The hopf fibration and hidden variables in quantum and classical mechanics

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Abstract. The quaternion is a natural representation of the magnetic moment of the fundamental particles. Under the hopf fibration the parameter space of the quaternion separates into an intrinsic and extrinsic parameter space, and accounts for the intrinsic and extrinsic spin of the fundamental particles. The intrinsic parameter space is the global, geometric and dynamic phases which are presented in this article in full generality. The equivalence between the quantum and classical equations of motion is established, and the global phase of the quaternion is shown to be a natural hidden variable which deterministically accounts for the results of the Stern-Gerlach experiment.

In one of his many great discourses on the quantum theory the formidable thinker, John Stewart Bell, once proposed that the prevailing theories of modern physics *relativity theory and the quantum theory* are akin to two great pillars [1, *ch* 18]. It so follows that if modern physics were a great temple, these two pillars would be the supporting columns of the roof. In order for the temple to be structurally sound, both the construction and position of these pillars must exhibit an inherent harmony with respect to each other, and the temple itself. Should one or the other be out of sync, it undermines the structural integrity of the temple and the entire building could collapse.

Yet the fact remains that the great pillars of modern physics are in conflict with each other, as they are entirely incompatible. Relativity theory, which is a theory applied to the macroscopic bodies, planets, stars and so forth, is a deterministic theory - and declares that nature at her core is deterministic and that her laws are that of arithmetic, and geometry. The quantum theory, which is a theory applied to the microscopic bodies, atoms, electrons and so forth, is a non-deterministic theory - and declares that nature at her core is non-deterministic and that her laws are probabilistic, that the states of the fundamental particles are not defined a priori, but require observation by an observer to 'create' our familiar classical world. This rather unsettling position is tentatively accepted today, as it is professed that there exists somewhere a barrier, a dividing line if you will, between the quantum realm and our classical reality [2]. That is to say that the fundamental particles are believed to obey laws that differ from those of the classical bodies, and that observation is a necessary ingredient for the quantum particle to make the transition across the quantum-classical border. A perspective which is aptly summarized in the Copenhagen Interpretation of quantum mechanics.

The Copenhagen Interpretation of quantum mechanics, is a probabilistic interpretation of deterministic equations, named after Danish physicist Niels Bohr who was among the principle proponents of this point of view. Among his supporters were Werner Heisenberg, Max Born, Wolfgang Pauli and John von Neumann. As with many concepts put forward by the quantum theory, a precise definition of the Copenhagen Interpretation is hard to come by. Nonetheless there is one aspect that is certain. *"The key feature of the Copenhagen Interpretation is the dividing line between quantum and classical"* [2]*

The quantum-classical boundary, combined with the probabilistic interpretation, gives rise to the concept of *the quantum superposition* - that the state of the fundamental particle is not defined before measurement, rather - it is a field of potentialities, in which the particle exists in many instances at once - until the point of observation when the field collapses, to become an actuality, thereby creating the measured state.

The other main postulate of the Copenhagen Interpretation is *Heisenberg's uncertainty principle*, and together these are summarized as follows.

(i) Heisenberg's uncertainty principle: Complimentary observables (such as position and momentum) cannot be measured with absolute precision simultaneously, and the lower bound of precision is given by Heisenberg's uncertainty relation.

(ii) The principle of superposition: The quantum particle exists in a weighted superposition of all possible states until such a time as a measurement occurs, at which point the wave-function 'collapses' into the measured state.

Of these two aspects of the Copenhagen Interpretation, Heisenberg's uncertainty principle is the least challenged. This partly due to the fact that it is unsurprising that any measurement of a subatomic system will perturb the system itself, thus sequential measurements of complimentary observables lose their meaning since each measurement changes the system in some small way.

The principle of superposition infers that *"The laws of nature formulated in mathematical terms no longer determine the phenomena themselves, but ... the probability that something will happen"* [3, *pg* 17]§. That is to say that quantum mechanics does not in any way describe the particle itself, it gives only the probability of an experimental result, the probability of finding the particle in a given state. That amounts to saying that the fundamental particles are not in themselves real, as the quantum realm is a world of potentialities rather than actualities. Only following an act of observation does the world of potentialities 'collapse' to create the particle's measured state.

"It is a fundamental quantum doctrine that a measurement does not, in general, reveal a preexisting value of the measured property. On the contrary, the outcome of a measurement is brought into being by the act of measurement itself, a joint manifestation of the state of the probed system and the probing apparatus. Precisely how the particular

* Wojciech H. Zurek

§ Werner Heisenberg

result of an individual measurement is brought into being - Heisenberg's 'transition from the possible into the actual' - is inherently unknowable. Only the statistical distribution of many such encounters is a proper matter for scientific inquiry." [4]*

But how can a particle exist in many states at once? How can a statement like that be proven, is it an artificial construct, devised to explain away the unknown, or is it simply the way nature is at her core? How can we be sure we are not completely misguided in promulgating these concepts - I mean, what if *"The appearance of probability is merely an expression of our ignorance of the true variables in terms of which one can find casual laws."* [5, *pg* 114]†

The Copenhagen Interpretation certainly renders quantum mechanics a very uncomfortable place to study physics, as progress steam rolls ahead without concern for the gaping hole between the predictions of the theory and the results of experimental measures, and this gaping hole constitutes the measurement problem of quantum mechanics, *"What exactly qualifies some physical system to play the role of 'measurer'? Was the wave-function of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared."* [6]§

Appeals to reason are not well received, and this is best exemplified in the case of Schrödinger's cat, which - in an ironic twist of fate - is today used to explain the weird and wonderful world of quantum mechanics. In order to elucidate the staggering consequences of accepting the principle of superposition as an aspect of reality, Schrödinger proposed a thought experiment in which he had a sealed box which housed his cat. Inside the box is a poisonous gas set to be released upon the decay of a radioactive element. The apparatus is allowed to sit for a period of time, in which there is a 50% probability that the nuclear element decays, releases the gas, and kills the cat. Before the box is opened one does not know whether the cat is alive or dead, and according to the Copenhagen Interpretation we must declare that the cat is in a superposition of being alive and dead. Of course from the cat's perspective, he is either alive *or* dead, but from the quantum mechanic's perspective he is both alive *and* dead. That is of course, until the box is opened and the wave-function of the cat 'collapses' and he is found to be either alive *or* dead.

To the philosopher in the street the solution is obvious - there is something radically wrong with the quantum mechanic's perspective, as if they were making a mountain out of a molehill. But the quantum mechanic is unperturbed. Safe in the knowledge that the paradox is only apparent, and will be resolved by the quantum theory at some point down the road. In the meantime we have Everett's many worlds interpretation [7] which claims to do away with the quantum-classical border, as the wave-function of the universe is thought to branch at every point of observation, and in the case of Schrödinger's cat - when the box is opened - the universe branches into one where the cat is alive and one where it is dead, ... and its all just a bit too much as we might as well

* David Mermin
† David Bohm
§ John Stewart Bell

be told that some monkey somewhere, in some branched universe, found a typewriter and wrote Hamlet.‡ Surely the measurement problem is not just a major failing of the quantum theory, but a gaping black hole in which any would-be mathematician worth their salt would lose-their-mind trying to make sense and/or use of the quantum theory.

"The only 'failure' of the quantum theory is its inability to provide a natural framework that can accommodate our prejudices about the workings of the universe." [2]* And what prejudices might they be? That nature might make sense, that there may be a inherent harmony to her workings, that the study of physics, which leads us to mathematics and geometry might actually afford us some appreciation of the workings of the natural world?

When the principle of superposition is combined with Heisenberg's uncertainty principle, we are resigned to the fact that *"In quantum mechanics there is no such concept as the path of a particle. ... The fact that an electron has no definite path means that ... for a system composed only of quantum objects, it would be entirely impossible to construct any logically independent mechanics"* [8, *pg* 2]† This testimony from the quantum theory demonstrates the little hope there is that a deterministic account of the fundamental processes will ever emerge from the theory. In this regard quantum mechanics is a unique physical theory as it is the only non-deterministic theory of the physical sciences. While disciplines like statistical mechanics, thermodynamics and general relativity each exhibit a distinct relationship to classical mechanics, the same cannot be said for the quantum theory as there is no clear relationship between quantum mechanics and classical mechanics. *"Quantum mechanics occupies a very unusual place among the physical theories: it contains classical mechanics as a limiting case, yet at the same time requires this limiting case for its own formulation"* [8, *pg* 3]†

These many issues of the quantum theory and the problems associated with it's non-deterministic interpretation were recognized early in the development of the theory, when it was acknowledged that the theory itself is incomplete [9]. To say quantum mechanics is incomplete is to say that there exists hidden variables which remain unaccounted for by the theory. The purported hidden variables are expected to remove the indeterminism of quantum mechanics, and - since the proposal of their existence in 1935 - the hidden variables of quantum mechanics have remained elusive.

It was the best part of 80 years (2015) since it was recognized by Einstein, Podolsky and Rosen that quantum mechanics is incomplete [9] when it was first proposed that *'the global phase of the qubit is a natural hidden variable.'* [10] In this article I show that the hidden variables of quantum mechanics are hidden spatial dimensions. Using a projection between dimensional spaces known as *the hopf fibration* - it is demonstrated that the 4th-dimension of the qubit "the global phase" is encoded in the 3d kinematics as a natural hidden variable.

Far from being just a *"2-level quantum system"* the qubit is a quaternion [11].

‡ *"Something is rotten in the state of Denmark."* Marcellus to Horatio, Act 1 Scene 4.
* Wojciech H. Zurek
† Lev Landau and Evgeny Lifshitz

Applied to the unit quaternion, the hopf fibration is a projection between the 3-sphere ($\mathbb{S}^3$ embedded in $\mathbb{R}^4$) and the 2-sphere ($\mathbb{S}^2$ embedded in $\mathbb{R}^3$). The 4- and 3-dimensional spaces are connected via the unit circle $\mathbb{S}^1$, which is a fibre bundle consisting of the global, geometric and dynamic phases.

$$\mathbb{S}^3 \xmapsto{\mathbb{S}^1} \mathbb{S}^2$$

The 4th-dimension cannot be directly *"seen"* in 3-dimensional space, we can perceive only it's "shadows" - which are the intrinsic spin of the fundamental particles. In this article we demonstrate that the global phase of the qubit exhibits the required properties to formulate a deterministic theory of particle spin for both fermions and bosons.

The article is structured as follows. We begin by reviewing the present theory of particle spin according to quantum mechanics, and show that the origins of non-determinism in the quantum theory is derived from the common use and acceptance of the Copenhagen Interpretation of the quaternion. We postulate that there is no scientific basis for this and that the present theory of particle spin rests on a Schrödinger's cat hypothesis, and is subsequently meaningless. In section 1 we review the fundamentals of the quaternion and introduce the cayley matrices. The hopf fibration is defined in section 2, as are the closed form expressions for the extrinsic and intrinsic parameters of the quaternion. The geometric phase is defined by solving the equation of parallel transport for all closed paths on the 2-sphere, and we show that the global phase is a sum of the geometric and dynamic phases. In section 3 it is shown that the hidden variables of quantum mechanics are the same hidden variables of classical mechanics. A numerical analysis of the global phase is presented in section 4, which prefaces the deterministic account of the *Stern-Gerlach experiment* in section 5.

Introduction

In quantum mechanics and quantum information theory the spin state of the spin-$\frac{1}{2}$ fermion is described using the Copenhagen Interpretation of the quaternion as follows; The qubit is expanded as a complex valued 2-vector $|\Psi\rangle \in \mathbb{C}^2$

$$|\Psi\rangle \;=\; \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle \tag{1}$$

The co-efficients α and β are complex numbers which satisfy $|\alpha|^2 + |\beta|^2 = 1$, and the basis vectors of the complex plane are chosen to represent the spin-up and spin-down states

$$|\uparrow\rangle \;=\; \begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad |\downarrow\rangle \;=\; \begin{pmatrix}0\\1\end{pmatrix}$$

This assignment of the basis vectors to the spin states of the fundamental particles is not exclusive in the quantum theory - as other variants do exist. For example; the basis vectors are also used to represent the horizontal and vertical polarizations of light $\{|H\rangle, |V\rangle\} \;=\; \{\binom{1}{0}, \binom{0}{1}\}$, and the ground and excited states of a 2-level atom $\{|g\rangle, |e\rangle\} = \{\binom{1}{0}, \binom{0}{1}\}$, and the left and right spatial modes of the double well $\{|L\rangle, |R\rangle\} = \{\binom{1}{0}, \binom{0}{1}\}$, and the binary 0 and 1 where current is either *off* or *on* $\{|0\rangle, |1\rangle\} = \{\binom{1}{0}, \binom{0}{1}\}$. While these are the more common representations found in the literature other unrelated examples do arise from time to time.

It is known from the Stern-Gerlach experiment that the spin states of the fundamental particles are mutually exclusive. The spin value of a particle is *either* spin-up *or* spin-down - never both. As a result the qubit (1) is interpreted as an equation which describes the probability of measuring a particle in either spin state. The coefficients of the qubit are known as 'probability amplitudes' as they describe the probability of a measured result being either spin-up or spin-down.

$$P_\uparrow \;=\; |\langle \uparrow |\Psi\rangle|^2 \;=\; |\alpha|^2 \qquad\qquad P_\downarrow \;=\; |\langle \downarrow |\Psi\rangle|^2 \;=\; |\beta|^2$$

The Copenhagen Interpretation of the qubit employs *the principle of superposition* which states that the particle exists in both spin states at the same time until the point of measurement, when the superposition collapses to return the measured value with a probability $P_\uparrow$ for spin-up and $P_\downarrow$ for spin-down.* This is the Schrödinger's cat hypothesis applied to particle spin and forms the present theoretical framework of the hypothesized quantum computer. Yet the fact remains that the qubit is not just a "2-level quantum system", the qubit is a unit quaternion. It is 4-dimensional.

The origin of the notorious *measurement problem of quantum mechanics* is rooted in the Copenhagen Interpretation of the quaternion. This is the source of non-determinism in our natural sciences, and necessarily places the observer squarely in the center of the universe - there to assist nature in making up her mind by continuously collapsing the wave-function. The quantum mechanic maintains this superposition, acting as both the narcissist and voyeur. To remove any further ambiguity, we now detail a simple calculation to prove the qubit is a unit quaternion.

There are two orthonormal representations of the qubit $|\Psi^\pm\rangle \in \mathbb{C}^2$, denoted by the *'kets'* $|\Psi^+\rangle = \binom{\alpha}{\beta}$, and $|\Psi^-\rangle = \binom{-\beta^*}{\alpha^*}$, where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. The *'bra'* representation of the qubit is the transpose conjugate $\langle\Psi^+| = (\alpha^* \;\; \beta^*)$, and $\langle\Psi^-| = (-\beta \;\; \alpha)$. Consequently the qubit satisfies the orthonormal relations $\langle\Psi^\pm|\Psi^\pm\rangle = 1$, and $\langle\Psi^\pm|\Psi^\mp\rangle = 0$. The qubit is parametrized in terms of the 3 angles of the 3-sphere as

$$|\Psi^+\rangle \;=\; e^{i\frac{\omega}{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad\qquad |\Psi^-\rangle \;=\; e^{-i\frac{\omega}{2}} \begin{pmatrix} -e^{-i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

The angles of the qubit are the polar angle θ, the azimuthal angle ϕ and the global phase ω. The polar angle is defined with respect to the axis which penetrates the poles of the 3-sphere and describes the *Rabi oscillations* of qubit. The 3-dimensional coordinates of the qubit are described by the polar and azimuthal angles of the bloch sphere.

* also known as the Born rule.

The global phase is neglected within the quantum theory as it is thought to be a meaningless gauge. *"the normalized wave function is determined only to within a constant phase factor of the form* $e^{i\frac{\omega}{2}}$ *(where* ω *is any real number). This indeterminacy is in principle irremovable; it is, however, unimportant, since it has no effect upon any physical results"* [8, *pg* 7].* Here we clarify once and for all that this reasoning is entirely unwarranted as the global phase identifies the qubit as a quaternion and therefore cannot be neglected.

The orthonormal states $|\Psi^{\pm}\rangle$, are the eigenvectors (columns) of the quaternion $\hat{\Psi}$,

$$\hat{\Psi} \;\equiv \big(|\Psi^+\rangle \;\; |\Psi^-\rangle\big) \;=\; \begin{pmatrix} e^{i\frac{\omega-\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\omega+\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\omega+\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{-i\frac{\omega-\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

All 2×2 matrices in this form are unit quaternions, and they constitute the Special Unitary group of 2×2 matrices SU(2).

In quantum mechanics and quantum information theory the global and azimuthal angles are not studied. From the perspective of quantum mechanics they are hidden variables, and quantum mechanics as a physical theory is incomplete [9]. The remainder of this article is a detailed account of these hidden variables.†

1. The quaternion

The quaternion was discovered in 1843 by William Rowan Hamilton [12] following his quest to generalize the description of 2-dimensional rotations in $\mathbb{R}^2$ generated by the complex numbers $\mathbb{C}$, to describe 3-dimensional rotations in a natural way [13, *ch* 11].

The quaternions are a 4-dimensional 'complex' number which describe rotations in 3-dimensions, in full generality [14]. Containing 1 'real' and 3 'complex' components, the quaternions are isomorphic to vectors in $\mathbb{R}^4$ in the same way that the complex numbers are isomorphic to vectors in $\mathbb{R}^2$. While the complex numbers $\mathbb{C}$ describe rotations in 2-dimensions, the quaternions $\mathbb{C}^2$ describe rotations in both 4-dimensions and 3-dimensions [15].

The hidden variables of quantum mechanics are hidden spatial dimensions found in the parameter space of the unit quaternion. The information contained in both the $\mathbb{S}^2$ path of the bloch vector, and the $\mathbb{S}^1$ fibre bundle, allows one to uncover the full $\mathbb{S}^3$ quaternion. We show that the hidden variables of classical mechanics are the same hidden variables of quantum mechanics. While quantum and classical mechanics are both broad disciplines in their own right, a connection between them is established by recognizing that they are each rooted in two fundamental groups SU(2) and SO(3). For the context of this article we define quantum mechanics relative to classical mechanics as analogous to the relationship between the SU(2) and SO(3) groups.

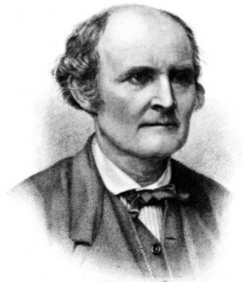
* Lev Landau and Evgeny Lifshitz

† This presentation concerns only the single qubit case and therefore does not immediately impact the Bell inequalities. In principle it is possible to extend this analysis - with recourse to the SU(4) group - for the purpose of resolving and clarifying the meaning of the Bell inequalities.

"Quantum Mechanics is to the Special Unitary Group of 2×2 matrices SU(2), as Classical Mechanics is to the Special Orthogonal Group of 3×3 matrices SO(3)."

The generators of both the SU(2) and SO(3) groups is the unit quaternion. In the SU(2) representation the quaternion is expanded in the basis of the 2×2 cayley matrices.

The cayley matrices are defined:



Arthur Cayley
1821 - 1895

$$\hat{\sigma}_i \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \hat{\sigma}_j \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \hat{\sigma}_k \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2}$$

where $\hat{\sigma}_1$ is the identity matrix, $i = \sqrt{-1}$ and

$$\hat{\sigma}_i^2 = \hat{\sigma}_j^2 = \hat{\sigma}_k^2 = \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_k = -\hat{\sigma}_1 \tag{3}$$

and $\hat{\sigma}_a^\dagger = -\hat{\sigma}_a$, for $a = i, j, k$. The cayley matrices are the natural basis matrices of the quaternion, and are here named after *Arthur Cayley* in honor of his many contributions to the development of pure mathematics and the quaternion. The cayley matrices have representations as 2×2 matrices in SU(2), and 4×4 matrices in SO(4), see Appendix A. For the scope of this article we need only consider the SU(2) cayley matrices.

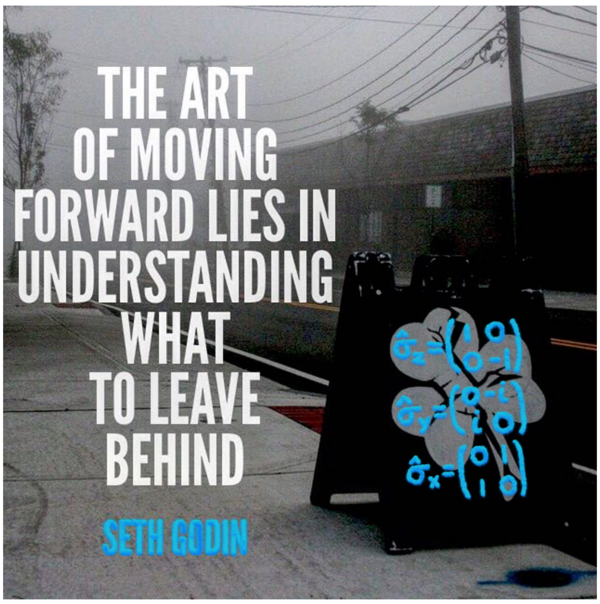
The quaternion is the fourtuple expanded in the SU(2) cayley basis as

$$\hat{Q} \equiv a\,\hat{\sigma}_1 + b\,\hat{\sigma}_i + c\,\hat{\sigma}_j + d\,\hat{\sigma}_k = \begin{pmatrix} a + \imath b & c + \imath d \\ -c + \imath d & a - \imath b \end{pmatrix} \tag{4}$$

and the transpose conjugate is $\hat{Q}^\dagger = a\hat{\sigma}_1 - b\hat{\sigma}_i - c\hat{\sigma}_j - d\hat{\sigma}_k$, with $a^2 + b^2 + c^2 + d^2 = 1$ and $\hat{Q}\hat{Q}^\dagger = \hat{\sigma}_1$, The SU(2) quaternion (4) is a unitary matrix since it's inverse is equal to the transpose conjugate $\hat{Q}^{-1} = \hat{Q}^\dagger$.

