Last time: Low-rank approximation

Goal: Given $A \in \mathbb{R}^{m\times n}$, $k < \min\{m,n\}$ (assume $m \geq n$)

Find $\min_{\operatorname{rank}(A_k)\leq k} \| A - A_k \|$ $(*)$

Here we take $\|\cdot\|$ to be 2-norm or Frobenius norm

Or equivalently, find the optimal range:

$$\min_{\substack{Q_k \in \mathbb{R}^{m\times k} \\ Q_k^T Q = I_k}} \| A - \underbrace{Q_k Q_k^T}_{=: P_Q} A \| \qquad (**)$$

$=: P_Q$ projection onto span of Q

Solution:

Let SVD of A be $A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$

where $U = [u_1 \; u_2 \; \cdots \; u_m] \in \mathbb{R}^{m\times m}$

$V = [v_1 \; v_2 \; \cdots \; v_n] \in \mathbb{R}^{n\times n}$

$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n) \in \mathbb{R}^{m\times n}$

The optimal rank-k approximation of A is given by

$$A_k := U_k \Sigma_k V_k^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

where $U_k = [u_1 \; u_2 \; \cdots \; u_k] \in \mathbb{R}^{m\times k}$

$V_k = [v_1 \; v_2 \; \cdots \; v_k] \in \mathbb{R}^{n\times k}$

$\Sigma_k = \operatorname{diag}(\sigma_1, \dots, \sigma_k) \in \mathbb{R}^{k\times k}$

The optimal $Q_k = U_k = [u_1 \; \cdots \; u_k]$

__Algorithm__ : (Randomized SVD)

__Stage A__ : 1) Generate iid Gaussian random matrix

Optimal range finder

$\Omega \in \mathbb{R}^{n\times(k+p)}$

2) Compute $Y = A\Omega \in \mathbb{R}^{m\times(k+p)}$

3) Compute QR fact. $Y = \hat{Q}_{k+p}R$, $\hat{Q}_{k+p} \in \mathbb{R}^{m\times(k+p)}$

__Stage B__ : 1) Compute $B = \hat{Q}_{k+p}^T A \in \mathbb{R}^{(k+p)\times n}$

Post-processing (truncated SVD)

2) Compute SVD : $B = \tilde{U}_{k+p}\hat{\Sigma}_{k+p}\hat{V}_{k+p}^T$

3) Compute $\hat{U}_{k+p} = \hat{Q}_{k+p}\tilde{U}_{k+p} \in \mathbb{R}^{m\times(k+p)}$

Output : $\hat{A}_k := \hat{U}_k\hat{\Sigma}_k\hat{V}_k^T = \sum_{i=1}^{k} \hat{\sigma}_i \hat{u}_i \hat{v}_i^T$

Error analysis : Let $P_Y := \hat{Q}_{k+p}\hat{Q}_{k+p}^T$

We split the final error into two parts

$$\|A - \hat{A}_k\| \le \underbrace{\|A - P_Y A\|}_{\substack{\text{approximation err} \\ \text{of ranger finder} \\ \text{(Stage A)}}} + \underbrace{\|P_Y A - \hat{A}_k\|}_{\substack{\text{truncation err} \\ \text{(stage B)}}}$$

Given $\hat{Q}_{k+p}$, Stage B is computing the best rank k approximation of $P_Y A := \hat{Q}_{k+p}\hat{Q}_{k+p}^T A$

b.c. $P_Y A = \hat{Q}_{k+p}\tilde{U}_{k+p}\hat{\Sigma}_{k+p}\hat{V}_{k+p}^T \longleftarrow$ SVD of $P_Y A$

and the output truncates to the top k modes

Thus $\|P_Y A - \hat{A}_k\|$ is easy to controll :

optimal rank k approx. of A

$$\| P_Y A - \hat{A}_k \| \underset{\uparrow\ \hat{A}_k \text{ is optimal}}{\le} \| P_Y (A - A_k) \| \underset{\uparrow\ \text{projection is a contraction (length shrinking)}}{\le} \| A - A_k \| \le \begin{cases} \sigma_{k+1}, & \|\cdot\| = \|\cdot\|_2 \\ \left(\sum_{j=k+1}^{n} \sigma_j^2\right)^{1/2}, & \|\cdot\| = \|\cdot\|_F \end{cases}$$

It remains to analyze the error in Stage A, i.e. $\| A - P_Y A \|$

- Special case: $\text{rank}(A) = k$, $\Omega \in \mathbb{R}^{n \times k}$

Then $Y = A\Omega = U_k \Sigma_k \underbrace{(V_k^T \Omega)}$

$\text{rank}(V_k^T \Omega) = k$ almost surely

so $\text{range}(Y) = \text{range}(U_k) = \text{range}(A)$ almost surely

$\Rightarrow \| A - P_Y A \| = 0$

- General case: $\text{rank}(A) > k$, oversampling $p > 1$

$$Y = A\Omega = [U_k, \bar{U}] \begin{bmatrix} \Sigma_k & \\ & \bar{\Sigma} \end{bmatrix} \begin{bmatrix} V_k^T \\ \bar{V}^T \end{bmatrix} \Omega$$

$$= \underbrace{U_k \Sigma_k (V_k^T \Omega)}_{=: Y_k} + \underbrace{\bar{U} \bar{\Sigma} (\bar{V}^T \Omega)}_{\text{perturbation } O(\sigma_{k+1})}$$

$\text{range}(Y_k) = \text{range}(U_k)$

Thm (Perturbation lemma)

