Supplemental materials

Sebastian M. Krause and Vinko Zlatić
Theoretical Physics Division, Rudjer Bošković Institute, Zagreb, Croatia

Michael M. Danziger
Department of Physics, Bar Ilan University, Ramat Gan, Israel

List of variables

	Networks
N	Number of nodes
$\bar{k}$	Average degree
k_i	Degree of node i
p_k	Degree distribution
α	Exponent of scale free degree distribution
g_0	Generating function of degree
g_1	Generating function of excess degree
	Colors
C	Number of colors
$c \in 1, 2, \dots C$	A color
r_c	Color distribution
n_{deg}	Degeneration of the highest color frequency
$\tilde{r}_{c,k}$	degree-dependent color distribution
	Standard percolation ingredients
$\mathcal{L}$	Set of nodes in the largest component (color blind)
u	Prob. of not being connected to giant comp. over a link
S	Size of giant component
$\phi_{\bar{c}}$	Fraction of nodes without color c
$\mathcal{L}_{\bar{c}}$	Set of nodes in the largest component, after nodes of color c deleted
$u_{\bar{c}}$	Prob. of not being connected to giant $\mathcal{L}_{\bar{c}}$ over a link
$S_{\bar{c}}$	Size of giant $\mathcal{L}_{\bar{c}}$
	Percolation over color avoiding paths
$\mathcal{L}_{\mathrm{color}}$	Candidate set of nodes for the largest avoidable colors component
S_{color}	Size of giant $\mathcal{L}_{\mathrm{color}}$
$B_{k,k'}$	Prob. that out of k links k' connect to giant component
$M_{k',\vec{\kappa}}$	Prob. that out of k' links κ_1 connect to color 1 etc.
$P_{\vec{\kappa}}$	Success probability having neighbors of colors acc. to $\vec{\kappa}$
$U_{\bar{c}}$	Prob. that a link fails connecting to $\mathcal{L}_{\mathrm{color}}$ which already connects to $\mathcal{L}$ and a node not having color c
$S_{\mathrm{color},\infty}$	Size of the set of all nodes being connected to giant component over two links or more
β	Critical exponent
$\bar{k}_{\mathrm{crit}}$	Critical value of average degree
k_{step}	Degree above which all nodes have the same color
γ	Fraction of nodes with highest degree

I. SIZE OF GIANT AVOIDABLE COLORS COMPONENT IN THE CONFIGURATION MODEL

We can find analytical results for S_{color} for random graph ensembles with randomly distributed colors in the limit of infinite graphs. These results can be used to estimate the situation in finite quenched networks. We are able to gain a general understanding including phase transitions. This can guide our understanding of real world networks.

We use the generalized configuration model graph ensemble with N nodes, where each degree sequences $\{k_i\}$ occurs with probability $\prod_i p_{k_i}$, with the degree distribution p_k. Additionally we want to assign to every node i a color $c_i \in 1, 2, \ldots, C$. For given degree sequence k_i, the color sequence $\{c_i\}$ has probability $\prod_i \tilde{r}_{c_i,k_i}$ with the degree-dependent color distribution $\tilde{r}_{c,k}$ ($\sum_c \tilde{r}_{c,k} = 1$ for every degree k separately). For a graph G_N out of the graph ensemble, $\mathcal{L}_{\text{color}}$ has a certain size $N_{\text{color}}(G_N)$. For the whole graph ensemble, we have to use the average value. By considering only giant contributions growing with network size, we have

$$S_{\text{color}} = \lim_{N\to\infty} \sum_{G_N} P(G_N) \frac{N_{\text{color}}(G_N)}{N}, \tag{1}$$

where $P(G_N) = \prod_i p_{k_i} \omega \prod_i \tilde{r}_{c_i,k_i}$ is the probability to have the graph G_N of size N, including ω, the probability of the connection scheme of G_N as a matching of half edges.

A. Question and connection to percolation theory

For calculating S_{color} in the random graph ensemble, we will follow ideas of Erdos and Renij and Newman. For calculating the size of the giant component, they used probabilities of connections for a single node in the graph ensemble. As we have to extend the method to a gradual procedure with conditional probabilities, it is useful to introduce the original method in detail with a shifted viewpoint.

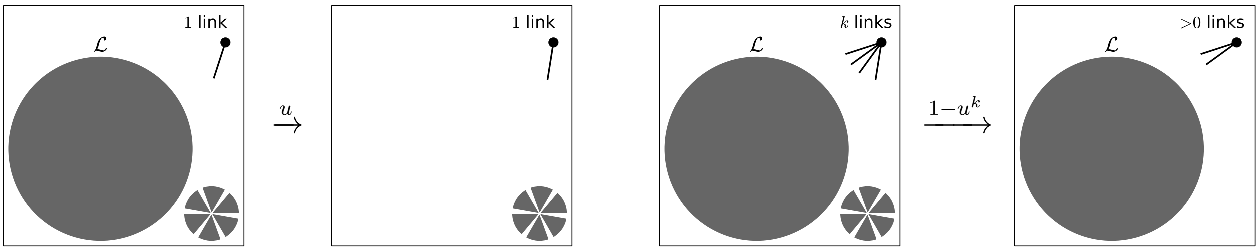

FIG. 1: We base our theory on the method to calculate the size of normal giant components, as illustrated in this figure. Using a self consistency equation, the probability u can be calculated. This is the probability, that a node is not connected to the giant component over a single link (see on the left). On the right, the probability for a node with k links is illustrated to have at least one link connecting to the giant component. u^k is the probability that all links fail.

