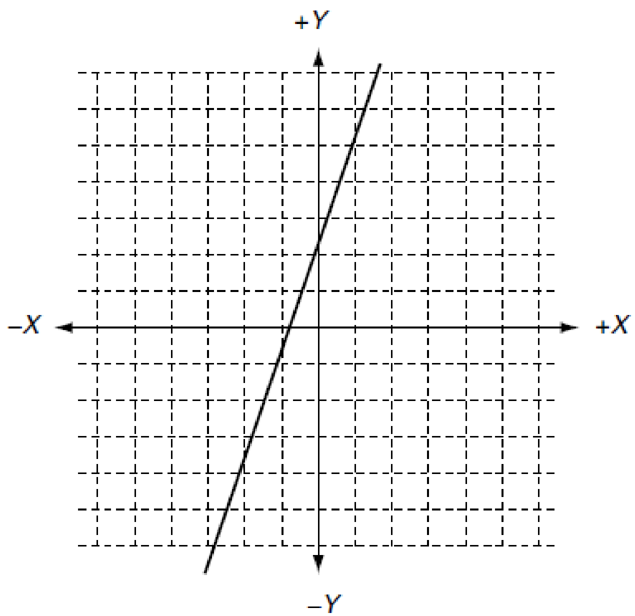
Math review

Coordinate

- Cartesian xy-plane: original point (origin) + basis vectors

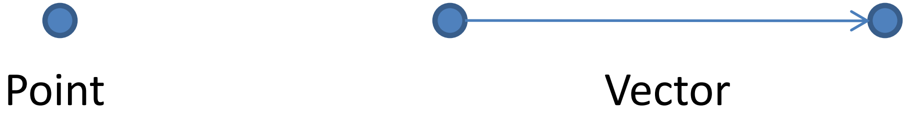


- Function graph: linear, quadratic, cubic, trigonometric

Point and Vector

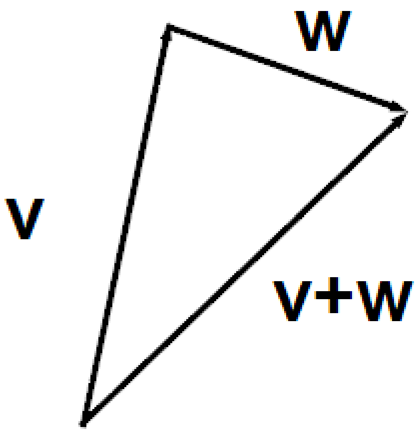
- Point: A pair of numbers in a plane
- Vector:

- A directed line segment.
- A pair of points (Geometrical view).
- A pair of numbers (Algebraic view).

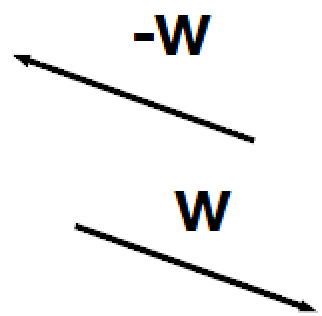


Manipulations (1)

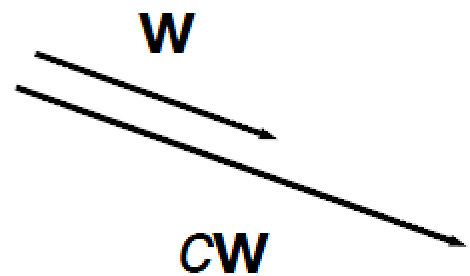
- Geometrical view



Addition



Negation



Multiplication by a scalar

Point - Point = Vector

Point + Vector = Point

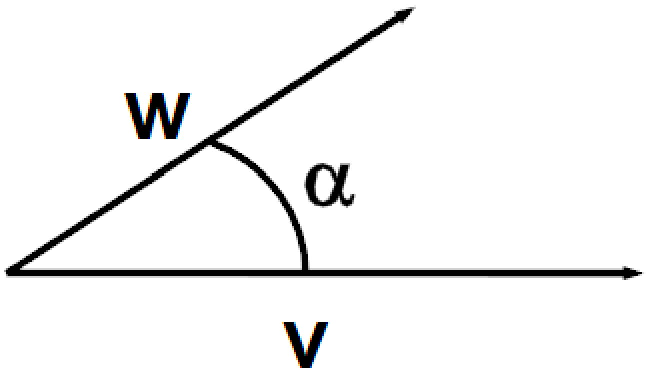
Manipulations (2)