In quantum mechanics the standard matrices used to describe the SU(2) group are the pauli matrices. These relate to the cayley matrices through multiplication by the complex number i as,

$$\{\hat{\sigma}_i, \hat{\sigma}_j, \hat{\sigma}_k\} = i\{\hat{\sigma}_z, \hat{\sigma}_y, \hat{\sigma}_x\}$$



The choice of using either the pauli or cayley matrices is simply a choice of convention. Since the cayley matrices allow for a more compact notation we abandon the use of the pauli matrices in favor of this more efficient algebra. In so doing we recast the standard equations of quantum mechanics in terms of these operators. For the remainder of the text all vector quantities that we are dealing with are quaternions expressed in the SU(2) cayley basis*. This helps to clarify that all mathematical forms under study such as the hamiltonian operator and bloch vector are simply quaternions by another name.

Throughout the text we make use of the terms *unitary matrix, spinor, hamiltonian* and *bloch vector*, to describe the quaternions in use. These names may have some differences with respect to the conventional meaning of these labels, therefore we define at the outset:

- $\hat{\Psi}(t)$: The **spinor** is a unit quaternion parametrized in the exponential cayley basis

$$\hat{\Psi}(t) \;=\; e^{-\hat{\sigma}_i \frac{\phi}{2}} e^{-\hat{\sigma}_j \frac{\theta}{2}} e^{\hat{\sigma}_i \frac{\omega}{2}} \tag{5}$$

$$\hat{\Psi}(t) \;=\; \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\phi-\omega}{2}\right) + i\sin\left(\frac{\phi-\omega}{2}\right)\right) & -\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\phi+\omega}{2}\right) - i\sin\left(\frac{\phi+\omega}{2}\right)\right) \\ \sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\phi+\omega}{2}\right) + i\sin\left(\frac{\phi+\omega}{2}\right)\right) & \cos\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\phi-\omega}{2}\right) - i\sin\left(\frac{\phi-\omega}{2}\right)\right) \end{pmatrix}$$

  The spinor extends from its initial state $\hat{\Psi}(t=0) = \hat{\Psi}_0$ via the unitary matrix

$$\hat{\Psi}(t) \;=\; \hat{U}(t)\;\hat{\Psi}_0 \tag{6}$$

- $\hat{U}(t)$: The **unitary** matrix is a time dependent unit quaternion, $a, b, c, d \in \mathbb{R}$

$$\hat{U}(t) \;=\; a(t)\hat{\sigma}_1 + b(t)\hat{\sigma}_i + c(t)\hat{\sigma}_j + d(t)\hat{\sigma}_k$$

  generated by using products of the exponential operators $e^{i\frac{a}{2}t\hat{\sigma}_\bullet}$ parametrized by t, such as

$$\hat{U}(t) \;=\; e^{i\frac{a_1}{2}t\hat{\sigma}_j} e^{i\frac{a_2}{2}t\hat{\sigma}_i} \;\cdots\; e^{i\frac{a_n}{2}t\hat{\sigma}_k}$$

  where $t \in [0, 2\pi]$ is the time, $a_\bullet$ is a real scalar, and the factor of $\frac{1}{2}$ is indicative of SU(2).

- $\hat{\mathcal{H}}(t)$: The **hamiltonian** operator is a pure quaternion†

$$\hat{\mathcal{H}}(t) \;\equiv\; \dot{\hat{U}}\hat{U}^\dagger \;=\; \frac{\mathcal{H}^i}{2}\hat{\sigma}_i + \frac{\mathcal{H}^j}{2}\hat{\sigma}_j + \frac{\mathcal{H}^k}{2}\hat{\sigma}_k \tag{7}$$

- $\hat{\mathcal{R}}(t)$: The **bloch** vector is a pure unit quaternion in $\mathbb{R}^3$

$$\hat{\mathcal{R}}(t) \;\equiv\; \hat{U}\hat{\mathcal{R}}_0\hat{U}^\dagger \;=\; \frac{\mathcal{R}^i}{2}\hat{\sigma}_i + \frac{\mathcal{R}^j}{2}\hat{\sigma}_j + \frac{\mathcal{R}^k}{2}\hat{\sigma}_k \tag{8}$$

  In spherical polar coordinates the components of the bloch vector are

$$\left(\mathcal{R}^i, \mathcal{R}^j, \mathcal{R}^k\right) \;=\; \left(\cos(\theta),\; \sin(\theta)\sin(\phi),\; \sin(\theta)\cos(\phi)\right)$$

Taking the first derivative of the spinor (6) and substituting for the hamiltonian (7), we arrive at *the Schrödinger equation.*

$$\dot{\hat{\Psi}} \;=\; \hat{\mathcal{H}}\;\hat{\Psi} \tag{9}$$

* Unless otherwise stated

† $\mathcal{H}^i \;=\; 2\left(c\dot{d} - \dot{c}d + a\dot{b} - \dot{a}b\right) \qquad \mathcal{H}^j \;=\; 2\left(\dot{b}d - b\dot{d} + a\dot{c} - \dot{a}c\right) \qquad \mathcal{H}^k \;=\; 2\left(b\dot{c} - \dot{b}c + a\dot{d} - \dot{a}d\right)$

Similarly by taking the first derivative of the bloch vector (8) and substituting for the hamiltonian (7), we arrive at *the von Neumann equation.*

$$\dot{\hat{\mathcal{R}}} = [\hat{\mathcal{H}}, \hat{\mathcal{R}}] \tag{10}$$

The bloch vector $\hat{\mathcal{R}}(\theta, \phi)$ describes a path on the 2-sphere parametrized by the polar θ and azimuthal ϕ angles - these are the extrinsic parameters of the quaternion. The global phase is the intrinsic parameter, which is not explicitly present in the von-Neumann equation of motion, as it is a natural hidden variable. We will later show that the global phase is encoded in the $\mathbb{S}^2$ path via the geometric and dynamic phases. From the von-Neumann equation (10) we recover the analytic forms of the extrinsic parameters

$$\dot{\theta} = \frac{\mathcal{H}^k\mathcal{R}^j - \mathcal{H}^j\mathcal{R}^k}{\sqrt{(\mathcal{R}^j)^2 + (\mathcal{R}^k)^2}} \qquad \dot{\phi} = -\mathcal{H}^i + \frac{\mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k}{(\mathcal{R}^j)^2 + (\mathcal{R}^k)^2}\,\mathcal{R}^i \tag{11}$$

2. The hopf fibration and the intrinsic parameter space

The hopf fibration [16] [17] is the mapping between the 3-sphere and the 2-sphere,

$$\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{S}^2$$

defined by any one of the hopf projections

$$\hat{\mathcal{R}} \equiv \hat{\Psi}\frac{\hat{\sigma}_i}{2}\hat{\Psi}^\dagger \qquad \hat{\mathcal{R}} \equiv \hat{\Psi}\frac{\hat{\sigma}_j}{2}\hat{\Psi}^\dagger \qquad \hat{\mathcal{R}} \equiv \hat{\Psi}\frac{\hat{\sigma}_k}{2}\hat{\Psi}^\dagger \tag{12}$$

These projections are developed in greater detail in Appendix C. For the scope of this article we consider only the $\hat{\sigma}_i$ hopf projection. The spinor $\hat{\Psi}$ describes a path on the 3-sphere $\mathbb{S}^3$ embedded in $\mathbb{R}^4$, while the bloch vector $\hat{\mathcal{R}}$ describes a path on the 2-sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$. $\mathbb{S}^1$ is a fibre bundle connecting the base spaces of the 3-sphere and 2-sphere. The components of the hopf fibration are,

- $\mathbb{S}^3$: Total space; The unit spinor $\hat{\Psi}(\omega, \theta, \phi)$ describes the 3-sphere.
- $\mathbb{S}^2$: Base space; The bloch vector $\hat{\mathcal{R}}(\theta, \phi)$ describes the 2-sphere.
- $\mathbb{S}^1$: Fibre bundle; The global phase $e^{\imath\omega}$ describes the unit circle.

The spinor $\hat{\Psi}(\omega, \theta, \phi)$ is parametrized by the global, polar and azimuthal angles, as

$$\hat{\Psi}(t) = e^{-\hat{\sigma}_i\frac{\phi}{2}}e^{-\hat{\sigma}_j\frac{\theta}{2}}e^{\hat{\sigma}_i\frac{\omega}{2}} \tag{13}$$

Although the global phase is not *explicitly* present in the bloch vector $\hat{\mathcal{R}}(\phi, \theta)$ as a parameter, it is encoded in the 3-D kinematics through the global, geometric and dynamic phases, as we now demonstrate.

2.1. The global phase

The closed form of the global phase is found from recasting the Schrödinger equation in its standard form.

$$\dot{\hat{\Psi}}\hat{\Psi}^{\dagger} \;=\; \hat{\mathcal{H}}$$

Equating the diagonal components of the above equation we recover

$$\dot{\omega}(t) \;\equiv\; \frac{\mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k}{(\mathcal{R}^j)^2 + (\mathcal{R}^k)^2} \qquad\qquad \omega(t) \;=\; \int_0^t dt'\,\dot{\omega}(t') \tag{14}$$

The global phase is a function of the elements of the hamiltonian and bloch vector, and is a measure of the total anholonomy of the path.

2.2. The dynamic phase

The dynamic phase is the integral of the expectation value of the hamiltonian over the closed path. The expectation value of the hamiltonian is the inner product,

$$\langle\Psi^{\pm}|\hat{\mathcal{H}}|\Psi^{\pm}\rangle \;=\; \pm\frac{\vec{\mathcal{H}}\cdot\vec{\mathcal{R}}}{2} \qquad\qquad \dot{\xi}(t) \;\equiv\; \vec{\mathcal{H}}\cdot\vec{\mathcal{R}}$$

The dynamic phase resembles the energy in the form of work for the path [18]

$$\xi(t) \;\equiv\; \int_0^t dt'\,\dot{\xi}(t') \tag{15}$$

2.3. Parallel transport and the geometric phase

In the pioneering study of geometric phases in quantum mechanics it was shown, through an analysis of adiabatically evolving quantum systems under the adiabatic approximation, that the global phase of the quantum state is the sum of the geometric and dynamic phases [19]. Immediately it was recognized that the global phase is a measure of the anholonomy of the spinor's $\mathbb{S}^2$ path, and that the geometric and dynamic phases constitute the elements of a fibre bundle [20].

The adiabatic approximation of the geometric phase is known as the *"The Berry Phase"*, and has stimulated a wealth of theoretical and experimental investigations into this geometric fibration [21]. Today the Berry phase and related studies concern the definition of the geometric phase in the parameter space of the hamiltonian. In this article we define the geometric phase in the parameter space of the spinor, which is entirely different than current studies. The geometric phase is defined as the solution to the equation of parallel transport [22] for all closed paths of the 2-sphere generated by the unit quaternion.

The tangent vector $\vec{\mathcal{V}}$ is parallel transported around the smooth curve described by $\vec{\mathcal{R}}$ via the *equation of parallel transport* as defined by [23]:

$$\frac{D\mathcal{V}^a}{Dt} \;\equiv\; \dot{\vec{\mathcal{V}}}\cdot\vec{e}_a \;=\; 0 \tag{16}$$

The 2-vector $\vec{\mathcal{V}}$ is expanded in the basis of the tangent plane $(\vec{e}_\phi, \vec{e}_\theta)$:

$$\vec{\mathcal{V}}(t) \;=\; \mathcal{V}^\phi \vec{e}_\phi + \mathcal{V}^\theta \vec{e}_\theta \tag{17}$$

The tangent plane is the normalized basis of the partial derivatives,

$$\vec{e}_\phi \;=\; \frac{\partial_\phi \vec{\mathcal{R}}}{\sqrt{\partial_\phi \vec{\mathcal{R}} \cdot \partial_\phi \vec{\mathcal{R}}}} \;=\; \begin{pmatrix} 0 \\ \cos{(\phi)} \\ -\sin{(\phi)} \end{pmatrix} \quad \vec{e}_\theta \;=\; \frac{\partial_\theta \vec{\mathcal{R}}}{\sqrt{\partial_\theta \vec{\mathcal{R}} \cdot \partial_\theta \vec{\mathcal{R}}}} \;=\; \begin{pmatrix} -\sin{(\theta)} \\ \cos{(\theta)}\sin{(\phi)} \\ \cos{(\theta)}\cos{(\phi)} \end{pmatrix} \tag{18}$$

From (17) and (18), the equation of parallel transport (16) is recast in the form

$$\dot{\mathcal{V}}^a \;=\; -\mathcal{A}^a{}_b \mathcal{V}^b$$

where $\mathcal{A}^a{}_b$ is the differential form defined by,

$$\mathcal{A}^a{}_b \;\equiv\; \vec{e}_a \cdot \dot{\vec{e}}_b \;=\; -\, \vec{e}_b \cdot \dot{\vec{e}}_a$$

Applied to the 2-sphere we obtain the coupled differential equations,

$$\begin{aligned} \begin{pmatrix} \dot{\mathcal{V}}^\phi \\ \dot{\mathcal{V}}^\theta \end{pmatrix} &= - \begin{pmatrix} \mathcal{A}^\phi{}_\phi & \mathcal{A}^\phi{}_\theta \\ \mathcal{A}^\theta{}_\phi & \mathcal{A}^\theta{}_\theta \end{pmatrix} \begin{pmatrix} \mathcal{V}^\phi \\ \mathcal{V}^\theta \end{pmatrix} \\ \begin{pmatrix} \dot{\mathcal{V}}^\phi \\ \dot{\mathcal{V}}^\theta \end{pmatrix} &= \begin{pmatrix} 0 & -\dot{\phi}\cos{(\theta)} \\ \dot{\phi}\cos{(\theta)} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}^\phi \\ \mathcal{V}^\theta \end{pmatrix} \end{aligned} \tag{19}$$

Therefore the tangent vector evolves from its initial state

$$\begin{pmatrix} \mathcal{V}^\phi(t) \\ \mathcal{V}^\theta(t) \end{pmatrix} \;=\; \begin{pmatrix} \cos{(\gamma)} & -\sin{(\gamma)} \\ \sin{(\gamma)} & \cos{(\gamma)} \end{pmatrix} \begin{pmatrix} \mathcal{V}^\phi_0 \\ \mathcal{V}^\theta_0 \end{pmatrix}$$

where $\gamma(t)$ is the geometric phase. The precession of the tangent vector (17) as it is parallel transported along a path in $\mathbb{S}^2$ is illustrated in figure 1.
<u>The Geometric Phase:</u>

$$\gamma(t) \;\equiv\; \int_0^t dt' \, \left[\dot{\phi}(t') \cos{(\theta(t'))}\right] \tag{20}$$

2.4. The $\mathbb{S}^1$ fibre bundle

From equations (11) and (20), we find $\dot{\gamma} = \dot{\omega} - \dot{\xi}$, and rearrange to express the global phase of the $\mathbb{S}^1$ bundle as the sum of the geometric and dynamic phases:

$$\omega \;=\; \gamma + \xi \tag{21}$$

The *global phase* ω is a measure of the total anholonomy of the path, the *geometric phase* γ describes the orientation of a parallel transported tangent vector along the path, and

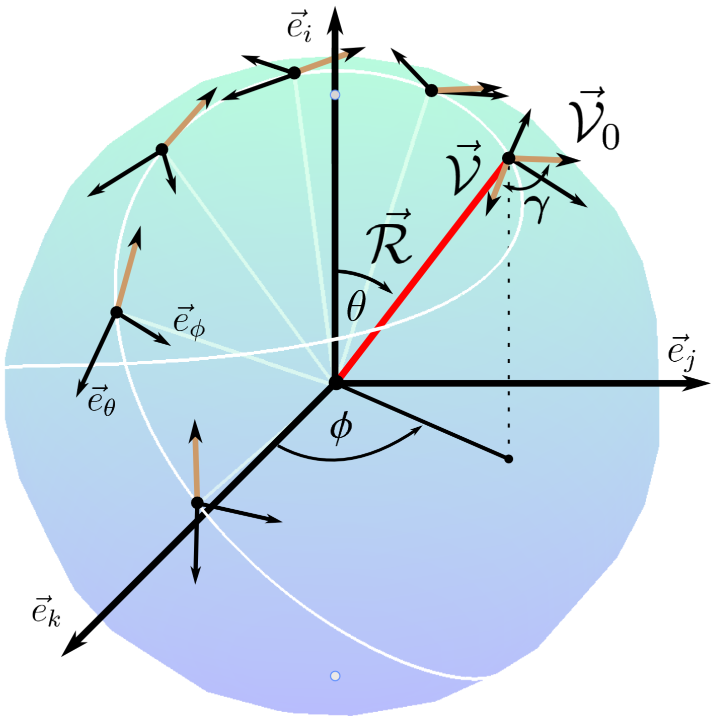


Figure 1. The 2-sphere $\mathbb{S}^2$. Shown is Cartesian frame $\{\vec{e}_i, \vec{e}_j, \vec{e}_k\}$ and the bloch vector $\vec{\mathcal{R}}$ which extends from the origin to the surface of the 2-sphere. The polar and azimuthal angles $\{\theta(t), \phi(t)\}$ respectively define the orientation of the bloch vector. The tangent frame $\{\vec{e}_\theta, \vec{e}_\phi\}$ maps the 2-dimensional surface of the bloch sphere. The tangent vector $\vec{\mathcal{V}}$, and tangent frame is parallel transported along the path (shown in white). The initial orientation of the tangent vector is $\vec{\mathcal{V}}_0$ and the final orientation is $\vec{\mathcal{V}}$. The angular difference between both is the geometric phase γ.

the *dynamic phase* ξ takes the form of the work-energy of the path. These hidden variables are the 4th dimensional shadows of the quaternion as seen in 3-dimensions.

$$\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{S}^2$$

The $\mathbb{S}^3$ and $\mathbb{S}^2$ total and base spaces are related through the $\mathbb{S}^1$ fibre bundle (unit circle), consisting of the global, geometric and dynamic phases. Equation (21) is a geometric principle of the unit quaternion in the same sense that Pythagoras's theorem is a geometric principle of the right angled triangle.

3. Hidden variables in classical mechanics

The global phase is a function of the elements of the hamiltonian and bloch vector. Here we move to the SO(3) representation of the same kinematic equations to demonstrate that the global phase is also a natural hidden variable of classical mechanics.

The bloch vector $\vec{\mathcal{R}}(t) = \mathcal{R}^a \vec{e}_a$ is expanded,

$$\begin{aligned}
\vec{\mathcal{R}}(t) &= \mathcal{R}^i \vec{e}_i + \mathcal{R}^j \vec{e}_j + \mathcal{R}^k \vec{e}_k \\
\vec{\mathcal{R}}(t) &= \begin{pmatrix} \mathcal{R}^i \\ \mathcal{R}^j \\ \mathcal{R}^k \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) \end{pmatrix}
\end{aligned} \tag{22}$$

The bloch vector traces a path on the unit 2-sphere $\mathbb{S}^2$, illustrated in figure 1. The Special Orthogonal group of 3×3 matrices is the group of unit quaternions of the form

$$\hat{U}(t) = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2-b^2-c^2+d^2 \end{pmatrix} \tag{23}$$

which satisfy $\hat{U}\hat{U}^{\mathrm{T}} = \hat{\sigma}_1$. The bloch vector extends from its initial state as,

$$\vec{\mathcal{R}}(t) = \hat{U}(t)\vec{\mathcal{R}}(0) \tag{24}$$

The SO(3) equation of motion is,

$$\dot{\vec{\mathcal{R}}} = \hat{\mathcal{H}}\,\vec{\mathcal{R}}$$

where the hamiltonian operator is defined*

$$\hat{\mathcal{H}}(t) \equiv \dot{\hat{U}}\hat{U}^{\mathrm{T}} = \mathcal{H}^i\hat{\pi}_i + \mathcal{H}^j\hat{\pi}_j + \mathcal{H}^k\hat{\pi}_k = \begin{pmatrix} 0 & -\mathcal{H}^k & \mathcal{H}^j \\ \mathcal{H}^k & 0 & -\mathcal{H}^i \\ -\mathcal{H}^j & \mathcal{H}^i & 0 \end{pmatrix} \tag{25}$$

The SO(3) hamiltonian operator is a skew symmetric matrix. SO(3) operators act on vectors in the same manner as the curl of the related 3-vector acts on a vector, i.e. $\hat{\mathcal{H}}(t)\vec{\mathcal{R}}(t) = \vec{\mathcal{H}}(t) \times \vec{\mathcal{R}}(t)$. The classical equation of motion [24, *pg* 106],

$$\dot{\vec{\mathcal{R}}} = \vec{\mathcal{H}} \times \vec{\mathcal{R}} \tag{26}$$

The classical equation of motion is the SO(3) representation of the Schrödinger equation (9), and is composed of the elements of the hamiltonian and bloch vector. Given that the global (14), geometric (20) and dynamic phases (15) are functions of the hamiltonian and bloch vector, they are here demonstrated as hidden variables of classical mechanics.