Lets denote with $\mathcal{L}$ the set of all nodes belonging to the largest component. In figure 1 on the outer left, a possible situation is illustrated. The largest component contains of a large part of the network, and the remaining nodes belong to smaller components. We have to calculate the size S of the giant component, meaning the average relative size of $\mathcal{L}$ in the network ensemble in the limit of infinite network size. For this we can define the average probability u that a node fails to connect to $\mathcal{L}$ over one particular link. This is illustrated in the figure with the left part. Again, the thermodynamic limit $N \to \infty$ is implied. With the definition of u at hand, we can calculate S in two steps: First, using a self consistency equation, u is calculated. The probability u is identical to the probability that the neighbor does not connect to the giant component over any of the remaining links,

$$u = g_1(u), \quad g_1(z) = \sum_k q_k z^k. \tag{2}$$

In this equation, g_1 is the generating function of excess degree $q_k = (k+1)p_{k+1}/\bar{k}$. For important degree distributions as e.g. Poisson or scale-free, the equation for u can only be solved numerically. The second step is an averaging over nodes with various degrees k. The probability to connect to the giant component over any of k links is $(1 - u^k)$, meaning that not all links fail at the same time. This is illustrated in the figure on the right. As a node which connects to the giant component belongs to it,

$$S = \sum_{k=0}^{\infty} p_k(1 - u^k) = 1 - g_0(u), \quad g_0(z) = \sum_k p_k z^k. \tag{3}$$

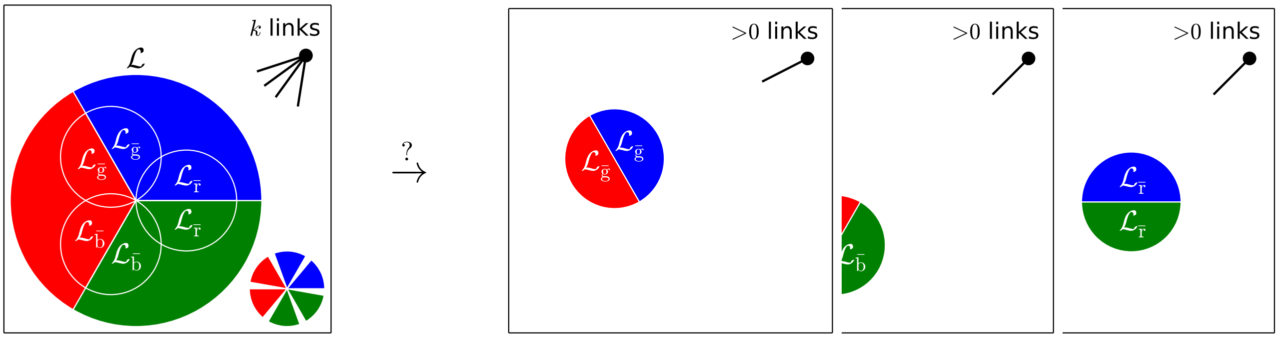

FIG. 2: We have to calculate the probability, if a node with k links is for every color c connected to the giant component $\mathcal{L}_{\bar{c}}$ with deleted color c. All connections over at least one link have to exist at the same time. We illustrate this question with the three colors red (c =r), green (c =g) and blue (c =b). If a link connects to $\mathcal{L}_{\bar{g}}$, it for sure does not connect to $\mathcal{L}_{\bar{c}}$ for one of the other colors. This kind of dependence forces us to use a stepwise calculation with conditional probabilities.

In analogy to the procedure described above, we will calculate S_{color} as the probability that a randomly chosen node belongs to $\mathcal{L}_{\text{color}}$. This has to be evaluated in the graph ensemble of infinite size. As we will perform an averaging over nodes with various degrees k, the following question has to be answered: What is the probability that a node with k links connects to a giant $\mathcal{L}_{\bar{c}}$ for all colors c at the same time. This is illustrated in figure 2. On the left, the situation for a graph with colors on the nodes is illustrated. Nodes of all colors might be in the largest component. After deleting all nodes of one color c, the remaining largest component $\mathcal{L}_{\bar{c}}$ might still contain a large part of all nodes in $\mathcal{L}$. The condition for the node belonging to $\mathcal{L}_{\text{color}}$ is illustrated on the right of the figure.

We will use the same two steps to attack this problem, as described for calculating the giant component above. First, we provide some single link probabilities which can be used as primitives for the further calculations. Second, we combine the single link probabilities to calculate S_{color}.

B. Single link probabilities

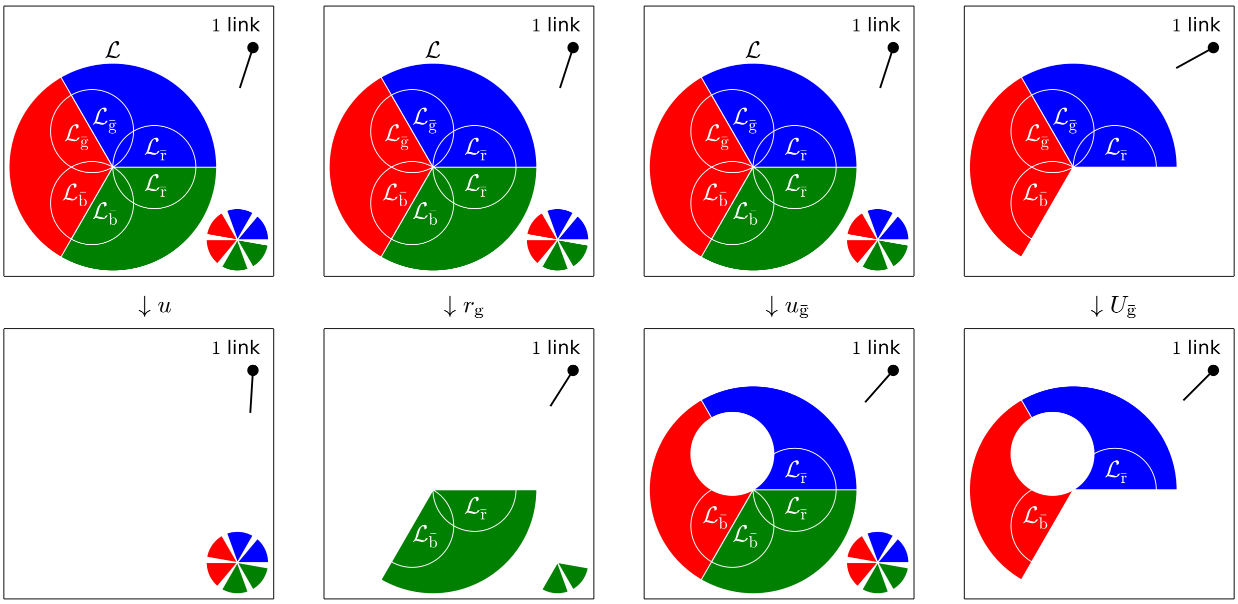

FIG. 3: Probabilities for a single link to connect to different parts of the network. We use these probabilities as primitives to calculate the probability for many links. While u, r_c and $u_{\bar{c}}$ can be calculated with standard methods invented for the configuration model before, the conditional probability $U_{\bar{c}}$ can be calculated as a combination of the others.

