3D Transformation

Transformation

Mapping T

$$\mathrm{T} : \mathrm{R}^3 \rightarrow \mathrm{R}^3$$
$$\mathrm{P} \mapsto \mathrm{P}'$$

Function

$$P' = T(P)$$

Parametric equation

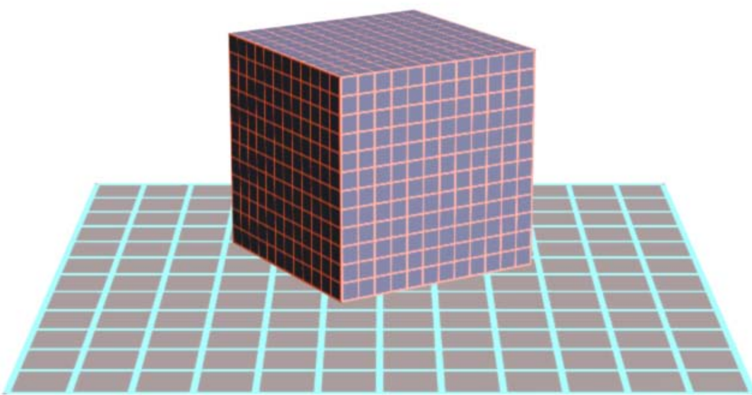
$$\begin{cases} P_{\mathrm{x}}' = \mathrm{T}_x(P_x, P_y, P_z) \\ P_y' = T_y(P_x, P_y, P_z) \\ P_z' = T_z(P_x, P_y, P_z) \end{cases}$$

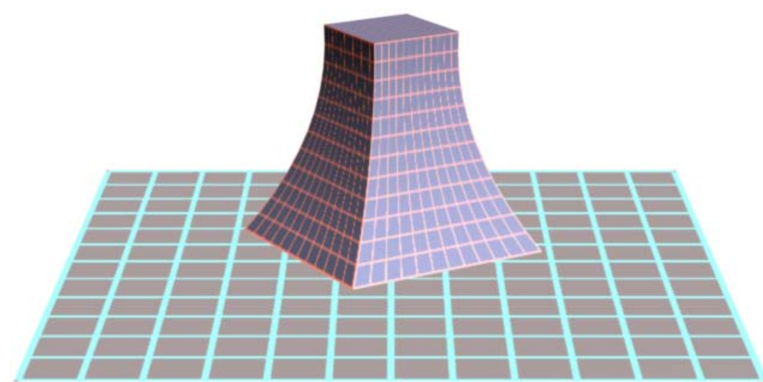
Tapering (Non-constant scaling)

Parametric equation:

$$\begin{cases} x' = T_x(x, y, z) = rx \\ y' = T_y(x, y, z) = ry \\ z' = T_z(x, y, z) = z \end{cases}$$

by r = f(z) = ?



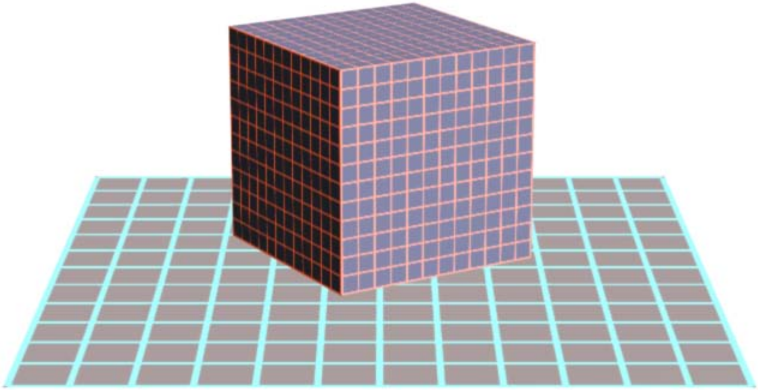


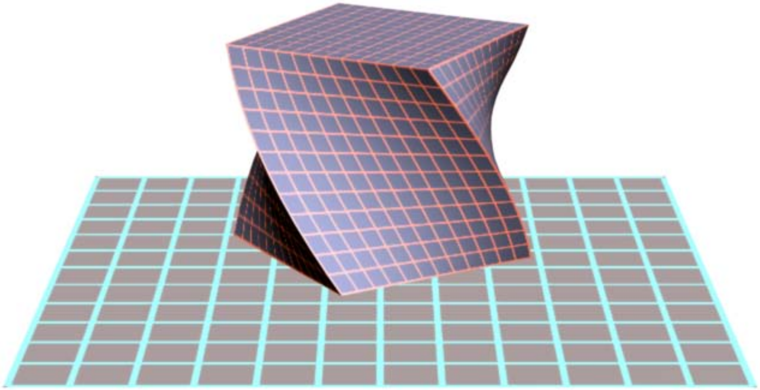
Twist

Parametric equation:

$$\begin{cases} x' = T_x(x,y,z) = x\cos\theta - y\sin\theta \\ y' = T_y(x,y,z) = x\sin\theta + y\cos\theta \\ z' = T_z(x,y,z) = z \end{cases}$$

by $\theta = f(z) = ?$





Affine Transformation

Parametric equation :

$$\begin{cases} P_x^{'} = m_{00}P_x + m_{10}P_y + m_{20}P_z + m_{30} \\ P_y^{'} = m_{01}P_x + m_{11}P_y + m_{21}P_z + m_{31} \\ P_z^{'} = m_{02}P_x + m_{12}P_y + m_{22}P_z + m_{32} \end{cases}$$

Matrix formulation :

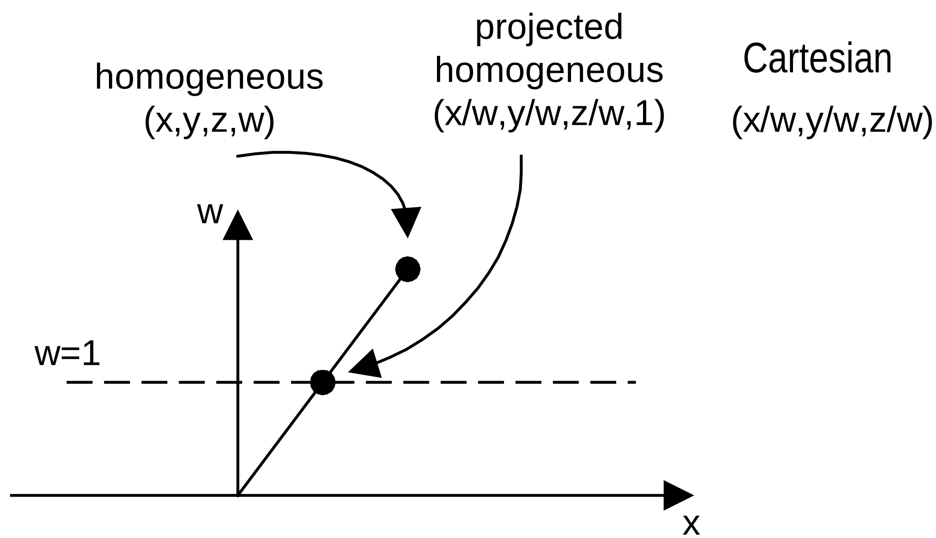
$$\begin{pmatrix} P_x^{'} & P_y^{'} & P_z^{'} & 1 \end{pmatrix} = \begin{pmatrix} P_x & P_y & P_z & 1 \end{pmatrix} \begin{pmatrix} \mathrm{m}_{00} & m_{01} & m_{02} & 0 \\ m_{10} & m_{11} & m_{12} & 0 \\ m_{20} & m_{21} & m_{22} & 0 \\ m_{30} & m_{31} & m_{32} & 1 \end{pmatrix}$$

Affine Transformation ‘s Properties

- The collinearity relation between points; i.e., three points which lie on a line continue to be collinear after the transformation
- Ratios of distances along a line; i.e., for distinct collinear points p_1, p_2, p_3, the ratio $| p_2 - p_1 | / | p_3 - p_2 |$ is preserved
- An affine transformation is composed of linear transformations (rotation, scaling or shear) and a translation (or "shift").

(by Eric W. Weisstein, Affine Transformation, MathWorld)