In Appendix D the fictitious forces of classical mechanics are derived from the unit quaternion. This is to compliment the derivation of equation (26) and shows that the fundamental equations of classical mechanics are rooted in the unit quaternion. The implication of these calculations shows that it is possible, in principle, to extend the analysis provided herein to recast the entire algebra of classical mechanics in terms of the quaternion [25].

* For a definition of the Lie algebra $\hat{\pi}_a$ see Appendix B

4. Numerical analysis

We have shown that under the hopf fibration

$$\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{S}^2$$

the spinor separates into an intrinsic and extrinsic parameter space. The extrinsic parameter space is the 2-sphere $\mathbb{S}^2$ described by the bloch vector $\hat{\mathcal{R}}$, and the intrinsic parameter space is the unit circle $\mathbb{S}^1$ described by the global phase $e^{\imath\frac{\omega}{2}}$. In this section we detail a numerical analysis of the intrinsic parameters. This section is divided into two parts.

Section 4.1: The global phase of all $\mathbb{S}^2$ closed paths is discretized as $\omega = 2n\pi$, for $n \in \mathbb{N}$. As the $\mathbb{S}^1$ fibration is described by the unit circle, there are two possible values for the closed path $e^{\imath\frac{\omega}{2}} = \pm 1$, depending on whether n is even or odd. This property of the global phase is demonstrated and used to classify the $\mathbb{S}^2$ paths.

Section 4.2: The parallel transport of the tangent vector along the $\mathbb{S}^2$ paths is graphically illustrated on the bloch sphere. The global phase can be represented on the Möbius band, and the intrinsic parameters of the $\mathbb{S}^1$ fibre bundle are plotted for the closed path.

4.1. The global phase of the closed path

All unitary matrices which satisfy the property $\hat{U}(0) = \hat{U}(2n\pi)$, for $n \in \mathbb{N}$ generate closed paths. The global phase of all $\mathbb{S}^2$ closed paths is discretized as $\omega = 2n\pi$. Since the $\mathbb{S}^1$ fibration is described by the unit circle, there are two possible values for the closed path $e^{\imath\frac{\omega}{2}} = \pm 1$, depending on whether n is even or odd. As the global phase of the closed path is discrete, this allows a natural characterization of the path as *fermionic or bosonic.*

- Fermionic Paths: For odd values of n the $\mathbb{S}^1$ fibration of the closed path is equal to minus one, $e^{\pm i\frac{\omega}{2}} = -1$. When the spinor $\hat{\Psi}$ completes one closed loop of the $\mathbb{S}^2$ path it acquires a minus sign and must complete a second orbit to return to its initial state. Since fermions require two rotations to return to their initial state, these paths are called the *"fermionic paths"*.
- Bosonic Paths: For even values of n, the $\mathbb{S}^1$ fibration of the closed path is equal to one, $e^{\pm i\frac{\omega}{2}} = 1$, and the spinor $\hat{\Psi}$ returns to its initial state on completion of one orbit of the path. Since bosons require one rotation to return to their initial state, these paths are called the *"bosonic paths"*.

The global phase of the closed $\mathbb{S}^2$ paths is discrete. To illustrate this property of the closed path we make use of two unitaries, one which generates exclusively *"fermionic paths"* and another which generates exclusively *"bosonic paths"*. There are many choices

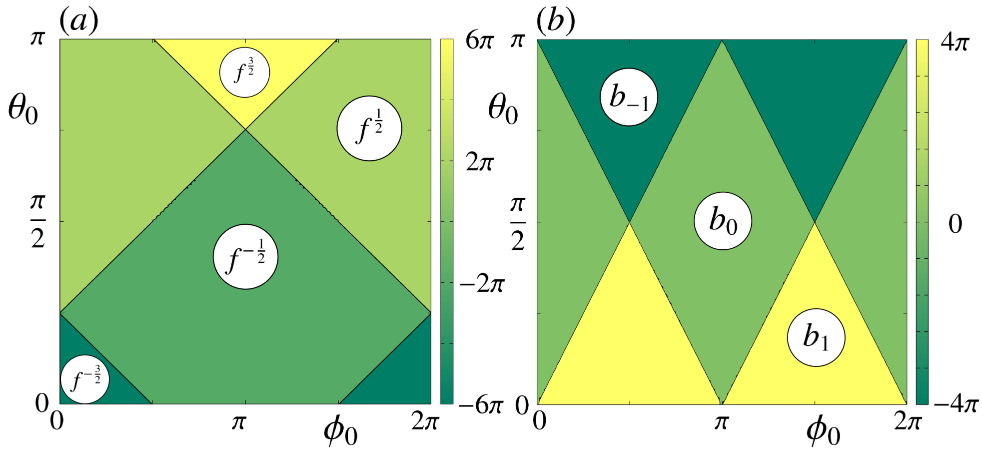


Figure 2. The global phase of the closed path as a function of the initial state for the fermionic unitary (27) in (a), and the bosonic unitary (28) in (b).

of unitaries which satisfy these requirements, and for our purposes it suffices to consider,

$$\text{Figure } 2(a): \quad \hat{U}(t) = e^{-\hat{\sigma}_i t} e^{\hat{\sigma}_k \frac{t}{2}} e^{\hat{\sigma}_i t} \qquad \text{‘Fermionic Path Generator’} \tag{27}$$

$$\text{Figure } 2(b): \quad \hat{U}(t) = e^{-\hat{\sigma}_i \frac{t}{2}} e^{-\hat{\sigma}_j t} e^{-\hat{\sigma}_i t} \qquad \text{‘Bosonic Path Generator’} \tag{28}$$

In figure 2 the global phase (14) is plotted as a function of the initial state $\{\theta_0, \phi_0\}$, for the fermionic unitary in (a) and the bosonic unitary in (b). It is seen that the allowed values of the global phase of the fermionic unitary are $\pm 2\pi, \pm 6\pi$, which are labeled $f^{\pm\frac{1}{2}}, f^{\pm\frac{3}{2}}$, and the allowed values of the bosonic unitary are $0, \pm 4\pi$, which are labeled $b_0, b_{\pm 1}$.

The ‘fermionic’ spinor (27) corresponds to a spin-$\frac{3}{2}$ particle with 4 allowed spin states. The ‘bosonic’ spinor (28) corresponds to a spin-1 particle with 3 allowed spin states. Using this picture to interpret spin physically, it is seen that a change in spin state corresponds to a change in initial state. Consequently for an ensemble of particles, a distribution of spin states corresponds to a distribution of initial states. In this way the intrinsic spin is described deterministically, and is understood as an observation of the 4th-dimensional shadow of the spinor.

The geometric property of the quaternion's $\mathbb{S}^2$ path that we are drawing attention to is the global phase tells us where we are 'globally' in the $\mathbb{S}^3$ path of the spinor. A global phase of $e^{\pm i\frac{\omega}{2}} = -1$ indicates that only half of the $\mathbb{S}^3$ has been traversed and another rotation in $\mathbb{S}^2$ is required to return to the initial state. These being the so called fermionic paths. Conversely a global phase of $e^{\pm i\frac{\omega}{2}} = 1$ is bosonic as one closed loop in $\mathbb{S}^2$ is sufficient for the $\mathbb{S}^3$ spinor to return to it's initial state. As we have shown, the unitaries can produce exclusively bosonic or fermionic paths as seen in figure 2, and as we now show they can produce a mixture of both - here referred to as mixed unitaries. The mixed unitary we consider is taken from the product of the fermionic and bosonic

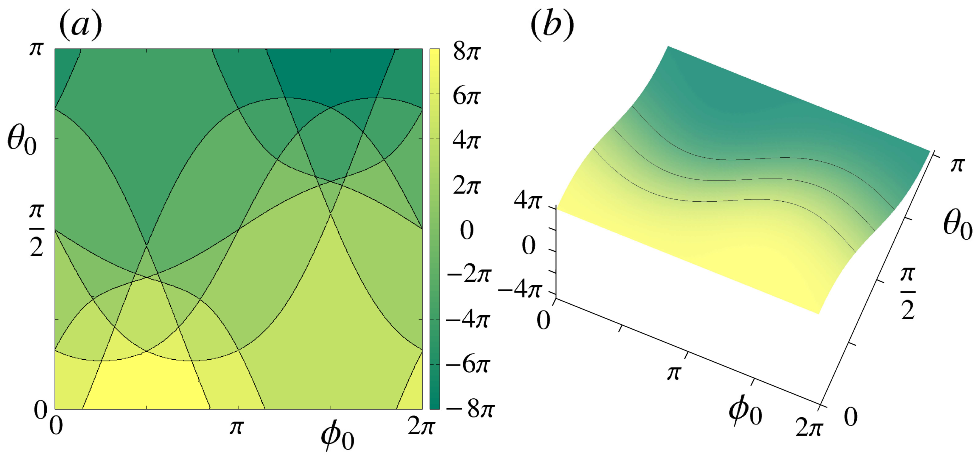


Figure 3. For the unitary (29) we have (a) The global phase of the closed path and (b) The dynamic phase of the closed path, as a function of the initial state.

unitaries of equations (27) and (28),

$$\hat{U}(t) \;=\; e^{-\hat{\sigma}_i t} e^{\hat{\sigma}_k \frac{t}{2}} e^{\hat{\sigma}_i \frac{t}{2}} e^{-\hat{\sigma}_j t} e^{-\hat{\sigma}_i t} \tag{29}$$

The global phase of the closed path according to (29) is shown in figure 3 (a) as a function of the initial state. The global phase (a) exhibits both bosonic and fermionic statistics since it takes the discreet values $0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \pm 8\pi$. For completeness the dynamic phase of the closed path is shown in (b), and it is observed as a smooth continuous function.

In this section we have shown that the parameter space of the quaternion contains the properties required to formulate a deterministic theory of particle spin. Interpreting the global phase physically as encoding the intrinsic spin of the fundamental particles, we have seen that a distribution of spin states is analogous to a distribution of initial states. This analysis shows that when correctly using the quaternion to model spin of a particle, the spin state is described before measurement.

4.2. The intrinsic parameters and the möbius band

When the spinor is viewed in $\mathbb{R}^3$ under the hopf fibration, the 4th-dimension is 'rolled up' in the $\mathbb{S}^1$ fibration which describes the unit circle $e^{i\frac{\omega}{2}}$. As the bloch vector follows the $\mathbb{S}^2$ path, the spinor rotates around an internal axis. The rate of rotation is given by the global phase. This internal rotation is easily represented on the Möbius band, which is parametrized by the global phase ω and the *'time'* t.

<u>The Möbius band:</u>

$$\begin{aligned} x(t) &= \left(R + l\cos\left(\tfrac{\omega}{2}\right)\right)\cos(t) \\ y(t) &= \left(R + l\cos\left(\tfrac{\omega}{2}\right)\right)\sin(t) \\ z(t) &= l\sin\left(\tfrac{\omega}{2}\right) \end{aligned}$$

where $t \in [0, 2\pi]$, l is the half-width of the band and R is the mid-circle radius.

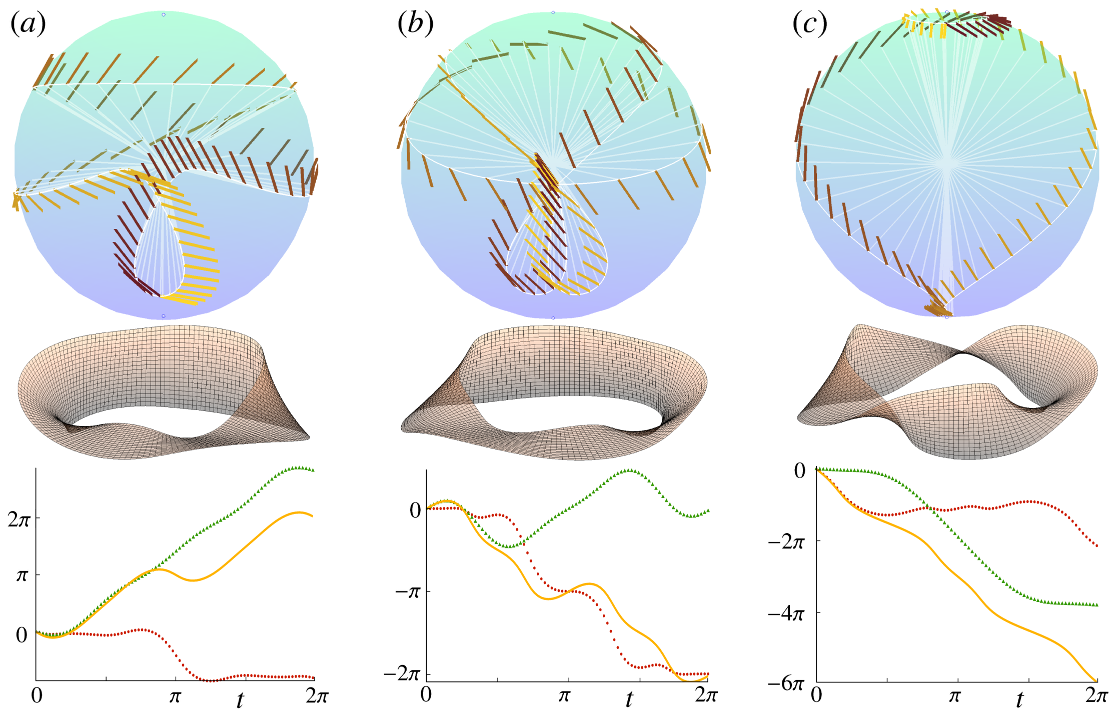


Figure 4. Above: The $\mathbb{S}^2$ path of the spinor (5) under the $\mathbb{S}^3 \xmapsto{\mathbb{S}^1} \mathbb{S}^2$ hopf fibration, and the accompanying geometric phase (20) for the unitary (27). Middle: The Möbius band. Below: The $\mathbb{S}^1$ fibre bundle consisting of the global phase solid-gold, the dynamic phase triangle-green and the geometric phase circle-red. The initial states are given by (a) $\{\theta_0, \phi_0\} = \{\frac{3\pi}{4}, 0\}$ (b) $\{\theta_0, \phi_0\} = \{\frac{\pi}{2}, \pi\}$ (c) $\{\theta_0, \phi_0\} = \{\frac{\pi}{10}, 0\}$.

In figures 4 and 5 the $\mathbb{S}^2$ path of the spinor (5) is shown for the unitaries (27) and (28). In the upper row of figures 4 and 5, the geometric phase (20) is graphically illustrated via the parallel transport of the tangent vector (17), whose color ranges from a dark red to gold as it progresses along the closed path.

The middle rows of figures 4 and 5 are the Möbius band representation of the $\mathbb{S}^1$ fibration. It is seen that in the case of the fermionic paths, the Möbius band has 1 half-turn in 4(a) and (b), and 3 half-turns in (c). Two orbits of the fermionic Möbius band are required to return to the initial state. For the bosonic paths the Möbius band has 1 full-turn in 5(a) and (c), and has no turns in (b). One orbit of the bosonic Möbius band is required to return to the initial state. The lower rows are plots of the $\mathbb{S}^1$ fibre bundle, which consists of the global phase (14), the geometric phase (20) and the dynamic phase (15). The initial states are listed in the figure captions.

In this analysis we have demonstrated that the global phase of the $\mathbb{S}^2$ closed paths is discrete. The discretization of the global phase allows a natural characterization of the $\mathbb{S}^2$ paths as fermionic or bosonic. The $\mathbb{S}^1$ fibration is a measure of the total anholonomy of the $\mathbb{S}^2$ path, and tells us where we are “globally” on the 3-sphere. Conceptually the spinor can be thought to rotate around an internal axis. This rotation is represented on the Möbius band and is here interpreted physically as encoding the intrinsic spin of

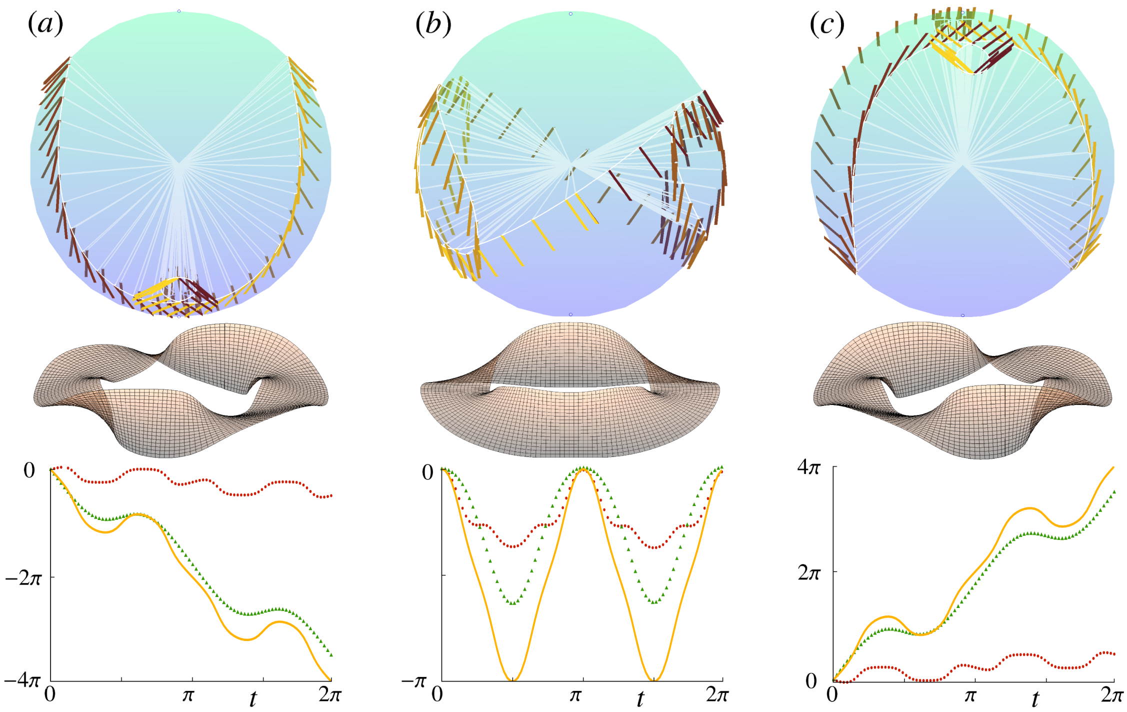


Figure 5. Above: The $\mathbb{S}^2$ path of the spinor (5) under the $\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{S}^2$ hopf fibration, and the accompanying geometric phase (20) for the unitary (28). Middle: The Möbius band. Below: The $\mathbb{S}^1$ fibre bundle consisting of the global phase solid-gold, the dynamic phase triangle-green and the geometric phase circle-red. The initial states are given by (a) $\{\theta_0, \phi_0\} = \{\frac{3\pi}{4}, \frac{3\pi}{2}\}$ (b) $\{\theta_0, \phi_0\} = \{\frac{\pi}{2}, \pi\}$ (c) $\{\theta_0, \phi_0\} = \{\frac{\pi}{4}, \frac{\pi}{2}\}$.

the fundamental particles. We have shown the $\mathbb{S}^1$ fibration is encoded in the $\mathbb{S}^2$ path via the geometric and dynamic phases.

5. The global phase and the Stern-Gerlach experiment

The Stern-Gerlach experiment is one of a number of significant experiments performed in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century on microscopic particles, whose results were unable to be accounted for by the classical mechanics of that era. The experiment of Stern and Gerlach [26] demonstrated that fundamental particles on the atomic scale possess an intrinsic angular momentum which takes discrete values, as they showed that an unpolarized beam of silver atoms, passing through an inhomogeneous magnetic field splits into two allowed spin states, spin up and spin down.

Prior to the experiment the magnetic moment of the silver atom was expected to be attracted/repelled by the inhomogeneous magnetic field in a manner analogous to a weightless bar magnet, which would result in a Gaussian distribution with a maximum along the axis of propagation. The surprising result that the beam of silver atoms splits into two distinct paths, demonstrated that the silver atom possessed an intrinsic spin. It was later established that intrinsic spin is an inherent property of the fundamental particles, as an analysis of the fine structure of atomic spectra [27] showed that the electron itself possesses an intrinsic spin, having two allowed intrinsic spin states, spin up and spin down.

In the following we account for the results of the Stern-Gerlach experiment, by interpreting the global phase of the spinor as encoding the intrinsic spin. In so doing we make a fundamental assumption: the magnetic moment of the silver atom is 4-dimensional and its precession is correctly described by the quaternion. To begin we outline the experiment in section 5.1 and offer an interpretation of the results using the global phase in section 5.2. Thereafter we propose an adapted version of the Stern-Gerlach experiment to quantify the accuracy of the experiment, in section 5.3.

5.1. The Stern-Gerlach experiment - outline

Absent a translation of the original article [26], we follow the description of the Stern-Gerlach experiment given by J.J. Sakurai in the opening chapter of his book [28], and adapt it suitably for our purposes.

A Stern-Gerlach apparatus is an inhomogeneous magnetic field which is produced by a pair of pole pieces, one of which has a very sharp edge.