We already gave equation 2 for calculating the probability u. In the case of colors on the nodes, as illustrated in figure 3 on the left, the colors can simply be ignored. We further define r_c as the probability to connect to a node of color c. This is illustrated in the second column of the figure. r_c has to include the fact that high degree nodes are reached more probably, therefore the colors on high degree nodes are found with higher probability:

$$r_c = \sum_k k p_k \tilde{r}_{c,k}/\bar{k}. \tag{4}$$

If $\tilde{r}_{c,k}$ does not depend on the degree k, $r_c = \tilde{r}_{c,k}$. We further introduce $u_{\bar{c}}$, the probability that a single link does not connect to a giant $\mathcal{L}_{\bar{c}}$. See the third column of the figure for an illustration. This can be calculated using percolation theory for degree-dependent (targeted) attack by solving

$$u_{\bar{c}} = 1 - f_1(1) + f_1(u_{\bar{c}}), \quad f_1(z) = \sum_k q_k (1 - \tilde{r}_{c,k+1}) z^k. \tag{5}$$

Again, if the color distribution is independent of the degree, we get the simpler expression $u_{\bar{c}} = r_c + (1 - r_c) g_1(u_{\bar{c}})$ (percolation with random attack).

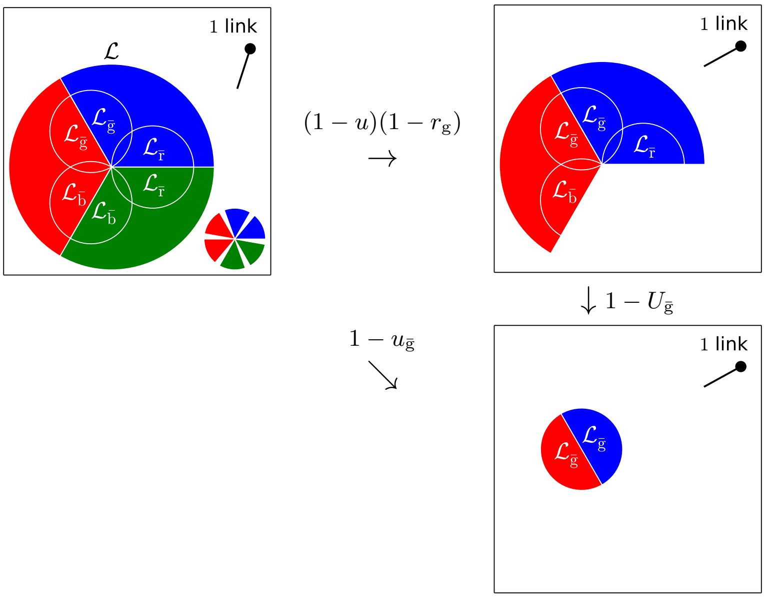

FIG. 4: This figure illustrates the calculation of $U_{\bar{g}}$ using the equality $(1-u)(1-r_{\mathrm{g}})(1-U_{\bar{g}}) = 1 - u_{\bar{g}}$. For that, we have assumed independence of the qualities of the link under consideration, especially of the color it connects to and if it connects to the giant component.

Unfortunately, $u_{\bar{c}}$ cannot be used directly for calculating S_{color}. If we look at the same link, the probabilities $u_{\bar{c}}$ are dependent for different colors. The most obvious argument is that always $\Pi_c(1 - u_{\bar{c}}) = 0$, as a link must at least miss one of the $\mathcal{L}_{\bar{c}}$. Instead, we will use the conditional probability $U_{\bar{c}}$, as illustrated with the outer right column of the figure. The precondition is that a link connects to the giant component and the node it connects to has not color c. $U_{\bar{c}}$ is the probability that such a link connects to $\mathcal{L}_{\bar{c}}$. For calculating it, we use the primitives introduced so far, as illustrated in figure 4. Assuming independence of the probabilities $(1-u)$ for connecting to the giant component and $(1-r_c)$ for not connecting to a node of color c, the precondition of $U_{\bar{c}}$ can be constructed. In this way, we can construct $(1 - u_{\bar{c}})$ using the probability we are searching for: $(1 - u_{\bar{c}}) = (1-u)(1-r_c)(1-U_{\bar{c}})$. With this we find

$$U_{\bar{c}} = 1 - \frac{1 - u_{\bar{c}}}{(1-u)(1-r_c)}. \tag{6}$$

Notice that the additional information of the explicit color, instead of only stating that the color is not c, does not alter the results, as a further restriction of the colors would meat the numerator and denominator identically and therefore would cancel out.

