Escape direction does not matter for some fish prey

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Supplemental Materials

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1 Performance Plateau when $K < 1$

In this is section, we provide details to show that when the prey is faster than the predator ($K < 1$), then there exists a performance plateau. That is, an equivalent greatest minimum distance is achieved for a large range of escape angles α. For escape angles in this range, the prey performs equally well.

1.1 Distance increases for a range of α values

Here we show that the minimum distance function is not affected by a large range of escape angles. For reference, the distance function is given by the following equation:

$$D^2 = ((X_0 - Ut) + Vt\cos\alpha)^2 + (Vt\sin\alpha)^2, \tag{1}$$

where X_0 is the starting position of the prey. Note that at $t = 0$, $D^2 = X_0^2$.

For this case, we are interested in escape angles between the two solutions given by Weihs and Webb (1984)

$$0 \le \alpha \le \arccos(K). \tag{2}$$

The above leads to an important inequality given by

$$K \le \cos\alpha \le 1. \tag{3}$$

This inequality plays a crucial role in the following proof. To show that the minimum distance does not change for this range of escape angles and $K < 1$, it suffices to show that the distance function is always increasing. That is, we want to show that the derivative of Eqn. 1 with respect to time is greater than or equal to zero.

Proof. Assume that $K < 1$ and that Eqn. 3 holds. Taking the derivative of Eqn. 1 with respect to time yields

$$\frac{\partial D^2}{\partial t} = 2(t(U^2 + V^2) - UX_0 + V(X_0 - 2tU)\cos\alpha) \tag{4}$$

Let $U = 1$. Since $K = U/V$, the assumption on the predator's speed leads to $V > 1$. Then equation (4) becomes

$$\frac{\partial D^2}{\partial t} = 2(1 + V^2 - 2V\cos\alpha)t + 2X_0(V\cos\alpha - 1), \tag{5}$$

which is a linear function in time. To show Eqn. 5 is nonnegative for $t \geq 0$, it suffices to show that the slope is positive and the intercept (when $t = 0$) is nonnegative.

- Positive slope;

  Eqn. 3 implies $V \cos\alpha \leq V$. Multiplying by -2 and adding $1+V^2$ on both sides yields

$$1 + V^2 - 2V\cos\alpha \geq 1 - 2V + V^2 = (V-1)^2 > 0. \tag{6}$$

- Nonnegative intercept;

  Here we simply need to show that $V\cos\alpha - 1 \geq 0$. From Eqn. 3 we have:

$$K \leq \cos\alpha \implies 1 = VK \leq V\cos\alpha \implies 0 \leq V\cos\alpha - 1. \tag{7}$$

Eqns. 6 and 7 together show that the distance is always increasing for $0 \leq \alpha \leq \arccos(K)$ and $K < 1$. □

An analogous argument applies for $-\arccos(K) \leq \alpha < 0$. The preceding proof shows that the minimum occurs at $t = 0$ and is $D^2 = X_0^2$. This defines a performance plateau for the prey as a wide range of angles yield equally successful escapes.

2 Initial Lateral Displacement

The distance function given by Eqn. 1 is based on the assumption that a predator is heading in the direction of the prey. To model a predator that does not perfectly align its motion with the prey's direction, as is presented in the main text for a predator robot, we introduced a lateral initial displacement to the distance function.

2.1 Distance function with initial lateral displacement

The distance function is now given by

$$D^2 = ((X_0 - Ut) + Vt\cos\alpha)^2 + (Y_0 + Vt\sin\alpha)^2. \tag{8}$$

In the following sections, we detail the steps for finding the optimal escape angle. Note that the introduction of Y_0 in the distance function allows us to rewrite the equation in polar coordinates.

