2.群的性质

其中 Z 为整数集,N 为自然数集。

Q1. {Z, +} 是否为群?若是,验证其满足群定义;若不是,说明理由。

是, 构成了整数加法群 G=(Z,+)

1. 封闭性
   $\forall z1, z2 \in Z, (z1+z2) \in Z;$
2. 结合律
   $\forall z1, z2, z3 \in Z, (z1+z2)+z3 = z1+(z2+z3);$
3. 幺元
   $\exists z0 \in Z, s.t. \forall z \in Z, z0+z=z+z0=z,$
   Therefore, z0 = 0;
4. 逆
   $\forall z \in Z, \exists z^{-1} \in Z$(逆), s.t. $z+z^{-1} = z0=0$
   对于整数加法群, 每一个元素的逆元是他的相反数 即 $z^{-1} = -z$

Q2. {N, +} 是否为群?若是,验证其满足群定义;若不是,说明理由。

不是

因为自然数的定义是不小于 0 的整数(非负整数) e.g. 0,1,2,3,4,5,6,7…

而自然数关于加法没有逆元

因为不是任意一个自然数都可以找到逆元使得 $z+z^{-1} = z0 = 0$;

仅有 0 本身可以找到一个逆元为 0

3.验证向量叉乘的李代数性质

Assume vector **a**, **b** to be

$$\vec{a} = (\mathrm{a1}, \mathrm{a2}, \mathrm{a3})^T = \mathrm{a1i} + \mathrm{a2j} + \mathrm{a3k}$$
$$\vec{b} = (\mathrm{b1}, \mathrm{b2}, \mathrm{b3})^T = b1i + b2j + b3k$$

1. 封闭性 $\forall \vec{a}, \vec{b} \in R^3$, $\vec{a}\times\vec{b} \in R^3$.

$$\vec{\mathrm{a}}\times\vec{b} = \begin{bmatrix} i & \mathrm{j} & k \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\mathrm{a3} & a2 \\ a3 & 0 & -a1 \\ -a2 & a1 & 0 \end{bmatrix}\begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$
$$= \begin{bmatrix} a2b3 - a3b2 \\ a3b1 - a1b3 \\ a1b2 - a2b1 \end{bmatrix} \in R^3$$

2. 双线性 $\forall \vec{\mathrm{x}}, \vec{y}, \vec{z} \in R^3$, a, b $\in$ R:
   $(a\vec{x}+b\vec{y})\times\vec{z} = a(\vec{x}\times\vec{z}) + b(\vec{y}\times\vec{z})$,
   $\vec{z}\times(a\vec{x}+b\vec{y}) = a(\vec{z}\times\vec{x}) + b(\vec{z}\times\vec{y})$

   According to the algebraic properties of cross product, we have

$(a\vec{x}+b\vec{y})\times\vec{z} = -\vec{z}\times(a\vec{x}+b\vec{y}) = -(\vec{z}\times(a\vec{x})+ \vec{z}\times(b\vec{y}))=-(a(\vec{z}\times\vec{x}) + b(\vec{z}\times\vec{y})) =a(\vec{x}\times\vec{z}) + b(\vec{y}\times\vec{z})$

   That is,

$$(a\vec{x}+b\vec{y})\times\vec{z} = a(\vec{x}\times\vec{z}) + b(\vec{y}\times\vec{z}),\ \vec{z}\times(a\vec{x}+b\vec{y}) = a(\vec{z}\times\vec{x}) + b(\vec{z}\times\vec{y})$$

3. 自反性 $\forall \vec{\mathrm{x}} \in R^3$, $\vec{x}\times\vec{x}=0$

$$\vec{x}\times\vec{x} = \vec{x}\,\vec{x}\sin\theta = 0\ (\theta=0°)$$

4. 雅克比等价 $\forall\ \vec{\mathrm{x}}, \vec{y}, \vec{z} \in R^3$, $\vec{\mathrm{x}}\times(\vec{\mathrm{y}}\times\vec{\mathrm{z}}) + \vec{\mathrm{y}}\times(\vec{\mathrm{z}}\times\vec{\mathrm{x}}) + \vec{\mathrm{z}}\times(\vec{\mathrm{x}}\times\vec{\mathrm{y}}) = 0$