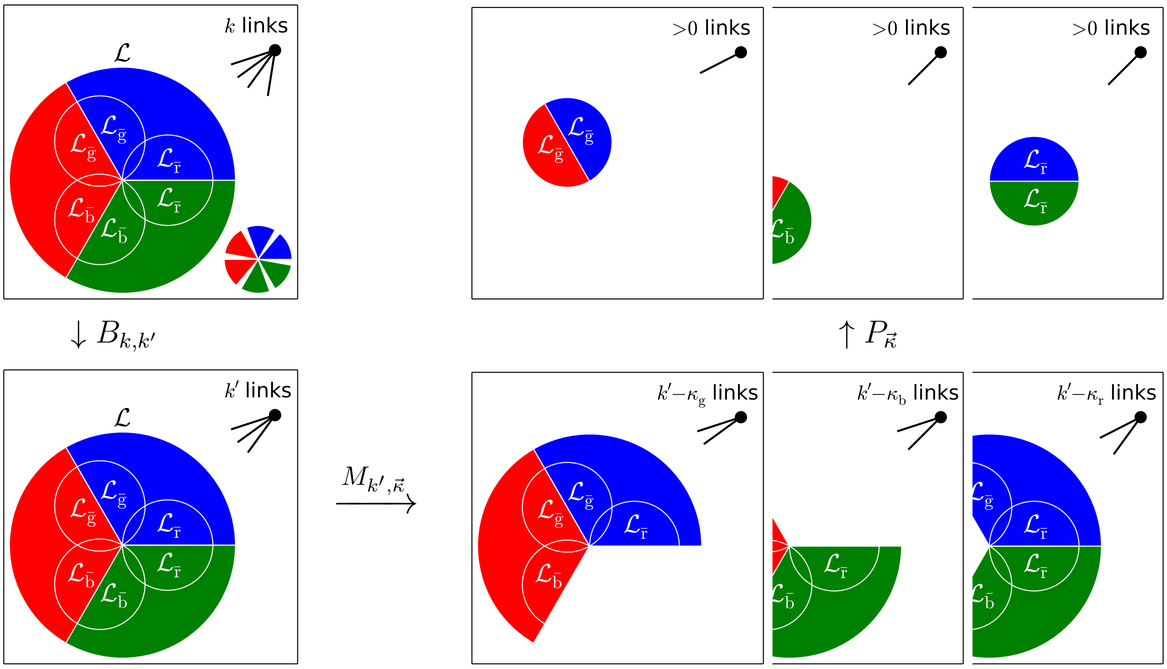

FIG. 5: For calculating the probability of a node with k links to belong to $\mathcal{L}_{\mathrm{color}}$, we have to average over different link constellations which this node might show. First, $B_{k,k'}$ is the probability that out of the k links k' connect to the giant component. It is calculated using u (compare figure 3 on the left). Second, $M_{k',\vec{\kappa}}$ gives the probability for a certain color distribution among the links. It is calculated using r_{g} etc. (compare figure 3, second from left). We assume that this second step is independent of the first step, what is confirmed with the final results. Third, $P_{\vec{\kappa}}$ gives the joint probability that for this color distribution $\mathcal{L}_{\bar{\mathrm{r}}}$, $\mathcal{L}_{\bar{\mathrm{b}}}$ and $\mathcal{L}_{\bar{\mathrm{g}}}$ are connected to at the same time. This is calculated using $U_{\bar{\mathrm{r}}}$ etc. (compare figure 3 on the right).

C. Averaging over link distributions

As in equation 3 for the giant component, we want to get an analytical result for S_{color} by averaging over possible link constellations of a randomly chosen node. Let us give the whole result and then explain it step by step afterwards:

$$S_{\mathrm{color}} = \sum_{k=0}^{\infty} p_k \sum_{k'=0}^{k} B_{k,k'} \sum_{\kappa_1,\ldots,\kappa_C=0}^{k'} M_{k',\vec{\kappa}} P_{\vec{\kappa}}, \tag{7}$$

$$B_{k,k'} = \binom{k}{k'} (1-u)^{k'} u^{k-k'}, \tag{8}$$

$$M_{k',\vec{\kappa}} = \frac{k'!}{\kappa_1! \times \cdots \times \kappa_C!} (r_1)^{\kappa_1} \times \cdots \times (r_C)^{\kappa_C} \, \delta_{k',\kappa_1+\cdots+\kappa_C}, \tag{9}$$

$$P_{\vec{\kappa}} = \prod_{c=1}^{C} [1 - (U_{\bar{c}})^{k'-\kappa_c}]. \tag{10}$$

The formulas include the single link probabilities u, r_c and $U_{\bar{C}}$ of equations (2), (4) and (6) (the last depends on (5)). An illustration of the procedure can be seen in figure 5. $B_{k,k'}$ is the binomial probability that out of the k links k' links connect to the giant component. $M_{k',\vec{\kappa}}$ gives the multinomial probability for a certain color distribution among the k' links connecting to the giant component. We assume that this second step is independent of the first step, what is confirmed with the final results. The numbers κ_c count the links which connect to a node of color c in the giant component. Finally, $P_{\vec{\kappa}}$ gives the joint probability that for the color distribution given by $\vec{\kappa}$ all giant $\mathcal{L}_{\bar{c}}$ are connected to at the same time. There is at least one link connecting to $\mathcal{L}_{\bar{c}}$ with probability $1-(U_{\bar{c}})^{k'-\kappa_c}$. The success probabilities for different colors have to be multiplied, as all $\mathcal{L}_{\bar{c}}$ have to be reached at the same time. We tested numerically that e.g. $U_{\bar{1}}$ and $U_{\bar{2}}$ are independent for a link connecting to the giant component and a third color.

II. EXAMINATION OF THE THEORY

A. k'-decomposition and an upper bound

For a further examination of S_{color}, it is useful to rearrange equation 7 with focus on k', the number of links connecting a single randomly chosen node to the giant component,

$$S_{\text{color}} = \sum_{k'=0}^{\infty} \underbrace{\sum_{k=k'}^{\infty} p_k B_{k,k'}}_{\pi_{k'}} \underbrace{\sum_{\kappa_1,\cdots=0}^{1} M_{k',\vec{\kappa}} P_{\vec{\kappa}}}_{\sigma_{k'}} = \sum_{k'=0}^{\infty} \pi_{k'} \sigma_{k'}. \tag{11}$$

The first term $\pi_{k'}$ gives the probability that a randomly chosen node connects over exactly k' nodes to the giant component. Using equation 8, it can be shown that $\sum_{k'=0}^{\infty} \pi_{k'} = 1$. This quality is meaningful, as every link should have either no or some connections to the giant component. Using the same equation, we can write $\pi_{k'}$ in terms of generating functions,

$$\pi_{k'} = \frac{(1-u)^{k'}}{k'!} \left. \frac{\mathrm{d}^{k'} g_0(z)}{\mathrm{d}z^{k'}} \right|_{z=u}. \tag{12}$$

By construction of the generating function we have $\pi_{k'} = p_k$ for $u = 0$, what should hold of cause. With equation 3 we find $S = 1 - \pi_0 = \sum_{k'=1}^{\infty} \pi_{k'}$.

