# Chapter 7: Sensitivity Analysis

# Introduction



- When solving nested structural optimization problems by generating a sequence of explicit first order approximations, such as MMA, one needs to differentiate the objective function and all constraint functions with respect to the design variables.
- The procedure to obtain these derivatives, or sensitivities, is called sensitivity analysis.
- In this chapter we will describe how to perform a sensitivity analysis for arbitrary objective and constraint functions and design variables.
- There are two main groups of methods: numerical methods, which are approximate, and analytical methods, which are exact.
- For examples on how to obtain the sensitivity of the compliance of a truss structures, discussed in lecture 2, with respect to the cross-sectional area of the bars, see P. W. Christensen, A. Klabring, „An Introduction to Structural Optimization (chapter 6)

# Numerical Methods



- We recall that the nested structural optimization problem may be written as

$$(\mathbb{SO})_{\text{nf}} \quad \begin{cases} \min_{\boldsymbol{x}} \hat{g}_0(\boldsymbol{x}) = g_0(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \\ \text{s.t.} \quad \hat{g}_i(\boldsymbol{x}) = g_i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \leq 0, \quad i = 1, \dots, l \\ \qquad \boldsymbol{x} \in \mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n : x_j^{\min} \leq x_j \leq x_j^{\max}, \ j = 1, \dots, n\} \end{cases}$$

where $\boldsymbol{x} \rightarrow \boldsymbol{u}(\boldsymbol{x})$ is an implicit function defined through the equilibrium equations

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}).$$

- In numerical sensitivity analysis methods, $\partial \widehat{g}_i / \partial xj, i = 0, \dots, l$, are approximated by finite differences, e.g. forward or central differences.

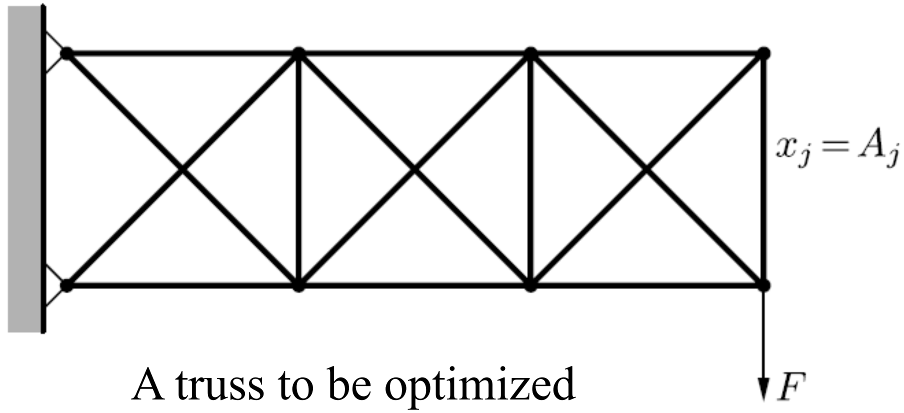
- The forward difference approximation of $\partial \widehat{g}_i / \partial x_j$ at a design $\boldsymbol{x}_k$ is

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} \approx D_f = \frac{\hat{g}_i(\boldsymbol{x}^k + h\boldsymbol{e}_j) - \hat{g}_i(\boldsymbol{x}^k)}{h}$$

where $\boldsymbol{e}_j = [0, \dots, 0, 1, 0, \dots, 0]^T$ and the 1 is in row $j$ .

# Illustration Example

- We calculate, at a design $\boldsymbol{x}^k$, the sensitivity of the compliance
$$\widehat{g_0}\,(\boldsymbol{x}) = g_0(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{F}(\boldsymbol{x})^T\,\boldsymbol{u}(\boldsymbol{x})$$
with respect to changes in the cross-sectional area $x_j = A_j$ of bar $j$ in a truss.



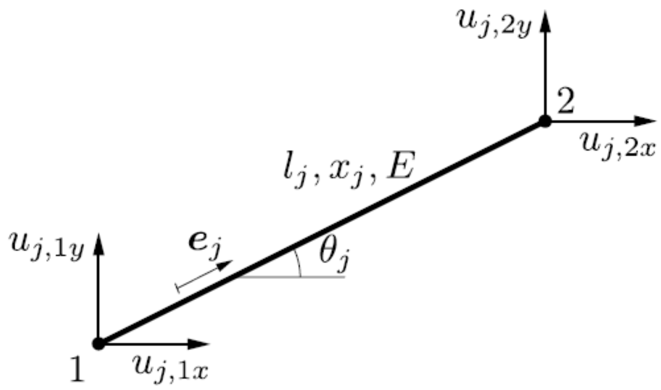
A truss to be optimized

**1.** the compliance $\widehat{g_0}\left(\boldsymbol{x}^k\right)$ is calculated.

**2.** a small number $h > 0$ is added to the cross-sectional area of bar $j$ .

**3.** find the compliance for this new design $\boldsymbol{x}^k + h\boldsymbol{e}_j$ by performing an FE analysis to solve the equilibrium equations and find $\boldsymbol{u}(\boldsymbol{x}^k + h\boldsymbol{e}_j)$ .

