Eigenvalues & Eigenvectors

The eigenvalue problem (EVP) for an $m \times m$ matrix A: find scalar λ and vector v s.t.

$$Av = \lambda v \qquad (1)$$

λ = "eigenvalue" $\qquad$ v = "eigenvector"

Along direction v, A acts like scalar mult. by λ.

When A has m linearly independent eigvecs we can rewrite (1) for all at once

$$A \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}}_{\Lambda}$$

Eigenvalue Decomposition (EVD)

$$\boxed{A = V \Lambda V^{-1} \qquad (2)}$$

We call A diagonalizable b/c $V^{-1}AV = \Lambda$.

Not every matrix is diagonalizable, but "almost every" matrix is diagonalizable (measure theoretic sense).

Example 1

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ vs. $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = 1,\ \lambda_2 = 1$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = ?$

not diagonalizable

$\lambda_1 = 1,\ \lambda_2 = 1$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

diagonalizable

Example 2

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\lambda_1 = 1$

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Real Symmetric (and Complex Hermitian) matrices always have

$\Rightarrow$ real eigenvalues

$\Rightarrow$ orthonormal basis of eigvecs so that $V^{-1} = V^*$

Connection to SVD

In Lecture 5 we saw that every matrix has an SVD: $m \times n$ matrix A is diagonalized by two orthogonal (unitary) matrices. Where do these come from?

Let $A = U\Sigma V^*$. Then,

$$AA^* = (U\Sigma V^*)(V\Sigma U^*) = U\Sigma^2 U^*$$

$$A^*A = (V\Sigma U^*)(U\Sigma V^*) = V\Sigma^2 V^*$$

The left and right singular vectors of A are the eigenvectors of the symmetric (Hermitian) matrices AA^* and A^*A, resp. The singular values are the positive roots of the corresponding eigenvalues.

When $A = A^*$, connection is simple:

$$A = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} - & v_1^* & - \\ & \vdots & \\ - & v_m^* & - \end{bmatrix}}_{V^* = V^{-1}} \qquad \text{(EVD)}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \begin{bmatrix} \text{sign}(\lambda_1) & & \\ & \ddots & \\ & & \text{sign}(\lambda_m) \end{bmatrix}}_{U \ (\text{still orthogonal})} \underbrace{\begin{bmatrix} |\lambda_1| & & \\ & \ddots & \\ & & |\lambda_m| \end{bmatrix}}_{\Sigma \ (\text{non-neg.})} \underbrace{\begin{bmatrix} - & v_1^* & - \\ & \vdots & \\ - & v_m^* & - \end{bmatrix}}_{V^*} \qquad \text{(SVD)}$$

<u>Why / When do we use eigenvalues?</u>

Like the SVD, the EVD diagonalizes A and decouples linear equations.

Unlike the SVD, the EVD does this using the <u>same basis</u> for inputs/outputs (but now not necessarily orthonormal).

This makes the EVD uniquely powerful for decoupling and analyzing equations with an iterative or evolving nature.

<u>Example:</u> Decoupling ODEs

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix}}_{A = V\Lambda V^{-1}} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

"m coupled ODEs with m unknowns $u_1(t), \dots, u_m(t)$"

i) Change to eigenvector coordinates

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{V} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \begin{bmatrix} \dot{c}_1 \\ \vdots \\ \dot{c}_m \end{bmatrix}$$

Soln. coordinates in eigenvector basis

$\dot{c}_k = \frac{d}{dt} c_k$

Inputs $\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ and Outputs $\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ in <u>same</u> basis V.

System is diagonalized & eqn's decouple

$$\frac{d}{dt}\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}}\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \iff \frac{dc_k}{dt} = \lambda_k c_k, \quad k=1,\dots,m$$

ii) Solve $\qquad c_k(t) = c_k(0)\, e^{\lambda_k t} \qquad k=1,\dots,m$

iii) Recouple $\begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_m t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} c_1(0) \\ \vdots \\ c_m(0) \end{bmatrix}}_{V^{-1}\begin{bmatrix} u_1(0) \\ \vdots \\ u_m(0) \end{bmatrix}}$

$$= e^{At}\, u_0$$

"matrix exponential" $\qquad$ initial condition $\begin{bmatrix} u_1(0) \\ \vdots \\ u_m(0) \end{bmatrix}$

This decoupling & recoupling via EVD is the beginning of powerful techniques for solving & analyzing ordinary & partial differential eqn's using matrix functions & functional calculus. $\square$

The eigenvalues of matrices can provide useful characterizations of asymptotic behavior for systems that evolve in time.

<u>E.g.</u> If $|\lambda_k| < 0$ for $k = 1, \dots, m$ in the ODE example above, then

$$\|u(t)\| \to 0 \quad \text{as } t \to \infty.$$

The system is asymptotically stable.

If $|\lambda_k| > 0$ for any $k = 1, \dots, m$, then

$$\|u(t)\| \to \infty \quad \text{as } t \to \infty.$$

The system is unstable.

This is the beginning of linear stability analysis.

<u>How do we compute eigenvalues?</u>

The eigenvalues of A are roots of the characteristic polynomial of A.

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0$$

IFF $$\underbrace{\det(A - \lambda I)}_{p_A(\lambda)\text{ - characteristic poly.}} = 0$$

The roots of polynomials can not always be written using finitely many algebraic operations (Abel, 1824): no analog of quadratic formula for polynomials of deg ≥ 5.

$\Rightarrow$ A general purpose EVD algorithm must be iterative in nature.

Iterative algorithms produce a sequence of approximate solutions. Ideally, this sequence converges to the true soln rapidly (but not necessarily in finitely many iterations).

Most EVD algorithms are based on the asymptotic behavior of powers of A:

$$\underset{\substack{\uparrow \\ \text{"initial vector"} \\ x}}{\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \underset{\substack{\uparrow \\ \text{"eig. coords"}}}{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}} \quad \Leftrightarrow \quad x = c_1 v_1 + \cdots + c_m v_m$$

$$A^k x = A^k (c_1 v_1 + \cdots + c_m v_m) = c_1 \lambda_1^k v_1 + \cdots + c_m \lambda_m^k v_m$$

Suppose $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|$ and $c_1 \neq 0$

$$\frac{1}{\lambda_1^k} A^k x = c_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k c_2 v_2 + \dots + \left(\frac{\lambda_m}{\lambda_1}\right)^k c_m v_m$$

$$= c_1 v_1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \qquad \text{as } k \to \infty.$$

Our basic idea will be to approximate the pair v_1 and λ_1 from powers of A.

Important points to address:

- other eigenpairs $(\lambda_2, v_2), \dots, (\lambda_m, v_m)$
- convergence rate as k increases
- numerical stability