Stern-Gerlach experiment:

- Silver (Ag) atoms are heated in an oven. The oven has a small hole through which some of the silver atoms escape. The beam of silver atoms goes through a collimator and is then subjected to an inhomogeneous magnetic field produced by a pair of pole pieces, one of which has a very sharp edge (Stern-Gerlach apparatus).
- The silver atom is made up of a nucleus and 47 electrons, where 46 out of the 47 electrons can be visualized as forming a spherically symmetrical electron cloud with no net angular momentum. To a good approximation, the heavy atom as a whole possesses a magnetic moment equal to the spin magnetic moment of the 47$^{\text{th}}$ electron.
- Adaptation: The magnetic moment of the heavy atom is 4-dimensional, described by a unit spinor which admits only two allowed values for the global phase of the closed path, $f^{\pm\frac{1}{2}}$. The direction of propagation of the silver atoms is along the $\vec{e}_i$ axis, since the north and south poles of the 2-sphere are singularity points which are located on the $\vec{e}_i$ axis.
- The atoms in the oven are randomly orientated, i.e. they have random initial states $\{\theta_0, \phi_0\}$.
- Adaptation: The inhomogeneous magnetic field measures the total magnetic moment of the silver atom, which consists of an intrinsic and extrinsic magnetic moment. Measurement of the intrinsic magnetic moment (the global phase) causes the splitting of the beam into an $f^{-\frac{1}{2}}$ beam, and an $f^{\frac{1}{2}}$ beam.

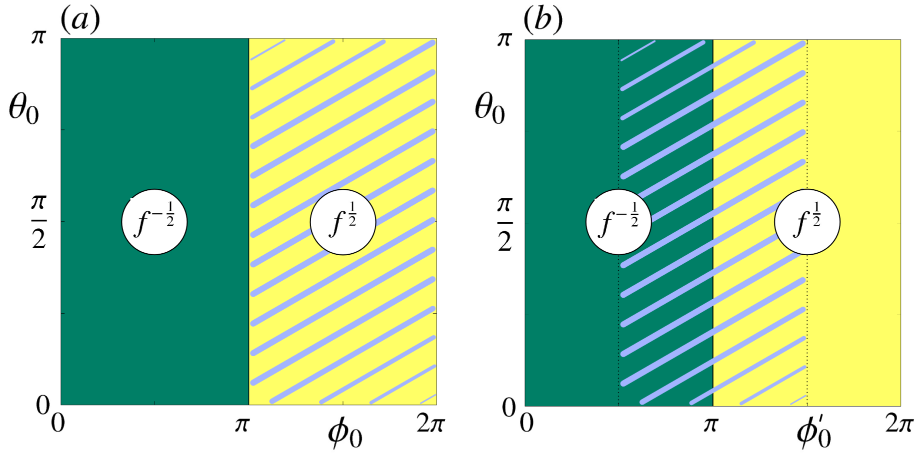


Figure 6. The global phase of the closed path according to the unitary (30). In (a) the inhomogeneous magnetic field is aligned along the $\vec{e}_k$ axis. The beam of silver atoms separates into the two allowed spin states. The spin down $f^{-\frac{1}{2}}$ beam is allowed to pass through a second inhomogeneous magnetic field, whereas the spin up $f^{\frac{1}{2}}$ beam is blocked, which is denoted by the shaded region. In (b) the inhomogeneous magnetic field is aligned along the $\vec{e}_j$ axis, which is at an angle of $\frac{\pi}{2}$ relative to the inhomogeneous magnetic field in (a). As a consequence the initial states are shifted by $\phi_0' = \phi_0 - \frac{\pi}{2}$, relative to (a). From the perspective of the $\vec{e}_j$ aligned inhomogeneous magnetic field, the approaching beam from (a) has two allowed spin states $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$, and we observe the splitting of the beam of silver atoms into spin up and spin down.

For the purposes of this analysis we assume that the magnetic moment of the silver atom is adequately described by the unitary,

$$\hat{U}(t) \;=\; e^{\hat{\sigma}_i \frac{t}{2}} e^{\hat{\sigma}_j \frac{t}{2}} \tag{30}$$

In figure 6 is the global phase of the closed path, as a function of the initial state $\{\theta_0, \phi_0\}$. It is seen that for $0 < \phi_0 < \pi$ the spinor is in a $f^{-\frac{1}{2}}$ spin state, whereas for $\pi < \phi_0 < 2\pi$ the spinor is in a $f^{\frac{1}{2}}$ spin state. We utilize this unitary to account for the results of the Stern-Gerlach experiment.

5.2. The Stern-Gerlach experiment - spinors

In the Stern-Gerlach experiment the beam of silver atoms is allowed to pass through an arrangement of 3 Stern-Gerlach apparatus'. The alignment of the inhomogeneous magnetic fields of the first and third apparatus are parallel, whereas the alignment of the second apparatus is perpendicular to the first and third.

- The inhomogeneous magnetic fields of the first and third Stern-Gerlach apparatus is aligned along the $\vec{e}_k$ direction. The second Stern-Gerlach apparatus has an inhomogeneous magnetic field aligned along the $\vec{e}_j$ direction, and the direction of propagation is the $\vec{e}_i$ direction.*

* The direction of propagation is chosen to be aligned along the same axis as the singularities of the

- The beam of silver atoms is allowed to pass through the first inhomogeneous magnetic field and separates into two beams $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$ according to the global phase of the closed path in figure 6 (a).
- The $f^{\frac{1}{2}}$ beam is blocked, as indicated by the shaded region, while the $f^{-\frac{1}{2}}$ is allowed to pass through the second Stern-Gerlach apparatus.
- Key Point: The second inhomogeneous magnetic field is shifted by an angle of $\frac{\pi}{2}$ relative to the first. Therefore it follows that the initial state of the spinor is shifted by an amount $\phi_0' = \phi_0 - \frac{\pi}{2}$. As a result, the initial values of the blocked and allowed states are also shifted by $\phi_0' = \phi_0 - \frac{\pi}{2}$, as illustrated in figure 6 (b). The figure shows that from the perspective of the second Stern-Gerlach apparatus, the intrinsic spin of the incoming beam is composed of both spin up and spin down states.
- The beam of silver atoms that enters the second Stern-Gerlach apparatus, now splits into an $f^{-\frac{1}{2}}$ beam and an $f^{\frac{1}{2}}$ beam according to figure 6 (b).
- The $f^{\frac{1}{2}}$ beam emerging from the second Stern-Gerlach apparatus is blocked while the $f^{-\frac{1}{2}}$ beam is allowed to pass through a third Stern-Gerlach apparatus, where the inhomogeneous magnetic field is aligned along the $\vec{e}_k$ direction.
- We expect that the beam emerging from the third Stern-Gerlach apparatus to be entirely spin down $f^{-\frac{1}{2}}$. What is reported however, is that the beam splits into two beams of $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$, contrary to expectation.

Sakurai does not state what the weighting of the third beam is, he simply relays that 'By improving the experimental techniques we cannot make the $f^{\frac{1}{2}}$ component out of the third apparatus disappear.' If it were a 90-10 weighting of the $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$, then it would be reasonable to assume that experimental error and spin flips would account for the observed discrepancy. In the next section we propose an adaptation of the Stern-Gerlach experiment to help quantify the accuracy of the experiment.

5.3. The Stern-Gerlach experiment - proposal

Since we are not privy to the weighting of the third beam, let us propose briefly an experimental measure that may help to shed some light on the issue.
Experimental Proposal:

- We consider an arrangement of three Stern-Gerlach apparatus', where in the first instance the inhomogeneous magnetic field of all three apparatus' is aligned along the $\vec{e}_k$ direction.
- The $f^{\frac{1}{2}}$ beam emerging from the first apparatus is blocked whereas the $f^{-\frac{1}{2}}$ beam is allowed to pass through the second apparatus.
- The $f^{-\frac{1}{2}}$ beam emerging from the second apparatus is allowed to pass through the third, and any spin flips resulting in an $f^{\frac{1}{2}}$ beam are accounted for and blocked.

2-sphere. This choice is taken due to symmetry - since the SG apparatuses are 90^o rotations around the axis of propagation, and for the purposes of the present discussion it works.

- The beam emerging from the third apparatus is measured to determine the weighting of the final $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$ beams. The final weighting is to be used as a standard from which one may determine the accuracy of the experiment.
- The experiment is repeated as above, where the alignment of the second inhomogeneous magnetic field is now rotated by an angle δ about the $\vec{e}_i$ axis.
- The angle of the second Stern-Gerlach apparatus is incrementally rotated (e.g. by an amount $\delta = \frac{\pi}{20}$) at the beginning of each run, until the inhomogeneous magnetic field is aligned along the $\vec{e}_j$ direction, as it was in the original experiment.

The weighting of the final beam is documented for each run. Since we expect the final beam to be entirely $f^{-\frac{1}{2}}$, this provides a useful metric from which one can deduce the accuracy of the experiment. The data acquired from the experiment described above will help to establish the validity of using Hamilton's quaternions to describe the dynamics of the fundamental particles. Pending the results of the above experiment, this proves that the magnetic moment of the fundamental particles is 4-dimensional.

6. Conclusions

The global phase (14) is the 4th-dimensional shadow of the quaternion and a natural hidden variable of both quantum mechanics, and classical mechanics. The global phase parametrizes the $\mathbb{S}^1$ unit circle which is a fibre bundle (21) consisting of the global, geometric and dynamic phases connecting $\mathbb{S}^3$ and $\mathbb{S}^2$

$$\mathbb{S}^3 \xmapsto{\mathbb{S}^1} \mathbb{S}^2$$

The global phase is a measure of the total anholonomy of the 3-dimensional path and offers a natural classification of the closed $\mathbb{S}^2$ paths as bosonic or fermionic. The bosonic paths, $e^{\imath\frac{\omega}{2}} = +1$ correspond to Möbius bands with an even number of half-turns, and the fermionic paths $e^{\imath\frac{\omega}{2}} = -1$ correspond to Möbius bands with an odd number of half-turns. Interpreted physically, the global phase describes the intrinsic spin of the bosons and fermions.

The principle result of this article is that the global phase can be interpreted to account for the results of the Stern-Gerlach experiment deterministically; and exhibits the properties required to develop a complete and deterministic account of spin for both the integer and half-integer spin particles, in a natural way. The quaternion is the general description of all "2-level quantum systems" found in quantum mechanics, describing everything from the kinematics of spin particles, to the hidden variables of the qubit in the quantum information sciences, to quantum tunneling, self-trapping and the density dynamics of a Bose-Einstein condensate in the double well [29].

Quantum mechanics remains the singular scientific discipline endorsing *the probability interpretation* of the complex numbers and the quaternion i.e. the Born Rule. In the quantum theory, the square magnitude of a complex number is believed to describe the probable result of an experimental measure. In the mathematical sciences there is a contrasting viewpoint; the magnitude of a complex number and the magnitude of a quaternion corresponds to the length measure of the respective 2- and 4-dimensional vectors. Recognizing the quaternion as the deterministic geometric object it is - the probabilistic interpretation of particle spin prescribed by the quantum theory is deemed inadequate, as - a deterministic theory of particle spin is available using the full parameter space of the quaternion. For lower dimensional systems that are described by the complex plane $\mathbb{C}$, it has been shown that a precise description of particle trajectory [30] is available using the DeBroglie-Bohm Pilot Wave theory [31]. These and related approaches completely remove indeterminism as the only degree of freedom is found in the initial state. As the mathematical algebra of the complex numbers $\mathbb{C}^n$ is continually developed it will inevitably lead to a natural physical interpretation of the intriguing results of Young's Double Slit Experiment [32].

The laws governing our 3-dimensional reality - according to classical mechanics - are seen to be rooted in 4-spatial dimensions via the quaternion. This is conclusive evidence that reality as we know it is not limited to 3-dimensions, at the very least reality is 4-dimensional and according to the hopf fibration extends as 2^n-dimensional. The multidimensional nature of reality has been revealed to us via the parameter space of the unit quaternion. *"We claim that Hamilton's conjecture, ... the concept that somehow quaternions are a fundamental building block of the physical universe, appears to be essentially correct in the light of contemporary knowledge."* [33]*

7. Outlook

In Maxwell's Treatise on the Electromagnetic Field [36], he originally formulated his field equations using the quaternions. Maxwell's equations, as we know them today, were simplified by Heaviside *et al.* for practical purposes. Today the connection between the *true* Maxwell's equations and the quaternion is not as well known. However, now that we are armed with the knowledge of the intrinsic parameter space of the unit quaternion it would serve greatly to revisit Maxwell's original treatise, with an aim to uncover the consequences of these parameters for the Electromagnetic Field, see Appendix A.

According to the Adams theorem, the extensions of the hopf fibration are limited to [37],

$$\mathbb{S}^3 \xmapsto{\mathbb{S}^1} \mathbb{S}^2$$

$$\mathbb{S}^7 \xmapsto{\mathbb{S}^3} \mathbb{S}^4$$

$$\mathbb{S}^{15} \xmapsto{\mathbb{S}^7} \mathbb{S}^8$$

Where the unit circle $\mathbb{S}^1$ describes the 1-sphere, the unit quaternion $\mathbb{S}^3$ describes the 3-sphere, the unit octonion $\mathbb{S}^7$ describes the 7-sphere, and the unit sedenion $\mathbb{S}^{15}$ describes the 15-sphere.

* Andre Gsponer and Jean-Pierre Hurni

The generalization of the hopf fibration to dimensional spaces beyond the quaternion has already captured significant attention for its potential power in characterizing mixed and entangled quantum states [38] [39] [40]. However, the extension of this work beyond the quaternion is a formidable task, as little is known about the hyper-complex numbers, the octonion and the sedenion. They are not only non-commutative but non-associative and also forbid square matrix representation. As long as the closed form representations of these groups remains unknown the extension of this work to the octonion and sedenion remains intractable. Efforts would be best served in mastering the unit quaternion, and above all else, the consequences and physical implications of the hidden variables for the Electromagnetic Field is highly sought [41].

"The present system of Quantum Mechanics would have to be objectively false, in order that another description of the elementary processes other than the statistical one be possible." [34, *pg* 55]* In light of the fact that the qubit is a unit quaternion - and given that the Copenhagen Interpretation of the quaternion has no place in any physical theory purporting to describe physical reality - we conclude that quantum mechanics is not only incomplete [9] but observably inadequate as there are indeed hidden variables unaccounted for by the theory. The hidden variables of quantum mechanics are found in the parameter space of the unit quaternion. *"You believe in the God who plays dice, and I in complete law and order in a world which objectively exists ... Even the great initial success of quantum theory does not make me believe in the fundamental dice-game ... No doubt the day will come when we will see whose instinctive attitude was the correct one."* [35, *pg 149 (Sept. 7th, 1944)*]†

We have shown that the present theory of particle spin - as per quantum mechanics - amounts to no less than a Copenhagen Interpretation of the quaternion - as it is a *Schrödinger's cat hypothesis* with it's foundations built upon the principle of superposition. Recognizing that this is a flawed logic it follows that all theoretical proposals based on the present quantum mechanical theory of spin - such as the quest to build a quantum computer - are entirely unfeasible.

Quantum mechanics is the singular scientific discipline endorsing the *Copenhagen Interpretation of the quaternion*, and the *probabilistic interpretation of the complex numbers.* The shortcomings of this line of reasoning has been emphasized since 1935 until modern times, yet progress in the field continues unabated, producing countless "developments" in the house of cards that is the mathematical algebra of quantum mechanics. And should a balance sheet be drawn up on *what the quantum theory actually says* about the nature of reality you can bet your bottom dollar the results would be inconclusive, open to interpretation and observer dependent. It is a well known fact - and one easily verified by experiment: 'ask 2 quantum mechanics the same question and you will get 3 different answers.' *"It's a bad sign in particular that those physicists who are happy about quantum mechanics, and see nothing wrong with it, don't agree with*

* John von Neumann

† Albert Einstein

*each other about what it means."** [7]

The good news is that the complex numbers and the quaternion are both very well known to modern science, and are used every day in computer vision, robotics, computer graphics, virtual reality, and related fields. While the known applications of the quaternion are very broad, they are *tastefully* confined to use in good old *classical* devices, in a *classical* world, that rely on *classical* Boolean logic. One interesting means of independently adjudicating the mathematics of quantum mechanics - and in particular the probabilistic interpretation of the complex numbers and the quaternion - is to have professional researchers who are knowledgable in the applications and mathematics of the complex numbers and the quaternion referee the entire field of quantum mechanics. This would facilitate achieving two objectives. Firstly that all misuses of the quaternion promulgated by the quantum theory would be acknowledged and accounted for. Secondly that all errors and inaccuracies related to the Copenhagen Interpretation of the quaternion be expunged from the theory - safe for the quantum theory expunging itself.

The phenomenon of entanglement and the associated concept of non-locality is one of the most novel and intriguing aspects of the quantum theory. This analysis is immediately applicable to the separable states and can in principle be extended to describe the entangled states, see Appendix F. To conclude we offer a quaternion's perspective of the phenomenon of entanglement and non-locality.

"It has been argued that quantum mechanics is not locally causal and cannot be embedded in a locally causal theory. That conclusion depends on treating certain experimental parameters, typically the orientations of polarization filters, as free variables. ... But it might be that this apparent freedom is illusory. Perhaps experimental parameters and experimental results are both consequences, or partially so, of some common hidden mechanism. Then the apparent non-locality could be simulated." [1, *ch* 12]†

Entanglement and the bipartite entangled state is described by the $\mathbb{C}^4$ spinor (quantum state), see Appendix G. In experiments involving the maximally entangled spin pair, two *'separate'* particles are demonstrated to exhibit *'non-local'* correlations. Observed in 3 dimensions is two spatially separated and non-locally connected particles, this is the 3d shadow of the multi-dimensional object. The appearance of two distinct and separate particles is an illusion, derived from looking at a multi-dimensional object from a 3-dimensional perspective. Non-locality is a 3-dimensional illusion.

"That the guiding wave, in the general case, propagates not in ordinary three-space but in a multi-dimensional configuration space is the origin of the notorious 'non-locality' of Quantum Mechanics." [1, *ch* 14]†

* 2016 Patrusky Lecture: Steven Weinberg on *"What's the matter with quantum mechanics?"*
† John Stewart Bell

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Appendix A. The electromagnetic field tensor and the isoclinic decomposition of SO(4)

The left cayley matrices are defined

$$\hat{\sigma}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \hat{\sigma}_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \hat{\sigma}_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A.1}$$

which satisfy the relation

$$\hat{\sigma}_i^2 = \hat{\sigma}_j^2 = \hat{\sigma}_k^2 = \hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k = -\hat{\sigma}_1$$

The right cayley matrices are defined

$$\hat{\rho}_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \hat{\rho}_j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \hat{\rho}_k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

which satisfy the relation

$$\hat{\rho}_i^2 = \hat{\rho}_j^2 = \hat{\rho}_k^2 = \hat{\rho}_i\hat{\rho}_j\hat{\rho}_k = -\hat{\rho}_1$$

The quaternion is expanded in the left and right cayley bases respectively as

$$\hat{U}_L = a\hat{\sigma}_1 + b\hat{\sigma}_i + c\hat{\sigma}_j + d\hat{\sigma}_k = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \tag{A.3}$$

$$\hat{U}_R = a\hat{\rho}_1 + b\hat{\rho}_i + c\hat{\rho}_j + d\hat{\rho}_k = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \tag{A.4}$$

The left and right isoclinic quaternions are commutative,

$$\left[\hat{U}_L, \hat{U}_R\right] = \left[\hat{U}_L, \hat{U}_R^\dagger\right] = \left[\hat{U}_L^\dagger, \hat{U}_R\right] = \left[\hat{U}_L^\dagger, \hat{U}_R^\dagger\right] = 0$$

The isoclinic decomposition of SO(4) is,

$$\hat{Q} = \hat{U}_L\hat{U}_R^\dagger \qquad \hat{Q}^\dagger = \hat{U}_L^\dagger\hat{U}_R$$

$$\hat{Q} = \begin{pmatrix} a^2-b^2-c^2-d^2 & -2ab & -2ac & -2ad \\ 2ab & a^2-b^2+c^2+d^2 & -2bc & -2bd \\ 2ac & -2bc & a^2+b^2-c^2+d^2 & -2cd \\ 2ad & -2bd & -2cd & a^2+b^2+c^2-d^2 \end{pmatrix}$$

The components of the magnetic field and electric field are respectively defined

$$
\begin{aligned}
\mathcal{B}^i &= 2\left(c\dot{d} - \dot{c}d\right) & \mathcal{E}^i &= 2\left(a\dot{b} - \dot{a}b\right) \\
\mathcal{B}^j &= 2\left(\dot{b}d - b\dot{d}\right) & \mathcal{E}^j &= 2\left(a\dot{c} - \dot{a}c\right) \\
\mathcal{B}^k &= 2\left(b\dot{c} - \dot{b}c\right) & \mathcal{E}^k &= 2\left(a\dot{d} - \dot{a}d\right)
\end{aligned}
$$

The covariant and contravariant forms of the electromagnetic field tensor are respectively

$$
\begin{aligned}
[\mathcal{F}_{\mu\nu}] &= \dot{\hat{Q}}\hat{Q}^\dagger & [\mathcal{F}^{\mu\nu}] &= \dot{\hat{Q}}^\dagger\hat{Q} \\
[\mathcal{F}_{\mu\nu}] &= \begin{pmatrix} 0 & \mathcal{E}^i & \mathcal{E}^j & \mathcal{E}^k \\ -\mathcal{E}^i & 0 & -\mathcal{B}^k & \mathcal{B}^j \\ -\mathcal{E}^j & \mathcal{B}^k & 0 & -\mathcal{B}^i \\ -\mathcal{E}^k & -\mathcal{B}^j & \mathcal{B}^i & 0 \end{pmatrix} & [\mathcal{F}^{\mu\nu}] &= \begin{pmatrix} 0 & -\mathcal{E}^i & -\mathcal{E}^j & -\mathcal{E}^k \\ \mathcal{E}^i & 0 & -\mathcal{B}^k & \mathcal{B}^j \\ \mathcal{E}^j & \mathcal{B}^k & 0 & -\mathcal{B}^i \\ \mathcal{E}^k & -\mathcal{B}^j & \mathcal{B}^i & 0 \end{pmatrix}
\end{aligned} \tag{A.5}
$$

