Linear Systems of Eq's

Part 3: Triangularization and $PA = LU$

P: Permutation

Recap Gaussian Elimination (w/out pivoting) makes A upper triangular by applying lower triangular transformations on the left.

$$\underset{A}{\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}} \xrightarrow{L_1} \underset{L_1 A}{\begin{bmatrix} x & x & x \\ 0 & \square & \square \\ 0 & \square & \square \end{bmatrix}} \xrightarrow{L_2} \underset{L_1 L_2 A = U}{\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & \square \end{bmatrix}}$$

"Convert A to upper triangular with elementary row operations."

- → Linear combinations of rows
- → Permutations of rows

$A = LU$, where $L = L_{m-1}^{-1} \ldots L_1^{-1}$

For generic dense matrices, LU factorization is <u>not</u> backward stable.

Thm If LU factorization of A exists & is computed by Gaussian elimination (w/out pivoting) in floating point arithmetic, then computed factors $\tilde{L}$ and $\tilde{U}$ satisfy

$$\tilde{L}\tilde{U} = A + \delta A \quad \text{where} \quad \frac{\|\delta A\|}{\|L\|\|U\|} = \mathcal{O}(\varepsilon_{mach}).$$

$\boxed{\text{Pf}}$ See Higham, Ch 9.2

When $\|L\|\|U\| \gg \|A\|$, $\|\delta A\|$ can be much larger than $\|A\|\mathcal{O}(\varepsilon_{mach})$. In this case, solving $Ax=b$ with $\tilde{L}$ and $\tilde{U}$ may be a disaster!

(See example at end of Lecture 6.)

Partial Pivoting (a partial solution)

Can we control $\|L\|$ and $\|U\|$?

Recall $L = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & \ddots & & \\ \vdots & \vdots & & \ddots & \\ l_{m1} & l_{m2} & & l_{m,m-1} & 1 \end{bmatrix}$ where $l_{jk} = \dfrac{x_{jk}}{x_{kk}}$

$(k \leq j \leq m)$

(x_{ij} - entries of j^{th} column of $L_k \cdots L_1 A$)

$$\overset{A}{\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}} \xrightarrow{L_1} \overset{L_1 A}{\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}} \xrightarrow{L_2} \overset{L_1 L_2 A = U}{\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}}$$

We call x_{kk} entries pivots.

Growth in $\|L\|$ (relative to $\|A\|$) occurs when pivot x_{kk} is very small - near zero.

<u>Idea</u>: use row permutations to avoid small pivots and control growth in $\|L\|$.

$$\overset{A}{\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}} \xrightarrow{P_1} \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \xrightarrow{L_1} \overset{L_1 A}{\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}}$$

select pivot (max entry on or below diagonal) — Row interchange — elimination

With partial pivoting, $x_{kk} \geq x_{jk}$ for $k \leq j \leq m$

so $\ell_{jk} \leq 1$ - no growth!

Note: can also incorporate column interchanges

$\Rightarrow$ Column pivoting, complete pivoting, etc. (see LNT Lec. 21)

$PA = LU$

How do permutations change $A = LU$?

$$L_{m-1} \cdots L_2 L_1 A = U \quad \Rightarrow \quad \underbrace{L_{m-1} P_{m-1} \cdots L_2 P_2 L_1 P_1}_{\text{Structure?}} A = U$$

Remarkably, we can repackage the transformations on the left in a special way.

Example ($m=3$)

$$L_2 P_2 L_1 P_1 A = U \iff L_2 \underbrace{P_2 L_1 P_2^{-1}}_{L_1'} \underbrace{P_2 P_1}_{P} A = U$$

$$L_1 = \begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ \ell_{31} & 0 & 1 \end{bmatrix} \iff \underbrace{P_2 L_1 P_2^{-1}}_{L_1'} = \begin{bmatrix} 1 & & \\ \times & 1 & \\ \times & 0 & 1 \end{bmatrix}$$

Since P_2 can only interchange rows $j \geq 2$ and P_2^{-1} interchanges the corresponding columns.

$$\Rightarrow \quad PA = LU \quad \text{where } L = L_1'^{-1} L_2^{-1} \text{ is lower triangular}$$

It is exactly as if we permuted rows of A and *then* did GE w/out pivoting on PA.

In general,

$$L_{m-1}P_{m-1}\cdots L_2P_2L_1P_1 = \underbrace{L'_{m-1}\cdots L'_2L'_1}_{L^{-1}}\underbrace{P_{m-1}\cdots P_2P_1}_{P}$$

where $L'_k = P_{m-1}\cdots P_{k+1}L_kP_{k+1}^{-1}\cdots P_{m-1}^{-1}$.

So Gaussian elimination w/partial pivoting gives us the factorization

$$PA = LU.$$

(Typically this more general form is what people mean when they say "LU factorization")

<u>Stability of GE w/partial pivoting?</u>

We can get a sense of stability for partial pivoting, by applying <u>Thm</u> (above) to PA. Since the entries of L are ≤ 1 now, $\|L\| = O(1)$. What about $\|U\|$?

⇒ Note that we expect stability if $\|U\| = \mathcal{O}(\|A\|)$.

Julia experiments

Worst case: $\dfrac{\|U\|}{\|A\|} = \mathcal{O}(2^{m-1})$

"Average" case: $\dfrac{\|U\|}{\|A\|} = \mathcal{O}(1)$

The mystery of Gaussian elimination:

In theory, Gaussian elimination (w/partial pivoting) is unstable. In practice, it behaves like a backward stable algorithm for almost every matrix that is encountered in "real world" applications.

Why? It appears that strong correlations in L and U damp growth of entries in U for all but a few exotic matrices A. See LNT Lecture 22 for more.