COMPUTATIONAL ASPECTS OF SUB-INDEX AND SUB-FACTORS OF GROUPS

In order to study the upper periodic subsets and some functional equations on algebraic structures, we studied factor subsets of magmas, semigroups and groups, and direct product of two subsets. Then, we arrived at a challenging problem about factorization of (arbitrary) groups by subsets (Kourovka Notebook). Regarding right and left factor subsets of groups, we found that the index concept of subgroups can be extended for factors and then arbitrary subsets. In line with that idea, we correspond to every subset A of a group G the left and right sub-factors and also four sub-indices $|G:A|^{\pm}$, $|G:A|_{\pm}$. We call A index stable if all of these sub-indices are equal (e.g., all subgroups and normal sub-semigroups). The topic has interesting applications for studying subsets of groups and it has some interactions to additive combinatorics and number theory. For defining them, we need to recall some concepts and notations from our paper.

We use G or $(G,\cdot)$ for a group, A and B for subsets, and A^{-1} for the set $\{a^{-1}: a \in A\}$. Also, A is called symmetric if $A = A^{-1}$, and it is obvious that a non-empty subset is a subgroup if and only if it is a symmetric sub-semigroup.

The product AB is called direct and it is denoted by $A \cdot B$ if the representation of each element x of AB as $x = ab$, $a \in A$, $b \in B$ is unique (we use $A \dot{+} B$ for additive notation that is *direct sum* of A and B). By the notation $G = A \cdot B$ we mean $G = AB$ so that the product AB is direct (a factorization of G by two subsets), and call A (resp. B) a left (resp. right) factor of G related to B (resp. A). We call A a factor (resp. two-sided factor) if it is a left or (resp. and) right factor. For example, every subgroup is a two-sided factor. A sub-semigroup of G is a factor if and only if it is a subgroup. Hence $\mathbb{Z}$, $\mathbb{Q}$ are factors of the additive group $(\mathbb{R},+)$ (as the sense of factorization by subsets), and $\mathbb{N}$, $\mathbb{Q}^+$ are not its factors. Now, we define the left and right differences of A as follows: $Dif_\ell(A) := A^{-1}A$, $Dif_r(A) := AA^{-1}$. Both $Dif_\ell(A)$ and $Dif_r(A)$ are symmetric and it is obvious that $A \neq \emptyset$ is a subgroup if and only if $Dif_\ell(A) = A$ (equivalently $Dif_r(A) = A$). We call a subset A a left (resp. right) difference-generating set of G if $Dif_\ell(A) = G$ (resp. $Dif_r(A) = G$). It is clear that every left (and right) difference-generating set is a generating set (but the converse is not true). In general, if A is a left or right difference-generating subset, then both $Dif_\ell(A)$ and $Dif_r(A)$ are left and right difference-generating subsets. If $Dif_\ell(A) = Dif_r(A)$, then we denote it by $Dif(A)$. For example, if A is symmetric, normal (i.e., $gA = Ag$ for all $g \in G$) or $A^{-1} \subseteq N_G(A)$, then $Dif(A) = A^{-1}A = AA^{-1}$ (hence $Dif(A) = A - A$, if G is abelian). In particular, if A is a normal sub-semigroup of G, then $Dif(A)$ is a subgroup. There are close relations between direct product of subsets and their differences. It is easy to see that the following statements are equivalent:

(i) $AB = A \cdot B$, (ii) $aB \cap a'B = \emptyset$; for all distinct elements $a, a' \in A$,

(iii) $Dif_\ell(A) \cap Dif_r(B) = \{1\}$, (iv) $Ab \cap Ab' = \emptyset$; for all distinct elements $b, b' \in B$.

If G is finite, then the property $|AB| = |A||B|$ is equivalent to the each of the above conditions.

Since $Dif_\ell : 2^G \to 2^G$ is a function which maps each $A \subseteq G$ to $A^{-1}A \subseteq G$, then we have the n-iteration of Dif_ℓ denoted by Dif_ℓ^n. Since $Dif_\ell(A)$ is symmetric and $(2^G,\cdot)$ is a semigroup, then $Dif_\ell^n(A) = Dif_\ell(A)^{2^{n-1}}$, for all $n \geq 1$.

