ECE 9045/9405: Computational Methods for the Power Grid

Fundamental Power System Concepts

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Readings/References

Boyd and Vandenberge

Topics Covered in Course

- Optimization Constructs
- **Fundamental Power System Constructs (Steady-State)**
- Power Grid Economics (Duality and Optimality Conditions)
- Multi-period Optimization in the Grid
- Game Theory for the Power Grid

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Useful Relations

- Representation of complex number $C\mathcal{C}$
  - Rectangular form: $C = a + \mathbf{j}b$
  - Polar form: $C = re^{\mathbf{j}\phi} = r\cos(\phi) + r\mathbf{j}\sin(\phi)$ where $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$
- Hermitian matrices are the extended form of symmetric matrices in complex domain $A = A^*$
- Transpose A^* of matrixes in complex domain ($A \in \mathcal{C}^{n\times n}$):
  - Apply a regular transpose to the matrix and then obtain the complex conjugate of each element
  - E.g. $A = \begin{bmatrix} 1 & 1+\mathbf{j} \\ 3-\mathbf{j} & 2 \end{bmatrix}$, $A^* = \begin{bmatrix} 1 & 3+\mathbf{j} \\ 1-\mathbf{j} & 2 \end{bmatrix}$
- Graph notations:
  - Set of n buses/nodes: $\mathcal{N} = \{b_1, \ldots, b_n\}$
  - Edges/links between nodes: $e_{ij} \in \mathcal{E}$
  - Weights of edges: $w_{ij} \in \mathcal{W}$ if $e_{ij} \in \mathcal{E}$

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Voltage, Current, Power

- Current $I(t)$ and voltage $V(t)$ vary with time
- Sinusoidal in **steady-state** with an angular frequency of ω radians, magnitudes $\bar{V}, \bar{I}$: $V(t) = \bar{V}cos(\omega t + \theta)$, $I(t) = \bar{I}cos(\omega t + \theta - \phi)$
- Relationship between period T, frequency f, and angular frequency ω: $\omega = \frac{2\pi}{T} = 2\pi f$
- **Instantaneous Power**:
  $P_{ins} = V(t)I(t) = \frac{\bar{V}\bar{I}}{2} cos(\phi)(1 + 2cos(2(\omega t + \theta))) + \frac{\bar{V}\bar{I}}{2} sin(\phi) sin(2(\omega t + \theta))$
- **Phasors**: $V(t) = \mathrm{Re}(\bar{V}e^{\mathbf{j}(\theta+\omega t)})$, $V = \frac{\bar{V}}{\sqrt{2}}e^{\mathbf{j}\theta}$; $I(t) = \mathrm{Re}(\bar{I}e^{\mathbf{j}(\theta-\phi+\omega t)})$, $I = \frac{\bar{I}}{\sqrt{2}}e^{\mathbf{j}(\theta-\phi)}$ (phasors V, I eliminate time and frequency components)
- **Complex power**: $S = VI^* = p + \mathbf{j}q = \frac{\bar{V}\bar{I}}{2}\cos(\phi) + \frac{\bar{V}\bar{I}}{2}\mathbf{j}\sin(\phi)$
- **Real power** $P = \frac{\bar{V}\bar{I}}{2}\cos(\phi)$: Does actual work (i.e. building heating, motor spinning)
- **Reactive power** $P = \frac{\bar{V}\bar{I}}{2}\sin(\phi)$: Sustains magnetic fields in induction motors and transformers (does not deliver energy)
- **Power factor**: $cos(\phi)$ how much power is real vs reactive

Voltage, Current, Power

- Power lines typically have multiple phases via separate lines
- **Balanced three phase operation**:

$$V_a = \bar{V}cos(\omega t + \theta); V_b = \bar{V}cos(\omega t + \theta + \frac{2\pi}{3}); V_c = \bar{V}cos(\omega t + \theta - \frac{2\pi}{3})$$

$$I_a = \bar{I}cos(\omega t + \theta); I_b = \bar{I}cos(\omega t + \theta + \frac{2\pi}{3}); I_c = \bar{I}cos(\omega t + \theta - \frac{2\pi}{3})$$

