

- Last time: Direct solvers for $Ax = b$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$

△ Pros:

- * Compute x exactly if arithmetic is exact
- * Rounding error analysis is well-studied

△ Cons:

- * High complexity:

GE/cholesky: $O(n^3)$ FLOPs (0.1~1ns per FLOP)

- When $n = 10^3$, take 0.1s
- When $n = 10^6$, take 10^8 s \approx 3 yrs

— Impractical in complexity and memory

- * Low complexity variants are tailored to specific matrix structures.

- * Sparse A ? — could lead to dense LU even A is sparse

- Today: Iterative solvers for $Ax = b$

- * Instead of solving $Ax = b$ directly, generate an approximate sequence x_1, x_2, x_3, \dots such that $x_k \xrightarrow{k \rightarrow \infty} x = A^{-1}b$

- * Each iteration is cheap to evaluate: matrix-vector multiplication only, Bx ,

where $B = A$ or constructed from A — share "similar" structure with A .

ex: A sparse (mostly zero)

say only m nonzero entries per row

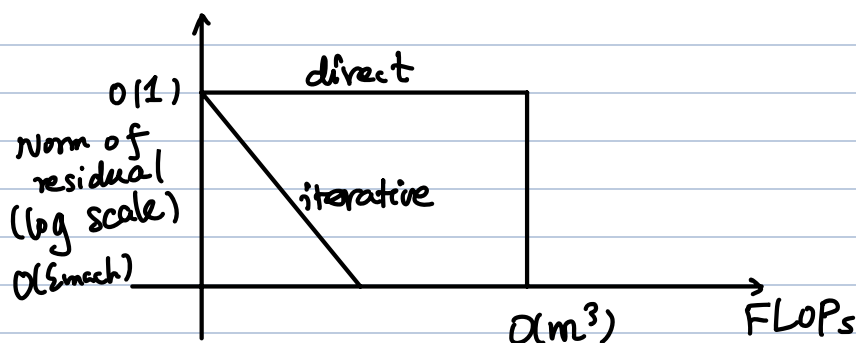
$\Rightarrow Ax$ cost $2mn$ FLOPs

ex: A is cyclic

$\Rightarrow Ax$ is $O(n \log n)$ FLOPs with FFTs

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

* Solving the problem approximately is not a problem because even direct solvers suffer from rounding error



* Rounding error analysis of iterative solvers is not well-developed — only analyze approximation error in this class.

• Iterative solvers

Let $x_1, \dots, x_k \in \mathbb{C}^n$ be known vectors

Compute the next vector through

$$x_{k+1} = F_k(x_k, x_{k-1}, \dots, x_{k-m})$$

\uparrow
(m+1)-step methods

Easiest case: $m=0$, F_k linear function

$$x_{k+1} = B_k x_k + f_k$$

Stationary iterative method: $B_k \equiv B \in \mathbb{C}^{n \times n}$, $f_k \equiv f \in \mathbb{C}^n$

↑
topic this week

More efficient iterative methods: Krylov subspace methods

↑
topic later

• Convergence of vectors and matrices

Def: • Let $\{x_k\}_{k=0}^{\infty} \subseteq \mathbb{C}^n$, if $\exists x^* \in \mathbb{C}^n$ s.t.

$$\|x_k - x^*\| \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

Then we say $\{x_k\}_{k=0}^{\infty}$ converges to x^* .

• Let $\{A_k\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m \times n}$, if $\exists A^* \in \mathbb{C}^{m \times n}$ s.t.

$$\|A_k - A^*\| \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

Then we say $\{A_k\}_{k=0}^{\infty}$ converges to A^* .

Rk: - By the equivalence of norms on finite dimensional space.

both definitions are independent of the choice of norm.

- Furthermore, by choosing $\|\cdot\|_{\infty}$ and $1\cdot\|\cdot\|_{\infty}$ we know both

convergence is equivalent to the convergence of entries.

The convergence of matrix can be characterized by testing on vectors

Prop Let $\{A_k\}_{k=0}^{\infty}$ then the following statements are equivalent

1) $\lim_{k \rightarrow +\infty} A_k = 0$

2) $\lim_{k \rightarrow +\infty} A_k x = 0, \quad \forall x \in \mathbb{C}^n$

Pf: 1) \Rightarrow 2) , $\|A_k x\| \leq \|A_k\| \|x\| \rightarrow 0$ as $k \rightarrow \infty$

2) \Rightarrow 1) , $A_k e_j \rightarrow 0 \quad \forall j=1, \dots, n$

thus each column of A_k converges to zero

and A_k converges to zero \square

For stationary iteration, we have

$$\begin{aligned} x_{k+1} &= Bx_k + f = B^2 x_{k-1} + Bf + f \\ &= \dots = \underline{B^{k+1}} x_0 + \sum_{i=0}^k B^i f \end{aligned}$$

We thus need to understand the convergence of $\{B^k\}_{k=1}^{\infty}$

The convergence of $\{B^k\}_{k=1}^{\infty}$ is closely related to the spectrum of B . We call the maximal magnitude of the eigenvalue of a matrix the spectral radius, denoted by

$$\rho(A) = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}$$

The spectral radius is closely related to the matrix norm.