Homogenous Coordinate



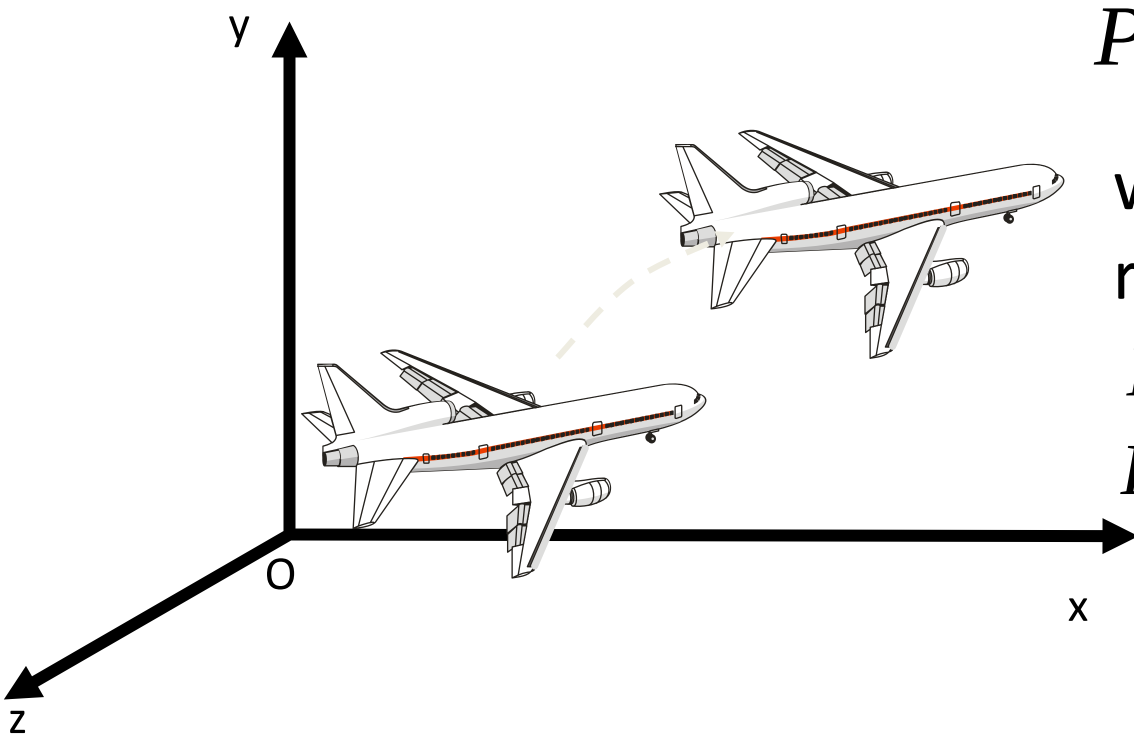
$$(x,y,z)_{\text{Cartesian}} \Rightarrow (x,y,z,1)_{\text{Homogeneous}}$$

$$(x,y,z,w)_{\text{Homogeneous}} \Rightarrow (x/w,y/w,z/w)_{\text{Cartesian}} \text{ with } w \neq 0$$

Translation

$$\mathrm{Tr}(\mathrm{Tr}_x, \mathrm{Tr}_y, \mathrm{Tr}_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{Tr}_x & \mathrm{Tr}_y & \mathrm{Tr}_z & 1 \end{bmatrix} \quad \text{or} \quad \left| \begin{array}{l} P_x' = P_x + Tr_x \\ P_y' = P_y + Tr_y \\ P_z' = P_z + Tr_z \end{array} \right.$$

$$P' = P.Tr\left(Tr_x, Tr_y, Tr_z\right)$$



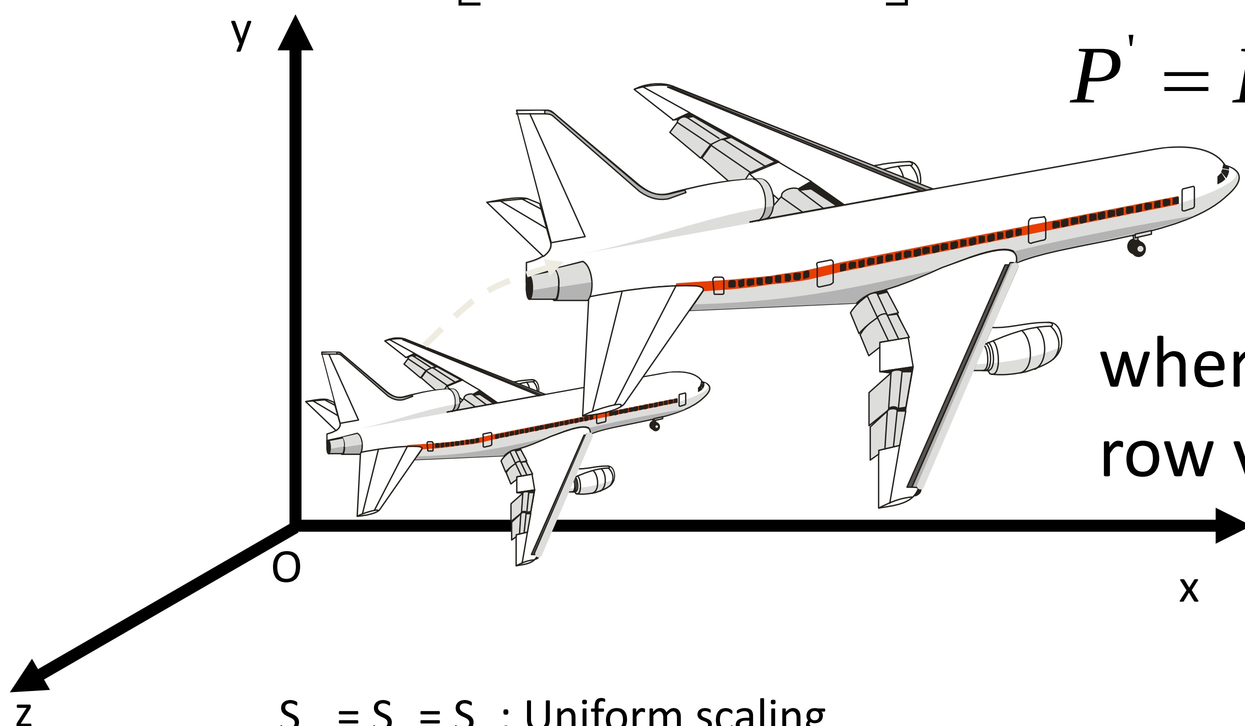
where *P* and *P′* are row vectors.