- Note that $I_a + I_b + I_c = 0$; $P = P_a + P_b + P_c = 3\frac{\bar{V}\bar{I}}{2}cos(\phi)$
- It is assumed that the three phases are balanced, and that the entire network can be represented via a single phase

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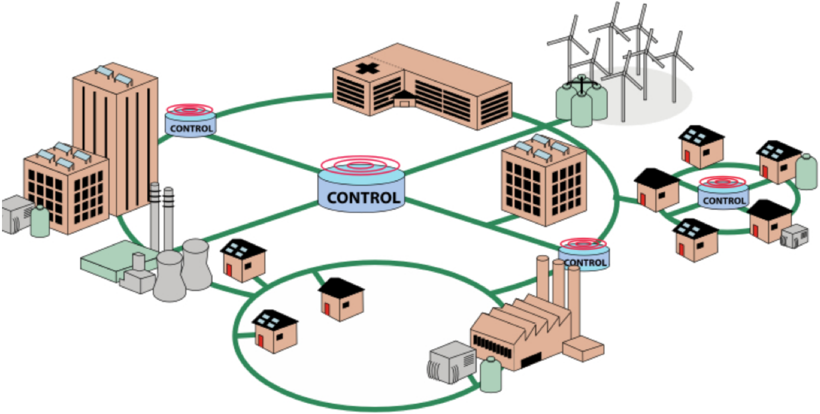
Components of a Power Grid

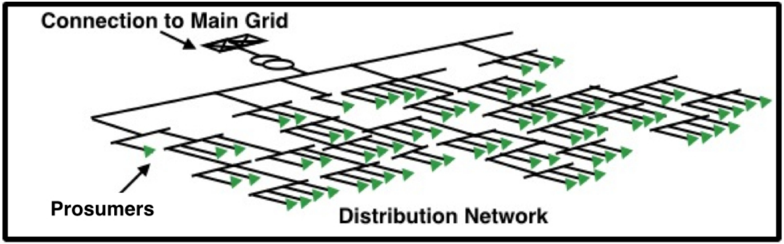
- Composed of generation systems (supply), loads (demand), lines, buses, protection devices, etc.
- State variables: voltages, current, power, etc.
- **Transmission Network** (TN): Transports power generated by bulk entities across wide geographical regions
  - At high voltages to minimize power loss and Alternating Current (AC) which is cheaper (e.g. transformer-side)
  - High-voltage Direct Current (HVDC) systems are gaining in popularity as these are efficient for transmission across long distances with greater costs for stepping up and down voltages (technologies are changing)
- **Distribution Network (DN)**: Delivers power from bulk grid to loads
  - Typically power flows from substation feeder to consumers
  - With consumers generating power, opposite is happening
- **Buses** $\mathcal{B}$ connect individual components (e.g. generation, loads, etc.) to the power network (TNs, and/or DNs)
- **Lines** $\mathcal{E}$ connect buses to one another and are associated with impedance Z (e.g. resistance and reactance) that contribute to power losses and inefficiencies

Evolving Nature of the Power Grid



Example of a Distribution Network

Note the radial/tree topology of the DN



Very Simple Transmission Network

Note the mesh topology of the TN (typically consists of thousands of buses)

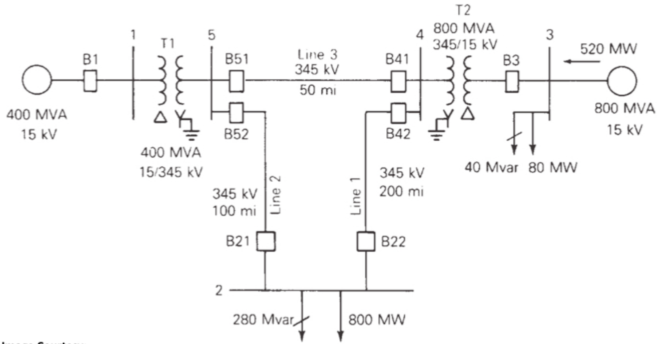
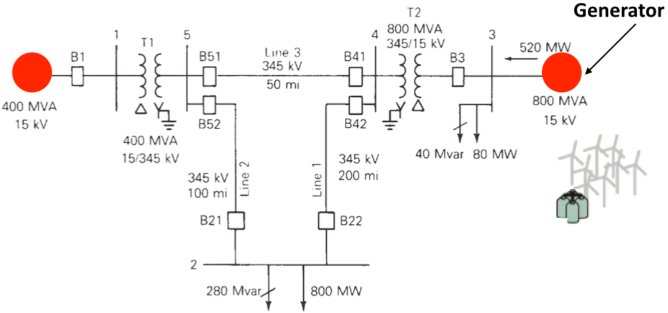