- Algebraic view

$$\mathbf{v} + \mathbf{w} = [\ v_x, v_y\ ] + [\ w_x, w_y\ ] = [v_x + w_x, v_y + w_y]$$
$$-\mathbf{w} = [\ -w_x, -w_y\ ]$$
$$\alpha\, \mathbf{w} = = [\alpha\, w_x, \alpha\, w_y\ ]$$

v- w ? w-v

Dot product

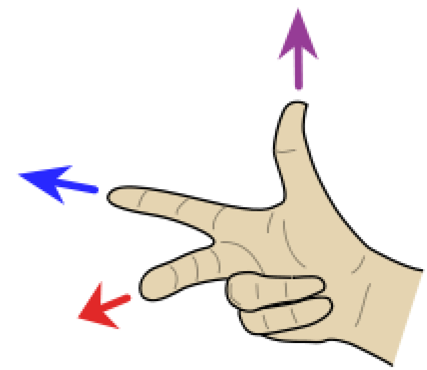


$(w \cdot v) = |w|\ |v| \cos\alpha$

$|v|$ = length of v

$(v \cdot w) = v_x w_x + v_y w_y$

Cross product



$h = v \times w$, $|h| = |v|\,|w| \sin(\alpha)$, $0 \le \alpha \le \pi$

$$(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$$

$$(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$$

$$\mathbf{v} \times \mathbf{w} = - \mathbf{w} \times \mathbf{v}$$

Note

$(v.w)$= scalar

$(v \times w)$ = vector

$h \times (v \times w) = (h.w)v - (h.v)w$

$(h \times k).(v \times w) = (h.v)(k.w) - (h.w)(k.v)$

Matrix

$$B = \begin{bmatrix} \boxed{1 \quad 0 \quad 0} \\ 0 \quad 1 \quad 0 \end{bmatrix}$$

Rectangular matrix

$$D = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

Square matrix

$$A = \begin{bmatrix} 9 & 0 & 2j \\ 0 & 3 & -1+j \\ 0 & 0 & 5 \end{bmatrix}$$

Upper triangular matrix

$$B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3+2j & 0 \\ 2-j & 4 & 5 \end{bmatrix}$$

Lower triangular matrix

Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrix

$$\vec{q} = A\vec{p} \Rightarrow \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \\ \vdots \\ q_N \end{bmatrix} = \begin{array}{r} a_1 \\ \\ \\ a_i \\ \\ \\ \end{array} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,i} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,i} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,i} & \cdots & a_{i,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,i} & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_i \\ \vdots \\ p_N \end{bmatrix}$$

Matrix

$$q_i = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,i} & \cdots & a_{i,N} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_i \\ \vdots \\ p_N \end{bmatrix}$$

$$q_i = (a_i . p)$$

Given $\vec{v} = (0, 4, 0)$, $\vec{q} = (0, 0, 1, 0)$

$$A = \vec{v}\vec{q}^T = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix

$A+B=B+A$

$A+(B+C)=(A+B)+C$

$a(A+B)=aA+aB$

$(a+b)A=aA+bA$

$usually\ \ AB \neq BA$

$A(BC)=(AB)C$

$a(AB)=(aA)B=A(aB)$

$A(B+C)=AB+AC$

$(A+B)^{\mathrm{T}}=A^{\mathrm{T}}+B^{\mathrm{T}}$

$(AB)^{\mathrm{T}}=B^{\mathrm{T}}A^{\mathrm{T}}$

A^{T}: transverse matrix of A

A^{-1}: inverse matrix of A

$A^{-1}A = AA^{-1}$ = Identity matrix

(A must be a square matrix)

$A^{\mathrm{T}}=A$: symmetric matrix

$A^{\mathrm{T}}=A^{-1}$: orthogonal matrix

$$A=\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \xRightarrow{?} A^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{?}{=} 2x_1^2+3x_2^2$$

Determinant

A must be a square matrix.

$$A = [a_{11}] \Rightarrow \det(A) = \left| a_{11} \right| = a_{11}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \det(A)$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{33} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant vs Cross product

$$\vec{v} = (v_1, v_2, v_3) \qquad \vec{r} = (r_1, r_2, r_3)$$

$$\hat{i} = (1,0,0),\ \hat{j} = (0,1,0),\ \text{and}\ \hat{k} = (0,0,1)$$

$$v \times r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

$$= (v_2 r_3 - v_3 r_2)\hat{i} + (v_3 r_1 - v_1 r_3)\hat{j} + (v_1 r_2 - v_2 r_1)\hat{k}$$

Determinant

(i) $\det(AB) = \det(A)\det(B)$

(ii) If A is an upper or lower triangular matrix, then $\det(A) = a_{1,1}a_{2,2}\cdots a_{N,N}$