The second term $\sigma_{k'}$ is the probability that a node connecting over k' links to the giant component belongs to S_{color}. As can be seen with equation 10, $\sigma_0 = \sigma_1 = 0$. On the other hand, with $k' \geq 2$ and probabilities $U_{\bar{c}}$ close to zero, $\sigma_{k'} \leq 1$ can come close to its upper limit. Therefore the upper limit

$$S_{\text{color}} \leq \sum_{k'=2}^{\infty} \pi_{k'} = 1 - \pi_0 - \pi_1 \equiv S_{\text{color},\infty} \tag{13}$$

is useful. Compared to the size of the giant component $S = 1 - \pi_0$, S_{color} misses at least all nodes which are connected to the giant component over $k' = 1$ links. This result is closely connected to k-core percolation with $k = 2$. Note that k-core percolation shows a continuous phase transition for $k = 2$, and only for $k > 2$ has the well known discontinuous behavior.

B. Closed form solutions

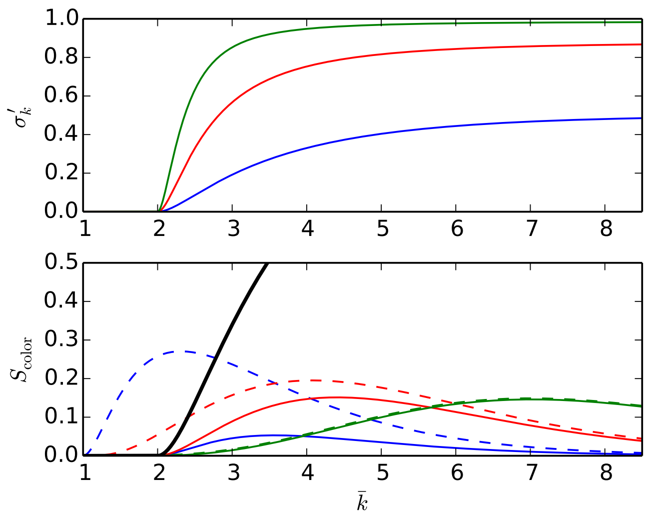

FIG. 6: S_{color} can be divided into contributions of nodes with different numbers k' of links to the giant component.

We now will calculate a closed form result for S_{color} on Poisson graphs with two colors. This is done to demonstrate how the extensive summations over k', k and $\vec{\kappa}$ can be performed analytically. In cases where this is not possible, a sampling of values $\vec{\kappa}$ has to be performed. The results can be tested against the analytically tractable situations and by comparing with numerical results. Additionally, our result will help us to gain first insights about the critical behavior.

For Poisson graphs, $\pi_{k'}$ can be given in a short and useful form for all k'. Instead of using generating functions, we simply rewrite

$$\pi_{k'} = \sum_{k=k'}^{\infty} \frac{\bar{k}^k e^{-\bar{k}}}{k!} \frac{k!}{(k-k')!k'!} u^{k-k'} (1-u)^{k'} \tag{14}$$

$$= \underbrace{\frac{\bar{k}^{k'} e^{-\bar{k}}}{k'!}}_{p_{k'}} e^{u\bar{k}} (1-u)^{k'} \underbrace{\sum_{k=k'}^{\infty} \frac{(u\bar{k})^{k-k'} e^{-u\bar{k}}}{(k-k')!}}_{1} \tag{15}$$

$$= p_{k'} e^{u\bar{k}} (1-u)^{k'}. \tag{16}$$

The most promising fact of this result is that it contains $p_{k'}$, and therefor the simply evaluable generating function of the Poisson distribution can be used below,

$$g_0(z) = e^{\bar{k}(z-1)}. \tag{17}$$

For evaluating $\sigma_{k'}$ with two colors, we rewrite

$$\sigma_{k'} = \sum_{\kappa_1=0}^{k'} \binom{k'}{\kappa_1} (r_1)^{\kappa_1} (r_2)^{k'-\kappa_1} \, [1-(U_{\bar{1}})^{k'-\kappa_1}][1-(U_{\bar{2}})^{\kappa_1}] \tag{18}$$

$$= \sum_{\kappa_1=0}^{k'} \binom{k'}{\kappa_1} \left[(r_1)^{\kappa_1} (r_2)^{k'-\kappa_1} - (r_1 U_{\bar{2}})^{\kappa_1} (r_2)^{k'-\kappa_1} - (r_1)^{\kappa_1} (r_2 U_{\bar{1}})^{k'-\kappa_1} + (r_1 U_{\bar{2}})^{\kappa_1} (r_2 U_{\bar{1}})^{k'-\kappa_1} \right] \tag{19}$$

$$= 1 - (r_1 + r_2 U_{\bar{1}})^{k'} - (r_2 + r_1 U_{\bar{2}})^{k'} + (r_1 U_{\bar{2}} + r_2 U_{\bar{1}})^{k'}. \tag{20}$$

In the last step, the binomial formula was used backward. Notice that this result is independent of the degree distribution. In figure 6, $\sigma_{k'}$ (upper panel), $\pi_{k'}$ (lower panel with dashed lines) and $\sigma_{k'}\pi_{k'}$ (solid lines) are given for $k'=2$ (blue), $k'=4$ (red) and $k'=7$ (green) on Poisson graphs. The fat black line shows S_{color}. We continue with the Poisson distribution and find with equation 6 and some rewriting

$$S_{\mathrm{color}} = e^{u\bar{k}} \sum_{k'=0}^{\infty} p_{k'} (1-u)^{k'} \sigma_{k'} \tag{21}$$

$$= e^{u\bar{k}} [g_0(1-u) - g_0(u_{\bar{1}} - u) - g_0(u_{\bar{2}} - u) + g_0(u_{\bar{1}} + u_{\bar{2}} - 1 - u)]. \tag{22}$$

For larger numbers of colors, giant $\mathcal{L}_{\bar{c}}$ will overlap. This is reflected in terms including products of the conditional probabilities $U_{\bar{c}}$ which cannot be further simplified. However, even for larger numbers of colors there are closed form expressions with generating functions. They contain increasing numbers of terms for increasing C, therefore they are probably only useful for a few colors. For two colors, it is even possible to rewrite the final result as a simple product:

$$S_{\mathrm{color}} = [1 - g_0(u_{\bar{1}})][1 - g_0(u_{\bar{2}})]. \tag{23}$$

C. Critical behavior for Poisson graphs

The terms $\sigma_{k'}$ imply demanding summations and products, and the critical behavior will be evaluated with a number of consecutive approximations. For give a first glance, it is useful to discuss the case with two colors first. For this we found a simple closed form expression which can be discussed without much effort.