Image Courtesy:

J. D. Glover, M. S. Sarma, and T. Overbye, Power system analysis and design, 5th ed. Boston: Thomson-Engineering, 2011.

Bulk Generation Systems in TN

Generation Systems can be synchronous plants (e.g. gas, nuclear, hydro) or renewable generation (e.g. wind/solar farms)



Generation Capacity: $\forall\ i\ \in\ \mathcal{G}$

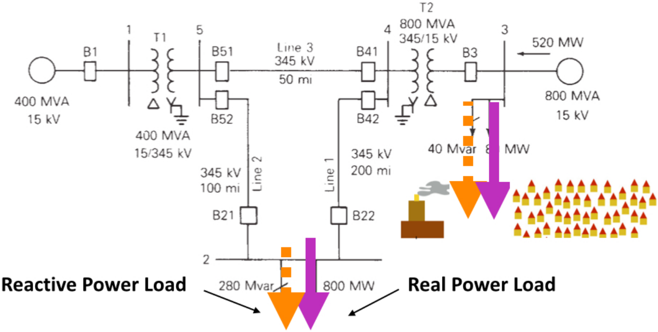
$$P^i_{min} \leq p^g_i \leq P^i_{max} \qquad \longleftarrow \textbf{Highly fluctuating}$$

$$Q^i_{min} \leq q^g_i \leq Q^i_{max}$$

$$p^g_i \in \mathcal{L} = \{L_1 \ldots L_n\} \qquad \longleftarrow \textbf{Non-convex}$$

Loads in TN

Loads can be industrial (connect directly) or consumers (connect via DNs)



Consumer Demand Curtailment: $\forall\ i\ \in\ \mathcal{D}$

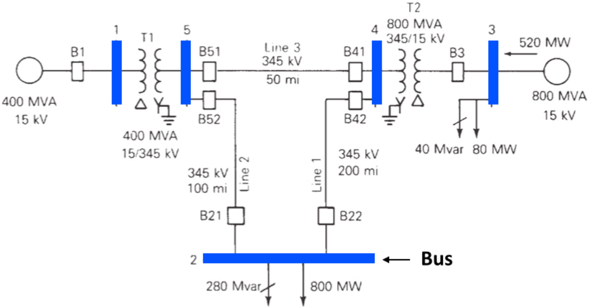
$$p_i^r \in \mathcal{L} = \{L_1 \ldots L_n\} \longleftarrow \textbf{Non-convex}$$

$$E(p_i^r) \leq E^i_{budget}$$

$$\sum_{i \in \mathcal{D}} p_i^r - C = 0$$

Buses in TN

Buses serve as a connection point/interface

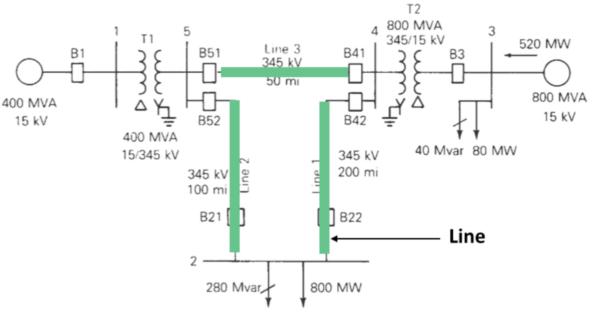


Voltage Limits at Buses: $\forall\ i\ \in\ \mathcal{B}$

$$|V_{min}|^2 \leq V_i V_i^* \leq |V_{max}|^2$$

Lines in TN

Lines connect multiple buses together



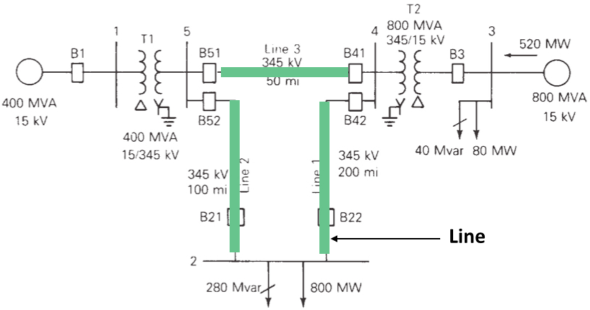
Overall Power Balance: $\forall\ n \sim m \in \mathcal{E}$

