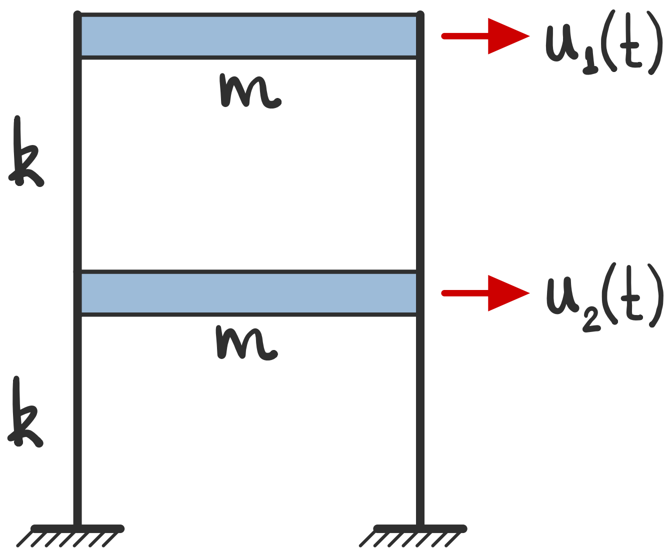
① A partir das informações dadas, calcule os modos de vibração e as frequências naturais.



$$\boxed{f_2/f_1 = 1+\varphi}$$

φ: "golden ratio"

$$\mathbb{D} = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}^{-1} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \frac{m}{k} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{(Golden ratio)} \qquad \begin{cases} \lambda_2 = (m/k)(2-\varphi) \\ \lambda_1 = (m/k)(1+\varphi) \end{cases}$$

$$\begin{cases} \omega_1^2 = \dfrac{1}{1+\varphi}(k/m) \\ \omega_2^2 = \dfrac{1}{2-\varphi}(k/m) \end{cases} \qquad \boxed{f_1/f_2 = \sqrt{\frac{2-\varphi}{1+\varphi}}} \; (=2-\varphi)$$

$$\begin{cases} \vec{\varphi}_1 = [(1+\sqrt{5}), 2]^T \rightarrow [\varphi, 1]^T \rightarrow [1, 1-\varphi]^T \\ \vec{\varphi}_2 = [(1-\sqrt{5}), 2]^T \rightarrow [1-\varphi, 1]^T \rightarrow [-1, \varphi]^T \end{cases}$$

Obs.: $(2-\varphi)(1+\varphi) = 1$

$$2+2\varphi-\varphi-\varphi^2 = 1$$
$$2+\varphi-\varphi^2 = 1$$

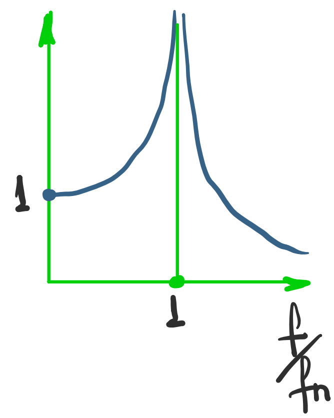
$$\left.\begin{aligned} 1/\varphi &= \varphi - 1 \\ 1/(1-\varphi) &= -\varphi \end{aligned}\right\} \quad \varphi^2 - \varphi = 1$$

∴

$\omega_1 = (\varphi-1)\sqrt{k/m}$	$\Phi = \begin{bmatrix} \varphi & -1 \\ 1 & \varphi \end{bmatrix}$
$\omega_2 = \varphi\sqrt{k/m}$	

(2) $S_{F_1}(f) = S_0 = cte.$ (white noise)

$S_{u_k}(f) = |H(f)|^2 S_0 \varphi_{1k}^2$ (no modo k)



$$|H(\beta,\xi)|^2 = \left\{K^2\left[(1-\beta^2)^2 + (2\xi\beta)^2\right]\right\}^{-1}$$

$$\int_0^\infty |H(\beta,\xi)|^2 \, d\beta = \frac{1}{K^2}\cdot\frac{8100}{\pi\,\xi^2}$$

$$\left(\begin{aligned} f &= \beta f_k \\ df &= f_k \, d\beta \end{aligned}\right)$$

$$\Rightarrow \sigma_{u_k}^2 = \frac{1}{K^2}\cdot\frac{8100}{\pi\,\xi^2}\cdot S_0\,\varphi_{1k}^2$$

