2.

Q1 在什么条件下,x 有解且唯一?

若矩阵的秩 $r(A)=r(\bar{A})=n$,则方程有唯一确定的解

$\bar{A}$为增广矩阵

$$\bar{A} = [A \vdots b]$$

$x = [x1\ x2\ \dots\ xn]^T$ 是 n 维的未知数向量)

Q2 高斯消元法的原理是什么

通过寻找同解的方程组, 并且通过回代解出方程

首先构造增广矩阵, 通过初等变换, 把增广矩阵变成行阶梯矩阵, 这个梯矩阵包含了同解方程的系数, 通过这个梯矩阵求得其中一个未知数的解,再把这个解带入到别的方程中求得其他未知数的解

(G-J 消元法是通过同解方程组的同解变形, 将方程组的系数矩阵变成单位矩阵, 这样可以直接获得每一个未知数的解而不需要迭代)

Q3QR 分解的原理是什么

Gram–Schmidt 正交化

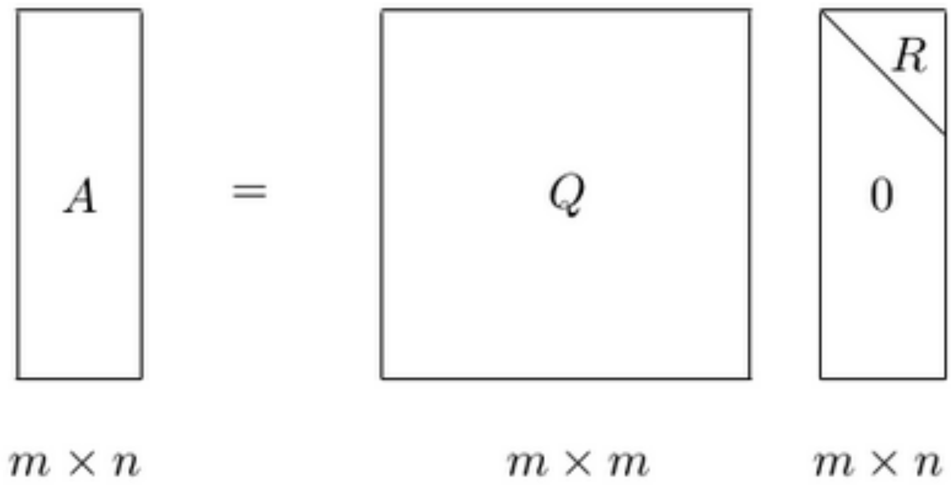
1.利用投影中所得残差向量正交于投影空间每个基的事实,产生于已知正交向量集中每一向量均正交的非零向量,从而得到正交基

2.对每一个基向量规范化,得到规范正交基

QR 分解

对于可逆矩阵 A 的列向量 进行 Gram–Schmidt 正交化

从而讲矩阵 A 分解为 A=QR



其中 Q 是正交矩阵, R 是上三角矩阵

则对于方程 Ax=b 可写成

QRx = b --> x=$R^{-1}Q^{T}$b

Q4Cholesky 分解的原理是什么?

Cholesky 分解是把一个对称正定的矩阵表示成一个下三角矩阵 L 和其转置的乘积的分解

$$A=LL^T$$

$$L=\begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \qquad L^T=\begin{pmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ 0 & l_{22} & \cdots & l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{nn} \end{pmatrix}$$

则对于方程 Ax=b

首先求解 LX' = b 得到 X'

再求解 L^TX = X' 得到 X

Q5

见代码

4.旋转的表达

Q1

Assume we have coordinate [e1, e2, e3] which is rotated to [e1', e2', e3'].
Then the vector $[a1, a2, a3]^T$ in the coordinate system is rotated to $[a1', a2', a3']^T$ in the rotated coordinate.