$$\sum_{i \in \mathcal{G}} p_i^g - \sum_{j \in \mathcal{D}} p_j^d = 0$$

$$\sum_{i \in \mathcal{G}} p_i^g - \sum_{j \in \mathcal{D}} p_j^d - \sum_{n \sim m \in \mathcal{E}} p_{n,m}^l = 0$$

Lines in TN

Lines connect multiple buses together



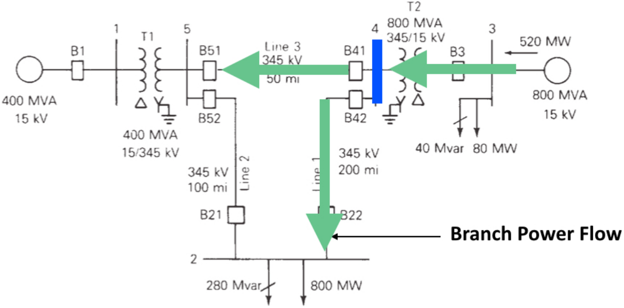
Overall power balance:

$$\sum_{i \in \mathcal{G}} p_i^g - \sum_{j \in \mathcal{D}} P_j^d - \sum_{n \sim m \in \mathcal{E}} P_{n,m}^l = 0$$

Not convex!

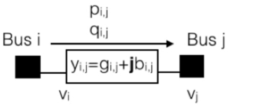
Lines in TN

Lines connect multiple buses together



Power Balance at Buses:

Non-convex



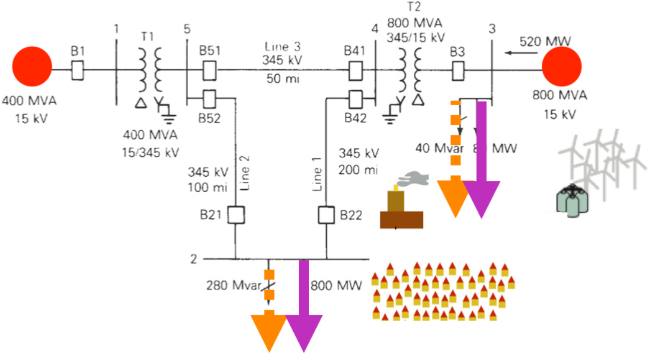
$$p_{i,j} + \mathbf{j}q_{i,j} = V_i I_{i,j}^* = Y_{i,j}^* V_i (V_i - V_j)^*$$

$$\sum_{n \in \mathcal{B}_i} p_n^g - \sum_{m \in \mathcal{B}_i} p_m^d = \mathrm{Re}\{\sum_{j}^{i \neq j} Y_{i,j}^* V_i (V_i - V_j)^*\}$$

$$\sum_{n \in \mathcal{B}_i} q_n^g - \sum_{m \in \mathcal{B}_i} q_m^d = \mathrm{Im}\{\sum_{j}^{i \neq j} Y_{i,j}^* V_i (V_i - V_j)^*\}$$

Objectives

Objectives are typically quadratic



Cost Functions/Objectives:

$$f(p^g) = \sum_{i \in \mathcal{G}} f_i(p_i^g)$$

$$f(p^g, p^d) = \sum_{i \in \mathcal{G}} f_i(p_i^g) + \sum_{j \in \mathcal{D}} f_j(p_j^r)$$

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Circuit Theory for the Power System

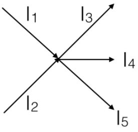
Describing the power system:

- Need a way to describe **relationship** between voltages, currents, power flows in the power system
- **Line admittance**: $Y_{ij} = g_{ij} + \mathbf{j}b_{ij}$ Measure of how easily a material allows current I to flow (inverse of **impedance** $Z_{ij} = r_{ij} + \mathbf{j}x_{ij}$) and $Y_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$
- **Nodal/Bus admittance matrix** $Y^b \in \mathbb{C}^{n \times n}$: Matrix that succinctly describes a power system with n buses
  - Obtained by applying **Kirchoff's current law** and **voltage law** to circuit with **steady-state** sinusoidal operation
  - Symmetric matrix with positive diagonals and non-positive off-diagonals
- S_n is the complex power at bus $n \in \mathcal{B}$, $V \in \mathbb{C}^n$ is a voltage vector

$$S_n = \sum_{m \in \mathcal{B}} V_n V_m^* Y_{nm}^b$$

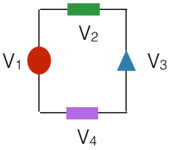
- The entire power system is defined by the above relation

Some Circuit Theory Preliminaries



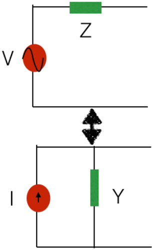
$I_1+I_2=I_3+I_4+I_5$

Kirchoff's Current Law



$V_1+V_2+V_3+V_4=0$

Kirchoff's Voltage Law



Norton's Theorem

Kirchoff's Current Law:

- Sum of current flowing into a node is equal to the current leaving it

Kirchoff's Voltage Law:

- Sum of voltage around a loop is 0

Norton's Theorem:

- Equivalence between parallel and series circuit ($I = \frac{V}{Z}$ and $Y = \frac{I}{Z}$)

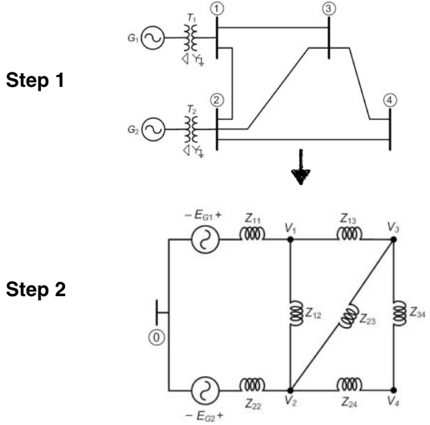
Computing the Nodal Admittance Matrix

Four steps for deriving the admittance matrix of a power system:

1. Obtain the **single line** diagram of the power system
2. Convert single line diagram to **impedance** diagram
3. Convert **voltage source** to equivalent **current source** representation
4. Infer Y from the **admittance** diagram

Computing the Nodal Admittance Matrix: Example

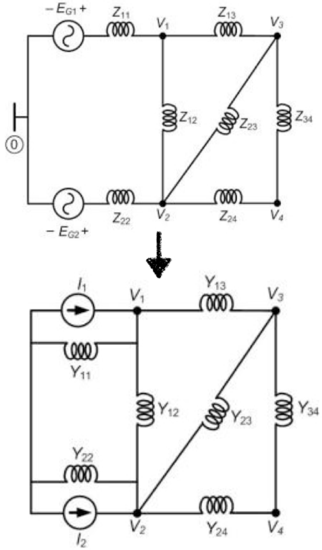
Step 1: Obtain the **single line** diagram of the power system



Step 2: Convert single line diagram to **impedance** diagram

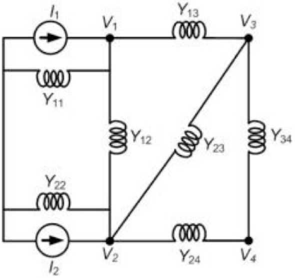
Computing the Nodal Admittance Matrix: Example

Step 3: Convert **voltage source** to equivalent **current source** representation (Norton's Theorem)



Computing the Nodal Admittance Matrix: Example

Step 4: Infer Y from the **admittance** diagram



Node 1:

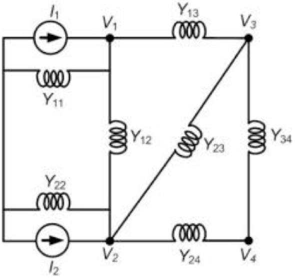
$$
\begin{aligned}
I_1 &= Y_{11}V_1 + Y_{12}(V_1 - V_2) + Y_{13}(V_1 - V_3) \\
&= (Y_{11} + Y_{12} + Y_{13})V_1 - Y_{12}V_2 - Y_{13}V_3
\end{aligned}
$$

