $x^k(1-x)^{n-k}$ as one of the factors in the product rule, one obtains:

$$\sum_{k=0}^{n} \binom{n}{k} \left[-nx^k(1-x)^{n-k} + x^{k-1}(1-x)^{n-k-1}(k-nx)^2\right] = 0 \quad \text{— (3)}$$

Apply (1) to (3): $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{k-1} (1-x)^{n-k-1} (k-nx)^2 = n$. Multiply this by $x(1-x)$ and then divide both sides by n^2;

- One obtains: $$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left(x-\frac{k}{n}\right)^2 = \frac{x(1-x)}{n} \quad \text{— (4)}$$

claim: $B_n(x) \to f(x)$ uniformly.

Pf. Using (1):

$$f(x) - B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left[f(x) - f\left(\frac{k}{n}\right)\right],$$

Taking absolute value both sides and using the fact that f is unif. cts. on $[0,1]$, for $\varepsilon>0$, $\exists\ \delta>0$ s.t. $\left|x-\frac{k}{n}\right| < \delta \Rightarrow \left|f(x)-f\left(\frac{k}{n}\right)\right| < \varepsilon/2$, for suff. large n.

Split the R.H.S. into two parts $\sum$ and $\sum'$ s.t. $\sum$ is the sum of those terms for which $|x-\frac{k}{n}|<\delta$ and $\sum'$ is the sum of the remaining terms.

(HW) $\sum < \varepsilon/2$. Consider $\sum'$: Since f is bdd., $|f(x)| \leq K$ for some $K>0$, $\forall x \in [0,1]$.

So, $\sum' \leq 2K \sum_k \binom{n}{k} x^k (1-x)^{n-k}$ where k is over all terms for which $|x-\frac{k}{n}| \geq \delta$.

$\{k : |x-\frac{k}{n}| \geq \delta\} =: S$

The identity (4) implies that $\delta^2 \sum_{k \in S} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{x(1-x)}{n}$.

$$\Rightarrow \sum_{k \in S} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{x(1-x)}{n \cdot \delta^2}.$$

Since $x(1-x) \leq 1/4$ $\forall x \in [0,1]$, $\sum_{k \in S} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4\delta^2 \cdot n}$.

Choose n suff. large s.t. $\frac{1}{4\delta^2 \cdot n} < \frac{\varepsilon}{4K}$. . . . . .

Q. Show that $C[0,1]$ is separable.

Remark: X: compact metric space.

$C(X) = \{ f : X \to \mathbb{R}$ cts. functions $\}$ w.r.t sup-norm.

Stone-Weierstrass Thm: (Real-valued): X: compact metric space

Let A be a subalgebra of $C(X)$.

If A separates pts. in X and vanishes at no pt. in X,

Then A is dense in $C(X)$.