2.2 Minimum distance in polar coordinates

To simplify the forthcoming analysis, we rewrite Eqn. 8 in polar coordinates (R, θ). We do this by setting $R_0^2 = X_0^2 + Y_0^2$, and $\theta_0 = \arctan(Y_0/X_0)^*$. This yields

$$D_0^2 = R_0^2 + (1+K^2)t^2V^2 - 2Vt(KVt\cos\alpha - R_0\cos(\alpha-\theta_0) + KR_0\cos\theta_0) \tag{9}$$

To find the time at which Eqn. 9 is minimized, we find the roots of the derivative of Eqn. 9 with respect to t which yields

$$t_{\min} = \frac{R_0}{V}\frac{[K\cos\theta_0 - \cos(\alpha-\theta_0)]}{1-2K\cos\alpha+K^2} \tag{10}$$

The above is negative when $K\cos\theta_0 < \cos(\alpha-\theta_0)$. Rewriting this inequality gives the range of α for which the distance is solely increasing. Explicitly, this is given by

$$\theta_0 - \arccos(K\cos\theta_0) < \alpha < \theta_0 + \arccos(K\cos\theta_0) \tag{11}$$

For these values of α, the minimum distance occurs at $t = 0$ and is thus equal to the initial distance R_0^2. This defines a performance plateau when $K < 1$.

If we now substitute $t_{\min}$ for t in Eqn. 9, we get the minimum distance as a function of α with parameters K and θ_0. This is given by the following:

$$\overline{D}^2_{\min} = \frac{D^2_{\min}}{R_0^2} = \frac{\left(\sin(\alpha-\theta_0) + K\sin\theta_0\right)^2}{K^2 - 2K\cos\alpha + 1} \tag{12}$$

*For this analysis, we assume $Y_0 \geq 0$, but the final results are presented for the more general case.

2.3 Greatest minimum distance: solving for α

To find the escape angle which yields the largest minimum distance, we solve the following equation

$$0 = \frac{\partial \overline{D}^2_{\min}}{\partial \alpha} = \frac{2(K\cos\alpha - 1)(K\cos\theta_0 - \cos(\alpha - \theta_0))(K\sin\theta_0 + \sin(\alpha - \theta_0))}{(K^2 - 2K\cos\alpha + 1)^2} \tag{13}$$

The solutions to this equation are found by finding where the numerator is equal to zero which is done by considering the following three cases:

Case 1: Solving the equation $K\cos\alpha - 1 = 0$ yields

$$\alpha_1 = \pm \arccos K^{-1}$$

This solution is valid for $K \geq 1$. This solution indicates that prey are equally effective if escaping at an optimal angle toward the left ($\alpha > 0$), or right ($\alpha < 0$) of the predator's heading.

Case 2: For the equation $K\cos\theta_0 - \cos(\alpha - \theta_0) = 0$ a more careful analysis (detailed below) is required since the solution can be a complex number for some combinations of K and θ_0. Rewriting the equation yields

$$\cos(\alpha - \theta_0) = K\cos\theta_0. \tag{14}$$

This equation gives the conditions that K and θ_0 must satisfy so that solutions are real–valued. Explicitly, the condition is given by the following

$$|K\cos\theta_0| \leq 1 \tag{15}$$

When K is between 0 and 1, Eqn. 15 is always satisfied. If, on the other hand, $K > 1$, then we must have that $|\cos\theta_0| \leq 1/K$. This leads to the following bound for θ_0:

$$\arccos(K^{-1}) \leq \theta_0 \leq \arccos(-K^{-1}) \tag{16}$$

Note that as the value of K increases, the allowable range for θ_0 decreases and approaches 90°. Keeping these restrictions in mind we proceed to solve Eqn. 14. The solutions are given by

$$\alpha_2 = \theta_0 + \arccos(K \cos\theta_0)$$
$$\alpha_3 = \theta_0 - \arccos(K \cos\theta_0)$$

For $K < 1$, the following equation defines the boundaries of the performance plateau.

$$|\alpha - \theta_0| \leq \arccos(K \cos\theta_0), \tag{17}$$

For $K > 1$ the solutions $\alpha_{2,3}$ are not optimal unless θ_0 satisfies Eqn. 16.

Case 3: Solving the equation $K \sin\theta_0 + \sin(\alpha - \theta_0) = 0$ yields

$$\alpha_4 = \theta_0 - \arcsin(K \sin\theta_0)$$
$$\alpha_5 = \pi + \theta_0 + \arcsin(K \sin\theta_0)$$

For $K < 1$, α_4 is contained within the bounds defined by Eqn. 11 and α_5 is a local minimum. When $K > 1$, the solutions are local minimums and thus not optimal.