Appendix B. The special orthogonal group of 3×3 matrices SO(3)

The special orthogonal group of 3×3 matrices SO(3) is defined by

$$
\begin{aligned}
\hat{U} &= \hat{U}_L\hat{U}_R \qquad\qquad \hat{U}^\dagger = \hat{U}_L^\dagger\hat{U}_R^\dagger \\
\hat{U} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2\,(bc-ad) & 2\,(bd+ac) \\ 0 & 2\,(bc+ad) & a^2-b^2+c^2-d^2 & 2\,(cd-ab) \\ 0 & 2\,(bd-ac) & 2\,(cd+ab) & a^2-b^2-c^2+d^2 \end{pmatrix}
\end{aligned}
$$

with $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{\sigma}_1$. The elements of the hamiltonian $\hat{\mathcal{H}}$ relate to the electric and magnetic field components as

$$
\mathcal{H}^i = \mathcal{B}^i + \mathcal{E}^i \qquad \mathcal{H}^j = \mathcal{B}^j + \mathcal{E}^j \qquad \mathcal{H}^k = \mathcal{B}^k + \mathcal{E}^k
$$

The hamiltonian in SO(3) is given by

$$
\hat{\mathcal{H}} = \dot{\hat{U}}\hat{U}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{H}^k & \mathcal{H}^j \\ 0 & \mathcal{H}^k & 0 & -\mathcal{H}^i \\ 0 & -\mathcal{H}^j & \mathcal{H}^i & 0 \end{pmatrix} \tag{B.1}
$$

Alternatively, the elements of the hamiltonian $\hat{\mathcal{G}}$ relate to the electric and magnetic field components as

$$
\mathcal{G}^i = \mathcal{B}^i - \mathcal{E}^i \qquad \mathcal{G}^j = \mathcal{B}^j - \mathcal{E}^j \qquad \mathcal{G}^k = \mathcal{B}^k - \mathcal{E}^k
$$

where

$$
\hat{\mathcal{G}} = \dot{\hat{U}}^\dagger\hat{U} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{G}^k & \mathcal{G}^j \\ 0 & \mathcal{G}^k & 0 & -\mathcal{G}^i \\ 0 & -\mathcal{G}^j & \mathcal{G}^i & 0 \end{pmatrix}
$$

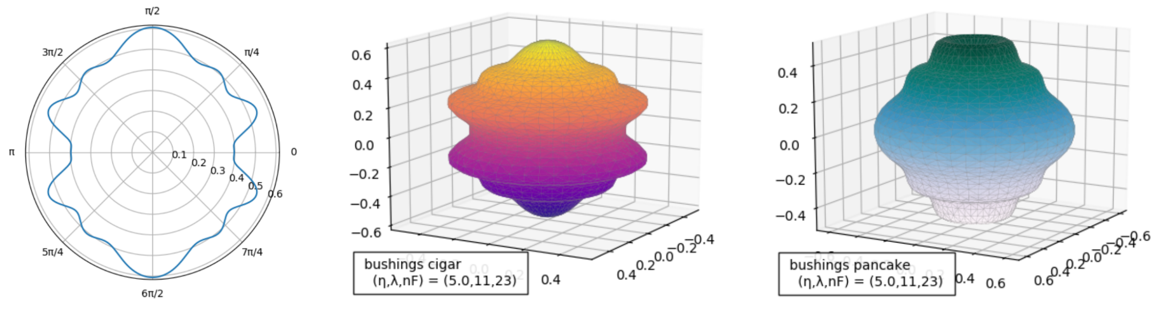


Figure C1. Bushings function (left) and the cigar and pancake surfaces of revolution.

The Lie algebra so(3) of the group SO(3) is spanned by the 3 matrices

$$\hat{\pi}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{\pi}_j = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \hat{\pi}_k = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.2}$$

The commutation relations among these generators are

$$[\hat{\pi}_i, \hat{\pi}_j] = \hat{\pi}_k \qquad [\hat{\pi}_j, \hat{\pi}_k] = \hat{\pi}_i \qquad [\hat{\pi}_k, \hat{\pi}_i] = \hat{\pi}_j$$

The spinor $\hat{\Psi}$ is expressed in terms of the Lie generators as

$$\hat{\Psi} = \exp\left[-\phi\hat{\pi}_i\right]\exp\left[-\theta\hat{\pi}_j\right]\exp\left[\omega\hat{\pi}_i\right]$$

$$\hat{\Psi} = \begin{pmatrix} \mathrm{c}(\theta) & -\mathrm{s}(\theta)\,\mathrm{s}(\omega) & -\mathrm{s}(\theta)\,\mathrm{c}(\omega) \\ \mathrm{s}(\theta)\,\mathrm{s}(\phi) & \mathrm{c}(\phi)\,\mathrm{c}(\omega)+\mathrm{s}(\phi)\,\mathrm{c}(\theta)\,\mathrm{s}(\omega) & \mathrm{s}(\phi)\,\mathrm{c}(\theta)\,\mathrm{c}(\omega)-\mathrm{c}(\phi)\,\mathrm{s}(\omega) \\ \mathrm{s}(\theta)\,\mathrm{c}(\phi) & -\mathrm{s}(\phi)\,\mathrm{c}(\omega)+\mathrm{c}(\phi)\,\mathrm{c}(\theta)\,\mathrm{s}(\omega) & \mathrm{c}(\phi)\,\mathrm{c}(\theta)\,\mathrm{c}(\omega)+\mathrm{s}(\phi)\,\mathrm{s}(\omega) \end{pmatrix}$$

where, $\mathrm{c}(\bullet) = \cos(\bullet)$ and, $\mathrm{s}(\bullet) = \sin(\bullet)$.

Appendix C. Moving frames and parallel transport

Here we apply the methods outlined in the main article to describe the 3 moving frames, the tangent frame, the darboux frame, and the frenet-serret frame, for surfaces of revolution in $\mathbb{R}^3$, [42] [43] [44] [45] [46, *ch* 10] [47, *ch* 6] [48, *ch* 3].

- *bushings surfaces of revolution*

The surface of revolution is generated from bushings function*.

$$r(\mu) = \frac{\gamma(n_F+1,\beta)}{\Gamma(n_F+1)} + e^{-\beta}\sum_{a=0}^{n_F}\frac{\beta^a}{a!}\frac{\gamma\left(\left\lfloor\frac{n_F-a}{\lambda}\right\rfloor+1,\frac{\alpha}{\lambda}\right)}{\Gamma\left(\left\lfloor\frac{n_F-a}{\lambda}\right\rfloor+1\right)} \tag{C.1}$$

* for references and python scripts see: github.com/mo-geometry/bushings function

$\gamma(...)$, $\Gamma(...)$ are the lower incomplete gamma function, and gamma function respectively. $\lfloor ... \rfloor$ is the floor function. $\eta \in \mathbb{R}$ is a positive real number, the anisotropy parameter λ is a positive integer, the shell (fermi) occupancy parameter n_{F} is a positive integer (-1, 0 inclusive), and $\mu \in [0, 2\pi]$. Two different surfaces of revolution are generated depending on the choice of rotation axis.

$$\begin{aligned}
\text{bushings cigar :} \qquad \alpha(\mu), \beta(\mu) &= \eta^2 \sin^2(\mu), \eta^2 \cos^2(\mu) \\
\text{bushings pancake :} \qquad \alpha(\mu), \beta(\mu) &= \eta^2 \cos^2(\mu), \eta^2 \sin^2(\mu)
\end{aligned}$$

The bushings surfaces are a convenient choice as they are continuously differentiable functions of μ. A polar plot of equation (C.1) for $(\eta, \lambda, n_{\mathrm{F}}) = (5.0, 11, 23)$ is shown in figure C1, and illustrated are the cigar and pancake surfaces of revolution.

We work in spherical polar coordinates where the principle axis is the x-axis $(i, j, k) = (x, y, z)$, and (μ, ν) are the polar and azimuthal angles. Bushings surface (C.1) is parametrized by

$$\vec{\mathcal{S}}(\mu, \nu) = r(\mu) \vec{\mathcal{R}}(\mu, \nu) = r(\mu) \begin{pmatrix} \cos(\mu) \\ \sin(\mu) \sin(\nu) \\ \sin(\mu) \cos(\nu) \end{pmatrix} \tag{C.2}$$

- *hopf projection*

The parameters of the bloch vector are determined by our choice of one of the 3 hopf projections*

$$\hat{\mathcal{I}} = \hat{\Psi}\, \hat{\pi}_i\, \hat{\Psi}^\dagger \qquad \hat{\mathcal{J}} = \hat{\Psi}\, \hat{\pi}_j\, \hat{\Psi}^\dagger \qquad \hat{\mathcal{K}} = \hat{\Psi}\, \hat{\pi}_k\, \hat{\Psi}^\dagger \tag{C.3}$$

The SO(3) spinor is defined by our choice of unitary (equations (27), (28), (29), (30))

$$\hat{\Psi}(t) = \hat{U}(t) \hat{\Psi}_0 \qquad \hat{\mathcal{H}}(t) = \dot{\hat{U}} \hat{U}^\dagger$$

with $\hat{\Psi}_0 = \hat{\Psi}(\phi_0, \theta_0, \omega_0)$. The spinor is expanded in the hopf parameter space as:

$$\hat{\Psi}(\phi, \theta, \omega) = \exp[-\phi \hat{\pi}_i] \exp[-\theta \hat{\pi}_j] \exp[\omega \hat{\pi}_i] = \begin{pmatrix} \mathcal{I}^i & \mathcal{J}^i & \mathcal{K}^i \\ \mathcal{I}^j & \mathcal{J}^j & \mathcal{K}^j \\ \mathcal{I}^k & \mathcal{J}^k & \mathcal{K}^k \end{pmatrix}$$

$$\hat{\Psi} = \begin{pmatrix} \mathrm{c}(\theta) & -\mathrm{s}(\theta)\,\mathrm{s}(\omega) & -\mathrm{s}(\theta)\,\mathrm{c}(\omega) \\ \mathrm{s}(\theta)\,\mathrm{s}(\phi) & \mathrm{c}(\phi)\,\mathrm{c}(\omega) + \mathrm{s}(\phi)\,\mathrm{c}(\theta)\,\mathrm{s}(\omega) & -\mathrm{c}(\phi)\,\mathrm{s}(\omega) + \mathrm{s}(\phi)\,\mathrm{c}(\theta)\,\mathrm{c}(\omega) \\ \mathrm{s}(\theta)\,\mathrm{c}(\phi) & -\mathrm{s}(\phi)\,\mathrm{c}(\omega) + \mathrm{c}(\phi)\,\mathrm{c}(\theta)\,\mathrm{s}(\omega) & \mathrm{s}(\phi)\,\mathrm{s}(\omega) + \mathrm{c}(\phi)\,\mathrm{c}(\theta)\,\mathrm{c}(\omega) \end{pmatrix}$$

where, $\mathrm{c}(\bullet) = \cos(\bullet)$ and, $\mathrm{s}(\bullet) = \sin(\bullet)$. The integrable spinor parameters are obtained from the $\hat{\pi}_i$ von-Neumann equation

$$\dot{\hat{\mathcal{I}}} = [\hat{\mathcal{I}}, \hat{\mathcal{H}}]$$

* also interesting is to consider the projections in the conjugate space: $\hat{\Psi}^\dagger \hat{\pi}_\bullet \hat{\Psi}$, for $\bullet = i, j, k$.

as:

$$\left(\dot{\phi},\ \dot{\theta},\ \dot{\omega}\right)\ \equiv\ \left(-\mathcal{H}^i+\frac{\mathcal{H}^j\mathcal{I}^j+\mathcal{H}^k\mathcal{I}^k}{\left(\mathcal{I}^j\right)^2+\left(\mathcal{I}^k\right)^2}\,\mathcal{I}^i,\ \frac{\mathcal{H}^k\mathcal{I}^j-\mathcal{H}^j\mathcal{I}^k}{\sqrt{\left(\mathcal{I}^j\right)^2+\left(\mathcal{I}^k\right)^2}},\ \frac{\mathcal{H}^j\mathcal{I}^j+\mathcal{H}^k\mathcal{I}^k}{\left(\mathcal{I}^j\right)^2+\left(\mathcal{I}^k\right)^2}\right) \tag{C.4}$$

Similarly the integrable parameters of the bloch vector are obtained from the von-Neumann equation

$$\dot{\hat{\mathcal{R}}}\ =\ [\hat{\mathcal{R}},\hat{\mathcal{H}}]$$

as:

$$(\dot{\nu},\ \dot{\mu})\equiv\left(-\mathcal{H}^i+\frac{\mathcal{H}^j\mathcal{R}^j+\mathcal{H}^k\mathcal{R}^k}{\left(\mathcal{R}^j\right)^2+\left(\mathcal{R}^k\right)^2}\,\mathcal{R}^i,\ \frac{\mathcal{H}^k\mathcal{R}^j-\mathcal{H}^j\mathcal{R}^k}{\sqrt{\left(\mathcal{R}^j\right)^2+\left(\mathcal{R}^k\right)^2}}\right) \tag{C.5}$$

with $\hat{\mathcal{R}}$ being equal to one of $\hat{\mathcal{I}},\hat{\mathcal{J}},\hat{\mathcal{K}}$. Evidently we recover the results from the main article with $\hat{\mathcal{R}}=\hat{\mathcal{I}}$, since ν,μ, correspond to the azimuthal, polar angles, and Ω is the global phase in $\mathbb{R}^3$.

- *moving frames*

Here we re-derive the results of the main article* with reference to the tangent, darboux, and frenet-serret frames, and thereafter extend the results for surfaces of revolution in $\mathbb{R}^3$. We define

$$n_\nu\ =\ r\sin(\mu)\qquad\qquad n_\mu\ =\ \sqrt{r^2+\partial_\mu r^2}\qquad\qquad n_v\ =\ \sqrt{n_\mu^2\dot{\mu}^2+n_\nu^2\dot{\nu}^2}$$

and the basis of the partial derivatives

$$\partial_\nu\vec{\mathcal{R}}\ =\ \begin{pmatrix}0\\ r\sin(\mu)\cos(\nu)\\ -r\sin(\mu)\sin(\nu)\end{pmatrix}\qquad\qquad \partial_\mu\vec{\mathcal{R}}\ =\ \begin{pmatrix}\partial_\mu r\cos(\mu)-r\sin(\mu)\\ (\partial_\mu r\sin(\mu)+r\cos(\mu))\sin(\nu)\\ (\partial_\mu r\sin(\mu)+r\cos(\mu))\cos(\nu)\end{pmatrix}$$

The total derivatives are expanded in terms of the partials as

$$\begin{aligned}\dot{\vec{\mathcal{R}}}\ &=\ \dot{\mu}\partial_\mu\vec{\mathcal{R}}+\dot{\nu}\partial_\nu\vec{\mathcal{R}}\\ \ddot{\vec{\mathcal{R}}}\ &=\ \ddot{\mu}\partial_\mu\vec{\mathcal{R}}+\dot{\mu}^2\partial_\mu^2\vec{\mathcal{R}}+2\dot{\nu}\dot{\mu}\partial_\nu\partial_\mu\vec{\mathcal{R}}+\ddot{\nu}\partial_\nu\vec{\mathcal{R}}+\dot{\nu}^2\partial_\nu^2\vec{\mathcal{R}}\end{aligned}$$

* The $\mathbb{S}^1$ fibre bundle composes the global, geometric and dynamic phases Ω,γ,ξ, related by $\Omega=\gamma+\xi$.

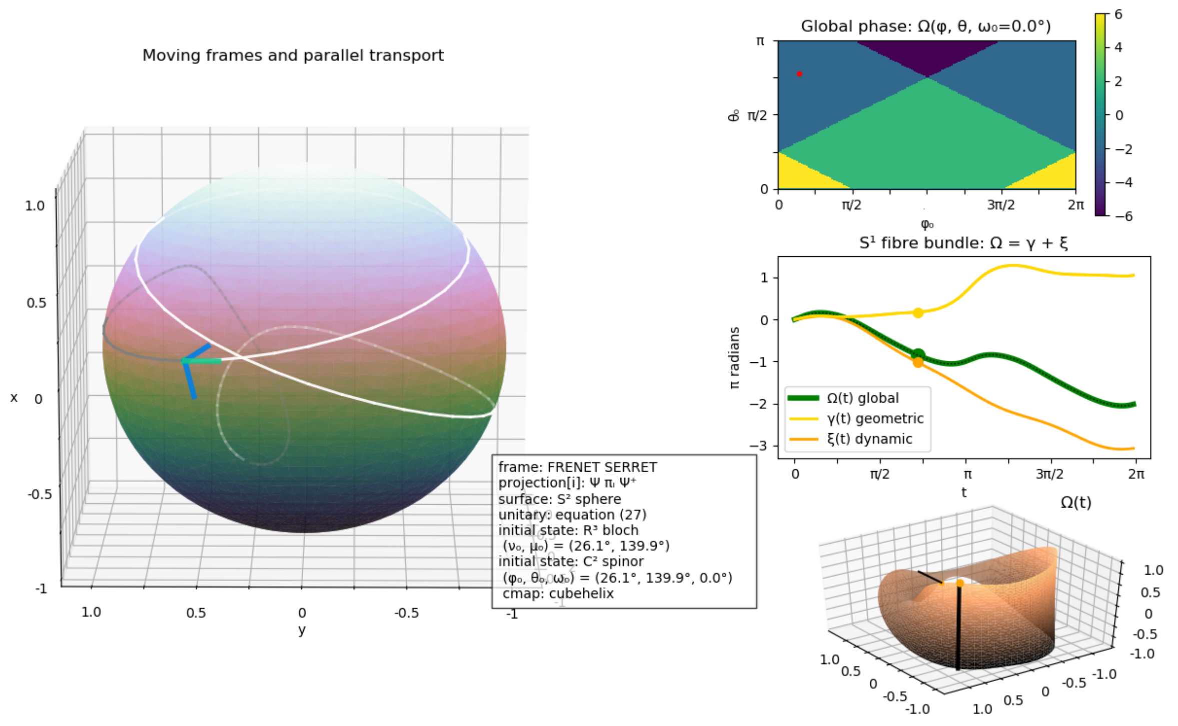


Figure C2. `matplotlib` display of the frenet-serret frame, details inset.