Therefore,

$$[e1, e2, e3]\begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} = [e1', e2', e3']\begin{bmatrix} a1' \\ a2' \\ a3' \end{bmatrix}$$

Then,

$$R = \begin{bmatrix} e1^T e1' & e1^T e2' & e1^T e3' \\ e2^T e1' & e2^T e2' & e2^T e3' \\ e3^T e1' & e3^T e2' & e3^T e3' \end{bmatrix} \text{(rotation from [e1', e2', e3'] to [e1, e2, e3])}$$

$$R^{-1} = \begin{bmatrix} e1'^T e1 & e1'^T e2 & e1'^T e3 \\ e2'^T e1 & e2'^T e2 & e2'^T e3 \\ e3'^T e1 & e3'^T e2 & e3'^T e3 \end{bmatrix} \text{(rotation from [e1, e2, e3] to [e1', e2', e3'])}$$

$$R^T = \begin{bmatrix} (e1^T e1')^T & (e2^T e1')^T & (e3^T e1')^T \\ (e1^T e2')^T & (e2^T e2')^T & (e3^T e2')^T \\ (e1^T e3')^T & (e2^T e3')^T & (e3^T e3')^T \end{bmatrix} = \begin{bmatrix} e1'^T e1 & e1'^T e2 & e1'^T e3 \\ e2'^T e1 & e2'^T e2 & e2'^T e3 \\ e3'^T e1 & e3'^T e2 & e3'^T e3 \end{bmatrix}$$

We can find that $R^{-1} = R^T$

According to the definition, we know that e1, e2, e3 are unit vectors normal to each other and e1', e2', e3' are rotated unit vectors normal to each other.

Therefore, we can assume

$$[e1, e2, e3] = A$$
$$[e1, e2, e3] = B$$

where A, B are both orthogonal matrices.

Then, we have

$$AR = B \rightarrow R = A^{-1}B$$
$$BR^{-1} = A \rightarrow R^{-1} = B^{-1}A$$

Therefore,

$$RR^{-1} = RR^T = A^{-1}B\,B^{-1}A = A^{-1}IA = I$$

That is,

$$RR^{-T} = I$$

Then, R is the orthogonal matrix.

$$\det R = 1$$

Q2

η denotes real number. Dimension = 1

$\varepsilon = [q1, q2, q3]^T \in \mathbb{R}^3$. Dimension = 3

Q3

$q1 = [\eta 1, x1, y1, z1]^T$ $q2 = [\eta 2, x2, y2, z2]^T$

$q^+ = \begin{bmatrix} \eta I + \varepsilon^x & \varepsilon \\ -\varepsilon^T & \eta \end{bmatrix}$ 4×4 matrix $q^{\oplus} = \begin{bmatrix} \eta I - \varepsilon^x & \varepsilon \\ -\varepsilon^T & \eta \end{bmatrix}$

where $\varepsilon^x = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$ (cross-product matrix)

$$q1^+ q2 = \begin{bmatrix} \eta 1 & -z1 & y1 & x1 \\ z1 & \eta 1 & -x1 & y1 \\ -y1 & x1 & \eta 1 & z1 \\ -x1 & -y1 & -z1 & \eta 1 \end{bmatrix} \begin{bmatrix} x2 \\ y2 \\ z2 \\ \eta 2 \end{bmatrix} = \begin{bmatrix} \eta 1x2 - z1y2 + y1z2 + x1\eta 2 \\ z1x2 + \eta 1y2 - x1z2 + y1\eta 2 \\ -y1x2 + x1y2 + \eta 1z2 + z1\eta 2 \\ -x1x2 - y1y2 - z1z2 + \eta 1\eta 2 \end{bmatrix}$$

