18.335 Problem Set 2

There are four problems below (Problem 1 - Problem 4). You may complete as many problems as you want, but please submit **only three** problems for grading. **Due March 30, 2025 at 11:59pm.** Late submission will only be accepted (with mild penalty) up to **three days** after the due date. You should submit your problem set **electronically** on the 18.335 Gradescope page. Submit **both** a *scan* of any handwritten solutions (I recommend an app like TinyScanner or similar to create a good-quality black-and-white "thresholded" scan) and **also** a *PDF printout* of the Julia notebook of your computer solutions.

`Problem 0: Pset Honor Code`

Include the following statement in your solutions:

> *I will not look at 18.335 pset solutions from previous semesters. I may discuss problems with my classmates or others, but I will write up my solutions on my own. <your signature>*

Problem 1: More on Matrices

Please prove the following facts:

(a) Let $A \in \mathbb{C}^{n\times n}$. Show that

$$|\det(A)|^2 \leqslant \prod_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\,.$$

Generate a random matrix and check this inequality on your own computer.

Hint: Use a suitable matrix factorization.

(b) Let $A \in \mathbb{C}^{n\times n}$. Let A be Hermitian with positive diagonal entries, i.e., $a_{ii} > 0$, $i = 1,\ldots,n$. Show that if A is either strictly diagonally dominant or irreducibly diagonally dominant, then A is positive definite. Generate a matrix satisfies these conditions, compute its eigenvalues, and verify that it is indeed positive definite.

Hint: Check the eigenvalues of A.

(c) Let $A \in \mathbb{C}^{n\times n}$ be an Hermitian positive definite matrix. Define $\|x\|_A = \sqrt{x^*Ax}$. Show that $\|\cdot\|_A$ is a norm over $\mathbb{C}^n$.

Hint: Use a suitable matrix factorization.

Problem 2: Solving a System involving Triangular Matrices

Let $U_1, U_2 \in \mathbb{R}^{n\times n}$ be upper triangular matrices and $U_1U_2 - \lambda I$ be invertible. Design an $O(n^2)$ time complexity (i.e., an operation count of $O(n^2)$) to solve the system:

$$(U_1U_2 - \lambda I)x = b$$

Then implement your algorithm on your own computer.

Hint: Expand the matrix equation componentwise. What patterns do you see?

Problem 3: Stationary Iterative Methods

(a) Let $A \in \mathbb{C}^{n\times n}$ be an Hermitian positive definite matrix. Split A as:

$$A = L + D + U\,,$$

where L is the strictly lower triangular part of A, U is the strictly upper triangular part of A, and D is the diagonal part of A. Define a symmetric Gauss-Seidel method as follows:

$$(D+L)x_{k+\frac{1}{2}} = -Ux_k + b$$
$$(D+U)x_{k+1} = -Lx_{k+\frac{1}{2}} + b\,.$$

Rewrite the method in the standard iterative form: $x_{k+1} = Bx_k + f$, and show that the spectral radius satisfies $\rho(B) < 1$, thereby establishing that the symmetric Gauss-Seidel method converges for all Hermitian positive definite matrices.

Hint: Use the proof of SOR as a guide.

(b) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0$. Prove that the iterative method

$$x_{k+1} = x_k + \omega(b - Ax_k)\,,\quad k = 0,1,\ldots$$

converges when $\omega \in (0, 2/\lambda_1)$, and find the optimal ω^* that achieves the fastest convergence rate. Implement the method on your own computer for a randomly generated Hermitian positive definite matrix. Experiment with different values of ω and verify that the convergence rate is indeed fastest for ω^* (You may plot the error the decay for different ω to verify this).

Hint: Use the proof of relaxed Jacobi as a guide.

Problem 4: Ill-conditioned problems

Let $H_n = (h_{ij}) \in \mathbb{R}^{n\times n}$ be the Hilbert matrix, defined as:

$$h_{ij} = \frac{1}{i+j-1}\,.$$

For this problem, you may want to choose an upper bound of approximately $n \approx 20$.

(a) Define $x_n = (1,\dots,1)^\top \in \mathbb{R}^n$ and set $b_n = H_n x$. Create an implementation of the Gauss elimination (with partial pivoting) and Cholesky factorization. Then use them to solve $H_n x_n = b_n$. Compute the relative error in any norm of your choice for different n: $\frac{\|x_n - \hat{x}_n\|}{\|x_n\|}$, where $\hat{x}_n$ is the computed solution. Compare your results with those obtained using the standard library functions `lu`$(\cdot)$ and `cholesky`$(\cdot)$ in Julia.

(b) Compute the condition number $\kappa_2(H_n)$ using the built-in function `cond`. Compare your computed results with the theoretical estimate: $\kappa_2(H_n) \approx \frac{(1+\sqrt{2})^{4n}}{\sqrt{n}}$. To improve the accuracy of your results, use the `SpecialMatrices` package to generate the inverse Hilbert matrix directly. Use arbitrary precision arithmetic to accurately compute the inverse of the Hilbert matrix before converting it to finite precision.

(c) It is known that H_n is a positive definite matrix. Use the Tikhonov regularization to improve the result in (a). Plot the relative error, measured in the chosen norm, as a function of the regularization parameter $\lambda > 0$. At what value of λ is the relative error minimized?

(d) (Not for credit) Use stationary iterative solvers to solve the system and compare their performance with the direct methods used above.

Feedback (optional)

Please let me know how you're finding the course and the first problem set. What are you hoping to get out of the class? How is the pace of lecture? Please rate the difficulty and volume of the first problem set. You can submit an anonymous survey here.