The tangent, darboux, and frenet-serret frames are defined:

tangent frame: [azimuthal, polar, surface normal]

$$(\vec{e}_\nu, \vec{e}_\mu, \vec{e}_n) = \left(\frac{\partial_\nu \vec{\mathcal{R}}}{n_\nu}, \frac{\partial_\mu \vec{\mathcal{R}}}{n_\mu}, \frac{\partial_\nu \vec{\mathcal{R}} \times \partial_\mu \vec{\mathcal{R}}}{|\partial_\nu \vec{\mathcal{R}} \times \partial_\mu \vec{\mathcal{R}}|} \right)$$

darboux frame: [velocity, bi-normal, surface normal]

$$(\vec{e}_v, \vec{e}_s, \vec{e}_n) = \left(\frac{\dot{\vec{\mathcal{R}}}}{n_v}, \frac{n_\mu}{n_\nu n_v} \dot{\mu} \partial_\nu \vec{\mathcal{R}} - \frac{n_\nu}{n_\mu n_v} \dot{\nu} \partial_\mu \vec{\mathcal{R}}, \frac{\partial_\nu \vec{\mathcal{R}} \times \partial_\mu \vec{\mathcal{R}}}{|\partial_\nu \vec{\mathcal{R}} \times \partial_\mu \vec{\mathcal{R}}|} \right)$$

frenet-serret frame: [velocity, acceleration, bi-normal FS]

$$(\vec{e}_v, \vec{e}_a, \vec{e}_b) = \left(\frac{\dot{\vec{\mathcal{R}}}}{n_v}, \frac{\ddot{\vec{\mathcal{R}}}}{|\ddot{\vec{\mathcal{R}}}|}, \frac{\dot{\vec{\mathcal{R}}} \times \ddot{\vec{\mathcal{R}}}}{n_v |\ddot{\vec{\mathcal{R}}}|} \right)$$

Figure C2 shows the `matplotlib` display for the frenet-serret frame.$\sharp$

- *parallel transport*

Orthonormal moving frames $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ are differentiable functions of t. A vector $\vec{\mathcal{V}}$ is expanded in the moving frame via

$$\vec{\mathcal{V}} = \mathcal{V}^a \vec{e}_a$$

$\sharp$ for python and matlab scripts see: github.com/mo-geometry/parallel transport where the upper and lower indices are summed. The first derivative

$$\dot{\vec{\mathcal{V}}} \;=\; \dot{\mathcal{V}}^a \vec{e}_a + \mathcal{V}^a \dot{\vec{e}}_a$$

The differential form

$$\mathcal{A}^a{}_b \;\equiv\; \vec{e}_a \cdot \dot{\vec{e}}_b \tag{C.6}$$

allows us to express the derivative of the basis vectors as the linear superposition

$$\dot{\vec{e}}_a \;=\; \mathcal{A}^b{}_a \, \vec{e}_b$$

and write the first derivative of a vector in the moving frame as

$$\dot{\vec{\mathcal{V}}} \;=\; \left(\dot{\mathcal{V}}^b + \mathcal{A}^b{}_a \mathcal{V}^a\right) \vec{e}_b \tag{C.7}$$

The vector is parallel transported if at all times t

$$\dot{\vec{\mathcal{V}}} \cdot \vec{e}_i \;=\; 0 \tag{C.8}$$

therefore we have

$$\dot{\mathcal{V}}^b + \mathcal{A}^b{}_a \mathcal{V}^a \;=\; 0$$

For the moving frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

$$\begin{pmatrix} \dot{\mathcal{V}}^1 \\ \dot{\mathcal{V}}^2 \\ \dot{\mathcal{V}}^3 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{A}^1{}_2 & \mathcal{A}^1{}_3 \\ \mathcal{A}^2{}_1 & 0 & \mathcal{A}^2{}_3 \\ \mathcal{A}^3{}_1 & \mathcal{A}^3{}_2 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}^1 \\ \mathcal{V}^2 \\ \mathcal{V}^3 \end{pmatrix} \;=\; 0$$

We reduce the indices of the differential forms via,

$$\mathcal{A}^a \;=\; \epsilon_{abc} \mathcal{A}^b{}_c$$

ϵ_{abc} is the Levi-Cevita symbol, which is equal to 1 when the indices are ordered, and equal to -1 when the indices are anti-ordered (0 otherwise).

$$\begin{pmatrix} \dot{\mathcal{V}}^1 \\ \dot{\mathcal{V}}^2 \\ \dot{\mathcal{V}}^3 \end{pmatrix} \;=\; \begin{pmatrix} 0 & -\mathcal{A}^3 & \mathcal{A}^2 \\ \mathcal{A}^3 & 0 & -\mathcal{A}^1 \\ -\mathcal{A}^2 & \mathcal{A}^1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}^1 \\ \mathcal{V}^2 \\ \mathcal{V}^3 \end{pmatrix} \tag{C.9}$$

Since $\vec{\mathcal{V}} = \mathcal{V}^a \vec{e}_a$ the above is expressed as a vector equation using the square brackets $[\;]_{\rm lf}$, to indicate that inside the brackets the basis is the cartesian frame, and post evaluation the coefficients are expanded in the local *moving* frame.

$$\left[\dot{\vec{\mathcal{V}}} - \vec{\mathcal{A}} \times \vec{\mathcal{V}}\right]_{\rm lf} \;=\; 0$$

- *geometric phase*

The tangent frame of bushings surface is defined

$$\vec{e}_\nu \equiv \begin{pmatrix} 0 \\ \cos(\nu) \\ -\sin(\nu) \end{pmatrix} \qquad \vec{e}_\mu \equiv \frac{1}{n_\mu}\begin{pmatrix} \partial_\mu r\cos(\mu) - r\sin(\mu) \\ (\partial_\mu r\sin(\mu) + r\cos(\mu))\sin(\nu) \\ (\partial_\mu r\sin(\mu) + r\cos(\mu))\cos(\nu) \end{pmatrix}$$

$$\vec{e}_n \equiv \frac{1}{n_\mu}\begin{pmatrix} \partial_\mu r\sin(\mu) + r\cos(\mu) \\ -(\partial_\mu r\cos(\mu) - r\sin(\mu))\sin(\nu) \\ -(\partial_\mu r\cos(\mu) - r\sin(\mu))\cos(\nu) \end{pmatrix}$$

The equation of parallel transport for the tangent frame is

$$\begin{pmatrix} \dot{\mathcal{V}}^\nu \\ \dot{\mathcal{V}}^\mu \\ \dot{\mathcal{V}}^n \end{pmatrix} = \begin{pmatrix} 0 & -\mathcal{A}^n & \mathcal{A}^\mu \\ \mathcal{A}^n & 0 & -\mathcal{A}^\nu \\ -\mathcal{A}^\mu & \mathcal{A}^\nu & 0 \end{pmatrix}\begin{pmatrix} \mathcal{V}^\nu \\ \mathcal{V}^\mu \\ \mathcal{V}^n \end{pmatrix} = 0$$

The elements of the parallel transport equation (C.9) with $(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (\vec{e}_\nu, \vec{e}_\mu, \vec{e}_n)$ for the tangent frame.

$$\mathcal{A}^\nu = \epsilon_{\nu\mu n}\mathcal{A}^\mu{}_n = \vec{e}_\mu \cdot \dot{\vec{e}}_n = \frac{\dot{\mu}}{n_\mu^2}\left(r^2 + 2\partial_\mu r^2 - r\partial_\mu^2 r\right)$$

$$\mathcal{A}^\mu = \epsilon_{\mu n\nu}\mathcal{A}^n{}_\nu = \vec{e}_n \cdot \dot{\vec{e}}_\nu = \frac{\dot{\nu}}{n_\mu}\left(\partial_\mu r\cos(\mu) - r\sin(\mu)\right)$$

$$\mathcal{A}^n = \epsilon_{n\nu\mu}\mathcal{A}^\nu{}_\mu = \vec{e}_\nu \cdot \dot{\vec{e}}_\mu = \frac{\dot{\nu}}{n_\mu}\left(\partial_\mu r\sin(\mu) + r\cos(\mu)\right)$$

Now as before we confine ourselves to the tangent plane, $\dot{\gamma} = \mathcal{A}^n$

$$\begin{pmatrix} \dot{\mathcal{V}}^\nu \\ \dot{\mathcal{V}}^\mu \end{pmatrix} = \begin{pmatrix} 0 & -\dot{\gamma} \\ \dot{\gamma} & 0 \end{pmatrix}\begin{pmatrix} \mathcal{V}^\nu \\ \mathcal{V}^\mu \end{pmatrix}$$

We define the geometric phase for surfaces of revolution (C.2):

$$\dot{\gamma}(t) \equiv \frac{\dot{\nu}}{n_\mu}\Big(\partial_\mu r\sin(\mu) + r\cos(\mu)\Big) \qquad \gamma(t) = \int_0^t dt'\ \dot{\gamma}(\mu(t'), t') \tag{C.10}$$

integrable solution:

$$\begin{pmatrix} \mathcal{V}^\nu(t) \\ \mathcal{V}^\mu(t) \end{pmatrix} = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix}\begin{pmatrix} \mathcal{V}_0^\nu \\ \mathcal{V}_0^\mu \end{pmatrix}$$

- *global phase*

The $\mathbb{S}^1$ fibre bundle is composed of the global, geometric and dynamic phases (Ω, γ, ξ). The global phase of the 2-sphere is defined

$$\dot{\Omega} \equiv \frac{\mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k}{(\mathcal{R}^j)^2 + (\mathcal{R}^k)^2} \qquad \Omega(t) = \int_0^t dt'\ \dot{\Omega}(t') \tag{C.11}$$

with $\hat{\mathcal{R}}(\mu, \nu)$ given by one of the hopf projections (C.3).

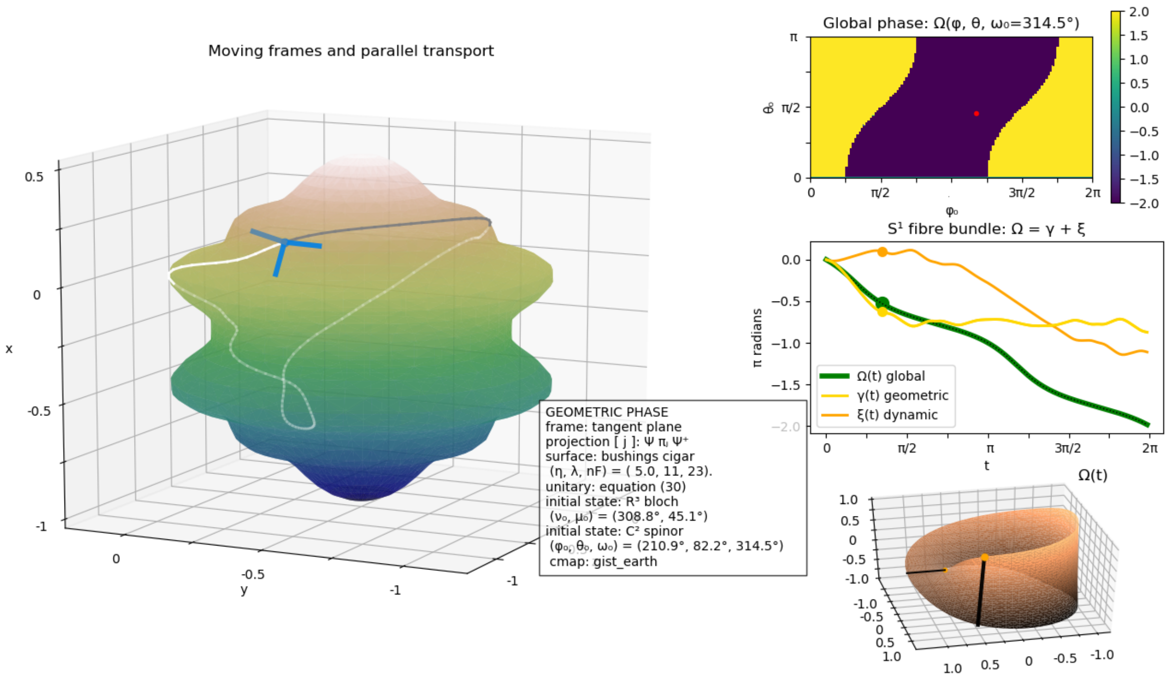


Figure C3. `matplotlib` display of the tangent vector & tangent frame, details inset.

- *dynamic phase*

The dynamic phase of the $\mathbb{S}^2$ path is the integral of the work

$$\dot{\xi} \;\equiv\; \vec{\mathcal{H}}(t)\cdot\vec{\mathcal{R}}(t) \;=\; \mathcal{H}^i\mathcal{R}^i + \mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k \qquad\qquad \xi(t) \;=\; \int_0^t dt'\;\dot{\xi}(t') \tag{C.12}$$

For surfaces of revolution in $\mathbb{R}^3$ the dynamic phase requires a more sophisticated analysis, perhaps in terms of contour integrals, than is currently available (to the author). Until such time we proceed in a heuristic manner and infer the form of the dynamic phase from proposition (21). The azimuthal angle is from (C.5) and (C.11) given by $\dot{\nu} = -\mathcal{H}^i + \dot{\Omega}\,\mathcal{R}^i$. In terms of the global phase the dynamic phase is developed (C.10)

$$\begin{aligned}
\dot{\xi} \;&=\; \dot{\Omega} - \dot{\gamma} \;=\; \dot{\Omega} - \frac{-\mathcal{H}^i + \dot{\Omega}\,\mathcal{R}^i}{n_\mu}\Big(\partial_\mu r\sin(\mu) + r\cos(\mu)\Big) \\
n_\mu\dot{\xi} \;&=\; n_\mu\dot{\Omega} - r\left(-\mathcal{H}^i\mathcal{R}^i + \dot{\Omega}\;\left(\mathcal{R}^i\right)^2\right) - \left(-\mathcal{H}^i + \dot{\Omega}\,\mathcal{R}^i\right)\partial_\mu r\sin(\mu) \\
n_\mu\dot{\xi} \;&=\; n_\mu\dot{\Omega} - r\left(\dot{\Omega} - \mathcal{H}^i\mathcal{R}^i - \mathcal{H}^j\mathcal{R}^j - \mathcal{H}^k\mathcal{R}^k\right) - \left(-\mathcal{H}^i + \dot{\Omega}\,\mathcal{R}^i\right)\partial_\mu r\sin(\mu)
\end{aligned}$$

The inferred expression for the dynamic phase:

$$\dot{\xi} \;=\; \frac{r}{n_\mu}\left(\mathcal{H}^i\mathcal{R}^i + \mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k\right) + \frac{\mathcal{H}^i}{n_\mu}\partial_\mu r\sin(\mu) + \frac{\dot{\Omega}}{n_\mu}\left(n_\mu - r - \mathcal{R}^i\partial_\mu r\sin(\mu)\right) \tag{C.13}$$

In this form the dynamic phase is part of the fibre bundle in the hopf projection to a surface of revolution in $\mathbb{R}^3$, that may be expressed? something like $\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{R}^3$.
For the unit sphere the hopf projection is more broadly understood:

$$\mathbb{S}^3 \overset{\mathbb{S}^1}{\longmapsto} \mathbb{S}^2$$

The *inferred* dynamic phase of (C.13) simplifies to $\dot{\xi} = \mathcal{H}^i\mathcal{R}^i + \mathcal{H}^j\mathcal{R}^j + \mathcal{H}^k\mathcal{R}^k$ for the unit sphere since $(r = 1, n_\mu = 1, \partial_\mu r = 0)$. The global, geometric and dynamic phases of the $\mathbb{S}^1$ fibre bundle satisfy (21)

$$\Omega \;=\; \gamma + \xi \tag{C.14}$$

for all projections (C.3) as demonstrated in the python scripts accompanying this article.

Appendix D. Fictitious forces

The first derivative (C.7) is expressed in vector form as

$$\left[\dot{\vec{\mathcal{V}}}\right]_{\mathrm{cf}} \;=\; \left[\dot{\vec{\mathcal{V}}} + \vec{\mathcal{A}} \times \vec{\mathcal{V}}\right]_{\mathrm{lf}}$$

the square brackets $[\;]_{\mathrm{cf}}$ indicate that the term is expanded in the cartesian frame (pre, and post evaluation), and 'lf' is the local *moving* frame (cartesian pre, and local post evaluation). The second derivative is expanded:

$$\ddot{\vec{\mathcal{V}}} \;=\; \left(\ddot{\mathcal{V}}^c + \dot{\mathcal{A}}^c{}_a\mathcal{V}^a + \mathcal{A}^c{}_a\dot{\mathcal{V}}^a + \mathcal{A}^c{}_b\dot{\mathcal{V}}^b + \mathcal{A}^c{}_b\mathcal{A}^b{}_a\mathcal{V}^a\right)\vec{e}_c \tag{D.1}$$

reducing indices as before yields the fictitious forces of classical mechanics [24, *pg* 112],

$$\left[\ddot{\vec{\mathcal{V}}}\right]_{\mathrm{cf}} \;=\; \left[\ddot{\vec{\mathcal{V}}} + \dot{\vec{\mathcal{A}}} \times \vec{\mathcal{V}} + 2\vec{\mathcal{A}} \times \dot{\vec{\mathcal{V}}} + \vec{\mathcal{A}} \times \left(\vec{\mathcal{A}} \times \vec{\mathcal{V}}\right)\right]_{\mathrm{lf}} \tag{D.2}$$

where,

- $\dot{\vec{\mathcal{A}}} \times \vec{\mathcal{V}}$: *'The Euler Force'*
- $2\vec{\mathcal{A}} \times \dot{\vec{\mathcal{V}}}$: *'The Coriolis Force'*
- $\vec{\mathcal{A}} \times \vec{\mathcal{A}} \times \vec{\mathcal{V}}$: *'The Centrifugal Force'*

This exercise shows that the fundamental equations of classical mechanics, such as *'the fictitious forces,'* are easily derived from the unit quaternion.

Appendix E. The Riemannian curvature tensor

Continuing in the spirit of the previous section, we develop the presented mathematical formalism to show that another well known result, the *'Riemannian curvature tensor'*, is easily derived from the unit quaternion.

The parameter space of the quaternion, and subsequently the moving frames, is essentially unbounded, e.g. in SO(3) we could define the multi-parameter spinor*

$$\hat{\Psi}\left(\mu^1,\ldots,\mu^n\right) = \exp\left[\mu^1\hat{\pi}_k\right]\exp\left[\mu^2\hat{\pi}_j\right]\exp\left[\mu^3\hat{\pi}_k\right]\cdots\exp\left[\mu^n\hat{\pi}_i\right]$$

thereafter perform one of the hopf projections (C.3). From the bloch vector $\hat{\mathcal{R}}$ (and if desired in combination with a surface of revolution $r(\mu^i)$), define the moving frame $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ in $\mathbb{R}^3$. The moving frame can have multiple (t dependent) parameters $\mu^\bullet(t)$.

$$\vec{e}_a(\mu^1,\ldots,\mu^n)$$

The partial and total derivatives are defined

$$\partial_\alpha\vec{e}_a \equiv \frac{\partial\vec{e}_a}{\partial\mu^\alpha} \qquad\qquad \dot{\vec{e}}_a \equiv \frac{d\vec{e}_a}{dt}$$

and

$$\dot{\vec{e}}_a = \dot{\mu}^\alpha\partial_\alpha\vec{e}_a$$

where repeated indices are summed over. The *affine connection* is defined [23, *pg* 63],♯

$$\Gamma^a{}_{\alpha b} \equiv \vec{e}_a\cdot\partial_\alpha\vec{e}_b \tag{E.1}$$

The differential form (C.6) relates to the affine connection as

$$\mathcal{A}^a{}_b = \vec{e}_a\cdot\dot{\vec{e}}_b = \dot{\mu}^\alpha\,\vec{e}_a\cdot\partial_\alpha\vec{e}_b = \dot{\mu}^\alpha\Gamma^a{}_{\alpha b}$$

As before we expand the vector is the basis of the moving frames

$$\vec{\mathcal{V}} = \mathcal{V}^a\vec{e}_a \qquad\qquad \dot{\vec{\mathcal{V}}} = \left(\dot{\mathcal{V}}^b + \mathcal{A}^b{}_a\mathcal{V}^a\right)\vec{e}_b$$

We write the first derivative (C.7) of $\vec{\mathcal{V}}$ using the affine connection (E.1).

$$\dot{\mu}^\alpha\nabla_\alpha\vec{\mathcal{V}} = \dot{\mu}^\alpha\left(\partial_\alpha\mathcal{V}^b + \Gamma^b{}_{\alpha a}\mathcal{V}^a\right)\vec{e}_b$$

The covariant derivative of $\vec{\mathcal{V}} = \mathcal{V}^a\vec{e}_a$

$$\nabla_\alpha\mathcal{V}^b = \partial_\alpha\mathcal{V}^b + \Gamma^b{}_{\alpha a}\mathcal{V}^a$$

The second order derivative of the vector is given by,

$$\ddot{\vec{\mathcal{V}}} = \dot{\mu}^\alpha\dot{\mu}^\beta\nabla_\beta\nabla_\alpha\mathcal{V}^a \qquad\qquad \ddot{\vec{\mathcal{V}}} = \dot{\mu}^\alpha\dot{\mu}^\beta\nabla_\alpha\nabla_\beta\mathcal{V}^a$$

These are developed as

$$\nabla_\beta\nabla_\alpha\mathcal{V}^a = \nabla_\beta\left(\partial_\alpha\mathcal{V}^a + \Gamma^a{}_{\alpha b}\mathcal{V}^b\right) \qquad\qquad \nabla_\alpha\nabla_\beta\mathcal{V}^a = \nabla_\alpha\left(\partial_\beta\mathcal{V}^a + \Gamma^a{}_{\beta b}\mathcal{V}^b\right)$$

* simple illustrative example

♯ In Hobson the affine connection is defined in terms of the dual basis, $\Gamma^a{}_{\alpha b} \equiv \underset{\rightarrow}{e}^a\cdot\partial_\alpha\vec{e}_b$. For the surfaces under consideration in this study there is no obvious? need for the dual basis, therefore $\underset{\rightarrow}{e}^a = \vec{e}_a$.

and

$$\begin{aligned}
\nabla_\alpha \nabla_\beta \mathcal{V}^a &= \partial_\alpha \partial_\beta \mathcal{V}^a + \partial_\alpha \Gamma^a{}_{\beta b} \mathcal{V}^b + \Gamma^a{}_{\alpha c} \partial_\beta \mathcal{V}^c + \Gamma^a{}_{\beta b} \partial_\alpha \mathcal{V}^b + \Gamma^a{}_{\alpha c} \Gamma^c{}_{\beta b} \mathcal{V}^b \\
\nabla_\beta \nabla_\alpha \mathcal{V}^a &= \partial_\beta \partial_\alpha \mathcal{V}^a + \partial_\beta \Gamma^a{}_{\alpha b} \mathcal{V}^b + \Gamma^a{}_{\beta c} \partial_\alpha \mathcal{V}^c + \Gamma^a{}_{\alpha b} \partial_\beta \mathcal{V}^b + \Gamma^a{}_{\beta c} \Gamma^c{}_{\alpha b} \mathcal{V}^b
\end{aligned}$$

The difference is the commutator

$$[\nabla_\alpha, \nabla_\beta] \mathcal{V}^a = \mathcal{R}^a{}_{b\alpha\beta} \mathcal{V}^b$$

where the *Riemannian Curvature Tensor* is defined [23, *pg* 158]

$$\mathcal{R}^a{}_{b\alpha\beta} \equiv \partial_\alpha \Gamma^a{}_{\beta b} - \partial_\beta \Gamma^a{}_{\alpha b} + \Gamma^a{}_{\alpha c} \Gamma^c{}_{\beta b} - \Gamma^a{}_{\beta c} \Gamma^c{}_{\alpha b}$$

Appendix F. Separable and entangled states

Consider the SU(2) spinors $\hat{\Psi}_A$ and $\hat{\Psi}_B$, which evolve from their initial states via the unitaries $\hat{U}_A$ and $\hat{U}_B$ respectively.

$$\hat{\Psi}_A(t) = \hat{U}_A(t) \hat{\Psi}_A(0) \qquad\qquad \hat{\Psi}_B(t) = \hat{U}_B(t) \hat{\Psi}_B(0)$$

Equations of motion:

$$\dot{\hat{\Psi}}_A = \hat{\mathcal{H}}_A \hat{\Psi}_A \qquad\qquad \dot{\hat{\Psi}}_B = \hat{\mathcal{H}}_B \hat{\Psi}_B$$

and the hamiltonian operators are defined

$$\hat{\mathcal{H}}_A(t) \equiv \dot{\hat{U}}_A \hat{U}_A^\dagger \qquad\qquad \hat{\mathcal{H}}_B(t) \equiv \dot{\hat{U}}_B \hat{U}_B^\dagger$$

- *separable states*

The separable state is the tensor product,

$$\hat{\Psi}_S \equiv \hat{\Psi}_A \otimes \hat{\Psi}_B$$

which evolves from its initial state as,

$$\hat{\Psi}_S(t) = \hat{U}_S(t) \hat{\Psi}_S(0)$$

and the unitary of the separable state is a 4×4 complex matrix given by the tensor product,

$$\hat{U}_S \equiv \hat{U}_A \otimes \hat{U}_B$$

The equation of motion:

$$\dot{\hat{\Psi}}_S = \hat{\mathcal{H}}_S \hat{\Psi}_S$$

The hamiltonian operator is a 4×4 matrix given by the sum of each of the product state hamiltonians.

$$\hat{\mathcal{H}}_S(t) \equiv \dot{\hat{U}}_S \hat{U}_S^\dagger = \hat{\mathcal{H}}_A \otimes \hat{\sigma}_1 + \hat{\sigma}_1 \otimes \hat{\mathcal{H}}_B$$

- *entangled states*

The entangled states $\hat{\Psi}_E(t)$ differ from the separable states in that they do not allow a tensor product decomposition of the spinors A and B, i.e. $\hat{\Psi}_E(t) \neq \hat{\Psi}_A \otimes \hat{\Psi}_B$. The equation of motion for the entangled state:

$$\dot{\hat{\Psi}}_E \;=\; \hat{\mathcal{H}}_E \hat{\Psi}_E$$

where $\hat{\mathcal{H}}_E, \hat{\Psi}_E$ are 4×4 complex matrices. The hamiltonian operator of the bipartite entangled state is the tensor product,

$$\hat{\mathcal{H}}_E \;\equiv\; \hat{\mathcal{H}}_A \otimes \hat{\mathcal{H}}_B$$

The spinor $\hat{\Psi}_E$ is a (poorly defined) complex valued 4×4 matrix. In the next section we explore this thought in more detail.