   According to Lagrange formula, we have

$$\vec{\mathrm{x}}\times(\vec{\mathrm{y}}\times\vec{\mathrm{z}}) = \vec{\mathrm{y}}(\vec{\mathrm{x}}\cdot\vec{\mathrm{z}}) - \vec{\mathrm{z}}(\vec{\mathrm{x}}\cdot\vec{\mathrm{y}})$$
$$\vec{\mathrm{y}}\times(\vec{\mathrm{z}}\times\vec{\mathrm{x}}) = \vec{\mathrm{z}}(\vec{\mathrm{y}}\cdot\vec{\mathrm{x}}) - \vec{\mathrm{x}}(\vec{\mathrm{y}}\cdot\vec{\mathrm{z}})$$
$$\vec{\mathrm{z}}\times(\vec{\mathrm{x}}\times\vec{\mathrm{y}}) = \vec{\mathrm{x}}(\vec{\mathrm{z}}\cdot\vec{\mathrm{y}}) - \vec{\mathrm{y}}(\vec{\mathrm{z}}\cdot\vec{\mathrm{x}})$$

$\vec{\mathrm{x}}\times(\vec{\mathrm{y}}\times\vec{\mathrm{z}}) + \vec{\mathrm{y}}\times(\vec{\mathrm{z}}\times\vec{\mathrm{x}}) + \vec{\mathrm{z}}\times(\vec{\mathrm{x}}\times\vec{\mathrm{y}}) = \vec{\mathrm{y}}(\vec{\mathrm{x}}\cdot\vec{\mathrm{z}}) - \vec{\mathrm{z}}(\vec{\mathrm{x}}\cdot\vec{\mathrm{y}}) + \vec{\mathrm{z}}(\vec{\mathrm{y}}\cdot\vec{\mathrm{x}}) - \vec{\mathrm{x}}(\vec{\mathrm{y}}\cdot\vec{\mathrm{z}}) + \vec{\mathrm{x}}(\vec{\mathrm{z}}\cdot\vec{\mathrm{y}}) - \vec{\mathrm{y}}(\vec{\mathrm{z}}\cdot\vec{\mathrm{x}}) = 0$

4.推导 SE(3) 的指数映射

$\mathbf{a}^\wedge\mathbf{a}^\wedge = \mathbf{a}\mathbf{a}^{\mathrm{T}} - \mathrm{I}$

$\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge = -\mathbf{a}^\wedge$

Taylor's formula

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + o\left(x^5\right)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^4\right)$$

$\phi = \theta\mathbf{a}$

$$\sum_{n=0}^{\infty} \frac{1}{(\mathrm{n}+1)!}(\theta\mathbf{a}^\wedge)^n$$

$= \mathrm{I} + \frac{1}{2!}\theta\mathbf{a}^\wedge + \frac{1}{3!}\theta^2\mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{4!}\theta^3\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{5!}\theta^4\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge + \ldots$

$= \mathbf{a}\mathbf{a}^{\mathrm{T}} - \mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{2!}\theta\mathbf{a}^\wedge + \frac{1}{3!}\theta^2\mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{4!}\theta^3\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{5!}\theta^4\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge + \ldots$

$= \mathbf{a}\mathbf{a}^{\mathrm{T}} + \left(\frac{1}{2!}\theta - \frac{1}{4!}\theta^3 + \frac{1}{6!}\theta^5 - \ldots\right)\mathbf{a}^\wedge - \left(1 - \frac{1}{3!}\theta^2 + \frac{1}{5!}\theta^4 - \ldots\right)\mathbf{a}^\wedge\mathbf{a}^\wedge$

$= \mathbf{a}\mathbf{a}^{\mathrm{T}} + \left(1 - \frac{\cos\theta}{\theta}\right)\mathbf{a}^\wedge - \left(\frac{\sin\theta}{\theta}\right)\mathbf{a}^\wedge\mathbf{a}^\wedge$

$= \mathbf{a}^\wedge\mathbf{a}^\wedge + \mathrm{I} + \left(1 - \frac{\cos\theta}{\theta}\right)\mathbf{a}^\wedge - \left(\frac{\sin\theta}{\theta}\right)\mathbf{a}^\wedge\mathbf{a}^\wedge$

$= \mathrm{I} + \left(1 - \frac{\cos\theta}{\theta}\right)\mathbf{a}^\wedge + \left(1 - \frac{\sin\theta}{\theta}\right)\mathbf{a}^\wedge\mathbf{a}^\wedge$

$= \mathrm{I} + \left(1 - \frac{\cos\theta}{\theta}\right)\mathbf{a}^\wedge + \left(1 - \frac{\sin\theta}{\theta}\right)(\mathbf{a}\mathbf{a}^{\mathrm{T}} - \mathrm{I})$

$= \frac{\sin\theta}{\theta}\mathrm{I} + \left(1 - \frac{\cos\theta}{\theta}\right)\mathbf{a}^\wedge + \left(1 - \frac{\sin\theta}{\theta}\right)\mathbf{a}\mathbf{a}^{\mathrm{T}}$

5. 伴随

Step 1.