Lemma 1) Let $\|\cdot\|$ be a submultiplicative matrix norm over $\mathbb{C}^{n \times n}$

then $\forall A \in \mathbb{C}^{n \times n}$, we have

$$\rho(A) \leq \|A\|$$

$\|\cdot\|$ is constructed

from A and \rightarrow 2) $\forall A \in \mathbb{C}^{n \times n}$ and $\varepsilon > 0$, \exists a subordinate norm $\|\cdot\|$

over $\mathbb{C}^{n \times n}$ such that $\|A\| \leq \rho(A) + \varepsilon$

Pf: 1) Let λ be an eigenvalue of A s.t. $\rho(A) = |\lambda|$

with eigenvector $x \in \mathbb{C}^n$. There exists $y \in \mathbb{C}^n$ s.t.

the matrix xy^T is nonzero, thus $\|xy^T\| \neq 0$

Now we have

$$\rho(A) \|xy^T\| \stackrel{\substack{\text{homogeneity} \\ \text{of norm}}}{=} \|Axy^T\| \stackrel{\substack{\text{submultiplicativity}}}{\leq} \|A\| \|xy^T\|$$

$$\Rightarrow \rho(A) \leq \|A\|$$

2) We shall use the Jordan decomposition of A :

$$J = P^{-1}AP$$

$$= \begin{bmatrix} \lambda_1 & \delta_1 & & & \\ & \lambda_2 & \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & \delta_{n-1} \\ & & & & \lambda_n \end{bmatrix}$$

$\delta_i = 0$ or 1 , $i = 1, \dots, n-1$, P nonsingular

Define $D_\varepsilon = \text{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})$

Then

$$J_\varepsilon = D_\varepsilon^{-1} J D_\varepsilon = \begin{bmatrix} \lambda_1 & \varepsilon \delta_1 & & & \\ & \lambda_2 & \varepsilon \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & \varepsilon \delta_{n-1} \\ & & & & \lambda_n \end{bmatrix}$$

Clearly, $\|J_\varepsilon\|_\infty = \max_{1 \leq i \leq n} (|\lambda_i| + \varepsilon |\delta_i|) \leq \rho(A) + \varepsilon$

Now consider the vector norm

$$\|x\| := \|D_\varepsilon^{-1} P^{-1} x\|_\infty \quad \swarrow \text{exercise.}$$

Then the subordinate norm $\|A\| = \|D_\varepsilon^{-1} P^{-1} A P D_\varepsilon\|_\infty$

$$= \|J_\varepsilon\|_\infty \leq \rho(A) + \varepsilon$$



The following theorem relates the convergence of $\{B^k\}_{k=0}^{\infty}$.

the spectral radius of B and norm of B :

Thm Let $B \in \mathbb{C}^{n \times n}$, then the following three statements are equivalent:

1) $\lim_{k \rightarrow \infty} B^k = 0$

2) $\rho(B) < 1$

3) there exists a subordinate norm such that $\|B\| < 1$.

Not for all norms

It is possible that $\|B\| > 1$ from some norms

Pf: 1) \Rightarrow 2) Let (λ, x) be an eigenpair of B .

$$\text{then } \|B^k x\| = |\lambda|^k \|x\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

implies that $|\lambda| < 1$

2) \Rightarrow 3) Proved in the lemma

3) \Rightarrow 1) $\|B^k\| \leq \|B\|^k \rightarrow 0$ as $k \rightarrow \infty$ \square

Remark: Requiring $\|B\| < 1$ for some specific $\|\cdot\|$ is a sufficient but not necessary condition for the convergence. The significance of 3) is that it gives you a hope to check convergence with commonly used matrix norms.