$K = (2\pi f_k)^2 M$ (rigidez modal e massa modal)

$$\mathbb{M}_k = \Phi^T \mathbb{M} \Phi = \begin{bmatrix} \varphi & 1 \\ -1 & \varphi \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \varphi & -1 \\ 1 & \varphi \end{bmatrix}$$

$$\Rightarrow \quad M_1 = M_2 = m(1+\varphi^2) = m(2+\varphi)$$

$$K_1 = \underbrace{\frac{1}{1+\varphi}(k/\cancel{m})}_{\omega_k^2} \cdot \cancel{m}(2+\varphi) \quad \therefore \quad K_1 = \frac{2+\varphi}{1+\varphi} \cdot k$$

$$K_2 = \overbrace{\frac{1}{2-\varphi}(k/\cancel{m})}^{\omega_k^2} \cdot \cancel{m}(2+\varphi) \quad \therefore \quad K_2 = \frac{2+\varphi}{2-\varphi} \cdot k$$

$$\begin{cases} \sigma_{u_1}^2 = \left(\dfrac{1+\varphi}{2+\varphi}\right)^2 \cdot \dfrac{1}{k^2} \cdot \dfrac{8100}{\pi \zeta^2} \cdot S_0 \cdot \varphi^2 \\[2ex] \sigma_{u_2}^2 = \left(\dfrac{2-\varphi}{2+\varphi}\right)^2 \cdot \dfrac{1}{k^2} \cdot \dfrac{8100}{\pi \zeta^2} \cdot S_0 \cdot 1 \end{cases}$$

$$\boxed{\sigma_{u_1} \approx 59{,}45 \, \frac{\sqrt{S_0}}{k\zeta}} \; /\!/$$

$$\boxed{\sigma_{u_2} \approx 5{,}36 \, \frac{\sqrt{S_0}}{k\zeta}} \; /\!/$$

Obs.: a resposta no segundo modo é apenas ~9% da resposta no primeiro modo!

Obs.: note a interessante (e coerente) dependência de σ_{uk} em relação a S_0, k e ζ.

O deslocamento máximo deve acontecer na massa de cima (g.d.l. "1"). Esse deslocamento tem a contribuição de 2 modos, então pode ser feita uma combinação quadrática simples.

$$\sigma_u = \left[(g_1 \sigma_{u_1} \varphi)^2 + (g_2 \sigma_{u_2} \cdot 1)^2 \right]^{1/2}$$

... onde g_1 e g_2 são os respectivos fatores de pico.

$$L = \sqrt{2 \ln \nu T} \quad \therefore \quad g = L + 0{,}5772/L$$

... onde a taxa de ultrapassagem é $\nu \approx f_k$ e o tempo de observação é arbitrário.

$\nu T = N$	2	10	100	1000
L	1,177	2,146	3,035	3,717
g	1,67	2,41	3,22	3,87

← número de períodos.

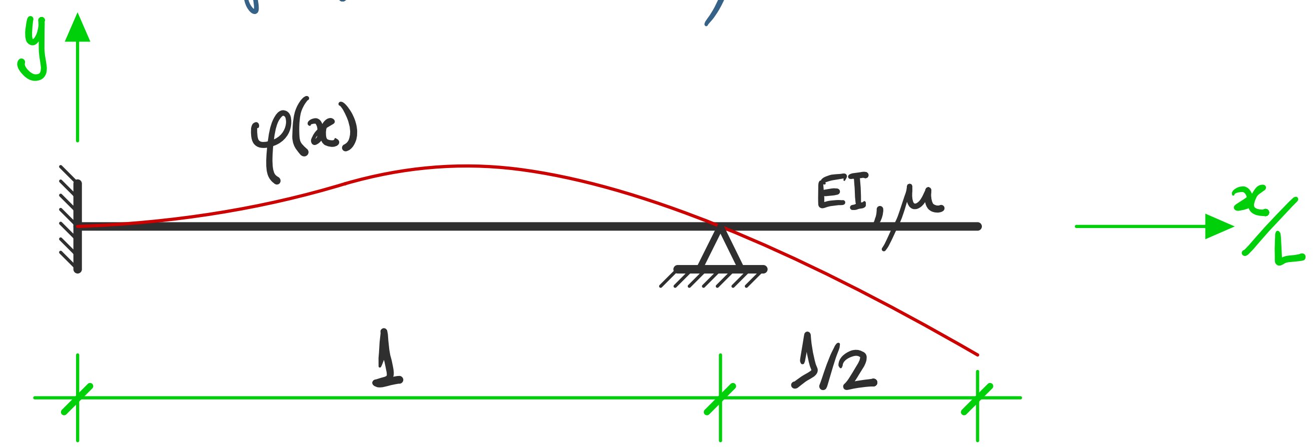
Adotando um fator de pico conservador, $g = 4$:

$$\sigma_u = \underset{g}{4} \cdot \frac{\sqrt{S_0}}{k\zeta} \left\{ (59{,}45 \cdot \varphi)^2 + \left[5{,}36 \cdot (-1) \right]^2 \right\}^{1/2}$$

$$\therefore \quad \boxed{\sigma_u \approx 385{,}4 \, \frac{\sqrt{S_0}}{k\zeta}}$$

isso é o mais importante!

③ Proponha uma função $\varphi(x)$ que represente de forma aproximada a forma modal desenhada abaixo, e calcule a respectiva frequência natural em função de L, EI e μ.



$$\varphi(x) = \sum_{k=1}^{4} c_k x^k = c_4 x^4 + c_3 x^3 + c_2 x^2$$

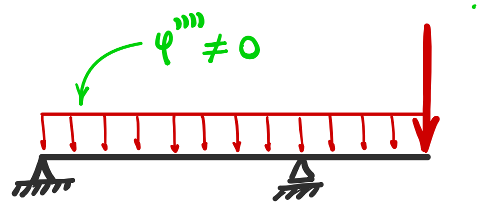
$$\left.\begin{aligned}
\varphi' &: \ 4c_4x^3 + 3c_3x^2 + 2c_2x \\
\varphi'' &: \ 12c_4x^2 + 6c_3x \\
\varphi''' &: \ 24c_4x + 6c_3 \\
\varphi'''' &: \ 24c_4
\end{aligned}\right\} \quad (0 \leq x \leq 3/2)$$

$$\left.\begin{aligned}
\varphi(0) = 0 \quad &\therefore \quad c_n = 0 \\
\varphi'(0) = 0 \quad &\therefore \quad c_{n-1} = 0
\end{aligned}\right\} \text{ engaste da esquerda}$$

$$\begin{array}{r}
\varphi(1) = 0 \\
\varphi''(3/2) = 0 \\
\varphi'''(3/2) = 1
\end{array}
\begin{bmatrix} 1 & 1 & 1 \\ 27 & 9 & 0 \\ 36 & 6 & 0 \end{bmatrix}
\begin{bmatrix} c_4 \\ c_3 \\ c_2 \end{bmatrix} =
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\begin{array}{l}
= 1/18 \\
= -1/6 \\
= 1/9
\end{array}$$

FORMA MODAL ADOTADA:

$$\boxed{\varphi(x) = \frac{16}{9}\left(x^4 - 3x^3 + 2x^2\right)}$$



(escala ajustada para que o deslocamento na ponta do balanço seja unitário!)

$$\varphi'(x) = \frac{16}{9}\left(4x^3 - 9x^2 + 4x\right)\cdot\frac{1}{L}$$

$$\varphi''(x) = \frac{16}{9}\left(12x^2 - 18x + 4\right)\cdot\frac{1}{L^2}$$

cuidado!

$$V = \frac{EI}{2}\int_0^{3/2}\left[\frac{16}{9}\left(12x^2 - 18x + 4\right)\frac{1}{L^2}\right]^2 L\,dx$$

$$T^* = \frac{\mu}{2}\int_0^{3/2}\left[\frac{16}{9}\left(x^4 - 3x^3 + 2x^2\right)\right]^2 L\,dx$$

$$\boxed{V = \frac{EI}{2L^3}\cdot\frac{2752}{135}} \qquad \boxed{T^* = \frac{\mu L}{2}\cdot\frac{33}{140}}$$

$$\therefore\ \omega_n^2 = V/T^* = \frac{2752}{135}\cdot\frac{140}{33}\cdot\frac{EI}{\mu L^4}$$

$$\boxed{\omega_n \approx \left(\frac{3{,}0495}{L}\right)^2\sqrt{\frac{EI}{\mu}}}$$

Um pouco menor do que bi-apoiada!