Node 2:

$$
\begin{aligned}
I_2 &= Y_{22}V_2 + Y_{12}(V_2 - V_1) + Y_{23}(V_2 - V_3) + Y_{24}(V_2 - V_4) \\
&= -Y_{12}V_1 + (Y_{22} + Y_{12} + Y_{23} + Y_{24})V_2 - Y_{23}V_3 - Y_{24}V_4
\end{aligned}
$$

Computing the Nodal Admittance Matrix: Example

Step 4: Infer Y from the **admittance** diagram



Node 3:

$$
\begin{aligned}
I_3 = 0 &= Y_{13}(V_3 - V_1) + Y_{23}(V_3 - V_2) + Y_{34}(V_3 - V_4) \\
&= -Y_{13}V_1 - Y_{23}V_2 + (Y_{13} + Y_{23} + Y_{34})V_3 - Y_{34}V_4
\end{aligned}
$$

Node 4:

$$
\begin{aligned}
I_4 = 0 &= Y_{24}(V_4 - V_2) + Y_{34}(V_4 - V_3) \\
0 &= -Y_{24}V_2 - Y_{34}V_3 + (Y_{24} + Y_{34})V_4
\end{aligned}
$$

Computing the Nodal Admittance Matrix: Example

$$I_1 = (Y_{11} + Y_{12} + Y_{13})V_1 - Y_{12}V_2 - Y_{13}V_3$$
$$I_2 = -Y_{12}V_1 + (Y_{22} + Y_{12} + Y_{23} + Y_{24})V_2 - Y_{23}V_3 - Y_{24}V_4$$
$$I_3 = 0 = -Y_{13}V_1 - Y_{23}V_2 + (Y_{13} + Y_{23} + Y_{34})V_3 - Y_{34}V_4$$
$$I_4 = 0 = -Y_{24}V_2 - Y_{34}V_3 + (Y_{24} + Y_{34})V_4$$

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} + Y_{12} + Y_{13} & -Y_{12} & -Y_{13} & 0 \\ -Y_{12} & Y_{22} + Y_{12} + Y_{23} + Y_{24} & -Y_{23} & -Y_{24} \\ -Y_{13} & -Y_{23} & Y_{13} + Y_{23} + Y_{34} & -Y_{34} \\ 0 & -Y_{24} & -Y_{34} & Y_{24} + Y_{34} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

- Non-positive off-diagonals ($Y^b_{ij} = -Y_{ij}$)
- Positive diagonals at index k general form:

$$Y^b_k = \sum_{j=1}^{N} Y_{kj}$$

Voltage and Current using Line Admittance

- Properties of a line $e_{ij} \in \mathcal{E}$:

$$\begin{aligned}
\textbf{Current:}\ \ & I_{ij} = (V_i - V_j)Y_{ij} \\
\textbf{Power:}\ \ & S_{ij} = p_{ij} + \mathbf{j}q_{ij} \\
& p_{ij} + \mathbf{j}q_{ij} = V_i I_{ij}^* \\
& p_{ij} + \mathbf{j}q_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^*
\end{aligned}$$

- Line power flow is not symmetric i.e. $S_{i,j} \neq S_{j,i}$

$$V_i(V_i^* - V_j^*)Y_{ij}^* \neq V_j(V_j^* - V_i^*)Y_{ji}^*$$

- V_i is a complex exponential $V_i = |V_i|e^{j\theta_i} = |V_i|(\cos(\theta_i) + \mathbf{j}\sin(\theta_i))$, the real and imaginary parts of complex power are:

$$\begin{aligned}
p_{ij} &= g_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\cos(\theta_i - \theta_j) - b_{ij}\sin(\theta_i - \theta_j)) \\
q_{ij} &= b_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\sin(\theta_i - \theta_j) + b_{ij}\cos(\theta_i - \theta_j))
\end{aligned}$$

- How non-convex is this??