For $\|\cdot\| = \|\cdot\|_2$ or $\|\cdot\|_F$,

and any $\Omega \in \mathbb{R}^{n \times (k+p)}$, s.t. $V_k^T \Omega \in \mathbb{R}^{k \times (k+p)}$ has full row rank, then

$$\|(I - P_Y) A\|^2 \le \|\bar{\Sigma}\|^2 + \| \bar{\Sigma} \underbrace{(\bar{V}^T \Omega)}_{\in \mathbb{R}^{(n-k)\times(k+p)} \Rightarrow\ \Omega_2 :=} \underbrace{(V_k^T \Omega)}_{\Omega_1 :=}^{\dagger} \|^2$$

$\dagger$: pseudoinverse

Ω_1, Ω_2 iid Gaussian entry

Use this lemma, we can prove average error bound:

$$\mathbb{E}\left[\| (I - P_Y) A \|_F^2 \right] \le \| \bar{\Sigma} \|_F^2 + \mathbb{E}\left[\| \bar{\Sigma} \Omega_2 \Omega_1^{\dagger} \|_F^2 \right]$$

Conditioned on Ω_1, we have

$$\mathbb{E}\left[\|\Sigma_2 \Omega_2 \Omega_1^{\dagger}\|_F^2 \mid \Omega_1 \right] = \|\bar{\Sigma}\|_F^2 \, \|\Omega_1^{\dagger}\|_F^2$$

$\mathbb{E}\left[\|SGT\|_F^2\right] = \|S\|_F^2 \, \|T\|_F^2$ (exercise)

Hence,

$$\mathbb{E}\left[\|(I - P_Y)A\|_F^2\right] \le \|\bar{\Sigma}\|_F^2 + \|\bar{\Sigma}_F\|_F^2 \; \mathbb{E}\left[\|\Omega_1^{\dagger}\|_F^2\right] \longleftarrow \text{Use Inverse Wishart distribution}$$

$$= \left(1 + \frac{k}{p-1}\right) \|\bar{\Sigma}\|_F^2$$

$$= \left(1 + \frac{k}{p-1}\right) \left(\sum_{j>k} \sigma_j^2\right)^{1/2}$$

$\|\Omega_1^{\dagger}\|_F^2 = \mathrm{tr}\left[(\Omega_1^T \Omega_1)^{-1}\right]$

Similar bound can be obtained for 2-norm

<u>Thm</u> $A \in \mathbb{R}^{m \times n}$. $k \geq 2$, $p \geq 2$ s.t. $k+p \le \frac{1}{2}\min\{m, n\}$

Then Stage A produces $m \times (k+p)$ orthonormal basis, s.t.

$$\mathbb{E}\left[\|A - \hat{Q}_{k+p} \hat{Q}_{k+p}^T A\|_2\right] \le \sigma_{k+1} \left[1 + \frac{4\sqrt{k+p}}{p-1} \sqrt{\min\{m,n\}} \right]$$

If $4 \log 4 \le p \le \min\{m, n\}$, then

$$\|A - \hat{Q}_{k+p} \hat{Q}_{k+p}^T A\|_2 \le \sigma_{k+1} \left[1 + 9\sqrt{k+p}\sqrt{\min\{m,n\}} \right]$$

with probability $\geq 1 - 3p^{-p}$ ∎

For $p = 10$. failure probability $\le 3 \times 10^{-10}$!

In the proof above, essentially we want

$$\|(\bar{V}^T \Omega)(V_k^T \Omega)^{\dagger}\| = O(1)$$

In the special case $k+p = n$, and $\Omega = I_n$

$$\|(\bar{V}^T \Omega)(V_k^T \Omega)^{\dagger}\| = \|\bar{V}^T V_k\| = 0$$

$\longleftarrow$ orthogonal basis

In general, we have

$$\bar{V}^T \Omega = \begin{bmatrix} - \bar{v}_1^T - \\ \vdots \\ - \bar{v}_{n-k}^T - \end{bmatrix} \Omega = \begin{bmatrix} \bar{v}_1^T \Omega \\ \vdots \\ \bar{v}_{n-k}^T \Omega \end{bmatrix} \in \mathbb{R}^{(n-k) \times (k+p)}$$

$$V_k^T \Omega = \begin{bmatrix} v_1^T \Omega \\ \vdots \\ v_k^T \Omega \end{bmatrix} \in \mathbb{R}^{k \times (k+p)}$$

In order for $\| \bar{V}^T \Omega (V_k^T \Omega)^+ \| \le \| \bar{V}^T \Omega \| \, \| V_k^T \Omega^+ \| = O(1)$

we need $\bar{v}_i^T \Omega \perp v_j^T \Omega$ approximately holds true $\forall i, j$

and $\| \bar{v}_i^T \Omega \| \approx \| \bar{v}_i \|$, $\| v_j^T \Omega \| \approx \| v_j \|$, $\forall i, j$

In other words, we can view Ω as a map from $\mathbb{R}^n$ to $\mathbb{R}^{k+p}$, and hope that the angles between $v_1, \cdots, v_k, \bar{v}_1, \cdots, \bar{v}_{n-k}$ are approximately preserved.

We want the choice of Ω is such that it "sketch" every given vector in an low-dimensional subspace with small distortion, without knowing the subspace in advance. Such choice of Ω is called <u>oblivious subspace embedding</u>.

Other than Gaussian matrices, there are other options such as subsampled randomized Hadamard (or Fourier) transforms, which allow fast evaluation of $A\Omega$.

i.e. $\Omega = \sqrt{\frac{n}{l}} DFR \in \mathbb{C}^{n \times l}$ (assume complex A for simplicity)

D: $n \times n$ diagonal iid uniform on unit circle

F: $n \times l$ unitary discrete FFT transform

R: $n \times l$ each column sample w/o replacement from I_n

when $l \sim k\log(k)$, we have

$0.4 \le \sigma_k(V^*\Omega)$, $\sigma_1(V^*\Omega) \le 1.48$