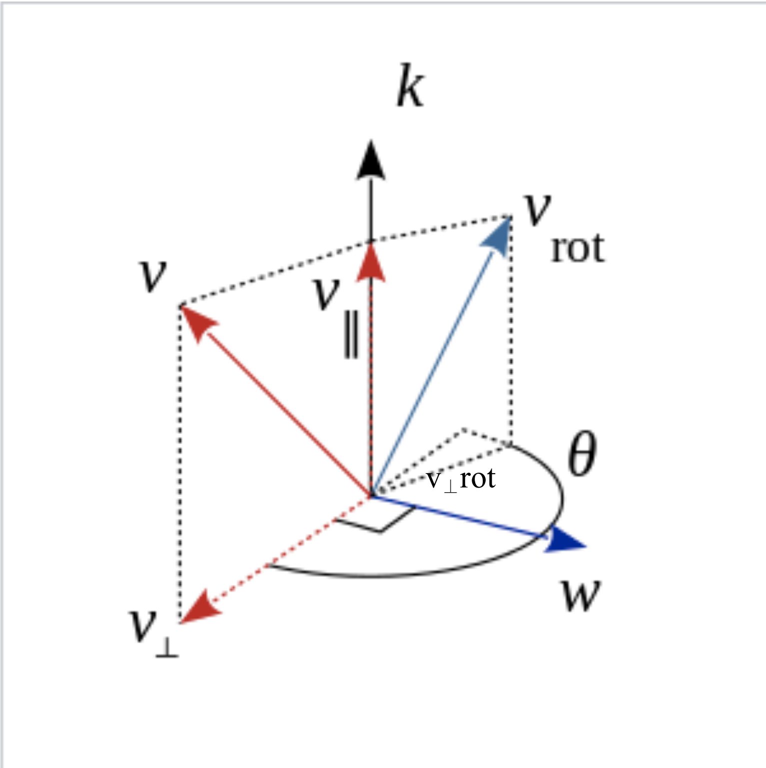
$$q2^{\oplus} q1 = \begin{bmatrix} \eta 2 & z2 & -y2 & x2 \\ -z2 & \eta 2 & x2 & y2 \\ y2 & -x2 & \eta 2 & z2 \\ -x2 & -y2 & -z2 & \eta 2 \end{bmatrix} \begin{bmatrix} x1 \\ y1 \\ z1 \\ \eta 1 \end{bmatrix} = \begin{bmatrix} \eta 2x1 + z2y1 - y2z1 + x2\eta 1 \\ -z2x1 + \eta 2y1 + x2z1 + y2\eta 1 \\ y2x1 - x2y1 + \eta 2z1 + z2\eta 1 \\ -x2x1 - y2y1 - z2z1 + \eta 2\eta 1 \end{bmatrix}$$

$$q1q2 = \begin{bmatrix} \eta 2x1 + z2y1 - y2z1 + x2\eta 1 \\ -z2x1 + \eta 2y1 + x2z1 + y2\eta 1 \\ y2x1 - x2y1 + \eta 2z1 + z2\eta 1 \\ -x2x1 - y2y1 - z2z1 + \eta 2\eta 1 \end{bmatrix}$$

where q1, q2 are both vectors of quaternions.

Q3 见代码

5.



Rodrigues' rotation formula rotates **v** by an angle *θ* around vector *k* by decomposing it into its components parallel and perpendicular to *k*, and rotating only the perpendicular component.

Assume k to be a unit vector of the rotation axis. Let vector v be any vector to rotate about k by angle θ.

The vector can be decomposed into components parallel and perpendicular to the axis k.

$$v = v_{||} + v_{\perp}$$

where

$$v_{||} = (v \cdot k)k$$

$$v_{\perp} = v - v_{||} = v - (v \cdot k)k = -k \times (k \times v)$$

$$k \times v = w$$

The component parallel to the rotation axis will not change after the rotation (both magnitude and rotation)

$$v_{||}\text{rot} = v_{||}$$

The perpendicular component will only change the direction (maintain the magnitude)

$$| v_{\perp}\text{rot} | = | v_{\perp} |$$

The angle between $v_\perp rot$ and $v_\perp$ is θ. Therefore, perpendicular component can be decomposed to

$$v_\perp rot = \cos\theta\, v_\perp + \sin\theta\, w$$

($v_\perp$ is normal to w, like 2D coordinate)

$$w = k \times v_\perp = k \times v$$

Then,

$$\begin{aligned} vrot &= v_{||} rot + v_\perp rot \\ &= v_{||} + \cos\theta(v - v_{||}) + \sin\theta\, k \times v \\ &= \cos\theta\, v + (1 - \cos\theta)\, v_{||} + \sin\theta\, k \times v \\ &= \cos\theta\, v + (1 - \cos\theta)\,(v \bullet k)k + \sin\theta\, k \times v \end{aligned}$$