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Summary of Variables, Constraints, Objectives

- Variables: $|V_i|, \theta_i, p_i, q_i$ (variables depend on the underlying problem considered)
- Magnitudes of bus voltages $i \in \mathcal{B}$: $\underline{V}_i \leq |V_i| \leq \bar{V}_i$
- Apparent power flow limit (magnitude of complex power) on lines $e_{ij} \in \mathcal{E}$: $p_{ij}^2 + q_{ij}^2 \leq \bar{s}_{ij}^2$
- Real and reactive power injections in buses $i \in \mathcal{B}$: $p_i = \sum_{e_{ij} \in \mathcal{E}} p_{ij}$ and $q_i = \sum_{e_{ij} \in \mathcal{E}} q_{ij}$
- Power injection capacities in $i \in \mathcal{B}$: $\underline{p}_i \leq p_i \leq \bar{p}_i$ and $\underline{q}_i \leq q_i \leq \bar{q}_i$
- Power flow across lines $e_{ij} \in \mathcal{E}$: $p_{ij} + \mathbf{i} q_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^*$
- Cost of real power generation: $\sum_{i \in \mathcal{G}} f_i(p_i)$ (typically non-convex)
- Resistive power losses: $\sum_{i \in e_{ij}\mathcal{E}} r_{ij} I_{ij}^2 = \sum_{g \in \mathcal{G}} p_g - \sum_{d \in \mathcal{D}} p_d$

Economic Dispatch

$$\mathcal{P}_{ED}: \quad \min_{P_g} \sum_{g \in \mathcal{G}} f(p_g)$$
$$s.t. \sum_{g \in \mathcal{G}} p_g = \sum_{d \in \mathcal{D}} p_d$$
$$\underline{p}_g \leq p_g \leq \bar{p}_g \; \forall g \in \mathcal{G}$$

- Used for planning purposes over long time horizons (e.g. day ahead, etc.)
- Overall power generated in the system must be equal to the overall power demand at minimal cost
- What are the shortcomings?

Demand Response

$$\mathcal{P}_{DR}: \quad \min_{P_d} \sum_{g \in \mathcal{D}} f(p_d)$$
$$s.t. \sum_{d \in \mathcal{D}} p_d = C$$
$$E(p_d) \leq E_d^{budget} \; \forall d \in \mathcal{D}$$
$$p_d \in \mathcal{L} = \{L_1, \ldots, L_n\}$$

- Used for reducing consumer demands for reducing aggregate demand C in the system
- Need to account for consumer comfort budget E_d^{budget} of each consumer
- Demand reduction can take discrete values
- What are the challenges? How can these be overcome?

DC Power Flow

$$\mathcal{P}_{FR} : \min_{p,\theta} \sum_{i \in \mathcal{B}} f(p_i, \theta_i)$$

s.t.

$$
\begin{aligned}
&\underline{p}_g \leq p_g \leq \bar{p}_g && \forall\ g \in \mathcal{G} \\
&p_i = p_i^g - p_i^d && \forall\ i \in \mathcal{B} \\
&p_i = \sum_{e_{ij} \in \mathcal{E}} -b_{ij}(\theta_i - \theta_j) && \forall\ i \in \mathcal{B}
\end{aligned}
$$

- No reactive power and voltage terms
- Following power flow constraints are approximated:

$$
\begin{aligned}
p_{ij} &= |V_i|^2 g_{ij} - |V_i||V_j| g_{ij} \cos(\theta_i - \theta_j) - |V_i||V_j| b_{ij} \sin(\theta_i - \theta_j) \\
q_{ij} &= -|V_i|^2 b_{ij} + |V_i||V_j| b_{ij} \cos(\theta_i - \theta_j) - |V_i||V_j| g_{ij} \sin(\theta_i - \theta_j)
\end{aligned}
$$

- Apply the following assumptions:
  1. $|g_{a,b}| << |b_{a,b}|$
  2. $\sin(\theta_a - \theta_b) \approx \theta_a - \theta_b$; $\cos(\theta_a - \theta_b) \approx 1$
  3. $|V_a| \approx 1$, $|V_b| \approx 1$ (in per unit system) $\forall\ a, b \in \mathcal{B}$