**4.** Insertion into The forward difference approximation equation



A general bar $j$ in the truss

- How to find the suitable h?
- It may be shown that the truncation error

$$\partial \hat{g}_i(\boldsymbol{x}^\kappa)/\partial x_j - D_f = O(h)$$

Source: Fig. 5.1 P. W. Christensen, A. Klabring, An Introduction to Structural Optimization

# Analytical Methods



- In order to obtain analytical expressions for $\partial \hat{g}_i(x^k)/\partial xj$, the chain rule is first applied:

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial g_i(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial x_j} + \frac{\partial g_i(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}(\boldsymbol{x}^k))}{\partial x_j}$$

where $\partial gi/\partial \boldsymbol{u}$ is a row matrix, and $\partial \boldsymbol{u}/\partial xj$ is a column matrix.

- We introduce two different analytical methods: the so-called direct and adjoint methods

# Direct Analytical Method



- In the direct analytical method, $\partial \boldsymbol{u}(\boldsymbol{x}^k)/\partial x_j$ is obtained by differentiation of the equilibrium equations $\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})$. The result is then inserted into analytical expression to get:

$$\frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k) + \boldsymbol{K}(\boldsymbol{x}^k)\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j}$$

which is rewritten as

***pseudo-load***

$$\boldsymbol{K}(\boldsymbol{x}^k)\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)$$

- In order to find $\partial \boldsymbol{u}(\boldsymbol{x}^k)/\partial x_j$ , we first need expressions for $\partial \boldsymbol{F}(\boldsymbol{x}^k)/\partial x_j$ and $(\partial \boldsymbol{K}(\boldsymbol{x}^k)/\partial x_j)\boldsymbol{u}(\boldsymbol{x}^k) \rightarrow$ see P. W. Christensen, A. Klabring, „An Introduction to Structural Optimization chapter 6.

# Direct Analytical Method (Computation Time)



$$\boldsymbol{K}(\boldsymbol{x}^k)\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)$$

- If the equilibrium equations have been solved by a direct solver rather than with an iterative one, so that $\boldsymbol{K}(\boldsymbol{x}^k)$ has been factorized (for instance by performing a Cholesky decomposition $\boldsymbol{K}(\boldsymbol{x}^k) = \boldsymbol{L}\boldsymbol{L}^T$ ), then only forward and backward substitutions are needed to solve the above eq. for $\partial \boldsymbol{u}(\boldsymbol{x}^k)/\partial x_j$ .
- Thus, the computation for a certain design variable *xj* is much cheaper than the solution to equilibrium equations.
- However, the above needs to be solved *n* times, so the amount of time needed to calculate the sensitivity for all design variables may still be considerable if there is a large number of design variables!

# Illustration Example



- The sensitivity of the compliance, $\widehat{g_0}(\boldsymbol{x}) = g_0(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{F}(\boldsymbol{x})^T \boldsymbol{u}(\boldsymbol{x})$, should be calculated using the direct analytical method. Differentiation of $g0$ gives:

$$\frac{\partial g_0(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial x_j} = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)^T}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k), \qquad \frac{\partial g_0(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial \boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{x}^k)^T.$$

from the pseudo-load eq. we have

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k)}{\partial x_j} = \boldsymbol{K}(\boldsymbol{x}^k)^{-1}\left(\frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)\right)$$

and by inserting the above into the analytical expression for $\partial \widehat{g}_i(x^k)/\partial xj$

$$\begin{aligned}\frac{\partial \hat{g}_0(\boldsymbol{x}^k)}{\partial x_j} &= \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)^T}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k) + \boldsymbol{F}(\boldsymbol{x}^k)^T \boldsymbol{K}(\boldsymbol{x}^k)^{-1}\left(\frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)\right) \\ &= 2\boldsymbol{u}(\boldsymbol{x}^k)^T \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \boldsymbol{u}(\boldsymbol{x}^k)^T \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k).\end{aligned}$$