Prove $\forall$ **a** $\in R^3$, $\mathbf{R}\mathbf{a}^\wedge\mathbf{R}^T = (\mathbf{R}\mathbf{a})^\wedge$

Identify an arbitrary vector **v**, then

$$(\mathbf{R}\mathbf{a})^\wedge \mathbf{v} = (\mathbf{R}\mathbf{a}) \times \mathbf{v} = (\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{R}^{-1} \mathbf{v}) = \mathbf{R}\,[\mathbf{a} \times (\mathbf{R}^{-1} \mathbf{v})] = \mathbf{R}\mathbf{a}^\wedge\mathbf{R}^T\mathbf{v}$$

where we have used the fact that for any rotation matrix **R** and vector **a**, **b** we have

$$(\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{b}) = \mathbf{R}\,(\mathbf{a} \times \mathbf{b})$$

Therefore,

$$(\mathbf{R}\mathbf{a})^\wedge = \mathbf{R}\mathbf{a}^\wedge\mathbf{R}^T.$$

Step 2.

For the Lie group SO(3), the adjoint is written Ad(R):

$$\mathbf{R}\exp(\mathbf{p}^\wedge) = \exp((\mathrm{Ad}(\mathbf{R})\mathbf{p})^\wedge)\mathbf{R}$$

Therefore,

$$\exp((\mathrm{Ad}(\mathbf{R})\mathbf{p})^\wedge) = \mathbf{R}\exp(\mathbf{p}^\wedge)\,\mathbf{R}^{-1}$$

If we take the logarithm of both sides, we have

$$(\mathrm{Ad}(\mathbf{R})\mathbf{p})^\wedge = \log(\mathbf{R}\exp(\mathbf{p}^\wedge)\,\mathbf{R}^{-1})$$

According to the properties of the logarithm of a matrix [1]

$$\log A = V\,(\log A')\,V^{-1}$$

$$A' = V^{-1}AV,$$

we have

$$\log(A) = \log(VA' V^{-1}) = V (\log A') V^{-1}$$

Then,

$$\log(\mathbf{R}\exp(\mathbf{p}^\wedge)\, \mathbf{R}^{-1}) = \mathbf{R}\mathbf{p}^\wedge\mathbf{R}^{-1} = \mathbf{R}\mathbf{p}^\wedge\mathbf{R}^T = (\mathbf{R}\mathbf{p})^\wedge \quad (\mathbf{p} \in R^3)$$

Therefore, Ad($\mathbf{R}$) = $\mathbf{R}$

Reference:

[1] https://en.wikipedia.org/wiki/Logarithm_of_a_matrix

6. 轨迹的描绘

Assume a point **pw** in the world coordinate and a point **pc** in the camera coordinate, then

$$\mathbf{pc} = \mathbf{Tcw}\ \mathbf{pw}$$

where **Tcw** represents the transformation from the world coordinate to the camera coordinate.

We also have

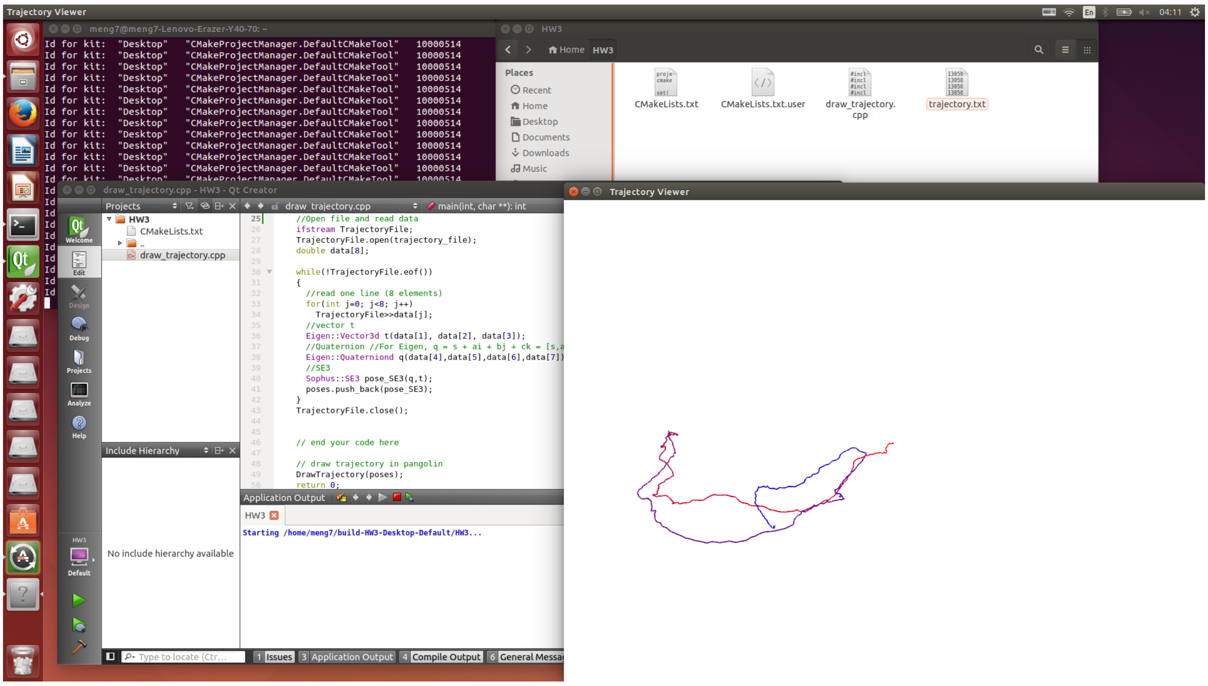
$$\mathbf{pw} = \mathbf{Twc}\ \mathbf{pc} = \mathbf{Tcw}^{-1}\ \mathbf{pc}$$