1. Two colors

With equations (5), (17) and (23) we have to analyze

$$S_{\text{color}} = [1 - g_0(u_{\bar{1}})][1 - g_0(u_{\bar{2}})], \tag{24}$$

$$u_{\bar{c}} = r_c + (1 - r_c)g_1(u_{\bar{c}}). \tag{25}$$

2. Critical connectivity with homogeneous color distribution

As long as $U_{\bar{c}} = 1$ for any color c, $P_{\vec{\kappa}} = \prod_{c=1}^{C}[1 - (U_{\bar{c}})^{k'-\kappa_c}]$ vanishes, and therefore $\sigma_{k'} = 0$ for every k'. For a homogeneous color distribution we have $r_c = r_1 = 1/C$ for all colors c, and accordingly $U_{\bar{c}} = U_{\bar{1}}$ is identical for all colors. As for $\bar{k} \leq 1$ the giant component has vanishing size, the conditional probability

$$U_{\bar{1}} = 1 - \frac{1 - u_{\bar{1}}}{(1-u)(1-r_1)} \tag{26}$$

is not defined in this region. However, we find $S_{\text{color}} = 0$ due to $\pi_{k'} = 0$. For $\bar{k} > 1$, $U_{\bar{c}} = 1$ whenever $u_{\bar{1}} = 1$, meaning that a single link has vanishing probability to connect to $G_{\bar{1}}$. We can find

$$u_{\bar{1}}(\bar{k}) = r_1 - (1 - r_1)u[\bar{k}(1 - r_1)], \tag{27}$$

as this solution inserted into the self consistency equation $u_{\bar{1}} = r_1 + (1-r_1)g_1(u_{\bar{1}})$ leads to the self consistency equation of $u = g_1(u)$. This enables us to calculate the critical value $\bar{k}_{\text{crit}}$, below which $S_{\text{color}} = 0$:

$$\bar{k}_{\text{crit}} = 1/(1 - r_1) = C/(C-1). \tag{28}$$

3. Critical behavior for a few colors

The normal giant component size S shows a special critical behavior shortly above the transition point, it scales linearly with $\bar{k} - 1$. Here we are interested in the behavior of S_{color} which is a function of $1 - u_{\bar{1}}$ which itself can be related to $1 - u = S$. For small arguments $(\bar{k} - \bar{k}_{\text{crit}})$,

$$1 - u_{\bar{1}}(\bar{k} > \bar{k}_{\text{crit}}) \approx (1 - r_1)^2 \left.\frac{\mathrm{d}(1-u)}{\mathrm{d}\bar{k}}\right|_{\bar{k}=1+0} (\bar{k} - \bar{k}_{\text{crit}}) \tag{29}$$

holds due to equation 27. Inserting into $U_{\bar{1}}$ we find using $1 - u(\bar{k} > 1) \approx \left.\frac{\mathrm{d}(1-u)}{\mathrm{d}\bar{k}}\right|_{\bar{k}=1+0} (\bar{k} - 1)$

$$\varepsilon \equiv 1 - U_{\bar{1}} \approx C(\bar{k} - \bar{k}_{\text{crit}}) \tag{30}$$

if additionally $\bar{k} - \bar{k}_{\text{crit}} \ll \bar{k} - 1$ holds ($1 - u_{\bar{1}}$ small compared to $1 - u$).

For calculating $\sigma_{k'}$, we first need to evaluate $P_{\vec{\kappa}}$ including expressions $1 - (U_{\bar{1}})^{k'-\kappa_c}$. Replacing with ε and applying an approximation we find $1 - (U_{\bar{1}})^{k'-\kappa_c} = 1 - (1-\varepsilon)^{k'-\kappa_c} \approx (k' - \kappa_c)\varepsilon$. This is true at least as long as $k'\varepsilon \ll 1$. With this we find $P_{\vec{\kappa}} \propto (\bar{k} - \bar{k}_{\text{crit}})^C$ independent of $\vec{\kappa}$, and finally

$$\sigma_{k'} \propto (\bar{k} - \bar{k}_{\text{crit}})^C. \tag{31}$$

We will see below that the largest k' giving a notable contribution to S_{color} is smaller then the number of colors C, therefore $k'\varepsilon \ll 1$ can be met and we finally find