# Adjoint Analytical Method



- By replacing the pseudo-load equation into analytical expression for $\widehat{g}_i(x^k)/\partial xj$, one obtains:

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial g_i}{\partial x_j} + \frac{\partial g_i}{\partial \boldsymbol{u}} \boldsymbol{K}(\boldsymbol{x}^k)^{-1} \left( \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j} \boldsymbol{u}(\boldsymbol{x}^k) \right),$$

where $g_i = g_i(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))$. In this expression, we define

$$\boldsymbol{\lambda}_i = \left( \frac{\partial g_i}{\partial \boldsymbol{u}} \boldsymbol{K}(\boldsymbol{x}^k)^{-1} \right)^T = \boldsymbol{K}(\boldsymbol{x}^k)^{-1} \left( \frac{\partial g_i}{\partial \boldsymbol{u}} \right)^T .$$

- In the adjoint method, one starts by solving

$$\boldsymbol{K}(\boldsymbol{x}^k)\boldsymbol{\lambda}_i = \left( \frac{\partial g_i}{\partial \boldsymbol{u}} \right)^T$$

for $\boldsymbol{\lambda}_i$ . This is then inserted into the above eq. to give the desired sensitivity as:

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial g_i}{\partial x_j} + \boldsymbol{\lambda}_i^T \left( \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j} \boldsymbol{u}(\boldsymbol{x}^k) \right)$$

# Comparision



- We compare the direct and the adjoint method to calculate $\hat{g}_i(x^k)/\partial xj$, for $j = 1,\ldots,n$ and $i = 0,\ldots,l$.
- In the direct method, one needs to solve the pseudo-lead eq.

$$\boldsymbol{K}(\boldsymbol{x}^k)\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)$$

once for each design variable $xj$ , i.e. $n$ times. The result is then inserted into

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial g_i(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial x_j} + \frac{\partial g_i(\boldsymbol{x}^k, \boldsymbol{u}(\boldsymbol{x}^k))}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}(\boldsymbol{x}^k))}{\partial x_j}$$

$l+1$ times for each $j$ .

- In the adjoint method $\boldsymbol{K}(\boldsymbol{x}^k)\boldsymbol{\lambda}_i = \left(\frac{\partial g_i}{\partial \boldsymbol{u}}\right)^T$

is solved for the objective function and each constraint function, i.e. $l+1$ times. The result is then inserted into

$$\frac{\partial \hat{g}_i(\boldsymbol{x}^k)}{\partial x_j} = \frac{\partial g_i}{\partial x_j} + \boldsymbol{\lambda}_i^T\left(\frac{\partial \boldsymbol{F}(\boldsymbol{x}^k)}{\partial x_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{x}^k)}{\partial x_j}\boldsymbol{u}(\boldsymbol{x}^k)\right)$$

$n$ times for each $i$ (= $0, \ldots, l$).

- Thus, we conclude that the adjoint method is to be preferred when there are fewer constraints than design variables, otherwise the direct method will be more efficient.

# Illustration Example



- Let us continue the last Example using the adjoint method to calculate the sensitivity of the compliance. Adjoint equation reads

$$K(x^k)\lambda = \left(\frac{g_0(x^k, u(x^k))}{\partial u}\right)^T = F(x^k)$$

This system of equations is identical to the equilibrium equations. Thus, since $\boldsymbol{K}(\boldsymbol{x}^k)$ is nonsingular, we conclude that $\boldsymbol{\lambda} = \boldsymbol{u}(\boldsymbol{x}^k)$! Obviously, there is no need to solve the above eq., as in this case we already have $\boldsymbol{u}(\boldsymbol{x}^k)$ available. Insertion into the eq. for the sensitivity gives

$$\begin{aligned}
\frac{\partial \hat{g}_0(x^k)}{\partial x_j} &= \frac{\partial F(x^k)^T}{\partial x_j} u(x^k) + u(x^k)^T \left(\frac{\partial F(x^k)}{\partial x_j} - \frac{\partial K(x^k)}{\partial x_j} u(x^k)\right) \\
&= 2u(x^k)^T \frac{\partial F(x^k)}{\partial x_j} - u(x^k)^T \frac{\partial K(x^k)}{\partial x_j} u(x^k).
\end{aligned}$$

As expected, the results of the direct and adjoint methods coincide.