For more detailed information, please refer to the papers [1] and [2]. Below, you'll find tables outlining concepts and their corresponding functions in GAP-codes, which can be used as usage instructions. The source code for these functions is stored in the "./lib" directory of the package.

Table 1: **Factor, sub-factors, sub-indices and index stability of group subsets**

Title	Notation in papers	Notation in GAP-codes	Definition or condition	Remarks
Multiplication of subsets	AB	$\mathrm{Mul}(G, A, B)$	$\{ab : a \in A, b \in B\}$	-
Inverse of subset	A^{-1}	$\mathrm{Inv}(G, A)$	$\{a^{-1} : a \in A\}$	-
Right, left differences of subsets	$Dif_r(A)$, $Dif_\ell(A)$	$\mathrm{RD}(G, A)$, $\mathrm{LD}(A)$	$A\,\mathrm{Inv}(G, A)$, $\mathrm{Inv}(G, A)A$	-
Right, left compliments of differences of subsets	$\mathcal{C}_r(A)$, $\mathcal{C}_\ell(A)$	$\mathrm{RC}(G, A)$, $\mathrm{LC}(G, A)$	$G \setminus \mathrm{RD}(G, A)$, $G \setminus \mathrm{LD}(G, A)$	-
Direct product of subsets	$A \cdot B$	$\mathrm{IsDirect}(G, A, B)$	$= AB$ with $\mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}$	equivalently $\|AB\| = \|A\|\|B\|$
Factorization by two subsets	$G = A \cdot B$	$\mathrm{IsFactorization}(G, A, B)$	$G = AB$ with $\mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}$	A (B) is a left (right) factor of G related to B (A)
The set of right, left factors of G related to A	$\mathrm{Fac}_r(G : A)$, $\mathrm{Fac}_\ell(G : A)$	$\mathrm{RF}(G, A)$, $\mathrm{LF}(G, A)$	$\{B \subseteq G \ : \ G = A \cdot B\}$, $\{B \subseteq G \ : \ G = B \cdot A\}$	The set of all $B \subseteq G$ s.t. $G = AB$ and $\mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}$
The set of right, left sub-factors of G related to A	$\mathrm{SubF}_r(G : A)$, $\mathrm{SubF}_\ell(G : A)$, $\mathrm{SubF}^1_r(G : A)$	$\mathrm{RSF}(G, A)$ or* $\mathrm{RBSF}(G, A)$, $\mathrm{LSF}(G, A)$, $\mathrm{RSF1}(G : A)$	$\Big\{B \subseteq G : \mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}, G = \mathrm{LD}(G, A)B\Big\}$, $\Big\{B \subseteq G : \mathrm{LD}(G, B) \cap \mathrm{RD}(G, A) = \{1\}, G = B\,\mathrm{RD}(G, A)\Big\}$, $\{B_1 \in \mathrm{SubF}_r(G, A) : 1 \in B_1\}$	The set of all $B \subseteq G$ s.t. G=LD(G,A) B and $\mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}$ * The functions $\mathrm{RSF}(A)$ and $\mathrm{RBSF}(A)$ utilize different algorithms
Right upper and lower index of A in G	$\|G : A\|^+$, $\|G : A\|^-$	$\mathrm{RidPlus}(G, A)$, $\mathrm{RidMinus}(G, A)$, $\mathrm{RidPM}(G : A)$	$\max\{\|B\| : B \in \mathrm{RSF}(G, A)\}$, $\min\{\|B\| : B \in \mathrm{RSF}(G, A)\}$, $(\|G : A\|^- , \|G : A\|^+)$	equivalently $\mathrm{RidPlus}(A) = \max\{\|B\| : \mathrm{LD}(G, A) \cap \mathrm{RD}(G, B) = \{1\}\}$ (only for right upper index)