Let **pc** = $[0, 0, 0]^T$ which represents the origin in the camera coordinate, then the related coordinate of the point in the world coordinate can be calculated

$$\mathbf{pw} = \mathbf{Twc}\ \mathbf{0} = \mathbf{twc}$$

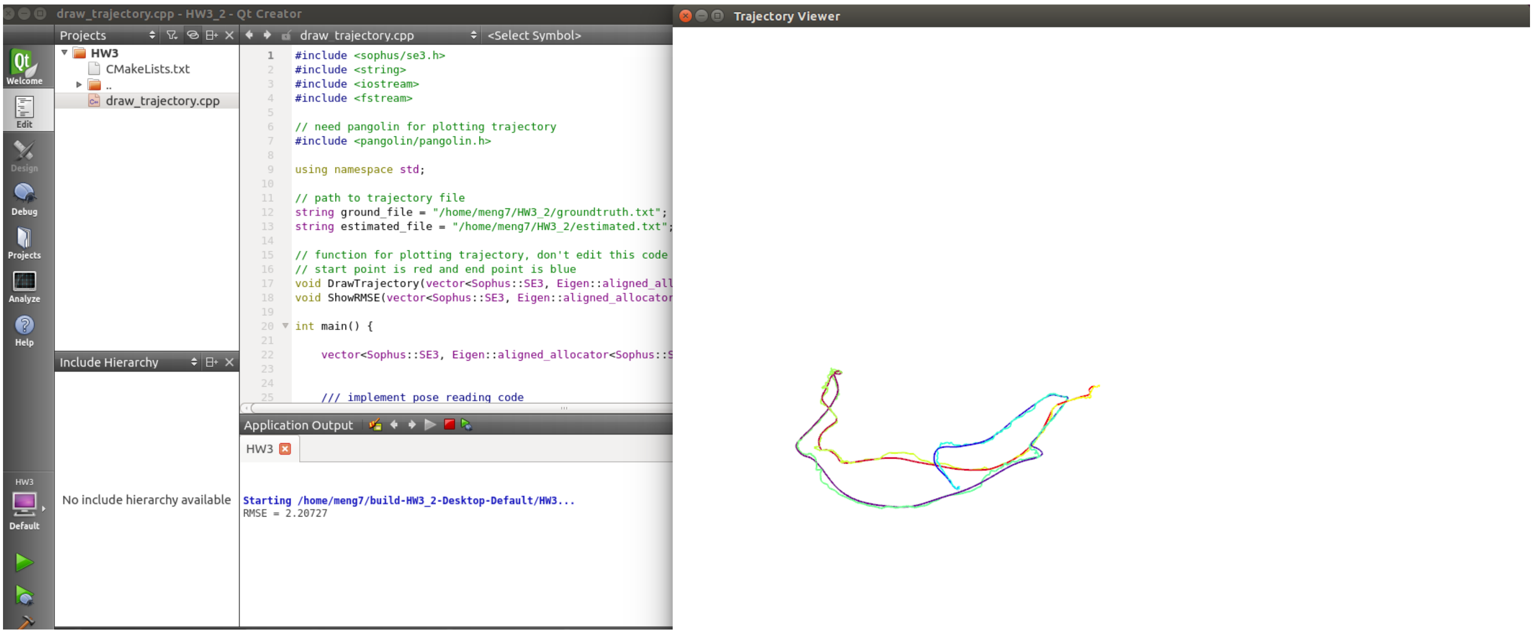
Therefore, the translation part **twc** of the transformation matrix actually represents the world coordinate point related to the origin of the camera coordinate. Furthermore, the translation parts of the transformation matrices show the trajectory of the camera.

Diagram of the trajectory



7. 轨迹的误差

Result is shown below



RMSE: