2D Transformation

Transformation

- Mapping T :

$$T : R^2 \rightarrow R^2$$
$$P(x, y) \mapsto Q(x', y')$$

- Functions (parametric equation):

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

- Matrix manipulation by *M*:

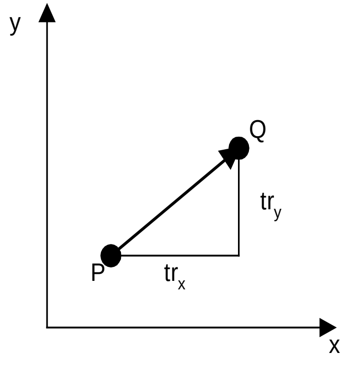
$$\begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} M + T, M \in R^{2\times 2}, T \in R^{1\times 2}$$

- Basic transformations: Translation, Rotation, and Scaling

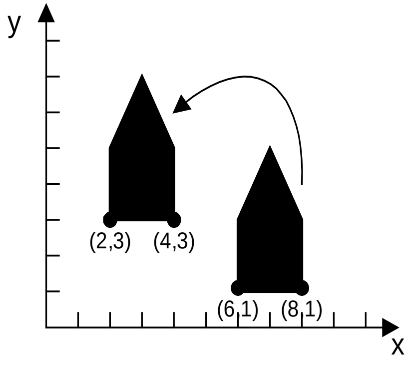
Affine Transformation

- Linear transformation :

$$\begin{cases} x' = ax + cy + e \\ y' = bx + dy + f \end{cases} \quad , a, b, c, d, e, f \in R, ad - bc \neq 0$$



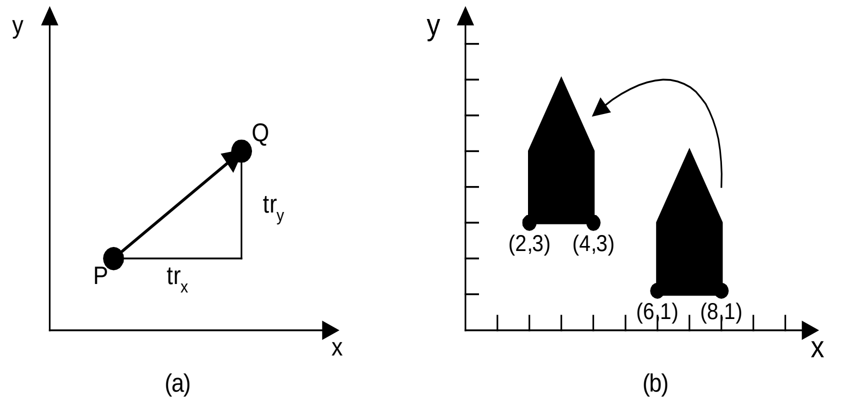
(a)



(b)

Translation

- Repositioning one object along a straight-line path from one coordinate location to another.



- Parametric equation:

$$\begin{cases} x' = x + tr_x \\ y' = y + tr_y \end{cases}$$

- Vector formulation:

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} tr_x & tr_y \end{bmatrix}$$

Translation

- A rigid-body transformation: it does not cause any deformation.
- Line segment is translated by applying translation to the line endpoints and redraw the line.
- Polygon is translated by applying translation to each vertex and redraw lines between pair of vertices.
- Circle ?

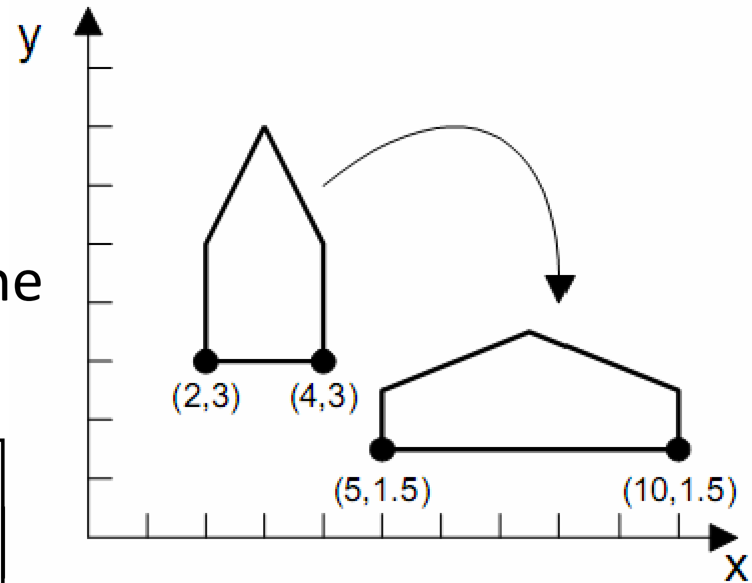
Scaling

- Scaling transformation changes the size of the object.