$$S_{\text{color}} \propto (\bar{k} - \bar{k}_{\text{crit}})^{\beta}, \tag{32}$$

$$\beta = C. \tag{33}$$

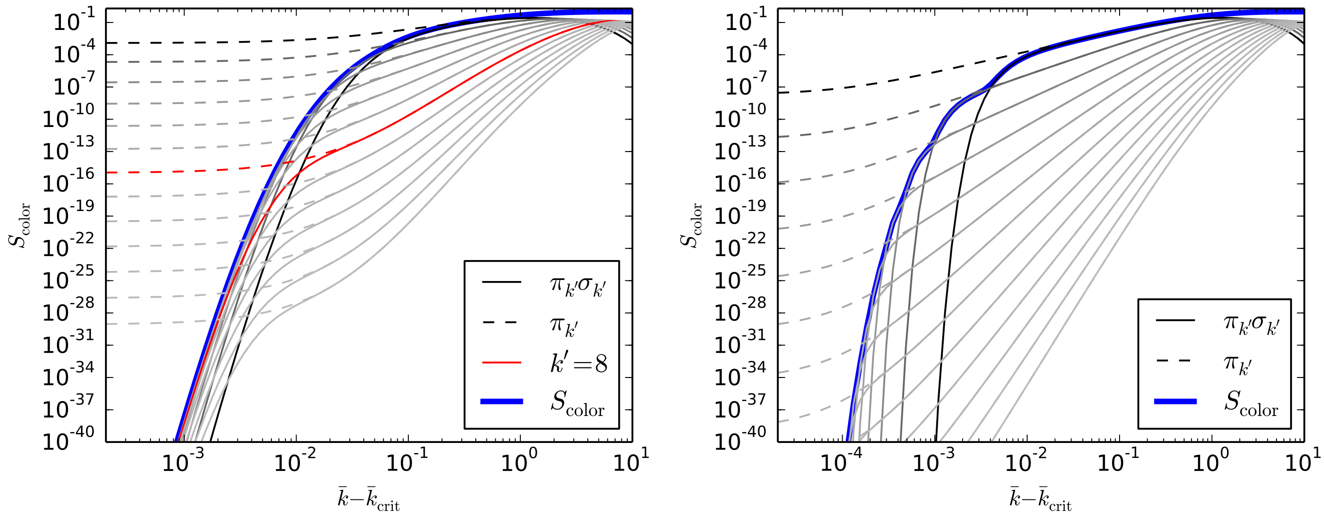

FIG. 7: S_color can be divided into contributions of nodes with different numbers k' of links to the giant component.

4. Staircase of crossovers for many colors

For large numbers of colors C, the functions $\sigma_{k'}$ have a rather extreme scaling behavior $\sigma_{k'} \propto (\bar{k} - \bar{k}_\mathrm{crit})^C$. This causes two open questions. First, as the functions $\sigma_{k'}$ are saturating to one for high values of $\bar{k} - \bar{k}_\mathrm{crit}$, there could be almost jump-like behavior. Second, contributions with increasing k' become important. Roughly speaking, this is due to the fact that $\sigma_{k'}$ is going to zero very fast with $\bar{k} - \bar{k}_\mathrm{crit} \to 0$, and therefore the exponential suppression in $\pi_{k'}$ with k' is compensated for.

For attacking both of these questions, let us apply an approximation. With a homogeneous color distribution $U_{\bar{c}} = U_{\bar{1}}$ and an expected value $\langle \kappa_c \rangle = k'/C \ll k'$, we find

$$P_{\vec{\kappa}} \overset{<}{\approx} (1 - (U_{\bar{1}})^{k'})^C, \tag{34}$$

$$\sigma_{k'} \overset{<}{\approx} (1 - (U_{\bar{1}})^{k'})^C. \tag{35}$$

As the approximate $P_{\vec{\kappa}}$ is independent of $\vec{\kappa}$, the summation over $M_{k',\vec{\kappa}}$ was easily executed. Now we are able to compare the contributions $\pi_{k'}\sigma_{k'}$ for $\bar{k} \to \bar{k}_\mathrm{crit}$, implying that $k'\varepsilon \ll 1$ holds. We have to find, whether above a certain k' the following holds:

$$\log(\frac{\pi_{k'+1}\sigma_{k'+1}}{\pi_{k'}\sigma_{k'}}) < 0. \tag{36}$$

We can find with

$$\log(\sigma_{k'}) \approx C \log[k'(1-u_{\bar{1}})/((1-u)(1-r_1)] \tag{37}$$

$$\approx C \log(k') + \{\text{independent of } k'\}, \tag{38}$$

$$\log(\pi_{k'}) = k' \log(\bar{k}) - \log(k'!) + k' \log(1-u) + \{\text{independent of } k'\}. \tag{39}$$

$$\tag{40}$$

and using $\log(\bar{k}_\mathrm{crit}) \approx 1/C$ and $\log(1-u(\bar{k}_\mathrm{crit})) \approx -\log(C)$ and $\log(1=x) \approx x$ for small x:

$$\log(\frac{\pi_{k'+1}\sigma_{k'+1}}{\pi_{k'}\sigma_{k'}}) \approx C/k' - \log(k') + 1/C - \log(C). \tag{41}$$

$$\tag{42}$$

This falls below zero approximately at $k' \log(Ck') = C$, and clearly contributions are exponentially dampened with large k'. Up to the k' with $k' \log(Ck') = C$, contributions $\pi_{k'}\sigma_{k'}$ grow. That is surprising, as nodes with increasing k' should be much less probable. However, if $\sigma_{k'}$ is very small that compensates for the other effect in a certain regime of k'. For estimating the critical region, where $S_{\rm color} \propto (\bar{k} - \bar{k}_{\rm crit})^C$, we use the upper bound for the leading term $k' = C$ and find with $k'\varepsilon = k'C(\bar{k} - \bar{k}_{\rm crit}) \ll 1$ finally

$$(\bar{k} - \bar{k}_{\rm crit}) \ll 1/C^2. \tag{43}$$

This region is not of practical importance for large C, as there are very small values $S_{\rm color}$.

Notice that this expression does not depend on $\vec{\kappa}$, what allows us to evaluate the summation over $M_{k',\vec{\kappa}}$ with result one. We have

$$S_{\text{color}} \leq \sum_{k'=2}^{\infty} \pi_{k'} \sigma_{k'}. \tag{44}$$

We will show below that $\sigma_{k'}$ has the following shape:

$$\sigma_{k'} \approx \begin{cases} 0 & \bar{k} < \bar{k}_{\text{crit}} \\ a(\bar{k} - \bar{k}_{\text{crit}})^C & \bar{k}_{\text{crit}} < \bar{k} \ll \bar{k}_{\text{sat}} \\ 1 & \bar{k}_{\text{sat}} \ll \bar{k} \end{cases}, \tag{45}$$

$$\bar{k}_{\text{crit}} = 1/(1-r_1) = C/(C-1), \tag{46}$$

$$a_{k'} = \ldots \tag{47}$$

$$\bar{k}_{\text{sat}}(k') = C^{-(k'-1)/k'} * (1 - C^{-1/k'}). \tag{48}$$

$$\tag{49}$$

Below $\bar{k}_{\text{crit}} = 1/(1-r_1) = C/(C-1)$ it is zero (links connected to the giant component can not communicate with avoidable colors). Above the critical value $\bar{k}_{\text{crit}}$, the critical behavior $\sigma_{k'} \propto (\bar{k} - \bar{k}_{\text{crit}})^C$ can be observed up to a saturation region. Independent of k', $\sigma_{k'}$ saturates to one, however, the argument $\bar{k}$ where saturation is reached depends on k'. We need to calculate the saturation regions before combining our results into the total critical behavior.