$$P = [P_x \quad P_y \quad P_z \quad 1]$$
$$P' = [P_x' \quad P_y' \quad P_z' \quad 1]$$

Scaling

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \left| \begin{aligned} P_x^{'} &= s_x P_x \\ P_y^{'} &= s_y P_y \\ P_z^{'} &= s_z P_z \end{aligned} \right.$$

$$P^{'} = P.S\left(S_x, S_y, S_z\right)$$



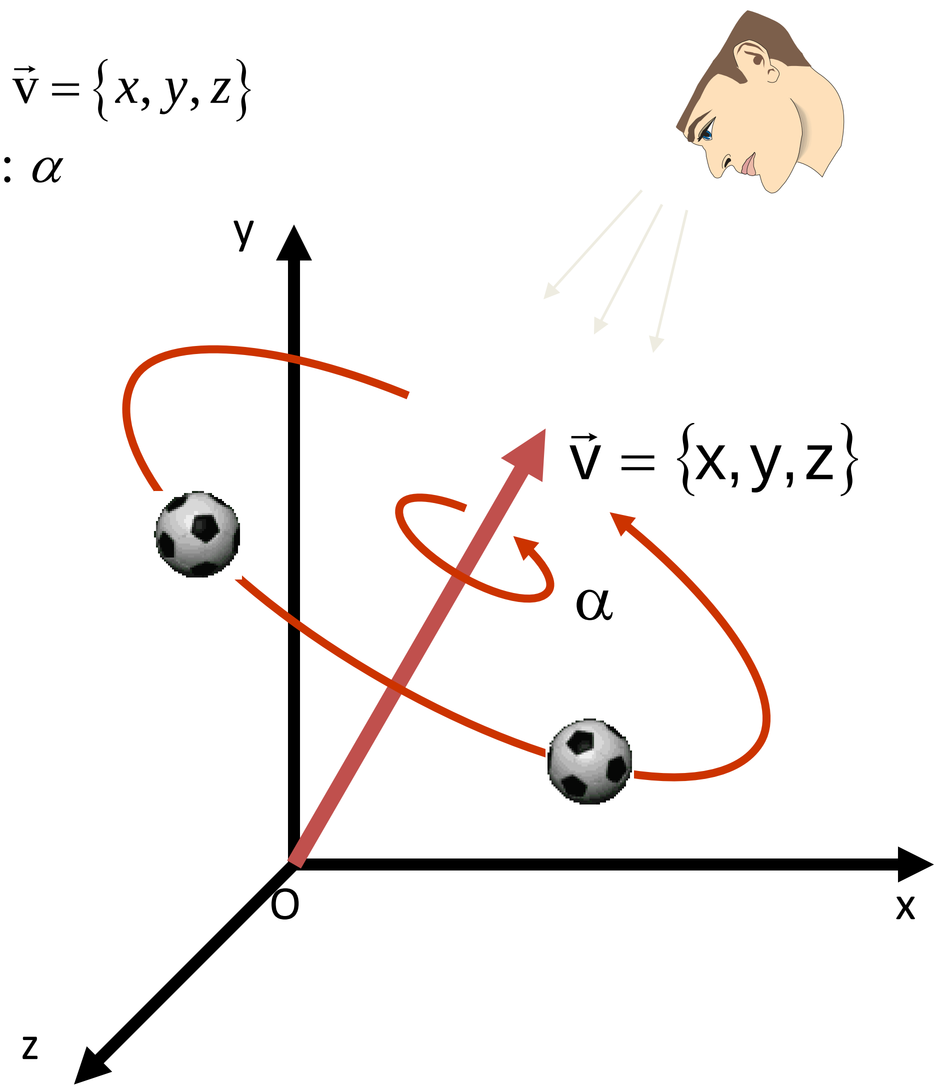
where P and P' are row vectors.