$$\begin{cases} x' = s_x . x \\ y' = s_y . y \end{cases}$$

S_x and S_y are scaling factor in the x and y directions, respectively.

$$\begin{bmatrix} x' & y' \end{bmatrix} = [x \quad y] \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$



Scaling

- $S_x = S_y$: Uniform scaling
- Reducing the size:

$$0 \le S_x \le 1, 0 \le S_y \le 1$$

- Enlargement:

$$|S_x|, |S_y| \ge 1$$

- Fixed point for scaling transformation

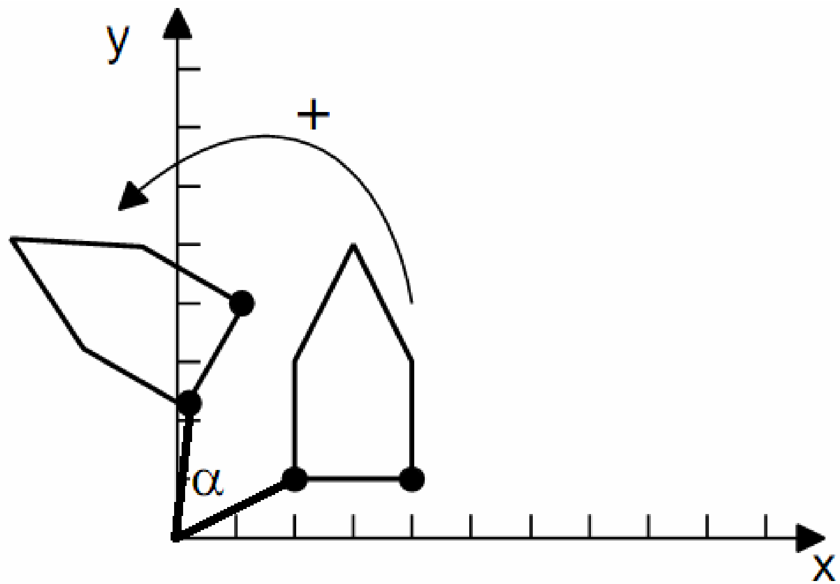
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} + \begin{bmatrix} x_f & y_f \end{bmatrix} \begin{bmatrix} 1-S_x & 0 \\ 0 & 1-S_y \end{bmatrix}$$

where (x_f, y_f) is the fixed point in the scaling transformation.

Rotation

- Repositioning objects along a circular path in the xy plane.
- We need to specify the rotation angle α and the rotation point (pivot point). The pivot point is usually located at the origin of the coordinate.
- Parametric equation:

$$\begin{cases} x' = \cos\alpha . x - \sin\alpha . y \\ y' = \sin\alpha . x + \cos\alpha . y \end{cases}$$



Rotation

- Matrix formulation

$$\begin{bmatrix} x' & y' \end{bmatrix} = [x \quad y] \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

- Pivot point is not the origin of coordinate

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x_r & y_r \end{bmatrix} + [x - x_r \quad y - y_r] \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

where (x_r, y_r) is the pivot point.

- Rotation is also a rigid-body transformation (without deformation)

Homogenous Coordinate

- We need to perform all basic transformation by matrix manipulation.
- Each position in the Cartesian coordinate (x,y) is mapped to its homogenous coordinate (x_h,y_h ,h), where

$$x = \frac{x_h}{h}, \qquad y = \frac{y_h}{h}$$

- To simplify for calculation, we can choose h=1.
- In general, the homogenous coordinate is presented by (*xh,yh,zh*), where $h \in R, h \neq 0$

Matrix formulation

- Translation

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tr_x & tr_y & 1 \end{pmatrix}$$

or $Q = P.M_T\left(tr_x, tr_y\right)$ and $M_T\left(tr_x, tr_y\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tr_x & tr_y & 1 \end{pmatrix}$

where *P* and *Q* are row vectors.

Matrix formulation

- Scaling

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} . \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or $Q = P.M_S(s_x, s_y)$ and $M_S(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$

where *P* and *Q* are row vectors.

Matrix formulation

- Rotation

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} . \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or $Q = P.M_R(\alpha)$ and $M_R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

where *P* and *Q* are row vectors.

Question

- Why do we need the homogenous coordinate and matrix formulation?

Composite Transformation

We can set up a matrix for any sequence of transformation by using matrix representation

- A sequence of Translation

$$Q = \{P.M_{T1}(tr_{x1}, tr_{y1})\}.M_{T2}(tr_{x2}, tr_{y2}) = P.\{M_{T1}(tr_{x1}, tr_{y1}).M_{T2}(tr_{x2}, tr_{y2})\}$$

$$P(x, y) \xrightarrow{M_1} P_1 \xrightarrow{M_2} \ldots \xrightarrow{M_n} Q(x', y')$$

$$P(x, y) \xrightarrow{\quad M_1 M_2 \ldots . M_n \quad} Q(x', y')$$

where *P* and *Q* are row vectors.

Translation

- Example:

$$M_{T1}(tr_{x1}, tr_{y1}).M_{T2}(tr_{x2}, tr_{y2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tr_{x1} & tr_{y1} & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tr_{x2} & tr_{y2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tr_{x1} + tr_{x2} & tr_{y1} + tr_{y2} & 1 \end{pmatrix}$$

$$M_{T1}(tr_{x1}, tr_{y1}).M_{T2}(tr_{x2}, tr_{y2}) = M_T(tr_{x1} + tr_{x2}, tr_{y1} + tr_{y2})$$

A sequence of Translation = Translation

Scaling

- A sequence of Scaling

$$Q = \{P.M_{S1}(s_{x1}, s_{y1})\}.M_{S2}(s_{x2}, s_{y2}) = P.\{M_{S1}(s_{x1}, s_{y1}).M_{S2}(s_{x2}, s_{y2})\}$$

where *P* and *Q* are row vectors.

$$P(x, y) \xrightarrow{M_{S1}(s_{x1}, s_{y1})} P_t \quad \xrightarrow{M_{S2}(s_{x2}, s_{y2})} Q(x', y')$$

$$M_{S1}(s_{x1}, s_{y1}).M_{S2}(s_{x2}, s_{y2}) = \begin{pmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{S1}(s_{x1}, s_{y1}).M_{S2}(s_{x2}, s_{y2}) = M_S(s_{x1}.s_{x2}, s_{y1}.s_{y2})$$

A sequence of Scaling = Scaling

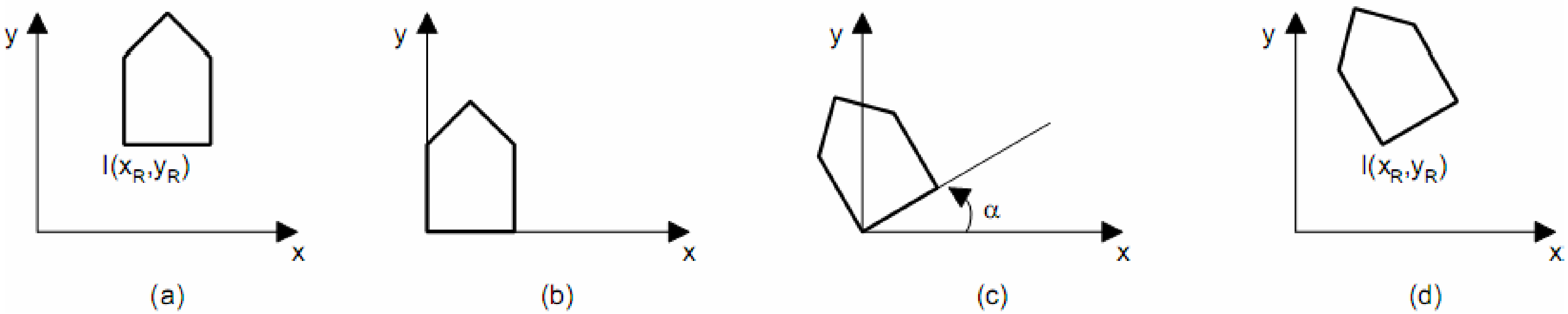
Rotation

- A sequence of Rotation

$$Q = \{P.M_{R1}(\alpha_1)\}.M_{R2}(\alpha_2) = P.\{M_{R1}(\alpha_1).M_{R2}(\alpha_2)\}$$

where P and Q are row vectors.