Left upper and lower index of A in G	$\|G:A\|_+$, $\|G:A\|_-$	LidPlus(G,A) , LidMinus(G,A) , LidPM(G,A)	$\max\{\|B\| : B \in \mathrm{LSF}(G,A)\}$, $\min\{\|B\| : B \in \mathrm{LSF}(G,A)\}$, ($\|G:A\|_-$, $\|G:A\|_+$)	equivalently RidPlus$(A) = \max\{\|B\| : \mathrm{LD}(G,B) \cap \mathrm{RD}(G,A) = \{1\}\}$ (only for left upper index)
Right (resp. left) index stable subsets	$\|G:A\|_r$ (resp. $\|G:A\|_\ell$)	Rid(G,A) (resp. Lid(G,A))	if RidPlus(G,A) = RidMinus(G,A) (resp. LidPlus(G,A) = LidMinus(G,A)) and denote it by Rid(G,A) (resp. Lid(G,A))	we say "A is right (resp. left) index stable" in G
Index stable subsets and their index	$\|G:A\|$	Id(A)	if Rid(A) = Lid(G,A) and denote it by Id(G,A) i.e., all sub-indices are equal	we say "A is index stable" in G
Upper (resp. lower) index stable subsets	-	IdPlus(A) (resp. IdMinus(A))	if RidPlus(G,A) = LidPlus(G,A) (resp. RidMinus(G,A) = LidMinus(G,A)) and denote it by IdPlus(G,A) (resp. IdMinus(G,A))	we say "A is upper (resp. lower) index stable" in G for example all subsets of abelian groups
Right (resp. left) k-index stable groups	G is right (resp. left) k-index stable	IsRightkIndexStable(G,k)	if all subsets $A \subseteq G$ with $\|A\| = k$ are right (resp. left) index stable	Note that a group is right k-index (resp. index) stable if and only if it is left k-index (resp. index) stable
k-index stable groups	G is k-index stable	IskIndexStable(G,k)	if all subsets $A \subseteq G$ with $\|A\| = k$ are index stable	-
Right (resp. left) index stable groups	G is right (resp. left) index stable	IsRightIndexStable(G)	A group G is called right (resp. left) index stable if all its subsets are right (resp. left) index stable.	It has been proved that right and left index stabilities are equivalent
Index stable groups	G is index stable	IsIndexStable(G)	A group G is called index stable if all its subsets are index stable.	All (right) index stable finite groups have been characterized

Table 2. **Infinite differences and difference length of subsets**

Title	Notation in papers	Notation in GAP-codes	Definition or condition	Remarks
Iterated right, left differences of subsets	$Dif_r^n(A)$, $Dif_\ell^n(A)$	$\mathrm{RD}(G,n,A)$, $\mathrm{LD}(G,n,A)$	$\mathrm{RD}(G,A)\mathrm{RD}(G,A)\cdots\mathrm{RD}(G,A)$; 2^{n-1}-times , $\mathrm{LD}(G,A)\mathrm{LD}(G,A)\cdots\mathrm{LD}(G,A)$; 2^{n-1}-times	for $n=0$ define $\mathrm{RD}(0,G,A)=\mathrm{LD}(0,G,A)=A$
Right , left infinite difference length of subsets	$dl_r^\infty(A)$, $dl_\ell^\infty(A)$	$\mathrm{Rdl}(G,A)$, $\mathrm{Ldl}(G,A)$	$\min\{n\geq 0:\mathrm{RD}(G,n,A)=\mathrm{RD}(G,n+1,A)\}$, $\min\{n\geq 0:\mathrm{LD}(G,n,A)=\mathrm{LD}(G,n+1,A)\}$	equivalently $\min\{n\geq 0:\mathrm{RD}(G,n,A)\leqslant G\}$, $\min\{n\geq 0:\mathrm{LD}(G,n,A)\leqslant G\}$
Limit of iterated differences of subsets	$Dif_r^\infty(A)$, $Dif_\ell^\infty(A)$	$\mathrm{RDD}(G,A)$, $\mathrm{LDD}(G,A)$	$\mathrm{RD}(G,\mathrm{Rdl}(G,A),A)$, $\mathrm{LD}(G,\mathrm{Ldl}(G,A),A)$	they are subgroups of G if $A\neq\emptyset$
Right , left Lagrange difference length of subsets	-	$\mathrm{Rlag}(G,A)$, $\mathrm{Llag}(G,A)$	Size of $\{1\leq n\leq\mathrm{Rdl}(G,A):$ $\lvert\mathrm{RD}(G,n,A)\rvert$ divides $\lvert G\rvert\}$, Size of $\{1\leq n\leq\mathrm{Rdl}(G,A):$ $\lvert\mathrm{RD}(G,n,A)\rvert$ divides $\lvert G\rvert\}$	-