According to the definition of the cross product, k × v can be represented as

$$\begin{bmatrix} (k\times v)x \\ (k\times v)y \\ (k\times v)z \end{bmatrix} = \begin{bmatrix} kyvz - kzvy \\ kzvx - kxvz \\ kxvy - kyvx \end{bmatrix} = \begin{bmatrix} 0 & -kz & ky \\ kz & 0 & -kx \\ -ky & kx & 0 \end{bmatrix} \begin{bmatrix} vx \\ vy \\ vz \end{bmatrix}$$

Let K denote the cross-product matrix.

$$K = \begin{bmatrix} 0 & -kz & ky \\ kz & 0 & -kx \\ -ky & kx & 0 \end{bmatrix}$$

Therefore,

$$Kv = k \times v$$

Similarly,

$$k \times (k \times v) = K(k \times v)$$

According to $v_\perp = v - v_{||} = -k \times (k \times v)$,

$$v_{||} = v - v_\perp = v + k \times (k \times v)$$

Then

$$\begin{aligned} vrot &= \cos\theta\, v + (1 - \cos\theta)v_{||} + \sin\theta Kv \\ &= \cos\theta\, v + (1 - \cos\theta)(v+K^2v) + \sin\theta Kv \\ &= v + K^2v - \cos\theta K^2 v + \sin\theta Kv \end{aligned}$$

That is,

$$Rv = v + (1 - \cos\theta)K^2 v + \sin\theta Kv$$

Therefore,

$$R = I + (1 - \cos\theta)\, K^2 + \sin\theta\, K$$

K is the skew-symmetric matrix of k

According to

$$(v \cdot k) = \begin{bmatrix} vx \\ vy \\ vz \end{bmatrix} \cdot \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix} = vxkx + vyky + vzkz$$

$$(v \cdot k)k = kk^T v,$$

we can calculate that,

$$R = \cos\theta\, I + (1 - \cos\theta)\, kk^T + \sin\theta\, k\hat{}$$

6.四元数运算性质的验证

Use the quaternion to represent a 3D point (x,y,z) as

$$p = [0, x, y, z] = [0, \mathbf{v}]$$

$$\text{where } v = [x, y, z]^T$$

Use unit vector $\mathbf{n} = [n1, n2, n3]^T$ and θ to represent the rotation axis and angel respectively.

Then, the rotation can be represented using a quaternion

$$q = [\cos\frac{\theta}{2}, n1 \sin\frac{\theta}{2}, n2\ \sin\frac{\theta}{2}, n3\sin\frac{\theta}{2}]$$

$$q^{-1} = q^* / ||q||^2$$

where $||q||^2 = 1$.

Then,

$$q^{-1} = q^* = [\cos\frac{\theta}{2}, -\mathbf{n}\sin\frac{\theta}{2}]$$

Let $r = qp = [\cos\frac{\theta}{2}, n1 \sin\frac{\theta}{2}, n2\ \sin\frac{\theta}{2}, n3\sin\frac{\theta}{2}]\ [0, x, y, z]$

$$= \begin{bmatrix} 0 - (n1x + n2y + n3z)\ \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} x + (n2z - n3y)\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} y + (n3x - n1z)\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} z + (n1y - n2x)\sin\frac{\theta}{2} \end{bmatrix}^T = [-\sin\frac{\theta}{2}\mathbf{n}^T\mathbf{v}, \cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n} \times \mathbf{v}]$$