- For one Rotation M_{R1}



$$P(x, y) \xrightarrow{M_T(-x_R, -y_R)} P_1 \xrightarrow{M_R(\alpha)} P_2 \xrightarrow{M_T(x_R, y_R)} Q(x', y')$$

Rotation

- The pivot point is not the origin

$$M_R(x_R, y_R, \alpha) = M_T(-x_R, -y_R).M_R(\alpha).M_T(x_R, y_R)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_R & -y_R & 1 \end{pmatrix} . \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_R & y_R & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ (1-\cos\alpha)x_R + \sin\alpha . y_R & -\sin\alpha . x_R + (1-\cos\alpha)y_R & 1 \end{pmatrix}$$

Rotation

$$M_{R1}(\alpha_1).M_{R2}(\alpha_2) = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ -\sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha_1 + \alpha_2) & \sin(\alpha_1 + \alpha_2) & 0 \\ -\sin(\alpha_1 + \alpha_2) & \cos(\alpha_1 + \alpha_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{R1}(\alpha_1).M_{R2}(\alpha_2) = M_R(\alpha_1 + \alpha_2)$$

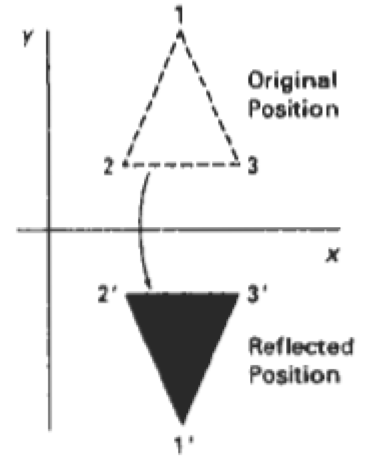
A sequence of Rotation = Rotation

(The pivot point is positioned at the origin)

Reflection

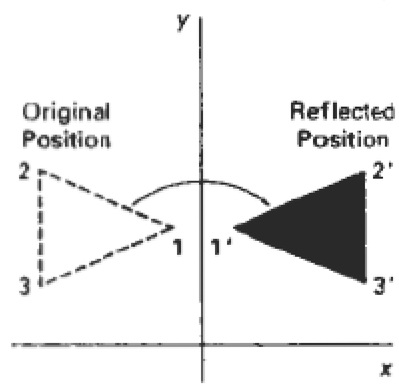
Rotating the object 180^0

$$M_{Rfx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$Q = P.M_{Rfx}$$

$$M_{Rfy} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$Q = P.M_{Rfy}$$

where *P* and *Q* are row vectors.

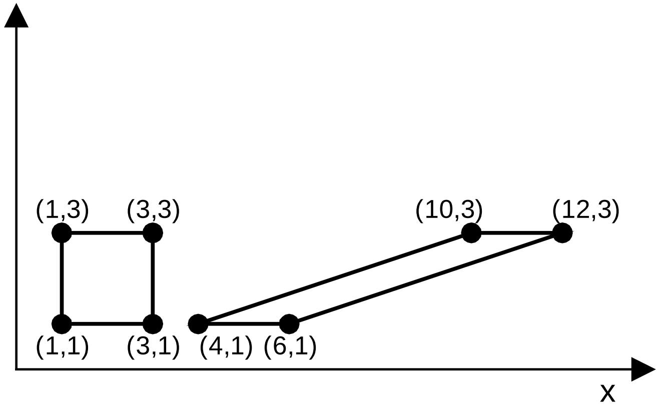
(Computer Graphic: C Version, 2nd Ed., D.Hearn, et.al., p. 201)

Shear

- Generate deformation of shape

- Shear on the x-axis: $Q = P.M_{Shx}$

$$M_{Shx} = \begin{pmatrix} 1 & 0 & 0 \\ sh_{xy} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Shear on the y-axis: $Q = P.M_{Shy}$

$$M_{Shy} = \begin{pmatrix} 1 & sh_{yx} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where *P* and *Q* are row vectors.

Inverse Transformation

- Undo one transformation
- Utilize inverse matrix

$$M = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ cf - de & be - af & 1 \end{pmatrix}$$

Inverse Transformation

$$M_T^{-1}(tr_x, tr_y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -tr_x & -tr_y & 1 \end{pmatrix} = M_T(-tr_x, -tr_y)$$

$$M_S^{-1}(s_x, s_y) = \frac{1}{s_x s_y}\begin{pmatrix} s_y & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_S\left(\frac{1}{s_x}, \frac{1}{s_y}\right)$$

$$M_R^{-1}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_R(-\alpha)$$

References

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[4] D.N.D.Tien, V.Q. Hoang, L. Phong, CG-Course Slide, HCM-University of Science.