Thm Let $B \in \mathbb{C}^{n \times n}$. $\|\cdot\|$ a submultiplicative norm

then $\lim_{k \rightarrow \infty} \|B^k\|^{1/k} = \rho(B)$

Pf: Since $\rho(B) \leq \|B\|$, we know that

why? $\rho(B) = (\rho(B^k))^{1/k} \leq \|B^k\|^{1/k}$

On the other hand, consider the matrix

$$B_\varepsilon = (\rho(B) + \varepsilon)^{-1} B, \quad \text{where } \varepsilon > 0$$

Clearly, $\rho(B_\varepsilon) < 1$, and thus $B_\varepsilon^k \rightarrow 0$ as $k \rightarrow \infty$

so we have

$$\|B^k\|^{1/k} = (\rho(B) + \varepsilon) \|B_\varepsilon^k\|^{1/k} < \rho(B) + \varepsilon$$

for sufficiently large k . Thus $\lim_{k \rightarrow \infty} \|B^k\|^{1/k} = \rho(B)$ \square

ex. Let $B = \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}$, $\lambda_1 = \lambda_2 = 1/2$, so $\rho(B) = 1/2$

$$B^k = \begin{pmatrix} 1/2^k & 0 \\ k/2^k & 1/2^k \end{pmatrix}, \quad \|B^k\|_\infty = \frac{1}{2^k} \left(1 + \frac{k}{2}\right)$$

$$\|B^k\|_\infty^{1/k} = \frac{1}{2} \left(1 + \frac{k}{2}\right)^{1/k} \rightarrow \frac{1}{2} = \rho(B) \quad \square$$

Consider stationary iteration

$$x_{k+1} = Bx_k + f$$

Suppose $x_k \rightarrow x^*$ as $k \rightarrow \infty$

Taking limit on both sides,

$$x^* = Bx^* + f$$

Let error $e_k = x_k - x^*$,

then $\{e_k\}_{k=0}^\infty$ satisfies

$$e_{k+1} = Be_k = B^k e_0$$

Corollary The stationary iteration converges if and only if one of the following conditions holds true

1) $\rho(B) < 1$

2) \exists a subordinate norm s.t. $\|B\| < 1$.

The limit x^* is unique under either condition.

Pf: Let $x^* = Bx^* + f$, $y^* = By^* + f$, then

$$\|x^* - y^*\| = \|B(x^* - y^*)\| \leq \|B\| \|x^* - y^*\| \Rightarrow \|x^* - y^*\| = 0 \quad \square$$

We can obtain different error bounds for the iteration

Thm Let $\|\cdot\|$ be a subordinate norm, then when

$\|B\| = q < 1$, we have

$$\|x_k - x^*\| \leq \frac{q}{1-q} \|x_k - x_{k-1}\| \quad (\text{a posteriori})$$

$$\|x_k - x^*\| \leq \frac{q^k}{1-q} \|x_1 - x_0\| \quad (\text{a priori})$$

Pf: From $x_{k+1} = Bx_k + f$

$$x_k - x^* = B(x_{k-1} - x^*)$$

$$= B(x_{k-1} - x_k) + B(x_k - x^*)$$

$$\Rightarrow \|x_k - x^*\| \leq q \|x_{k-1} - x_k\| + q \|x_k - x^*\|$$

$$\Rightarrow \|x_k - x^*\| \leq \frac{q}{1-q} \|x_{k-1} - x_k\| \quad (\text{a posteriori})$$

$$= \frac{q}{1-q} \|B(x_{k-2} - x_{k-1})\|$$

$$\leq \frac{q^2}{1-q} \|x_{k-2} - x_{k-1}\|$$

$$\leq \dots \leq \frac{q^k}{1-q} \|x_1 - x_0\| \quad (\text{a priori})$$

□

• From $e_k = B^k e_0$ we know $\|e_k\| \leq \|B^k\| \|e_0\|$

on average, the contraction rate in each step is $\|B^k\|^{1/k} \rightarrow \rho(B)$

Usually use $\rho(B)$ to compare the convergence rate of the iteration.

- How to construct B ?

matrix splittings: $A = (A - C) + C$ (C invertible)

$$b = Ax = (A - C)x + Cx$$

$$\Rightarrow x = \underbrace{C^{-1}(C - A)}_B x + \underbrace{C^{-1}b}_f$$

common choices:

$$A = \underset{\substack{\uparrow \\ \text{strictly} \\ \text{lower } \Delta \\ \text{part}}}{L} + \underset{\substack{\uparrow \\ \text{diagonal} \\ \text{part}}}{D} + \underset{\substack{\uparrow \\ \text{strictly} \\ \text{upper } \Delta \\ \text{part}}}{U}$$

Want C easily invertible

- $C = D$ Jacobi iteration
- $C = L + D$ Gauss-Seidel iteration.
- $C = L + D/\omega$ Successive over-relaxation (SOR)
- $C = D/\omega$ Relaxed Jacobi iteration.