5. *Evaluation of the saturation function*

$U_{\bar{1}}(\bar{k})$ is a monotone decreasing function with values in the interval $[0,1]$

$$\sigma_{k'} \leq (1 - (U_{\bar{1}})^{k'})^C, \tag{50}$$

$$U_{\bar{1}} = 1 - \frac{1 - u_{\bar{1}}}{(1-u)(1-r_1)} \tag{51}$$

This includes results of the self consistency equations $u = g_1(u)$ and $u_{\bar{1}} = r_1 + (1-r_1)g_1(u_{\bar{1}})$ (with $r_1 = 1/C$). We can find the critical onset by expanding $g_1(x) = \exp[(x-1)\bar{k}]$ for arguments $x \approx 1$:

$$u \approx a \times (\bar{k} - 1) \tag{52}$$

$$u_{\bar{1}} \approx a(1-r_1) \times (\bar{k} - \bar{k}_{\text{crit}}) \tag{53}$$

$$\bar{k}_{\text{crit}} = 1/(1-r_1) = C/(C-1) \tag{54}$$

with a constant a.

D. Critical behavior for Poisson graphs

With general color distributions r_c ($\sum_c r_c = 1$), the color with the largest probability r_c dominates the behavior, as it corresponds to the largest conditional link failure probability $U_{\bar{c}}$ in equation 10. For Poisson graphs, $U_{\bar{c}}$ falls below one at $\bar{k} = 1/(1-r_c)$, and as long as one $U_{\bar{c}}$ is one, equation 10 gives always zero. Therefor the critical value is

$$\bar{k}_{\text{crit}} = 1/(1 - \max_c r_c). \tag{55}$$

We can understand the critical exponent with expanding equation 10 using $U_{\bar{c}} = 1 - \varepsilon$ for the highest color frequency. We will show below using results from standard percolation that $\varepsilon \propto (\bar{k} - \bar{k}_{\text{crit}})$. First of all, we have $(U_{\bar{c}})^{\sum_{c' \neq c} \kappa_{c'}} \approx 1 - (\sum_{c' \neq c} \kappa_{c'}) \times \varepsilon$, and therefore we find $P_{\vec{\kappa}} \propto \varepsilon^{n_{\text{deg}}}$ (unless $P_{\vec{\kappa}} = 0$ in the case $\sum_{c' \neq c} \kappa_{c'} = 0$ for any c). As equation **??** is therefore a superposition of either vanishing terms or terms with leading order $\varepsilon^{n_{\text{deg}}}$, we have

$$\beta = n_{\text{deg}}. \tag{56}$$

To complete the discussion of the critical behavior, we have to show that $1 - U_{\bar{c}} \propto (\bar{k} - \bar{k}_{\rm crit})$ for a certain region of critical behavior. We know from standard percolation theory for Poisson graphs that $S \propto (\bar{k} - 1)^1$, and therefore $1 - u(\bar{k}) \approx a \times (\bar{k} - 1)$ for $\bar{k}$ exceeding the critical value about small values. With inserting into equation 5 it can be shown that $u_{\bar{c}}(\bar{k}) = 1 - \phi_{\bar{c}} + \phi_{\bar{c}} u(\bar{k}\phi)$. Using this in equation 6 ($\phi_{\bar{c}} = 1 - r_c$), we find

$$U_{\bar{c}} = 1 - \frac{1 - u(\bar{k}\phi_{\bar{c}})}{1 - u(\bar{k})} \tag{57}$$

$$\approx 1 - \phi_{\bar{c}} \frac{\bar{k} - \bar{k}_{\rm crit}}{\bar{k} - 1} \tag{58}$$

which drops linearly from one in the critical region

$$0 < \bar{k} - \bar{k}_{\rm crit} \ll \bar{k}_{\rm crit} - 1. \tag{59}$$

In figure **??** we have $\bar{k}_{\rm crit} - 1 = 1/2$, and critical behavior up to about $\bar{k} - \bar{k}_{\rm crit} = 1/10$.

As a final remark let us discuss the critical behavior for largest color frequencies which are not perfectly degenerated. Lets assume two colors with close by values $\bar{k}_1 < \bar{k}_2$, where $U_{\bar{c}}$ drops below one. With equation 10 and the definition $\varepsilon = \bar{k} - \bar{k}_2$ we have above $\bar{k}_2$ that $P_{\vec{\kappa}} \propto \varepsilon \times (\varepsilon + (\bar{k}_2 - \bar{k}_1))$. This is dominated by a linear term for small ε and by a quadratic term for larger values, the crossover is at about

$$\bar{k} - \bar{k}_2 = \bar{k}_2 - \bar{k}_1. \tag{60}$$

Exceeding the critical value $\bar{k}_{\rm crit} = \bar{k}_2$ about more than the distance $\bar{k}_2 - \bar{k}_1$, $S_{\rm color}$ behaves as if the color frequencies would be degenerated.

With these results altogether we can understand the behavior of $S_{\rm color}$ for small frequencies of all colors. There is a deviation from $S_{\rm color,\infty}$ due to $S_{\rm color} = 0$ below $\bar{k}_{\rm crit}$, and above there is a region of slow critical growth with large critical exponent β. This region acts as an effective shift of the critical parameter. Further increasing $\bar{k}$, $U_{\bar{c}}$ saturates smoothly to zero for all colors. Accordingly, $S_{\rm color}$ has a smooth rise to finite values, but not governed by a certain critical exponent. It comes closer to $S_{\rm color,\infty}$ without reaching it, as there are other effects (e.g. nodes with exactly two links can connect to two nodes of the same color). The smaller the highest color frequencies are, the closer $S_{\rm color}$ is to $S_{\rm color,\infty}$.