Non-symmetric Eigenvalue Problems

Recap: For symmetric eigenvalue problem

$$A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

V $\qquad$ V $\qquad$ Λ

orthogonal matrix $\qquad$ real eigenvalues

$V^{-1} = V^*$

QR Algorithm computes approximations to V and Λ in two phases.

Phase 1: Tridiagonal Reduction

$$\underbrace{O_{n-2}^T \cdots O_2^T O_1^T}_{O^T} A \underbrace{O_1 O_2 \cdots O_{n-2}}_{O} = \begin{bmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \ddots & \ddots & \ddots \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

sequence of orthogonal transformations, i.e., Householder

T: tridiagonal mat.

Phase 2: Tridiagonal QR iterations (w/shifts)

$$A^{(0)} = T, \quad Q^{(k)} R^{(k)} = A^{(k-1)} - \mu^{(k)} I, \quad A^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$$

$\mu^{(k)} = A_{nn}^{(k-1)}$ - Rayleigh Quotient Shifts accelerates convergence $A^{(k)} \to \Lambda$.

Note: The full QR Algorithm has a few more bells and whistles that improve efficiency and numerical accuracy:

- **Wilkinson Shifts** improve global convergence properties compared to Rayleigh-Quotient Shifts (See textbook lecture 29).
- **Implicit Shift** strategies calculate $A^{(k)} = Q^{(k)T} A^{(k-1)} Q^{(k)}$ w/out explicitly factoring the shifted matrix $A^{(k-1)} - \mu^{(k)} I$. This is more efficient, improves stability, and is the beginning of sophisticated multi-shift iterations which are central to modern implementations.
- **Deflation** breaks $A^{(k)}$ into submatrices whenever <u>any</u> subdiagonal entry becomes sufficiently small. This saves computational effort on eigenvalues that have converged.

Stability Since both Phase 1 & Phase 2 rely exclusively on Orthogonal transformations, the 2-phase QR algorithm is backward stable (see Thm. 26.1 & 29.1, LNT).

Non-symmetric EVP

The EVD of a non-symmetric diagonalizable matrix is

$$A = V \Lambda V^{-1}$$

V need not be orthogonal in general, $V^{-1} \neq V^*$. I.e., A may <u>not</u> be diagonalized by orthogonal similarity transforms.

Schur Decomposition

On the other hand, every $n \times n$ matrix A can be triangularized by orthogonal similarity transforms:

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{\substack{Q \\ = \text{orthogonal}}} \underbrace{\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\substack{U \\ = \text{upper triangular}}} \underbrace{\begin{bmatrix} - & q_1^* & - \\ - & q_2^* & - \\ & \vdots & \\ - & q_n^* & - \end{bmatrix}}_{Q^*}$$

The eigenvalues of A appear on the diagonal of the upper triangular matrix U. This is the Schur Decomposition.

<u>Idea:</u> For non-symmetric matrices, use QR-type iteration to compute Schur decomposition by orthogonal similarity.

=> Orthogonal transformations preserve numerical stability
=> eigenvalues appear along diagonal of triangular Schur factor.
=> reconstruct eigenvectors of A from the Schur basis Q (Pset 3, Problem 4).

QR for non-symmetric matrices

Two phase algorithm with tridiagonal T replaced by upper Hessenberg matrix

$$H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

zeros

H has only zeros below the first subdiagonal.

$$A \xrightarrow[\text{Hessenberg Reduction}]{\text{Phase 1}} \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}}_{H} \xrightarrow[\text{Hessenberg QR iter.}]{\text{Phase 2}} \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix}}_{U}$$

Both phases are entirely analogous to their symmetric counterpart, with T replaced by H.

Computing the SVD

The SVD diagonalizes general non-symmetric matrices with different orthogonal transformations applied on the left and the right. In Lecture 11, we saw that these come from the EVP for

$$A^*A \qquad \text{and} \qquad AA^*$$

which are symmetric positive definite.

In theory, we could compute the SVD of A by applying the QR algorithm to A^*A and AA^*, but there is a more efficient, stable two-phase approach.

Phase 1: Bidiagonalization

The extra degrees of freedom associated with <u>separate</u> left and right transformations allow us to bidiagonalize $A \in \mathbb{R}^{m \times n}$

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \xrightarrow{U_1^* \cdot} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \xrightarrow{\cdot V_1} \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$$

$$\xrightarrow{U_2^* \cdot} \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \xrightarrow{\cdot V_2} \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

x - updated nonzero   0 - new zero

$$\Rightarrow \quad B = U_n^* \cdots U_2^* U_1^* A V_1 V_2 \cdots V_{n-2}$$

is a bidiagonal matrix.

<u>**Phase 2:**</u>

Similar to QR iterations, orthogonal transformations are applied iteratively to left and right of B to obtain sequence of bidiagonal matrices $B^{(1)}, B^{(2)}, \ldots$ that converge to Σ.

Remark The symmetric matrix (recall Pset 2 Problem 5.4)

$$H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$$

typically plays a key role in the analysis and derivation of Phase 2, but is not explicitly formed in practice.