$S_x = S_y = S_z$: Uniform scaling

Rotation

Specified parameters :

$$\begin{cases} \text{Rotation axis } : \vec{\mathrm{v}} = \{x, y, z\} \\ \text{Rotation angle} : \alpha \end{cases}$$



Oz-Axis Rotation

Specified parameter:

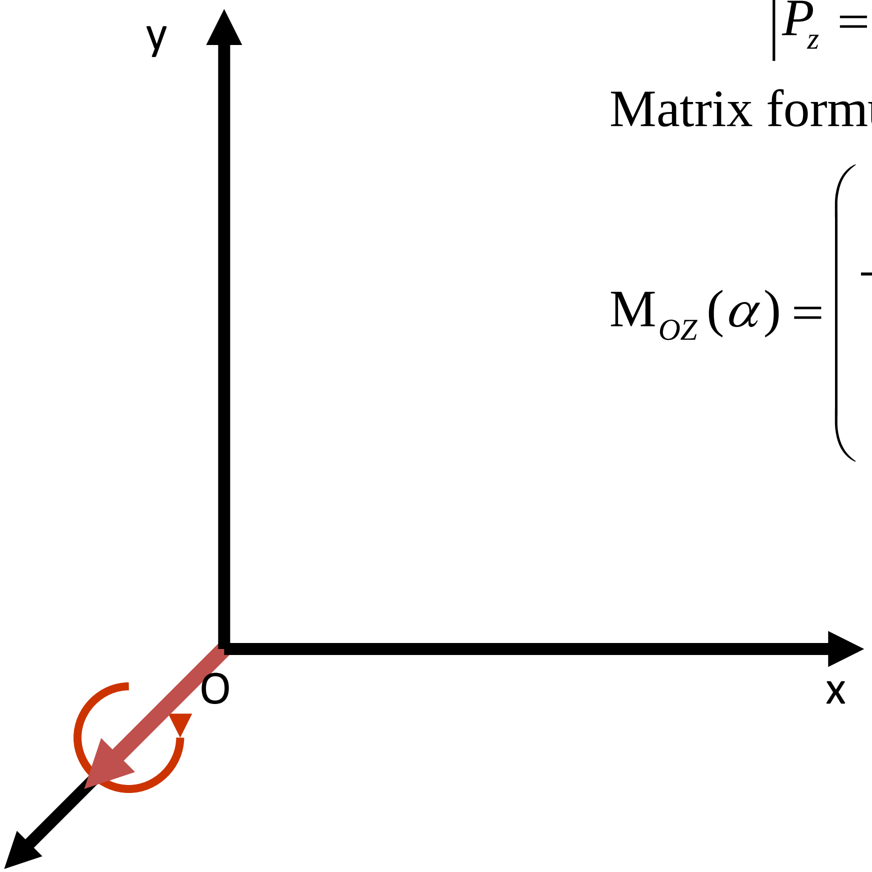
$$\begin{cases} \text{Rotation axis} \ : \vec{v} = \{0,0,1\} \\ \text{Rotation angle} : \alpha \end{cases}$$

Parametric equation:

$$\begin{cases} P_x^{'} = \cos\alpha P_x - \sin\alpha P_y \\ P_y^{'} = \sin\alpha P_x + \cos\alpha P_y \\ P_z^{'} = P_z \end{cases}$$

Matrix formulation:

$$\mathrm{M}_{OZ}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$P^{'} = P.M_{OZ}(\alpha)$$

$$P = [P_x \quad P_y \quad P_z \quad 1]$$

$$P^{'} = [P_x^{'} \quad P_y^{'} \quad P_z^{'} \quad 1]$$

Ox-Axis Rotation

Specified parameters:

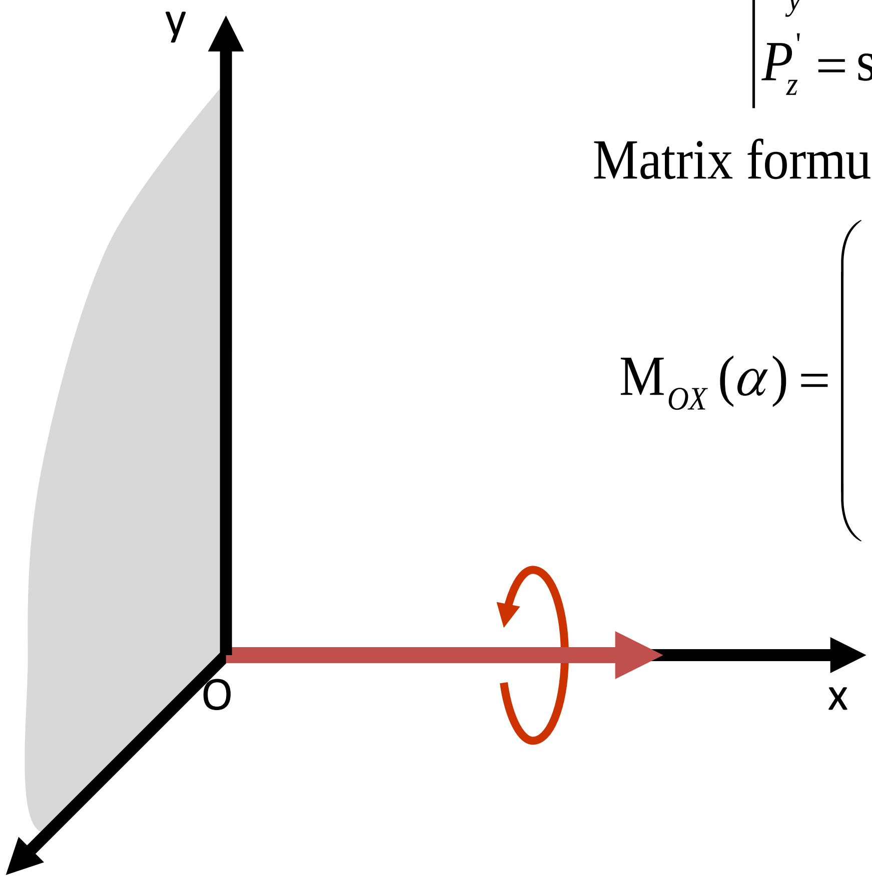
$$\left|\begin{array}{l}\text{Axis rotation } : \vec{v} = \{1,0,0\} \\ \text{Axis angle} : \alpha\end{array}\right.$$

Parametric equation:

$$\left|\begin{array}{l}P_x^{'} = P_x \\ P_y^{'} = \cos\alpha P_y - \sin\alpha P_z \\ P_z^{'} = \sin\alpha P_y + \cos\alpha P_z\end{array}\right.$$

Matrix formulation:

$$\mathrm{M}_{OX}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$P^{'} = P.M_{OX}(\alpha)$$

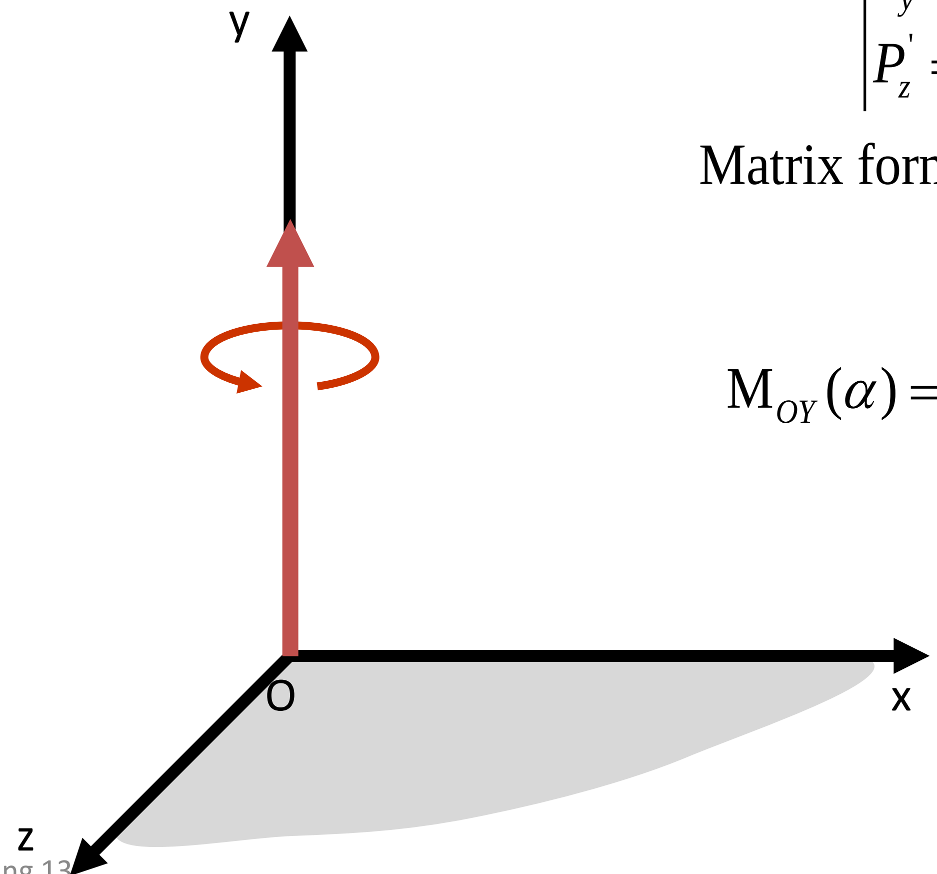
$$P = [P_x \quad P_y \quad P_z \quad 1]$$

$$P^{'} = [P_x^{'} \quad P_y^{'} \quad P_z^{'} \quad 1]$$

Oy-Axis Rotation

Specified parameters :

$$\left| \begin{array}{l} \text{Axis rotation } : \vec{\mathrm{v}} = \{0,1,0\} \\ \text{Axis angle} : \alpha \end{array} \right.$$



