Problem A

Suppose we have n vertices labeled $1, 2, \ldots, n$ arranged as a line. Consider a robot that can hop from one vertex to another. Suppose that hopping from i to j requires *energy* equal to $E(i, j) = \frac{(i-j)^2}{3}$ (thus longer jumps take significantly higher amount of energy).

The goal is to go from vertex 1 to n using the minimum total energy. Let $ans(i)$ denote the minimum energy required to go from vertex i to n.

Question 1 (4 points). Consider the recurrence:

$ans(i) = \min_{j>i} E(i, j) + ans(j)$

for computing the desired answer. Consider a recursive procedure that implements the above recurrence. What is its running time? Next, suppose we remember the answers once computed (and look up instead of making a recursive call). What is the running time of this new procedure?

(a) $O(n^2)$ and $O(n)$ respectively

(b) $\exp(n)$ and $O(n)$ respectively

(c) $O(n^2)$ and $O(n^2)$ respectively

(d) $\exp(n)$ and $O(n^2)$ respectively

In the provided box, give the answer (a-d) along with *one line* of explanation. (No explanation will yield 3/4 points.)

Question 2. (4 points). In the recurrence above, suppose instead of searching over all $j > i$, we restrict the search to $j \in [i + 1, i + r]$. I.e., the recurrence is:

$ans(i) = \min_{j \in [i+1, i+r]} E(i, j) + ans(j)$

What is the smallest value of r for which this yields a correct answer? Explain in a couple of sentences. What is the running time of the new procedure?

Question 3 (2 points). Can you view the problem above as that of finding the shortest path in an appropriate graph? Answer with a couple of lines of explanation (no need to be very formal).

Problem B

Suppose $G = (V, E)$ is an undirected graph. We consider the *coloring* problem, where we are given a set of colors $\{1, 2, \ldots, k\}$, and the goal is to assign colors to vertices so as to minimize the number of "bad edges". An edge ij is said to be bad if the two end points receive the same color.

Question 4 (4 points). Consider a *random* coloring, i.e., to every vertex, we assign a random color from $\{1, 2, \ldots, k\}$. What is the expected number of bad edges? (You must write down the answer along with a clear description of how you arrived at it. (*Hint:* Define appropriate random variable, express it as sum of "binary" random variables, ...)

Question 5 (4 points). Which of the following is an upper bound on the probability that the number of bad edges (strictly) exceeds $\frac{4|E|}{k}$?

(a) 0 (b) 0.25 (c) 0.5 (d) 1

Say what all apply, and give one line of explanation.

Problem C

Suppose we have a set X of n points divided into k equal sized clusters. Now suppose we sample $2k$ points uniformly at random from X, with replacement. We say that a cluster is *unlucky* if none of its points is selected in the sample.

Question 6 (4 points). What is the expected number of unlucky clusters?

(a) 0 (b) $O(\sqrt{k})$ (c) $\Theta(k)$ (d) None of the above

Select an answer and provide a couple of lines of reasoning. Not providing any reasoning will yield 2/4 points. *Hint:* you may use $\left(1 - \frac{1}{k}\right)^k \approx \frac{1}{e}$.

Problem D

Recall the optimization formulation we saw for the Set Cover problem: we have a set of n candidates and a set of m skills. For each candidate, we are given $S_i \subseteq [m]$ (where $[m]$ is shorthand for $\{1, 2, \ldots, m\}$), which is the set of skills that candidate i possesses. The goal is to choose the fewest number of candidates to hire, while satisfying the requirement that for each skill, at least one person with that skill is hired.

Question 7 (4 points). Consider a variant of the problem, where each candidate has a minimum "wage requirement". I.e., let $w_1, w_2, \ldots, w_n$ be positive integers such that hiring person i costs w_i. How would you modify the optimization formulation so as to incorporate this? (We now wish to minimize the total wages.) You may state **only the differences** between the new formulation and the one we saw in class.

Question 8 (4 points). Let us go back to the standard setting (without w_i's). Consider an instance in which $m = n = 4$ (we have 4 candidates and 4 skills of interest). Suppose that for $1 \leq i \leq 4$, candidate i possesses all but the ith skill.

For this instance, what is the best possible value of the set cover objective?

Now, consider the linear programming *relaxation* for Set Cover for this instance:

minimize $x_1 + x_2 + x_3 + x_4$ subject to: $x_2 + x_3 + x_4 \geq 1$ $x_3 + x_4 + x_1 \geq 1$ $x_4 + x_1 + x_2 \geq 1$ $x_1 + x_2 + x_3 \geq 1$ $0 \leq x_i \leq 1$ for $i = 1, 2, 3, 4$.

Construct a feasible solution to the relaxation with objective value < 1.4. (Slightly larger values can fetch partial credit.) [You just need to write down the x_i values, no need for explanation.]