$$rq^{-1} = [-\sin\frac{\theta}{2}\mathbf{n}^T\mathbf{v}, \cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n} \times \mathbf{v}]\ [\cos\frac{\theta}{2}, -\mathbf{n}\sin\frac{\theta}{2}]$$

$$= [-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathbf{n}^T\mathbf{v} + (\cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n} \times \mathbf{v})^T\mathbf{n}\sin\frac{\theta}{2},$$
$$\sin\frac{\theta}{2}\mathbf{n}^T\mathbf{v}\,\mathbf{n}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}(\cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n}\mathbf{v}) - (\cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n} \times \mathbf{v}) \times \mathbf{n}\sin\frac{\theta}{2}]$$

$$= [(\mathbf{n} \times \mathbf{v})^T\mathbf{n}\sin^2\frac{\theta}{2},$$
$$\sin\frac{\theta}{2}\mathbf{n}^T\mathbf{v}\,\mathbf{n}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}(\cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n}\times\mathbf{v}) - (\cos\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\mathbf{n} \times \mathbf{v}) \times \mathbf{n}\sin\frac{\theta}{2}]$$

where $(\mathbf{n} \times \mathbf{v})^T$ represent the transpose of the vector normal to **n** and **v.**

Therefore,

$$(\mathbf{n} \times \mathbf{v})^T \mathbf{n} = 0$$

$$p' = qpq^{-1} = q^+p^+q^{-1} = q^+q^{-1\oplus}p$$

$$q = [n1 \sin\frac{\theta}{2}, n2\ \sin\frac{\theta}{2}, n3\sin\frac{\theta}{2}, \cos\frac{\theta}{2}] = [\sin\frac{\theta}{2}\mathbf{n}, \cos\frac{\theta}{2}]$$

$$q^{-1} = q^* = [-n1 \sin\frac{\theta}{2}, -n2\ \sin\frac{\theta}{2}, -n3\, sin\frac{\theta}{2}, \cos\frac{\theta}{2}] = [-\sin\frac{\theta}{2}\mathbf{n}, \cos\frac{\theta}{2}]$$

$$q^+ = \begin{bmatrix} \eta I + \varepsilon^{x} & \varepsilon \\ -\varepsilon^T & \eta \end{bmatrix}$$

$$q^{-1\oplus} = \begin{bmatrix} \eta I + \varepsilon^{x} & -\varepsilon \\ \varepsilon^T & \eta \end{bmatrix}$$

$$q^+q^{-1\oplus} = \begin{bmatrix} \eta^2 I + (\varepsilon^x)^2 + \eta I\varepsilon^x + \varepsilon^x\eta I + \varepsilon\varepsilon^T & \eta I\varepsilon + \varepsilon^x\varepsilon + \varepsilon\eta \\ -\varepsilon^T\eta I + \varepsilon^T\varepsilon^x - \eta\varepsilon^T & -\varepsilon^T\varepsilon^x + \eta^2 \end{bmatrix}$$

$$\varepsilon^x = \sin\frac{\theta}{2}\begin{bmatrix} 0 & -n3 & n2 \\ n3 & 0 & -n1 \\ -n2 & n1 & 0 \end{bmatrix}$$

$$\varepsilon^x\varepsilon^x = \sin^2\frac{\theta}{2}\begin{bmatrix} 0 & -n3 & n2 \\ n3 & 0 & -n1 \\ -n2 & n1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -n3 & n2 \\ n3 & 0 & -n1 \\ -n2 & n1 & 0 \end{bmatrix}$$

$$= \sin^2\frac{\theta}{2}\begin{bmatrix} -n3^2 - n2^2 & n2n1 & n3n1 \\ n1n2 & -n3^2 - n1^2 & n3n2 \\ n1n3 & n2n3 & -n2^2 - n1^2 \end{bmatrix}$$

$$\varepsilon\varepsilon^T = \sin^2\frac{\theta}{2}\begin{bmatrix} n1^2 & n2n1 & n3n1 \\ n1n2 & n2^2 & n3n2 \\ n1n3 & n2n3 & n3^2 \end{bmatrix}$$