Parametric equation:

$$\left| \begin{array}{l} P_x^{'} = \cos\alpha P_x + \sin\alpha P_z \\ P_y^{'} = P_y \\ P_z^{'} = -\sin\alpha P_x + \cos\alpha P_z \end{array} \right.$$

Matrix formulation:

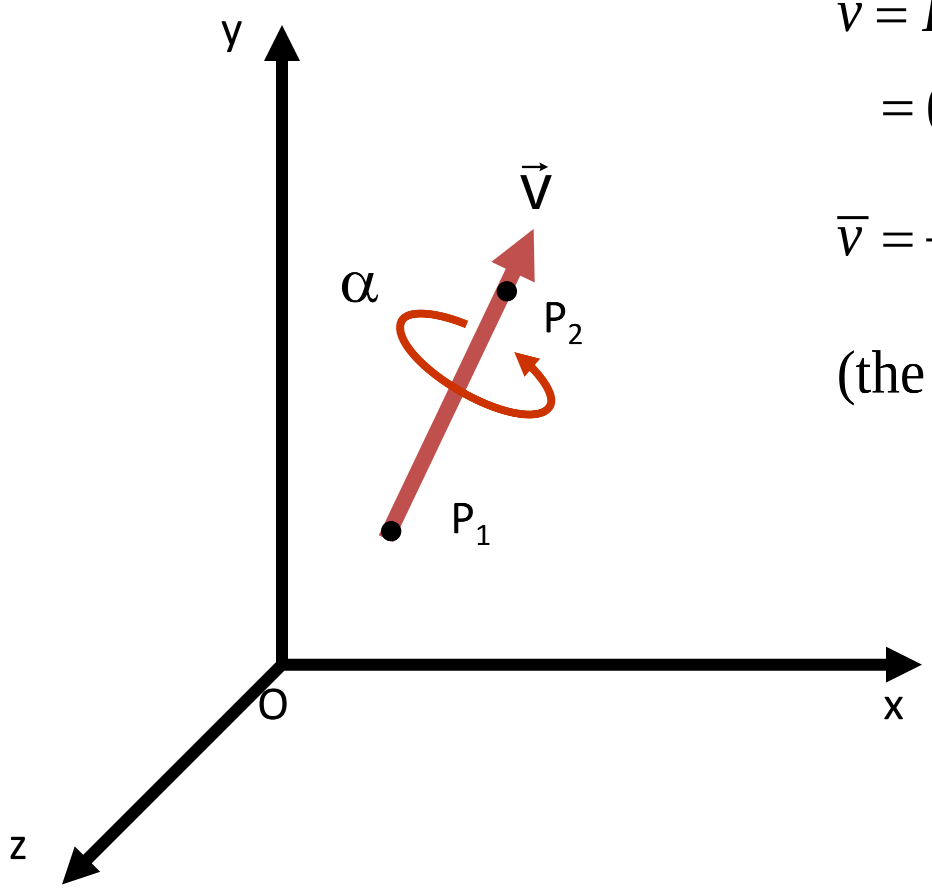
$$\mathrm{M}_{OY}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{'} = P.M_{OY}(\alpha)$$