(iii) $\det(A^T) = \det(A)$

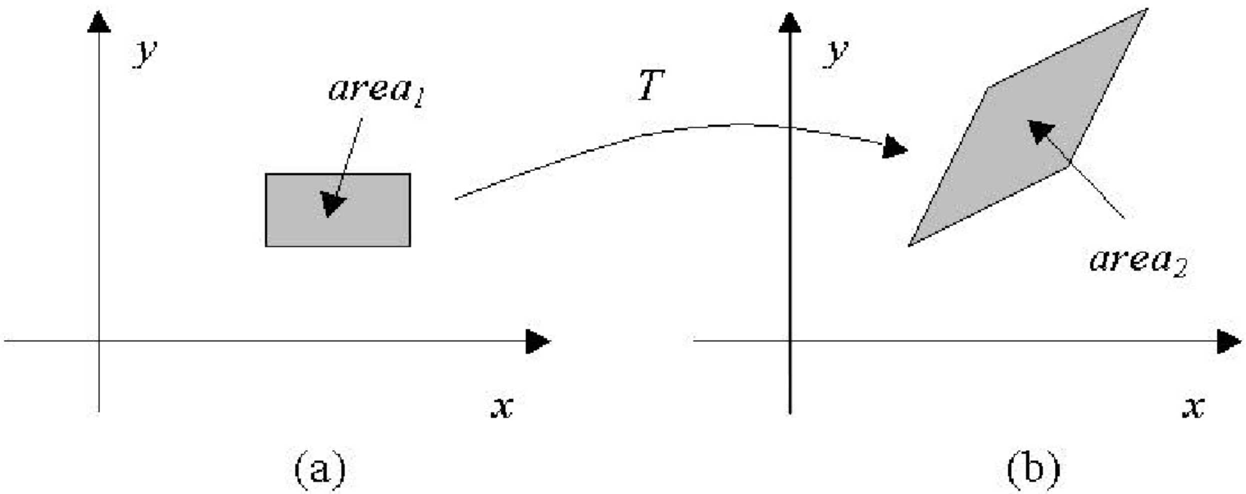
(iv) $\det(aA) = a^N \det(A)$

(v) The vectors defined by each row of A are linearly dependent if and only $\det(A) = 0$

(vi) If all the elements in one of the rows (or columns) of A are zero, then $\det(A) = 0$

(vii) If A is orthogonal (unitary) then $|\det(A)| = 1$

Determinant



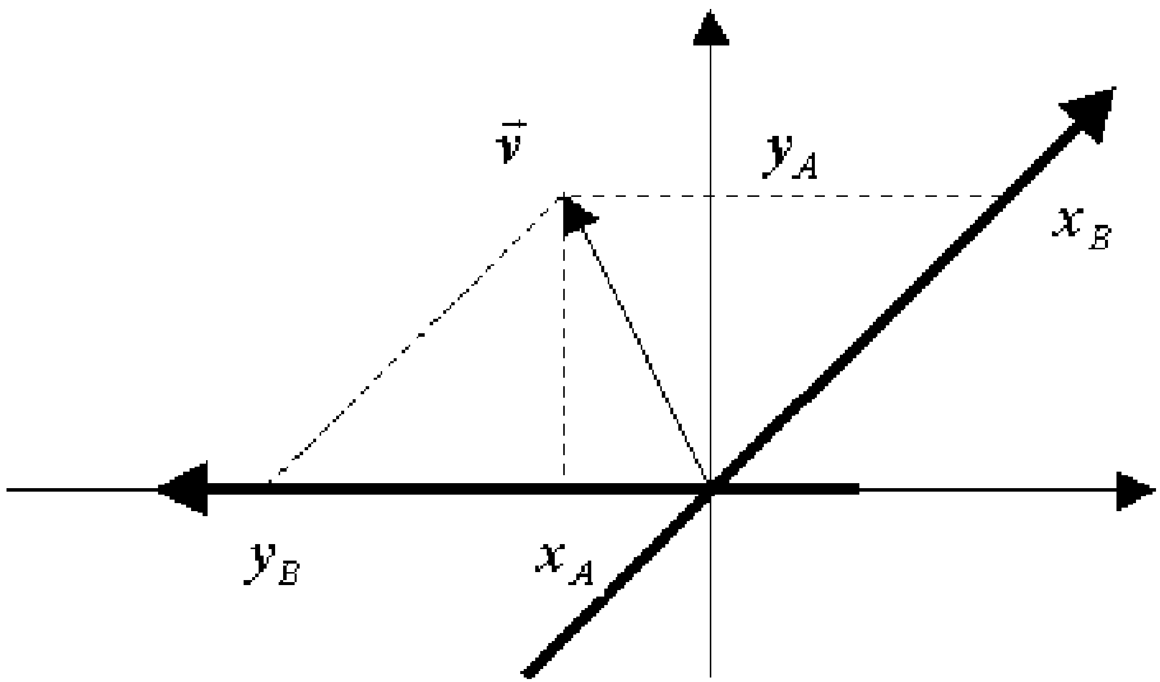
The transformation T, defined by the square matrix A, taking a rectangle in (a) into a polygon in (b)

$$\det(A) = \frac{area_2}{area_1}$$

Rotation matrix: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ⇒ det(R) = ?

Question

Find the coordinates of the vector $\vec{v}_A = (-1, 2)^T$ (represented with respect to the canonical basis) in the new basis defined by $B = \left\{ \vec{b}_1 = (1, 1)^T ; \vec{b}_2 = (-2, 0)^T \right\}$.



Hint

$$\vec{v} = \mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_N \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_i & \dots & a_N \end{bmatrix}^T =$$

$$= \left(a_1, a_2, \dots, a_i, \dots, a_N\right) = a_1\hat{e}_1 + a_2\hat{e}_2 + \dots + a_N\hat{e}_N$$

Answer

$$C = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$$

$$\vec{v}_B = C^{-1}\vec{v}_A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

References

[1] E. Tosun, CG-Course Slide, New York University.

[2] L.D.F. Costa, R.M. Cesar Jr, Shape Analysis and Classification: Theory and Practice, CRC. Press, 2000.