Table 3: **Index and sub-index pervasive properties for groups**

Title	Notation in papers	Notation in GAP-codes	Definition or condition	Remarks
The set of all right (resp. left) index stable subsets of G	$\iota\sigma_r(G)$ (resp. $\iota\sigma_l(G)$)	$\mathrm{RISS}(G)$ (resp. $\mathrm{LISS}(G)$)	$\{A\subseteq G$: $\mathrm{RidPlus}(G,A){=}\mathrm{RidMinus}(G,A)\}$ (resp. $\{A\subseteq G$: $\mathrm{LidPlus}(G,A){=}\mathrm{LidMinus}(G,A)\}$)	i.e. $\{A\subseteq G$: A is right index stable$\}$ (resp. $\{A\subseteq G$: A is left index stable$\}$)
All index stable subsets of G	$\iota\sigma(G)$	$\mathrm{ISS}(G)$	$\{A\subseteq G$: $\mathrm{RidPlus}(G,A)=\mathrm{RidMinus}(G,A)=\mathrm{LidPlus}(G,A)=\mathrm{LidMinus}(A)$ $\}$	i.e. $\{A\subseteq G$: A is index stable$\}$
The set of all right (resp. left) indices of G	$\mathrm{Num}_{\iota_r}(G)$ (resp. $\mathrm{Num}_{\iota_l}(G)$)	$\mathrm{IndR}(G)$ (resp. $\mathrm{IndL}(G)$)	$\{\mathrm{Rid}(G,A)$: A is right index stable$\}$ (resp. $\{\mathrm{Lid}(G,A)$: A is left index stable$\}$)	i.e. $\{\lvert G:A\rvert_r : A\in\iota\sigma_r(G)\}$ (resp. $\{\lvert G:A\rvert_l : A\in\iota\sigma_l(G)\}$
The set of all right (resp. left) sub-indices of G	$\mathrm{Num}_{s\iota_r}(G)$ (resp. $\mathrm{Num}_{s\iota_l}(G)$)	$\mathrm{IndslR}(G)$ (resp. $\mathrm{IndslL}(G)$)	$\{\mathrm{RidMinus}(G,A):A\subseteq G\}\cup\{\mathrm{RidPlus}(G,A):A\subseteq G\}$ (resp. $\{\mathrm{LidMinus}(G,A):A\subseteq G\}\cup\{\mathrm{LidPlus}(G,A):A\subseteq G\}$)	i.e $\{\lvert G:A\rvert^\pm : A\in P(G)\}$ (resp. $\{\lvert G:A\rvert_\pm : A\in P(G)\}$)
The set of all indices of subsets of G	$\mathrm{Ind}(G)$	$\mathrm{Ind}(G)$	$\{\mathrm{Id}(A)$: A is index stable$\}$	i.e $\{\lvert G:A\rvert : A\in\iota\sigma(G)\}$

The set of all sub-indices of subsets of G	$\mathrm{Num}_{s\iota}(G)$	$\mathrm{Indsl}(G)$	$\mathrm{IndslR}(G) \cup$ $\mathrm{IndslL}(G)$	i.e. $\{\lvert G : A\rvert_{\pm}^{\pm} : A \in P(G)\}$
Possible numbers for sub-indices	$\mathrm{Num}_{\frac{1}{2}}(G)$	$\mathrm{Num}(G)$	$\{1, \cdots, \lfloor \frac{\lvert G\rvert}{2} \rfloor\} \cup \{\lvert G\rvert\}$	-
Right (resp. left) index pervasive groups	G is right (resp. left) index pervasive	IsRightIndex Pervasive(G) (resp. IsLeftIndex Pervasive(G))	if $\mathrm{RInd}(G) = \mathrm{Num}(G)$ (resp. $\mathrm{RInd}(G) = \mathrm{Num}(G)$)	-
Index pervasive groups	G is index pervasive	IsIndex Pervasive(G)	if $\mathrm{Ind}(G) = \mathrm{Num}(G)$	-
Sub-index pervasive groups	G is sub-index pervasive	IsSubIndex Pervasive(G)	if $\mathrm{SInd}(G) = \mathrm{Num}(G)$	-

References

[1] M.H. Hooshmand, *Index, sub-index and sub-factor of groups with interactions to number theory*, J. Alg. Appl., 19:6(2020), 1-23.

[2] M.H. Hooshmand, *Subindices and subfactors of finite groups*, Commun. Algebra, 51:6(2023), 2644-2657.