$$\eta I\varepsilon^x + \varepsilon^x\eta I = \sin\theta\begin{bmatrix} 0 & -n3 & n2 \\ n3 & 0 & -n1 \\ -n2 & n1 & 0 \end{bmatrix}$$

$$R = \mathrm{im}(q^+q^{-1\oplus}) = \eta^2 I + (\varepsilon^x)^2 + \eta I\varepsilon^x + \varepsilon^x\eta I + \varepsilon\varepsilon^T$$

$$=\cos^2\frac{\theta}{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+$$

$$\sin^2\frac{\theta}{2}\begin{bmatrix}-n3^2-n2^2+n1^2 & 2n2n1 & 2n3n1\\ 2n1n2 & -n3^2-n1^2+n2^2 & 2n3n2\\ 2n1n3 & 2n2n3 & -n2^2-n1^2+n3^2\end{bmatrix}$$

$$+\sin\theta\begin{bmatrix}0&-n3&n2\\n3&0&-n1\\-n2&n1&0\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}(-n3^2-n2^2+n1^2) & 2\sin^2\frac{\theta}{2}n2n1-\mathrm{n3}\sin\theta & 2\sin^2\frac{\theta}{2}n3n1+\mathrm{n2}\sin\theta\\ \mathrm{n3}\sin\theta+\sin^2\frac{\theta}{2}2n1n2 & \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+(-n3^2-n1^2+n2^2) & -\mathrm{n1}\sin\theta+\sin^2\frac{\theta}{2}2n3n2\\ -\mathrm{n2}\sin\theta+\sin^2\frac{\theta}{2}2n1n3 & \mathrm{n1}\sin\theta+\sin^2\frac{\theta}{2}2n2n3 & \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}(-n2^2-n1^2+n3^2)\end{bmatrix}$$

$$=\begin{bmatrix}1-2\sin^2\frac{\theta}{2}n2^2-2\sin^2\frac{\theta}{2}n3^2 & 2\sin^2\frac{\theta}{2}n2n1-\mathrm{n3}\sin\theta & 2\sin^2\frac{\theta}{2}n3n1+\mathrm{n2}\sin\theta\\ \mathrm{n3}\sin\theta+2\sin^2\frac{\theta}{2}n1n2 & 1-2\sin^2\frac{\theta}{2}n1^2-2\sin^2\frac{\theta}{2}n3^2 & -\mathrm{n1}\sin\theta+2\sin^2\frac{\theta}{2}n3n2\\ -\mathrm{n2}\sin\theta+2\sin^2\frac{\theta}{2}n1n3 & \mathrm{n1}\sin\theta+2\sin^2\frac{\theta}{2}n2n3 & 1-2\sin^2\frac{\theta}{2}n1^2-2\sin^2\frac{\theta}{2}n2^2\end{bmatrix}$$

Therefore, for q = q0 + q1i + q2j + q3k

$$R=\begin{bmatrix}1-2q2^2-2q3^2 & 2q1q2-2q0q3 & 2\mathrm{q1q3}+2\mathrm{q0q2}\\ 2\mathrm{q1q2}+2\mathrm{q0q3} & 1-2q1^2-2q3^2 & 2\mathrm{q2q3}-2\mathrm{q0q1}\\ 2\mathrm{q1q3}-2\mathrm{q0q2} & 2\mathrm{q2q3}+2\mathrm{q0q1} & 1-2q1^2-2q2^2\end{bmatrix}$$

7. 熟悉 C++11

1.for(auto& a: avec) cout<<a.index<<" ";

使用了简化 for 循环 可以用于遍历数组,容器,string 以及由 begin 和 end 函数定义的序列(即有 Iterator)

2. 上述代码 使用了新的关键词 auto 为了自动类型推导

3. std::sort(avec.begin(), avec.end(), [](const A&a1, const A&a2) {return a1.index<a2.index;});

使用了 Lambda 表达式 它可以用于创建并定义匿名的函数对象,以简化编程工作。