$$P = [P_x \quad P_y \quad P_z \quad 1]$$

$$P^{'} = [P_x^{'} \quad P_y^{'} \quad P_z^{'} \quad 1]$$

General Rotation

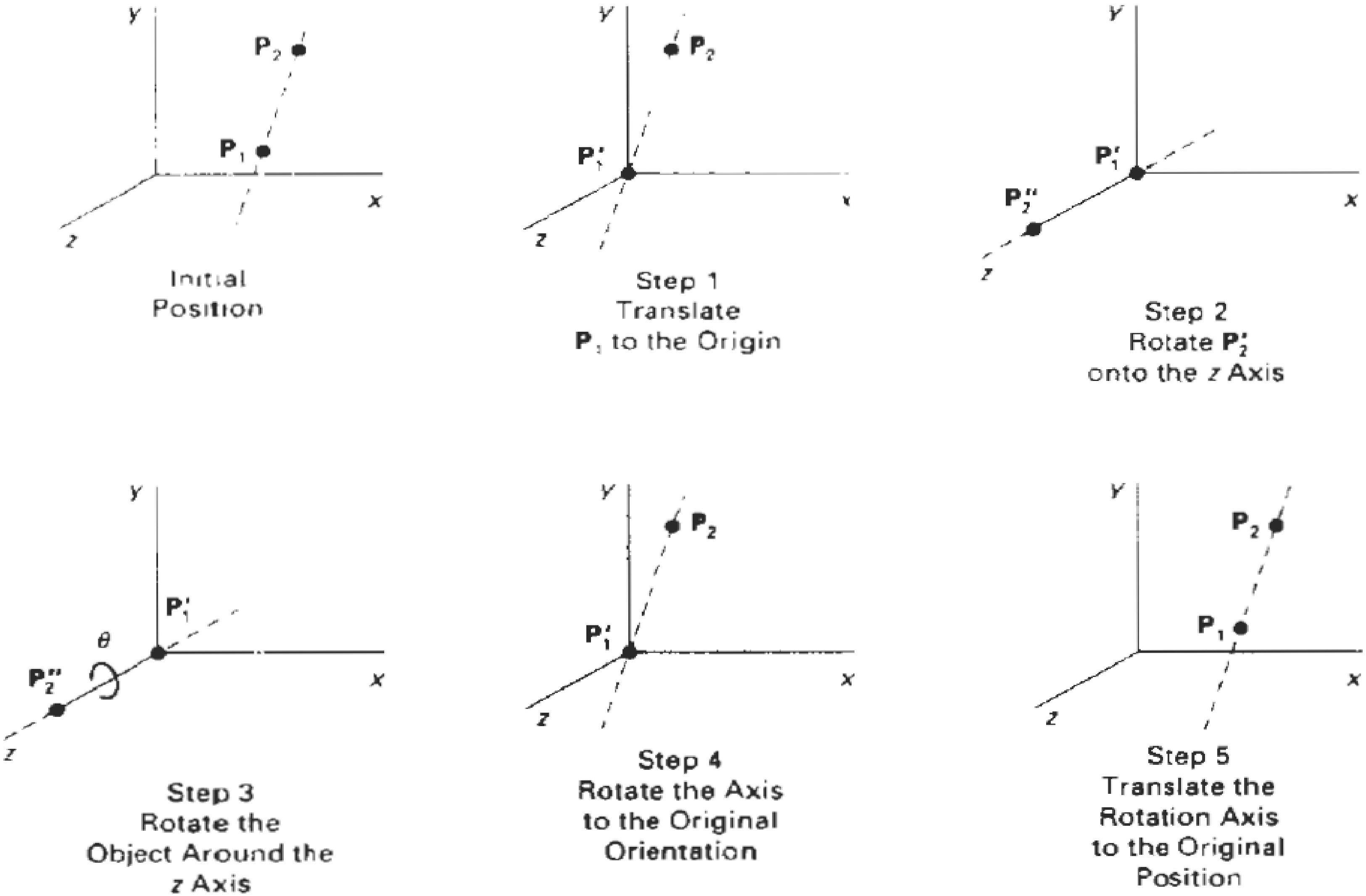


$$v = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\overline{v} = \frac{v}{\| v \|} = (a, b, c),$$

(the unit vector along the rotation axis)

General Rotation



Questions

- Calculate the composite transformation for the general rotation.

$$R(\theta) = T^{-1}.R_x^{-1}(\alpha).R_y^{-1}(\beta).R_z(\theta).R_y(\beta).R_x(\alpha).T$$

where $T = ?,\ \alpha = ?,$ and $\beta = ?$

Reflection Transformation

$$Mr(x) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plane yOz

$$Mr(y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plane zOx

$$Mr(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Plane xOy

$$M_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X-Axis

$$M_y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Y-Axis

$$M_z = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Z-Axis

Shear

$$Sh = \begin{bmatrix} 1 & h_{yx} & h_{zx} & 0 \\ h_{xy} & 1 & h_{zy} & 0 \\ h_{xz} & h_{yz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

References

[1] H. Kiem, D.A. Duc, L.D. Duy, V.H. Quan, Cơ Sở Đồ Họa Máy Tính, NXB. Giáo Dục, 2005.

[2] D. Hearn, M.P. Baker, Computer Graphic: C Version in 2nd Ed., Prentice Hall, 1996.

[3] B.T. Len, CG-Course Slide, HCM-University of Science.

[4] D.N.D.Tien, V.Q. Hoang, L. Phong, CG-Course Slide, HCM-University of Science.