Optimal Power Flow

$$\mathcal{P}_{OPF}: \min_{p_g} \sum_{g \in \mathcal{G}} f_g(p_g)$$

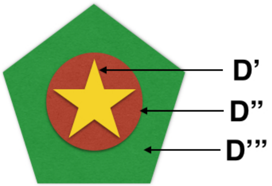
s.t.

$$\begin{aligned}
&\underline{c}_g^p \leq p_g \leq \bar{c}_g^p && \forall\, g \in \mathcal{G} \\
&\underline{c}_g^q \leq q_g \leq \bar{c}_g^q && \forall\, g \in \mathcal{G} \\
&|V_{min}|^2 \leq V_i V_i^* \leq |V_{max}|^2 && \forall\, i \in \mathcal{B} \\
&p_i + \mathbf{j} q_i = \sum_{e_{ij} \in \mathcal{E}} s_{ij} && \forall\, i \in \mathcal{B} \\
&s_{ij} = \sum_{e_{ij} \in \mathcal{E}} V_i (V_i - V_j)^* Y_{ij}^* && \forall\, i \in \mathcal{B} \\
&p_{ij}^2 + q_{ij}^2 \leq \bar{s}_{ij} && \forall\, i \in \mathcal{B}
\end{aligned}$$

- OPF is comprehensive as underlying steady-state physical constraints are accounted
- Which one of the above constraints are problematic/non-convex?

Relaxations Applied to Optimal Power Flow

- The objective of the OPF is typically convex
- As constraint set $\mathcal{D}$ is problematic, relaxations are applied to $\mathcal{D}$
- How good is the relaxation? Depends on the tightness of the relaxation



SDP Relaxation Applied to OPF

- Let V_{ij} be $V_i V_j^*$ and construct the matrix V based on V_{ij}
- For $[V_1 \dots V_n]^T [V_1 \dots V_j]^*$ to be a unique solution, $V \succeq 0$ (diagonal elements are positive and real) and $\text{rank}(V) = 1$
- When $\text{rank}(V) = 1$, only one eigenvalue λ is non-zero and $\lambda x = Vx$ where $\lambda = [V_1 \dots V_n]^* [V_1 \dots V_n]^T$ and $x = [V_1 \dots V_n]^T$
- The power flow constraint is now $p_{ij} + \mathbf{j} q_{ij} = (V_{ii} - V_{ij}) Y_{ij}^*$
- Optimal solution is in matrix form and decomposition (e.g. Cholesky) can be applied to extract $V_i \ \forall i \in \mathcal{B}$
- Resulting modified constraints are:

$$
\begin{aligned}
&|V_{min}|^2 \leq V \leq |V_{max}|^2 \\
&p_i + \mathbf{j} q_i = \sum_{e_{ij} \in \mathcal{E}} s_{ij} && \forall\ i \in \mathcal{B} \\
&s_{ij} = \sum_{e_{ij} \in \mathcal{E}} (V_{ii} - V_{ij})^* Y_{ij}^* && \forall\ i \in \mathcal{B} \\
&V \succeq 0, \text{rank}(V) = 1
\end{aligned}
$$

- Remove the $\text{rank}(V) = 1$ constraint to obtain an SDP relaxation

SOCP Relaxation Applied to OPF

- For large-scale optimization problems, SDP relaxations are computationally intensive to solve and the variable space which is squared is difficult to manage
- The matrix V is typically sparse (i.e. many 0 entries) and many of the new variables are unnecessary
- SOC relaxation taps onto the alternate definition of positive semi-definite matrix which specifies that the principal minors of the matrix V must be non-negative
- Principal minor is defined as the determinant of a sub-matrix whose diagonal coincides with the main matrix
- Constraints $V \succeq 0$ and rank(V) are replaced by the determinants of the 2 by 2 and 1 by 1 principal sub-matrices as follows (real constraints):

$$V_{ij} V_{ij}^* \leq V_{ii} V_{jj}, V_{ii} \geq 0 \; \forall \; e_{ij} \in \mathcal{E}$$

- The number of variables is now equal to the number of lines in the system and this can be solved by efficient SOC interior point methods

Exactness of Relaxations

- If the optimal solution of the relaxed problem satisfies all constraints of the original problem, then the solution is exact
- SDP and SOCP relaxations are exact when the underlying system is radial and $\underline{p}_i$ and $\underline{q}_i$ are $-\infty$