Appendix G. Entangled states: what is $\mathbb{C}^4$?

This section is an attempt at gaining a greater degree of understanding of the $\mathbb{C}^4$ vector [49], i.e. the one that is typically quoted in quantum information articles as:

$$|\Psi\rangle \;=\; \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. We proceed with the vision of being able to speak about $\mathbb{C}^4$ in a manner equivalent to our understanding of $\mathbb{C}^2$, which is reasonably well defined. The columns of $\hat{\psi}$ are the vectors $|\psi^\pm\rangle \in \mathbb{C}^2$.

$$|\psi^+\rangle \;=\; \begin{pmatrix} \alpha \\ -\beta^* \end{pmatrix} \qquad |\psi^-\rangle \;=\; \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix} \qquad \hat{\psi} \;=\; \begin{pmatrix} |\psi^+\rangle & |\psi^-\rangle \end{pmatrix} \;=\; \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

The spinor is of unit length when $|\alpha|^2 + |\beta|^2 = 1$, and $\hat{\psi} \in \mathrm{SU}(2)$, and the basis is orthonormal $\hat{\psi}^\dagger \hat{\psi} = \hat{\sigma}_1$. The spinor is *unitary* as it's inverse is equal to the transpose conjugate, $\hat{\psi}^{-1} = \hat{\psi}^\dagger$ A spinor of non-unit length $|\alpha|^2 + |\beta|^2 \neq 1$, is always expressible as an SU(2) spinor multiplied by a real valued *length measure* coefficient.

It is desirable to know the same for $\mathbb{C}^4$. We would like to construct the SU(4) matrix, and to do so we assume its form is analogous to the SU(2) matrix. From the pair of spinors,

$$\hat{A} \;=\; \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \qquad \hat{B} \;=\; \begin{pmatrix} \gamma & \delta \\ -\delta^* & \gamma^* \end{pmatrix} \qquad \hat{A}\hat{A}^\dagger + \hat{B}\hat{B}^\dagger \;=\; \hat{\sigma}_1$$

we construct the 4x4 matrix

$$\hat{\Psi} \;=\; \begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B}^\dagger & \hat{A}^\dagger \end{pmatrix} \tag{G.1}$$

This matrix is non-unitary (first problem) since $\hat{\Psi}^{-1} \neq \hat{\Psi}^{\dagger}$.

$$\hat{\Psi}\hat{\Psi}^{\dagger} = \begin{pmatrix} \hat{\sigma}_1 & -[\hat{A},\hat{B}] \\ [\hat{A}^{\dagger},\hat{B}^{\dagger}] & \hat{\sigma}_1 \end{pmatrix} \qquad \hat{\Psi}^{\dagger}\hat{\Psi} = \begin{pmatrix} \hat{\sigma}_1 & [\hat{A}^{\dagger},\hat{B}] \\ -[\hat{A},\hat{B}^{\dagger}] & \hat{\sigma}_1 \end{pmatrix}$$

with

$$[\hat{A},\hat{B}] = [\hat{A}^{\dagger},\hat{B}^{\dagger}] = -[\hat{A}^{\dagger},\hat{B}] = -[\hat{A},\hat{B}^{\dagger}]$$

The commutator is

$$[\hat{A},\hat{B}] = \begin{pmatrix} \beta^*\delta - \beta\delta^* & \delta(\alpha-\alpha^*) - \beta(\gamma-\gamma^*) \\ \delta(\alpha-\alpha^*) - \beta(\gamma-\gamma^*) & -\beta^*\delta + \beta\delta^* \end{pmatrix}$$

The spinors commute when $\hat{A} = \hat{B}$ (the trivial case), so not too interesting! More interesting however, is to look at the general form of the 4x4 matrix (G.1) we have constructed.

$$\hat{\Psi} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta^* & \alpha^* & -\delta^* & \gamma^* \\ -\gamma^* & \delta & \alpha^* & -\beta \\ -\delta^* & -\gamma & \beta^* & \alpha \end{pmatrix} \tag{G.2}$$

The rows of this matrix are $\mathbb{C}^4$ vectors, as are the columns. Careful examination of the off diagonal elements shows that the rows and columns are related by the transpose conjugate and a change of sign. To better illustrate this property we define the complex quaternion, $\hat{q} = \alpha\hat{\sigma}_1 + \beta\hat{\sigma}_i + \gamma\hat{\sigma}_j + \delta\hat{\sigma}_k$, according to the first row and the quaternion conjugate, $\hat{q}^{\dagger} = \alpha^*\hat{\sigma}_1 - \beta^*\hat{\sigma}_i - \gamma^*\hat{\sigma}_j - \delta^*\hat{\sigma}_k$, corresponds to the first column. From (G.2) we extract the ket vectors*

$$|\Psi_1\rangle = \begin{pmatrix} \alpha \\ -\beta^* \\ -\gamma^* \\ -\delta^* \end{pmatrix} \qquad |\Psi_2\rangle = \begin{pmatrix} \beta \\ \alpha^* \\ \delta \\ -\gamma \end{pmatrix} \qquad |\Psi_3\rangle = \begin{pmatrix} \gamma \\ -\delta^* \\ \alpha^* \\ \beta^* \end{pmatrix} \qquad |\Psi_4\rangle = \begin{pmatrix} \delta \\ \gamma^* \\ -\beta \\ \alpha \end{pmatrix}$$

Perhaps now you can guess at what I'm trying to get at. The objective is to define an orthonormal basis in $\mathbb{C}^4$ such that the corresponding spinor of (G.2) is orthonormal $\hat{\Psi}\hat{\Psi}^{\dagger} = \hat{\sigma}_1$. By definition the basis vectors are of unit length $\langle\Psi_i|\Psi_i\rangle = 1$ for $i = 1, 2, 3, 4$. However the proposed basis is not orthonormal, since $\langle\Psi_i|\Psi_j\rangle \neq 0$ for all $i \neq j$.

Now I will try state the problem generally without restriction to $\mathbb{C}^4$.

Proposition: Every operator of the special unitary groups of $n \times n$ matrices SU(n) has its general form described by the set of n orthonormal basis vectors $\{|\Psi_i\rangle; i = 1,\ldots,n; |\Psi_i\rangle \in \mathbb{C}^n\}$. The basis is known for $n = 2$ but appears to be undefined (to my

* The bra vectors are the transpose conjugate $\langle\Psi_i| = |\Psi_i\rangle^{\dagger}$.

knowledge) for $n \geq 3$.

$$\underline{n=2}: \qquad \hat{\Psi} = \begin{pmatrix} |\Psi_1\rangle & |\Psi_2\rangle \end{pmatrix} \qquad |\Psi_1\rangle = \begin{pmatrix} \alpha \\ -\beta^* \end{pmatrix} \qquad |\Psi_2\rangle = \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix}$$

$$\hat{\Psi} \in \mathrm{SU}(2) \qquad |\Psi_1\rangle, |\Psi_2\rangle \in \mathbb{C}^2 \qquad \alpha, \beta \in \mathbb{C}$$

$$\underline{n=3}: \qquad \hat{\Psi} = \begin{pmatrix} |\Psi_1\rangle & |\Psi_2\rangle & |\Psi_3\rangle \end{pmatrix} \qquad \hat{\Psi}\hat{\Psi}^\dagger = \hat{\sigma}_1$$

$$\hat{\Psi} \in \mathrm{SU}(3) \qquad |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle \in \mathbb{C}^3$$

$$\underline{n=4}: \qquad \hat{\Psi} = \begin{pmatrix} |\Psi_1\rangle & |\Psi_2\rangle & |\Psi_3\rangle & |\Psi_4\rangle \end{pmatrix} \qquad \hat{\Psi}\hat{\Psi}^\dagger = \hat{\sigma}_1$$

$$\hat{\Psi} \in \mathrm{SU}(4) \qquad |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle \in \mathbb{C}^4$$

Greatly appreciated would be the closed form representations of $|\Psi_i\rangle$ for $n = 3$ or $n = 4$, or some explanations leading to a resolution of my confusions outlined herein.

• *further thoughts*

Consider the unit quaternion $a, b, c, d \in \mathbb{R}$ expanded in the left cayley basis (A.1).

$$\hat{q} = a\hat{\sigma}_1 + b\hat{\sigma}_i + c\hat{\sigma}_j + d\hat{\sigma}_k = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

We can generate a 'complex' quaternion by $\hat{q}' = e^{i\vartheta}\hat{q}$, and it is *unitary* since $\hat{q}'\hat{q}'^\dagger = e^{i\vartheta}e^{-i\vartheta}\hat{q}\hat{q}^\dagger = \hat{\sigma}_1$. While the complex coefficient adds an additional degree of freedom it is not a genuine 'complex quaternion' as it simply rotates the quaternion in the complex plane $\mathbb{C}$, and can be thought of as a global gauge factor. It adds an extrinsic degree of freedom. We require a complex quaternion with more intrinsic degrees of freedom, i.e. a higher dimensionality, than the real valued quaternion. Consider for example the complex quaternion

$$\begin{aligned} \hat{Q} &= a\, e^{i\lambda_a}\hat{\sigma}_1 + b\, e^{i\lambda_b}\hat{\sigma}_i + c\, e^{i\lambda_c}\hat{\sigma}_j + d\, e^{i\lambda_d}\hat{\sigma}_k \\ \hat{Q}^\dagger &= a\, e^{-i\lambda_a}\hat{\sigma}_1 - b\, e^{-i\lambda_b}\hat{\sigma}_i - c\, e^{-i\lambda_c}\hat{\sigma}_j - d\, e^{-i\lambda_d}\hat{\sigma}_k \end{aligned}$$

We write the phases as integer multiples of π

$$\lambda_a = n_1\pi \qquad \lambda_b = n_2\pi \qquad \lambda_c = n_3\pi \qquad \lambda_d = n_4\pi \tag{G.3}$$

where $n_1, n_2, n_3, n_4 \in \mathbb{Z}$, the positive and negative integers. Consequently

$$\hat{Q}\hat{Q}^\dagger = a'\hat{\sigma}_1 + b'\hat{\sigma}_i + c'\hat{\sigma}_j + d'\hat{\sigma}_k$$

with $a' = a^2 + b^2 + c^2 + d^2 = 1$ and the b', c', d' terms are zero for integers n_1, n_2, n_3, n_4

$$
\begin{aligned}
b' &= -2i\left(ab\sin\left((n_1 - n_2)\pi\right) + cd\sin\left((n_3 - n_4)\pi\right)\right) = 0\\
c' &= -2i\left(ac\sin\left((n_1 - n_3)\pi\right) - bd\sin\left((n_2 - n_4)\pi\right)\right) = 0\\
d' &= -2i\left(bc\sin\left((n_2 - n_3)\pi\right) + ad\sin\left((n_1 - n_4)\pi\right)\right) = 0
\end{aligned}
$$

$\hat{P}$ is the same complex quaternion expanded in the right cayley basis (A.4), $n_{a,b} = n_a + n_b$.

$$
\hat{P}\hat{Q} = \begin{pmatrix}
1 & 0 & 0 & 0\\
0 & a^2 + b^2 - c^2 - d^2 & 2\left(bc\, e^{n_{2,3}\pi} - ad\, e^{in_{1,4}\pi}\right) & 2\left(bd\, e^{in_{2,4}\pi} + ac\, e^{in_{1,3}\pi}\right)\\
0 & 2\left(bc\, e^{in_{2,3}\pi} + ad\, e^{in_{1,4}\pi}\right) & a^2 - b^2 + c^2 - d^2 & 2\left(cd\, e^{in_{3,4}\pi} - ab\, e^{in_{1,2}\pi}\right)\\
0 & 2\left(bd\, e^{in_{2,4}\pi} - ac\, e^{in_{1,3}\pi}\right) & 2\left(cd\, e^{in_{3,4}\pi} + ab\, e^{in_{1,2}\pi}\right) & a^2 - b^2 - c^2 + d^2
\end{pmatrix}
$$

This approach may give some hints to the form of the SU(3) operator.

Appendix H. April fools' edition

This work is presented as a compliment to the hopf fibration - as it demonstrates there is more to the quaternion and more to the hopf fibration than has been previously thought. More accurate is probably 'than has been previously acknowledged.'

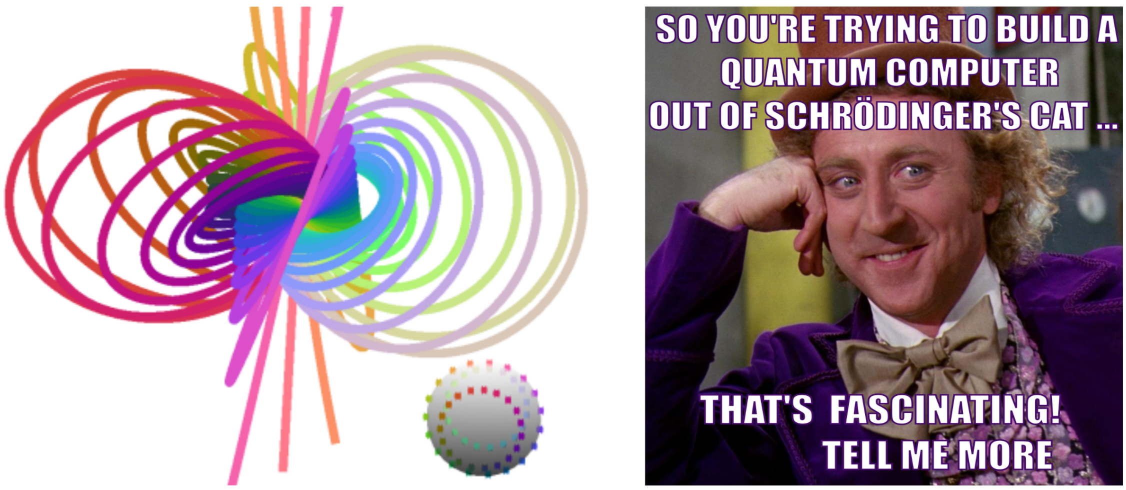
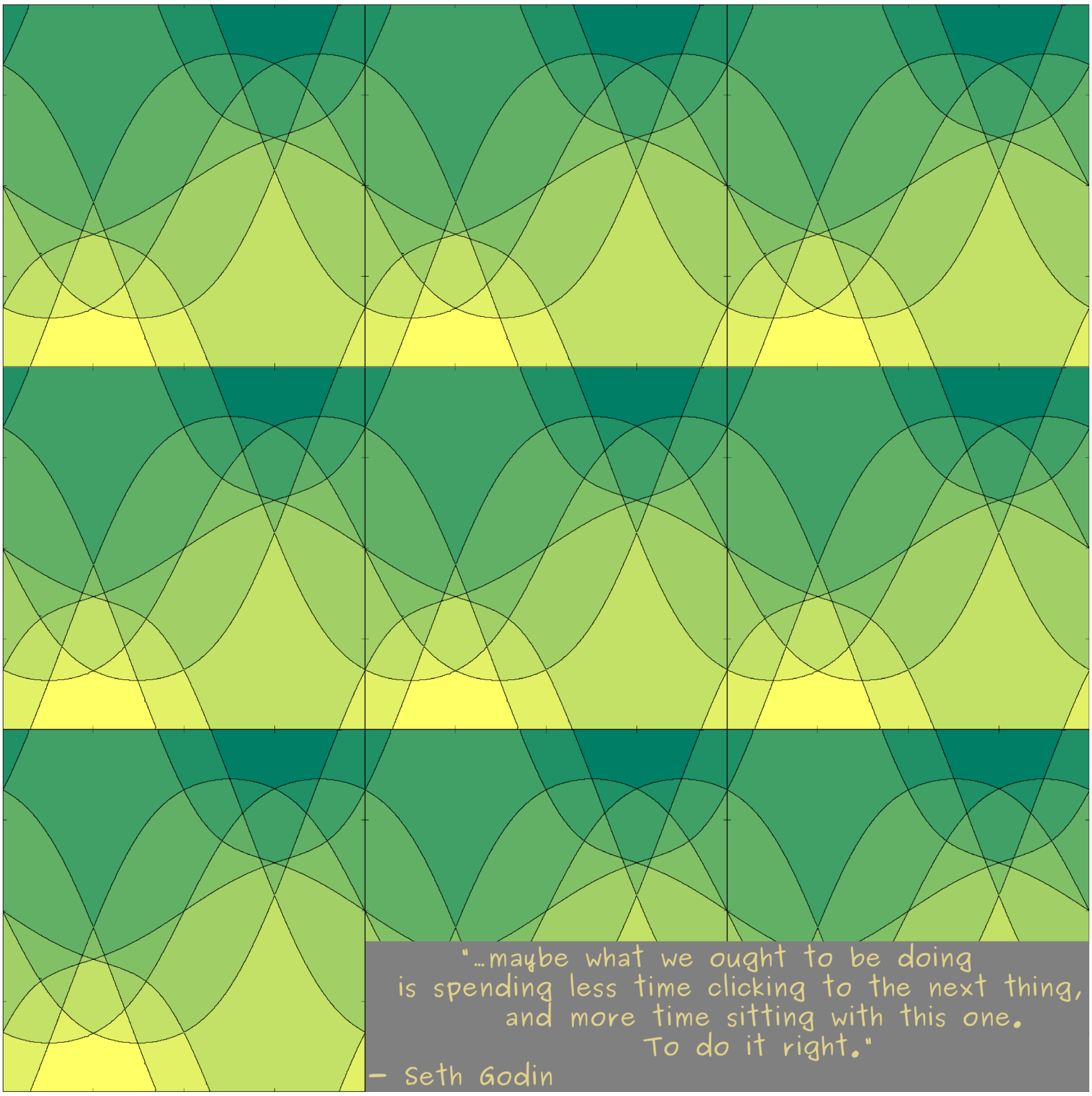


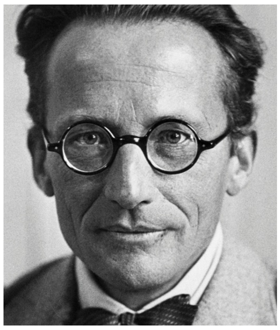
Figure H1. *Left*: Illustration of the hopf fibration. *Right*: Gene Wilder [1933 - 2016] appreciates the wonders of the human mind.

The main critiques of this article *"The hopf fibration and hidden variables in quantum and classical mechanics"* raised by the NWO edition* have been addressed and resolved (for the most part) in Appendix C. A natural next step can be to investigate the path integral formalism of the dynamic phase. As a result the NWO is no more and what remains are the very many April fools'. It wouldn't be polite to continue the joke at their expense, and so it is now most prudent to go and tell your local quantum mechanic 'the qubit is a quaternion' and bring this work to their attention!

* (i) and (ii) listed in appendix H of arXiv:1601.02569v10


"…maybe what we ought to be doing
is spending less time clicking to the next thing,
and more time sitting with this one.
To do it right."
– Seth Godin

Appendix I. The 4^{th}-dimension and the end of quantum mechanics



"Let me say at the outset, that in this discourse, I am opposing not a few special statements of quantum mechanics held today (1950s). I am opposing as it were the whole of it, I am opposing its basic views that have been shaped 25 years ago, when Max Born put forward his probability interpretation which was accepted by almost everybody * *.... I don't like it and I'm sorry I ever had anything to do with it.*† *"*
- Erwin Schrödinger

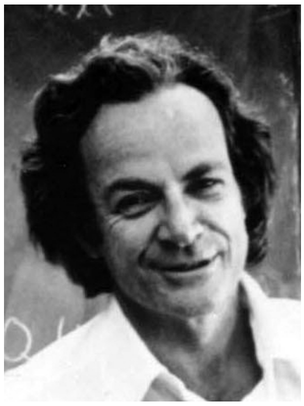
The purpose of this section is to offer a commentary on this article, quantum mechanics, and the quantum computer. The scientific portion of the text is largely complete and what follows are some personal reflections and opinions. Throughout the journey of this work I have felt like I have beared a responsibility to the scientific community, which I have thus far upheld and will now conclude.

This is the 4^{th} and final version of this article (2017 edition). I have encountered some difficulty getting it reviewed and published. The discovery of the hidden variables is credited to Wharton and Koch [10], which is an IOP article published in the Journal of Mathematical Physics A (J. Phys. A). The first version of this article (arxiv.org/abs/1601.02569v2) was submitted to J. Phys. A for review and rejected by the editors. This is not a complaint, and I'm not particularly surprised since I was the first to reject the article myself (arxiv.org/abs/1601.02569v3). A brief moment of weakness in the aftermath of finishing the first version. Its not that I didn't believe what I wrote - I believe everything I have written wholeheartedly - at the time I just did not want to. So that became a part of the story and it is what it is.

The 3^{rd} iteration of this article (arxiv.org/abs/1601.02569v7) was resubmitted to J. Phys. A. for review and again rejected by the editorial team. I inquired as to why an expert on the quaternion was not consulted. I received a reply but was given no specific reason other than some mumble about novelty. I didn't understand at the time, and still don't. Given that the discovery of the hidden variables is published in J. Phys. A, my feeling is that they bear the responsibility of having their closed form reviewed. This *"heads in the sand"*♯ attitude does not get us anywhere. A full review by experts in the field would give the scientific community an opportunity to point out omissions or shortcomings in the text, to verify the accuracy and interpretations of the results, and to potentially improve the presentation. Unfortunately they have reneged on this and I'm not particularly of the mind to go shopping around for some random journal to publish this work in. Published or unpublished its all the same to me, the important thing is that this is the final version. We carry on.

* Erwin Schrödinger, "The Interpretation of Quantum Physics" Ox Bow Press, Woodbridge CN (1995).
† Epigraph, without citation, in John Gribbin "In Search of Schrdingers Cat: Quantum Physics and Reality" Bantam Books (1984).
♯ John Stewart Bell, as quoted by Jeremy Bernstein page 84 'Quantum Profiles' Princeton University Press (1991).

My perspective of the modern academic scientific community - that of an outsider looking in at the publish or perish culture that is the modern norm - I see a neurosis.



"There's all kinds of myths and pseudoscience all over the place. Now, I might be quite wrong maybe they do know all those things, but I don't think I'm wrong, see I have the advantage of having found out how hard it is to get to really know something. How careful you have to be about checking the experiments. How easy it is to make mistakes and fool yourself. I know what it means to know something. And therefore I see how they get their information and I cannot believe that they know it! They haven't done the work necessary, they haven't done the checks necessary, they haven't done the care necessary! I have a great suspicion .... "[*]
- Richard Feynman

I know what it means to know something. I know the qubit is a quaternion and I cannot believe - for a hot minute - that anybody who has convinced themselves that the Copenhagen interpretation of the quaternion means something, is going to build a quantum computer. I don't believe in quantum computing. Now, I might be quite wrong maybe they do know all those things, but I don't think I'm wrong.

According to Wikipedia; quantum computers *"make direct use of quantum-mechanical phenomena, such as superposition and entanglement, to perform operations on data."*

I've already covered the superposition, so let's delve a little into the quantum theory of entanglement. The basis vectors of the entangled state are the tensor products;

$$|\uparrow\rangle_A \otimes |\uparrow\rangle_B = |\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |\uparrow\rangle_A \otimes |\downarrow\rangle_B = |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|\downarrow\rangle_A \otimes |\downarrow\rangle_B = |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |\downarrow\rangle_A \otimes |\downarrow\rangle_B = |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

These vectors represent the spin states of an entangled pair $|\psi\rangle_A$ and $|\psi\rangle_B$. According to quantum mechanics, the maximally entangled state is described by the bell state.

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle) \qquad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$$

The theory says that if the composite system is described by $|\Psi^{\pm}\rangle$, and particle A is measured in a spin-up state then particle B is in a spin-up state with certainty.

[*] Richard Feynman - The Pleasure of Finding Things Out - Horizon (link).

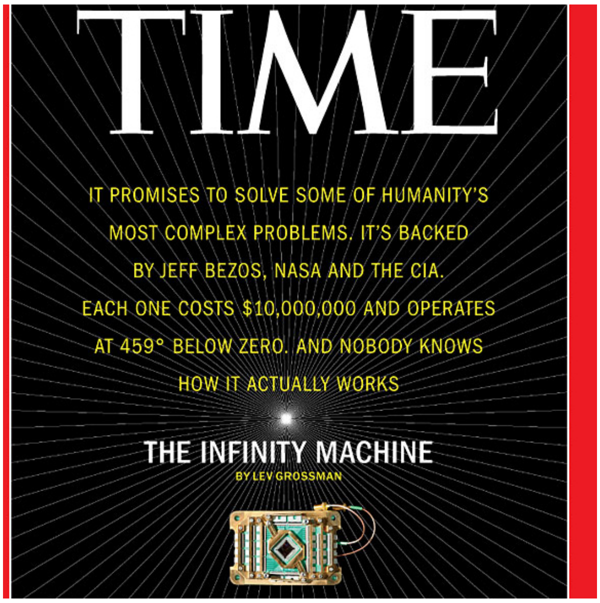
Conversely, if the composite system is $|\Phi^{\pm}\rangle$ and particle A is measured in a spin-up state, then particle B is in a spin-down state with certainty. The probable result of the first measurement being spin-up or spin-down is a 50-50 coin toss, since

$$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Once the spin state of particle A is known, then the spin of particle B is determined with certainty. In order to convert from spin-states to binary information, all that is required is to colour in the kets with a 0 and 1 in place of the arrows. And from this *"the quantum theory of entanglement"*, we are promised a quantum computer.

Claims of having built a quantum computer has emerged from different quarters. The story typically changes, we built a quantum computer - it's contested as not being genuine quantum computing - we didn't build a quantum computer, it's a feasible platform.

On May 11, 2011, D-Wave Systems announced D-Wave One, described as *"the world's first commercially available quantum computer"*. The story has changed quite a bit since then, first it's not a genuine quantum computer, then it's adiabatic quantum computing, then it's a quantum annealer. And nobody knows how it actually works.

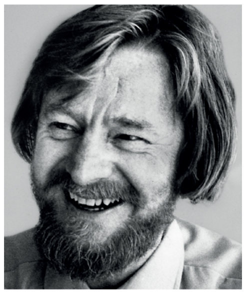


The founder of D-Wave describes the D-Wave machine as *"an alter to an alien god"** that *"exploits parallel universes"*. We are told to imagine *"that there really are parallel universes out there. Now imagine you have 2, that are exactly identical in every respect, ... with only one difference, and that's the value of a little thing called a qubit on this chip. ... In a quantum computer that device can be in a strange situation where these 2 parallel universes have a nexus - a point in space where they overlap - and when you increase the number of these devices, everytime you add one of these qubits you double the number of parallel universes that you have access to. Until such time as you get to a chip like this ...."* the D-Wave chip.

Did that make any sense to you? It certainly didn't make any sense to me. With a sales pitch like that, I am amazed they managed to accumulate a resume of customers that include Lockheed Martin, Google, NASA and the Los Alamos National Laboratory. Absolutely stunned. According to sources *"the boys stuck a compoota in a fridge running a gradient descent algorithm and sold it for $20 million! Best swindle ever."*†

* Geordie Rose - Quantum Computing: Artificial Intelligence Is Here (link) @12:37.
† Some might say.

Apparently most people on this planet were born yesterday. I however, was not born yesterday. We have Microsoft, IBM and Google and a host of other companies developing quantum programming platforms[+] in anticipation of the arrival of the heralded quantum computer. The contradiction is startling.¶ A programming platform that runs on a classical computer to simulate quantum computing. Oh and by the way, IBM have simulated a 56 qubit quantum computer♯. What happened to the quantum-classical border? Did that just recently disappear? *"Nobody knows where the boundary between the classical and quantum domain is situated. More plausible to me is that we will find there is no boundary.*[*] If I recall correctly quantum mechanics is not compatible with classical mechanics, just like quantum logic is not compatible with Boolean logic. If a quantum computer is the same as a classical computer - what are you even doing? How long do we have to listen to the quantum mechanics contradict themselves?



"This progress is made in spite of the fundamental obscurity in quantum mechanics. The progress .... is made by sleepwalkers, is it wise to shout 'wake up'? I am not sure that it is. So I speak now in a very low voice."∗[b] The dichotomy never ends! Does anyone with some level of credibility know what's going on? ....then a child in the crowd, too young to understand the desirability of keeping up the pretence blurts out that the emperor is wearing nothing at all.... and so entered the 4th-dimension and the end of quantum mechanics.††

Meanwhile in 'modern' quantum mechanics, google have set themselves the goal of demonstrating quantum supremacy by the end of 2017‡ and the clock is ticking! An ambitious one to say the least but most welcome, as it is about time a deliverable is met. The history of quantum computing is a *never ending series of hype, empty promises and excuses.* The trusty old reliable *"decoherence"* that supposedly destroys the quantum superposition, as the environment is constantly measuring the quantum state. Stick it in a super-cooled vacuum chamber. Still no quantum computer. Then they're blaming the Earth's magnetic field. Stick it in a Faraday cage. Still no quantum computer. Then they're blaming the classical electric current powering the device. Still no quantum computer. *The only thing left to do is put the quantum mechanic out of their misery and pull the plug on the whole operation.*

We are promised the world and the repeated rhetoric is that the quantum computer will solve humanity's greatest problems, even some we haven't thought of yet! The last time I checked humanity's greatest problems are simple, and can be solved if people would just pay attention. Homelessness and soaring house prices - people are dying on

[+] Microsoft Quantum development kit (link).
¶ Nature - Quantum computing hits theoretical quagmire (link).
♯ IBM Simulates a 56-Qubit Machine (link). arXiv:1203.5813 (link).
[*] John Bell [1] Ch 4, page 41. [b] John Bell [1] Ch 18, pg 170.
†† This is the story .... of a wee Irish lad, who went to the 4th-dimension to learn about physics (link).
‡ Google plans to demonstrate the supremacy of quantum computing (link). arXiv:1709.06678 (link).

the streets. Why are we paying to use money nowadays? And then there is the garbage pile we are making of the environment, desolation of the animals, land and seas, and the endless creation of non-biodegradable plastics.♯ Marketing and packaging; take, make and throw away. Toxic air, toxic water and toxic minds. *"Instead of war on poverty they got a war on drugs"*† These are real problems that can be addressed.

Still no quantum computer.

It would be fitting - come January 1st 2018 - if and when the deliverables are not met, and we are once again fed more hype, more empty promises and more excuses, that every last drop of funding for the quantum computer is pulled, and reallocated to addressing and resolving some of humanity's social problems. That resources be channeled into doing something that actually matters for once. Something that will make a real difference in people's lives.

Today the human race is breaking world records in a new field - creating and selling garbage. It is happening all across the board and quantum mechanics is no different, where scientific confetti papers are produced at a blundering pace. While it's nice to have a sense of humour about things‡, I don't find it amusing in the least bit - the situation is now critical.

"I have ran into a brick wall trying to get this article reviewed by the same journal that published the discovery of the hidden variables [10]. Now I stand here questioning the integrity of the scientific process, and the ethics of those authority figures and organisations guarding the gates of modern science. In order for the scientific process to work, and for the field of quantum mechanics to maintain some authenticity, questions and critiques must be given a voice. Otherwise it is no longer a scientific discipline, the whole thing degrades into a cult. My hand is now forced as I have no other option than to declare, on behalf of the quaternion, logic and reason that as of today quantum mechanics is finished as a scientific discipline."*

If you are a quantum mechanic; and are waking up to the reality of what I am saying, your first port of call should not be to turn to the zombie next to you and ask their opinion. That would be an amateur mistake. You will blow your cover immediately. Anyone who knows anything about the zombie apocalypse knows, that if the zombies find out you're not a zombie, they'll come get ya, and turn you back into a zombie. It is a situation easily avoided. Your best maneuver is to get out of dodge asap. Tuck this paper into your back pocket and slip out the back door immediately.

♯ *"Plastic fibers in the water and seas"* Guardian (link).
† Tupac Shakur "Changes" Greatest Hits (1998) (link).
‡ *"History of Quantum Mechanics or the Comedy of Errors"* Jean Bricmont. arXiv:1703.00294 (link).
* 14th of December 2017 - on the anniversary of the 47 Ronin.

Appendix J. Sensor fusion and the principle of superposition - an analogy

While I am entirely convinced the vast majority of the concepts promoted in quantum mechanics are inherently untrue, I am well aware there are many good works within the field. For this reason alone, it would be fitting that a commission be established to review the theory - *every paper published since 1935* - in order to separate the diamonds from the rough, so to speak. Should it take 10 years to complete this review, then let it take 10 years. Should it take 15 then let it take 15.

The following is a logical line of reasoning that can be used by this commission in order to find those diamonds. Consider the question

"is there any truth to be found in the principle of superposition?"

Putting the interpretations of quantum mechanics to one side for the moment, I draw an analogy from "Sensor Fusion" to offer a *classical* interpretation of the superposition principle.

Sensor Fusion is a technique that uses inertial measurement units to determine rotational and translational motion of a device in the global coordinate system. There are 3 inertial measurement units used in sensor fusion. The gyroscope, accelerometer and magnetometer.

- A gyroscope is used to measure the angular accelerations of the device.
- An accelerometer is used to measure the linear acceleration of the device and the acceleration due to gravity.
- A magnetometer is used to measure the device magnetic field, local magnetic fluxes and the Earth's magnetic field.

This discussion concerns only the accelerometer and magnetometer. The global coordinates of a device can be determined when both of these sensors give perfect readings.* When the device is held stationary, and no local magnetic fields are present,

- A perfectly calibrated accelerometer measures only the acceleration due to the Earth's gravitational field.
- A perfectly calibrated magnetometer measures only the Earth's magnetic field.

The accelerometer and magnetometer are expressed in the ijk device co-ordinate frame

$$\hat{a} \;=\; a_i\hat{\sigma}_i + a_j\hat{\sigma}_j + a_k\hat{\sigma}_k \qquad\qquad \hat{m} \;=\; m_i\hat{\sigma}_i + m_j\hat{\sigma}_j + m_k\hat{\sigma}_k$$

Since the sensors are perfectly calibrated and free from local accelerations and magnetic fluxes, they are of unit norm.

$$a_i^2 + a_j^2 + a_k^2 \;=\; 1 \qquad\qquad m_i^2 + m_j^2 + m_k^2 \;=\; 1$$

* In sensor fusion algorithms all 3 IMUs are used to compensate and correct for inaccuracies acquired by one or more measurement units over a given time sequence.

The gravitational field vector $\hat{g}$ and the magnetic field vector $\hat{b}$ are expressed in the ijk Earth co-ordinate frame

$$\hat{g} = \hat{\sigma}_j \qquad \hat{b} = \sin(\alpha)\,\hat{\sigma}_i + \cos(\alpha)\,\hat{\sigma}_j$$

The "up" direction is the j-axis, and α is the angle between the vectors describing the Earth's gravitational field $\hat{g}$ and the Earth's magnetic field $\hat{b}$.

- The accelerometer reading $\hat{a}$ is the earth's gravitational field expressed in the device frame.
- The magnetometer reading $\hat{m}$ is the earth's magnetic field expressed in the device frame.

The co-ordinate system of the device *the device frame*, is related to the Earth's co-ordinate system *the earth frame* through the transformation,

$$\hat{a} = \hat{Q}\hat{g}\hat{Q}^{\dagger} \qquad \hat{m} = \hat{Q}\hat{b}\hat{Q}^{\dagger} \tag{J.1}$$

$\hat{Q}$ is the unit quaternion expressed in the left-cayley basis,

$$\hat{Q} = q_1\hat{\sigma}_1 + q_i\hat{\sigma}_i + q_j\hat{\sigma}_j + q_k\hat{\sigma}_k$$

$$\hat{Q} = \begin{pmatrix} q_1 & -q_i & -q_j & -q_k \\ q_i & q_1 & -q_k & q_j \\ q_j & q_k & q_1 & -q_i \\ q_k & -q_j & q_i & q_1 \end{pmatrix}$$

There are 2 scenarios that are of interest.

- <u>2 sensors:</u> When both the accelerometer and magnetometer are used, there is a unique quaternion which defines the global coordinates of the device.
- <u>1 sensor:</u> When only one of the sensors are used, i.e. the accelerometer *or* magnetometer, then there is an infinite but bounded set of solution quaternions which can be used to describe the global coordinates of the device.

For example, if only the accelerometer is used then the *up-down* direction is known, but the cardinal directions *north-south-east-west* remain unknown. Conversely if only the magnetometer is used, then the *north-south* directions are known while the *up-down-east-west* directions remain unknown.

Using only one inertial measurement unit is similar to the situation when a principle of superposition is applied to a quantum system. In this case the *polar* angle is the known coordinate, whereas the *global and azimuthal* angles remain unknown. Specifying only the *polar* angle corresponds to an infinite but bounded set of unit quaternions, that may be used to describe the quantum particle.

Having an infinite but bounded set of quaternions to define the device's global coordinates, does not mean that the device exists in all possible orientations at the same time until the point of measurement. This is a fallacy of reasoning. The device has only one correct orientation. Similarly, the quantum state expressed as a superposition corresponds to an infinite but bounded set of quaternions. This does not mean that the particle described by the quantum state exists in all possible orientations at the same time until the point of measurement. This is a fallacy of reasoning. The particle has only one correct orientation.

Global coordinates

From (J.1) we obtain the relations

$$\begin{aligned} a_i &= 2(q_i q_j - q_1 q_k) \\ a_j &= q_1^2 - q_i^2 + q_j^2 - q_k^2 \\ a_k &= 2(q_j q_k + q_1 q_i) \end{aligned}$$

for the elements of the accelerometer, and

$$\begin{aligned} m_i &= a_i \cos(\alpha) + \left(q_1^2 + q_i^2 - q_j^2 - q_k^2\right) \sin(\alpha) \\ m_j &= a_j \cos(\alpha) + 2(q_i q_j + q_1 q_k) \sin(\alpha) \\ m_k &= a_k \cos(\alpha) + 2(q_i q_k - q_1 q_j) \sin(\alpha) \end{aligned}$$

for the elements of the magnetometer.

Expressing the quaternion $\hat{Q}$ as a rotation matrix $\hat{R}$, we immediately resolve for 6 of the 9 components.

$$\begin{aligned} \hat{R} &= \begin{pmatrix} q_1^2 + q_i^2 - q_j^2 - q_k^2 & 2(q_i q_j - q_1 q_k) & 2(q_i q_k + q_1 q_j) \\ 2(q_i q_j + q_1 q_k) & q_1^2 - q_i^2 + q_j^2 - q_k^2 & 2(q_j q_k - q_1 q_i) \\ 2(q_i q_k - q_1 q_j) & 2(q_j q_k + q_1 q_i) & q_1^2 - q_i^2 - q_j^2 + q_k^2 \end{pmatrix} \\ \hat{R} &= \begin{pmatrix} \frac{m_i - a_i \cos(\alpha)}{\sin(\alpha)} & a_i & A \\ \frac{m_j - a_j \cos(\alpha)}{\sin(\alpha)} & a_j & B \\ \frac{m_k - a_k \cos(\alpha)}{\sin(\alpha)} & a_k & C \end{pmatrix} \end{aligned}$$

The elements of the unknown vector A, B, C are given by the normalized cross product of the known 2 vectors. The complete rotation matrix is

$$\hat{R} = \begin{pmatrix} \frac{m_i - a_i \cos(\alpha)}{\sin(\alpha)} & a_i & \frac{a_k m_j - a_j m_k}{\sin(\alpha)} \\ \frac{m_j - a_j \cos(\alpha)}{\sin(\alpha)} & a_j & \frac{a_i m_k - a_k m_i}{\sin(\alpha)} \\ \frac{m_k - a_k \cos(\alpha)}{\sin(\alpha)} & a_k & \frac{a_j m_i - a_i m_j}{\sin(\alpha)} \end{pmatrix}$$

and the elements q_1, q_i, q_j, q_k of the quaternion in its standard form

$$\hat{Q} = q_1 \hat{\sigma}_1 + q_i \hat{\sigma}_i + q_j \hat{\sigma}_j + q_k \hat{\sigma}_k$$

is easily extracted from the above.

Appendix K. Fibrations

"New scientific ideas never spring from a communal body, however organized, but rather from the head of an individually inspired researcher who struggles with his problems in lonely thought and unites all his thought on one single point which is his whole world for the moment."

- Max Planck.

and I smoked so many rollies

and I drank so much coffee

and I neglected my body and tortured my soul

and I struggled as I realized

my enemy is my teacher.

And I listened. And I responded.

And I did what I had to do.

And ever so slowly it unfolded before me.

With a pace, and a quickening, and an intensity.

That had an air of inevitability about it.

As it were always going to happen.

As it were, written in the stars.

And in that moment I saw what Hamilton saw.

And I understood why he was so taken by the quaternion.

And why he dedicated his life to its study and dissemination.

For through his discovery of the unit quaternion

Hamilton glimpsed the mind of the creator.

And recognized in that instant

that all of reality is a grand symphony.

Written in an as of yet unknown key.

And that key is, in the most profound way

a play

on the number 0.

- Brian O'Sullivan.